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ADVANCED ANALYTIC GEOMETRY

BY

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PREFACE

The purpose of this book is to introduce the student to the analytic side of projective geometry. It leads him from the concepts and methods of elementary analytic geometry to the ideas of cross-ratio, triangle of reference, general projective transformation, etc. Very little mention is made of solid analytic geometry. Being introductory in nature, this book does not aim to give complete discussions of transformation groups and subgroups, of invariants, or of other advanced topics.

This book is supposed to follow a first course in plane analytic geometry, but it contains material for a more advanced course also. A semester course in general plane analytic projective geometry could be taught from this textbook by taking a few sections here and there in the first eight chapters and then going into the second half of the book. On the other hand, a more elementary course in advanced analytic geometry could be given by using the first half of the book almost exclusively.

There are references in the second half of the book to the following sections in the first half: §§3, 4, 9, 11, 12, 13, 14, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 36, 37, 42, 43, 44, 45, 46, 49, 50, 51, 56, 58, 60, 62, 63. Most of these references, however, are brief and can be looked up by the student when he comes to them in the text. In order to give a more advanced semester course using the second half of the book the teacher would probably have to assign for a hasty reading all or parts of the following sections from the first half: §§6, 7, 8, 9, 11, 12, 13, 14, 18, 21, 23, 24, 29, 30, 31, 35, 40, 50. The first five of these sections could be read before undertaking §64, and the rest could be taken parallel with the text beyond §64.

The author wishes to express his gratitude to Professor R. D. Carmichael for reading the manuscript and offering many valuable suggestions. The author wishes also to express his thanks to Mr. Joseph J. Eachus and to Mr. John E. Hart for their assistance with the manuscript and with the proof.

A. D. C.

SYRACUSE UNIVERSITY
May, 1938

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ADVANCED ANALYTIC GEOMETRY

PART I

INTRODUCTION TO AFFINE PLANE ANALYTIC PROJECTIVE GEOMETRY

CHAPTER I

VARIOUS COORDINATE SYSTEMS

1, 2. Frames of reference. DEFINITION. Any collection of points and lines (or other curves) that is made use of to set up a system of coordinates for the points of a plane is called a *frame of reference* for this plane.

A pair of rectangular or oblique axes, their point of intersection (0,0), and the two so-called *unit points* on the axes, namely (0,1) and (1,0), constitute such a frame of reference. The pole, the unit angle, the initial line, and the point (1,0) on this line form a frame of reference for polar coordinates in the plane.

For the sake of brevity we shall designate as the *ordinary* frame of reference in a plane the frame consisting of a pair of rectangular axes with the same-sized unit on each axis.

The formulas for the distance between two points, for the angle between two lines, for the area of a triangle, also the equation of a circle, and many other formulas and equations in analytic geometry *presuppose* the use of an *ordinary* frame of reference.

We shall call the unit on the x-axis the *x-unit*, and the unit on the y-axis the *y-unit*. In an ordinary frame of reference we take the x- and y-units of the same size. In fact, even when using oblique axes, more often than not we take (0,1) and (1,0) each at the same distance from the origin.

However, let us note that we sometimes use units of different sizes. For example, to draw the curve $y = 100\,x^2$ we should probably take the y-unit one-tenth the size of the x-unit.

In analytic projective geometry, on the other hand, we use either oblique or rectangular axes; and we take any two arbitrary points on these axes (neither point at the origin) as (1,0) and (0,1). We sometimes take an arbitrary point not on the axes as (1,1), and from this point determine (1,0) and (0,1) by lines parallel to the axes.

Also we free ourselves from the conventions of always labeling the horizontal axis as the x-axis and the vertical as the y-axis, and always taking the positive direction on the horizontal axis to the right and on the vertical axis upward. We note that in the calculus we sometimes vary these directions. Thus in the study of a falling body and of hydraulic pressure, the x-axis is often taken vertical with positive direction downward.

Later on (see §70), we shall introduce a still more general frame of reference for the plane called a *triangle of reference.* It will be seen that *any* triangle ABC can be used as a triangle of reference; and that if we join any point P in the plane to the vertices A and B of this triangle, the lines AP and BP will cut the sides CB and CA respectively in points whose coordinates are assigned to P. (Compare this with the fact that for rectangular or oblique axes we can determine the abscissa of a point P by drawing through P a line m parallel to the y-axis and finding where m cuts the x-axis; similarly we can determine the ordinate of P by drawing through P a line l parallel to the x-axis.)

Rectangular and oblique axes will turn out to be only special cases of a triangle of reference.

Just as in a plane, so on a line and in space, when we assign coordinates to points we use a frame of reference. For example, when we attach to the points P_1, P_2, P_3, $\cdots$ of a line l the coordinates x_1, x_2, x_3, $\cdots$ respectively, we utilize a frame of reference composed of *any two distinct points* on l to which we give the coordinates 0 and 1.

EXERCISES

1. Plot the points $(-7, 6)$ and $(3, -5)$ referred to oblique axes with different-sized units and with the positive directions on the x-axis to the left and on the y-axis downward.

2. Draw a figure and show that the formula for the coordinates of a point dividing a line-segment in a given ratio does not require rectangular axes or even the same length of unit for each axis. Hint: Note the derivation of this formula that is given in Fine and Thompson's " Coordinate Geometry."

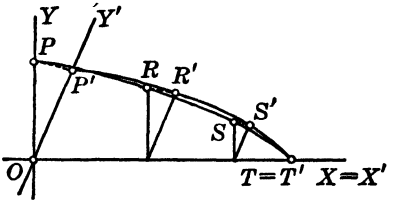
3. Look through a book on elementary analytic geometry and find what formulas and equations do not require an ordinary frame of reference.

4. Draw the following curves referred to rectangular axes with the y-unit one-half the size of the x-unit:

$$x^2 + y^2 = 1, \quad x^2 + \frac{y^2}{4} = 1, \quad y = x$$

3. **Oblique axes.** In order to free ourselves from the habit of looking upon the coordinates x and y as referred always to an ordinary frame of reference, we shall consider more in detail the use of oblique axes. We shall term as *ordinary oblique axes* those with x- and y-units of equal length.

We note first of all a simple way to draw curves referred to oblique axes by drawing them first referred to rectangular axes (as in the adjoining figure) having the same origin and x-axis, and then using a compass to lay off the oblique ordinate corresponding to each abscissa. (In this figure the dotted circular arcs are to represent the path of the compass while constructing the primed points from the unprimed.)



In oblique coordinates, $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ no longer gives the distance between two points, $\frac{m_1 - m_2}{1 + m_1 m_2}$ is not the formula for the tangent of the angle between two lines, $y = m_1 x + b_1$ and $y = m_2 x + b_2$, m_1 and m_2 are not the slopes* of these two lines, dy/dx is not the slope* of a tangent to a curve,

$$x' \cos \alpha + y' \sin \alpha - p$$

is not the distance from a line to a point, $x^2 + y^2 = r^2$ is not the equation of a circle. (We might prolong this list indefinitely.)

However, $x/a + y/b = 1$ is still the intercept form for the equation of a line in oblique coordinates, $x = \frac{k_1 x_2 + k_2 x_1}{k_1 + k_2}$ and $y = \frac{k_1 y_2 + k_2 y_1}{k_1 + k_2}$ are (just as in rectangular coordinates) the coordinates of the point dividing the segment of the line from

* By the slope of a line we mean here the tangent of the angle the line makes with the positive direction of the x-axis.

(x_1,y_1) to (x_2,y_2) in the ratio k_1/k_2, a first-degree equation still gives a straight line and a second-degree equation a conic — and all these facts are still true even for the case when the units on the two axes are not of the same length. (We leave the proofs of these facts for the reader to discuss, in the exercises.)

EXERCISES

1. Verify the statements made in the last paragraph of the text. Hint: Look up the derivations of these formulas and equations in an elementary textbook.

2. Prove that the slope of a line $y = mx + b$ referred to ordinary oblique axes is $m' = \dfrac{m \sin \omega}{1 + m \cos \omega}$, where ω is the angle between the two axes. Hint: Draw a figure showing this line as joining two points $P_1(x_1, mx_1 + b)$ and $P_2(x_2, mx_2 + b)$, then draw a right triangle with P_1P_2 as its hypotenuse and with its base parallel to the x-axis; from this right triangle find m'.

3. Find the tangent of the angle between two lines referred to ordinary oblique axes.

4. Derive the distance formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2 \cos \omega \, (x_1 - x_2) \, (y_1 - y_2)}$$

for ordinary oblique axes, where ω is the angle between the two axes. Hint: Use the figure of Ex. 2, or use the trigonometric formula for the length of a side of an oblique triangle.

5. Prove that a circle referred to ordinary oblique axes has an equation of the form

$$x^2 + y^2 + 2xy \cos \omega + 2ax + 2by + c = 0$$

Hint: Use Ex. 4.

6. Derive the formula

$$A = \tfrac{1}{2} \sin \omega \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

for the area of a triangle (x_1,y_1), (x_2,y_2), (x_3,y_3) referred to ordinary oblique axes with ω as the angle between the two axes.

7. Draw roughly the curves

$$y^2 = 4x, \quad xy = 1, \quad x^2 - y^2 = 1, \quad y^2 = x^3$$

referred to ordinary oblique axes, using the hint in the text.

8. Show that the formulas and equations of Exs. 2, 3, 4, 5, 6 reduce to the corresponding formulas and equations for an ordinary frame of reference if we put $\omega = \pi/2$. (This illustrates how rectangular axes may be looked upon as a special case of oblique axes. Note how much simpler all such formulas and equations for an ordinary frame of reference are. This, of

course, is one reason why we habitually use rectangular axes. Can you think of other reasons for preferring an ordinary frame of reference?)

9. In the triangle (1,3), (−2,−4), (1,−2), where $\omega = 60°$, find the equations of the sides and the lengths of these sides.

10. If $\omega = \pi/4$, find the equation of the line with y-intercept 2 and with slope 3, also the equation of the line through (1,1) and perpendicular to $y = 2x$.

11. Find the equation of the parabola with focus $(p,0)$ and directrix $x = -p$, where the coordinates are referred to ordinary oblique axes with angle ω.

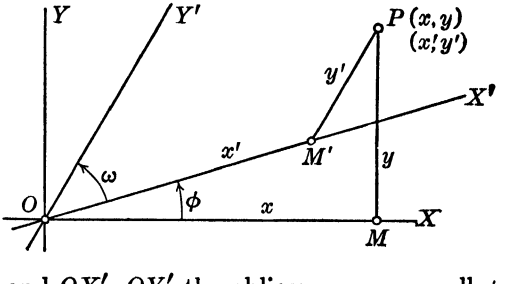
12. Using the foci $F_1(ae,0)$ and $F_2(-ae,0)$ where $ae = \sqrt{a^2 - b^2}$, also using the definition of an ellipse as the locus of a point P such that $F_1P + F_2P = 2a$, derive the equation of this ellipse referred to ordinary oblique axes with angle ω.

13. Taking $ae = \sqrt{a^2 + b^2}$ and $F_1P - F_2P = 2a$, do the same for a hyperbola as in Ex. 12 for an ellipse.

14. Show that for $\omega = \pi/2$ the equations of Exs. 11, 12, and 13 reduce to the typical equations of these conics that were derived in elementary analytic geometry.

4. Transformations from rectangular to oblique axes. In Exs. 2, 3, 4, 5, and 6 of §3 we see that the equations of some well-known geometric loci and the formulas that give some familiar geometric properties are quite different when the coordinates refer to oblique axes from what they are when an ordinary frame of reference is used. We also note that some formulas and equations are the same both for an ordinary frame of reference and for ordinary oblique axes.

We can discuss the question of such differences that arise when using various kinds of axes most readily by considering how we must *modify formulas* and *equations* when going from *rectangular* to *oblique* coordinates. Therefore we want to derive the equations that effect such a change of coordinates.



We suppose first of all that the rectangular and oblique axes have the same origin. (See the adjoining figure.) For the case where the origins are distinct, see Ex. 13 in §6. We let OX, OY be the rectangular axes and OX', OY' the oblique axes, we call ϕ

the angle XOX' and ω the angle $X'OY'$; then the angle $Y'OY$ is $\pi/2 - \omega - \phi$, whether or not $\omega + \phi < \pi/2$. We take any point P with coordinates x,y and x',y', we label the foot of each of the ordinates y and y' respectively M and M'. Projecting the broken line $OM'P$ onto OX and then onto MP, we get

$$(1) \quad x = x' \cos \phi + y' \cos (\omega + \phi), \quad y = x' \sin \phi + y' \sin (\omega + \phi)$$

Solving (1) for x' and y' in terms of x and y, we obtain

$$(1') \quad x' = x \frac{\sin (\omega + \phi)}{\sin \omega} - y \frac{\cos (\omega + \phi)}{\sin \omega},$$

$$y' = -x \frac{\sin \phi}{\sin \omega} + y \frac{\cos \phi}{\sin \omega}$$

We note that if OX' is the same line as OX with the same positive direction, then $\phi = 0$, and the above transformation to oblique coordinates reduces to

$$(2) \quad x = x' + y' \cos \omega, \quad y = y' \sin \omega$$

$$(2') \quad x' = x - y \cot \omega, \quad y' = y \csc \omega$$

(See also Ex. 7 in the exercises.)

DEFINITION. We call (1') and (2') the *inverses* of (1) and (2), respectively. We also call (1) and (2) the *inverses* of (1') and (2').

These inverse transformations change *back* from *oblique* to *rectangular* coordinates (or vice versa). Thus if we perform (1) on the circle $x^2 + y^2 = r^2$, we obtain the equation $x'^2 + y'^2 + 2\,x'y' \cos \omega = r^2$; and if we perform (1') on the second equation, we come back to the first.

Equations (1) and (2) are useful for transforming a *formula* or *equation* from rectangular to oblique coordinates, but (1') and (2') are better adapted than (1) and (2), respectively, to finding the oblique coordinates of a *point* given in rectangular coordinates. For example, if $\omega = \phi = 30°$, then (1') becomes $x' = \sqrt{3}\,x - y$, $y' = -x + \sqrt{3}\,y$, and the point $(\sqrt{3},\sqrt{3})$ referred to an ordinary frame of reference has now the oblique coordinates $(3 - \sqrt{3}, -\sqrt{3} + 3)$.

The transformations (1), (1'), (2), (2') are all of the first degree

in the variables x, y, x', y' and so cannot raise the degree* of the equation of a locus. Also no one of these transformations can lower the degree* of any such equation, for then its inverse would have to raise the degree of the transformed equation in order to return us to the original equation of the locus. This argument proves the following

THEOREM. *In oblique coordinates, as well as in rectangular, every line has a first-degree equation and conversely every first-degree equation is the equation of a line; every conic has a second-degree equation and conversely every second-degree equation is the equation of a conic.* (See also Ex. 6 in §6.)

ILLUSTRATIVE EXAMPLE. If we apply (2) to the formula

$$A = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

for the area of a triangle referred to an ordinary frame of reference we get

$$A = \tfrac{1}{2}\begin{vmatrix} x_1' + y_1' \cos\omega & y_1' \sin\omega & 1 \\ x_2' + y_2' \cos\omega & y_2' \sin\omega & 1 \\ x_3' + y_3' \cos\omega & y_3' \sin\omega & 1 \end{vmatrix} = \tfrac{1}{2}\sin\omega \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}$$

which is the formula for the area of a triangle referred to ordinary oblique axes. (To simplify this determinant we remove the factor $\sin\omega$ from the second column, then multiply the second column by $\cos\omega$ and subtract it from the first.)

EXERCISES

1. Show that the derivation of (1) is valid even for $\omega + \phi > \pi/2$.
2. Obtain (1′) from (1) and (2′) from (2), carrying through the details omitted in the text.
3. Show that, for $\omega = \pi/2$, equations (1) and (1′) reduce to the ordinary rotation of axes and its inverse. Compare (7) and (7′) in §6.
4. Find the forms that (1) and (1′) take if $\phi = \pi$, i.e., if OX' is the same line as OX only with a different positive direction.
5. The points (1,0), (0,1), (1,1), (3,5), (−6,0) are referred to rectangular axes. Find the coordinates of these points referred to oblique axes that are obtained from the rectangular axes by (1) with $\omega = \phi = 30°$.
6. If the points of Ex. 5 are given as referred to the oblique axes of this example, find their coordinates referred to the rectangular axes.
7. Derive the transformation (2) directly from a figure.

* We remind the student that the *degree* of the equation of a locus $ax^{\alpha_1}y^{\beta_1} + bx^{\alpha_2}y^{\beta_2} + \cdots = 0$ means the *largest* value of the numbers $d_1 = \alpha_1 + \beta_1$, $d_2 = \alpha_2 + \beta_2$, etc. Thus $xy = 1$ is a second-degree equation; $y^2x = x^2 - 2$, a third-degree equation.

8. Obtain the results in Exs. 2, 3, 4, 5, 6, 11, 12, and 13 of §3 by using the transformation (2) on the corresponding equations and formulas for an ordinary frame of reference.

9. Subject the equations $y^2 = 4x$, $xy = 1$, $x^2 - y^2 = 1$ to the transformation (1) with $\omega = \phi = \pi/4$.

10. In ordinary oblique* coordinates with $\omega = 60°$ find the tangent of the angle between the two lines $y = x + 2$ and $y = -x + 2$; find the distance from (1,1) to (2,2); find the area of the triangle with vertices (0,0), 1,1), (3,4): find the equation of the circle through the three points (0,0) (1,0), (0,1). Note that we here use unprimed variables for oblique coordinates.

11. Find the center and radius of the circle in Ex. 10, also of the general circle in Ex. 5 of §3.

12. If $\omega = 45°$, find the equation of the circle with center (1,1) and radius $\sqrt{2}$.

13. If $\omega = 120°$, draw $y = x^3$ and $y = x^4$. See note in Ex. 10.

14. Derive the formula in ordinary oblique coordinates for the perpendicular distance from a line $ax + by + c = 0$ to a point $P'(x',y')$. Hint: Use (2′) on this line, apply the corresponding formula for rectangular axes to this transformed equation of the line, then apply (2) to the result to return to oblique coordinates. Note that x,y now are oblique coordinates (compare Ex. 10), so in (2′) and (2) we should replace $x'y'$ by x,y and x,y by some such variables as x'',y'' before we use these transformations in this problem. Check your answer by putting $\omega = \pi/2$ in the formula you obtain and seeing if it then reduces to

$$d = \frac{ax' + by' + c}{\pm\sqrt{a^2 + b^2}}$$

Why does this check the answer?

15. If $\phi = 30°$, $\omega = 60°$, change $3x^2 - y^2 = 1$ to oblique coordinates by means of (1). Interpret your result geometrically. Also change $x'y' = 1$ to rectangular axes by (1′) with $\phi = 30°$, $\omega = 60°$.

16. Consider the hyperbola $x^2/16 - y^2/9 = 1$ with asymptotes $x/4 \pm y/3 = 0$. Using these asymptotes as x'- and y'-axes, find $\sin\phi$, $\cos\phi$, $\sin(\omega + \phi)$, $\cos(\omega + \phi)$, and by means of (1) change the above equation into oblique coordinates. Note that the coefficients of (1) must satisfy the equations $\sin^2\phi + \cos^2\phi = \sin^2(\omega + \phi) + \cos^2(\omega + \phi) = 1$.

17. In rectangular coordinates the equation

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0$$

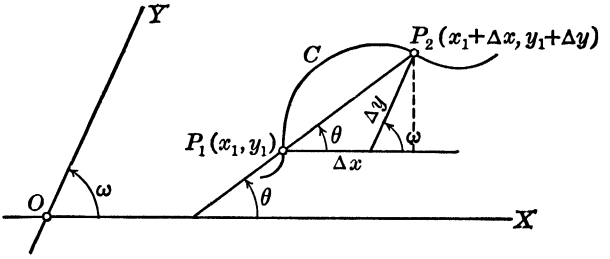
where $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$ is that of an ellipse, parabola, or hyperbola according as $h^2 - ab < 0$, $h^2 - ab = 0$, $h^2 - ab > 0$, respectively. Find the corresponding conditions for oblique coordinates. Hint: Use (2′) on this equation as in Ex. 14. Check your answer by putting $\omega = \pi/2$.

* By *ordinary oblique coordinates* we mean, of course, coordinates referred to ordinary oblique axes; and similarly for *ordinary rectangular coordinates*.

5. **Tangents to curves in oblique coordinates.** In order to find the geometrical meaning of dy/dx in ordinary oblique coordinates, we note from the figure below that

$$\tan\theta = \frac{\Delta y \sin\omega}{\Delta x + \Delta y \cos\omega}$$

where θ is the angle between the secant P_1P_2 to the curve C and the x-axis. (We use unprimed variables for oblique coordinates, as in Ex. 10 of §4.)



Dividing numerator and denominator of this fraction by Δx and letting $\Delta x \to 0$ (which makes $\Delta y \to 0$), we get the slope* of the tangent to a curve in ordinary oblique coordinates, namely

$$m = \frac{\dfrac{dy}{dx}\sin\omega}{1 + \dfrac{dy}{dx}\cos\omega}$$

(Compare Ex. 2 in §3.)

However, we show now that the *equation* of the tangent to a curve $y = f(x)$ in *oblique* coordinates at a point $P_1(x_1,y_1)$ is the *same* as in *rectangular* coordinates, namely

$$(3) \qquad y - y_1 = \left.\frac{dy}{dx}\right|_{\substack{x=x_1\\y=y_1}} (x - x_1)$$

PROOF: From the above figure we see that the equation of the secant P_1P_2 is

$$\frac{y - y_1}{y_1 - (y_1 + \Delta y)} = \frac{x - x_1}{x_1 - (x_1 + \Delta x)}$$

* Here the slope is understood as the limit of $\tan\theta$ as $\Delta x \to 0$.

since this equation is of the first degree in x and y and so must be the equation of a line (compare the theorem in §4). Also this equation is satisfied by the coordinates of P_1 and P_2 and so must be the equation of the secant P_1P_2. (Note this type of argument; it is used frequently.) Rearranging the equation of P_1P_2, we obtain $y - y_1 = (x - x_1)\,\Delta y/\Delta x$. Letting $\Delta x \to 0$ (and so $\Delta y \to 0$), we obtain the equation of the tangent to the curve $y = f(x)$ at P_1, namely (3).

EXERCISES

1. Find the equation of the tangent to the curve $y^2 = 4x$ (referred to ordinary oblique axes with $\omega = 60°$) at the point (1,2), and find the slope of this tangent.

2. Show that the equation of the tangent to the so-called *general conic*

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0 \tag{4}$$

at the point (x',y') can be arranged in the form

$$axx' + byy' + c + f(y + y') + g(x + x') + h(x'y + xy') = 0 \tag{5}$$

Compare the derivation of (5) that is given in Fine and Thompson's "Coordinate Geometry."

CHAPTER II

PRELIMINARY DISCUSSION OF LINEAR TRANSFORMATIONS

6. **Translations and rotations.** In elementary analytic geometry we studied what were called *translations* and *rotations* of axes, using an ordinary frame of reference.

DEFINITION. A *translation* is a *linear* transformation of ordinary rectangular coordinates given by equations of the form

$$x = x' + \alpha, \quad y = y' + \beta \tag{6}$$

$$x' = x - \alpha, \quad y' = y - \beta \tag{6'}$$

Note that we omit the word axes from the above definition, and compare §11. The derivation of equations (6) and (6′) is valid also for oblique axes. We leave this fact for the student to prove in the exercises.

DEFINITION. A *rotation* is a *linear* transformation of ordinary rectangular coordinates given by equations of the form

$$x = x' \cos \phi - y' \sin \phi, \quad y = x' \sin \phi + y' \cos \phi \tag{7}$$

$$x' = x \cos \phi + y \sin \phi, \quad y' = -x \sin \phi + y \cos \phi \tag{7'}$$

In the derivation of (6), α and β are the coordinates of the new origin referred to the old x- and y-axes. In the case of (7), ϕ is the angle between the x- and the x'-axes, but the origin is not changed. In Ex. 3 of §4 we see that (7) and (7′) are the special cases of (1) and (1′), respectively, where $\omega = \pi/2$.

We call attention here to the fact that such equations as (1), (1′), (2), (2′), (6), (6′), (7), and (7′) are *type forms* such that the variables* x,y or x',y' or all the variables x, y, x', y' may be replaced by other sets of variables and yet the transformations be considered as *unaltered*; but *different* values of α, β, ω, ϕ give *different* transformations. Thus $x = x' + 2$, $y = y' - 3$ is the same translation as $x'' = x''' + 2, y'' = y''' - 3$ but is a different translation from $x = x' + 3$, $y = y' - 2$.

* When no ambiguity will result we use the word *variables* as well as the word *coordinates* as the name for x and y, x' and y', etc.

In fact when we follow one transformation by another we *must* perforce replace x,y in the type form of the second transformation by x',y' (with which variables the first transformation leaves us) and replace x',y' in the type form of the second transformation by some other pair of variables, such as x'',y''.

ILLUSTRATIVE EXAMPLE. As an illustration of the last paragraph, suppose we wish to follow $x = x' + \alpha_1$, $y = y' + \beta_1$ by $x = x' + \alpha_2$, $y = y' + \beta_2$. We write the second translation as $x' = x'' + \alpha_2$, $y' = y'' + \beta_2$, then substitute these values for x',y' in the first translation, and we have

$$x = x'' + \alpha_1 + \alpha_2, \quad y = y'' + \beta_1 + \beta_2$$

This last translation can be written in the type form (6) as

$$x = x' + (\alpha_1 + \alpha_2), \quad y = y' + (\beta_1 + \beta_2) \tag{8}$$

$$x' = x - (\alpha_1 + \alpha_2), \quad y' = y - (\beta_1 + \beta_2) \tag{8'}$$

EXERCISES

1. Derive (6), (6'), (7), (7').

2. Show that the derivation of (6) and (6') is valid also for oblique axes. Compare the derivation in Fine and Thompson's " Coordinate Geometry."

3. Rotate the axes so as to rid $xy = 1$ of the xy-term.

4. Translate the axes so as to rid $x^2 + y^2 + 2ax + 2by + c = 0$ of the x- and y-terms.

5. What are the coordinates of (1,1) after subjecting the axes to (7) with $\phi = \pi/4$? A point has the coordinates (1,1) after applying (7) with $\phi = \pi/4$. What were its original coordinates?

6. Prove that (6) and (7) cannot lower or raise the degree of the equation of a locus. Compare the theorem in §1.

7. Derive the equations for the rotation of oblique axes, namely,

$$x = (x' \sin(\omega - \phi) - y' \sin\phi) \csc\omega, \quad y = (x' \sin\phi + y' \sin(\omega + \phi)) \csc\omega$$

Hint: Apply (2) to both sides of (7), simplify the results, and put them into type form. Check this answer by putting $\omega = \pi/2$.

8. Show that the relative sizes of the x- and y-units do not affect the validity of (6) but do affect the validity of (7).

9. Put (1'), (2'), (6'), and (7') into type forms resembling (1), (2), (6), and (7), respectively. Hint: Write (6') as $x' = x'' + (-\alpha)$, $y' = y'' + (-\beta)$. Then (6') has the form of (6), only with α replaced by $-\alpha$ and β by $-\beta$. We can even write (6') as

$$x = x' + (-\alpha), \quad y = y' + (-\beta)$$

if no confusion with (6) results therefrom. Similarly (7') can be put in the form (7), only with ϕ replaced by $-\phi$.

10. Derive (1) by first rotating the axes by (7) and then applying (2).

11. Derive the equations of Ex. 8 directly from a figure.

12. Look up in such a book as Fine and Thompson's " Coordinate Geometry " the reduction of the general conic (4) to a typical form by (6) and (7).

13. Obtain a general transformation from rectangular to oblique coordinates by following (6) by (7) and then by (2). Compare §4. Check your answer by putting $\omega = \pi/2$, then $\alpha = \beta = 0$.

7. **Products of transformations.** DEFINITION. A transformation obtained by following one transformation by a second transformation is called the *product* of the two transformations.

Thus (8) is the product of two translations (6) with constants α_1,β_1 and α_2,β_2, respectively. To get (8) we can take these translations in either order. Hence (8) is said to be a *commutative* (or *permutable*, or *abelian*) product of transformations.

It is easy to show (we leave this for the exercises) that the *rotations* (7) are *commutative* with one another, that is, if we perform two such rotations with angles ϕ_1 and ϕ_2 in either order, we get the same resultant of the form (7) with $\phi = \phi_1 + \phi_2$.

However, we see that a combination of a translation of the origin to a new point (α,β) followed by a rotation about this new origin through an angle ϕ has the equations

$$(9) \quad x = x' \cos\phi - y' \sin\phi + \alpha, \quad y = x' \sin\phi + y' \cos\phi + \beta$$

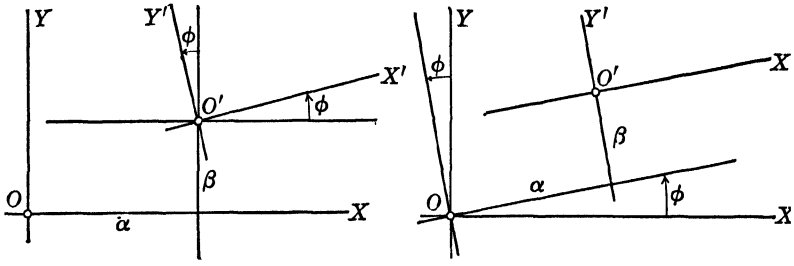
$$(9') \qquad \begin{aligned} x' &= x \cos\phi + y \sin\phi - \alpha \cos\phi + \beta \sin\phi, \\ y' &= -x \sin\phi + y \cos\phi + \alpha \sin\phi - \beta \cos\phi \end{aligned}$$

On the other hand, if we perform (7) first and then (6), we arrive at the transformation

$$(10) \qquad \begin{aligned} x &= x' \cos\phi - y' \sin\phi + \alpha \cos\phi - \beta \sin\phi, \\ y &= x' \sin\phi + y' \cos\phi + \alpha \sin\phi + \beta \cos\phi \end{aligned}$$

$$(10') \quad x' = x \cos\phi + y \sin\phi - \alpha, \quad y' = -x \sin\phi + y \cos\phi - \beta$$

Since (10), (10′) are not the same as (9), (9′), we see that it makes a difference whether we translate our axes first and then rotate them, or rotate our axes first and then translate them. Since (9) is the product of (6) by (7) and (10) is the product of (7) by (6) in the sense of the preceding paragraph, we find that the products of translations and rotations are *non-commutative* (*non-permutable* or *non-abelian*). It is evident from the following figure why (9), (9′) are not the same as (10), (10′).



EXERCISES

1. Prove analytically that translations are commutative.
2. Prove analytically that rotations around a point are commutative.
3. Derive (9), (9′), (10), (10′).
4. Show geometrically why translations are commutative, why rotations around a point are commutative.
5. Show that the products of (1) and (6) are non-commutative, also the products of (2) and (6).
6. Do with (1) and (7), also with (2) and (7), as in Ex. 5 you did for (1) and (6).
7. Show that we can never have a translation commutative with a rotation. Hint: Equate the constant terms in (9) with the corresponding ones in (10), and show that we must then have either $\phi = 0$ or $\alpha = \beta = 0$.

8. **Inverses of transformations and the identical transformation.** We saw in Ex. 9 of §6 that (6′), the inverse of (6), can be written as a distinct transformation of a type like (6), namely

$$(6'') \qquad x = x' + (-\alpha),\ y = y' + (-\beta)$$

We can look upon (6″) as a *translation by its own right* and not merely as the inverse of (6).

Similarly, we can write (7′) as

$$(7'') \qquad \begin{aligned} x &= x' \cos(-\phi) - y' \sin(-\phi), \\ y &= x' \sin(-\phi) + y' \cos(-\phi) \end{aligned}$$

and we can look upon (7″) as a rotation that has nothing in particular to do with (7). In fact we can consider (6) as the inverse of (6″) and (7) as the inverse of (7″). Similar remarks apply to (1′), (2′), (8′), (9′), and (10′).

If we follow any transformation by its inverse, we shall obtain $x = x''$, $y = y''$, which may be written

$$(11) \qquad x = x',\ y = y'$$

We call (11) the *identical* transformation or the *identity*, since it does not alter the coordinates of any point or change any equation or formula except to replace unprimed variables by primed variables (but we consider such a replacement as really not a change at all). We leave for the exercises the proof of this statement that (11) arises from the product in either order of a transformation by its inverse, in so far as the transformations (1), (2), (6), (7), (9), and (10) are concerned.

EXERCISE

Prove the statement made in the last sentence of the text.

9. **An algebraic notation for transformations.** We often represent a transformation of axes (coordinates, variables) by the letter T, its inverse by T^{-1}, the product of T into itself by T^2, the identity by I. If T_1 and T_2 are two transformations whose product in the order given is T_3, we write this fact as a sort of algebraic equation

$$T_1T_2 = T_3$$

(where the sign = means "*is the same transformation as*"). Ordinarily we have

$$T_2T_1 = T_4 \neq T_3$$

Compare §7. We have also

$$T^{-1}T = TT^{-1} = T^{1-1} = T^0 = I$$

The algebraic expressions $T_1 + T_2$ and T_1/T_2 have no meaning for transformations. The algebra in the last paragraph is *purely formal* and we must not treat it like ordinary algebra. However, the products of transformations are associative, just as are the products in ordinary algebra, i.e.,

$$T_1(T_2T_3) = (T_1T_2)T_3$$

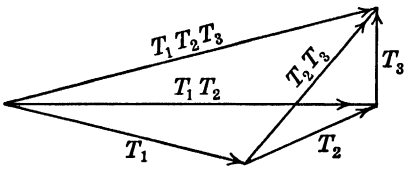
This fact will be proved more generally in §13.

For translations and rotations this associative property is evident analytically from the identical equations

$$\begin{aligned}(\alpha_1 + \alpha_2) + \alpha_3 &\equiv \alpha_1 + (\alpha_2 + \alpha_3),\\ (\beta_1 + \beta_2) + \beta_3 &\equiv \beta_1 + (\beta_2 + \beta_3),\\ (\phi_1 + \phi_2) + \phi_3 &\equiv \phi_1 + (\phi_2 + \phi_3)\end{aligned}$$

Therefore by the product $T_1T_2T_3$ we mean either $(T_1T_2)T_3$ or

$T_1(T_2T_3)$, i.e., we obtain the same resultant transformation if we first take the product of T_1 by T_2 and follow this by T_3 as we do if we first take T_1 and follow this by the product of T_2 by T_3. Geometrically, if we represent the path of the origin in a translation by an arrow, the adjoining figure will show why translations are associative, since we obtain the translation $T_1T_2T_3$ either by using T_1T_2 and then T_3 or T_1 and then T_2T_3.



If $T_3 = T_2^{-1}$ in the product $T_1T_2T_3T_4$, we have

$$T_1T_2T_2^{-1}T_4 = T_1IT_4 = T_1T_4$$

since $T_1I = IT_1 = T_1$. Therefore the inverse of $T_1T_2T_3$ is

$$(T_1T_2T_3)^{-1} = T_3^{-1}T_2^{-1}T_1^{-1}$$

since

$$(T_1T_2T_3)\,(T_1T_2T_3)^{-1} = T_1T_2T_3T_3^{-1}T_2^{-1}T_1^{-1} =$$
$$T_1T_2IT_2^{-1}T_1^{-1} = T_1T_2T_2^{-1}T_1^{-1} = T_1IT_1^{-1} = T_1T_1^{-1} = I$$

EXERCISES

1. Show geometrically that rotations around a point are associative.
2. Show geometrically by arrows (like the figure in the text) that translations are commutative.
3. State fully how the equations $\alpha_1 + (\alpha_2 + \alpha_3) \equiv (\alpha_1 + \alpha_2) + \alpha_3$, etc., show that rotations and translations are associative.
4. Prove analytically that (1), (2), (6), (7), etc., are associative.
5. Prove that the inverse of $T_1^2T_2^{-1}T_3^3$ is $T_3^{-3}T_2T_1^{-2}$, where by T_3^{-3} and T_1^{-2} we mean the inverses respectively of T_3^3 and T_1^2.
6. Prove that $T^\alpha T^\beta = T^\beta T^\alpha$ no matter what positive or negative integers are represented by α and β.

10. A slightly more general transformation of coordinates. In the previous sections we have studied translations, rotations, and changes from rectangular to oblique axes. In all these transformations *no alteration* was made in the *relative sizes* of the x- and y-units. In this section we shall consider some transformations that actually change the sizes of these two units. The simplest case is

$$x = \gamma x', \quad y = \delta y', \text{ where } \gamma\delta \neq 0 \tag{12}$$

$$x' = \frac{1}{\gamma}x, \quad y' = \frac{1}{\delta}y \tag{12'}$$

Thus $x = 3x'$, $y = 2y'$ gives a new x-unit thrice the size of the old x-unit and a new y-unit twice the size of the old one, because $x = 1$ gives $x' = 1/3$, $y = 1$ gives $y' = 1/2$. The equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be reduced to the form $x'^2 + y'^2 = 1$ by means of (12) with $\gamma = a$ and $\delta = b$.

Conversely, if we want, for example, to get a new x-unit one-half as long as the old one and a new y-unit twice as long as the old one, then $x' = 1$ on the x-axis must be the same point as $x = 1/2$ and $y' = 1$ on the y-axis must be the same point as $y = 2$. Hence we must have in (12) $\gamma = 1/2$ and $\delta = 2$.

We note that (12) does not alter the *position* of the axes, but merely changes the *sizes* of the units. We call (12) a *stretching* (or *contracting*) of the axes. If we use (12) we cannot study lengths of line-segments or angles between lines, or many of the properties of conic sections, because the new x- and y-units are not the same as the old.

If we combine (12) with the preceding transformations of coordinates, we obtain still more general changes of axes. Thus (12) followed by (6) gives us

$$x = \gamma x' + \gamma\alpha, \quad y = \delta y' + \delta\beta$$

Since (6) followed by (12) gives

$$x = \gamma x' + \alpha, \quad y = \delta y' + \beta$$

therefore the product of (6) and (12) is non-commutative.

On the other hand, we can analyze certain more general transformations of axes into products of those we have already studied. Thus

$$x = 2x' + 3, \quad y = 3y' + 9$$

can be written

$$x = 2(x' + \tfrac{3}{2}), \quad y = 3(y' + 3)$$

and in the latter form we see that the equations are those of a stretching $x = 2x'$, $y = 3y'$ followed by a translation $x = x' + \frac{3}{2}$, $y = y' + 3$. Again we can analyze this transformation as a translation $x = x' + 3$, $y = y' + 9$ followed by a stretching $x = 2x'$, $y = 3y'$.

EXERCISES

1. Show analytically that the products of (12) with any of the preceding transformations are non-commutative.

2. Analyze in two ways each of the transformations:

(*a*) $\qquad x = 6x' + 4, \quad y = 7y' - 3$

(*b*) $\qquad x = 2x' - 2y' + 5, \quad y = 3x' + 3y' + 6$

Hint: Write (*b*) as

$$x = 2\sqrt{2}\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right) + 5, \quad y = 3\sqrt{2}\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) + 6$$

Note that $\sin 45° = \cos 45° = 1/\sqrt{2}$.

(*c*) $\qquad x = 2x' - 2\sqrt{3}y' + 6, \quad y = \sqrt{3}x' + y' - 2$

3. Show that by rotations, translations, and stretchings the equation of every non-degenerate conic, i.e., (4) with $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$, can be reduced to the form

$$x^2 + y^2 = \pm 1, \quad \text{or} \quad x^2 - y^2 = 1, \quad \text{or} \quad y^2 = 4x$$

Hint: What are the *normal* (or *type*) forms to which these equations were reduced in elementary analytic geometry by means of rotations and translations?

4. Analyze the transformation

$$x = 2x' \cos\phi - 3y' \sin\phi + 4, \quad y = 2x' \sin\phi + 3y' \cos\phi - 3$$

5. Determine γ and δ in (12) so as to reduce $x^2/4 - y^2/9 = 1$ to the form $x'^2/9 - y'^2/16 = 1$.

6. Choose γ and δ in (12) so that the new y-unit shall be $\frac{5}{3}$ of the new x-unit.

7. If $\gamma = 7$, $\delta = -6$ in (12), find the new coordinates of the point (1,3).

8. Show that the equation of every line not through the origin can be put in the form $x' + y' = 1$ by (12).

11. **Two interpretations of transformations of coordinates.** We call attention to the fact that we have been gradually changing our attitude toward the transformations we have been considering. We frankly derived (1) and (2) from a figure, looking upon them as changes from rectangular to oblique axes. But when we came to the discussion of translations and rotations in §6, we defined them analytically as transformations of coordinates (or of variables).

In elementary analytic geometry we studied translations and rotations from the viewpoint of *changes of the axes of reference.* Thus, if we found that a certain conic was an ellipse, we translated the origin to the center of the ellipse, then we rotated the axes around this new origin until they coincided with the major

and minor axes of the conic, and its equation took on the simpler form $x^2/a^2 + y^2/b^2 = 1$.

In analytic projective geometry we may adopt the same viewpoint as above toward transformations of coordinates; but more often we look upon such transformations as keeping the axes and their unit points where they were at first and *changing the positions of the curves* or other sets of points that are under consideration, i.e., x and y, x' and y' refer to exactly the *same* frame of reference.

DEFINITION. The name *alias* has been wittily and fittingly suggested for a transformation considered as a *change of axes*, and the name *alibi* for a transformation looked upon as altering the positions of points in the plane while keeping the frame of reference the *same*. To repeat, an *alias* changes the equation of a given locus to another form, but an *alibi* replaces the given locus by a new locus with another equation.

The above-mentioned reduction of the equation of an ellipse to a so-called normal (canonical or typical) form can be described as an *alibi* in the following manner: If the center of an ellipse is at the origin and its major and minor axes lie on the coordinate axes, then it is shown in elementary analytic geometry that the equation of this ellipse has the form $x^2/a^2 + y^2/b^2 = 1$. Now if we take any ellipse in the plane, we can translate and rotate this curve in such a way as to make it coincide with one of the above specially placed ellipses for some pair of values of a and b.

We remark that under the aspect of an *alibi* a transformation of coordinates *does not* change the *axes* looked upon as a *frame of reference* but *does* change the *axes* looked upon as *mere lines* in the plane. Thus (6) sends $x = 0$ to $x' = -\alpha$. An illustration of this two-fold role of the axes is the way we might measure distances on the ground from two sticks sunk in the earth, then we might remove these sticks and measure the distances from the grooves that are left in the earth. Similarly, a transformation under the guise of an *alias does* change the axes looked upon as a frame of reference but *does not* change these axes looked upon as mere lines in the plane.

EXERCISES

1. Why are the names *alias* and *alibi* so fitting when applied to the two different ways of looking at a transformation of coordinates?

2. Show geometrically, looking upon the transformations as *alibis*:

(*a*) Translations are commutative.
(*b*) Rotations around a point are commutative.
(*c*) Rotations and translations are non-commutative.
(*d*) Rotations and stretchings are non-commutative.
(*e*) Translations and stretchings are non-commutative.

Hint: First draw a figure and describe rotations, translations, and stretchings geometrically as *alibis*.

3. Describe as an *alibi* the transforming of any parabola into $y^2 = 4\,px$, and of any circle into $x^2 + y^2 = r^2$. Describe these two reductions as *aliases*.

4. Describe as an *alibi*, then as an *alias*, the reduction of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{to} \quad x^2 + y^2 = 1$$

by means of (12).

5. Go through the complete reduction of the general conic to a normal form by rotations and translations.

6. Test and reduce to a normal form the conic

$$x^2 + 4\,xy - 2\,y^2 + 2\,x - 2\,y + 10 = 0$$

7. Derive the *standard* (or *type*) forms for the equations of the circle, ellipse, parabola, hyperbola. Do the ellipse and hyperbola in two ways: first, use the sum or difference respectively of the distances of a general point on the conic from the two foci; second, use a focus, its corresponding directrix, and the eccentricity of the conic.

8. Describe a translation and a rotation as *alibis*.

CHAPTER III

INTRODUCTION TO AFFINE LINEAR TRANSFORMATIONS

12. **Affine linear transformations.** The rotations and translations and all the other transformations of coordinates that we have considered up till now in this book are special cases of the following:

DEFINITION. An *affine linear transformation* is one whose equations have the general form

$$x = a_1x' + a_2y' + a_3, \quad y = b_1x' + b_2y' + b_3, \tag{13}$$

$$\text{where } \Delta \equiv \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0$$

$$x' = \frac{1}{\Delta}(b_2x - a_2y - a_3b_2 + a_2b_3), \tag{13'}$$

$$y' = \frac{1}{\Delta}(-b_1x + a_1y - a_1b_3 + a_3b_1)$$

For example, (6) belongs under (13) with $a_1 = 1$, $a_2 = 0$, $a_3 = \alpha$, $b_1 = 0$, $b_2 = 1$, $b_3 = \beta$. Note that we do not let $\Delta = 0$ in (13), otherwise we have the relation between x and y that is given by

$$b_2x - a_2y = a_3b_2 - a_2b_3$$

for every pair of values of x' and y', which means geometrically that to every point (x',y') in the plane there would correspond a point (x,y) on this line.

We call (13) *linear* because its equations are of the first degree in both the primed and the unprimed coordinates. Such a transformation as

$$x = \frac{x'}{x'^2 + y'^2}, \quad y = \frac{y'}{x'^2 + y'^2}$$

would not belong under (13). We call (13) *affine* because it sends *finite* points (i.e., points with finite coordinates) into finite

points. Such a transformation as

$$x = \frac{1}{x'}, \quad y = \frac{y'}{x'}$$

is not affine because it sends into the point (0,1) a point with coordinates $x = \infty$, $y = \infty$. (We shall see later that such a point will be assumed to lie at an infinite distance in the plane.)

EXERCISES

1. If $\Delta \equiv \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$ in (13), show that we have

$$b_1 x - a_1 y = a_3 b_1 - a_1 b_3$$

and that this is the same line as the one found in the text.

2. Obtain (13′) from (13).

3. Show that (1), (2), (7), (10), (11), (12) all fit under (13), and find Δ for each transformation.

4. Show that $x = 1/x', y = y'/x'$ sends $x^2 - y^2 = 1$ into $x'^2 + y'^2 = 1$.

5. Show that (13) cannot raise or lower the degree of the equation of a locus. Compare §4; also Ex. 6 in §6.

13. **The matrix of an affine linear transformation.** DEFINITION. The *matrix* of the affine linear transformation (13) is the square array

$$D = \begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{Vmatrix} \tag{14}$$

In reality D is the same sort of array of rows and columns as though it were an unexpanded determinant, but we use the word matrix since we do *not* look upon D as having an *expansion.* The determinant of D is Δ in §12, or Δ written as

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{vmatrix}$$

The matrix D is *uniquely determined* by a transformation, i.e., there is one and only one matrix D for each transformation. Thus (6) has

$$D = \begin{Vmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{Vmatrix}$$

On the other hand, D *uniquely determines* a transformation. For example

$$D = \begin{Vmatrix} 3 & 2 & 1 \\ -7 & 7 & 3 \\ 0 & 0 & 1 \end{Vmatrix}$$

clearly represents the transformation

$$x = 3\,x' + 2\,y' + 1, \quad y = -7\,x' + 7\,y' + 3$$

We define the product of two such matrices D and D' in the same way as we define the product of two determinants.

DEFINITION. For two matrices D and D' we mean by the *product* DD' a matrix with the same number of rows and columns as D (and D') and with the element in its ith row and jth column obtained by multiplying in order each element of the ith row of D by the corresponding element of the jth column of D' and adding together the resulting products.

If we take another transformation like (13), namely

$$(15) \qquad x = a_1'x' + a_2'y' + a_3', \quad y = b_1'x' + b_2'y' + b_3'$$

we have for (15) the matrix

$$D' = \begin{Vmatrix} a_1' & a_2' & a_3' \\ b_1' & b_2' & b_3' \\ 0 & 0 & 1 \end{Vmatrix}$$

The product DD' is therefore

$$(16) \quad DD' = \begin{Vmatrix} a_1a_1' + a_2b_1' & a_1a_2' + a_2b_2' & a_1a_3' + a_2b_3' + a_3 \\ b_1a_1' + b_2b_1' & b_1a_2' + b_2b_2' & b_1a_3' + b_2b_3' + b_3 \\ 0 & 0 & 1 \end{Vmatrix}$$

But the product $D'D$ is

$$(17) \quad D'D = \begin{Vmatrix} a_1'a_1 + a_2'b_1 & a_1'a_2 + a_2'b_2 & a_1'a_3 + a_2'b_3 + a_3' \\ b_1'a_1 + b_2'b_1 & b_1'a_2 + b_2'b_2 & b_1'a_3 + b_2'b_3 + b_3' \\ 0 & 0 & 1 \end{Vmatrix}$$

We leave for the reader to prove in the exercises that (13) followed by (15) gives an affine linear transformation with matrix $D'' = DD'$, whereas the product of (15) by (13) is a transformation with matrix $D''' = D'D$. Since (16) and (17) are ordinarily distinct (i.e., $DD' \neq D'D$ usually), we see that in general the affine linear transformations (13) are non-commutative. Also we shall find matrices very useful in finding products of such transformations.

ILLUSTRATIVE EXAMPLE. The two transformations

$$T_1\colon\ x = 3x' - 2y' + 4,\quad y = x' + 2y' - 3 \text{ with } D_1 = \begin{Vmatrix} 3 & -2 & 4 \\ 1 & 2 & -3 \\ 0 & 0 & 1 \end{Vmatrix}$$

and

$$T_2\colon\ x = 2x' - 3y' - 1,\quad y = 3x' + y' - 2 \text{ with } D_2 = \begin{Vmatrix} 2 & -3 & -1 \\ 3 & 1 & -2 \\ 0 & 0 & 1 \end{Vmatrix}$$

can be combined to form the products T_1T_2 with

$$D_1D_2 = \begin{Vmatrix} 0 & -11 & 5 \\ 8 & -1 & -8 \\ 0 & 0 & 1 \end{Vmatrix} \text{ and } T_2T_1$$

with

$$D_2D_1 = \begin{Vmatrix} 3 & -8 & 0 \\ 10 & -4 & -15 \\ 0 & 0 & 1 \end{Vmatrix}$$

We can show this either by actually forming these two products T_1T_2 and T_2T_1 or by taking their matrices D_1D_2 and D_2D_1.

In §9 we saw that rotations and translations are associative, i.e., $(T_1T_2)T_3 = T_1(T_2T_3)$ if T_1, T_2, T_3 are three rotations around the same point or are three translations. It is easily shown (we leave this for the exercises) that three matrices D_1, D_2, D_3 of the above form are associative when multiplied together, i.e., $(D_1D_2)D_3 = D_1(D_2D_3)$; hence any three affine linear transformations (13) are associative as to multiplication.

EXERCISES

1. Prove that (13) followed by (15) gives a transformation of the form (13) with matrix DD', whereas (15) followed by (13) gives one with matrix $D'D$.
2. Carry out the details of the illustrative example in the text.
3. Carry out the details of the last paragraph in the text.
4. If D' is the matrix of (13′), show that

$$DD' = D'D = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

Note that (13) (13′) = (13′) (13) = I. We can therefore put $D' = D^{-1}$.

5. Use matrices to find the product (*a*) of two translations, (*b*) of two rotations, (*c*) of a rotation and a translation (both orders of multiplication).
6. Find the conditions on the coefficients of (13) in order that $T_1T_2 = T_2T_1$ where T_1 and T_2 are of the form (13).
7. Find the conditions on the coefficients of (13) in order (*a*) that $(10)^2 = I$, (*b*) that $(10)^3 = I$.
8. Make up numerical examples to illustrate Exs. 6 and 7.

14. The number of points that determine an affine linear transformation. DEFINITION. A pair of points (lines, or other

geometric loci) that are transformed into one another by an affine linear transformation (considered as an *alibi*) are said to be a pair of *corresponding* (or *homologous*) points (lines, etc.).

DEFINITION. Two sets of homologous points (lines, etc.) are said to *determine uniquely* an affine linear transformation if there is one and only one such transformation sending one set into the other.

We shall now consider how many pairs of homologous points it takes to determine uniquely the transformations so far studied. We see at once that there is one and only one translation that sends any given point $P_1(x_1,y_1)$ into any other given point $P_1'(x_1',y_1')$ because, if we substitute these values of x,y and x',y', respectively, in (6), we obtain unique values for α and β (namely $\alpha = x_1 - x_1'$ and $\beta = y_1 - y_1'$).

If now we take any three distinct points $P_1(\alpha_1,\beta_1)$, $P_2(\alpha_2,\beta_2)$, $P_3(\alpha_3,\beta_3)$ and wish to determine a transformation (13) that will send P_1, P_2, P_3 into any other three distinct points $P_1'(\alpha_1',\beta_1')$, $P_2'(\alpha_2',\beta_2')$, $P_3'(\alpha_3',\beta_3')$, respectively, we get

$$\begin{aligned}
\alpha_1 &= a_1\alpha_1' + a_2\beta_1' + a_3, \quad \beta_1 = b_1\alpha_1' + b_2\beta_1' + b_3,\\
\alpha_2 &= a_1\alpha_2' + a_2\beta_2' + a_3, \quad \beta_2 = b_1\alpha_2' + b_2\beta_2' + b_3,\\
\alpha_3 &= a_1\alpha_3' + a_2\beta_3' + a_3, \quad \beta_3 = b_1\alpha_3' + b_2\beta_3' + b_3
\end{aligned}$$

These are three *linear* (i.e., first-degree) equations in the three unknowns a_1, a_2, a_3 and three linear equations in the three unknowns b_1, b_2, b_3. We learned in algebra that for these three equations in a_1, a_2, a_3 (or in b_1, b_2, b_3) to have a *unique finite* solution, we must have

$$\begin{vmatrix} \alpha_1' & \beta_1' & 1 \\ \alpha_2' & \beta_2' & 1 \\ \alpha_3' & \beta_3' & 1 \end{vmatrix} \neq 0$$

Geometrically this means that the three points P_1', P_2', P_3' must not be *collinear* (i.e., must not lie on the same line).

If we use the transformation in the form (13′) and consider the equations we obtain as containing the unknowns b_2/Δ, $-a_2/\Delta$, and $(-a_3b_2 + a_2b_3)/\Delta$, we find that for a unique finite solution we must have also

$$\begin{vmatrix} \alpha_1 & \beta_1 & 1 \\ \alpha_2 & \beta_2 & 1 \\ \alpha_3 & \beta_3 & 1 \end{vmatrix} \neq 0$$

i.e., the three points P_1, P_2, P_3 must not be collinear. Therefore we have proved the following

THEOREM. *The general transformation* (13) *is uniquely determined by three pairs of homologous points* P_1 *and* P_1', P_2 *and* P_2', P_3 *and* P_3' *provided that both* P_1, P_2, P_3 *and* P_1', P_2', P_3' *are non-collinear.*

(Of course special types of affine linear transformations require fewer pairs of homologous points to determine them, as we saw above for the translation.) In particular, we can send by (13) any three non-collinear points P_1, P_2, P_3 into (1,0), (0,0), (0,1), respectively; i.e., looking upon (13) as an *alias*, we can change our rectangular or oblique frame of reference to any rectangular or oblique frame of reference we please by means of an affine linear transformation.

In elementary analytic geometry we proved that we could send any ordinary frame of reference into any other ordinary frame of reference (if the units on the corresponding axes have the same sizes) by means of rotations and translations alone. The above result is much more general and of great use in what follows.

ILLUSTRATIVE EXAMPLE. As an illustration of the preceding theorem we note that, to send (0,0), (0,1), (1,0) into (1,1), (−1,−1), (0,1) by (13), we must have

$$0 = a_1 + a_2 + a_3, \quad 0 = b_1 + b_2 + b_3, \quad 0 = -a_1 - a_2 + a_3,$$
$$1 = -b_1 - b_2 + b_3, \quad 1 = a_2 + a_3, \quad 0 = b_2 + b_3$$

Solving these equations we obtain

$$a_3 = 0, \quad b_3 = \tfrac{1}{2}, \quad a_2 = 1, \quad b_2 = -\tfrac{1}{2}, \quad a_1 = -1, \quad b_1 = 0$$

Hence

$$x = -x' + y', \quad y = -\tfrac{1}{2}y' + \tfrac{1}{2}$$

is the unique transformation of the form (13) that we desire.

EXERCISES

1. Fill in the details of the proof in the text that we must have

$$\begin{vmatrix} \alpha_1 & \beta_1 & 1 \\ \alpha_2 & \beta_2 & 1 \\ \alpha_3 & \beta_3 & 1 \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} \alpha_1' & \beta_1' & 1 \\ \alpha_2' & \beta_2' & 1 \\ \alpha_3' & \beta_3' & 1 \end{vmatrix} \neq 0$$

2. Prove the fact mentioned in the text concerning the equivalence of certain ordinary frames of reference under rotations and translations. (By *equivalent* we mean able to be transformed into one another.)

3. Make up an example like the illustration in the last paragraph of the text.

4. Show geometrically that a pair of homologous points determines a translation. Hint: As an *alibi*, a translation can be looked upon as moving the whole plane a given distance in a given direction. Prove this.

5. Determine the coefficients of (13) so as to send (1,1), (2,3), $(-1,-1)$ into (1,0), (0,0), (0,1), respectively.

6. Prove analytically and geometrically that a rotation is uniquely determined by a single pair of homologous points P and P', provided P and P' lie on a circle with the origin as center. Compare Ex. 4.

7. In order to study a conic C we often take on C a point P as (0,1), the tangent to C at P as $x = 0$, another point R on C as (1,0), and the tangent at R as $y = 0$. Show, from the next to the last paragraph in the text, why it is permissible to take C in this position.

8. Prove the theorem in the text by sending $P_1(\alpha_1,\beta_1)$, $P_2(\alpha_2,\beta_2)$, $P_3(\alpha_3,\beta_3)$ into (0,1), (0,0), (1,0), respectively. How does this prove the theorem generally?

15. The equivalence of conics under affine linear transformations. DEFINITION. Two conics C and C' are said to be *equivalent* under a transformation T if we can send C into C' (or C' into C) by means of T.

We saw in §10 that the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be transformed into the circle by $x = ax'$, $y = by'$. We shall now show analytically that (13) will not send a circle or ellipse into a hyperbola or into a parabola. (Geometrically, this is almost self-evident, because affine transformations send finite points into finite points, and a circle or ellipse is a closed curve whereas a hyperbola or parabola is not a closed curve.) We leave to the exercises for the student the task of showing that (13) cannot send a hyperbola into a parabola.

Consider the circle $x^2 + y^2 = 1$, to which every circle or ellipse can be reduced by (13). Subjecting this circle to the transformation (13), i.e., substituting in the equation of this circle the values of x and y in terms of x' and y' that are given by the equations of (13), we obtain the conic

$$(a_1^2 + b_1^2)x'^2 + (a_2^2 + b_2^2)y'^2 + 2(a_1a_2 + b_1b_2)x'y' + 2(a_1a_3 + b_1b_3)x' + 2(a_2a_3 + b_2b_3)y' + a_3^2 + b_3^2 = 1$$

For this latter conic to be a hyperbola or a parabola we must have

$$(a_1a_2 + b_1b_2)^2 - (a_1^2 + b_1^2)(a_2^2 + b_2^2) > 0 \text{ or } = 0$$

respectively. But this expression in the coefficients of (13) is the same as $-\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 < 0$. Hence the theorem follows.

EXERCISES

1. By using the equation $xy = 1$, prove that (13) cannot send a hyperbola into a parabola. Why is there no loss of generality in taking the hyperbola as $xy = 1$? Compare the text.

2. Since (13) cannot send a hyperbola into a parabola, how does it follow that (13) cannot send a parabola into a hyperbola? Hint: The inverse of (13) is a linear affine transformation and can be put in the same form as (13).

3. Check the algebra in the text.

4. Derive the test that discriminates between ellipses, parabolas, and hyperbolas referred to an ordinary frame of reference, namely $h^2 - ab < 0$ or $= 0$ or > 0, respectively, for the general conic (4). If necessary, refer to some book on elementary analytic geometry.

5. We note that $(3x + 2y - 1)^2 - (x - y)^2 = 1$ is a hyperbola because $3x + 2y - 1 = x', x - y = y'$ reduces its equation to the form $x'^2 - y'^2 = 1$. Now show that

$$(x - y)(3x + y) = 1$$

is a hyperbola,

$$(3x - 2y)^2 + (5x - 3y + 1)^2 = 1$$

is an ellipse,

$$(3x - 1)^2 = 2(x - 3y + 4)$$

is a parabola.

What theorems in the text and in a previous example justify this proof?

6. Solve each of the transformations of Ex. 5 for x and y in terms of x' and y'.

7. The discussion in the text looked upon (13) as an *alibi*. Reword this discussion from the viewpoint of (13) as an *alias*.

16, 17. **Degenerate conics.** DEFINITION. A conic is said to be *degenerate* if its equation is factorable into a pair of real or imaginary linear factors. Geometrically, the locus of a degenerate conic is a pair of real or imaginary lines, or a *double line* (i.e., a line taken twice, this case occurring when the equation of the conic has two real and equal factors).

We shall now derive the test for the degeneracy of the general conic (4), namely the vanishing of the so-called *discriminant* of this conic whose formula is

$$\Gamma = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \tag{18}$$

(We shall treat only the case where $b \neq 0$ in (4), leaving the other cases for the exercises.)

Solving (4) for y in terms of x, we obtain

$$(19) \qquad y = \frac{-(hx + f) \pm \sqrt{(hx + f)^2 - b(ax^2 + 2gx + c)}}{b}$$

If the equation (4) is to be factorable, the expression under the radical must be a perfect square, or a constant, or vanish identically. But this quadratic expression in x can be arranged as

$$(h^2 - ab)x^2 + 2(hf - bg)x + f^2 - bc$$

If this quadratic is a perfect square we must have

$$(hf - bg)^2 - (h^2 - ab)(f^2 - bc) = 0$$

This last equation is in reality $-b\Gamma = 0$. Since $b \neq 0$, we see that $\Gamma = 0$ is the condition desired.

If the expression under the radical is to be a constant or vanish identically, we must have

$$h^2 - ab = hf - bg = 0 \quad \text{and} \quad f^2 - bc \neq 0 \quad \text{or} \quad f^2 - bc = 0$$

We leave it for the student to show that here again we have $\Gamma = 0$. Therefore we have proved that for all cases where $b \neq 0$, (4) is a degenerate conic if and only if $\Gamma = 0$. Note that $\Gamma \equiv abc + 2fgh - af^2 - bg^2 - ch^2$ may have terms missing because some of the coefficients of (4) are zero.

We note here two ways to factor (4) if the conic is degenerate. Suppose the terms

$$ax^2 + 2hxy + by^2$$

are factored into

$$(\alpha_1 x + \beta_1 y)(\alpha_2 x + \beta_2 y)$$

where α_1, β_1, α_2, β_2 are real or imaginary. If we equate (4) identically to

$$(\alpha_1 x + \beta_1 y + l)(\alpha_2 x + \beta_2 y + m)$$

we can determine l and m so that (4) is factored. Another way to factor (4) is, of course, to use (19).

ILLUSTRATIVE EXAMPLE. The conic

$$3x^2 - xy - 2y^2 + 10y - 12 = 0$$

is degenerate, because we have

$$\Gamma = \begin{vmatrix} 3 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -2 & 5 \\ 0 & 5 & -12 \end{vmatrix} = 0$$

Now we can solve the equation of this conic for y in terms of x and so factor it. Or we can put

$$3x^2 - xy - 2y^2 + 10y - 12 \equiv (3x + 2y + l)(x - y + m)$$

and determine l and m from the equations

$$-l + 2m = 10, \quad l + 3m = 0$$

which gives us $l = -6, m = 2$.

EXERCISES

1. Check all the algebra in the text.
2. Prove that, if $\Gamma = 0$, then (4) is a pair of real lines, a double line, or a pair of imaginary lines according as $h^2 - ab \begin{matrix} > \\ = \\ < \end{matrix} 0$ respectively.
3. Derive the condition $\Gamma = 0$ for (4) to be a degenerate conic when $b = 0$ $a \neq 0$; when $a = b = 0$, $h \neq 0$; when $a = b = h = 0$.
4. Show that in general a degenerate conic can be reduced by (13) to

$$x'y' = 0, \quad \text{or} \quad x'^2 + y'^2 = 0, \quad \text{or} \quad x'^2 = 0$$

5. Show that we cannot reduce by (13) a non-degenerate conic to a degenerate conic or one of the types of degenerate conics to another. Hint: Lines go into lines by means of (13).
6. Make up examples of degenerate conics, and find Γ for each such conic; also factor each such conic.
7. Show that $x^2 + 2xy + 4y - 4 = 0$ is a degenerate conic. Factor this equation in two different ways.
8. By equating (4) identically to $(\alpha_1 x + \beta_1 y + l)(\alpha_2 x + \beta_2 y + m)$ as in the text, derive the condition $\Gamma = 0$ for (4) to be a degenerate conic.

18. **Preliminary discussion of invariants of affine linear transformations.** DEFINITION. An *algebraic invariant* of a linear transformation T is an algebraic expression E, such that after operating with T on the variables in E the new expression E' has exactly the same form as E (only with new variables replacing the old), or $E' \equiv \alpha E$ where α is a power of the determinant of the matrix of T. (See §13.) In the first case E is called an *absolute* invariant, and in the second case a *relative* invariant.

DEFINITION. A *numerical invariant* of a linear transformation T is a number N associated intimately with a geometrical locus or an algebraic expression such that N is unchanged by T

DEFINITION. A *geometric invariant* of a linear transformation T is a geometric property G of a locus (or loci) L such that G is still true for the locus (or loci) L' into which T sends L.

ILLUSTRATIONS. The degree of the algebraic equation of a curve is a numerical invariant under (13). (See the theorem in §4, also Ex. 6 in §6, also Ex. 5 in §12.) The geometric property of a locus being an ellipse is invariant (or sometimes called *fixed*) under (13). (See §15.) The algebraic expression $x^2 + y^2$ is an absolute invariant under rotations, because we have

$$\begin{aligned} x^2 + y^2 &= (x' \cos\phi - y' \sin\phi)^2 + (x' \sin\phi + y' \cos\phi)^2 \\ &= x'^2 + y'^2 \end{aligned}$$

(Note that for many geometric invariants we must presuppose an ordinary frame of reference, because otherwise such ideas as length of line segments, areas of triangles, etc., have no meaning.) If we rotate or translate the points of a plane, it is easy to see geometrically that lengths of line segments, angles between lines, and areas of triangles are preserved (i.e., kept invariant).

Let us consider the analytic proofs of these facts. Under (6) we have

$$\begin{aligned} &\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(x_1' + \alpha - x_2' - \alpha)^2 + (y_1' + \beta - y_2' - \beta)^2} \\ &= \sqrt{(x_1' - x_2')^2 + (y_1' - y_2')^2} \end{aligned}$$

Hence we see that if we translate two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ to $P_1'(x_1',y_1')$ and $P_2'(x_2',y_2')$, respectively, then if we measure the distance $d = P_1P_2$ and subject the variables in d to the translation (6), we get the same numerical result as though we had applied the formula for distance to the segment $P_1'P_2'$.

Another way to look at this analytically is to measure the distance d' by the formula and then use (6′) to show that $d' = d$, thus

$$\begin{aligned} d' &\equiv \sqrt{(x_1' - x_2')^2 + (y_1' - y_2')^2} \\ &\equiv \sqrt{(x_1 - \alpha - x_2 + \alpha)^2 + (y_1 - \beta - y_2 + \beta)^2} \\ &\equiv d \end{aligned}$$

Analytic proofs of the invariance of geometric properties under other transformations are like the above discussion.

In the preceding paragraph the transformations were looked upon as *alibis*, the axes were not changed, therefore the formula for distance remained unaltered. If transformations are looked upon as *aliases*, then of course distances, angles, etc., are not changed,

since all the points in the plane remain unmoved. In this case we prove the *invariance* of the *formulas* for distances, angles, etc.

Next we shall consider the transformation (12). Applying (12) to any two points we find that

$$\begin{aligned} d &\equiv \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &\equiv \sqrt{(\gamma x_1' - \gamma x_2')^2 + (\delta y_1' - \delta y_2')^2} \not\equiv d' \end{aligned}$$

unless $\gamma = \pm 1$, $\delta = \pm 1$. Hence (12) does not in general leave distances invariant.

Let us examine what (12) does to the angle between any two lines

$$y = m_1 x + b_1 \quad \text{and} \quad y = m_2 x + b_2$$

Performing (12) on the variables x and y, we obtain the two new lines

$$\delta y' = \gamma m_1 x' + b_1 \quad \text{and} \quad \delta y' = \gamma m_2 x' + b_2$$

If we measure the angle between these last two lines we obtain

$$\tan \theta' = \frac{m_1' - m_2'}{1 + m_1' m_2'} = \frac{(\gamma/\delta)m_1 - (\gamma/\delta)m_2}{1 + (\gamma/\delta)m_1(\gamma/\delta)m_2} \neq \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

except for $\gamma = \delta$.

For *certain pairs* of points we *do* have under (12) the equality

$$(x_1' - x_2')^2 + (y_1' - y_2')^2 = \gamma^2 (x_1' - x_2')^2 + \delta^2 (y_1' - y_2')^2$$

and so $d = d'$, namely whenever

$$\frac{(x_1' - x_2')^2}{(y_1' - y_2')^2} = \frac{1 - \delta^2}{\gamma^2 - 1}$$

But from the above equation in the form

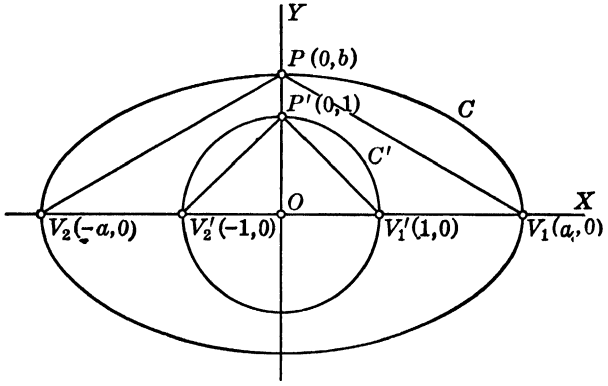
$$(x_1' - x_2')^2 (\gamma^2 - 1) + (y_1' - y_2')^2 (\delta^2 - 1) = 0$$

we see that $d \neq d'$ if (x_1', y_1') and (x_2', y_2') are any two points at all unless $\gamma^2 - 1 = \delta^2 - 1 = 0$, i.e., $\gamma = \pm 1$ and $\delta = \pm 1$ as we saw above.

Similarly for certain pairs of lines we do have $\tan \theta = \tan \theta'$ if $m_1 m_2 = \dfrac{\delta^2 - \gamma\delta}{\gamma\delta - \gamma^2}$; but we see from this equation that only if $\delta^2 - \gamma\delta = \gamma\delta - \gamma^2 = 0$ or $\gamma = \delta$ (as we saw above) do we have $\tan \theta = \tan \theta'$ for any and every pair of lines, since in this case m_1 and m_2 are not connected by any relation because the equation

$m_1m_2 = (\delta^2 - \gamma\delta)/(\gamma\delta - \gamma^2)$ then becomes $m_1m_2 = 0/0$ (which is indeterminate).

Geometrically we can show by the figure below that (12) does not in general keep lengths or angles invariant. We suppose the ellipse $C(x^2/a^2 + y^2/b^2 = 1)$ where $a^2 > 1$, $b^2 > 1$ has been sent by (12) into the circle $C'(x'^2 + y'^2 = 1)$. The line segment PV_1 goes into $P'V_1'$, which is smaller than PV_1, and PV_2 goes into $P'V_2'$; also, the obtuse angle V_2PV_1 goes into the right angle $V_2'P'V_1'$.



Next we show analytically that (6) leaves the area of any triangle invariant. Subjecting

$$A = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

to a translation we get

$$A = \tfrac{1}{2}\begin{vmatrix} x_1' + \alpha & y_1' + \beta & 1 \\ x_2' + \alpha & y_2' + \beta & 1 \\ x_3' + \alpha & y_3' + \beta & 1 \end{vmatrix} = \tfrac{1}{2}\begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix} = A'$$

(the area of the new triangle), and this proves the invariance of A. On the contrary (12) gives us

$$A = \tfrac{1}{2}\begin{vmatrix} \gamma x_1' & \delta y_1' & 1 \\ \gamma x_2' & \delta y_2' & 1 \\ \gamma x_3' & \delta y_3' & 1 \end{vmatrix} = \gamma\delta A' \neq A'$$

unless $\gamma = 1/\delta$.

EXERCISES

1. Show that for (12) to give $d = d'$ we must also have

$$(x_1 - x_2)^2 (1/\gamma^2 - 1) + (y_1 - y_2)^2 (1/\beta^2 - 1) = 0$$

Hint: Use (12′) on d'.

2. Derive the condition given in the text for $\tan \theta = \tan \theta'$ under (12), namely

$$m_1 m_2 = \frac{\delta^2 - \gamma\delta}{\gamma\delta - \gamma^2}$$

3. Prove that, if two transformations T_1 and T_2 have the same variant J, then their products T_1T_2 and T_2T_1 and their inverses T_1^{-1} and T_2^{-1} also have J as an invariant.

4. Show that (13) sends a tangent to a curve into a tangent to another curve. Hint: Consider each tangent as the limiting position of a secant.

5. Prove that rotations have the invariants d, $\tan \theta$, A.

19. **The discriminant and other invariants of a conic.** If we translate the general conic (4) we obtain

$$(4') \quad ax'^2 + by'^2 + (c + a\alpha^2 + b\beta^2 + 2f\beta + 2g\alpha + 2h\alpha\beta) + 2(f + b\beta + h\alpha)y' + 2(g + a\alpha + h\beta)x' + 2hx'y' = 0$$

The discriminant of (4′) is

$$\Gamma' = \begin{vmatrix} a & h \\ h & b \\ g + a\alpha + h\beta & f + b\beta + h\alpha \end{vmatrix} \begin{matrix} g + a\alpha + h\beta \\ f + b\beta + h\alpha \\ c + a\alpha^2 + b\beta^2 + 2f\beta + 2g\alpha + 2h\alpha\beta \end{matrix} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \Gamma$$

(We can show that $\Gamma' = \Gamma$ by subtracting from the last row the first row multiplied by α and the second row multiplied by β, then subtracting from the last column the first column multiplied by α and the second column multiplied by β.) From this result we see that (6) keeps invariant the discriminant of any conic.

If we perform (12) on the variables of (4) we get a new conic

$$(4'') \quad a'x'^2 + b'y'^2 + c' + 2f'y' + 2g'x' + 2h'x'y' = 0$$

where $a' = a\gamma^2$, $b' = b\delta^2$, $c' = c$, $f' = f\delta$, $g' = g\gamma$, $h' = h\delta$

We find that $\Gamma' = \gamma^2\delta^2\Gamma \neq \Gamma$ unless $\gamma = \pm 1/\delta$.

To discuss invariants under (13) we pursue a policy of *divide*

and conquer, that is, we write (13) as the product of the following transformations

$$(20)\quad T_1\colon x = x' + a_3,\quad y = y' + b_3;\quad T_2\colon x = \frac{a_1b_2 - a_2b_1}{b_2}x' + \frac{a_2}{b_2}y',\quad y = y';\quad T_3\colon x = x',\quad y = b_1x' + b_2y'$$

in the order $T_1T_2T_3$. We have shown that T_1 does not alter Γ of (4).

If we subject the variables of (4) to a transformation of the form

$$(21)\qquad x = \alpha x' + \beta y',\quad y = y'$$

we obtain a conic (4″) with

$$a' = a\alpha^2,\quad b' = b + a\beta^2 + 2\,h\beta,\quad c' = c,\quad f' = f + g\beta,\quad g' = g\alpha,\quad h' = a\alpha\beta + h\alpha$$

and with discriminant

$$\Gamma' = \begin{vmatrix} a\alpha^2 & a\alpha\beta + h\alpha & g\alpha \\ a\alpha\beta + h\alpha & b + a\beta^2 + 2\,h\beta & f + g\beta \\ g\alpha & f + g\beta & c \end{vmatrix} = \alpha^2 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \alpha^2\Gamma$$

It is now evident that $x = x'$, $y = b_1x' + b_2y'$ sends (4) into a new conic with $\Gamma' = b_2^2\Gamma$. Therefore we see from (20) that (13) sends (4) into a new conic with discriminant

$$(22)\qquad \Gamma' = (a_1b_2 - a_2b_1)^2\Gamma = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{vmatrix}^2 \Gamma$$

From (22) we have the

THEOREM. *The discriminant of a conic is a relative invariant under linear affine transformations.*

As a numerical example of the last paragraph we note that $x = 3\,x' - 2\,y' + 1$, $y = x' + y' - 2$ sends $x^2 - 2\,xy + 6\,y^2 + 4\,x + 8\,y - 7 = 0$ into a conic C' with discriminant

$$\Gamma' = \begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix}^2 \begin{vmatrix} 1 & -1 & 2 \\ -1 & 6 & 4 \\ 2 & 4 & -7 \end{vmatrix} = (25)(-91) = -2275$$

EXERCISES

1. Prove that rotations and translations have the absolute invariants $h^2 - ab$ and $a + b$ for the conic (4).

2. Prove the theorem in the text by using (13) in its unfactored form on the conic (4).

3. Fill in the algebraic details of the discussion that led up to (22).

4. Fill in the algebraic details of the numerical illustration in the text.

5. Resolve (13) into the product (20), i.e, factor (13). Hint: Write (13) as

$$x = x' + a_3, \quad y = y' + b_3$$

followed by

$$x' = \alpha_1 x'' + \beta_1 y'', \quad y' = y''$$

then followed by

$$x'' = x''', \quad y'' = b_1 x''' + b_2 y'''$$

and determine α_1 and β_1 so as to have this product the same as (13).

6. Fill in the algebraic details in the discussion of the effect of (21) on Γ.

7. Prove (what is stated as evident in the text) that T_3 of (20) sends Γ into $b_2^2\Gamma$. How could we conclude from symmetry that this fact was evident, after discussing (21)?

8. Resolve (13) into a product $T_3'T_2'T_1'$, where T_1', T_2', T_3' are of the same type as T_1, T_2, T_3, respectively, in (20).

9. Resolve $x = 3x' - 2y' + 4$, $y = x' + y' - 3$ into a product like (20).

10. Show that $x = 2x' + y' - 1$, $y = x' + y' + 3$ keeps the discriminant of (4) and the area of any triangle absolutely invariant.

11. Make up two numerical examples like the one in the text.

12. Make up numerical cases of (13), not (6) or (7), that (*a*) keep Γ absolutely invariant; (*b*) keep A absolutely invariant; (*c*) keep $h^2 - ab$ absolutely invariant.

13. Prove that (13) sends the area A of a triangle into

$$A' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{vmatrix} A$$

14. Show that (13) transforms $h^2 - ab$ of (4) into

$$h'^2 - a'b' = (a_1b_2 - a_2b_1)^2 (h^2 - ab)$$

so that $h^2 - ab$ is a relative invariant under (13).

15. Show how the fact that $h^2 - ab$ goes into $(a_1b_2 - a_2b_1)^2 (h^2 - ab)$ under (13) proves that (13) cannot send a hyperbola into an ellipse or a parabola, or an ellipse into a parabola. Hint: Note that $(a_1b_2 - a_2b_1)^2 > 0$.

16. From the relative invariance of Γ under (13) show that (13) cannot send a non-degenerate conic into a degenerate conic, or conversely.

17. Show that under (13) we have $\Gamma' \gtreqless 0$ according as $\Gamma \gtreqless 0$.

18. From Ex. 17 show that if $\Gamma > 0$ for (4), then the conic is reducible to $x^2 + y^2 + 1 = 0$ and so has no real points on it.

20. **Transformations that have given invariants.** Conversely to the discussions in §§18, 19 of invariants of given transformations, we shall now consider briefly the problem of *how to determine the coefficients* of the transformation (13) so that it shall keep *invariant* lengths of line-segments, or areas of triangles, or the like.

If we perform (13) on the distance formula we obtain

$$d = \sqrt{\{(a_1^2 + b_1^2)\,(x_1' - x_2')^2 + (a_2^2 + b_2^2)\,(y_1' - y_2')^2 + 2(a_1a_2 + b_1b_2)\,(x_1' - x_2')\,(y_1' - y_2')\}}$$

This algebraic expression is not the same as

$$d' = \sqrt{(x_1' - x_2')^2 + (y_1' - y_2')^2}$$

i.e., distance is not invariant under (13) unless

$$a_1^2 + b_1^2 = a_2^2 + b_2^2 = 1, \quad a_1a_2 + b_1b_2 = 0$$

(but a_3 and b_3 are arbitrary).

From these equations of condition we see that we can consider a_1 and b_1 as the cosine and sine of some angle ϕ, also a_2 and b_2 as the sine and cosine of some angle θ. But then

$$a_1a_2 + b_1b_2 = \cos\phi \sin\theta + \sin\phi \cos\theta = \sin(\theta + \phi) = 0$$

so $\theta + \phi = n\pi$ (where n is any positive or negative integer) or $\theta = n\pi - \phi$. Replacing a_1, a_2, b_1, b_2 by their values,

$$\cos\phi, \sin(n\pi - \phi) = \pm\sin\phi, \quad \sin\phi, \cos(n\pi - \phi) = \mp\cos\phi$$

respectively, we conclude that, for a transformation T of the type (13) to have d as an invariant, T must be of the form (9′) or (9), or (10′) or (10), that is T must be the product of a rotation and a translation. (Note that T may also be either a rotation alone or a translation alone.)

Next we observe that (13) sends (4) into a conic (4″) with

$$a' = aa_1^2 + bb_1^2 + 2ha_1b_1, \quad b' = aa_2^2 + bb_2^2 + 2ha_2b_2$$
$$h' = aa_1a_2 + bb_1b_2 + h(a_1b_2 + a_2b_1)$$

Therefore

$$a' + b' = a(a_1^2 + a_2^2) + b(b_1^2 + b_2^2) + 2h(a_1b_1 + a_2b_2) = a + b$$

if, and only if,

$$a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1, \quad a_1b_1 + a_2b_2 = 0$$

By an argument exactly similar to that used in the last paragraph

we can prove that $a' + b' = a + b$ if, and only if, the transformation is the product of a rotation and a translation. We leave the details of this argument to the student in the exercises.

Let us find out what is the condition on the coefficients of a transformation T of the form (13) for T to leave the area of any triangle invariant. Using (13), we find that

$$A = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \tfrac{1}{2}\begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 1 \end{vmatrix} = A'$$

the area of the new triangle if, and only if, $a_1b_2 - a_2b_1 = 1$. Compare Ex. 13 in §19.

We see that T may keep the area of every triangle invariant and yet not keep distances invariant. However, we see from §19 that $a_1b_2 - a_2b_1 = 1$ is also the condition for T to keep Γ and $h^2 - ab$ of the conic (4) absolutely invariant.

ILLUSTRATIVE EXAMPLE. The transformation

$$x = 3x' - 5y' + 1, \quad y = 2x' - 3y' + 7$$

which is evidently not the product of a rotation and a translation, has $a_1b_2 - a_2b_1 = -9 + 10 = 1$, and so keeps areas of triangles, Γ and $h^2 - ab$, absolutely invariant.

If we submit the variables of the two lines

$$y = m_1x + k_1, \quad y = m_2x + k_2$$

to the transformations (13), we have two new lines with slopes

$$m_1' = \frac{m_1a_1 - b_1}{b_2 - m_1a_2}, \quad m_2' = \frac{m_2a_1 - b_1}{b_2 - m_2a_2}$$

Hence for $\tan\theta' = \dfrac{m_1' - m_2'}{1 + m_1'm_2'}$ to be the same as $\pm\tan\theta = \pm\left(\dfrac{m_1 - m_2}{1 + m_1m_2}\right)$,

we find we must have

$$a_1^2 + a_2^2 = b_1^2 + b_2^2 = \pm(a_1b_2 - a_2b_1) \neq 0, \quad a_1b_1 + a_2b_2 = 0$$

We leave the details of this discussion for the exercises.

EXERCISES

1. Why can we use the words "*if, and only if,*" in the text? In other words, show that the conditions on the coefficients that are derived in the text are both *necessary* and *sufficient.*

2. Give the complete discussions for the invariance of $a + b$, also for the invariance (except possibly for sign) of $\tan\theta$.

3. Give numerical illustrations of transformation (13) that leave invariant Γ, $h^2 - ab$, and A.

4. Show that the conditions for (13) to preserve distance are *necessary* and *sufficient*.

21. The effect of affine linear transformations on the equations of lines. It is worth while finding out what the *effect* is of (13), or of any one of its special cases, on the *equations of lines*. Thus (6) does not alter m in

$$y = mx + b$$

but replaces b by $b' = b + m\alpha - \beta$. On the contrary (7) changes this line into

$$y'(\cos\phi + m\sin\phi) = x'(m\cos\phi - \sin\phi) + b$$

Since x, x', y, y' all refer to the same frame of reference, we can ignore the primes on the variables. Hence we can describe the change in the equation of the line by saying that (7) replaces the coefficient m by $m\cos\phi - \sin\phi$, does not change the constant term, but replaces 1 (the coefficient of y) by $b\cos\phi + m\sin\phi$.

In general we note that (13) sends the line

$$ux + vy + w = 0 \tag{23}$$

into

$$(a_1u + b_1v)x' + (a_2u + b_2v)y' + (a_3u + b_3v + w) = 0 \tag{23'}$$

If we write (23') in the form

$$u'x' + v'y' + w' = 0 \tag{23''}$$

we see that (13) sends the general line (23) with coefficients (this name is to include the constant term) u, v, w into another line with x',y' (which are the same variables as x,y) and u', v', w', where the following relations exist between u', v', w' and u, v, w:

$$\rho u' = a_1u + b_1v, \quad \rho v' = a_2u + b_2v, \quad \rho w' = a_3u + b_3v + w \tag{24'}$$

$$\sigma u = b_2u' - b_1v', \quad \sigma v = -a_2u' + a_1v', \quad \sigma w = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} u' + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} v' + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} w' \tag{24}$$

where $\sigma = 1/\rho \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ and ρ is an arbitrary non-vanishing

constant that is introduced because $\rho u'$, $\rho v'$, $\rho w'$ give us the same line (23″) as u', v', w'.

We may look upon (24) as a transformation of the coefficients of the line (23) caused (or *induced*) by the transformations of the coordinates of points given by (13). We must keep in mind that every transformation of the *coordinates* of points causes a transformation of the *coefficients* of lines. In the transformed equation of a line we can replace x' and y' by x and y because the axes of reference are left unchanged, and we plot any line by finding two points (ordinarily the intercepts on the axes).

We note that the matrix of (24′) is

$$\begin{Vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 1 \end{Vmatrix}$$

This fact gives us an easy way to find the transformation of the coordinates of lines that is induced by any affine linear transformation of the coordinates of points. Compare (14) in §13. For example, (7) has the matrix

$$\begin{Vmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

hence it causes a transformation

$$\rho u' = u\cos\phi + v\sin\phi, \quad \rho v' = -u\sin\phi + v\cos\phi, \quad \rho w' = w$$

EXERCISES

1. Give the details omitted in the text, such as the derivation of (23′) and (24).
2. Find the effect of each of the transformations (1), (2), (6), (7), (9), (10), (12) on the line (23), first by actual substitution for the coordinates in the equation of the line and, second, by the method of the last paragraph in the text.
3. Obtain (24) by performing (13′) on the variables of (23″).
4. What does the transformation

$$x = 2x' + y' - 1, \quad y = x' + y' + 3$$

do to the line (23)?

22. A first hint of plane duality. We saw in §14 that three pairs of homologous points P_1 and P_1', P_2 and P_2', P_3 and P_3'

uniquely determine a transformation (13) so long as P_1, P_2, P_3 and also P_1', P_2', P_3' are non-collinear. We shall now prove the

THEOREM. *Three pairs of homologous lines l_1 and l_1', l_2 and l_2', l_3 and l_3' uniquely determine a transformation (24′), and hence (13), so long as l_1, l_2, l_3 and also l_1', l_2', l_3' are non-concurrent (i.e., do not all pass through the same point).*

DEFINITION. Two such theorems as the above and the one in §14 that can be obtained from one another by the *interchange* of the words *point* and *line*, *collinear* and *concurrent*, are called the *plane duals* of each other.

To prove the preceding theorem we suppose l_1 to have the coefficients u_1, v_1, w_1; l_1' to have u_1', v_1', w_1', etc. Since l_1, l_2, l_3 go into l_1', l_2', l_3', respectively, we must have from (24′)

$$\rho_1 u_1' = a_1u_1 + b_1v_1,\ \rho_1 v_1' = a_2u_1 + b_2v_1,\ \rho_1 w_1' = a_3u_1 + b_3v_1 + w_1,$$
$$\rho_2 u_2' = a_1u_2 + b_1v_2,\ \rho_2 v_2' = a_2u_2 + b_2v_2,\ \rho_2 w_2' = a_3u_2 + b_3v_2 + w_2,$$
$$\rho_3 u_3' = a_1u_3 + b_1v_3,\ \rho_3 v_3' = a_2u_3 + b_2v_3,\ \rho_3 w_3' = a_3u_3 + b_3v_3 + w_3$$

We use the three distinct constants ρ_1, ρ_2, ρ_3 because no two of the lines l_1', l_2', l_3' necessarily have the same arbitrary multiplier for their equations.

