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THE PRINCIPLES OF ELECTROMAGNETISM

BY

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PREFACE

I HAVE long wished to write a book which would develop the principles of dynamo electric machinery, in such a way that the reader could retrace the steps from a simplified method of approach used in dynamo design, right back to the fundamental tenets of electromagnetism. Finding that the digressions on fundamental principles were so lengthy that the continuity of the main theme was broken, I decided to write a book on the 'Principles of Electromagnetism', which would be a preliminary and companion volume to a book on the 'Principles of the Direct Current Dynamo' which I hope to publish shortly. This book on the Dynamo will be self-contained, but when the steps justifying a given process are more than can be described in some half-page of print, the residue will be covered by explicit references to the present work.

Once I had realized this book must be separated from my initial project, it was necessary to consider what form would give it a coherence and distinguish it from a mere foreword to a book on the Dynamo.

As a student, my experience of the text-books on the Dynamo was that they were very plausible without being convincing. That the procedure was substantially correct was quite evident, because by its use dynamos were built whose performance agreed very closely with that predicted from the calculations which formed the technical part of the design. The disagreement between experiment and calculation was always appreciable and the stereotyped procedure of calculation seldom showed explicitly why and where these discrepancies must exist. It was clear that drastic simplifications had been made, but it was not clear what these simplifications were, nor what would be the method of estimating the approximate magnitude of their effect.

These difficulties led me to study the fundamental basis of electromagnetism from such classic books as those by Clerk Maxwell, Sir James Jeans, and Oliver Heaviside. I then found many things difficult to reconcile with the established practice of electrotechnics: an uncomfortable situation for a lecturer. I am writing the book on the Dynamo on the method familiar to the electrical engineer, but am constantly tracing the steps which relate it to classic principles.

On taking up the study of radio communication I found a new subject much less stereotyped than dynamo practice, and here the text-books exhibited gaps, to fill which I had again to seek help from the great writers. Hence it has seemed natural to include in this book some of the problems I have required in practical work and have had to extract laboriously from the classic authors. Thus from starting to write a book on the Dynamo I have in fact written a fairly comprehensive work on the Principles of Electromagnetism.

This book is intended primarily for those students who have an engineer's turn of mind, by which I mean a mind whose delight is to apply knowledge to the solution of some practical and constructional problem, but I hope it may find some readers among students of general physics. I am a little surprised to find myself writing so much with so little explicit reference to ways and means of application, but I have had consistently to curb my natural tendency to make things directly applicable, since to do so one would have to introduce or assume technical details which would be quite irrelevant: I have tried to remember that this book is a preliminary to others of a different character. For this reason the engineering student will find the examples at the end of each chapter rather academic. I suppose this book will be used chiefly by engineering students, and for their benefit I will explain its relation to their course of study: for this I must use as examples the Engineering Course at Oxford or at Cambridge.

The first two chapters contain an amplification of all those things which for many years I gave in the third term of my first-year lecture-course at Cambridge. The undergraduate at Oxford or Cambridge should know thoroughly the whole of these two chapters and should have worked all the examples by the end of his third term of residence. During his first long vacation he should read as far as p. 130 of Chapter III, and by the end of his fourth term should know the whole of Part 1 of Chapter III, and have worked all the examples which start on p. 189.

The remainder of the book is suited only to those students who are interested primarily in the electrical side of engineering. For these, all should master Part 2 of Chapter III during the first term of their third year. Those who take up the advanced work (such as schedule B) on dynamos will benefit themselves by gaining some acquaintance with Chapter IV and especially from p. 224 onwards. Those who take up advanced work on radio communication should study Chapter V. I hope the post-graduate student will find many helpful things in Chapters IV and V. It was my experience that the electrical engineering text-books of my student days seemed to contain just enough for some small section of a course and were afterwards discarded for ever. If I am able to carry out my intention of following up this book with one on the Principles of the D.C. Dynamo and another on the Principles of Polyphase Machinery and perhaps others, the series will form a coherent and cross-referenced work which I hope some readers may regard as their friend through the whole of their student days.

I quite realize that the average student will not regard this as an easy book, but he ought not to be put off because it contains some advanced problems which he may never study. I think he will find that careful reading of the copious letterpress will help him appreciably to answer the numerical problems which will confront him in examinations, even if he omits some of the more intricate analysis: much that I have written has been suggested by my experience in tuition, as well as lecturing, during the last fourteen years in which I have been teaching first at Cambridge and then at Oxford. From its very nature this book cannot contain any original contributions to knowledge, but I trust it is not a mere slavish repetition of the lectures I have myself attended.

I tender my acknowledgements to Messrs. the Telegraph and Guttapercha Co. for permission to reproduce Figs. 83, 84, and 86, and to Messrs. the Cambridge Instrument Co. for Fig. 79. Also to Mr. B. C. Hague for permission to reproduce from his masterly work *Electromagnetic Problems in Electrical Engineering* many of the figures in Chapter IV.

My grateful thanks are due to Mr. R. R. M. Mallock for many valuable criticisms of the manuscript and for the many hours of helpful argument I have had with him on these subjects during the last eight years. Also to Mr. V. Belfield and Mr. H. D. Ellis who have assisted in reading the proofs.

E. B. M.

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CONTENTS

I

ELEMENTS OF MAGNETISM

1. Preliminary ideas

A bar magnet is too familiar to need description. If the end of a bar magnet is presented to one end of another bar magnet which is mounted as a compass needle, that end of the compass needle will either be repelled or attracted. This experiment shows that magnetism is of two kinds, commonly called North and South. If a bar magnet is broken into two portions, each portion is found to be a complete magnet with north and south ends whose strength is unaltered by the breakage. The breaking process may be continued indefinitely with the same result and it is never possible to separate the north from the south portion: the two kinds of magnetism are complementary and inseparable. Simple experiments, for example with iron filings, show that the magnetism is mainly concentrated in small regions near the ends of the bar, and these regions are called poles; it is found that like poles repel.

Any magnet can be supposed to consist of a bundle of filamentary magnets whose poles are concentrated in points: the elementary filaments may differ in length, but the majority have approximately the same length so that most of the polar points are grouped in a small region. Each filamentary magnet of finite length may be supposed to be built up from a large number of very short filamentary magnets placed end to end: the north pole of one coinciding with the south pole of the next, so that unneutralized poles are left only at the extreme ends. These hypothetical ultimate elements must be considered so short that they cannot be broken in half. They are a device to make magnets atomic in structure, but there is no need to insist that their dimensions shall be comparable with those of the molecules of the matter of which they are composed.

Since any magnet is thus supposed to be built up of very short magnetic particles which have poles concentrated at their end points, we shall examine the properties of a magnetic particle and deduce those of a finite magnet by superposition.

Rough experiments with a bar magnet suggest that the force due to a given pole varies inversely as the square of the distance from it.

Direct experimental verification of the law of force is troublesome because it is physically impossible to separate a north pole from its complementary south pole, and further it is impossible to obtain a magnet which has its poles concentrated at definite points; so the distance from the pole cannot be measured precisely. There are further practical difficulties which are due to the tendency for fresh magnetism to be induced in both the magnets used for the measurement. Many ingenious experiments have been devised, by Searle and others, which employ long ball-ended magnets which approach closely to the ideal conception of a magnet with poles concentrated at points. Such experiments suggest the law is inverse square: accordingly we develop an analysis which presumes the law of force is precisely the inverse square* and test experimentally if this analysis is capable of explaining precisely the effects observed from real magnets. If it is capable of doing so, and experience shows that it is, we have inductive proof of the inverse square law.

The magnetic force at a specified point in the neighbourhood of magnets, or in other words the magnetic force at a specified point in a magnetic field, is defined as the force which would be experienced by an isolated magnetic pole of unit strength placed at that point. This definition is unrealizable physically because a pole cannot be isolated: in reality the measurement would have to be made by observing the deflexion of a small spring-controlled pivoted magnet placed at the point. The experimental difficulties are not a valid objection to the definition, which is a very simple one, and easy to visualize as a process to be performed. For purposes of clear thought it is often convenient to imagine an isolated unit pole, but in doing so we need never forget that nature has not provided isolated poles. The idea is a useful simplification which does not really do violence to the basic principles of a real problem.†

Experience shows that no arrangement of permanent magnets can form a source of power. A unit pole which has been brought to any point in the field has a certain store of potential energy which is

* Unit pole is defined to be such that poles of strength m and m' repel one another with a force $\frac{mm'}{r^2}$ dynes, where r is the distance in cm. between them: the strength of a north pole is positive and of a south pole is negative.

† It is assumed that the test pole does not alter the strength of the poles producing the field which is being examined, because throughout this chapter all magnets are supposed to be truly permanent; that is, to have poles whose strength cannot be altered by magnetic induction.

a definite characteristic of that point in the particular field. In the same way a unit mass which has been brought to a certain point on a hill has a definite store of potential energy which is independent of the path or the manner by which it was brought there.

2. Potential of a point in a magnetic field

It is often convenient to express the force as a function of the work done in moving a small pole against the field: as a preliminary example consider the field due to a pole of strength m placed at the origin. The field is everywhere radial and at a distance r, the force is a repulsion of value
$$H = m/r^2.$$

If a unit pole is moved a distance dr against the field, external work must be supplied; let this amount of work be dV. Then
$$dV = Hdr,$$
or
$$\frac{dV}{dr} = H = m/r^2.$$

Hence H can be expressed in terms of a function V, which is called the potential of the field, and by integrating the equation we can find this function: on doing so we obtain
$$V = m/r + C,$$
where C is the constant of integration and is independent of r.*

Since V depends only on r and not on a direction θ, the potential of a given point in the field is unique and does not depend on the path by which the point is approached. The work done in taking a unit pole from one point to another is measured by the difference of potential between these two points. Since the potential at a point depends only on r, there is no net work done in taking a pole round any circuital path. If this statement is not at once obvious to the reader, let him consider it as follows. Any circuital path can be made up of elementary steps which are either directed along a line of force or perpendicular thereto. No work is done along the perpendicular steps because on them the movement is perpendicular to the force. The net work for the whole set of the steps along the lines is zero, because for every outward step there is a companion backward step.

* The integral of $1/r^2$ is $-1/r$, but here the negative sign has been ignored for a reason which is explained below.

We now have to consider the meaning of C, the constant of integration. The only known fact is the force at a given point, and the mathematical tool V is invented so that $dV = Hdr$: this defines only difference of potential and cannot possibly decide the constant C which is required to determine the absolute size of V. Any convenient value can be assigned to C without altering the difference of V between two points. It is usual to make C zero and then $V = m/r$. When this is done V becomes the work in bringing a unit pole from infinity to the point considered. The reader may be surprised at the arbitrary choice of C and feel anxious lest it is an assumption having physical significance. But V is only a mathematical tool invented for convenience, and no experiment can do more than measure differences of V: consequently we cannot violate physically observable facts by measuring V from the most convenient datum. These remarks have important significance in relation to another derived tool called the vector potential of a current (see Chap. IV) and especially in relation to the 'retarded vector potential' which is required to solve problems about electromagnetic waves (see Chap. V and Moullin, *Radio Frequency Measurements*, Chap. I).

So far we have considered the potential of a single pole, but any collection of poles can be built up from a series of single poles, and hence the potential of any system is $V = \sum m/r$. The convenience of the potential should now be obvious: in order to find the resultant force at any point it would be necessary to find the force from each component pole and then make a vector addition of all these component forces. The total potential can be found by scalar addition, and then a single process of differentiation yields the force.

Since a force may be either an attraction or a repulsion it is necessary to consider the sign of V. Work is considered positive if it has to be supplied externally, and negative if it is supplied by the field; force is considered positive if it is in the direction of r increasing. So if a force H moves a unit pole a distance dr, the work is negative, and hence

$$-dV = Hdr,$$

and so

$$H = -\frac{dV}{dr},$$

$$\therefore \quad V = \frac{m}{r}. \tag{1}$$

If H_1, H_2, and H_3 are components of magnetic force in the direction of the axes of x, y, and z, then

$$\left.\begin{aligned} H_1 &= -\frac{\partial V}{\partial x} \\ H_2 &= -\frac{\partial V}{\partial y} \\ H_3 &= -\frac{\partial V}{\partial z} \end{aligned}\right\}. \tag{1 a}$$

The relation between force and space rate of change of potential is independent of the law of force and is applicable to any system of central forces: we will now consider the particular application to magnetism, for which the law of force is inverse square.

3. Potential and field of a magnetic particle

We can now consider the potential and the field due to an elementary magnetic particle situated at the origin and pointing along the axis of x. Let the poles be situated at the points A and B, separated from one another by a distance d. Then the potential at the point (r, θ), see Fig. 1, is

$$V = \frac{m}{PB} - \frac{m}{PA} \tag{2}$$

$$\doteqdot \frac{m(AP-BP)}{OP^2}$$

$$\doteqdot \frac{md\cos\theta}{r^2}, \text{ if } \frac{d}{r} \ll 1,$$

$$\equiv \frac{M\cos\theta}{r^2}. \tag{3}$$

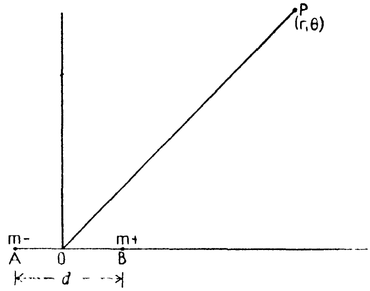


Fig. 1

$M = md$ is called the magnetic moment of the particle: it will be noticed that the force depends on the magnetic moment and neither m nor d can be separated from a measurement of the force at P.

Let H_r be the radial component of force, then

$$H_r = -\frac{\partial V}{\partial r} = \frac{2M\cos\theta}{r^3}. \tag{4}$$

If H_t is the tangential component of force, then

$$H_t = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{M\sin\theta}{r^3}. \tag{5}$$

So at a given distance r, the force at a point on the line through the particle and perpendicular to its axis is half the force at a point on the axis of the particle.

Since m and d occur only as a product, the expression for V will be valid for any bunch of magnetic particles placed at the origin and all pointing in the same direction: not all members of the bunch need have the same value of m or of d or of their product. A bar magnet has been regarded as a bunch of filament-magnets, for all of which d had roughly the same value: so the potential of a bar magnet at a very distant point (r, θ) is $V = M\cos\theta/r^2$, where $M = \sum md$. It

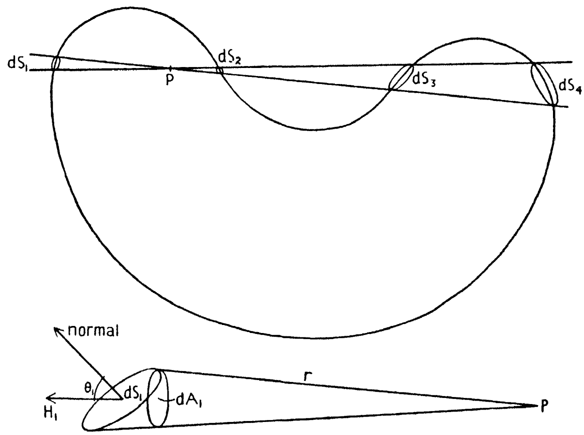


FIG. 2

is easy to extend this argument and show that a magnetized body of any shape must have the same effect at all distant points as would a magnetic particle with a certain definite moment and direction.

4. Gauss's theorem

We now require the very important theorem of Gauss, which applies to any attractive system in which the forces vary as the inverse square of the distance. Consider an isolated magnetic pole situated at the point P (see Fig. 2) and let it be surrounded by a closed surface of any shape whatever. From P draw a double cone of small angle: let this double cone cut through the closed surface and let the area, of the surface enclosed by the cone at the points of penetration be dS_1, dS_2, etc. Let H_1 be the value of the magnetic force in the sense of the outward-drawn normal to dS_1, H_2 to dS_2, etc., and let θ_1, etc., be the inclination of the normals to the axis of the cone.

Now
$$H_1 dS_1 = \frac{m\cos\theta\, dS_1}{r^2}$$
$$= \frac{m\, dA_1}{r^2}$$
$$= m\, d\Omega,$$
where $d\Omega$ is the solid angle* of the cone. Consider the sum of the products HdS for all points of cutting. At dS_3 and dS_4 the outward-drawn normals are in opposite directions, whereas the forces are in the same direction. Hence $H_3dS_3+H_4dS_4 = 0$. But at dS_1 and dS_2 the force is in the direction of the outward-drawn normal and so $H_1dS_1+H_2dS_2 = 2m\, d\Omega$. If every possible cone is drawn from P the whole surrounding surface will be covered with patches like dS_1 and the product HdS taken over the whole surface will be
$$\iint H\, dS = 2m \sum d\Omega$$
$$= 4\pi m.$$
So we arrive at the following theorem. If any number of poles are enclosed within any closed surface, and if H the normal component of magnetic force is found at every point of the surface and multiplied by the corresponding small element of surface at that point, and if these products are added up over the whole surface, the result of the addition will always be equal to 4π times the net pole strength enclosed. This is true whatever be the size or shape of the surrounding surface.

So far this theorem seems very abstract and difficult to visualize, but it is of paramount importance since it permits field strengths to be represented by a given number of lines of force per unit area. We will first regard it in various ways so as to realize its physical meaning. First imagine that at some point in an infinite ocean there is a pipe emitting water at a constant rate: imagine the pipe outlet surrounded by an envelope of wire netting of any size or shape and made of infinitely fine wire. To measure the flow of water from the pipe it would be necessary to measure the magnitude and direction of velocity through every mesh, multiply the outward normal component of velocity through that mesh by the area of that mesh, repeat the process at every mesh, and add the total. The result would give the rate of flow of water from the pipe. The same answer would be

* The solid angle of a cone is defined as the area it surrounds on the surface of a sphere of unit radius centred at the vertex of the cone.

obtained whatever the shape and size of the netted surface. The answer would be the same because water is incompressible, and could not appear or disappear except through another pipe. Instead of a water-pipe, suppose there is a source of light; then there must be the same flux of light through any closed surface. So magnetic force from a pole (or electric force from a charge) has some analogy with, say, the velocity of water. Note, however, that the writer does not wish to imply any idea of flow, but has merely described a certain process of charting which would be very obvious for water or light but which does not seem so obvious for magnetic or electric forces. We will now try to invent some idea which will allow Gauss's theorem to be used, for evidently it has great potentialities.

A line of force in a field is too familiar and obvious an idea to need

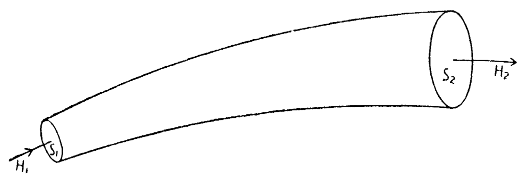


FIG. 3

laborious description and definition: it is a chart showing the direction of the force in space. In any given region, imagine a large number of such lines have been drawn, not necessarily all in one plane. Some of these lines could be selected to form generators of the sides of a tube. Let such a tube be illustrated by Fig. 3. Let a short length of this tube be closed by end pieces S_1 and S_2, set square to the tube. Regard the closed surface so formed as a Gauss surface. Over the curved walls of the tube H is zero, because by construction the direction of the force is along the walls of the tube. Hence

$$\iint H\,dS = H_1S_1 - H_2S_2.$$

If the tube does not enclose a pole, then $H_1S_1 = H_2S_2$. The same property would hold good along a tube of water-flow, and there H would represent a velocity. Fields are mapped by tubes of force and it is helpful to imagine they have real existence like stretched threads. Then their number per unit area is a measure of the field strength at the point. A tube of force conceived in this manner has the fundamental property, due to the relation $H_1S_1 = H_2S_2$, that it is indestructible and unterminating except on a pole. Every pole gives rise to a certain number of tubes of force which stretch outwards till they converge ultimately on other complementary poles. Engineers always speak of these tubes as lines of force and a field of unit strength is described as a field having one line per unit area, and in it there is unit force on unit pole. The extended idea of lines of force must not be taken too literally, because the finite number obviously suggests interspaces between the lines, and this is absurd: the absurd possibility is less blatant if tubes are pictured instead of lines. The reader has now been warned against forming too rigid a picture of lines or tubes as real physical entities. They are a con-

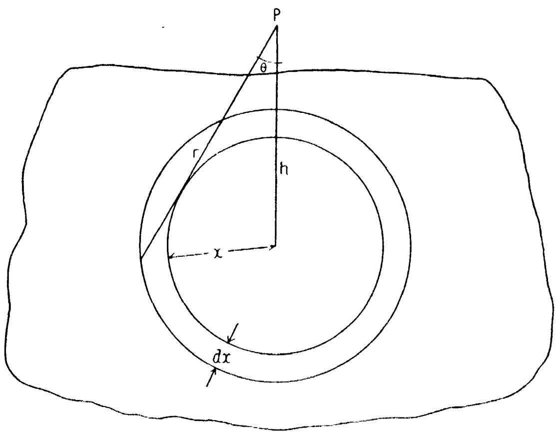


FIG. 4

venient way of describing the magnitude and direction of a field, but they must not be forced to do more than this: the model must not be pressed beyond its legitimate scope. When an expression is used such as 'here the lines of force are many or few', we remember just how much and just how little is implied by the statement.

5. The force due to certain groupings of poles

(*a*) *Planar distribution.* Let the like poles of many very long magnets be arranged so as to form a plane area of very great extent: this is how the flat end of a bar magnet would appear when viewed from a point very close to it. Let the pole strength per unit area be I. Consider the force at a point P (Fig. 4), distant h from the surface, due to the poles included in the ring whose inner and outer radii are respectively x and $x+dx$. Then dF, the outward component of force due to the poles in this ring, is

$$dF = \frac{2\pi x dx I \cos\theta}{r^2}$$

$$= 2\pi I h \tan\theta\, h \sec^2\theta d\theta \frac{\cos^3\theta}{h^2}$$

$$= 2\pi I \sin\theta\, d\theta.$$

So the force due to a circular plane which subtends an angle ϕ at P is

$$F = 2\pi I \int\limits_0^\phi \sin\theta\, d\theta$$

$$= 2\pi I(1-\cos\phi). \qquad (6)$$

If the radius of the end is several times h, then $\cos\phi$ is nearly zero and $F = 2\pi I$. If there is a small hole at the centre of the plane, subtending an angle 2α at P, the force becomes $2\pi I\cos\alpha \doteqdot 2\pi I(1-\frac{1}{2}\alpha^2)$. Thus, provided the distance of P from the surface is several times the diameter of the hole, the effect of the hole is very small.

Magnets must have two poles; what is the effect of the poles at the other ends of these long magnets which are grouped so as to give a plane surface at their ends? The force due to the other surface is almost zero if the magnets are very long, because for that end $\cos\phi$ is nearly unity. So the force very near to the end of a bar magnet is very nearly equal to $2\pi I$. Now suppose the bar magnet is curled round into a ring as shown in Fig. $5a$, with the two flat surfaces parallel and very near together. At a point between the end faces, not too close to the edges, the force due to the poles which end on the flat surfaces will be approximately equal to $4\pi I$, since there is a pull of $2\pi I$ from one surface and a push of $2\pi I$ from the other surface.

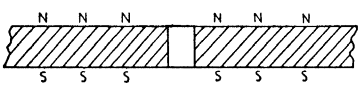


FIG. 5*b*

Now consider the force inside a small hole which has been drilled through a thin magnetized plate, as shown in Fig. $5b$. The push from the top surface will be reduced inappreciably by the absence of the poles which have been removed by the hole, except for points which are only just below the surface. The same is true for the force from the bottom surface, and therefore the force inside a hole like that shown in Fig. $5b$ is sensibly equal to $4\pi I$ throughout the greater part of its length.

(*b*) *Work done in taking a unit pole through a hole in a magnetized plate.* We will now find the work done in moving a unit pole along the axis of a circular hole drilled through a uniformly magnetized plate. A cross-section through the plate and the hole is shown in Fig. 5*c*.

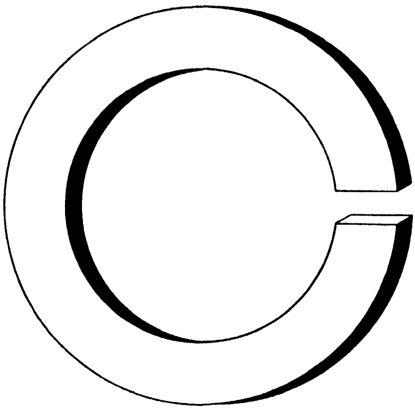


FIG. 5*a*

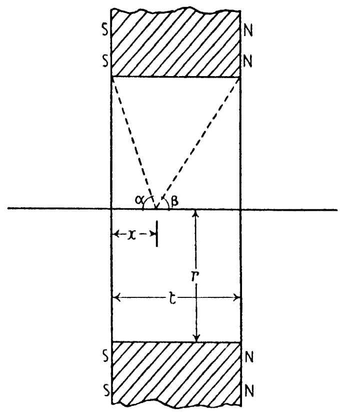


FIG. 5*c*

Then
$$F_x = 2\pi I(\cos\alpha+\cos\beta)$$
$$= 2\pi I\left(\frac{x}{\sqrt{x^2+r^2}}+\frac{t-x}{\sqrt{(t-x)^2+r^2}}\right)$$
$$\therefore \int_0^t F_x\,dx = 2\pi I\Big[\sqrt{x^2+r^2}-\sqrt{(t-x)^2+r^2}\Big]_0^t$$
$$= 4\pi I\{\sqrt{t^2+r^2}-r\}$$
$$= 4\pi It\left\{\sqrt{1+\frac{r^2}{t^2}}-\frac{r}{t}\right\}$$
$$\doteqdot 4\pi It\left\{1-\frac{r}{t}\right\} \text{ if } t \gg r.$$

This approaches very quickly to the limiting value $4\pi It$ when the radius of the hole is made very small compared with the thickness t. We have seen that the force inside the hole is substantially constant and equal to $4\pi I$, and the work done is very nearly equal to this force multiplied by the distance t. We have proved that the net work done is zero when a pole is taken round any circuital path in a field arising from any system of poles; so the work done in bringing the pole from infinity to the hole and then away again to infinity on the other side must be equal to $-4\pi It$, since it is $4\pi It$ through the hole.

(c) *Force due to a uniform magnetic shell.* Consider a thin iron plate of uniform thickness but of any size or shape: the plate may be flat or it may be dented in any manner. Let the plate be magnetized so that the whole of one side is north polar with uniform intensity* I, and the whole of the other side has therefore uniform intensity $-I$. Such a plate is called a uniform magnetic shell, and it can be imagined to be built up from an infinite number of short bar magnets stuck together side by side. A uniformly magnetized shell may be difficult to construct in practice, but the idea contains nothing incompatible with the physical concepts of magnetism. Let Fig. 6 represent some section through any uniform magnetic shell. Consider the potential due to an elementary magnetic particle at B whose polar ends contribute an area dS to the shell. If the pole strength per unit area is I and the thickness t, then $It = M$, is the magnetic moment per unit area and is known as the strength of the shell. Then by equation (3) the potential at P due to the element at B is

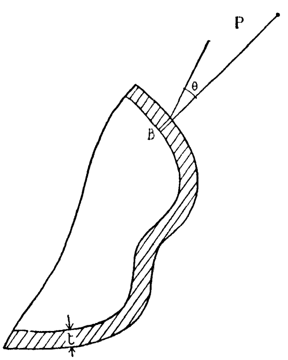


FIG. 6

$$dV = \frac{MdS\cos\theta}{r^2}.$$

But $dS\cos\theta$ is the element of area projected perpendicular to the ray r, and $dS\cos\theta/r^2$ is the element of solid angle subtended at P by the element dS of area at B. Hence the potential V at P of the whole shell is

$$V = M\Omega, \tag{7}$$

where Ω is the solid angle subtended at P by the boundary edge of the shell. We have arrived at the conclusion, surely an unexpected one, that the force at any external point depends only on the boundary edge and the strength and not on the shape or surface area of the shell. The shell could receive any number of bends or dents without altering the force at P, so long as the bounding edge was not altered thereby. All shells having the same boundary and the same strength give the same force at any assigned point P. The difference of potential between any two external points is M times the difference of solid angle subtended by the boundary of the shell from these two points.

* Intensity means pole strength per unit area of plate.

The work done in taking a unit pole from a point on one side of a shell, and just outside the surface, to a corresponding point on the other side of the shell must approach very closely to $4\pi M$. This may be seen in either of two ways. First of all, suppose a fine hole is bored through the shell and the unit pole is taken round a complete circuital path. The net work round the whole path must be zero, and we have just shown that the work done in going through the hole approaches $4\pi It = 4\pi M$: hence the work done in the part of the circuit exclusive of the hole must approach $-4\pi M$. Or consider the change of solid angle, as the point is brought away from the surface of the shell and swept round to the other side; the rays of the cone must sweep through the whole surface of a unit sphere and so the change of solid angle approaches 4π.

ELECTROMAGNETISM

6. Initial experiments

About the year 1805 Oersted discovered that a magnetic field is produced if the terminals of a voltaic cell are joined by a wire. For this, and other reasons, we say that an electric current is flowing through the wire and through the cell. This was the first discovery that magnetic fields could be produced by electric currents. All the region round the electric circuit is surrounded by a magnetic field, so it seems probable that some arrangement of permanent magnets could be contrived which would produce a field indistinguishable from that produced by the current. It must be possible to express the form of the circuit and the strength of the current in terms of equivalent pole strength. The laws by which this equivalence is expressed were formulated by Ampère.

We will describe some experiments which are similar to those classic experiments from which Ampère formulated his laws. Connect a piece of wire to the terminals of a cell and at a distant point observe the magnetic field by some suitable instrument, say a magnetometer. The field strength will be found to increase as the loop of wire is made to enclose a larger and larger area. When the wire is stretched out straight and doubled back on itself, enclosing no area, the field strength becomes zero. So the field due to a circuit of given perimeter is a function of the area it encloses.

Now arrange the circuit as shown in Fig. 7. The portions AB and ED produce no field since they enclose no area, and sensibly all the field is due to the loop BCD. The field at a very distant fixed point will be found to vary directly as the area BCD: for a given area of BCD the field at all distant points is the same as would be produced by a magnetic particle placed at the loop and with its axis perpendicular to the plane of the loop. For a given loop and point of observation, the field is found to be a function of the current strength, and this may be varied by changing, for example, the number of cells.

For the purpose of the present discussion we shall say that the

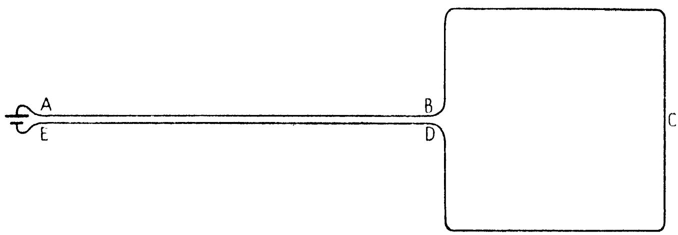


FIG. 7

current strength is defined as being proportional to the strength of the field. Thus if a change in the battery makes an n-fold increase of field strength, the current is said to have increased n-fold. So if M is the moment of the equivalent magnet and A is the area of the small circuit, then the strength of the current* is such that $Ai = M$.

Equivalence to a magnetic particle can hold good only at very great distances, distances which are sufficient to make the indefiniteness of the term 'distance from the circuit' (which is a finite area and not a point) unimportant. It is not possible to test the equivalence very exactly because of the difficulty of defining this distance. The

* This definition and description of current contains no reference to the idea of flow of electric charges along the wire, and consequently is repugnant to a true electrician. If current is described in this way, then experiment shows that current strength is proportional to the number of charges flowing per second, the charges being described in terms of electrostatic theory. Capacity effects are not dealt with till the last chapter of this book and so it is not essential to discuss the idea of electric charge. However, the writer hopes the reader will recognize this description of current as a logical and possible one, although undesirable because of other known facts of electricity which do not concern us at present. The description is used as a convenient way of avoiding a digression and of cutting a long story short. The reader is supposed to have been trained in electricity and so he will understand the implications of the last paragraph and will recognize the desirability of the procedure in the present circumstances.

result of the experiment must be regarded as an inherent probability of what occurs in the limit: on this supposition an analysis will be developed, and this will suggest further experiments leading to inductive proofs.

7. Ampère's equivalent magnetic shell

The assumed equivalence does not show how to find the force near a circuit of finite size: to do this we require the ingenious device due to Ampère. Ampère showed that a circuit could be regarded as built up from a large number of elementary circuits, each complete and containing its own battery. Each elementary circuit must be placed

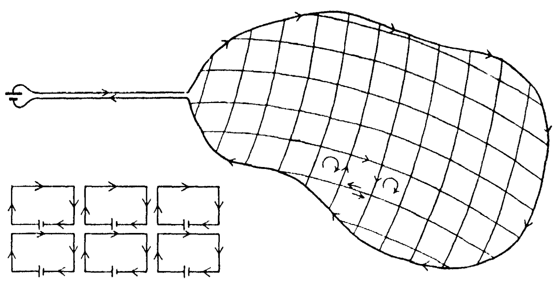


FIG. 8

contiguous to its neighbour, like tiles on a floor. The whole area is covered over and the process is stopped only at the periphery of the real circuit. The arrangement is shown diagrammatically in Fig. 8, which also shows an enlarged view of some contiguous elements, or meshes as they may be called. Except round the periphery of the circuit there are always two wires side by side with equal and opposite currents: by the initial experiment these sensibly coincident wires produce no magnetic field. So at every point, the field of the original circuit and of the built-up system are identical. By making the meshes very small, any point in space becomes relatively distant from every mesh,* and so every mesh may be replaced by an equivalent magnetic particle. When every mesh has been replaced by a magnetic particle, the result will be a magnetic shell. The field of the circuit is therefore the same as the field of the magnetic shell bounded by the circuit and having a moment per unit area such that $M = i$. But the shell need not be flat, it may have any number of bulges and dents so long as it has the correct boundary edge. So any shell

* The meshes need not be in one plane.

having the correct strength and the correct bounding edge will produce the same field as the circuit.

Now consider the work done in taking a unit pole from any point, round some path which threads the circuit, and back again to the starting-point. This process is represented by Fig. 9. Let us set out to calculate the work round some circuital path which threads some particular electric circuit we have set up. We must first calculate the magnitude and direction of the force at every point along the assigned path, and to do this we must choose some convenient magnetic shell

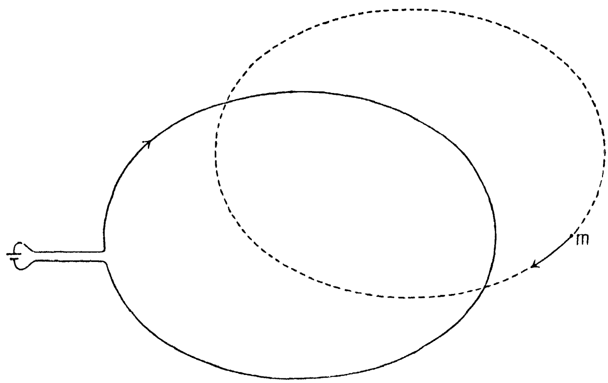


FIG. 9

bounded by the electric circuit. We do not in fact place a real shell in position but we imagine some convenient shell, and proceed for every point of the path to integrate the force from the elementary poles which form its surface. After a very laborious procedure we can now imagine we have found the force at every point on the path, except over the very short length which was occupied by the chosen shell. We can now choose some other shell, and if we repeat the process we shall obtain the same values as we found previously; but the new shell will allow us to complete the small gap occupied by the first. We find the force in this piece is finite and the length of path indefinitely short, so the work contributed by this piece is negligible. But we know the work done from one side of a shell to another is $4\pi M = 4\pi i$, so the work done round the whole circuital path threading a current must be $4\pi i$ and not zero. The work round a path which threads through a real shell is zero because the work is recovered in passing through the shell. But there is no real shell associated with a current, and $4\pi i$ units of work must be performed every time a pole is taken round a circuital path which threads the current. This work is not recoverable by the pole and is not a stored energy. As engineers, we naturally speculate about the mechanism by which the work is done or is absorbed. It happens through the agency of the e.m.f. induced in the circuit by the magnetic flux of the moving pole. While the pole is moving the current i must alter in value unless an external adjustment is being made simultaneously to maintain it at a steady value. It is out of place, and definitely apart from the author's scheme, to discuss the induced e.m.f. at present; this is reserved for Chapter II. Doubtless the reader knows something about the induced voltage, and so he can make a mental note of the process without deflecting his attention from the main lines of the argument.