Let us take the three of the above equations that have w in them. We see they are of the following type:

$$a_3u_1 + b_3v_1 + \delta w_1 = \rho_1 w_1',\quad a_3u_2 + b_3v_2 + \delta w_2 = \rho_2 w_2',$$
$$a_3u_3 + b_3v_3 + \delta w_3 = \rho_3 w_3'$$

where $\delta \equiv 1$. If we look upon these last three equations as having the unknowns a_3, b_3, δ, we see we must have

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \neq 0$$

if these equations are to have a unique solution in a_3, b_3, $\delta \equiv 1$. But this shows that the three lines l_1, l_2, l_3 must be non-concurrent.

If now we use (24) instead of (24′) and call

$$\lambda = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix},\quad \mu = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix},\quad \nu = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

the unknowns in the three equations that contain w, we see (by an argument like the above, which we leave to the student in the

exercises) that we must have

$$\begin{vmatrix} u_1' & v_1' & w_1' \\ u_2' & v_2' & w_2' \\ u_3' & v_3' & w_3' \end{vmatrix} \neq 0$$

i.e., the three lines l_1', l_2', l_3' must be non-concurrent.

ILLUSTRATIVE EXAMPLE. If we want to send $x = 0$ into $x + y = 0$, $y = 0$ into $x - y = 0$, $x + 2y - 1 = 0$ into $x - 2y - 1 = 0$, we see we have

$$u_1 = 1, \quad v_1 = w_1 = 0; \quad u_1' = v_1' = 1, \quad w_1' = 0; \quad u_2 = w_2 = 0, \quad v_2 = 1;$$
$$u_2' = -v_2' = 1, \quad w_2' = 0; \quad 2u_3 = v_3 = -2w_3 = 2; \quad 2u_3' = -v_3' = -2w_3' = 2$$

Using (24′) we get

$$\rho_1 = a_1, \quad \rho_1 = a_2, \quad 0 = a_3; \quad \rho_2 = b_1, \quad -\rho_2 = b_2, \quad 0 = b_3;$$
$$\rho_3 = a_1 + 2b_1, \quad -2\rho_3 = a_2 + 2b_2, \quad -\rho_3 = a_3 + 2b_3 - 1 = -1$$

Therefore (24′) must have the form

$$\rho u' = \rho_1 u + \rho_2 v, \quad \rho v' = \rho_1 u - \rho_2 v, \quad \rho w' = w$$

where $\rho_1 + 2\rho_2 = 1$, $\rho_1 - 2\rho_2 = -2$, so $\rho_1 = -\frac{1}{2}$, $\rho_2 = \frac{3}{4}$. Hence (24′) can be written

$$\rho u' = -\tfrac{1}{2}u + \tfrac{3}{4}v, \quad \rho v' = -\tfrac{1}{2}u - \tfrac{3}{4}v, \quad \rho w' = w$$

Since $$a_1 = a_2 = -\tfrac{1}{2}, \quad b_1 = -b_2 = \tfrac{3}{4}, \quad a_3 = b_3 = 0$$

we see that (13) takes the form

$$x = -\tfrac{1}{2}x' - \tfrac{1}{2}y', \quad y = \tfrac{3}{4}x' - \tfrac{3}{4}y'$$

(We note here that the above theorem can be obtained geometrically from the theorem in §14 in the following manner. The vertices of the triangle whose sides are l_1, l_2, l_3 must go into the corresponding vertices of the triangle whose sides are l_1', l_2', l_3'. From this fact follows the above theorem. Also we can use this note to solve the above illustrative example more readily.)

Another solution of this problem comes from the hint that the vertices (0,0), $(0,\frac{1}{2})$, (1,0) of the triangle with sides $x = 0$, $y = 0$, $x + 2y - 1 = 0$ must go into the corresponding vertices (0,0), $(\frac{1}{3},-\frac{1}{3})$ $(-1,-1)$ of the triangle with sides $x + y = 0$, $x - y = 0$, $x - 2y - 1 = 0$. Using (13) directly we get the equations

$$0 = 0 + 0 + a_3, \quad 0 = 0 + 0 + b_3, \quad 1 = -a_1 - a_2 + a_3,$$
$$0 = -b_1 - b_2 + b_3, \quad 0 = \tfrac{1}{3}a_1 - \tfrac{1}{3}a_2 + a_3, \quad \tfrac{1}{2} = \tfrac{1}{3}b_1 - \tfrac{1}{3}b_2 + b_3$$

so $a_3 = b_3 = 0$, $b_1 = -b_2$, $a_1 = a_2 = -\frac{1}{2}$, $b_2 = -\frac{3}{4}$ and the required transformation is $x = -\frac{1}{2}x' - \frac{1}{2}y'$, $y = \frac{3}{4}x' - \frac{3}{4}y'$.

We can solve the illustrative example in still a *third* way. Since $x = 0$ is to go into $x' + y' = 0$, we must have $x = \alpha(x' + y')$. Since $y = 0$ is to go into $x' - y' = 0$, we must have $y = \beta(x' - y')$. Since $x + 2y - 1 = 0$ is to go into $x' - 2y' - 1 = 0$, we must have

$$x + 2y - 1 = \alpha(x' + y') + 2\beta(x' - y') - 1 = x' - 2y' - 1$$

From this last equation we have $\alpha + 2\beta = 1$, $\alpha - 2\beta = -2$, whence $\alpha = -\frac{1}{2}$ and $\beta = \frac{3}{4}$, and again we obtain the transformation $x = -\frac{1}{2}x' - \frac{1}{2}y'$, $y = \frac{3}{4}x' - \frac{3}{4}y'$. (The logic of this solution is more easily seen if we look upon the transformation as an *alias*; because then, for example, such an equation as $x = 0$ must give the same locus as $x' + y' = 0$, hence we must have $x = \alpha(x' + y')$, as above.)

EXERCISES

1. What coefficients must the equations in (13) have so as to send $x = 0$ into $y = 0$, $x + 3y - 4 = 0$ into $2x - y + 5 = 0$, and $x + y - 1 = 0$ into $3x - 2y + 7 = 0$? Do this problem in three ways, as was done in the last paragraphs in the text.

2. Carry through the analytical proof sketched in the text that l'_1, l'_2, l'_3 must be non-concurrent.

3. Fill in the details of the geometric derivation of the theorem in the text from the theorem in §14.

4. If the point of contact of a tangent to a conic is the plane dual of the tangent, what is the plane dual of the figure formed by a hexagon circumscribed to a conic and with its diagonals drawn?

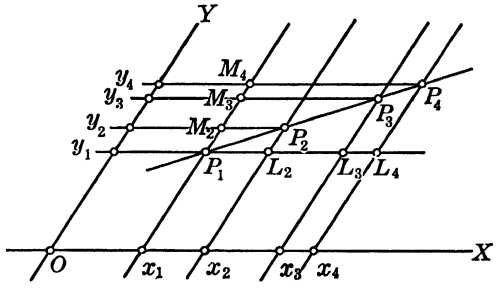
5. Explain the *logic* in the last paragraph of the text, looking upon the transformation as an *alibi*.

CHAPTER IV

INTRODUCTION TO THE STUDY OF CROSS-RATIO

23. Cross-ratio, an important invariant of affine linear transformations. We have seen in §§18 and 20 that the general affine linear transformation (13) does not leave invariant lengths of line-segments, angles between lines, or areas of triangles. There is a very important invariant of (13) that we shall now consider, called a *cross-ratio* (*anharmonic ratio* or *double ratio*) of four collinear points. This invariant does *not* presuppose an ordinary frame of reference.

We suppose there are given four distinct collinear points $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_3(x_3,y_3)$, $P_4(x_4,y_4)$ on a line $y = mx + b$ (see the adjoining figure). Through P_1 take the line P_1L_2 parallel to OX and the line P_1M_2 parallel to OY; construct the points x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, L_2, L_3, L_4, M_2, M_3, M_4 as in this figure. From plane geometry we have



$$\frac{P_1P_2}{P_1L_2} = \frac{P_2P_3}{L_2L_3} = \frac{P_3P_4}{L_3L_4} = \frac{P_1P_4}{L_1L_4} = c$$

where c is a non-vanishing constant. But we have

$$P_1L_2 = x_2 - x_1, \quad L_2L_3 = x_3 - x_2, \quad L_3L_4 = x_4 - x_3, \quad L_1L_4 = x_4 - x_1$$

Similarly we have

$$\frac{P_1P_2}{P_1M_2} = \frac{P_2P_3}{M_2M_3} = \frac{P_3P_4}{M_3M_4} = \frac{P_1P_4}{M_1M_4} = c' \neq 0$$

Therefore we have

$$\frac{P_1P_2}{P_3P_2}\,\frac{P_3P_4}{P_1P_4} = \frac{x_2 - x_1}{x_2 - x_3}\,\frac{x_4 - x_3}{x_4 - x_1} = \frac{y_2 - y_1}{y_2 - y_3}\,\frac{y_4 - y_3}{y_4 - y_1} \tag{25}$$

DEFINITION. This product of ratios $\frac{P_1P_2}{P_3P_2}\frac{P_3P_4}{P_1P_4}$ is called a *cross-ratio* of the four collinear points P_1, P_2, P_3, P_4.

DEFINITION. The product of the ratios in the abscissas (ordinates) in (25) will now be defined as a cross-ratio. This is more general than the previous definition, because it contains no reference to lengths of line-segments. See §§77, 106.

There are five other cross-ratios of these four points, namely, $P_1P_3/P_2P_3 \cdot P_2P_4/P_1P_4$, $P_1P_3/P_4P_3 \cdot P_4P_2/P_1P_2$, etc. We shall now prove the

THEOREM. *Cross-ratio is an absolute invariant under affine linear transformations.*

PROOF: We can write (13′) in the form

$$(13'') \qquad x' = \alpha_1 x + \alpha_2 y + \alpha_3, \quad y' = \beta_1 x + \beta_2 y + \beta_3$$

Since rotations and translations leave invariant the lengths of line-segments and therefore surely leave (25) invariant, we can rotate and translate the line $P_1P_2P_3P_4$ into the position of $y = 0$ before we consider the effect of (13) upon (25), and yet lose no generality in our argument. Hence these four points (now on $y = 0$) go by (13″) into four points with the coordinates

$$x_1' = \alpha_1 x_1 + \alpha_3,\ y_1' = \beta_1 x_1 + \beta_3;\ x_2' = \alpha_1 x_2 + \alpha_3,\ y_2' = \beta_1 x_2 + \beta_3$$
$$x_3' = \alpha_1 x_3 + \alpha_3,\ y_3' = \beta_1 x_3 + \beta_3;\ x_4' = \alpha_1 x_4 + \alpha_3,\ y_4' = \beta_1 x_4 + \beta_3$$

respectively. If we take the corresponding cross-ratio of these four new points P_1', P_2', P_3', P_4', we get

$$\frac{P_1'P_2'}{P_3'P_2'}\frac{P_3'P_4'}{P_1'P_4'} = \frac{x_2' - x_1'}{x_2' - x_3'}\frac{x_4' - x_3'}{x_4' - x_1'} = \frac{\alpha_1 x_2 + \alpha_3 - \alpha_1 x_1 - \alpha_3}{\alpha_1 x_2 + \alpha_3 - \alpha_1 x_3 - \alpha_3}$$
$$\frac{\alpha_1 x_4 + \alpha_3 - \alpha_1 x_3 - \alpha_3}{\alpha_1 x_4 + \alpha_3 - \alpha_1 x_1 - \alpha_3} = \frac{x_2 - x_1}{x_2 - x_3}\frac{x_4 - x_3}{x_4 - x_1} = \frac{P_1P_2}{P_3P_2}\frac{P_3P_4}{P_1P_4}$$

Hence we see that the corresponding cross-ratios of the two sets of points are exactly equal.

Note that the invariance of cross-ratios means that if a cross-ratio of four given collinear points is 2, say, then the *corresponding* cross-ratio of the four points into which (13) sends the original four has the *same* value 2.

Another way to interpret the proof given in the text is the following, without supposing the line l to have been rotated and

translated into the position of $y = 0$. Suppose we want to send the line l into a line l' by (13) and to show that cross-ratios are kept invariant. The discussion in the text shows that any transformation T sending $y = 0$ to l (or l') keeps cross-ratios invariant, hence the inverse T^{-1} keeps them invariant. Therefore we can break up the transformation sending l to l' into the product of a transformation T_1 sending l to $y = 0$ and a transformation T_2 sending $y = 0$ to l'. Since T_1 and T_2 keep cross-ratios invariant (by the proof in the text), the product $T' = T_1T_2$ does the same thing.

There are *twenty-four* ways of arranging the four points when taking different cross-ratios; however, there are only *six* (or less than six) distinct values for these cross-ratios (as we leave for the student to prove in the exercises), namely (if one such cross-ratio has the value r)

$$(26) \qquad r, \frac{1}{r}, 1 - r, \frac{1}{1-r}, \frac{r}{r-1}, \frac{r-1}{r}$$

For example, the following cross-ratios are evidently equal to one another

$$\frac{P_1P_2}{P_3P_2}\frac{P_3P_4}{P_1P_4}, \quad \frac{P_2P_1}{P_4P_1}\frac{P_4P_3}{P_2P_3}, \quad \frac{P_3P_4}{P_1P_4}\frac{P_1P_2}{P_3P_2}, \quad \frac{P_4P_3}{P_2P_3}\frac{P_2P_1}{P_4P_1}$$

yet these cross-ratios correspond to four distinct ways of arranging the four points.

ILLUSTRATIVE EXAMPLE. The four points on the x-axis with abscissas 1, 2, 3, 4 have the six cross-ratios

$$\frac{2-1}{2-3}\frac{4-3}{4-1} = -\tfrac{1}{3}, \quad \frac{3-1}{3-2}\frac{4-2}{4-1} = \tfrac{4}{3}, \quad \frac{3-1}{3-4}\frac{2-4}{2-1} = 4,$$

$$\frac{2-1}{2-4}\frac{3-4}{3-1} = \tfrac{1}{4}, \quad \frac{4-1}{4-3}\frac{2-3}{2-1} = -3, \quad \frac{4-1}{4-2}\frac{3-2}{3-1} = \tfrac{3}{4}$$

The four points (0,6), (1,9), (−1,3), (−2,0) on the line $y = 3x + 6$ have one cross-ratio $\dfrac{1-0}{1+1}\dfrac{-2+1}{-2-0} = \dfrac{9-6}{9-3}\dfrac{0-3}{0-6} = \tfrac{1}{4}$, and the five other cross-ratios $4, \tfrac{3}{4}, \tfrac{4}{3}, -3, -\tfrac{1}{3}$.

EXERCISES

1. Show why the cross-ratios in the next to the last paragraph in the text are equal to one another and also how these four cases arise when considering all the possible cross-ratios of four given collinear points.

2. Prove the fact that there are only at the most six distinct cross-ratios of four points, and that these six are related to one another as in (26).

3. Prove by plane geometry that $P_1P_2/P_1L_2 = P_2P_3/L_2L_3 = \ldots$, etc.

4. The so-called inversion $x' = x/(x^2 + y^2)$, $y' = y/(x^2 + y^2)$ sends points on a line l through the origin into other points on the same line l. Show that cross-ratio is preserved. Hint: Take l as $y = y' = 0$. Why is there no lack of generality in this choice of l?

5. Find all six cross-ratios of $-1, 1, -2, 2$; of $-2, 1, 2, 4$; of (0,0), (1,2), $(-2,-4)$, (3,6) on the line $y = 2x$. For the points on $y = 2x$ find one of the cross-ratios in three ways: (*a*) using the abscissas of the points; (*b*) using the ordinates; (*c*) using the distance formula on P_1P_2, P_3P_2, P_3P_4, P_1P_4. Find all the cross-ratios of the points on $y = 2x$, first by direct computation, second by using (26) on one of the cross-ratios.

6. By using the distance formula prove analytically the equality in (25).

7. Make up a numerical example of the six cross-ratios of four points on a line l not passing through the origin.

8. If the transformation T sending $y = 0$ to l preserves cross-ratios (i.e., keeps them invariant), why then does T^{-1} do the same?

24. **Harmonic sets of points.** If we try all possible ways for the cross-ratios in (26) to be equal to one another for special values of r, we find there are just two distinct ways, namely, when $r = 1/r$ and so $r = -1$ (we cannot have $r = 1$ for four distinct points, since then $1 - r = 0$) and also when $r = 1/(1 - r)$ and so $r = (1 \pm \sqrt{3}\, i)/2$ where $i = \sqrt{-1}$. The second case is impossible for four real points. The first case gives us:

DEFINITION. A *harmonic set* of four collinear points consists of four such points that have -1 as one cross-ratio. (The other cross-ratios of these four points are then 2 and $\frac{1}{2}$.)

By all odds the most important sets of four collinear points are these harmonic sets. If the four points of a harmonic set are arranged from left to right in the order P_1, P_2, P_3, P_4, we see that the cross-ratio $P_1P_2/P_3P_4 \cdot P_3P_4/P_1P_4$ must be negative and so must have the value -1. We note that P_1P_3 can be looked upon as a line-segment divided internally by P_2 and externally by P_4 in the same ratio.

The Greeks called this set of four points the *golden section.* The two pairs of points P_1P_3 and P_2P_4 are said to *separate* one another *harmonically.* The point $P_1(P_3)$ is called the *harmonic conjugate* of $P_3(P_1)$ with respect to the pair of points P_2, P_4 and similarly for P_2 and P_4 with respect to P_1, P_3. If we are careful to arrange our points and choose our cross-ratio as above, we shall always get -1 as the value of our cross-ratio if our set of four distinct

collinear points is a harmonic set. However, the values $\frac{1}{2}$ or 2 for a cross-ratio at once stamp four collinear points as a harmonic set as surely as does the value -1.

Harmonic sets of points appear very frequently in analytic geometry. Thus for the ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a > b$ and $ae = \sqrt{a^2 - b^2}$, we have on the x-axis the harmonic set

$$(-a,0),\ (ae,0),\ (a,0),\ (a/e,0)$$

with a cross-ratio $\dfrac{ae + a}{ae - a}\dfrac{a/e - a}{a/e + a} = -1$, and also the harmonic set

$$(-a/e,0),\ (-a,0),\ (-ae,0),\ (a,0)$$

with a cross-ratio $\dfrac{-a + a/e}{-a + ae}\dfrac{a + ae}{a + a/e} = -1.$

Geometrically, we can say that on the major axis of an ellipse the two vertices, either focus, and the point where the corresponding directrix cuts the major axis form a harmonic set. Evidently the same theorem holds true for a hyperbola.

Again, the line $x + y = 1$ cuts the lines $x = 0$, $y = mx$, $y = 0$, $y = -mx$ in the points $(0,1)$, $(1/(1 + \mathrm{m}), m/(1 + m))$, $(1,0)$, $(1/(1 - m), -m/(1 - m))$ that have one cross-ratio $\dfrac{1/(1 + m) - 0}{1/(1 + m) - 1}\dfrac{1/(1 - m) - 1}{1/(1 - m) - 0} = -1$ and so evidently form a harmonic set for every value of m.

EXERCISES

1. Show why the cross-ratio $r = \frac{1}{2}(-1 \pm \sqrt{3}\,i)$ is impossible for four real points and also why $r = 1$ is impossible for four distinct points.

2. Make up a numerical example of a harmonic set of four points on a line l not through the origin.

3. Show that $r = -1$ and $r = \frac{1}{2}(-1 \pm \sqrt{3}\,i)$ are actually the only values of r that give cross-ratios that are equal in pairs.

4. Show that there are only three cross-ratios for a harmonic set of points, namely, -1, $\frac{1}{2}$, and 2.

5. Show that for a harmonic set $P_1P_2/P_3P_2 \cdot P_3P_4/P_1P_4$ must have the value -1.

6. In the last paragraph of the text we took the ellipse in the form $x^2/a^2 + y^2/b^2 = 1$. Why was no loss of generality incurred thereby? Why was the proof valid also for hyperbolas?

7. Prove that $y = 0$, $y = mx$, $x = 0$, $y = -mx$ cut any line $y = \alpha x + \beta$ in a harmonic set of points.

25. **A first hint of infinite points on lines.** Let us consider the equations (4) and (5) again. We make the following

DEFINITION. If the point $P'(x',y')$ does not lie on the conic (4), then (5) is called the *polar* of P' with respect to (4) and P' is called the *pole* of (5) with respect to (4).

We shall now prove the

THEOREM. *Any line l through a point P' cuts a conic* (4) *and the polar* (5) *of P' with respect to this conic in three points that form with P' a harmonic set of points.*

We take P' as (0,0) and l as $y = 0$. The polar of (0,0) with respect to (4) is $gx + fy + c = 0$. The line $y = 0$ cuts (4) in the points

$$\left(\frac{-g + \sqrt{g^2 - ac}}{a},0\right) \quad \text{and} \quad \left(\frac{-g - \sqrt{g^2 - ac}}{a},0\right)$$

and also $y = 0$ cuts the polar of (0,0) in the point $(-c/g,0)$. The point (0,0) and the three above-mentioned points form a harmonic set, since one cross-ratio of these four points is

$$\frac{0 - \dfrac{-g + \sqrt{g^2 - ac}}{a} \quad \dfrac{-c}{g} - \dfrac{-g - \sqrt{g^2 - ac}}{a}}{0 - \dfrac{-g - \sqrt{g^2 - ac}}{a} \quad \dfrac{-c}{g} - \dfrac{-g + \sqrt{g^2 - ac}}{a}} = -1$$

In the above paragraph, if $c \neq 0$ but $g \to 0$, then $(-c/g,0)$ approaches the form $(\pm\infty,0)$ according as $c \gtrless 0$ and $g \to 0$ through positive or negative numbers. We interpret this peculiar situation by saying that $y = 0$ cuts the polar of (0,0) in a so-called *infinite point*. Our reason for this interpretation lies in the fact that, if g is made smaller and smaller without becoming zero, the two lines $gx + fy + c = 0$ and $y = 0$ cut one another in a point that goes farther and farther out in the plane. Note that as $g \to 0$ these two lines become parallel lines.

Again let us consider the four points $P_1 = -a$, $P_2 = 0$, $P_3 = a$, $P_4 = x'$ on the x-axis, with one cross-ratio

$$r = \frac{0 + a \quad x' - a}{0 - a \quad x' + a} = -\frac{1 - a/x'}{1 + a/x'}$$

As $x' \to \infty$, $r \to -1$. To interpret this result we say that the two *ends* of a *line-segment* on a line l, the *midpoint* of this segment,

and an *infinite point* on l form a *harmonic set.* To make this interpretation more plausible we note that the farther out the point $P_4 = x'$ goes on l, the more nearly do the two segments P_1P_4 and P_3P_4 approach equality in length, whereas P_1P_2 and P_2P_3 are already numerically equal in length but opposite in sign. With this interpretation, we say that the two vertices and the center of any ellipse or hyperbola form with an infinite point on the major axis a harmonic set of points.

Finally we see that for the parabola $y^2 = 4\,px$ we have the focus $(p,0)$, the directrix $x = -p$, and the vertex $(0,0)$. But the four points $-p,\ 0,\ p,\ x' \to \infty$ form a harmonic set. Therefore, to keep this result in agreement with the similar theorem about ellipses and hyperbolas, we assume that parabola has an infinite vertex on its axis as well as the vertex $(0,0)$.

Mathematicians are continually making assumptions and interpretations like the above in order to take care of apparently exceptional cases. We must be on the lookout for such cases in this course.

EXERCISES

1. Find the cross-ratios of $1, 0, 2, x$ as $x \to \infty$.

2. Show why there is no loss of generality in proving the theorem because of choosing P' as $(0,0)$ and l as $y = 0$. Hint: Rotate and translate P' and l to these positions.

3. Prove that (13) sends a point P, a conic C, and the polar of P with regard to C into P', C' and the polar l' of P' with regard to C. Hint: Use cross-ratios.

4. Prove that a focus and the corresponding directrix of a conic are pole and polar with respect to this conic. Hint: Take the conics in the so-called normal (type or canonical) forms $x^2/a^2 \pm y^2/b^2 = 1$ and $y^2 = 4\,px$.

5. Try to find the polar of the center of a conic. Hint: Take the center at $(0,0)$ and the conic as $x^2/a^2 \pm y^2/b^2 = 1$ or $x^2 + y^2 = r^2$.

6. To interpret the result of Ex. 5 we note that the polar of $P'(x',y')$ with respect to $x^2/a^2 \pm y^2/b^2 = 1$ can be put in the form $\dfrac{x}{a^2/x'} \pm \dfrac{y}{b^2/y'} = 1$. If now $x' \to 0$ and $y' \to 0$, what happens to this polar? Hence how do we interpret Ex. 5?

7. Why is it we do not lose generality in our proofs in Exs. 4, 5, 6 by taking the conics in the normal forms instead of in the more general form (4)?

8. Find the pole of $3x + 2y - 4 = 0$ with respect to the conic $x^2/9 + y^2/16 = 1$. Check up on the theorem in the text for this pole, polar, and conic.

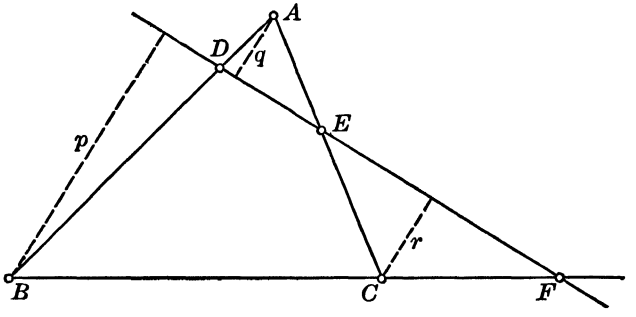
26. A geometric construction for a harmonic set of points. We shall now derive a *geometric construction* for a *harmonic set* of

points on a line. First we must prove a theorem due to an ancient Greek geometer, namely,

MENELAUS' THEOREM. *If DEF is a line cutting a triangle ABC in the points D on AB, E on AC, and F on BC, then we have*

$$AD \cdot BF \cdot CE = AE \cdot CF \cdot BD$$

Let us take the figure below. We drop the perpendicular p (from B), q (from A), and r (from C) onto the line DEF. Then

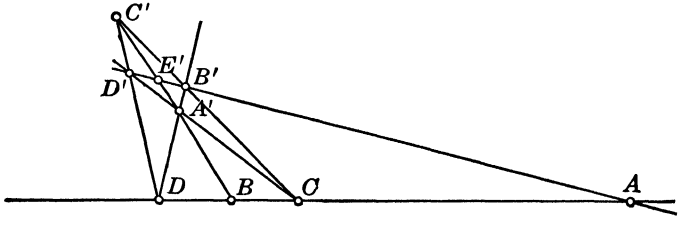


$BD/AD = p/q$ (from similar right triangles), $CE/AE = r/q$, $BF/CF = p/r$. Therefore

$$\frac{AD}{BD}\frac{BF}{CF}\frac{CE}{AE} = \frac{q}{p}\frac{p}{r}\frac{r}{q} = 1 \quad \text{or} \quad AD \cdot BF \cdot CE = AE \cdot CF \cdot BD$$

as was to be proved.

Let us now consider the following quadrilateral A', B', C', D', with its pairs of opposite sides produced until they meet, and



with its diagonals drawn. We shall see later on that this completed figure is called a *complete quadrangle*, and is of great importance.

We shall prove that the four points, A, B, C, D form a harmonic set (i.e., that $AC/BC \cdot BD/AD = -1$). We do this by using Menelaus' theorem four times for the triangle $AE'B$ and the four

transversals $B'A'D$, $D'A'C$, $C'B'C$, $C'D'D$, respectively. From this theorem we obtain

$$(a) \qquad E'B' \cdot AD \cdot A'B = E'A' \cdot BD \cdot AB'$$
$$(b) \qquad E'D' \cdot A'B \cdot CA = E'A' \cdot BC \cdot AD'$$
$$(c) \qquad E'B' \cdot AC \cdot BC' = E'C' \cdot BC \cdot AB'$$
$$(d) \qquad E'D' \cdot BC' \cdot AD = E'C' \cdot BD \cdot AD'$$

Dividing (a) by (c) we get

$$(e) \qquad \frac{AD}{AC}\frac{A'B}{BC'} = \frac{BD}{BC}\frac{E'A'}{E'C'}$$

Dividing (b) by (d) we get

$$(f) \qquad \frac{A'B}{BC'}\frac{AC}{AD} = \frac{E'A'}{E'C'}\frac{BC}{BD}$$

Dividing (e) by (f) we get

$$(g) \qquad \left(\frac{AD}{AC}\right)^2 = \left(\frac{BD}{BC}\right)^2$$

Hence we have

$$\frac{AC}{BC}\frac{BD}{AD} = -1$$

looking upon AC, BC, BD, AD as directed line-segments, since BC has its direction opposite to that of the other three segments.

Note that dividing (a) by (b) and (c) by (d), or dividing (a) by (d) and (b) by (c), gives us harmonic sets on the other two sides of the triangle $AE'B$. Also note that the two pairs of points AB and CD separate each other harmonically. (Compare §24.)

From the above result there readily follows a method of constructing a harmonic set of points given *three* of them *arranged in a certain order*. Suppose in the preceding figure we are given D, B, C in this order and wish to construct A as the harmonic conjugate of B with respect to C and D. (Compare §24.) Through D we can draw two arbitrary lines DA' and DD'. Through C we can draw one arbitrary line cutting DA' in a point A' and DD' in a point D'. Then the line BA' cuts DD' in a point C', and CC' cuts DA' in a point B'. The line $D'B'$ cuts the line DBC in the desired point A.

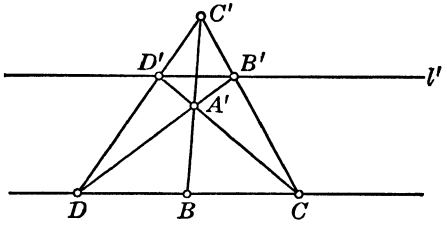
Analytically we can see as follows that A is a unique point (if D, B, C are given in this order and A is to be the harmonic

conjugate of B with regard to C and D). Suppose D is (0,0), B is (1,0), C is $(a,0)$. We wish to determine $A(x,0)$ so that

$$\frac{AC}{BC}\frac{BD}{AD} = \frac{a-x}{a-1}\frac{0-1}{0-x} = -1$$

This equation gives us the unique value $x = a/(2-a)$, so there is only one such point A $(a/(2-a),0)$.

We remark that if $a \to 2$ in the last paragraph, A approaches the form $(\infty,0)$. In this case B is the midpoint between D and C, hence (since $BD/BC = -1$) we must have $AC/AD = 1$. But no finite point A can satisfy this condition (since $C \neq D$). Therefore we must either debar this case or assume the existence of an infinite point on the line DBC. (Compare §25.) If we draw the figure for this case we find that $D'B'$ is parallel to DBC.



From the last sentence above we have a method of *bisecting a line-segment DC* (see the adjoining figure). Draw any line l' parallel to the line DC. Through D draw two arbitrary lines cutting l' in the points D' and B'. Then $D'C$ cuts DB' at A' and CB' cuts DD' at C'. The line $C'A'$ bisects the segment DC at B.

EXERCISES

1. Explain fully why, in the last paragraph, $C'A'$ bisects DC at B.
2. In the analytic proof in the test of the uniqueness of the point A why is there no loss of generality incurred by taking the line DBC as $y = 0$, also D as (0,0) and B as (1,0)?
3. Give a construction for drawing through a point (say D') a line l' parallel to a given line l (say DBC) not passing through D'.
4. Show that on the other two sides of the triangle $AE'B$ in the first figure of the text there are formed harmonic sets of points.
5. Given three collinear points DBC, find A as
   (*a*) the harmonic conjugate of C with regard to D and B,
   (*b*) the harmonic conjugate of D with regard to B and C,
   (*c*) the harmonic conjugate of B with regard to D and C.
6. Prove Menelaus' theorem for a line DEF cutting the triangle ABC of the text (*a*) beyond B and C; (*b*) beyond B but between A and C.
7. Taking E' as (0,0), $E'B$ as $x = 0$, $E'A$ as $y = 0$, A as (2,0), B as (0,2), C as (1,1), find the coordinates of the other points in the second figure in the text and the equations of the other lines.

8. Use Ex. 7 to prove the results in the text analytically. Why is there no loss of generality in the choice of the coordinates of the points and the equations of the lines in Ex. 7?

27. Harmonic sets of concurrent lines. In Ex. 7 of §24 we discovered that the four concurrent lines $x = 0$, $y = 0$, $y = mx$, $y = -mx$ (for m arbitrary) cut any line $y = \alpha x + \beta$ in a harmonic set of points.

DEFINITION. A set of *four concurrent lines* that cuts any arbitrary line in a harmonic set of points is called a *harmonic set of lines.*

We prove next a more general theorem, namely, that *if we have any two lines* $l_1 \equiv a_1x + b_1y - c_1 = 0$* *and* $l_2 \equiv a_2x + b_2y - c_2 = 0$ *and take four lines of the pencil*† *of lines*

$$l_1 + \lambda l_2 \equiv (a_1x + b_1y - c_1) + \lambda(a_2x + b_2y - c_2) = 0$$

with parameters‡ $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *such that*

$$\frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_3}\frac{\lambda_4 - \lambda_3}{\lambda_4 - \lambda_1} = -1$$

then these four lines form a harmonic set of lines. Also, conversely, if four lines of this pencil form a harmonic set, then one of the cross-ratios§ *of their parameters has the value* -1.

Without loss of generality we can consider the points in which these four lines cut $y = 0$, namely $((c_1 + \lambda_i c_2)/(a_1 + \lambda_i a_2), 0)$, where $i = 1, 2, 3, 4$. Calling these four points $P_i(x_i, 0)$ where $i = 1, 2, 3, 4$ and taking a cross-ratio we find that

$$\frac{x_2 - x_1}{x_2 - x_3}\frac{x_4 - x_3}{x_4 - x_1} = \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_3}\frac{\lambda_4 - \lambda_3}{\lambda_4 - \lambda_1}$$

(We leave the details to the exercises.)

* The expression $l \equiv ax + by - c = 0$ is brief for "the line l whose equation is $ax + by - c = 0$."

† We carry over from elementary analytic geometry the notion of a *pencil* of lines (i.e., *all* the lines through the point of intersection of *two given lines*) whose equation $l_1 + \lambda l_2 = 0$ has an arbitrary constant λ called the *parameter.*

‡ Different values of λ in $l_1 + \lambda l_2 = 0$ give different lines of the pencil.

§ Note how we speak of the cross-ratios of four parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, meaning thereby such expressions as $(\lambda_2 - \lambda_1)/(\lambda_2 - \lambda_3) \cdot (\lambda_4 - \lambda_3)/(\lambda_4 - \lambda_1)$ which in the sense of §23 would be cross-ratios of four points $(\lambda_1, 0)$, $(\lambda_2, 0)$, $(\lambda_3, 0)$, $(\lambda_4, 0)$.

This result shows that (*a*) if the four points on $y = 0$ form a harmonic set with the above cross-ratio (say) equal to -1, then the corresponding cross-ratio of the parameters of the four lines also has the value -1; and (*b*) if the above cross-ratio of the parameters equals $-1, \frac{1}{2}$, or 2, then the four points of intersection of these lines with $y = 0$ form a harmonic set. (Note that the four lines $x = 0$, $y = 0$, $y = mx$, $y = -mx$ are lines of the pencil $y + \lambda x = 0$ with parameters $\lambda_1 = \infty$*, $\lambda_2 = 0$, $\lambda_3 = -m$, $\lambda_4 = m$, respectively.)

Another way to prove the above theorem would be to reduce the pencil of lines to the form $y' + \lambda x' = 0$ by the transformation $a_1x + b_1y - c_1 = y'$, $a_2x + b_2y - c_2 = x'$. By doing this we lose no generality because (13) preserves the cross-ratios of points; also the above transformation has no effect at all on the parameter λ (i.e., an old line $l_1 + \lambda l_2 = 0$ goes into $y' + \lambda x' = 0$). If we cut any line $y' = mx' + b$ by four of these lines with parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we get the points $(-b/(\lambda_i + m), b/(\lambda_i + m))$ where $i = 1, 2, 3, 4$. Taking the same cross-ratio of these four points that we took in the preceding paragraph, we find its value is $(\lambda_2 - \lambda_1)/(\lambda_2 - \lambda_3) \cdot (\lambda_4 - \lambda_3)/(\lambda_4 - \lambda_1)$.

Note that we have proved furthermore that *every line is cut by a harmonic set of lines in a harmonic set of points, and conversely that a harmonic set of points on any line l is joined to any point P not on l by a harmonic set of lines.* These two *converse*† statements are also the *plane duals* of one another. (See §22.) Also, we have shown above that (whether or not we are dealing with a harmonic set) the cross-ratios of the above-mentioned points equal the corresponding cross-ratios of the parameters of the four lines of the pencil.

ILLUSTRATIVE EXAMPLE. As a numerical illustration of the preceding discussion we note that the four lines $(3x + 4y - 1) + \lambda(x - y - 2) = 0$ where $\lambda = 1, 2, 3, 4$, cut $y = 0$ in the respective points $(\frac{3}{4},0)$, (1,0), (7/6,0), (9/7,0). We have one cross-ratio

$$\frac{1 - \frac{3}{4}}{1 - \frac{7}{6}} \frac{\frac{9}{7} - \frac{7}{6}}{\frac{9}{7} - \frac{3}{4}} = \frac{2 - 1}{2 - 3} \frac{4 - 3}{4 - 1} = -\frac{1}{3}$$

* By the line with $\lambda_1 = \infty$ we mean the line $1/\lambda_1 y + x = 0$ where $\lambda_1 \to \infty$, or putting $\lambda_1 = \lambda'/\lambda''$ and clearing $y + \lambda'/\lambda'' x = 0$ of fractions we mean $\lambda'' y + \lambda' x = 0$, where $\lambda'' \to 0$ but $\lambda' \neq 0$.

† The *statement* and *proof* of converse theorems constitute an important and often difficult task.

EXERCISES

1. If $l_1 = 0$ and $l_2 = 0$ are two lines, prove that the pencil consisting of all the lines through the point of intersection of these two lines is given by the equation $l_1 + \lambda l_2 = 0$.

2. Prove Ex. 7 of §24 by using the idea of a pencil of lines and the cross-ratio of the four parameters.

3. Why is there no loss of generality incurred in the first proof of the text by taking $y = 0$ as the line containing the harmonic set of points?

4. Fill in the details of the two proofs in the text. Why do these discussions prove so many more facts than the ones we started to prove?

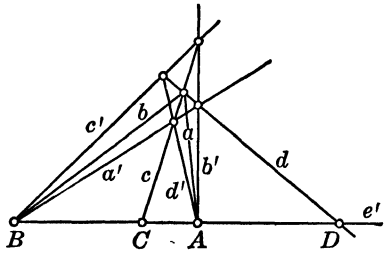
5. Dualize the figure formed of four lines of a pencil cut by two distinct lines in two sets of four points.

6. Make up a harmonic set of lines, find the points in which they cut $y = 0$, and take cross-ratios.

7. Make up an example dual to Ex. 6, and solve it.

8. Make up and solve examples like Exs. 5 and 6 for four concurrent lines that do not form a harmonic set.

28. **A geometric construction for a harmonic set of lines.** In this section we shall *dualize* (i.e., give the plane dual) of the geometric construction given in §26 for a harmonic set of points. (See the adjoining figure.) Instead of the five points A', B', C', D', E' in §26, we have now five lines a', b', c', d', e'. Instead of the four points C, D, B, A in §26, determined as follows — C by the intersection of the pair of opposite sides $C'B'$, $D'A'$ of the quadrilateral $A'B'C'D'$; B by the intersection of the diagonal $C'A'$ of the quadrilateral with the line $ADBC$; etc. — we have four lines c, d, b, a, determined as follows — c as the line joining the pairs of opposite vertices $c'b'$; $d'a'$* of the quadrilateral $a'b'c'd'$; b as the line from the point of intersection of two opposite sides a',c' of the quadrilateral to the point $adbc$; etc. To show that the four lines c, d, b, a are really a harmonic set of lines we have merely to notice that these lines intersect the line e' in a harmonic set of points C, D, B, A determined thereon by the same quadrilateral $a'b'c'd'$.



* Note how we specify the *point of intersection* of c' and b' as $c'b'$ and the point of intersection of the four lines a, d, b, c as $adbc$. This is the plane dual of the way we specify in §26 the line joining the points C' and B' as $C'B'$, etc.

Just as for a harmonic set of points, so here we call the line a the *harmonic conjugate* of b with respect to c and d. If a' and c' (say) are parallel lines, we must draw b parallel to them. We may assume here an infinite point B of intersection of a', c', and b. This special case need not arise in §26. (Why?)

Note how the figure composed of the four sides a', b', c', d' (and the four vertices) of a quadrilateral, together with the two points of intersection A and B of the two pairs of opposite sides a', c' and b', d', is exactly the *plane dual* of the figure in §26. The figure in this section is called a *complete quadrilateral.*

EXERCISES

1. Describe completely the figure in this section as the dual of the figure in §26.
2. Give the complete description of the four lines c, d, b, a.
3. Dualize Ex. 5 in §26 and solve this dual example. Hint: First dualize the construction given in the text of §26.
4. Show that at A and B in the figure in the text we have two more harmonic sets of lines. This is the plane dual of Ex. 4 in §26. Hint: The four points on c form a harmonic set. (Why?)
5. Show that the lines joining C to ab, C to dc', C to $c'd'$, C to $a'b'$ form a harmonic set. Hint: The four points on d form a harmonic set. (Why?)
6. Construct as in Ex. 5 a harmonic set of lines concurrent at D. Prove these are a harmonic set of lines.
7. Answer the query (Why?) in the next to the last paragraph of the text.
8. Taking the point $abcd$ as (0,0), the line d as $x = 0$, the line c as $y = 0$, the line e as $x + y = 2$, the line a as $y = x$, find the equations of the other lines in the figure in the text and the coordinates of the other points. (This is the dual of Ex. 7 in §26.)
9. Use Ex. 8 to prove the results in the text analytically. Why is there no loss of generality in the choice of the equations of the lines in Ex. 8?

29. **A first hint of projection and section.** In §27 we discussed four concurrent lines of a pencil of lines $l_1 + \lambda l_2 = 0$ with parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and four collinear points P_1, P_2, P_3, P_4 that were so situated that $P_i (i = 1, 2, 3, 4)$ lay on the line with parameter λ_i. This leads us to a definition.

DEFINITION. If a set of collinear points $P_i (i = 1,2, \ldots)$ are such that each lies on a line $l_i (i = 1,2, \ldots)$ of a pencil of lines concurrent in a point P (called the *center* of the pencil of lines), then the points P_i are said to be *projected* from the point P by the lines l_i. *Conversely* (and *dually*) the lines l_i are said to be cut in a *section* by the collinear points P_i.

These ideas of *projection* and *section* are fundamental in the

purely geometric (*synthetic*, as opposed to *analytic*) side of *projective geometry*.* We shall deal with these ideas more fully later on. We merely call attention to them here where they arise for the first time. Since our approach is from the *analytic* side of projective geometry, therefore these ideas of projection and section do not appear at the *beginning* of our discussion.

The above set of collinear points and set of concurrent lines are said also to be *perspective* with one another. Also (because of plane duality) the set of points $P_i (i = 1,2, \ldots)$ on the line l are said to form a pencil (or *range*) of points with the line l as *axis*.

DEFINITION. A pencil π of points (lines) with axis (center) $l(P)$† is said to be *projective* with a pencil π' of points (lines) with axis (center) $l'(P')$ if π' is obtained from π as the result of a series of projections (sections) and sections (projections). Thus, if π is a pencil of points and we have π projected from a point P_1, then the pencil of lines with center at P_1 is cut in a section by l_1, then the pencil of points on l_1 is projected from a point P_2, and the pencil of lines with center at P_2 is cut by l' in the pencil of points π' — we say that π and π' are projective with each other.

EXERCISES

1. By using parentheses as in the text, give in one definition the definitions of a complete quadrangle and of a complete quadrilateral. See §§26, 28.

2. By using parentheses as in the text, give in one description the geometric constructions of a harmonic set of points and of a harmonic set of lines. See §§26, 28.

3. Prove that if a pencil π of points (lines) is projective with a pencil π', then any harmonic set of points (lines) in π must correspond to a harmonic set of points (lines) in π'. Hint: See §27.

30. **Parametric coordinates for collinear points; their cross-ratios.** Suppose we have a line through two given distinct points $P'(x',y')$ and $P''(x'',y'')$,‡ namely

$$\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ x'' & y'' & 1 \end{vmatrix} = 0 \tag{27}$$

* See §§31, 95 for definitions of *affine projective geometry* and *general analytic projective geometry*.

† In this definition we use the parentheses in order to give two dual definitions at once. The terms inside the sets of parentheses are to go together and those outside are to go together.

‡ Here x',y' and x'',y'' designate two *distinct definite points* and do not mean a *change of variables*.

In (27) we cannot have $\begin{vmatrix} x' & 1 \\ x'' & 1 \end{vmatrix} = \begin{vmatrix} y' & 1 \\ y'' & 1 \end{vmatrix} = 0$ because then $x' = x''$, $y' = y''$, and $P' = P''$ (contrary to the hypothesis that P' and P'' are distinct points). Suppose then that $\begin{vmatrix} x' & 1 \\ x'' & 1 \end{vmatrix} \neq 0$.

We can now solve the two equations

$$\lambda x' + \mu x'' = x, \quad \lambda + \mu = 1$$

for λ and μ. If we multiply the second row of (27) by the value of λ that we obtain and multiply the third row by the value of μ, then subtract these two rows from the first, we obtain

$$\begin{vmatrix} 0 & y - \lambda y' - \mu y'' & 0 \\ x' & y' & 1 \\ x'' & y'' & 1 \end{vmatrix} = -(y - \lambda y' - \mu y'') \begin{vmatrix} x' & 1 \\ x'' & 1 \end{vmatrix} = 0$$

Since $\begin{vmatrix} x' & 1 \\ x'' & 1 \end{vmatrix} \neq 0$, we must have $y - \lambda y' - \mu y'' = 0$.

This result shows us that the coordinates of any point $P(x,y)$ on the line (27) can be written in the so-called *parametric* form

$$x = \lambda x' + \mu x'', \quad y = \lambda y' + \mu y'' \tag{28}$$

Since $\lambda = 1 - \mu$, we can also write (28) as

$$x = x' + \mu(x'' - x'), \quad y = y' + \mu(y'' - y') \tag{28'}$$

where μ is called a *parameter*. (Note that $\mu = 0$ gives us P' and $\mu = 1$ gives us P''.)

If we take the cross-ratio $(x_2 - x_1)/(x_2 - x_3) \cdot (x_4 - x_3)/(x_4 - x_1)$ of four points $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_3(x_3,y_3)$, $P_4(x_4,y_4)$ on (27) with parameters μ_1, μ_2, μ_3, μ_4, respectively, we find that it has the value $(\mu_2 - \mu_1)/(\mu_2 - \mu_3) \cdot (\mu_4 - \mu_3)/(\mu_4 - \mu_1)$. Hence we see *that the cross-ratios of four collinear points that have their coordinates expressed in the parametric form of* (28') *are the same as the cross-ratios of their four parameters.* Compare the dual result in §27.

ILLUSTRATIVE EXAMPLE. Any point on the line joining (1,1) and (3,3) can be given by the parametric coordinates $x = 1 + 2\mu$, $y = 1 + 2\mu$. If we take the above-mentioned cross-ratio of the four points $(-1,-1)$, (0,0), (2,2), (4,4) with parameters -1, $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, respectively, we obtain

$$\frac{0+1}{0-2}\frac{4-2}{4+1} = \frac{-\frac{1}{2}+1}{-\frac{1}{2}-\frac{1}{2}}\frac{\frac{3}{2}-\frac{1}{2}}{\frac{3}{2}+1} = -\frac{1}{5}$$

Another way to obtain the parametric coordinates (28′) for the points on a line joining $P'(x',y')$ to $P''(x'',y'')$ is to write this line as

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'}$$

We know from algebra that, if two fractions are equal, both their numerators and their denominators are proportional. Hence we have $y - y' = \mu(y'' - y')$ and $x - x' = \mu(x'' - x')$, from which we get (28′).

To bring out the dual relation between sets of points on a line and sets of lines through a point, we might obtain results similar to those in §27 in the following manner. We start with the fact (proved in elementary analytic geometry) that for three lines

$$l \equiv ux + vy + w = 0,\quad l' \equiv u'x + v'y + w' = 0,\quad l'' \equiv u'' + v''y + w'' = 0$$

to be concurrent we must have

$$\text{(29)} \qquad \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix} = 0$$

By an argument exactly similar to that in the preceding paragraphs of this section we find for the coefficients of any line in the pencil of lines through the point of intersection of l' and l'' the parametric equations

$$\text{(30)} \qquad \tau u = \rho u' + \sigma u'', \quad \tau v = \rho v' + \sigma v'', \quad \tau w = \rho w' + \sigma w''$$

$$\text{(30′)} \qquad \lambda u = u' + \nu u'', \quad \lambda v = v' + \nu v'', \quad \lambda w = w' + \nu w''$$

where τ, λ are *arbitrary constants* which appear because a line $ux + vy + w = 0$ is also given by $kux + kvy + kw = 0$ where k is an arbitrary constant (not zero), also $\lambda = \tau/\rho$, $\nu = \sigma/\rho$.

We can obtain results for lines like those in this section for points if we consider a pencil of lines through the point of intersection of two lines $u'x + v'y + 1 = 0$ and $u''x + v''y + 1 = 0$. Then (29) has the form

$$\begin{vmatrix} u & v & 1 \\ u' & v' & 1 \\ u'' & v'' & 1 \end{vmatrix} = 0$$

and (30′) has a form

$$u = u' + \sigma(u'' - u'), \quad v = v' + \sigma(v'' - v')$$

where $ux + vy + 1 = 0$ is the equation of any line in this pencil. Note that this discussion debars lines through the origin because such lines cannot be put in the form $ux + vy + 1 = 0$. (We leave the details of the discussions in the last two paragraphs for the student in the exercises.)

EXERCISES

1. Fill in the details of the discussions in the last two paragraphs of the text.
2. Obtain (28′) from the equation of the line in the form

$$\frac{y - y'}{y' - y''} = \frac{x - x'}{x' - x''} = \mu$$

3. Derive the parametric coordinates

$$x = x' + r\cos\theta, \quad y = y' + r\sin\theta$$

for the points on a line through $P'(x',y')$ and making an angle θ with the x-axis. Show that

$$\frac{x_2 - x_1}{x_2 - x_3}\,\frac{x_4 - x_3}{x_4 - x_1} = \frac{r_2 - r_1}{r_2 - r_3}\,\frac{r_4 - r_3}{r_4 - r_1}$$

What is the geometrical meaning of r? Why are these parametric coordinates valid only for an ordinary frame of reference?

4. Why are the parametric coordinates (28′) valid for any frame of reference?

5. Derive the equation (29).

6. In (28′) where do the points lie relative to P' and P'' if $\mu < 0$, if $0 < \mu < 1$, if $\mu > 1$? Hint:

$$\mu = \frac{x - x'}{x'' - x'} = \frac{y - y'}{y'' - y'} = \frac{\sqrt{(x - x')^2 + (y - y')^2}}{\sqrt{(x'' - x')^2 + (y'' - y')^2}}.$$ (Prove this.)

7. If P' is (2,3) and P'' is (3,5), find four definite points on the line $P'P''$ (in parametric coordinates) and find one of their cross-ratios.

8. Do the same as in Ex. 7 for the line joining $(-1,-1)$ and (4,7).

9. Derive (28) by means of the formula for the coordinates of a point $P(x,y)$ that divides the segment of the line joining $P'(x',y')$ and $P''(x'',y'')$ in the ratio μ/λ. Hint: In this formula you can take μ and λ so that $\mu + \lambda = 1$. (Why?)

CHAPTER V

PRELIMINARY DISCUSSION OF GROUPS OF LINEAR TRANSFORMATIONS AND THEIR ASSOCIATED GEOMETRIES

31. **Introduction to groups of transformations.** In §§7, 8, 9, 12 we have already defined what is meant by the *products* of two linear transformations T_1 and T_2, namely T_1T_2 and T_2T_1, and have shown that ordinarily $T_1T_2 \neq T_2T_1$ so that multiplication is in general non-commutative (non-permutable, non-abelian) for transformations. Also we have defined the *inverse* T^{-1} of a transformation and have proved that $T^{-1}T = TT^{-1} = I$ (the *identical* transformation). We saw that $(T_1T_2)T_3 = T_1(T_2T_3)$, i.e., the multiplication of transformations is *associative.*

In the case of the translations (6) we saw that if T_1 and T_2 are two translations, then $T_1T_2 = T_2T_1 = T_3$ and, also, T_3 is a third translation, i.e., a transformation that can be put in the type form (6). Besides, T^{-1} (the inverse of a translation T) can also be classified under (6) with $-\alpha$ for α and $-\beta$ for β. Moreover, $I(x = x', y = y')$ can be looked upon as being of the form (6) with $\alpha = \beta = 0$.

Again we saw in §20 that certain linear affine transformations keep invariant the areas of triangles. If T_1 and T_2 are two such transformations evidently T_1T_2 and T_2T_1 are two other transformations that keep invariant the areas of triangles. Likewise T_1^{-1}, T_2^{-1}, and I preserve these areas. This is an example of a geometric way of describing certain sets (or collections) of transformations (by means of invariants). In this case the geometric condition gives rise to the corresponding analytic condition on (13), namely that $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 1$.

DEFINITION. A *group* of linear transformations (defined* *analytically*) is a *set* of transformations whose *equations* have a

* By "*defining*" we mean distinguishing or specifying what transformations belong to the group, i.e., giving criteria by which to decide whether or not a transformation is in the group.

given type form, such that the *products* of any two transformations of the set are found in the set and, also, the *inverse* of any transformation of the set and the *identical* transformation are found in the set. If, moreover, $T_1T_2 = T_2T_1$ for any two transformations T_1 and T_2 of this set, the group is called a *commutative* (or *abelian*) group.

DEFINITION. A *group* of linear transformations (defined *geometrically*) is a *set* of transformations $T_i(i = 1,2, \ldots)$, each of which has a *given geometric invariant* associated with it, such that T_iT_j and T_jT_i belong to the set for all values of i and j that give transformations in the set; also T_i^{-1} and I are found in the set.

We note that ordinarily we seek to find analytic conditions on the coefficients of the transformations of a group defined *geometrically*, so that we may also define this group *analytically*. Thus the group that preserves areas of triangles can be defined analytically by $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 1$.

The translations (6) form an abelian group, so also do the rotations (7). The transformations (9) form a non-commutative group. The transformations (12) form an abelian group (as the student will prove in the exercises). The transformations (13) form a non-abelian group called the general affine linear group.

The groups mentioned in the last paragraph are called *infinite groups* (or groups of *infinite order*) because each of them contains an infinite number of transformations. The group

$$x = ix', \quad y = jy' \tag{31}$$

where $i,j = \pm 1$ is a *finite* group of only four transformations (called *terms* or *members* of the group), and hence is said to be of *order four*.

Sometimes it is very difficult to write down the equations of a group of transformations defined geometrically, as for example the group of those sending a certain curve into itself. Again two *different* geometric invariants may define the *same* group, e.g., the group leaving areas of triangles invariant and the group leaving Γ and $h^2 - ab$ for the conic (4) invariant we saw (where?) were both the same, namely, the group with $\Delta = 1$.

We note that there are groups of transformations *not considered in this book* (since we shall confine our attention entirely to linear transformations). An important group omitted from our study

is that consisting of all the *inversions* with respect to circles with centers at the origin, with equations

$$x = \frac{r^2 x'}{x'^2 + y'^2}, \quad y = \frac{r^2 y'}{x'^2 + y'^2}$$

Finally, we call attention to the fact that certain *sets* of transformations do *not* form *groups*. For example, the equations

$$x = a_2 y', \quad y = b_1 x' + b_2 y'$$

do not define a group.

ILLUSTRATIVE EXAMPLE. Suppose we want the group of linear affine transformations that leave the curve $y^2 = 4x$ invariant (i.e., that send points on this curve into other points on the same curve). Performing (13) on $y^2 = 4x$ we have

$$b_1^2 x'^2 + b_2^2 y'^2 + 2\, b_1 b_2 x' y' + 2(b_1 b_3 - 2\, a_1) x' + 2(b_2 b_3 - 2a_2) y' + (b_3^2 - 4a_3) = 0$$

This equation must be of the form $c(y'^2 - 4x') = 0$ where $c \neq 0$, so we must have

$$b_1 = 0, \quad b_2 \neq 0, \quad a_1 = b_2^2, \quad b_2 b_3 - 2a_2 = 0, \quad b_3^2 - 4\, a_3 = 0$$

Hence the required group has equations of the form

$$x = b_2^2 x' + \tfrac{1}{2}\, b_2 b_3 y' + \tfrac{1}{4}\, b_3^2, \quad y = b_2 y' + b_3$$

EXERCISES

1. Prove that all the transformations of (13) that leave the origin fixed form a group, defined by $a_3 = b_3 = 0$.
2. Show that $x = y'$, $y = x'$ and $x = x'$, $y = y'$ form a group of order 2 (the order being the number of transformations in the group); similarly, $x = -x'$, $y = -y'$ and $x = x'$, $y = y'$ form a group of order 2.
3. If we take $\phi = \pi/n$ (where n is a given integer) in (7), prove that we have a group of order $2n$.
4. Prove that (12) gives an abelian group.
5. Look up all the facts quoted in the text.
6. Prove that (13) gives a non-abelian group.
7. Prove that $x = a_2 y'$, $y = b_1 x' + b_2 y'$ do not define a group.
8. Prove that the inversions quoted in the text form an abelian group.
9. Prove analytically that the equations in the last line of the text actually form a group according to the first definition in the text.
10. Find the linear equations that give the group leaving invariant (*a*) the curve $x^2 - y^2 = 1$; (*b*) the curve $xy = 1$. Hint: Compare the last paragraph in the text.
11. Prove analytically that the equations obtained in Ex. 10 actually form a group according to the first definition in the text.
12. Find the group of linear transformations (*a*) leaving the point $(-1, -1)$ fixed; (*b*) leaving the x-axis fixed.

13. Show analytically that the equations obtained in Ex. 12 actually give us a group.

14. Make up cases of sets of linear transformations that do not form groups.

32. Subgroups of groups of linear transformations. DEFINITION. A group of linear transformations, *all* of whose terms belong to a *larger* group, is said to be a *subgroup* of this larger group.

For example, the translations and rotations are subgroups of the group (9), whereas the latter group is a subgroup of the group defined by $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 1$, and this last group is a subgroup of the general linear affine group (13).

DEFINITION. A group that is made up entirely of the *powers** of a transformation T is called a *cyclic* group. Here T is said to *generate* the group.

Thus the group given in Ex. 3 of §31 is a cyclic group of finite order, whereas the group obtained by putting $\phi = 2\pi/\sqrt{2}$ in (7) is a cyclic group of infinite order.

EXERCISES

1. Prove the statements in the text. Show that the set of powers of one transformation T actually forms a group (which is abelian).

2. Prove that all the translations of (6) that have both α and β integers form a subgroup of (6). If we take the translations with α and β both positive integers, show that these do not form a subgroup.

3. Make up some finite and infinite subgroups of (13).

4. Make up some infinite subgroups of (13) defined geometrically and find their equations. Then show that these groups satisfy the first (analytic) definition of a group.

5. Make up some sets of transformations of (13) that are not subgroups.

6. Prove that (1), (2), (12) are subgroups of (13).

7. Find all the transformations in the group formed of the powers of $x = x'$, $y = x' - y'$.

8. Prove that all the transformations of (13) that are commutative with a given transformation T of (13) form a subgroup. Hint: If $TT_1 = T_1T$, then $T = T_1TT_1^{-1}$; hence $T_1^{-1}T = TT_1^{-1}$, and so the inverse of T_1 is also commutative with T.

33. Invariant points and lines of linear transformations. *Every transformation* (13) *has invariant* (*fixed*) *points and lines.* We *prove* this fact by actually *finding* these invariant points and

* The *powers* of transformations were defined in §9.

lines. Any fixed point of (13) must have coordinates that satisfy the equations (since for these points $x' = x$, $y' = y$)

$$x = a_1x + a_2y + a_3, \quad y = b_1x + b_2y + b_3$$

or

$$x(a_1 - 1) + a_2y + a_3 = 0, \quad b_1x + y(b_2 - 1) + b_3 = 0$$

Hence the single fixed (finite)* point of (13) is

$$(32) \quad x = \frac{\begin{vmatrix} -a_3 & a_2 \\ -b_3 & b_2 - 1 \end{vmatrix}}{\begin{vmatrix} a_1 - 1 & a_2 \\ b_1 & b_2 - 1 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 - 1 & -a_3 \\ b_1 & -b_3 \end{vmatrix}}{\begin{vmatrix} a_1 - 1 & a_2 \\ b_1 & b_2 - 1 \end{vmatrix}}$$

unless $\begin{vmatrix} a_1 - 1 & a_2 \\ b_1 & b_2 - 1 \end{vmatrix} = 0.$

From (24′) we see that the fixed lines of (13) must have coefficients that satisfy the equations (since for these lines $u' = cu$, $v' = cv$, $w' = cw$, where c is an arbitrary constant not zero)

$$\dagger\rho u = a_1u + b_1v, \quad \rho v = a_2u + b_2v, \quad \rho w = a_3u + b_3v + w$$

or

$$u(a_1 - \rho) + b_1v = 0, \quad a_2u + v(b_2 - \rho) = 0$$
$$a_3u + b_3v + w(1 - \rho) = 0$$

These are three homogeneous linear equations in the three unknowns u, v, w. For these equations to have a solution not all zeros we must have

$$(33) \quad \begin{vmatrix} a_1 - \rho & b_1 & 0 \\ a_2 & b_2 - \rho & 0 \\ a_3 & b_3 & 1 - \rho \end{vmatrix} = (\rho^2 - (a_1 + b_2)\rho + a_1b_2 - a_2b_1) \cdot (1 - \rho) = 0$$

The equation (33) is a cubic in ρ, so there are three fixed lines of (13) given by the three roots. The value $\rho = 1$ gives an anomalous result because the three equations

$$u(a_1 - 1) + b_1v = 0, \quad a_2u + v(b_2 - 1) = 0, \quad a_3u + b_3v + 0w = 0$$

can have no other solution than $u = 0$, $v = 0$, w arbitrary ($\neq 0$). We interpret the line $0u + 0v + w = 0$ as consisting entirely of infinite points. (Compare Ex. 6 in §25.)

* Later on (see §85), we shall see that we are led to *assume* the existence also of two fixed infinite points for (13).

† Here ρ is really $c\rho$ in terms of the ρ of (24′) because $u' = cu$, etc.

The intercepts of a line $ux + vy + w = 0$ are $a = -w/u$, $b = -w/v$. If $u \to 0$ and $v \to 0$ while $w \neq 0$, we see that the line moves farther and farther out in the plane. This fact justifies our interpretation of the fixed line of (13) given by $\rho = 1$.

Note that a transformation (13) may not only leave a line l *fixed* (as to *position*) but may even leave it *pointwise invariant* (i.e., leave *every* point on l *fixed*). Thus $x = x' + a_2y'$, $y = b_2y'$ leaves $y = y' = 0$ pointwise invariant.

ILLUSTRATIVE EXAMPLE. As an illustration we note that

$$x = 3x' + 2y' - 1 \quad y = x' - y' + 2$$

has the fixed point

$$x = \frac{\begin{vmatrix} 1 & 2 \\ -2 & -1-1 \end{vmatrix}}{\begin{vmatrix} 3-1 & 2 \\ 1 & -1-1 \end{vmatrix}} = -\frac{1}{3}, \quad y = \frac{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix}} = +\frac{5}{6}$$

From (33) we get

$$(\rho^2 - 2\rho - 5)(1 - \rho) = 0$$

We leave to the student the finding of these invariant lines in the exercises.

EXERCISES

1. Find the fixed lines of the illustration in the last paragraph of the text.
2. Find the subgroup of (13) that leaves the y-axis pointwise invariant.
3. Find the subgroup of (13) that leaves the point (1,1) linewise invariant. Hint: Every line $y - 1 = m(x - 1)$ must go into itself.
4. Find the subgroup of (13) that leaves the origin linewise invariant. Hint: Every line $y = mx$ must go into itself.
5. Find the fixed point and lines of the transformation

$$x = x' - y' + 3, \quad y = 5x' - 3y' + 1$$

6. Find the fixed point and lines of the translation (6) and of the rotation (7).
7. Explain the results of Ex. 6 for the translation (6).
8. If T is any transformation of a subgroup of (13) and S is any transformation of (13), if $S^{-1}TS = T'$ (another transformation of the subgroup) where S is allowed to run over all the transformations of (13), then the subgroup containing T is called a *self-conjugate subgroup* under (13). Prove that (6) is a self-conjugate subgroup under (13). Hint: See §§12, 13. Use matrices.

34. The parabolic metric group. At the end of §20 we derived the conditions for (13) to keep the angles between lines invariant, namely

$$a_1^2 + a_2^2 = b_1^2 + b_2^2 = \pm(a_1b_2 - a_2b_1) \neq 0, \quad a_1b_1 + a_2b_2 = 0$$

From these conditions we get (since $a_2^2b_2^2 = a_1^2b_1^2$)

$$a_1^2a_2^2 + a_2^4 - b_1^2a_2^2 - a_1^2b_1^2 = 0$$

or

$$(a_1^2 + a_2^2)\,(a_2^2 - b_1^2) = 0$$

Thus we infer that $a_2 = \pm b_1$, $a_1 = \mp b_2$ (since $a_1b_1 + a_2b_2 = 0$ and $a_1^2 + a_2^2 \neq 0$). Hence the linear affine transformations that leave angles invariant can be written

$$x = \alpha x' - \beta y' + \gamma_1, \quad y = \epsilon(\beta x' + \alpha y') + \gamma_2 \tag{34}$$

where $\epsilon = \pm 1$. These are called *similarity* transformations.

The group (34) is called the *parabolic metric group*. This group is closely connected with the ordinary plane (Euclidean) geometry. For instance, the rotations, translations, and also (31) together form a subgroup of (34) given by

$$x = \alpha x' - \beta y' + \gamma_1, \quad y = \epsilon(\beta x' + \alpha y') + \gamma_2 \tag{35}$$

where $\epsilon = \pm 1$, $\alpha^2 + \beta^2 = 1$.

The group (35) is called the group of *displacements* and *symmetries*. This is the group of *rigid motions* of figures that are used in elementary plane geometry to prove figures are *congruent* or *symmetric*, etc.

The larger group (34) gives figures that are similar to one another, because (34) preserves angles and gives $d' = (\alpha^2 + \beta^2)d$ where d is the distance between two points.

EXERCISES

1. Fill in all the details in the text, such as the complete proof that $a_2 = \pm b_1$ and $a_1 = \mp b_2$; that (35) includes (6), (7), and (31) and no other transformations; that (34) gives $d' = (\alpha^2 + \beta^2)d$.

2. Show that $a_1^2 + a_2^2 = b_1^2 + b_2^2 = \pm(a_1b_2 - a_2b_1) \neq 0$ follows from $a_1^2 + a_2^2 = b_1^2 + b_2^2$, $a_1b_1 + a_2b_2 = 0$.

3. Prove that (35) is a self-conjugate subgroup of (34). See Ex. 8 in §33.

4. Prove analytically that (34) gives a group; that (35) gives a group.

5. Show that, if $\Delta > 0$, then (34) preserves also the sense of angles; but, if $\Delta < 0$, then (34) changes θ into $\theta' = -\theta$.

6. Show that (34) for $\Delta > 0$ is a self-conjugate subgroup under (34). Hint: First show that (34) for $\Delta > 0$ forms a group.

7. Show that (35) for $\Delta = 1$ is a self-conjugate subgroup both under (35) and under (34) with $\Delta > 0$. Hint: First show that (35) with $\Delta = 1$ forms a group.

35. **Geometries associated with groups of transformations.** Associated with each group of transformations we find certain invariant algebraic expressions or equations, or other invariants, some of them absolute invariants, others relative invariants. (Compare §§18, 19, 20.) Thus the group (9) keeps invariant distances between points, angles between lines, areas of triangles. The group with $\Delta = 1$ keeps invariant the areas of triangles.

DEFINITION. An invariant is said to *belong* to a group G if there is *no larger* group G' containing G and also having this invariant.

According to this definition the invariant distance belongs to (35), the invariant angle belongs to (34) with $\Delta > 0$, the invariant area of a triangle belongs to the group with $\Delta = 1$. We now can define a *geometry* as follows.

DEFINITION. A *geometry* is said to be *associated* with two groups G and G' if it consists of the study of all the invariants that belong to this group G (as well as all other invariants under G) and if G' is the *smallest* group containing G and in which G is a *self-conjugate subgroup*.*

The most important geometry is the Euclidean geometry, which is associated with the groups (35) and (34).

Similarly we can define an *affine projective geometry* associated with the general affine group (13), which consists of the study of the invariants (both absolute and relative) under (13).

We have also a *rotation* geometry and a *translation* geometry, but these are of less importance for us. We might call Euclidean geometry a *sub-geometry* of affine projective geometry and rotation (translation) geometry a *sub-geometry* of Euclidean geometry.

We have seen that (13) has *fewer* invariants than any of its subgroups; however, it has cross-ratio as an invariant. We wish to emphasize here the fact that any *subgroup* G' of (13) will have *all* the invariants of (13), *plus* some more invariants that are peculiar to any subgroups of (13) of which G' is a further subgroup, *plus* still more invariants that are peculiar to G' alone. Thus the group (9) will have the invariants of (13), also those of the subgroup of (13) that leaves areas of triangles invariant, also those of (34), those of (35), and finally invariants *all its own.*

We saw (see §12, Ex. 5) that (13) sends a line into a line, a conic into a conic, etc. It sends, therefore, points of intersection

* For a definition of " self-conjugate subgroup " see Ex. 8 in §33.

of curves (lines) into points of intersection of the same or of other curves (lines). Also (13) sends hyperbolas into hyperbolas, parabolas into parabolas, and ellipses into ellipses or circles.

We can see, as follows, that (13) sends tangents to a curve into tangents to the same or another curve. We can define a tangent to a curve as a line whose equations, when solved with the equation of the curve, gives the coordinates of points (x_1,y_1), (x_2,y_2), etc., such that $x_1 = x_2$ and $y_1 = y_2$. But (13) sends $x_1 = x_2$ into $x_1' = x_2'$ and $y_1 = y_2$ into $y_1' = y_2'$ and, therefore, sends a tangent into a tangent. (Compare Ex. 4 in §18.)

We want to emphasize again the importance of *relative* invariants (as well as *absolute* invariants). For instance, Γ and $h^2 - ab$, connected with the conic (4), are relative invariants under (13), the first giving by its vanishing a degenerate conic and the second serving to distinguish the types of non-degenerate conics.

In most of our discussion of invariants we had to presuppose an ordinary frame of reference; otherwise the algebraic invariants lacked geometrical meaning. (Moreover, we looked upon the transformations of coordinates as *alibis*.) However, cross-ratio remains invariant under (13), a hyperbola remains a hyperbola, a parabola remains a parabola, an ellipse remains an ellipse (calling a circle a special case of an ellipse), even though the axes of reference should happen to be oblique with different-sized units on the two axes.

EXERCISES

1. Find some invariants that belong to (9).
2. Give some invariants belonging to the subgroup of (13) that has $a_1b_2 - a_2b_1 = 1$.
3. Show that (13) cannot send two curves that are tangent to each other into two curves that are not tangent to each other.
4. Show that (13) sends a line l_1 cutting a curve C in n coincident points at P into a line l cutting a curve C' in n coincident points at P'.
5. Show that (13) sends *points of inflection** into *points of inflection*. Hint: Use Ex. 4.
6. Find the subgroup of (13) that sends $y = mx + b_1$ and $y = -x/m + b_2$ into two perpendicular lines.
7. Find the subgroup of (13) that sends circles into circles.

* For a definition of *points of inflection* see any textbook on the calculus or §§43, 47.

CHAPTER VI

PRELIMINARY DISCUSSION OF IMAGINARY ELEMENTS IN GEOMETRY

36. **Imaginary points.** If we solve $x = -1$ simultaneously with $y^2 = 4x$, we find the two imaginary points of intersection $(-1,2i)$ and $(-1,-2i)$ where $i = \sqrt{-1}$. If we factor the equation $x^2 + 2xy + y^2 + 1 = 0$, we obtain the two imaginary lines $x + y + i = 0$ and $x + y - i = 0$. If we factor the equation $x^4 + 2x^2y^2 + y^4 + 1 = 0$, we obtain the two imaginary conics $x^2 + y^2 + i = 0$ and $x^2 + y^2 - i = 0$. Finally, we note the imaginary conic $x^2 + y^2 + 1 = 0$, whose equation is real but whose locus has no real points on it.

In the previous paragraph we have given examples of so-called *imaginary* points, lines, and curves. Now we shall deal more systematically with these imaginary elements of geometry. In this section we shall take up imaginary points.

DEFINITION. An *imaginary* point is a point one or both of whose coordinates are imaginary numbers, e.g., $(i,0)$ where $i = \sqrt{-1}$, and $(1 + i, 2 - 3i)$.

Each imaginary point lies on *one* and *only one* real line (it cannot lie on two real lines, because the intersection of two real lines must be a real point). We shall *prove* the above fact by actually finding the equation of the real line on which the imaginary point lies. (*Note this method of proof.*) Thus $(i,0)$ lies on the x-axis and $(1 + i, 2 - 3i)$ lies on the line $3x + y = 5$.

To determine the real line on which any such imaginary point $(a + ib, c + id)$ lies (where a, b, c, d are real numbers) we take the general equation of a straight line (23) and substitute in this equation the coordinates of the imaginary point, thus getting

$$(ua + vc + w) + i(ub + vd) = 0$$

It is shown in algebra that (since a, b, c, d, u, v, w are all real) from this equation follows

$$ua + vc + w = 0, \quad ub + vd = 0$$

Solving for u and v in terms of w, we have

$$u = \frac{\begin{vmatrix} -w & c \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad v = \frac{\begin{vmatrix} a & -w \\ b & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

if $ad - bc \neq 0$. If $ad - bc = 0$, i.e., if $a/b = c/d$, we find $w = 0$ and $u/v = -d/b$. In either case the real line through this imaginary point $(a + ib, c + id)$ has an equation of the form

$$-dx + by + ad - bc = 0 \tag{36}$$

On any *real* line there lie an *infinitely large number* of *imaginary* points. For example, on $y = 0$ are all the points $(a + ib, 0)$ where a and b are any real numbers. The real points on $y = 0$ have the coordinates $(a,0)$. Therefore we can say that there are in a sense *more* imaginary points on $y = 0$ than there are real, meaning thereby that we can take a fixed value b' for b and make correspond to each real point $(a_1,0)$, $(a_2,0)$, etc., an imaginary point $(a_1 + ib', 0)$, $(a_2 + ib', 0)$, etc., and still we have not exhausted the imaginary points on $y = 0$ because b can be given any real value. (Since any real line l can be transformed into $y = 0$, the above discussion applies equally well to every real line.)

If $P(a + ib, c + id)$ lies on a real line (23), then $P'(a - ib, c - id)$ lies on the same line, as is easy to prove.

DEFINITION. Any two points P and P' are called *conjugate imaginary* points if their abscissas (or ordinates) are conjugate imaginary numbers and their ordinates (or abscissas) are conjugate imaginary numbers (or real and equal).

The points P and P' in the last paragraph are conjugate imaginary points. So also are $(2 + i, 1)$ and $(2 - i, 1)$, also $(i,0)$ and $(-i,0)$, $(3 + 2i, 1 - i)$ and $(3 - 2i, 1 + i)$, but not $(1 + i, 2 + 3i)$ and $(1 + i, 2 - 3i)$. There are *innumerably more* pairs of *imaginary* points on a real line than there are pairs of *conjugate imaginary* points; for instance, $y = 0$ has all such imaginary pairs as $(i,0)$ and $(2 - i, 0)$, $(3 + i, 0)$ and $(2 - 7i, 0)$, or, in general, $(a_1 + ib_1, 0)$ and $(a_2 + ib_2, 0)$, where $a_1 \neq a_2$ and $b_1 \neq b_2$ or $a_1 = a_2$ and $b_1 \neq b_2$ or $a_1 \neq a_2$ and $b_1 = b_2$.

We cannot locate the exact position of an imaginary point on a real line because we use real points and lines in our frames of reference. Note also that there are *pairs of imaginary points*

that do *not* lie on *real* lines, e.g., the pair $(i,0)$ and $(0,i)$. If we allow the coefficients of our line (23) to be imaginary, we can see that $x + y - i = 0$ is the equation of what we might call an imaginary line determined by $(i,0)$ and $(0,i)$. (See the next section.)

EXERCISES

1. Check all the algebra in the text.

2. Prove that if a, b, c, d, u, v, w are real, then from $(ua + vc + w) + i(ub + vd) = 0$ follow the equations $ua + vc + w = 0$ and $ub + vd = 0$.

3. Fill in the details in the derivation of (36).

4. Prove that if $P(a + ib, c + id)$ lies on (23), so also does $P'(a - ib, c - id)$.

5. Find two pairs of conjugate imaginary points on the line $3x - 5y + 2 = 0$; also two pairs of ordinary imaginary points.

6. Make up two pairs of imaginary points that do not lie on real lines, and find the two imaginary lines determined by these pairs. Hint: Use the *determinant* form of the equation of a line.

7. Find the points of intersection of $x^2 + y^2 = 1$ and $2y = x^2 - 3$. Hint: To find $x' = \sqrt{a + bi}$ we put $\alpha + \beta i = \sqrt{a + bi}$ where α and β are real; hence $\alpha^2 + 2\alpha\beta i - \beta^2 = a + bi$, so $\alpha^2 - \beta^2 = a$ and $2\alpha\beta = b$. Why?

8. Review in an algebra textbook how to manipulate complex numbers. Note that $\sqrt{-2}\ \sqrt{-3} = \sqrt{2}\ \sqrt{3}\ i^2 = -\sqrt{6}$. Prove by *induction* that $(x + iy)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

9. Find the real line through $(3 + 2i, 1 - 5i)$; through $(3 + i, 1 + 2i)$.

10. Find some imaginary points (and the tangents at these points) on the curves $x^2/16 \pm y^2/9 = 1$, $y^2 = 4x$, $xy = 1$, $x^2 + y^2 = 1$, $y^2 = x^3$, $y = x^4$. Hint: For finding the tangents, use (3). Show from the proof of (3) why you can use (3) for an imaginary point of contact.

11. Find the six lines of the complete quadrilateral with vertices $(3 + i, 2 - i)$, $(3 - i, 2 + i)$, $(1 + 2i, i)$, $(1 - 2i, -i)$. See §26.

12. Prove analytically that the intersection of any two real lines is a real point, taking care of all possible cases.

13. In the next to the last paragraph of the text, show that all possible types of pairs of ordinary imaginary points are covered by the cases given there.

14. Find the conditions on a pair of imaginary points $(a_1 + ib_1, c_1 + id_1)$ and $(a_2 + ib_2, c_2 + id_2)$ that these two points lie on a real line. Hint: Expand

$$\begin{vmatrix} x & y & 1 \\ a_1 + ib_1 & c_1 + id_1 & 1 \\ a_2 + ib_2 & c_2 + id_2 & 1 \end{vmatrix} = 0$$

and put in the conditions that the resulting equation have only real coefficients.

15. Show why $w = 0$, in the first paragraph of the text, when $ad - bc = 0$.

37. Imaginary lines. We have introduced into our analytic geometry new entities called imaginary points. We shall now

define other new entities (*dual* to these points) that we shall call *imaginary lines.*

DEFINITION. By an *imaginary line* we mean a line whose equation has as one or more of its constants an imaginary number, e.g., $ix + (3 + i)y + 2 = 0$.

The line $(\alpha a + \alpha bi)x + (\beta a + \beta bi)y + (\gamma a + \gamma bi) = 0$ we call *real* because we can divide its equation through by $a + bi$ and obtain the real equation $\alpha x + \beta y + \gamma = 0$.

We can make any one of the constants real in the equation of an imaginary line by multiplying this equation by the number that is conjugate imaginary to this constant; for example, we can make the coefficient of y in $(2 - i)x + (1 + 2i)y + (3 + 4i) = 0$ real by multiplying this equation by $1 - 2i$.

Now we shall prove (what is exactly dual to the theorem about imaginary points) that there is *one* and *only one real* point on each *imaginary* line. (There is not more than one real point, because two real points determine a real line.) We can prove this fact for the general imaginary line

$$(37) \qquad (\alpha + i\beta)x + (\gamma + i\delta)y + (\epsilon + i\zeta) = 0$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ are real, by solving simultaneously the equations

$$\alpha x + \gamma y + \epsilon = 0 \quad \text{and} \quad \beta x + \delta y + \zeta = 0$$

and *actually finding* the real point with coordinates

$$x' = \frac{\begin{vmatrix} -\epsilon & \gamma \\ -\zeta & \delta \end{vmatrix}}{\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix}}, \quad y' = \frac{\begin{vmatrix} \alpha & -\epsilon \\ \beta & -\zeta \end{vmatrix}}{\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix}}$$

if $\alpha\delta - \beta\gamma \neq 0$. If $\alpha\delta - \beta\gamma = 0$, either $-\epsilon\delta + \gamma\zeta = -\alpha\zeta + \beta\epsilon = 0$ or one or both of the coordinates x', y' are infinitely large. In the former instance we have $\alpha/\beta = \gamma/\delta = \epsilon/\zeta$ and, if we divide the equation (37) by $\alpha + i\beta$, the line l proves to be real, contrary to hypothesis. In the latter instance the two lines $\alpha x + \gamma y + \epsilon = 0$ and $\beta x + \delta y + \zeta = 0$ are parallel; hence we can debar this case or else (which is preferable) *assume* the *existence* of an infinite point of intersection of these two lines, which point will be the one real point on (37).

ILLUSTRATIVE EXAMPLE. The line $(3 + i)x + (1 - i)y + i = 0$ has the real point $x' = -\frac{1}{4}$, $y' = \frac{3}{4}$ on it. The line $(3 + i)x + (3 + i)y - i = 0$ has an infinite real point.

Through any *real* point there passes an *infinity* of *imaginary* lines. (Compare the dual theorem for real lines.) Thus through (0,0) we have all the lines $y = (\mu + i\nu)x$, where μ and ν are any real numbers ($\nu \neq 0$).

If the line (37) passes through a real point P so also does the line

$$(37') \qquad (\alpha - i\beta)x + (\gamma - i\delta)y + (\epsilon - i\zeta) = 0$$

as is readily shown. The lines (37) and (37′) are called *conjugate imaginary* lines. If we multiply their equations together, we obtain the real quadratic equation

$$(\alpha x + \gamma y + \epsilon)^2 + (\beta x + \delta y + \zeta)^2 = 0$$

whose locus is a degenerate conic consisting of the pair of conjugate imaginary lines. Conversely, if (4) is a degenerate conic it must be either a pair of real lines, a double line (i.e., a pair of real lines l,l', where $l = l'$), or a pair of conjugate imaginary lines.

Just as with imaginary points, a pair of imaginary lines may *not* be a pair of *conjugate imaginary* lines and still intersect in a *real* point, as for example $y = ix$ and $y = 2ix$. We can say that there are in a sense *more* imaginary lines through a real point than there are *real* lines; compare the set of real lines $y = \alpha x$ and set of imaginary lines $y = (\alpha + i\beta)x$ through (0,0). We cannot draw an imaginary line because our frames of reference are real. Finally there are pairs of imaginary lines like $y = ix$ and $x = i + 1$ that do *not* intersect in a *real* point.

ILLUSTRATIVE EXAMPLE. To show how imaginary points and lines arise when we are studying real points, lines, and curves we shall consider the two curves $x^2 - y^2 = -3$ and $x^2 + y^2 = 1$, which intersect in the four points $P_1(i,2)$, $P_2(-i,2)$, $P_3(i,-2)$, $P_4(-i,-2)$. The line P_1P_2 is real (with equation $y = 2$), P_3P_4 is real ($y = -2$), P_1P_3 is imaginary ($x = i$), P_2P_4 is imaginary ($x = -i$), P_1P_4 is $y = -2ix$, P_2P_3 is $y = 2ix$.

Note that the pairs of points P_1,P_2 and P_3,P_4 are pairs of conjugate imaginary points. There are also two pairs of conjugate imaginary lines in this *configuration*,* namely $y = 2ix$ and

* By a *configuration* we mean any collection of points, lines, etc., that are considered as taken together in any discussion.

$y = -2ix$ and also $x = i$ and $x = -i$, the former pair intersecting in the real finite point (0,0) and the latter pair being interpreted as intersecting in an infinite real point.

We call attention to the way in which the discussion of imaginary lines *parallels* that of imaginary points. We can bring out this plane duality in the above paragraphs more clearly if we repeat some of the statements, enclosing in parentheses the words that refer to lines. An imaginary point (*line*) is a point (*line*) one or more of whose coordinates (*constants*) are imaginary numbers. An imaginary point (*line*) lies on (*passes through*) one real line (*point*) but cannot lie on (*pass through*) two real lines (*points*), because two real lines (*points*) intersect in (*determine*) a real point (*line*). If an imaginary point P (*line* l) lies on (*passes through*) a real line p (*point* L), then the conjugate imaginary point P' (*line* l') lies on p (*passes through* L).

EXERCISES

1. Fill in all the algebraic details omitted in the text, especially in the illustrative examples.

2. Prove analytically that two real lines intersect in a real point.

3. Give the complete details in the proof that an imaginary line (37) has one real point on it.

4. Prove that if (37) passes through a real point P so also does (37′)

5. In the text, just before (37′), why is $\nu \neq 0$?

6. Give an exact and complete definition of conjugate imaginary lines.

7. Prove that if (4) is a degenerate conic, it must be a pair of real lines, a double line, or a pair of conjugate imaginary lines.

8. Just as we did for points on $y = 0$, give a general formula (with all special cases) for all the pairs of imaginary lines through (0,0), distinguishing the pairs of conjugate imaginary lines.

9. Give several examples of pairs of imaginary lines that do not intersect in real points.

10. Find the conditions on the two imaginary lines

$$(\alpha_1 + \imath\beta_1)x + (\gamma_1 + \imath\delta_1)y + (\epsilon_1 + i\zeta_1) = 0$$

and

$$(\alpha_2 + \imath\beta_2)x + (\gamma_2 + i\delta_2)y + (\epsilon_2 + i\zeta_2) = 0$$

in order that they intersect in a real point. Hint: Solve these equations simultaneously and put the conditions on the point of intersection that it have real coordinates.

11. Prove analytically that, if the product of the equations of the two general imaginary lines l, l' in Ex. 10 is a real quadratic equation in x and y, then l and l' are conjugate imaginary lines. Hint: Multiply these two equations together and put the conditions on the resulting quadratic equation in x and y that it have only real constants.

12. Prove that the equation of every imaginary line (37) that has an infinite real point on it can be put in the form $\alpha' x + \gamma' y + (\epsilon' + i\zeta') = 0$, where α', γ', ϵ', ζ', are all real. Hint: The two lines $\alpha x + \gamma y + \epsilon = 0$ and $\beta x + \delta y + \zeta = 0$ must be parallel. (Why?) Now divide the resulting equation of the imaginary line by the imaginary factor that is common to the coefficients of x and y.

13. Find the real point on $(2 + i)x + (3 - 2i)y + (7 - 3i) = 0$; on $3x + iy + (2 - i) = 0$.

14. Take the equation of Ex. 13 and multiply it (*a*) by $(2 - i)$; (*b*) by $(3 + 2i)$, (*c*) by $(7 + 3i)$.

38. **Imaginary curves.** By an *imaginary curve* (*not* a line) we mean either one whose equation has one or more imaginary constants (i.e., coefficients or constant term) in it, or one whose equation, although real, is not satisfied by the coordinates of any real points. Thus $ix^2 + y^2 - 1 = 0$ is an example of the former type and $x^2 + y^2 + 1 = 0$ of the latter.

As was the case with the line, a curve may have imaginary coefficients that may be made real by removing a common imaginary factor. For example, $ix^2 + iy^2 - i = 0$ is the real circle $x^2 + y^2 - 1 = 0$. Imaginary conics of the second type are called imaginary ellipses because for their equations $h^2 - ab < 0$.

No imaginary algebraic curve* of *odd* degree can be of the *second type* because, if we solve $y = 0$ simultaneously with the equation of this curve, we get an odd degree equation in x (with real constants), but we have seen in algebra that every such equation has at least one real root (since imaginary roots occur in pairs). The imaginary quartic $x^4 + y^4 + x^2 + y^2 + 1 = 0$ is of the second type.

Note that the first type of imaginary conics did not appear in our previous classification of conics because in (4) and (13) all the constants were supposed to be *real.* In fact, *unless otherwise stated,* in this book all variables and constants are supposed to have *real* values assigned them.

Again, we wish to point out that $x = x'$, $y = iy'$ transforms $x^2/a^2 + y^2/b^2 = 1$ into $x'^2/a^2 - y'^2/b^2 = 1$, a thing that (13) could not do; but this transformation sends the real points of the ellipse into imaginary points on the hyperbola and, furthermore, into the real points on the hyperbola sends certain of the imaginary points on the ellipse.

There can be real points on the first type of imaginary curves.

* For the definition of an *algebraic curve* see §40.

Thus $ix^2 + y^2 - 1 = 0$ has (0,1) and (0,−1) on it. If we have an imaginary curve $Q_1 + iQ_2 = 0$, where Q_1 and Q_2 are real algebraic expressions in x and y of the mth and nth degrees,* respectively, then the real points on this curve are obtained by solving simultaneously $Q_1 = 0$ and $Q_2 = 0$, i.e., these points are the points of intersection of the two real curves $Q_1 = 0$ and $Q_2 = 0$.

We prove further on (where?) that these last two curves intersect in general in mn points (real or imaginary or both real and imaginary, distinct or equal); hence the imaginary curve $Q_1 + iQ_2 = 0$ can have as many as mn real points on it. Thus the conic

$$\left(\frac{x^2}{9} + \frac{y^2}{16} - 1\right) + i\left(\frac{x^2}{16} + \frac{y^2}{9} - 1\right) = 0$$

has four real points on it. (Find them.)

EXERCISES

1. Prove that $h^2 - ab < 0$ for all so-called imaginary ellipses. Hint: Consider all the possible cases $h^2 - ab > 0$, $= 0$, < 0. Note that $\Gamma \neq 0$.

2. Make up examples of imaginary non-degenerate conics and quartics of both varieties. Find any real points on them.

3. Find in two ways the equation of the conic through (0,1), (0,−1), (1,0), (−1,0), and (i,2 i).

4. Find in two ways the equation of the conic through the five points ($0,i$), ($0,-i$), (i,0), ($-i$,0), (1,2).

5. Find the real points on $(3+i)x^2 + (7-i)y^2 + (8-i)x + (2-3i)y + (5-i) = 0$; on $x^3 + iy^3 + ix + y = 0$.

6. Why is it impossible for $Q_1 + i(x^2 + y^2 + 1) = 0$ to have any real points on it, no matter what Q_1 is?

7. Make up examples of imaginary conics with four real points on them, three real points on them, two, one, no real points on them. Hint: Take $Q_1 = 0$ and $Q_2 = 0$ intersecting in the required number of real points. See further along in this book (where?) for numerical examples of $Q_1 = 0$ and $Q_2 = 0$ that intersect as required.

8. Find the equation of the conic tangent to the x-axis at (i,0), tangent to the y-axis at (2 i,0) and passing through the point (1,1).

9. Prove the facts from algebra that are quoted in the text, namely, that imaginary roots of algebraic equations occur in pairs and also that an odd degree equation in x has at least one real root.

10. In the next to the last paragraph of the text show which imaginary points on the ellipse go into real points on the hyperbola. Hint: Perform $x' = x$, $y' = 1/iy = -iy$ on the hyperbola.

* For the *degree* of an algebraic expression in x and y see §40.

39. **Inside points with respect to a conic.** Any real non-degenerate conic divides all the points of the plane into *three* categories: those *on* the conic, those *inside* the conic (according to the following definition), and those *outside* the conic.

DEFINITION. A point P is said to be *inside* with respect to a given real, non-degenerate conic C if the two tangents to C from P are imaginary lines (they will be shown to be conjugate imaginary lines).

Let us consider (0,0) and $y^2 = x + 1$. Solving $y = mx$ simultaneously with $y^2 = x + 1$, we obtain the equation $m^2x^2 - x - 1 = 0$, with roots $x = (1 \pm \sqrt{1 + 4m^2})/2m^2$. For $y = mx$ to be tangent to the parabola, we must have $1 + 4m^2 = 0$ or $m = \pm 1/2i$. Hence (0,0) is a point inside this conic.

To treat the general case we can suppose P is (0,0). Solving $y = mx$ simultaneously with the equation (4), we get

$$x^2(a + bm^2 + 2hm) + 2x(g + fm) + c = 0$$

with roots

$$x = \frac{-(g + fm) \pm \sqrt{(f^2 - bc)m^2 + 2(fg - hc)m + (g^2 - ac)}}{(a + bm^2 + 2hm)}$$

For $y = mx$ to be tangent to (4) we must have m a solution of the equation obtained by equating the radical to zero. If the roots of this last equation are imaginary, they must evidently be conjugate imaginary.

EXERCISES

1. Why is there no loss of generality in the last paragraph of the text caused by taking P at (0,0)?

2. Find the actual values of m in the last paragraph of the text.

3. Find the tangents from (0,0) to $x^2 + y^2 - 2x - 2y - 4 = 0$; to $y^2 = 4x + 4$, to $x^2 + y^2 = 1$, to $x^2/16 \pm y^2/9 = 1$. Find the points of contact of these tangents. How do you interpret the results for the last two curves in the light of our assumptions concerning infinite points?

4. Find the equations and the points of contact for the tangents from (0,0) to $y^2 = x + 1$; to (4).

CHAPTER VII

ELEMENTARY DISCUSSION OF *n*TH DEGREE CURVES

40. **Introduction to *n*th degree curves (*n*-ics).** In the preceding chapters we have dealt mostly with some of the *tools* of affine analytic projective geometry, such as frames of reference and affine linear transformations of coordinates. In this chapter we shall consider some of the *geometric material* on which we use these tools, namely, curves with algebraic equations of various degrees in the variables.

In *elementary analytic geometry* we studied principally *straight lines* and *conics*. We also drew the graphs of a few curves with equations of higher degrees, such as $y^2 = x^3$, $y = x^3$, $y = x^4$, etc. In *analytic projective geometry** we shall spend much time on lines and conics, but we shall consider also so-called *higher degree curves*.

DEFINITION. We define an *nth degree curve* (or an *n-ic*) as a curve whose equation is algebraic and when cleared of fractions and of negative and fractional exponents and of radicals has at least *one* term of the form $x^\alpha y^\beta$ with $\alpha + \beta = n$ (where n is a positive integer) and *no* term $x^\gamma y^\delta$ with $\gamma + \delta > n$.

For example, $x^2y = 1$ is a third-degree curve (or *cubic*), $x^3y + 3x = 1$ is a fourth-degree curve (or *quartic*), $y = 1/x^4$ is a fifth-degree curve (or *quintic*), $y^5 = x^6$ is a sixth-degree curve (or *sextic*), etc. A *straight line* is classed as a *curve* of the *first degree* and a *conic* as a *curve* of the *second degree*. The curve $y^{-2} = \sqrt[3]{x+3}$ must be put into the form $y^6(x+3) = 1$ in order to determine its degree. The equation $y = \sqrt{x+1}$ when squared gives the parabola $y^2 = x + 1$. Note that the above equation with a radical gives us only one part of the curve, the other part being given by $y = -\sqrt{x+1}$.

If the equation of an n-ic is factorable, the curve is called *degenerate* (or *composite*) and consists of the loci obtained by equating to zero the factors of its equation. Thus the quartic

* See §§35, 95.

$y^2 = x^4$ is degenerate and consists of the two parabolas $y = x^2$ and $y = -x^2$. In this book we shall not consider such curves as $y = \log x$, $y = e^x$, $y = \sin x$, etc. (which are called *transcendental curves* to distinguish them from n-ics, called *algebraic curves*).

The following equation gives us the *general cubic*

$$(38) \qquad ax^3 + by^3 + c + dx^2y + exy^2 + fx^2 + gx + hy^2 + jy + kxy = 0$$

The *general* equation of an n-ic can be written

$$(39) \qquad \begin{aligned} &a + (b_0x + b_1y) + (c_0x^2 + 2\,c_1xy + c_2y^2) \\ &\quad + (d_0x^3 + 3\,d_1x^2y + 3\,d_2xy^2 + d_3y^3) \\ &\quad + (e_0x^4 + 4\,e_1x^3y + 6\,e_2x^2y^2 + 4\,e_3xy^3 + e_4y^4) \\ &\quad + \cdots + \left(l_0x^n + nl_1x^{n-1}y + \frac{n(n-1)}{2!}l_2x^{n-2}y^2\right. \\ &\quad \left. + \cdots + l_ny^n\right) = 0 \end{aligned}$$

EXERCISES

1. Fit under (38) the cubics $y = 1/x^2$, $y = x^3$, $y^2 = x^3$, $y^2 = x(x-1)(x-2)$. Hint: $y^3 = 1 - x^3$ or $x^3 + y^3 - 1 = 0$ belongs under (38) with $a = b = 1$, $c = -1$, $d = e = f = g = h = j = k = 0$.

2. Fit under (39) the curves $y = x^4$, $y^2 = x^5$, $y^3 = 1/x$, $x^4 + y^4 = 4xy$.

3. Draw rough graphs of the curves $y = x^3$, $y^2 = x^3$, $y^2 = x^2(x+1)$, $y^2 = x^2(x-1)$, $y^2 = x(x-1)(x+1)$, $y^2x = 1$, $y^2x = x + 1$, $y^2x = x^2 - 1$.

4. Draw roughly $y = x^4$, $y^3 = x^4$, $xy^3 = 1$, $y = (x^2-1)^2$, $y = (x^2+1)^2$ $y = \dfrac{x^2-1}{(x+1)(x+2)}$, $y^2 = \dfrac{1}{x^2-1}$, $y^2 = \dfrac{1}{(x-1)^2}$.

41. **Some types and typical forms of n-ics.** An important problem in connection with these n-ics is the study of the types of nth degree curves and of the so-called *typical* (or *standard*) forms to which the equations of n-ics can be reduced by linear transformations of the variables (looking upon these transformations as *aliases*).

There are *five* distinct types of *non-degenerate* conics in Euclidean geometry, namely, the circle, ellipse, hyperbola, parabola, and imaginary ellipse (such as $x^2 + y^2 + 1 = 0$*); there are *three*

* The student should distinguish between giving mere *illustrative examples* of simplified equations of curves and listing completely the *typical equations* to which the equations of *all* the curves of a given degree can be reduced.

types of *degenerate* conics, namely, a pair of real lines (such as $xy = 0$), a pair of conjugate imaginary lines (such as $x^2 + y^2 = (x + iy)(x - iy) = 0$), and a double line (such as $x^2 = 0$).

In the general affine geometry *circles* and *ellipses* belong to the *same* type of conics, because their equations are reducible to one another under (13). Moreover, in the Euclidean geometry the typical forms of the equations of the conics contain *arbitrary* constants, such as a,b,p,r in $x^2/a^2 \pm y^2/b^2 = 1$, $y^2 = 4px$, $x^2 + y^2 = r^2$.

If a hyperbola (for example) has an equation reducible to $x^2/a^2 - y^2/b^2 = 1$, with $a^2 = a'^2$, $b^2 = b'^2$, then this hyperbola is not reducible to the typical form with any other pair of values for a^2 and b^2 (except that $x = y'$, $y = x'$ gives the typical form with $a^2 = b'^2$, $b^2 = a'^2$).

On the other hand, in affine geometry the typical equations of the conics contain *no* arbitrary constants. Thus $y^2 = 4px$ is reducible to $y'^2 = 4x'$ by $x = x'/p$, $y = y'$; $x^2/a^2 \pm y^2/b^2 = 1$ is reducible to $x'^2 \pm y'^2 = 1$ by $x = ax'$, $y = by'$; and $x^2 + y^2 = r^2$ to $x'^2 + y'^2 = 1$ by $x = rx'$, $y = ry'$.

There are a great many *more* types of cubics than of conics, still more of quartics, etc. We have learned in elementary analytic geometry how to reduce the general conic (4) to one of the typical (or *normal*, or *canonical*) forms. The reduction of a cubic to one of the type forms is just touched upon in this book and is exhaustively treated only in more advanced books on plane algebraic curves. The reduction of all such curves as those of the seventh degree (for instance) to type forms has never been accomplished.

Of course the reduction of curves to typical forms is only one of many problems we consider; among other problems considered is the study of *properties* of curves (as with the conics we studied foci, directrices, eccentricity, asymptotes, diameters, etc.).

To show how to obtain typical forms for the equations of curves, let us consider in the general affine geometry the degenerate cubics that consist of three distinct real lines l_1, l_2, l_3. *First* we suppose the three lines are concurrent in a point P. If we take P as $(0,0)$, l_1 as $x = 0$, and l_2 as $y = 0$, then the equation of the cubic has the form $xy(y - mx) = 0$, $m \neq 0$. (Why?) If we put $x = x'/m$, $y = y'$, then drop the primes from the variables, we obtain the typical equation $xy(y - x) = 0$.

If l_1, l_2, l_3 are all parallel, we can take l_1 as $x = 0$, l_2 as $x - 1 = 0$, and our cubic becomes $x(x - 1)(x - \alpha) = 0$ (where α is an arbitrary constant). Note that if we take the three points in which these lines cut the x-axis, and for a fourth point take a point at infinity P' on the x-axis, we have four points (0,0), (1,0), (α,0), (x',0) where $x' \to \pm\infty$.* Taking a cross-ratio of these four points we have

$$\frac{\alpha - 0}{1 - 0}\frac{1 - x'}{\alpha - x'} = \alpha\frac{1/x' - 1}{\alpha/x' - 1} \to \alpha \quad \text{as} \quad x' \to \pm\infty$$

From this result we see that there are only six possible values of α that can give cubics equivalent to (i.e., reducible to) one of the above cubics with $\alpha = \alpha'$, namely, $\alpha = \alpha'$, $1/\alpha'$, $1 - \alpha'$, $1/(1 - \alpha')$, $\alpha'/(\alpha' - 1)$, $(\alpha' - 1)/\alpha'$, because the cross-ratios of the above four points are preserved by (13). For instance, $x = \alpha x'$, $y = y'$ changes the above cubic $x(x - 1)(x - \alpha) = 0$ to $x'(x' - 1/\alpha)(x' - 1) = 0$ or $x(x - 1)(x - 1/\alpha) = 0$. Compare §23.

Next we suppose the three lines are not concurrent and (*a*) l_1 and l_2 are parallel; (*b*) no two of the lines are parallel. In case (*a*) we can take l_1 as $x = 0$, l_2 as $x - 1 = 0$, l_3 as $y = 0$ and our typical cubic is $xy(x - 1) = 0$. In case (*b*) we can take l_1 as $x = 0$, l_2 as $y = 0$, l_3 as $x + y - 1 = 0$, and our cubic is $xy(x + y - 1) = 0$.

The above special choices of the lines l_1, l_2, l_3 amount to transformations of coordinates considered as *aliases*. For example, in case (*b*) if l_1 and l_2 intersect in P_3, l_1 and l_3 in P_2, l_2 and l_3 in P_1, we can send P_3 to (0,0), P_2 to (0,1), and P_1 to (1,0) by (13) and so obtain the equation $xy(x + y - 1) = 0$. See §14.

EXERCISES

1. Show how case (*b*) in the text can be treated under (24′).

2. Prove that all imaginary non-degenerate conics of the first type (see §38) can be reduced to $x^2/a^2 + y^2/b^2 + 1 = 0$ in Euclidean geometry and to $x^2 + y^2 + 1 = 0$ in affine geometry. Hint: Rotate and translate the axes so as to rid (4) of the xy, x, and y terms.

3. Explain more fully the relation between cross-ratio and the constant α in $x(x - 1)(x - \alpha) = 0$.

* By $x' \to \pm\infty$ we mean x' becoming greater than any positive number *no matter how great* ($x' \to +\infty$), or x' becoming less than any negative number *no matter how low* ($x' \to -\infty$).

4. Interpret as transformations of coordinates all the special choices for l_1, l_2, l_3 that are given in the text.

5. Find transformations to reduce $x(x-1)(x-\alpha') = 0$ to all the possible typical forms, i.e., with $1-\alpha'$, $1/(1-\alpha')$, $\alpha'/(\alpha'-1)$, $(\alpha'-1)/\alpha'$, $1/\alpha'$ replacing α'.

6. Prove the statement in the text that in Euclidean geometry $x^2/\alpha'^2 \pm y^2/b'^2 = 1$ cannot be reduced to $x^2/a^2 \pm y^2/b^2 = 1$ unless $a^2 = a'^2$, $b^2 = b'^2$, or $a^2 = b'^2$, $b^2 = a'^2$.

7. Reduce to typical forms under (13) the equations of all degenerate cubics that consist each of a double line and another line. Hint: These cubics must have the form

$$(a_1x + \beta_1y + \gamma_1)^2(\alpha_2x + \beta_2y + \gamma_2) = 0 \quad \text{or} \quad (\alpha_1x + \beta_1y + \gamma_1)^2 \cdot (\alpha_1x + \beta_1y + \gamma_2) = 0$$

(Why?) Now put $\alpha_1x + \beta_1y + \gamma_1 = x'$, $\alpha_2x + \beta_2y + \gamma_2 = y'$, etc.

8. Reduce to typical equations under (13) all the degenerate quartics that consist each of a double line and a pair of real lines.

9. Reduce to typical equations under (13) all the degenerate cubics that consist each of a real line and a pair of conjugate imaginary lines. Hint: Reduce the latter to $x^2 + y^2 = 0$ or $x^2 + 1 = 0$.

42. **Conditions determining an nth degree curve.** In elementary analytic geometry we saw that *five distinct points* determine a conic, because each point gives a linear equation connecting the six coefficients of the general conic (4); but we can divide (4) through by any coefficient that does not vanish, hence the five equations given by the five points uniquely determine five of these coefficients in terms of the sixth.

Thus, if we want (4) to pass through (1,0), (−1,0), (0,1), (0,−1), (2,1), we have

$$a + c + 2g = 0, \quad a + c - 2g = 0, \quad b + c + 2f = 0,$$
$$b + c - 2f = 0, \quad 4a + b + c + 2f + 4g + 4h = 0$$

whence $g = f = 0$, $a = b = -h = -c$; and the required conic is $x^2 + y^2 - 1 - 2xy = 0$. This conic is a pair of lines because (1,0), (−1,0), and (2,1) are collinear.

A nicer way to work the problem in the last paragraph is to take the lines $x = 0$, $y = 0$, $x + y - 1 = 0$, $x + y + 1 = 0$ that pass through four of the given points and write the equation

$$\lambda xy + (x + y - 1)(x + y + 1) = 0$$

which is a conic passing through the four given points, for every value of λ. (Why?) Putting $x = 2$, $y = 1$ we get $\lambda = -4$, and

expanding the parentheses we obtain again $x^2 + y^2 - 1 - 2xy = 0$.

In this last method the lines l_1, l_2, l_3, l_4 must be paired in $l_1l_2 + \lambda l_3l_4 = 0$ in such a way that l_1 cuts l_3 and l_4 in two given points, also l_2 cuts l_3 and l_4 in two other given points. For example, the equation $\lambda x(x + y - 1) + y(x + y + 1) = 0$ would not help us to solve the above problem. (Why?)

In general we note that $uv + \lambda ws = 0$,* where $u = 0$, $v = 0$, $w = 0$, $s = 0$ are straight lines, is the equation of a conic through the four points of intersection of $u = 0$ and $w = 0$, $u = 0$ and $s = 0$, $v = 0$ and $w = 0$, $v = 0$ and $s = 0$, for every value of λ. If $w \equiv s$, $uv + \lambda w^2 = 0$ is the equation of a conic tangent to $u = 0$ and $v = 0$ at their points of intersection with $w = 0$, because $u = 0$ (when solved simultaneously with this equation) gives $w^2 = 0$ and $v = 0$ gives $w^2 = 0$. As an illustration, $xy + \lambda(x + y - 1)^2 = 0$ is the equation of a conic tangent to $x = 0$ at (0,1) and tangent to $y = 0$ at (1,0).

Again $uvw + \lambda s^2t = 0$* (where $t = 0$ is also a straight line) is the equation of a cubic tangent to $u = 0$, $v = 0$, $w = 0$ at their points of intersection with $s = 0$, and also passing through the points of intersection of the first three lines with $t = 0$. The quartic $uvws + \lambda t^2r^2 = 0$ ($r = 0$ also a line) has in evidence four tangents $u = 0$, $v = 0$, $w = 0$, $s = 0$, each of them with two points of contact (on $t = 0$ and $r = 0$). Such tangents are called *bi-tangents*. To interpret such an equation as

$$(3x + y - 1)^2 (x + y - 1) = (x - y)(x + y)(3x + 2y - 1)$$

we put it in the form $u^2v = wst$.

Finally, if $u = 0$, $v = 0$, $w = 0$, $s = 0$ are the equations of conics, then $uv + \lambda ws = 0$ is the equation of a quartic through the sixteen points of intersection of the pairs of conics $u = 0$ and $w = 0$, $u = 0$ and $s = 0$, $v = 0$ and $w = 0$, $v = 0$ and $s = 0$ (as we shall see later on, two conics intersect in general in four points). The equation $uv + \lambda w^2 = 0$ is a quartic tangent to the conics $u = 0$ and $v = 0$ at their points of intersection with the conic $w = 0$. (Why?)

We now illustrate still *another way* of obtaining the equations of curves that satisfy given conditions. Suppose we want the

* Such an equation is sometimes said to be written in *abridged notation*. See §115.

conic tangent to the x-axis at (1,0), tangent to the y-axis at (0,1), and also passing through the point (3,3). If we solve $y = 0$ simultaneously with (4), we should obtain an equation of the form $a(x - 1)^2 = 0$; hence we must have $2g/a = -2$ and $c/a = 1$. If we solve $x = 0$ with (4), we should get $b(y - 1)^2 = 0$, so we must have $2f/b = -2$, $c/b = 1$. Therefore, $c = a = b$, $g = f = -a$, and (4) becomes (after dividing through by a)

$$x^2 + y^2 + 2hxy/a - 2x - 2y + 1 = 0$$

This conic must pass through (3,3); hence $9 + 9 + 18h/a - 6 - 6 + 1 = 0$ or $h = -7a/18$, and the conic we desire is

$$x^2 + y^2 - \tfrac{7}{9}xy - 2x - 2y + 1 = 0$$

Lastly we wish to find *how many* points determine an *n-ic* (we saw that five points determine a conic). There are *ten* coefficients in (38); hence, in general, *nine* points determine a *cubic* because we can divide its equation by any non-vanishing coefficient. This statement has exceptions because (as we shall see later) two cubics $K_1 = 0$ and $K_2 = 0$ intersect each other generally in nine points; therefore the equation $K_1 + \lambda K_2 = 0$ represents an *infinite* number of cubics through *nine* points (each value of λ giving a distinct cubic).

Let us consider the general equation of an n-ic (39). The number of coefficients in (39) is the same as the sum of the arithmetic progression $1 + 2 + 3 + \cdots + (n + 1)$, namely

$$\tfrac{1}{2}(n + 1)(n + 2)$$

Therefore the number of points necessary to determine such a curve is $\frac{1}{2}(n + 1)(n + 2) - 1 = \frac{1}{2}n(n + 3)$. For a conic we have $\frac{1}{2}\,2(2 + 3) = 5$ points, for a cubic $\frac{1}{2}\,3(3 + 3) = 9$ points, for a quartic $\frac{1}{2}\,4(4 + 3) = 14$ points, for a quintic $\frac{1}{2}\,5(5 + 3) = 20$ points, for a sextic $\frac{1}{2}\,6(6 + 3) = 27$ points. Note again that there are exceptions to this rule; for instance, all the infinite number of quartics given above by $uv + \lambda ws = 0$ have sixteen common points of intersection.

EXERCISES

1. Check all the algebra in the text.
2. Answer all the queries (Why?) in the text.
3. In the second paragraph of the text find a different set of lines l_1, l_2, l_3, l_4 such that by using $l_1l_2 + \lambda l_3l_4 = 0$ we can solve the problem.

4. Show why a tangent t and its point of contact P count for two points in determining an n-ic. Hint: Take t as the limiting position of a secant PP' when $P' \to P$.

5. Give exact definitions of the statements " how many points determine an n-ic " and " hence, in general, nine points determine a cubic " in the next to last paragraph of the text. Compare §14.

6. Find in two ways the equation of the conic through the five points (1,0), (−1,0), (0,1), (0,−1), (1,1).

7. Find in two ways the equation of the conic tangent to the y-axis at (0,3), tangent to the x-axis at (3,0), and passing through the point (1,1).

8. Interpret the equations $y^2(x+1) = x(x-1)(x+2)$, $xy(x+y) = (x+y-1)^2$, $(x^2+y^2-1) = (x+y-1)(2x-y+1)$, $(x^2+y^2-1) = (x+y-1)^2$, $(x-1)^2(x-2)^2 = y(y-1)(y+3)(y+4)$.

9. Find the equation of the conic tangent to $x+y-1=0$ at (1,0), tangent to $x+y+1=0$ at (0,−1), and (*a*) passing through (0,0); (*b*) passing through (3,3).

10. What types of conics are all those in the above example?

11. Make up an example of quintics that are exceptions to the rule that twenty points determine a quintic.

12. Make up an equation of a cubic tangent to three concurrent lines at three collinear points and cutting these lines again at three other collinear points.

43. **Points of intersection of a line with an n-ic.** We have been accustomed to saying that any line l intersects a given conic C in *two* points, or is *tangent* to C, or does *not* cut C at all. From now on we shall say that l cuts C in *two* points that are *real* and *distinct*, or *real* and *equal*, or *imaginary* and *distinct*. (Of course, if l is imaginary, it may cut C in two distinct imaginary points or two equal imaginary points.)

ILLUSTRATIVE EXAMPLE. The lines $x=0$, $x=1$, $x=2$ cut $x^2+y^2=1$ in the above three ways. The imaginary lines $x=i$ and $y=\sqrt{2}\,ix+i$ cut this circle, respectively, in two distinct imaginary and in two coincident (or equal) imaginary points.

We should note here that a tangent to a cubic or a higher degree curve ordinarily cuts the curve *again* in one or more points. Thus the line $y=3x-2$ is tangent to the cubic $y=x^3$ at (1,1) and cuts this curve again at (−2,−8). The line $y=0$ touches $y=x^2(x-1)^2$ at the two points (0,0) and (1,0) and so is a bitangent to this quartic. (See §42.)

We show as follows that *in general a line l cuts an n-ic in n points.* We can take l as $y=mx+b$ where $m \neq 0$, and the n-ic in the form (39). Eliminating y (or x) between l and the

n-ic, we get *ordinarily* an nth degree equation in x (or y), which we learned in algebra has n roots $x_1,x_2, \ldots, x_n(y_1,y_2, \ldots, y_n)$; hence l cuts the n-ic in the n points (x_1,y_1), (x_2,y_2), $\ldots$, (x_n,y_n), which may be real or imaginary, distinct or equal.

In algebra we learned that the *imaginary* roots of any equation in one variable (with real coefficients) occur in *pairs* of the form $a + ib$, $a - ib$ (i.e., pairs of conjugate imaginary roots). Therefore a real line l cuts an n-ic in one or more *pairs* of *conjugate imaginary points*, if not entirely in real points. Also we see that if a real line l is tangent to the n-ic in one imaginary point P, it must be tangent to the n-ic also in the conjugate imaginary point P'. (Why?) Thus $y = 0$ touches $y(x + y + 1) = (x^2 + y^2 + 1)^2$ in two such points P, P'.

A line may cut an n-ic in two, three, four, $\ldots$, n *coincident* points at a given point P. For example $y = 0$ cuts $y = x^3$ in three coincident points at (0,0), cuts $y = x^4$ and $y = x^4(x - y)$ each in four coincident points there, and finally cuts $y = x^n$ in n coincident points at the origin. The origin is called a point of inflection (or more briefly an *inflection*) for $y = x^3$, also a *hyperinflection* for the curves $y = x^4$ and $y = x^n$, in each case with $y = 0$ as tangent. A cubic cannot have a point of hyperinflection, because no line can cut a cubic in more than three points. For the same reason no cubic can have a bi-tangent.

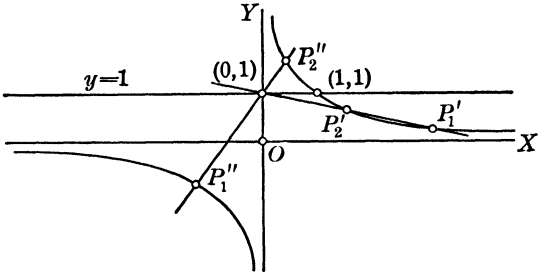
Now the question arises how to interpret geometrically the fact that sometimes, as in the following example, the equation in x (or y) that we get by eliminating y (or x) between l and the n-ic (39) is of *lower degree* than the nth.

The line $y = 1$ cuts the curve $xy = 1$ apparently in the one point (1,1) given by $x = 1$. However, every line $y = mx + 1$ (where $m \neq 0$) through (0,1) cuts $xy = 1$ in the points

$$P_1\left(\frac{-1 - \sqrt{1 + 4m}}{2m}, 1\right) \quad \text{and} \quad P_2\left(\frac{-1 + \sqrt{1 + 4m}}{2m}, 1\right)$$

As $m \to 0$ the coordinates of P_1 approach the form $(\infty,1)$ and the coordinates of P_2 the indeterminate form (0/0,1). To evaluate this indeterminate form 0/0 we multiply numerator and denominator of $\dfrac{-1 + \sqrt{1 + 4m}}{2m}$ by $-1 - \sqrt{1 + 4m}$ and so obtain $-2/(-1 - \sqrt{1 + 4m})$, which approaches the value 1 as $m \to 0$; therefore, P_2 approaches the point (1,1) as $m \to 0$.

To interpret the fate of P_1 in the preceding paragraph we note in the adjoining figure that if we draw such a line as $y = mx + 1$ cutting $xy = 1$ above (1,1) and rotate this line around the point (0,1) down to the position of $y = 1$, the point P_1 (now called P_1'') moves indefinitely far out to the left along the lower branch of this curve. On the other hand, if we draw this line so as to cut $xy = 1$ in two points below (1,1) and to the right, then rotate the line up to the position of $y = 1$, the point P_1 (now P_1') moves indefinitely far out to the right along the upper branch of the hyperbola.



Hence we say that the line $y = 1$ cuts the curve $xy = 1$ in one finite and one infinite point.* We do not say that $y = 1$ cuts this curve in one finite and two infinite points (one such point on the upper branch of the curve and one on the lower) because that would make a line cut a conic in three points instead of two.

From the last sentence we see that we have to assume that the two branches of the hyperbola intersect each other in an infinite point on the x-axis and an infinite point on the y-axis (inasmuch as these two axes are the asymptotes of this curve).

EXERCISES

1. Check all the statements in the text, filling in any necessary algebraic details.

2. Answer the query (Why?) in the text.

3. Discuss the way the lines $y = \pm 1$ cut the parabola $y^2 = 4x$. Show how this affects our idea of how the parabola behaves at infinity (i.e., what we must assume concerning infinite points on the parabola).

4. Show that any line l must cut an n-ic in $n - 2r$ real points, where r is an integer, positive or zero.

5. How does $y = 0$ cut the curves $y = x^5(x + y - 1)^2$, $(x - 1)^2 (x - 2)^2 = y^3$, $(x^2 + 1)^2 = y^3$? How does $x = 1$ cut the second curve? How do $y = 0$ and $x = 1$ cut the curve $(x - 1)^3 (x - 2)^3 = y^4$?

* Here and in §§25, 49 we are merely making clear the reasonableness of the assumption that infinite points exist. Their existence *cannot* be *proved*.

6. Show that a quintic cannot have a tangent with three distinct points of contact.

7. Give an example of a sextic with a tangent such as is described in Ex. 6. Hint: Take $y = 0$ as this tangent.

44. Asymptotes to n-ics. In a manner similar to that employed in the last section we can interpret all the cases where a line apparently cuts an n-ic in *fewer* than n points (i.e., where l does cut the n-ic in less than n finite points). For example, $x = 0$ seemingly cuts $y = x^3$ only at the origin; hence we assume that $x = 0$ cuts this curve again in two infinite points P'_∞ and P''_∞ and also that $P'_\infty = P''_\infty$ (because later on we make the hypothesis that all the infinite points in a plane lie on a line l_∞ called the *line at infinity*). See §49.

Again $y = 0$, when solved simultaneously with $xy = 1$, gives us $0 = 1$; therefore to explain this anomaly we assume that $y = 0$ cuts this curve in two infinite coincident points. The above assumption agrees with the fact that $y = k$ cuts $xy = 1$ in one infinite point and one finite point $P(1/k,k)$, but as $k \to 0$ the point P also becomes an infinite point (i.e., P moves indefinitely far out on the curve). We know from elementary analytic geometry that $y = 0$ is one of the asymptotes of $xy = 1$.

Note that every line $x = h$ cuts $y = x^3$ in one finite point (h,h^3) and therefore in two infinite coincident points. (Why coincident?) If we use an assumption (made later on) that any two parallel lines in the plane intersect in an infinite point, we interpret this curve as having a so-called double point at infinity on the y-axis. See §48. (Compare the double point at the origin on $y^2 = x^3$, where every line $y = mx$ cuts the curve in two coincident points given by $x^2 = 0$.)

Next we study the curve $x^2 - y^2 = 1$. Any line $y = mx + a$ cuts this hyperbola in two points whose abscissas satisfy the equation

$$x^2(1 - m^2) - 2\,amx - a^2 - 1 = 0,$$

$$\text{i.e., } x = \frac{2\,am \pm \sqrt{4\,a^2m^2 + 4(1 - m^2)\,(a^2 + 1)}}{2(1 - m^2)}$$

For this line to be a tangent we must have

$$4\,a^2m^2 + 4(a^2 + 1)\,(1 - m^2) = 0$$

If we let $m \to \pm 1$, we must let $a \to 0$ for $y = mx + a$ to remain

tangent to the curve; but then the points of contact of these two tangents have abscissas (see above) of the form $1/0 = \infty$ (we see this also by solving $y = \pm x$ with $x^2 - y^2 = 1$).

From this we see that $y = \pm x$ can be looked upon as the limiting positions of any two tangents to the curve $x^2 - y^2 = 1$ as the points of contact of these tangents move out to infinity. Such lines we call *asymptotes* to the curve. (We do not call $x = 0$ an asymptote to the cubic $y = x^3$, because we picture the infinite point on this line as a double point of the cubic.)

Asymptotes as well as other tangents may cut the n-ics *again* in one or more finite points, may be bi-tangents, or tangents at inflections, double points, etc. (finite or infinite). As illustrations, the asymptote $x + y + 1 = 0$ touches the curve $x^3 + y^3 = 3xy$ at an infinite inflection, $y = 0$ touches the quartic $yx^3 = 1$ at an infinite hyperinflection, $y = 0$ is an asymptote of the quartic $y^2x^2 = (x - 1)^2 + y^3$ and at the same time a bi-tangent with one finite point of contact (1,0), the line $x + y - 1 = 0$ is tangent to $(x + y - 1)^2x = 1$ at an infinitely distant double point.

The above geometric interpretations of infinite points of intersection of lines and n-ics can be rendered more plausible by the following algebraic discussion; at the same time we develop a method of finding the asymptotes of an n-ic. Let us consider the general equation of the nth degree in x

$$(40)\quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_rx^{n-r} + \cdots + a_{n-3}x^3 + a_{n-2}x^2 + a_{n-1}x + a_n = 0$$

If $a_n = a_{n-1} = a_{n-2} = \cdots = a_{n-i} = 0$, $a_{n-i-1} \neq 0$, this equation has $i + 1$ roots all equal to zero and so equal to each other.

Now we put $x = 1/z$ in (40), multiply the equation by the highest power of z that occurs in any denominator, and get

$$(41)\quad a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots + a_rz^r + \cdots + a_{n-3}z^{n-3} + a_{n-2}z^{n-2} + a_{n-1}z^{n-1} + a_nz^n = 0$$

If $a_0 = a_1 = a_2 = a_3 = \cdots = a_i = 0$, $a_{i+1} \neq 0$, then (41) has $i + 1$ zero roots in z; and (40) has therefore by interpretation $i + 1$ infinite roots $x' = \infty$, since as $z \to 0$, $x = 1/z \to \infty$.

Suppose now we want to study the curve $x^2 - 3xy + 2y^2 = 1$ for asymptotes. We first see if there are any vertical asymptotes (i.e., we find out if there are any values of x that when substituted in this equation cause the resulting equation in y to have two

equal infinite roots); then we solve $y = mx + b$ simultaneously with this equation, obtaining

$$x^2(1 - 3m + 2m^2) + b(-3 + 4m)x + (2b^2 - 1) = 0$$

Solving $1 - 3m + 2m^2 = 0$ and $b(-3 + 4m) = 0$ simultaneously for m and b, we get $m = 1, \frac{1}{2}$ and $b = 0$; therefore the two asymptotes are $y = x$ and $y = \frac{1}{2}x$.

We observe that every line $y = x + a$ parallel to the asymptote $y = x$ cuts the above curve in just one finite point $P((1 - 2a^2)/a, (1 - 2a^2)/a)$, and similarly for every line parallel to the other asymptote; also that as $a \to 0$ the point P becomes an infinite point.

EXERCISES

1. Why do we suppose that $x = h$ cuts $y = x^3$ in two coincident infinite points?

2. Give an exact definition of an asymptote to an n-ic.

3. Prove the facts mentioned in the text about the asymptotes to the curves given as illustrations.

4. Test for asymptotes (and find any such asymptotes) the curves

$$x^3 + y^3 = 3xy, \quad xy = -1, \quad y^2 = 4px, \quad x^2 + y^2 + 2gx + 2fy + c = 0,$$

$$x^2 + y^2 = r^2, \quad \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

Interpret geometrically the results you get. Observe that the imaginary asymptotes of any circle are parallel to those of any other circle, for all values of g, f, and c.

5. Test for asymptotes the general conic (4). When are your results real; when imaginary? How about the cases when $h^2 - ab = 0$?

6. By a focus of an n-ic we mean a point of intersection of two tangents t_1 and t_2 to the curve where t_1 passes through the infinite imaginary point I_1, and t_2 through I_2 (where I_1 and I_2 are the two points of intersection of the asymptotes of different circles, see Ex 3). For a line l to be parallel to the asymptote of a circle (i.e., pass through I_1 or I_2) we must have the equation of l of the form $y = \pm ix + c$. If we solve $y = \pm ix + c$ simultaneously with $y^2 = 4px$, $x^2/a^2 \pm y^2/b^2 = 1$, then put the condition on c that these lines be tangent to the conics, then find the points of intersection of the pairs of these lines, we shall find the foci of these curves. Do so.

7. Find the foci of $y = x^3$, $y = x^4$, $x^3 + y^3 = 3xy$; of (4); of $x^2 + y^2 + 2gx + 2fy + c = 0$.

8. Show that, as $a \to b$ in $x^2/a^2 + y^2/b^2 = 1$, the foci approach the center of the conic.

45. **Points of intersection of m-ics and n-ics.** We shall now prove that an m-ic and an n-ic intersect *in general* in mn points

(finite or infinite, real or imaginary, distinct or coincident). We shall consider only the case of a conic and a cubic, but our method will be seen to be general. Consider the general conic (4) with its coefficients primed and the general cubic (38). We can suppose without loss of generality that the origin is not a point of intersection of the two curves and that no point of intersection lies on $x = 0$. If we solve $y = mx$ simultaneously with (4) and (38), we obtain two equations in x, namely

$$f(x) \equiv (a' + b'm^2 + 2h'm)x^2 + (2g' + 2f'm)x + c' = 0$$
$$\phi(x) \equiv (a + bm^3 + dm + lm^2)x^3 + (f + hm^2 + km)x^2 + (g + jm)x + c = 0$$

We can write these two equations

$$f(x) \equiv \alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0$$
$$\phi(x) \equiv \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0 = 0$$

where $\alpha_2 = a' + b'm^2 + 2h'm$, etc.

Multiplying $f(x) = 0$ by x and calling the result a new equation, then multiplying $f(x) = 0$ by x^2, next multiplying $\phi(x) = 0$ by x, finally replacing x^4 by r, x^3 by s, x^2 by t, x by u, 1 by v in the constant terms of the equations, every time x^4, x^3, x^2, x occur in $f(x) = 0$, $\phi(x) = 0$, $xf(x) = 0$, $x^2f(x) = 0$, and $x\phi(x) = 0$, we obtain the following five linear homogeneous equations in the five unknowns r, s, t, u, v

$$\alpha_2 r + \alpha_1 s + \alpha_0 t = 0,\ \alpha_2 s + \alpha_1 t + \alpha_0 u = 0,\ \alpha_2 t + \alpha_1 u + \alpha_2 v = 0,$$
$$\beta_3 s + \beta_2 t + \beta_1 u + \beta_0 v = 0,\ \beta_3 r + \beta_2 s + \beta_1 t + \beta_0 u = 0$$

where $\alpha_i (i = 0,1,2)$ and $\beta_j (j = 0,1,2,3)$ are respectively of the ith and jth degrees in m.

The necessary and sufficient condition for these five equations to have a solution in r, s, t, u, v not consisting entirely of zeros is (from algebra)

$$D \equiv \begin{vmatrix} \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\ 0 & 0 & \alpha_2 & \alpha_1 & \alpha_0 \\ 0 & \beta_3 & \beta_2 & \beta_1 & \beta_0 \\ \beta_3 & \beta_2 & \beta_1 & \beta_0 & 0 \end{vmatrix} = 0$$

If $f(x) = 0$ and $\phi(x) = 0$ have a common root $x = x'$ for some value of $m = m'$, i.e., if the line $y = m'x$ passes through a point of intersection $P'(x',y')$ of the conic and the cubic, then the five

equations in r, s, t, u, v certainly have a common solution $r = x'^4$, $s = x'^3$, $t = x'^2$, $u = x'$, $v = 1$, not consisting entirely of zeros (since no point lies on $x = 0$); therefore $D = 0$. This shows that $D = 0$ is a *necessary* condition for $f(x) = 0$ and $\phi(x) = 0$ to have a common root.

To show that $D = 0$ is also a *sufficient* condition for $f(x) = 0$ and $\phi(x) = 0$ to have a common root, we multiply the first column of D by x^4 and add the result to the fifth column of D; we multiply the second column by x^3 and add to the fifth; the third by x^2 and add to the fifth; the fourth by x and add to the fifth; and we obtain

$$D \equiv \begin{vmatrix} \alpha_2 & \alpha_1 & \alpha_0 & 0 & x^2 f(x) \\ 0 & \alpha_2 & \alpha_1 & \alpha_0 & x f(x) \\ 0 & 0 & \alpha_2 & \alpha_1 & f(x) \\ 0 & \beta_3 & \beta_2 & \beta_1 & \phi(x) \\ \beta_3 & \beta_2 & \beta_1 & \beta_0 & x\phi(x) \end{vmatrix} = 0$$

Now expanding $D = 0$ and transposing to the right-hand side of this equation all the terms having $\phi(x)$ as a factor, we get an equation of the form (since for the variable x, $D \equiv 0$)

$$f(x)\ (A_0x^2 + A_1x + A_2) \equiv \phi(x)\ (B_0x + B_1)$$

where A_0, A_1, A_2, B_0, B_1 are algebraic expressions containing powers of m. Any value $x = x'$ causing $\phi(x)$ to vanish must cause $A_0x^2 + A_1x + A_2$ to vanish or $f(x)$ to vanish. But $\phi(x) = 0$ is of the third degree and so has three roots; therefore at least one root of $\phi(x) = 0$ must be a root of $f(x) = 0$, since $A_0x^2 + A_1x + A_2 = 0$ is only of the second degree.

From these arguments our conclusion follows that every value of $m = m'$ that causes D to vanish (and so makes $f(x) = 0$ and $\phi(x) = 0$ have a common root) gives us a line $y = m'x$ joining the origin to a point of intersection $P'(x',y')$ of the conic and the cubic. But $D = 0$ is of the sixth degree in m; hence in general there are six such lines and therefore six such points of intersection.

The above discussion, when generalized to the case of an m-ic and an n-ic, proves that two such curves generally intersect in mn points that may be real or imaginary, finite or infinite, distinct or coincident. For example, the two curves $xy = 1$ and $x(x + y - 1) = 3$ have the common asymptote $x = 0$. If we solve the equations of these curves simultaneously, we find

that the two hyperbolas have the common finite points $(2,\frac{1}{2})$ and $(-1,-1)$. We therefore conclude that the two curves have two coincident infinite points of intersection on $x = 0$ (i.e., they are tangent to each other at infinity, with $x = 0$ as the common tangent).

It is often very difficult to count up the mn points of intersection of an m-ic and an n-ic, especially if they are tangent to each other at certain points, or have a double point in common, etc. A complete discussion of all these types of intersection must be left for a book on higher plane curves.

ILLUSTRATIVE EXAMPLE. The curves $xy = 1$ and $x^2 + y^2 = 4$ cut each other in four real and distinct finite points; $x^2 + y^2 = 1$ and $y^2 = 4x$ cut each other in two real and two imaginary finite ponits; $y^2 = 4x$ and $xy = 1$ intersect in one real and two imaginary finite points and therefore are assumed to cut each other in one real infinite point.

Strange to say, the infinite point of intersection of the last two curves in the illustrative example must be supposed to lie on the x-axis, since the y-axis touches the parabola $y^2 = 4x$ at the origin and the two axes are the asymptotes of the hyperbola $xy = 1$. This remark shows how our *geometric intuitions* may be violated by the assumption of the existence of *infinite points*, because the parabola certainly appears to draw continually away from the x-axis and yet must be assumed to cross this axis at infinity.

Next we note that if we say two parallel lines intersect in an infinite point, then, since the asymptotes of any two circles are parallel (see §44), we must suppose that all the circles in the plane intersect each other in two common imaginary infinite points I_1 and I_2 (called the *circular points at infinity*). This agrees with the facts that any two circles intersect in only two finite points, also that three points determine a circle, whereas it ordinarily takes five points to determine a conic.

The hyperbolas $xy = 1$ and $xy = 2$ do not intersect in finite points, but since they both have the axes for asymptotes we assume they are tangent to each other at two infinite points on these axes, and in this way we account for their four points of intersection by saying they are coincident in pairs and infinite.

The two parabolas $y^2 = 4x$ and $y^2 = 8x + 1$ do not intersect in finite points; but we must assume that these curves cut each other in four coincident infinite points on the x-axis, because

every line $x = h$ and also every line $y = mx$ (where $m \neq 0$) cut each parabola in two finite points.

We say we *assume* the *existence* of all the above-mentioned infinite points because we cannot (so to speak) go out to infinity and see them; but if we *postulate* their existence we can clear up many difficulties. Also, later on, we shall see how to do what amounts to bringing these infinite points into the finite part of the plane for study and observation. (Compare the transformation $x = 1/x'$, $y = y'/x'$ in §12.)

EXERCISES

1. Why is there no loss of generality caused by assuming in the text that the m-ic and the n-ic do not intersect on $x = 0$?
2. Describe the proof of the general case that an m-ic and an n-ic intersect in mn points.
3. Why is $D \equiv 0$ with respect to x in the text?
4. Go through the algebraic details omitted in the text, especially finding the points of intersection in the illustrative examples.
5. Prove (what is quoted from algebra) that $D = 0$ (in the text) is the necessary and sufficient condition for the existence of a solution in r, s, t, u, v not all zeros.
6. How do concentric circles cut each other?
7. Prove that if an m-ic and an n-ic have any imaginary points of intersection, these form pairs of conjugate imaginary points. Hint: What values of the slope m give such imaginary points?
8. Find the points of intersection of $x^2 + y^2 = 1$ and $y^2 = 4x$, of $y = x^3$ and $y^2 = 4x$, of $y^2 = x^3$ and $x^2 + y^2 = 2$, of $y^2 + 2xy + 2x = 0$ and $y^2 - 2xy + 2x = 0$, of $y^2 + 2x + x^2 = 0$ and $y^2 + 2x - x^2 = 0$, of $y = x^4$ and $y = x^3$, of $y^2 = x^3$ and $y = x^4$.
9. Reproduce the proof in the text as applied to two conics, namely, (4) and another conic like (4) except for primed coefficients.

46. **Tangents to n-ics.** We shall give two definitions of a tangent to a curve, one algebraic and the other geometric. The first definition will be useful later on. The second definition has been used to derive (3).

DEFINITION. *Algebraically,* a *tangent* to a curve is a line such that when its equation is solved simultaneously with that of the curve we obtain at least two solutions (x',y') and (x'',y'') such that $x' = x''$ and $y' = y''$.

Thus $y = 0$ is tangent to $y = x^2$ at (0,0) because $y = 0$ gives $x' = x'' = y' = y'' = 0$ when solved simultaneously with $y = x^2$. Similarly, solving $y = 0$ with $y = x^3$ we get $x' = x'' = x''' =$

$y' = y'' = y''' = 0$. Every line $y = mx$ cuts $y^2 = x^3$ at (0,0) in two coincident points and so comes under our definition of a tangent; but in this case we call $y = 0$ the tangent at (0,0) because this line is the only one that cuts the cubic in three coincident points instead of two.

DEFINITION. *Geometrically*, by a *tangent* to a curve at a point we mean the limiting position of a secant through two points $P'(x',y')$ and $P''(x' + \Delta x,\ y' + \Delta y)$ on the curve as $P'' \to P'$ along the curve (i.e., as $\Delta x \to 0$ and $\Delta y \to 0$), if this limiting position exists.

Note that both these definitions apply to *oblique* as well as to *rectangular* axes, also to axes with different-sized units, because no idea of the slope of a line is used. Using the two-point form for the equation of a line (which form is valid for all these above-mentioned frames of reference), we have already derived the equation of a tangent (3).

If the curve is given by $f(x,y) = 0$, the tangent at P' is

$$\frac{\partial f}{\partial x'}(x - x') + \frac{\partial f}{\partial y'}(y - y') = 0 \tag{42}$$

where $\partial f/\partial x'$ and $\partial f/\partial y'$ mean $\partial f/\partial x$ and $\partial f/\partial y$ with x and y replaced by x' and y'. If $\partial f/\partial x' = \partial f/\partial y' = 0$, the point P' is a so-called *multiple point* on the curve. Thus for $y^2 = x^3$, $\partial f/\partial x' = \partial f/\partial y' = 0$ for $x' = y' = 0$. (We study such points in §48.)

If the curve is given in *parametric coordinates* by $x = \phi(t)$, $y = \psi(t)$, the equation of the tangent at $P'(\phi(t'), \psi(t'))$ is

$$\phi'(t')\ (x - x') + \psi'(t')\ (y - y') = 0 \tag{43}$$

where $dx/dt = \phi'(t)$, $dy/dt = \psi'(t)$.

An easy way to find the tangents $y = mx + b$ to a given conic that have a given value for m (m being the *slope** only for an *ordinary* frame of reference) is to solve $y = mx + b$ simultaneously with the equation of the conic and determine b so that the resulting equation in x (or y) alone shall have a *double root* (i.e., two equal roots). For example, solving $y = 3x + b$ with $x^2 + y^2 = 1$, we obtain $10x^2 + 6bx + b^2 - 1 = 0$; hence, for this line to be a tangent, we must have $36b^2 - 40(b^2 - 1) = 0$ or $b = \pm\sqrt{10}$, so the two tangents are $y = 3x \pm \sqrt{10}$. Similarly for the tan-

* See §3 for a definition of *slope*.

gents from a point $P'(x',y')$ we solve $y - y' = m(x - x')$ with the conic and determine m (as we did b).

This method can be used for higher degree curves, but it is more difficult because of the complexity of the conditions for higher degree equations in x (or y) to have double roots.

For the sake of reference we quote here (from the theory of equations) the conditions that the following cubic and quartic equations in x shall have double roots.

The *cubic*

$$a_0x^3 + 3\,a_1x^2 + 3\,a_2x + a_3 = 0 \tag{44}$$

has a *double root* if

$$a_0^2a_3^2 - 6\,a_0a_1a_2a_3 + 4\,a_0a_2^3 + 4\,a_1^3a_3 - 3\,a_1^2a_2^2 = 0 \tag{45}$$

The *quartic*

$$a_0x^4 + 4\,a_1x^3 + 6\,a_2x^2 + 4\,a_3x + a_4 = 0 \tag{46}$$

has a *double root* if

$$(a_0a_4 - 4\,a_1a_3 + 3\,a_2^2)^3 = 27 \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}^2 \tag{47}$$

We can readily prove (45) from the fact that (44) and its derivative $3\,a_0x^2 + 6\,a_1x + 3\,a_2 = 0$ must have a common root in order that (44) have a double root. Multiplying these two equations, (44) and its derivative, by x, x^2, . . ., etc., and proceeding as in §45, we obtain the condition

$$\begin{vmatrix} a_0 & 3\,a_1 & 3\,a_2 & a_3 & 0 \\ 0 & a_0 & 3\,a_1 & 3\,a_2 & a_3 \\ 0 & 0 & 3\,a_0 & 6\,a_1 & 3\,a_2 \\ 0 & 3\,a_0 & 6\,a_1 & 3\,a_2 & 0 \\ 3\,a_0 & 6\,a_1 & 3\,a_2 & 0 & 0 \end{vmatrix} = 0$$

which (when expanded) gives us (45). A similar derivation would give us (47). We leave the details of all this to the exercises.

ILLUSTRATIVE EXAMPLES. Suppose we want to *determine* b so that $y = x + b$ shall be *tangent* to the cubic $y = 2\,x^3 + x^2$. Solving simultaneously, we obtain $2\,x^3 - 2\,bx - b^2 = 0$. Here

$$a_0 = 2, \quad a_1 = 0, \quad a_2 = -2\,b/3, \quad a_3 = -b^2$$

Substituting these values for the constants in (45), we get $b^3 = 0$ and $b^3 = -\frac{1}{2}\frac{6}{7}$. Therefore the four tangents are $y = x$, $y = x - 2\sqrt[3]{2}/3$, and $y = x + \alpha$ where α is either one of the imaginary roots of $b^2 - 2\sqrt[3]{2}/3\, b + 4\sqrt[3]{4}/9 = 0$ (the second factor of $b^3 + \frac{1}{2}\frac{6}{7}$ equated to zero). Note that $b^3 = 0$ means three coincident tangents are accounted for by $y = x$. (The origin is a double point for the cubic with $y = x$, $y = -x$ as tangents.)

Next suppose we want $y = x + b$ to be tangent to $4\,y = x^4$. The resulting equation in x is $x^4 - 4\,x - 4\,b = 0$, where $a_0 = 1$, $a_1 = a_2 = 0$, $a_3 = -1$, $a_4 = -4\,b$. From (47) we get $b = \pm 2\sqrt{3}/9$, so the two tangents are $y = x \pm 2\sqrt{3}/9$.

If we want $y = x + b$ to be tangent to $5\,y = x^5$, we have $x^5 - 5\,x - 5\,b = 0$, with the derivative $x^4 - 1 = 0$; hence $x = \pm 1, \pm i$ (which are the abscissas of the points of contact of the desired tangents) and $b = \frac{1}{5}(\alpha^5 - 5\,\alpha)$ where $\alpha = \pm 1, \pm i$.

EXERCISES

1. Complete the derivations of (45) and of (47).

2. Prove that $f(x) = 0$ has a double root if $f(x) = 0$ and $f'(x) = 0$ have a common root.

3. Show why (45), (47), and Ex. 2 give necessary and sufficient conditions.

4. Show that (0,0) is a double point on $y^2 = x^3$ by solving with $y = mx$ (m arbitrary).

5. Determine b so that $y = 2\,x + b$ shall be tangent to the sextic $3\,y = x^6$.

6. Derive (42) and (43).

7. Show that if $a_0 = 0$ in (45) and $a_0 = a_1 = 0$ in (47) these conditions for double roots become the ordinary condition that the resulting quadratic equations (44) and (46) shall have double roots.

8. Show that there are ordinarily *six* tangents to a *cubic* from a point $P'(x',y')$. Hint: Take P' as (0,0); solve $y = mx$ simultaneously with (38), use (45) on the resulting equation in x. What is the degree in m of the new equation obtained by using (45)? How does this prove the theorem? Why is there no loss of generality in taking P' as (0,0)?

9. Show that there are ordinarily *twelve* tangents to a *quartic* from a point.

10. Show that (0,0) is a double point on (38) if and only if $c = g = h = 0$. Hint: Solve $y = mx$ with (38). The resulting equation in x must start with the term in x^2 for every value of m. (Why?)

11. Show that if $c = j = h = 0$ in (38), (0,0) is a point of inflection with $x = 0$ as tangent.

12. Show how many tangents to (38) there are from $P'(0,0)$ if P' is an ordinary point on (38); if P' is an inflection (use Ex. 11).

13. Why can there be no tangents to the cubic (38) from P' if P' is a double point on (38)?

14. Find the tangents to $y = x^3$ parallel to $y = x$; the tangents to $y = x^4$ parallel to $y = 2\,x$.

15. Find the tangents to $xy = 1$ that are parallel to $y = -3\,x$; parallel to $y = 3\,x$.

16. Find the tangent to $x^3 + y^3 = 3xy$ at $(\frac{3}{2}, \frac{3}{2})$; the tangent to $y = x(x-1)(x-2)$ at the point where $x = \frac{1}{2}$; at the point where $x = \frac{3}{2}$.

17. How are *parallel* tangents to a curve included under the term *tangents through a point?*

18. Describe how the coordinate axes and lines parallel to them cut the curves $y = x^4$, $x^2y = 1$, $x^3y = x^2 - y^2$.

19. Show analytically that (3,0) is an inside point with respect to $x^2 - y^2 = 1$ and an outside point with respect to $x^2 + y^2 = 1$. Compare §39.

47. Points of inflection on n-ics. DEFINITION. A point of inflection (or briefly an *inflection*) on an n-ic is a point where the tangent intersects the curve in *three coincident points*, but every other line through P cuts the curve there in only one point. A point of *hyperinflection* is such a point as P, only with the tangent intersecting the curve there in four or more coincident points.

Thus (0,0) is an inflection for $y = x^3$ and $y = x^3(x + y - 1)$ with $y = 0$ as tangent, but a hyperinflection for $y = x^4$, $y = x^5$, $y = x^4(2x + 1)$, . . . , $y = x^n$ (for n a positive integer > 3).

An interesting way to *test cubics* for *inflections* is as follows. Consider the cubic $y^2 = x^3$. Solving this cubic with $y = mx + b$, we get

$$x^3 - m^2x^2 - 2bmx - b^2 = 0$$

If $y = mx + b$ is to be the tangent at an inflection, then this equation in x alone must have three equal roots, i.e., it must be of the form

$$x^3 + 3cx^2 + 3c^2x + c^3 = 0 \tag{48}$$

From this we get $3c = -m^2$, $3c^2 = -2bm$, $c^3 = -b^2$. Hence we have $9c^4 = 4b^2m^2 = 4 \cdot c^3 \cdot 3c$, giving us $c^4 = 0$, so $m = b = 0$, and there is no other possible tangent apparently than $y = 0$. But (0,0) is a double point on $y^2 = x^3$ with $y = 0$ as tangent. (We shall see later on that we must suppose this cubic to have an infinite point of inflection on the y-axis with a tangent that consists entirely of infinite points. One way to see this is to note that the transformation $x = x'/y'$, $y = 1/y'$ sends $y = x^3$ into $y'^2 = x'^3$.)

The method is to eliminate m and b between the three equations, solve the resulting equation in c, then substitute the roots back into the equations that determine m and b. We saw above that we may also stumble onto double points. (Why?)

We now show that the ordinary method of the calculus for the

finding of finite inflections on curves will apply also to *oblique* coordinates. This method consists in solving simultaneously the curve $f(x,y) = 0$ and the equation $d^2y/dx^2 = 0$ where $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$. From (2) and (2′) we see that (for oblique coordinates with the same x-axis and origin) we have

$$dx' = dx - dy \cot \omega, \quad dy' = dy \csc \omega, \quad \frac{dy'}{dx'} = \frac{\frac{dy}{dx} \csc \omega}{1 - \frac{dy}{dx} \cot \omega}$$

Hence we have

$$\frac{d^2y'}{dx'^2} = \frac{d}{dx'}\left(\frac{dy'}{dx'}\right) = \frac{d}{dx}\left(\frac{dy'}{dx'}\right)\frac{dx}{dx'} = \frac{d}{dx}\left(\frac{dy'}{dx'}\right)\frac{1}{\frac{dx'}{dx}}$$

$$= \frac{\frac{d^2y}{dx^2} \csc \omega}{(1 - \frac{dy}{dx} \cot \omega)^3}$$

Therefore if $d^2y/dx^2 = 0$, then $d^2y'/dx'^2 = 0$; and, conversely, if $d^2y'/dx'^2 = 0$, then $d^2y/dx^2 = 0$. This shows that in order to find points of inflection on a curve $f(x,y) = 0$ referred to oblique axes, we do the same as for rectangular axes (i.e., we solve $f(x,y) = 0$ simultaneously with $d^2y/dx^2 = 0$). For example, $y = x^3 - 3x^2$ gives $d^2y/dx^2 = 6x - 6$, so the only finite point of inflection is $(1,-2)$.

EXERCISES

1. Check over all the algebra and calculus in the text.
2. Why is there no loss of generality in the last paragraph of the text incurred by using (2) instead of (1), or of (1) and a translation? Hint: One way is to show what (6) and (7) do to d^2y/dx^2.
3. Test for inflections (by the algebraic method in the text) the cubics $x^3 + y^3 = 3xy$, $y = x^3 + x^2$, $y = x^3 - x^2$, $y = x^3$.
4. Use the calculus test on the cubics of Ex. 3.
5. Make up a method of testing quartics for hyperinflections similar to the algebraic method of testing cubics for inflections. Use this method on $y^3 = x^4$, $y = x^4 - x^3$, $y = x^4 - x^2$.
6. Look up in a calculus textbook the discussion of points of inflections on curves.

48. **Multiple points on n-ics.** We noted in §46 that the equation (42) or (3) of a tangent to a curve $f(x,y) = 0$ at a point $P'(x',y')$ becomes *indeterminate* if $\partial f/\partial x = \partial f/\partial y = 0$ for $x = x'$ and $y = y'$. In this case P' is called a *multiple point* on the curve.

To discuss multiple points on n-ics we must consider *Taylor's formula* for the expansion of a function of *two* variables. We shall first *derive* this formula. If we put $x = x' + \lambda\,\Delta x$, $y = y' + \lambda\,\Delta y$ in $f(x,y)$, we obtain $f(x' + \lambda\,\Delta x,\ y' + \lambda\,\Delta y)$. This last is a function of λ alone if x', y', Δx, Δy are constants and we put $f(x' + \lambda\,\Delta x,\ y' + \lambda\,\Delta y) = \phi(\lambda)$. Expanding this function $\phi(\lambda)$ in powers of λ by Taylor's theorem for a function of *one* variable, we obtain

$$f(x' + \lambda\,\Delta x,\ y' + \lambda\,\Delta y) \equiv \phi(\lambda) = \phi(0) + \phi'(0)\lambda + \frac{\phi''(0)}{2!}\lambda^2 + \cdots + \phi^{(n)}(0)\lambda^n$$

But we have

$$\phi(0) = f(x',y'),\quad \phi'(\lambda) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y,\quad \phi'(0) = \frac{\partial f}{\partial x'}\Delta x + \frac{\partial f}{\partial y'}\Delta y \text{ (since } x = x' + \lambda\,\Delta x,\quad y = y' + \lambda\,\Delta y), \ldots,$$

$$\phi^{(n)}(0) = \left(\frac{\partial f}{\partial x'}\Delta x + \frac{\partial f}{\partial y'}\Delta y\right)^{(n)}$$

where (n) is *symbolic* and this last expression is short for

$$\frac{\partial^n f}{\partial x'^n}\overline{\Delta x}^n + n\frac{\partial^n f}{\partial x'^{n-1}\,\partial y'}\overline{\Delta x}^{n-1}\,\Delta y + \frac{n(n-1)}{2!}\frac{\partial^n f}{\partial x'^{n-2}\,\partial y'^2}\overline{\Delta x}^{n-2}\overline{\Delta y}^2 + \cdots + \frac{\partial^n f}{\partial y'^n}\overline{\Delta y}^n$$

(wherein $\partial^n f/\partial x'^n$ means $\partial^n f/\partial x^n$ with x and y replaced by x' and y', similarly for the other derivatives).

From this we obtain Taylor's expansion for a function of two variables

$$(49)\quad f(x' + \lambda\,\Delta x, y' + \lambda\,\Delta y) = f(x',y') + \lambda\left(\frac{\partial f}{\partial x'}\Delta x + \frac{\partial f}{\partial y'}\Delta y\right) + \frac{\lambda^2}{2!}\left(\frac{\partial^2 f}{\partial x'^2}\overline{\Delta x}^2 + 2\frac{\partial^2 f}{\partial x'\,\partial y'}\Delta x\,\Delta y + \frac{\partial^2 f}{\partial y'^2}\overline{\Delta y}^2\right)$$

$$+\frac{\lambda^3}{3!}\left(\frac{\partial^3 f}{\partial x'^3}\overline{\Delta x}^3+3\frac{\partial^3 f}{\partial x'^2\,\partial y'}\overline{\Delta x}^2\,\Delta y+3\frac{\partial^3 f}{\partial x'\,\partial y'^2}\Delta x\,\overline{\Delta y}^2\right.$$
$$\left.+\frac{\partial^3 f}{\partial y'^3}\overline{\Delta y}^3\right)+\cdots+\frac{\lambda^n}{n!}\left(\frac{\partial f}{\partial x'}\Delta x+\frac{\partial f}{\partial y'}\Delta y\right)^{(n)}$$

The last term in (49) may also be written $\lambda^n f(\Delta x,\Delta y)$.

The equation of the secant through $P'(x',y')$ and a point $P''(x'+\Delta x, y'+\Delta y)$ has the form $(y-y')/(x-x')=\Delta y/\Delta x$. From this equation we find

$$(50)\qquad y-y'=\lambda\,\Delta y,\quad x-x'=\lambda\,\Delta x\quad\text{or}\quad y=y'+\lambda\,\Delta y,\quad x=x'+\lambda\,\Delta x$$

Instead of solving $(y-y')/(x-x')=\Delta y/\Delta x$ simultaneously with the equation of the n-ic $f(x,y)=0$ so as to get the points of intersection of this secant and the curve, we can substitute x and y from (50) in $f(x,y)=0$, giving us

$$f(x'+\lambda\,\Delta x,\ y'+\lambda\,\Delta y)=0$$

This last equation, which is really (49) equated to zero, is an equation in λ alone, since x', y', Δx, Δy are all constants temporarily. The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of this equation give us the coordinates

$$(x'+\lambda_1\,\Delta x,\ y'+\lambda_1\,\Delta y),\quad (x'+\lambda_2\,\Delta x,\ y'+\lambda_2\,\Delta y),\ldots,$$
$$(x'+\lambda_n\,\Delta x,\ y'+\lambda_n\,\Delta y)$$

of the points of intersection of the secant and the curve.

Ordinarily, since $f(x,y)=0$ is an n-ic, (49) is of the nth degree in λ. Thus we have proved *again* (incidentally) that a *straight line* cuts an *nth degree* curve in *n points*.

Now if $P'(x',y')$ is a multiple point on $f(x,y)=0$, then $f(x',y')=\partial f/\partial x'=\partial f/\partial y'=0$ and the equation in λ has at least one double root, namely $\lambda^2=0$, which means that every secant through P' cuts the n-ic at P' in at least two coincident points.

If one (or more) of the second partial derivatives of $f(x,y)$ does not vanish at P', this point is called a *double point* on the curve (or a *multiple point of order two*).

If $\partial^2 f/\partial x'^2=\partial^2 f/\partial x'\partial y'=\partial^2 f/\partial y'^2=0$, but one (or more) of the third partial derivatives does not vanish at P', every secant through P' cuts the curve at P' in three coincident points, and is called a *triple point* (or a *multiple point of order three*). Similarly, a curve may have multiple points of still higher orders.

If P' is a double point on the curve, we replace $\lambda\,\Delta x$ by $x - x'$ and $\lambda\,\Delta y$ by $y - y'$ in the equation

$$\frac{\partial^2 f}{\partial x'^2}\lambda^2\overline{\Delta x}^2 + 2\frac{\partial^2 f}{\partial x'\,\partial y'}\lambda^2\,\Delta x\,\Delta y + \frac{\partial^2 f}{\partial y'^2}\lambda^2\overline{\Delta y}^2 = 0$$

and get the equation

$$\frac{\partial^2 f}{\partial x'^2}(x - x')^2 + 2\frac{\partial^2 f}{\partial x'\,\partial y'}(x - x')\,(y - y') + \frac{\partial^2 f}{\partial y'^2}(y - y')^2 = 0 \tag{51}$$

This equation gives two straight lines that are said to be the *tangents* to the curve at the point P', since the values

$$\frac{y - y'}{x - x'} = r_1, \quad \frac{y - y'}{x - x'} = r_2$$

that respectively satisfy (51), or in other notation

$$y - y' = \lambda\,\overline{\Delta y}', \; x - x' = \lambda\,\overline{\Delta x}' \quad \text{and} \quad y - y' = \lambda\,\overline{\Delta y}'', \; x - x' = \lambda\,\overline{\Delta x}''$$

when solved simultaneously with the curve cause (49) when equated to zero to have at least a triple root $\lambda^3 = 0$, i.e., these two lines cut the n-ic at P' in three or more coincident points.

According as $\left(\frac{\partial^2 f}{\partial x'\,\partial y'}\right)^2 - \frac{\partial^2 f}{\partial x'^2}\frac{\partial^2 f}{\partial y'^2} \gtreqless 0$, the two lines (51) are real and distinct, real and coincident, or conjugate imaginary, respectively; and P' is called a *crunode*, *cusp*, or *acnode* (or *isolated point*) in these three cases. For example, $y^2 = x^3 + x^2$ has a crunode at (0,0) with tangents $y = \pm x$; $y^2 = x^3$ has a cusp at (0,0) with $y = 0$ as tangent; $y^2 = x^3 - x^2$ has an acnode at (0,0) with tangents $y = \pm ix$.

Note that if (0,0) is on the n-ic (39), the terms of *lowest degree* in x and y when equated to zero give the *tangent* (or *tangents*) to the n-ic at the origin. (Why?) Thus (if $a = 0$) in (39)

$$b_0x + b_1y = 0, \text{ or } c_0x^2 + 2\,c_1xy + c_2y^2 = 0 \text{ (if } b_0 = b_1 = 0)$$

or

$$d_0x^3 + 3\,d_1x^2y + 3\,d_2xy^2 + d_3y^3 = 0$$

$$\text{(if } b_0 = b_1 = c_0 = c_1 = c_2 = 0)$$

etc., give us the tangents to (39) at (0,0); and (0,0) is, respectively, an ordinary point, a double point, a triple point, etc.

ILLUSTRATIVE EXAMPLE. To test the curve

$$x^3 - 8y^3 - 6x^2y + 12xy^2 - x^2 + 2xy - y^2 = 0$$

we solve this equation simultaneously with

$$\frac{\partial f}{\partial x} \equiv 3x^2 - 12xy + 12y^2 - 2x + 2y = 0, \quad \frac{\partial f}{\partial y} \equiv -24y^2 - 6x^2 + 24xy + 2x - 2y = 0$$

and get $x' = y' = 0$. Here $\frac{\partial^2 f}{\partial x'^2} = \frac{\partial^2 f}{\partial y'^2} = -2$ and $\frac{\partial^2 f}{\partial x\, \partial y} = 2$, so (0,0) is a cusp with tangent (51) reducing to $y = x$ taken twice.

If P' is a triple point on the n-ic, the curve has at P' three so-called *tangents* given by the equation

$$\frac{\partial^3 f}{\partial x'^3}(x - x')^3 + 3\frac{\partial^3 f}{\partial x'^2\, \partial y'}(x - x')^2(y - y') + 3\frac{\partial^3 f}{\partial x'\, \partial y'^2}(x - x')(y - y')^2 + \frac{\partial^3 f}{\partial y'^3}(y - y')^3 = 0 \tag{52}$$

each of which tangents cuts the curve in at least four coincident points at P'.

Similarly there are four tangents at a quadruple point, five tangents at a quintuple point, . . ., n tangents at an n-ple point. Also these tangents may be real or imaginary, distinct, or some or all of them coincident. Also some of the tangents at a multiple point of order r may cut the curve there in more than $r + 1$ coincident points. Thus for the curve $xy = x^4 + y^4 - y^3$ the origin is a node with $y = 0$ as a tangent intersecting the curve there in four coincident points instead of three.

We call attention to the fact that our definition of the tangent as the limiting position of secant $P'P''$ actually gives us tangents at a double point P' that cut the curve there in at least three coincident points (the limiting position of the double point P' and at least one other point of intersection of the secant with the curve as $P'' \to P'$); a similar remark applies to a multiple point of higher order.

We can discuss a multiple point P' very readily by translating this point to the origin.

Finally we note that our discussion of *multiple points* is valid for *oblique* axes as well as for *rectangular*, and even for different-sized units on the axes.

As *illustrations* of our discussion we see that $y^2 = x^5$ has a cusp at (0,0) with tangent $y = 0$ that cuts the curve there in five coincident points (and so is said to have quintuple contact there with the curve). The curve $y(y - x) = x^3y + x^5$ has a crunode at the origin whose tangents $y = x$ and $y = 0$ have respectively quadruple and quintuple contact there with the curve. Note that every other line $y = mx$ through the origin cuts this curve in points given by $m(m - 1)x^2 = mx^4 + x^5$, i.e., in two points at (0,0) and in three other points given by $x^3 + mx^2 = m(m - 1)$; also as $m \to 1$ two more of these points approach the origin (because this last equation approaches the form $x^3 + x^2 = 0$), but as $m \to 0$ three more of these points approach the origin. The tangents at the origin to the curve $y^3 = x^3(x + 1)$ are given by $y^3 - x^3 = 0$ and so they are $y = x$, $y = (-1 \pm \sqrt{3}\, i)\, x/2$. The tangents to $(x^2+ y^2)\ (x^2 - y^2)y = x^5$ at the origin are $y = \pm x$, $y = \pm ix$, $y = 0$.

Suppose a quartic curve has a triple point P' with three real tangents, two of them coincident. If we take P' as (0,0) and the tangents as $x^2 = 0$ and $y = 0$, then in the n-ic (39) we have $n = 4$, $a = b_0 = b_1 = c_0 = c_1 = c_2 = d_0 = d_2 = d_3 = 0$, $d_1 \neq 0$, $e_0e_4 \neq 0$.

EXERCISES

1. Give an exact definition of a multiple point of order r on an n-ic.
2. Prove that the last term in (49) may be written $\lambda^n f(\Delta x,\Delta y)$. Hint: Put $\lambda = 1/\mu$, expand $(1/\mu)f(\mu x' + \Delta x, \mu y' + \Delta y)$, then replace μ by $1/\lambda$.
3. How do we get from $(y - y')/(x - x') = \Delta y/\Delta x$ the equations $y - y' = \lambda\, \Delta y$, $x - x' = \lambda\, \Delta x$?
4. Prove that each of the tangents given by (52) cuts the curve in at least four coincident points.
5. Check all the algebra in the text. Answer the queries (Why?).
6. Why cannot a cubic have two double points?
7. Show that the necessary and sufficient conditions for the origin to be a point of inflection on the n-ic (39) are $a = 0$ and $b_0x + b_1y$ a factor of $c_0x^2 + 2\, c_1xy + c_2y^2$.
8. Suppose (39) is a quartic with a triple point at the origin whose tangents are $x = 0$, $y = 0$, $y = x$. What about the coefficients of (39)?
9. If the equation in λ in the text for an n-ic $f(x,y) = 0$ is of degree lower than the nth, how must we interpret this peculiarity?

10. What sort of multiple point is the origin for the curves $x^3y^4(x^2+y^2)(x^2-4y^2)=(x+y)^{11}$, $y^3=x^4$, $x^4+y^4=x^5$? Find the tangents at the origin.

11. Test the following curves for multiple points, determine the nature of these points, the equations of their tangents, and the order of contact of these tangents:

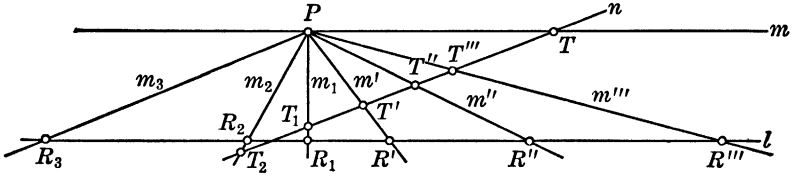
$$4(x-1)^3+(y-3x+2)^2=0, \quad (x^2+y^2-1)^3+27x^2y^2=0,$$
$$(x^2-1)^2y=(y^2-2)^2x, \quad x^4-2ay^3+3a^2y^2-2a^2x^2+a^4=0$$

12. Prove that a quartic curve cannot have a triple point P' and a double point P'', or more than two collinear double points. Hint: In how many points would the line $P'P''$ cut the quartic?

13. Make up examples of curves with multiple points at the origin whose tangents have differing orders of contact.

14. Prove that a quartic cannot have four double points. Hint: In how many points would a conic through the four double points and a fifth point on the quartic cut this curve?

49. **Parallel lines and the line at infinity in a plane.** Let us consider the following figure to see what is meant by two lines l and m in a plane being parallel. Suppose a line to rotate in counter-clockwise direction around a point P and in its various positions m', m'', m''', . . ., m_3, m_2, m_1, . . . to cut a fixed line l in the points R', R'', R''', . . ., R_3, R_2, R_1, . . . , respectively. As this line rotates, its point of intersection R with l passes through the positions R', R'', R''', etc., to the right, then comes back from the left through R_3, R_2, R_1, etc.



In *plane geometry* and in *elementary analytic geometry* it is *assumed* that when the rotating line reaches the position m (where it is said to be parallel to l), then its point of intersection R with l *ceases to exist.* Note that this is a mere assumption because we cannot follow this point of intersection R out there and investigate its behavior.

In *projective geometry* we assume that R does then exist in a unique position R_∞ (of course R_∞ is *not* a finite point), and we call R_∞ the point at infinity on l.

For many reasons we do not assume the existence of *two* such

points at infinity on l (one in each direction). If we did so, then two distinct parallel lines would intersect in two (infinite) points; but two finite points determine a line, and we should want *infinite* points (which are the limiting positions of finite points) to behave like *finite* points.

Also from the above figure (where n is taken parallel to m_3) we see another reason for assuming the existence of one and only one infinite point on a line. By means of the lines through P there is established a *one-to-one correspondence* between the points of l and those of n (compare §29) such that to *each* point R', R'', R''', . . ., R_2, R_1, . . . on l there corresponds a *unique* point T', T'', T''', . . ., T_2, T_1, . . . on n (and *conversely*), with the *exceptions* of T and R_3. We can *remove these exceptions* by assuming the existence on l of one and only one infinite point R_∞ to correspond to T, and the existence on n of one and only one infinite point T_∞ to correspond to R_3.

Note that the above argument is all based on the *assumption* that through a *given* point P there is *one* and *only one* line m *parallel* to a given line l. If we assume there are *two* such parallel lines through P (one obtained by rotating m counter-clockwise around P, the other by rotating m clockwise), then we should postulate the existence of *two* infinite points on each line.

Two lines l and l' that are *not parallel* must be supposed to have *distinct* points R_∞ and R'_∞ at infinity. We assume the locus of all these points R_∞ in a plane to be a *straight line* l_∞ that lies entirely at infinity (i.e., consists entirely of infinite points) because any line cuts this locus in only one point. We call l_∞ the *line at infinity* in the plane.

We can look upon l_∞ as the *limiting position* of *any* line $\alpha x + \beta y - 1 = 0$ as $\alpha \to 0$ and $\beta \to 0$, i.e., as the intercepts of this line become infinitely great (that is, grow beyond all bounds). (Compare §25.) In preceding sections we have spoken about a line consisting entirely of infinite points (compare §47); now we shall call such a locus the line l_∞ that consists entirely of infinite points or the infinite line l_∞.

This assumption of the existence of R_∞ on a line l and of l_∞ in a plane π is amply justified by its usefulness, and also it does not conflict with any other part of mathematics. From now on we shall say that *any* two lines in a plane *intersect* in a point *R finite* or *infinite*.

EXERCISES

1. Look over the preceding sections of this book for any mention of infinite points and of infinite lines, and note how all such previous discussions naturally point toward the assumption of the existence of l_∞.

2. Generalize to a space of three dimensions the above discussion of infinite points. Show why we assume the existence of a plane at infinity π_∞.

3. Explain how the assumption that there is an infinite point on each line does not contradict the theorem from plane geometry that parallel segments included between parallel lines are equal.

4. Using the assumption of the existence of l_∞ and referring to §44, prove that an n-ic can have at the most n asymptotes.

5. Explain why an n-ic may have less than n asymptotes. How about the parabola, the cubics $y = x^3$ and $x^3 + y^3 = 3xy$?

CHAPTER VIII

CONICS AND LINEAR FAMILIES OF CONICS

50. **Poles and polars with respect to conics.** We have already seen that the tangent to the conic (4) at the point $P'(x',y')$ is (5) or (after rearranging the terms)

$$(53)\quad (ax' + hy' + g)x + (hx' + by' + f)y + (gx' + fy' + c) = 0$$

There is *no* such simple formula for the tangents to the general *n-ics* (39).

If $P'(x',y')$ does not lie on the conic (4), we call the line (53) the *polar* of P' with respect to the conic and P' the *pole* of (53). Compare §25.

The condition that the polar (53) of P' shall pass through a point $P''(x'',y'')$ is

$$(54)\quad ax''x' + by''y' + c + f(y'' + y') + g(x'' + x') + h(x'y'' + x''y') = 0$$

Interchanging x' and x'', y' and y'' in (54) does *not* change the equation but *does* give us the condition that the polar of P'' passes through P'. Therefore we have the theorem:

THEOREM. *If the polar of a point P' with respect to a conic passes through P'', then the polar of P'' passes through P'.*

If P' lies on the conic (4), the *polar* (53) is then the *tangent* at P', but the equation (54) and its results are still valid.

Let us take the point of intersection P of two tangents t_1 and t_2 (real or imaginary) with points of contact P_1 and P_2, respectively. Since P lies on t_1 (the polar of P_1), therefore the polar p of P must pass through P_1, similarly p must pass through P_2. Hence *the polar of P is the line joining the points of contact of the tangents to the conic from P.* If P is outside the conic, this gives us a method of constructing the polar of P (since then t_1 and t_2 are real).

If P is inside the conic, any chord through P cuts the conic in two points P_1 and P_2 whose tangents t_1 and t_2 intersect on the polar of P (since the polar of their point of intersection is the given

chord through P). Taking *two* such chords we can determine the *polar* of P as the *line joining the two poles of these chords.*

Note that *the polars of all the points on a line p are lines through a point P, and conversely.* (Why?)

Let us consider the *close connection* between the relation of *pole* and *polar* and the relation called *plane duality.* We can dualize the construction of the polar of an outside point and we get exactly the construction of the pole of a line that cuts the conic in two real points, and similarly for the construction of the polar of an inside point. (We leave the details for the exercises.)

In fact, to take the polar of a figure or theorem with respect to a conic (if we use infinite points and lines as well as finite) is exactly similar to taking the plane dual of a figure or theorem; because in either case we replace the words point by line, line by point, collinear points by concurrent lines, concurrent lines by collinear points. However, in taking the plane dual we do *not* presuppose any conic and the pole and polar relation as a machinery. In all probability the pole and polar relation originally suggested the idea of plane duality.

ILLUSTRATIVE EXAMPLE. The polar of the *focus* $(p,0)$ with respect to $y^2 = 4px$ is $y \cdot 0 = 2p(x + p)$, which is the *directrix*. The pole of $3x + 2y = 1$ with respect $x^2/16 + y^2/9 = 1$ is obtained by taking the polar form $xx'/16 + yy'/9 = 1$ and from this getting $x'/16 = 3$, $y'/9 = 2$, so the desired pole is $(48,18)$.

Let us take up another use of tangents to conics and of poles and polars. Suppose we want $(0,0)$ to have the polar $x + y - 1 = 0$ with respect to (4) and $(1,0)$ the polar $x = 0$. The polar of $(0,0)$ is

$$gx + fy + c = 0$$

so we have $g = f = -c$. The polar of $(1,0)$ is

$$(a + g)x + (h + f)y + (g + c) = 0$$

so we have $h + f = g + c = 0$ and $a + g \neq 0$. Therefore (4) has the simpler form

$$ax^2 + by^2 + g(-1 + 2y + 2x - 2xy) = 0$$

Next let us reduce the equation of a hyperbola to a simple form by referring its equation to its asymptotes as rectangular or oblique axes. (This is equivalent to making an affine linear transformation of coordinates considered as an alias.) In this case $x = 0$ when solved with (4) must give an equation of the form $0 \cdot y^2 + 0 \cdot y + c = 0$ (from the definition of an asymptote), hence we must have $b = f = 0$. Also $y = 0$ must give $0 \cdot x^2 + 0 \cdot x + c = 0$, hence $a = g = 0$. Our hyperbola has now the equation $2hxy + c = 0$,

where $ch \neq 0$. If we put $x = 1/2\,x'$, $y = -c/hy'$, and divide the equation through by c, we get the well-known form $x'y' = 1$.

DEFINITION. If a point P'' lies on the polar of P' with respect to a conic C (and therefore P' lies on the polar of P'' with respect to C), then P' and P'' are called *conjugate points* with respect to C. Also, dually, if a line l'' passes through the pole of a line l' with respect to C (and therefore l' passes through the pole of l''), then l' and l'' are said to be *conjugate lines* with respect to the conic C.

We shall take up again, later on, the discussion of pole and polar and of conjugate points with respect to a conic. We shall then show (see also in the exercises below), among other things, that the pole of any *diameter* lies on its *conjugate diameter*; hence the name conjugate diameters (since these two diameters are conjugate lines with respect to the conic).

ILLUSTRATIVE EXAMPLE. Let us find the point on $y = x$ conjugate to (1,1) with respect to $x^2 + y^2 = 4$. The polar of (1,1) is $x + y = 4$, which cuts $y = x$ in the desired point (2,2). *Dually*, let us find the line through (1,1) conjugate to $x + y = 2$ with respect to this circle. The pole of $x + y = 2$ is (2,2), hence the desired line is $y = x$.

EXERCISES

1. Prove that the necessary and sufficient condition for the polar of $P'(x',y')$ with respect to (4) to pass through P' is that P' lie on (4).

2. What form does the equation (4) take if the polar of (1,1) is $y = -x$ and the polar of $(1,-1)$ is $y = x$?

3. Taking the asymptotes of a hyperbola (4) as $y = x$ and $y = -x$, reduce its equation to the form $x^2 - y^2 = 1$.

4. A triangle is said to be *self-polar* with respect to a conic C if its sides are the polars of the opposite vertices with respect to C. Prove that *if two vertices of a triangle and the sides opposite them are pole and polar with respect to C, then the triangle is self-polar.*

5. Find the pole of $2x + 3y - 4 = 0$ with respect to $x^2/25 + y^2/16 = 1$; with respect to $xy = 1$.

6. Find the point on $y = 0$ conjugate to (3,0), to $(7 - i,0)$, with respect to $x^2/25 + y^2/16 = 1$. Find the line through (0,0) conjugate to $y = 0$ with respect to the conic $x^2 - 2xy + 4y^2 - 7x + 2y = 3$.

7. Make up a pair of conjugate points and a pair of conjugate lines with respect to the conic $7x^2 - 9y^2 = 1$.

8. We have a pair of conjugate points P,P' with respect to a conic C and lying on a line l that cuts C in the two points P_1, P_2. Where does it show (incidentally) in a previous section of the text that P, P_1, P', P_2 form a harmonic set? How does it follow that if $P_1 = P_2$ (i.e., if l is tangent to C), then $P' = P_1 = P_2$ and P is any point on l?

9. Prove that a pair of conjugate points P,P' with respect to a conic C must either lie both outside C, or one inside C and one outside C.

10. Prove that *the polars of all the points on a line p with respect to a conic C are lines through a point of p, and conversely.*

11. In the reduction (in the text) of the hyperbola to the form $x'y' = 1$, state fully why $x = 0$ must give $0 \cdot y^2 + 0 \cdot y + c = 0$.

12. Find two distinct self-polar triangles with respect to the conic $xy = -1$.

13. Dualize the construction derived in the text for the polar of an outside (inside) point with respect to a conic.

51. **Some applications of poles and polars with respect to a conic.** In §25 we saw that the polar of a focus of a conic with respect to this conic is a directrix; hence from §50 we observe that the tangents at the two extremities of any focal chord (i.e., chord through the focus) intersect on the directrix corresponding to this focus. Also the polar of the *center* (0,0) of a conic $x^2/a^2 \pm y^2/b^2 = 1$ (or $x^2 + y^2 = r^2$) is a line with equation of the form $0 \cdot x/a^2 + 0 \cdot y/b^2 = 1$ (or $0 \cdot x + 0 \cdot y = r^2$). We *interpret* this equation as giving a line consisting entirely of infinite points (i.e., l_∞). Therefore the tangents at the ends of a diameter must be parallel.

Again we note that to find the pole of $x = c$ with respect to $x^2/a^2 \pm y^2/b^2 = 1$ we take the polar of $P'(x',y')$ in the form $x \pm yy'/b^2x' = a^2/x'$. This equation is to be $x = c$, hence $a^2/x' = c$, $y' = 0$ and the required pole is $(a^2/c, 0)$. As $c \to 0$ this pole approaches the point $(\infty,0)$, which we interpret as the point at infinity on $y = 0$. A similar result holds for the pole of $x = 0$. Therefore the two axes and the line form a sort of self-polar triangle with respect to these conics. (See Ex. 4 in §50.)