The difference of magnetic potential between two points in the field of a circuit is always equal to the difference of solid angle subtended by the edge of the shell at these two points, but the absolute value of the potential is indeterminate. Thus suppose we start at a point P in the field and perform a circuital tour which threads the current and arrive back at the point P; an amount of work $4\pi i$ has been done, and so in a sense there is this difference of potential between two points which are coincident. The potential at any point is multi-valued and may have an infinite series of values each differing by $4\pi i$. But the force is equal to the space rate of change of potential and is not multi-valued, so there is no confusion or violation of observed physical facts. The potential is a derived mathematical tool which is defined from a space rate of change. It is a function which has a definite gradient but an uncertain value because the datum-level is open to choice. If the reader understands that the potential is a tool invented to simplify calculations he will not stray into philosophical difficulties about its value. The difficulties are not real and are only apparent to those who do not realize the method of definition. To describe the potential of a point as the work done in bringing up a pole from infinity is a mere convention which includes certain arbitrary decisions about the constants of integration.

If the equivalence of an elementary particle and an elementary circuit is correct, then $4\pi i$ units of work must be performed every time a unit pole is carried round a circuital path which threads the current. This is a logical deduction from the premisses, but the result seems surprising until we understand the induced e.m.f. The initial experiments are necessarily rather rough and incapable of very exact verification. But from them we deduce the work law which will lead to further deductions which are capable of precise verification. If these further deductions prove correct we shall then presume the equivalence between an elementary circuit and a magnetic particle is correct. Precise experiments show that $4\pi i$ units of work are done in taking a unit pole round a closed path which threads a current, and so this may be taken as the initial experimental relation between electricity and magnetism.

It is possible to start with the work law as the initial assumption and then proceed to the Heaviside current element, considered later, and omit all mention of equivalent shells. But this is not the process by which the science grew up, and it also obscures the fact that any circuit is equivalent to a single magnet in its effect at a very distant point: this equivalence is useful to bear in mind.

The work law is of fundamental importance to every one and of overwhelming importance to the dynamo engineer, since it is one of his most used weapons for attacking a problem. His problems are generally too complicated to permit of exact solution, and the work law is often his only weapon. The work law is generally written in symbols as follows:

$$\int H\,dl = 4\pi i. \tag{8}$$

Here the integral sign stands for integration round any closed path, and H stands for the component of magnetic force in the direction of the movement dl.

The work law of electromagnetism is a close counterpart of Gauss's theorem in magneto- or electrostatics. The work law gives no information about the alteration in magnitude and direction of the field from point to point and tells only of the net average effect. It can be used as a tool to solve problems only when the direction and magnitude at every point of the field is known already: in many dynamo problems this is known approximately by that form of experience often described as common sense.

8. Two particular magnetic fields

(*a*) *Field near a long straight wire.* There are two fields whose distribution is obvious from symmetry. One is that round a straight wire of very great length, such as a single telegraph wire. From symmetry the lines of force must be very nearly circles; they cannot be perfect circles because there must be a return conductor somewhere in the far distance. First suppose the return conductor is so distant as to make no measurable effect; we shall see later how far off it must be to reduce the error to any assigned amount. The wire and current are shown diagrammatically in Fig. 10, and also a typical line of force, having a radius R. From symmetry the magnetic field H is the same at all points of the circumference.

$$\therefore \quad \int H\,dl = H\int dl$$

$$= H \times 2\pi R.$$

$$\therefore \quad 2\pi RH = 4\pi i$$

$$\therefore \quad H = \frac{2i}{R}. \qquad (9)$$

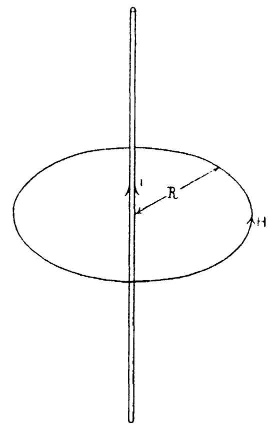


FIG. 10

So the field strength varies inversely as the distance from the centre of the wire. The direction of the current and the direction of the magnetic field are related to one another as the direction of motion and of rotation of a right-handed screw. In this problem we know from common sense that the lines of force must be circles and that H must be constant along the circumference of any particular line of force; given this all-important information, H can be calculated from the work law.

The same problem may be solved directly by replacing the circuit by an equivalent magnetic shell. Any shell having the correct edge will produce the correct magnetic field: one edge lies along the wire of infinite length, and the other edge is somewhere at a great distance. The shell is shown diagrammatically in Fig. 11 (p. 21), where A is a section of the wire and B is the broken edge of a shell which extends to infinity; the other edge may be anywhere at infinity: let it lie in a horizontal plane through A. Round the point P describe a sphere of unit radius; then the solid angle subtended at P by the shell is equal to the area of the sphere cut off by a horizontal plane through P and the plane through P containing the wire and making an angle θ with the horizontal.

So
$$V = i \times \frac{\theta}{2\pi} \times 4\pi = 2i\theta.$$

$$\therefore \quad \frac{\partial V}{\partial r} = 0 \quad \text{and} \quad \frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{2i}{r}.$$

So the lines of force are circles centred on the wire, and the magnetic force varies inversely as the distance of the point from the wire.

(*b*) *Magnetic field inside the conductor.* Experiment shows that a steady current is distributed uniformly over the cross-section of any wire, whether or not the wire is situated in an external magnetic field.* So from symmetry the current in a circular wire must produce a magnetic field inside the wire in which the lines of force are circles centred on the axis of the wire. If the wire of radius R carries a current i, then the uniform current density is $\frac{i}{\pi R^2}$. Take a unit pole round a circle of radius a, then the current enclosed by it is $i\frac{a^2}{R^2}$.

$$\therefore \quad H \times 2\pi a = 4\pi i \frac{a^2}{R^2}.$$

$$\therefore \quad H = \frac{2ia}{R^2}. \tag{10}$$

So the internal field increases uniformly from the axis to the circumference; outside the wire it decreases and varies inversely as the distance from the centre of the wire.

It is a little difficult to visualize the field inside a solid conductor because it is hard to imagine measuring the force on a test pole placed inside it. The test pole must be imagined as placed inside a cavity which is too small to disturb the current flow appreciably.

If the conductor is tubular, the current in it produces absolutely no force inside the tube. If the outside conductor of a concentric main is a tube, then the force external to the main is zero everywhere and the force inside the main is due to the inner current only. In practice the outside conductor is formed of wire strands and not by a solid drawn tube, so the external force is nearly but not quite zero.

(*c*) *Field inside a toroidal current sheet.* Consider a tubular anchor ring, as shown in cross-section and plan by Fig. 12, and let the current flow round the circumference of the cross-section as shown by the

* That is, ignoring the very small 'Hall effect'.

arrows in the upper figure. This arrangement may be supposed to be produced by making a slit round the ring and by connecting a succession of batteries in parallel to the contiguous edges of the slit; the arrangement is indicated in Fig. 12. From symmetry it follows

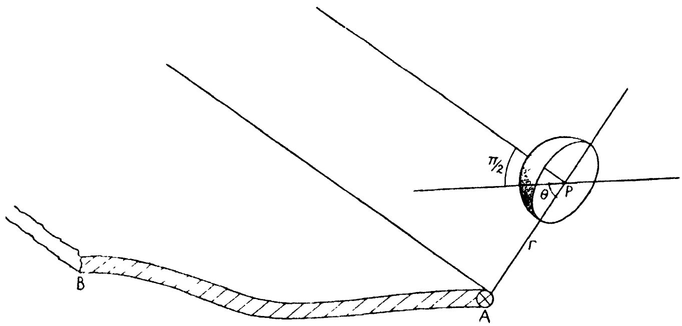


FIG. 11

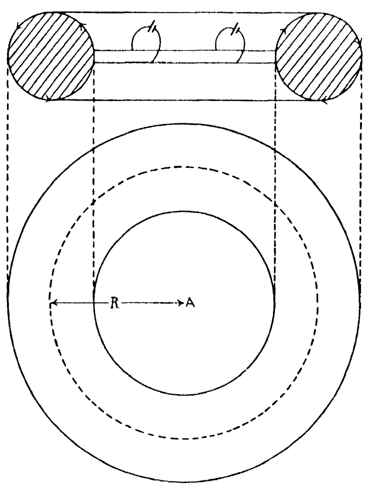


FIG. 12

that the lines of force must be circles centred at A: the work law shows there is no field outside the tube. Let the total current flowing round the tube be I, then carrying a unit pole round a line of force of radius R, we have $2\pi RH = 4\pi I$. The natural way to make a close approximation to a uniform toroidal current sheet is to wind a ring uniformly with a large number N of fine wires. The winding is then a helix of small pitch and the coil is usually called a ring solenoid.

Let H be the field strength at radius a, then

$$2\pi aH = 4\pi iN,$$

$$\therefore \quad H = \frac{4\pi iN}{2\pi a}. \tag{11}$$

The field inside the ring solenoid will differ inappreciably from that inside a toroidal current sheet; the field outside will not be zero but will be that of a single circle of wire of radius a. This may be understood by imagining the helical winding is avoided, by using an enormous number of flat single turns each joined to the next by short steps following the mean circumference. In the limit this arrangement approaches a toroidal current sheet together with a single-turn circle.

9. Oliver Heaviside's rational current element

We have now dealt with all the problems in which the distribution of magnetic field can be seen by inspection and the magnitude calculated by direct application of the work law. The device of the equivalent magnetic shell has led to the method of calculating the force at any point due to any circuit, and it has led to the all-important work law. The device can now be dismissed from everyday use, for it has served its turn as a stepping-stone to the work law.

Ampère's device consists in building up a circuit from elementary or atomic circuits. Since the essence of a circuit is a boundary edge, it seems natural to subdivide a circuit into elements of length. But this is physically absurd because current can flow only in a circuit; a steady current flowing in an open-ended conductor is physically absurd and impossible. Yet in the end, the Ampère device shows that the force depends only on the edge of the shell, which is the circuit, and indeed it could not possibly depend on anything else. So regarding this physically it seems as if there ought to be some method of cutting up the circuit into elements of length. The elements cannot be made as shown in Fig. 13 because the current returning through the battery and connexions would neutralize that round the edge of the circuit, and the net magnetic field would be zero everywhere. The correct form of current element was devised by Oliver Heaviside (see Heaviside, *Collected Papers*, vol. ii, pp. 500–4) and is as follows. Consider a short length of insulated wire containing a battery: let the short length of wire, with its battery, be submerged in an infinite cean of conducting material, say salt water or a solution of copper sulphate. Such an arrangement is shown in Fig. 14; an alternative arrangement would be the familiar cylindrical battery used in electric torches. Current will flow from the bare end of the wire into the conducting ocean, and after a distributed flow it will collect again into the other bare end of the wire. The system is a uniform current along the length of the wire and a return current filling all space. If a second element is fitted end on to the first, one bare end will be covered up and the outflow transferred to the bare end of the second element. Many elements can be placed end to end so as to imitate

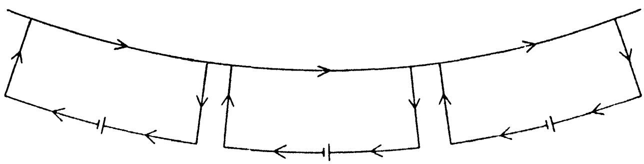


FIG. 13

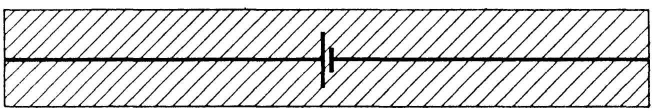


FIG. 14

the shape of the original wire circuit. When the last element is put in position the circuit becomes closed on itself and the last pair of bare ends is covered up. Flow in the surrounding conducting ocean then ceases suddenly and current flows in the wire only: a complete circuit can be imagined built up from elementary circuits formed in this way. If we can find the magnetic force at any point due to one of these elements by itself, then the force due to a circuit can be found by adding the contributions from all the elements into which the circuit is divided.

Current will stream out from one bare end with perfect radial symmetry, like lines of force from a magnetic pole or from an electric charge, and it will converge on the other end with the same symmetry. Every line of magnetic force must be a circle centred on the axis of the element. In Fig. 15 let AB be the element and let the circle of radius R and centre C be some line of force; let the magnetic force on this circle be H. Then by the work law

$$2\pi RH = 4\pi \times \text{current flowing through circle of radius } R \text{ centred at } C.$$

But the net current through this circle is the difference between the outflowing radial current emanating from B and the inflowing radial current converging on A.

The current density at a distance r from B is $\frac{i}{4\pi r^2}$ and the current through the ring is the current flowing through the cap subtending an angle θ at the centre of a sphere of radius r centred at B.

Hence the outgoing current through the ring

$$= \frac{i}{4\pi r^2} \times 2\pi r^2(1-\cos\theta)$$

$$= \frac{i}{2}(1-\cos\theta).$$

Similarly the inflowing current

$$= \frac{i}{2}\{1-\cos(\theta-\delta\theta)\}.$$

Hence the net current through the ring

$$= \frac{i}{2}\{\cos(\theta-\delta\theta)-\cos\theta\}$$

$$= \frac{i}{2}\delta\theta\sin\theta.$$

$$2\pi RH = \frac{4\pi i}{2}\delta\theta\sin\theta.$$

$$\therefore \quad H = \frac{i\delta\theta\sin\theta}{R}$$

$$= \frac{i\delta\theta}{r}$$

$$= \frac{i\delta l\sin\theta}{r^2}. \qquad (12)$$

Thus the field of a Heaviside element, immersed in its ocean of conducting medium, varies inversely as the square of the distance of the point from the element and directly as the length of the element measured perpendicular to the radius vector. A complete circuit of wire may be supposed built up of Heaviside elements, and so each element of a circuit may be supposed to contribute to the magnetic force an amount which varies inversely as the square of the distance from that element and directly as the length of that element measured perpendicular to the radius vector. Summation of these contributions is often more convenient than calculating the solid angle subtended by the circuit.

But the work law was required to find the force due to a Heaviside element: so this convenient device could not have been arrived at without the preliminary steps of Ampère's magnetic shell.

It is impossible to prove that a given element dl at some assigned place in a circuit does in fact make a contribution $\frac{idl \sin \theta}{r^2}$ to the total magnetic force at some point of space, because if that element

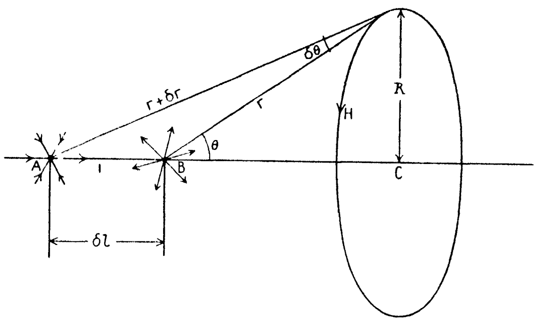


FIG. 15

was removed the current would cease and the whole magnetic field would disappear. We can prove only that the total force can be calculated correctly assuming that every element of circuit does make the supposed contribution. If circuits were composed of Heaviside elements, then the field at a given point would be reduced by the supposed amount on removing a given element of length; but then the current flow would be changed from that in a wire circuit to a flow in the wire and in all the surrounding medium.

The law of force for a current element was arrived at by Ampère, many years before Heaviside described a rational explanation for it. Other laws of force have been proposed which give the correct answer for the whole circuit, so we can say only that the Ampère law may be correct but is not necessarily so. But the law of force for a Heaviside element immersed in its infinite medium is necessarily correct.

Classical text-books make a long story about the supposed difficulties of the current element, but there is nothing mysterious about it when the problem is considered physically. Current flow must cease as soon as an element is removed from a circuit, and Ampère's initial experiments cannot possibly contain information from which to deduce the contribution of individual elements: his experiments dealt only with the effect of a complete circuit and could not possibly deal with anything else. It is useless to speculate about the effect of electricity moving in a particular piece of circuit until we have discovered further laws of electromagnetism. The Maxwell hypothesis of displacement current is perhaps sufficient to prove the reality of the force contributed from an element of circuit.

10. The magnetic field of parallel wires, circles, and solenoids

We will now apply the Ampère law of force for an element, to calculate the field of some complete circuits of practical interest.

(*a*) *Force at a point in the plane of two equal and oppositely directed currents.* The system is shown in Fig. 16, and the wires are supposed to stretch to infinity at both ends. The summation may be made for each wire separately and the two summations added together. The left-hand wire may be considered to contribute at A a force

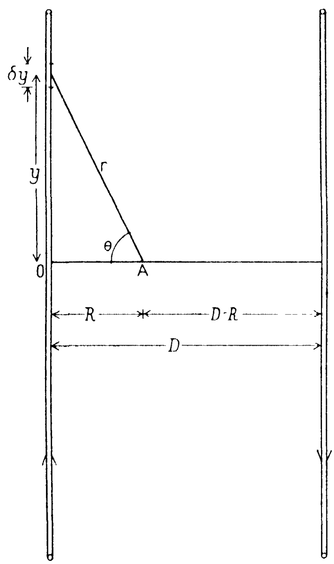


Fig. 16

$$H_1 = i\int\limits_{-\infty}^{+\infty} \frac{dy\cos\theta}{r^2}$$

$$= i\int\limits_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} R\sec^2\theta\, d\theta\, \frac{\cos\theta\cos^2\theta}{R^2}$$

$$= \frac{i}{R}\int\limits_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \cos\theta\, d\theta$$

$$= \frac{2i}{R}.$$

The total force at A due to the whole circuit is

$$H = 2i\left(\frac{1}{R}+\frac{1}{D-R}\right).$$

If A is not between the wires

$$H = 2i\left(\frac{1}{R}-\frac{1}{D+R}\right).$$

So if $D-R \gg R$, say $D-R = 1{,}000R$, the force approaches the value $H = 2i/R$ found already for a long straight wire.

(*b*) *Force at the middle point of a long rectangle.* It follows from the previous analysis that the force at O, Fig. 17, is

$$\begin{aligned} H &= \frac{4i}{b}\sin\theta+\frac{4i}{l}\cos\theta \\ &= \frac{4i}{b}\sin\theta\left(1+\frac{b}{l}\cot\theta\right) \\ &= \frac{4i}{b}\frac{l}{\sqrt{b^2+l^2}}\left(1+\frac{b^2}{l^2}\right) \\ &\doteq \frac{4i}{b}\left(1-\frac{1}{2}\frac{b^2}{l^2}\right)\left(1+\frac{b^2}{l^2}\right) \\ &\doteq \frac{4i}{b}\left(1+\frac{1}{2}\frac{b^2}{l^2}\right). \end{aligned}$$

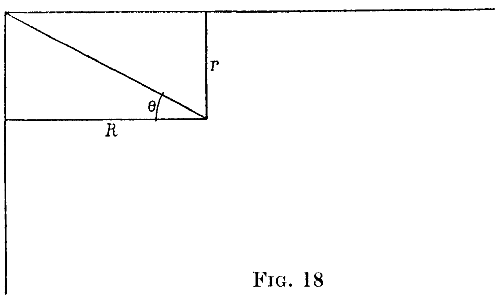


FIG. 18

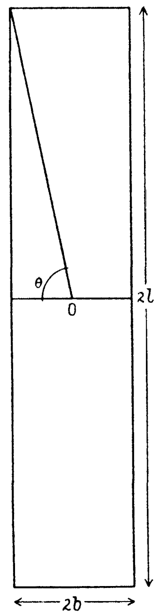


FIG. 17

If $l/b > 10$ the force differs from $4i/b$ by less than 0·5 per cent. This gives an indication of how distant the ends must be, in order that the wires may be treated as infinite in length.

(*c*) *Force near the corner of a large rectangle.* Consider the force at a point distant R, r from the sides (Fig. 18) of a rectangle whose sides are very long compared with R.

Then
$$\begin{aligned} H &= \frac{i}{R}(\sin\theta+\sin\tfrac{1}{2}\pi)+\frac{i}{r}(\cos\theta+\sin\tfrac{1}{2}\pi) \\ &= \frac{2i}{R}(1+\sin\tfrac{1}{4}\pi), \text{ when } R=r, \\ &= 1{\cdot}701\times\frac{2i}{R}. \end{aligned}$$

If each wire had been treated as doubly infinite, then the force would have been written erroneously as $H = 2\times 2i/R$. A long straight wire contributes a force $2i/R$ at points near its middle, but it contributes a force i/R at a point very near one end.

(*d*) *Work law applied to a pair of oppositely directed parallel currents.* Consider two equal parallel currents of magnitude i, separated a distance $2D$ (see Fig. 19). Then it follows that the force in the median plane is perpendicular to the plane of the wires and has the value

$$H = 4i\sin\theta/r$$

$$= \frac{4iD}{r^2}$$

$$= \frac{4iD}{x^2+D^2}.$$

$$\int_{-\infty}^{+\infty} H\,dx = 4iD\int_{-\infty}^{+\infty} \frac{1}{x^2+D^2}\,dx$$

$$= 4i\left[\tan^{-1}\frac{x}{D}\right]_{-\infty}^{+\infty} = 4\pi i.$$

So in this, as in all other problems, the field calculated from the formula $dH = \dfrac{idl\sin\theta}{r^2}$ yields a result consistent with the work law.

(*e*) *Lines of force for parallel currents.* It follows from the geometry of similar triangles that the force at a point distant r_1 from one wire and r_2 from the other wire is $H = \dfrac{2D}{r_1 r_2}$. It can also be shown that every line of force is a circle: the circles are not concentric, but the two wires are their inverse points. The lines of force of the field are shown in Fig. 20 (see A. Russell, *Alternating Currents*, vol. i, Chap. 15).

(*f*) *Field on the axis of a circular turn of wire.* An element of arc $Rd\theta$ contributes at P (Fig. 21) a magnetic force $dH = \dfrac{iRd\theta}{r^2}$ perpendicular to r and the plane containing r and the element. This force has an axial component $dH\sin\phi$: from symmetry the force from the whole circle is along the axis, and so its value is

$$H = 2\pi Ri\cdot\frac{\sin\phi}{r^2}$$

$$= \frac{2\pi R^2 i}{r^3}$$

$$= \frac{2\pi R^2 i}{(x^2+R^2)^{\frac{3}{2}}}. \tag{13}$$

At the centre of the circle this has the value

$$H = \frac{2\pi i}{R}. \tag{14}$$

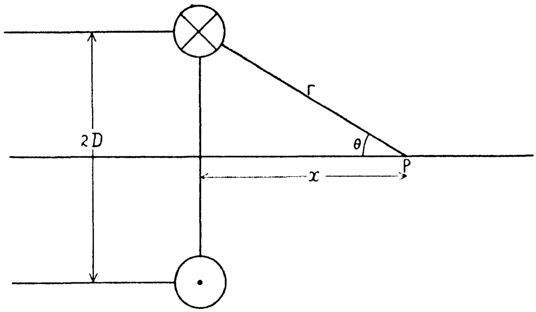


FIG. 19

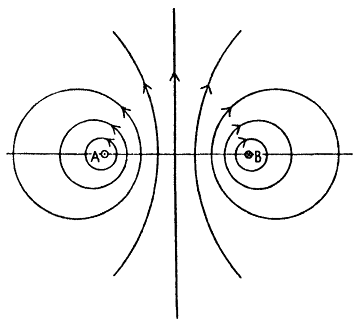


FIG. 20

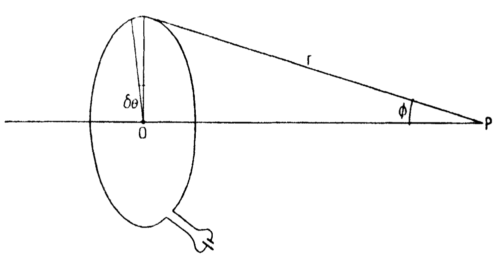


FIG. 21

When $x \gg R$, $H \doteqdot \frac{2\pi i R^2}{x^3}$; so at a great distance the circuit has the same field as a magnetic particle of moment $\pi R^2 i$, and this agrees with the initial assumption of Ampère.*

* Compare formula (4).

When $x \ll R$, $$H \doteqdot \frac{2\pi i}{R}\left(1-\frac{3}{2}\frac{x^2}{R^2}\right).$$

Fig. 22 shows how the magnitude of the field varies with the distance along the axis: at a distance $2R$ from the centre, the field has fallen to 10 per cent. of its value at the centre, and at a distance $5R$ it has fallen to less than 1 per cent. Again applying the work law,

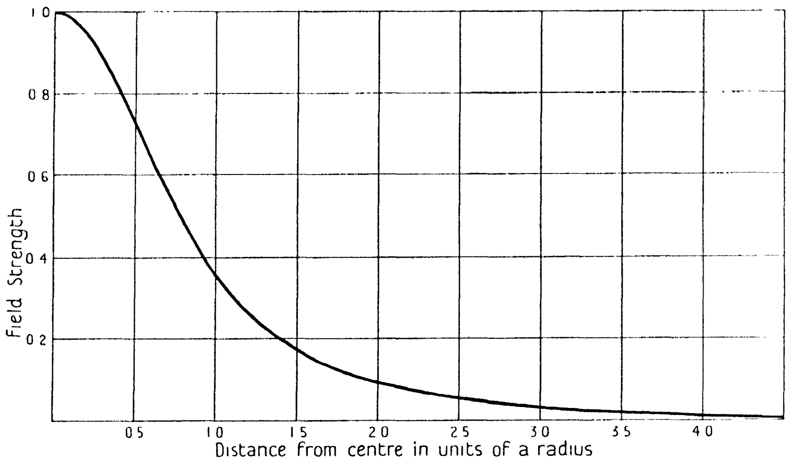


FIG. 22. Field along the axis of a circular current

we have

$$\int_{-\infty}^{+\infty} H\,dx = 2\pi Ri\int_{-0}^{0}\frac{\sin^3\phi}{R^2}\times(-R\operatorname{cosec}^2\phi)\,d\phi$$

$$= 2\pi i\big[\cos\phi\big]_{-0}^{+0} = 4\pi i.$$

Intricate mathematics is required to calculate the field at points not on the axis, and the problem is not of sufficient practical importance to develop the mathematics in this book.* The field in the plane of the coil, at a distance r from the centre, may be calculated by one or other of the following series:

$$H = \frac{2\pi i}{R}\left(1+\frac{3}{4}\frac{r^2}{R^2}+\frac{45}{64}\frac{r^4}{R^4}+\dots\right) \text{ when } r < R,$$

or $$H = \frac{2\pi i}{R}\cdot\frac{R^3}{2a^3}\left(1+\frac{9}{8}\frac{R^2}{r^2}+\frac{75}{64}\frac{R^4}{r^4}+\dots\right) \text{ when } r > R.$$

* For the necessary analysis see Clerk Maxwell, *Electricity and Magnetism*, vol. ii, Chap. 14.

These series assume the wire is indefinitely thin, and accordingly the field becomes infinite when $r = R$. To obtain the field close to the wire a different formula must be used. It can be shown that the field in the plane of the circle, at a small distance c from the centre of the wire, is given by the expression

$$H = 2i\left(\frac{1}{c} \pm \frac{1}{2R}\right).$$

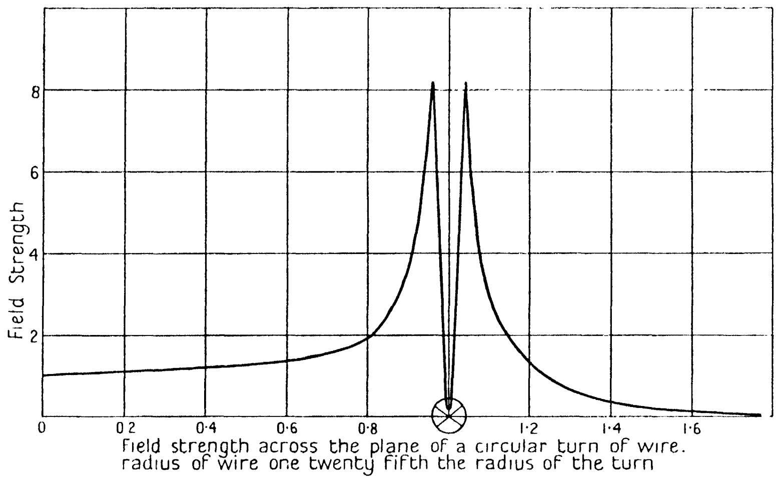


FIG. 23

This is the field which would exist at a small distance c from one of two parallel wires which were spaced a distance apart equal to a diameter of the circle. The application of this formula to a circle is expressed more conveniently in the form

$$H = \frac{2\pi i}{R}\left(\frac{\frac{1}{2} \pm \frac{R}{c}}{\pi}\right).$$

In Fig 23 is shown how the field varies along the radius for a circle of wire whose radius is one twenty-fifth of the radius of the circular turn. It may be seen that there is a large area round the axis in which the field strength is roughly constant, and at a distance $0{\cdot}4R$ from the centre the increase is only 12 per cent. The lines of force for a circular current are shown in Fig. 24.

(*g*) *Field on the axis of a solenoid.* Let the solenoid be wound with a helix of small pitch having T turns per unit length. Consider a

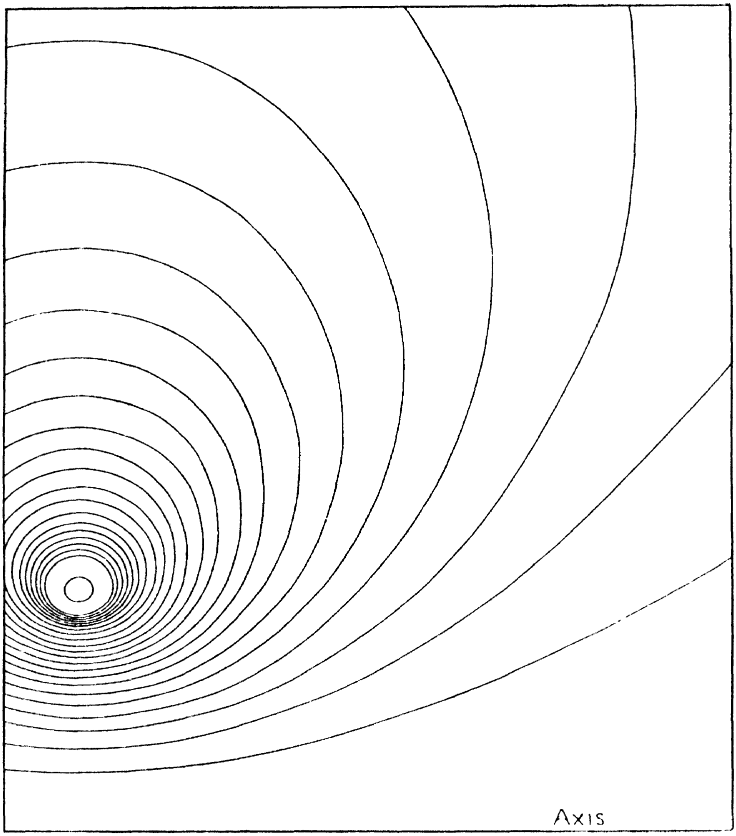


FIG. 24. Lines of force of circular current

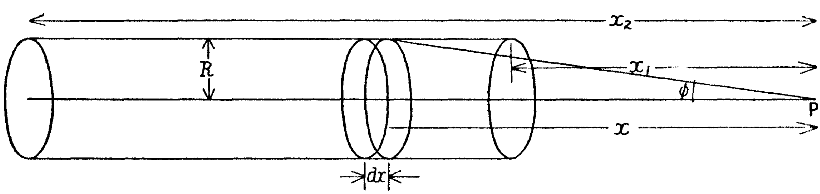


FIG. 25

strip dx, Fig. 25, distant x from P. This contributes at P a force

$$dH = 2\pi RiTdx\,\frac{\sin^3\phi}{R^2}$$

$$= \frac{2\pi iT}{R}\sin^3\phi\,dx$$

$$= -2\pi iT\sin\phi\,d\phi$$

$$\therefore\quad H = 2\pi iT(\cos\phi_2-\cos\phi_1). \tag{15}$$

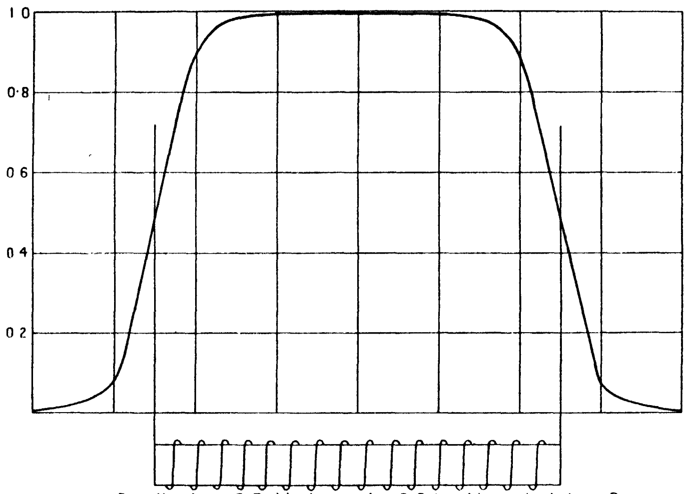


FIG. 26

When P is at the middle of the solenoid $H = 4\pi iT\cos\phi$, and when P is at an end $H = 2\pi iT\cos\phi_1$. When the solenoid is very long the force at the centre tends to the value $H = 4\pi iT$ and to half this value at an end.

When P is very distant, that is to say, x much greater than R or l,

$$H = 2\pi iT\left\{\frac{x_1+l}{\sqrt{(x_1+l)^2+R^2}}-\frac{x_1}{\sqrt{x_1^2+R^2}}\right\}$$

$$\doteqdot \frac{2\pi iTR^2}{2}\left\{\frac{1}{x^2}-\frac{1}{(x+l)^2}\right\}$$

$$\doteqdot \frac{2\pi R^2 iTl}{x^3}.$$

This is the same field as a magnetic particle of moment πR^2Tli.*

* Compare equation (4), p. 5.

Fig. 26 shows the variation of H along the axis of a solenoid whose length is ten times its diameter. The force at the middle of a long solenoid varies very little across the cross-section, and in the limit it is uniform everywhere and has the value $4\pi iT$.

(*h*) *Field along the axis of a steep pitch solenoid.* In the previous example we have supposed the pitch of the helix was very small. Let the solenoid consist of n complete turns, wound on a cylinder of radius R, and let the angle of the helix be α.

Any element ds of the wire may be put into two components, one parallel to the axis and one circumferential. The circumferential component contributes a magnetic field along the axis of the solenoid and the field due to the axial component is radial. We shall calculate only the axial component of field at the middle point of the solenoid: this is due to an infinite number of elementary circumferential arcs distributed along a helix round the cylinder. The position of any one such elementary arc may be described by Fig. 21, and accordingly we have

$$dH = \frac{iRd\theta}{r^2}\sin\phi$$

$$= \frac{iR^2d\theta}{(R^2+x^2)^{\frac{3}{2}}}.$$

But $$Rd\theta = ds\cos\alpha \text{ and } dx = ds\sin\alpha,$$

$$\therefore\quad Rd\theta = dx\cot\alpha$$

$$\therefore\quad H = 2Ri\cot\alpha\int\limits_0^{\pi Rn\tan\alpha}\frac{dx}{(R^2+x^2)^{\frac{3}{2}}}$$

$$= -\frac{2i\cot\alpha}{R}\left[\frac{x}{(R^2+x^2)^{\frac{1}{2}}}\right]_0^{\pi Rn\tan\alpha}$$

$$= \frac{2\pi ni}{R}\,\frac{1}{(1+\pi^2n^2\tan^2\alpha)^{\frac{1}{2}}}.$$

Now $l = 2\pi Rn\tan\alpha$, and $n = lT$,

$$\therefore\quad H = \frac{2\pi lTi}{R}\,\frac{1}{\left(1+\dfrac{l^2}{4R^2}\right)^{\frac{1}{2}}} = 4\pi iT\frac{l}{\sqrt{l^2+4R^2}}$$

$$= 4\pi iT\cos\phi.$$

So we arrive at the rather surprising result that the axial component depends only on the turns per unit length and involves the pitch in no other way. So equation (15) is correct whether the pitch be large or small. But the radial component of field is zero if the pitch is zero and has the value $2i/R$ in the limiting case when the pitch is infinite.