To obtain the result concerning conjugate diameters that is quoted in §50, we take $y = mx + c$. We put the polar of $P'(x',y')$ with respect to $x^2/a^2 \pm y^2/b^2 = 1$ in the form $\pm b^2xx'/a^2y' + y = \pm b^2/y'$. This must be the line $-mx + y = c$, hence $\pm b^2/y' = c$ and $\pm b^2x'/a^2y' = -m$. This last equation shows that the pole of $y = mx + c$ lies on the line through the origin whose equation is $y = \pm b^2/a^2mx$. As $c \to 0$ and $y = mx + c$ approaches the diameter, then $y'(= \pm b^2/c) \to \infty$ but the pole $P'(x',y')$ still lies on $y = \mp b^2/a^2mx$. Therefore we see that the diameters with slopes m and m' connected by $mm' = \mp b^2/a^2$ are conjugate lines with respect to the conic $x^2/a^2 \pm y^2/b^2 = 1$.

Finally, we shall use the result in the first paragraph of this section to find the center of the conic (4). If $P'(x',y')$ is the center

of (4), its polar (53) must be of the form

$$0 \cdot x + 0 \cdot y + (gx' + f'y + c) = 0$$

(Why?) Hence we must have

$$ax' + hy' + g = 0, \quad hx' + by' + f = 0$$

giving us the center

$$x' = \frac{-bg + fh}{ab - h^2}, \quad y' = \frac{hg - af}{ab - h^2}$$

if $ab - h^2 \neq 0$. (Compare this brief derivation of the center with the one given in elementary analytic geometry.)

If $ab - h^2 = 0$, either $x' = \infty$ or $y' = \infty$ or both are infinite (i.e., the center is an infinite point); or we have

$$-bg + fh = hg - af = ab - h^2 = 0, \quad x' = \frac{0}{0}, \quad y' = \frac{0}{0}$$

To interpret geometrically the last possibility we remark that the discriminant Γ of the conic (4) can then be written

$$\Gamma \equiv c(ab - h^2) + g(-bg + fh) + f(hg - af) = 0$$

Therefore the conic is degenerate. In fact, it can be shown that the conic is then a double line.

For a parabola ($\Gamma \neq 0$, $ab - h^2 = 0$) we find above that the center is an infinite point. We assume, therefore, that the center P' lies on the conic and the polar (l_∞) of the center P' touches the conic at P'. This agrees with the fact that the parabola has no asymptotes. (Why?) Thus we assume that the parabola $y^2 = 4px$ has an infinite center on the x-axis, since for this curve $hg - af = 0$ so $y' = 0/0$, but $-bg + fh = 2p$ so $x' = \infty$. Compare Ex. 5 under §49.

EXERCISES

1. Look up the derivation of the center of a conic as given in elementary analytic geometry. Find two different derivations.

2. Answer all queries (Why?) in the text.

3. As was done in the text for $x = 0$, so, similarly, find the pole of $y = 0$ with respect to $x^2/a^2 \pm y^2/b^2 = 1$.

4. Show that if the x- and y-axes are taken as conjugate lines with respect to the conic (4) and the polar of the origin is l_∞, then the equation of this conic can be reduced to the form $x^2/a^2 + y^2/b^2 = 1$. (This is the converse of the discussion in the text.) Hint: What is the pole of $x = 0$ (of $y = 0$) with respect to (4) and the polar of (0,0)?

5. Find the centers of the conics

$$x^2 - xy + 2y^2 + 3x - 2y = 4, \quad x^2 + y^2 + 2gx + 2fy + c = 0$$

Do this in two ways, first by the formula and then by the condition that l_∞ must be the polar of the center.

52. A use of poles and polars to derive the discriminant of a conic. Now we shall give another derivation of the discriminant of a conic (4). Compare §19. First we prove that the polar of *every* point $P_i'(x_i',y_i')$ where $i = 1,2,3$, etc., does *not* pass through a point $P''(x'',y'')$ if the conic (4) is *non-degenerate* but *does* pass through such a point P'' if the conic (4) is *degenerate.*

We prove this fact for the *normal* forms of these conics. Thus, if the three polars of three non-collinear points P_1', P_2', P_3' with respect to $x^2/a^2 + y^2/b^2 = 1$ pass through $P''(x'',y'')$, we must have

$$\frac{x_1'x''}{a^2} + \frac{y_1'y''}{b^2} = 1, \quad \frac{x_2'x''}{a^2} + \frac{y_2'y''}{b^2} = 1, \quad \frac{x_3'x''}{a^2} + \frac{y_3'y''}{b^2} = 1$$

For these three equations to be satisfied by $P''(x'',y'')$ we must have

$$\begin{vmatrix} x_1'/a^2 & y_1'/b^2 & -1 \\ x_2'/a^2 & y_2'/b^2 & -1 \\ x_3'/a^2 & y_3'/b^2 & -1 \end{vmatrix} \equiv -\frac{1}{a^2b^2} \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix} = 0$$

But then the points P_1', P_2', P_3' are collinear (contrary to hypothesis). We leave to the exercises the similar discussions for $y^2 = 4px$, $x^2/a^2 - y^2/b^2 = 1$, $\alpha^2x^2 + \beta^2y^2 + \gamma^2 = 0$ (the imaginary non-degenerate conic).

Next we consider the polars of $P'(x',y')$ with respect to the degenerate conics

$$2xy = 0, \quad x^2 + y^2 = 0, \quad x^2 = 0$$

namely

$$xy' + x'y = 0, \quad xx' + yy' = 0, \quad xx' = 0$$

For every point $P'(x',y')$ these polars with regard to any one of these degenerate conics all pass through the same point (0,0).

The above discussion shows that *a necessary and sufficient condition for a conic C to be degenerate is that the polar of every point $P_i'(x_i',y_i')$ where $i = 1,2,3$, etc., with respect to C shall pass through a common point $P''(x'',y'')$.* Consider the general conic (4), the polar (53), and the condition (54) that (53) shall pass

through P''. Rearranging (54), we obtain

$$(54') \qquad (ax'' + hy'' + g)x' + (hx'' + by'' + f)y' + (gx'' + fy'' + c) = 0$$

This equation (54′) must be satisfied by every pair of values of x' and y'. Therefore we must have

$$ax'' + hy'' + g = 0, \quad hx'' + by'' + f = 0, \quad gx'' + fy'' + c = 0$$

The necessary and sufficient condition for these three linear equations in two unknowns to have a common solution x'', y'' is

$$\Gamma \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Thus we arrive again at the discriminant of (4).

EXERCISES

1. In the text we took the normal (standard) forms for the equations of the non-degenerate (and degenerate) conics to prove the polars of all points P' non-concurrent (concurrent). Why was there no loss of generality?
2. Why was the condition in the test for (4) to be degenerate (namely, that all polars are concurrent) both necessary and sufficient?
3. Discuss $y^2 = 4px$, $x^2/a^2 - y^2/b^2 = 1$, $\alpha^2x^2 + \beta^2y^2 + \gamma^2 = 0$ for the polars of P_1' as $x^2/a^2 + y^2/b^2 = 1$ was discussed in the text.
4. Prove the algebraic condition quoted in the text for (54′) to be satisfied by every pair of values of x' and y'.

53. **Intersections of conics.** Ordinarily two conics C_1 and C_2 intersect in four points P_1, P_2, P_3, P_4. If we enumerate all the possible ways for C_1 and C_2 to intersect when the points P_1, P_2, P_3, P_4 are all real, we have (*a*) all the points *distinct*, (*b*) $P_1 = P_2$ and $P_3 \neq P_4$, (*c*) the points *coincident* in *pairs*, (*d*) *three* of the points *coincident*, (*e*) *all four* points *coincident*.

We can show that the five possible types of intersection of two conics exist by giving a numerical example of each case, as follows:

(*a*) $x^2/9 + y^2/16 = 1$ and $x^2/16 + y^2/9 = 1$

(*b*) $y^2 = 2x + 4$ and $y^2 = 4x + 4$

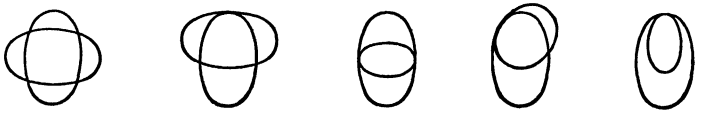
(*c*) $x^2 + y^2 = 16$ and $x^2/16 + y^2/9 = 1$

(*d*) $y^2 = 4x$ and $y^2 = 4x + 2y$

(*e*) $y^2 = 4x$ and $x^2 + y^2 = 4x$

Some of the points of intersection in the numerical illustrations are finite and some infinite. If we allow P_1, P_2, P_3, P_4 to be imaginary as well as real, we have many more cases to distinguish. Thus $x^2 + y^2 = 1$ and $x^2/9 + y^2/16 = 1$ intersect in four distinct and finite imaginary points, whereas $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ touch each other in two infinite imaginary points.

The five types of intersection are illustrated schematically by these figures.



EXERCISES

1. Find the points of intersection for all the pairs of conics given in the text.
2. Tabulate all the possible cases for the intersection of conics, distinguishing the cases where all the points are real and finite, all are real but some are infinite, some or all are imaginary and finite, some or all are imaginary and some are infinite.
3. Give as many numerical examples as you can for the cases in Ex. 2.

54. **Pencils of circles.** We have already studied in elementary analytic geometry so-called *pencils* of *lines* and *pencils* of *circles*, namely all the lines given by

$$(55) \qquad (a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$$

where λ is an arbitrary constant called a *parameter*, and all the circles given by

$$(56) \qquad (x^2 + y^2 + 2\,g_1x + 2\,f_1y + c_1) + \lambda(x^2 + y^2 + 2\,g_2x + 2\,f_2y + c_2) = 0$$

These are also called *linear one-parameter families* of *lines* and of *circles* respectively. These families consist, respectively, of all the lines through the point of intersection of two given lines and of all the circles through the points of intersection of two given circles.

To include the second given line or circle in the above families we must be allowed to divide their respective equations through by λ and then let $\lambda \to \infty$ (so that $1/\lambda \to 0$). The two given lines (circles) are called the *fundamental lines* (*circles*) of the pencil.

Note that any pair of circles in the pencil (56) have the *same*

radical axis (given by $\lambda = -1$), namely,

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0 \tag{57}$$

which may be considered as forming with the hypothetical line at infinity l_∞ a *degenerate circle* of the pencil. We can say that the two lines of this degenerate circle pass through the points of intersection of the circles of the pencil. Compare §45.

This is a useful convention that we now adopt, namely, of agreeing to *consider a linear equation in x and y, when found among (i.e., in a set of) quadratic equations, as the limiting case of a pair of lines where one line has (as we may say) been allowed to move out to infinity.* Thus analytically $x + y - 1 = 0$ may be looked upon as the limiting case of

$$(\alpha x + \beta y - 1)(x + y - 1) = 0$$

as $\alpha \to 0$ and $\beta \to 0$. This convention resembles that used in the study of asymptotes in §44. How?

When we speak of the points of intersection of the circles of a pencil of circles we must stretch our terms slightly so as to cover a case like

$$(x^2 + y^2 - 1) + \lambda(x^2 + y^2 - 6x + 8) = 0$$

where the circles intersect only in imaginary points, or like

$$(x^2 + y^2 - 1) + \lambda(x^2 + y^2 - 4x + 3) = 0$$

where the circles are tangent to one another, or like

$$(x^2 + y^2 - 1) + \lambda(x^2 + y^2 - 4) = 0$$

where the circles do not intersect in finite points and the radical axis must be supposed to be l_∞. Compare Ex. 6 in §45.

Note that all the concentric circles $x^2 + y^2 = r^2$ belong to the last pencil above, because this pencil can be written $x^2 + y^2 = (1 + 4\lambda)/(1 + \lambda)$ and we can take $(1 + 4\lambda)/(1 + \lambda) = r^2$. Any value of the radius $r = r'$ has a corresponding unique value of the parameter $\lambda' = (r'^2 - 1)/(4 - r'^2)$; and conversely to any value of the parameter $\lambda = \lambda'$ there corresponds a unique value of the radius given by $r' = \sqrt{(1 + 4\lambda')/(1 + \lambda')}$ if we allow r to have imaginary values (when $(1 + 4\lambda')/(1 + \lambda') < 0$, i.e., $\lambda' < 0$).

Again all the circles given by

$$x^2 + y^2 + 2gx = 0$$

(namely, with centers on the x-axis and passing through the origin) form a pencil of circles

$$(x^2 + y^2 + 2x) + \lambda(x^2 + y^2 - 2x) = 0$$

where $g = (1 - \lambda)/(1 + \lambda)$ or $\lambda = (1 - g)/(1 + g)$.

EXERCISES

1. Go through all the algebraic details in the text; find the points of intersection of the circles in the pencils quoted in the text.
2. Prove the statement made in the text that the pencil of lines (circles) consists of all the lines (circles) through the point (points) of intersection of the fundamental lines (circles).
3. Show that two circles of a pencil may intersect like cases (*a*), (*b*), (*c*) but not (*d*) and (*e*) in §53 (if we allow for imaginary points of intersection). Make up numerical examples to illustrate these facts.
4. Write the circles $(x - x_0)^2 + (y - y_0)^2 = r^2$, where x_0 and y_0 are constants, as a pencil of circles with a parameter λ. Do the same for $x^2 + y^2 + 2gx + 2gy = 0$.

55. **Degenerate circles in a pencil of circles.** In the pencil of circles (56) we take the discriminant Γ of what we may call a *general circle* in the pencil (i.e., a circle in whose equation we have not substituted a definite value of λ) and, equating this discriminant to zero, we have

$$(58) \qquad \begin{vmatrix} 1+\lambda & 0 & g_1+\lambda g_2 \\ 0 & 1+\lambda & f_1+\lambda f_2 \\ g_1+\lambda g_2 & f_1+\lambda f_2 & c_1+\lambda c_2 \end{vmatrix} = (1+\lambda)$$

$$\{(c_2 - f_2^2 - g_2^2)\lambda^2 + (c_1 + c_2 - 2f_1f_2 - 2g_1g_2)\lambda + (c_1 - f_1^2 - g_1^2)\} = 0$$

We can call the determinant in (58), when not equated to zero, the discriminant of the pencil (56).

The equation (58) when solved for λ gives the values of the parameter for which the corresponding circle in (56) is degenerate. Since (58) is a cubic in λ, there are ordinarily *three* such degenerate circles. (Note that such an equation as $x^2 + y^2 = 0$, which is really a pair of conjugate imaginary lines, can be looked upon as a degenerate circle.)

Without actually solving (58) for its roots, we can decide from geometrical considerations what the degenerate circles of (56) must be. Each degenerate circle must pass through all the points of intersection of the circles in (56). (Why?) The root $\lambda = -1$ gives us the common radical axis, which with l_∞ makes up one degenerate circle. If the circles of (56) intersect in two distinct points P_1 and P_2, the two other degenerate circles must be the two pairs of lines P_1I_1, P_2I_2 and P_1I_2, P_2I_1 (where I_1 and I_2 are the circular points at infinity). (Why?) If $P_1 = P_2 = P$, then there must be just one other degenerate circle, namely, the pair of lines PI_1, PI_2.*

If the circles have no finite points of intersection, there is just the one degenerate circle l_∞ taken twice. In this case the circles must be concentric and touch one another at I_1 and I_2. (Why?)

ILLUSTRATIVE EXAMPLE. The pencil of circles $x^2 + y^2 + 1 + 2\lambda x = 0$ illustrates the first case mentioned in the last paragraph. One degenerate circle is the line-pair* $x = 0$ and l_∞; also $\lambda = \pm 1$ gives the two other degenerate circles $(x \pm 1)^2 + y^2 = 0$.

The pencil of circles $x^2 + y^2 + 2\lambda x = 0$ illustrates the second case. One degenerate circle is the line-pair $x = 0$ and l_∞; also $\lambda^2 = 0$ gives the other degenerate circle $x^2 + y^2 = 0$.

Finally the pencil of circles $x^2 + y^2 + 2\lambda = 0$ illustrates the third case, with l_∞ as the only degenerate circle.

EXERCISES

1. Check the algebraic details in the text, especially the illustrative examples (finding the points of intersection of the circles, etc.).

2. Answer all the queries (Why?) in the text.

3. Prove there is always a real circle in (56). Hint: $\lambda = -c_1/c_2$ gives such a real circle, unless $f_1/f_2 = g_1/g_2 = c_1/c_2$. But then the pencil can be put in the form $(x^2 + y^2) + \lambda(2gx + 2fy + c) = 0$. (How?) Then we have a real circle if we take $-\lambda c + \lambda^2(g^2 + f^2) > 0$, that is $\lambda > c/(g^2 + f^2)$, except for the case $g = f = 0$. In this last case take $\lambda < 0$ if $c > 0$ (if $c < 0$, $\lambda = 1$ gives a real circle).

4. Find the degenerate circles and the points of intersection of the circles for the following pencils:

$$(x^2 + y^2 + 4x - 21) + \lambda(x^2 + y^2 - 8y - 9) = 0,$$
$$(x^2 + y^2 - 9) + \lambda(x^2 + y^2 + 6x - 8y + 24) = 0$$

56. Transformations on pencils of circles, linear in the variables and bilinear in the parameter. We are able to reduce the equation

* A pair of lines looked upon as a degenerate conic is often called a *line-pair*.

(56) for a pencil of circles to a simpler form by using (13) on the variables and by choosing new fundamental circles out of the pencil.

We shall show that this choice of new fundamental circles amounts to a so-called *bilinear* transformation of the parameter with equation of the form

$$\lambda = \frac{\alpha\lambda' + \beta}{\gamma\lambda' + \delta} \tag{59}$$

Suppose we have a pencil of circles $C_1 + \lambda C_2 = 0$ and we choose a new pair of fundamental circles $C_1' \equiv a'C_1 + b'C_2$, $C_2' \equiv a''C_1 + b''C_2$ so that our pencil has the form $C_1' + \lambda' C_2' = 0$. If we write $C_1' + \lambda' C_2' = 0$ in terms of C_1 and C_2, we get

$$\begin{aligned} C_1' + \lambda' C_2' &\equiv (a'C_1 + b'C_2) + \lambda'(a''C_1 + b''C_2) \\ &\equiv (a' + \lambda' a'')C_1 + (b' + \lambda' b'')C_2 = 0 \end{aligned}$$

If we divide the last form of this equation (containing C_1 and C_2) by the coefficient of C_1, the resulting pencil must be exactly the same as $C_1 + \lambda C_2 = 0$. Hence we have the following relation between λ and λ', namely, $\lambda = (b''\lambda' + b')/(a''\lambda' + a')$, which is an equation of the form (59).

Illustrative Example. The transformation $\lambda = \lambda' - 1$ puts (56) in the form

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) + \lambda'(x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0$$

If we wish to put (56) in a new form $C_1' + \lambda' C_2' = 0$ where C_2' lacks the x^2 and y^2 terms and C_1' lacks the constant term, we can use (59) and get

$$(\delta C_1 + \beta C_2) + \lambda'(\gamma C_1 + \alpha C_2) \equiv C_1' + \lambda' C_2' = 0$$

If we take $\gamma = 1$, $\alpha = -1$, then C_2' will have no x^2 and y^2 terms; also, if we take $\delta = c_2$, $\beta = -c_1$, then C_1' will lack a constant term. Put in the bilinear form, our transformation of parameter has the equation $\lambda = -(\lambda' + c_1)/(\lambda' + c_2)$.

Since (13) has Γ as a relative invariant, we see that an *affine linear transformation of variables has no effect whatever on the cubic equation* (58).

We can now reduce (56) to a simpler form by a combination of (13) and (59). If we put

$$x + g_1 = x', \quad y + f_1 = y'$$

then (56) has the form

$$(x'^2 + y'^2 + c_1') + \lambda(x'^2 + y'^2 + 2g_2'x' + 2f_2'y' + c_2') = 0$$

where $$c_1' = c_1 - g_1^2 - f_1^2,\ g_2' = g_2 - g_1,\ f_2' = f_2 - f_1,$$
$$c_2' = c_2 + g_1^2 + f_1^2 - 2g_1g_2 - 2f_1f_2$$

If now we put $\lambda = \lambda'/(1-\lambda')$ we reduce the pencil still further to the form

$$(56') \qquad (x'^2 + y'^2 + c_1') + \lambda'(2g_2'x' + 2f_2'y' + c_2' - c_1') = 0$$

EXERCISES

1. Give in full detail the reasons why (13) has no effect on the equation (58).
2. In the last sentence of the text, derive the transformation $\lambda = \lambda'/(1-\lambda')$.
3. Using the simplified form (56′) for a pencil of circles, show that the pencil has an imaginary non-degenerate circle (i.e., with imaginary radius) if we can determine λ' so that $\lambda'^2(g_2'^2 + f_2'^2) + \lambda' c_1' - (c_2' + c_1') < 0$. If $c_2' + c_1' > 0$ then $\lambda'^2(g_2'^2 + f_2'^2) + \lambda' c_1' - (c_2' + c_1') = 0$ certainly has real roots, and λ' can surely be chosen so as to satisfy the above inequality. (Why?) If $c_2' + c_1' < 0$ and $c_1'^2 + 2(g_2'^2 + f_2'^2)(c_2' + c_1') < 0$, then no value of λ' can be found to satisfy this inequality (and so the pencil then has no such imaginary non-degenerate circle). (Why?)
4. Using (56′), find the condition that the pencil of circles has a degenerate circle of the form $(x' - x_0')^2 + (y' - y_0')^2 = 0$. Hint: Find the radius of the general circle of (56′) and equate this to zero and solve for λ'; then the roots λ' must be real.
5. Reduce (56) to the form

$$(56'') \qquad (x'^2 + y'^2 + c_1') + \lambda(2g_2'x' + c_2') = 0$$

Hint: Translate the center of C_1 to the origin, rotate the line joining the centers to the position of the x-axis, and then use (59).

6. Check all the algebraic manipulations in the text, especially such as finding $c_1' = c_1 - g_1^2 - f_1^2$, etc.

57. Pencils of conics. Just as we have pencils of circles (56), so we have more generally *pencils of conics* defined by an equation of the form.

$$(60) \qquad (a_1x^2 + b_1y^2 + c_1 + 2f_1y + 2g_1x + 2h_1xy) + \lambda(a_2x^2 + b_2y^2 + c_2 + 2f_2y + 2g_2x + 2h_2xy) = 0$$

We write (60) as $C_1 + \lambda C_2 = 0$ and we call $C_1 = 0$ and $C_2 = 0$ the *fundamental conics* of the pencil and λ the *parameter*. Also we speak of (60) as a *one-parameter linear family of conics*.

We show now that (60) *consists of all the conics through the four points of intersection of* C_1 *and* C_2. (By the points of intersection we may mean finite or infinite points, real or imaginary, distinct or coincident, according to the way the two fundamental conics cut each other.) *In the first place*, any point $P'(x',y')$ on the two conics $C_1 = 0$ and $C_2 = 0$ satisfies the equation $C_1 + \lambda C_2 = 0$

for every value of λ, since the values $x = x'$, $y = y'$ make $C_1 \equiv 0$ and $C_2 \equiv 0$ separately. *In the second place,* five points uniquely determine a conic. Hence if we want a particular conic through the four points of intersection of $C_1 = 0$ and $C_2 = 0$ and through a fifth point $P''(x'',y'')$ or subject to some other similar condition, we can determine $\lambda = \lambda''$ (say) so that $C_1 + \lambda'' C_2 = 0$ shall pass through P'' or be subject to the other similar condition; then this conic (which belongs to the pencil) must be the conic we desire (since there is only *one* such conic satisfying the given conditions).

The parabolas $y^2 = 4\,px$ form a pencil of conics with equation $(y^2 - 4\,x) + \lambda(y^2 - 8\,x) = 0$, where $p = (1 + 2\,\lambda)/(1 + \lambda)$ or $\lambda = (1 - p)/(p - 2)$. The *confocal conics* given by $x^2/(a^2 + \lambda) + y^2/(b^2 + \lambda) = 1$ do *not* form a pencil of conics since, clearing this equation of fractions, we find that the parameter λ appears to both the first and second degrees.

EXERCISES

1. Check the statements made in the text about $y^2 = 4\,px$.
2. Prove that *the polars of any point P' with respect to all the conics of a given pencil of conics are concurrent lines.* Hint: Use (55).
3. Using (53) show that *there are in general five types of pencils of conics* (not distinguishing between real and imaginary points, or between finite and infinite points). Make up two numerical examples of each such type of pencil.
4. Show that $x^2/\alpha + y^2 = 1$ and $xy = k$ can be given as pencils of conics, and find the relation between λ and α or λ and k. Describe geometrically these two pencils.
5. Show that the parabolas $y^2 = 4\,px$ must be looked upon as having two pairs of coincident points of intersection on the x-axis (one pair finite and the other pair infinite).
6. Prove that in any pencil of conics (60) there are ordinarily two parabolas. Hint: A parabola would have $(h_1 + \lambda h_2)^2 - (a_1 + \lambda a_2)(b_1 + \lambda b_2) = 0$.
7. Note that there are no parabolas in $(xy - 1) + \lambda(xy - 2) = 0$. (Why?) Explain this fact geometrically.
8. What must be the condition (or conditions) on a_1, a_2, b_1, b_2, etc., in (60) in order that the pencil have no parabola? In order that the pencil consist entirely of parabolas?
9. Under what conditions may (60) have a circle?
10. Make up numerical examples of pencils of conics to illustrate Exs. 8 and 9.

58. **Degenerate conics in a pencil of conics.** Our discussion of pencils of conics is nothing more or less than a generalization of our discussion of pencils of circles. Thus *to find the degenerate*

conics of a pencil of conics (60) *we equate to zero the discriminant of the pencil* (compare §55) and get

$$(61)\qquad \begin{vmatrix} a_1+\lambda a_2 & h_1+\lambda h_2 & g_1+\lambda g_2 \\ h_1+\lambda h_2 & b_1+\lambda b_2 & f_1+\lambda f_2 \\ g_1+\lambda g_2 & f_1+\lambda f_2 & c_1+\lambda c_2 \end{vmatrix} \equiv \Gamma_1 +$$

$$\lambda\left\{\begin{vmatrix} a_2 & h_1 & g_1 \\ h_2 & b_1 & f_1 \\ g_2 & f_1 & c_1 \end{vmatrix} + \begin{vmatrix} a_1 & h_2 & g_1 \\ h_1 & b_2 & f_1 \\ g_1 & f_2 & c_1 \end{vmatrix} + \begin{vmatrix} a_1 & h_1 & g_2 \\ h_1 & b_1 & f_2 \\ g_1 & f_1 & c_2 \end{vmatrix}\right\} +$$

$$\lambda^2\left\{\begin{vmatrix} a_2 & h_2 & g_1 \\ h_2 & b_2 & f_1 \\ g_2 & f_2 & c_1 \end{vmatrix} + \begin{vmatrix} a_2 & h_1 & g_2 \\ h_2 & b_1 & f_2 \\ g_2 & f_1 & c_2 \end{vmatrix} + \begin{vmatrix} a_1 & h_2 & g_2 \\ h_1 & b_2 & f_2 \\ g_1 & f_2 & c_2 \end{vmatrix}\right\} +$$

$$\Gamma_2\lambda^3 = 0$$

where Γ_1 and Γ_2 are the discriminants of the two fundamental conics C_1 and C_2, respectively.

Since (61) is a cubic equation in λ, there are *in general three degenerate conics* in the pencil; but if (61) has a double root there are only two such degenerate conics; and if (61) is a perfect cube, only one degenerate conic. Thus for the pencil $x^2 + \lambda(y^2 + 2x) = 0$ we have from (61) the perfect cube $-\lambda^3 = 0$, so there is only one degenerate conic $x^2 = 0$ in this pencil.

Some of the degenerate conics in (60) may be imaginary. The cubic (61) may have one or more infinite roots, or two imaginary roots. Note that $(2y^2 + x^2) + \lambda(2y^2 + x^2 + 2x) = 0$ has a degenerate conic $2x = 0$, which we interpret as a line-pair ll' where l' is l_∞. (Compare the pencils of circles.)

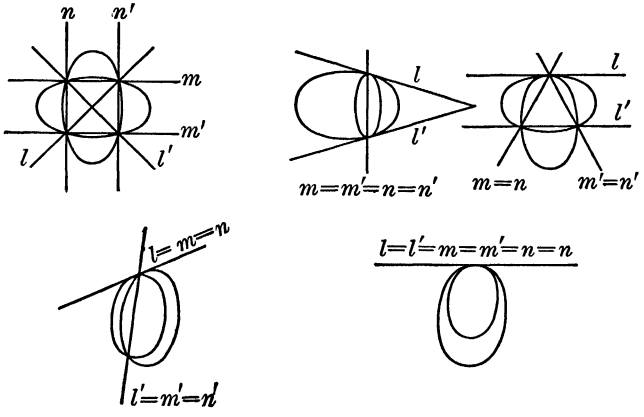
If the discriminant of the pencil of conics (60) vanishes identically, all the conics of the pencil are degenerate, and we have what is called a degenerate pencil of conics. Such a pencil is $x^2 + \lambda y^2 = 0$.

Geometrically we can often decide what the degenerate conics of a pencil are, or given the degenerate conics how the conics of the pencil intersect each other, because a degenerate conic must pass through all these points of intersection and not cut the conics elsewhere. For example, if the conics all intersect in four coincident points, since the pencil must have at least one degenerate conic (finite or infinite), this must be the common tangent (taken as a double line).

Such a pencil with only one degenerate conic (a double line)

is $x^2 + \lambda(y^2 - 4x) = 0$, whose conics intersect in four coincident points at the origin with the y-axis as tangent and as the sole degenerate conic ($x^2 = 0$) of the pencil. To interpret the pencil $(y^2 - 4x + 1) + \lambda(y^2 - 4x) = 0$ we must suppose that $\lambda = -1$ gives us the line l_∞ as a double line, also that all the conics (which are all parabolas) in this pencil cut each other in four coincident infinite points on the x-axis.

The following figures (compare §53) give schematically five types of pencils of conics (represented only by their fundamental and degenerate conics) where the points of intersection of the conics are supposed to be real and finite, and where the degenerate conics are ll', mm', nn'.



The pencil $x^2 + \lambda(y^2 - 4x) = 0$ illustrates the fifth figure. In the fourth figure the conics intersect in three coincident points and a fourth point. An example of this case is $(y^2 + 2x + 2xy) + \lambda(y^2 + 2x) = 0$. Illustrations of the other three cases in order are $(x^2/9 + y^2/16 - 1) + \lambda(x^2/16 + y^2/9 - 1) = 0$, $(4x^2 + y^2 - 4) + \lambda(y^2 - 4x - 4) = 0$, and $(x^2 + y^2 - 2x) + \lambda(y^2 - x) = 0$.

We emphasize the fact that the five figures given above *exhaust all* the *possibilities* where the points of intersection of the conics in each pencil are *real* and *finite*; but the equations of the pencils that are given are merely *illustrations* and are not typical forms to which the equations of all such pencils can be reduced by affine linear transformations on the variables combined with bilinear transformations on the parameters.

EXERCISES

1. Go through all the details in the text.
2. Find the points of intersection, the discriminants, and the degenerate conics of all the illustrative pencils of conics given in the text.
3. Make up more examples of pencils of conics to illustrate the five cases given geometrically in the text.
4. Describe the following pencils of conics:

(*a*) $(x^2 + 2y^2) + \lambda(2x^2 + y^2) = 0$
(*b*) $2xy + \lambda(x^2 - y^2) = 0$
(*c*) $(y^2 - 4x) + \lambda(x^2 - 4y) = 0$
(*d*) $(x^2 + 2xy) + \lambda(y^2 + 2y) = 0$

59. **Transformations on pencils of conics, linear in the variables, bilinear in the parameter.** Just as with pencils of circles, so with pencils of conics we can change the *fundamental conics* in any pencil. For instance, the pencil $(2x^2 + y^2 - 1) + \lambda(2x^2 + y^2 - 2x) = 0$ can be written $(2x^2 + y^2 - 1) + \lambda'(-2x + 1) = 0$. We also can find an equation relating λ to λ', namely, $\lambda = \lambda'/(\lambda' + 1)$.

To each new choice of fundamental conics for a pencil there corresponds (as above) a bilinear transformation (59) of the parameter (classifying such an equation as $\lambda = \lambda' - 1$ as bilinear with $\alpha = \delta = 1$, $\beta = -1$, $\gamma = 0$). Conversely, a bilinear change of the parameter of a pencil gives us a new set of fundamental conics for this pencil. For example, $\lambda = \lambda' - 1$ changes $(y^2 - 4x) + \lambda(-x^2 + y^2 - 4x) = 0$ into the really simpler form $x^2 + \lambda'(-x^2 + y^2 - 4x) = 0$, both equations giving us exactly the same pencil of conics, only with different sets of fundamental conics.

On the other hand, if we perform (13) looked at as an *alibi* on the variables of a pencil of conics, we do not get the same pencil but a new pencil equivalent to the old one under (13). Looked at as an *alias*, (13) changes the variables and coefficients of the pencil of conics (60) but does not change the equation (61) except to multiply this whole equation by a constant. (Why?)

Finally we shall consider a case of the reduction of a certain set of degenerate pencils to some standard forms.

Suppose each of the degenerate pencils in question (with equations $C_1 + \lambda C_2 = 0$) has a double line l. By means of (59) we can take l as the fundamental conic $C_1 = 0$. By (13) we can reduce the equation of l to the form $x^2 = 0$; then we use (59)

to rid C_2 of the term in x^2. Our pencil now is of the form

$$(60') \qquad x^2 + \lambda(by^2 + c + 2fy + 2gx + 2hxy) = 0$$

If $b \neq 0$ in (60'), we can put

$$x = x', \quad y + \frac{f}{b} + \frac{h}{b}x = y'$$

then drop the primes from the variables, again change λ by (59) so as to rid C_2 of the term in x^2, and we obtain

$$(60'') \qquad x^2 + \lambda(by^2 + c' + 2g'x) = 0$$

where c' and g' have values that are probably different from c and g respectively.

The discriminant of (60'') is

$$\Gamma \equiv \begin{vmatrix} 1 & 0 & g'\lambda \\ 0 & b\lambda & 0 \\ g'\lambda & 0 & c'\lambda \end{vmatrix} = bc'\lambda^2 - bg'^2\lambda^3$$

But the pencil is supposed to be degenerate; hence we must have $\Gamma \equiv 0$, so $c' = g' = 0$ (since $b \neq 0$, by hypothesis). Putting $\lambda = \lambda'/b$ we obtain the typical pencil

$$x^2 + \lambda' y^2 = 0$$

If $b = 0$ in (60'), we see from its discriminant

$$\Gamma \equiv \lambda^3(2fgh - bg^2 - ch^2) + \lambda^2(bc - f^2) \equiv 0$$

that (since $b = 0$) we must have $f = 0$ and also $ch^2 = 0$. If $c = 0$, $h \neq 0$, we put

$$x = x', \quad y = y' - \frac{g}{h}$$

and (after dropping the primes from the variables* and putting $\lambda = \lambda'/g$) we get a second typical pencil

$$x^2 + 2\lambda' xy = 0$$

If $b = 0$, $c = h = 0$, $g \neq 0$, we put $\lambda = \lambda'/g$ and get a third typical pencil

$$x^2 + 2\lambda' x = 0$$

* In the rest of this discussion we suppose the dropping of the primes from the variables to be done, without explicitly stating the fact.

If $b = 0, c \neq 0, h = g = 0$, we put $\lambda = \lambda'/c$ and get

$$x^2 + \lambda' = 0$$

If $b = 0$, $h = 0$, $cg \neq 0$, we put $\lambda = \lambda'/c$; then $x = x'/$ $y = y'$, then $\lambda' = \lambda''/g^2$, and then we multiply the resultir equation by g^2 and get the final typical pencil

$$x^2 + \lambda''(2x + 1) = 0$$

This exhausts all the possible cases of such degenerate penci (with a double line), and also gives us the typical forms to whic all such pencils can be reduced by (13) and by (59). Now we wa to see if the five pencils of conics in the preceding paragraph a really non-equivalent to each other under (13), i.e., to see wheth or not there is some transformation (13) that will send one these typical pencils into another.

The first pencil has two finite double lines and so cannot l transformed into one of the others (since each of them has on one finite double line). The second pencil has no pair of lines where l' is l_∞, and so this pencil must be non-equivalent to tl next three pencils. The fourth pencil has a double line (give by $\lambda' = \infty$) that coincides with l_∞; hence this pencil cannot l transformed into the third or the fifth. If we try to send tl fifth pencil into the third by (13), we must send $x^2 = 0$ in $x^2 = 0$ (i.e., $x'^2 = 0$), so we must have in (13) $a_2 = a_3 =$ $a_1 \neq 0$. But then we see that (13) cannot possibly send $2x +$ $= 0$ of the fifth pencil into a conic lacking the constant tern hence these two pencils are non-equivalent.

EXERCISES

1. Check all the details in the text.

2. Reduce to typical forms the degenerate pencils of conics that ha no double lines, but have at least one real line-pair per pencil. Hint: Tal C_1 (in $C_1 + \lambda C_2 = 0$) as $2xy$, then put the pencil in the form

$$2xy + \lambda(ax^2 + by^2 + c + 2fy + 2gx) = 0$$

Here we must have $\Gamma \equiv 0$. Now consider the cases $a \neq 0$ (or $b \neq 0$ $a = b = 0$ but $f \neq 0$ (or $g \neq 0$), etc. Note that the case of $a = 0$, $b \neq$ is reducible to that of $a \neq 0$ by the transformation $x = y'$, $y = x'$; al the case of $a = b = 0$, $f = 0$, $g \neq 0$ is reducible to $a = b = 0$, $f \neq 0$ b $x = y'$, $y = x'$. Finally test the typical pencils for non-equivalence.

3. Find the change of parameter corresponding to the choice of $x^2 + y^2 = 0$ and $2x = 0$ as new fundamental conics in the pencil

$$(2x^2 + y^2 - 2x) + \lambda(2x^2 + y^2 - 4x) = 0$$

4. Determine the coefficients in (59) so as (*a*) to rid C_1 of x^2 and C_2 of y^2 at the same time in the pencil $(x^2/9 + y^2/16 - 1) + \lambda(x^2/16 + y^2/9 - 1) = 0$; (*b*) to rid C_1 of y^2 and C_2 of x^2; (*c*) to rid C_1 of -1 and C_2 of x^2.

5. Show that (59) has only three essential constants. In all our uses of (59) we obtain equations like

$$\alpha l_1 + \gamma m_1 = n_1, \quad \alpha l_1' + \gamma m_1' = n_1', \quad \delta l_2 + \beta m_2 = n_2, \quad \delta l_2' + \beta m_2' = n_2$$

(Why?) Show that we can make four such changes (using four such equations) if $n_1 \neq 0$ (or $n_1' \neq 0$); also if $n_2 \neq 0$ (or $n_2' \neq 0$). Otherwise we can make only three such changes, or only two (if $n_1 = n_1' = n_2 = n_2' = 0$).

60. **Nets of conics.** By a *two-parameter linear family* of conics (or net of conics) we mean all the conics given by the equation

$$\begin{aligned}(62)\quad \lambda C_1 + \mu C_2 + C_3 \equiv \lambda(&a_1x^2 + b_1y^2 + c_1 + 2f_1y + 2g_1x \\ &+ 2h_1xy) + \mu(a_2x^2 + b_2y^2 + c_2 + 2f_2y + 2g_2y \\ &+ 2h_2xy) + (a_3x^2 + b_3y^2 + c_3 + 2f_3y + 2g_3x \\ &+ 2h_3xy) = 0\end{aligned}$$

where λ and μ are arbitrary parameters and the three fundamental conics $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ of the net must not all belong to the same pencil of conics.

(If *three* conics do *not* belong to the same *pencil* they are said to be *linearly independent*, i.e., there is no linear relation connecting their coefficients, such as $a_3 = \lambda a_1 + \mu a_2$, $b_3 = \lambda b_1 + \mu b_2$, etc.).

To count $C_1 = 0$ as belonging to the net (62) we must allow λ to take the value ∞; similarly to have $C_2 = 0$ in the net we must take $\mu = \infty$.

All the conics $\pm x^2/a^2 \pm y^2/b^2 = 1$ form a net of conics $\lambda x^2 + \mu y^2 + 1 = 0$ with $\lambda = \mp 1/a^2$, $\mu = \mp 1/b^2$. This net consists of all the hyperbolas and ellipses (also circles), real and imaginary, with centers at the origin and axes on the coordinate axes. Note that the net has three double lines $x^2 = 0$, $y^2 = 0$, $1 = 0$ (a double line that coincides with l_∞, and is given by $\lambda = \mu = 0$). We can describe this net geometrically as consisting of all the conics that have a sort of self-polar triangle whose sides are the coordinate axes and l_∞. Compare §51.

Again, all the circles that pass through the origin (i.e., with general equation of the form $x^2 + y^2 + 2gx + 2fy = 0$) form a net of conics $\lambda(x^2 + y^2) + 2\mu x + 2y = 0$. This net has (among

other degenerate conics) a conjugate imaginary line-pair $x^2 + y^2 = 0$ and a degenerate pencil of line-pairs like $2x = 0$ (given by $\lambda = 0$) that have one line coinciding with l_∞.

We remark here that the family of all circles having a given radius r' has an equation of the form $(x - \lambda)^2 + (y - \mu)^2 = r'^2$ and so cannot be a net of conics because λ and μ appear to the second degree as well as to the first.

We recall that it takes five points to determine a conic. If we suppose three non-collinear points as given, then the coefficients of (4) can be expressed linearly in terms of two arbitrary parameters, i.e., all the conics through three non-collinear points form a net of conics. For example, if we want (4) to pass through the points (0,0), (1,0), (0,1), we must have

$$c = 0, \quad a + 2g = 0, \quad b + 2f = 0$$

If we assume $a \neq 0$ and divide (4) by a, we get a net of conics

$$(x^2 - x) + \lambda(y^2 - y) + 2\mu xy = 0$$

where $\lambda = b/a$, $\mu = h/a$. To include in this net the conics with $a = 0$ we must permit λ or μ to have the value ∞.

EXERCISES

1. If the three fundamental conics of (62) belong to the same pencil, show that we must have all the three-rowed determinants of the matrix

$$\left\| \begin{matrix} a_1 & b_1 & c_1 & f_1 & g_1 & h_1 \\ a_2 & b_2 & c_2 & f_2 & g_2 & h_2 \\ a_3 & b_3 & c_3 & f_3 & g_3 & h_3 \end{matrix} \right\|$$

vanish identically, and conversely. The conics are then said to be linearly dependent. Hint: Every conic (4) of a pencil $C_1 + \lambda C_2 = 0$ has $a = a_1 + \lambda a_2$, $b = b_1 + \lambda b_2$, etc. (Why?)

2. Show how the equation $y^2 = \alpha x + \beta$ (i.e., all the parabolas having the x-axis as their axis) can be written as a net of conics with $y^2 = x$, $y^2 = -x + 4$, and $y^2 = 4x - 4$ as fundamental conics. Find the relations between λ,μ and α,β.

3. Find the net of conics through the three points (0,0), (1,1), (1,−1).

61. Cubics associated with a net of conics, the discriminant and the Jacobian. The values of λ and μ that give degenerate conics in the net (62) must satisfy the following equation

$$(63) \qquad K \equiv \begin{vmatrix} \lambda a_1 + \mu a_2 + a_3 & \lambda h_1 + \mu h_2 + h_3 & \lambda g_1 + \mu g_2 + g_3 \\ \lambda h_1 + \mu h_2 + h_3 & \lambda b_1 + \mu b_2 + b_3 & \lambda f_1 + \mu f_2 + f_3 \\ \lambda g_1 + \mu g_2 + g_3 & \lambda f_1 + \mu f_2 + f_3 & \lambda c_1 + \mu c_2 + c_3 \end{vmatrix} = 0$$

We call K the *discriminant* of the net. If $K \equiv 0$ we call (62) a *degenerate* net, since every conic it contains is degenerate. Such a net is $x^2 + \lambda y^2 + 2\mu xy = 0$. We call attention to the fact that (13) *does not affect the equation* (63) *except to multiply every coefficient by a constant.* (Why?) However (59) *does transform* (63) *into an equivalent cubic.*

If we take another plane π and plot λ and μ as the coordinates of points in this plane, we get in this $\pi(\lambda,\mu)$ plane a cubic curve (63), to *each point* of which there corresponds a *degenerate conic* in the net (62) in the x,y-plane. Conversely, to every degenerate conic in (62) there corresponds a point on the cubic curve (63).

Any line in the π plane has an equation of the form $\lambda = a\mu + b$(or $\mu = c$). Substituting such a value for λ(or μ) in (62), we see that to this *line* there corresponds a *pencil* of conics in the x,y-plane, namely

$$(bC_1 + C_3) + \mu(C_2 + aC_1) = 0 \text{ [or } (C_3 + cC_2) + \lambda C_1 = 0]$$

Conversely, any pencil of conics in the net (62), except the pencil $\lambda C_1 + \mu C_2 = 0$, is obtained by putting $\lambda = a\mu + b$ (or $\mu = c$) in (62); hence to a pencil in this net there corresponds a line in the λ,μ-plane.

To obtain a pencil of the form $\lambda' C_1 + \mu' C_2 = 0$ in the net (62) we must put $\lambda = \lambda'\lambda''$, $\mu = \mu'\lambda''$, divide the equation of the net by λ'', then let $\lambda'' \to \infty$. But then we have $\lambda = \mu = \infty$. Therefore we interpret this pencil as corresponding to the line at infinity l_∞ in the π plane.

If the cubic (63) is *degenerate* (*composite*), to any straight line forming part of the cubic there corresponds a degenerate pencil of conics in the net (62), and conversely. If the cubic (63) turns out to be a quadratic or linear expression or a mere constant, we interpret this as a degenerate cubic with one *component* (or two, or three components, respectively) coinciding with the line l_∞ in the π plane. Such a cubic is associated with the net $x^2 + \lambda y^2 + \mu = 0$, namely, $\lambda\mu = 0$.

If the net (62) has a double line we can reduce it by (13) and (59) to $C_3 \equiv x^2 = 0$. Then $b_3 = c_3 = h_3 = f_3 = g_3 = 0$ in (62), so the cubic curve (63) has no constant term or first-degree terms in x and y; therefore if this cubic is non-degenerate it must have a node or cusp at (0,0). See §48. The converse of this theorem

is not true, as is shown by the net

$$(y^2 - x^2) + 2\lambda x + \mu(x^2 + 2y) = 0$$

whose cubic $\lambda^2 - \mu^2 + \mu^3 = 0$ has a crunode at (0,0) with tangents $\lambda = \pm\mu$. The net $(y^2 + x^2) + 2\lambda x + \mu(x^2 + 2y) = 0$ has the cubic $\lambda^2 + \mu^2 + \mu^3 = 0$, which has an acnode at (0,0) with tangents $\lambda = \pm i\mu$. The net $x^2 + 2\lambda y + \mu(1 + 2xy) = 0$ has the cubic $\lambda^2 + \mu^3 = 0$, which has a cusp at (0,0) with tangent $\lambda = 0$.

Note that if $\lambda = 0$ is a tangent to the cubic (63) at a point of inflection (0,0), then solving $\lambda = 0$ with (63) should give us $\mu^3 = 0$; i.e., the pencil $C_3 + \mu C_2 = 0$ in the net (62) must have just the one degenerate conic $C_3 = 0$ (given by $\mu^3 = 0$)

The polar of any point $P'(x',y')$ with respect to a conic of the net (62) has the equation

$$\begin{aligned}[\lambda(a_1x' + h_1y' + g_1) + \mu(a_2x' + h_2y' + g_2) \\ + (a_3x' + h_3y' + g_3)]x + [\lambda(h_1x' + b_1y' + f_1) \\ + \mu(h_2x' + b_2y' + f_2) + (h_3x' + b_3y' + f_3)]y \\ + [\lambda(g_1x' + f_1y' + c_1) + \mu(g_2x' + f_2y' + c_2) \\ + (g_3x' + f_3y' + c_3)] = 0\end{aligned}$$

For this polar to pass through a *given* point $P''(x'',y'')$, for *every value* of λ and μ we must have (why?):

$$\begin{aligned}(a_1x' + h_1y' + g_1)x'' + (h_1x' + b_1y' + f_1)y'' \\ + (g_1x' + f_1y' + c_1) = 0, \\ (a_2x' + h_2y' + g_2)x'' + (h_2x' + b_2y' + f_2)y'' \\ + (g_2x' + f_2y' + c_2) = 0, \\ (a_3x' + h_3y' + g_3)x'' + (h_3x' + b_3y' + f_3)y'' \\ + (g_3x' + f_3y' + c_3) = 0\end{aligned}$$

For these three equations to have a common solution in x'' and y'', we must have (dropping the primes from x' and y')

$$(64)\qquad J \equiv \begin{vmatrix} a_1x + h_1y + g_1 & h_1x + b_1y + f_1 & g_1x + f_1y + c_1 \\ a_2x + h_2y + g_2 & h_2x + b_2y + f_2 & g_2x + f_2y + c_2 \\ a_3x + h_3y + g_3 & h_3x + b_3y + f_3 & g_3x + f_3y + c_3 \end{vmatrix} = 0$$

This equation (64) gives us another cubic curve (this time in the x,y-plane) associated with the net (62), and such that every point $P'(x',y')$ on (64) has all its polars with respect to the conics of the net concurrent. This cubic is called the *Jacobian* of the

net (62). Since the three equations from which we obtain (64) have no λ or μ in them, the transformation (59) must have no effect at all on (64). On the other hand (13) sends J into an equivalent cubic. As an illustration of (64), we see that the net $x^2 + 2\lambda y + \mu(2xy + 1) = 0$ has the Jacobian

$$\begin{vmatrix} x & 0 & 0 \\ 0 & 1 & y \\ y & x & 1 \end{vmatrix} \equiv x(1 - xy) = 0$$

EXERCISES

1. Find the Jacobians of all the nets of conics that have been given in §§60, 61.

2. Interpret as cubic curves $\mu = 0$, $\mu^2 = 0$, $\lambda^2 + \mu^2 - 1 = 0$.

3. State and prove the converse of the statement made in the text that if the cubic (63) is degenerate, to any straight line forming part of this cubic there corresponds a degenerate pencil in the net (62).

4. Prove that if the cubic (63) has a finite cusp P, then the net (62) must have a double line to correspond to P. Hint: Take P as (0,0) with $\lambda = 0$ as tangent, i.e., in (63) there will be no constant term and no terms in λ, μ, $\lambda\mu$, μ^2. Now suppose it possible for C_3 to have the form xy or $x^2 + y^2$ or x and show that there arises a contradiction, but that C_3 can be x^2 or a constant. Why can we do all this apparent specializing of (62) and (63) without any loss of generality in our proof? The case of an infinite cusp on (62) will be taken care of later.

5. Find and describe the curves in the λ,μ-plane corresponding to the nets

$$\lambda x^2 + 2\mu y + (y^2 + 2xy \pm 1) = 0, \quad \lambda x^2 + 2\mu y + (2xy + 1) = 0,$$
$$x^2 + \lambda y^2 + \mu(2xy + 2x + 2y + 1) = 0,$$
$$2xy + \lambda(y^2 + 2x) + \mu(x^2 + 2y) = 0,$$
$$2xy + \lambda(2xy + 2x) + \mu(x^2 + y^2 - 1) = 0$$

6. Are there any parabolas or circles in the nets of conics in Ex. 5?

7. Check the cubics given in the text.

8. Find any degenerate pencils of conics in the nets in Ex. 5 and in the nets in the text.

9. Find the Jacobians of the nets in Ex. 5.

10. In solid analytic geometry we have what are called *quadric surfaces*, with equations of the form

$$Q \equiv ax^2 + by^2 + cz^2 + d + 2fyz + 2gzx + 2hxy + 2kx + 2ly + 2mz = 0$$

The condition that the quadric be a cone or degenerate is

$$D \equiv \begin{vmatrix} a & h & g & k \\ h & b & f & l \\ g & f & c & m \\ k & l & m & d \end{vmatrix} = 0$$

We have pencils of quadrics and other linear families of quadrics. Associated with a pencil of quadrics $Q_1 + \lambda Q_2 = 0$ we have (from D) a fourth-degree equation in λ to give us the degenerate quadrics or cones in the pencil. (Compare pencils of conics.)

Associated with nets of quadrics we have quartic curves in the λ,μ-plane. (Compare nets of conics.) Find the degenerate quadrics or cones in the pencils

$$(x^2 + y^2 + z^2 + 1) + \lambda(x^2 - y^2 - z^2 + 1) = 0,$$
$$2(xy + xz + yz) + \lambda(x^2 + y^2 + z^2 - 1) = 0,$$
$$(x^2 + yz) + \lambda(y^2 + 2zx) = 0$$

Find the curves in the λ,μ-plane associated with the nets of quadrics

$$(x^2 - y^2 + z^2) + 2\lambda(xy + xz + yz) + \mu(x^2 - 2x + 2y + 2z + 1) = 0,$$
$$x^2 + \lambda(y^2 - 2zx) + \mu(z^2 - 2xy) = 0,$$
$$2xy + 2\lambda xz + \mu(x^2 + y^2 - z^2) = 0$$

What sort of point on the associated quartic curve corresponds to a double plane (like $x^2 = 0$) in a net of quadrics; to a pair of planes (like $2xy = 0$ or $x^2 + y^2 = 0$)?

Derive the equation of the Jacobian of a net of quadrics. Find the Jacobians of the above nets of quadrics.

Show that all the quadrics in space belong to the same nine-parameter linear family of quadrics. Describe geometrically a pencil of quadrics.

62. Transformations of nets of conics, linear in the variables, bilinear in the parameters. A change of the fundamental conics of (62) causes a change of parameters (just as for pencils of conics). Any new set of fundamental conics $C_1' = 0$, $C_2' = 0$, $C_3' = 0$ must belong to the net, so we must have

$$C_1' = \alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3, \quad C_2' = \beta_1 C_1 + \beta_2 C_2 + \beta_3 C_3,$$
$$C_3' = \gamma_1 C_1 + \gamma_2 C_2 + \gamma_3 C_3$$

Hence the new form of the net is

$$\lambda' C_1' + \mu' C_2' + C_3' \equiv \lambda'(\alpha_1 C_1 + \cdots) + \mu'(\beta_1 C_1 + \cdots)$$
$$+ (\gamma_1 C_1 + \cdots) \equiv (\alpha_1\lambda' + \beta_1\mu' + \gamma_1)C_1 + (\alpha_2\lambda'$$
$$+ \beta_2\mu' + \gamma_2)C_2 + (\alpha_3\lambda' + \beta_3\mu' + \gamma_3)C_3 = 0$$

This last form must be the original form of the net, therefore we must have (after dividing the above equation by $\alpha_3\lambda' + \beta_3\mu' + \gamma_3$)

$$(65) \qquad \lambda = \frac{\alpha_1\lambda' + \beta_1\mu' + \gamma_1}{\alpha_3\lambda' + \beta_3\mu' + \gamma_3}, \quad \mu = \frac{\alpha_2\lambda' + \beta_2\mu' + \gamma_2}{\alpha_3\lambda' + \beta_3\mu' + \gamma_3}$$

The transformation of parameters (65) is called a *bilinear** (or *linear fractional*) transformation.

* Compare (59) in §56.

To interpret in the λ,μ-plane any new choice of fundamental conics for the net (62) we must see what (65) means geometrically. It is easy to see that (65) *sends lines into lines, conics into conics, cubics into cubics.* To the line $l' \equiv \alpha_3\lambda' + \beta_3\mu' + \gamma_3 = 0$ there corresponds the line l_∞. (Why?) Note that to the point with $\lambda' = \infty, \mu' = 0$ there corresponds the point

$$\lambda = \frac{\alpha_1\infty + \gamma_1}{\alpha_3\infty + \gamma_3} = \frac{\alpha_1 + \gamma_1/\infty}{\alpha_3 + \gamma_3/\infty} = \frac{\alpha_1}{\alpha_3}, \quad \mu = \frac{\alpha_2 + \gamma_2/\infty}{\alpha_3 + \gamma_3/\infty} = \frac{\alpha_2}{\alpha_3}$$

To the point with $\lambda' = 0, \mu' = \infty$ there corresponds the point

$$\lambda = \frac{\beta_1 + \gamma_1/\infty}{\beta_3 + \gamma_3/\infty} = \frac{\beta_1}{\beta_3}, \quad \mu = \frac{\beta_2 + \gamma_2/\infty}{\beta_3 + \gamma_3/\infty} = \frac{\beta_2}{\beta_3}$$

Therefore we say that to the line l_∞ corresponds the line

$$l \equiv \begin{vmatrix} \lambda & \mu & 1 \\ \alpha_1/\alpha_3 & \alpha_2/\alpha_3 & 1 \\ \beta_1/\beta_3 & \beta_2/\beta_3 & 1 \end{vmatrix} = \frac{1}{\alpha_3\beta_3}\,[(\alpha_2\beta_3 - \alpha_3\beta_2)\lambda + (\beta_1\alpha_3 - \alpha_1\beta_3)\mu + (\alpha_1\beta_2 - \alpha_2\beta_1)] = 0$$

The two lines l and l' are called the *vanishing lines* of the transformation (65). We shall see later on that (65) is merely another way of writing what is called the *general projective transformation* in the plane. Also note that the general affine projective transformations in the λ,μ-plane are the subgroup of (65) that have $\alpha_3 = \beta_3 = 0$, $\gamma_3 \neq 0$. We shall see in the exercises that (65) preserves cross-ratio.

The transformation (13) does not alter (63), but (65) transforms (63) into what are called *projectively equivalent* cubics. From this discussion we see that with every net (62) there are associated not one but an infinite number of projectively equivalent cubics in the λ,μ-plane (corresponding to the different possible choices of the fundamental conics in the net). (Note that the cubics $\lambda^2 = \mu^3$ and $\lambda = \mu^3$ are projectively equivalent cubics, because the first is reducible to the second by $\lambda = 1/\lambda', \mu = \mu'/\lambda'$.)

We see that two nets cannot be reduced to one another by (13) and (65) unless their cubics (63) are projectively equivalent. But the converse is not true because two such nets as

$$2\,\lambda xy + 2\,\mu x + 2\,y = 0 \quad \text{and} \quad \lambda x^2 + \mu y^2 + 1 = 0$$

both have the cubic $\lambda\mu = 0$; but the second net has double lines

in it (whereas the first net has not) and so cannot be reduced to the first. The two nets

$$\lambda x^2 + \mu y^2 + 1 = 0 \quad \text{and} \quad \lambda x^2 + 2\mu x + 2y = 0$$

have, respectively, the cubics $\lambda\mu = 0$ and $\lambda\mu^2 = 0$; hence we see these nets are not reducible to one another because of their associated cubic curves in the λ,μ-plane. The net $(x + y - 1)^2 + \lambda(2x + 3y)^2 + \mu = 0$ is reducible to $x'^2 + \lambda y'^2 + \mu = 0$ by the transformation $x + y - 1 = x'$, $2x + 3y = y'$.

EXERCISES

1. The net $x^2 + \lambda y^2 + \mu(x^2 + 1) = 0$ has $K \equiv \lambda(\mu + 1)$. Explain why K is apparently not a cubic. Find all the degenerate conics in this net.

2. Prove that (65) preserves cross-ratio. Hint: Take $\lambda = a\mu + b$. Suppose this line goes to $\mu' = 0$. Put $\mu' = 0$ in (65). Take the cross-ratio $(\lambda_4 - \lambda_1)/(\lambda_2 - \lambda_1) \cdot (\lambda_2 - \lambda_3)/(\lambda_4 - \lambda_3)$ of four points on $\lambda = a\mu + b$. Why is there no loss of generality in this proof?

3. Explain exactly what is meant in the text by the statement about an infinite number of cubics.

4. Are any of the nets in Ex. 5 of §61 reducible to one another?

5. In the nets $\lambda x^2 + \mu y^2 + C_3 = 0$ and $\lambda x^2 + 2\mu xy + C_3 = 0$ determine the coefficients of C_3 so that we have the discriminant $K \equiv 0$; then reduce these nets to their simplest forms by (13) and (65).

6. Show that the discriminant Γ of any double line (4) has all its first minors zero. State the converse and prove it.

7. Show that if in the discriminant Γ of (4) we have $ab - h^2 = ac - g^2 = bc - f^2 = 0$, then $\Gamma = 0$ and all its first minors vanish.

8. What is the converse of the statement that if a net has a double line, its cubic in the λ,μ-plane has a node or a cusp corresponding to this double line? Is this converse true? Why or why not?

9. We can write the net $\lambda x^2 + \mu y^2 + 1 = 0$ as $\lambda'(x^2 + y^2 + 1) + \mu'(2x^2 - y^2) + 1 = 0$ or as $\lambda'(-x^2 + 1) + \mu'(-y^2 + 1) + (x^2 + y^2) = 0$. Use (65) to put the net into these two forms. Find in two ways the new cubics in the λ,μ-plane that we get for the net by choosing these new sets of fundamental conics.

10. What do the transformations $\lambda = \lambda' - \mu' + 2$, $\mu = \lambda' + 2\mu'$ and $\lambda = 2\lambda'$, $\mu = 3\mu' - 1$ do to the fundamental conics of the net $2\lambda xy + 2\mu x + 2y = 0$ and to its cubic in the λ,μ-plane?

63. **Three-and-four-parameter linear families of conics.** Besides pencils and nets of conics we have *three-and-four-parameter linear families of conics.* If we look upon the coefficients of (4) as arbitrary parameters we see that *all* the conics in the plane belong to the *same five-parameter linear family* of conics with an

equation that can be written

$$x^2 + \lambda y^2 + \mu + 2\nu y + 2\rho x + 2\sigma xy = 0$$

The family of *all the circles* in the plane has three arbitrary constants g, f, and c, and so forms a *three-parameter linear family* of conics

$$x^2 + y^2 + 2\lambda y + 2\mu x + \nu = 0$$

Geometrically, the family of all the circles can be described as consisting of all the conics passing through two given points (the two circular points at infinity I_1 and I_2).

All the conics through one point form a four-parameter linear family of conics; all the conics through two points form a three-parameter family. Instead of having the conics pass through given points we might put any other sort of linear conditions upon the coefficients of (4) and obtain linear families of conics. For instance, all the conics that have the origin as center belong to the three-parameter linear family

$$x^2 + \lambda y^2 + 2\mu xy + \nu = 0$$

It is in general difficult to describe geometrically nets and three- and four-parameter families of conics. Later on we shall discuss a property called *apolarity* and use it for this purpose.

Let us finally consider the family of conics that are all tangent to the x-axis at the origin. In (4) we have $a \neq 0$, $c = g = 0$, because $y = 0$ when solved simultaneously with (4) must give $x^2 = 0$. Dividing (4) by a and replacing arbitrary coefficients by parameters we get

$$x^2 + \lambda y^2 + 2\mu y + 2\nu xy = 0$$

which is a three-parameter linear family of conics with discriminant μ^2. There is a pencil of circles in this family given by $\lambda = 1, \nu = 0$; all the parabolas in the family are given by $\nu^2 - \lambda = 0$ and so do not form a linear family. There are two degenerate nets in the family (and these nets contain all the degenerate conics of the family), namely

$$x^2 + \lambda y^2 + 2\nu xy = 0 \quad \text{and} \quad y^2 + 2\mu' y + 2\nu' xy = 0$$

To obtain the latter net from the family we must put $\mu = \mu'\lambda$, $\nu = \nu'\lambda$, divide the equation of the family by λ, then let $\lambda \to \infty$.

EXERCISES

1. If $x^2 + \lambda y^2 + 2\mu xy + \nu = 0$ is written $C_1 + \lambda C_2 + \mu C_3 + \nu C_4 = 0$, what sort of conic is $C_4 = 0$?

2. Show that all the parabolas in a plane do not form a linear family of conics. Hint: $\Gamma \neq 0$, $h^2 - ab = 0$ in (4) give the parabolas.

3. Prove that $x^2 + \lambda y^2 + 2\mu xy + \nu = 0$ does actually give all the conics in the plane with center at the origin. Hint: Use the formula for the center of a conic, or else put on the condition that (0,0) is the pole of l_∞.

4. Find the discriminants and degenerate conics in the families in the text.

5. Find the family of all degenerate conics that have the y-axis as one component.

6. Find the family of conics (a) tangent to the y-axis at the origin; (b) tangent to $x = 0$ at (0,1).

7. Find the circles, parabolas, hyperbolas, and ellipses in the families of conics in the text.

8. Find the family of conics tangent to $x = 0$ at (0,1) and tangent to $y = 0$ at (1,0).

9. Find the discriminants, degenerate conics, circles, and parabolas for the families

$$x^2 + \lambda y^2 + 2\mu xy + \nu(1 - 2x - 2y) = 0,$$
$$2xy + 2\lambda x + 2\mu y + \nu(x^2 + y^2 - 1) = 0$$

PART II

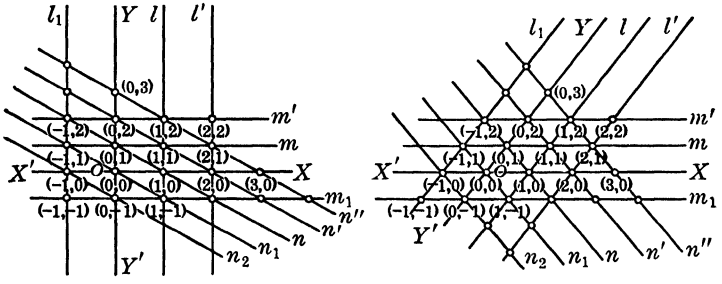
INTRODUCTION TO GENERAL PLANE ANALYTIC PROJECTIVE GEOMETRY

CHAPTER IX

INTRODUCTION TO THE TRIANGLE OF REFERENCE

64. **Rectangular and oblique axes from a new viewpoint.** In this section we shall set up a system of coordinates for points in a plane referred to rectangular (or oblique) axes. We shall do this in such a way as to *avoid all ideas of measure of distance or of angle*, and yet arrive at the same coordinates for the points as we should get by the ordinary methods of elementary analytic geometry. Our purpose is to have at hand a construction for the coordinates of points that will generalize to the so-called *triangle of reference*. (See §70.)

Consider the following pair of rectangular (or oblique) axes $X'OX, Y'OY$.



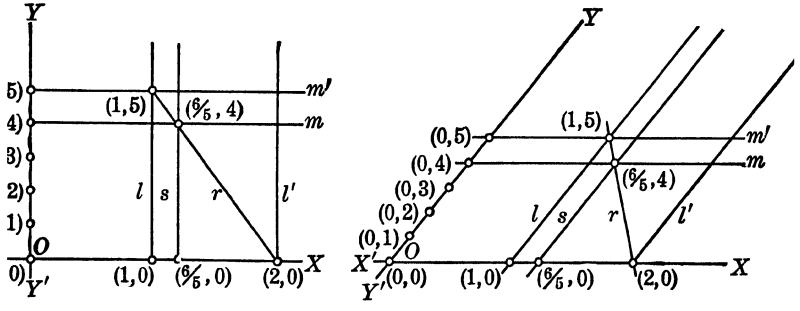
Take two arbitrary points on $X'OX$ and $Y'OY$ and call them (1,0) and (0,1), respectively. Through (1,0) draw a line l parallel to $Y'OY$ and through (0,1) draw a line m parallel to $X'OX$, label the point of intersection of l and m (1,1). Join (1,0) and (0,1) by a line n. Through (1,1) draw a line n' parallel to n, meeting $X'OX$ in a point we call (2,0) and meeting $Y'OY$ in a point we call

(0,2). Through (2,0) draw l' parallel to l, through (0,2) draw m' parallel to m, call the point $l'm'$* (2,2), call the point $l'm$ (2,1). Through (2,1) draw n'' parallel to n, meeting $X'OX$ in a point we call (3,0) and meeting $Y'OY$ in a point we call (0,3). In this way we can continue our constructions and determine all the points on $X'OX$ and $Y'OY$ and in the first quadrant whose coordinates are positive integers.

We extend our construction into the other quadrants in the following manner. Through O take a line n, parallel to n and meeting l in a point $(1,-1)$ and meeting m in a point $(-1,1)$. Through $(1,-1)$ draw m_1 parallel to $X'OX$ and meeting OY' in $(0,-1)$. Through $(-1,1)$ draw l_1 parallel to $Y'OY$ and meeting OX' in $(-1,0)$. Then l_1 and m_1 intersect in $(-1,-1)$. Now we can proceed to get all the points on the axes and in the whole plane whose coordinates are, both of them, integers (positive or negative, counting zero as an integer).

Note that, instead of taking (0,1) and (1,0) both arbitrarily, we might have taken an arbitrary point that is not on $X'OX$ or $Y'OY$ for (1,1) and then constructed (0,1) and (1,0) by lines through (1,1) parallel to the axes. Note that the point $(-1,0)$ can be determined either as the point of intersection of l_1 and OX' or as the point of intersection of OX' and a line n_2 through $(0,-1)$ and parallel to n; but in either case we get the same point for $(-1,0)$, because of certain well-known properties of parallels included between parallels. (Explain fully the last statement.)

To construct points with fractional coordinates we proceed as in the following figure:



* Note that since we call the line joining two points P and P' the line PP', so we shall (dually) call the point of intersection of two lines l and m the *point lm*. Compare §22.

Suppose we desire to construct the point $(\frac{6}{5},0)$. We assume the points (1,0), (2,0), (0,1), (0,2), (0,3), (0,4), (0,5) to be constructed. Through (1,0) and (2,0) we draw l and l' parallel to $Y'OY$; through (0,4) and (0,5) we draw m and m' parallel to $X'OX$; m' then meets l in (1,5). Join (1,5) and (2,0) by the line r cutting m at $(\frac{6}{5},4)$. Finally the line s through $(\frac{6}{5},4)$ parallel to $Y'OY$ cuts OX at $(\frac{6}{5},0)$.

More generally the point $(u/v,0)$ can be obtained as follows. Suppose u is prime to v and p is the next integer smaller than u/v, so that $u/v = p + t/v$. Through the points $(p,0)$ and $(p+1,0)$ draw l and l' parallel to $Y'OY$ and, also, through $(0,v-t)$ and $(0,v)$ draw m and m' parallel to $X'OX$; then m' cuts l in (p,v). Join (p,v) to $(p+1,0)$ by r cutting m in $(u/v,v-t)$. Through $(u/v,v-t)$ draw s parallel to $Y'OY$ cutting OX in $(u/v,0)$.

The points with irrational coordinates can be obtained as the limiting points of sequences of points with rational coordinates. Thus $(\sqrt{2},0)$ is the limiting point of the sequence (1,0), (2,0), (1.4,0), (1.5,0), (1.41,0), (1.42,0), (1.414,0), (1.415,0), etc. Roughly speaking, this means that by the above method we can obtain a rational point as close as we please to the irrational point $(\sqrt{2},0)$ whose existence we postulate.

EXERCISES

(Use both rectangular and oblique axes)

1. Prove that the points (2,0), (3,0), $(\frac{6}{5},0)$, $(u/v,0)$, $(p,0)$, $(-1,0)$, (0,2), etc., are the same as those obtained by the methods of elementary analytic geometry; i.e., prove that the segments from (1,0) to (2,0), from (2,0) to (3,0), etc., in the above construction are equal in length, also that the point $(\frac{6}{5},0)$ is actually one-fifth the distance from (1,0) to (2,0), etc. Hint: Use plane geometry and equal triangles, or parallels between parallels.

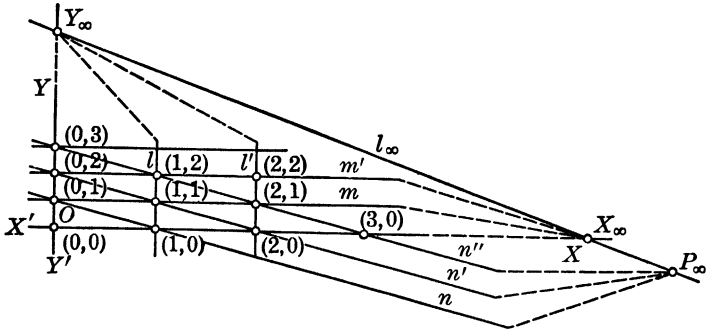
2. Generalize the construction in the text and show how to find the points $(p,0)$ and $(0,p)$ where p is any positive integer. Hint: Use $(p-1,0)$, $(0,p-1)$, $(1,p-1)$, $(p-1,1)$.

3. Construct the points $(-\frac{3}{8},0)$, $(\frac{5}{7},\frac{3}{7})$, $(-6,-3)$ starting from (1,0) and (0,1).

4. Construct the point $(-2,-2)$ given: (*a*) (1,0) to the left of 0 and (0,1) below 0; (*b*) (1,1) in the second quadrant; (*c*) (1,1) in the fourth quadrant.

65. The line at infinity and the axes of reference. We shall now reword our discussion of axes of reference in §64 so as to fit in with our concept of the line at infinity l_∞. (Compare §49.)

We draw a schematic figure (for rectangular axes only) of these axes and l_∞, where parallel lines are represented as being solid and parallel for a distance, then broken off and dotted for a distance to indicate the omission of the rest of these lines, then drawn converging to a point on l_∞. We choose arbitrary points



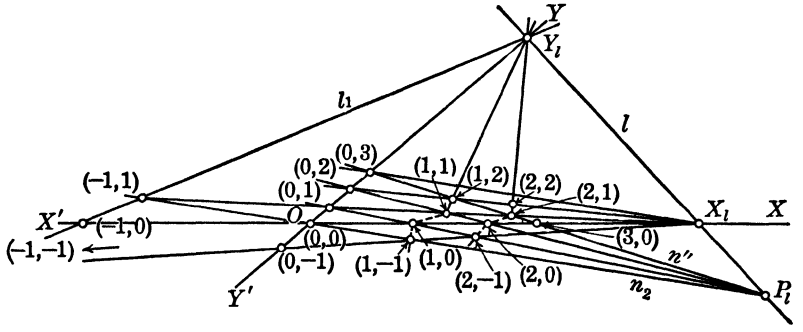
on $X'OX$ and $Y'OY$ for the so-called *unit points** (1,0) and (0,1), respectively. Then we join (1,0) to Y_∞ (the point of intersection of $Y'OY$ and l_∞) by a line l and join (0,1) to $X_\infty(X'OX, l_\infty)$ by a line m, the lines l and m intersecting in (1,1). (Note how we use the phrase *join (1,0) to* Y_∞ *by a line* l as meaning draw a line through (1,0) parallel to $Y'OY$; also how we abbreviate to $X_\infty(X'OX, l_\infty)$ the phrase "X_∞, which is the point of intersection of $X'OX$ and l_∞.") The line n joining (0,1) and (1,0) cuts l_∞ at P_∞. We join (1,1) to P_∞ by a line n' that cuts $Y'OY$ in (0,2) and cuts $X'OX$ in (2,0). Thus we see that the axes of reference and the line l_∞ form a sort of triangle. (Compare §70.)

In fact, if we take these axes and any line $l \equiv x/a + y/b - 1 = 0$ where $ab \neq 0$, then let this line move out beyond all finite bounds in such a way that the intercepts a and b become immeasurably great (this is written algebraically $a \to \infty$ and $b \to \infty$), our assumption of the existence of l_∞ compels us to conclude that l must approach the position l_∞ (since the two intercepts of l approach X_∞ and Y_∞).

We take such a line l and the axes of reference in the following figure and draw points that we label (1,0), (2,0), (2,2), etc. We note that as $l \to l_\infty$ the points (1,0), (2,0) (2,2), etc., in this

* From now on we shall often refer to these points and (1,1) as the *unit points.*

figure approach the positions of (1,0), (2,0), 2,2), etc., as constructed in §64.



We have no reason to suppose offhand that such a line in the above figure as n'' will pass through the three previously constructed points P_l, (2,1), (1,2) if it passes through two of them, or that n_2, which joins P_l to $(0,-1)$, will cut $X'OX$ in the same point $(-1,0)$ as the line l_1, which joins Y_l to $(-1,1)$, cuts $X'OX$. Our next three sections not only contain material valuable in itself but also lead up to the proofs of the collinearity of such points as (1,2), (2,1), P_l or $(0,-1)$, $(-1,0)$, P_l. (Note that there are no dotted lines in the second figure in the text because this figure represents a genuine triangle and is not schematic like the first figure.)

In §64 we saw that the correspondingly designated points were collinear because of certain properties of parallel lines, but there are no parallel lines in the above second figure. This figure is essentially what we shall study in §70 under the name of a triangle of reference.

EXERCISES

1. Draw the first figure in the text, only with (0,1) below O but (1,0) and l_∞ where they are in the text.

2. Draw oblique axes and l_∞ as we drew rectangular axes and in the text.

3. Reword the constructions in the text of §64 for the points $(-1,0)$, $(0,-1)$, $(-1,-1)$, $(\frac{p}{5},0)$, $(p,0)$, $(u/v,0)$, using the concept of l_∞ as was done in the above text for the points (0,2) and (2,0).

4. Draw an oblique triangle OX_lY_l and locate the points (0,1), (0,2), (0,3), etc., as in the last paragraph of the text; then move $l(X_lY_l)$ much further out (keeping OX and OY fixed) and, using the same (0,1) and (1,0), relocate the points (0,2), (0,3), etc.

5. Draw a figure like the last one in the text, only with (1,0) to the left of O, but with (0,1) and l the same as before.

6. Show that the points $(0,p)$ and $(p,0)$ never reach Y_l and X_l, respectively, in the last figure of the text, no matter how large a (finite) value we give to p.

66. **Desargues' theorem on perspective triangles.** Before taking up the so-called triangle of reference we must establish several lemmas and introduce some new concepts. The necessity for these lemmas was discussed in the last paragraph of §65.

DEFINITION. Two triangles ABC and $A'B'C'$ are said to be *perspective from a point* P (called the *center of perspectivity*) if the pairs of corresponding vertices A and A', B and B', C and C' are collinear with P.

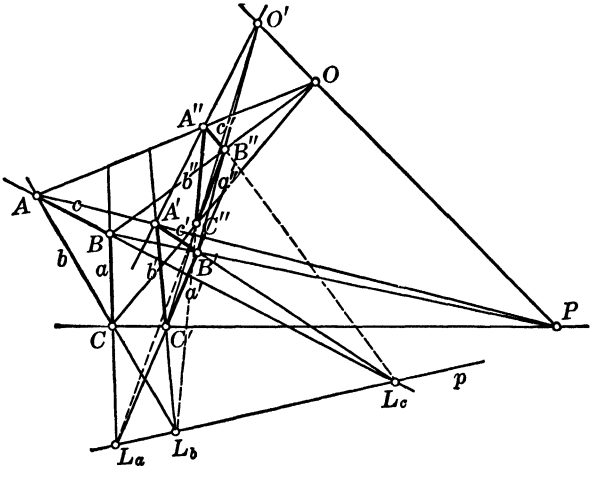
DEFINITION. Two triangles abc and $a'b'c'$ are said to be *perspective from a line* p (called the *axis of perspectivity*) if the pairs of corresponding sides a and a', b and b', c and c' are concurrent in points on p.

Note that if the two pairs of triangles are in the *same* plane these two definitions are the *plane duals* of each other. (See §22.)

We shall now prove the following theorem (due to Desargues).

THEOREM. *If two triangles are perspective from a center they are also perspective from an axis.*

We shall suppose the two triangles to lie in the same plane, leaving the other cases (also the converse of this theorem) for the exercises. Let us consider the following figure.



In the figure on page 144 we take two triangles ABC and $A'B'C'$ (in the same plane) perspective from the center P; we are to prove they are perspective from an axis p. Through P we draw any line not in the plane of the triangles and on this line we take two arbitrary points O and O' (neither point at P). We join OA, OB, OC and $O'A'$, $O'B'$, $O'C'$. Since OA and $O'A'$ lie in the same plane (determined by OP and AP), they meet in a point A'' (finite or infinite). Similarly, OB and $O'B'$ meet in B'', and OC and $O'C'$ meet in C''.

Consider the triangle $A''B''C''$ with sides a'', b'', c''. Since a'' and a are in the same plane (determined by OB and OC), they are concurrent. Similarly, b'' and b are concurrent, also c'' and c. But these pairs of lines must meet in points on the line p that is common to the plane of ABC (and $A'B'C'$) and the plane of $A''B''C''$. Also a'' and a' are concurrent on p, likewise b'' and b', c'' and c'. But a'' meets p in the point L_a, so a'' must meet both a and a' at L_a (hence a and a' meet at L_a). In the same way b'' meets both b and b' at L_b, c'' meets both c and c' at L_c. Therefore we see that a and a', b and b', c and c' are concurrent in pairs at points on p (i.e., the two given triangles are perspective from the axis p).

EXERCISES

1. Draw several cases of Desargues' theorem: (a) where P is at infinity; (b) where p is at infinity; (c) where a (but not b or c) is at infinity.

2. Draw the complete figure in the text, only with O on one side of P and O' on the other.

3. State and prove the plane dual of Desargues' theorem. (See §22.)

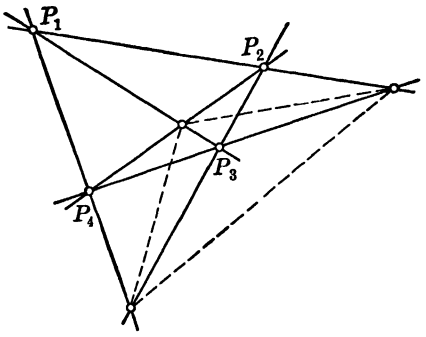
4. Prove Desargues' theorem when the two triangles are not in the same plane, first where their two planes are parallel, second where their two planes intersect in finite points. Hint: These cases are incidentally disposed of in the text.

5. State and prove the converse of Desargues' theorem when the two triangles are not in the same plane, first when their two planes are parallel and second when their two planes intersect in finite points.

6. State and prove Desargues' theorem for any two complete quadrilaterals. (For definition see §28.) Hint: Split the quadrilaterals up into triangles.

67. **Complete quadrangles and complete quadrilaterals.** In projective geometry we make much use of two plane figures that are generalizations of the quadrilateral of plane geometry. First, suppose this quadrilateral to be looked upon as having only four

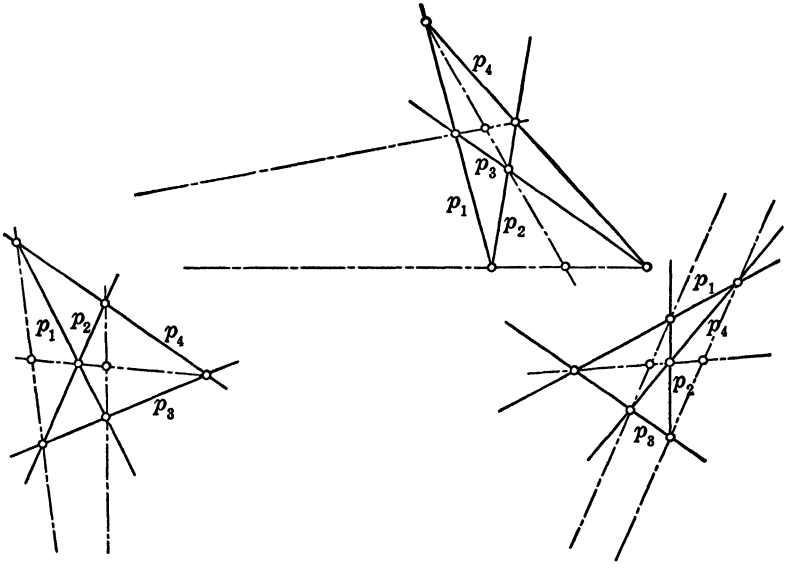
vertices P_1, P_2, P_3, P_4 (see the adjoining figure), but six sides (the four sides we studied in plane geometry plus the two diagonals); then we call the figure a *complete quadrangle.* (Compare §26.)



The points of intersection of the pairs of opposite sides of this complete quadrangle we call *diagonal points,* and the triangle of which these points are vertices we call the *diagonal triangle* of the complete quadrangle.

Next we look upon the quadrilateral of plane geometry as having only four sides p_1, p_2, p_3, p_4 (see the figure below) but six vertices (the four vertices we considered in plane geometry plus the two points of intersection of the pairs of opposite sides), and we call this figure a *complete quadrilateral.*

The lines joining pairs of opposite vertices of the complete quadrilateral we call *diagonal lines,* and the triangle of which these lines are the sides we call the *diagonal triangle* of the complete quadrilateral.



The complete quadrilateral *may have two of its opposite sides cross each other between the other pair of opposite sides, or may have a re-entrant angle,* whereas these types of quadrilaterals were not considered in plane geometry. (See the preceding figure.)

Note that the complete quadrangle and complete quadrilateral (if they are in the same plane) are *each* the *plane dual* of the *other.* Note also that *no three* of *the vertices* (*sides*) of the complete quadrangle (quadrilateral) can be *collinear* (*concurrent*).

Since we are using the concept of l_∞, the complete quadrangle or quadrilateral may have two sides parallel, or one or two vertices on l_∞, or a vertex (or side) of its diagonal triangle on l_∞. An important example of a complete quadrangle is the one whose vertices are (1,0), (0,1), (−1,0), (0,−1). (See §64.) Similarly, we have the complete quadrilateral whose sides are the lines joining (1,0) to (0,1), (0,1) to (−1,0), (−1,0) to (0,−1), and (0,−1) to (1,0) with equations

$$x + y - 1 = 0, \quad -x + y - 1 = 0, \quad x + y + 1 = 0, \quad x - y + 1 = 0$$

EXERCISES

1. Find the sides (vertices), diagonal points (lines), of the complete quadrangle (quadrilateral) mentioned in the last sentence of the text. Find the equation of any finite* line and the coordinates of any finite point.

2. Do as in Ex. 1 for the complete quadrangle (quadrilateral) made up from the quadrilateral whose vertices are (1,1), (−1,1), (−1,−1), (1,−1).

3. Given four points (a_1,b_1), (a_2,b_2), (a_3,b_3), (a_4,b_4) as the vertices of a complete quadrangle. Find the equations of its sides and of the sides of its diagonal triangle; find the vertices of its diagonal triangle. Hint: Use determinants to find the equations of any lines.

4. Given four lines $a_ix + b_iy - 1 = 0$, where $i = 1,2,3,4$. Work the dual of Ex. 3.

5. Draw figures to illustrate the special cases of complete quadrangles (quadrilaterals) mentioned in the last paragraph of the text.

6. How is l_∞ peculiarly situated with respect to the complete quadrangle with vertices (1,0), (0,1), (−1,0), (0,−1)? Compare §26.

7. How is l_∞ peculiarly situated with respect to the complete quadrilateral with sides $\pm x \pm y - 1 = 0$?

8. Do as in Ex. 1 for the complete quadrangle (quadrilateral) made up from the quadrilateral $(0,i)$, $(i,0)$, $0,-i)$, $(-i,0)$, where $i = \sqrt{-1}$.

* By *finite* line we do not mean a finite segment but an ordinary line as distinguished from l_∞.

68. **Quadrangular sets of points (lines); harmonic sets.** DEFINITION. Any line l (not a side of a given complete quadrangle) will cut the sides of this quadrangle in six or five or four collinear points, which are said to form a *quadrangular set* of points on l.

If l contains two diagonal points of the quadrangle (i.e., if l is a side of the diagonal triangle), this set of points (now four in number) is called a *harmonic set.* Compare §§26, 67; also the position of l_∞ in Ex. 6 of §67.

DEFINITION. Similarly (and *dually*), any point P (not a vertex of a given complete quadrilateral) will form with the vertices of this quadrilateral a set of six or five or four concurrent lines which are said to form a *quadrangular set* of lines at P.

If P is the point of intersection of two diagonal lines of the quadrilaterial (i.e., if P is a vertex of the diagonal triangle), this set of lines (now four in number) is called a *harmonic set.* Compare §28.

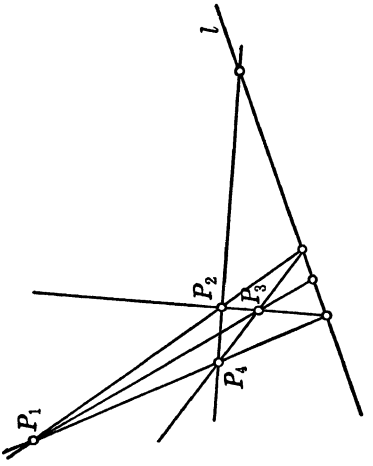
Note that in these previous sections all the points and lines were supposed to be finite, whereas now the infinite line l_∞ (or a point or points on l_∞) may figure in our quadrangular sets of points (or lines). Thus in the above definitions of quadrangular sets of points and lines either l or P may be on l_∞,* or one of the points on l may be on l_∞; also the given complete quadrangle (or quadrilateral) may have one or more of its elements (points and lines) on l_∞.

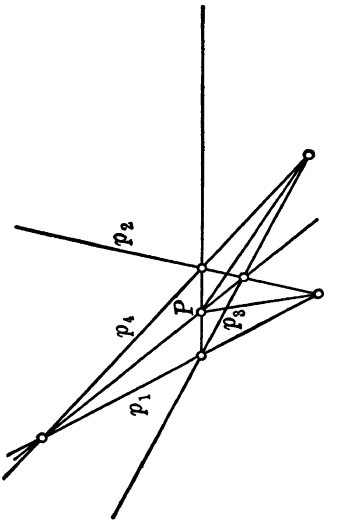
Note also the proof given in §28 that the projection from a point P of a harmonic set of points on a line p is a harmonic set of lines concurrent at P. This proof is valid also in the present connection, if we extend our notions of points and lines to include l_∞ and its points.

The figures drawn below show the different types of quadrangular sets of points (lines) for l (or P), real and finite, and the points (lines) of the complete quadrangles (quadrilaterals) as well as of the quadrangular sets, all real and finite. (Note that our definitions apply equally well to imaginary points and lines as to real.) (See §36.)