11. Force on conductors and circuits in a magnetic field

Since a circuit produces a field equivalent to a magnetic shell, the circuit must be acted on by a force of translation or rotation or both, when it is placed in an external magnetic field. The force acts on

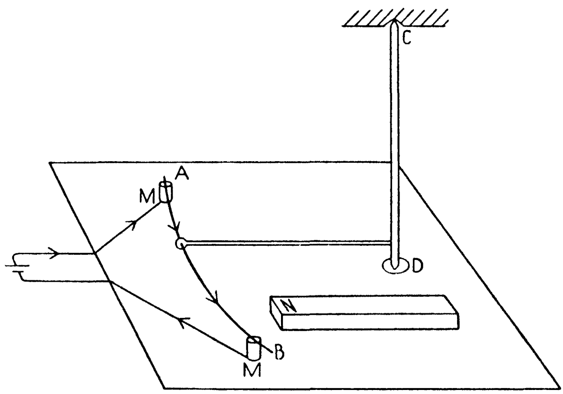


FIG. 27

the conductors, so it is natural to begin the inquiry by investigating the mechanical force on an element of conductor placed in a known magnetic field. This force can be measured experimentally because an element of circuit can be isolated mechanically from the remainder of the circuit, for example by mercury cups. It is found that the force is entirely perpendicular to the conductor: this was proved by Ampère by means of an experimental apparatus shown diagrammatically in Fig. 27. A circular arc of wire AB was mounted in a supporting frame so that it was free to turn about the vertical axis CD. Current was led in and out of the arc through mercury cups at M. Ampère found that no arrangement of magnets produced any tendency for the arc to turn about the axis CD, thus proving there is no component of force in the direction of the current, and consequently that the force is perpendicular to the current.

If a straight conductor of length l, carrying a current i, is placed across the direction of a uniform magnetic field of strength H, it is found that the force F on the current is

$$F = Hil. \tag{16}$$

The force is perpendicular to the conductor and to the direction of the magnetic field, and the relative directions are shown in Fig. 28. The relative direction of the field and the force may be settled by drawing the resultant of the applied field and the field due to the current, and then the direction of the force is obvious. The process should be rendered clear by Fig. 29.

(*a*) *Force between two long parallel wires.* Consider two long parallel wires carrying equal and oppositely directed currents, Fig. 30. Wire A lies in a field of value $H = 2i/D$, perpendicular to the plane containing the two wires, so the repulsive force between them is $F = 2i^2/D$ per unit length. When the wires have a finite diameter, not all the current is in a field $H = 2i/D$, because this field is not uniform over the cross-section of wire A and is less at the point α than it is at the point β. So it looks as if the formula $F = 2i^2/D$ may be incorrect for wires whose diameter is not very small compared with D. But this formula is always correct for circular wires, and this may be proved as follows. Let the wire A be indefinitely thin and let wire B have any finite radius R. Then all the current in wire A lies in a field $H = 2i/D$, even though the wire B has a finite radius R. So a wire of finite radius produces a force $F = 2i^2/D$ on a very thin wire distant D from its centre. But action and reaction must be equal and opposite, and so a fine wire must produce a force $F = 2i^2/D$ on a parallel wire having any finite radius R. But the field penetrating wire B is unaltered if wire A is expanded from a filament to a wire of finite size. Consequently the force between any two long parallel wires, having the same or different radii, is

$$F = 2i^2/D, \tag{17}$$

where D is the distance between their centres.

(*b*) *Force between two parallel wires of finite length.* Consider two wires AB and CD of finite length l as shown in Fig. 31: the circuit is completed by the portions AEC and BFD, where AE and BF are in one plane and bent at right angles to AB; CE and DF at right angles to CD. The legs AE, EC and BF, FD will produce a field tending to bow the wires upward, but the force spreading AB and CD apart will not be affected by the current in the legs AE, etc.* Not all the length AB will lie in a field $H = 2i/D$, because the wires AB and CD each have a finite length l.

* We are ignoring the force between AE and EC.

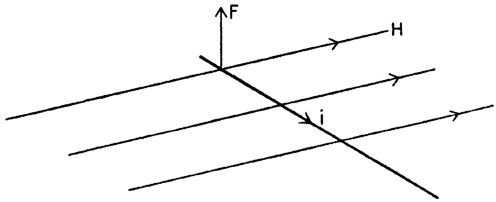


Fig. 28

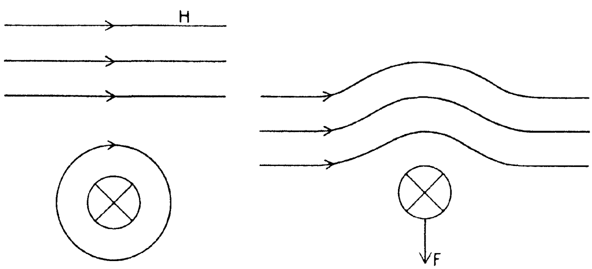


Fig. 29

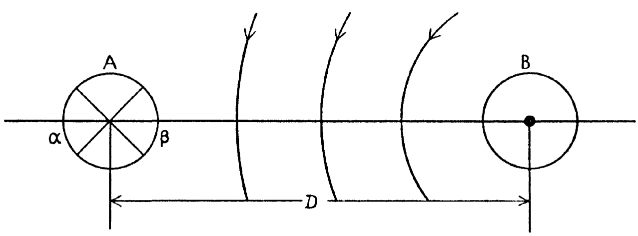


Fig. 30

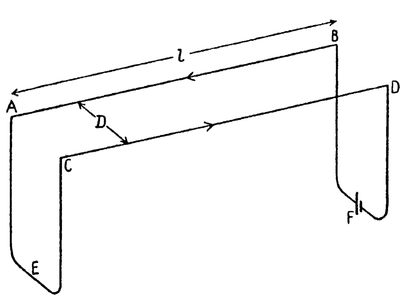


Fig. 31

It follows from § 10 (*a*) that the magnetic force at a distance D from a wire and at a distance x from its end (see Fig. 32) is given by the equation

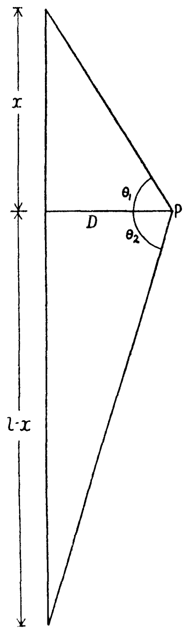


FIG. 32

$$H = \frac{i}{D}(\sin\theta_1+\sin\theta_2)$$

$$= \frac{i}{D}\left\{\frac{x}{\sqrt{x^2+D^2}}+\frac{l-x}{\sqrt{(l-x)^2+D^2}}\right\}.$$

Now

$$F = \int_0^l Hi\,dx$$

$$= \frac{i^2}{D}\int_0^l \left\{\frac{x}{\sqrt{x^2+D^2}}+\frac{l-x}{\sqrt{(l-x)^2+D^2}}\right\}dx$$

$$= \frac{i^2}{D}\left[\sqrt{x^2+D^2}-\sqrt{(l-x)^2+D^2}\right]_0^l$$

$$= \frac{2i^2}{D}\left[\sqrt{l^2+D^2}-D\right]$$

$$\doteqdot \frac{2i^2l}{D}\left(1-\frac{D}{l}+\frac{1}{2}\frac{D^2}{l^2}\right) \text{ when } l \gg D.$$

So if $l = 10D$, the force is about 10 per cent. less than for a corresponding length of infinite wires.*

(*c*) *Hoop tension in a circle of wire.* When current flows in a circular turn of wire, the field at the centre of the wire is not zero, and so there is a force tending to expand the diameter of the turn. It has been stated previously† that the field at the centre of the wire which forms a circle of diameter D is $H = 2i/D$. Hence there is a uniform radial force round the inside of the circumference equal to $2i^2/D$ per unit length. This will cause a tension T in the wire, and considerations of equilibrium show that $2T = \dfrac{2i^2D}{D}$, hence $T = i^2$.

(*d*) *Couple on a circle placed in a uniform field of force.* Consider a circle, of radius R, which carries a current i and is placed with its plane parallel to the direction of a uniform magnetic field of strength H (see Fig. 33). Consider the force on an element $R\delta\theta$ situated at θ. The length of this element projected perpendicular to the field is

* The particular form of this formula is due to W. F. Dunton; see *Journal of Scientific Instruments*, 1927, vol. 4, p. 440. † See p. 31.

$R\cos\theta\,d\theta$, so the force on this element is $HiR\cos\theta\,d\theta$. In Fig. 33, the force on all elements in the semi-circumference ACB is down into the paper and on all elements in ADB it is up out of the paper. So there is a couple tending to turn the circle about the axis AB. The moment arm of any representative element is $R\cos\theta$, and so

$$T = Hi\int\limits_0^{2\pi} R^2\cos^2\theta\,d\theta = \frac{HiR^2}{2}\int\limits_0^{2\pi}(1-\cos 2\theta)\,d\theta = \pi R^2Hi.$$

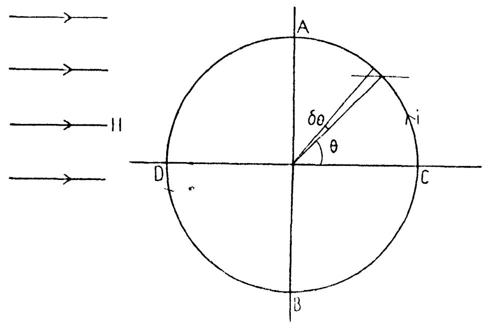


FIG. 33

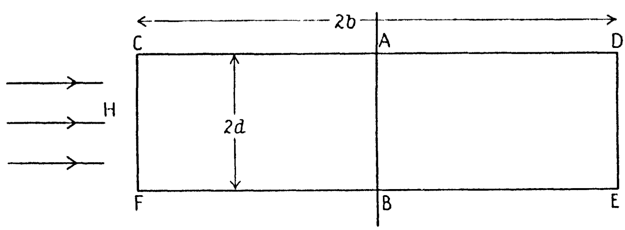


FIG. 34

If the plane of the circle makes an angle ϕ with the direction of the field, the component of force perpendicular to the moment arm is $HiR\cos\phi$, and so

$$T = Hi\pi R^2\cos\phi. \tag{18}$$

(*e*) *Couple on a rectangle in a uniform field.* Let the sides of the rectangle be $2b$ and $2d$, and let it be free to turn on an axis AB as shown in Fig. 34. There is no force on CD or EF because these sides lie in the direction of the field. The force on CF or DE is $2diH$, and so

$$T = iH\times 2d\times 2b$$
$$= iH\times \text{area of rectangle}.$$

So the couple on a circle or a rectangle placed in a uniform field depends on the area of the circuit. This result is true for a coil of any shape and may be proved as follows. Let the circuit be crenelated into elementary steps which are perpendicular to and parallel to the field, then the force is on the perpendicular elements only. If some perpendicular element of length dy is distant x from the axis about which the coil can twist, it will contribute a couple $Hi\,dy\,x$. Whence $T = Hi\int x\,dy = Hi\times$area of the circuit. So we see that the couple on the circle is unaltered if it is pivoted about any chord which is perpendicular to the field.

(*f*) *Couple on a coil placed inside a very long solenoid.* Let a coil of N turns and area A carrying a current I be placed inside a long solenoid which carries a current i and is wound with n turns per unit length. The solenoid produces a uniform field $H = 4\pi in$, and so $T = 4\pi in\times INA$.

If the coil is circular and of radius R and pivoted about a diameter,

$$T = 4\pi^2R^2IinN.$$

(*g*) *Couple between two circular coils with planes perpendicular.* This problem requires intricate mathematics unless one coil is very small compared with the other. If the small coil is sensibly on the axis of the large coil it will be in a substantially uniform field. Let the small coil carry a current i and have n turns each of radius r. Then

$$T = \frac{2\pi IR^2N}{(x^2+R^2)^{\frac{3}{2}}}\times\pi r^2ni. \tag{19}$$

When $x = 0$,

$$T = \frac{2\pi^2Nnr^2Ii}{R}. \tag{20}$$

More exact treatment shows that formula (20) is closely correct even when the coils are comparable in size. It can be shown that the next approximation is

$$T = \frac{2\pi^2Nnr^2Ii}{R}\cos\theta\left\{1-\frac{9}{16}\frac{r^2}{R^2}(1-5\sin^2\theta)\right\},$$

where θ is the angle between the plane of one coil and the normal to the plane of the other; in practice θ is usually zero. This formula shows that the approximation is correct to 3 per cent. even when r/R is as large as $1/4$.

The force between two coils has an important application in alternating current ammeters. A common system is to use two coils each of radius R and placed a distance $2x$ apart, and to suspend a small coil midway between them. Helmholz pointed out that if $2x = R$, then formula (19) will be correct to a very high order of accuracy for finite values of r/R.

12. Force on a circle from a magnetic pole placed on its axis

Let a pole of strength m be placed on the axis of a circle of radius R and at a distance x from its centre; see Fig. 35. Every element

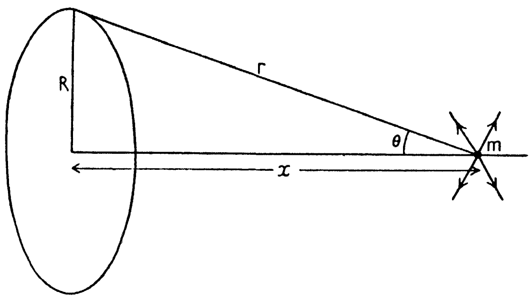


FIG. 35

of circumference lies in a field m/r^2, and this field is in the direction of r. The component of this field in the plane of the circle is $m\sin\theta/r^2$.

$$F = 2\pi Ri\frac{m\sin\theta}{r^2} = \frac{2\pi^2 Rim}{(R^2+x^2)^{\frac{3}{2}}}.$$

This is the force of the pole on the circle, but previous work shows it is also the force of the circle on the pole. So action and reaction are equal and opposite, as they must be.

13. General expression for the force on any coil in any field

The force between two coils depends on the product of the current in one and the field due to the other coil; since the field of a current can be simulated by a suitable magnetic shell, it seems probable that the force between two coils is the same as the force between the two equivalent magnetic shells. But this must not be assumed too readily because the force on a shell would act all over the surface whereas the force on a coil acts only on the wire. Also it is not necessary to assume the equivalence, because we have the experimental law that the force on a wire is perpendicular to the wire and equal to the product of the current and the component of field perpendicular to the wire. But the concept of equivalent shells has led to a particular general expression for the force, which will be found in all mathematical treatises and which the reader should recognize when it is met with. To show how the expression has come about we will first find the force on a magnetic shell which is placed in a given field.

Consider any uniform magnetic shell of strength M; the magnetic potential at a point Q from which the boundary of the shell subtends a solid angle Ω is $M\Omega$. So in bringing a magnetic pole of strength m from infinity to the point Q, work of amount $mM\Omega$ will be done. But $m\Omega$ is that fraction of the total number of lines of force emanating from the pole of strength m which thread through the boundary of the shell when the pole is at Q. If there are other poles at other points, the total work is equal to the sum of the contributions from each pole. So the total work done in bringing up any number of poles against the repulsion of the shell is proportional to the number of lines of force from these poles which would be embraced by the boundary of the shell. Hence the work done in bringing a shell to a given position in a given field is M times the number of lines of force embraced by the boundary of the shell when it is in that position: thus $V = M\Omega$. Let the mechanical force on the shell, attraction or repulsion, be P, and let the shell be moved a small distance dx in the direction of the force; then

$$P\,dx = dV$$

$$P = \frac{dV}{dx} = M\frac{d\phi}{dx}.$$

If this expression holds good for a circuit carrying a current i, we have $M = i$ and so shall expect to find that $P = i\dfrac{d\phi}{dx}$.

But perhaps this is rather an unexpected result, since the force on a coil depends only on the field strength in which each element is situated and does not depend on the body of flux threading through the coil: the expression $\dfrac{d\phi}{dx}$ is apt to suggest that the flux through the coil has an effect.

We will now derive a general expression for the force on a coil, and do so without using the concept of equivalent shell. Consider Fig. 36 in which $ABCD$ represents any circuit and $A'B'C'D'$ represents the same circuit which has been moved bodily and parallel to itself through a small distance dx. At some point on the wire let H

be the component of field strength in a direction perpendicular to the wire and also to OX, the direction of movement. Then the force on an element dl is $Hidl$ and the total force, is $P = \int Hi\,dl = i \int H\,dl$. Now a closed Gauss surface is formed by the two flat disks $ABCD$ and $A'B'C'D'$ together with the parallel-sided ribbon of width dx. Let the flux which enters $ABCD$ be ϕ, and let that which leaves $A'B'C'D'$ be $\phi - \frac{\partial\phi}{\partial x}dx$. Since there are no poles enclosed by the surface, as much flux leaves it as enters it, and hence the flux passing out through the ribbon must be $\frac{\partial\phi}{\partial x}dx$. But if H is the component of field normal to the ribbon at some point on it, the flux leaving the ribbon is $\int H\,dxdl = dx \int H\,dl$,

$$\therefore \quad dx \int H\,dl = \frac{\partial\phi}{\partial x}dx$$

$$\therefore \quad \int H\,dl = \frac{\partial\phi}{\partial x}$$

$$\therefore \quad P = i\frac{\partial\phi}{\partial x}.$$

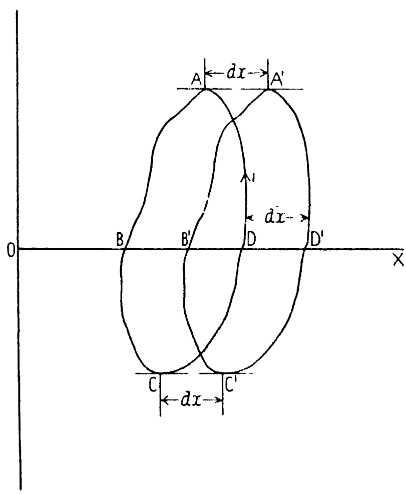


FIG. 36

This is a general expression for the force on a coil situated in a field described by ϕ: we see it is the same force as there would be on the equivalent magnetic shell.

If a coil is in a magnetic field there will be a translational force and also a couple tending to turn the coil about some axis. It may be shown by a similar process of reasoning that the expression for the couple is

$$T = i\frac{d\phi}{d\theta}.$$

We will use this expression to calculate the couple on a circular coil of radius R placed in a uniform magnetic field of strength H. The flux through the coil is $\phi = \pi R^2 H \sin\theta$, where θ is the angle between the field and the plane of the coil. Whence

$$T = \pi R^2 Hi\frac{d}{d\theta}(\sin\theta) = \pi R^2 Hi\cos\theta. \qquad (18)$$

This is the same expression as found previously by integrating the contributions from every element of circuit. The method by which we have proved the general expression shows that the general expression merely conceals the process of integration round the circuit. The sign in the equation for the couple on a coil can be made part of a generalized convention to give the sense of the couple. But the engineer must learn to think of the physical conditions of all problems at every stage of their solution, and he will do better to settle the sense of the couple by thinking out the direction of the force on representative elements of the circuit. It is only in very complex problems involving many coils where it is essential to fall back on a convention of signs and directions

The flux ϕ may be due to permanent magnets or it may be due to another coil carrying a current I. Suppose it is due to another coil and let M_{12} lines of force thread coil 2 when there is unit current in coil 1; then $\phi = M_{12}I$. So the force which coil 1 exerts on coil 2 is $P = Ii\frac{dM_{12}}{dx}$: likewise the force which coil 2 exerts on coil 1 is $P = Ii\frac{dM_{21}}{dx}$. But since action and reaction are equal and opposite, it follows that

$$\frac{dM_{12}}{dx} = \frac{dM_{21}}{dx}.$$

Whence

$$M_{12} = M_{21}.$$

The constant of integration in this equation must be zero, since both M_{12} and M_{21} are necessarily zero when x becomes infinite. M is called the coefficient of mutual induction, or the mutual inductance of the two circuits. The proposition that $M_{12} = M_{21}$ is of great importance and must never be overlooked: it is a proposition which is far from obvious, as may be seen from some simple examples. Thus consider Fig. 37, which shows a circular coil of radius R inside a very long solenoid wound with N turns per unit length. The field strength inside the circular turn is $H = 4\pi N$ per unit current in the solenoid, and hence the flux through the single turn is $\phi = 4\pi N\pi R^2$: if the circular coil has n close turns then $M = 4\pi^2 NnR^2$.

Now approach the problem from the point of view of Fig. 37 (b) in which the current is in the circular coil. The lines of force spread out through the walls of the solenoid and every turn of the solenoid embraces a different amount of flux. A troublesome piece of integration, involving elliptic functions, is required to calculate the flux through any particular turn: having found the flux through a representative turn, an addition must be made of the flux through every turn along the whole length of the solenoid. If this laborious process was performed correctly, the answer would be $M = 4\pi^2 NnR^2$, because of the general property that $M_{12} = M_{21}$. Many other examples may be thought of in which it is difficult to see, without performing the integration, that this property holds good: one such example is

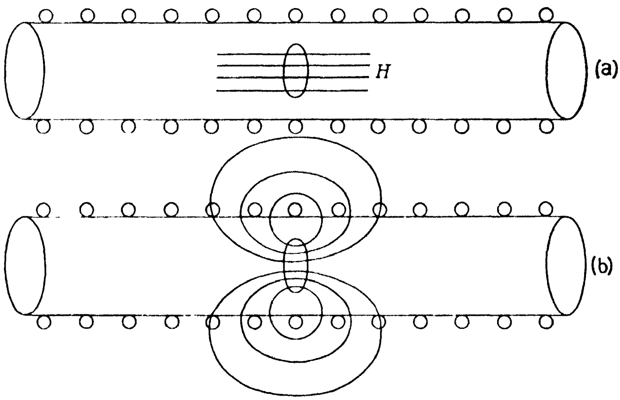


FIG. 37

a circular coil near a pair of long parallel wires. So when the reader has to calculate M for a particular problem, he should remember that if the integration seems intractable for evaluating M_{12}, it may be quite simple for evaluating M_{21}. If a coil B looks unpleasant, in a mathematical sense, when viewed from coil A, then go to coil B and see if coil A presents a more pleasing prospect. The property that $M_{12} = M_{21}$ presumably supplies the answer to some definite integrals which could not otherwise be evaluated.

Tables of M are available for circles and solenoids, and by their help it is possible to calculate exactly some force and couple problems which arise occasionally.

(*a*) *Force between two circular coils in parallel planes.* Consider two parallel circles having radii R and r respectively and placed on a common axis as shown in Fig. 38: r is supposed very much less than R. The field through r will have a vertical component which is small compared with the horizontal component through the coil. The horizontal component will produce a force tending to expand the diameter of the small coil, but the vertical component will produce a force tending to move the two coils along the common axis, either towards or away from one another. Whether the force is attraction or repulsion depends on the relative directions of the two currents: the direction of the force can be settled by drawing the resultant field round a short piece of the small coil. If this is done it will be found that the force is attraction when both the currents are right handed or both left handed.

When both coils are in the same plane H_v is zero, and so the force is zero; when the coils are separated by an infinite distance the force must again be zero. So there is some critical distance at which the force is a maximum. But if we try to calculate H_v we are led at once into intricate mathematics which is beyond the scope of this book, so it might seem that we are unable to calculate the force. But now we find a use for the formula $P = Ii\frac{dM}{dx}$. The magnetic force on the axis of the large coil is

$$H = \frac{2\pi R^2 NI}{(R^2+x^2)^{\frac{3}{2}}}.$$

So if the second coil is very small the flux through it is

$$\phi = \frac{2\pi^2 R^2 r^2 NnI}{(R^2+x^2)^{\frac{3}{2}}}$$

$$P = 2\pi^2 R^2 r^2 NnIi\,\frac{d}{dx}\left\{\frac{1}{(R^2+x^2)^{\frac{3}{2}}}\right\}$$

$$= -2\pi^2 R^2 r^2 NnIi \times \frac{3x}{(R^2+x^2)^{\frac{5}{2}}}. \qquad (21)$$

To find when P is a maximum we must find when $\frac{dP}{dx} = 0$; this occurs when

$$R^2 = 4x^2,$$

that is, when

$$R = \pm 2x.$$

$$\therefore \quad P_{\max} = \frac{96\pi^2 NnIi}{5^{5/2}}\,\frac{r^2}{R^2}$$

$$= 16{\cdot}9NnIi\,\frac{r^2}{R^2}.$$

When $x = R$, $\qquad P = 10{\cdot}4NnIi\,\frac{r^2}{R^2};$

when $x \ll R$, $\qquad P \doteqdot 60NnIi\,\frac{x}{R}\left(1-\frac{5}{2}\,\frac{x^2}{R^2}+\frac{35}{8}\,\frac{x^4}{R^4}\right)\frac{r^2}{R^2}.$

These expressions give a means of calculating H_v at a point near the axis of the coil, for $P = 2\pi rniH_v$, whence

$$H_v = \frac{3\pi R^2 rNIx}{(R^2+x^2)^{5/2}}.$$

This example again shows the importance of realizing that $M_{12} = M_{21}$, for though it is a simple matter to calculate the flux which the large coil sends through the small coil, it is a less simple matter to calculate the flux which the small coil sends through the large coil. But in this problem the calculation can be performed, because the small coil can be replaced by a magnetic particle of moment $M = \pi r^2 i$. The

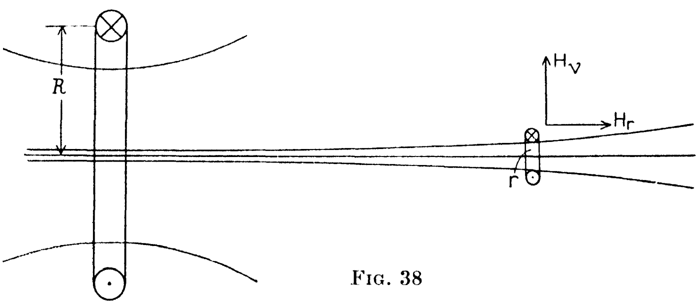


FIG. 38

radial magnetic force at a point (r, θ)* due to a magnetic particle is $H_r = 2M\cos\theta/r^3$. Let the circumference of the large circle be distant a from the centre of the small circle, and let a radius subtend there an angle ϕ. The flux through the large circle is the radial flux passing through the cap of a sphere whose radius is a, the radius of the cap being R.

So $$d\phi = 2\pi y a d\theta \frac{2M\cos\theta}{a^3}$$

$$\therefore \quad \phi = \frac{4\pi M}{a^2}\int_0^\phi y\cos\theta\, d\theta = \frac{4\pi M}{a}\int_0^\phi \sin\theta\cos\theta\, d\theta$$

$$= \frac{2\pi M}{a}\sin^2\phi$$

$$= \frac{2\pi MR^2}{a^3} = \frac{2\pi^2 R^2 r^2 i}{(R^2+x^2)^{3/2}},$$

and so by integration $M_{12} = M_{21}$.

* See formula (4), p. 5.

This example shows how problems may sometimes be solved by viewing them from a different aspect. First the vertical component of the field was obtained by calculating the force between the coils, and then the flux through the big coil was calculated by remembering that the small coil could be replaced by its equivalent magnetic shell, which happened to be a magnetic particle. Either problem would have been very troublesome to solve by any one whose sole weapon was the formula $dH = idl\cos\theta/r^2$.

14. Energy of a magnetic field

Energy is required to produce a magnetic field, but no energy is required to maintain it. The energy of a whole field can be calculated but the process of calculation will not be given here. If the field strength is H in a given volume element $d\tau$ in the field (the field being *in vacuo* or in a material not containing magnetizable material), then it can be shown that

$$E = \iiint \frac{H^2}{8\pi}\,d\tau. \tag{22}$$

The triple sign of integration means that the summation is to be performed throughout all space. This formula is equivalent to saying that the whole energy of the field is the same as if energy was distributed through the field at a rate of $\frac{H^2}{8\pi}$ per unit volume. Of course it is impossible to say that this amount of energy is contained in every unit volume because nothing is known of the mechanism whereby the energy is stored. Magnetic fields are imagined to occur in a hypothetical medium called the aether: this imperceptible medium is sometimes likened to an elastic jelly, and if such a medium really exists the energy is stored as strain energy in this jelly. But even if this is so it is impossible to prove that $\frac{H^2}{8\pi}$ units of energy are stored in any particular unit volume; it is not possible to say more than that the total energy of the whole field is not inconsistent with this hypothesis. It is common to speak of this amount of energy being stored in specific unit volumes, just as it is common to speak of some particular element of current making a contribution to the field of amount $dH = idl\sin\theta/r^2$. Energy of amount $\frac{H^2}{8\pi}$ per unit volume is a useful notation which does not necessarily contain any physical significance.

15. Self-inductance of a circuit

The energy of a magnetic field associated with a current must be proportional to the square of the current, because H at any point is proportional to the current i. The self-inductance of a circuit is defined as twice the energy of the magnetic field associated with a coil per unit of current. If this quantity is denoted by L, then

$$E = \tfrac{1}{2}Li^2.$$

This definition is not easy to visualize, and the engineer is much more familiar with a different one, which defines the inductance of a circuit as the flux turns which link it per unit current in itself.

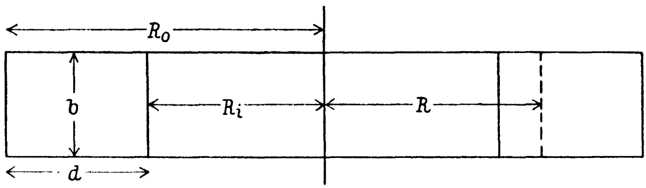


FIG. 39

This definition is very easy to think of: thus the inductance of a circle is the total number of lines of force crossing the plane of the circle and enclosed within the circumference. The second definition includes a certain difficulty because the wires must have a finite cross-section, and so the boundary of the circuit is indefinite. Consider, for example, a single-turn circle; should the flux be reckoned as that enclosed by the inside or by the mean or by the outside periphery? The difficulty of the indefiniteness can be got over by more careful definition of the term circuit,* and then the two definitions are identical. The reason for the identity will become apparent in the next chapter. If the second definition is taken without paying attention to the exact definition of the term circuit, then the inductance calculated will be too small by the amount of the energy contributed by the field inside the wires. In many problems of heavy current engineering, especially when the coils have iron cores, the energy of the field inside the wires is a negligible fraction of the whole and the indefiniteness of the term is insignificant. We will now give some examples of calculating self-inductance.

(*a*) *Inductance of a toroidal coil of rectangular cross-section.* Let the

* See, for example, F. B. Pidduck, *A Treatise on Electricity*, p. 182, or Moullin, *Radio Frequency Measurements*, p. 327.

coil be wound with total turns N and let it have a cross-section as shown in Fig. 39.

$$H = \frac{2N}{R},$$

$$\phi = \int_{R_i}^{R_0} Hb\,dR = 2Nb \int_{R_i}^{R_0} \frac{dR}{R} = 2Nb \log_e \frac{R_0}{R_i}.$$

The same flux threads all the turns, so,

$$\begin{aligned} L &= \phi N \\ &= 2N^2 b \log_e \frac{R_0}{R_i} \\ &= 2N^2 b \log_e \frac{R_i + d}{R_i} \\ &\doteqdot \frac{2N^2 bd}{R_i}, \text{ if } R_i \gg d, \\ &= \frac{4\pi N^2 bd}{2\pi R_i} \\ &= 4\pi T^2 Al, \end{aligned} \tag{23}$$

where T is the turns per unit length, l is the circumference, and A is the area of the rectangular cross-section. The inductance of a very long straight solenoid approaches the value $4\pi T^2 A$ per unit length. The inductance of a single-layer circular solenoid of radius R and length l may be expressed in the form

$$L = 2KR^2T^2l,$$

where K is a parameter depending on the ratio l/R and whose value may be found from Fig. 40. This formula for L ignores the energy of the flux inside the wire.

(*b*) *Long solenoid with many layers of fine wire.* Let the solenoid be wound with T turns per layer per unit length, and let the winding have a radial depth b as shown in Fig. 41: then the total number of wires per unit length of the solenoid is T^2b. There is no field outside the solenoid, and inside the solenoid it is all parallel to the axis. Applying the work law to the rectangle marked with arrows in Fig. 41, we enclose a current T^2xi, and hence the field at a depth x from the outside of the winding is $H = 4\pi T^2 xi$, and the flux through a turn whose radius is $(R_0 - x)$ is

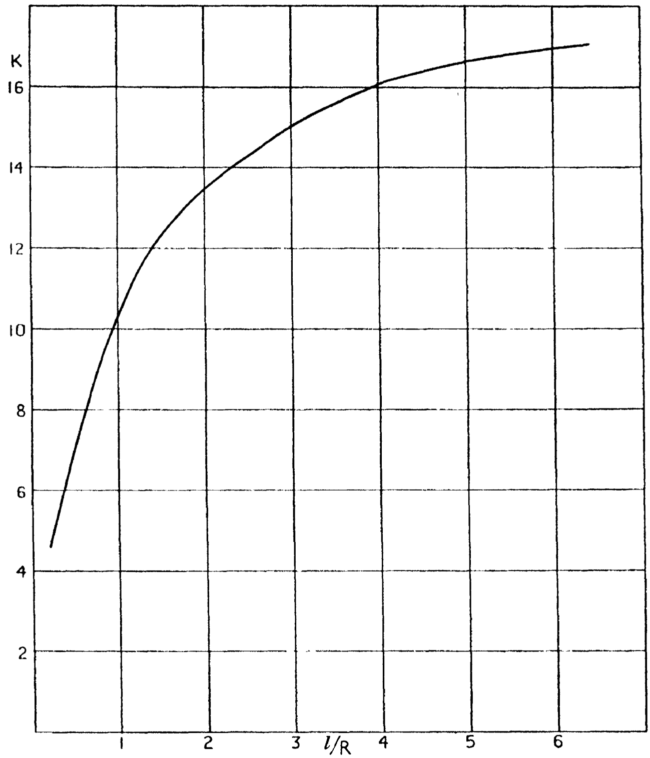


Fig. 40. Coefficient for calculating the inductance of single-layer solenoids

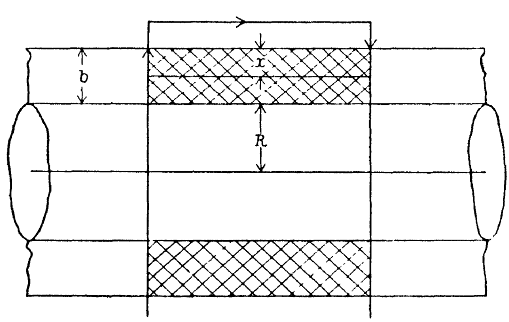


Fig. 41

$$\phi = 4\pi^2T^2bR_i^2+8\pi^2T^2\int\limits_x^b (R_0-x)x\,dx$$

$$= 4\pi^2T^2\left\{bR_i^2+2\left[\frac{R_0x^2}{2}-\frac{x^3}{3}\right]_x^b\right\}$$

$$= 4\pi^2T^2\{b(R_i^2+R_0b)-\tfrac{2}{3}b^3-R_0x^2+\tfrac{2}{3}x^3\}.$$

$$\therefore\quad L = 4\pi^2T^4l\int\limits_0^b \{b(R_i^2+R_0b)-\tfrac{2}{3}b^3-R_0x^2+\tfrac{2}{3}x^3\}\,dx$$

$$= 4\pi^2T^4l\left[\{b(R_i^2+R_0b)-\tfrac{2}{3}b^3\}x-\frac{R_0x^3}{3}+\frac{x^4}{6}\right]_0^b$$

$$= 4\pi^2T^4b^2R_i^2\left(1+\frac{2R_0b}{3R_i^2}-\frac{1}{2}\frac{b^2}{R_i^2}\right)l$$

$$= 4\pi^2N^2R_i^2\left(1+\frac{2}{3}\frac{b}{R_i}+\frac{1}{6}\frac{b^2}{R_i^2}\right)l, \tag{24}$$

where N is the total number of turns per unit length of the solenoid. If the inductance had been approximated to by multiplying $4\pi N^2$ by the mean area, we should have had

$$L = 4\pi^2N^2R_i^2\left(1+\frac{b}{R_i}+\frac{1}{4}\frac{b^2}{R_i^2}\right)l,$$

and this estimate would have exceeded the true value.

(*c*) *Energy of the magnetic field inside a round wire.* The field per unit current inside and at a distance a from the centre of a wire of radius R is $H = 2a/R^2$, hence the energy per unit length is

$$E = \int\limits_0^R \frac{H^2}{8\pi}\times 2\pi a\,da$$

$$= \frac{1}{R^4}\int\limits_0^R a^3\,da$$

$$= \tfrac{1}{4}. \tag{25}$$

It is interesting to notice that the energy per unit length is independent of the radius of the wire.

(*d*) *Self-inductance of a concentric cable.* A cross-section of the cable is shown by Fig. 42. The energy may be considered to consist of three contributions, namely: that inside the central core which has just been shown to be $\tfrac{1}{4}$; the energy contribution from the field in the annular space between the core and the sheath; and the energy due to the field inside the sheath. The field outside the whole cable is zero everywhere. The field in the annular space at a distance r from the centre of the core is $H = 2/r$, hence

$$E_2 = \int_a^b \left(\frac{2}{r}\right)^2 \frac{2\pi r\, dr}{8\pi}$$
$$= \log_e b/a.$$

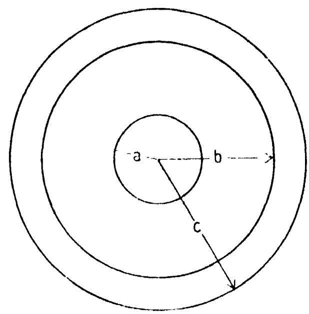


FIG. 42

Per unit current, the current density in the outside sheath is

$$\frac{1}{\pi(c^2-b^2)}.$$

In taking a unit pole round a circle of radius r, inside the outside sheath, the net current curled round is

$$\left(1-\frac{r^2-b^2}{c^2-b^2}\right) = \frac{c^2-r^2}{c^2-b^2}.$$

$$\therefore \quad H = \frac{2}{r}\left(\frac{c^2-r^2}{c^2-b^2}\right)$$

$$\therefore \quad E_3 = \frac{1}{8\pi}\int_b^c \left(\frac{c^2-r^2}{c^2-b^2}\right)^2 \frac{4}{r^2}\, 2\pi r\, dr$$

$$= \frac{1}{(c^2-b^2)^2}\int_b^c \left(\frac{c^4}{r} - 2rc^2 + r^3\right) dr$$

$$= \frac{1}{(c^2-b^2)^2}\left[c^4 \log_e r - r^2c^2 + \frac{r^4}{4}\right]_b^c$$

$$= \frac{1}{(c^2-b^2)}\left(\frac{c^4}{c^2-b^2}\log_e \frac{c}{b} - \frac{3c^2-b^2}{4}\right).$$

$$\therefore \quad E \equiv E_1+E_2+E_3$$

$$= \tfrac{1}{4}+\log_e \frac{b}{a} + \frac{1}{(c^2-b^2)}\left(\frac{c^4}{c^2-b^2}\log_e \frac{c}{b} - \frac{3c^2-b^2}{4}\right).$$

$$\therefore \quad L = 2\log_e \frac{b}{a} + \tfrac{1}{2} + \frac{1}{(c^2-b^2)}\left(\frac{2c^4}{c^2-b^2}\log_e \frac{c}{b} - \frac{3c^2-b^2}{2}\right). \qquad (26)$$

Usually the area of the sheath equals the area of the core, so that $c^2-b^2 = a^2$, and then

$$L = 2\log_e\frac{b}{a}+\tfrac{1}{2}+\frac{1}{a^2}\left\{\frac{2(a^2+b^2)^2}{a^2}\log_e\frac{\sqrt{a^2+b^2}}{b}-\frac{3a^2+2b^2}{2}\right\}$$

$$= 2\log_e\frac{b}{a}+\left(1+\frac{b^2}{a^2}\right)^2\log_e\left(1+\frac{a^2}{b^2}\right)-\frac{b^2}{a^2}-1. \tag{27}$$

For a given voltage between the core and the sheath, it may easily be shown that the maximum electric stress in the insulation is a minimum when $b/a = e$, then it follows that

$$L = (1-e^2)+(1+e^2)^2\log_e\left(1+\frac{1}{e^2}\right)$$

$$\doteq 3+\frac{1}{e^2}$$

$$= 3{\cdot}133.$$

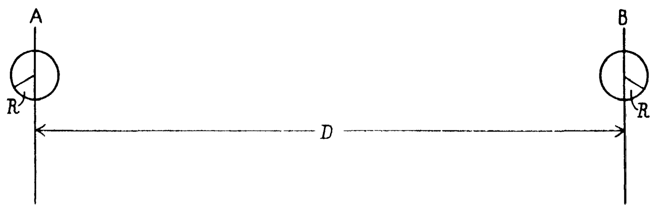


FIG. 43

(*e*) *Self-inductance of a two-core cable.* Let each core have a radius R, and let the central distance between the cores be D, as shown in Fig. 43. The exact solution will not be attempted. It follows that the flux between the cylinders is

$$L = 4\log_e\frac{D-R}{R}.$$

In the absence of A's field, which penetrates B, the energy inside B would be $\frac{1}{4}$, but the penetrating field alters this energy. Ignoring the energy of the penetrating field we should have

$$L = 4\log_e\left(\frac{D-R}{R}+1\right).$$

It can be shown that the exact value is

$$L = 4\left(\log_e\frac{D}{R}+1\right) \tag{28}$$

and one method of proving this is given on p. 56.

(*f*) *Inductance of a single-turn circle of wire.* Let the diameter of the circle be D and of the wire be d. To calculate the inductance we require the use of elliptic integrals and the analysis will not be attempted here.* It can be shown that

$$L = 2\pi D\left[\left(1+\frac{d^2}{8D^2}\right)\log_e\frac{8D}{d}+\frac{d^2}{24D^2}-1{\cdot}75\right]. \tag{29}$$

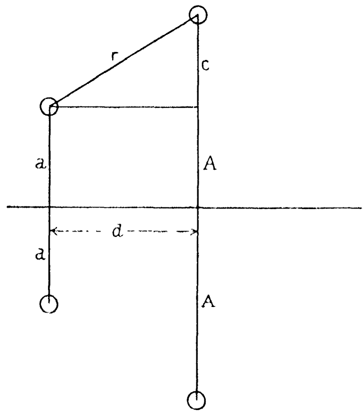


FIG 44

(*g*) *Mutual inductance between circles in parallel planes.* See Fig. 44. It can be shown that

$$M = 4\pi a\left\{\left(1+\frac{c}{2a}+\frac{c^2+3d^2}{16a^2}-\frac{c^3+3cd^2}{32a^3}\right)\log_e\frac{8a}{r}-\right.$$
$$\left.-\left(2+\frac{c}{2a}-\frac{3c^2-d^2}{16a^2}+\frac{c^3-6cd^2}{48a^3}\right)\right\}.$$

If $a = A$,

$$M = 4\pi a\left\{\left(1+\frac{3d^2}{16a^2}\right)\log_e\frac{8a}{d}-\left(2+\frac{d^2}{16a^2}\right)\right\}. \tag{30}$$

This formula is correct to 1 per cent. if $d/a = 1$, and is correct to one part in a million if d/a is less than 0·1.

(*h*) *The force deforming a circuit.* Consider two coils in series with one another, one of which is free to rotate: for example, a variometer tuning inductance. Here the inductance of the whole circuit is a function of the angle between the two coils.

Now $$E = \tfrac{1}{2}Li^2,$$

$$\therefore\quad \frac{dE}{d\theta} = \tfrac{1}{2}i^2\frac{dL}{d\theta}. \qquad \therefore\quad T = \tfrac{1}{2}i^2\frac{dL}{d\theta}.$$

* See Pidduck, *A Treatise on Electricity*, Chap. 8, p. 258.

In the particular example cited, we may write

$$L = L_1 + L_2 + M,$$

and then

$$\frac{dL}{d\theta} = \frac{dM}{d\theta}.$$

Now consider a deformation due to expanding the area, as, for example, increasing the centre distance of a two-core cable. If P is the repulsive force between the wires, then evidently

$$P = \tfrac{1}{2}i^2\frac{dL}{dx}$$

$$= \tfrac{1}{2}i^2\frac{d}{dD}\left\{4\log_e\frac{D}{R}+1\right\}$$

$$= \frac{2i^2}{D}.$$

This is another way of proving that the force depends only on the central distance between the wires and not on their radii.* But we can use this knowledge to prove the formula for the inductance, the analysis for which was too complicated to obtain directly. Thus

$$P = \frac{2i^2}{D} = \tfrac{1}{2}i^2\frac{dL}{dD}.$$

$$\therefore\quad \frac{dL}{dD} = \frac{4}{D}.$$

$$\therefore\quad L = 4(\log_e D + \alpha),$$

where α is a constant which other analysis shows is $1-\log R$. The above process shows definitely that L depends on $\log D$, and it is another example showing how an intricate piece of analysis can sometimes be avoided by approaching a problem in a different manner.

16. Note on units

So far nothing has been said about the units in which current is to be measured. In § 6 we saw that the magnetic field of a small circuit was equivalent to that of a magnetic particle whose moment was jointly proportional to the area of the small circuit and to the current, and we made the statement that $M = Ai$. This statement arbitrarily defined the unit of current. This led to the expression $F = \dfrac{2\pi im}{R}$ for the force on a pole of strength m placed at the centre of a circle

* Compare p. 36.

of radius R: if the pole is of unit strength and the circle of unit radius, then $F = 2\pi i$ dynes. So if the force is 2π dynes, the current is of unit strength. This unit is called the 'electromagnetic absolute unit of current'.

But current is commonly measured in units called Amperes, and the size of the unit has been chosen so that ten amperes are equal to one electromagnetic absolute unit. So when currents are measured in amperes, the symbol i in the foregoing formulae must be replaced by $i/10$. For example, the work law becomes $\int H\,dl = \frac{4\pi i}{10}$, and the force at a distance r from the centre of a long straight wire is $H = \frac{2i}{10r}$ dynes on unit pole.

The formulae for inductance yield an answer in electromagnetic absolute units of inductance often called centimetres: the practical unit of inductance is called the Henry and is chosen so that one henry is equal to 10^9 electromagnetic absolute units. All the foregoing formulae for inductance must be divided by 10^9 to yield an answer in henrys. We may therefore define self-inductance as the flux turns per ampere divided by 10^8. For example, it is required to find the inductance in henrys per kilometre of length of two parallel wires each 1 cm. radius and placed 10 cm. apart centre to centre.

$$L = 4\left(\log_e \frac{D}{R} + 1\right) \text{ absolute units per unit length}$$

$$= \frac{4(\log_e 10 + 1)}{10^9} \text{ H. per cm. of length}$$

$$= \frac{4(\log_e 10 + 1)}{10^9} \times 10^5 \text{ H. per km. of length}$$

$$= \frac{13 \cdot 2}{10^4} \text{ H./km.}$$

$$= 1 \cdot 32 \text{ mH./km.}$$

EXAMPLES TO CHAPTER I

1. Define the terms *pole strength, magnetic force at a point.* Show how to find these by experiment.

Two magnets of pole strength 40 and 25, and 10 cm. long, are arranged in a square $ABCD$, the one being AB with north pole at A, the other CD with north pole at C. Find the magnetic force at the centre of the square and at the mid-point of the side BC. (I.C.E., 1909.)

[The force at the middle of the square is parallel to AB and in the direction AB. It is $F = 2\times\left(\frac{\sqrt{2}}{10}\right)^2\cos 45^\circ\,(40-25) = 0{\cdot}425$ dynes. Force at mid-point of BC. The force due to the poles at B and C are in the same direction and towards B: the resultant is $F_1 = \frac{1}{5^2}(40+25) = 2{\cdot}6$ dynes. The force due to the poles at A and D give a component from B to C which is $F_2 = \frac{1}{125}\times\frac{5}{\sqrt{125}}(40+25)$ $= 0{\cdot}232$ dynes. So net force in direction CB is $2{\cdot}36$ dynes. These poles give a component perpendicular to BC and in the direction AB which is F_3 $= \frac{1}{125}\,\frac{10}{\sqrt{125}}\,(40-25) = 0{\cdot}107$ dynes.]

2. Two thin magnetized rods of uniform cross-section are fixed together so that they cross one another at right angles at their middle points, where they are carried on a vertical pivot at right angles to both magnets. Their lengths are 12 and 8 cm., their masses are 3 and 2 grammes, and their pole strengths are 20 and 10 units respectively. Find the periodic time of a small horizontal oscillation, if $H = 0{\cdot}18$ c.g.s. unit (I.C.E., 1926).

[In the position of rest let the longer magnet make an angle ϕ with the magnetic meridian. Take moments about the pivot, then

$$20H\times 6\sin\phi = 10H\times 4\cos\phi \qquad \therefore\quad \tan\phi = \tfrac{1}{3}.$$

Let ϕ be increased by a small angle θ, then the restoring torque T is

$$\begin{aligned} T &= 240H\sin(\phi+\theta)-80H\cos(\phi+\theta)\\ &\doteqdot 80H(3\sin\phi-\cos\phi+3\theta\cos\phi+\theta\sin\phi)\\ &= 266H\theta\cos\phi,\ \text{since}\ \tan\phi = \tfrac{1}{3},\\ &= 48\theta\cos\phi.\\ I &= \frac{3\times 36}{3}+\frac{2\times 16}{3} = 46{\cdot}6\ \text{gm. cm.}^2\ \text{units.}\\ \therefore\quad 48\theta\cos\phi &= -46{\cdot}6\ddot{\theta}, \end{aligned}$$

whence

$$T = \frac{2\pi}{0{\cdot}99} = 6{\cdot}3\ \text{seconds.}\qquad \text{Ans.}$$

3. A straight conductor of diameter d carries a current i: what is the magnitude and direction of the magnetic force at a point distant r from the axis of the conductor (1) when the point is outside (2) when the point is inside the conductor?

Two parallel straight conductors, each 2 cm. in diameter and 4 metres between centres carry the same current flowing in opposite directions.

Determine the coefficient of self-induction, measured in henrys, for one kilometre of length of the double wire.

[For (1) and (2) see formulae (9) and (10). For inductance see formula (28) and Fig. 43. $L = 4l\log_e\left(\frac{D}{R}+1\right) = \frac{4\times 10^5}{10^9}\log_e 401$ H $= 2\cdot 4$ mH. ANS.]

4. A two-core cable carries a current of 1,000 amps. and the cores are 6 cm. apart, centre to centre. Find the magnetic force at a point mid-way between the two cores. ANS. 133·3 dynes/unit pole.

5. A current of 10,000 amps. is flowing in a long tube whose walls are 1 cm. thick and whose outside diameter is 3 cm. What is the magnetic force inside the tube? ANS. Think carefully and then see p. 20.

6. The mean circumference of a toroid coil is 20 cm. and it is wound with 10 turns per cm. and carries a current of 10 amps. Find the magnetic force at a point on the mean circumference. ANS. See formula (11), 40 dynes/unit pole.

7. How is the *ampere* defined from electromagnetic considerations?

Calculate the force in dynes on a bar magnet of pole strength 15 units and length 10 cm., placed along the axis of a short coil containing 30 turns of diameter 30 cm., and carrying a current of 15 amps., the centre of the magnet being 10 cm. from the plane of the coil. (I.C.E., 1922.)

ANS. 40 dynes. See formula (13).

8. Calculate the magnetic force at the centre of one end of a solenoid 60 cm. long and 15 cm. diam., wound uniformly with 1800 turns of wire carrying a current of 2 amps. ANS. 36·4 dynes/unit pole; see formula (15).

9. A small horizontal magnetic compass needle is free to rotate about a vertical axis. A horizontal bar magnet of effective length 4 in. is set up with its axis collinear with that of the needle, the centres of magnet and needle being 8 in. apart. The frequency of oscillation of the needle after it has been disturbed is noted. The magnet is now replaced by a circular coil of 80 turns 1 ft. in diameter, the axis and centre of the coil coinciding with the former axis and centre of the magnet. A current flows in the coil in the sense to produce a field in the same sense as that due to the magnet, and when it is adjusted to 0·495 amps. the frequency of the needle is the same as before. Calculate the pole strength of the magnet.

ANS. 43 units. The coil and the magnet must produce the same field strength at the small needle: use formula (13) and inverse square law.

10. Two circular coils in series, each containing 10 turns 15 cm. in diameter, are placed with their axes coincident and horizontal, their planes being 15 cm. apart. Calculate the magnetic force at a point on the axis midway between them when the coils carry a current of I amps. A small coil of area 5 sq. cm. and containing 100 turns is suspended so that its centre is at the above-mentioned point and its plane is perpendicular to the planes of the coils. The small coil carries a current i. Calculate the couple on the coil in dyne cm. units. (I.C.E., 1927.) ANS. $H = 0\cdot 6I$ and $T = 30Ii$.

11. A wattmeter consists of a fixed series coil containing 40 turns of radius 8 cm. at the centre of which a volt coil containing 160 turns of radius 1 cm. is hung by a bifilar suspension which gives a restoring couple of $5\cdot 5\times 10^{-3}$ dyne cm. per radian of deflexion. Assuming that the field strength throughout the region occupied by the volt coil is the same as that at the centre of the

series coil, find the constant in watts per radian when the total resistance of the volt circuit is R ohms. (M.S. Tripos, 1921.)

ANS. $T = 160Ii$. $3{\cdot}48 \times 10^{-5}R$.

12. A vertical wire is carrying 100 amps. A square coil of 200 turns and 40 cm. side is pivoted about a vertical axis in the plane of the coil, two sides of the square being parallel to and equidistant from the axis, which is 15 cm. from the wire. Find the maximum possible couple on the coil when it is carrying 2 amps. (I.C.E., 1921.) ANS. $3{\cdot}08 \times 10^4$ dyne cm.

13. A circular coil has 4 close turns each of radius 10 cm., and carries a current of 5 amps.: find the force in dynes on a unit pole placed on the axis of the coil and distant 10 cm. from its centre. This coil is placed in a uniform magnetic field with its axis perpendicular to the direction of the field. Find the torque in dyne cm. on the coil if the strength of the field is 10 c.g.s. units. (I.C.E., 1st year, 1926.) ANS. 0·446, $6{\cdot}3 \times 10^3$ dyne cm.

14. The armature of a two-pole electric motor is 1 ft. in diameter and 6 in. long: it has 100 conductors each of which carries a current of 50 amps. The pole faces cover two-thirds of the armature circumference: the flux density under the poles is uniform and equal to 5,000 lines per sq. cm., and the flux density not under the poles is negligible. Find the greatest possible torque on the armature. (I.C.E., 1st year, 1926.) Use formula (16) and add the torques contributed by the separate wires. ANS. 5·85 lb. ft.

15. The go and return wires of a 440 V.D.C. power line are supported parallel to each other and 10 in. apart, on poles. The power is 30 kW. Find the force in poundals per foot run exerted on each wire by the other. (I.C.E., 1st year, 1927.) See formula (17). ANS. $7{\cdot}85 \times 10^{-3}$.

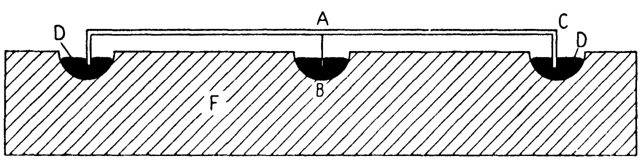


FIG. 45

16. A metal disk A (see figure) is provided with an axial vertical pivot resting in a mercury cup B, and a cylindrical edge C dipping into mercury D in an insulated container F. A p.d. E is maintained by a battery across B and D, between which points the resistance is R. There is a uniform vertical magnetic field of strength H: the disk is of radius a; and the mechanical torque resisting motion is $k\omega$, where k is a constant, and ω is the angular velocity of the disk, show that $\omega = \dfrac{2Ha^2E}{H^2a^4+4kR}$. (I.C.E., 1st year, 1927.)

[Use formula (16) for torque and formula (4) of Chap. II to calculate the e.m.f. due to rotation. Note this example illustrates the principle of the most common form of electric supply meter, first developed by Chamberlain and Hookham. It is interesting to consider the reaction of the disk on the magnetic field, which may be supposed to be produced by placing the motor at the middle of a long solenoid with a vertical axis. The reaction on the field of the solenoid cannot produce any couple tending to rotate the solenoid about a vertical axis: the solenoid could be free to rotate and would not require to be held when the motor was transmitting a torque through its shaft. Where then is the equal and opposite torque reaction? The opposite torque is on the container F which must be held from rotating and it is produced by the force on the wire which necessarily joins B and D, through a battery. The magnetic field is essential to the working of the apparatus and yet there are no torque reactions on the magnet producing the field: it is perhaps reminiscent of a catalyser in certain chemical reactions.]

17. A circular coil of radius r and having n close turns is pivoted on an axle through its centre, but the plane of the circle is set at 45 degrees to the axle. The coil is placed inside a long solenoid, having N turns per unit length and the axle about which it turns is inclined at 45 degrees to the axis of the solenoid. The solenoid and the coil are connected in series and carry a current i, find the maximum couple tending to rotate the axle.

II

SECOND LAW OF ELECTRODYNAMICS

1. Induced currents and Faraday's law of induced e.m.f.

Consider two neighbouring coils A and B (see Fig. 46). The circuit of B is completed by a galvanometer G while the circuit of A is completed by a battery C, which produces a steady current in A of value I. Circuits A and B are insulated from one another, and so the battery C cannot produce a current in coil B. But it is found that when I is altered, a current is indicated by G, and this current persists only during such time as the current I is changing in A. The transitory current indicated by G is called an induced current, and the effect was discovered by Faraday in 1831. The same effect is produced if circuit B is moved relatively to circuit A, even though the current I is unaltered; in this experiment current is indicated by G only while the relative motion is in progress. These experiments suggest that current is induced in B by the change of magnetic field penetrating through the circuit. The same effect is produced if the circuit A is replaced by a permanent magnet; current is induced in B during any relative movement of B and the permanent magnet. This shows conclusively that current is induced by change of magnetic field through a circuit and it is immaterial by what means the magnetic field is produced or how it is changed. The current induced by a given rate of change of flux through a given circuit is found to vary inversely as the resistance of the circuit. This shows that the induced voltage, which causes the flow of current, is independent of the material of the circuit, and therefore a given rate of change of flux creates a voltage round the circuit which depends only on the form of the circuit.

The phenomenon of induced voltage was discovered by Faraday, but the laws relating its magnitude to the rate of change of flux were enunciated by Neumann about 1845.

If the magnetic flux through a circuit is ϕ then E, the e.m.f. round the circuit, is

$$E = -\frac{d\phi}{dt}. \tag{1}$$

The negative sign is to relate the direction of the e.m.f. with the direction of the flux and its rate of change, and will be discussed later.

The existence of the induced e.m.f. can be inferred from the first law of electromagnetism, though it requires separate experimental proof. We have seen that $4\pi i$ units of work are required to take a unit pole round a circuit which threads a current. This work is not stored in the circuit and is not recoverable: experiment shows the sink of work is in the circuit itself and discloses itself by raising the temperature of the wire. Electric current can do work only against an electric force, and hence the passage of a pole through a circuit must create an electromotive force; separate experiment shows this does occur. An electric motor is a familiar example of the

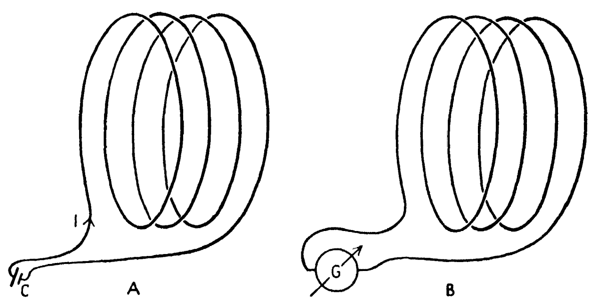


FIG. 46

induced e.m.f. If the motor is rotating and giving an output of work, the electric current must be working against some electric force which is a function of the rotation. It is working against the 'back' e.m.f. induced by the rate of change of the magnetic flux through the coils of the armature.

The numerical relation between the e.m.f. and the rate of change of flux can be calculated by the help of the following hypothetical experiment. Suppose there is a uniform magnetic field H perpendicular to the plane of the paper and let there be two metal rails AB and CD, as in Fig. 47, joined at one end by a battery and at the other end by a cross-piece EF free to roll on the rails; then current will flow round the circuit $AEFD$. All the conductors are in the superposed uniform magnetic field and there is a force on all of them due to this field; the force on EF will cause it to roll along the rails. Let the rolling be resisted by a constant force P, so that EF moves with constant speed v.

Now $$P = Hil,$$

and the rate of working is $$W = Pv = Hilv.$$

But the induced e.m.f. is equal to the rate of change of flux through the circuit.

So $$E = H \times \frac{d}{dt}\,(\text{area } AEFD) = Hlv.$$

To drive the current against this e.m.f. the battery must work at the rate $Ei = Hlvi$. But from the above $P = Hil$, and so $Ei = Pv$. Thus all the work done against the external force P is derived from the battery. If E and i are expressed respectively in volts and amperes, then

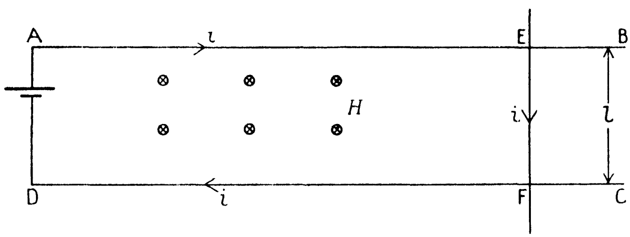


FIG. 47

$$W = Ei \text{ watts}$$
$$= 10^7 Ei \text{ ergs/sec.}$$

and $$P = \frac{Hil}{10} \text{ dynes,}$$

$$\therefore\quad 10^7 E = \frac{Hlv}{10},$$

$$\therefore\quad E = \frac{Hlv}{10^8}. \tag{2}$$

Hence the e.m.f. expressed in volts equals 10^{-8} times the rate of change of flux through the circuit.*

Though we know the total e.m.f. round the circuit, we do not know how much electric force is generated at any point of the circuit. If the electric force at some point is F, then the Faraday law of induced e.m.f. may be written

$$\int F\,dl = -\frac{d\phi}{dt}, \tag{3}$$

* A very similar experiment is described in § 8 of this chapter, and reference to this will show that the movement of EF causes an additional e.m.f. to be generated round the circuit. The additional e.m.f. is a function of the current in the circuit only and is independent of the superposed field H: likewise the force P on the conductor is the force which results from the superposed field and is less than the total force which is exerted on EF.

a form of expression resembling the work law

$$\int H\,dl = 4\pi i.$$

Both the laws of electromagnetism are essentially circuital. But whereas H can be calculated at any point by means of Ampère's device of the shell, no simple device has yet been found for calculating F. This still remains outside human knowledge, though we shall make a close approach to it in Chapter V.

The electric force existing in any specified element of circuit cannot be calculated any more than the magnetic force contributed by an element of circuit can be calculated; in each case it is the total effect which is known.

If the magnetic field in any given region is changing, there will be an e.m.f. round any circuit threaded by the flux. It seems unlikely this e.m.f. is brought into existence by the metal circuit and therefore presumably there is an e.m.f. round any closed line in space which is threaded by changing flux, whether or not a metal wire coincides with the closed line in space. Hence any space filled with changing magnetic force is also filled with changing electric force; the electric force disappears as soon as the magnetic field ceases to vary. Not only is a current induced in any closed conducting circuit through which a flux is changing, but any isolated electric charge in the varying magnetic field is acted on by an electric force.

If the reader becomes familiar with Maxwell's hypothesis of displacement current he will have a broader view of the problem, but for the present it is sufficient to remember that every changing magnetic field is, *ipso facto*, accompanied by an electric field.

2. Cutting rule for induced e.m.f.

It is very common to describe the Faraday law of induced e.m.f. in terms of the rate at which the boundary of the circuit cuts across the lines of force.

Thus suppose the rectangular circuit $ABCD$ in Fig. 48, p. 67 is moving in the direction of the arrow in a steady magnetic field which is perpendicular to the plane of the circuit: the field strength changes along OX but not along OZ. Let the strength of the magnetic field, along the axis OX be represented by the curve in Fig. 49, p. 67; at the moment shown the wire AB is situated in a field of strength h_1 and the wire CD in a field of strength h_2, and the flux through the circuit is proportional to the shaded area. Let the circuit move an infinitesimal distance dx in a time dt, so as to bring it to the position $A'D'$. Then flux of amount $h_1 l dx$ has passed out of the circuit and flux of amount $h_2 l dx$ has come into the circuit. The net change of flux through the circuit is $(h_1-h_2)l dx$. This change occurs in time dt, and so

$$E = (h_2-h_1)l\frac{dx}{dt}$$

$$= (h_2-h_1)lv,$$

where v is the speed with which the circuit cuts through the field. So there would be the same e.m.f. round the circuit if the action of cutting a conductor of length l, with speed v, across a field h, produced in that conductor a p.d. equal to hlv.

This cannot be proved experimentally, but it is usual to suppose it is true, and so we obtain the important equation

$$E = Hlv. \tag{4}$$

This is very similar in form to the formula for the mechanical force on a conductor, which is

$$F = Hli.$$

3. Examples of induced e.m.f.

(*a*) *Rectangular coil revolving in a uniform field.* Let the coil have a breadth b and a length l (see Fig. 50) and revolve with uniform angular velocity ω, in a uniform field of strength H. The flux through the coil in Fig. 50*b* is

$$\phi = Hlb\cos\theta.$$

$$\therefore \quad E = \frac{d\phi}{dt}$$

$$= -Hlb\sin\theta\frac{d\theta}{dt}$$

$$= -Hlb\omega\sin\omega t. \tag{5}$$

So the induced e.m.f. is alternating and simple harmonic and has the same frequency as the rotation of the coil.

We will now use the cutting rule to calculate the e.m.f. The lines of force are *cut* by conductors AB and CD only; conductors BC and AD do not cut across the lines. The conductors AB and CD move perpendicular to themselves with a uniform velocity $\frac{1}{2}b\omega$, but their component velocity perpendicular to the field is $\frac{1}{2}b\omega\sin\theta$. According to the cutting rule the conductor AB has generated in it a voltage

$\frac{Hbl}{2}\omega\sin\theta$ and an equal voltage is generated in CD; hence

$$E = 2\times\frac{Hbl}{2}\omega\sin\theta$$

$$= Hbl\omega\sin\omega t. \qquad (5)$$

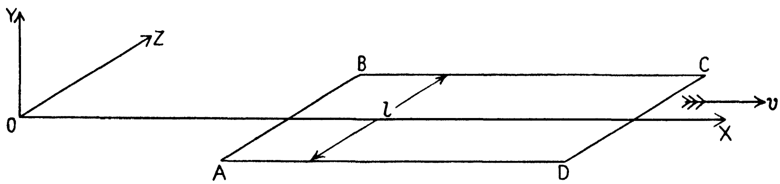


FIG. 48

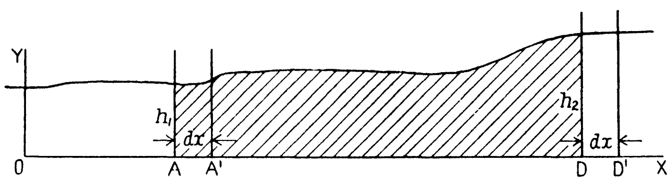


FIG. 49

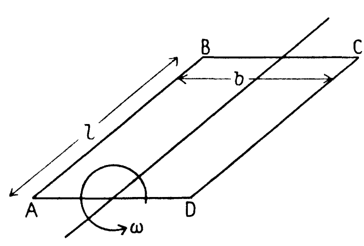


FIG. 50 *a*

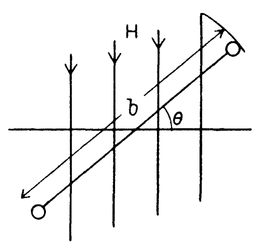


FIG. 50 *b*

So the two methods of approaching the problem are consistent and give the same result.*

* No doubt it will be observed that the two derivations of (5) differ in sign. The sign in such an equation relates to the direction of the e.m.f., whereas here we are calculating only its magnitude. The reader is expected to determine the direction by applying a cutting rule, equivalent to the rule given on p. 36 of Chap. I. If the threading rule is used to give both magnitude and direction, then we must write $E = -\frac{d\phi}{dt}$. In general this negative sign is omitted, because the writer prefers to separate the processes of calculating the magnitude of the e.m.f. and deciding its direction.

(*b*) *Rectangle revolving near a long straight current.* The rectangle and long current are shown in Figs. 51 *a* and 51 *b*. The flux threading the coil in Fig. 51 *b* is

$$\frac{\phi}{i} = l\int_{r_1}^{r_2} H\,dr$$

$$= 2l\log_e\frac{r_2}{r_1}$$

$$= 2l\log_e\frac{\sqrt{c^2+b^2+2bc\cos\theta}}{\sqrt{c^2+b^2-2bc\cos\theta}}$$

$$= l\log_e(c^2+b^2+2bc\cos\theta)-l\log_e(c^2+b^2-2bc\cos\theta).$$

$$\therefore\quad \frac{d\phi}{dt} = \frac{4bcl\omega(b^2+c^2)\sin\theta}{(b^2+c^2)^2-4b^2c^2\cos^2\theta}\dot{i}$$

$$= \frac{2\dfrac{b^2+c^2}{2bc}l\omega\sin\theta}{\left(\dfrac{b^2+c^2}{2bc}\right)^2-\cos^2\theta}\dot{i}.$$

$$\therefore\quad E \equiv \frac{2al\omega\sin\theta}{a^2-\cos^2\theta}\dot{i}. \tag{6}$$

Since θ is a function of time, the induced voltage is not simple harmonic. The expression in formula (6) can be expanded in a Fourier series, and when this is done we have

$$E = \frac{4ali\omega}{\sqrt{k^2-1}}\left(1+\sum_1^\infty \frac{2}{\alpha^n}\cos 2n\theta\right)\sin\theta,$$

where $$k = 2a^2-1$$

and $$\alpha = k+\sqrt{k^2-1}$$

$$= 2a^2-1+2a\sqrt{a^2-1}.$$

$$\therefore\quad \frac{E}{i} = \frac{2\omega l}{\sqrt{a^2-1}}\left[\sin\theta+\sum_1^\infty\frac{1}{\alpha^n}\{\sin(2n+1)\theta-\sin(2n-1)\theta\}\right]$$

$$= \frac{2}{\sqrt{a^2-1}}\,\frac{\alpha-1}{\alpha}\,\omega l\left[\sin\omega t+\frac{1}{\alpha}\sin 3\omega t+\frac{1}{\alpha^2}\sin 5\omega t+\ldots+\right.$$

$$\left.+\frac{1}{\alpha^{n-1}}\sin(2n-1)\omega t\right].$$

This shows that the e.m.f. is compounded of a simple harmonic term whose frequency is that of the rotation of the coil together with an infinite series of simple harmonic terms whose frequencies are three, five, seven, etc., times the frequency of the fundamental component. This system of breaking up the voltage into a fundamental component, together with terms whose frequencies are integral multiples of the fundamental, is used a great deal in the analysis of

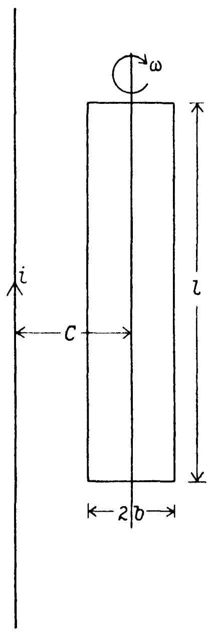


FIG. 51 *a*

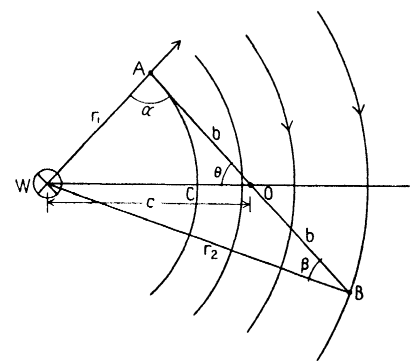


FIG. 51 *b*

alternating currents. The reader should notice that the additional frequencies are odd multiples of the fundamental: this is a property of all alternating current dynamos in which the plane of the coil contains the axis of rotation. The dynamo engineer would describe such a coil as a full-pitch winding: if the axis of rotation was outside the plane of the coil, the induced voltage would contain harmonics of even order, but nevertheless the even terms would have a much smaller amplitude than the adjacent odd terms.

It is instructive to form some idea of the relative magnitudes of the harmonic terms, or, in other words, to see how α depends on the ratio c/b.

Now $\alpha = 2a^2-1+2a\sqrt{a^2-1}$

$$= \frac{b^4+c^4+2b^2c^2}{2b^2c^2}-1+\frac{b^2+c^2}{bc}\sqrt{\frac{(b^2+c^2)^2}{4b^2c^2}-1}$$

$$= \frac{b^4+c^4}{2b^2c^2}+\frac{c^4-b^4}{2b^2c^2}$$

$$= \frac{c^2}{b^2},$$

whence it follows that

$$\frac{E}{i} = \frac{4bl\omega}{c}\left[\sin\omega t+\frac{b^2}{c^2}\sin 3\omega t+\frac{b^4}{c^4}\sin 5\omega t+\ldots\right]. \tag{7}$$

It should be noticed that the fundamental component is the voltage which would be generated by a coil of area $A = 2bl$, revolving in a uniform magnetic field of strength $H = 2i/c$: if c/b is greater than ten, then the third harmonic component is less than 1 per cent. of the fundamental.