Note that we can consider a quadrangular set of points as cut on the line l by three lines through any one vertex of the complete quadrangle (say P_3) and by the sides of the triangle formed by the

* This is sometimes described as *being at infinity* in the plane.


l
P1
P2
P3
P4


p2
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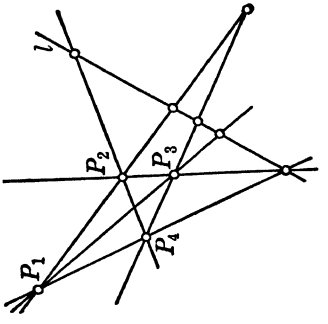
P
p3
p1


l
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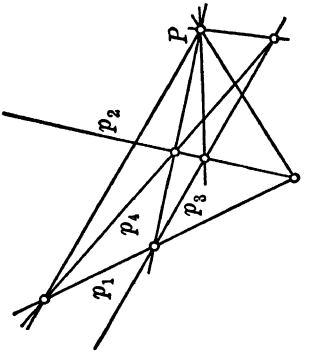
P
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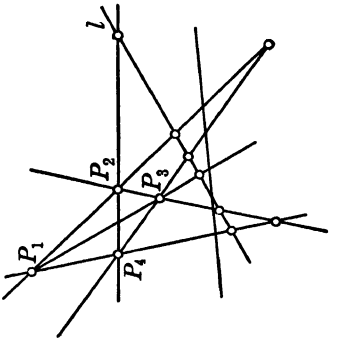
l
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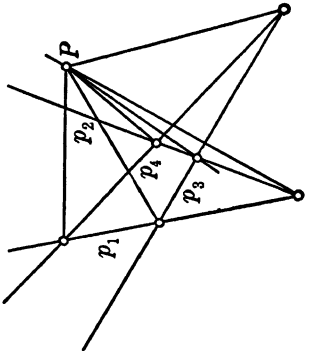
P4


P
p2
p4
p3
p1

three other vertices of the complete quadrangle (say P_1, P_2, P_4). We shall call the first set of three points on l a *point triple* and the second set a *triangle triple.* The quadrangular set of points on l can be looked at in four ways as made up of a point triple and a triangle triple, because there are four vertices (P_1, P_2, P_3, P_4) that can be chosen in turn to determine a point triple on l. Note that if the quadrangular set R_1, R_2, R_3, R_4, R_5, R_6 is a harmonic set, such that $R = R_1 = R_2$ and $R' = R_4 = R_5$, then the four possible point triples reduce to two (RR_3R' and $RR'R_6$), whereas the triangle triples are respectively $RR'R_6$ and RR_3R'.

EXERCISES

1. Dualize to the case of a quadrangular set of lines the ideas of point triple and triangle triple.
2. In §28, where a harmonic set of points is shown to be projected from a point by a harmonic set of lines, what is peculiar about the three lines that project a point triple?
3. Find the quadrangular set of points cut upon the line $x + y = 1$ by the complete quadrangle with vertices $(0,i)$, $(i,0)$, $(0,-i)$, $(-i,0)$.
4. Find the quadrangular set of points determined on $y = 0$ by the complete quadrangle $(0,2)$, $(-2,+1)$, $(0,-2)$, $(-2,-1)$.
5. Find the quadrangular set of lines determined at $(0,0)$ by the complete quadrilateral

$$x + y - 1 = 0, \quad x - y - 1 = 0, \quad -x + y - 1 = 0, \quad x + y + 1 = 0$$

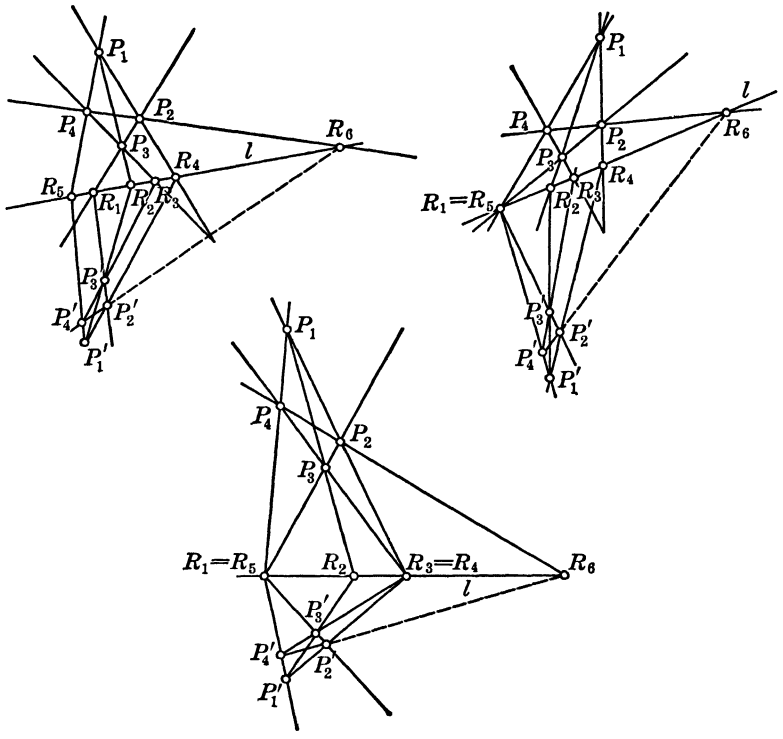
then the quadrangular set determined by the complete quadrilateral

$$y = ix + 1, \quad y = -ix + 1, \quad y = 2x - 1, \quad y = -2x - 1$$

69. **The fundamental theorem for quadrangular sets of points.** Now we shall prove the fundamental theorem for quadrangular sets of points, namely:

THEOREM. *Given any quadrangular set of points on a line l lettered R_1, R_2, R_3, R_4, R_5, R_6 (where one or two pairs of these points may be pairs of coincident points) such that R_1, R_2, R_3* form a point *triple and R_4, R_5, R_6 a triangle triple (relative of course to some given complete quadrangle that determines this quadrangular set on l). If five of the sides of any other complete quadrangle cut l in any five of these six points so that again we have R_1, R_2, R_3 a point triple, etc., relative to this new quadrangle, then the sixth side of this second complete quadrangle will pass through the sixth point of the given quadrangular set.*

PROOF: Consider the figure below:



We suppose that the quadrangular set is formed by P_1, P_2, P_3, P_4 (as in the figure) and that P_1', P_2', P_3', P_4' has its sides passing through the points R_1, R_2, R_3, R_4, R_5 in such a way that $P_1'P_2'$ goes through R_4, $P_1'P_3'$ through R_2, $P_2'P_3'$ through R_1, $P_4'P_3'$ through R_3, $P_1'P_4'$ through R_5. To prove that $P_2'P_4'$ passes through R_6.

The two triangles $P_1P_2P_3$ and $P_1'P_2'P_3'$ are perspective from the axis l; hence they are perspective from a center O (not drawn in the figure) by the converse (and plane dual) of Desargues' theorem. Also $P_1P_4P_3$ and $P_1'P_4'P_3'$ are perspective from the same axis l; hence they are perspective from a center O' (not drawn). But O and O' can both be determined as the intersection of P_1P_1' and P_3P_3'; therefore $O \equiv O'$. Therefore the triangles $P_1P_2P_4$ and $P_1'P_2'P_4'$ are perspective from a center O, so they are perspective from an axis; but this axis must be l, since P_1P_4 and $P_1'P_4'$ intersect in R_5 whereas P_1P_2 and $P_1'P_2'$ intersect in R_4 (and R_4 and R_5 determine l). From

this we conclude that P_2P_4 and $P'_2P'_4$ must intersect on l, so $P'_2P'_4$ passes through R_6.

Q.E.D.

In the above proof the two complete quadrangles must be so placed as to determine the same point triple R_1, R_2, R_3 and the same triangle triple; otherwise the above proof breaks down. A carefully drawn figure will convince the student that the theorem is still true even if R_1, R_2, R_3 is a *point triple* with respect to the complete quadrangle P_1, P_2, P_3, P_4 but a *triangle triple* with respect to P'_1, P'_2, P'_3, P'_4. However, the proof of this last fact must wait until much later. See §121.

We call attention to the fact that in drawing a complete quadrangle P'_1, P'_2, P'_3, P'_4 that will intersect l in the given quadrangular set of points R_1, R_2, R_3, R_4, R_5, R_6 we can take P'_3 as an *arbitrary point not on* l, join P'_3 to R_1, R_2, and R_3, then on the line $R_3P'_3$ pick an arbitrary point distinct from R_3 and P'_3 for the point P'_4; but from here on the rest of the complete quadrangle is uniquely determined. Another way to determine P'_1, P'_2, P'_3, P'_4 is to pick *three arbitrary lines* for $R_3P'_3$, $R_1P'_3$, and $R_5P'_4$, and from these construct the rest of the sides and vertices. A third way is to pick an *arbitrary* line meeting l in a point on a corresponding side of the first quadrangle and a point not on this line as a vertex.

EXERCISES

1. Why is there no loss of generality in the proof in the text due to choosing R_1, R_2, R_3 as a point triple and R_6 as the point in dispute?
2. Dualize the last paragraph of the text.
3. Describe two other arbitrary ways of selecting a complete quadrangle P'_1, P'_2, P'_3, P'_4.
4. Draw figures for the proof in the text for the cases where l passes through one or two diagonal points of the complete quadrangle and for the cases where R_6 is or is not a diagonal point.
5. Find analytically another complete quadrangle that will give the same quadrangular set of points as the set in Ex. 3 of §68; as the set in Ex. 4 of §68.
6. Find analytically another complete quadrilateral that will give the same quadrangular set of lines as each of the sets in Ex. 5 of §68.
7. Draw the figure in the text showing the center O.
8. If $(-1,0)$, $(0,0)$, $(1,0)$, $(2,0)$, $(3,0)$ form five points of a quadrangular set with the first three points a point triple, find the sixth point of this set. Is this sixth point unique? Hint: Choose a complete quadrangle.
9. Draw a figure for the theorem in the text (*a*) where l is l_∞; (*b*) where R_6 is on l_∞; (*c*) where R_4 is on l_∞.

10. Suppose R_2, R_3, R_4, R_5, R_6 in the figure of the text are given as points of intersection with l of the sides of the two complete quadrangles. Prove that the quadrangles intersect also in R_1.

11. Show why the proof in the text breaks down if a point triple with respect to the first complete quadrangle is a triangle triple for the second.

12. Given the five points of a quadrangular set $(-i,0)$, $(0,0)$, $(i,0)$, $(-1,0)$, $(1,0)$, where the first three points form a triangle triple, find the sixth point of this set.

13. State and prove the corresponding (dual) theorem about quadrangular sets of lines, drawing figures to illustrate the three special cases similar (and dual) to those given in the text of §68 for quadrangular sets of points. Hint: If the point L is the dual of the line l in the text, then any two complete quadrilaterals determining five of the six lines of the set at L must be perspective from L. Why?

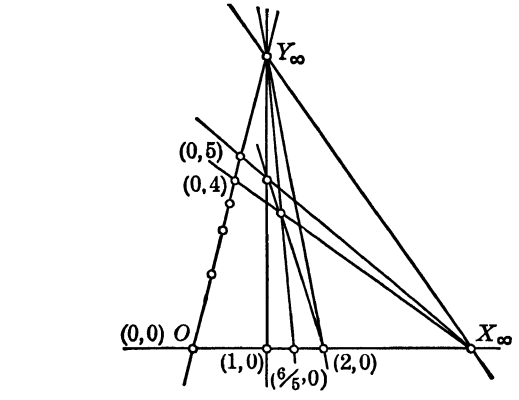
70. **The triangle of reference.** We are now in a position to consider a coordinate system where the coordinates are referred to what is called a triangle of reference. In §65 we showed that by using l_∞ we have for oblique or rectangular axes a sort of triangle of reference. But in this section we shall consider as a triangle of reference one all of whose sides (vertices) are lines (points) that lie in the finite part of the plane, i.e., are finite lines (points).

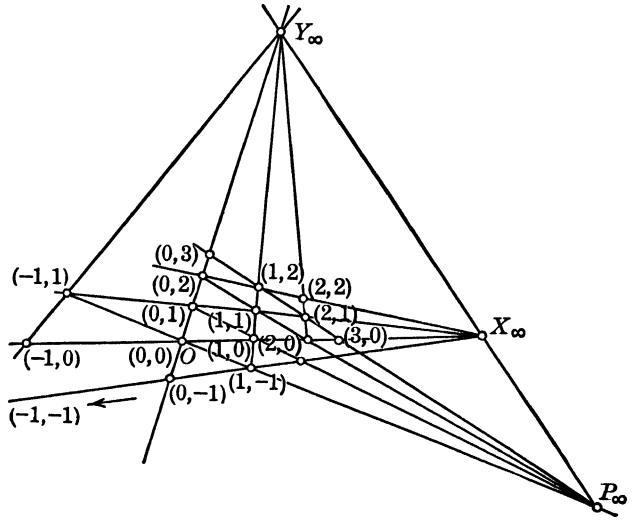
The constructions now given for finding the coordinates of points (and for locating points) referred to this triangle will be seen to be generalizations of those given in §65. By means of §§66, 67, 68, 69 we can show that the necessary lines are concurrent so as to make every point have a unique set of coordinates referred to this triangle of reference. (Compare the second figure in the text of §65 and the discussion there.) Consider the figures on page 154.

We have lettered these figures so that the discussion in §65 applies equally well here if we replace the name l_∞ by the line $X_\infty Y_\infty$; hence we shall not repeat here the description of how to obtain the points $(2,0)$, $(0,2)$, $(-1,-1)$, $(\frac{6}{5},0)$, etc. We note, however, that the first triangle in §65 is *purely schematic*, whereas the present triangles are *not* schematic at all (for instance, the lines through Y_∞ are not supposed to be parallel lines as in §65). These triangles are like the last triangle in §65. The sides of a triangle of reference are, of course, extended beyond the vertices.

In the second figure we obtain $(-1,0)$ as the point of intersection of the line joining Y_∞ and $(-1,1)$ with the line OX_∞. But the line through P_∞ and $(0,-1)$ should pass through $(-1,0)$ if the

determination of the point $(-1,0)$ is to be unique. (Similarly we must have the determination of other points unique, and so the line joining (1,2) and (2,1) must pass through P_∞, etc. This can all be shown in the following manner.)





The complete quadrangle with vertices P_∞, Y_∞, $(-1,1)$, (0,1) determines on OX_∞ the harmonic set of points $(-1,0)$, (0,0), (1,0), X_∞. The complete quadrangle with vertices P_∞, Y_∞, $(1,-1)$, $(0,-1)$ has its sides (by construction) passing through all the points of this harmonic set except that we must prove that

the line joining P_∞ to $(0,-1)$ passes through $(-1,0)$. Note that in the given harmonic set $(0,0)$ and X_∞ each counts as a pair of coincident points; also for each of these two complete quadrangles X_∞, $(1,0)$, $(0,0)$ form a point triple, determined for one quadrangle by the vertex $(0,1)$ and for the other quadrangle by $(1,-1)$. Therefore the fundamental theorem on quadrangular sets applies here, and we see that the three points P_∞, $(0,-1)$, and $(-1,0)$ are collinear.

Q.E.D.

EXERCISES

1. Prove that $(0,0)$, $(1,1)$, $(2,2)$, ... (n,n), where n is any positive or negative real number, are collinear. Hint: Use the fundamental theorem for quadrangular sets of points on the complete quadrangles with vertices $(0,1)$, $(0,0)$, $(1,0)$, $(1,1)$ and $(0,2)$, $(0,0)$, $(2,0)$, $(2,2)$, etc. First it must be shown that the line joining $(n,0)$ and $(0,n)$ passes through P_∞ if n is a fraction or an irrational number.

2. Prove that the line joining $(1,2)$ and $(2,1)$ passes through P_∞ ... ; the line joining $(1,p)$ and $(p,1)$ passes through P_∞ (where p is any positive or negative number).

3. Prove that the points $(0,0)$, $(2,1)$, $(4,2)$, $(6,3)$, ... are collinear; also that the points $(0,0)$, $(p,1)$, $(2\,p,2)$, $(3\,p,3)$, ... are collinear (where p is any positive or negative real number).

4. Show how the uniqueness of the determination of $(-1,0)$ and similarly of other points referred to axes of reference (as in §65) is disposed of.

5. Draw the following points referred to a triangle of reference $(-3,0)$, $(\frac{5}{3},0)$, $(3,-1)$.

6. Draw a triangle of reference (*a*) with $(1,0)$ to the left of O; (*b*) with $(0,1)$ below O and $(1,0)$ to the left of O. Determine in each instance the points $(2,0)$ and $(-1,0)$. In these instances and in the instance in the text label the quadrants (or what are called the *triangular regions* of the plane with respect to the triangle of reference), i.e., mark the regions of the plane I where for each point both x and y are positive, II where x is negative and y is positive, III where both x and y are negative, IV where x is positive and y is negative.

7. Show that choosing the point $(1,1)$ arbitrarily (but not on a side of the triangle of reference) is equivalent to choosing both $(0,1)$ and $(1,0)$ arbitrarily. Show cases where $(1,1)$ is to the right of $X_\infty Y_\infty$, to the left of OY_∞, below OX_∞, and in other positions. (Compare §65.) Mark the quadrants for each case.

8. Show that X_∞ and Y_∞, also any other point on the side $X_\infty Y_\infty$, cannot have both coordinates finite (by the construction given in the text).

9. In a given triangle of reference (*a*) take $(0,1)$ midway between O and Y_∞; (*b*) take $(0,1)$ nearer to Y_∞ than to O. In each case find the points $(0,2)$, $(0,-1)$, and $(0,-2)$. Treat in a similar manner the point $(1,0)$ for two triangles of reference as above with $(0,1)$. How do harmonic sets and parallel lines enter into this example? Compare §§25, 26.

71. A note on bilinear (or linear fractional) transformations. In §§56, 62 we had our first introduction to bilinear (or linear fractional) transformations of the parameters λ and μ of a net of conics (or λ of a pencil of conics). The equations (65) gave us the general bilinear transformation in the λ,μ-plane.

In this section we shall consider briefly bilinear transformations of the coordinates x and y of points in the x,y-plane, given by the equations

$$(66) \qquad x = \frac{a_1x' + b_1y' + c_1}{a_3x' + b_3y' + c_3}, \quad y = \frac{a_2x' + b_2y' + c_2}{a_3x' + b_3y' + c_3},$$

where $\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, with $M \equiv \begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix}$

defined as the matrix* of (66).

If $\Delta = 0$ for (66), any line $ax + by + c = 0$ goes into a line

$$a(a_1x' + b_1y' + c_1) + b(a_2x' + b_2y' + c_2) + c(a_3x' + b_3y' + c_3) = 0$$

But this last line for every value of a, b, and c passes through the point of intersection of the three lines

$$a_1x' + b_1y' + c_1 = 0, \quad a_2x' + b_2y' + c_2 = 0, \quad a_3x' + b_3y' + c_3 = 0$$

(These last three lines intersect in a point, because $\Delta = 0$.) In this case (66) is said to be a singular transformation. Hence we suppose $\Delta \neq 0$.

We are introducing bilinear transformations at this juncture in order to use them in changing from a triangle of reference to axes of reference (or from axes to a triangle).

Solving (66) for x' and y' in terms of x and y, we get

$$(66') \quad x' = \frac{(b_2c_3 - b_3c_2)x - (b_1c_3 - b_3c_1)y + (b_1c_2 - b_2c_1)}{(a_2b_3 - a_3b_2)x - (a_1b_3 - a_3b_1)y + (a_1b_2 - a_2b_1)}$$

$$y' = \frac{-(a_2c_3 - a_3c_2)x + (a_1c_3 - a_3c_1)y - (a_1c_2 - a_2c_1)}{(a_2b_3 - a_3b_2)x - (a_1b_3 - a_3b_1)y + (a_1b_2 - a_2b_1)}$$

The coefficients in (66′) are the first minors of Δ. If we use

* Compare §13.

cofactors (i.e., a cofactor is the minor of the element in the ith row and jth column multiplied by $(-1)^{i+j}$), we can write (66′) in the form

$$(66'') \qquad x' = \frac{A_1x + A_2y + A_3}{C_1x + C_2y + C_3}, \quad y' = \frac{B_1x + B_2y + B_3}{C_1x + C_2y + C_3}$$

where A_i, B_j, C_k $(i, j, k = 1,2,3)$ are the cofactors of a_i, b_j, c_k, respectively, in Δ.

It is easy to see that (66) sends straight lines into straight lines, conics into conics, cubics into cubics, etc. Also (66) preserves cross-ratio. (See §23.) But (66) is not affine, because the line $c_1x + c_2y + c_3 = 0$ goes into l_∞ and l_∞ goes into $a_3x' + b_3y' + c_3 = 0$ by (66). These two finite lines are called the *vanishing lines* of (66). (Compare §51.)

Note that (66) has *eight essential constants* (we can divide the numerator and the denominator of each fraction by a non-vanishing constant). This shows that four pairs of corresponding points determine (66) uniquely. (Compare §14.) Suppose we want the points (α_1,β_1), (α_2,β_2), (α_3,β_3), (α_4,β_4) to correspond to (α_1',β_1'), (α_2',β_2'), (α_3',β_3'), (α_4',β_4'), respectively, under (66). Substituting in (66) these unprimed and primed coordinates for the unprimed and primed variables, then clearing of fractions, we obtain

$$a_1\alpha_1' + b_1\beta_1' - a_3\alpha_1\alpha_1' - b_3\alpha_1\beta_1' + c_1 - c_3\alpha_1 = 0$$
$$a_2\alpha_1' + b_2\beta_1' - a_3\beta_1\alpha_1' - b_3\beta_1\beta_1' + c_2 - c_3\beta_1 = 0$$

and six other equations similar to these.

Suppose the above eight equations have their terms arranged in the order of the unknowns a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3. Consider three of the equations in a_1, b_1, c_1 arranged as follows:

$$a_1\alpha_1' + b_1\beta_1' + c_1 = a_3\alpha_1\alpha_1' + b_3\alpha_1\beta_1' + c_3\alpha_1$$
$$a_1\alpha_2' + b_1\beta_2' + c_1 = a_3\alpha_2\alpha_2' + b_3\alpha_2\beta_2' + c_3\alpha_2$$
$$a_1\alpha_3' + b_1\beta_3' + c_1 = a_3\alpha_3\alpha_3' + b_3\alpha_3\beta_3' + c_3\alpha_3$$

Suppose that a_3, b_3, c_3 have been already determined and we wish to solve these three equations for a_1, b_1, c_1. Then we see that in order to have a solution not all zeros the determinant of the coefficients of a_1, b_1, c_1 must not vanish. By taking all possible sets of three equations out of the four equations in a_1, b_1, c_1, and arguing

as in the last sentence, we see we must have

$$\begin{vmatrix} \alpha_1' & \beta_1' & 1 \\ \alpha_2' & \beta_2' & 1 \\ \alpha_3' & \beta_3' & 1 \end{vmatrix} \neq 0, \quad \begin{vmatrix} \alpha_1' & \beta_1' & 1 \\ \alpha_2' & \beta_2' & 1 \\ \alpha_4' & \beta_4' & 1 \end{vmatrix} \neq 0, \quad \begin{vmatrix} \alpha_1' & \beta_1' & 1 \\ \alpha_3' & \beta_3' & 1 \\ \alpha_4' & \beta_4' & 1 \end{vmatrix} \neq 0,$$

$$\begin{vmatrix} \alpha_2' & \beta_2' & 1 \\ \alpha_3' & \beta_3' & 1 \\ \alpha_4' & \beta_4' & 1 \end{vmatrix} \neq 0$$

From the non-vanishing of these determinants we see that no three of the four points with primed coordinates can be collinear.

Using (66′) instead of (66) we can show similarly that no three of the four points with unprimed coordinates can be collinear. A brief way of stating these results is to say that (66) *is uniquely determined when we send the vertices* P_1, P_2, P_3, P_4 *of one complete quadrangle into the vertices* P_1', P_2', P_3', P_4' of any other or of the same complete quadrangle. (Compare §§26, 67.)

The student should notice how the matrices (and determinants) of the coefficients of (66) and (66″) resemble one another, for this resemblance makes these transformations easy to remember.

We emphasize the fact that, since (66) is not affine, one or more of the vertices of a complete quadrangle in the above discussion may be on l_∞. As yet we cannot readily handle such cases analytically,* so we shall confine our illustrations mostly to cases where the vertices are finite.

We could, however, treat an infinite point P_∞ as the intersection with l_∞ of a line $y = mx$, where m is called the direction of P_∞. Thus, if we want (1,1) to go into the point P_∞' on $y' = 3x'$, we can divide the numerator and denominator of each fraction in (66) by x' and get

$$x = \frac{a_1 + b_1 y'/x' + c_1/x'}{a_3 + b_3 y'/x' + c_3/x'}, \quad y = \frac{a_2 + b_2 y'/x' + c_2/x'}{a_3 + b_3 y'/x' + c_3/x'}$$

Putting $x = 1$, $y = 1$, $y'/x' = 3$, $x' = \infty$, we get

$$1 = \frac{a_1 + 3\,b_1}{a_3 + 3\,b_3}, \quad 1 = \frac{a_2 + 3\,b_2}{a_3 + 3\,b_3}$$

Again, if we want the point P_∞ on $y = 4x$ to go into (3,2), we can

* This is the reason for introducing homogeneous coordinates later on. See §75.

either use (66′) or we can divide the first equation in (66) by the second and get

$$\frac{y}{x} = \frac{a_2x' + b_2y' + c_2}{a_1x' + b_1y' + c_1}$$

then, putting $y/x = 4$, $x' = 3$, $y' = 2$ and also $a_3 \cdot 3 + b_3 \cdot 2 + c_3 = 0$ (since we must have $x = \infty$, $y = \infty$ for $x' = 3$, $y' = 2$), we get the two equations

$$4 = \frac{3\,a_2 + 2\,b_2 + c_2}{3\,a_1 + 2\,b_1 + c_1}, \quad 3\,a_3 + 2\,b_3 + c_3 = 0$$

Illustrative Example. Suppose we want to send (0,1), (1,0), (0,−1), (−1,0) to (1,1), (1,−1), (−1,1), (−1,−1). Substituting the coordinates of the first four points for the unprimed variables in (66) and the coordinates of the second four points for the primed variables, we get

$$0 = \frac{a_1 + b_1 + c_1}{a_3 + b_3 + c_3}, \quad 1 = \frac{a_2 + b_2 + c_2}{a_3 + b_3 + c_3}, \quad 1 = \frac{a_1 - b_1 + c_1}{a_3 - b_3 + c_3},$$

$$0 = \frac{a_2 - b_2 + c_2}{a_3 - b_3 + c_3}, \quad 0 = \frac{-a_1 + b_1 + c_1}{-a_3 + b_3 + c_3}, \quad -1 = \frac{-a_2 + b_2 + c_2}{-a_3 + b_3 + c_3},$$

$$-1 = \frac{-a_1 - b_1 + c_1}{-a_3 - b_3 + c_3}, \quad 0 = \frac{-a_2 - b_2 + c_2}{-a_3 - b_3 + c_3}$$

Solving these equations for the coefficients and substituting the solutions in (66), we find the required transformation is

$$x = \frac{-y' + 1}{2\,x'}, \quad y = \frac{y' + 1}{2\,x'}$$

We wish to emphasize the fact that (looked upon as an *alibi*) the transformation (66) may have to do with points referred to a *triangle of reference*.

We leave for the student to prove in the exercises that (66) forms a group* and also, that, if M_1 is the matrix of one transformation of this group and M_2 that of another, then the product of these two transformations has for its matrix M_1M_2 or M_2M_1. Here we wish to point out that the general affine group (13) can be looked upon as a subgroup of (66), defined geometrically by the fact that (13) keeps l_∞ invariant and defined analytically from (66) by $a_3 = b_3 = 0, c_3 \neq 0$.

EXERCISES

1. Find (66′) from (66); fill in all the other algebraic details in the text, such as proving that no three of the points P_1, P_2, P_3, P_4 can be collinear, completing the solution of the illustrative example, etc.

* For a definition of the concept of group see §31.

2. Prove that the transformations (66) form a group. Compare §31. Also show that if M_1 and M_2 are the matrices of two such transformations, the matrix of the product of these two transformations is M_1M_2 or M_2M_1.

3. Prove that (66) preserves cross-ratio. Compare §23.

4. Prove that (66) sends lines into lines, conics into conics, cubics into cubics, n-ics into n-ics.

5. Determine (66) so as to send (0,1), (1,0), (0,−1), (−1,0) into (0,i), (0,−i), (−i,0), (i,0), respectively.

6. Determine (66) so as to send (−1,−1) into the point P'_∞ on the line $y' = 2x'$.

7. Determine (66) so as to send the point P_∞ on $y = -x$ into (2,2).

8. Interpret l_∞, P_∞, being at infinity, etc., when the points of the plane are referred to a triangle of reference.

72. The effect of bilinear transformations on the equations of lines. Just as with affine transformations in §21, so here we wish to consider the effect of (66) on the general straight line (23). Substituting for x and y in (23) their values in terms of x' and y' from (66), we get (after clearing of fractions)

$$(a_1u + a_2v + a_3w)x' + (b_1u + b_2v + b_3w)y' + (c_1u + c_2v + c_3w) = 0$$

Writing this last equation in the form $u'x' + v'y' + w' = 0$, we find the transformation (66) causes the following relations between u', v', w' and u, v, w

$$(67') \qquad \begin{aligned} \rho u' &= a_1u + a_2v + a_3w, \quad \rho v' = b_1u + b_2v + b_3w, \\ \rho w' &= c_1u + c_2v + c_3w \end{aligned}$$

where ρ is an arbitrary non-vanishing constant that is introduced because the equation of a straight line can be multiplied through by any such constant. Solving (67′) for u, v, w we get

$$(67) \qquad \begin{aligned} \sigma u &= A_1u' + B_1v' + C_1w', \quad \sigma v = A_2u' + B_2v' + C_2w', \\ \sigma w &= A_3u' + B_3v' + C_3w' \end{aligned}$$

where the capital letters are the *cofactors** of the small letters in the determinant Δ of (66); also where $\sigma = \Delta/\rho$.

By an argument similar to the one in §71 (concerning complete quadrangles) or the ones in §§14, 21 we see that four pairs of corresponding lines (no three of each set of four lines being concurrent) uniquely determine (67), and so uniquely determine (66). That is, (67) *is uniquely determined when we send the sides* P_1, P_2,

* The cofactor of a term a_{ij} in the ith row and jth column of a determinant is the first minor of a_{ij} with respect to this determinant multiplied by $(-1)^{i+j}$.

P_3, P_4 of one complete quadrilateral into the sides P'_1, P'_2, P'_3, P'_4 of any other or of the same complete quadrilateral. Notice how this is the *plane dual* of the above results concerning complete quadrangles and (66). (Compare §22.)

We remark how the result concerning complete quadrilaterals follows from that concerning complete quadrangles, because if we take two complete quadrilaterals we can choose four vertices (no three of them collinear) from each quadrilateral and reduce the question of the corresponding quadrilaterals to that of these corresponding sets of four vertices. On the other hand, we can reduce the question of corresponding quadrangles under (66) to that of corresponding quadrilaterals by choosing from each quadrangle four sides (no three of them concurrent).

Since (66) and so (67) are not affine, we see that one or more vertices of one of the corresponding complete quadrilaterals may be on l_∞. To find the vanishing lines under (66) or to determine (66) partly by assigning certain lines for vanishing lines, it is easier not to use (67) but to note that from (66) $a_3x' + b_3y' + c_3 = 0$ is the line into which l_∞ goes and from (66″) $C_1x + C_2y + C_3 = 0$ is the line that goes into l_∞.

EXERCISES

1. Check the algebra in the text, such as finding (67′), solving (67′) for u, v, w so as to get (67).

2. Prove the dual of the theorem in §71 concerning the equivalence of complete quadrangles under (66).

3. Prove that

$$\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

of (67) is a power of Δ of (66).

4. Determine (67′) and (66) so as to send the complete quadrilateral with the sides

$$x + y = 1, \quad x - y = 1, \quad -x + y = 1, \quad -x - y = 1$$

into the complete quadrilateral with the sides

$$2x' + y' = 1, \quad 2x' - y' = 1, \quad x' + 2y' = 1, \quad x' - 2y' = 1$$

respectively. Do this problem in two ways. (See the hint in the next to the last paragraph of the text.)

5. Determine (67) or (66) so as to send (*a*) l_∞ into $x' + y' - 1 = 0$; (*b*) $2x - y + 1 = 0$ into l_∞. Do this problem in two ways. (See the hint in the last paragraph of the text.)

73. **A note on transformations of points on a line and between lines.** In this section we wish to study very briefly the effect of the transformations (13) and (66) on the points of a line l that is sent into itself or between the points of two corresponding lines l and l'. We shall go into this subject more fully later on, but we want some of the results and concepts now.

If (13) sends the x-axis into itself, then $l \equiv y = 0$ must be the same line as $l' \equiv y' = 0$; hence we must have in (13) $b_1 = b_3 = 0$, $b_2 a_1 \neq 0$. In this case, if we are considering only the effect on $y = 0$, we can omit $y_2 = b_2 y'$ from (13) and write the rest (putting $y' = 0$)

$$x = a_1 x' + a_3 \tag{68}$$

If (13) sends $l \equiv x = 0$ into $l' \equiv y' = 0$, then we must have $a_1 = a_3 = 0$, $a_2 b_1 \neq 0$, and we can omit $x = a_2 y'$ from (13) and write the result (putting $y' = 0$)

$$y = b_1 x' + b_3 \tag{69}$$

If (66) sends $l \equiv y = 0$ into $l' \equiv y' = 0$, we must have $a_2 = c_2 = 0$, $b_2 \neq 0$. We can omit $y = b_2 y'/(a_3 x' + b_3 y' + c_3)$ from (66) and write the result (putting $y' = 0$)

$$x = \frac{a_1 x' + c_1}{a_3 x' + c_3} \tag{70}$$

Similarly, if (66) sends $l \equiv x = 0$ into $l' \equiv y' = 0$, we must have $a_1 = c_1 = 0$, $b_1 \neq 0$. We can omit $x = b_1 y'/(a_3 x' + b_3 y' + c_3)$ from (66) and write the result (putting $y' = 0$)

$$y = \frac{a_2 x' + c_2}{a_3 x' + c_3} \tag{71}$$

The transformations (68), (69,) (70), (71) give us the effect of (13) or (66) on certain lines of the plane, but tell us nothing about the rest of the plane. If we want to study any other line in the plane, we can first send it by some transformation T into $x = 0$ or $y = 0$. These transformations between points of the same line (or of two different corresponding lines) are said to be *induced* by (13) and (66), respectively.

Note that these transformations might be studied for *their own sake*, quite apart from the rest of the plane. We have already run

across a bilinear transformation of the parameter of a pencil of conics in §59 that has a form like (70).

We see that (68) has just *two* constants, so two pairs of corresponding points determine this transformation. Thus, if (68) is to send $x = 1$ to $x' = 2$ and $x = 2$ to $x' = 3$, we must have $1 = a_1 2 + a_3$, $2 = a_1 3 + a_3$, so $a_1 = 1$ and $a_3 = -1$, and the required transformation is $x = x' - 1$.

We note, on the other hand, that (70) has *three* essential constants; therefore three pairs of corresponding points determine such a transformation. (We classify as a pair of corresponding points also a pair P,P' where $P \equiv P'$.) Suppose now that we want (70) to send 1 to -1, 0 to 2, 3 to -3. We have, on substitution of the coordinates of these points,

$$1 = \frac{-a_1 + c_1}{-a_3 + c_3}, \quad 0 = \frac{2\,a_1 + c_1}{2\,a_3 + c_3}, \quad 3 = \frac{-3\,a_1 + c_1}{-3\,a_3 + c_3}$$

or $$c_1 = -2\,a_1, \quad a_3 = -\tfrac{2}{3}a_1, \quad c_3 = -\tfrac{11}{3}\,a_1$$

so the desired transformation is

$$x = \frac{x' - 2}{-\frac{2}{3}\,x' - \frac{11}{3}} = \frac{-3\,x' + 6}{2\,x' + 11}$$

Next suppose we want to send 1 to ∞, ∞ to -1, 0 to 2, we have $x = (a_1 + c_1/x')/(a_3 + c_3/x')$, hence

$$1 = \frac{a_1 + c_1/\infty}{a_3 + c_3/\infty} = \frac{a_1}{a_3}, \quad -a_3 + c_3 = 0, \quad 0 = 2\,a_1 + c_1$$

or $$a_3 = c_3 = a_1, \quad c_1 = -2\,a_1$$

and our transformation is $x = (x' - 2)/(x' + 1)$.

EXERCISES

1. Prove that every transformation of the form (68) has one double (or invariant) point and every one of the form (70) has two such points. Compare §33.
2. Prove that (68) form a group; also (70) form a group.
3. Check all the algebra in the text.
4. Prove that (68) and (70) keep cross-ratio invariant. See §23.
5. Determine (68) to send $1 + i$ to 0 and 2 to $3 - i$.
6. Determine (70) to send 0 to ∞, ∞ to 0, 1 to -1; to send i to 2, 0 to $1 - i$, ∞ to 1.
7. Interpret ∞ when the points of the plane are referred to a triangle of reference.

74. **Change from a triangle of reference to axes of reference.** In §§71, 72, 73 we were looking upon bilinear transformations of coordinates as *alibis*. We may also look upon these transformations as *aliases*.

Suppose now we have the points of the plane referred to a triangle of reference (with x and y coordinates) and we wish to change the frame of reference to axes of reference (with x' and y' coordinates). Suppose the line $X_\infty Y_\infty$ of the triangle of reference in §70 is the line

$$\alpha x' + \beta y' - 1 = 0$$

referred to axes of reference, also that OX_∞ lies on the x'-axis and OY_∞ lies on the y'-axis.

If we want to send $X_\infty Y_\infty$ by (66) to l_∞, to keep $x = 0$ on $x' = 0$ and to keep $y = 0$ on $y' = 0$, then we must have

$$b_1 = c_1 = a_2 = c_2 = 0, \quad a_3 x' + b_3 y' + c_3 \equiv c(\alpha x' + \beta y' - 1)$$

where $c \neq 0$. (Compare the hint in the last paragraph of §72.) Therefore we see that the transformation

$$(72) \qquad x = \frac{a_1 x'}{c(\alpha x' + \beta y' - 1)}, \quad y = \frac{b_2 y'}{c(\alpha x' + \beta y' - 1)}$$

sends the line $X_\infty Y_\infty$ to l_∞ in such a way that X_∞ goes to the point of intersection of the x'-axis and l_∞, also Y_∞ to the point of intersection of the y'-axis and l_∞.

One more pair of corresponding points not on the axes (or two more pairs of corresponding points on the axes, see §73) will determine this transformation (72) uniquely, since we have two arbitrary constants remaining (namely, a_1/c and b_2/c).* Note that $x = a_1 x'/c(\alpha x' - 1)$ is the transformation induced on the x'-axis (see §73) and $y = b_2 y'/c(\beta y' - 1)$ is the transformation induced on the y'-axis by the above bilinear transformation (72) in the plane.

If now we compare §§64, 65, and 70 we see that (72) has sent the points and lines necessary to determine the x and y coordinates of a point P referred to the triangle of reference into the points and lines necessary to determine the x' and y' coordinates of this

* This result is in keeping with the fact that the arbitrary choice of the one unit point (1,1) is equivalent to choosing arbitrarily the two unit points (0,1) and (1,0).

same point P referred to x'- and y'-axes plus l_∞. Therefore the above equations in (72) give us the relations between the primed and the unprimed coordinates of P.

ILLUSTRATIVE EXAMPLE. For example, the transformation

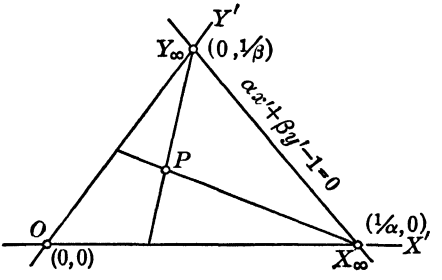
$$x = \frac{x'}{x' + y' - 1}, \quad y = \frac{y'}{x' + y' - 1}$$

sends the triangle of reference with sides $x' = 0$, $y' = 0$, $x' + y' - 1 = 0$ referred to a set of x'- and y'-axes plus l_∞ into this set of axes and l_∞ in such a way that the point $P(1,1)$ has the same coordinates referred to both frames of reference. Solving this transformation for x' and y' in terms of x and y we get

$$x' = \frac{x}{x + y - 1}, \quad y' = \frac{y}{x + y - 1}$$

which equations show that l_∞ referred to the triangle of reference has the equation $x + y - 1 = 0$. (Why?) The point (2,3) referred to the triangle has the coordinates $x' = 2/(2 + 3 - 1) = \frac{1}{2}$, $y' = \frac{3}{4}$ referred to the axes. Note that the complete quadrangle with vertices (0,0), (0,1), (1,0), (1,1) goes by the above transformation into the complete quadrangle with vertices (0,0), $(0,\infty)$, $(\infty,0)$, (1,1), respectively. Compare §71.

The above change from a triangle of reference to axes of reference is so important that we shall consider it again from a more geometric point of view. In the adjoining figure suppose that $X_\infty Y_\infty$ is the line $\alpha x' + \beta y' - 1 = 0$ referred to the axes OX', OY' and also one vertex of the triangle of reference is O (0,0).



Let us take any point P referred to the two frames of reference. The line $X_\infty Y_\infty$ behaves for the triangle of reference the way l_∞ does for the axes. Hence any line of the pencil of lines with OY_∞ and $Y_\infty X_\infty$ as fundamental lines has an equation of the form $x - \lambda \cdot 1 = 0$. (Compare §60.) Also any line of the pencil with OX_∞ and $X_\infty Y_\infty$ as fundamental lines has an equation of the form $y - \mu \cdot 1 = 0$. The same lines referred to the axes have, respectively, equations of the form

$$x' - \lambda'(\alpha x' + \beta y' - 1) = 0, \quad y' - \mu'(\alpha x' + \beta y' - 1) = 0$$

Solving these last two equations simultaneously for x' and y' we get

$$x' = \frac{\lambda'}{\alpha\lambda' + \beta\mu' - 1}, \quad y' = \frac{\mu'}{\alpha\lambda' + \beta\mu' - 1}$$

The equations of the lines in primed coordinates can be obtained from $x - \lambda \cdot 1 = 0$ and $y - \mu \cdot 1 = 0$, respectively, by replacing x by c_1x', y by c_2y', and 1 by $c_3(\alpha x' + \beta y' - 1)$, where $c_1c_2c_3 \neq 0$. Therefore we have $\lambda' = c_3/c_1\lambda$, $\mu' = c_3/c_2\mu$. Therefore, since $\lambda = x$, $\mu = y$ we have $\lambda' = \gamma x$, $\mu' = \delta y$, where $\gamma = c_3/c_1$, $\delta = c_3/c_2$. Substituting these values for λ' and μ' in the equations giving x' and y', we get

$$x' = \frac{\gamma x}{\alpha\gamma x + \beta\delta y - 1}, \quad y' = \frac{\delta y}{\alpha\gamma x + \beta\delta y - 1}$$

Solving these last equations for x and y in terms of x' and y', we get

$$x = \frac{x'/\gamma}{\alpha x' + \beta y' - 1}, \quad y = \frac{y'/\delta}{\alpha x' + \beta y' - 1}$$

which is again a transformation of the form (72).

In the above discussions the sides of the triangle of reference were chosen in special positions relative to the axes of reference. To obtain the general case we have merely to perform (13) on the triangle in order to get it into the above special position. However, the product of (13) by (72) gives a transformation of the type (66). (Why?) Therefore we see that (66) can be looked upon (in the guise of an *alias*) as a change from a triangle of reference to axes of reference (or from axes to triangle).

EXERCISES

1. Solve (72) for x', y' in terms of x and y.
2. Find the conditions that must be imposed on (72) in order (*a*) to keep (1,1) fixed; (*b*) to keep (0,1) and (1,0) fixed.
3. Determine (66) to send the triangle of reference $x = x' = 0, y = y' = 0$, $x' + y' - 1 = 0$ into the axes $x' = 0, y' = 0, l_\infty$ and (1,1) into (1,1). Do this in two ways.
4. Show that (13) followed by (72) gives (66).
5. Why would there be no loss of generality involved in choosing $X_\infty Y_\infty$ of the triangle of reference (for the second discussion in the text) as the line $x' + y' - 1 = 0$?
6. A triangle of reference has $x' = 0$, $y' = 0$, $2x' - 3y' - 1 = 0$ as its sides $x = 0$, $y = 0$, $X_\infty Y_\infty$; also $x' = 2$, $y' = 3$ as its unit point (1,1).

Find the coordinates of the point $x = 1, y = 2$ referred to the axes; of the point $x' = -2, y' = 1$ referred to the triangle. Find the equation of $x + y - 1 = 0$ referred to the axes, and of $x' + y' - 1 = 0$ referred to the triangle.

7. Prove that a curve whose equation is of the nth degree referred to axes of reference still has an equation of the nth degree when referred to a triangle of reference.

8. The equations

$$x = \frac{3x'}{2x' + y' - 1}, \quad y = \frac{2y'}{2x' + y' - 1}$$

change from a triangle of reference to axes plus l_∞. Locate the sides and the unit points of this triangle with respect to the axes.

9. Solve Ex. 8, looking upon the transformation as changing from axes (unprimed coordinates) to a triangle of reference (primed coordinates).

10. Interpret

$$x = \frac{x' + y' - 1}{2x' - 4y' + 1}, \quad y = \frac{x' - y' + 1}{2x' - 4y' + 1}$$

as changing from axes to triangle, finding the sides of the triangle and its unit points with respect to the axes, also the axes (and l_∞) and their unit points with respect to the triangle.

11. Construct a triangle of reference and draw the loci $2x - y - 1 = 0$ and $2x + 3y - 6 = 0$.

CHAPTER X

INTRODUCTION TO HOMOGENEOUS COORDINATES

75. Homogeneous coordinates for points. In order to be able to assign coordinates to the points on l_∞ (when we are using *axes* of reference) or to the points on $X_\infty Y_\infty$ (when we are using a *triangle* of reference) we introduce what are known as *homogeneous coordinates.* (Compare the footnote in §71.) From now on the type of coordinates (x and y, or x' and y') of a point that we have used thus far will be known as *non-homogeneous coordinates.*

To obtain the homogeneous coordinates x_1,x_2,x_3 of any point $P(x,y)$ we put $x = x_1/x_3$, $y = x_2/x_3$ and write the point as $P(x_1,x_2,x_3)$. We note that the point (kx_1,kx_2,kx_3) where $k \neq 0$ is the same as the point (x_1,x_2,x_3) because $kx_1/kx_3 = x_1/x_3 = x$, $kx_2/kx_3 = x_2/x_3 = y$.

If $x_3 \to 0$ while x_1 or x_2 (or both x_1 and x_2) does not vanish, we see that $x \to \infty$ or $y \to \infty$ (or both $x \to \infty$ and $y \to \infty$), i.e., the point $P(x,y) \equiv P(x_1,x_2,x_3)$ approaches the line l_∞ (if we are using axes of reference) or the line $X_\infty Y_\infty$ (if we are using a triangle of reference). Hence we say that $x_3 = 0$ is the equation of l_∞ (using axes) or of $X_\infty Y_\infty$ (using the triangle).

No definite point in the plane is given by (0,0,0) because the non-homogeneous form (0/0,0/0) is *wholly indeterminate.* Therefore we suppose that x_1, x_2, x_3 are not all zero at the same time.

The x-axis (or side OX_∞ of the triangle) has the equation $x_2 = 0$, the y-axis (or side OY_∞) has the equation $x_1 = 0$. The points $(x_1,0,x_3)$ lie on $x_2 = 0$, the points $(x_1,x_2,0)$ on $x_3 = 0$, the points $(0,x_2,x_3)$ on $x_1 = 0$. The origin (or vertex O of the triangle) is $(0,0,x_3)$, the point at infinity on the x-axis (or the vertex X_∞ on the triangle) is $(x_1,0,0)$, the point at infinity on the y-axis (or the vertex Y_∞ on the triangle) is $(0,x_2,0)$. The unit points have coordinates of the form $(x_1,0,x_1)$, $(0,x_2,x_2)$, (x_1,x_1,x_1). We call $(x_1,x_1,0)$ the unit point on $x_3 = 0$. Note that this last point is where the line $y = x$ (in non-homogeneous coordinates) through the two points (0,0) and (1,1) cuts l_∞ (or $X_\infty Y_\infty$). (Why?) We can write the last seven points mentioned above as (0,0,1), (1,0,0), (0,1,0), (1,0,1), (0,1,1), (1,1,1), and (1,1,0), respectively. (Why?)

If $P(x_1,x_2,x_3)$ is not on l_∞ (or $X_\infty Y_\infty$) we can multiply its coordinates by $k = 1/x_3$ and write the point as $P(x,y,1)$. There is no occasion to employ the symbol ∞ when we are using homogeneous coordinates, because *all* the points in the plane (even those on l_∞ or $X_\infty Y_\infty$) now can be designated by *finite* values of x_1, x_2, and x_3. This fact is one of the gains in using homogeneous coordinates.

If no ambiguity can arise from confusion with non-homogeneous coordinates, we often use x,y,z instead of x_1,x_2,x_3 for the homogeneous coordinates of a point, ordinarily reserving $z = 0$ for the equation of l_∞ (or $X_\infty Y_\infty$). In this case we can change from non-homogeneous to homogeneous coordinates by the equations $x = x/z$, $y = y/z$, remembering that the new x and y are not the same as the old x and y.

Note that, if we are considering simultaneously a triangle of reference ($x = 0, y = 0, z = 0$) and axes of reference plus $l_\infty (x' = 0, y' = 0, z' = 0)$, we do not ordinarily have $x = 0$ the same line as $x' = 0$ or $y = 0$ the same as $y' = 0$, and $z = 0$ usually has an equation of the form $\alpha x' + \beta y' - 1 = 0$ referred to the axes. Also, conversely, l_∞ has an equation of the form $x/a + y/b - 1 = 0$ referred to the triangle of reference, where a and b are the intercepts (*finite* only in coordinates, but in reality *infinite* points) of l_∞ on the sides OX_∞ and OY_∞ of the triangle.

We might use $x_1 = 0$ or $x_2 = 0$ ($x = 0$ or $y = 0$) instead of $x_3 = 0$ ($z = 0$) for l_∞ or $X_\infty Y_\infty$. As illustrations of points with homogeneous coordinates we note that (2,3) is the same point as (4,6,2) or $(-\frac{2}{3},-1,-\frac{1}{3})$ or (200, 300, 100) or $(2\sqrt{3}, 3\sqrt{3}, \sqrt{3})$; also (7,6,5) is the same point as $(\frac{7}{5},\frac{6}{5})$.

To illustrate how formulas appear in homogeneous coordinates, we take that for cross-ratio, namely (25) and replace x_i by x_i/z_i and y_i by y_i/z_i where $i = 1,2,3,4$; then we multiply the numerator and denominator by $z_1z_2z_3z_4$ and we have

$$(73) \qquad \frac{x_1z_2 - x_2z_1}{x_1z_4 - x_4z_1} \frac{x_3z_4 - x_4z_3}{x_3z_2 - x_2z_3}$$

EXERCISES

1. Put into homogeneous coordinates (x,y,z) the formulas for the distance between two points, for the area of a triangle, for the point of division of a line-segment.

2. Using homogeneous coordinates show by a cross-ratio that $-a, 0, +a, \infty$ form a harmonic set. Hint: Write ∞ as the point (1,0), etc.

3. Find a cross-ratio of (1,2,3), (2,3,1), (3,5,4), $(-1,-1,2)$, first using homogeneous coordinates and (73), then using non-homogeneous coordinates.

4. Express in three ways in homogeneous coordinates each of the points $(-1,-2)$, $(1,\infty)$, $(\frac{1}{2},-\frac{1}{3})$.

76. Equations in homogeneous coordinates. An equation in non-homogeneous coordinates (variables) can be written as a homogeneous equation when the coordinates (variables) are made homogeneous, as the following example will show. (This is one reason for the name homogeneous coordinates.) In the equation $xy = 1$ we can put $x = x_1/x_3$, $y = x_2/x_3$, clear the equation of fractions and get $x_1x_2 = x_3^2$; or, using $x = x/z$, $y = y/z$, we get $xy = z^2$. Note how clearly the form $xy = z^2$ for the hyperbola $xy = 1$ shows the way this curve cuts l_∞, because $x = 0$ (or $y = 0$) when solved with $xy = z^2$ gives us $z^2 = 0$, so the two axes are the asymptotes to the curve.

Again, if we want to have the transformation (66) in homogeneous coordinates, we put $x = x/z$, $y = y/z$, $x' = x'/z'$, $y' = y'/z'$ and multiply numerator and denominator of each fraction by z'. Then we have (from certain properties of fractions)

$$(74) \qquad \rho x = a_1x' + b_1y' + c_1z', \quad \rho y = a_2x' + b_2y' + c_2z', \quad \rho z = a_3x' + b_3y' + c_3z'$$

where ρ is an arbitrary non-vanishing constant. Solving for x',y',z' in terms of x,y,z we get

$$(74') \qquad \sigma x' = A_1x + A_2y + A_3z, \quad \sigma y' = B_1x + B_2y + B_3z, \quad \sigma z' = C_1x + C_2y + C_3z$$

where $\sigma = \Delta/\rho$, Δ being the determinant of (66) and A_i, B_j, C_k $(i,j,k = 1,2,3)$ are the cofactors of a_i, b_j, c_k in Δ.

We show how to solve simultaneously homogeneous equations. Thus $x^2 + y^2 = 5z^2$, $x^2 - y^2 = z^2$ give $2x^2 = 6z^2$, $2y^2 = 4z^2$, so $x = \pm\sqrt{3}\,z$, $y = \pm\sqrt{2}\,z$. Hence the four points of intersection of these two curves are $(\pm\sqrt{3}\,z, \pm\sqrt{2}\,z, z)$ where $z \neq 0$, or (dividing by z) the four points are $(\pm\sqrt{3}, \pm\sqrt{2}, 1)$. Similarly, the two lines $3x - 2y + 9z = 0$, $x + y - 2z = 0$ intersect in a point given by $5x + 5z = 0$ and $5y - 15z = 0$, so the point of intersection is $(-z,3z,z)$ or $(-1,3,1)$.

Finally, if we put $x = x/z$, $y = y/z$ (or $x = x_1/x_3$, $y = x_2/x_3$)

in the equation of the general conic (4), then multiply the equation through by z^2 (or x_3^2), we get the equation of such a conic in homogeneous coordinates

$$(75) \qquad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

or

$$(75') \quad ax_1^2 + bx_2^2 + cx_3^2 + 2fx_2x_3 + 2gx_3x_1 + 2hx_1x_2 = 0$$

Note the *symmetry* in the way (75) is written, namely, the order of the terms is x^2, y^2, z^2, yz, zx, xy.

Again we note that if we put $x = \infty$, $y = \infty$ in (4) we get an indeterminate form

$$a\infty^2 + b\infty^2 + c + 2f\infty + 2g\infty + 2h\infty^2 = 0$$

Putting $x = x/z$, $y = y/z$ in (4), multiplying by z^2 to get (75), then putting $z = 0$ in (75), *all amount to evaluating this indeterminate form.*

EXERCISES

1. Explain fully how we get (74) from (66) and the properties of fractions that we used.

2. How does the fact that $y = 0$ solved with $xy = z^2$ gives $z^2 = 0$ show that the x-axis is an asymptote of the hyperbola $xy = 1$?

3. Get (74′) from (74).

4. Put all the transformations of variables into homogeneous form. Hint: The translation becomes

$$\rho x = x' + hz', \quad \rho y = y' + kz', \quad \rho z = z'$$

(Why?) Return to the non-homogeneous form to check your results.

5. Put into homogeneous coordinates (first using x,y,z then using x_1,x_2,x_3) the normal forms for the equations of ellipses, hyperbolas, and parabolas; the general equation of the circle; the general equation of a cubic; the general equation of an n-ic; the different type forms for the equation of a straight line; the equations

$$y^2 = x^3, \quad y = x^3, \quad y^3 = x^4, \quad y = x^4, \quad y^2 = x^2(x \pm 1),$$
$$y^2 = x(x-1)(x-\alpha)$$

6. Solve simultaneously:

$$x_1 + 3x_2 - x_3 = 0 \quad \text{and} \quad 2x_1 - x_2 + 5x_3 = 0$$
$$x_1^2 + x_2^2 = x_3^2 \quad \text{and} \quad x_1 - x_2 = x_3$$

Next put these equations into non-homogeneous form and solve them simultaneously.

7. Put into homogeneous coordinates the following pencils of conics, and find and describe their degenerate conics. (Compare §58.)

$$(x^2 + y^2 - 1) + \lambda(x^2 + y^2 - 2) = 0, \quad (y^2 - 4x) + \lambda(y^2 - 8x) = 0,$$
$$(x^2 + y^2 - 1) + \lambda(x^2 + y^2 + 2x - 1) = 0$$

8. Look through the first part of this book for all the cases where we have to interpret an equation like $0x^2 + 0y^2 + 1 = 0$ or infinite points and lines come up in some other way. In each case change to homogeneous coordinates and go through the argument again. Hint: Then such an equation as $0x^2 + 0y^2 + 1 = 0$ becomes $z^2 = 0$; also an equation like $2x = 0$ (where a degenerate conic is to be expected) becomes $2xz = 0$.

9. Using homogeneous coordinates find the equation of the circle through the three points (2,3,1), (1,0,1), (0,1,1); also find the equation of the conic through the five points (0,0,1), (0,1,0), 1,0,0), (1,1,1), (1,−1,2).

10. Prove that the equation of a line through two points (x_1,y_1,z_1) and (x_2,y_2,z_2) is

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

Hint: Transform to homogeneous coordinates the determinant form for the equation of a line in non-homogeneous coordinates. Or else argue that the equation of a line must be homogeneous and linear, hence this is the desired equation.

77. Homogeneous coordinates and l_∞; non-homogeneous coordinates for points on l_∞. Let us consider the equation $y = mx$ of a line l through the origin. In homogeneous coordinates (x_1,x_2,x_3) this line has the equation $x_2 = mx_1$. Solving $x_2 = mx_1$ simultaneously with $x_3 = 0$, we determine the point of intersection of l and l_∞. Hence l and l_∞ intersect in the point $P_\infty(x_1,mx_1,0)$ where $x_1 \neq 0$.

Note that for an ordinary frame of reference m is the slope of the line $y = mx$, so m is called the *direction* of the point P_∞ on l_∞ (compare §§49, 71). If we divide the coordinates of P_∞ by x_1, we can write this point as $P_\infty(1,m,0)$. We may look upon m as a *non-homogeneous* coordinate for the point P_∞, just as $(0,b,1)$ has b as a *non-homogeneous* coordinate and $(a,0,1)$ has a.

The line $x_1 = 0$ (i.e., $x = 0$) cuts l_∞ in the point $(0,x_2,0)$ for whose non-homogeneous coordinate we can write ∞; the line $x_2 = 0$ (i.e., $y = 0$) cuts l_∞ in the point $(x_1,0,0)$ with 0 as its non-homogeneous coordinate. (If we divided the homogeneous coordinates of all points on l_∞ by x_2 instead of x_1 then $(x_1,0,0)$ would be ∞ and $(0,x_2,0)$ would be zero, but these non-homogeneous coordinates would be the reciprocals of the directions of the corresponding

points on l_∞ and not the directions of these points.) The line $x_1 = x_2$ (i.e., $x = y$) cuts l_∞ in the unit point $(x_1,x_1,0)$ with 1 as its non-homogeneous coordinate.

By the above discussion, we have assigned to every point P_∞ on l_∞ (by using the line joining P_∞ to the origin) a set of homogeneous coordinates and also a non-homogeneous coordinate. Note how this resembles the determination of the coordinates of points on the x- and y-axes in §§64, 65 and of points on the sides OX_∞ and OY_∞ of the triangle of reference in §70.

We remark that the transformation $x_1 = x_3'$, $x_2 = x_2'$, $x_3 = x_1'$ sends a point $(x_1,mx_1,0)$ on l_∞ into a point $(0,mx_3',x_3')$ on the y-axis. We have defined (in §§23, 106) the cross-ratios of four collinear points in terms of their abscissas or their ordinates (*with no idea of lengths of line-segments entering into the definition*). If we send (by some transformation like the one above) these four finite points into four points on l_∞, our formula (25) for their cross-ratios becomes what we shall call the formula for the cross-ratios of four points on l_∞.

Thus the four points $(x_1,-x_1,0)$, $(x_1,0,0)$, $(x_1,x_1,0)$, $(0,x_2,0)$ with non-homogeneous coordinates $-1,0,1,\infty$ are said to have a cross-ratio $(1-0)/(-1-0)\cdot(-1-\infty)/(1-\infty) = -1$ and, therefore, are defined as a harmonic set on l_∞. We would get the same cross-ratio if we used (73) on the homogeneous coordinates of these points. Note how this cross-ratio is related to that of the four lines $y = -x$, $y = 0$, $y = x$, $x = 0$ as given in §27. Later on, we shall discuss cross-ratio again, more in detail.

Again let us see how homogeneous coordinates help us interpret the intersections of curves with l_∞. Thus $y^2z = x^3$ shows that $y^2 = x^3$ should be interpreted as having a point of inflection at the intersection of the y-axis and l_∞, with l_∞ as tangent (because $z = 0$ gives $x^3 = 0$ when solved with this equation). Also $yz^2 = x^3$ shows that $y = x^3$ should be looked upon as having a cusp at infinity on the y-axis with l_∞ as tangent. (Why?) Compare §44.

Note that $x = x'$, $y = z'$, $z = y'$ sends $y^2z = x^3$ into $y'z'^2 = x'^3$; therefore we must consider $y = x^3$ and $y^2 = x^3$ as *equivalent* cubics under (74), only that one has a cusp at infinity and an inflection at the origin while the other has an inflection at infinity and a cusp at the origin. These two cubics are *not* equivalent to each other under (13).

EXERCISES

1. Send $x^2 + y^2 = 1$ into $x^2 - y^2 = 1$, using (74) and homogeneous coordinates.

2. Find a cross-ratio of (3,2,0), (1,3,0), (−1,1,0), (0,1,0), first using homogeneous coordinates, then using non-homogenerous coordinates.

3. Find and describe the intersections of the following curves with l_∞:

$$x^2/a^2 \pm y^2/b^2 = 1, \quad x^2 + y^2 = r^2, \quad y^2 = 4\rho x, \quad y^2 = x^5, \quad y^3 = x^5,$$
$$y^4 = x^5, \quad x^2y = 1, \quad y = x^4, \quad xy(x^2 - y^2) = 1$$

Use homogeneous coordinates (x_1,x_2,x_3 for some of the curves and x,y,z for the rest). Also put the coordinates of the points of intersection of these curves with l_∞ into non-homogeneous form.

4. Find the intersections of l_∞ with the general conic (75), the general circle, the general cubic, the general n-ic. For (75) and the general circle put the coordinates of these points of intersection with l_∞ into the non-homogeneous form.

78. Homogeneous coordinates and a triangle of reference; non-homogeneous coordinates for points on $X_\infty Y_\infty$. Since (66) sends linear equations into linear equations, therefore, even when the points of the plane are referred to a triangle of reference, $y = mx$ is the equation of a line through $O(0,0)$ and conversely every line through O has such an equation.

We are now in a position to discuss completely the subject of homogeneous coordinates and a triangle of reference, giving coordinates (homogeneous and non-homogeneous) to the points on $X_\infty Y_\infty$. The discussion is exactly similar to that given in §77 for axes of reference and l_∞. We shall merely note here that to a point P on $X_\infty Y_\infty$, where a line $y = mx$ ($x_2 = mx_1$) cuts $X_\infty Y_\infty$, we give the homogeneous coordinates $(x_1,mx_1,0)$ or $(1,m,0)$, and to P we give the non-homogeneous coordinate m.

We wish to emphasize the fact that two lines $y = mx + b_1$ and $y = mx + b_2$ referred to a triangle of reference intersect on $X_\infty Y_\infty$ and so are *not* parallel. Also the equation $2xy = z^2$ shows that $2xy = 1$ is tangent to OX_∞ and OY_∞ at X_∞ and Y_∞, but does *not* show that this curve is a *hyperbola.*

If $z = 0$ is the line $\alpha x' + \beta y' - 1 = 0$ referred to axes of reference where $x = 0$ is $x' = 0$ and $y = 0$ is $y' = 0$, the curve $2xy = z^2$ has the equation (putting $x = x'$, $y = y'$, $z = \alpha x' + \beta y' - 1$)

$$\alpha^2 x'^2 + \beta^2 y'^2 + 2(\alpha\beta - 1)x'y' - 2\alpha x' - 2\beta y' + 1 = 0$$

referred to these axes with $(1/(\alpha + \beta - 1), 1/(\alpha + \beta - 1))$ as

unit point (1,1) for the axes. Therefore this curve is a hyperbola, parabola, or ellipse according as

$$(\alpha\beta - 1)^2 - \alpha^2\beta^2 = -2\alpha\beta + 1 \gtreqless 0$$

As illustrations of this result we see that $2xy = z^2$ is a hyperbola for $\alpha = \beta = \frac{1}{2}$, a parabola for $\alpha = \frac{1}{2}$ and $\beta = 1$, an ellipse for $\alpha = \beta = 1$. Note that even the above is a special choice for the triangle of reference, since $x = 0$ is $x' = 0$ and $y = 0$ is $y' = 0$.

EXERCISES

1. Show that in the last paragraph of the text the point $(1/(\alpha + \beta - 1), 1/(\alpha + \beta - 1))$ referred to the triangle of reference is the point (1,1) referred to the axes.

2. Given a triangle of reference, as in the last paragraph of the text, find when two lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are parallel, also when $y^2 = 4px$ (and $x^2/a^2 + y^2/b^2 = 1$) gives an ellipse or a parabola or a hyperbola.

3. Give the full discussion of homogeneous (and non-homogeneous) coordinates for points on $X_\infty Y_\infty$.

4. In the last paragraph of the text is the transformation $x = x'$, $y = y'$, $z = \alpha x' + \beta y' - 1$ looked upon as an alias or as an alibi?

5. Find the inverse of the transformation in Ex. 4.

79. Change of homogeneous coordinates from a triangle of reference to axes plus l_∞. Suppose we have a triangle of reference ($x = 0$, $y = 0$, $z = 0$) and we wish to refer the points of the plane to a set of oblique or rectangular axes plus l_∞. The transformation (74) will evidently do this for us.

However, let us see how homogeneous coordinates will *simplify* the discussion in §74. Thus, for the *first* method of attack given in the text there, we see we can put

$$\rho x = a_1x', \quad \rho y = b_2y', \quad \rho z = c(\alpha x' + \beta y' - z')$$

and then proceed with the argument. For the *second* method of attack the lines that determine the coordinates of P referred to the triangle of reference have, respectively, the equations $x - \lambda z = 0$, $y - \mu z = 0$. Substituting in these equations

$$\rho x = c_1x', \quad \rho y = c_2y', \quad \rho z = c_3(\alpha x' + \beta y' - z')$$

we have the equations

$$x' - \lambda'(\alpha x' + \beta y' - z') = 0, \quad y' - \mu'(\alpha x' + \beta y' - z') = 0$$

where $\lambda' = c_3\lambda/c_1 = c_3/c_1 \cdot x/z$, $\mu' = c_3/c_2 \cdot y/z$. Now we can proceed as in §74.

In general, we suppose that we have a triangle of reference ($x = 0$, $y = 0$, $z = 0$) whose three sides have, respectively, the equations

$$a_1'x' + b_1'y' + c_1'z' = 0,\ a_2'x' + b_2'y' + c_2'z',\ a_3'x' + b_3'y' + c_3'z' = 0$$

referred to some axes plus l_∞. We can change from the triangle to the axes by putting

$$\rho x = \alpha(a_1'x' + b_1'y' + c_1'z'),\quad \rho y = \beta(a_2'x' + b_2'y' + c_2'z'),$$
$$\rho z = \gamma(a_3'x' + b_3'y' + c_3'z')$$

where α, β, γ are to be determined by another pair of corresponding points. We *prove* this by noting that *one* and *only one* transformation (74) will send this triangle and the point (1,1,1) not on the triangle to these axes and the new point (1,1,1). The above, which is of the form (74), certainly accomplishes this because $x = 0$ gives $a_1'x' + b_1'y' + c_1'z' = 0$, etc., so the *above* must be the *unique* transformation we are seeking.

EXERCISES

1. Determine α,β,γ in the last paragraph of the text so that $P(1,1,1)$ has the same coordinates both for the triangle and for the axes. Hint: Put $x = y = z = x' = y' = z' = 1$.

2. Change to a set of axes from a triangle of reference ($x = 0, y = 0, z = 0$) with sides $x' + y' - z' = 0$, $x' - y' - z' = 0$, $2x' + y' + z' = 0$ respectively referred to these axes. and let $P(1,1,1)$ have the same coordinates for each frame of reference.

3. Use the idea of corresponding complete quadrangles (see §71) to explain the presence of α,β,γ in the equations in the last paragraph of the text.

4. Under the transformation from a triangle of reference ($x = 0$, $y = 0$, $z = 0$) to axes:

$$\rho x = x' + 2y' - 3z',\quad \rho y = -x' + y' + z',\quad \rho z = 2x' + 3y' - z'$$

what forms do the following equations take?

$$x' + 3y' + z' = 0,\quad x'y' = z'^2,\quad xy = z'^2,\quad x'^2 + y'^2 = z'^2,\quad yz^2 = x^3$$

What are the primed coordinates of the unprimed unit point (1,1,1)? What are the unprimed coordinates of the primed unit point?

5. Under the transformation of Ex. 4 find the coordinates of (3,0,1), (−1,2,1), (−3,2,1) that are here referred to the axes, when these points are referred to the triangle of reference. Where does l_∞ cut the triangle of reference?

6. Under Ex. 4 find the coordinates of $(-1,0,1)$, $(-1,-1,1)$, $(3,-5,1)$ that are here referred to the triangle, when they are referred to the axes. Hint: Use the transformation in the form where x',y',z' are solved for in terms of x,y,z.

80. **The equations of curves referred to a triangle of reference.** The transformations (74) between axes and a triangle of reference are linear in the variables; so also are their inverses. Therefore a line, conic, cubic, n-ic, when referred to a triangle of reference, still have, respectively, equations of the first, second, third, nth degrees in the variables. (Compare §§4, 18.) Conversely, any equations of the first, second, third, nth degrees in the variables (referred to a triangle of reference) are of the *same* degrees when referred to axes and so must give, respectively, lines, conics, cubics, n-ics. (Compare §40.)

This gives us an easy proof that

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0 \tag{76}$$

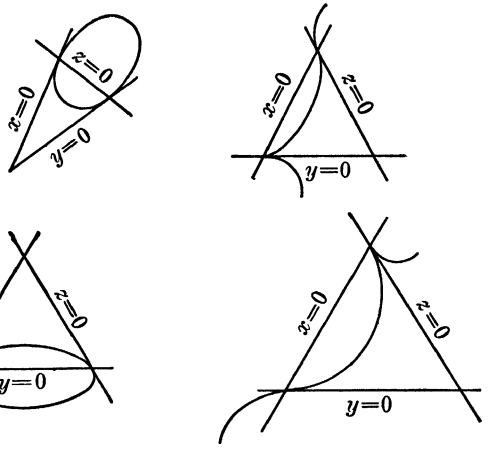
is the equation of the line through the two points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ referred to a triangle of reference, because this equation is certainly satisfied by the coordinates of P_1 and P_2, and being of the first degree it is the equation of a straight line.

We saw in §4 that the transformations from rectangular to oblique axes are also linear in their variables. This all shows us that $x^2 + y^2 = r^2z^2$ (or $x^2 + y^2 = r^2$), for instance, gives a circle referred to an ordinary frame of reference, an ellipse referred to oblique axes, and still some sort of conic when referred to a triangle of reference. (Compare in §3 the text and also Ex. 5.)

It is very difficult to draw a real *graph* of a curve referred to a triangle of reference. However, we can draw *sketches* of these curves that will help us to visualize the situation when $z = 0$ gives l_∞ instead of the side $X_\infty Y_\infty$ of the triangle of reference. Thus the equations

$$xy = z^2, \quad y^2z = x^3, \quad y^2 = 4\,pxz, \quad yz^2 = x^3$$

give us the following sketches when referred to a triangle of reference:



EXERCISES

1. Justify the sketches in the text.
2. Draw sketches for the following curves referred to a triangle of reference:

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = z^2, \quad y^2z = x^2(x \pm z), \quad y^2z = x(x - z)\ (x - \alpha z)$$

3. Show that $\rho x = x'$, $\rho y = y'$, $\rho z = \alpha x' + \beta y' - z'$ multiplies the discriminant Γ of the general conic (75) by a non-vanishing constant.
4. By means of Ex. 3 and (13) show that for a triangle of reference $\Gamma = 0$ is still the condition for (75) to be a degenerate conic.
5. Draw sketches for the following curves referred to a triangle of reference:

$$y = x^3, \quad y^2 = \frac{x^2}{x - 1}, \quad y^3 = x^4, \quad x^2y = 1$$

6. What is the equation in homogeneous coordinates of the polar of a point $P'(x',y',z')$ with respect to the general conic (4)?

CHAPTER XI

SOME THEOREMS ABOUT THE LINE AT INFINITY

81. **Intersections of curves with l_∞; asymptotes.** Curves may intersect l_∞ in many different ways, such as in two or more real and distinct points, in imaginary points, in pairs of coincident points,* in three coincident points,† in four coincident points, etc., or in several of these ways at once.

For example, $xy = z^2$ cuts l_∞ at (0,1,0) and (1,0,0); $x^2 + y^2 = z^2$ cuts l_∞ at $(1,i,0)$ and $(1,-i,0)$, where $i = \sqrt{-1}$; $y^2 = 4xz$ touches l_∞ at (1,0,0); $y^2z = x^3$ has l_∞ as an inflectional tangent at (0,1,0); $yz^2 = x^3$ has l_∞ as the tangent at a cusp (0,1,0); $y^3z = x^4$ has l_∞ as a tangent that cuts the curve in four coincident points at (0,1,0). These facts are all obtained by solving $z = 0$ simultaneously with the equations of the curves. Of course the curves are supposed to be referred to axes plus l_∞ and not to a triangle of reference.

The *asymptotes* of a curve (compare §44) we may now define as the *tangents* to this curve whose *points of contact lie on* l_∞; if l_∞ is such a tangent, we do not count l_∞ as an asymptote. If a bitangent l (see §§42, 43) has one of its points of contact on l_∞, we take the definition of a tangent as given in §46.

Asymptotes may be real or imaginary, plain or inflectional, etc. The hyperbola $xy = z^2$ has the axes as asymptotes, $x^2 - y^2 = z^2$ has $y = \pm x$ as asymptotes, the ellipse $x^2/a^2 + y^2/b^2 = z^2$ has $y = \pm ib/ax$ as asymptotes, the parabola $y^2 = 4pxz$ has no asymptotes because l_∞ touches this curve at (1,0,0).

To find the *asymptotes* of any n-ic we first put its equation in *homogeneous* form (say with x,y,z as variables), and *solve* $z = 0$ *simultaneously with it.* By this means we get the *slopes* (or *directions*) $y/x = m_1, m_2, \ldots, m_r$ of the asymptotes. Now we solve each of the equations $y = m_ix + b_iz$ $(i = 1,2, \ldots)$ simul-

* The line l_∞ is then a tangent or a line through a double point.

† The line l_∞ is then an inflectional tangent, or a tangent at a double point, or a line through a triple point.

taneously with the equation of the curve, thus getting in each case an equation in x and z (if we have eliminated y). Finally, we equate to zero the coefficient of the highest power of x that appears in this last equation, and so determine b_i that we have $y = m_i x + b_i z$ as the equation of an asymptote. (Compare §44.) From the choice of m_i we see that the equation in x and z will not have a term of the nth degree in x, because $y = m_i x$ is parallel to an asymptote.

ILLUSTRATIVE EXAMPLE. We consider the conic $x^2 - y^2 + 4x - 2y = 1$, which in homogeneous form is

$$x^2 - y^2 + 4xz - 2yz = z^2$$

Solving with $z = 0$, we get $x^2 - y^2 = 0$, so the directions of the asymptotes are $m_1 = 1$ and $m_2 = -1$. Solving $y = x + b_1 z$ simultaneously with the equation of the curve we get

$$0\,x^2 + (-2b_1 + 2)\,xz - (b_1^2 + 2b_1 + 1)\,z^2 = 0$$

hence $b_1 = 1$ gives us one asymptote $y = x + z$. Now solving $y = -x + b_2 z$ with the curve we get

$$0\,x^2 + (2b_2 + 6)\,xz - (b_2^2 + 2b_2 + 1)\,z^2 = 0$$

therefore $b_2 = -3$ gives us the other asymptote $y = -x - 3z$.

To study *infinite* points on curves we can put $x = z'$, $y = y'$, $z = x'$ and bring these points into the *finite part* of the plane. In non-homogeneous form this transformation is $x = 1/x'$, $y = y'/x'$ (a bilinear transformation). Compare §12. The line $x = 0$ goes to $z' = 0$ (i.e., to l_∞), and so $x = 0$ is a vanishing line of the transformation (compare §71); also l_∞ comes to $x' = 0$. This discussion is from the viewpoint of an alibi.

EXERCISES

1. Check the facts (and the algebra) about all the curves discussed in the text, and draw these curves.
2. Find how these curves cut l_∞:

$$y^3 = x^5, \quad y^2(x - y)^2 = x^3, \quad x^2y^3 = 1$$

3. Find the asymptotes of

$$x^2 + y^2 - 2x + 4y = 1, \quad xy - y^2 + 2x = 0, \quad x^3 - y^3 = 3xy$$

of the general conic (4); of the general circle.

4. Prove that an n-ic has at the most n asymptotes.

5. Make up a numerical example of a non-degenerate n-ic with n asymptotes, for $n = 6$ and $n = 7$.

82. **Use of l_∞ to discuss types of conics.** We consider the general equation (75) of a conic in *homogeneous coordinates*. The

line l_∞ cuts (75) in points given by the equation

$$ax^2 + by^2 + 2hxy = 0$$

According as $h^2 - ab \gtreqless 0$, the two points of intersection of (75) with l_∞ are real and distinct, real and equal, or imaginary and distinct, respectively.

If $b \neq 0$, the two points on l_∞ are $(1, (-h + \sqrt{h^2 - ab})/b, 0)$ and $(1, (-h - \sqrt{h^2 - ab})/b, 0)$.

If $b = 0$, $h \neq 0$, the two points are $(1, -a/2h, 0)$ and $(0,1,0)$.

If $b = h = 0$, the two points coincide at $(0,1,0)$. (Note how we discuss these points on l_∞ thoroughly for all possible cases, such as $b \neq 0$, then $b = 0$ and $h \neq 0$, then $b = h = 0$.)

According as $h^2 - ab \gtreqless 0$, we have in (75) a hyperbola, parabola, or ellipse if the conic is non-degenerate; but if (75) is degenerate, we have, respectively, a pair of real and distinct lines (called a *real line-pair*), a pair of real and coincident lines (called a *double line*), or a pair of conjugate imaginary lines (called a *conjugate imaginary line-pair*). Note how naturally the expression $h^2 - ab$ arises in connection with l_∞, and compare its discussion in elementary analytic geometry.

Illustrative Examples. The conic

$$3x^2 + 27y^2 + z^2 + 6yz + 8zx + 18xy = 0$$

cuts l_∞ in the points given by

$$3x^2 + 18xy + 27y^2 \equiv 3(x + 3y)^2 = 0$$

Therefore this conic is a parabola, since its discriminant is $\begin{vmatrix} 3 & 9 & 4 \\ 9 & 27 & 3 \\ 4 & 3 & 1 \end{vmatrix} \neq 0.$

EXERCISES

1. Derive the expression $h^2 - ab$ by rotations and translations, as is done in elementary analytic geometry.

2. Taking the polar of $P'(x',y',z')$ with respect to (75) and putting on it the condition that this polar reduce to $z = 0$ (i.e., that P' be the center of the conic), find the coordinates of the center of (75). Compare §51.

3. What sort of conic is $3x^2 - 2y^2 - 4z^2 + 8yz + 20zx - 14xy = 0$? Find its center and asymptotes.

83. Circles and their circular points at infinity, minimal lines, perpendicular lines. If the conic (75) is a circle, we now show

that it passes through two imaginary points on l_∞ namely $(1,i,0)$ and $(1,-i,0)$ called the *circular points at infinity*, no matter what the center and radius of this circle may be. (Compare §81.) We can put the circle in the form

$$x^2 + y^2 + 2fyz + 2gzx + cz^2 = 0 \tag{77}$$

referred to an ordinary frame of reference plus l_∞. Solving this equation with $z = 0$ we get $x^2 + y^2 = 0$, i.e., the two points $(1,i,0)$ and $(1,-i,0)$.

These circular points at infinity are of great importance in geometry. Any line (except l_∞) through one of these points is called a *minimal* line. Through every point $P'(x',y')$ in the plane there pass two minimal lines $y - y' = \pm i(x - x')$. For $y = mx + b$ to be a minimal line we must have $m = \pm i$. (Why?)

Let us consider two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ on the minimal line $y = \pm ix + b$. If we try to apply the distance formula to these two points, we get $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2 (1 + i^2)} = 0$. From this property these lines receive the name minimal lines. Conversely, to have the distance between two points on a line vanish, we must have $m^2 + 1 = 0$; i.e., this line must be a minimal line.

Again, if we try to find the line perpendicular to $y = \pm ix + b$, we see we must have its slope $m = -1/\pm i = \pm i$, so these minimal lines are called *self-perpendicular*. Conversely, for a line $y = mx + b$ to be self-perpendicular, we must have $m = -1/m$ or $m = \pm i$, so the line must be a minimal line. Of course these are *purely formal* applications of the formulas for distance and for perpendicularity to entities for which they have in fact *no meaning*.

Now we show that a *pair of perpendicular lines cut l_∞ in a pair of points that form a harmonic set with the circular points at infinity*. The lines $y = mx + bz$ and $y = -1/mx + b'z$ are perpendicular and cut l_∞ in the points $(1,m,0)$ and $(1,-1/m,0)$. Taking a cross-ratio of these two points and the circular points, we get (using non-homogeneous coordinates for points on l_∞)

$$\frac{m - i}{-1/m - i}\,\frac{-1/m + i}{m + i} = \frac{m - i}{-1 - mi}\,\frac{-1 + mi}{m + i}$$

$$= \frac{im + 1}{-(im + 1)}\,\frac{im - 1}{im - 1} = -1$$

so this is a harmonic set of points.

Conversely, we consider the two lines, $y = mx + bz$, $y = m'x + b'z$ and we suppose they cut l_∞ in points that form with the circular points a harmonic set. We have then

$$\frac{m - i}{m' - i}\frac{m' + i}{m + i} = -1$$

hence $$mm' + 1 = 0, \quad m' = -\frac{1}{m}$$

so the two lines are perpendicular to each other.

EXERCISES

1. Show that if the general conic (75) passes through one circular point, it passes through the other circular point, and the conic is a circle.
2. Using the formula $\tan\theta = (m_1 - m_2)/(1 + m_1m_2)$, find the angle between two minimal lines; also find the angle between any line $y = mx + b$ and a minimal line.
3. Find the distance between a point on a minimal line and a point not on a minimal line; also between two points on two distinct minimal lines $y = \pm ix + b$; also the distance of any point from such a line.
4. In the equation $\alpha x + \beta y + \gamma z = 0$ determine α,β,γ so that this line shall pass through $(1,i,0)$, through $(1,-i,0)$.
5. Prove that if $\tan\theta = \pm i$ for the angle between two lines, then one of these lines is a minimal line.

84. Rotations and translations and l_∞. Putting the translation (6) into homogeneous form we get

$$\rho x = x' + hz', \quad \rho y = y' + kz', \quad \rho z = z'$$

Compare Ex. 4 in §76. This formula shows that the line l_∞ goes into *itself* by a *translation* (looked upon as an *alibi*) and also *every* point $(x,y,0)$ on l_∞ goes into $(\rho x,\rho y,0)$, i.e., into itself; so l_∞ is said to be *pointwise invariant* under a translation. (Compare §33.)

Every line $y = kx/h + bz$ through the point $(1,k/h,0)$ goes into the line $y' = kx'/h + bz'$ (i.e., into itself) by a translation. We call $(1,k/h,0)$ the *center* of the translation. (Compare §97.) Also every line $y = mx + bz$ where $m \neq k/h$ goes into a parallel line $y' = mx' + (b + mh - k)z'$ by the translation. This last result we can also obtain from the fact that l_∞ is *pointwise* invariant under a translation.

On the other hand, a *rotation* (being affine) keeps l_∞ invariant as to *position*, but *not* pointwise invariant. This we see from

the fact that a point $(x,y,0)$ on l_∞ goes into $(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, 0)$ by a rotation (7). The *invariant* (or *fixed* or *double*) points on l_∞ are given by

$$\frac{y}{x} = \frac{-\sin\theta + (y/x)\cos\theta}{\cos\theta + (y/x)\sin\theta}, \quad \text{or} \quad \frac{y}{x} = \pm i,$$

i.e., these are the *circular points* $(1,i,0)$ and $(1,-i,0)$. (Compare §83.)

Conversely, suppose we want to determine the coefficients of the general affine transformation (13) so as to keep the circular points on l_∞ fixed. We put the affine transformation into the homogeneous form

$$(78) \quad \rho x = a_1x' + a_2y' + a_3z', \quad \rho y = b_1x' + b_2y' + b_3z', \quad \rho z = c_3z'$$

We want $(1,i,0)$ to go to $(1,i,0)$, so we have

$$\rho = a_1 + a_2i, \quad \rho i = b_1 + b_2i, \quad \text{or} \quad \rho = a_1 + a_2i = -b_1i + b_2$$

so

$$a_1 = b_2, a_2 = -b_1$$

Therefore we see that the *rotations* are only a *subgroup* of a subgroup G of (13), where G is characterized by the fact that all its transformations leave the circular points at infinity invariant.

Next we want to show that *the only affine transformation leaving l_∞ pointwise invariant and also sending every line through a given point $(h,k,0)$ into itself is a translation with center at $(h,k,0)$*. Since every point $(x,y,0)$ goes into itself, we must have in (78) $a_2 = b_1 = 0$, $a_1b_2 \neq 0$, also $a_1 = b_2$. Since every line $y = kx/h + bz$ goes into itself, the transformed equation

$$b_2y' + b_3z' = \frac{k}{h}(a_1x' + a_3z') + bc_3z'$$

must be of the form $y' = k/hx' + bz'$; hence from this equation we have

$$b_2 = c_3 = a_1, \quad \frac{k}{h}a_3 - b_3 = 0 \quad \text{or} \quad \frac{a_3}{b_3} = \frac{h}{k}$$

(i.e., $a_3 = ch$, $b_3 = ck$, where $c \neq 0$). But the resulting transformation

$$\sigma x = x' + ch/a_1z', \quad \sigma y = y' + ck/a_1z', \quad \sigma z = z'$$

where $\sigma = \rho/a_1$, is a translation with center at $(h,k,0)$.

One more pair of corresponding points will uniquely determine this translation (because of the presence of the single arbitrary constant c/a_1). Any linear transformation that keeps a line p pointwise invariant and sends into itself any line l through a fixed point P on p is called an *elation* with *center* P and *axis* p. (Compare §97.) A *translation* is evidently an *elation* with l_∞ as axis and $(h,k,0)$ as center.

EXERCISES

1. Check all the algebra in the text; fill in all the missing details in the discussions.

2. Determine the coefficients of (78) so that $(1,i,0)$ goes into $(1,-i,0)$ and $(1,-i,0)$ into $(1,i,0)$.

3. Prove that a translation sends a minimal line into a minimal line, and therefore sends circles into circles.

4. Why does the subgroup G derived in the text send circles into circles? Can we describe it geometrically as the only subgroup that sends circles into circles? Compare Ex. 2.

5. What do the transformations $x = ax'$, $y = by'$ do to l_∞; to other points in the plane? Hint: Consider what happens to a line $y = mx$.

6. Prove that a rotation sends circles into circles.

7. A *homology* is defined like an elation, except that the center P does *not* lie on the axis p. (Compare §97.) Show that $x = ax'$, $y = ay'$ is a homology with center at the origin and l_∞ as axis; also one more pair of corresponding points determines uniquely such a transformation. Compare §97.

8. What does a rotation do to points on the minimal lines $y = \pm ix$? Show this analytically. How about points on the other minimal lines $y = \pm ix + b$?

9. Show geometrically that the translations (the rotations around a point P) form a group by considering their effect on l_∞ (on l_∞ and P).

10. Show geometrically that the transformations $x = ax'$, $y = by'$ form a group by considering their effect on l_∞ and the origin. See Ex. 5.

11. Determine c/a_1 so that the translation in the last paragraph of the text (with $h = 2$, $k = 3$) sends $(1,1,1)$ into $(2,1,4)$.

12. Determine a and b in $x = ax'$, $y = by'$ so that $(1,1)$ goes into $(-1,-1)$; so that $(1,0)$ goes to $(-2,0)$ and $(0,3)$ to $(0,4)$. Note that this last condition is equivalent to sending $(1,3)$ to $(-2,4)$; show geometrically the reason for this fact.