The voltage may also be calculated by the cutting rule as follows: Wire A cuts across a field $2i/r_1$ with velocity $b\omega\sin\alpha$, and wire B cuts across a field $2i/r_2$ with velocity $b\omega\sin\beta$ (see Fig. 51 *b*),

$$\therefore\ \frac{E}{i} = 2b\omega l\left(\frac{\sin\alpha}{r_1}+\frac{\sin\beta}{r_2}\right)$$

$$= 2b\omega l\sin\theta\left(\frac{c}{r_1^2}+\frac{c}{r_2^2}\right)$$

$$= \frac{4bcl\omega(b^2+c^2)\sin\theta}{(b^2+c^2)-4b^2c^2\cos^2\theta}$$

$$= \frac{2al\omega\sin\theta}{a^2-\cos^2\theta}. \tag{6}$$

So E may be calculated either by the threading rule or by the cutting rule.

(*c*) *Rectangle moving with constant speed v away from a straight current.* With the same notation as in Fig. 51, it follows that when the wire and the coil are in one plane

$$E = 2il\left(\frac{1}{c-b}-\frac{1}{c+b}\right)v = \frac{4blvi}{c^2-b^2}. \tag{8}$$

Or by the threading rule

$$\phi = 2li\int\limits_{c-b}^{c+b}\frac{1}{r}\,dr = 2li\log_e\frac{c+b}{c-b}.$$

$$\therefore\quad \frac{d\phi}{dt} = 2li\left(\frac{1}{c+b}-\frac{1}{c-b}\right)\frac{dc}{dt}.$$

$$\therefore\quad E = \frac{4blvi}{c^2-b^2}. \tag{8}$$

If the coil is rotating at the same time that it is moving as a whole, we may find E by differentiating the expression for ϕ in (b) above: if this is done, we find

$$\frac{E}{i} = \frac{2al\omega\sin\theta}{a^2-\cos^2\theta}+\frac{4b(b^2-c^2)lv\cos\theta}{(b^2+c^2)^2-4b^2c^2\cos^2\theta}$$

$$= \frac{\left(2a\omega\sin\theta+\dfrac{b^2-c^2}{c^2}v\cos\theta\right)l}{a^2-\cos^2\theta}. \tag{9}$$

(d) *Swinging of a galvanometer coil.* If a galvanometer coil is swinging in its magnetic field, an e.m.f. is induced, and this causes a current to flow if the coil circuit is closed. The current produces a retarding torque which damps the motion, and the kinetic energy is transformed, by means of the current, to heat energy in the resistance. Let the uniform and radial magnetic field have strength H, and let the rectangular coil have breadth b, length l, moment of inertia I, and n turns; let the resistance of the coil and external circuit be R. If at time t the angular velocity of the coil is ω, then by the cutting rule we have

$$E = 2Hl\frac{b}{2}\omega = Hlb\omega.$$

This e.m.f. will produce a current i, such that

$$i = E/R = \frac{Hlb\omega}{R}.$$

This current produces a retarding torque T, such that

$$\begin{aligned} T &= 2Hil\frac{b}{2} \\ &= Hilb \\ &= \frac{(Hlb)^2\omega}{R}. \end{aligned}$$

If the strength of the suspension is λ, the equation of motion is

$$I\ddot{\theta}+\frac{(Hlb)^2}{R}\dot{\theta}+\lambda\theta = 0$$

or

$$\ddot{\theta}+A\dot{\theta}+\frac{\lambda}{I}\theta = 0,$$

where

$$A \equiv \frac{(Hlb)^2}{IR}.$$

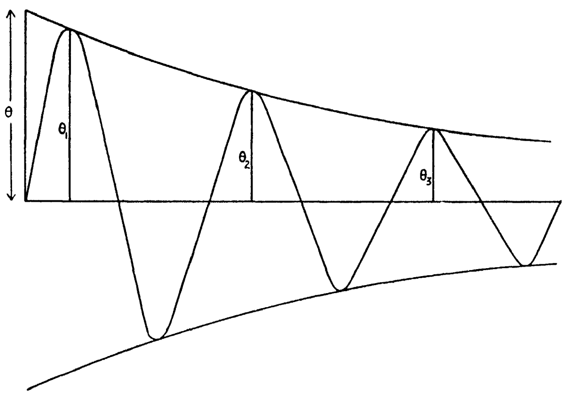


Fig. 52

The solution of this is

$$\theta = \alpha e^{-\frac{At}{2}} \sin\left(\sqrt{\frac{\lambda}{I}-\frac{A^2}{4}}t+\beta\right),$$

where α and β are constants to be determined from the initial conditions. Usually the damping is small, and then we can write

$$\theta \doteqdot \alpha e^{-\frac{A}{2}t} \sin\left(\sqrt{\frac{\lambda}{I}}t+\beta\right). \tag{10}$$

When the galvanometer is used ballistically, a quantity of electricity is discharged through it, and this gives the coil an impulse. This impulse causes the coil to swing with damped harmonic motion and the initial swing is less than it would have been if the damping had been zero. Thus suppose the motion is

$$\theta = \alpha e^{-\frac{A}{2}t} \sin\sqrt{\frac{\lambda}{I}}t,$$

which is represented by Fig. 52. The first maximum occurs at time $t = \frac{1}{4}T$, where the time period is $T = 2\pi\sqrt{\frac{I}{\lambda}}$.

So
$$\theta_1 = \alpha e^{-\frac{A}{2}\frac{T}{4}}$$
$$\theta_2 = \alpha e^{-\frac{A}{2}\frac{5T}{4}}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\theta_n = \alpha e^{-\frac{A}{2}\frac{(4n-3)}{4}T}.$$

The quantity $e^{\frac{A}{2}\frac{T}{4}}$ is defined as the damping factor and is represented by the symbol Δ. It is that quantity by which the first swing must be multiplied in order to obtain the swing which would occur with zero damping; thus $\alpha = \Delta\theta_1$.

Now
$$\frac{\theta_1}{\theta_n} = e^{(4n-4)\frac{A}{2}\frac{T}{4}}$$
$$= \Delta^{4n-4}.$$
$$\therefore \quad \Delta = \left(\frac{\theta_1}{\theta_n}\right)^{\frac{1}{4(n-1)}}. \tag{11}$$

But
$$\Delta = e^{\frac{A}{2}\frac{T}{4}}$$
$$= e^{\frac{(Hlb)^2}{2IR}\frac{\pi}{2}\sqrt{\frac{I}{\lambda}}}.$$
$$\therefore \quad \log\Delta = \frac{\pi(Hlb)^2}{4R\sqrt{\lambda I}}. \tag{12}$$

So $\log\Delta$ varies inversely as the circuit resistance and directly as the square of the field strength and the area of the coil.

The effect of the e.m.f. induced by the movement of the coil is noticeable in any heavily shunted galvanometer: the growth of the current in the coil is impeded by this e.m.f. and the needle takes an appreciable time to crawl up to its final deflexion. The effect may best be illustrated by a numerical example, as follows: A moving-coil galvanometer gives a full-scale deflexion of two radians for a current of $100\,\mu$A and has a resistance of $60\,\Omega$. The control spring has a strength of 3 dyne cm. per radian, and the movement has a periodic time of 6·28 seconds when undamped. It is shunted so as to make the full-scale deflexion correspond to a current of one ampere; find the deflexion three seconds after starting a current of one ampere through the shunted instrument.

A diagram of connexions is shown in Fig. 53: the current i through the galvanometer is impeded by the e.m.f. generated by the moving coil. The value of this e.m.f. is $E = HA\dot{\theta}\,10^{-8}$V., and the torque moving the coil is $T = HAi/10$ dyne cm., where H is the field strength in which the coil of area A is moving.

Finally $\theta = 2$ and $i = 100\,\mu$A,

$$\therefore \quad 6 = \frac{HA}{10^5}.$$

$$T = 2\pi\sqrt{\frac{I}{\lambda}} \quad \text{and} \quad T = 6{\cdot}28,$$

$$\therefore \quad I = \lambda = 3.$$

$$RJ = ri + \frac{HA}{10^8}\dot{\theta} \quad \text{and} \quad i+J = x,$$

$$\therefore \quad Rx = (R+r)i + \frac{HA}{10^8}\dot{\theta}.$$

$$\frac{HAi}{10} = I\ddot{\theta} + \lambda\theta,$$

$$\therefore \quad I\ddot{\theta} + \frac{H^2A^2}{10^9(R+r)}\dot{\theta} + \lambda\theta = \frac{HAx}{10}\,\frac{R}{R+r}.$$

$$\therefore \quad 3\ddot{\theta} + \frac{36\times 10^{10}}{6\times 10^{10}}\dot{\theta} + 3\theta = \frac{6\times 10^5\times 1}{10}\times\frac{1}{10^4}.$$

$$\therefore \quad \ddot{\theta} + 2\dot{\theta} + \theta = 2.$$

$$\therefore \quad \theta = 2 + \alpha t e^{-t} + \beta e^{-t}.$$

Since $\theta = 0$ and $\dot{\theta} = 0$ when $t = 0$,

$$\therefore \quad \alpha = \beta = -2.$$

$$\therefore \quad \theta = 2(1 - te^{-t} + e^{-t}).$$

$$\therefore \quad \text{when } t = 3, \qquad \theta = 2\times 0{\cdot}8.$$

So in three seconds this galvanometer has attained only 80 per cent. of its full deflexion and creep would be detectable for several minutes. This often presents a serious difficulty in shunting a very sensitive galvanometer, and it is often essential to increase enormously the resistance of the galvanometer so that it shall reach its final deflexion in a reasonable time. Another example of the same principle is the habit of short-circuiting an unclamped galvanometer before packing it for transit.

(*e*) *E.m.f. induced by an alternating field.* Suppose that in Fig. 51 the rectangle is at rest and that the current in the long wire is not a steady current but alternates simple harmonically so that $i = I\sin pt$.

Then, as before,

$$\phi = 2il \log_e \frac{r_2}{r_1}$$

$$= 2Il \log_e \frac{r_2}{r_1} \sin pt.$$

$$\therefore \quad E = -\frac{d\phi}{dt} = -2pIl \log_e \frac{r_2}{r_1} \cos pt. \tag{13}$$

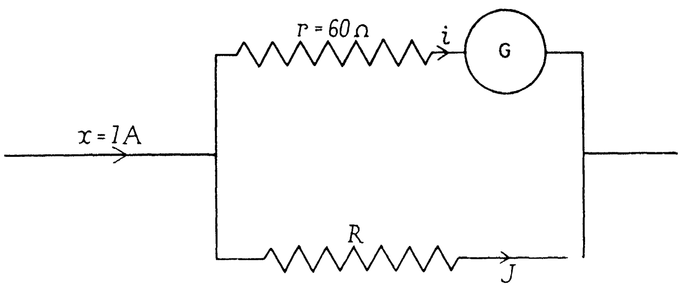


FIG. 53

In attempting to solve this problem by the cutting rule we are confronted by an impasse because we do not know the speed at which the lines of force are moving, or indeed if they are moving at all. The picture of the lines of force expanding and contracting round the wire readily suggests itself and this gives the idea of a velocity; but what is the velocity ?

The difficulty is fundamental and arises because we have not considered in this book all the properties of the electromagnetic field: the Maxwell hypothesis of displacement currents has so far been ignored (see Chap. V). If the current is steady, the magnetic field at a distance r from a straight wire is $H = 2i/r$, but if the current is alternating, the field is not expressed exactly by $H = \frac{2I}{r} \sin pt$. It can be shown that the e.m.f. calculated as above is indefinitely nearly correct for frequencies less than a few thousand cycles per second, but the e.m.f. cannot be calculated by the cutting rule. The cutting rule is applicable only to the movement of a circuit across an unvarying field of force. The cutting rule is of the utmost importance to the engineer, and in some dynamo problems it is necessary to invent intricate analysis to avoid the use of the threading rule.

(*f*) *Coil rotating in a uniform field which is pulsating harmonically.* Suppose the field in Fig. 50 is not steady but pulsates harmonically, so that $h = H \sin pt$.

Then
$$\phi = Hlb \sin pt \cos \theta$$

$$\therefore \quad E = -\frac{d\phi}{dt} = Hlb\{-p \cos pt \cos \omega t + \omega \sin \omega t \sin pt\}. \tag{14}$$

If it happens that $\omega = p$, then

$$E = -\omega Hlb(\cos^2\omega t - \sin^2\omega t)$$
$$= -\omega Hlb \cos 2\omega t, \tag{15}$$

and then the e.m.f. has twice the frequency of the rotation. The e.m.f. consists of two distinct components, one due to the rotation which *can* be calculated by the cutting rule, and one due to pulsation which *must* be calculated by the threading rule. The two components are often described as the dynamo e.m.f. and the transformer e.m.f. respectively.

4. General expression for the e.m.f. induced in one coil by a change of current in another coil

Let coil A carry a current I, and let it cause a flux MI to thread coil B, where M is the coefficient of mutual inductance discussed in Chapter I. Then,

$$E = -\frac{d}{dt}(MI)$$
$$= -M\frac{dI}{dt}. \tag{16}$$

If coil B is revolving or moving, M is not constant, and then

$$E = -M\frac{dI}{dt} - I\frac{dM}{dt}. \tag{17}$$

But the reader should remember that the flux threading coil B is not exactly equal to MI, where M is the mutual inductance calculated for steady currents. If it were so, it would not be possible for an alternating current in a coil to send measurable radio signals round the world. The value of M for alternating currents is a function of frequency and distance, but for low frequencies and moderate distances it differs by an inappreciable amount from that calculated for steady currents. Thus if the current alternates at 50 cycles/sec. the effective mutual inductance between a coil in England and a coil in New York would differ from the steady current value by less than

0·1 per cent. So for separations of a few feet, the difference is far beyond the accuracy of any measurement and it is appropriate to use the steady current value. This note is inserted to remind the reader of the order of approximation he is making; if he extends his interests to radio communication he need not feel his early training was incorrect.

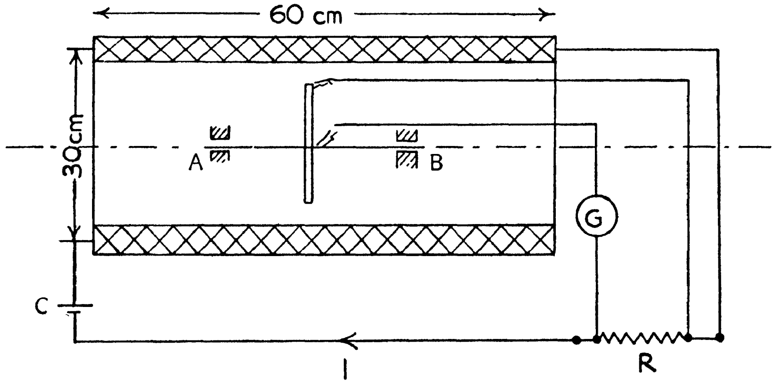


FIG. 54

5. The Lorentz method of determining the ohm

The Lorentz method of determining the ohm is an interesting example of an ingenious application of the law of induced e.m.f. The method is shown diagrammatically in Fig. 54. A copper disk can revolve on an axle AB which is placed on the axis of the solenoid. Current from the battery C passes through the solenoid and through the resistance R. The plane of the disk is transverse to the magnetic field inside the solenoid: when the disk revolves it cuts this field, and a p.d. is produced between the axle and the circumference. Light brushes press on the axle and on the circumference, and from them wires are taken to the ends of the resistance R. The disk is rotated in the direction which makes the induced p.d. tend to drive current through the galvanometer G in the reverse direction to that which the p.d. RI would send current. At some speed the induced p.d. will be exactly equal to the p.d. RI, and then the galvanometer deflexion is zero. The induced p.d. depends on the speed and the field strength, and therefore it is proportional to the speed and the current I. Hence the speed for balance is independent of the current I, and the resistance R is determined by observing the speed n at which balance occurs. The system will be understood better by writing down the condition for balance. Let the solenoid be wound with N turns per unit length, and let the semi-angle subtended by the end at the middle of the axis be ϕ. Then the field along the axis AB is

$$H = \frac{4\pi IN\cos\phi}{10}$$

(see Chap. I, formula (15)). If the solenoid is long, this field is sensibly unaltered at a considerable radial distance away from the axis, and the small variations can be calculated from a table of mutual inductance of solenoids. Let the disk have a radius r, then it is assumed that all the disk is sensibly in the same field strength H. The circumferential speed is $2\pi rn$: consider any imaginary radial line drawn on the disk; this line cuts the field at a speed which varies from zero to $2\pi rn$ and the mean cutting speed is πrn. So, applying the cutting rule,

$$V = \frac{H\times r\times \pi rn}{10^8}$$
$$= \frac{4\pi^2 r^2 NnI\cos\phi}{10^9}.$$

So in the balance condition

$$\frac{4\pi^2 r^2 NnI\cos\phi}{10^9} = RI,$$

$$\therefore \quad R = \frac{4\pi^2 r^2 Nn\cos\phi}{10^9}\Omega. \tag{18}$$

As a numerical example, suppose $N = 100$, $n = 50$, and $\cos\phi = 1$ and $r = 1$ cm., then $R = \frac{2\pi^2\times 10^6}{10^9} = 19{\cdot}7\,\text{m}\Omega$.

It is interesting to notice that in this formula for R, resistance appears to have the dimensions of a velocity: it also has the dimensions of a velocity in a system of units in which magnetic permeability is considered to have no dimensions.

6. Metal ring falling through a uniform radial magnetic field

Consider an electromagnet of the form shown diagrammatically in Fig. 55. Lines of force spread out radially from the central core and cross the uniform air gap: the field in this gap is consequently uniform and radial.

Let a circular metal ring be placed over the top of the central core and released. In falling it will cut the radial field and generate an e.m.f. round the ring, and this e.m.f. will produce a current. The reaction between the current and the field produces a force tending to support the ring, which therefore falls more slowly than it would under gravity alone. Let the field strength be H and the ring have a cross-sectional area A and circumference l, specific resistance ρ and

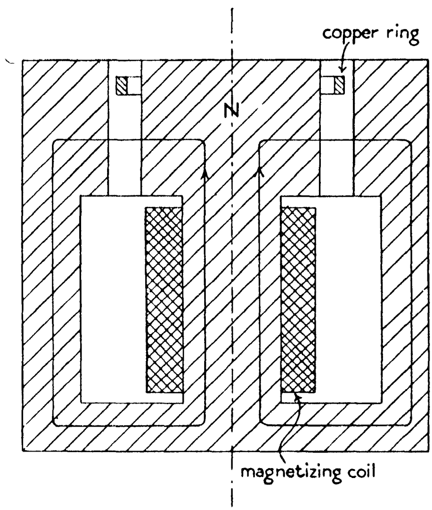


FIG. 55

density σ. Let the downward velocity be v at time t. Then

$$E = Hlv,$$

$$I = \frac{Hlv}{R} = \frac{HlvA}{\rho l} = \frac{HvA}{\rho},$$

$$F = HIl = \frac{H^2vAl}{\rho}.$$

$$\therefore \quad m\frac{dv}{dt} = mg - \frac{H^2vAl}{\rho}.$$

$$\therefore \quad \frac{dv}{dt} + \frac{H^2}{\sigma\rho}v = g.$$

$$\therefore \quad v = \frac{g\sigma\rho}{H^2}\left(1 - e^{-\frac{H^2}{\sigma\rho}t}\right).$$

$$\therefore \quad s = \frac{g\sigma\rho}{H^2}\left\{t - \frac{\sigma\rho}{H^2}\left(1 - e^{-\frac{H^2}{\sigma\rho}t}\right)\right\}.$$

The limiting velocity is $\frac{g\sigma\rho}{H^2}$, and a numerical example will show that this may have a surprisingly small value. Thus, suppose a copper ring, for which $\rho = 1{\cdot}8\,\mu\Omega/\text{cm.}^3$ and $\sigma = 8{\cdot}8$: if $H = 5{,}000$ lines sq. cm., then $v = \frac{981\times1{\cdot}8\times8{\cdot}8\times10^9}{10^6\times25\times10^6} = 6{\cdot}2$ mm./sec. So the ring would fall so slowly as to appear to be almost levitating in the air gap. The limiting velocity will sensibly have been reached in a time

$$t = \frac{4\sigma\rho}{H^2} = \frac{64\times10^9}{25\times10^6} = 2{,}560 \text{ secs.} = 42{\cdot}6 \text{ min.}$$

and in this time it will have fallen a distance $3g\left(\frac{\sigma\rho}{H}\right)^2 = 12{\cdot}3$ m. If the magnetic field alternates simple harmonically, the ring may ascend or remain stationary.

7. Self-induced e.m.f.

The magnetic flux through a circuit which carries a current i is $\phi = Li$: if the current changes, the flux changes with it, and so there is an e.m.f., $E = -L\frac{di}{dt}$, induced round the circuit by the change of its own flux. Thus a voltage is required to increase the current through a coil, and the voltage must be maintained while the change is in progress. Work is required to increase the current, and this energy is stored somehow by the coil and is not dissipated in heat. The mechanism of the storage is not understood, but it is often described as a storage in the ether. The stored energy can be calculated in terms of L and i as follows. At time t when the current is i, the battery must apply a p.d. $E = L\frac{di}{dt}$ and its rate of working at the moment is Ei; whence

$$\begin{aligned} W &= \int_0^T Ei\,dt \\ &= L\int_0^T i\frac{di}{dt}\,dt \\ &= L\int_0^I i\,di = \tfrac{1}{2}LI^2. \end{aligned} \tag{19}$$

It was stated in § 14 of Chapter I that the energy of a magnetic field was $E = \iiint \frac{H^2}{8\pi}\,dxdydz$. By means of equation (19) we shall verify this statement for a ring solenoid coil: for such a coil there is no field outside the winding and the integration throughout all space reduces itself to an integration inside the solenoid. Let the ring have a cross-sectional area A, a mean length l and a uniform flux density H over the cross-section. Then

$$\iiint \frac{H^2}{8\pi}\,dxdydz = \frac{H^2}{8\pi}\iiint dxdydz$$

$$= \frac{H^2}{8\pi}Al$$

$$= \frac{(16\pi IT)^2 Al}{8\pi}$$

$$= \frac{4\pi TA\times Tl\times I^2}{2}$$

$$= \tfrac{1}{2}LI^2.$$

When the threading rule states that 'the e.m.f. is equal to the rate of change of flux through the circuit', the term flux must of course mean net flux; the resultant of an external imposed flux and that due to the current in the coil. Let a circuit of resistance R and self-inductance L be penetrated by a flux ϕ when the current in the coil is zero. When the current in the coil is i, the net flux through it is $(\phi - Li)$, and it is the rate of change of this net flux which produces the e.m.f. required to force the current against the resistance R.

Hence $$\frac{d}{dt}(\phi - Li) = Ri.$$

$$\therefore\quad \frac{d\phi}{dt} = L\frac{di}{dt} + Ri. \tag{20}$$

This equation shows that the initial experiments described in § 1 could never be used to find accurately the law of induced e.m.f. because the term L is never zero. It is only when R is increased so as to make Ri dominate $L\frac{di}{dt}$ that i tends to the value $\frac{1}{R}\frac{d\phi}{dt}$. By this means the law of induced e.m.f. can be inferred and then tested accurately by means of equation (20), using a calculated value of L.

The induced current will flow in such a direction as to reduce the net flux through the circuit, and in the limiting condition of zero resistance the net flux would be zero.

Thus suppose a circular coil of zero resistance is penetrated by a flux: when this flux changes, a current will be induced in the coil of such a value as to make the net flux zero at every instant. This does not mean that there is no magnetic force at any point in the plane of the coil but merely that as many lines pass from back to front as pass from front to back; perhaps this statement will be better understood by reference to Fig. 56.

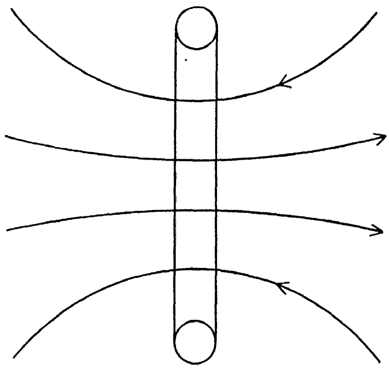


FIG. 56

Suppose a long solenoid is placed with its axis parallel to a uniform magnetic field. A current in the solenoid will produce an internal field which is uniform everywhere. So if a penetrating field changes, a current will instantly be induced of such a value as to neutralize the change completely.

An observer with a magnetic pole inside the solenoid would be unable to tell if the external field was varying. If the penetrating field grew from zero value there would never be any net magnetic force inside the solenoid because the induced current will always exactly neutralize the field which would penetrate if the solenoid was on open circuit. All this is exactly true only if the resistance is zero, and in practice there is always some net flux inside, of sufficient value to generate the e.m.f. to drive the induced current against the resistance. Suppose the imposed flux pulsates harmonically with a constant amplitude but with a frequency which can be increased continuously. Then, as the frequency rises from zero, the penetrating flux will fall asymptotically to a constant amplitude which is a very small fraction of that which would exist if the coil was on open circuit. This is an example of magnetic screening in which an alternating flux is neutralized by an equal and opposite flux due to an induced current. It is a simple matter to screen a given closed space from an alternating field; in Chapter IV (p. 200) it is shown that it is very difficult to screen a given closed space from a steady magnetic field.

7 a. Growth of a current in an inductive circuit

Let a coil of inductance L and resistance R be connected at time $t = 0$ to a battery of voltage V (see Fig. 57). The current will not rise instantly to the value $i = V/R$ because its rate of increase will be impeded by the self-induced voltage in L: the net e.m.f. round the circuit is $V - L\frac{di}{dt}$, and so we have

$$V = L\frac{di}{dt} + Ri.$$

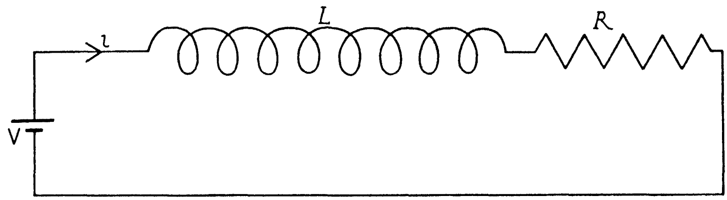


Fig. 57

A solution of this equation may readily be found to be of the form $i = A + Be^{mt}$, where A, B, and m are constants to be determined; thus

$$V = LBme^{mt} + RA + RBe^{mt},$$

whence it follows that $A = V/R$ and $m = -R/L$.

So
$$i = V/R + Be^{-\frac{R}{L}t}.$$

When $t = 0$, $i = 0$ and therefore $L\frac{di}{dt} = V$ initially; when $t = 0$, $\frac{di}{dt} = mB = -RB/L$.

$$\therefore \quad B = -\frac{V}{R}.$$

$$\therefore \quad i = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\doteqdot \frac{V}{R}\left(\frac{R}{L}t - \frac{R^2}{2L^2}t^2\right), \quad \text{when } t \ll 1,$$

$$= \frac{V}{L}t\left(1 - \frac{R}{2L}t\right).$$

The form of the current curve is shown in Fig. 58; L/R is called the time-constant of the circuit. For the current to rise to within 2 per cent. of its final value a time $t = 4L/R$ must elapse. If the battery in Fig. 57 is short-circuited the current will not fall instantly to zero: the energy associated with the inductance will be given back and dissipated in the circuit resistance. The e.m.f. induced by the decreasing current causes the current flow to persist until the energy has been dissipated. Suppose the battery is short-circuited at time

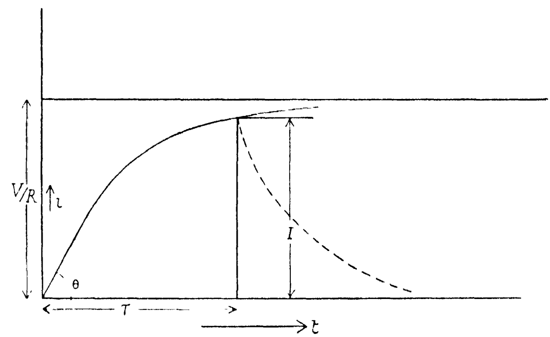


FIG. 58

τ (see Fig. 58). The equation representing the current flow for subsequent time is

$$L\frac{di}{dt}+Ri = 0;$$

whence $$i = Ae^{-\frac{R}{L}t}.$$

When $t = 0$, $$i = I,$$

and so $$i = Ie^{-\frac{R}{L}t}.$$

This is illustrated by the dotted curve in Fig. 58. If the current is forced to become zero very rapidly, as, for example, by opening a switch, then $L\frac{di}{dt}$ is necessarily big and a large voltage will be induced in the coil. This voltage across the opening contacts of the switch may be sufficient to cause ionization and the production of an arc. This is the cause of the spark which may be seen at the contacts of an electric bell or on opening the field circuit of a dynamo.

When discussing screening effects in the last section it was stated that in the absence of resistance the current would grow instantly to a value which would make the net field inside the coil zero. Current cannot rise instantly to a finite value, for this would entail an infinite rate of increase and an infinite self-induced voltage: the rate of rise is necessarily affected by capacity effects which are associated with every inductance.

A numerical example. A two-core cable is connected to a battery which maintains a constant p.d. of 200 V. The cable is 100 m. long and the cores are 1 cm. diameter and placed 4 cm. apart centre to centre: if the cable is short-circuited at the far end, find the time taken for the current to rise to its final value and also the force tending to separate the two cores. Specific resistance of copper 1·68 $\mu\Omega$/cm.³

$$R = \frac{\rho l}{A} = \frac{1\cdot 68}{10^6} \times \frac{10^4 \times 4}{\pi}\ \Omega \text{ per cable}$$

$$= \frac{2\cdot 14}{10^2}\ \Omega.$$

$$\text{Final current} = \frac{200 \times 10^2}{4\cdot 28}\ \text{amp.}$$

$$= 4{,}680\ \text{amp.}$$

$$L = 4\left(\log_e \frac{D}{r} + 1\right) \times \frac{1}{10^9}\ \text{H./cm.*}$$

$$= \frac{9\cdot 5}{10^9}\ \text{H./cm.}$$

$$\frac{L}{R} = \frac{9\cdot 5 \times 10^4 \times 10^2}{10^9 \times 4\cdot 28} = \frac{2\cdot 22}{10^3}.$$

The current will rise to within 2 per cent. of its final value in a time $t = 4L/R$, that is, in $8\cdot 9 \times 10^{-3}$ seconds.

The final value of the force is

$$P = \frac{2I^2}{D}$$

$$= 11 \times 10^4\ \text{dynes/cm.}$$

$$= 112\ \text{gm./cm.}$$

$$= 25\cdot 5\ \text{lb./cm.}$$

8. An example of self-induced e.m.f.

Consider the end of a very long rectangular circuit $ABCD$ in which the cross-piece BC can move downwards. Let $AB = h$, where $h \gg D$, see Fig. 59.

* See Chap. I, formula (28).

We will calculate the flux through the rectangle, and consider first the effect of one wire.

$$H_P = \frac{I}{x}\left(1+\sin\theta\right)$$
$$= \frac{I}{x}\left(1+\frac{y}{\sqrt{x^2+y^2}}\right).$$

Hence the flux through the strip of width dx and length y is

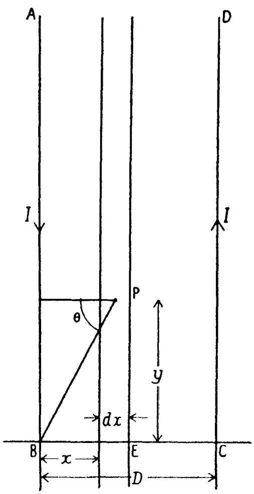


FIG. 59

$$d\phi = \frac{I}{x}\int_0^h \left(1+\frac{y}{\sqrt{x^2+y^2}}\right) dy$$
$$= \frac{I}{x}\left[y+\sqrt{y^2+x^2}\right]_0^h$$
$$\doteqdot \frac{I}{x}(2h-x),$$

$\therefore$ flux through $ABCD$ due to current in both wires is

$$\phi = 2I\int_r^D \left(\frac{2h}{x}-1\right) dx$$
$$\doteqdot 4Ih\left[\log\frac{D}{r}-\frac{D}{2h}\right].$$

Now let BC move downwards with uniform velocity v,

$$\frac{d\phi}{dt} = \frac{d\phi}{dh}\frac{dh}{dt} = 4Iv\log\frac{D}{r}.$$

So the self-induced e.m.f. is $E = 4Iv\log\frac{D}{r}$.

This problem might have been approached in another way. A point E in the cross-piece is situated in a field

$$H = I\left(\frac{1}{x}+\frac{1}{D-x}\right).$$

As the bar moves down it may be supposed to cut across this field with speed v, and so applying the cutting rule we should obtain

$$E = Iv\int_r^{D-r} \left(\frac{1}{x}+\frac{1}{D-x}\right) dx \doteqdot 2Iv\log\frac{D}{r}.$$

So the e.m.f. calculated by the cutting rule is only half the self-induced e.m.f. This is because the flux through the whole circuit is changing because of the motion of the cross-piece. The cutting rule must not be used when the flux at any given point of space inside the circuit is changing with time.

It is simpler to calculate the force on the cross-piece directly by the methods of Chapter I, but it can be calculated from the induced e.m.f. as follows. The input from the battery is EI, and this goes to store energy in the magnetic field and also to do work on the cross-piece.

So
$$EI = \tfrac{1}{2}I^2\frac{dL}{dt}+Fv$$
$$= \tfrac{1}{2}EI+Fv.$$
$$\therefore\quad Fv = \tfrac{1}{2}EI.$$
$$\therefore\quad F = 2I^2\log\frac{D}{r}.$$

Calculating F directly, we have
$$F = 2\int\limits_r^D \frac{I}{x}\,dx\,I$$
$$= 2I^2\log\frac{D}{r}.$$

But if* E had been calculated incorrectly by the cutting rule we should have supposed that
$$F = I^2\log\frac{D}{r}.$$

9. The coil-driven loud speaker

The system is shown diagrammatically in Fig. 60: a uniform radial field crosses a cylindrical air-gap and enters a central iron core, and a coil, shown by AB, is free to move horizontally in this air gap. The coil is attached rigidly to the cone diaphragm, which is indicated in the figure; we will suppose the cone suspended on threads which do not impose any appreciable constraint on its horizontal motion. The magnetic field in the air-gap is uniform and steady, but the current in the coil alternates simple harmonically and is $i = I\sin pt$.

* See W. F. Dunton, *Journal of Scientific Instruments*, 1927, vol. 4, p. 440, who points out the incorrect result which is obtained from using the cutting rule.

Let the strength of the field be H, the mass of the moving system M, and the length of wire on the coil l. The force on the coil at any instant is $Hil = HIl \sin pt$. Since the force is harmonic, the motion also will be harmonic; let its amplitude be X. There is supposed to be no elastic constraint or frictional force on the moving system, and therefore the acceleration is proportional to the force. The motion

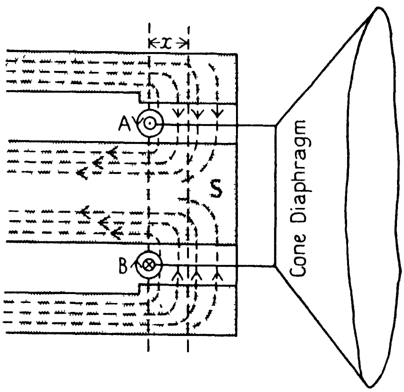


FIG. 60

is harmonic and so the acceleration is a maximum when the coil is at the ends of its travel. Consequently the coil is at an end of its travel at the instant when the current is a maximum, and the current is zero when the coil is in the mid position. This may be expressed in symbols as follows:

$$F = Hil.$$

$$\therefore \quad -M\frac{d^2x}{dt^2} = HIl \sin pt.$$

$$\therefore \quad -\frac{dx}{dt} = \frac{HIl}{pM} \cos pt.$$

$$\therefore \quad x = \frac{HIl}{p^2M} \sin pt$$

$$\equiv X \sin pt.$$

Suppose in Fig. 60 that the coil is just about to start moving to the right, then the direction of the current in it must be as shown. Starting from the moment when the coil is at this end of its travel, the curve of current and time is as shown in Fig. 61, where ordinates above the line represent current flowing in the direction marked in

Fig. 60. When the coil moves it cuts across the radial field and a voltage is generated in it. Application of the screw rule will show that when the coil is moving to the right this voltage has a direction opposite to that shown by the arrow-point and tail marked at A and B respectively in Fig. 60: this voltage is shown, in its correct phase, by the dotted sine curve in Fig. 61. If the coil has a velocity v at

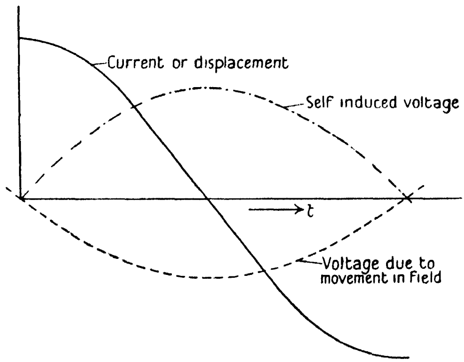


FIG. 61

time t, then

$$E = Hlv$$
$$= Hl\frac{dx}{dt}$$
$$= -\frac{H^2l^2}{pM} I \cos pt.$$

But the total flux through the coil is that due to the radial field together with that produced by the current in the coil; that is to say, its own self-induction flux. Two lines of self-induction flux are shown round the wires at A and B: at the instant shown it passes through the coil in the opposite direction to the flux due to the permanent magnet. As the coil moves to the right both these fluxes are decreasing and therefore the self-induced voltage is opposed to that due to the movement of the coil in the radial field. The self-induced voltage is shown, in its correct phase, by the chain-dotted sine curve in Fig. 61. That the two voltages are opposed is sometimes expressed by saying that the movement of the coil gives it a negative, or capacitative, reactance; these are terms which will be familiar to the student of alternating currents.

Accordingly, the p.d. which must be applied to the coil, whose resistance is R, is

$$\begin{aligned} e &= L\frac{di}{dt} - Hlv + Ri \\ &= pLI\cos pt - \frac{H^2l^2}{pM}I\cos pt + RI\sin pt \\ &= \left(pL - \frac{H^2l^2}{pM}\right)I\cos pt + RI\sin pt. \end{aligned}$$

The first term will be zero when $p^2LM = H^2l^2$, and a voltage of given value will produce the greatest current when the frequency is such as to satisfy this equation.

We will illustrate this by a numerical example: A coil of a loud speaker had an inductance of 0·2 H. and was wound with 300 m. of wire and moved in a uniform radial field of 5,000 lines per sq. cm. A voltage of constant value was applied to the coil, and this produced the greatest current when the frequency was 500 cycles/sec.: if this current was 10 mA., find the mass of the moving system and the amplitude of the motion.

$$n = 500. \qquad \therefore \quad p = 10^3\pi.$$

$$\begin{aligned} M &= \frac{H^2l^2}{p^2L} = \frac{25\times10^6\times9\times10^8}{\pi^2\times10^6\times0\cdot2\times10^9} \\ &= 11\cdot3 \text{ gm.} \\ X &= \frac{HlI}{p^2M} = \frac{5\times10^3\times3\times10^4\times10\sqrt{2}\times10^{-3}}{\pi^2\times10^6\times11\cdot3} \\ &= 0\cdot188 \text{ mm.*} \end{aligned}$$

10. Eddy currents induced in a thin circular tube

Suppose a long circular tube, of radius R and thickness t (see Fig. 62), is penetrated by a uniform magnetic field h which pulsates harmonically so that $h = H\sin pt$. Consider the circuit formed by the two long strip elements AB and CD together with closing pieces at each end at infinity. The flux through this circuit pulsates and so there is an e.m.f. round it which will cause a current to flow parallel to the axis of the cylinder: if at some instant it is flowing into the paper in AB, then at the same instant it is flowing out from the paper in CD.

* If L is in H., I in amps, then $X = \dfrac{HIl}{10p^2M}$ cm. and $E = \dfrac{Hlv}{10^8}$ V.: or alternatively L may be expressed in electromagnetic absolute units by multiplying by 10^9.

Over the semi-circumference EFG current is flowing parallel to the axis and likewise in the semi-circumference EHG, but the currents in the two semi-circumferences are oppositely directed at every instant. At E and G the current changes direction, and therefore the current density at these two points must always be zero. The current density increases from zero at E and G to a maximum at F and H. Now the long tube would ordinarily be considered as a single conductor, yet currents are generated in it because it has a finite radius. Such currents are usually called eddy currents, a term

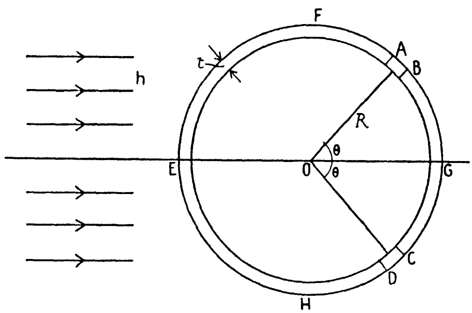


FIG. 62

which distinguishes them from currents flowing round a wire circuit. The problem of eddy currents in conductors and in the cores of transformers and dynamos is very important to the engineer. This example is a suitable introduction to many such problems which arise in practice.

Let the current density at the point (R, θ) in Fig. 62 be σ, and let the specific resistance of the tube be ρ.

The flux passing between the strips AB and CD is

$$\begin{aligned} \phi &= 2hR\sin\theta \\ &= 2HR\sin\theta\sin pt. \\ \therefore \quad E &= -2pHR\sin\theta\cos pt. \end{aligned}$$

Applying Ohm's law to the circuit, we have

$$2\sigma\rho = 2pHR\sin\theta\cos pt.$$

$$\therefore \quad \sigma = \frac{pHR}{\rho}\sin\theta\cos pt.$$

So the current density increases as the sine of the angle between the point considered and the diametral plane parallel to the flux.

The current distribution along the generators of the cylinder will produce a magnetic field, which is uniform everywhere inside the cylinder. Before showing that it is uniform we will calculate its value at the centre of the cross-section. The current density $I\sin\theta$ in the long strip AB produces a field $\frac{2I\sin\theta\, Rt\, d\theta}{R}$ at O, and this has a component $2It\sin^2\theta\, d\theta$ in the direction EG; whence

$$H = 4It\int_0^{\pi}\sin^2\theta\, d\theta = 2\pi tI = \frac{2\pi pHRt}{\rho}.$$

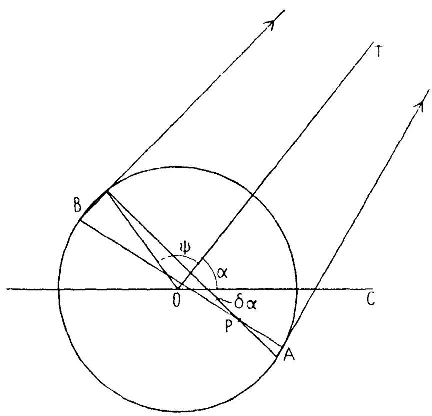


FIG. 63

There are various ways of showing that the field is uniform and parallel to EG at every point inside the cylinder, but we shall use a proof due to Professor C. Godfrey.*

Consider the field at P, Fig. 63, due to current density σ' and σ'' in the elementary arcs at A and B respectively. Let OT be drawn perpendicular to the chord through P which bisects the elementary arcs at A and B.

Then the current in A produces at P a field $\frac{2\sigma' ds' t}{AP}$ in the direction OT. Hence the arcs at A and B produce at P a field

$$dH = 2t\left(\frac{\sigma' ds'}{AP} + \frac{\sigma'' ds''}{BP}\right)\cos\alpha,$$

in the direction OC.

* See *Journal Inst. Elect. Engrs.*, 1923, vol. lxi, p. 938.

Now
$$\frac{APd\alpha}{ds'} = \sin\psi$$
$$= \frac{BPd\alpha}{ds''}$$

and
$$\sigma' = I\sin(\psi-\alpha)$$

and
$$\sigma'' = I\sin(\psi+\alpha).$$

$$\therefore\quad dH = 2It\{\sin(\psi-\alpha)+\sin(\psi+\alpha)\}\frac{\cos\alpha\,d\alpha}{\sin\psi}$$
$$= 4It\cos^2\alpha\,d\alpha.$$

$$\therefore\quad H = 4It\int_0^{\pi}\cos^2\alpha\,d\alpha$$
$$= 2It\pi.$$

This expression is independent of the position of P, and so the field inside the cylinder is uniform everywhere and directed along OC. This is an important property of a sine distribution of current round a cylinder, and it should be remembered.

But since the induced current produces an internal magnetic field, the net flux through the circuit shown in Fig. 62 is less than it would be if the field was steady. Let the distribution of current density be $i = I\sin\theta\sin(pt+\alpha)$: we will now find I and α in terms of H and p.

The internal uniform field due to the current is
$$h = 2\pi It\sin(pt+\alpha).$$

So the net internal field is
$$h' = H\sin pt - 2\pi It\sin(pt+\alpha).$$
$$\therefore\quad \phi = 2R\sin\theta\{(H-2\pi It\cos\alpha)\sin pt - 2\pi It\sin\alpha\cos pt\};$$

whence
$$i = \frac{Rp\sin\theta}{\rho}\{(H-2\pi It\cos\alpha)\cos pt + 2\pi It\sin\alpha\sin pt\}$$
$$= I\sin\theta\{\sin\alpha\cos pt + \cos\alpha\sin pt\};$$

whence
$$\frac{Rp}{\rho}(H-2\pi It\cos\alpha) = I\sin\alpha$$

and
$$\frac{2\pi Rpt}{\rho}\sin\alpha = \cos\alpha.$$

$$\therefore \quad \frac{RpH}{\rho} = I\left(\sin\alpha + 2\pi\frac{Rpt}{\rho}\cos\alpha\right)$$

$$= I\sin\alpha\left(1+\frac{4\pi^2R^2p^2t^2}{\rho^2}\right).$$

$$\therefore \quad 2\pi tI = \frac{2\pi Rpt}{\rho}\frac{H}{\sqrt{1+\dfrac{4\pi^2R^2p^2t^2}{\rho^2}}}$$

$$\doteqdot H\left(1-\frac{\rho^2}{8\pi^2R^2p^2t^2}\right) \quad \text{when } \frac{2\pi Rpt}{\rho} \gg 1$$

or

$$\doteqdot \frac{2\pi RtpH}{\rho}\left(1-\frac{2\pi^2R^2p^2t^2}{\rho^2}\right) \quad \text{when } \frac{2\pi Rpt}{\rho} \ll 1.$$

We see that as the frequency increases, the internal field tends to equal the penetrating field, and then the net internal field is zero. This is in accordance with the general proposition stated earlier, that the induced current produces a field which tends to neutralize the penetrating field.

We may say that the induced eddy current tends to screen the interior space from the penetrating field. So if it is desired to screen a given region from an alternating magnetic field, the region should be surrounded by a thick copper tube. In general it is much easier to screen a space from an alternating magnetic field than from a steady magnetic field. As a numerical example we will suppose a copper tube is 10 cm. mean radius and 4 mm. thick: it is required to find the internal magnetic field when this alternates at a frequency of 50 cycles/sec. The specific resistance of copper is 1,680 electromagnetic absolute units of resistance per cm. cube ($1{\cdot}68\ \mu\Omega/\text{cm.}^3$), whence it may be found that

$$\frac{2\pi pRt}{\rho} = 4{\cdot}8.$$

$$\therefore \quad \frac{H-2\pi It}{H} = \frac{1}{2\times 4{\cdot}8^2} = 0{\cdot}0216.$$

So such a tube reduces the internal field to 2·16 per cent. of the unscreened value, even though the frequency is but 50 cycles/sec.

11. Thin circular cylinder revolving in a uniform field

Let the thin cylinder, shown in Fig. 64, be revolving with angular velocity Ω in a steady uniform magnetic field H. Then an elementary strip, such as AB, is cutting across the field with a velocity $R\Omega\cos\theta$, and so by the cutting rule there is a voltage generated in it of value $HR\Omega\cos\theta$; a current must flow of such a value as to give a resistance drop equal and opposite to the voltage produced by cutting the lines of force. Hence we have

$$\sigma\rho = HR\Omega\cos\theta.$$

$$\therefore\quad \sigma = \frac{HR\Omega}{\rho}\cos\theta.$$

So again there is a sine distribution of current density, but here the density is a maximum at C and D, and zero at E and F. There will

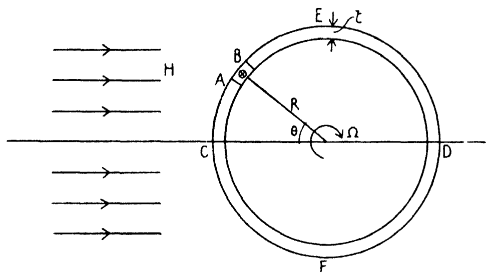


FIG. 64

be a force between the induced current and the field which will produce a torque resisting the motion. The value T of this torque per unit length of the cylinder may be calculated as follows:

$$T = 2\int_{-\pi/2}^{+\pi/2} H\sigma R\,d\theta\,tR\cos\theta$$

$$= \frac{2H^2R^3t\Omega}{\rho}\int_{-\pi/2}^{+\pi/2}\cos^2\theta\,d\theta$$

$$= \frac{\pi H^2R^3t\Omega}{\rho}.$$

$$\text{Rate of working} = T\Omega$$

$$= \frac{\pi H^2R^3t\Omega^2}{\rho}.$$

12. Eddy currents in a solid circular cylinder

The heavy-current engineer is concerned mainly with the energy loss produced by eddy currents, and we will introduce this problem by considering the eddy currents induced in a solid circular cylinder which is penetrated by a uniform magnetic field of intensity $h = H \sin pt$: a cross-section of the system is illustrated in Fig. 65.

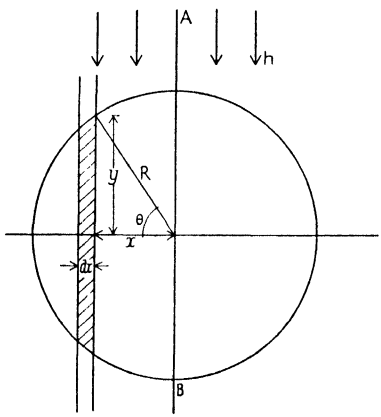


FIG. 65

From symmetry there is no current flow across the median plane AB and at every instant the current flow over the whole of one half-cylinder is in the same direction and in the opposite direction over the other half-cylinder. Picture the cylinder built up of imaginary planks of width dx, one of which is distant x from the median plane. Since the flux passing between the median plane and this plank is the same at all points along the plank, the current density in it is uniform. Applying the threading rule to the flux between AB and the plank at x, we have

$$\sigma\rho = pHx \cos pt.$$

$$dW = \sigma^2\rho \text{ per unit volume}$$

$$= \frac{p^2H^2x^2}{\rho}\cos^2 pt\ 2y\,dx \text{ per plank.}$$

$$\therefore \quad \text{Mean rate of loss} = \frac{p^2H^2x^2}{2\rho}\ 2y\,dx \text{ per plank.}$$

$$\therefore \quad W = \frac{2p^2H^2}{\rho}\int_0^r x^2y\,dx$$

$$= \frac{2R^4p^2H^2}{\rho}\int_0^{\pi/2}\cos^2\theta\sin^2\theta\,d\theta$$

$$= \frac{R^4p^2H^2}{2\rho}\int_0^{\pi/2}\sin^2 2\theta\,d\theta$$

$$= \frac{\pi R^4p^2H^2}{8\rho} \text{ per unit length.}$$

This formula is substantially correct only when the field of the induced currents is small compared with the penetrating field. The induced currents produce a field which tends to neutralize that which is penetrating the cylinder, and so as the frequency increases the current density tends to become a surface distribution. The full calculation is much too involved for this book (see also pp. 238 and 243), but it may be shown that the simple formula just found is substantially correct so long as $\frac{\pi p R^2}{\rho}$ is not much greater than unity. If the field alternates at 50 cycles/sec., it may be applied to copper rods whose radius is not greater than 1·3 cm.

13. Eddy loss in a circular cylinder with axis parallel to a uniform magnetic field

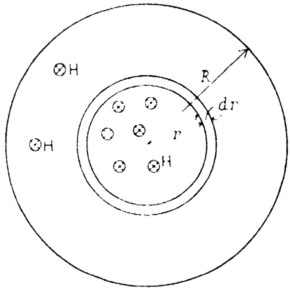


FIG. 66

For example, let the cylinder be the core of a long solenoid; then the field will be parallel to the axis and will be uniform. Symmetry shows that the eddy current paths are circles centred on the axis and that the current density is zero at the centre of the cross-section. Then the flux enclosed by a circle of radius r (see Fig. 66) is

$$\phi = \pi r^2 H \sin pt.$$

$$\therefore \quad 2\pi r \sigma \rho = p\pi r^2 H \cos pt.$$

$$\therefore \quad \sigma = \frac{prH}{2\rho} \cos pt.$$

The loss per unit volume $= \sigma^2 \rho$

$$= \frac{p^2 r^2 H^2}{4\rho} \cos^2 pt.$$

$\therefore$ Mean power loss per unit volume $= \frac{p^2 r^2 H^2}{8\rho}$.

$$\therefore \quad W = \frac{p^2 H^2}{8\rho} \int_0^R 2\pi r \, dr \, r^2$$

$$= \frac{\pi p^2 H^2 R^4}{16\rho}.$$

This formula is also not strictly correct because it ignores the field of the induced currents. It is substantially correct so long as $\frac{p\pi R^2}{\rho}$ is not much greater than unity, and so is applicable to copper rods smaller than 1·3 cm. radius when the frequency of alternation is 50 cycles/sec.

14. Eddy currents in a round wire due to an alternating current flowing in it

The current $i = I \sin pt$ produces a magnetic field inside the wire which increases uniformly from zero at the centre to $H = \dfrac{2I}{a}\sin pt$ at the surface of the wire (see Chap. I, p. 20).

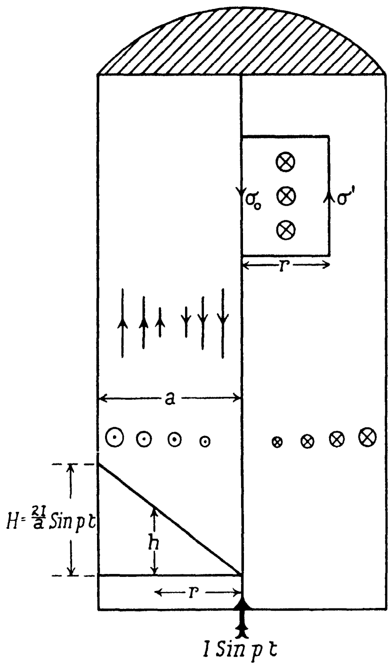


FIG. 67

The lines of force are everywhere circles concentric with the axis of the wire, and this pulsating field will generate an eddy current in the wire. The eddy current, which is not uniformly distributed, must be combined with the uniformly distributed current I to form a resultant current which increases towards the surface of the wire. So current is distributed uniformly over the section of a wire only if it is a steady current. Consider the flux through a rectangle of unit length and of breadth r and having one side in the axis of the wire (Fig. 67): there will be an e.m.f. round this rectangle which will cause a current to flow. If at a given instant it is flowing downwards along the axis, it must be flowing upwards at the surface of the wire and so must have zero value at some radius. Let the eddy current density be σ_0 along the axis and σ at radius r. Then

$$(\sigma+\sigma_0)\rho = \int_0^r \frac{\partial h}{\partial t}\,dr$$

$$= \int_0^r \frac{\partial}{\partial t}\left(\frac{2Ir\sin pt}{a^2}\right)dr, \text{ since } h = \frac{2Ir}{a^2}\sin pt,$$

$$= \frac{pIr^2}{a^2}\cos pt.$$

Now the total eddy current flowing up the wire must equal at every instant the total eddy current down the wire.

$$\therefore \quad 2\pi \int_0^a \sigma r \, dr = 0.$$

$$\therefore \quad \frac{pIa^2}{4\rho} \cos pt = \frac{\sigma_0 a^2}{2}.$$

$$\therefore \quad \sigma = \frac{pI \cos pt}{\rho} \left(\frac{r^2}{a^2} - \frac{1}{2} \right).$$

Whence the eddy-current density is zero at a radius $r = a/\sqrt{2}$. The resultant current density is

$$\sigma = \frac{I}{\pi a^2} \sin pt + \frac{pI}{\rho} \left(\frac{r^2}{a^2} - \frac{1}{2} \right) \cos pt.$$

$$\text{Rate of loss per unit volume} = \sigma^2 \rho.$$

$$\therefore \quad \text{Mean rate of loss per unit volume} = \frac{I^2 \rho}{2\pi^2 a^4} + \frac{p^2 I^2}{2\rho} \left(\frac{r^2}{a^2} - \frac{1}{2} \right)^2.$$

$$\begin{aligned} \text{The rate of loss in the whole wire} &= \int_0^a \sigma^2 \rho 2\pi r \, dr \\ &= \frac{I^2}{2} \left(\frac{\rho}{\pi a^2} + \frac{\pi^2 p^2 a^2}{12\rho} \right) \\ &= \frac{I^2 R}{2} \left(1 + \frac{p^2}{12 R^2} \right). \end{aligned}$$

So the effective resistance of a wire is a function of the frequency. The effect is very important at high frequencies and is often referred to as the skin effect.

In deriving this formula we have again ignored the field due to the eddy currents; it is substantially correct so long as $\frac{\pi p a^2}{\rho}$ is not greater than unity.

15. Eddy currents in the cores of dynamos and transformers

The conductors of a dynamo are placed in slots cut in the surface of an iron core called an armature. An e.m.f. is generated in the winding because the armature revolves in a magnetic field; if an external circuit is provided, current will flow from the winding. But since the iron armature revolves with the conductors, an e.m.f. is also generated in the iron, and eddy currents will flow because the iron core provides a closed circuit for them. Such eddy currents are objectionable because they cause a waste of energy. The e.m.f. which produces the eddy currents cannot be reduced, but the currents can be reduced by forcing them to flow in high-resistance paths. The path is made to have a high resistance by using an iron alloy which has a high resistance (Stalloy) and also by building the core from thin disks lightly insulated from one another. Consider Fig. 68, which shows very diagrammatically a longitudinal cross-section through a dynamo. In Fig. 68*a* the core is supposed to be solid and the eddy-current paths are indicated roughly by the dotted circuits: from symmetry the eddy currents must cut the plane AB at right angles. In Fig. 68*b* the core is supposed made of two plates insulated from one another by the plane AB. Eddy currents cannot now cross the plane AB, and so they have to follow paths which are indicated roughly by the dotted circuits. The energy loss is reduced because the effective resistance of the eddy path is increased by splitting the core into two disks. An armature is built from a very large number of disks which are each about 1/50th inch thick, so the eddy-current paths take the form indicated by Fig. 69. The eddy circuits are completed by an axial flow of current most of which occurs near the periphery of the disks. Therefore the main portion of the resistance is in the direction of the radial flow, and the cross-section of the path varies inversely as the thickness t of the stampings. Suppose the core is built from N stampings: then the e.m.f. per disk is $1/N$th of the total e.m.f., and the resistance of the path varies as $1/N$. So the loss per stamping varies as $E^2/R = \dfrac{kE^2}{N^3R}$.

$$\text{The loss in } N \text{ stampings} = \frac{NkE^2}{N^3R} = \frac{kE^2}{N^2R}.$$

So the eddy loss varies as the square of the thickness of the plates. Consequently the eddy-current loss can be reduced enormously by using thin stampings. The stampings are insulated from one another by a covering of very thin paper about two thousandths of an inch thick. When the thickness of the stampings becomes comparable with the thickness of the paper there is a considerable reduction of effective cross-section of iron to carry the flux,* and this imposes

* So far we have not considered magnetic force and flux density inside iron and this remark is not fully intelligible until Chap. III has been read. The reader will do well to picture for the time that the plates are of brass or copper and not of iron. The general argument is not affected.

a limit to the amount of subdivision which is desirable. The eddy-current loss in the core of an average dynamo is from 0·5 per cent. to 1 per cent. of the output of the machine.

In the expression $\frac{kE^2}{N^2R}$, E stands for the eddy e.m.f. Since this

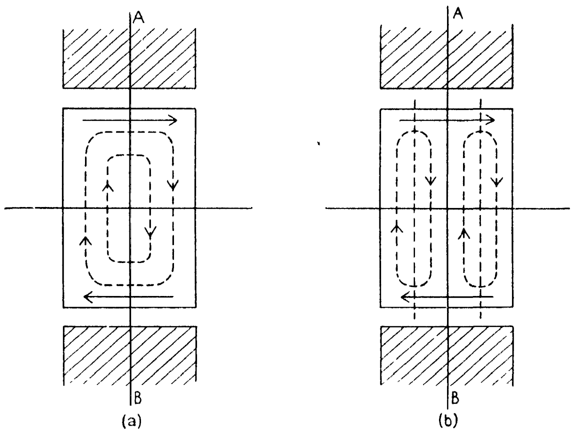


FIG. 68

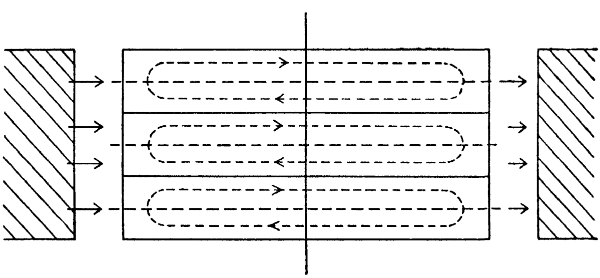


FIG. 69

e.m.f. is due to the rotation of the armature, there must be a constant factor relating E to the voltage V generated by the dynamo. So the eddy-current loss varies as the square of the voltage of the machine and the square of the thickness of the plates. Since the voltage depends on the flux-density B and the speed n, we can write

$$W = kB^2n^2t^2,$$

but the expression

$$W = kV^2t^2$$

is a more useful form.

Eddy currents heat the armature core and so raise its resistance, therefore the losses tend to decrease as the machine warms up.

It is also necessary to laminate the core of a transformer. Thus consider Fig. 70, which represents the cross-section of the iron core of a transformer: round the core is a winding which carries an alternating current. This current produces a magnetic flux through the core, and so every stamping is threaded by an alternating flux which produces an eddy current whose path is indicated roughly in Fig. 70.

We will now calculate the eddy-current loss in a large thin iron plate whose thickness is threaded by a flux density $b = B\sin pt$.

Consider Fig. 71, which shows a section through a plate which extends very far into the paper and very far to the right and left of the broken edges: the flux density is indicated by the arrow-tails and is directed perpendicular to the plane of the paper. The plane, whose trace is AB, is a plane of symmetry of current flow: if the current is flowing from left to right above AB it is flowing from right to left below AB. From symmetry the current flow is mainly parallel to AB: near the edges of the plates the current must flow perpendicular to AB, but the plate is supposed to be so wide compared with the thickness t that the amount occupied with vertical flow is negligible compared with that occupied by horizontal flow. Consider a rectangle of unit width and of depth $2x$ (see Fig. 71): the flux through this rectangle is $\phi = 2x.1.B\sin pt$, and so the e.m.f. round the rectangle is $E = -2pxB\cos pt$. If the current density is σ at a height x above AB, the p.d. per unit width of plate at height x is $\sigma\rho$. Hence

$$2\sigma\rho = 2pxB\cos pt.$$

$$\sigma = \frac{pBx}{\rho}\cos pt.$$

So the current density increases uniformly from zero in the plane AB to a maximum at the surface of the plate. The energy loss in time dt in a slab of thickness dx and unit width is

$$\begin{aligned} dE &= \sigma^2\rho\, dxdt \\ &= \frac{p^2x^2B^2}{\rho}\, dx\cos^2 pt\, dt. \end{aligned}$$

$$\therefore \quad \text{Power loss } dW = \frac{p^2x^2B^2}{\rho}\, dx \frac{\int_0^{2\pi/p} \cos^2 pt\, dt}{2\pi/p} = \frac{p^2x^2B^2}{2\rho}\, dx.$$

$$\therefore \quad \text{Total power loss} = \frac{p^2B^2}{2\rho}\int_{-t/2}^{t/2} x^2\,dx$$

$$= \frac{p^2B^2t^3}{24\rho} = \frac{\pi^2B^2n^2t^3}{6\rho}.$$

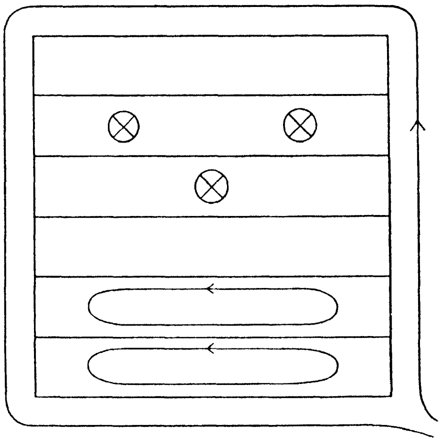


FIG. 70

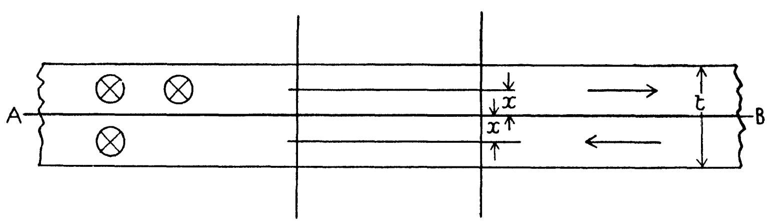


FIG. 71

In unit thickness of a packet of stampings there are $1/t$ plates, and so the loss per unit volume is

$$W = \frac{1}{t}\times\frac{\pi^2B^2n^2t^3}{6\rho} = \frac{\pi^2B^2n^2t^2}{6\rho}.$$

If the flux does not change simple harmonically, but in some complex manner having a periodic time T, then

$$W = \frac{t^2}{12\rho}\,\frac{\int_0^T \left(\frac{db}{dt}\right)^2 dt}{T} = \frac{t^2}{12\rho}\left(\text{mean square } \frac{db}{dt}\right).$$

Thus suppose b increases uniformly from $-B$ to $+B$ in time $T/2$, and then decreases uniformly from $+B$ to $-B$ in time $T/2$, and so on. Then $\frac{db}{dt} = \frac{4B}{T}$, and $W = \frac{t^2}{12\rho} \times 16B^2n^2 = \frac{8B^2n^2t^2}{6\rho}$, and so for the same maximum value of B the constant π^2 is reduced to 8. The problem is approached more elegantly by means of a Fourier series: thus let

$$b = B_1(\sin pt + B_3 \sin 3pt + B_5 \sin 5pt + ...),$$

then
$$\frac{db}{dt} = pB_1(\cos pt + 3B_3 \cos 3pt + 5B_5 \cos 5pt + ...).$$

$$\text{Mean square } \frac{db}{dt} = \frac{p^2B_1^2}{2}(1+9B_3^2+25B_5^2+...).$$

$$\therefore \quad W = \frac{\pi^2n^2t^2}{6\rho} B_1^2(1+9B_3^2+25B_5^2+...).$$

For the triangular wave considered previously $B_1 = \frac{8B_{\text{max}}^2}{\pi^2}$ and $B_n = B_1/n^2$.

$$\therefore \quad W = \frac{\pi^2n^2t^2}{6\rho} \frac{64B^2}{\pi^4}\left(1+\frac{1}{9}+\frac{1}{25}+...\right)$$

$$= \frac{8}{6}\frac{B^2n^2t^2}{\rho}, \quad \text{since} \quad \left(1+\frac{1}{9}+...\right) = \frac{\pi^2}{8}.$$

EXAMPLES TO CHAPTER II

1. The vertical component of the earth's magnetic field in England is about 0·47 c.g.s. units, and the gauge of a railway is 4·71 feet. A millivoltmeter is connected between the insulated rails of an isolated track on which an express train is travelling, and reads 2 mV. Find the speed of the train assuming that the resistance of the rails and axles is negligible. The total resistance of the circuit formed by the instrument, rails, and axles is 10 Ω. Find the extra horse-power which the engine must develop to maintain its speed after the voltmeter has been connected. ANS. 66·2 m.p.h.: $5{\cdot}37 \times 10^{-10}$ H.P.

2. A square coil, whose sides are one metre in length, is wound with 1,000 turns of wire and has a resistance of 50 Ω. It is revolved uniformly at a speed of 10 revolutions per second, about a diagonal which is in bearings and placed so as to be horizontal. If the vertical component of the earth's magnetic field is 0·47 c.g.s. units, find the heat developed in a resistance of 1,000 Ω connected to the terminals of the coil. ANS. $4{\cdot}2 \times 10^{-3}$ W.

3. A rectangular coil, 30 cm. long and 20 cm. wide, is arranged, as shown in Fig. 51 (*a*), parallel to a long wire which carries a current of 1,000 amp. The coil has 100 turns and is revolved uniformly at a speed of 20 revolutions per second: the axis of revolution is 15 cm. from the centre of the long wire. Calculate the maximum value of the induced e.m.f. and the amplitude of the fundamental component of the Fourier series which represents the induced voltage. ANS. 0·7 V., 0·99 V.

4. If the coil in Question 3 is placed so that the coil and the long wire are in one plane and then projected with a speed of 100 cm./sec., calculate the induced voltage at the moment when the nearer long side is 50 cm. from the current. ANS. 3·42 mV.

5. The coil of a galvanometer measures 4 cm. × 2 cm. and is wound with 100 turns of wire: it is in a uniform radial magnetic field of 1,000 lines sq. cm. The time period of a complete swing is 12 seconds and the moment of inertia of the coil is 10 gm. cm.^{2} units. Calculate the damping factor when the total resistance of the circuit is 1,000 Ω. ANS. $\Delta = 1{\cdot}1$.

6. The first swing of a ballistic galvanometer is 100 scale divisions and the sixth swing on the same side is 20 divisions. Calculate the damping factor. ANS. 1·08.

7. The galvanometer described on p. 73 is shunted so as to give full-scale deflexion for a current of 0·1 amp. Calculate the deflexion 1 second after switching on the current. ANS. 0·52 radians.

8. If the coil described in Question 3 above is at rest, and if the current in the long wire alternates simple harmonically, at a frequency of 50 cycles/sec., and has a maximum amplitude of 1,000 amp., calculate the maximum value of the induced e.m.f. (*a*) when the plane of the coil contains the current, (*b*) when the plane of the coil is 45° away from the plane containing the axis and the current, (*c*) when the plane of the coil is 90° away from the plane containing the axis and the current. ANS. (*a*) 3·04 V.; (*b*) 1·48 V.; (*c*) 0.

9. An apparatus for determining the ohm by the Lorentz method is similar to that depicted in Fig. 54. The solenoid is 50 cm. long and 25 cm. in mean

diameter and is wound with 500 turns of wire. The copper disk is 6 cm. in radius and balance occurs when the speed is 3,000 r.p.m. Find the value of the resistance R and the magnitude of the uniform magnetic field if the current is 10 amp. ANS. 0·634 mΩ.

10. An aluminium ring, density 2·7 and specific resistance 2·8 microhms/cm.^{3}, is falling through a uniform radial magnetic field, as shown in Fig. 55, of 10,000 lines sq. cm. Calculate the limiting value of the velocity with which it falls. ANS. 0·74 mm./sec.

11. Each of twelve poles of a certain dynamo is wound with 600 turns and carries a flux of 10^7 lines when the current is 10 amp. The twelve poles are connected in series and the total resistance is 25 Ω. Calculate the current flowing 0·1 seconds after connecting the winding to a battery of 400 V. p.d. and the time constant of the circuit. ANS. 4·7 amp., 0·286 secs.

12. If, in Example 11, a field current of 10 amp. is reduced uniformly to zero in 0·1 second, calculate the voltage induced in the winding. ANS. 720 V.

13. A very long solenoid, of diameter 10 cm., is wound with 10 layers of wire, each layer having 10 turns per cm. The wire is of copper and has a cross-sectional area of 10^{-3} sq. in. (S.W.G., No. 20). Calculate the time constant of the coil (see p. 52).

ANS. $L = 12$ mH/cm., $R \doteqdot 0{\cdot}9\,\Omega$/cm., $L/R = 13{\cdot}4 \times 10^{-3}$ secs.

14. In the arrangement depicted in Fig. 59, the cross-piece BC is allowed to move uniformly at the rate of 100 cm./sec., the wires are 2 mm. diameter, and are placed 10 cm. apart. Calculate the e.m.f. which must be supplied by the battery, in order to maintain a current of 10 amp. flowing in the circuit.

ANS. 18·4 μV.

15. A coil-driven loud speaker, as illustrated in Fig. 60, has a field strength of 3,000 lines sq. cm. and is wound with 100 m. of wire: if the mass of the moving system is 10 gm., calculate the 'apparent capacitance of the coil'.

ANS. 11·1 μF.

16. A long, vertical, uniform solenoid of T turns is threaded by one limb of a U-shaped ball-ended permanent magnet of pole strength m. The magnet is placed so that the ball is level with the top of the solenoid and then it is allowed to drop freely under gravity. If the solenoid is part of a circuit of large resistance R, show that the heat developed in the circuit during the fall tends to the value

$$\frac{32\pi^2 m^2 T^2}{3R}\sqrt{\frac{2g}{l}}$$

as R is increased, where l is the length of the solenoid and g is the gravitational acceleration. Why is the formula incorrect if R is not large?