13. An *orthogonal line reflection* is defined as a harmonic* homology whose center is on l_∞, for example, $x = x'$, $y = -y'$, $z = z$. Prove geometrically that a rotation is a product of two orthogonal line reflections.

85. The general affine linear transformations and l_∞. Let us consider briefly the effect on l_∞ of the general affine linear transformation (78) in §84. This transformation sends any point

* See §97.

$P(x,y,0)$ into

$$P'\left(\frac{b_2x - a_2y}{a_1b_2 - a_2b_1}, \frac{a_1y - b_1x}{a_1b_2 - a_2b_1}, 0\right)$$

To find the *fixed* (*double* or *invariant*) points on l_∞ of this transformation, we solve the equation

$$\frac{y}{x} = \frac{a_1y/x - b_1}{b_2 - a_2y/x}$$

getting

$$\frac{y}{x} = \frac{-(a_1 - b_2) \pm \sqrt{(a_1 - b_2)^2 + 4\,a_2b_1}}{2\,a_2}$$

if $a_2 \neq 0$, or $y/x = b_1/(a_1 - b_2)$ and $y/x = \infty$ if $a_2 = 0$.

According as $(a_1 - b_2)^2 + 4\,a_2b_1 \gtreqless 0$, (78) has two real and distinct double points on l_∞, two real and equal, or two distinct and imaginary. From analogy with the way the conics cut l_∞, we might call the corresponding affine transformations *hyperbolic*, *parabolic*, and *elliptic* respectively.

For a translation $a_1 = b_1 = 1$, $a_2 = b_2 = 0$, so the equation for double points on l_∞ becomes $0\,y^2/x^2 + 0\,y/x - 0 = 0$, which is satisfied by every value of y/x and indicates that every point on l_∞ is invariant under the translation.

Note that we may look upon the transformation

$$\rho x = a_1x' + a_2y', \quad \rho y = b_1x' + b_2y' \tag{79}$$

obtained by putting $z = z' = 0$ in (78), as a transformation among the points of l_∞ induced by the transformation (78). Compare §§102, 73. This transformation of the line l_∞ may be studied for its own sake, quite apart from its connection with (78). In the three cases mentioned above, this transformation on l_∞ is called hyperbolic, parabolic, or elliptic.

ILLUSTRATIVE EXAMPLE. We note that

$$\rho x = 2\,x' - y' + 3\,z', \quad \rho y = x' + 2\,y' - z', \quad \rho z = 3\,z'$$

has $(a_1 - b_2)^2 + 4\,a_2b_1 = (2 - 2)^2 - 4 < 0$ and so is elliptic. Also this transformation induces on l_∞ the transformation

$$\rho x = 2\,x' - y', \quad \rho y = x' + 2\,y' \quad \text{or (with } w = y/x) \quad w = \frac{1 + 2\,w'}{2 - w'}$$

From the bilinear form of this transformation on l_∞ we see that the double points on l_∞ are $w = \pm i$. Compare §§103, 73 (Ex. 1).

EXERCISES

1. Fill in all the algebraic details in the text.
2. By the method of the text find the double points on l_∞ of a rotation.
3. Find the effect on l_∞ of $x = -ax' + by' + c$, $y = bx' + ay' + d$.
4. Make up two examples of affine transformations with the following double points on l_∞: (*a*) (1,1,0) and (1,−1,0); (*b*) (0,1,0) and (1,0,0).
5. Make up examples of hyperbolic, parabolic, and elliptic affine transformations.

86. **The real plane as part of a complex plane; conjugate imaginary points and lines.** In previous sections we came upon imaginary points and lines, e.g., the circular points at infinity, the minimal lines, the imaginary asymptotes of ellipses, etc. Compare especially §37.

We shall now show that the real plane can be looked upon as *part* of a so-called *complex plane*. First of all we note that the *real* numbers can be considered as *part* of the *complex* numbers of the form $a + ib$, where a and b are any real numbers, in the following way:

If $b = 0$, then $a + ib$ is a real number.
If $a = 0$, then $a + ib$ is a pure imaginary number.
If $ab \neq 0$, then $a + ib$ is a complex imaginary number.

Now consider the point $P(a + ib,\ c + id)$, or $P(a + ib,\ c + id,\ e + if)$ in homogeneous coordinates. If $b = d = 0$ (or $b = d = f = 0$), then P is a real point; otherwise it is an imaginary point, except when its homogeneous coordinates have a form like $(1 + i,\ 2 + 2i,\ 3 + 3i)$, which can be changed to (1,2,3) through division by $1 + i$.

Consider next the line

$$(a + bi)x + (c + di)y + (e + fi)z = 0 \tag{80}$$

If $b = d = f = 0$ or $a + bi = \alpha(c + di) = \beta(e + fi)$, then we have what we call a real line. Otherwise (80) is an imaginary line. Such a line as $(1 - i)x + (2 - 2i)y + (5 - 5i)z = 0$ is real, because upon dividing its equation by $1 - i$ we get $x + 2y + 5z = 0$.

Ordinarily we consider only *part* of a real line, namely, the real points on this line. But every real line has imaginary points on it. (Compare §36.) Thus $x = 0$ has on it all the imaginary points $(0,\ c + id,\ e + if)$ as well as the real points $(0,c,e)$. Simi-

larly through (0,0,1) pass the real lines $y = mx$ and the imaginary lines $y = (\lambda + i\mu)x$.

Again, if we allow the conic (75) to have imaginary coefficients and the variables x,y,z to take on imaginary values, we have the conics in this complex plane. Similarly, we can describe the complex cubics and the other complex n-ics and other curves.

We describe the complex plane that contains the real plane as consisting of all the points, lines, conics, cubics, n-ics, etc., that are noted in the preceding paragraphs. Of special interest to us are the imaginary points on any real lines or curves that we are studying, and the imaginary lines through any real points or tangent to any real curves we are considering. Take, for example, the circular points on l_∞ and the imaginary asymptotes of an ellipse.

(The discussions of points and of lines are plane dual to each other. Compare §22. Therefore in the next few sentences we enclose in parentheses the words that refer only to lines.)

Among these imaginary points (lines) on real lines (through real points) we find pairs of points (lines) that are called *conjugate imaginary* points (lines) because their coordinates (equations) are conjugate imaginary numbers (have conjugate imaginary coefficients); e.g.,

$$(a + ib,\ c + id,\ e + if) \quad \text{and} \quad (a - ib,\ c - id,\ e - if);$$

$$(a + ib)x + (c + id)y + (e + if)z = 0$$

and

$$(a - ib)x + (c - id)y + (e - if)z = 0$$

If we multiply together the equations of these two conjugate imaginary lines we get

$$(ax + cy + ez)^2 + (bx + dy + fz)^2 = 0$$

which is a real quadratic expression; hence we can look upon a pair of conjugate imaginary lines as being a *degenerate* case of the real conic (75).

Note that, if $b = 0$ in $a + ib$, then $a - ib$ is the same as $a + ib$, namely, a. Thus we see that $(2,3,4 + i)$ and $(2,3,4 - i)$ are conjugate imaginary points; also $x - y + (3 + 2i)z = 0$ and $x - y + (3 - 2i)z = 0$ are conjugate imaginary lines.

In projective geometry we say that a line cuts an n-ic in n points that may be finite or infinite, real or imaginary, distinct or coincident, or a combination of these cases. Thus l_∞ cuts every circle

in the same two imaginary (circular) points. Of course we cannot plot these imaginary points and lines. Compare §36.

EXERCISES

1. Prove that if the product of the equations of two imaginary lines l,l' (in homogeneous coordinates) is a real quadratic equation, then l and l' are conjugate imaginary lines.

2. State and solve the problem concerning imaginary points that is the plane dual of Ex. 1.

3. Prove that the imaginary line (80) always has one real point on it.

4. State and prove the plane dual of Ex. 3.

5. Find the equation of the line through the two points $(i,2,1)$ and $(3+i, 2-i, 1+2i)$. Hint: Use (76).

6. Find in two different ways the equation of the conic through the five points $(i,0,1)$, $(-i,0,1)$, $(1,i,1)$, $(1,1,i)$, $(1+i,2i,1)$.

7. Find the sides (also the vertices and sides of the diagonal triangle) of the complete quadrangle (a) with vertices $(i,0,1)$, $(-i,0,1)$, $(3+2i,1,0)$, $(3-2i,1,0)$; (b) with vertices $(i,0,1)$, $(i,1,0)$, $(0,i,1)$, $(1,i+1,1)$. See §67.

8. Solve the plane dual of Ex. 7 where the numbers in the parentheses are now the coefficients in the equations of lines; thus one line is $ix+0y+1z=0$.

CHAPTER XII

INTRODUCTION TO LINE COORDINATES AND PLANE DUALITY

87. Homogeneous and non-homogeneous line coordinates. Let us consider the equation of the straight line in homogeneous coordinates

$$u_1x_1 + u_2x_2 + u_3x_3 = 0 \tag{81'}$$

or

$$ux + vy + wz = 0 \tag{81}$$

with every term put on the left-hand side of the equation.

The coefficients u_1,u_2,u_3 (or u,v,w) taken in the proper order determine uniquely the line (81′) or (81) just as the coordinates x_1,x_2,x_3 (or x,y,z) taken in the proper order determine uniquely a point. (Compare §§21, 22.) Thus the line $3x + 2y + 4z = 0$ is uniquely determined by its coefficients 3,2,4 given in this order.

Since we can multiply the equation of a line through by any constant $k \neq 0$, we can use $-3,-2,-4$ or $\frac{3}{2},1,2$, or 6,4,8, etc., to give the same line. We shall call the coefficients of (81′) or (81) the *homogeneous line coordinates* of the line, and we write them inside brackets as $[u_1,u_2,u_3]$ or $[u,v,w]$.

Note that the equation $ux + vy + wz = 0$ when u,v,w are considered as constants gives us the condition that any point $P(x,y,z)$ shall lie on the given line $l[u,v,w]$. This same equation, when x,y,z are considered as constants, gives us the condition that any line $l[u,v,w]$ shall pass through a given point $P(x,y,z)$. Hence the above equation (81) is either *that of a line in point coordinates* (*if u,v,w are constants*) *or the equation of a point in line coordinates* (*if x,y,z are constants*).

For example, the equation $2u + 3v + 4w = 0$ is satisfied by every line $l[u,v,w]$ that passes through the point $P(2,3,4)$ as we see by substituting 2,3,4 for x,y,z in (81); therefore this is the equation in line coordinates of the point $P(2,3,4)$. The equation

of the point of intersection of two lines $l_1[u_1,v_1,w_1]$ and $l_2[u_2,v_2,w_2]$ is

$$\begin{vmatrix} u & v & w \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = 0 \tag{82}$$

because this is a first-degree equation in u,v,w and is satisfied by the line coordinates of l_1 and l_2. The point of intersection of $2x - 3y + 4z = 0$ and $x + y - 2z = 0$ is

$$\begin{vmatrix} u & v & w \\ 2 & -3 & 4 \\ 1 & 1 & -2 \end{vmatrix} \equiv 2u + 8v + 5w = 0$$

or in point coordinates (2,8,5). This makes an easy way to find such a point of intersection of two lines.

Similarly, any higher degree equation in u_1,u_2,u_3 (or u,v,w) gives us a so-called *line locus*, i.e., a family of lines whose line coordinates satisfy this equation; just as any equation in x_1,x_2,x_3 (or x,y,z) gives us a *point locus*, i.e., a family of points whose point coordinates satisfy this equation. Thus the line locus $uv = w^2$ has on it such lines as $[1,1,1]$ or $x + y + z = 0$, $[-1,-1,1]$ or $-x - y + z = 0$, $[2,\frac{1}{2},1]$ or $2x + \frac{1}{2}y + z = 0$. To find lines of this locus we can give w and v arbitrary values and determine u by the equation $u = w^2/v$.

To find the common lines of two line loci we solve their equations simultaneously. For example, if we solve simultaneously the equations $u^2 + v^2 = w^2$ and $2uv = w^2$ we find these two loci have in common the lines

$$[\pm\sqrt{2},\pm\sqrt{2},2] \quad \text{or} \quad \pm\sqrt{2}\,x \pm \sqrt{2}\,y + 2z = 0$$

Notice how all this discussion is just the plane dual of the discussion of point loci. (Compare §§22, 86.)

The most general equation for a second-degree line locus is

$$Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0 \tag{83}$$

If $\Lambda = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$, we see at once, by a comparison with the discriminant Γ of (4) or (75), that (83) is factorable (and then is called degenerate). *Also* (83) *has only five essential constants, so in general five lines uniquely determine this locus.* Com-

pare §42. To find the locus (83) that contains the five lines [0,0,1], [0,1,0], [1,0,0], [1,1,1], [2,−1,1], we substitute these line coordinates in (83) and get the equations

$$C = 0,\ B = 0,\ A = 0,\ F + G + H = 0,\ -F + 2G - 2H = 0$$

Hence the desired locus is $-4\,vw + wu + 3\,uv = 0$.

We shall also set up what are called *non-homogeneous line coordinates* for any line *not* through the *origin*. Compare the exceptional character here of lines through the origin with the fact that we must except the points on l_∞ (or on the side $X_\infty Y_\infty$ of a triangle of reference) when we set up non-homogeneous point coordinates. Such a line (not through the origin) can be written $-x/a - y/b + 1 = 0$, and we take $u = -1/a$, $v = -1/b$ as non-homogeneous coordinates for this line.

Such non-homogeneous line coordinates are useful in drawing any line (just as non-homogeneous point coordinates are useful in plotting a point). Thus the line [5,3,4] in non-homogeneous coordinates is $[\frac{5}{4},\frac{3}{4}]$, and its intercepts are $a = -\frac{4}{5}$, $b = -\frac{4}{3}$. More generally the line $[u,v,w]$ is $[u/w,v/w]$, and its intercepts are $a = -w/u$, $b = -w/v$. Note the resemblance to changing from homogeneous to non-homogeneous coordinates for finite points in the plane (or, if a triangle of reference is used, for points not on $X_\infty Y_\infty$).

EXERCISES

1. Check all the algebra in the text.

2. Use another way to find the locus $-4\,vw + wu + 3\,uv = 0$ from that given in the text. Hint: Compare §42.

3. How does comparison with the discriminant of (4) show that (83) is degenerate if $\Lambda = 0$? Hint: Replace a by A, x by u, b by B, y by v, etc., in (75), then the derivation of the discriminant of (4) is valid here if we use the method that consists of solving (4) for y in terms of x (given in §16). Why?

4. Find in two ways the line locus (83) having on it the lines $[i,0,1]$, $[-i,0,1]$, $[0,i,1]$, $[0,-i,1]$, $[1,i,1]$. Hint: Dualize the two methods used in finding the equation of a conic through five given points.

5. Find the vertices (also the vertices and sides of the diagonal triangle) of the complete quadrilateral: (*a*) with sides [1,0,1], [0,1,1], [1,1,0], [1,1,1]; (*b*) with sides $[i,1,0]$, $[-i,1,0]$, $[1,i,1]$, $[1,-i,1]$.

6. Draw the line loci (in homogeneous coordinates) $uw = w^2$, $u^2 - v^2 = w^2$; (in non-homogeneous coordinates) $u = v^3$, $u = v^4$, $v^2 = u$.

7. Solve simultaneously and interpret your solution geometrically:

$$3\,u + 2\,v + 4\,w = 0,\quad u - v + w = 0;\quad \text{and}\quad u^2 + v^2 = w^2,\quad u = 3\,w$$

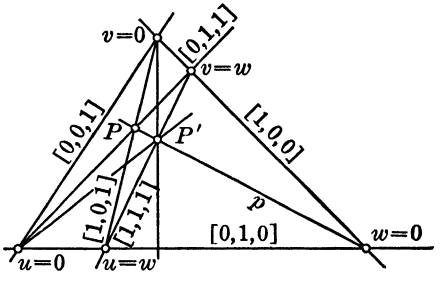
8. Interpret geometrically $H^2 - AB \gtreqless 0$ for (83). Compare $h^2 - ab \gtreqless 0$ for (4).

9. Find the lines through (0,0,1) that belong to the loci $uv = w^2$, $u^2 - v^2 = w^2$.

88. **Triangle of reference for line coordinates; axes and line coordinates.** We might now, in a manner dual to the way we proceeded with axes (or a triangle of reference) for point coordinates, set up what we call *centers* (or a *triangle of reference*) for *line coordinates.* Instead of the axes $x = 0$, $y = 0$ we should have as two centers the points $u = 0$, $v = 0$ (for non-homogeneous line coordinates).

To set up a triangle of reference for homogeneous line coordinates we can take any three non-collinear points as vertices and give them, respectively, the equations $u = 0$, $v = 0$, $w = 0$. Then we can take any line through $u = 0$ (not a side of the triangle of reference) and call it [0,1,1], also a second such line through $v = 0$ and call it [1,0,1]; or we can take any line not through a vertex of the triangle of reference and call it [1,1,1]. Note how this is dual to our discussion of unit points in §70.

The line [1,1,1] cuts the side [1,0,0] of the triangle (see the adjoining figure) in the point $v = w$, and the side [0,1,0] in the point $u = w$. The line joining $u = 0$ to $v = w$ is [0,1,1], and the line joining $v = 0$ to $u = w$ is [1,0,1]. To find the lines [0,2,1] and [2,0,1] we dualize the construction given in §§64, 65 for obtaining the points (0,2) and (2,0); namely, we join the point of intersection P of the lines [0,1,1] and [1,0,1] to the point $w = 0$ by the line p. (Dually we found in §70 the point of intersection P_∞ of the line joining (0,1) to (1,0) and the line $X_\infty Y_\infty$ or $z = 0$.) Then the lines joining P' (the point of intersection of p and [1,1,1]) to $v = 0$ and $u = 0$ are the lines [2,0,1] and [0,2,1], respectively. (Dually we joined the point (1,1) to P_∞ and labeled with (2,0) and (0,2), respectively, the points of intersection of this line with $y = 0$ and $x = 0$.) In a similar manner we could dualize



all the constructions in §70 and get the lines [3,0,1], [−1,0,1], [$\frac{6}{5}$,0,1], etc.

It is more useful to have the *same* triangle of reference for both point and line coordinates. In this case we see that the point (0,0,1) is $w = 0$, (1,0,0) is $u = 0$, (0,1,0) is $v = 0$. The line [1,1,1] is $x + y + z = 0$, which cuts [0,1,0] or $y = 0$ in the point (−1,0,1) or $u = w$ and cuts [1,0,0] or $x = 0$ in the point (0,−1,1) or $v = w$. We see that in the non-homogeneous system of line coordinates the point (0,0,1) or $w = 0$ plays a role similar to that of the line [0,0,1] or $z = 0$ in non-homogeneous point coordinates.

Note that the last paragraph applies without the change of a word — except the words "triangle of reference" to the words "axes plus l_∞" — to the case where the plane is referred to axes plus l_∞. In this case $w = 0$ is the origin, $u = 0$ is the intersection of the x-axis with l_∞, $v = 0$ is the intersection of the y-axis with l_∞.

EXERCISES

1. Complete the discussion of a triangle of reference for non-homogeneous line coordinates (dualizing the discussion for point coordinates in §70) by locating the lines [3,0], [0,3], [−1,0], [0,−1], [$\frac{6}{5}$,0].
2. Do the same as in Ex. 1, only for axes plus l_∞.

89. **Cross-ratios of non-homogeneous line coordinates.** The logical thing to consider next is what we mean by the cross-ratios of four concurrent lines. Compare §23. Suppose the four lines (in non-homogeneous line coordinates) are $[u_1,v_1]$, $[u_2,v_2]$, $[u_3,v_3]$, $[u_4,v_4]$ and they intersect in the point $v = mu + b$. We define one of the cross-ratios of these four lines by the fraction

$$\frac{u_1 - u_2}{u_3 - u_2}\frac{u_3 - u_4}{u_1 - u_4} \equiv \frac{v_1 - v_2}{v_3 - v_2}\frac{v_3 - v_4}{v_1 - v_4} \tag{84}$$

By the dual of the argument given in §23 we can show that (84) is invariant under transformations of the form

$$u = \alpha_1 u' + \alpha_2 v' + \alpha_3, \quad v = \beta_1 u' + \beta_2 v' + \beta_3 \tag{85}$$

where $\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \neq 0$, which leaves $w = 0$ invariant as to position (and so are the duals of affine transformations, which leave $z = 0$ invariant as to position). If we submit (84) to the transformation

(85), we get

$$\frac{\alpha_1(u_1' - u_2') + \alpha_2(v_1' - v_2')}{\alpha_1(u_3' - u_2') + \alpha_2(v_3' - v_2')} \frac{\alpha_1(u_3' - u_4') + \alpha_2(v_3' - v_4')}{\alpha_1(u_1' - u_4') + \alpha_2(v_1' - v_4')}$$

But $v_i' = m'u_i' + b'$, where $i = 1,2,3,4$ and where $v' = m'u' + b'$ is the point into which (85) sends $v = mu + b$. Substituting these values for the v's into the above fraction, we get

$$\frac{(\alpha_1 + \alpha_2 m')}{(\alpha_1 + \alpha_2 m')} \frac{(u_1' - u_2')}{(u_3' - u_2')} \frac{(\alpha_1 + \alpha_2 m')}{(\alpha_1 + \alpha_2 m')} \frac{(u_3' - u_4')}{(u_1' - u_4')}$$

which shows that (84) is invariant under (85).

Next we want to show that the four points P_1, P_2, P_3, P_4 in which the above four lines cut any other line $[u_5,v_5]$ have the same cross-ratios as the above-defined cross-ratios of the four given lines. These four points are

$$\begin{vmatrix} u & v & 1 \\ u_i & v_i & 1 \\ u_5 & v_5 & 1 \end{vmatrix} \equiv (v_i - v_5)u + (u_5 - u_i)v + (u_iv_5 - u_5v_i) = 0$$

where $i = 1,2,3,4$ or P_i $((v_i - v_5)/(u_iv_5 - u_5v_i),\ (u_5 - u_i)/(u_iv_5 - u_5v_i))$. Taking the corresponding cross-ratio $(x_1 - x_2)/(x_3 - x_2) \cdot (x_3 - x_4)/(x_1 - x_4)$ we get

$$\frac{\dfrac{v_1 - v_5}{u_1v_5 - u_5v_1} - \dfrac{v_2 - v_5}{u_2v_5 - u_5v_2}}{\dfrac{v_3 - v_5}{u_3v_5 - u_5v_3} - \dfrac{v_2 - v_5}{u_2v_5 - u_5v_2}} \cdot \frac{\dfrac{v_3 - v_5}{u_3v_5 - u_5v_3} - \dfrac{v_4 - v_5}{u_4v_5 - u_5v_4}}{\dfrac{v_1 - v_5}{u_1v_5 - u_5v_1} - \dfrac{v_4 - v_5}{u_4v_5 - u_5v_4}}$$

If we simplify this complicated fraction, then substitute $v_i = mu_i + b$ where $i = 1,2,3,4$ we find that the resulting expression is $(u_1 - u_2)/(u_3 - u_2) \cdot (u_3 - u_4)/(u_1 - u_4)$. Compare §27.

EXERCISES

1. Prove that $\begin{vmatrix} u & v & 1 \\ u_i & v_i & 1 \\ u_5 & v_5 & 1 \end{vmatrix} = 0$ gives the equation of the point of intersection of the two lines $[u_i, v_i]$ and $[u_5, v_5]$. Compare §87.

2. Check all the algebraic work in the text.

3. Use the method of the text to prove that (13) leaves the cross-ratios of points invariant.

4. Show that if (13) leaves (0,0,1) invariant, this transformation causes on the line $ux + vy + wz = 0$ a transformation (24) that can be put in the form (85).

5. Find a cross-ratio of the four lines $y = m_i x + b$, where $i = 1,2,3,4$. Find the four points in which these lines cut any line $x/\alpha + y/\beta - 1 = 0$, and the corresponding cross-ratio of these four points.

6. Show that $x = 0$, $y = 0$, $y = mx$, $y = -mx$ form a harmonic set of lines. (Compare §27.) Find a cross-ratio of the four points in which these lines cut l_∞.

7. Make up numerical examples to illustrate the text.

90. **Points of contact of line loci; line coordinates for tangents to curves.** An ordinary tangent to a curve (i.e., point locus) at a point P' can be defined as the limiting position of a secant through two points P' and P'' on the curve as $P'' \to P'$ along the curve. Compare §46. Similarly (and dually) we define:

DEFINITION. The *point of contact* of a line l' of a line locus is the limiting position of the point of intersection of two lines l' and l'' of this line locus as $l'' \to l'$ along this locus (i.e., through the set of lines that make up the line locus).

The locus of these points of contact is in reality a point locus, and the lines of the line locus are the tangents to the point locus (since the line l joining the points of contact P' and P'' of the two lines l' and l'' described above approaches the same limiting position l' as $l'' \to l'$, and so $P'' \to P'$; also l is a secant of the point locus formed by the points of contact of the lines of the line locus). We can therefore describe a point P' of a curve as the limiting position of the point of intersection of two tangents t' and t'' to this curve as $t'' \to t'$.

We shall now dualize from §46 the derivation of the equation of a tangent to the curve $f(x,y) = 0$ and obtain the equation (in non-homogeneous line coordinates) of a point of contact of the line locus $\phi(u,v) = 0$. Consider any two lines $[u',v']$ and $[u' + \Delta u, v' + \Delta v]$ of this locus. The point of intersection of these two lines is

$$\frac{v - v'}{v' - (v' + \Delta v)} = \frac{u - u'}{u' - (u' + \Delta u)} \quad \text{or} \quad v - v' = \frac{\Delta v}{\Delta u}(u - u')$$

since this is a first-degree equation that is plainly satisfied by the coordinates of the two lines. If we let $\Delta u \to 0$ (so that $\Delta v \to 0$), we get the equation of the point of contact of $[u',v']$, namely,

$$v - v' = \left.\frac{dv}{du}\right]_{\substack{u=u' \\ v=v'}} (u - u') \tag{86}$$

EXERCISES

1. Find the points of contact of [1,1] on the curves $v^2 = u^3$, $vu = 1$.
2. Find the two points of contact of [1,1] for each of the two curves $u^2 + v^2 = 1$ and $uv = -2$. Hint: Dualize the method of finding the tangents to a given conic from a given point not on this conic.
3. Dualize and interpret geometrically the discussion in §§47, 48 of points of inflection and multiple points.
4. Do the same as in Ex. 3 for asymptotes to curves; also for poles and polars with respect to conics.

91. **The conics and other n-ics in line coordinates; the class of an n-ic.** Given the equation of an n-ic in point coordinates it is possible (though often very difficult) to find the equation in line coordinates of the line locus composed of the tangents to this point locus. We call this latter equation the *equation* of the given curve in *line coordinates.*

The degree n of the equation of the above-mentioned curve in point coordinates is called the degree of the curve. The degree m of the equation of this curve in line coordinates is called the *class* of the curve.

We showed in §43 that the degree of an n-ic is the same as the number of points in which any straight line l cuts the curve, counting in imaginary points, counting two coincident points twice, etc. The *class* m of a curve will now be shown to be the same as the *number of tangents to the curve from any general point* P *not on the curve.* This we see by taking P as $w = 0$, then solving $w = 0$ simultaneously with the mth degree equation of the curve in homogeneous line coordinates. We get thereby an mth degree homogeneous equation in u and v to determine the tangents passing through the point $w = 0$.

The above choice in the proof of $w = 0$ for the general point P merely amounts to a change of the triangle (or axes) of reference and so does not harm the generality of the proof, because linear transformations of coordinates cannot alter the number of tangents to a curve through a given point. As an illustration of this theorem we see that from $w = 0$ there are three tangents to the curve $(u^2 - v^2)(u + 2v) = 3uvw$, namely, $[1,1,0]$, $[1,-1,0]$, and $[1,-\frac{1}{2},0]$. We shall prove later on (in §124) that the class m of an n-ic usually is given by the equation $m = n(n-1) - 2\delta - 3\kappa$ where δ is the number of nodes on the n-ic and κ the number of cusps.

If we want to *determine* the equation of an n-ic in line coordinates, we can proceed as follows. Solving $ux + vy + wz = 0$ for z, we get $z = -(ux + vy)/w$. Substituting this value of z in the equation of the n-ic, $f(x,y,z) = 0$, we get an equation in y/x, namely, $\phi(y/x) = 0$ This equation $\phi(y/x) = 0$ gives us the slopes of the lines from (0,0,1) to the points of intersection of the n-ic with $ux + vy + wz = 0$. If this last line is tangent to the curve, then at least two of the lines from (0,0,1) must coincide; hence $\phi(y/x) = 0$ must have at least a double root in y/x.

If we have a general equation of condition for $\phi(y/x) = 0$ to have a multiple root, we can use this condition to get an equation in u,v,w as variables, i.e., the equation of the n-ic in line coordinates. Otherwise we can eliminate y/x between $\phi(y/x) = 0$ and $\phi'(y/x) = 0$ (where ϕ' means the derivative of ϕ with regard to y/x) and thus obtain the equation of the n-ic in line coordinates.

As examples of the above method of getting the equation of an n-ic in line coordinates, let us consider the following curves. In the circle $x^2 + y^2 = r^2z^2$ we put $z = -(ux + vy)/w$ and get

$$x^2(w^2 - r^2u^2) + y^2(w^2 - r^2v^2) - 2r^2uvxy = 0$$

The condition that this quadratic equation have a double root is

$$r^4u^2v^2 - (w^2 - r^2u^2)(w^2 - r^2v^2) \equiv w^2(-w^2 + r^2u^2 + r^2v^2) = 0$$

which is the desired equation in line coordinates. Here $w = 0$ is the equation of (0,0,1) and gives us the two asymptotes of the circle. We note that we are told by the rest of the last equation that any line

$$ux + vy \pm r\sqrt{u^2 + v^2}\, z = 0$$

where u and v are arbitrary, is a tangent to the circle $x^2 + y^2 = r^2z^2$.

In the cubic $y^2z = x^3$ we put $z = -(ux + vy)/w$ and get

$$wx^3 + uxy^2 + vy^3 = 0 \quad \text{or} \quad \frac{wx^3}{y^3} + \frac{ux}{y} + v = 0$$

From the condition (45) for a cubic in one variable x/y to have a double root, we obtain (since here $a_0 = w$, $a_1 = 0$, $a_2 = u/3$, $a_3 = v$) the equation

$$w^2v^2 + \frac{4wu^3}{27} = 0 \quad \text{or} \quad w(27v^2w + 4u^3) = 0$$

as the equation of this cubic in line coordinates. Note that such a line as $x + y - \frac{4}{27}z = 0$, i.e., $[1,1,-\frac{4}{27}]$ is a tangent to this cubic.

Another way to solve this problem of the cubic is to take the derivative with respect to x/y of

$$\frac{wx^3}{y^2} + \frac{ux}{y} + v = 0, \quad \text{which is} \quad \frac{3\,wx^2}{y^2} + u = 0$$

We have now

$$\frac{x}{y}\left(\frac{wx^2}{y^2} + u\right) = -v \quad \text{or} \quad \pm\sqrt{\frac{-u}{3\,w}}\left(w \cdot \frac{-u}{3\,w} + u\right) = -v$$

hence the equation of the cubic in line coordinates is $4\,u^3/27\,w = -v^2$ (which is obtained by squaring the last equation).

Next we consider the quartic $y^3z = x^4$. We get

$$wx^4 + uxy^3 + vy^4 = 0 \quad \text{or} \quad \frac{wx^4}{y^4} + \frac{ux}{y} + v = 0$$

Using the condition (47) for a quartic equation in one variable to have a double root, we get (since $a_0 = w, a_1 = a_2 = 0, a_3 = u/4$, $a_4 = v$) the equation

$$(w \cdot v)^3 = 27 \begin{vmatrix} w & 0 & 0 \\ 0 & 0 & u/4 \\ 0 & u/4 & v \end{vmatrix}^2 \quad \text{or} \quad w^3v^3 = \frac{27 \cdot u^4w^2}{256}$$

as the so-called line equation of our quartic curve. By the other method we have $4\,wx^3/y^3 + u = 0$. So $x/y(wx^3/y^3 + u) + v = 0$ gives us $\sqrt[3]{-u/4\,w}(w \cdot -u/4\,w + u) = -v$; hence (cubing this last equation) we have again the equation $27u^4/256 = v^3w$.

Finally we shall derive the equation of the *general conic* (75) in *line coordinates*. We do this by considering the condition that a line shall be tangent to this conic. We can write the equation (5), when put in homogeneous coordinates, of the tangent to this conic at a point $P'(x',y',z')$ as $ux + vy + wz = 0$. Hence, since also P' must lie on this tangent, we have

$$-\rho u + ax' + hy' + gz' = 0, \quad -\rho v + hx' + by' + fz' = 0,$$
$$-\rho w + gx' + fy' + cz' = 0, \quad 0 \cdot \rho + ux' + vy' + wz' = 0$$

These equations may be looked upon as four linear homogeneous equations in the four unknowns ρ, x', y', z'. In order that these

equations may have a solution not all zero, we must have

$$(87) \qquad \begin{vmatrix} -u & a & h & g \\ -v & h & b & f \\ -w & g & f & c \\ 0 & u & v & w \end{vmatrix} = 0$$

If we expand this determinant we get the equation of (75) in line coordinates, namely (83) where $A,B,C, \ldots$ are the cofactors of $a,b,c, \ldots$ in the discriminant Γ of (75).

Note that such a conic as $x^2 = 0$ has no equation in line coordinates, whereas $u^2 = 0$ has no equation in point coordinates. As an illustration the conic $x^2 + y^2 - r^2z^2 = 0$ has the line equation

$$\begin{vmatrix} -u & 1 & 0 & 0 \\ -v & 0 & 1 & 0 \\ -w & 0 & 0 & -r^2 \\ 0 & u & v & w \end{vmatrix} \equiv -u \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & -r^2 \\ u & v & w \end{vmatrix} + v \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -r^2 \\ u & v & w \end{vmatrix}$$

$$-w \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & w \end{vmatrix} \equiv r^2u^2 + r^2v^2 - w^2 = 0$$

EXERCISES

1. Check all the algebra in the text, filling in missing details.

2. Prove that the general cubic curve (38) ordinarily is of class six. Hint: Substitute $z = -(ux + vy)/w$, etc.

3. Prove that the general *quartic* curve ordinarily is of class *twelve*.

4. Find the equation in point coordinates of the line conic (83) by dualizing the discussion in the text. Hint: The coefficients $a,b,c, \ldots$ are proportional to the cofactors of $A,B,C, \ldots$ in the discriminant Λ of (83). Prove this fact, or look up its proof in Bôcher's "Higher Algebra."

5. Find in three ways the equations in line coordinates of:

$$y^2 = 4px, \quad x^2 + y^2 + 2gx + 2fy + c = 0, \quad \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1, \quad xy = 1$$

Hint: Change to homogeneous point coordinates.

6. Find in two ways the equations in line coordinates of:

$$y = x^3, \quad y^2 = x^2(x \pm 1), \quad y^2 = x(x - 1)\ (x - \alpha), \quad y = x^4, \ x^3y = 1$$

7. Find the equations in point coordinates of:

$$v^2 = 4pu, \quad u^2 + v^2 + 2gu + 2fv + c = 0, \quad \frac{u^2}{a^2} \pm \frac{v^2}{b^2} = 1, \quad uv = 1$$

8. Find the equations in point coordinates of:

$$u^2v = 1, \quad v = \frac{u}{(u+1)}, \quad u^3 + v^3 = 3uv$$

9. Give the classes of all the curves in the text and in the above examples.

10. Prove that if $\phi(y/x) = 0$ is to have a double root in y/x, then $\phi(y/x)$ and $\phi'(y/x)$ must have a common factor.

92. **Plane duality.** In §22 we defined duality in a plane (called also plane duality). Since then we have kept pointing out duality as it arose and using the principle of plane duality in our discussions. Now we shall prove that plane duality is not only a characteristic of geometrical definitions and theorems in plane geometry, but (when l_∞ is included in the plane) it gives us also a *method of proof of many theorems.* That is, we shall show:

THEOREM. *If we interchange the words point and line (collinear and concurrent, etc.) in any projective theorem* in plane analytic geometry, where the line at infinity is included in the plane, we shall have a new theorem whose validity follows from that of its dual theorem and therefore requires no further proof.*

For all the results that are obtained by purely analytic means, the above theorem follows at once from the fact that, in any of the equations or formulas we use in plane analytic geometry, the homogeneous variables x,y,z may be replaced by u,v,w; the algebraic manipulations will still be valid; and the analytical results will be theorems or expressions in line coordinates concerning line loci, where the dual theorems and expressions were in point coordinates and had to do with point loci.

The line at infinity must be taken as part of the plane in order to use duality because, whereas any two distinct points determine a line, any two distinct lines determine (i.e., intersect in) a point if, and only if, we assume that two parallel lines intersect on l_∞.

If x,y,z and u,v,w are referred to a triangle of reference, we cannot use duality in our analytic work until we have shown that the duals of the purely geometric discussions of §§66, 67, 68, 69 (that were at the very foundation of the discussion of a triangle of

* Roughly speaking, such a theorem is one that does not involve lengths of line-segments or measures of angles. This is because in this discussion we are not differentiating between finite and infinite points and also we are not singling out for special treatment the so-called circular points at infinity. See §83.

reference) are valid. The dual of Desargues' theorem on two perspective triangles was proved in Ex. 3 of §66. The dual of the fundamental theorem on quadrangular sets of points is contained in Ex. 13 of §69. Therefore we can dualize our analytic results, even when the point and line coordinates are referred to a triangle of reference.

EXERCISES

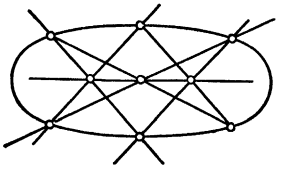
1. Show how the dual of the theorem on quadrangular sets of points enters into the discussion of a triangle of reference for line coordinates as given in §88. Hint: Dualize parts of §§64, 65, 70.

2. Dualize the theorem that any two conics intersect in four points, using the idea of tangents to point conics.

3. Dualize a bi-tangent. See §42.

4. Dualize the definition of a tangent at a point of inflection on a curve. Hint: Such a tangent is the limiting position of a chord $PP'P''$ as $P' \to P$ if at the same time $P'' \to P$.

5. State and dualize the theorem illustrated in the following figure. (This theorem is called Pascal's theorem about a hexagon inscribed in a conic.)



93. **Plane duality from the purely geometric (synthetic) standpoint.** We shall not give the complete proof of the validity of plane duality for results that are obtained by purely geometric means (called *synthetic* as opposed to *analytic*). Note that the validity of duality for analytic discussions follows from its validity for purely geometric discussions because of the way coordinates are assigned to points and lines in this book. However, it is not true that conversely the validity of plane duality for purely geometric discussions follows from its validity for analytic discussions.

We shall state here that *all* the geometry in this book could have been built upon the foundation of the following *assumptions* (see Veblen and Young, " Projective Geometry," Vol. II, pp. 2, 3):

(α) If A and B are distinct points, there is one and only one line passing through A and B.

(β) If A,B,C are points not all on the same line, and D and E ($D \neq E$) are points such that B,C,D are on a line and C,A,E are

on a line, there is a point F such that A,B,F are on a line and also D,E,F.

(γ) There are at least three points on every line.

(δ) There exists at least one line.

(ϵ) All points are not on the same line.

(ζ) All points are not on the same plane. (For the definitions of a plane and of a three-dimensional space see further on in this section.)

(η) If S_3 is a three-dimensional space, every point is in S_3. (Or we may assume that all points are not in the same S_3; and if S_4 is a four-dimensional space, every point is in S_4. Or we may assume the existence of still higher dimensional spaces.)

(θ) In the setting up of the axes plus l_∞ and of the triangle of reference in §§64, 65, 70, the points on the axes and l_∞ (or on the sides of the triangle of reference) that we obtained are such that their non-homogeneous coordinates are in one-to-one correspondence with the real numbers, i.e., we exhaust all the points on these lines that are determined as described when we exhaust the real numbers as labels.

Note that (θ) does not preclude the existence of imaginary points on these axes and l_∞ (or on the sides of the triangle of reference) but merely states that we cannot locate such points geometrically by the means at our disposal.

Also we note that a point is an undefined element in the above assumptions. A line is defined as a set of points by the assumptions (α)–(ϵ). A plane (and similarly a three-dimensional space, etc.) is to be defined in terms of points and lines in the following manner:

DEFINITION. Given a line p determined by two points A and B using assumptions (α) and (δ), we take a point P (not on p) whose existence is guaranteed by assumption (ϵ). The *plane* π (that is said to be determined by P and p) is then defined as consisting of *all* points *collinear* with P and *some* point of p.

We see that we must assume the *existence* of an *infinitely distant* point on p in order to include in π the points on the line through P that is *parallel* to p.

By means of assumptions (α) through (η) and the above definitions we could now establish the necessary theorems about points, lines, and planes that would be essential in the proof of

such fundamental results as Desargues' theorem on perspective triangles. These theorems are such as the following:

(a) Every point on a line l in a plane π lies in π.

(b) Any two lines in π intersect in a point of π.

(c) The plane π is uniquely determined by any three non-collinear points of π (not alone by A,B,P as in the above definition).

(d) The plane π cuts any other plane π' in a line (which line is l_∞ if π and π' are parallel).

(e) The plane π cuts any line l not on π in a single point (which point is on l_∞ if π and l are parallel).

(f) Three planes that do not intersect in a line must intersect in a point (which may lie on l_∞). See Veblen and Young, "Projective Geometry," Vol. I, pp. 15–25.

The *only* place in this book so far where we *step out of the plane* in our discussion is in the derivation of *Desargues' theorem* on perspective triangles in §66. The above-mentioned assumptions (α)–(η) and elementary theorems (a)–(f) are sufficient to prove this theorem.

The plane duals of the above assumptions α, β, γ, δ, ϵ, θ are as follows:

(α') If l and m are distinct lines, there is one and only one point of intersection of l and m.

(β') If l,m,n are lines not all through the same point and p and r ($p \neq r$) are lines such that m,n,p are concurrent and n,l,r are concurrent, there is a line s such that l,m,s and also p,r,s are concurrent.

(γ') There are at least three lines through every point.

(δ') There exists at least one point.

(ϵ') All lines are not through the same point.

(θ') In setting up the centers plus the origin (or triangle of reference) for line coordinates in §88 the lines through the centers (or through the vertices of the triangle of reference) are such that their non-homogeneous coordinates are in one-to-one correspondence with the real numbers.

We shall prove most of the above duals of the original assumptions but shall refer the student to Veblen and Young, *loc. cit.*, for the derivation of the above-mentioned elementary theorems on points, lines, and planes.

Using these assumptions and their above duals, the elementary

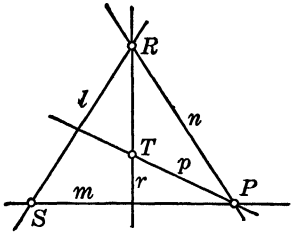
theorems mentioned above, Desargues' theorem and its dual, and also the fundamental theorem on quadrangular sets of points and its dual, we can then proceed to prove the theorem of plane duality by the following statement:

PROOF. Any proposition deducible from the above assumptions and from the elementary theorems mentioned above (which theorems are in turn deducible from these assumptions) is obtained from these assumptions and theorems by a certain *sequence of formal logical inferences.* The same sequence of formal logical inferences may be applied to the *duals* of the above assumptions and theorems and the result will be *valid.* But this last result will be merely the original proposition with the words point and line, concurrent and collinear, etc., interchanged. Hence we can prove either one of two dual theorems, and the other theorem follows at once by plane duality. (Compare Veblen and Young, *loc. cit.*, Vol. I, pp. 28, 29.)

We shall now prove most of the duals of the above assumptions. We leave the statement and the proof of the duals of the above elementary theorems (a)–(f) to the student in the exercises.

For (α') we refer to the theorem (b) that any two lines in a plane intersect in a point. See Veblen and Young, p. 18.

For (β') we note the adjoining figure. The points P, R, S exist, by (α'). The given lines p and r intersect in T, by (α'). The points S and T determine the desired line s, by (α).



To prove (γ') we take a point P. There exists at least one line l by (δ). I: If l does not pass through P it has at least three points, R, S, T by (γ). The points P and R, P and S, P and T determine the three required lines, by (α). II: If l passes through P, it has another point R, by (γ). Also there is a point S not on l, by (ϵ). Moreover, R and S determine a line m, by (α). On m there is another point T, by (γ). The points P and R, P and S, P and T determine the three required lines, by (α).

The proposition (δ') follows from (δ) and (γ).

To prove (ϵ') we take any given point P and a line l through it. The existence of l is assured by (δ), for if l does not pass through P then (ϵ') is proved by the existence of l. On l we have a point R,

by (γ). There is, by (ϵ), a point S not on l The line RS, given by (α), is the required line not passing through P, because if RS passed through P then (α) would be violated.

Finally we prove (θ') as follows: The two-point form for the equation of a line (especially the determinant form, in homogeneous or non-homogeneous coordinates) shows that the coefficients (hence the line coordinates) of this line are real if the two points are real. Hence (θ') follows from (θ).

EXERCISES

1. Draw figures for assumptions (β) and (β').
2. Generalize the definition of a plane π that is given in the text and obtain the definition of an S_3, of an S_4.
3. State and prove the duals of the elementary theorems (a)–(f) mentioned in the text as being at the foundation of a proof of Desargues' theorem.
4. To prove (γ') in the text we take a point P. Prove the existence of P by earlier assumptions.
5. Show how Desargues' theorem follows from assumptions (α)–(η) and the elementary theorems (a)–(f). This fact is merely stated in the text.

94. **Points and lines and plane duality from a strictly analytic viewpoint.** Another way to approach the subject of plane duality is to define a *point* P as an entity given *analytically* by a set of coordinates x,y,z and a *line* l as an entity given by an *equation* of the form $ux + vy + wz = 0$, where $[u,v,w]$ are called the homogeneous line coordinates of l. Here (kx,ky,kz) for $k \neq 0$ is the same point as (x,y,z) and $(0,0,0)$ is no point at all, similarly for $[ku,kv,kw]$ and $[0,0,0]$.

Next we show that the assumptions (α)–(η) and Desargues' theorem are valid for the above-defined entities, if we call (x,y,z,t) a point in space and

$$ux + vy + wz + rt = 0$$

a plane in space. Then we replace (θ) by the assumption that x,y,z,t and u,v,w,r are real numbers (or complex numbers or some other sort of numbers; compare §§86, 136, 138).

Assumption (α) is satisfied by the above entities, because the equation $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$ gives us such a line through any two points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

We shall leave the validity of (β) for the student to prove in the exercises.

To show (γ) is valid we note that any point $(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2)$ satisfied the above equation of the line through P_1 and P_2.

Assumption (δ) is satisfied because $x = 0$ is such a line.

To prove (ϵ) valid we note that $(x_1,y_1,2z_1)$ is not on the line through P_1 and P_2.

For (ζ) we note that (0,0,0,1) is not on the plane $ux + vy + wz + rt = 0$ if $r \neq 0$.

From this discussion of the assumptions, and a similar strictly analytical treatment of the elementary theorems (a)–(f) which we leave for the exercises, Desargues' theorem follows as in the book. Or, to keep our ideas more analytical, we could now proceed to prove Desargues' theorem analytically for the above entities that we call points and lines.

EXERCISES

1. Prove Desargues' theorem and the fundamental theorem for quadrangular sets of points analytically for the entities called points and lines in the text.

2. State (η) for the points and lines in this section. How do we insure the validity of (η) in this case?

3. Prove the elementary theorems (a)–(f) analytically for the points and lines in this section.

CHAPTER XIII

INTRODUCTION TO GENERAL ANALYTIC PROJECTIVE GEOMETRY

95. General linear (or projective) transformations in point and line coordinates, general analytic projective geometry. In this section we come to the most general *linear* (or *projective*) transformation of the homogeneous variables x,y,z, namely (see §76):

$$(88) \qquad \rho x = a_1x' + a_2y' + a_3z', \quad \rho y = b_1x' + b_2y' + b_3z', \quad \rho z = c_1x' + c_2y' + c_3z'$$

where we introduce ρ (an arbitrary non-vanishing constant) because of the homogeneous character of the variables x,y,z and x',y',z'. The inverse of (88) is

$$(88') \qquad \sigma x' = A_1x + B_1y + C_1z, \quad \sigma y' = A_2x + B_2y + C_2z, \quad \sigma z' = A_3x + B_3y + C_3z$$

where A_1, B_1, etc., are the cofactors of a_1, b_1, etc., respectively, in the following determinant Δ of the coefficients of (88).

$$\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and $\sigma = \Delta/\rho$.

We do not consider the cases where $\Delta = 0$, because for these cases

$$\rho(A_1x + B_1y + C_1z) \equiv (a_1A_1 + b_1B_1 + c_1C_1)x' + (a_2A_1 + b_2B_1 + c_2C_1)y' + (a_3A_1 + b_3B_1 + c_3C_1)z' \equiv 0 \cdot x' + 0 \cdot y' + 0 \cdot z' = 0$$

This last equation means geometrically that no matter what point $P'(x',y',z')$ we take in the plane, the corresponding point $P(x,y,z)$ lies on the one line $A_1x + B_1y + C_1z = 0$. In this case (88) is called singular, and has no inverse. We shall suppose Δ (the so-called discriminant of the transformation) does not vanish.

If we follow (88) by a similar transformation with matrix $M' \equiv \begin{Vmatrix} a_1' & a_2' & a_3' \\ b_1' & b_2' & b_3' \\ c_1' & c_2' & c_3' \end{Vmatrix}$ and discriminant $\Delta' \neq 0$, also with ρ' in place of ρ, we obtain the product (see §§13, 71) of (88) by this new transformation:

$$
\begin{aligned}
(89)\quad \tau x &= (a_1a_1' + a_2b_1' + a_3c_1')x' + (a_1a_2' + a_2b_2' + a_3c_2')y' \\
&\quad + (a_1a_3' + a_2b_3' + a_3c_3')z' \\
\tau y &= (b_1a_1' + b_2b_1' + b_3c_1')x' + (b_1a_2' + b_2b_2' + b_3c_2')y' \\
&\quad + (b_1a_3' + b_2b_3' + b_3c_3')z' \\
\tau z &= (c_1a_1' + c_2b_1' + c_3c_1')x' + (c_1a_2' + c_2b_2' + c_3c_2')y' \\
&\quad + (c_1a_3' + c_2b_3' + c_3c_3')z'
\end{aligned}
$$

with matrix $M'' \equiv M \cdot M'$, where M is the matrix of (88), with discriminant $\Delta'' \equiv \Delta \cdot \Delta' \neq 0$, and with $\tau = \rho\rho' \neq 0$. Here the product of two matrices is defined to be like the product of two unexpanded determinants.

If we perform first the transformation with matrix M' and then (88), we obtain a new product with matrix $M_1'' \equiv M' \cdot M$, and discriminant $\Delta_1'' \equiv \Delta' \cdot \Delta \neq 0$. Ordinarily $M'' \neq M_1''$ even though $\Delta'' = \Delta_1''$, so the new transformation is usually not the same as (89). Hence we see that two general linear transformations of the form (88) are ordinarily *not commutative* (*permutable*). If we represent (88) by T and the transformation with M' by T', we express this fact by $TT' \neq T'T$. (See §9.)

The identical transformation (*identity*) in homogeneous coordinates

$$(90)\qquad \rho x = x', \quad \rho y = y', \quad \rho z = z'$$

belongs to the set of transformations represented by (88), as we see by putting $a_1 = 1$, $a_2 = a_3 = 0$, $b_1 = b_3 = 0$, $b_2 = 1$, $c_1 = c_2 = 0$, $c_3 = 1$.

The *inverse* (88′) of (88) has (if we interchange x and x', y and y', z and z') the same form as (88) only with a_1 replaced by A_1, a_2 by B_1, a_3 by C_1, etc. Also the *product* (89) has the form of (88) with $a_1a_1' + a_2b_1' + a_3c_1'$ instead of a_1, $a_1a_2' + a_2b_2' + a_3c_2'$ instead of a_2, etc. Moreover, the product in the other order of (88) and the transformation with matrix M' has the form of

(88) with $a_1a_1' + b_1a_2' + c_1a_3'$ instead of a_1, $a_2a_1' + b_2a_2' + c_2a_3'$ instead of a_2, etc.

Hence the general projective (linear) transformations form a *group* (see §31), with the affine linear transformations as a *subgroup.* There is a *general projective geometry* corresponding to this general projective group of transformations (compare §35) in which we study the invariants under this group (compare §§18, 20). The affine geometry and Euclidean geometry we may call *sub-geometries* under this general projective geometry (compare §32).

If we subject the variables x,y,z of the general line $ux + vy + wz = 0$ to the transformation (88), we obtain a new line $u'x' + v'y' + w'z' = 0$ with the following relations between u',v',w' and u,v,w (compare §21).

$$(91') \qquad \tau u' = a_1u + b_1v + c_1w, \quad \tau v' = a_2u + b_2v + c_2w, \quad \tau w' = a_3u + b_3v + c_3w, \quad \tau \neq 0$$

or

$$(91) \qquad \sigma u = A_1u' + A_2v' + A_3w', \quad \sigma v = B_1u' + B_2v' + B_3w', \quad \sigma w = C_1u' + C_2v' + C_3w', \quad \sigma = \Delta/\tau$$

We may look upon (91) as a transformation of line coordinates *induced* by (88), or simply as (88) in *line coordinate form.* Or we may consider (91) *on its own merits,* quite apart from any associated transformation in point coordinates.

NUMERICAL ILLUSTRATIONS. We note the transformation

$$T : \rho x = x' + y' - z', \quad \rho y = x' - y' + z', \quad \rho z = -x' + y' + z'$$

which has $\Delta \equiv \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -4$ and so is not singular. Also T^{-1} (the inverse of T) has the following form (solving T for x',y',z' in terms of x,y,z then interchanging x and x', y and y', z and z'):

$$T^{-1} : \sigma x = -2x' - 2y', \quad \sigma y = -2x' - 2z', \quad \sigma z = -2y' - 2z'$$

where $\sigma = -4/\rho$. Also T in line coordinates is

$$\tau u' = u + v - w, \quad \tau v' = u - v + w, \quad \tau w' = -u + v + w$$

or

$$\sigma u = -2u' - 2v', \quad \sigma v = -2u' - 2w', \quad \sigma w = -2v' - 2w'$$

Again we take a transformation

$$T_1 : \rho' x = x' - y' - z', \quad \rho' y = -x' - y' + z', \quad \rho' z = -x' + y' - z'$$

with $\Delta_1 \equiv \begin{vmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} = 4$. From (89) we see that TT_1 is

$$T' : \tau x = x' - 3y' + z', \quad \tau y = x' + y' - 3z', \quad \tau z = -3x' + y' + z'$$

On the other hand the product $T_1 T$ is

$$T'' : \tau' x = x' + y' - 3z', \quad \tau' y = -3x' + y' + z', \quad \tau' z = x' - 3y' + z'$$

The matrix of T' is $\begin{Vmatrix} 1 & -3 & 1 \\ 1 & 1 & -3 \\ -3 & 1 & 1 \end{Vmatrix}$ whereas the matrix of T'' is $\begin{Vmatrix} 1 & 1 & -3 \\ -3 & 1 & 1 \\ 1 & -3 & 1 \end{Vmatrix}$. We see at once that $T_1 T \neq TT_1$.

EXERCISES

1. Given the determinant

$$\bar{\Delta} \equiv \begin{vmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 & 0 \\ -1 & 0 & 0 & a_1' & a_2' & a_3' \\ 0 & -1 & 0 & b_1' & b_2' & b_3' \\ 0 & 0 & -1 & c_1' & c_2' & c_3' \end{vmatrix}$$

show that $\bar{\Delta} = \Delta \cdot \Delta'$. (Hint: Expand $\bar{\Delta}$ in terms of the first column.) Now multiply the first column by a_1' and add it to the fourth column, multiply the second by b_1' and add to the fourth, multiply the third by c_1' and add to the fourth; treat the fifth and sixth columns in fashion similar to the way the fourth was treated. Expand the resulting determinant in terms of the first column and we have Δ'' of (89). Note that this method can be generalized so as to give the formula for the *product of two determinants of any order.* See Fine's " College Algebra."

2. Check all the algebra in the last two paragraphs of the text. Find the inverse and both line coordinate forms for each of the transformations T, T', and T''.

3. Derive (88′) from (88), (91′) from (88), (91) from (91′), also (91) from the effect of (88′) on $u'x' + v'y' + w'z' = 0$.

4. Derive (89). Also find the product $T''T$ spoken of in the text.

5. Show that if $\Delta = 0$ in (88), the line $A_1 x + B_1 y + C_1 z = 0$ is the same as $A_2 x + B_2 y + C_2 z = 0$ and as $A_3 x + B_3 y + C_3 z = 0$. Hint: Multiply the equations of (88) respectively by A_2, B_2, C_2 and add them together. Then multiply these equations of (88) by A_3, B_3, C_3, respectively, and add them. But there can be only one such line, because to each point $P'(x',y',z')$ there corresponds only one point $P(x,y,z)$ by (88). Hence the above three equations in x,y,z must all give the same line.

6. What does (88) do to l_∞? What line goes by (88) into l_∞? (This is looking upon the transformation as an alibi and the coordinates as referred to axes plus l_∞. But we might consider the transformation as an alias, also we might have the coordinates referred to a triangle of reference. Compare §11.)

7. Prove that if (88) has a matrix that is symmetric with respect to the main diagonal (i.e., $a_2 = b_1$, $a_3 = c_1$, $b_3 = c_2$), then (88) represents commutative (permutable) transformations, i.e., $TT_1 = T_1T$ where T and T_1 are of this special form for (88). Note that this is only a sufficient (not a necessary) condition for such transformations to be commutative. See §17.

8. Find the necessary and sufficient conditions for (88) to give commutative transformations. Hint: The corresponding coefficients of TT' and $T'T$ must be proportional.

9. Show how the rotations and translations satisfy the conditions of Ex. 8, but the general Euclidean transformations (35) do not satisfy these conditions for commutativity (permutability). Hint: Put rotations and translations, etc., into homogeneous coordinates and add on a third equation $\rho z = z'$ to each transformation.

10. Make up two commutative transformations (not rotations or translations), find their equations in line coordinates, also find their inverses and their products (in either order).

11. Do the same as in Ex. 10 for two non-commutative transformations. Note that $(T_1T_2)^{-1} = T_2^{-1}T_1^{-1}$.

12. If we want to send $x^2 + y^2 = 1$ into $x^2 - y^2 = 1$ by (88), we write these equations in homogeneous form as $x^2 + y^2 = z^2$ (or $z^2 - x^2 = y^2$) and $x^2 - y^2 = z^2$. Then we see that the transformation $z = x'$, $x = y'$, $y = z'$ sends the circle into the hyperbola. Now send $xy = 1$ into $y^2 = x$, $x^2 - y^2 = 1$ into $y^2 = x$. (Hint: $x^2 - y^2 = (x - y)(x + y)$.) Send $y^2 = x^3$ into $y = x^3$; $y^3 = x^4$ into $y = x^4$. Note that (13) cannot give these results. Why?

13. Determine (88) so as to send (0,0,1), (0,1,0), (1,0,0), (1,1,1) into (0,1,0), (1,0,0), (1,1,1), (0,0,1), respectively. Hint: Use (88), also $\rho = \rho_1$, ρ_2, ρ_3, ρ_4, respectively. Compare §100. Note the relation of (88) to bilinear transformations (66).

14. Determine (88) so as to send $z = 0$, $y = 0$, $x = 0$, $x + y + z = 0$ into $y = 0$, $x = 0$, $x + y + z = 0$, $z = 0$, respectively. Do this in two ways, first using (88), then using (91′).

15. Take a transformation T and show algebraically that T and its inverse T^{-1} have the same fixed points and lines.

16. Prove that the discriminant Γ of (75) is invariant under (88). Compare §19.

96. Invariant points and lines under general projectivities. To find the *invariant* (or *double*, or *fixed*) points and lines under any general projective transformation (or projectivity) we solve the following equations, obtained from (88) and (89′), for x', y', z'

and u',v',w', respectively (compare §33):

$$(92)\quad \rho x' = a_1x' + a_2y' + a_3z',\quad \rho y' = b_1x' + b_2y' + b_3z',\quad \rho z' = c_1x' + c_2y' + c_3z'$$

$$(93)\quad \sigma u' = a_1u' + b_1v' + c_1w',\quad \sigma v' = a_2u' + b_2v' + c_2w',\quad \sigma w' = a_3u' + b_3v' + c_3w'$$

The condition that (92) have a solution x',y',z' not all zero is

$$(94)\quad \begin{vmatrix} a_1 - \rho & a_2 & a_3 \\ b_1 & b_2 - \rho & b_3 \\ c_1 & c_2 & c_3 - \rho \end{vmatrix} = 0$$

which is a third-degree equation in ρ with constant term $\Delta \neq 0$ (hence $\rho = 0$ is not a root of this equation). Thus we see that *there are in general three invariant points under* (88).

The condition that (93) have a solution u',v',w' not all zero is

$$(95)\quad \begin{vmatrix} a_1 - \sigma & b_1 & c_1 \\ a_2 & b_2 - \sigma & c_2 \\ a_3 & b_3 & c_3 - \sigma \end{vmatrix} = 0$$

which is the same third-degree equation as (94). Hence there are *in general three invariant lines under* (88). Also, if a pair of invariant points are imaginary (due to imaginary roots in ρ), then a pair of invariant lines are imaginary, because equations (94) and (95) are in reality the *same* equation. Similarly, in other ways, there is complete (plane) *duality* between the invariant *points* and *lines* of a projectivity (88).

Note that by invariance we here mean invariance as to *position*. For example, the affine transformations (13) usually leave l_∞ invariant only as to position, but move points along l_∞. (Compare §85.) Thus the rotations move all the points on l_∞ except the circular points. (Compare §§84, 83.)

On the other hand, some invariant lines may have every point on them invariant (i.e., the lines be *pointwise* invariant) just as (dually) some invariant points may have every line through them invariant as to position (i.e., the points be *linewise* invariant). For example, any translation keeps l_∞ pointwise invariant (see §84) and also keeps its center (on l_∞) linewise invariant. The transformation $x = ax'$, $y = ay'$ keeps the origin linewise invariant because every line $y = mx$ goes into $y' = mx'$; but

$x = ax'$, $y = by'$ keeps the origin invariant only as to position because a line $y = mx$ goes into $y' = max'/b$.

Let us now find the invariant points and lines of the rotation (in homogeneous coordinates)

$$(96) \quad \rho x = x' \cos\theta - y' \sin\theta, \quad \rho y = x' \sin\theta + y' \cos\theta, \quad \rho z = z'$$

or in line coordinates

$$(97) \quad \sigma u' = u \cos\theta + v \sin\theta, \quad \sigma v' = -u \sin\theta + v \cos\theta, \quad \sigma w' = w$$

The equation (94) in ρ is here

$$\begin{vmatrix} \cos\theta - \rho & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \rho & 0 \\ 0 & 0 & 1 - \rho \end{vmatrix} \equiv (1 - \rho)(\rho^2 - 2\rho\cos\theta + 1) = 0$$

The value $\rho = 1$ gives us

$$x'(1 - \cos\theta) + y' \sin\theta = 0, \quad x' \sin\theta + y'(1 - \cos\theta) = 0, \quad 0 \cdot z' = 0$$

so (0,0,1) is the double point of (96) corresponding to $\rho = 1$, and (dually) the line [0,0,1] or l_∞ is the corresponding invariant line. The values $\rho = \cos\theta \pm i \sin\theta$ give us the points $(1,\mp i,0)$ and the lines $[1,\mp i,0]$ or $y = \mp ix$, i.e., the circular points at infinity and the minimal lines through the origin.

Next let us consider the translation

$$(98) \quad \rho x = x' + hz', \quad \rho y = y' + kz', \quad \rho z = z'$$

$$(99) \quad \sigma u' = u, \quad \sigma v' = v, \quad \sigma w' = hu + kv + w$$

The equation in ρ is

$$\begin{vmatrix} 1 - \rho & 0 & h \\ 0 & 1 - \rho & k \\ 0 & 0 & 1 - \rho \end{vmatrix} \equiv (1 - \rho)^3 = 0$$

The value $\rho = 1$ gives us

$$0 \cdot x' - hz' = 0, \quad 0 \cdot y' - kz' = 0, \quad 0 \cdot z' = 0$$

which equations are satisfied by every point $(x',y',0)$, i.e., every point on l_∞ is invariant as to position. The equation in σ is $(1 - \sigma)^3 = 0$. The value $\sigma = 1$ gives the equations

$$0 \cdot u' = 0, \quad 0 \cdot v' = 0, \quad 0 \cdot w' + hu' + kv' = 0$$

which show us that every line with $u'/v' = -k/h$ is invariant, i.e., the center $(1,h/k,0)$ is linewise invariant. (Compare §84.)

Suppose we want the invariant points and lines of the transformation T in the next to the last paragraph of §97. The equation in ρ is here

$$\begin{vmatrix} 1-\rho & 1 & -1 \\ 1 & -1-\rho & 1 \\ -1 & 1 & 1-\rho \end{vmatrix} \equiv -(\rho^3 - \rho^2 - 4\rho + 4) = 0$$

with roots $\rho = 1,2,-2$. The root $\rho = 1$ gives us the point $(1,1,1)$; $\rho = 2$ gives $(-1,0,1)$; $\rho = -2$ gives $(1,-2,1)$. The roots of the equation in τ (for invariant lines) are $\tau = 1,2,-2$. The value $\tau = 1$ gives the line $[1,1,1]$ or $x + y + z = 0$; $\tau = 2$ gives $[-1,0,1]$ or $-x + z = 0$; $\tau = -2$ gives $[1,-2,1]$ or $x - 2y + z = 0$.

EXERCISES

1. Check all the algebraic manipulations in the text.
2. Find the invariant (double) points and lines in all the transformations in the text of §95.
3. Derive (93) from (92), (97) from (96), (99) from (98).
4. Put (13) into homogeneous form and find its invariant points and lines by the method of this section.
5. All the discussion in the text of invariant points and lines is from the standpoint of a transformation considered as an alibi. Interpret this discussion from the standpoint of an alias.

97. Collineations, homologies, and elations. Since (88) sends lines into lines (i.e., collinear points into collinear points), the general projectivities given by (88) are sometimes called *collineations* (or *projective collineations*). Two fundamental types of collineations are the *homology* and the *elation*. (Compare the last paragraph of the text in §84, also Ex. 7.)

DEFINITION. A *homology* is a collineation having a line l that is pointwise invariant and a point P (not on l) that is linewise invariant. The line l and the point P are called the *axis* and *center*, respectively, of the homology.

A simple case of a homology is $x = \alpha x'$, $y = \alpha y'$ whose center is the origin and whose axis is l_∞. In §96 we said that, since (94) is a cubic in ρ, a collineation has in general only three invariant points. The homology, which has a line of invariant (fixed)

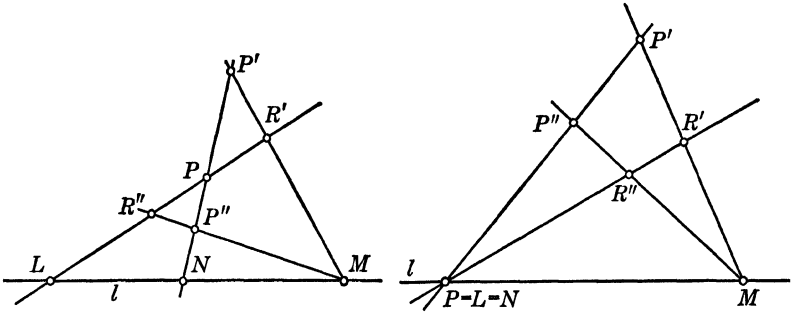
points, causes us to modify the above statement to read three *non-collinear* invariant points.

DEFINITION. An *elation* is a projectivity (collineation) that has a pointwise invariant line l (its *axis*) and a linewise invariant point P (its *center*), where P *lies on* l.

A simple case of elation is the translation $x = x' + h$, $y = y' + k$, whose axis is l_∞ and whose center is at the point on l_∞ with (or in) the direction k/h.

An important type of homology is the so-called *harmonic homology*, which sends any point P into P' and P' into P, and so is of *period two* (i.e., the *square** of this projectivity is the identity, since this square sends P to P' and then back to P). Compare §9. A simple case of harmonic homology is $x = -x', y = -y'$ (with the origin as center and l_∞ as axis) which sends any point $P(a,b)$ into $P'(-a,-b)$ and sends $P'(-a,-b)$ into $P(a,b)$. Note that the points P,P', the origin, and the point at infinity on the line PP' form a harmonic set, since we have the cross-ratio $(a - 0)/(a - \infty) \cdot (-a - \infty)/(-a - 0) = -1$. Compare §24.

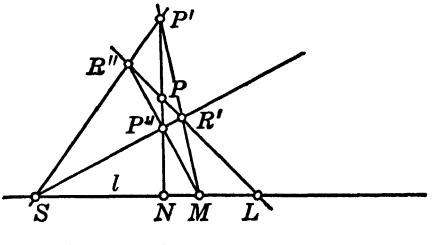
We see by the following figure *that a homology (or elation) is uniquely determined in general by its center P and axis l and one pair of corresponding points P′,P″ (where of course P,P′,P″ must be collinear).*



We take any point R' (in the figure R' is not on the line $P'P''$). The corresponding point R'' must lie on the line $R'P$; also the line $P'R'M$ must go by the homology (or elation) into the line $P''M$; hence R'' is uniquely determined by the two lines $P''M$ and PR'.

* See §9. By the product of T_1 by T_2 we mean T_1T_2 in this order. If $T_1 = T_2$, we obtain T_1T_1 or T_1^2.

On the other hand, if we are dealing with a *harmonic homology*, we see from the adjoining figure that the *center* and *axis* determine *uniquely* such a projectivity because of the presence of harmonic sets of points in the construction, which sets we shall note. (Hence comes the name harmonic homology.)



PROOF. If we take the center P and axis l of a harmonic homology and two points P' and R' and wish to find the points P'' and R'' corresponding to P' and R', respectively, we note that $P'R'$ and $P''R''$ must intersect in M on l. But P'' must go to P' (by the very definition of this harmonic homology) and R' to R'', hence $P''R'$ and $R''P'$ must intersect in a point S on l; so P', P, P'', N must form a harmonic set, also R'', P, R', L must form a harmonic set. But the three points P', P, N determine the harmonic set P', P, P'', N (taken in this order), so a harmonic homology is uniquely given by its center P and axis l.

Note that (for a harmonic homology) to a point T between P and l there must correspond a point T' on the side of P away from l, such that P and l determine with T',T on the line $T'PT$ a harmonic set of points in which P lies between T and T'. The facts that $x = x' + h$, $y = y' + k$ is determined by one pair of corresponding points and also that $x = ax'$, $y = ay'$ is determined by one such pair of points are analytic examples of the results of the above geometric discussion of homologies and elations.

EXERCISES

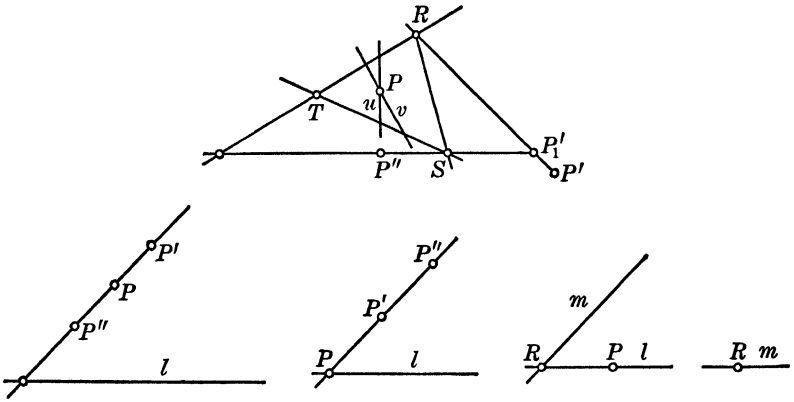
1. In the homology and elation illustrated by figures in the text (given P,P',P'',l), suppose R' is on $PP'P''$. Show how to determine R''. Hint: Take a point T' not on $PP'P''$.

2. In the various homologies and elations given by figures in the text, show that given P, P', P'', R', R'', l we can determine S'' for an S' either from P, P', P'', l or from P, R', R'', l. Why do we get the same S'' each time? Hint: Look out for quadrangular sets of points.

3. Check over all the work in the text.

4. Construct pairs of corresponding points for the following collineations: a homology (*a*) with l at l_∞; (*b*) with P on l_∞; an elation (*a*) with l at l_∞; (*b*) with P on l_∞; a harmonic homology (*a*) with l at l_∞; (*b*) with P on l_∞.

98. **Types of collineations.** In the following manner we show that there are just *five* possible types of general projectivities (collineations), using the possible numbers and arrangements of invariant points and lines (ignoring the sub-cases where some or all of these points and lines are imaginary). In this classification we treat l_∞ like any other line in the plane. Note the following figures:



In the first place, note three non-collinear invariant points R, S, T. The three invariant lines must be the sides of the triangle RST. (Why?) If *none* of the sides of RST are *pointwise* invariant, we have a *first type* of collineation An example of this type is $x = ax'$, $y = by'$, $a \neq b$, where R, S, T are the origin and the two points at infinity on the axes.

If *one* of the sides of the triangle RST in the above discussion is *pointwise* invariant, we have the case of a *homology* (which is represented by the second of the above five figures). We cannot have two of the sides of RST (say RT and TS) pointwise invariant, otherwise every point P in the plane (being the intersection of two lines, u, v that must be invariant as to position) will be invariant and the projectivity will be the identical transformation (or the identity).

Next, we suppose there are just *two invariant points* P, R and *two invariant lines* l, m; then l must be the line RP (say) and m must pass through R (say). If l is *pointwise* invariant, then (dually) R must be *linewise* invariant, and we have an *elation.* The third figure above represents an elation.

If l in the last paragraph is *not pointwise invariant* (and so R

is not linewise invariant), we have a *fourth* type of collineation (represented graphically by the fourth of the preceding figures). A typical example of this case is $x = \beta x' + \beta y'$, $y = \beta y'$, where l is $y = 0$ and m is l_∞.

Finally, we suppose there is *only one invariant point* R and *only one invariant line* m through R. An example of this case is $x = x' + \alpha$, $y = \beta x' + y' + \gamma$, where m is l_∞ and R is the point at infinity on the y-axis. These types of projectivities are discussed in Veblen and Young, "Projective Geometry," Vol. I, pp. 106–108, 271–276.

EXERCISES

1. Check all the algebra in the text, also all the statements made there.
2. Discuss the cases of collineations where some or all of the invariant elements are imaginary. Can we have all these elements imaginary?
3. Look up in Veblen and Young, *loc. cit.*, the complete discussion, analysis, and reduction to typical analytical forms for all five cases of collineations.
4. Make up numerical examples of the five types of collineations.
5. What sorts of collineations are the following?

   (*a*) $\rho x' = x, \quad \rho y' = y, \quad \rho z' = \alpha y + z$

   (*b*) $\rho x' = x, \quad \rho y' = ax + y, \quad \rho z' = bx + cy + z$

   (*c*) $\rho x' = \alpha x, \quad \rho y' = \beta x + \gamma y, \quad \rho z' = \delta x + \epsilon y + \zeta z$

   (*d*) $\rho x' = \alpha x, \quad \rho y' = \beta y, \quad \rho z' = \gamma y + \beta z$

6. Construct pairs of corresponding points for a collineation of the first type (*a*) with ST at l_∞; (*b*) with R on l_∞ but S and T finite.

99. The resolution of a collineation into a product of homologies and elations; reduction of a collineation to a typical form. It is shown in Veblen and Young, *loc. cit.*, that *every projectivity (collineation) can be resolved into a series of homologies and elations* (i.e., the collineation can be analyzed as a product of homologies and elations); also that by a suitable choice of the invariant elements the five types of collineations can be reduced to simple analytical forms.

Use is made (in the discussion referred to above) of a theorem (proved in the next section; compare also §71) that *four pairs of corresponding points* PP', RR', SS', TT' (where no three of the four primed points, or of the four unprimed points, are collinear) *determine uniquely a collineation.* Thus, for the first type of collineation with three invariant non-collinear points, it takes one pair of corresponding (homologous) points not on a side of the triangle RST to determine uniquely such a collineation. (Why?)

Compare the fact that, if in the case of $x = ax'$, $y = by'$ we say that $P(\alpha,\beta)$ shall go to $P'(\alpha',\beta')$, then we have $\alpha = a\alpha'$, $\beta = b\beta'$ and a,b are uniquely determined.

In the figure for the first type of collineation we suppose P' and P'' to correspond in the collineation. We join P' to R and P'' to S by lines that intersect in the point P_1'. Now we see that a homology with center R and axis ST sending P' to P_1', followed by a homology with center S and axis RT sending P_1' to P'', will give us the transformation desired (since P' goes to P'' and R, S, T are kept invariant).

Note that the construction in the last paragraph shows the uniqueness of the determination of the collineation by R, S, T and P', P'', even without the theorem in the next section; because each homology is uniquely determined by a pair of corresponding points (plus the center and axis), therefore the desired collineation is uniquely determined by P' and P'' (plus R, S, T). This statement means that any other point U' has a unique corresponding point U_1' with respect to the first homology, whereas U_1' has a unique corresponding point U'' by the second homology; therefore U' has a unique homologous point U'' by the product of these two homologies (in the given order). The analysis of the other types of collineations we leave for the student to look up in Veblen and Young, *loc. cit.*

We shall give one reduction of a projectivity to a typical (or canonical) analytic form. Let us take the first case with R, S, T as the vertices of a triangle of reference. Then in the general projectivity (88) the point (0,0,1) must go into itself, so $a_3 = b_3 = 0$, $c_3 \neq 0$; also (0,1,0) is invariant, so $a_2 = c_2 = 0$, $b_2 \neq 0$; likewise (1,0,0) is invariant, so $a_1 \neq 0$, $b_1 = c_1 = 0$. The cubic (94) in ρ is now

$$\begin{vmatrix} \rho - a_1 & 0 & 0 \\ 0 & \rho - b_2 & 0 \\ 0 & 0 & \rho - c_3 \end{vmatrix} = 0$$

The roots of this cubic must be distinct, hence we must have a_1, b_2, c_3 distinct; and our transformation is

$$\rho x = a_1 x', \quad \rho y = b_2 y', \quad \rho z = c_3 z'$$

If $z' = 0$ is l_∞, we can put this transformation in the well-known form $x = ax'$, $y = by'$. The reductions of the other cases to

simpler forms we leave for the student to study in Veblen and Young.

As a numerical illustration let us discuss the transformation

$$\rho x = x', \quad \rho y = x' - y', \quad \rho z = x' + y' - z'$$

The equation in ρ is

$$\begin{vmatrix} 1-\rho & 0 & 0 \\ 1 & -1-\rho & 0 \\ 1 & 1 & -1-\rho \end{vmatrix} \equiv (1-\rho)(1+\rho)^2 = 0$$

where $\rho = 1$ gives the fixed point (4,2,3); $\rho = -1$ gives us $0\,x' = 0$, $x' - 0\,y' = 0$, $x' + y' - 0\,z' = 0$ so the point (0,0,1) is invariant. The equations of this transformation in line coordinates are

$$\sigma u' = u + v + w, \quad \sigma v' = -v + w, \quad \sigma w' = w$$

The value $\sigma = 1$ gives the line [1,0,0] or $x = 0$, the value $\sigma = -1$ gives the line [1,−2,0] or $x - 2\,y = 0$. Taking the inverse of the transformation or

$$\tau x' = x, \quad \tau y' = x - y, \quad \tau z' = 2\,x - y - z$$

we see that a point $(0,y,z)$ goes into $(0,-y,-y-z)$ so $x = 0$ is not pointwise invariant. Also a point $(2y,y,z)$ goes into $(2y,y,3y-z)$ so $x - 2\,y = 0$ is not pointwise invariant. Therefore the above transformation belongs to the fourth type pictured and discussed above in §98.

EXERCISES

1. In the reduction of a collineation to a canonical form that is given in the text, determine a_1, b_2, c_3 so that the result will be a typical form for a homology. Hint: One side of the triangle RST must be pointwise invariant.
2. Check all the algebra in the text, also all the statements made there.

100. **Complete quadrangle (quadrilateral) as determining a projectivity.** We shall now show that if we say a complete quadrangle P_1, P_2, P_3, P_4 is to go into another (or the same) complete quadrangle P_1', P_2', P_3', P_4' by a general projectivity (88), then this transformation is completely determined as to the coefficients of its equations. (Compare §71.)

Proof. Suppose the points $P_i(\alpha_i,\beta_i,\gamma_i)$ where $i = 1,2,3,4$ are to go into (0,0,1), (0,1,0), (1,0,0), (1,1,1), respectively. Substituting the coordinates of the former points for x,y,z and

the coordinates of the latter points for x',y',z' in the equations of (88) and using a different ρ for each pair of corresponding points (compare §22), we get

$$\begin{aligned}
\rho_1\alpha_1 &= a_3, \quad \rho_1\beta_1 = b_3, \quad \rho_1\gamma_1 = c_3 \\
\rho_2\alpha_2 &= a_2, \quad \rho_2\beta_2 = b_2, \quad \rho_2\gamma_2 = c_2 \\
\rho_3\alpha_3 &= a_1, \quad \rho_3\beta_3 = b_1, \quad \rho_3\gamma_3 = c_1 \\
\rho_4\alpha_4 &= a_1 + a_2 + a_3, \quad \rho_4\beta_4 = b_1 + b_2 + b_3, \\
&\qquad \rho_4\gamma_4 = c_1 + c_2 + c_3
\end{aligned}$$

Substituting the values of a_1,b_1,c_1, a_2,b_2,c_2, a_3,b_3,c_3 from the first nine equations into the last three equations, we have

$$\begin{gathered}
\rho_4\alpha_4 = \rho_3\alpha_3 + \rho_2\alpha_2 + \rho_1\alpha_1, \quad \rho_4\beta_4 = \rho_3\beta_3 + \rho_2\beta_2 + \rho_1\beta_1, \\
\rho_4\gamma_4 = \rho_3\gamma_3 + \rho_2\gamma_2 + \rho_1\gamma_1
\end{gathered}$$

Dividing each of these last equations through by ρ_4 we have three non-homogeneous linear equations in the three unknowns ρ_1/ρ_4, ρ_2/ρ_4, ρ_3/ρ_4. These equations have a unique solution not containing zero, since none of the determinants

$$\begin{vmatrix} \alpha_i & \alpha_j & \alpha_k \\ \beta_i & \beta_j & \beta_k \\ \gamma_i & \gamma_j & \gamma_k \end{vmatrix} = 0$$

where $i,j,k = 1,2,3,4$; but $i \neq j$, $j \neq k$, $k \neq i$ (due to the fact that no three of the four points P_1, P_2, P_3, P_4 are collinear). Using the solution of these last three equations in the first nine equations, we obtain $a_1, b_1, \ldots, c_3$ uniquely except for the common factor $1/\rho_4$ (since we have $\rho_1/\rho_4\alpha_1 = a_3/\rho_4$, etc.).

In this proof an invariant point P counts for a pair of corresponding points. If we look upon the triangle of reference (or the axes plus l_∞) as having a fixed position in the plane, the above discussion shows that the vertices of any complete quadrangle can be sent into the vertices of the triangle of reference plus the unit point (1,1,1) by one and only one collineation; and, conversely, the vertices of the triangle and (1,1,1) can be sent into the vertices of any complete quadrilateral by one and only one collineation. Now to show that any complete quadrangle P_1, P_2, P_3, P_4 can be sent into any other complete quadrangle P_1', P_2', P_3', P_4' by one and only one collineation, we note from the above proof that we can send the first quadrangle into the vertices of the triangle of reference plus (1,1,1), then we can send the vertices of this

triangle and (1,1,1) into P_1', P_2', P_3', P_4'; and the product of these two transformations gives us the unique collineation sending P_1, P_2, P_3, P_4 directly into P_1', P_2', P_3', P_4'.

ILLUSTRATIVE EXAMPLE. Let us send (1,1,1), (1,−1,1), (−1,1,1), (−1,−1,1) into (0,0,1), (0,1,0), (1,0,0), (1,1,1), respectively. From (88) we get

$$\rho_1 = a_3, \quad \rho_1 = b_3, \quad \rho_1 = c_3; \qquad \rho_2 = a_2, \quad -\rho_2 = b_2, \quad \rho_2 = c_2;$$
$$-\rho_3 = a_1, \quad \rho_3 = b_1, \quad \rho_3 = c_1; \qquad -\rho_4 = a_1 + a_2 + a_3,$$
$$-\rho_4 = b_1 + b_2 + b_3, \quad \rho_4 = c_1 + c_2 + c_3$$

Substituting a_1, b_1, c_1, etc., from the first nine equations into the last three, we get (dividing through by ρ_4)

$$\frac{-\rho_3}{\rho_4} + \frac{\rho_2}{\rho_4} + \frac{\rho_1}{\rho_4} = -1, \quad \frac{\rho_3}{\rho_4} - \frac{\rho_2}{\rho_4} + \frac{\rho_1}{\rho_4} = -1, \quad \frac{\rho_3}{\rho_4} + \frac{\rho_2}{\rho_4} + \frac{\rho_1}{\rho_4} = 1$$

Solving these three last equations, we get

$$\frac{\rho_1}{\rho_4} = -1, \quad \frac{\rho_2}{\rho_4} = 1, \quad \frac{\rho_3}{\rho_4} = 1, \quad \therefore \quad \frac{a_3}{\rho_4} = \frac{\rho_1}{\rho_4} = -1, \quad \frac{b_3}{\rho_4} = -1, \quad \frac{c_3}{c_4} = -1,$$
$$\frac{a_2}{\rho_4} = -\frac{b_2}{\rho_4} = \frac{c_2}{\rho_4} = 1, \quad \frac{-a_1}{\rho_4} = \frac{b_1}{\rho_4} = \frac{c_1}{\rho_4} = 1$$

Substituting these values in (88) and dividing the resulting equations by ρ_4, we get the required transformation

$$\rho' x = -x' + y' - z', \quad \rho' y = x' - y' - z', \quad \rho' z = x' + y' - z'$$

where $\rho' = \rho/\rho_4$.

Dually to the above discussion, we see that a complete quadrilateral l_1, l_2, l_3, l_4 can be sent into any other (or the same) complete quadrilateral l_1', l_2', l_3', l_4' by one and only one collineation. We leave this dual discussion to the student in the exercises.

EXERCISES

1. Dualize the analytic discussion in the text.

2. In the first paragraph of the text why does the converse of the theorem follow at once from the proof given there?

3. Send (1,1,2), (2,−1,1), (1,−2,1), (1,2,3) to (0,0,1), (1,0,0), (0,1,0), (1,1,1), respectively, by a collineation of the form (88). Now send the last four points into (1,1,1) (1,2,1), (−1,−1,1), (0,0,1), respectively, by another collineation. Finally show that the product of the two transformations in the above order sends the first set of four points to the third set of four points.

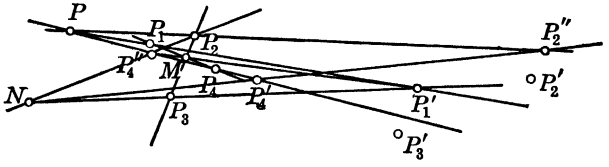
4. Check the algebra and fill in the details in the text.

5. Find the projectivity sending (0,0,1), (0,1,0), (1,0,0), (1,1,1) into (1,1,−1), (0,0,1), (1,0,0), (0,1,0), respectively; into (0,i,1), (i,1,0), (i,0,1), (1,1,1), respectively.

6. Discuss fully for invariant points, types of transformations, and equations in line coordinates, the collineation worked out in the text and those in Exs. 3 and 5.

7. Show both geometrically and analytically why four invariant points (no three of them collinear) determine the identity.

101. **Geometrical construction of collineations.** Now let us consider a geometrical construction of a collineation that will actually send a given real complete quadrangle into another given real complete quadrangle. We shall consider only the case where the two quadrangles have no common vertices and have the most general possible positions in the plane relative to each other. We leave the student to look up the other cases in Veblen and Young, *loc. cit.*, Vol. I, p. 74.

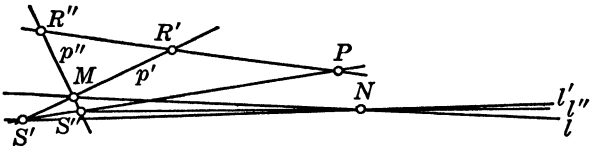


Consider the above figure. With the line P_2P_3 as axis of a homology and the point P of intersection of P_1P_1' and P_4P_4' as center, we send P_1 into P_1'. Then P_4 goes into P_4'', the point of intersection of the lines PP_4P_4' and $P_1'M$ (where M is the point of intersection of P_2P_3 and P_1P_4). Why? Now, by means of a homology with $P_1'P_3$ as axis and P as center, we send P_4'' into P_4'. Then P_2 goes into P_2'', the point of intersection of the lines PP_2 and $P_4'N$ (where N is the point of intersection of P_2P_4'' and $P_1'P_3$). Why? Next, by means of a homology with $P_1'P_4'$ as axis and a suitable center (not shown in the figure), we send P_2'' into P_2'; but P_3 goes by this homology into some point P_3''.