17. A copper cylinder 15 cm. mean radius and 5 mm. thick is set transverse to a uniform magnetic field. Find the fractional strength of the magnetic field inside the solenoid when the frequency of alternation is (a) 5 cycles/sec., (b) 50 cycles/sec. ANS. (a) 33 %, (b) 0·63 %.

18. If in Question 17 the uniform magnetic field has a strength of 1,000 lines sq. cm. and is steady, find the couple required to revolve the cylinder uniformly at 120 r.p.m. ANS. 4×10^7 dyne cm. units.

19. Calculate the increase of effective resistance for a copper wire 1 cm. diameter, to alternating currents of frequency 50 cycles/sec. ANS. 0·18 %.

20. An eddy-current brake consists of a copper disk rotating between the poles of an electromagnet: it is used to test a constant speed motor. With a given output the temperature of the disk is 70° C., atmospheric temperature being 15° C. Find the ratio of the flux density which must be cut by the disk to absorb twice the original horse-power to the original flux density, the temperature coefficient of copper being 0·004 per degree Centigrade. It is to be assumed that each test is long enough to allow the temperature of the brake to reach a steady value. ANS. 1·53.

III

IRON IN A MAGNETIC FIELD

PART I

1. Force inside a magnet

Since the poles of a magnet cannot be separated, all magnets must be regarded as a collection of elementary particles, each of which is a complete magnet. An elementary magnetic filament is a line of these particles set end to end, so that adjacent poles neutralize one another completely; by this device a finite magnet can be regarded as having equivalent poles concentrated at definite points. The subdivision of magnets into particles is a necessity, and not a mere convenience like the indefinite subdivision of electric charge. Though we are forced to regard every magnet as an infinite collection of small magnets, yet the external force is calculated in terms of two concentrated poles, without reference to the infinite number of poles of which it must be composed. We have so far considered and defined the magnetic force at points external to the iron of the magnet, but as our definition depended on force acting on a pole at the point, it cannot apply directly to points inside the iron unless a small cavity is scooped out of the iron. In making the necessary cavity, some of the molecular chains must be broken, and these will leave unneutralized poles which will contribute to the force in the cavity. The force in the cavity will depend on the strength of the magnetic particles which compose the iron; this strength is called the intensity of magnetization of the iron. If a vast number of magnetic particles are placed end to end and all pointing in the same direction, the result will be equivalent to a bar magnet which exhibits polarity over its end faces only: a real magnet may have unneutralized poles inside the magnet as well as on its surface.

The conditions inside a small cavity scooped out of the iron may be likened to those existing among a shoal of small compass-needles, a few of which have been removed to make a space for the exploration. The cavity is very large compared with an individual magnetic particle, but very small compared with the bulk of the whole magnet. Inside the cavity there will be a magnetic force whose magnitude will depend partly on local conditions arising from the exposed poles of the broken magnetic chains. The number of exposed poles will depend on the shape of the cavity which may have been cut across the chains or parallel to them. Now let the cavity be enlarged only in the direction of the force, the result being analogous to a worm-hole in wood. The direction of such a tunnel will follow the average direction of the magnetic particles, or in other words will be parallel to the magnetic chains. In such a tunnel, component poles in the chain will contribute no force because each one is neutralized by its immediate neighbour. The force in the tunnel will be due only to poles on the surface of the magnet and to any poles which were naturally within the magnet and to the poles of external magnets. In a piece of iron magnetized by induction it is the force from the external poles which maintains the orientation of the magnetic particles and arranges them into chains, and prevents them from going back to that arrangement which gives no resultant external magnetic effect anywhere; the arrangement natural to unmagnetized iron. The force experienced in such a tubular cavity is called the 'magnetic force' and is denoted by the symbol H.

Now let a thin, flat, disk-shaped cavity be cut at some point in the iron, the plane of the disk being perpendicular to the direction of the magnetic force. The flat walls of this cavity are formed by the exposed poles of broken magnetic chains. Since the cavity is very thin, these walls will behave sensibly as infinite planes to an isolated pole placed between them. Such a pole will be repelled by one face and attracted by the other. If the pole strength per unit area of one face is I, the repulsive force due to this face will be $2\pi I$ (see Chap. I, § 5) and the attractive force due to the other face is also $2\pi I$. So the broken magnetic chains contribute a force $4\pi I$ which is in addition to the magnetizing force H which arises from poles of the magnet and from poles and currents external to the magnet.

So the total force B on unit pole in the disk cavity is

$$B = H+4\pi I.$$

Now I, which is a measure of the ordered arrangement of the particles, must be a function of the magnetic force H which maintains the ordered arrangement. The relation between H and I is expressed by the equation

$$I = kH,$$

where k is called the magnetic susceptibility, and in all kinds of iron* k is not a constant but is a function of H.

* The word iron is used to denote any magnetic material: the only substances which are appreciably magnetic are iron, nickel, and cobalt and their alloys.

The total force B in a disk-shaped cavity is called the magnetic induction and it is usual to relate B and H by the equation

$$B = \mu H,$$

where μ is called the magnetic permeability; μ is not constant but is a function of H.

The magnetic force inside a magnet has very little meaning until a cavity has been made, because if the molecular interspaces are supposed to be explored by a minute test pole, the force experienced must vary between enormous limits for movements of molecular dimensions. Once a cavity has been cut, H or B can be observed and can be represented by lines or tubes of force. Every tube of force due to the I on the face of a cavity must end on a pole which has a chain of counterparts which can be followed till the final end of that molecular chain is reached and the tube branches out into open space. Hence, if an infinitely thin cavity is cut right across the mid plane of a magnet, the total flux of B crossing this gap will equal the total flux of tubes from either end of the magnet. Or again, suppose a certain area is chosen on the end of a magnet and the tubes ending on this area are counted. If a flat cavity is then made just inside the end, the same number of tubes will cross it, and if this cavity is then supposed to be pushed right through the magnet from end to end, the same number of tubes will always be present and the appearance would be that the tubes were followed through the magnet. So tubes of force outside the magnet may be completed by imaginary tubes inside the magnet, called tubes of induction. They are imaginary in the sense that they cannot be considered to exist till a cavity has been made. If the iron is pictured as a homogeneous material, not having a granular structure of magnetic particles, then these tubes or lines of induction can be regarded as lines of force running through the material and made circuital outside the magnet. According to this view the magnetic flux through a given area inside the iron is μ times as great as it would be if the iron were removed and H remained unaltered everywhere. This view regards the iron as having the arbitrary property of increasing the magnetic flux associated with a given magnetizing force, and then μ is a mere physical property of the material and is not thought of as a statistical method of dealing in bulk with a vast number of poles. The elastic properties of different materials provide a close analogy; the same tensile stress produces different amounts of yield in different materials, and the whole effect is described by the constant called Young's Modulus, which is analogous to the permeability.

Some writers object to treating B and H as quantities of the same character, though they do not object to the equation $B = H+4\pi I$. Until forces are considered in materials which necessarily contain scattered elementary magnets, there is no reason to distinguish between B and H. If the granular structure of iron is ignored, then the flux from a given pole does depend on the medium as well as on the pole strength, but when the problem is regarded as a pole reckoned as an entity and a vast collection of poles reckoned collectively by means of the factor μ, the desirability for distinction disappears.

The engineer will picture lines of force surrounding a magnet and picture these lines as completed by magnetic chains inside the magnet; he may call these chains lines of induction if he likes, but he may certainly draw the lines of force going through the magnet. Once the chain notion has been grasped clearly, no mistake need be made.

2. Magnetic potential inside a magnet

Now B has been defined as the force on a test pole which is placed inside a disk-shaped cavity. This is perfectly definite and explicit, but the reader should remember that it was the formation of the cavity which created the definite force B. Thus B is the force inside a cavity inside the iron, but it is not the force inside the iron not in a cavity. Though the cavity is small, it must be large compared with molecular dimensions. But the force inside the iron surely ought to mean the force in the molecular interspaces. If the force in these interspaces could be explored with a test pole it might still be considered to have two components. One component would be due to H as defined already, and the other component would be due to the polar ends of the immediately neighbouring magnetic particles. The second component would change enormously in value over distances small compared with the size of a molecule, and the average value of the force would not be B but would be H.

The true potential at a point in the molecular interspaces must change rapidly with minute changes in the position of the point, but the average value of the potential in a small volume region must depend only on H. If we speak of an equipotential surface within the iron we can refer only to average values and are bound to ignore the violent jumps which must occur in very small distances. Thus a true equipotential surface in the iron might be like that shown in Fig. 72, where the distance x is very small, say, a ten-thousandth of a millimetre. The only thing to do is to smooth out the irregularities, as shown by the thick line, and regard the thick line as part of the equipotential surface. So in speaking of the potential V within a magnet we mean explicitly that

$$-\frac{\partial V}{\partial s} = H.$$

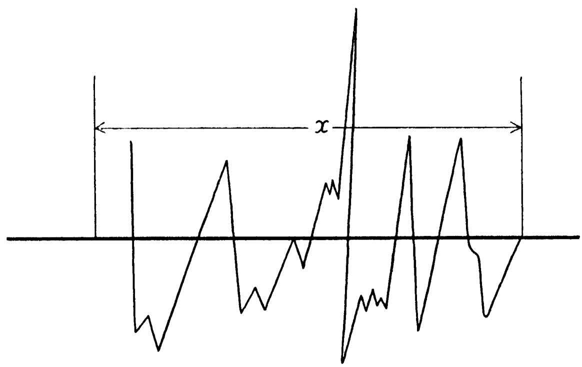


FIG. 72

To describe B it is necessary to remove some iron and form a disk cavity. Since it is a cavity free of iron, there can be no poles inside it, so if any Gauss surface (see Chap. I, § 4) is formed inside the cavity, that surface cannot enclose any free poles. So by Gauss's theorem as many lines of force must pass out of the surface as pass into it, and therefore the lines of B are continuous everywhere; this is often described by calling B a solenoidal vector. (See p. 195.)

Outside the magnet we picture lines of force which end on the surface poles, and we picture these are continued as lines of B inside the iron.

We have pictured magnets as a bundle of filaments each of which is a chain of particles; there has generally been a tacit supposition that these chains persisted right up to the surface of the magnet and none ended within the volume. In scooping out a disk cavity magnetic chains have to be severed, and there was a tacit assumption that each face of the cavity contained the same number of broken chains. But if some chains do not persist to the surface of the iron there will be polar ends within the volume, and one face of the cavity may contain some chain ends which were there naturally and were not produced by severance. But H is defined as that fraction of the total force within the cavity which is due to currents or to polar ends not created by scooping the cavity. Hence, if a magnet contains any poles within the volume, H will change abruptly in very small distances; H will not be continuous and will not be a solenoidal vector. Because B is a solenoidal vector it does not necessarily follow that H is also.

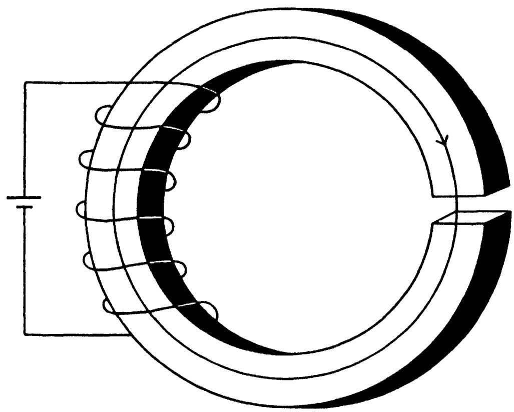


FIG. 73

The permeability μ has been defined by the equation $B = \mu H$, and we know that for iron μ is very far from constant and is a function of H. If μ were constant, H would always be the same fraction of B, and since B is always solenoidal H would be also. Thus we realize that an iron magnet usually has a volume as well as a surface distribution of polarity, and that a constant permeability implies the impossibility of anything but a surface distribution of poles.

The magnetic qualities of a specimen of iron are obtained by testing a ring specimen: in this, symmetry prevents either a surface or a volume distribution of poles, and in these circumstances H happens to become a solenoidal vector.

3. Work law applied in a magnetic medium

Consider an iron bar wound with a magnetizing coil, as shown, for example, in Fig. 73. Consider the work done in taking a unit pole round any closed path, mainly in the iron. The magnetic field at any point inside the iron is due to the magnetizing coil and the field of the exposed ends of magnetic chains, most of which are concentrated on the flat ends; the same is true of the magnetic force at any point not in the iron. The net work done against the forces contributed by the poles is zero, just as it would be for any system of permanent magnets (see Chap. I, § 2). The net work done against the forces contributed by the coil is $4\pi i$. So the work law

$$\int H\,dl = 4\pi i$$

is true universally and does not depend on the permeability of the medium, nor is it affected by a change of μ from point to point.

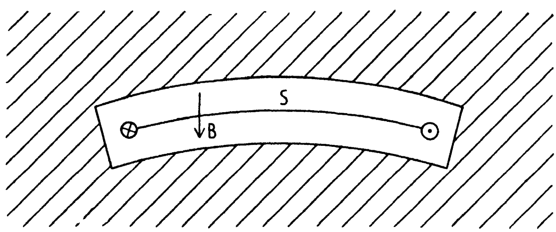


FIG. 74

4. Force on a conductor buried in iron

A cavity must be formed in which to embed a conductor in iron, and in this cavity the magnetic force is $H+4\pi I = B$. So the force on the conductor is Bi per unit length and not Hi. Thus consider Fig. 74 in which S is a surface with a circuit for its rim and an air cavity is cleared round the circuit to make a place to contain it. The direction of the field is shown by the arrow in the diagram. Each conductor lies in a field $B = H+4\pi I$, and so the force on each is Bi per unit length. The potential energy of the coil is i times the flux through it, and the flux through the coil is clearly increased by the iron. So the self-inductance of a coil buried in an infinite block of iron is μ times its self-inductance in air.

5. Permanent magnetism and the relation between B and H

So far we have supposed the magnetic chains retain an ordered formation as the result of a superposed external magnetic field; but this cannot be entirely correct, for if it were, a permanent magnet could not exist. Thus a bar magnet has a surface distribution of polarity which results from the terminal ends of magnetic chains, and these chains remain unbroken. The chains lie in a superposed field resulting from the surface distribution of polarity, and consideration will show that this field has a direction which tends to reverse the links of the chain. Hence molecular chains have some tendency to retain their formation once they have been formed, and this effect is called permanent or remanent magnetism. The physical mechanism which causes it cannot be fully explained; were there no such effect, permanent magnets could not exist.

We will now consider the relation between the magnetic force and the magnetic intensity it produces. The magnetic force has been defined as the force within a long worm-hole cavity bored along the direction of magnetization; this force arises from the surface poles on the piece of iron which is being magnetized and also from any external superposed field. In an experiment to determine the relation between H and I or between H and B it is necessary to avoid the surface distribution of poles, because their contribution to H cannot well be calculated since the distribution of polarity is unknown. It is essential to use an anchor ring of iron and to magnetize it by a uniformly wound toroidal coil. Then from the symmetry of the coil and its core there will be no surface poles, and all the molecular chains will be closed on themselves and form circular rings. Then the magnetic force will be due only to the current in the coil, and it is calculable exactly in terms of the current strength and the number of turns of wire.* The stronger the current the more nearly will the links of the chain be brought into line, end to end with one another, and I will approach asymptotically to a limiting or saturation value.

6. Induced voltage method of obtaining the cyclic curve

Consider the circuit and arrangement shown in Fig. 75. The current through the magnetizing winding can be taken from a positive maximum through zero to a negative maximum, by moving the sliding contact down the resistance R. The magnetization of the iron will be changed by this process and consequently the flux through

* The reader should realize clearly that in the equation $B = \mu H$, H refers to the magnetic force in a worm-hole cavity in the iron and does not refer to the magnetic force which would be found at the same spot if all the iron were removed. With a ring solenoid the value of H would not be altered by removing the iron, because the symmetry of the ring precludes surface or volume polarity.

the secondary S will be altered. The change of flux will produce in S a voltage equal at any instant to the rate of change of the flux, and this voltage will cause a current to flow in the circuit and be indicated by the galvanometer G. Now B was defined as the force in a disk-shaped cavity cut across the direction of H, and here the direction of cutting would be radial. Ignoring the small difference between the inside and outside radius of the anchor ring, symmetry shows that B will be constant at all points of any radial cross-section. So the flux through the secondary S must be B lines per unit area and the total flux ϕ through the area A is $\phi = BA$.

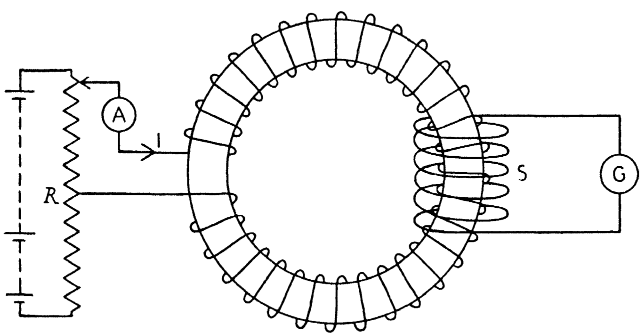


FIG. 75

It must be possible to change the current in such a way as to make $\frac{d\phi}{dt} = A\frac{dB}{dt}$ constant, though this will not result from changing the current at a uniform rate. If the current is changed in such a manner as to make $\frac{dB}{dt}$ constant,* then G will show a steady deflexion during the change. We shall suppose the observer moves the sliding contact in such a way as to maintain a steady deflexion of G; if the deflexion flags, the movement must be accelerated and vice versa. Let the movement be made so as to produce a steady voltage V, then

$$V = A\frac{dB}{dt} \times 10^{-8}.$$

* The reader should be particular to realize that the induced voltage is proportional to $\frac{dB}{dt}$ and not to $\frac{dH}{dt}$. In the process of forming the magnetic chains, the components *links* must swing round and pass their lines of force through S.

Hence
$$B = B_m - \frac{V}{10^8 A} t,$$

where B_m is the initial value of B. If the current is changed from I to $-I$ in time τ, then B will change from B_m to $-B_m$. Hence it follows that
$$B_m = \frac{V}{2 \times 10^8 A} \tau.$$

Hence, by measuring V and A and observing τ, it is possible to find the value of B which results from a known value of H, calculated from the particular value of I which was used. The reader may wonder why it is necessary to change the current from $+I$ to $-I$ instead of from $+I$ to zero: permanent magnetism makes it necessary, and this will become clear shortly.

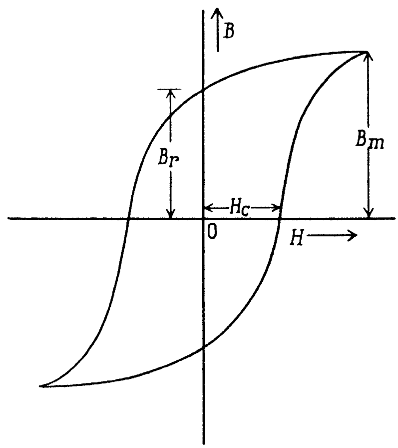


FIG. 76

The requisite rate of movement of the sliding contact is far from uniform; it must be rapid in the initial stages and slow in the final stages of the process. Suppose the current is changed from $+I$ to $-I$ and back again to $+I$, and that a record is kept of current and time during the process. Since B can be calculated after a lapse of time t from the equation
$$B = B_m - \frac{V}{10^8 A} t,$$

and since I is also recorded as a function of time, it is possible to plot a curve of B and current, and hence of B and H. If this is done the result will be typified by Fig. 76, and it may be seen that B is not proportional to H and that the value of B for a given H depends on whether B is decreasing or increasing. When H is zero B is not zero, but has a value B_r, and this measures the permanent or remanent magnetism of the particular grade of iron, when it has been magnetized to that particular maximum B. It will also be noticed that H must be given a negative value H_c in order to remove the once existing positive B_m. The whole curve is called a cyclic curve and H_c is called the coercive force: the positive and negative halves of the curve are exactly similar in shape and size. If another cyclic curve is obtained for a larger value of B_m it will have the same general character but the hook will be more pronounced. It is found that the cyclic curve for a given value of B_m does not become unique until the current has been changed from I to $-I$ and back to $+I$ several times: the iron is then said to have attained the cyclic condition, and then this condition is not disturbed until B_m is changed

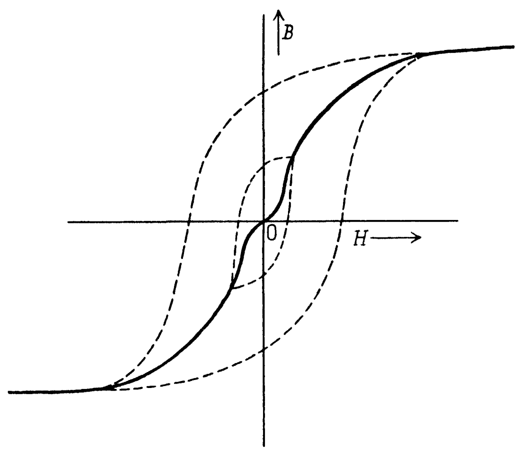


FIG. 77

to a larger value. The cyclic curve is also called a hysteresis loop, and the whole effect of permanent magnetism and a loop is referred to as magnetic hysteresis.

7. Magnetization curves

We have just seen that the relation between B and H is not definite but depends on the previous history of the iron; it is also not unique because the magnetization follows the hysteresis loop. It is usual to exhibit the magnetic qualities of iron by a curve relating the maximum B to the maximum H when the iron is in a cyclic state for the particular cycle having that maximum B; such a curve is the locus of the tips of an infinite number of cyclic curves and is typified by Fig. 77, in which two cyclic curves are also shown dotted. This curve is much less laborious to obtain than the whole loop: it could be obtained by the method described by Fig. 76 by adjusting the current to some particular value I_1 and noting the time taken to change this to $-I_1$, while maintaining a constant induced voltage V; this suffices to determine the value of B_m corresponding to I_1. The current should now be increased to some value I_2 and the iron taken round several cycles of magnetization to put it into the cyclic state. The value of B_m corresponding to I_2 can then be determined and the process continued for various suitable currents.

8. The flux meter

The method just described for determining the cyclic curve and reversal curve is both instructive and practicable, but on the whole

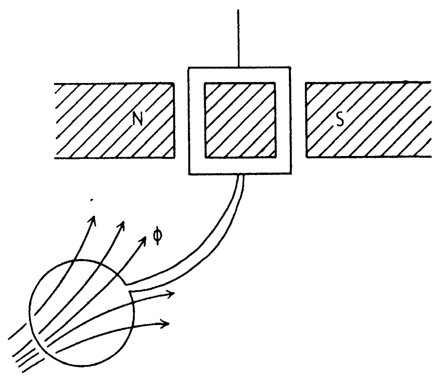


FIG. 78

it is inconvenient. The usual method is to deduce the flux change by observing the fling of a ballistic galvanometer connected in place of G in Fig. 75. It is not proposed to go into the details and technique of magnetic testing, but we will now describe a very useful instrument called a flux meter. It is a pointer instrument whose appearance is very like a milliammeter (see Fig. 103) and whose scale is engraved to read directly in change of flux turns. It can be used as conveniently as an ordinary ammeter, and once it has been described we need say little more in this book about methods of measuring flux. The reader will then know of an instrument for reading flux directly and can picture any necessary measurements being made by it. In practice there are some measurements which are made more suitably by other means, just as some measurements of current are made more suitably by means other than a simple ammeter.

The flux meter consists essentially of a coil mounted in a uniform radial magnetic field, and to that extent it is indistinguishable from a moving-coil galvanometer. But instead of the coil being mounted on pivots with hair springs, it is suspended on a silk fibre which exerts no control on its movement. The ends of the coil are brought to two terminals by connexions which must also be so flexible that they do not restrain the coil from twisting. The system is shown in diagrammatic cross-section in Fig. 78; the terminal connexions are made by two open helices of soft silver wire, wound respectively right- and left-handed, and the net effect of these two springs can be made negligible. Consequently, if any current, however small, is maintained in the coil, the coil will twist until it has moved out of the magnetic field. Fig. 79 is a photograph showing the suspension system of an actual instrument. The two silver spirals are clearly visible and the silk fibre runs along their axis. The pointer is seen above the clamp which runs through a square frame attached to the coil. Let the terminals of the coil be connected to an external search coil through which a flux can be changed: this is shown diagrammatically in Fig. 78. Let the total resistance of the whole circuit, moving coil and search coil, be R, and let L be the total self-inductance of the whole circuit. If the flux ϕ is changed, there will be an induced e.m.f. and a current will flow during the changing process. This current will produce a couple tending to turn the suspended coil, and any movement of this coil will generate an e.m.f. because it is cutting the field of the permanent magnet. Let the coil of n turns each of area A be suspended in a uniform radial field of strength B: then, by Chap. I, § 11 (*e*),

$$T = inAB,$$

and by Chap. II, § 2,

$$E = nAB\frac{d\theta}{dt}$$

$$= \frac{T}{i}\frac{d\theta}{dt}.$$

If the search coil has N turns, the equation of current flow is

$$N\frac{d\phi}{dt} = L\frac{di}{dt} + Ri + E.$$

If the moment of inertia of the coil is J,

$$T = J\frac{d^2\theta}{dt^2}$$

$$= J\frac{d\omega}{dt}.$$

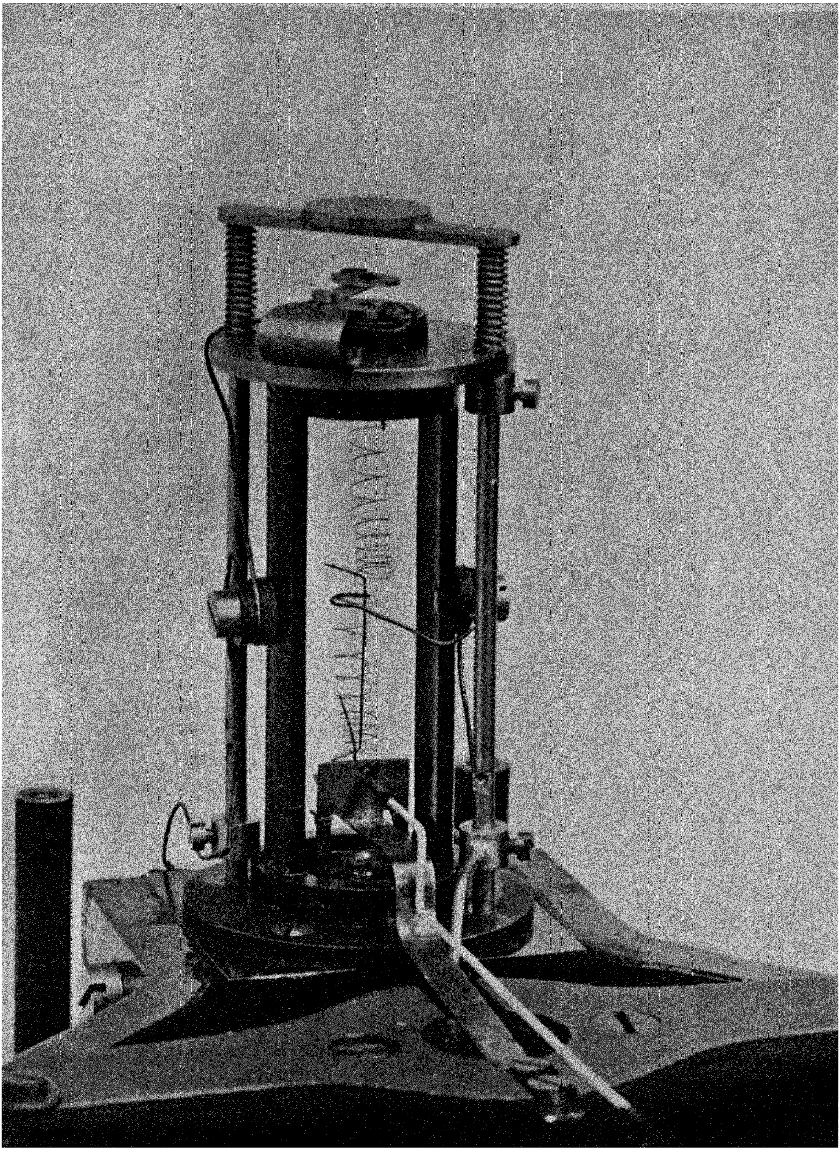


Fig. 79

$$\therefore \quad i = \frac{J}{nAB}\frac{d\omega}{dt}.$$

$$\therefore \quad N\frac{d\phi}{dt} - nAB\frac{d\theta}{dt} = L\frac{di}{dt} + \frac{RJ}{nAB}\frac{d\omega}{dt}.$$

$$\therefore \quad \int\limits_0^t \left(N\frac{d\phi}{dt} - nAB\frac{d\theta}{dt}\right) dt = L\int\limits_0^t \frac{di}{dt}\,dt + \frac{RJ}{nAB}\int\limits_0^t \frac{d\omega}{dt}\,dt.$$

$$\therefore \quad N(\phi_1 - \phi_2) - nAB(\theta_1 - \theta_2) = L(i_1 - i_2) + \frac{RJ}{nAB}(\omega_1 - \omega_2).$$

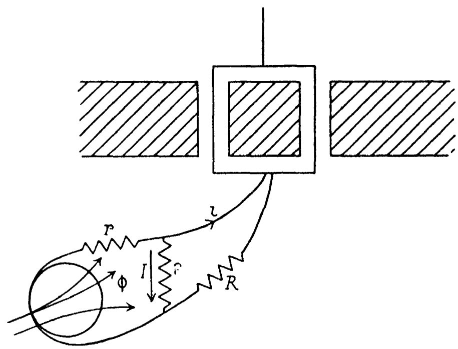


FIG. 80

But both i and ω are zero at the start and at the end of the change of ϕ, hence

$$N(\phi_1 - \phi_2) = nAB(\theta_1 - \theta_2).$$

Hence we find that the angular movement of the suspended coil is proportional to the change of flux through the search coil of N turns, and thus the scale of the flux meter can be engraved to read directly in flux turns. If a given number of lines are withdrawn from the search coil, then the suspended coil will turn until it has embraced as many lines of force as were withdrawn from the search coil; the total flux through the whole circuit remains constant. The instrument has no definite zero, and the pointer remains at the place it came to rest when the previous measurement was made; consequently the relevant reading is equal to the difference between the initial and final position of the pointer.

The range of the instrument can be extended by shunting the search coil: thus, let the suspended coil have a resistance R and the search coil a resistance r and the shunt a resistance ρ. If for simplicity the inductance is ignored, we have (see Fig. 80)

$$N\frac{d\phi}{dt}-r(I+i) = Ri-E$$
$$= \rho I,$$
$$\therefore \quad I = \frac{Ri-E}{\rho}.$$

Hence
$$N\frac{d\phi}{dt}+\frac{\rho+r}{\rho}E = \left(\frac{r+\rho}{\rho}\right)Ri.$$
$$\therefore \quad N\frac{d\phi}{dt}+\frac{\rho+r}{\rho}nAB\frac{d\theta}{dt} = \left(\frac{r+\rho}{\rho}\right)\frac{RJ}{nAB}\frac{d\omega}{dt}.$$

Hence
$$N(\phi_1-\phi_2) = \frac{\rho+r}{\rho}nAB(\theta_1-\theta_2).$$

So the shunting effect depends only on the ratio of the resistance of the shunt to the resistance of the search coil and is independent of the resistance of the suspended coil.

The unshunted flux meter has the remarkable property that the readings are quite unaffected by the resistance of the search coil.

9. Measurement of flux by ballistic galvanometer

Consider the circuit of Fig. 75 and let G be a ballistic galvanometer. Let the current in the magnetizing winding be changed suddenly from I to $-I$, then the flux through the secondary S will be changed rapidly from $B_m A$ to $-B_m A$, and during this process of change a current will flow through G. If the change takes place in a time which is small compared with the time period of the galvanometer, the coil will not have moved appreciably until the current has ceased to flow; the transient current gives an impulse to the coil which slowly swings out to a maximum and performs a damped simple harmonic motion (see Chap. II, Fig. 52). It is well known* that the amplitude of the motion is proportional to the quantity of electricity which passes through the coil during the transient flow of current; we shall now show that this quantity is proportional to the change of flux through the secondary S. Let the flux through the secondary S, of N turns, be ϕ at time t, and let R be the resistance and L the inductance of the whole circuit. Then

$$N\frac{d\phi}{dt} = Ri+L\frac{di}{dt}.$$

* See, for example, C. G. Lamb, *Notes on Magnetism*, § 19.

$$\therefore \quad N\int_0^t \frac{d\phi}{dt}\,dt = R\int_0^t i\,dt + L\int_0^t \frac{di}{dt}\,dt.$$

$$\therefore \quad N(\phi_1-\phi_2) = RQ,$$

where $Q = \int_0^t i\,dt$ is the quantity of electricity which passes through the galvanometer; $L\int_0^t di$ is zero because the transient current starts and finishes at zero value. Hence Q is proportional to the change of flux, which consequently is measured by the fling of the galvanometer.

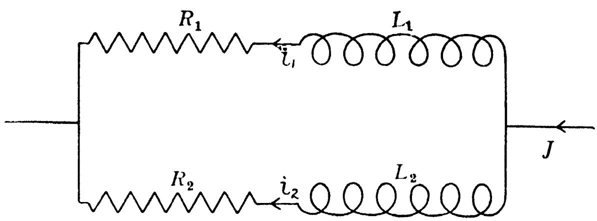


FIG. 81

meter. This fling may be calibrated in terms of flux by means of a standard field, which may be either a calibrated permanent magnet or a mutual inductance of calculable value. If R is measured in ohms and Q in coulombs, then

$$Q = \frac{N(\phi_1-\phi_2)}{R\times 10^8}.$$

The technical details of this method of measurement need not be described here. (See, for example, *Notes on Magnetism*, by Dr. C. G. Lamb.) It is, however, worth noting that a ballistic galvanometer may be shunted by a resistance and the shunting ratio does not depend on the inductance of the galvanometer or of the shunt. Thus, referring to Fig. 81, we have

$$L_1\frac{di_1}{dt}+R_1 i_1 = L_2\frac{di_2}{dt}+R_2 i_2$$

$$= L_2\left(\frac{dJ}{dt}-\frac{di_1}{dt}\right)+R_2(J-i_1).$$

$$\therefore \quad L_2\frac{dJ}{dt}+R_2 J = (L_1+L_2)\frac{di_1}{dt}+(R_1+R_2)i_1.$$

$$\therefore \quad L_2 \int_0^t \frac{dJ}{dt}\,dt + R_2 \int_0^t J\,dt = (L_1+L_2) \int_0^t \frac{di_1}{dt}\,dt + (R_1+R_2) \int i_1\,dt.$$

$$\therefore \quad R_2 Q = (R_1+R_2)q_1.$$

$$\therefore \quad q_1 = \frac{R_2}{R_1+R_2} Q.$$

In this figure $L_1 R_1$ may represent the galvanometer, whose coil has necessarily a considerable inductance: we have shown that the total quantity of electricity divides up between the galvanometer and the shunt in the same way that a steady current would divide.

10. Examples of magnetic reversal curves

Now that we are equipped with a flux meter it is a simple matter to obtain the reversal curve for any particular specimen of iron. An iron ring must be prepared with suitable dimensions, say 10 cm. inside diameter and a square section of 1 cm. side. This ring must be wound uniformly with a magnetizing winding having a known number of turns, and a suitable secondary must be provided which need not be wound uniformly. The flux meter must be connected to the secondary, and the magnetizing winding must be connected through an ammeter and a reversing key to a resistance potential divider. The process consists in plotting the flux-meter reading against the current reversed through the magnetizing winding, and the details of reducing the measurements will be apparent from a numerical example.