We saw in §99 that a collineation with three given invariant non-collinear points is uniquely determined by one pair of corresponding points. Also we analyzed such a collineation into the product of two homologies. Hence we can find such a transformation (of the first type) that will keep P_1', P_2', P_4' fixed and will send P_3'' into P_3'. The product of all these homologies is a collineation sending P_1, P_2, P_3, P_4 into P_1', P_2', P_3', P_4'.

We shall now dualize the definition and construction of a homology (given in §97) and so get a method of constructing a

line l'' corresponding to a line l' under a homology. A homology was defined as having a pointwise invariant line (its axis) and a linewise invariant point (its center); hence the dual of a homology is a homology.



From the above figure we see that from P,l, and the pair of corresponding points R',R'' we can determine a pair of corresponding lines p',p''. Now, given a line l', to find its corresponding line l'' under the homology, we join S' (the point of intersection of p' and l') to P by a line cutting p'' at S''; then the line joining S'' to N (the point of intersection of l' and l) is the required line l''. Why?

EXERCISES

1. By a series of homologies, send a complete quadrangle P_1, P_2, P_3, P_4 into another one P_1', P_2', P_3', P_4'. Take a point T and find its corresponding point T' under the series of homologies.
2. Do the same as in Ex. 1 only by a series of elations.
3. Do as in Ex. 1: (*a*) for the two quadrangles P_1, P_2, P_3, P_4 and P_1, P_2, P_3', P_4'; (*b*) for P_1, P_2, P_3, P_4 and $P_1' = P_2$, $P_2' = P_3$, $P_3' = P_4$, $P_4' = P_1$; (*c*) for P_1, P_2, P_3, P_4 and P_1, P_2, $P_3' = P_4$, $P_4' = P_3$.
4. Dualize the definition and construction of an elation and find the construction of a line l'' corresponding to a given line l' under an elation.
5. Finish the first figure in the text.
6. Answer the queries (Why?) in the text.
7. Dualize the geometric discussion of a collineation given in the first paragraph of the text.
8. Do Ex. 5 in §100 by a series of homologies.
9. Show how a series of collineations of the first type can be combined so as to send P_1, P_2, P_3, P_4 into P_1', P_2', P_3', P_4'.

102. **Transformations of points on a line and between lines, in homogeneous coordinates.** In this section we shall study what a general projectivity (88) does to the *points* of a line l that it keeps *invariant as to position*, also how it sends the points of a line l' that is *not* kept invariant into those of the line l'' that corresponds to l' under the collineation. (Compare §73.)

First we take the invariant line as $z = z' = 0$. Since (88) must send $z = 0$ into $z' = 0$, we have $c_1 = c_2 = 0$. Putting $z = z' = 0$ in (88), we have

$$\rho x = a_1x' + a_2y', \quad \rho y = b_1x' + b_2y' \tag{100}$$

where $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0$; or in non-homogeneous form (replacing y/x by y and y'/x' by y')

$$y = \frac{ay' + b}{cy' + d} \tag{101}$$

where $a = b_2$, $b = b_1$, $c = a_2$, $d = a_1$.

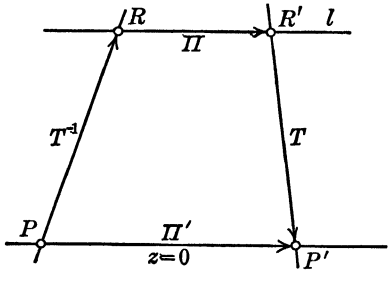
Note that (101) is the same as (70), which came out of a discussion of bilinear transformations in a plane. We can ignore the rest of the plane and consider only this one line.

Note that (100) has just three essential coefficients, since we can divide both equations by any non-vanishing coefficient. Each pair of corresponding points $(x_i,y_i,0)$, $(x_i',y_i',0)$ is in non-homogeneous form y_i,y_i' and gives us from (101) one linear equation between these coefficients; hence three pairs of corresponding (homologous) points determine such a projectivity between points on a line.

If the invariant line is not $z = z' = 0$ but is $l \equiv \alpha x + \beta y + \gamma z = 0$, we can put

$$T: \qquad x' = x', \quad y' = y, \quad z' = \alpha x + \beta y + \gamma z$$

which sends l into $z' = 0$. Suppose a collineation Π keeps $l \equiv \alpha x + \beta y + \gamma z = 0$ invariant, then $\Pi' = T^{-1}\Pi T$ has a similar effect on $z = z' = 0$ to that which Π has on l, because T^{-1} sends $z = 0$ to l, Π keeps l invariant as to position, T sends l back to $z = 0$. Hence we can study the effect of Π' on $z = 0$, then by means of T^{-1} we can send the pairs of corresponding points on $z = 0$ under Π to the pairs of corresponding points on l under Π, as the adjoining schematic figure shows. (In this figure R,R' correspond under Π and P,P' correspond under Π. Since T^{-1}



sends P to R and T sends R' to P', therefore T^{-1} sends the pair P,P' to the pair R,R'.)

DEFINITION. If $\Pi' = T^{-1}\Pi T$, then Π' is called the *transform* of Π by T.

When studying how (100) changes points of l' into those of another line $l'' \neq l'$, we first suppose l' to be $z = 0$ and l'' to be $x' = 0$. Since $z = 0$ must go into $x' = 0$ under (88), we have $c_2 = c_3 = 0$; and, when we put $x' = z = 0$ in (88), we get:

$$\rho x = a_2 y' + a_3 z', \quad \rho y = b_2 y' + b_3 z' \tag{102}$$

or in non-homogeneous form (replacing x/y by x and y'/z' by y')

$$x = \frac{ay' + b}{cy' + d} \tag{103}$$

where $a = a_2$, $b = a_3$, $c = b_2$, $d = b_3$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. Note that (103) is the same as (71).

Just as with the case of an invariant line l so in (103) three pairs of homologous points determine this projectivity between the points of the two lines l' and l''. We see that for $c \neq 0$, (103) may be written

$$x = \frac{a}{c} + \frac{bc - ad}{c(cy' + d)}$$

Hence we cannot have $bc - ad \equiv -\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$; otherwise a single point $x = a/c$ on l' corresponds to every point on l'' (since $x = a/c + 0/c(cy' + d)$ is satisfied by any value of y'). If $c = 0$ in (103), again $bc - ad \neq 0$; otherwise $a = 0$ or $d = 0$ and (103) is $x = b$ or $x = \infty$.

If l' is not $z = z' = 0$ but is $\alpha x + \beta y + \gamma z = 0$, and l'' is $\alpha' x + \beta' y + \gamma' z = 0$ and not $x = x' = 0$, we can put

$$T\colon\ x' = \alpha' x + \beta' y + \gamma' z, \quad y' = y, \quad z' = \alpha x + \beta y + \gamma z$$

which transformation sends l', l'' into $z' = 0$, $x' = 0$, respectively. Now we can argue as we did for the invariant line l.

The transformations (100) and (102) were supposed to have been caused (induced) by a collineation in the plane. But we can study these projectivities on a line or between lines by and for themselves, without reference to any collineation in the plane. Thus we might look upon (88) as induced by a projectivity in

space of three dimensions, but we ordinarily study (88) for its own sake and apart from any such reference to space.

ILLUSTRATIVE EXAMPLE. Let us now find the transformation (100) of points on a line $l(z = 0)$ that sends (1,1,0) to (0,1,0), (0,1,0) to (1,0,0), (1,0,0) to (1,1,0).

Ignoring the rest of the plane, we can write these points each with only two coordinates x and y; thus we can write (1,1,0) as (1,1). Since (1,1) goes to (0,1), we have $\rho' = a_2$, $\rho' = b_2$; (0,1) goes to (1,0), so $0 = a_1$, $\rho'' = b_1$; (1,0) goes to (1,1), so $\rho''' = a_1 + a_2$, $0 = b_1 + b_2$. Hence

$$b_1 = -b_2 = -\rho', \quad a_1 = 0, \quad a_2 = \rho'$$

and the desired transformation is (after dividing by ρ')

$$\rho x = y', \quad \rho y = -x' + y'$$

If we wish to use (101) on the above problem we see that 1 goes to ∞, ∞ to 0, 0 to 1. (Why?) Hence $a/c = 1$, $d = 0$, $a + b = 0$ in (101). (Why?) And the desired transformation is

$$y = \frac{y' - 1}{y'}$$

Of course, we can get this bilinear form from the homogenous form by dividing the second equation by the first, then replacing y/x by y and y'/x' by y'.

EXERCISES

1. Dualize the discussion in the text.
2. Find the inverses of all the transformations in the text.
3. Determine (101) to send: (*a*) 1 to -1, 2 to 3, 4 to -2; (*b*) 0 to ∞, ∞ to 2, 1 to 0.
4. Solve Ex. 3 using homogeneous coordinates and (100).

103. Invariant (double) points of projectivities on a line; hyperbolic, parabolic, elliptic transformations. To find the invariant (double) points of a projectivity (101) on a line l we put $y' = y$ and then we have the equation $cy^2 + (d - a)y - b = 0$ to solve. (Compare §33.) Hence according as

$$(d - a)^2 + 4bc \gtreqless 0$$

we have, respectively, two real and distinct double points, two real and coincident double points, two conjugate imaginary double points. See §85, also Ex. 5.

Examples of these three cases are

$$y = \frac{1}{y'}, \quad y = y' + 1, \quad y = -\frac{1}{y'}$$

with the respective double points ± 1, ∞, $\pm i$. These three types of projectivities on a line are called, respectively, *hyperbolic*, *parabolic*, and *elliptic*, after analogy with the manner in which the three principal types of non-degenerate conics cut l_∞. As an illustration, let us find the double points on l_∞ in the projectivity induced thereon by the rotation (7). Here we have in (101) $a = \cos\theta$, $b = \sin\theta$, $c = -\sin\theta$, $d = \cos\theta$. The double points are given therefore by $y^2 + 1 = 0$.

The double points of (100) are given by solving simultaneously

$$(\rho - a_1)x - a_2 y = 0, \quad -b_1 x + (\rho - b_2)y = 0$$

For these two equations to have a solution not all zero we must have

$$\begin{vmatrix} \rho - a_1 & -a_2 \\ -b_1 & \rho - b_2 \end{vmatrix} \equiv \rho^2 - (a_1 + b_2)\rho + a_1 b_2 - a_2 b_1 = 0$$

(Compare §96.) Hence

$$\rho = \frac{(a_1 + b_2) \pm \sqrt{(a_1 + b_2)^2 - 4\,(a_1 b_2 - a_2 b_1)}}{2}$$

so the nature of these double points depends on whether $(a_1 - b_2)^2 + 4\,a_2 b_1 \gtreqless 0$.

EXERCISES

1. Find the double points of $y = \dfrac{-5\,y' + 11}{-2\,y' + 5}$; of $\rho x = 3\,x' + 2\,y'$, $\rho y = -4\,x' - y'$.

2. Make up cases of (100) and of (101) where the transformations are hyperbolic, parabolic, and elliptic, respectively.

104. **Involutions of points on a line.** DEFINITION. A projectivity on a line that sends each point P into a point P' and also sends P' into P is called an *involution*, and P, P' are called a pair of *conjugate points* in this involution.

The involutions are by all odds the most important projectivities on a line. Let us find what *forms* (100) and (101) *must have* to be *involutions*. Suppose P is α and P' is α' (where $\alpha \neq \alpha'$) and (101) is an involution. We have from (101)

$$\alpha = \frac{a\alpha' + b}{c\alpha' + d} \quad \text{also} \quad \alpha' = \frac{a\alpha + b}{c\alpha + d}$$

giving us

$$c\alpha\alpha' + d\alpha - a\alpha' - b = 0 \quad \text{and} \quad c\alpha\alpha' + d\alpha' - a\alpha - b = 0$$

Subtracting these two equations, we get $(\alpha - \alpha')(d + a) = 0$. Since $\alpha \neq \alpha'$, we have $d = -a$. Therefore the general equation of an involution on a line in non-homogeneous coordinates is

$$y = \frac{ay' + b}{cy' - a} \tag{104}$$

or in homogeneous form

$$\rho x = a_1 x' + a_2 y', \quad \rho y = b_1 x' - a_1 y' \tag{105}$$

Note that we have proved above incidentally that *just one pair of points P, P' in a transformation* (100) *or* (101) *such that P corresponds to P' and P' to P makes this projectivity on l an involution.*

The double points of (104) are given by $cy^2 - 2ay - b = 0$. The discriminant of this quadratic is

$$4(a^2 + bc) \equiv -4 \begin{vmatrix} a & b \\ c & -a \end{vmatrix} \neq 0$$

(otherwise the projectivity is singular). Therefore we can have *hyperbolic* or *elliptic* involutions, but *not parabolic.* Thus $y = 1/y'$ is hyperbolic and $y = -1/y'$ is elliptic.

We can reduce every hyperbolic involution to the form $y = 1/y'$ by choosing the coordinates on the line in the following manner. We take the double points for (104) as 1 and -1. Therefore we have

$$1 = \frac{a + b}{c - a} \quad \text{or} \quad c - 2a - b = 0,$$

$$-1 = \frac{-a + b}{-c - a} \quad \text{or} \quad c + 2a - b = 0$$

so $a = 0$, $c = b$; and the involution becomes $y = 1/y'$. This also shows that a *hyperbolic involution is uniquely determined by its double points.* (The same thing is true of an elliptic involution.)

Note that, if a pair of conjugate points in this hyperbolic involution $y = 1/y'$ is $\alpha, 1/\alpha$, we have the cross-ratio

$$\frac{\alpha - 1}{\alpha + 1} \frac{1/\alpha + 1}{1/\alpha - 1} = -1$$

which proves that *any pair of conjugate points of a hyperbolic involution form with the double points a harmonic set of points*

such that the double points separate the conjugate pair. (Compare §24.) Here we see a geometric reason why the double points uniquely determine such an involution. (We leave for the student in the exercises the reduction of any elliptic involution to the form $y = -1/y'$ and the proof of the exactly similar theorems for elliptic involutions.)

The inverse of an involution (104) is (104) again. Hence an involution is said to be of *period two.* (See §31, Ex. 2.)

From (104) we see that *two pairs of conjugate points determine an involution.* (Why?) Suppose we want 1,2 and 3,4 to be pairs of conjugate points under (104). This gives us

$$2 = \frac{a+b}{c-a} \quad \text{or} \quad 2c - 2a = a + b$$

also

$$4 = \frac{3a+b}{3c-a} \quad \text{or} \quad 12c - 4a = 3a + b$$

hence $2\,c/b - 3\,a/b = 1$, $12\,c/b - 7\,a/b = 1$, giving us $a/b = -\frac{5}{11}$, $c/b = -\frac{2}{11}$; so the desired involution is $y = (-5\,y' + 11)/(-2\,y' + 5)$.

EXERCISES

1. Fill in the details of the algebra in the text.
2. Solve in homogeneous coordinates the problem in the last paragraph of the text.
3. Prove that if (101) is its own inverse, it must be an involution.
4. Prove that if the square of (101) is the identity, then (101) must be an involution.
5. Work Exs. 3 and 4 for the homogeneous form (100).
6. Derive (105) from (100).
7. Derive $y = 1/y'$ in homogeneous form from (105).
8. Show how to construct the conjugate of P' with respect to a hyperbolic involution whose double points are P_1 and P_2, using a complete quadrangle. (Compare §26.) Take first the case where P_1, P_2, p' are all finite; then the case where P' is an infinite point; then the case where P_2 is an infinite point.
9. Prove for the general form (104) of the involution that the double points and any pair of conjugate points form a harmonic set.
10. Find the involution with 1,3 and 2,4 as pairs of conjugate points.
11. By taking $+i$ and $-i$ as the double points in an elliptic involution, reduce this involution to the type form $y = -1/y'$. Prove that the double points determine uniquely such an involution. Also prove that any pair of conjugate points form with the double points a harmonic set.
12. Find the involution (*a*) having 0 and ∞ as double points; (*b*) having $1 + i$ and $1 - i$ as double points.

13. Prove that if every pair of homologous points under (101) forms with the double points of (101) a harmonic set, then (101) is an involution.

14. Find the conditions on (101) that it shall be (*a*) of period three; (*b*) of period four. Hint: Multiply (101) by itself three and four times, respectively.

15. Find the product of (101) by $y = (a'y' + b')/(c'y' + d')$ in either order. When are these projectivities commutative?

16. Show that the projectivities (101) form a group. Compare §31.

17. Find and describe the involution sending 1 to -1 and 0 to ∞; the involution sending 1 to -1, and 2 to -2; the involution sending α to $-\alpha$ and 0 to 2α; the involution sending α to $-\alpha$ and β to $-\beta$.

18. Show that the concentric circles $x^2 + y^2 = r^2$ cut any line through the origin in pairs of conjugate points of an involution with the origin as one double point and the point at infinity as the other double point.

19. Show that the vertices of each of the confocal conics $x^2/(a^2 + \lambda) + y^2/(b^2 + \lambda) = 1$ are pairs of conjugate points of an involution whose double points consist of one of the common foci and the point of intersection of the corresponding directrix with the axis. Compare §50.

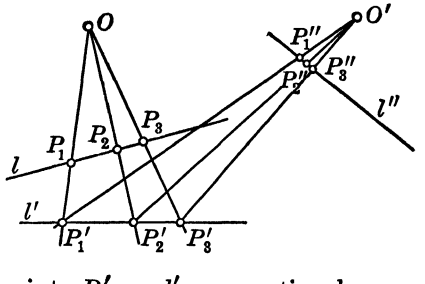
20. Show that the circular points at infinity are the double points of an elliptic involution wherein the pairs of points where pairs of perpendicular lines cross l_∞ form pairs of conjugate points.

21. Taking $y \equiv \alpha = 1 - \epsilon$ (or $-1 - \epsilon$) in $y = 1/y'$, show that as $y \to 1$ (or -1), i.e., $\epsilon \to 0$, so does $y' \to 1$ (or -1). What property of harmonic sets does this prove?

22. Take $\alpha = i - \epsilon$ (or $-i - \epsilon$) in $y = -1/y'$ and proceed as in Ex. 21.

23. Show that if $\alpha < \beta$, then the involution $y = 1/y'$ gives us $\alpha' > \beta'$, and the involution $y = -1/y'$ gives $\alpha' > \beta'$ with both α' and β' negative (if $\alpha > 0$ and $\beta > 0$). For $y = 1/y'$ show that β and β' lie between α and α', but for $y = -1/y'$ show that α' lies between β and β'. For the elliptic involution the pairs α,α' and β,β' are said to separate each other.

105. Geometric description of projectivities between lines. DEFINITION. The points on a line l are said to be *perspective* with the points on another line l' by a *center* O (not on l or l') if pairs of corresponding points P,P' are collinear with O. (See the adjoining figure. Compare §101.)



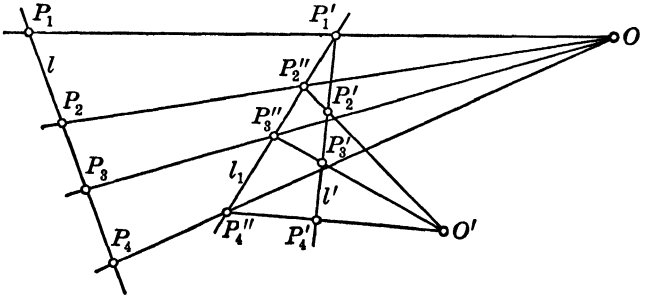
DEFINITION. If we make the points $P_i (i = 1,2,3 \ldots)$ on l perspective by a center O with the points $P_i (i = 1,2,3 \ldots)$ on l', then the points P_i' on l' perspective by a center $O' \neq O$ with the points P_i'' on l'' (we may have $l'' = l$), . . ., the points $P_i^{(n-1)}$ on a

line $l^{(n-1)}$ perspective by a center $O^{(n-1)} \neq O^{(n-2)}$ with the points $P_i^{(n)}$ on a line $l^{(n)}$, we say the points on l are *projective* with those on $l^{(n)}$.

Thus in the figure on page 232 the points on l are *perspective* with those on l', but *projective* with those on l''.

We shall now show geometrically how the above-defined projective correspondence between two lines l and l' can be determined by *three pairs* of corresponding (homologous) points. That is, if we say P_1, P_2, P_3 on l are to correspond to P_1', P_2', P_3' on l', we set up a mechanism of centers and auxiliary lines that will effect this projectivity and also will send any fourth point P_4 on l into a unique point P_4' on l' (and conversely). If $l' \neq l$, we can set up this projective relation between l and l' by two centers O,O'.

In the following figure we have given l, $l' \neq l$, P_1, P_2, P_3, P_1', P_2', P_3'. Through P_1' we take an arbitrary line $l_1 \neq l'$, on the line P_1P_1' we take an arbitrary center O that makes P_1, P_2, P_3 on l perspective with P_1', P_2'', P_3'' on l_1. We now take O' (the point of intersection of the two lines $P_2''P_2'$ and $P_3''P_3'$) as our second center of perspectivity that sends P_2'', P_3'', P_1' to P_2', P_3', P_1'. By O, O', and the auxiliary line l_1 we have set up a projective relation between l and l' such that for any point P on l there is a corresponding point P' on l' (and conversely).



There is so much arbitrariness in the above construction involved in the choice of O and of l_1 that the question naturally arises as to whether other choices of O or of l_1 (or of both O and l_1) would give different points P' to correspond to P. We cannot *prove* this one way or another; but careful drawings of this figure using different choices for O and l_1 would lead us to believe that we

always get the same point P' to correspond to P. Therefore we make the following

ASSUMPTION. *A projective correspondence between the points of two lines l and l' (where we may have $l = l'$) is uniquely determined by three pairs of homologous points.*

Now we shall show that the projective correspondence between the points of two lines l and l' described above *geometrically* is the same as the projectivity given *analytically* in §102 by (100) or (101), and conversely.

Suppose in the above figure l is $x = x' = 0$, l' is $y = y' = 0$, P_1 is (0,1,0), P_2 is (0,1,1), P_3 is (0,a,1), P_1' is (1,0,0), P_2' is (1,0,1), P_3' is (a',0,1), P_4 is any fourth point (0,y_1,1) on l and its corresponding point P_4' is (x_1',0,1). We wish to find the analytic relation between x_1' and y_1. The line P_2P_2' is $x + y - z = 0$, the line P_1P_1' is $z = 0$. We take the line $y - z = 0$ as l_1, and O as (1,$-$1,0). The point P_2'' is

$$\begin{vmatrix} u & v & w \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{vmatrix} \equiv 0 \cdot u - v - w = 0, \quad \text{or} \quad (0,1,1)$$

The line P_3O is

$$\begin{vmatrix} x & y & z \\ 1 & -1 & 0 \\ 0 & a & 1 \end{vmatrix} \equiv -x - y + az = 0$$

The point P_3'' is (a, -1,1,1). The line $P_2''P_2'$ is P_2P_2' or $x + y - z = 0$. The line $P_3''P_3'$ is

$$\begin{vmatrix} x & y & z \\ a-1 & 1 & 1 \\ a' & 0 & 1 \end{vmatrix} \equiv x + (a' - a + 1)y - a'z = 0$$

The point O' is

$$\begin{vmatrix} u & v & w \\ 1 & 1 & -1 \\ 1 & a' - a + 1 & -a' \end{vmatrix} \equiv u(-a' + a' - a + 1) + v(a' - 1) + w(a' - a + 1 - 1) = 0$$

or $(-a + 1, a' - 1, a' - a)$. The line P_4O is $-x - y + y_1z = 0$, so P_4'' is ($y_1 - 1$,1,1). The line $P_4''O'$ is

$$\begin{vmatrix} x & y & z \\ -a+1 & a'-1 & a'-a \\ y_1 - 1 & 1 & 1 \end{vmatrix} = 0$$

which cuts $y = 0$ in the point

$$P_4'\left(\frac{y_1(1-a') + (a'-a)}{1-a}, 0\right)$$

But P_4' has the coordinates $(x_1',0,1)$, hence we have the analytic relation between points on l and their corresponding points on l' given by

$$x' = \frac{y(1-a') + (a'-a)}{1-a}$$

or $$\rho z' = (1-a)z, \quad \rho x' = (1-a')y + (a'-a)z$$

This analytic relation is exactly of the form induced by collineations. See §102.

Conversely, we wish to show that an analytic transformation of coordinates between two lines l, l' induced by a collineation in the plane is geometrically a projective relation between the points of l and l' established by two centers O and O' or a perspective relation from one center.

We suppose the lines are $l \equiv x = x' = 0$ and $l' \equiv y = y' = 0$, that $P_1(0,1,0), P_2(0,1,1), P_3(0,a,1)$ are sent to $P_1'(1,0,0), P_2'(1,0,1)$, $P_3'(a',0,1)$, respectively, by the transformation (103) in the form

$$x' = \frac{\alpha y + \beta}{\gamma y + \delta}$$

Therefore we use non-homogeneous coordinates so P_1 is ∞, P_2 is 1, P_3 is a, P_1' is ∞, P_2' is 1, P_3' is a'. Hence we have

$$\infty = \frac{\alpha + \beta/\infty}{\gamma + \delta/\infty} \text{ so } \gamma = 0; \quad 1 = \frac{\alpha+\beta}{\gamma+\delta} \text{ so } \delta = \alpha + \beta \text{ since } \gamma = 0;$$

$$a' = \frac{\alpha a + \beta}{\gamma a + \delta} \text{ so } (\alpha+\beta)a' = \alpha a + \beta \quad \text{or} \quad \frac{\alpha}{\beta} = \frac{1-a'}{a'-a}$$

and $$\frac{\delta}{\beta} = \frac{\alpha}{\beta} + 1 = \frac{1-a}{a'-a}$$

Therefore our desired transformation is $x' = \{y(1-a') + (a'-a)\}/(1-a)$, which we have already shown to be the analytic form of a projective relation established by two centers O and O'. If $a' = a$, this transformation is $x' = y$ and is geometrically a perspectivity with center O.

If l and l', P_1, P_2, P_1', P_2' are not in the special positions of the

above paragraphs, we can perform a collineation T and get them into these special positions. (Note that the choices of l_1 and O are perfectly arbitrary, anyhow, as the assumption shows.) Now if we perform T^{-1} we get $l, l', P_1, P_2, P_1', P_2'$ back to their old positions. But T^{-1} sends collinear points and concurrent lines into other collinear points and concurrent lines; hence in their original positions the points of l and l' were projectively related by two centers because they are so related in their new positions.

Note that if $l = l'$ we can suppose l is $z = 0$. We can make l perspective with $l_1 \equiv x = 0$ by

$$\rho y = y', \quad \rho x = z'$$

as we saw above. Then we have the case $l_1 \neq l'$. But the product of this transformation by the one above gives us another bilinear transformation.

EXERCISES

1. Given two lines l, l' and a projectivity determined by 0, 0', l_1 where P_1, P_2, P_3 correspond to P_1', P_2', P_3'. Make several other choices of O and l_1 and show that each such choice gives us the same point P_4' on l' to correspond to a given point P_4 on l.

2. Find the coordinates of two centers O and O' that will send (0,1,0), (0,1,1), (0,a,1) into (1,0,1), (1,0,0), (a',0,1), respectively. If (0,y_1,1) goes to (x_1',0,1) find the analytic expression for this projectivity between $x = x' = 0$ and $y = y' = 0$.

3. Prove that if $l = l'$, three centers will set up a projectivity on the points of l.

4. Find the coordinates of three centers to send (0,0,1), (0,1,0), (0,1,1) into (0,1,0), (0,1,1), (0,0,1), respectively; also find the analytic expression for this projectivity in the form (100).

5. In $x' = 1/y$ find the coordinates of two centers to give this transformation geometrically.

6. In $y' = y + 1$ find the coordinates of three centers to give this transformation geometrically.

7. Do the same as in Ex. 1 for a projectivity between points of the same line, using now three centers, O, O', O''.

8. Dualize the discussion and the results in the text.

9. Work the duals of Exs. 1, 7.

106. **Cross-ratio.** We have seen that affine projectivities (13) have an important invariant called *cross-ratio* (sometimes called anharmonic ratio or double ratio). Compare §§23, 77. We shall now show that the *general projectivities* (88) have this *same* invariant.

We must define cross-ratio in such a way as not to bring in the idea of lengths of line-segments and yet so as to reduce to the definition given for affine transformations, and to the definition in terms of lengths of line-segments that is useful for the ordinary frame of reference. *Our definition of cross-ratio must be valid for infinite points as well as for finite points, for a triangle of reference as well as for axes of reference.* See §§23, 77.

Suppose we are given four collinear points $P_i(x_i,y_i,z_i)$ where $i = 1,2,3,4$ on a line $y = mx + bz$. (We leave the cases of a line $x = \alpha z$ and a line $z = 0$ for the exercises.) We define a cross-ratio of these four points as the expression

$$\frac{x_1 - x_2}{x_3 - x_2}\frac{x_3 - x_4}{x_1 - x_4}$$

where x_i stands for x_i/z_i. If we replace y_i/z_i by y_i, we see that we have also

$$\frac{y_1 - y_2}{y_3 - y_2}\frac{y_3 - y_4}{y_1 - y_4} = \frac{mx_1 + b - mx_2 - b}{mx_3 + b - mx_2 - b}\frac{mx_3 + b - mx_4 - b}{mx_1 + b - mx_4 - b}$$
$$= \frac{x_1 - x_2}{x_3 - x_2}\frac{x_3 - x_4}{x_1 - x_4}$$

Also, taking z_i for z_i/x_i or for z_i/y_i, we have

$$\frac{z_1 - z_2}{z_3 - z_2}\frac{z_3 - z_4}{z_1 - z_4} = \frac{x_1 - x_2}{x_3 - x_2}\frac{x_3 - x_4}{x_1 - x_4}$$

Again, taking y_i' for y_i/x_i, we have

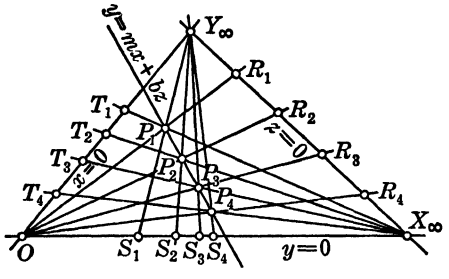
$$\frac{y_1' - y_2'}{y_3' - y_2'}\frac{y_3' - y_4'}{y_1' - y_4'} = \frac{x_1 - x_2}{x_3 - x_2}\frac{x_3 - x_4}{x_1 - x_4}$$

In homogeneous form the above cross-ratios are respectively

$$\frac{x_1z_2 - x_2z_1}{x_3z_2 - x_2z_3}\frac{x_3z_4 - x_4z_3}{x_1z_4 - x_4z_1} = \frac{y_1z_2 - y_2z_1}{y_3z_2 - y_2z_3}\frac{y_3z_4 - y_4z_3}{y_1z_4 - y_4z_1}$$
$$= \frac{y_1x_2 - y_2x_1}{y_3x_2 - y_2x_3}\frac{y_3x_4 - y_4x_3}{y_1x_4 - y_4x_1}$$

Thus we see that a cross-ratio of four collinear points P_1, P_2, P_3, P_4 on a line $y = mx + bz$ has the same value as that of the four points $R_i(i = 1,2,3,4)$ in which the lines OP_i cut $z = 0$ (in the

adjoining figure), or of the four points S_i in which the lines $Y_\infty P_i$ cut $y = 0$, or of the four points T_i in which the lines $X_\infty P_i$ cut $x = 0$. (This figure may be regarded either as a triangle of reference or as a schematic picture of axes plus l_∞.)



Note that for the cross-ratios in non-homogeneous form (compare §§23, 77) we use x/z as a non-homogeneous coordinate for a point on $y = 0$, y/z on $x = 0$, y/x or x/y on $z = 0$. But we might equally well use z/x on $y = 0$ or z/y on $x = 0$.

There are in all six cross-ratios for any four collinear points. (Compare §23.) Note how the above definition of cross-ratio reduces for affine geometry to that given first in §23, also to the definition in terms of lengths of line-segments for the ordinary frame of reference.

Now we prove that *a general projectivity* (88) *leaves cross-ratio invariant.* We first suppose that P_1, P_2, P_3, P_4 lie on $y = 0$. The collineation (88) sends $y = 0$ into $b_1x' + b_2y' + b_3z' = 0$. We suppose $b_2 \neq 0$ (if $b_2 = 0$, then $b_1 \neq 0$ or $b_3 \neq 0$ and we can take the cross-ratio of the points S_i' or T_i' instead of the points R_i' for that of the points P_1', P_2', P_3', P_4' that correspond to P_1, P_2, P_3, P_4 under the collineation). Hence, if we put $y' = (-b_1x' + b_3z')/b_2$ in (88), we see that we have the following relations between the coordinates of points on $y = 0$ and the coordinates of the corresponding points on $b_1x' + b_2y' + b_3z' = 0$:

$$\rho x = \left(a_1 - a_2\frac{b_1}{b_2}\right)x' + \left(a_3 - a_2\frac{b_3}{b_2}\right)z',$$

$$\rho z = \left(c_1 - c_2\frac{b_1}{b_2}\right)x' + \left(c_3 - c_2\frac{b_3}{b_2}\right)z'$$

hence

$$\frac{x}{z} = \frac{(a_1b_2 - a_2b_1)x'/z' + (a_3b_2 - a_2b_3)}{(c_1b_2 - c_2b_1)x'/z' + (c_3b_2 - c_2b_3)}$$

or more briefly

$$x = \frac{ax' + b}{cx' + d}$$

where x is for x/z and x' for x'/z', also $a = a_1b_2 - a_2b_1$, $b = a_3b_2 - a_2b_3$, $c = c_1b_2 - c_2b_1$, $d = c_3b_2 - c_2b_3$. Hence

$$x_i = (ax_i' + b)/(cx_i' + d) \quad \text{for} \quad i = 1,2,3,4$$

Putting these values for x_i in the above cross-ratio, we get

$$\frac{x_1 - x_2}{x_3 - x_2}\frac{x_3 - x_4}{x_1 - x_4} = \frac{\dfrac{ax_1' + b}{cx_1' + d} - \dfrac{ax_2' + b}{cx_2' + d}}{\dfrac{ax_3' + b}{cx_3' + d} - \dfrac{ax_2' + b}{cx_2' + d}}\,\frac{\dfrac{ax_3' + b}{cx_3' + d} - \dfrac{ax_4' + b}{cx_4' + d}}{\dfrac{ax_1' + b}{cx_1' + d} - \dfrac{ax_4' + b}{cx_4' + d}}$$

$$= \frac{x_1' - x_2'}{x_3' - x_2'}\frac{x_3' - x_4'}{x_1' - x_4'}$$

This result proves that (88) leaves cross-ratio invariant when sending $y = 0$ into any line $y = mx + bz$ (which line in the above argument was $b_1x' + b_2y' + b_3z' = 0$). Conversely, when sending any line l to $y = 0$, (88) therefore leaves cross-ratio invariant. Why? To prove the general case where (88) sends l to l' we can send l to $y = 0$ and then send $y = 0$ to l'. Therefore we see that in all cases (88) preserves cross-ratio. (Note this whole argument is from the standpoint of a collineation as an *alibi* so x,y,z and x',y',z' refer to exactly the same triangle of reference or axes of reference plus l_∞.)

ILLUSTRATIVE EXAMPLE. As an illustration of the above theory we see that

$$x' = x + y - z, \quad y' = x - y + z, \quad z' = -x + y + z$$

sends the points

$$(1,0,1), \quad (2,0,1), \quad (3,0,1), \quad (4,0,1) \text{ on } y = 0$$

into the points $(0,2,0)$, $(1,3,-1)$, $(2,4,-2)$, $(3,5,-3)$, respectively, on $x' = z'$. Using y_i'/x_i' of the points R_i' for the cross-ratio of these four points on $x' = z'$ we get

$$\frac{\infty - 3/1}{4/2 - 3/1}\frac{4/2 - 5/3}{\infty - 5/3} = -\frac{1}{3}$$

Using x_i/z_i of the points S_i for the corresponding cross-ratio of the homologous points on $y = 0$, we get again

$$\frac{1 - 2}{3 - 2}\frac{3 - 4}{1 - 4} = -\frac{1}{3}$$

EXERCISES

1. Make up a definite collineation T and take a definite line l and send it to a line l' by T. On l take four definite points. Find a cross-ratio of the four points on l and the corresponding cross-ratio of the four homologous points on l'. Show that these two cross-ratios have the same numerical value.

2. Put all the cross-ratios of the text into homogeneous form.

3. Work the illustrative example of the text using homogeneous forms for the cross-ratios.

4. Treat the cases in the theory of the text where P_1, P_2, P_3, P_4 lie on a line $x = \alpha z$ and where they lie on $z = 0$. Treat the case where $b_2 = 0$ in the proof of the theorem in the text.

5. By using (101) and (103) show that cross-ratio is an invariant under (88).

6. If $y = 0$ goes into itself by (88), i.e., if $b_1 = b_3 = 0$, show that cross-ratio is preserved.

7. Fill in the algebraic details in the text.

8. Dualize the discussion and results in the text.

9. Prove geometrically and analytically that (88) sends tangents into tangents, points of inflection into points of inflection, double points into double points. Hint: Consider a tangent as the limiting position of a secant. Analytically, an ordinary tangent cuts its curve in two coincident points $P'(x',y',z')$ and $P''(x'',y'',z'')$ where $P' = P''$ and so $x' = kx''$, $y' = ky''$, $z' = kz''$ where $k \neq 0$. Now what does (88) do to x', x'', y', y'', z', z''?

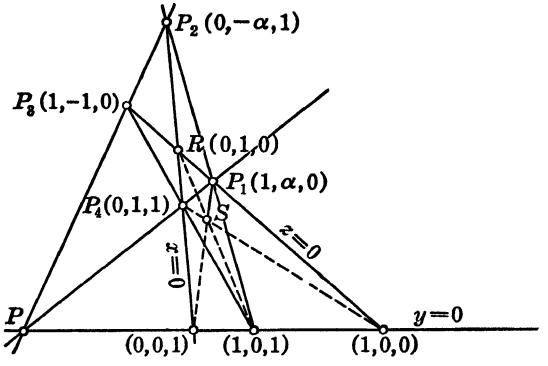
10. Analyze a *rotation* into *two (imaginary) homologies*. Hint: A rotation keeps fixed $(0,0,1)$, $(1,i,0)$, $(1,-i,0)$ and therefore is of the first type of collineations. Compare §84.

107. **Harmonic sets of points and of lines.** In §26 we defined a harmonic set of points on a line l by means of a complete quadrangle as consisting of two diagonal points belonging to the quadrangle and of the two other points of intersection of l with the remaining sides of the quadrangle.

Dually we defined a harmonic set of lines intersecting in a point P as consisting of two diagonal lines of a complete quadrilateral plus the two other lines joining P to the remaining vertices of the quadrilateral.

In §24 we defined analytically for affine geometry a harmonic set of points as having the cross-ratios -1, $\frac{1}{2}$, 2. In §26 we gave a geometric construction by means of the complete quadrangle for such a harmonic set. In general projective geometry, on the other hand, *we define a harmonic set of points geometrically as in* §26. Now we show analytically that any such set has the cross-ratios -1, $\frac{1}{2}$, 2.

We take a complete quadrangle as in the following figure. The three lines in the figure that consist of two sides of the quadrangle and a side of the diagonal triangle we take respectively as $x = 0$, $z = 0$, $y = 0$. We call P_4 (0,1,1) and one diagonal point on $y = 0$ we call (1,0,1).



We wish to prove that the other diagonal point P on $y = 0$ is $(-1,0,1)$, i.e., that the quadrangular set on $y = 0$ has one cross-ratio equal to -1 and so is a harmonic set according to the analytic definition as well as to the geometric.

PROOF. The line P_3P_4 is $x + u - z = 0$, P_1P_3 is $z = 0$, P_2P_4 is $x = 0$. Hence P_3 is the point $(1,-1,0)$. Suppose P_1 is $(1,\alpha,0)$. The line P_1P_2 is then

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 1 \\ 1 & \alpha & 0 \end{vmatrix} \equiv y + \alpha z - \alpha x = 0$$

Hence P_2 is the point $(0,-\alpha,1)$. The line P_2P_3 is

$$\begin{vmatrix} x & y & z \\ 1 & -1 & 0 \\ 0 & -\alpha & 1 \end{vmatrix} \equiv x - \alpha z - y = 0$$

The line P_1P_4 is

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 1 \\ 1 & \alpha & 0 \end{vmatrix} \equiv y - z - \alpha x = 0$$

These last two lines must meet on $y = 0$ (according to the figure), but $y = 0$ cuts P_2P_3 in the point $(-\alpha,0,1)$ whereas $y = 0$ cuts

P_1P_4 in the point $(-1/\alpha,0,1)$. We must have, therefore, $-1/\alpha = -\alpha$, so $\alpha = 1$. This shows that the other diagonal point P is $(-1,0,1)$, which forms with $(1,0,1)$, $(0,0,1)$, $(1,0,0)$ a harmonic set in the analytic meaning (i.e., with a cross-ratio equal to -1).

Note that $-1/\alpha = -\alpha$ gives us also $\alpha = -1$; this value of α (however) is debarred, because then $(-1/\alpha,0,1)$ would be the point $(1,0,1)$. Since cross-ratio is invariant under (88), the above special choice of the triangle of reference does not affect the generality of the discussion.

We could in the above figure take $y = 0$ as the same line, take P_1P_2 as $z = 0$, P_2P_3 as $x = 0$, so the two diagonal points on $y = 0$ are now $(0,0,1)$ and $(1,0,0)$. We could take P_4 as $(1,1,1)$, then the point on $y = 0$ between the two diagonal points must be $(1,0,1)$. In this case P_3 must be $(0,1,1)$ and P_1 must be $(1,1,0)$. Hence the line P_1P_3 (which is now $x - y + z = 0$) must cut $y = 0$ in the point $(-1,0,1)$, which is the fourth point in the quadrangular set on $y = 0$.

Here again we see that the four points on $y = 0$ form a harmonic set in the analytic sense. Again our special choice of the three vertices of the triangle of reference and of P_4 as vertices for our complete quadrangle does not cause a loss of generality in the discussion, because some transformation of (88) sends the four vertices of any given complete quadrangle into the four vertices of any other given complete quadrangle and also (88) preserves cross-ratio. We notice that in the figure and in the first discussion $(-1,0,1)$ and $(1,0,1)$ are the coordinates of the diagonal points on $y = 0$, whereas in the second discussion $(0,0,1)$ and $(1,0,0)$ are the coordinates of these same diagonal points.

Again let us note that, if in the above figure we join $(0,0,1)$ to $P_1(1,\alpha,0)$ and $(1,0,0)$ to $P_4(0,1,1)$, these two lines intersect in the point $S(1,1,1)$ since $\alpha = 1$. Also the point of intersection of P_1P_3 and P_2P_4 is $R(0,1,0)$; hence the line RS is $x - z = 0$, which line must cut $y = 0$ in the point $(1,0,1)$. But the complete quadrangle R, P_1, S, P_4 has the same quadrangular set on $y = 0$ as P_1, P_2, P_3, P_4 except that now $(0,0,1)$ and $(1,0,0)$ are diagonal points, whereas for P_1, P_2, P_3, P_4 the diagonal points are $(-1,0,1)$ and $(1,0,1)$. This shows up an important fact, namely, that if T_1, T_2, T_3, T_4 are a harmonic set of points on a line l where T_1,T_3 separate T_2,T_4, then this set of points can be determined by a complete quadrangle where either T_1,T_3 or T_2,T_4 form the di-

agonal points. Another way to state this result is to say that if T_3 is the harmonic conjugate of T_1 with respect to T_2 and T_4, then T_4 is also the harmonic conjugate of T_2 with respect to T_1 and T_3.

Conversely to the above discussion, we should now prove that every harmonic set of points defined analytically is also a harmonic set according to the geometric definition. To do this we can take the set on $y = 0$, choosing three of the set (in this order) as $(-1,0,1)$, $(0,0,1)$, $(1,0,1)$, then the fourth point on the right must be $(1,0,0)$ because one cross-ratio (in non-homogeneous form) is

$$\frac{-1-0}{1-0}\,\frac{1-x}{-1-x} = -1 \quad \text{or} \quad x = \infty$$

if the fourth point is x or in homogeneous coordinates $(1,0,0)$. Now we see that the lines $x = 0$, $z = 0$, $x + y - z = 0$, $y + z - x = 0$, $x + y + z = 0$, $x - y + z = 0$ form the six sides of a complete quadrangle that defines this harmonic set geometrically.

EXERCISES

1. Check all the algebra in the text.
2. Dualize the discussion in the text.
3. Carry through the details in the last paragraph of the text. Why is there no loss of generality in the discussion there? Hint: How many pairs of corresponding points determine a transformation between two lines?
4. Find two other complete quadrangles to give the harmonic set $(-1,0,1)$, $(0,0,1)$, $(1,0,1)$, $(1,0,0)$: (*a*) with $(-1,0,1)$ and $(1,0,1)$ as diagonal points; (*b*) with $(0,0,1)$ and $(1,0,0)$ as diagonal points.

108. The absolute involution. A pair of perpendicular lines $y = mx + b_1 z$ and $y = -x/m + b_2 z$ cut l_∞ in the two points $(1,m,0)$ and $(1,-1/m,0)$. If we put the points $(x,y,0)$ on l_∞ in the form $(1,y,0)$ where y replaces y/x we see that the equation $y = -1/y'$ connects the pairs of points on l_∞ in which pairs of perpendicular lines cut l_∞. Compare §83.

But the equation $y = -1/y'$ gives on l_∞ an elliptic involution with the circular points as double points. This involution is called the absolute involution.

The Euclidean transformations (35) keep the circular points on l_∞ fixed or interchange them and so send any pair of conjugate points of the absolute involution into another pair of conjugate points of the same involution. Compare §§34, 35. Hence these transformations are said to leave the absolute involution invariant.

We note, however, that if in (13) $a_1 = b_2$ and $b_1 = -a_2$, these affine transformations keep the circular points fixed and therefore keep the absolute involution invariant. Compare §34. Hence the Euclidean transformations are only a *subgroup* of the larger subgroup G of the affine group (13), where G is defined as consisting of all the affine transformations keeping the absolute involution invariant. For the Euclidean transformations we must have also $a_1^2 + b_1^2 = a_2^2 + b_2^2 = 1$.

EXERCISES

1. Why is it that a transformation keeping the circular points fixed must for this reason keep the absolute involution invariant?
2. Show analytically that the affine transformations with $a_1 = b_2, b_1 = -a_2$ form a group. Compare §31.
3. Show analytically that the affine transformations with $a_1 = b_2, b_1 = -a_2$, $a_1^2 + b_1^2 = a_2^2 + b_2^2 = 1$ form a group.
4. Show that the Euclidean group is uniquely defined as consisting of all the transformations that leave the absolute involution invariant and also leave the area of any triangle invariant. Compare §34.

CHAPTER XIV

INTRODUCTION TO CORRELATIONS AND POLARITIES

109. **Tangents to curves.** In §§46, 50 we discussed tangents to curves, also poles and polars with respect to a conic, for affine geometry. Before taking up these topics for general projective geometry and homogeneous coordinates (referred either to a triangle of reference or to axes plus l_∞), we must first discuss briefly the tangents to curves in general.

Given any curve in homogeneous coordinates x,y,z, the equation of a secant to this curve through the two points $P'(x',y',z')$ and $P''(x' + \Delta x, y' + \Delta y, z' + \Delta z)$ on the curve is evidently

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x' + \Delta x & y' + \Delta y & z' + \Delta z \end{vmatrix} \equiv \begin{vmatrix} x & y & z \\ x' & y' & z' \\ \Delta x & \Delta y & \Delta z \end{vmatrix} = 0$$

since this is a first-degree equation and is satisfied by the coordinates of P' and P''.

If x,y,z are assumed to be functions of a parameter t, we divide this equation by Δt, let $\Delta t \to 0$, and we have the equation of the tangent at $P'(x',y',z')$

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ \dfrac{dx'}{dt'} & \dfrac{dy'}{dt'} & \dfrac{dz'}{dt'} \end{vmatrix} = 0 \tag{106}$$

where t is the parameter and dx'/dt', dy'/dt', dz'/dt' are the derivatives dx/dt, dy/dt, dz/dt at the point $P'(x',y',z')$, for which point t has the value t'.

Let us digress a moment to prove *Euler's theorem* that, if $f(x,y,z) = 0$ is an n-ic in the homogeneous coordinates x,y,z, we have

$$x' \frac{\partial f}{\partial x'} + y' \frac{\partial f}{\partial y'} + z' \frac{\partial f}{\partial z'} \equiv nf(x',y',z') \tag{107}$$

where $\partial f/\partial x'$, $\partial f/\partial y'$, $\partial f/\partial z'$ mean $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$, respectively, for $x = x', y = y', z = z'$.

PROOF. If we put $x = x't$, $y = y't$, $z = z't$ in $f(x,y,z)$, then take $d/dt\,[f(x,y,z)]$, we get

$$\frac{d}{dt} f(x,y,z) = \frac{d}{dt} t^n f(x',y',z') = nt^{n-1} f(x',y',z')$$

$$\equiv \frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial z} z'$$

If now we take $t = 1$, we have $\partial f/\partial x'$, $\partial f/\partial y'$, $\partial f/\partial z'$ for $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ and therefore (107) follows.

Again we note that from $\omega \equiv f(x,y,z) = 0$ we have

$$\frac{d\omega}{dt} \equiv \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0 \tag{108}$$

where t is any variable (x or y or z or a parameter).

If $P'(x',y',z')$ lies on the n-ic $f(x,y,z) = 0$, we have the two equations

$$\frac{\partial f}{\partial x'}\frac{dx'}{dt'} + \frac{\partial f}{\partial y'}\frac{dy'}{dt'} + \frac{\partial f}{\partial z'}\frac{dz'}{dt'} = 0, \tag{109}$$

$$x'\frac{\partial f}{\partial x'} + y'\frac{\partial f}{\partial y'} + z'\frac{\partial f}{\partial z'} = 0$$

From these last two equations we find that the coefficients of x,y,z in (106), which are proportional respectively to

$$y'\frac{dz'}{dt'} - z'\frac{dy'}{dt'}, \quad z'\frac{dx'}{dt'} - x'\frac{dz'}{dt'}, \quad x'\frac{dy'}{dt'} - y'\frac{dx'}{dt'}$$

are also proportional to $\partial f/\partial x'$, $\partial f/\partial y'$, $\partial f/\partial z'$. Hence the tangent $f(x,y,z) = 0$ at $P'(x',y',z')$ may also be written

$$\frac{\partial f}{\partial x'} x + \frac{\partial f}{\partial y'} y + \frac{\partial f}{\partial z'} z = 0 \tag{110}$$

Subtracting from (110) the second equation of (109) we get still another form for the equation of the tangent, namely the homogeneous equation

$$\frac{\partial f}{\partial x'}(x - x') + \frac{\partial f}{\partial y'}(y - y') + \frac{\partial f}{\partial z'}(z - z') = 0 \tag{111}$$

EXERCISES

1. Fill in the details for the derivation of (110) from (106) and (109).
2. Find the tangent to $y^2z = x^3$ at (1,1,1) from (106); then from (110); then from (111).

110. Poles and polars with respect to a conic. If we apply (110) to the general equation of the conic in homogeneous form (75), we get the equation (5) again in homogeneous coordinates.

Whether or not $P'(x',y',z')$ is on the conic (75), (5) is called the *polar* of P' with respect to this conic. The properties of *pole* and *polar* that are given in §50 for affine geometry are valid also in general projective geometry, since the proofs given in §50 carry over and apply to (75) in the general projective geometry.

Dually the pole of any line $l'[u',v',w']$ with respect to the general line conic (83) is

$$(112) \qquad (Au' + Hv' + Gw')u + (Hu' + Bv' + Fw')v + (Gu' + Fv' + Cw')w = 0$$

or

$$(112) \qquad Auu' + Bvv' + Cww' + F(v'w + vw') + G(wu' + w'u) + H(u'v + uv') = 0$$

DEFINITION. We call a triangle *self-polar* with respect to a conic if its vertices are the poles of the opposite sides with respect to this conic. (Compare §50, Ex. 4.)

From §50 and (112) we see that if two vertices of a triangle and their opposite sides are poles and polars with respect to a conic, then the triangle is self-polar with respect to this conic.

Let us take for the *triangle of reference* a triangle *self-polar* with respect to the general conic (75). Then $z = 0$ must be the polar of (0,0,1); hence we must have $f = g = 0$ in (75) and (5). Also (0,1,0) and $y = 0$ are pole and polar so $h = f = 0$. Therefore our conic takes the simple form

$$(113) \qquad ax^2 + by^2 + cz^2 = 0, \quad abc \neq 0$$

Now we prove that *any line l through a point $P'(x',y',z')$ cuts a conic and the polar l' of P' with respect to this conic in three points that form with P' a harmonic set.* In §25 we proved this for the affine geometry. We can take P' as (0,0,1) and the conic in the form (113) without loss of generality, because (88) preserves

cross-ratio and also every point P' is a vertex for countless self-polar triangles. (Why?) If we take any line $y = mx$ through P' and solve its equation simultaneously with that of the conic (113), we get the points of intersection $(1, m, \pm\sqrt{(-a - bm^2)/c})$, whereas $y = mx$ cuts the polar $z = 0$ of P' in the point $(1,m,0)$ and finally P' is $(0,0,1)$. We put z for z/x and take the cross-ratio

$$\frac{z_1 - z_2}{z_3 - z_2}\frac{z_3 - z_4}{z_1 - z_4}$$

$$= \frac{\sqrt{(-a - bm^2)/c} - 0}{-\sqrt{(-a - bm^2)/c} - 0}\,\frac{-\sqrt{(-a - bm^2)/c} - \infty}{\sqrt{(-a - bm^2)/c} - \infty} = -1$$

which shows that these four points do form a harmonic set.

EXERCISES

1. Why must we have $abc \neq 0$ for (113)? Answer the question (Why? in the text.

2. Dualizing the derivation of (113) in the text, derive the simple form $Au^2 + Bv^2 + Cw^2 = 0$ for a line conic.

3. Dualize the last theorem in the text, and its proof.

4. Show how, while deriving (113), we incidentally proved it for general projective geometry.

5. Prove again for general projective geometry the properties of poles and polars given in §51, especially the method of finding the center of a conic.

6. How are the coordinate axes and l_∞ situated with respect to the conics $x^2/a^2 \pm y^2/b^2 = \pm 1$?

7. Show that any two conjugate diameters of a conic are conjugate with respect to this conic, i.e., one diameter passes through the pole of the other. From this fact derive the properties of conjugate diameters. Hint: Take the conics in the type forms, then use homogeneous coordinates. Compare §51.

111. **Polarities; their relation to duality.** We take a general pole and polar with respect to the conic (75) and write them respectively in point and line coordinates. Now, if we drop the primes from the variables, we get the equations (using the condition that the line $ux + vy + wz = 0$ shall be the polar of $P(x,y,z)$):

$$\rho u = ax + hy + gz, \quad \rho v = hx + by + fz,$$

$$\rho w = gx + fy + cz, \quad \Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0 \tag{114}$$

We can look upon (114), *quite apart from its connection with the general conic* (75), as a transformation sending *points* into *lines* and *lines* into *points*. We call (114) a *polarity*. We can find the inverse of (114) very readily from (88′), namely,

$$(114') \qquad \sigma x = Au + Hv + Gw, \quad \sigma y = Hu + Bv + Fw, \quad \sigma z = Gu + Fv + Cw$$

where $\sigma = \Delta/\rho$ and Δ is the discriminant of the general conic.

If we try to find the points $P'(x',y',z')$ that lie on their own polars under the transformation (114), we see we must have $ux' + vy' = wz' = 0$. Expanding this equation (since u,v,w have primed point coordinates), we get merely the condition that P' shall lie on the general conic (75). Hence we call (75) the fundamental conic of the polarity (114).

If we take the conic $x^2 + y^2 + z^2 = 0$, we find that this is the fundamental conic of the polarity

$$(115) \qquad \rho u = x, \quad \rho v = y, \quad \rho w = z$$

If we use this polarity on the algebraic equations, formulas, etc., that lead up to a theorem (or other result), we see that we get the duals of this theorem and of the algebraic parts of its proof. For example, solving simultaneously the equations of the two conics

$$x^2 + y^2 = 16\,z^2, \quad 2\,xy = 9\,z^2$$

we find that these two curves intersect in the four common points $((\pm 5 \pm \sqrt{7})/2,\ (\pm 5 \mp \sqrt{7})/2,\ 1)$ and $((\pm 5 \mp \sqrt{7})/2,\ (\pm 5 \pm \sqrt{7})/2,\ 1)$ where the upper signs in each parentheses go together and so also the lower signs. Applying (115) we find that the two conics

$$u^2 + v^2 = 16\,w^2, \quad 2\,uv = 9\,w^2$$

have the four common tangents $[(\pm 5 \pm \sqrt{7})/2,\ (\pm 5 \mp \sqrt{7})/2,\ 1]$ and $[(\pm 5 \mp \sqrt{7})/2,\ (\pm 5 \pm \sqrt{7})/2,\ 1]$.

EXERCISES

1. If $\Delta = 0$ in (114), what geometrical effect does this vanishing of Δ have on the points and lines that correspond under (114)? Compare $\Delta = 0$ for (88).

2. Prove that (114) keeps cross-ratio invariant. What does this mean geometrically?

3. Show that (114) sends a point P' into a line l' and l' into P'.

4. Solve (114) for x,y,z and get (114′).

5. Find the equation in line coordinates (also find the point coordinates) of the point into which (114′) sends the line $\alpha x + \beta y + \gamma z = 0$.

6. Prove that (114) sends all the points on a line into all the lines through a point, and conversely.

7. Make up examples of polarities.

8. Find the fundamental conic of

$$\rho u = x + y, \quad \rho v = x + y + z, \quad \rho w = y + z$$

9. Find the polarities that have, respectively, the fundamental conics

$$2xy + 2xz + 2yz = 0, \quad \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = z^2, \quad y^2 = 4px$$

112. **Correlations in general.** The polarity (114) is only a special case of more general linear point-to-line (and line-to-point) transformations, called *correlations*, that are given by the equations

$$(116) \qquad \rho u = a_1x + a_2y + a_3z, \quad \rho v = b_1x + b_2y + b_3z, \quad \rho w = c_1x + c_2y + c_3z$$

where

$$\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0$$

with the inverse

$$(116') \qquad \sigma x = A_1u + B_1v + C_1w, \quad \sigma y = A_2u + B_2v + C_2w, \quad \sigma z = A_3u + B_3v + C_3w$$

A polarity is distinguished among the correlations (116) by the fact that *its determinant (matrix) is symmetric with respect to the main diagonal*, i.e., $a_2 = b_1$, $a_3 = c_1$, $b_3 = c_2$.

It is easy to prove (see the exercises) that *a polarity has only one fundamental conic, but a general correlation has two such conics* — one of them is in point coordinates and one in line coordinates — and these two are not the point and line equations of the same conic unless the correlation is a polarity.

Since (116′) is linear in both the point and the line coordinates, the correlations (and so of course the polarities) send the *points on a line* into *lines through a point* (for they send the equation of a line in point coordinates into the equation of a point in line coordinates), and conversely.

Also (116) can be obtained by following (88) by (115). Since (88) and also (115), as is evident, keep cross-ratio invariant, *therefore* (116) *keeps cross-ratio invariant.*

EXERCISES

1. Does Ex. 1 of §111 hold true here?
2. Show that (116) sends a point P' into a line l' but does not send l' into P'.
3. Show that if (116) sends one point P' into l' and l' into P', then this correlation is a polarity.
4. Derive (116′) from (116).
5. Prove that the two fundamental conics of a general correlation are distinct. Hint: Take the correlation first as (116), then as (116′). Find the relation between the two fundamental conics of a general correlation.
6. Find the fundamental conics of

$$\rho u = x + y + z, \quad \rho v = x - y - z, \quad \rho w = y + z$$

7. Make up examples of correlations and find their fundamental conics.
8. Using Ex. 5, make up the equations of a point conic and of a line conic that can serve as the fundamental conics of a correlation. Find the corresponding correlation.
9. Prove geometrically that a correlation sends points to lines and lines to points. See Veblen and Young, Vol. I.

CHAPTER XV

SOME THEOREMS ON CONICS

113. **Pencils of points and pencils of lines; parametric coordinates for points and for lines.** DEFINITION. Often we wish to consider the *points* on a line instead of the line itself as a *whole.* In this case we call the points of a line a *pencil of points.* Similarly, and dually, all the lines through a point we call a *pencil of lines.* The line on which a pencil of points lies is called the *axis* of the pencil. The point through which all the lines of a pencil pass is called the *center* of the pencil.

If we take any two lines through a point with equations

$$u_1x + v_1y + w_1z = 0 \quad \text{and} \quad u_2x + v_2y + w_2z = 0$$

respectively, we note that any line through this point is given by the equation (with a parameter λ)

$$(u_1 + \lambda u_2)x + (v_1 + \lambda u_2)y + (w_1 + \lambda w_2)z = 0$$

or (with homogeneous parameters λ_1 and λ_2) by the equation

$$(\lambda_1 u_1 + \lambda_2 u_2)x + (\lambda_1 v_1 + \lambda_2 v_2)y + (\lambda_1 w_1 + \lambda_2 w_2)z = 0$$

Therefore any line through the point of intersection of $l_1[u_1,v_1,w_1]$ and $l_2[u_2,v_2,w_2]$ has parametric coordinates given by

$$(117) \qquad \rho u = u_1 + \lambda u_2, \quad \rho v = v_1 + \lambda v_2, \quad \rho w = w_1 + \lambda w_2$$

or

$$(117') \qquad \rho u = \lambda_1 u_1 + \lambda_2 u_2, \quad \rho v = \lambda_1 v_1 + \lambda_2 v_2, \quad \rho w = \lambda_1 w_1 + \lambda_2 w_2$$

where λ, λ_1, λ_2 are called the parameters and $l_1[u_1,v_1,w_1]$ and $l_2[u_2,v_2,w_2]$ are called the *fundamental lines* of the pencil of lines.

Dually, if we take any two points (in line coordinates) on a line, with equations

$$x_1u + y_1v + z_1w = 0 \quad \text{and} \quad x_2u + y_2v + z_2w = 0$$

respectively, we note that any point on this line has an equation

of the form

$$(x_1 + \mu x_2)u + (y_1 + \mu y_2)v + (z_1 + \mu z_2)w = 0$$

or $$(\mu_1 x_1 + \mu_2 x_2)u + (\mu_1 y_1 + \mu_2 y_2)v + (\mu_1 z_1 + \mu_2 z_2)w = 0$$

Therefore any point on the given line through $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ has parametric coordinates given by

$$\sigma x = x_1 + \mu x_2, \quad \sigma y = y_1 + \mu y_2, \quad \sigma z = z_1 + \mu z_2 \tag{118}$$

or

$$\sigma x = \mu_1 x_1 + \mu_2 x_2, \quad \sigma y = \mu_1 y_1 + \mu_2 y_2, \quad \sigma z = \mu_1 z_1 + \mu_2 z_2 \tag{118'}$$

where $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ are called the *fundamental points* of the pencil of points.

We take the following cross-ratio of four collinear points (expressed in parametric coordinates) with parameters μ', μ'', μ''', μ^{IV} and we get

$$\begin{aligned}
&\frac{x' - x''}{x''' - x''} \frac{x''' - x^{IV}}{x' - x^{IV}} \\
&= \frac{x_1 + \mu' x_2 - x_1 - \mu'' x_2}{x_1 + \mu''' x_2 - x_1 - \mu'' x_2} \frac{x_1 + \mu''' x_2 - x_1 - \mu^{IV} x_2}{x_1 + \mu' x_2 - x_1 - \mu^{IV} x_2} \\
&= \frac{\mu' - \mu''}{\mu''' - \mu''} \frac{\mu''' - \mu^{IV}}{\mu' - \mu^{IV}}
\end{aligned}$$

We have a similar dual result for four concurrent lines. Hence *the cross-ratios of four collinear points (or of four concurrent lines) are the same as the cross-ratios of their parameters.*

For example, on the line $y = 0$ with (1,0) and (2,0) as fundamental points, the points

$$(3,0), \quad (5,0); \qquad (7,0), \quad (9,0)$$

can be written, respectively

$$(1 + 1 \cdot 2, 0), \quad (1 + 2 \cdot 2, 0), \quad (1 + 3 \cdot 2, 0), \quad (1 + 4 \cdot 2, 0)$$

Take the following cross-ratio, we have

$$\frac{3-5}{7-5}\frac{7-9}{3-9} = \frac{1-2}{3-2}\frac{3-4}{1-4} = -\frac{1}{3}$$

Using parametric coordinates we shall now show that *the cross-ratios of four points of a pencil of points are the same as the*

corresponding cross-ratios of the four lines of a pencil of lines that pass through the four given points, respectively.

PROOF. Suppose the four points are $P_i(x_i,y_i,z_i)$ where $i = 1,2,3,4$ and the four lines are $l_i[u_i,v_i,w_i]$. Suppose we have

$$u_1x_1 + v_1y_1 + w_1z_1 = 0 \quad \text{and} \quad u_2x_2 + v_2y_2 + w_2z_2 = 0$$

i.e., the lines l_1 and l_2 pass through the points P_1 and P_2, respectively. Taking this pair of lines and this pair of points as the fundamental lines and the fundamental points of the pencil of lines and the pencil of points, respectively, we have (117) and (116). If the line with parameter λ_i is to pass through the point with parameter μ_i, we must have

$$(u_1 + \lambda_i u_2)\ (x_1 + \mu_i x_2) + (v_1 + \lambda_i v_2)\ (y_1 + \mu_i y_2) + (w_1 + \lambda_i w_2)\ (z_1 + \mu_i z_2) = 0$$

From this equation we get

$$\lambda_i = -\frac{u_1x_2 + v_1y_2 + w_1z_2}{u_2x_1 + v_2y_1 + w_2z_1}\mu_i$$

which is of the form $\lambda_i = \alpha\mu_i$. But now we see that we have

$$\frac{\lambda' - \lambda''}{\lambda''' - \lambda''}\frac{\lambda''' - \lambda^{IV}}{\lambda' - \lambda^{IV}} = \frac{\mu' - \mu''}{\mu''' - \mu''}\frac{\mu''' - \mu^{IV}}{\mu' - \mu^{IV}}$$

i.e., the cross-ratios of the four lines have the same values as the corresponding cross-ratios of the four points.

An important result of this theorem is the fact that a *harmonic set of points* on a line l subtends from any point P not on l a *harmonic set of lines*. Geometrically, a harmonic set of points is determined by a *complete quadrangle*, but a harmonic set of lines is determined by a *complete quadrilateral*; therefore the above result is quite astonishing. (Compare, however, §28.)

The above theorem could have been proved (without the use of parametric coordinates) by taking the pencil of lines with Y_∞ for its center and the pencil of points with $y = 0$ as its axis, using a triangle of reference. Then the equation of any line through a point $(x_i,0)$ is $x = x_i$; or in homogeneous line coordinates such a line is $[1,0,-x_i]$. Taking w for w/u in the coordinates of these lines we have

$$\frac{w_1 - w_2}{w_3 - w_2}\frac{w_3 - w_4}{w_1 - w_4} = \frac{x_1 - x_2}{x_3 - x_2}\frac{x_3 - x_4}{x_1 - x_4}$$

This shows again that these lines have the same cross-ratios as their points of intersection with a line l.

Any change of the fundamental lines (or points) in the pencil of lines (or points) amounts to multiplying l_1 (or P_1) by one constant and l_2 (or P_2) by another constant and adding the results for a new first fundamental line (or point) and similarly for a second fundamental line (or point). *Hence such changes of fundamental lines (similarly for points) are given by the equations*

$$\tau\lambda_1' = a\lambda_1 + b\lambda_2, \quad \tau\lambda_2' = c\lambda_1 + d\lambda_2 \tag{119}$$

or in non-homogeneous form

$$\lambda' = \frac{c + d\lambda}{a + b\lambda} \tag{119'}$$

where $\begin{vmatrix} c & d \\ a & b \end{vmatrix} \neq 0.$

For points we have equations in μ instead of λ. Conversely, (119) changes the fundamental lines. *Note that* (119) *is of the form* (102) *and so keeps cross-ratio invariant.* As an illustration, suppose in the pencil of points

$$(\mu_1 7 + \mu_2 3, \ \mu_1 2 - \mu_2 4, \ \mu_1 1 + \mu_2 3)$$

we choose $(10,-2,4)$ and $(4,6,-2)$ as new fundamental points. Then we can write the pencil as

$$(\mu_1' 10 + \mu_2' 4, \ -\mu_1' 2 + \mu_2' 6, \ \mu_1' 4 - \mu_2' 2)$$

and (119) has the form (with μ for λ)

$$\tau\mu_1' = \mu_1 + \mu_2, \quad \tau\mu_2' = \mu_1 - \mu_2$$

EXERCISES

1. Why must we have $\begin{vmatrix} c & d \\ a & b \end{vmatrix} \neq 0$ for (119')?
2. Why is no loss of generality incurred in the first proof of the theorem in the text by taking the fundamental points and lines as they are taken?
3. Why is there no loss of generality in the second proof of the theorem in the text when Y_∞ is taken as the center of the pencil of lines and $y = 0$ is taken as the axis of the pencil of points?
4. Fill in any details omitted in the text.

5. The pencil of points

$$\rho x = 3 + 2\mu, \quad \rho y = 5 - \mu, \quad \rho z = 6 + 3\mu \quad \text{or} \quad (3 + 2\mu, 5 - \mu, 6 + 3\mu)$$

can be written as

$$\rho x = 8 + 5\mu', \quad \rho y = 9 + 4\mu', \quad \rho z = 15 + 9\mu'$$

or

$$(8 + 5\mu', 9 + 4\mu', 15 + 9\mu')$$

Find the coefficients for (119′) to accomplish this change of fundamental points. Hint: Put in homogeneous parametric form.

6. How does $\lambda' = (3 - \lambda)/(2 + \lambda)$ change the pencil of lines $[3 - \lambda, 2 + 3\lambda, -1 + \lambda]$?

7. Make up examples like Exs. 5 and 6.

8. Find the cross-ratios of the lines $x = 0$, $y = 0$, $x^2 + \alpha y^2 = 0$.

9. Show how to get parametric point coordinates for a pencil of points with $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ as fundamental points, from the equation of the line P_1P_2, namely,

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

Hint: Prove that a necessary and sufficient condition for this determinant to vanish is that we can multiply the second row by μ_1 and the third row by μ_2, add the results, and, subtracting this sum from the first row, obtain a row of zeros at the top of the determinant.

114. **A point (line) conic as determined by two projective pencils of lines (points).** First of all, we must find out about the effect of the general projectivity (88) on the *parametric* point and line coordinates. Consider any line

$$[u_1 + \lambda u_2, v_1 + \lambda v_2, w_1 + \lambda w_2]$$

of a pencil of lines. In point coordinates this line has the equation

$$(u_1x + v_1y + w_1z) + \lambda(u_2x + v_2y + w_2z) = 0$$

The general collineation (88) sends this line into

$$(u_1'x' + v_1'y' + w_1'z') + \alpha\lambda(u_2'x' + v_2'y' + w_2'z') = 0$$

where $\sigma u_1' = a_1u_1 + b_1v_1 + c_1w_1$, similarly for $\sigma v_1'$ and $\sigma w_1'$; also $\sigma' u_2' = a_1u_2 + b_1v_2 + c_1w_2$, similarly for $\sigma' v_2'$ and $\sigma' w_2'$; also $\alpha = \sigma'/\sigma$.

That is, if we put this result in line coordinates, (88) sends $[u_1 + \lambda u_2, \ v_1 + \lambda v_2, \ w_1 + \lambda w_2]$ into $[u_1' + \alpha\lambda u_2', \ v_1' + \alpha\lambda v_2', \ w_1' + \alpha\lambda w_2']$. Thus we see that, if we have two projective pencils of lines and we take for the fundamental lines in one of the pencils $l_1[u_1,v_1,w_1]$ and $l_2[u_2,v_2,w_2]$, while we take for the fundamental lines

of the second pencil the lines $l_1'[u_1',v_1',w_1']$ and $l_2'[u_2',v_2',w_2']$ that correspond under (88) to l_1 and l_2, respectively, then to establish the projectivity between the two given pencils $[u_1 + \lambda u_2,\ v_1 + \lambda v_2,\ w_1 + \lambda w_2]$ and $[u_1' + \lambda' u_2',\ v_1' + \lambda' v_2',\ w_1' + \lambda' w_2']$ we have merely to put

$$\lambda' = \alpha\lambda \tag{120}$$

Note that, if we have any other fundamental lines for the second pencil, we must first operate on this pencil with (119′) so as to have fundamental lines correspond to fundamental lines under the projectivity (88), then put $\lambda' = \alpha\lambda$, and we have the projectivity established. But the product of (119′) by (120) has the same form as (119′); hence *we can interpret this transformation of the parameter as effecting a projective transformation between the lines of the same or of different pencils of lines.*

(Remember that we are here looking upon the projectivity as an alibi; hence $x = x' = 0$, $y = y' = 0$, $z = z' = 0$ are the sides of one and the same triangle of reference or are the same axes plus l_∞, according to which frame of reference we are using.)

Now we have

$$\lambda = -\frac{u_1x + v_1y + w_1z}{u_2x + v_2y + w_2z}$$

from the first pencil of lines in the above discussion and since we have (after replacing x' by x, y' by y, z' by z in the second pencil of lines)

$$\lambda' = -\frac{u_1'x + v_1'y + w_1'z}{u_2'x + v_2'y + w_2'z}$$

The equation (120), or (119′) if the fundamental lines do not correspond under the projectivity, gives us a quadratic equation in x,y,z (i.e., a point conic).

Therefore we can describe a point conic as the locus of the points of intersection of the corresponding lines of two projective pencils of lines. Dually, a line conic can be described as the locus of the lines joining the corresponding points of two projective pencils of points.

For example, if the pencil of lines

$$(x + y - z) + \lambda(2x + y - z) = 0$$

is made projective with the pencil

$$(x + 2y - z) + \lambda'(x - 2y - z) = 0$$

by the equation $\lambda' = 3\lambda$, we have the point conic given by

$$-\frac{x+2y-z}{x-2y-z} = -3\frac{x+y-z}{2x+y-z}$$

or $$x^2 - 8y^2 + 2z^2 + 6yz - 3zx - 8xy = 0$$

Conversely, we want to prove that, if we take any two points P and P' on a conic and consider as pairs of corresponding lines in the two pencils of lines with centers at P and P', respectively, all the pairs of lines l,l' that intersect on the conic, then we have a projectivity between these two pencils of lines.

PROOF. If we take P as (1,0,0) and P' as (0,1,0) and also take (0,0,1) on the conic, then our conic has for its equation

(121) $$2hxy + 2gxz + 2fyz = 0$$

The two pencils of lines with centers at P and P' can be written, respectively,

$$y + \lambda z = 0 \quad \text{and} \quad x + \lambda' z = 0, \quad \therefore \lambda = -\frac{y}{z} \quad \text{and} \quad \lambda' = -\frac{x}{z}$$

Putting these values of y/z and x/z in (121), since the corresponding lines must intersect on the conic, we get

$$2h\lambda\lambda' - 2g\lambda' - 2f\lambda = 0, \quad \text{or} \quad \lambda' = \frac{f\lambda}{h\lambda - g}$$

which gives us a projectivity between the two pencils of lines. We do not get (120), hence the pairs of fundamental lines do not correspond under the projectivity.

Since three pairs of corresponding lines determine uniquely a projectivity between two pencils of lines (by the dual of the assumption in §105), therefore the line PP' cannot correspond to itself; otherwise, if l is the line joining the points of intersection of two pairs of corresponding lines l_1,l_1' and l_2,l_2', then the two pencils of lines with centers at P and P' are perspective (with l as axis). If the transformation is actually a perspectivity, then the conic we obtain degenerates into a pair of lines, l and PP'.

To obtain a non-degenerate conic we must have the transformation between the two pencils of lines a projectivity; therefore the line PP' considered as a line in the pencil with center at P must correspond to the tangent to the conic at P', and this same

line PP' considered as a line in the pencil with center at P' must correspond to the tangent to the conic at P. (If PP' corresponded to a line l_1 through P that cut the conic again at P'', then l_1 would correspond to two lines PP' and $P'P''$, contrary to the fact that a projectivity is a one-to-one correspondence.)

As an illustration of a degenerate conic, we see that $\lambda' = 2\lambda$ makes $x = 0$ in $x + \lambda z = 0$ correspond to $x = 0$ in $x + \lambda' y = 0$, and the resulting conic is $xz - 2xy = 0$. Here $x = 0$ is the common line of the two pencils of lines, whereas $z = 2y$ is the axis of the perspectivity set up between the two pencils by $\lambda' = 2\lambda$.

EXERCISES

1. Recast the argument in the first part of the text in terms of line coordinates.

2. Dualize the discussion in the text and get a line conic. What is the degenerate case of such a line conic?

3. If a non-degenerate conic is to be a hyperbola, the two projective pencils of lines that generate this conic must have two pairs of corresponding lines that are pairs of parallel lines. Why? How about these two pencils of lines if the conic is to be a parabola, or an ellipse? Why cannot there be three pairs of parallel corresponding lines?

4. Given the two pencils of lines

$$[3 + 2\lambda, 4 + 5\lambda, 6 - \lambda] \quad \text{and} \quad [1 - 2\lambda', 2 + \lambda', \lambda']$$

find five points on the conic determined by the projectivity

$$\lambda' = \frac{3 - \lambda}{2 + \lambda}$$

between these two pencils, without getting the equation of the conic in point coordinates.

5. Find the equation in point coordinates of the conic in Ex. 4. Check the five points found in Ex. 4 by substituting their coordinates in this equation.

6. Find the equation of the conic given by $x + \lambda z = 0$, $y + \lambda' z = 0$, and $\lambda' = 2\lambda/(\lambda + 2)$.

7. Find the projectivity (in parameters) between the pencils of lines with centers at (0,1,0) and (1,0,0) on the conic $xy = z^2$; on the conic $xy - xz + z^2 = 0$.

8. Do as in Ex. 7 for the pencils of lines with centers at (1,1,1) and $(-1,-1,1)$ on the conic $xy = z^2$.

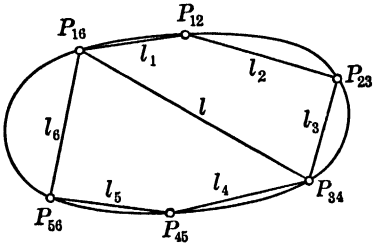
9. Why is there no loss of generality in the text due to taking the pencils of lines with centers at (1,0,0) and (0,1,0), and also taking the conic in the form (121)?

115. **The Pascal line (Brianchon point) of a hexagon inscribed in (circumscribed about) a conic.** Before discussing hexagons inscribed in and circumscribed about a conic, we want to *abridge* our notation somewhat. Compare §42.

If we represent a line $l_i \equiv \alpha_i x + \beta_i y + \gamma_i z = 0$ by the equation $l_i = 0$, we see that the equation

$$l_1 l_3 + \lambda l_2 l = 0$$

is the equation of a conic (see the adjoining figure) through the four points P_{12}, P_{23}, P_{34}, P_{16} for every value of λ. We can determine $\lambda = \lambda'$ so that this conic will also pass through the point P_{45}.



Suppose P_{56} is also on this conic. Now the equation

$$l_4 l_6 + \mu l_5 l = 0$$

is the equation (for every value of μ) of a conic through the four points P_{16}, P_{56}, P_{45}, P_{34}. We can determine $\mu = \mu'$ so that this second conic passes through the point P_{12}. Hence these two conics $l_1 l_3 + \lambda' l_2 l = 0$ and $l_4 l_6 + \mu' l_5 l = 0$ must be the same conic (having five points in common).

We suppose the above two equations of the same conic are such that when expanded they are identical as to coefficients and not simply with their coefficients proportional (this we can secure by dividing the second equation through by a suitable constant). Next we subtract the two equations and get

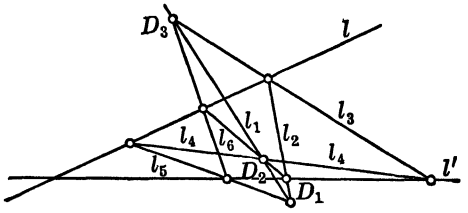
$$l_1 l_3 - l_4 l_6 - l(\mu' l_5 - \lambda' l_2) \equiv 0 \quad \text{or} \quad l_1 l_3 - l_4 l_6 \equiv l(\mu' l_5 - \lambda' l_2)$$

since the two equations are identical.

The left side of the second of the above identities must be factorable, with l as one factor and $\mu' l_5 - \lambda' l_2$ as the other factor. But l_1 cuts l_6 on l and l_3 cuts l_4 on l; hence l_1 must cut l_4 in a point P' and l_3 cut l_6 in a point P'' such that these two points P' and P'' lie on the line $\mu' l_5 - \lambda' l_2 = 0$ (which line also passes through the point of intersection of l_2 and l_5). This line is called the Pascal line of the inscribed hexagon. By duality we get the Brianchon point of a hexagon circumscribed about a conic.

In the above discussion the conic might be a pair of lines, and

the proof would still be valid. Suppose the hexagon is inscribed in a pair of lines l,l' (as in the following figure), then the three points D_1, D_2, D_3 are collinear (by Pascal's theorem).



In the hexagon P_1, P_2, P_3, P_4, P_5, P_6 inscribed in a conic we shall call the pairs of sides P_1P_2 and P_4P_5, P_2P_3 and P_5P_6, P_4P_5 and P_6P_1, pairs of opposite sides (since a single vertex separates the vertices of the sides of each pair; thus P_3 separates P_2 and P_4, P_6 separates P_5 and P_1, so P_1P_2 and P_4P_5 are opposite sides). We can now state Pascal's and Brianchon's theorems as follows:

THEOREM. *If a hexagon is inscribed in a conic, the three pairs of opposite sides intersect in three collinear points (on the Pascal line).*

THEOREM. *If a hexagon is circumscribed about a conic, the three pairs of opposite vertices lie on three concurrent lines (that intersect in the Brianchon point).*

Note that if we change the order in which we join the vertices P_1, P_2, P_3, P_4, P_5, P_6 we get a *new* hexagon with a *new* Pascal line. Thus the hexagon with sides $l_1(P_1P_3)$, $l_2(P_3P_5)$, $l_3(P_5P_2)$, $l_4(P_2P_4)$, $l_5(P_4P_6)$, $l_6(P_6P_1)$ has a different Pascal line from that in the above figure.

EXERCISES

1. In the conic $x^2/9 + y^2/16 = 1$ show analytically that Pascal's theorem is satisfied by the hexagon with vertices (taken in this order):

$$(0,4),\ (3,0),\ (0,-4),\ (-3,0),\ (3/\sqrt{2},4/\sqrt{2}),\ (-3/\sqrt{2},-4/\sqrt{2})$$

Find the Pascal line.

2. Change the order of joining up the vertices in Ex. 1 and find the new Pascal line.

3. We can reduce the general conic to $xy = z^2$. (Compare §122.) Take a general hexagon $(x_i,\ 1/x_i,\ 1)$ inscribed in this conic (where $i = 1,2,3,4,5,6$) and prove Pascal's theorem analytically for this hexagon, using the actual equations of the opposite sides. Does this constitute a general proof of Pascal's theorem? Why, or why not?

4. Draw illustrations of Pascal's theorem and its dual, using hyperbolas, ellipses, and parabolas.

5. Draw an illustration of Pascal's theorem (and one of its dual theorems), then join up the six vertices (take the intersections of the six sides) in different orders, show how to get other hexagons and different Pascal lines (Brianchon points).

6. Dualize the discussion in the text and obtain the Brianchon point of a circumscribed hexagon.

7. Describe the case where $P_{16} \equiv P_{12}$, i.e., where l_1 is a tangent. Dualize this case.

8. Show how to use the Pascal line to get a sixth point P_6 on a conic determined by five points P_1, P_2, P_3, P_4, P_5 where P_6 is on a given line l through P_1. Dualize this discussion.

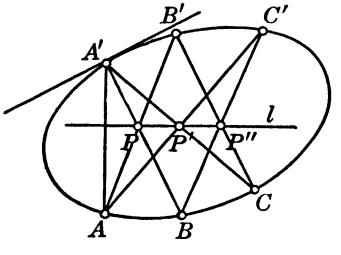
9. Show how to get the tangent at P_{12} on the conic of Ex. 7. Dualize this result.

10. Discuss the Pascal line for the other cases of coincidence of pairs of the six vertices of the hexagon, such as $P_2 \equiv P_3$ and $P_4 \equiv P_5$, etc.

11. Work Ex. 8 for the other cases of Ex. 10.

12. Dualize Ex. 10.

116. **Projectivities on a conic; center and axis of homology.** The conic $xy = z^2$ is sent into itself by every projectivity of the form $\rho x = ax'$, $\rho y = 1/ay'$, $\rho z = z'$ as well as by numerous other types of projectivities. Compare §20. Each of these projectivities T sets up a correspondence between the points of the conic such that A and A', B and B', C and C', etc. (see the adjoining figure) are pairs of homologous points.



We see that the pencil of lines $A'A$, $A'B$, $A'C$, etc., is projective under this collineation T with itself in such a way that $A'A$ corresponds to the tangent at A', $A'B$ to $A'B'$, $A'C$ to $A'C'$, etc. But we saw in §114 that since all these points lie on a conic, therefore the pencil of lines with center at A is projective with the pencil of lines with center at A' in such a way that AA' corresponds to the tangent at A', AB' to $A'B'$, AC' to $A'C'$, etc.

Combining these two projectivities, we find that the two pencils of lines with centers at A and A' are projective with each other in such a way that AA' corresponds to $A'A$, AB' to $A'B$, AC' to $A'C$, ... But in this last projectivity AA' is self-correspond-

ing, so the correspondence must in reality be a perspectivity with axis $l(PP')$ where P is the intersection of $A'B$ and AB' and P' the intersection of $A'C$ and AC'. Compare §105.

We see that l is the Pascal line of the hexagon A, B', C, A', B, C'; therefore BC' intersects $B'C$ on l in a point P''. A similar discussion shows that any other pair of corresponding points D and D' on the conic must be such that AD' and $A'D$ intersect on l, also BD' and $B'D$, CD' and $C'D$.

DEFINITION. The line l whose existence is proved in the last paragraph is called the *axis of homology* for the projective relation between the *points* on the conic. Dual to l we have a point P related similarly to the *tangents* at corresponding points on the conic, and called the *center of homology* for this projectivity on the conic.

EXERCISES

1. If T is $\rho x = y'$, $\rho y = x'$, $\rho z = z'$ and the conic is $xy = z^2$, find the axis of homology.

2. Find the center of homology for Ex. 1. Hint: Put into line coordinates.

3. Give the details of the discussion in the text showing that AD' and $A'D$, BD' and $B'D$, CD' and $C''D$ intersect on l. Hint: Consider the hexagons A, B', D, A', B, D' and A, C'', D, A', C, D'.

4. Dualize the discussion in the text and get the center of homology P.

5. Prove that P and l are pole and polar with respect to the conic. Compare §50.

6. Prove that a projectivity on a conic is uniquely determined by three pairs of corresponding points. Hint: If P and P', R and R', S and S' are these pairs of homologous points, the tangent p at P goes to the tangent p' at P', the tangent r at R goes to r' at R', the point $T(p,r)$ goes to $T'(p',r')$, S goes to S'. Show that this uniquely determines the conic, using Pascal's theorem on a line through S.

7. Dualize Ex. 6.

8. Prove that a projectivity on a conic is uniquely determined by its axis of homology and one pair of corresponding points.

9. Dualize Ex. 8.

10. Prove that every double point of a projectivity on a conic must be a point of intersection of the axis of homology with the conic.

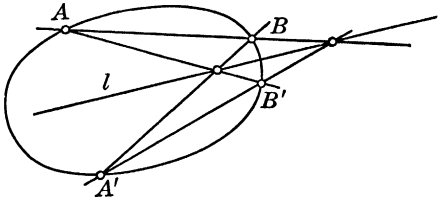
11. When is a projectivity on a conic parabolic?

12. Dualize Ex. 10.

117. **Involutions on a conic.** If a projectivity on a conic is an involution, then (see the following figure) A must correspond to A', also A' to A, B to B', and B' to B. Therefore the lines AB' and $A'B$, AB and $A'B'$ must intersect on the axis of homology l.

Why? Also AA' and BB' intersect in the center of homology P. Why?

Again, suppose a projectivity Π on a conic C sends A to A' and A' to A''. If I_1 is the involution on C with A' as a double point, and AA'' as a conjugate pair, then $I_1\Pi$ sends A to A' and



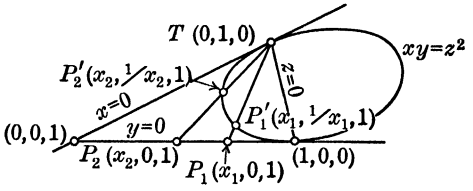
A' to A. Hence $I_1\Pi$ is an involution I_2. Why? But $I_1\Pi = I_2$ gives us $\Pi = I_1I_2$. (How?) Therefore we have the theorem:

THEOREM. *Any projectivity* Π *on a conic is the product of two involutions* I_1, I_2 *either of which has an arbitrary point* A' *(not a double point of* Π*) as a double point.*

EXERCISES

1. Answer the (How's?) and (Why's?) in the text. Hint: Since $I_1\Pi$ sends A to A' and A' to A, if B and B' are any other pair of corresponding points, AB' cuts $A'B$ on the axis of homology l, also AB cuts $A'B'$ on l. (Why?) This shows that B corresponds to B' as well as B' to B. (Why?) Hint: If $I_1\Pi = I_2$, then $I_2I_1\Pi = \Pi$. (Why?)
2. If I_2 has AA'' as a conjugate pair and A' as a double point, show that $\Pi I_2'$ sends A' to A'' and A'' to A', so $\Pi I_2' = I_1'$ and $\Pi = I_1'I_2'$.
3. For Π of the form $\rho x = y'$, $\rho y = x'$, $\rho z = z'$ and the conic $xy = z^2$, find I_1 and I_2 analytically so that $\Pi = I_1I_2$.

118. One-to-one correspondences between the points on a conic and the points on a line. *It is often found useful to set up a*



one-to-one correspondence between the points P *of a line* l *and the points* P' *of a conic* C. Then we can prove readily some theorems about the points P' on C and by this one-to-one correspondence

interpret these theorems in terms of theorems about the points P on the line l. In the above figure we see one way of setting up this one-to-one correspondence for the conic $xy = z^2$ and the line $y = 0$, by means of lines through the point $T(0,1,0)$. We see that (1,0,0) corresponds to itself and (0,0,1) corresponds to 0,1,0).

It is readily seen that the relation between the coordinates of the points $P(x,y,z)$ on $(y = 0)$ and $P'(x',y',z')$ on $C(xy = z^2)$ is given by

$$T: \qquad \rho x = x', \quad \rho y = \frac{z'^2}{y'} - x', \quad \rho z = z'$$

which is not linear in the variables and so is not a projectivity.

However, even though T is not a projectivity, we note that T makes a projectivity on the line correspond to a projectivity on the conic, and so of course an involution to an involution (because three pairs of homologous points determine a projectivity either for the line or for the conic).

EXERCISES

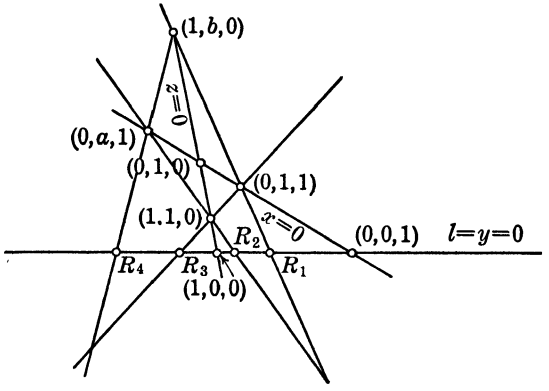
1. Using the one-to-one correspondence given in this section show that any projectivity on a line is the product of two involutions.
2. Dualize the discussion in the text.

CHAPTER XVI

SOME THEOREMS ABOUT COMPLETE QUADRANGLES

119. **Two theorems on the complete quadrangle.** First we wish to prove:

THEOREM. *The pairs of opposite sides of any complete quadrangle P_1, P_2, P_3, P_4 cut any given line l in pairs of conjugate points of an involution.*



PROOF. We take the quadrangle and line l as in the above figure. The line

$$\begin{vmatrix} x & y & z \\ 0 & a & 1 \\ 1 & 1 & 0 \end{vmatrix} \equiv y - az - x = 0$$

cuts $y = 0$ (i.e., l) in the point $R_2(-a,0,1)$. The line

$$\begin{vmatrix} x & y & z \\ 1 & b & 0 \\ 0 & 1 & 1 \end{vmatrix} \equiv bx + z - y = 0$$

cuts $y = 0$ in $R_1(1,0,-b)$. The point R_3 is $(-1,0,1)$. The line

$$\begin{vmatrix} x & y & z \\ 1 & b & 0 \\ 0 & a & 1 \end{vmatrix} \equiv bx + az - y = 0$$

cuts $y = 0$ in $R_4(-a,0,b)$. Writing the six points on l in non-homogeneous coordinates, we have the quadrangular set

$$(0,0),\ (\infty,0);\quad (-\frac{1}{b},0),\ (-a,0);\quad (-1,0),\ (-\frac{a}{b},0)$$

We wish to show that these points are pairs of conjugate points in the same involution on l. Since 0 and ∞ are to be a conjugate pair, such an involution must have an equation of the form $x = \alpha/x'$. Since $(-a,0)$ must correspond to $(-1/b,0)$ in this involution, we have $-1/b = -\alpha/a$; so $\alpha = a/b$, and the involution must have the equation $x = a/(bx')$. This involution sends $(-a/b,0)$ to $(-1,0)$ since $-1 = a/b(-a/b)$.

Q.E.D.

Next we wish to prove:

THEOREM. *If a complete quadrangle is inscribed in a conic, this conic cuts any line l in a pair of conjugate points of the involution determined on l by this complete quadrangle.* (See the above theorem.)

PROOF. Referring to the complete quadrangle of the preceding theorem, we see that such a conic has the equation

$$xz - \lambda(y - az - x)\ (bx - y + z) = 0$$

because this equation is of the second degree and evidently gives a curve passing through the four vertices of the above complete quadrangle, namely,

$$(0,1,1),\ (1,1,0),\ (0,a,1),\ (1,b,0)$$

But this conic (for every value of λ) cuts $y = 0$ in points given by

$$\lambda bx^2 + (\lambda ab + \lambda + 1)xz + \lambda az^2 = 0$$

i.e., in the two points

$$\frac{x}{z} = \frac{-\lambda ab - \lambda - 1 \pm \sqrt{(\lambda ab + \lambda + 1)^2 - 4\lambda^2 ab}}{2\lambda b}$$

Now the above involution $x = a/(bx')$ (determined by the complete quadrangle) sends

$$x = \frac{-\lambda ab - \lambda - 1 + \sqrt{(\lambda ab + \lambda + 1)^2 - 4\lambda^2 ab}}{2\lambda b}$$

into

$$x = \frac{2a\lambda}{(-\lambda ab - \lambda - 1) + \sqrt{(\lambda ab + \lambda + 1)^2 - 4\lambda^2 ab}}$$
$$= \frac{-\lambda ab - \lambda - 1 - \sqrt{(\lambda ab + \lambda + 1)^2 - 4\lambda^2 ab}}{2\lambda b}$$

Q.E.D.

Any two points P, P' are said to be *conjugate* with respect to a conic C if the *polar* of P with respect to C *passes through* P'. We now prove that *on any line l the pairs of conjugate points with respect to C form pairs of an involution of which the double points are the points of intersection of l with C.*

Taking the *triangle of reference self-polar* with respect to C, we have C in the form

$$ax^2 + by^2 + cz^2 = 0$$

We suppose that l is $z = 0$. The polar of any point $P(x',y',0)$ with respect to C is $axx' + byy' = 0$. This polar cuts $z = 0$ in the point $P'(1, -ax'/by',0)$. The polar of $(1,0,0)$ is $x = 0$, which cuts $z = 0$ in $(0,1,0)$. In non-homogeneous coordinates the above pairs of conjugate points are

$$0, \infty; \quad y_1, -\frac{a}{by_1}$$

where $y_1 = y'/x'$.

Evidently the involution $y = -a/by'$ sends 0 to ∞ and y_1 to $-a/by_1$. Also the double points of this involution are given by $y^2 = -a/b$. But $z = 0$ cuts the conic C in these same points.

EXERCISES

1. Dualize all the discussion in the text.

2. Prove the converse of the first theorem; namely, that if six points on a line l are pairs of conjugate points of an involution, then these six points form a quadrangular set. Hint: Take l as $y = 0$; the points as $(0,0)$, $(\infty,0)$, $(-1/b,0)$, $(-a,0)$, $(-1,0)$, $(-a/b,0)$; and the involution as $x = a/bx'$. Take any point P not on l as $(0,1,0)$. Now show how a complete quadrangle can be constructed that will determine this set of points on l.

3. Treat the two special cases of the first theorem in the text, namely, where $R_1 = R_2$ but $R_3 \neq R_4$, and where $R_1 = R_2$ also $R_3 = R_4$.

4. Show why the two proofs in the text do not lose generality because of the special choice of the triangle of reference that is involved in them.

5. Prove that a quadrangular set of points is sent by a collineation into a quadrangular set of points and by a correlation into a quadrangular set of lines. Hint: What happens to the complete quadrangle?

6. We have seen (where?) that three pairs of homologous points determine a projectivity of points on a line. If in the first theorem of the text we determine such a projectivity on l so as to send $R_2, (1,0,0), R_3$, into $R_1, (0,0,1), R_4$, how does this show that the given quadrangular set of points can be determined by a complete quadrangle for which the role of point triple and triangle triple is interchanged from that in the figure? Hint: Consider what that collineation that induces the projectivity on l does to the quadrangle $(1,b,0)$, $(0,a,1)$, $(1,1,0)$, $(0,1,1)$.

7. Prove that all the conics of any pencil of conics through four given points cut a line l in pairs of conjugate points of an involution. Hint: Use the second theorem in the text.

8. Prove that a collineation (or correlation) sends an involution into an involution. Hint: Use the first theorem in the text.

9. Fill out all the details omitted in the text.

10. How does the generation of a conic by means of two projective pencils of lines show that five points determine a conic? Hint: Take the two centers P, P' of these pencils of lines. How many pairs of lines determine a projectivity between these two pencils? How about P and P'?

11. If in Ex. 10 we use the line PP' and the tangent at P' as one pair of homologous lines in the projectivity between the two pencils of lines with centers at P and P', show that two more points determine the conic. Thus, to say that a conic is tangent to a given line l at a given point P is equivalent to giving two points on the conic. Note that the centers of the two pencils that generate a conic may be any two points on the conic. (Why?)

12. If in Ex. 10 we use PP' and the tangent at P' as one pair of homologous lines, also $P'P$ and the tangent at P as another pair, then one more point determines the conic. (Why?) Hence show that to have given the asymptotes of a hyperbola leaves one more point to determine uniquely the hyperbola.

120. A theorem about a complete quadrangle inscribed in a conic.

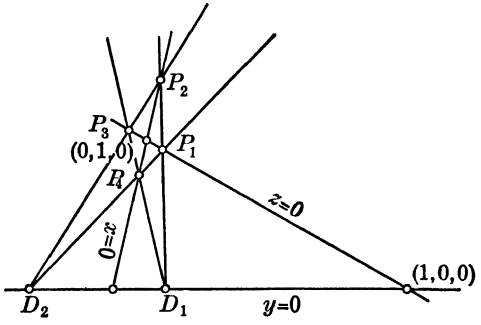
We shall prove now

THEOREM. *If a complete quadrangle P_1, P_2, P_3, P_4 is inscribed in a conic, then this complete quadrangle and the complete quadrilateral formed by the tangents to the conic at the vertices of this quadrangle both have the same diagonal triangle.*

We take this quadrangle with its four vertices on the sides $x = 0$ and $z = 0$ of the triangle of reference (as in the figure on page 270) and $y = 0$, passing through the two remaining diagonal points.

Hence P_1, P_3, $(0,1,0)$, $(1,0,0)$ form a harmonic set (note the complete quadrangle D_1, D_2, P_2, P_4); also P_2, P_4, $(0,1,0)$, $(0,0,1)$ form a harmonic set (note the quadrangle D_1, D_2, P_1, P_3). There-

fore if we take P_1 as (1,1,0), then P_3 is (−1,1,0); if we take P_4 as (0,1,1), then P_2 is (0,−1,1). Then P_1P_2 is $x - z - y = 0$ and cuts $y = 0$ in $D_1 \equiv (1,0,1)$.



But D_1, D_2, (0,0,1), (1,0,0) form a harmonic set, so D_2 is (−1,0,1). Accordingly the vertices of the diagonal triangle of this given complete quadrangle are (0,1,0), (−1,0,1), (1,0,1) and the sides are

$$y = 0, \quad x + z = 0, \quad x - z = 0$$

The line P_3P_4 is $x - z + y = 0$, P_1P_4 is $x + z - y = 0$, P_2P_3 is $y + z + x = 0$.

Any conic through P_1, P_2, P_3, P_4 has an equation of the form

$$(x + z - y)\ (x + y + z) + 2\lambda xz = 0$$

or

$$C \equiv x^2 - y^2 + z^2 + 2(1 + \lambda)xz = 0$$

The tangent to C at any point $P'(x',y',z')$ is

$$xx' - yy' + zz' + (1 + \lambda)\ (x'z + xz') = 0$$

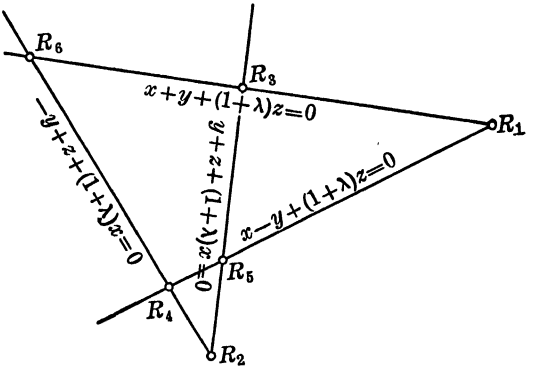
The tangents at (1,1,0), (−1,1,0), (0,1,1), and (0,−1,1), respectively, are

$$x - y + (1 + \lambda)z = 0, \quad x + y + (1 + \lambda)z = 0,$$
$$-y + z + (1 + \lambda)x = 0, \quad y + z + (1 + \lambda)x = 0$$

We draw a picture of these lines. (See page 271.)

The pairs of opposite vertices R_1 and R_2, R_3 and R_4, R_5 and R_6 of this complete quadrilateral could now be found. But it is easier to note that $y = 0$ is the line R_1R_2 (obtained by combining suitably each of the pairs of opposite sides of the quadrilateral);

$x = -z$ is the line R_6R_5; $x = z$ is the line R_3R_4. Hence the complete quadrilateral has the same diagonal triangle as the complete quadrangle.



EXERCISES

1. Dualize the discussion in the text.
2. Show why the proof in the text does not lose generality from the special choice of the triangle of reference.
3. Find the vertices of the complete quadrilateral in the text, and so find its diagonal triangle.
4. Prove that the converse of the theorem in the text is true.
5. Prove the theorem in the text by first reducing the equation to the form

$$2xy + 4yz + 2zx = 0$$

then taking the complete quadrangle as (0,0,1), (0,1,0), (1,0,0), (−1,1,1). Hint: To reduce C to this form take the triangle of reference on the conic, then use a suitable transformation of the form

$$x = \alpha x', \quad y = \beta y', \quad z = \gamma z'$$

Why is there no loss of generality in this proof?

6. Check all the algebra in the text.

121. **Four theorems on quadrangular sets and on cross-ratio.** We prove now:

THEOREM. *A transformation T between points of the same line (or of different lines) that preserves cross-ratio is a projectivity.*

PROOF. Suppose that T sends x_2, x_3, x_4 into y_2, y_3, y_4, respectively, and preserves cross-ratio. Then any point x must go into a point y such that we have

$$\frac{y - y_2}{y_3 - y_2}\frac{y_3 - y_4}{y - y_4} = \frac{x - x_2}{x_3 - x_2}\frac{x_3 - x_4}{x - x_4}$$

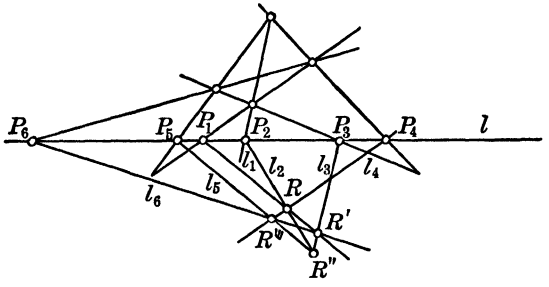
If we solve the above equation for y in terms of x, we get an equation of the form (103), i.e., a projectivity.

Note, however, that this projectivity (103) may be *induced* by a transformation that is *not* a collineation. For example,

$$\rho x = a_1x' + a_2y' + a_3z'^2, \quad \rho y = b_1x' + b_2y' + b_3z'^3, \quad \rho z = c_3z'^4$$

induces on the line $z = z' = 0$ a projectivity, but this transformation in the plane is not a collineation because of the presence in its equations of the terms in z'^2, z'^3, and z'^4 so the transformation does not send all straight lines into straight lines.

Next we consider a quadrangular set of points P_1, P_2, P_3, P_4 on a line l (as in the following figure) with P_1, P_2, P_3 as a point



triple and P_4, P_5, P_6 as a triangle triple. Compare §69. We shall now prove:

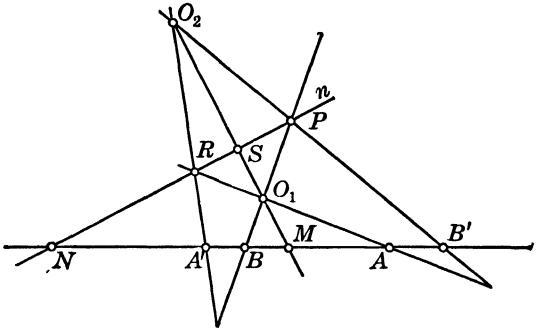
THEOREM. *There exists a complete quadrangle determining the same quadrangular set P_1, P_2, P_3, P_4, P_5, P_6, but for which the former point triple P_1, P_2, P_3 is now a triangle triple and the former triangle triple P_4, P_5, P_6 is now a point triple.*

PROOF. Through P_1, P_2, P_3 draw three arbitrary lines l_1, l_2, l_3, respectively, intersecting in the three points $R(l_1l_2)$, $R'(l_1l_3)$, $R''(l_2l_3)$. Since P_1 and P_5, P_2 and P_6, P_3 and P_4 are the pairs of the involution cut on l by the first quadrangle, we want to have the new quadrangle determine the same involution by its pairs of opposite sides. Hence we join R' to P_6 by a line l_6 and R to P_4 by a line l_4, then l_6 and l_4 intersect in the fourth vertex R''' of the required complete quadrangle. Now the line $R''R'''$ must pass through P_5 because the two complete quadrangles determine the same involution on l.

We next prove:

THEOREM. *A necessary and sufficient condition for the points M, N, A, B on a line l to be projective with the points M, N, A', B' on l (where $M \neq N$) is that the points M, A, B, N, B', A' form a quadrangular set where M and N, A and B', B and A' are the pairs of points determined on l by the pairs of opposite sides of a complete quadrangle; also M,A,B is a point (or triangle) triple and N,B',A' is a triangle (or point) triple.*

PROOF. *In the first place,* if these are points in a quadrangular set as in the following figure, then the two centers O_1 and O_2 with the auxiliary line n generate the above projectivity.

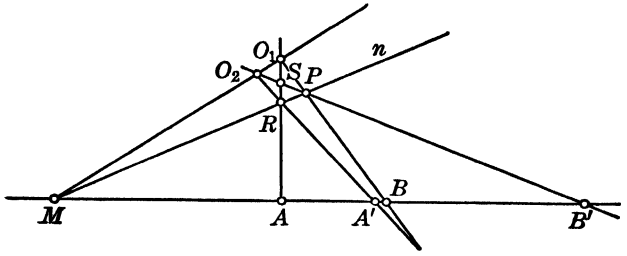


In the second place, if the points on l are projective as stated above, we take any line n through N and an arbitrary line O_1O_2 through M. With any point O_1 on O_1O_2 as center, we send B to P, A to R, N to N, M to S (the points P, R, S lying on n), by a perspectivity. With O_2 (determined as the intersection of $A'R$ and O_1O_2) as a new center, we send R to A', S to M, N to N, by a perspectivity. But three pairs of corresponding points determine a projectivity, therefore O_2P must pass through B', and we have the complete quadrangle O_1, O_2, P, R that determines the given set of points as a quadrangular set as described in the theorem.

Q.E.D.

Finally, we note that *a necessary and sufficient condition for M, M, A, B on a line l to be projective with M, M, A', B' on l is that these points form a quadrangular set as in the figure on page* 274. The proof is so like the one given above that we leave it for the student in the exercises.

Note that these last two theorems give us a way to determine *geometrically* corresponding points in a hyperbolic or a parabolic projectivity on a line. Compare the use of *harmonic sets* to determine corresponding points of a hyperbolic involution. (See §104.)



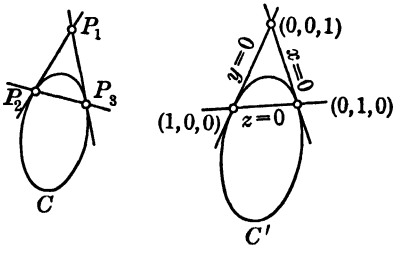
EXERCISES

1. Carry out the proof in the last paragraph.
2. Dualize the discussions in the text.
3. Set up geometrically several pairs of corresponding points in a given hyperbolic projectivity on a line l, in a given parabolic projectivity, in a given hyperbolic involution, where in each case the double points (or point) are given.
4. In the first paragraph of the text solve the equation for y in terms of x.
5. In the proof of the second theorem in the text explain in full why the line $R''R'''$ must pass through P_5.
6. Interpret the result in the second theorem for the case of a harmonic set. Hint: Note the positions of the diagonal points of the two complete quadrangles that enter into the theorem.

CHAPTER XVII

FURTHER DISCUSSION OF n-ICS

122. **Reduction of a conic and of a cubic to simpler forms.** To illustrate a use of the properties of general projectivities, we shall send the general conic C (75) into a conic with a simpler equation. We take a point P_1 outside C and the points of contact P_2 and P_3 of the two tangents to C from P_1 (see the adjoining figure), and we send these three points into the points (0,0,1), (1,0,0), (0,1,0), respectively, by a collineation. This sends C into some conic C' tangent to $y = 0$ at (1,0,0) and tangent to $x = 0$ at (0,1,0).



Let us now determine what is the form of the equation of C'. In regard to the equation (75), if this is to be the equation of C', $x = 0$ must give $z^2 = 0$ when solved simultaneously with (75), hence $b = f = 0$; $y = 0$ must give $z^2 = 0$, hence $a = g = 0$. Therefore the equation of C' must have the form

$$cz^2 + 2\,hxy = 0$$

where $ch \neq 0$.

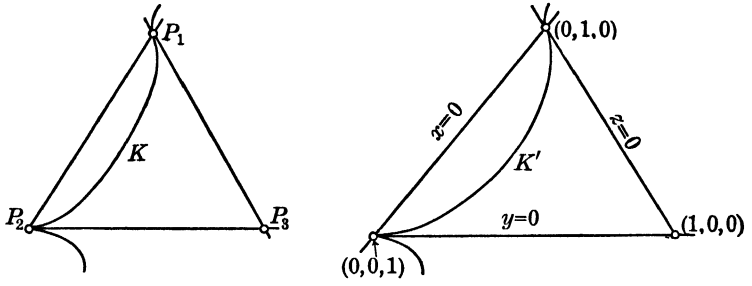
We can suppose (1,1,1) is on C' (this amounts to the choice of a fourth pair of corresponding points in the projectivity, and two homologous complete quadrangles determine uniquely such a projectivity); hence $c + 2\,h = 0$. The conic C' that is desired has therefore the equation $xy = z^2$. Another way to make this final reduction is to put

$$x = x', \quad y = \frac{-c}{2\,h}y', \quad z = z'$$

in $cz^2 + 2\,hxy = 0$, and we get $x'y' = z'^2$.

Next let us take a *cubic* K with a *cusp* and an *inflection* on it. (See the figure on page 276.) Suppose P_1 is the inflection with a tangent P_1P_3; P_2 is the cusp with P_2P_3 as its tangent. We send

the cubic K by (88) into a cubic K' with (0,0,1) as a cusp and $y = 0$ as its tangent, with (0,1,0) as an inflection and $z = 0$ as its tangent. Then P_1, P_2, P_3 must go into (0,1,0), (0,0,1), (1,0,0), respectively, since a projectivity sends a cusp into a cusp, an inflection into an inflection, a tangent into a tangent.



We want to find the form of the equation of K' by considering the general cubic (38). In (38) $z = 0$ must give $x^3 = 0$, hence $b = d = e = 0$; $y = 0$ must give $x^3 = 0$, hence $c = f = g = 0$. Every line $y = mx$ when solved with (38) must give an equation of the form

$$x^2(\alpha x + \beta z) = 0$$

since (0,0,1) is a double point, hence $j = 0$. Now our cubic K' has the form

$$ax^3 + yz(hy + kx) = 0$$

But (0,0,1) is a cusp with $y = 0$ as a tangent, hence $k = 0$ (otherwise $hy + kx = 0$ would be another tangent at this double point). We put (since $ah \neq 0$)

$$x = x', \quad y = y', \quad z = -\frac{a}{h}z'$$

and our desired cubic K' is $y'^2z' = x'^3$.

EXERCISES

1. Why do we have $ah \neq 0$ in the cubic in the text?
2. Show that $y = x^3$ has a cusp at infinity; that $y^2 = x^3$ has an inflection at infinity.
3. Reduce the cubic with a crunode and an inflection to the form

$$x^3 + y^2z + xyz = 0$$

Hint: Make use of the last three or more sentences in the text. A crunode is a double point with real tangents.

4. Reduce the cubic with an acnode (i.e., a double point with imaginary tangents) and an inflection to the form

$$x^3 + x^2z + y^2z = 0$$

Hint: Here (0,0,1) is the acnode with $y = \pm ix$ as tangents, and (0,1,0) is the inflection with $z = 0$ as tangent. If (38) is to have $x^2 + y^2 = 0$ as the tangents at (0,0,1), we must have $f = h$. Why?

5. Reduce the general conic to

$$x^2 + y^2 \pm z^2 = 0$$

Hint: Take a self-polar triangle as the triangle of reference; then put $x = \alpha x', y = \beta y', z = \gamma z'$ and determine α, β, γ so as to get the above equation.

6. Interpret the discussion in the text as an *alias*.

7. Explain in full why $j = 0$ in the text when reducing the cubic with a cusp.

8. In the three simple forms of cubics given in the examples above and in the text, show that every line $x = mz$ cuts the cubic and the line l (defined below) in a harmonic set of points. For the cubic with a cusp, l is the tangent; otherwise we determine l so as to form with $x = 0$ and the two tangents at the double point a harmonic set of lines.

9. Find l (of Ex. 8) for the following cubics and apply Ex. 7 to these cubics:

$$y^2z = x^2(x \pm z), \quad y^2z = x(x - z)\ (x - \alpha z)$$

10. If the general quartic (39) has three double points, take these at the vertices of the triangle of reference. Now what form does the equation of the quartic have? If these three double points are cusps with $y = x, z = x, y = z$ as tangents (each tangent with quadruple contact), what form does the equation of the quartic have?

123. **Polars of points with respect to n-ics.** Suppose an n-ic has the equation $f(x,y,z) = 0$. Let us put

$$x = x' + x''t, \quad y = y' + y''t, \quad z = z' + z''t$$

in $f(x,y,z)$ and we get

$$f(x' + x''t, y' + y''t, z' + z''t)$$

which is a function of t alone if $x', y', z', x'', y'', z''$ are looked upon as constants temporarily. (Compare §48.) Expanding this function of t in powers of t by *Taylor's formula*, we get

$$\phi(t) \equiv f(x' + x''t, y' + y''t, z' + z''t) = \phi(0) + \phi'(0)t + \frac{\phi''(0)}{2!}t^2 + \cdots + \frac{\phi^{(n)}(0)}{n!}t^n$$

But we have

$$\phi(0) = f(x',y',z'); \quad \phi'(0) = \left(\frac{\partial f}{\partial x'}x'' + \frac{\partial f}{\partial y'}y'' + \frac{\partial f}{\partial z'}z''\right)$$

where $\partial f/\partial x'$ means $\partial f/\partial x$ with x, y, z replaced by x', y', z' and similarly for $\partial f/\partial y'$, $\partial f/\partial z'$. Also we have

$$\begin{aligned}\phi''(0) &= \frac{\partial^2 f}{\partial x'^2}x''^2 + \frac{\partial^2 f}{\partial y'^2}y''^2 + \frac{\partial^2 f}{\partial z'^2}z''^2 + 2\frac{\partial^2 f}{\partial y'\,\partial z'}y''z'' \\ &\quad + 2\frac{\partial^2 f}{\partial z'\,\partial x'}z''x'' + 2\frac{\partial^2 f}{\partial x'\,\partial y'}x''y'' \\ &= \left(\frac{\partial f}{\partial x'}x'' + \frac{\partial f}{\partial y'}y'' + \frac{\partial f}{\partial z'}z''\right)^{(2)}\end{aligned}$$

where the exponent (2) is *symbolic* and is defined by the equation. Finally, we have in the same symbolic notation

$$\phi^{(i)}(0) = \left(\frac{\partial f}{\partial x'}x'' + \frac{\partial f}{\partial y'}y'' + \frac{\partial f}{\partial z'}z''\right)^{(i)}$$

for $i = 1,2,3,\ldots n$.

If we replace x',y',z' by x,y,z and x'',y'',z'' by x',y',z', we have from the preceding paragraph Taylor's formula for three variables, namely,

$$\begin{aligned}(122)\qquad f(x + x't,\, y + y't,\, z + z't) &= f(x,y,z) \\ &\quad + t\left(\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'\right) + \cdots \\ &\quad + \frac{t^i}{i!}\left(\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'\right)^{(i)} + \cdots \\ &\quad + t^n f(x',y',z')\end{aligned}$$

We note that the last term in the expansion is really $t^n f(x',y',z')$ since $f(x,y,z)$ is homogeneous; see (107). Compare §109.

The ith polar curve of $P'(x',y',z')$ with respect to the n-ic $f(x,y,z) = 0$ is defined from (122) as the curve whose equation is

$$(123)\qquad \left(\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'\right)^{(i)} = 0$$

Note that for $n = 2$ and $i = 1$ (123) becomes the ordinary polar of P' with respect to the conic (75). As an example of the above discussion, the first polar of (1,1,1) with respect to the cubic $y^2z - x^3 = 0$ is

$$-3x^2 + 2yz + y^2 = 0$$

the second polar is

$$-6x + 2z + 2y = 0$$

Note that the ith polar of (1,0,0) is $\partial^i f/\partial x^i = 0$, that of (0,1,0) is $\partial^i f/\partial y^i = 0$, that of (0,0,1) is $\partial^i f/\partial z^i = 0$. Thus the third polar of (1,0,0) with respect to $y^4z + x^5 - x^3y^2 = 0$ is $60\,x^2 - 6\,y^2 = 0$.

EXERCISES

1. Write out in full

$$\left(\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'\right)^{(i)}$$

for $i = 3,4,5$.

2. Check over all the work in the text; fill in the details.
3. Prove that

$$\left(\frac{\partial f}{\partial x'}x' + \frac{\partial f}{\partial y'}y' + \frac{\partial f}{\partial z'}z'\right)^{(i)} = n(n-1)\,(n-2)\cdot\cdots(n-i+1)f(x',y',z')$$

Hint: Put $x = x't$ in $f(x,y,z)$, then take $\partial f/\partial t$ and we get

$$\left(\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'\right) = nt^{n-1}f(x',y',z')$$

since $f(x,y,z) = t^n f(x',y',z')$. Put $t = 1$ and we get the above result for $i = 1$. Now repeat the argument for

$$\left(\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'\right)$$

in place of $f(x,y,z)$.

124. **The class m of an n-ic.** DEFINITION. By the *class* m of an n-ic we mean the *number* m of tangents to an n-ic from any general point P in the plane not lying on the n-ic and not on the tangent at an inflection or on the tangent at a double point, or in other such special position.

Note that in §43 we saw that the *degree* n of an n-ic is the same as the *number* of points of intersection of a general line with this n-ic, besides being the degree of the equation of the curve in point coordinates. Dually, we see that the *class* m of an n-ic is also the *degree* of the equation of this n-ic in *line coordinates.* See §91.

In this section we show that the class m of an n-ic is associated with the number $n(n-1)$ of points of intersection of the n-ic and the first polar of a general point P with respect to this n-ic. First we discuss further the polars of the n-ic.

We can write the general equation (39) of an n-ic in homogeneous coordinates in the form

$$
\begin{aligned}
(124)\qquad f(x,y,z) \equiv\ & az^n + (b_0x + b_1y)z^{n-1} \\
& + (c_0x^2 + 2\,c_1xy + c_2y^2)z^{n-2} \\
& + (d_0x^3 + 3\,d_1x^2y + 3\,d_2xy^2 + d_3y^3)z^{n-3} \\
& + (e_0x^4 + 4\,e_1x^3y + \cdots + e_4y^4)z^{n-4} \\
& + \cdots + (l_0x^n + \cdots + l_ny^n) = 0
\end{aligned}
$$

The first polar of (0,0,1) with respect to (124) is

$$
(125)\qquad \frac{\partial f}{\partial z} \equiv naz^{n-1} + (n-1)\,(b_0x + b_1y)z^{n-2} + \cdots = 0
$$

The $(n-2)$th polar (or polar conic) of (0,0,1) with respect to (124) is

$$
\begin{aligned}
(126)\qquad \frac{\partial^{n-2} f}{\partial z^{n-2}} \equiv\ & \frac{n!}{2}az^2 + (n-1)!\,(b_0x + b_1y)z \\
& + (n-2)!\,(c_0x^2 + 2\,c_1xy + c_2y^2) = 0
\end{aligned}
$$

The first polar of (1,0,0) with respect to (124) is

$$
(127)\qquad \frac{\partial f}{\partial x} = b_0z^{n-1} + (2\,c_0x + 2\,c_1y)z^{n-2} + \cdots = 0
$$

Note that, if we put $t = 1/\tau$ in $f(x' + x''t,\ y' + y''t,\ z' + z''t)$, we get $1/\tau^n\ f(x'' + x'\tau,\ y'' + y'\tau,\ z'' + z'\tau)$. Expanding this last function, omitting $1/\tau^n$, by Taylor's formula in powers of τ, then replacing x',y',z' by x,y,z and x'',y'',z'' by x',y',z', also replacing τ by $1/t$, we get

$$
\begin{aligned}
(128)\qquad f(x + x't,\ y + y't,\ z + z't) =\ & f(x',y',z') \\
& + t\left(\frac{\partial f}{\partial x'}x + \frac{\partial f}{\partial y'}y + \frac{\partial f}{\partial z'}z\right) + \frac{t^2}{2!}\left(\frac{\partial^2 f}{\partial x'^2}x^2 + \frac{\partial^2 f}{\partial y'^2}y^2\right. \\
& \left. + \frac{\partial^2 f}{\partial z'^2}z^2 + 2\frac{\partial^2 f}{\partial y'\,\partial z'}yz + 2\frac{\partial^2 f}{\partial z'\,\partial x'}zx + 2\frac{\partial^2 f}{\partial x'\,\partial y'}xy\right) \\
& + \cdots + f(x,y,z)
\end{aligned}
$$

Comparing (128) with (122) we see that the polar conic of

$P'(x',y',z')$ with respect to $f(x,y,z) = 0$ is

$$\frac{\partial^2 f}{\partial x'^2}x^2 + \frac{\partial^2 f}{\partial y'^2}y^2 + \frac{\partial^2 f}{\partial z'^2}z^2 + 2\frac{\partial^2 f}{\partial y'\,\partial z'}yz + 2\frac{\partial^2 f}{\partial z'\,\partial x'}zx + 2\frac{\partial^2 f}{\partial x'\,\partial y'}xy = 0 \tag{129}$$

If $P'(x',y',z')$ lies on the n-ic $f(x,y,z) = 0$, we have $f(x',y',z') = 0$, and the tangent to the n-ic at P' is

$$\frac{\partial f}{\partial x'}x + \frac{\partial f}{\partial y'}y + \frac{\partial f}{\partial z'}z = 0 \tag{130}$$

Why? Compare §§109, 123. If P' is a double point, i.e., if $\partial f/\partial x' = \partial f/\partial y' = \partial f/\partial z' = 0$, the tangents at P' are

$$\left(\frac{\partial f}{\partial x'}x + \frac{\partial f}{\partial y'}y + \frac{\partial f}{\partial z'}z\right)^{(2)} = 0 \tag{131}$$

which is also the polar conic of P' with respect to the n-ic $f(x,y,z) = 0$.

If P' is a point of inflection on the n-ic, then $\partial f/\partial x'x + \partial f/\partial y'y + \partial f/\partial z'z$ is a factor of $(\partial f/\partial x' + \partial f/\partial y'y + \partial f/\partial z'z)^{(2)}$ (why?), and therefore the polar conic of P' is degenerate. If P' is a double point, its polar conic is again degenerate. If P' is a triple point, or other multiple point of higher order than the second, its polar conic vanishes identically. (Why?)

If $y = 0$ is a tangent from (1,0,0) to (124) with (0,0,1) as point of contact, then we must have in (124) $a = b_0 = 0$. Why? But then (127) passes through (0,0,1). Hence we see that the first polar of any point P' with respect to an n-ic passes through the points of contact of tangents to the n-ic from P'. Therefore, in general, the class of an n-ic is $m = n(n-1)$, or less, since the first polar is an $(n-1)$-ic. Compare §123.

If (0,0,1) is a node on (124), we have $a = b_0 = b_1 = 0$. Then (124) starts with a term in z^{n-2} for its highest power in z. It can be shown (compare Hilton's "Plane Algebraic Curves," pp. 11, 96) that in this case (124) and (127) intersect at (0,0,1) in two coincident points. If (0,0,1) is a cusp on (124) with $x = 0$ as tangent, we must have $a = b_0 = b_1 = c_1 = c_2 = 0$, $c_0 \neq 0$. (Why?) In this case x is a factor of the highest terms in z that occur both in (124) and in (127). It can be shown that now (124)

and (127) intersect in three coincident points at (0,0,1). But (1,0,0) was any point in the plane. Therefore we have the following formula for the class of an n-ic

$$m = n(n-1) - 2\delta - 3\kappa \tag{132}$$

where m is the class of the n-ic (the number of tangents to the n-ic from a general point), n is the degree of the curve, δ is the number of nodes on the curve, κ is the number of cusps on the curve.

ILLUSTRATIVE EXAMPLE. As an illustration of the above theory suppose we want the points of contact of the tangents to the curve

$$(x^2 + y^2)z = 2x^3$$

from the point (1,1,1). The first polar of (1,1,1) is

$$y^2 - 5x^2 + 2z(x + y) = 0$$

This polar meets the curve in the points given by

$$(y^2 - 2xy + x^2)(y^2 + 2xy - x^2) = 0$$

The factor $y^2 - 2xy + x^2 = 0$ gives us the point (1,1,1) twice over, and this means that the tangent at (1,1,1) counts for two tangents from (1,1,1). The points of contact of the desired tangents from (1,1,1) are therefore $(1, -1 \pm 2^{\frac{1}{2}}, 1 \pm 2^{-\frac{1}{2}})$.

EXERCISES

1. Find how many of the points $(1, -1 \pm 2^{\frac{1}{2}}, 1 \pm 2^{-\frac{1}{2}})$ actually lie on the given cubic in the last paragraph of the text.

2. Check over all the work in the text, filling in algebraic details.

3. Prove that a double point P' on an n-ic is a double point for every polar of P' with respect to this n-ic and has the same tangents for every polar. Hint: Take P' as (0,0,1), find $\partial^t f/\partial z^t$ from (124). If (0,0,1) is a double point on (124), then $a = b_0 = b_1 = 0$ and the tangents at (0,0,1) are given by $c_0x^2 + 2c_1xy + c_2y^2 = 0$. Why? Compare §48.

4. Prove the generalization of Ex. 3 for an r-ple point on the n-ic.

5. Prove that a point of inflection P' on an n-ic is a point of inflection with the same tangent for every polar of P'. Hint: Take P' as (0,0,1) with $y = 0$ as the tangent. Then $a = b_0 = c_0 = 0$ in (124). Why?

6. Generalize Ex. 5 for a point with a tangent of t-ple contact.

7. Use Ex. 5 to prove that the first polar of a point of inflection P' on a cubic degenerates into the tangent at P' and another line. Hint: A non-degenerate conic cannot have a point of inflection.

8. Find the points of contact of the tangents from (1,1,1) to $x^3 + y^3 = 2z^3$, from (0,1,−1) to $x^3 + y^3 + z^3 = 5xyz$, from (11, 16, 9) to $x^3 + y^3 = 3xyz$.

9. Assuming that the dual of a node is a bi-tangent (i.e., a tangent with

two points of contact such as $y = 0$ for the curve $y = x^2(x-1)^2$) and also that the dual of a cusp is an inflectional tangent, show from (132) that we have

$$n = m(m-1) - 2\tau - 3\iota \tag{133}$$

where τ is number of bi-tangents to the n-ic and ι the number of inflections on the n-ic.

10. Determine the class of each of the curves:

$$y^2z = x^3, \quad y^2z = x^2(x \pm z), \quad y^2z = x(x-z)\,(x-\alpha z),$$
$$y^2 = x^3(x-1), \quad y^2z^2 = x^2(x^2 - z^2)$$

11. Answer all the queries (Why?) in the text.

125. **The Hessian of an n-ic; inflections on an n-ic.** If (0,0,1) is a *point of inflection* (or briefly an inflection) on (124) with $y = 0$ as tangent, we must have $a = b_0 = c_0 = 0$. Why? But then the *polar conic* (126) of (0,0,1) is *degenerate* and consists of a pair of lines

$$y\{(n-1)!\,b_1z + 2(n-2)!\,c_1x + (n-2)!\,c_2y\} = 0$$

one of them ($y = 0$) being the tangent to the curve at the inflection.

If (0,0,1) is a node or cusp on the n-ic (124), then $a = b_0 = b_1 = 0$ and the polar conic (126) of (0,0,1) is again degenerate.

The discriminant of (126) is (if we take (0,0,1) on the n-ic and suppose a change of variables has made $b_0 = 0$):

$$\begin{vmatrix} (n-2)!\,c_0 & (n-2)!\,c_1 & 0 \\ (n-2)!\,c_1 & (n-2)!\,c_2 & (n-1)!\,b_1/2 \\ 0 & (n-1)!\,b_1/2 & 0 \end{vmatrix}$$
$$\equiv -(n-2)!\left(\frac{(n-1)!}{2}\right)^2 b_1^2c_0$$

This *discriminant vanishes* if $b_1 = 0$ or $c_0 = 0$ (or $b_1 = c_0 = 0$), i.e., if the point (0,0,1) is a *node, cusp,* or *inflection.*

The condition that (129) be a degenerate conic is (after replacing x', y', z' by x, y, z):

$$\mathrm{H} \equiv \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial x\,\partial z} \\ \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial y^2} & \dfrac{\partial^2 f}{\partial y\,\partial z} \\ \dfrac{\partial^2 f}{\partial x\,\partial z} & \dfrac{\partial^2 f}{\partial y\,\partial z} & \dfrac{\partial^2 f}{\partial z^2} \end{vmatrix} = 0 \tag{134}$$

The equation (134) gives us a curve called the *Hessian* of the n-ic $f(x,y,z) = 0$. Every point on (134) has a degenerate polar conic with respect to the n-ic. We see that (134) cuts the n-ic in the points of inflection and the nodes or cusps on the n-ic. Since (134) is of the degree $3(n-2)$, we see that there are in general $3\,n(n-2)$ or less inflections on the n-ic.

The number of coincident points of intersection of an n-ic and its Hessian that occur at a node or a cusp can be determined as follows. This number is evidently independent of n, so there are two constants A and B such that

$$\iota = 3\,n(n-2) - A\delta - B\kappa$$

where ι is the number of inflections, n the degree of the curve, δ the number of nodes on the n-ic, κ the number of cusps on the n-ic. (See Hilton, *loc. cit.*, p. 101.) By duality we have

$$\kappa = 3\,m(m-2) - A\tau - B\iota$$

(See Ex. 9 in §124.) But we have also the two equations (132) and (133) from the text in §124 and Ex. 9 in §124.

Eliminating m, τ, ι from the above-mentioned four equations in n, m, δ, κ, τ, ι, we have

$$(A-6)\{(n^2-2n-2\delta-3\kappa)(n^2-2\delta-3\kappa)+4\delta+6\kappa\}$$
$$+(3A-2B-2)\{-3n^2+6n+A\delta+(B-1)\kappa\}=0$$

Since this last equation must hold for all values of n, we have (equating to zero the coefficients of the powers of n):

$$A-6=3A-2B-2=0, \quad \text{or} \quad A=6, B=8$$

Therefore we have the following equation giving the number of inflections on an n-ic,

$$\iota = 3\,n(n-2) - 6\,\delta - 8\,\kappa \tag{135}$$

From (135) we see that every cubic has at least one inflection, because at the very worst a cubic can have a cusp, so for this case $\iota = 3\cdot 3(3-2) - 8 = 1$. If the cubic has a node we see that $\iota = 3\cdot 3(3-2) - 6 = 3$. Again we note that a cubic has at least one real inflection because a third-degree curve (the cubic) and another third-degree curve (its Hessian) have at least one real point in common. (See §45.)

As an illustration we note that the Hessian of

$$x^3 + y^3 + z^3 + 6\,mxyz = 0 \text{ is } m^2(x^3 + y^3 + z^3) = (2\,m^3 + 1)xyz$$

which meets the cubic where $xyz = 0$. Hence the inflections of this cubic are the points of intersection of the cubic with $x = 0$, $y = 0$, and $z = 0$.

EXERCISES

1. Show how, if we take (0,0,1) on the n-ic (124), we can get $b_0 = 0$ by a suitable transformation of the variables.
2. Give in detail the reason why a cubic has at least one real inflection.
3. Find all the inflections on the cubic in the last paragraph of the text, and show that they lie by threes on (real or imaginary) straight lines.
4. Eliminate m, τ, ι from the four equations referred to in the text and give all the details in the derivation of (135).
5. Show that the dual of (135) is

$$\kappa = 3\,m(m-2) - 6\,\tau - 8\,\iota \tag{136}$$

6. From (132), (133), (135), (136) obtain the following five equations (which with the above four complete the list of so-called Plücker equations):

$$\tfrac{1}{2}\,n(n+3) - \delta - 2\,\kappa = \tfrac{1}{2}\,m(m+3) - \tau - 2\,\iota$$
$$\tfrac{1}{2}\,(n-1)\,(n-2) - \delta - \kappa = \tfrac{1}{2}\,(m-1)\,(m-2) - \tau - \iota$$
$$\iota - \kappa = 3(m-n)$$
$$2(\tau - \delta) = (m-n)\,(m+n-9)$$
$$n^2 - 2\,\delta - 3\,\kappa = m^2 - 2\,\tau - 3\,\iota$$

7. Find the Hessians of the following curves:

$$y^p z^q = x^{p+q}, \quad (x+y+z)^3 + 6\,kxyz = 0$$
$$x^3 + y^3 + z^3 = h(x+y+z)^3, \quad y^2z = x^3, \quad y^2z = x^2(x \pm z)$$
$$y^2z = x(x-z)\,(x-\alpha z), \quad yz^3 = x^4, \quad yz^4 = x^5$$

8. Show that if (0,0,1) lies on the n-ic (124), the polar conic (126) cannot be a double line unless (0,0,1) is a cusp. Hint: We have $a = 0$, hence (126) must (to be a double line) have the form $(\alpha x + \beta y)^2 = 0$. Why?

126. **Inflections on cubics and their harmonic polars.** If (124) is a cubic with a point of inflection at (0,0,1) and $yz = 0$ as the polar conic (126) of this inflection, we have $a = b_0 = c_0 = c_1 = c_2 = 0$. Why? Any line $y = mx$ when solved with this cubic gives then

$$b_1 mxz^2 + (d_0 + 3\,d_1 m + 3\,d_2 m^2 + d_3 m^3)x^3 \equiv x(x^2 - \alpha z^2) = 0$$

Therefore on $y = mx$ we have the four points

$$(0,0,1),\ (1,m,0),\ (\sqrt{\alpha},m\sqrt{\alpha},1),\ (-\sqrt{\alpha},-m\sqrt{\alpha},1)$$

where $(1,m,0)$ is the point of intersection of $z = 0$ and $y = mx$, $(0,0,1)$ is the inflection, the other two points are on the cubic..

These four points form a harmonic set, as we see by taking the cross-ratios of the associated points on $y = 0$, namely (in non-homogeneous form x/z), 0, ∞, $\sqrt{\alpha}$, $-\sqrt{\alpha}$. See §24. Note that $(0,0,1)$ and $(1,m,0)$ separate the other two points on $y = mx$. We call $z = 0$ the *harmonic polar* of the point of inflection $(0,0,1)$ on the cubic.

Let us take a *harmonic homology* T (see §97) with a point of inflection P on a cubic curve as *center* and the harmonic polar l of P as *axis*. This transformation T will send the cubic into itself (as $x = -x'$, $y = -y'$ sends $y = x^3$ into itself) because every line through P cuts the cubic in two other points P_1, P_2 and cuts the harmonic polar l of P in a point P' such that P, P_1, P', P_2 form a harmonic set where P, P' separate P_1, P_2.

Also, since a collineation sends a point of inflection into a point of inflection, we see that if this cubic has another point of inflection R other than P, the cubic must have a third point of inflection R' on the line RP. Why?

The above theorem can be proved also as follows. Suppose the cubic has two points of inflection. We take these inflections on $z = 0$ with $x = 0$ and $y = 0$ as tangents. Then our cubic must have an equation of the form

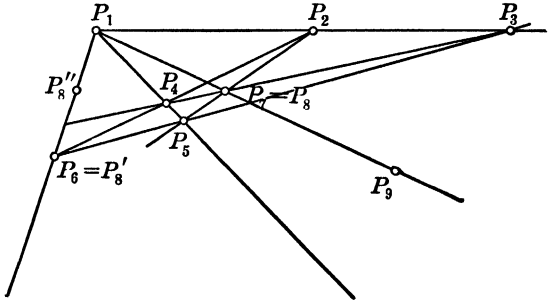
$$xy(ax + by + cz) = dz^3$$

(Why?) This equation shows that the line $ax + by + cz = 0$ must be the tangent at a third point of inflection that lies on $z = 0$ (i.e., is collinear with the other two points of inflection).

From the figure on page 287 we shall show that a cubic (even though it may have nine points of inflection) can have only three real points of inflection (and these three must be collinear, from the preceding discussion). See Hilton, " Plane Algebraic Curves," page 216.

Suppose the cubic has a fourth real inflection P_4, the line P_1P_4 then has another real inflection P_5 on it (from the above paragraphs). The line P_2P_4 has another real inflection P_6 on it. The line P_3P_4 has another real inflection P_7 on it; also the line

P_2P_5 has a third real inflection P_8 on it. If $P_7 = P_8$, the line P_3P_5 has another real point of inflection P_8' on it. If $P_8' = P_6$, the line P_1P_6 has another real inflection P_8'' on it, which inflection cannot coincide with a previously determined inflection. (Why?) Also P_1P_7 has a real inflection P_9, which must be distinct from the



other inflections. (Why?) But then $P_8''P_9$ must have a tenth (real) inflection on it, which is utterly impossible. This discussion does not preclude the possible existence of six imaginary inflections. Why?

EXERCISES

1. Explain fully why a collineation sends a point of inflection into a point of inflection.

2. Draw a figure to prove that a cubic cannot have more than three real inflections, supposing $P_7 \neq P_8$ in the figure of the text; next draw a figure assuming $P_7 = P_8$ but $P_8' \neq P_6$.

3. Answer all the queries (Why?) in the text; also fill in all algebraic details.

4. Find all the points of inflection on the cubic $y^2z = x^2(x \pm z)$. Show that a cubic with an acnode has three real inflections, but a cubic with a crunode has one real inflection and two imaginary inflections. See Ex. 3 in §127.

5. Solve the Hessian of $y^2z = x(x \pm z)\ (x - \gamma z)$ simultaneously with the curve. Do the same for $y^2z = x(x^2 + 2\gamma xz + z^2)$.

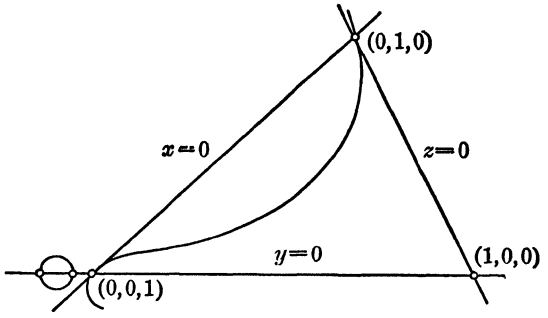
6. Prove that if a quartic curve has three collinear points of hyperinflection (where the tangent intersects the curve in four coincident points), it has a fourth point of hyperinflection collinear with the other three. Hint: Use the second method of proof in the text that the points of inflection on a cubic lie by threes on straight lines.

7. Prove that a cubic with three asymptotes (no two of them inflectional tangents) meets these asymptotes in three collinear points. (Compare Ex. 6.)

127. The reduction of a cubic curve to a normal form. From the theory in §126 we have at hand a method of reducing a cubic

with *no node* or *cusp* to an equivalent cubic with a simpler equation.

This cubic has at least one real point of inflection (see §125) which we take as (0,1,0) with $z = 0$ as tangent and $y = 0$ as harmonic polar. (See the following figure.)



If the cubic cuts $y = 0$ at (0,0,1), this must be the point of contact of a tangent from (0,1,0). (Why?) If (38) is the cubic pictured above, $z = 0$ must give $x^3 = 0$, so $b = d = e = 0$; $x = 0$ must give y^2z, so $c = j = 0$. Every line $z = mx$ must give an equation of the form $x(x^2 + \alpha y^2) = 0$. (Why?) But $z = mx$ gives us

$$x^3(a + fm + gm^2) + x^2y(km) + xy^2(hm) = 0$$

hence $k = 0$. Now we have the cubic in the form

$$-hy^2z = x(gz^2 + fxz + ax^2)$$

If $y = 0$ cuts the cubic in three real points, we take (1,0,1) as one of these points, so our cubic must have the form

$$\alpha y^2z = x(x - z)\ (x + \beta z)$$

(Why?) If $\alpha/\beta > 0$, we put

$$x = x', \quad z = \frac{1}{\beta}z', \quad y = \sqrt{\frac{\beta}{\alpha}}y'$$

If $\alpha/\beta < 0$, we put

$$x = -x', \quad z = \frac{1}{\beta}z', \quad y = \sqrt{\frac{-\beta}{\alpha}}y'$$

In either case we get an equation of the form

$$y^2z = x(x \pm z)\ (x - \gamma z)$$

A cross-ratio of the points $(\gamma,0,1)$, $(\pm1,0,1)$, $(0,0,1)$, $(1,0,0)$ on the harmonic polar $y = 0$ of $(0,1,0)$ is

$$\frac{0 \mp 1}{\infty \mp 1} \frac{\infty - \gamma}{0 - \gamma} = \frac{\mp 1}{1} \frac{1}{-\gamma} = \frac{\pm 1}{\gamma}$$

From this we see that the above cubic cannot be sent by a collineation into another cubic with the same form of equation only with γ replaced by γ' unless we have

$$\gamma' = \gamma, \frac{1}{\gamma}, 1 - \gamma, \frac{1}{1 - \gamma}, \frac{\gamma}{\gamma -}, \frac{\gamma - 1}{\gamma}$$

since cross-ratios are preserved by such a transformation.

EXERCISES

1. Answer the queries (Why?) in the text. Fill in the algebraic details.
2. Why can we take (1,0,1) on the cubic in the text?
3. Using the method of the text reduce a cubic with a cusp to $y^2z = x^3$; a cubic with a crunode to $y^2z = x^2(x + z)$; a cubic with an acnode to $y^2z = x^2(x - z)$; a cubic with no double point but with an inflection whose harmonic polar cuts the curve in only one real point to the form $y^2z = x(x^2 + 2\gamma xz + z^2)$ where $\gamma^2 - 1 < 0$.
4. Show by cross-ratios that $y^2z = x(x - z)(x - \gamma z)$ is not reducible to $y^2z = x(x + z)(x + \gamma' z)$ where $\gamma' = \gamma$.
5. Find a cross-ratio of the four points where $y = 0$ cuts the cubic $\gamma^2 z = x(x^2 + 2\gamma xz + z^2)$ of Ex. 3 and cuts $z = 0$ (the tangent at the inflection).
6. Explain in detail the last paragraph of the text. Why are the given points on $y = 0$ intimately connected with the cubic?
7. If a cubic has three real inflections with concurrent tangents, show that the cubic can be put in the form

$$y(y + z)(y - z) = mx^3$$

If the tangents are non-concurrent, show that the cubic can be put in the form

$$(x + y + z)^3 = mxyz$$

8. Solve the Hessian of each cubic in Ex. 7 simultaneously with the cubic.
9. Can m be determined in either or both of the cubics in Ex. 7 so that the cubic shall have an acnode? Why not a crunode? See §122.
10. If a cubic has a crunode, show that it can be put in the form $y(y^2 + z^2) = mx^3$, and determine m so as to give the cubic a crunode.
11. In the net of conics (compare §§60, 135)

$$\nu x^2 + 2\mu yz + \lambda(by^2 + cz^2 + 2gzx + 2hxy) = 0$$

determine b, c, g, h so that the associated cubic in λ, μ, ν shall have the form $\mu^2\nu = \lambda^3$.

12. Determine b, c, g, h in Ex. 11 so that the cubic in λ, μ, ν shall have the form $\mu^2\nu = \lambda^2(\lambda \pm \nu)$.

CHAPTER XVIII

FURTHER DISCUSSION OF LINEAR FAMILIES OF CONICS

128. **Apolarity.** Let us take a pencil of conics

$$(137) \qquad (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + \lambda C = 0$$

where $C = 0$ is like the first conic except that its coefficients are primed. Let us set the discriminant Γ of this pencil equal to zero, and we have (see §58)

$$\Gamma \equiv \begin{vmatrix} a + \lambda a' & h + \lambda h' & g + \lambda g' \\ h + \lambda h' & b + \lambda b' & f + \lambda f' \\ g + \lambda g' & f + \lambda f' & c + \lambda c' \end{vmatrix} = 0$$

If we expand Γ and equate to zero the coefficient of λ^2 in this expansion, we have

$$(138) \qquad aA' + bB' + cC' + 2fF' + 2gG' + 2hH' = 0$$

where A', B', C', F', G', H' are the coefficients of the equation of the conic C in line coordinates. (See §91.)

DEFINITION. If a point conic and a line conic (i.e., a conic with its equation put into point coordinates and a conic with its equation put into line coordinates) have the coefficients of their equations connected by the relation (138), they are said to be *apolar* to one another.

We shall use apolarity to describe geometrically nets and other linear families of conics. (See §60.) First we note that $\Gamma = 0$ is merely multiplied by a constant when the two fundamental conics of the pencil are subjected to a collineation. (See §19.) Therefore *apolarity is an invariant of the general projective group* (88)

Now let us consider the *geometric meaning* of (138). First, if $C \equiv u^2 = 0$, we have $a = 0$ in (138). Therefore we see that, if $C = 0$ is a point P, any apolar point conic (i.e., with $a = 0$) passes through P.

Next, if $C \equiv 2uv = 0$, we have $h = 0$ in (138). But it is easy to see that the two points, $u = 0$ and $v = 0$, i.e. (1,0,0) and

(0,1,0), are conjugate with respect to any point conic apolar to C (i.e., with $h = 0$). (See §50.)

Third, if $C \equiv u^2 + v^2 = 0$, we have $a + b = 0$ in (138). We leave for the exercises the proof that in this case the two points $iu + v = 0$ and $iu - v = 0$, i.e. $(1,-i,0)$ and $(1,i,0)$, are conjugate with respect to any point conic apolar to C (i.e., with $a + b = 0$).

Fourth, if $C = 2uv + 2vw + 2wu = 0$, we have $h + g + f = 0$ in (138). Let us take the complete quadrilateral circumscribed to C, with sides

$$x = 0, \quad y = 0, \quad z = 0, \quad hgx + fhy + gfz = 0$$

and pairs of opposite vertices

$$(0,0,1) \quad \text{and} \quad (1,-g/f,0), \ (1,0,0) \quad \text{and} \quad (0,-g/h,1), \\ (0,1,0) \quad \text{and} \quad (-f/h,0,1)$$

We leave for the student to show that the pairs of opposite vertices of this complete quadrilateral are pairs of conjugate points with respect to any point conic that is apolar to C (i.e., with $h + g + f = 0$).

Fifth, if $C \equiv u^2 + v^2 + w^2 = 0$, we have $a + b + c = 0$ in (138). If we put

$$u = u' + iv', \quad v = v' + iw', \quad w = w' + iu'$$

we transform $C = 0$ into $2u'v' + 2v'w' + 2w'u' = 0$. Now we can see that, by the preceding case, there exists a complete quadrilateral tangent to C and such that its pairs of opposite vertices are pairs of conjugate points with respect to any point conic that is apolar to C.

DEFINITION. A complete quadrilateral whose pairs of opposite vertices are pairs of conjugate points with respect to a point conic is said to be *self-polar* with respect to this conic.

Any line conic can be reduced to one of the five forms

$$u^2 = 0, \quad 2uv = 0, \quad u^2 + v^2 = 0, \quad 2uv + 2vw + 2wu = 0, \\ u^2 + v^2 + w^2 = 0$$

(See §122.) Therefore the above geometric descriptions take care of all possible cases of apolar conics.

Note that under the fourth and fifth cases there are an infinite number of quadrilaterals circumscribed to the line conic and

self-polar with respect to a given apolar point conic. We may choose as the three sides of any such quadrilateral any three tangents to the line conic.

EXERCISES

1. Complete the proofs for the five cases of apolar conics discussed in the text.
2. Find another self-polar quadrilateral for the fourth case in the text.
3. Find a self-polar quadrilateral for $u^2 + v^2 + w^2 = 0$ in the fifth case in the text.
4. Explain fully how the fifth case in the text was reduced to the fourth case.
5. Dualize the whole discussion in the text.
6. Reduce the general line conic to the five forms given in the text.

129. **Apolar linear families of conics.** Let us consider the following equation:

$$(139)\quad (\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_i a_i)(\mu_1 A_1' + \mu_2 A_2' + \cdots + \mu_j A_j') + (\lambda_1 b_1 + \cdots + \lambda_i b_i)(\mu_1 B_1' + \cdots + \mu_j B_j') + (\lambda_1 c_1 + \cdots + \lambda_i c_i)(\mu_1 C_1' + \cdots + \mu_j C_j') + 2(\lambda_1 f_1 + \cdots + \lambda_i f_i)(\mu_1 F_1' + \cdots + \mu_j F_j') + 2(\lambda_1 g_1 + \cdots + \lambda_i g_i)(\mu_1 G_1' + \cdots + \mu_j G_j') + 2(\lambda_1 h_1 + \cdots + \lambda_i h_i)(\mu_1 H_1' + \cdots + \mu_j H_j') = 0$$

where i, j are to be given various values. We can draw some important conclusions from this equation.

First, if $j = 1$, we see that if each of the fundamental conics of a linear family of point conics is apolar to a line conic C, then every conic of this family is apolar to C.

Second, if $j > 1$, we see that if the conics of a linear family of point conics are apolar to each of the fundamental conics of a linear family of line conics, then every conic of the first family is apolar to every conic of the second family.

Again we note that (138) is a single linear relation connecting the coefficients of a point conic if the apolar line conic C is given. Hence we see that apolar to one line conic we have a four-parameter linear family of point conics. Apolar to a pencil of line conics we have a three-parameter linear family of point conics. Apolar to a net of line conics we have a net of point conics.

Also, if we project a family F_1 of line conics (calling one conic a zero-parameter family) into an equivalent family F_1', then the

apolar family F_2 of point conics (apolar to F_1) goes into the family F_2' of point conics apolar to F_1'. (Why?)

By means of the above theory we can find normal forms to which all three- and four-parameter families of point conics can be reduced, by taking the families apolar to the normal forms of pencils of line conics and the normal forms of line conics respectively. (Why?) Also we can describe geometrically two-, three-, and four-parameter linear families of point conics by means of apolarity. (See §63.)

Thus apolar to $u^2 = 0$ we have the family

$$\lambda y^2 + \mu z^2 + 2\,\nu yz + 2\,\rho zx + 2\,\sigma xy = 0$$

where we replace b by λ, c by μ, f by ν, g by ρ, h by σ. Apolar to $2\,uv = 0$ we have the family

$$\lambda x^2 + \mu y^2 + \nu z^2 + 2\,\rho yz + 2\,\sigma zx = 0$$

Apolar to $u^2 + v^2 = 0$ we have the family

$$\lambda(x^2 - y^2) + \mu z^2 + 2\,\nu yz + 2\,\rho zx + 2\,\sigma xy = 0$$

Apolar to $2\,uv + 2\,vw + 2\,wu = 0$ we have the family

$$\lambda x^2 + \mu y^2 + \nu z^2 + 2\,\rho(yz - zx) + 2\,\sigma(zx - xy) = 0$$

Apolar to $u^2 + v^2 + w^2 = 0$ we have the family

$$\lambda(x^2 - y^2) + \mu(y^2 - z^2) + 2\,\nu yz + 2\,\rho zx + 2\,\sigma xy = 0$$

To obtain the last family we have the general conic apolar to $u^2 + v^2 + w^2 = 0$ in the form (since $a + b + c = 0$)

$$ax^2 + by^2 + (-a - b)z^2 + 2\,fyz + 2\,gzx + 2\,hxy = 0$$

Taking $a = \lambda$, $b = \mu$, $f = \nu$, $g = \rho$, $h = \sigma$, we get the above family.

Note that the above are *five typical forms to which all four-parameter linear families can be reduced.* From §128 we can obtain the geometrical descriptions of these families. We leave this to the student in the exercises.

EXERCISES

1. Find the family of conics apolar to $u^2 + 2\,vw = 0$.
2. Give the geometric descriptions of the typical four-parameter linear families of conics derived in the text.
3. Fill in the details in the text.

130. **Apolarity for three-parameter linear families of conics.** To be able to derive by apolarity typical forms for three-parameter linear families of conics we should have at our disposal typical forms for pencils of line conics. These are well known* but we shall not quote them all here. Rather than this we shall quote just a few typical pencils of line conics and find and describe geometrically their apolar families of point conics.

Apolar to the pencil $\lambda u^2 + \mu v^2 = 0$ we have the three-parameter family of point conics

$$\lambda z^2 + 2\,\mu yz + 2\,\nu zx + 2\,\rho xy = 0$$

This family consists of all the conics through two given points, here $u^2 = 0$ and $v^2 = 0$, i.e. (1,0,0) and (0,1,0).

Apolar to the pencil $\lambda u^2 + 2\,\mu uv = 0$ we have the family

$$\lambda y^2 + \mu z^2 + 2\,\nu yz + 2\,\rho zx = 0$$

Here the point $u^2 = 0$ must lie on every conic of the family, also $uv = 0$ must be a pair of conjugate points with respect to every conic of the family (in fact, every conic $u(\lambda u + 2\,\mu v) = 0$ must be a pair of conjugate points with respect to every point conic of the apolar family). This means that the line containing $u = 0$ and $v = 0$ (namely, $z = 0$) must be tangent to every conic of this family at the point $u^2 = 0$ or (1,0,0).

The pencil $2\,\lambda uv + 2\,\mu uw = 0$ gives us the apolar family

$$\lambda x^2 + \mu y^2 + \nu z^2 + 2\,\rho yz = 0$$

The points (1,0,0) and (0,1,0), (1,0,0) and (0,0,1) must be pairs of conjugate points with respect to the conics of this three-parameter family. But a line joining two points conjugate to a given point P must be the polar of P. (Why?) Therefore we can describe the above family of point conics geometrically as consisting of all the conics that have a given point P and line p as pole and polar.

Apolar to $\lambda(u^2 - v^2) + \mu(u^2 - w^2) = 0$ we have the family

$$\lambda(x^2 + y^2 + z^2) + 2\,\mu yz + 2\,\nu zx + 2\,\rho xy = 0$$

We can describe this family geometrically as consisting of all the conics that have a common self-polar quadrilateral whose pairs of

* See Veblen and Young, Vol. I, pp. 287–293.

opposite vertices are given by $u^2 - v^2 = 0$, $u^2 - w^2 = 0$, and $v^2 - w^2 = 0$; i.e., these are the points (1,1,0) and (1,−1,0), (1,0,1) and (−1,0,1), (0,1,1) and (0,−1,1).

EXERCISES

1. Show how to obtain the last family of point conics given in the text. Find the sides of the common self-polar quadrilateral.

2. Check all the geometric descriptions in the text by another means. Thus it is evident that $z = 0$ gives $y^2 = 0$ for every conic in the family and so this is the common tangent at (1,0,0).

3. Find and describe the family apolar to $2\lambda uv + \mu(u^2 - v^2) = 0$. Hint: This pencil has two imaginary double points. What are they?

4. Find and describe the family apolar to $\lambda u^2 + \mu(v^2 + 2uw) = 0$. Hint: The point $u^2 = 0$ is a point of contact on $v^2 + 2uw = 0$. Why?

5. Find and describe the family apolar to $2\lambda uv + \mu(v^2 + 2uw) = 0$. Hint: The point $u = 0$ is a point of contact on $v^2 + 2uw = 0$, and the point $v = 0$ lies on the tangent at $u = 0$. Why?

6. Find and describe the family apolar to $\lambda u^2 + 2\mu vw = 0$.

7. Find and describe the family apolar to $2\lambda uw + 2\mu(vw + wu) = 0$. Hint: The point $v + u = 0$ is on the line joining $v = 0$ and $u = 0$. Why?

8. Find and describe the family apolar to $\lambda(u^2 + v^2) + \mu(v^2 + w^2) = 0$.

9. Find and describe the family apolar to $\lambda u^2 + \mu(v^2 \pm w^2) = 0$.

10. Find and describe the family apolar to $2\lambda uv + \mu(v^2 \pm w^2) = 0$. Hint: The point $v = 0$ is on the line joining the two points given by $v^2 \pm w^2 = 0$.

11. Check all the geometric descriptions in the above problems by some other method.

12. Dualize the text.

131. **Apolarity and nets of conics.** We cannot *derive* typical forms for nets of conics by means of *apolarity*, as we did for three- and four-parameter linear families of conics. For that matter, we could not derive the type forms for conics or for pencils of conics by apolarity. We can, however, *describe* nets of conics geometrically by means of *apolarity*.

Thus, apolar to the net of line conics $\lambda u^2 + \mu v^2 + \nu w^2 = 0$ we have the net of point conics

$$2\lambda yz + 2\mu zx + 2\nu xy = 0$$

This net consists of all the point conics circumscribed to a given triangle.

Apolar to $2\lambda uv + 2\mu vw + 2\nu wu = 0$ we have the net

$$\lambda x^2 + \mu y^2 + \nu z^2 = 0$$

This net consists of all the point conics with a given self-polar triangle.

Apolar to $\lambda w^2 + 2\mu uv + \nu(u^2 - v^2) = 0$ we have the net

$$\lambda(x^2 + y^2) + 2\mu xz + 2\nu yz = 0$$

This net consists of all the point conics through three non-collinear points (one real point and two conjugate imaginary points). If we interpret $z = 0$ as the line at infinity, we see that this net consists of all the circles that pass through the origin.

EXERCISES

1. Fill in all the details in the text.
2. Find and describe the net apolar to $\lambda u^2 + \mu v^2 + 2\nu uw = 0$.
3. Find and describe the net apolar to $\lambda u^2 + 2\mu uv + 2\nu vw = 0$.
4. Find and describe the net apolar to $\lambda(u^2 + v^2) + 2\mu uw + 2\nu vw = 0$.
5. Check the descriptions in the text and in these problems by some other means.
6. Dualize the text and these problems.

132. **The derivation of some typical pencils of conics.** We have shown (where?) that there are only five types of pencils of conics, ignoring the differences between real and imaginary (finite and infinite) points. We shall derive a few typical pencils of conics.

First, suppose the conics of the pencil intersect in four real points P_1, P_2, P_3, P_4, no three of them collinear. We can take the triangle of reference as the diagonal triangle of the complete quadrilateral with four of its vertices P_1, P_2, P_3, P_4. We can take as P_1 the point (1,0,1) and as P_2 the point (0,1,1). Why? Then P_3 is (−1,0,1) and P_4 is (0,−1,1). Why? Two conics through these four points are $z^2 - y^2 = 0$ and $x^2 - z^2 = 0$. Therefore our pencil can be written

$$(x^2 - z^2) + \lambda(z^2 - y^2) = 0$$

Next, we suppose the conics of the pencil touch each other at two real points P_1 and P_2. We can take P_1 as (1,0,0) and P_2 as (0,1,0), also the tangent at P_1 as $y = 0$ and the tangent at P_2 as $x = 0$. Why? But then each conic has the form $\alpha xy + \beta z^2 = 0$. Why? Therefore our pencil can be written

$$z^2 + 2\lambda xy = 0$$

If the conics of the pencil touch each other at a real point P_1 and intersect at two other real points P_2 and P_3, we take P_1 as (0,0,1) with $y = x$ as the common tangent, also P_2 as (0,1,0) and P_3 as (1,0,0). Every conic of the pencil must have an equation of the form $2\alpha xy + 2\beta xz + 2\gamma yz = 0$. Why? But $y = x$ must give $x^2 = 0$. Why? Since $y = x$ gives us $2\alpha x^2 + 2\beta xz + 2\gamma xz = 0$, we must have $\beta = -\gamma$. Hence, taking β/α as λ, we have the pencil

$$2xy + 2\lambda(xz - yz) = 0$$

Suppose now that the conics of the pencil have three coincident points in common at a real point P_1 and intersect at a second real point P_2. The only degenerate conic in the pencil is a pair of lines $l_1 l_2$ where l_1 is the tangent at P_1 and l_2 is the line P_1P_2. (See §58.) Taking the pair of lines $l_1 l_2$ as $xy = 0$, also P_1 as (0,0,1) and P_2 as (0,1,0), we have the pencil

$$2xy + \lambda(ax^2 + 2fyz + 2hxy) = 0$$

since (0,0,1) and (0,1,0) are on every conic and $y = 0$ is tangent to every conic at (0,0,1). By the bilinear transformation $\lambda = (1/a)\lambda'/(1 - h\lambda')$ we can rid the second fundamental conic in this pencil of the xy-term. Then the transformation

$$x = x', \quad y = y', \quad z = \frac{a}{f}z'$$

reduces the pencil to the form:

$$2xy + \lambda(x^2 + 2yz) = 0$$

Finally, we suppose the pencil consists of conics having four coincident points in common at a real point P. The pencil then has just one degenerate conic, a double line l which is the common tangent at P. Why? Taking l as $x^2 = 0$ and P as (0,0,1), also transforming λ (as we did above) to rid the second fundamental conic of the x^2-term, we have the pencil

$$x^2 + \lambda(by^2 + 2fyz + 2gzx + 2hxy) = 0$$

with discriminant

$$\Gamma \equiv \begin{vmatrix} 1 & h\lambda & g\lambda \\ h\lambda & b\lambda & f\lambda \\ g\lambda & f\lambda & 0 \end{vmatrix} \equiv \lambda^3(2fgh - bg^2) - f^2\lambda^2$$

We must have $\Gamma \equiv \alpha\lambda^3$. Why? Hence $f = 0$, $bg \neq 0$. We now put

$$x = x', \quad y = y', \quad gz + hy = bz'$$

and writing λ instead of $b\lambda$, we have the pencil

$$x^2 + \lambda(y^2 + 2zx) = 0$$

EXERCISES

1. Show why in each of the discussions in the text there is no loss of generality. Hint: Two homologous complete quadrangles determine a projectivity. See §100.

2. Answer the queries (Why?) in the text.

3. Fill in the details in the text. In the last paragraph give the bilinear transformation in λ necessary to rid the second fundamental conic of the x^2-term.

4. If in the first case in the text P_1, P_2, P_3, P_4 are two pairs of conjugate imaginary points, show how to reduce the pencil to the form

$$(x^2 + z^2) + \lambda(z^2 + y^2) = 0$$

5. Treat the case where P_1, P_3 are real points and P_2, P_4 are conjugate imaginary points. Hint: Take P_2, P_4 as $(1,i,0)$ and $(1,-i,0)$. Then the pencil consists of circles (if $z = 0$ is l_∞). Take P_1, P_3 as $(0,1,1)$ and $(0,-1,1)$. One degenerate conic is $xz = 0$. Why? Take for the second fundamental conic the unit circle $x^2 + y^2 - z^2 = 0$. Why is there no loss of generality?

6. Treat the case where the conics are tangent to each other at two conjugate imaginary points. Prove that the pencil is then reducible to

$$z^2 + \lambda(x^2 + y^2) = 0$$

7. Show that if the conics touch one another at a real point and intersect again in a pair of conjugate imaginary points P_2, P_3, we can reduce the pencil to

$$2xz + \lambda(x^2 + y^2) = 0$$

Hint: Take $x = 0$ as the common tangent at $(0,0,1)$ and P_2, P_3 as $(1,i,0)$ and $(1,-i,0)$. Then $z = 0$ in the general conic (75) must give $x^2 + y^2 = 0$, so $h = 0$ and $a = b \neq 0$. Also $x = 0$ must give $y^2 = 0$, so $c = f = 0$, $g \neq 0$. Divide the conic by g, take $\lambda = a/g$.

8. Describe the pencil of conics $(x^2 + y^2) + \lambda(z^2 + 2xy) = 0$.

9. Dualize the text and the above examples.

10. Find and describe the families of point conics that are apolar to the typical pencils of line conics dual to the pencils of point conics in the text and in the above examples, i.e., apolar to $(u^2 - w^2) + \lambda(w^2 - v^2) = 0$, etc.

133. **The derivation of some typical degenerate pencils of conics.** By a degenerate pencil of conics we mean a pencil all of whose conics are degenerate (hence the discriminant Γ of the pencil vanishes identically). Compare §19.

Let us derive some of these typical degenerate pencils. First, suppose the pencil has a double line l in it. We take l as $x^2 = 0$ and put the pencil in the form

$$x^2 + \lambda(by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

(If the second fundamental conic has a term in x^2, the bilinear transformation $\lambda = \lambda'/(1 - a\lambda')$ will rid this conic of the x^2-term.) The discriminant of this pencil is

$$\Gamma \equiv \begin{vmatrix} 1 & h\lambda & g\lambda \\ h\lambda & b\lambda & f\lambda \\ g\lambda & f\lambda & c\lambda \end{vmatrix} \equiv \lambda^3(2fgh - bg^2 - ch^2) + \lambda^2(bc - f^2)$$

Since we must have $\Gamma \equiv 0$, we have $bc - f^2 = 0$ and $2fgh - bg^2 - ch^2 = 0$. Since $bc - f^2 = 0$, we have either

$$b = c = f = 0 \quad \text{or} \quad by^2 + cz^2 + 2fyz \equiv (\alpha y + \beta z)^2$$

Why? The first case gives us

$$(A) \qquad x^2 + \lambda(2gzx + 2hxy) = 0$$

If $h \neq 0$ in (A) we put

$$x = x', \quad gz + hy = y', \quad z = z'$$

and get a typical degenerate pencil (dropping the primes from the variables)

$$x^2 + 2\lambda xy = 0$$

If $h = 0$, $g \neq 0$ in (A), we put

$$x = x', \quad y = z', \quad z = y'$$

and get the case of $h \neq 0$ all over again. We cannot have $h = g = 0$.

In the second case, where $bcf \neq 0$ but $bc - f^2 = 0$, we put (if $\alpha \neq 0$ in $\alpha y + \beta z$)

$$x = x', \quad \alpha y + \beta z = y', \quad z = z'$$

and get the pencil (dropping primes from the variables)

$$(B) \qquad x^2 + \lambda(y^2 + 2g'zx + 2h'xy) = 0$$

with $\Gamma \equiv \begin{vmatrix} 1 & h'\lambda & g'\lambda \\ h'\lambda & \lambda & 0 \\ g'\lambda & 0 & 0 \end{vmatrix} \equiv -g'^2\lambda^3$, hence $g' = 0$.

If $h' = 0$ in (B), we have the typical pencil

$$x^2 + \lambda y^2 = 0$$

If $h' \neq 0$ in (B), we put

$$x = \frac{1}{h'} x', \; y = y'; \quad \lambda = \frac{1}{h'^2}\lambda'$$

then drop all the primes, multiply the equation by h'^2, and get

$$x^2 + \lambda(y^2 + 2xy) = 0$$

If $\alpha = 0$, $\beta \neq 0$ in $\alpha y + \beta z$, we put $x = x'$, $y = z'$, $z = y'$ and get the above cases again.

EXERCISES

1. Find α and β in $by^2 + cz^2 + 2fyz \equiv (\alpha y + \beta z)^2$. Find g' and h' in (B) in terms of b, c, f, g, and h.

2. Show how to derive the transformation $\lambda = \lambda'/(1 - a\lambda')$ to rid the second fundamental conic of the x^2-term.

3. Reduce to typical forms the degenerate pencils containing no double lines. Hint: Put the pencil in the form

$$2xy + \lambda(ax^2 + by^2 + cz^2 + 2fyz + 2gzx) = 0$$

whose discriminant Γ must vanish identically.

4. Consider the degenerate pencil

$$(x^2 + y^2) + \lambda(by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

which cuts $z = 0$ in points given by

$$x^2 + 2\lambda hxy + (1 + \lambda b)y^2 = 0$$

We can find some value of λ so that $\lambda^2 h^2 - \lambda b - 1 > 0$, unless $h = b = 0$. Why? If $h = b = 0$ the pencil has the discriminant

$$\Gamma \equiv \begin{vmatrix} 1 & 0 & \lambda g \\ 0 & 1 & f\lambda \\ \lambda g & f\lambda & c\lambda \end{vmatrix} \equiv c\lambda + \lambda^2(-g^2 - f^2)$$

But we must have $\Gamma \equiv 0$, so $c = g = f = 0$. (Why?) Hence we cannot have $h = b = 0$. Why? Hence there are conics that cut $z = 0$ in real points. Why? Hence there are always pairs of real lines in a degenerate pencil. Why? This shows why we can put the pencil of Ex. 3 into the form we took. (How so?)

134. The derivation of some typical degenerate nets of conics. DEFINITION. If every conic in a net of conics is a degenerate conic, we call the net a *degenerate net.*

The method of derivation of typical degenerate nets of conics is somewhat different from that used for non-degenerate nets, so we first treat some cases of degenerate nets.

First, we suppose the degenerate net has two double lines l_1 and l_2. We take l_1 as $x^2 = 0$ and l_2 as $y^2 = 0$, and put the net in the form

$$\lambda x^2 + \mu y^2 + \nu(cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

with discriminant

$$\Gamma \equiv \begin{vmatrix} \lambda & h\nu & g\nu \\ h\nu & \mu & f\nu \\ g\nu & f\nu & c\nu \end{vmatrix} \equiv c\lambda\mu\nu + \nu^3(2fgh - ch^2) - g^2\mu\nu^2 - f^2\lambda\nu^2$$

Since the net is degenerate, we must have $\Gamma \equiv 0$, hence $c = g = f = 0$ and our net becomes

$$\lambda x^2 + \mu y^2 + 2\nu xy = 0$$

Next we suppose the degenerate net has not two double lines but has a pencil reducible to $\lambda x^2 + 2\mu xy = 0$. We can put the net into the form

$$\lambda x^2 + 2\mu xy + \nu(by^2 + cz^2 + 2fyz + 2gzx) = 0$$

with discriminant

$$\Gamma \equiv \begin{vmatrix} \lambda & \mu & g\nu \\ \mu & b\nu & f\nu \\ g\nu & f\nu & c\nu \end{vmatrix} \equiv \lambda\nu^2(bc - f^2) + \mu^2\nu(2fg) - c\mu^2\nu + \nu^3(-bg^2)$$

We must have $\Gamma \equiv 0$; hence $c = 0$, so $bc - f^2 = 0$ gives $f = 0$, also $bg^2 = 0$. If $c = f = g = 0$, we have a second double line (contrary to hypothesis). Therefore we have $c = f = b = 0$, $g \neq 0$, and our net becomes

$$\lambda x^2 + 2\mu xy + 2\nu xz = 0$$

EXERCISES

1. Suppose the degenerate net has no double line, and show this case is impossible. Hint: Then the net reduces to

$$2\lambda xy + 2\mu zx + \nu C = 0 \quad \text{or} \quad \lambda(x^2 + y^2) + 2\mu xy + \nu C = 0$$

Why?

2. Show that no degenerate net can have a conic reducible to $x^2 + y^2 = 0$.

135. **The derivation of some typical non-degenerate nets of conics.** We saw in §61 that associated with every net of conics there are cubic curves in the λ,μ,ν-plane. Two nets with *non-equivalent* cubic curves in the λ,μ,ν-plane *cannot be equivalent.* We saw that to every double line in a net of conics, there corresponds a double point on the associated cubic K.

Let us consider some derivations of typical non-degenerate nets of conics where we make use of the associated cubic K. First, suppose the net has a double line l to which there corresponds on K a cusp. First we reduce K to the form

$$\mu^2\nu = \lambda^3$$

by transformations on λ, μ, ν. Then to the cusp (0,0,1) on K there corresponds the double line l. We can reduce l to $x^2 = 0$ by transformations on x, y, z (which transformations, we saw, do not affect K at all). If the point of inflection (0,1,0) on K corresponds to a real line-pair, we can put the net in the form

$$(A) \quad \lambda(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + 2\mu yz + \nu x^2 = 0$$

because the pencil $\lambda = 0$ (with discriminant $\mu^2\nu$) must be of the form reducible to the second case in §132.

The discriminant of (A) is

$$K \equiv \begin{vmatrix} a\lambda+\nu & h\lambda & g\lambda \\ h\lambda & b\lambda & f\lambda+\mu \\ g\lambda & f\lambda+\mu & c\lambda \end{vmatrix} \equiv \lambda^3(abc+2fgh-af^2-bg^2-ch^2)$$
$$+ \lambda^2\nu(bc-f^2) + \lambda^2\mu(2gh-2af) - \mu^2\nu - 2f\lambda\mu\nu - a\lambda\mu^2 = 0$$

We must have K of the form $\lambda^3 - \mu^2\nu = 0$. Hence we have

$$a = f = 0, \quad bc = 0, \quad gh = 0, \quad abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$$

First, suppose $b = 0$ and $c \neq 0$, hence $g = 0$ and $h \neq 0$. Why? Our net is now

$$\lambda(cz^2 + 2hxy) + 2\mu yz + \nu x^2 = 0$$

We put

$$x = \frac{c}{h}x', \; y = y', \; z = z'; \quad \lambda = \frac{1}{c}\lambda', \; \mu = \mu', \; \nu = \frac{h^2}{c^2}\nu'$$

and, dropping all the primes, we get the typical net

$$\lambda(z^2 + 2xy) + 2\mu yz + \nu x^2 = 0$$

If $b \neq 0$ and $c = 0$, then $g \neq 0$ and $h = 0$. Why? But now we put $x = x', y = z', z = y'$ and reduce this case to that where $b = 0$, $c \neq 0$.

If the point of inflection (0,1,0) on K corresponds to a pair of conjugate imaginary lines, we can put the net in the form

$$(B) \quad \lambda(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + \mu(y^2 + z^2) + \nu x^2 = 0$$

using Ex. 6 of §132. The discriminant of (B) is

$$K \equiv \begin{vmatrix} a\lambda + \nu & h\lambda & g\lambda \\ h\lambda & b\lambda + \mu & f\lambda \\ g\lambda & f\lambda & c\lambda + \mu \end{vmatrix} \equiv \lambda^3(abc + 2fgh - af^2 - bg^2 - ch^2) + \lambda^2\mu(ab + ac - g^2 - h^2) + \lambda\mu^2(a) + \lambda^2\nu(bc - f^2) + \lambda\mu\nu(b + c) + \mu^2\nu = 0$$

But K must be of the form $\mu^2\nu - \lambda^3 = 0$. Hence we have

$$a = 0, \quad g^2 + h^2 = 0, \quad b + c = 0, \quad bc - f^2 = 0,$$
$$\cancel{abc} + 2fgh - \cancel{af^2} - bg^2 - ch^2 \neq 0$$

Therefore $g = h = 0$. But this makes $2fgh - bg^2 - ch^2 = 0$. Thus we see that (B) cannot give us a net whose cubic K has a cusp.

EXERCISES

1. Find K for the typical net derived in the text. Show how to reduce K to $\lambda^3 - \mu^2\nu = 0$.

2. Derive a typical net with a double line in it where (a) $K = 0$ is a cubic with a crunode; (b) $K = 0$ is a cubic with an acnode. Hint: Put $K = 0$ into the forms $\mu^2\nu = \lambda^2(\lambda \pm \nu)$. Use (A) and (B) in the text.

3. Derive a typical net with no double line in it, but where $K = 0$ has (a) a crunode and (b) an acnode. Hint: Put $K = 0$ into the forms $\mu^2\nu = \lambda^2(\lambda \pm \nu)$. Put the net into the forms:

$$(A) \quad \lambda(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + 2\mu(xz - yz) + 2\nu xy = 0$$

$$(B) \quad \lambda(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + 2\mu xz + \nu(x^2 + y^2) = 0$$

See the third typical pencil in the text of §132, also Ex. 7 in §132.

4. In the text and in Ex. 3 why must the pencils given by $\lambda = 0$ be reducible to the forms $2\mu yz + \nu x^2 = 0$, $\mu(y^2 + z^2) + \nu x^2 = 0$, $2\mu(xz - yz) + 2\nu xy = 0$, and $2\mu xz + \nu(x^2 + y^2) = 0$, respectively. Hint: $\lambda = 0$ must give a pencil with discriminant $\mu^2\nu$. Why? Take the discriminant of every typical pencil in §132 and the examples there. These are all the types of such pencils that exist. Why?

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