Thus suppose the iron ring has a mean diameter of 11 cm. and a cross-sectional area of 1·25 sq. cm., and is wound uniformly with a magnetizing winding of 330 turns and has a secondary winding of 100 turns. From the symmetry of the system the field is entirely inside the iron and there is no surface or volume distribution of poles, and hence H inside the iron is exactly the same as it would be if the iron were removed. The H is due entirely to the magnetizing current and can be calculated from the work law (see Chap. I, § 8 (*c*)). Let H be the value of the magnetizing force along the mean circumference, then $H \times 11\pi = 4\pi I \frac{330}{10}$, whence $H = 12I$.* We now have

* It is true that this is the mean value of H over the cross-section: along the inner circumference it is $12 \times \frac{11}{10 \cdot 6} I = 12 \cdot 5I$ and along the outer circumference it is $11 \cdot 5I$.

the relation between H and current and so can replot the curve of flux-meter reading and current in terms of flux-meter reading and H. Let the flux-meter calibration be 15×10^3 flux turns per division, and let the flux-meter reading change by 200 divisions when the magnetizing current is changed from $+0{\cdot}5$ amp. to $-0{\cdot}5$ amp. The flux density over the cross-section of the iron is sensibly constant, let its value be B: when the current is reversed this is changed from plus B

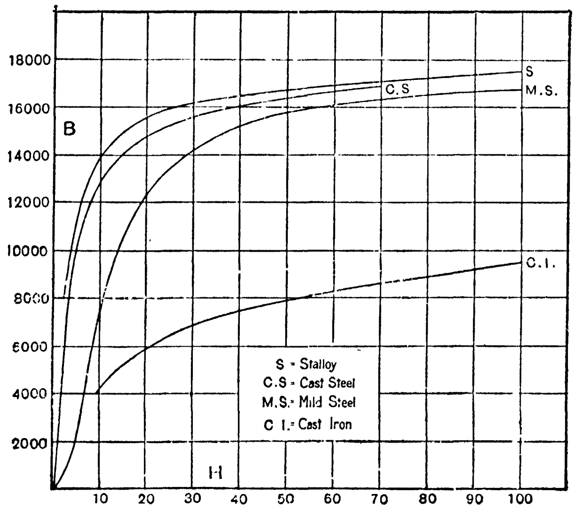


FIG. 82

to minus B, and so the change of flux turns through the secondary winding is $2B \times 1{\cdot}25 \times 10^2$. But the flux meter registers a change of $200 \times 15 \times 10^3$ flux turns, whence it follows that $B = 12{,}000$ lines sq. cm. We have now found the scale factor for converting flux-meter reading into B and so can plot the experimental curve in terms of B and H.

Various grades and alloys of iron can be made so as to give sensibly the same magnetic properties wherever and whenever they are made, but the magnetic qualities are very sensitive to the exact composition of the alloy. If (B, H) curves are obtained for, say, cast iron, wrought iron, hard steel, and the various alloy steels made for technical purposes, great contrast is found between their magnetic properties.

Fig. 82 shows the (B, H) reversal curves for cast iron, cast steel, mild steel, and an important silicon alloy known as Stalloy, and the difference between their properties is very noticeable. It is out of place in this book to discuss the magnetic properties of the different magnetic materials in common use; this will be reserved for the companion volume on dynamos. But it will be noticed from Fig. 82 that Stalloy and mild steel are magnetized with difficulty beyond, say, $B = 16{,}000$ lines sq. cm., and it is almost impossible to obtain this flux density in cast iron. The reader should realize at once that a flux density greater than 16,000 is very high for any kind of iron and can be obtained only at the expense of an excessive value of H. The

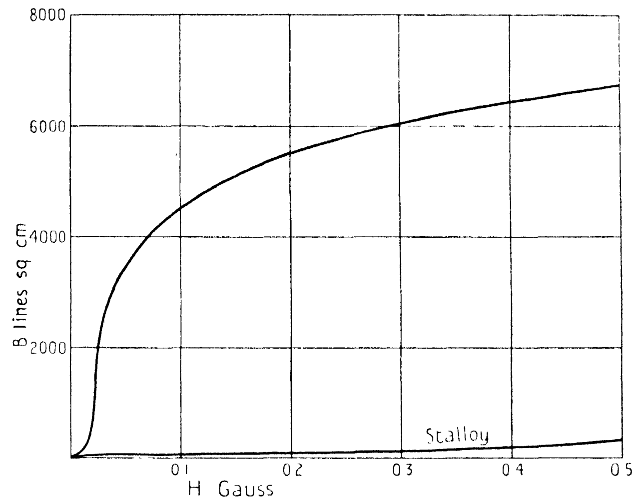


FIG. 83. (B, H) curve for Mumetal No. 6

improved magnetic qualities possessed by the various alloys are not so much an increment of the maximum obtainable B but a reduction of the H required to obtain a B of moderate value: a striking example of this is provided by the recently discovered nickel iron alloys. The (B, H) curve for the nickel iron alloy known as Mumetal No. 6 is shown in Fig. 83, and for comparison a Stalloy curve is drawn on the same graph: Fig. 84 shows a (B, H) curve obtained on a spirally wound strip core of Mumetal 0·015 in. thick, and represents about the best which can be done with this type of material.

The magnetic qualities of iron are profoundly affected both by mechanical stress and by temperature. If the flux density is small, say, less than 5,000 lines sq. cm., the magnetic qualities improve slightly with rising temperature, up to temperatures of about 650° C. But for large flux densities, say greater than 12,000 lines sq. cm., the magnetic qualities deteriorate slightly with rising temperature. In either case the change due to a temperature rise of 200° C. is negligible, and iron may be considered to have a negligible temperature coefficient of magnetism for temperatures attained in a dynamo or transformer. But between 700° and 800° C. there is a rapid decrease of magnetic qualities, and by 800° C. all iron has become non-magnetic. If a mass of hot, glowing iron is allowed to cool, there is a certain

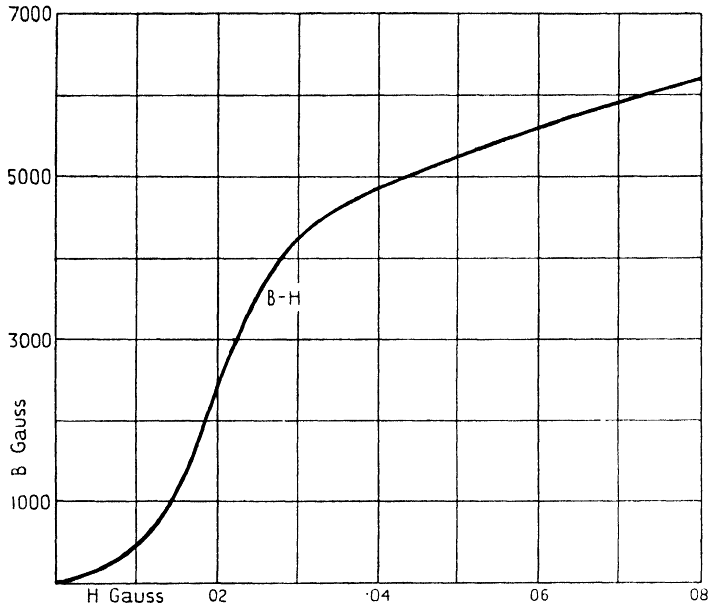


FIG. 84. (*B*, *H*) curve for spirally wound strip core of Mumetal 0·015″ thick

temperature at which the dull red iron begins to glow again: this is called the recalescence point, and the temperature is that at which the iron becomes magnetizable. Iron containing 12 per cent. of manganese and 1 per cent. of carbon is unmagnetizable, and this alloy does not show a recalescence point.

Mechanical stress in general decreases the magnetic qualities of iron, which, however, can be regained by annealing: this effect is often noticeable in the slot punchings for armatures, and the material Mumetal is very sensitive to mechanical treatment. In general it should be remembered that punching and machining tends to deteriorate the magnetic qualities by an amount which is by no means always negligible, as the dynamo-builder well knows.

The heat tempering of steels also has a marked effect on their magnetic qualities. Hardening decreases the remanent magnetism and increases the coercive force: for this reason it is essential to make permanent magnets of glass-hard steel. It might be thought that permanent magnets should be made of soft iron, because this material has the largest remanent magnetism; but this is associated with a vanishingly small coercive force which would be obliterated by weak fields in which the magnet might be situated. In order that a magnet shall retain its properties through the vicissitudes of its existence it

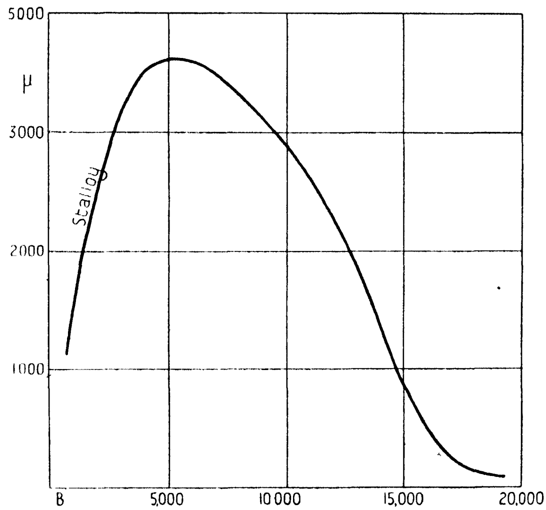


FIG. 85. (B, μ) curve for Stalloy

must have a very large coercive force. Tungsten and cobalt steels are used for permanent magnets: after magnetizing, the steel is subjected to a process of ageing by temperature changes and mechanical vibration, and such treatment seems to settle the material into a state which is very stable and difficult to upset except by drastic means.

11. Magnetic permeability

In § 1 of this chapter, permeability was defined by the equation $B = \mu H$, and it was stated that μ was not a constant. We have now seen that μ is a complicated function of H which is represented graphically by the cyclic curve, such as Fig. 76. It is usual to restrict the meaning of μ to the ratio B/H obtained from the reversal curve or what we shall commonly call the (B, H) curve of the material. The curve of B and μ for Stalloy and Mumetal, derived from Figs. 82 and 83, is shown in Figs. 85 and 86.

Since permeability varies so enormously with B, μ it is not a very useful factor in calculations, and a (B, μ) curve is useful only for a casual glance to show where the maximum occurs and the steepness of the peak. The reader should remember that Stalloy has a maxi-

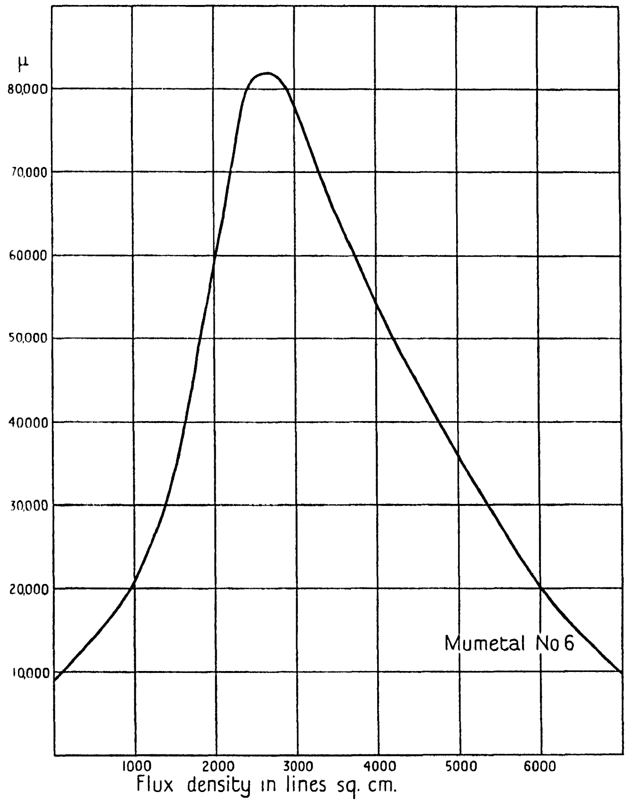


FIG. 86. (B, μ) curve for Mumetal No. 6

mum permeability of about 4,000, but beyond this he should avoid numerical values of μ. Permeability is often a useful term for brief description and conveys a useful idea. Thus we may find it convenient to say: 'It is a pity to work Stalloy at a low flux density, such as $B = 1{,}000$, because we are not then utilizing its full permeability; if so low a value of B is necessary we should use Mumetal which has a far greater permeability in this region.' The reader should studiously avoid working with numerical values of μ and should always work from the (B, H) curve.

12. The work absorbed by hysteresis

All iron has a tendency to remain a permanent magnet, and this tendency means inevitably that energy is absorbed when iron is taken round a magnetic cycle. The energy appears as heat in the iron, but the mechanism of conversion into heat is not understood; it may be described vaguely as molecular friction. But it is easy to show that work must be done in taking iron round a magnetic cycle, for consider Fig. 87 which represents an iron cylinder which is free to turn on an axle through O and which is mounted between the

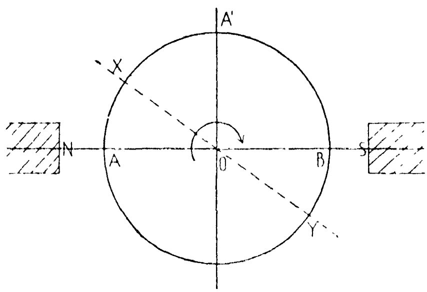


FIG. 87

poles N and S of a magnet. Suppose the cylinder is at rest, then a south pole will be induced at A and a north pole at B.

If the cylinder is turned through half a revolution, the point A will have moved to B and then there will be a north pole at A and a south pole at B. But we have seen that an appreciable demagnetizing force is required to reduce the magnetism to zero, and so the point A will retain some memory of its south polarity after it has passed the point A' in Fig. 87. Hence, when the cylinder is revolving, the centre of the south pole will not be at a point such as A but at a point such as that marked X in Fig. 87. Since X is not in line with N and Y not in line with S, there will be a backward drag on the revolving cylinder. Work must be done on the cylinder to keep it revolving uniformly, and this work goes to heat the cylinder: the couple required to turn the cylinder is independent of the speed of rotation. The apparatus just described is reminiscent of the Ewing hysteresis tester.*

It will now be shown that the work done is proportional to the area of the hysteresis cycle. Consider an iron ring wound uniformly

* See, for example, Dr. C. G. Lamb, *Notes on Magnetism*, p. 58.

with T turns and having a cross-sectional area A, and let the magnetizing current be taken from I to $-I$ and back again to $+I$. The arrangement is shown diagrammatically in Fig. 88. During the change of current there will be a self-induced voltage e, against which the current has to do work; ignoring the effect of resistance, the battery is working at a rate ei and the total work done is $W = \int_0^T ei\,dt$.

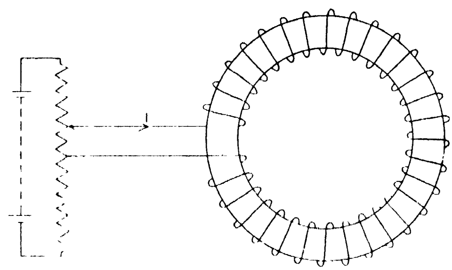


FIG. 88

Now $$e = \frac{AT}{10^8}\frac{db}{dt}\,V,$$

and $$H = \frac{4\pi iT}{10l},\ \text{if } i \text{ is in amperes.}$$

So $$\int_0^T ei\,dt = \int_0^T \frac{AT}{10^8}\frac{db}{dt} \times \frac{10lH}{4\pi T}\,dt$$

$$= \frac{Al}{4\pi \times 10^7}\int_0^T H\frac{db}{dt}\,dt$$

$$= \frac{Al}{4\pi \times 10^7}\int_0^B H\,db.$$

When taken round the whole cycle $\int H\,db$ is the area of the cyclic curve, and so

$$W = \frac{1}{4\pi} \times (\text{area of cyclic curve}) \times (\text{volume of iron}).$$

So h, the work in ergs absorbed in hysteresis per cubic centimetre per cycle, is equal to $h = \dfrac{\text{area of cycle}}{4\pi}$.

If the toroid had a non-magnetic core, say wood or air, then the reversal curve would be a straight line, since B is proportional to H, and the area of the loop is zero. The same amount of work is then given back by the field as was required to create it; but the field in an iron core gives back less work than was used to create it.

In the induced voltage test described in § 6 the current was changed in such a way as to keep the induced voltage constant; during part of the cycle this e.m.f. is opposing the current and during part of the cycle it is helping the current to flow. The secondary induced voltage was maintained at a constant value V, and since the same flux threads all turns whether primary or secondary, the e.m.f. induced in the magnetizing winding is equal to $\frac{S}{T}V$.

Now
$$W = \int_0^t ei\,dt = \frac{S}{T}V\int_0^t i\,dt$$
$$= \frac{S}{T}V \times \text{time} \times \text{mean current}.$$

So the induced voltage method of plotting the hysteresis cycle also gives simultaneously a method of determining the area of the cycle.*

The area of the hysteresis cycle for a given maximum flux density differs very greatly according to the variety of iron used.

The hysteresis loss in dynamos and transformers is very important because it means a loss of efficiency and also a temperature rise which heats the windings and may deteriorate the insulation: the maximum output of most machines depends mainly on the safe temperature rise.

It is very necessary to know the hysteresis loss for a cycle having any given maximum flux density, and to do this it is necessary to obtain several hysteresis cycles for a given kind of iron and to plot their area against the maximum flux density of the cycle; we then have a curve showing h, the hysteresis loss per cycle per unit volume, as a function of the maximum flux density. The process of obtaining the cyclic curve and finding its area is very laborious, and in practice the energy loss is measured more satisfactorily by the use of a wattmeter and alternating currents.

* The mean current is not zero, because it is positive for a longer period than it is negative.

13. The relation between hysteresis loss and maximum flux density

A curve relating h, the hysteresis loss per cubic cm. per cycle, and B_{max} is shown in Fig. 89. The form of this curve is found to be independent of the composition of the iron, and curves for all kinds of iron can be made to coincide by a suitable change of scale for h. This apparently fortuitous property is a great practical convenience and permits us to express h by the formula

$$h = \eta K \text{ ergs/c.c./cycle},$$

where K is a function of the flux density which is exhibited by Fig. 90, and η is a constant depending on the kind of iron: some representative values of η are shown in Table 1 below.

TABLE 1

Hysteresis constant for various kinds of iron

Material.	*Hysteresis constant η.*
Dynamo sheet steel	2×10^{-3} to 3×10^{-3}
4 per cent. silicon steel	1×10^{-3}
2·5 per cent. silicon steel	$2{\cdot}2\times10^{-3}$
Very soft iron	2×10^{-3}
Cast iron	11×10^{-3} to 16×10^{-3}
Cast steel	3×10^{-3} to 12×10^{-3}

The reader should remember that for iron used in machine construction h is of the order of 2×10^{-3}, and that for cast iron it is about five times as great.

The form of the lower portion of Fig. 89 suggests that the curve may perhaps be represented by an equation of the type $h = \eta B^n$, and logarithmic plotting shows that K varies as $B^{1{\cdot}6}$. This relationship is generally called 'Steinmetz law' and apparently was noticed first by Professor C. P. Steinmetz. Accordingly we have the useful relationship

$$h = \eta B_{max}^{1{\cdot}6} \text{ ergs/c.c./cycle},$$

which is valid for all kinds of iron or steel so long as B_{max} is less than about 17,000 lines sq. cm. This simple empirical relationship is very useful for making rapid estimates of the alteration of loss due to a given change of B.

For example, suppose a transformer is rated for 200 V. and it is desired to use it on a p.d. of 250 V., what will be the increase of temperature rise? A transformer is essentially an inductance, and

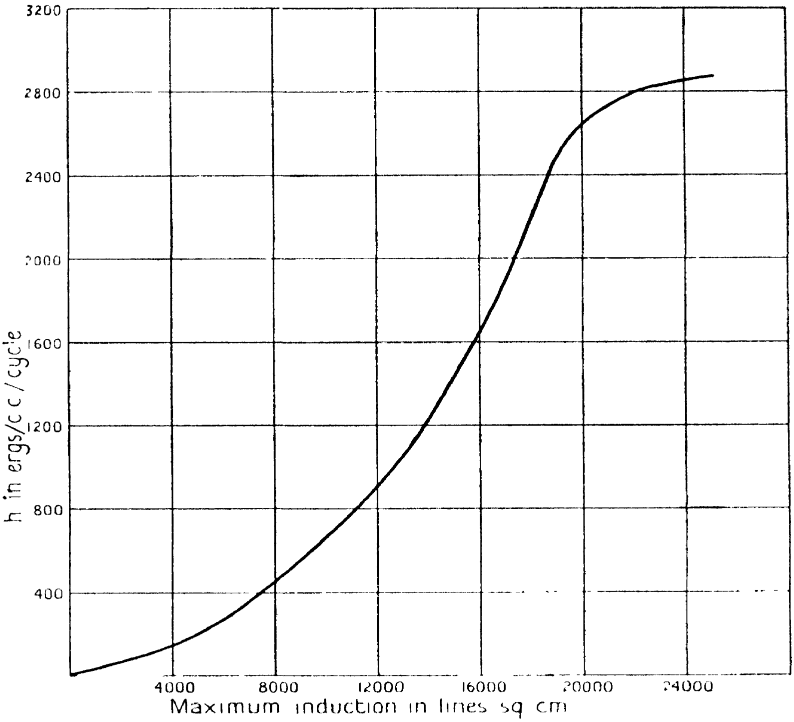

3200
2800
2400
2000
1600
1200
800
400
h in ergs/c c / cycle
4000
8000
12000
16000
20000
24000
Maximum induction in lines sq cm

FIG. 89

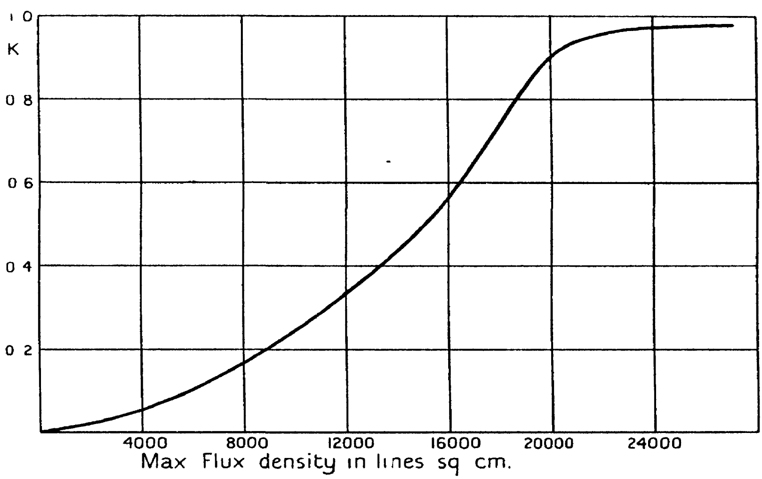

1 0
K
0 8
0 6
0 4
0 2
4000
8000
12000
16000
20000
24000
Max Flux density in lines sq cm.

FIG. 90

so the rate of change of flux turns through its windings must equal the applied voltage: hence the reader need scarcely be versed in alternating currents to realize that the maximum flux density will increase in the ratio of the voltages. Hence $B_{\max}$ increases in the ratio 1·25 and h increases in the ratio $(1{\cdot}25)^{1{\cdot}6} = 1{\cdot}43$. The engineer will know that the hysteresis loss in a transformer is about 40 per cent. of the total: so the new loss will be approximately

$$(0{\cdot}6+1{\cdot}43\times 0{\cdot}4) = 1{\cdot}17$$

of the original loss and thus the increase of temperature rise will be about 17 per cent.

It is instructive to calculate the temperature rise due to hysteresis. We will take $\eta = 10^{-3}$ and $B = 10{,}000$ lines sq. cm., density of iron 7·8 and specific heat 0·11.

Then $$h = 10^{-3}\times 10^{6{\cdot}4} \text{ ergs/c.c./cycle}$$
$$= 2500.$$

Then if θ is the temperature rise per cycle,

$$7{\cdot}8\times 0{\cdot}114\theta = \frac{2500}{4{\cdot}18\times 10^7}.$$

$$\therefore\quad \theta = \frac{6{\cdot}6}{10^5} \text{ degrees C.}$$

When an iron specimen is taken round a single cycle having $B_{\max} = 10^4$ lines sq. cm., there will be a temperature rise of about 10^{-4} degrees centigrade. The temperature rise due to a single cycle of magnetism is much less than can be measured: whether the temperature rise occurs suddenly at one point of the cycle or whether it is distributed over the whole process is not known. If the cyclic change is repeated uniformly at the rate of 50 cycles/sec., the temperature of the iron will rise 0·198° C. per minute.

The hysteresis loss depends on the manner in which the cyclic change is produced. In the arrangement shown in Fig. 87, the field remains fixed in space and the iron cylinder is revolved, thus moving any given volume of iron through the whole cycle of flux density: in this cycle the direction of magnetization in a given element of iron revolves uniformly. In taking a ring specimen of iron through a magnetic cycle (see Fig. 75), the direction of magnetization of any given element does not rotate but is fixed in space. The loss in the first system is often called the rotating hysteresis loss, and in the second system the alternating hysteresis loss. The hysteresis loss in soft iron when carried round a cycle by each of these means is shown in Fig. 91.

For very high flux densities, say greater than 18,000 lines sq. cm.,

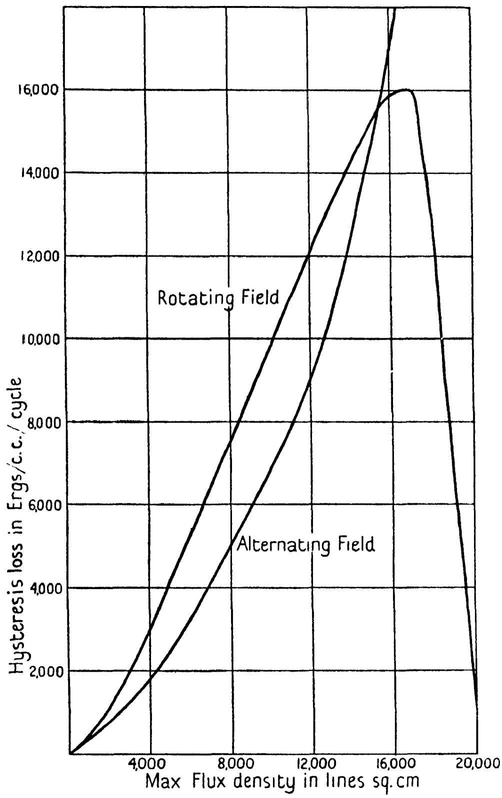


FIG. 91

the rotating hysteresis loss falls rapidly, and this effect is what would be expected from Ewing's molecular theory.

True rotating hysteresis seldom occurs in practice. It might seem it would occur in dynamos, but this is not so because the armature core is always pierced by a large hole, leaving an annulus of iron to carry the flux: conditions in this annulus are the same as in the ring coil. Some elements of the iron may go through a rotating cycle, while for the bulk it is purely alternating. Hence the hysteresis loss in an armature cannot be calculated precisely even though the quality of the iron is known; approximate calculations must be adjusted from the results of previous experience. In practice it is not possible to derive much advantage from the decrease of h which occurs in very intense rotating fields.

The Steinmetz law, $h = \eta B^{1\cdot 6}$, applies to alternating hysteresis loss.

14. Loops on the cyclic curve

Suppose a specimen of iron is in the cyclic state for a cycle shown by PQ in Fig. 92. We will suppose the magnetizing current is being changed from a negative maximum towards the positive maximum. When it has reached the positive value OA, let it be decreased to OB and then increased again to OA and then finally to the positive maximum OC. It will be found that the flux density follows a small subsidiary cycle such as that shown at the loop in Fig. 92. The shape of this loop cannot be found from the normal reversal curve of the iron and must be determined experimentally. A cycle of this character must be gone through in the iron of inductances which carry an alternating current superposed on a steady current: for example, in the smoothing inductances used in the filter circuits of rectifier units which supply a continuous current from an alternating supply.

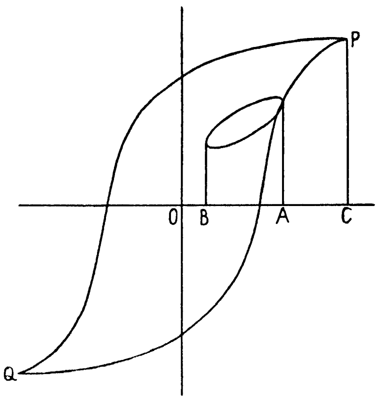


FIG. 92

15. Eddy currents in iron

When iron is taken through a magnetic cycle, eddy currents must be created, and these have been discussed in § 15 of Chap. II. It was shown there that the eddy loss in thin iron plates can be calculated from the formula

$$W_e = \frac{\pi^2}{6}\,\frac{B^2 t^2 n^2}{\rho \times 10^{16}}\ \text{W./c.c.}$$

The hysteresis loss per unit volume of iron is independent of the form of the specimen, but the eddy current loss can be reduced indefinitely by using thinner laminations and by increasing the specific resistance of the material. Not only have silicon and nickel iron alloys the property of high permeability but also they have a very high specific resistance, and this reduces the eddy loss. Some relevant values of specific resistance and density are given in Table 2 below.

TABLE 2

Material.	*Spec. res. in* $\mu\Omega$/cm.³	*Density.*
Soft iron	10	—
Armature iron	12	7·7
Stalloy	56–66	7·53
Mumetal No. 6	40–45	8·61
Special Mumetal	80	—

This table shows that for the same eddy loss, stampings in Stalloy can be about $\sqrt{6} = 2{\cdot}46$ times as thick as stampings of soft iron: this is a considerable advantage because a smaller fraction of the total volume is occupied by the paper insulation between the plates. Common thicknesses for stampings are 15 mils (0·4 mm.) and 20 mils (0·51 mm.).

The total loss in laminated iron can be expressed by the equation

$$W = \left(\eta B^{1\cdot6} n + \frac{\pi^2}{6}\frac{B^2 n^2 t^2}{\rho \times 10^9}\right)\frac{1}{10^7}\,\text{W./c.c.}$$

$$\doteqdot \left(\eta B^{1\cdot6} n + \frac{\pi^2}{6}\frac{B^2 n^2 t^2}{\rho \times 10^9}\right)\frac{5{\cdot}9}{10^6}\,\text{W./lb.}$$

Giving η the value 10^{-3} and ρ the value 60 $\mu\Omega$/cm.³, we have

$$W = \left(B^{1\cdot6} n + \frac{B^2 n^2 t^2}{36}\right)\frac{5{\cdot}9}{10^9}\,\text{W./lb.},$$

which shows that for frequencies of the order of 50 cycles/sec. and thicknesses of the order of 0·5 mm. the eddy loss will be much less than the hysteresis. When $n = 50$ cycles/sec., $t = 0{\cdot}5$ mm., and $\rho = 60\,\mu\Omega$/cm.³, then

$$W_e = \frac{B^2}{10^9}\,\text{W./lb.}$$

Manufacturers of magnetic materials usually publish curves showing the total loss at a given frequency for a given thickness of plate. Such a curve for 0·5 mm. Stalloy plates at 50 cycles/sec. is shown in Fig. 93, and the lower curve in this figure is the eddy loss calculated from the formula $W_e = \frac{B^2}{10^9}$ W./lb. It may be seen that the eddy loss in such plates accounts for about 10 per cent. of the total. A similar curve for the nickel iron alloy, Mumetal No. 6, is shown in Fig. 94.

As a numerical example, take $B = 10{,}000$ lines sq. cm. and the hysteretis constant $\eta = 10^{-3}$. Then $W_h = 0{\cdot}75$ and $W_e = 0{\cdot}1$, making $W = 0{\cdot}85$, and this happens to agree precisely with the value shown by the Stalloy curve: when $B = 15{,}000$, $W = 1{\cdot}44+0{\cdot}225 = 1{\cdot}665$, whereas the curve gives $1{\cdot}85$.

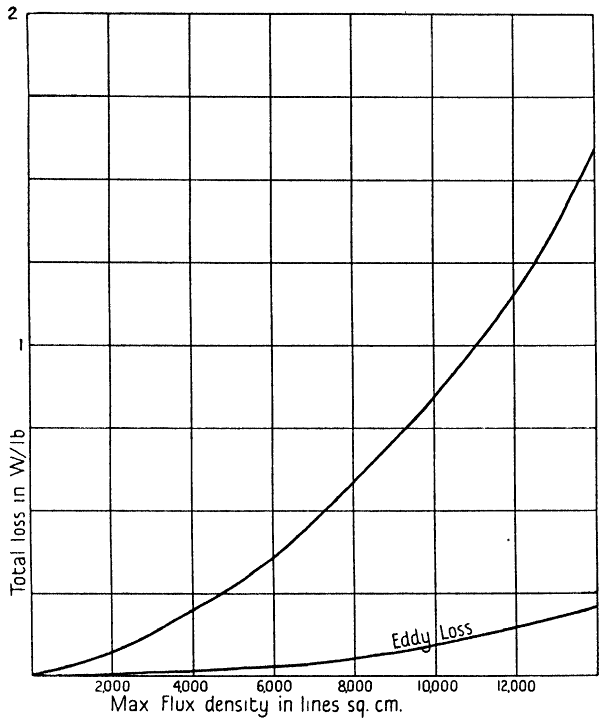


FIG. 93. Energy loss in W./lb. for Stalloy plates 0·5 mm. thick at 50 cycles/sec.

If the plates are used in the armature of a dynamo it is always found that the eddy loss is much larger than that calculated as above, and instead of the constant having the value $\frac{1}{6}\pi^2 = 0{\cdot}165$ it is more nearly twice this value. The discrepancy is due in part to the rotation of the flux which occurs in some parts of the iron and also to the burring of the inside of the slots which makes successive laminations imperfectly insulated from one another.

Since in plates of given thickness magnetized to a given flux density the total loss has one component varying with n and another varying with n^2, we have a simple way of separating experimentally the hysteresis loss from the eddy current loss. Thus suppose a ring specimen is magnetized by alternating current and that the total loss is measured by a wattmeter: if the voltage and the frequency are changed in the same ratio the maximum flux density will remain

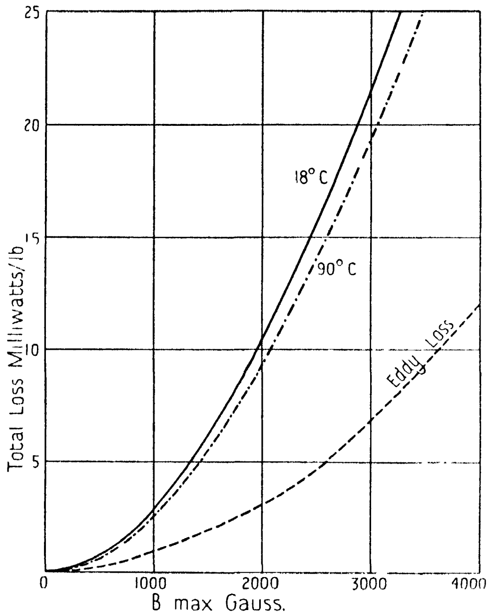


FIG. 94. Energy loss in Mumetal No. 6 at 50 cycles/sec.

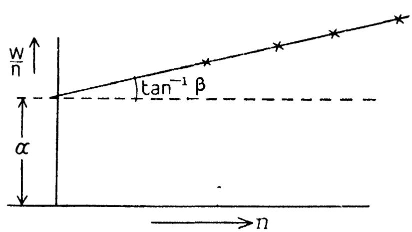


FIG. 95. Curve relating W/n and n for constant flux density B

unaltered. Thus, since B is constant, the total loss will be related to the frequency by an equation of the form

$$W = \alpha n + \beta n^2$$

or

$$W/n = \alpha + \beta n.$$

So if W/n is plotted against n the result should be a straight line as shown in Fig. 95, from which α and β can be determined.

The eddy loss varies as n^2 only so long as the field due to the eddy currents is negligible compared with the applied field. In a material with constant permeability μ, it can be shown that the eddy loss varies sensibly as n^2 so long as the factor $2\pi t\sqrt{\frac{\mu n}{\rho}}$ is less than unity. Since the permeability of iron is very far from constant the eddy loss in iron plates cannot be calculated exactly. However, we may approximate to the criterion by giving μ the maximum possible value. Thus for Stalloy we have

$$2\pi t\sqrt{\frac{3000n}{60000}} < 1$$

or

$$t\sqrt{n} < 0{\cdot}71;$$

if $t = 0{\cdot}5$ mm., then n must be less than 200 cycles/sec., or if $n = 50$ cycles/sec., then t must be less than 1 mm. For Mumetal we have

$$2\pi t\sqrt{\frac{80000}{40000}n} < 1$$

or

$$t\sqrt{n} < 0{\cdot}118.$$

That is, if $n = 50$ cycles/sec., t must be less than $0{\cdot}166$ mm. $= 7$ mils.

At frequencies much above the criterion stated, the eddy loss will be found to vary as $\sqrt{n}$: this is of practical importance when the iron is used for the core of transformers used to carry currents of speech frequency.

16. Ring coil with concentrated winding

The iron specimens used to determine the (B, H) curves were uniformly wound anchor rings, because the symmetry of this arrangement ensures that all the flux is inside the iron and there is no polarity on its surface. For this reason B had the same value all round the ring and H was calculable exactly because at every point it was due to the current only. But when an iron ring is to be magnetized for almost any purpose, except determining the (B, H) curve, the exciting coil is generally concentrated on a short piece of the circumference and it is unusual to spread it uniformly. The symmetry of the arrangement has now been lost and there certainly will be some field outside the iron: this necessitates surface polarity on the iron, and till its amount is known H cannot be calculated. So it is impossible to calculate the field at any specified point inside the iron; but because iron happens to have a permeability of the order of thousands, it is possible to make a very good approximation. We will begin by considering a hypothetical experiment. Suppose an iron ring is wound uniformly and that a flux meter is connected to a concentrated search coil: suppose the loosely wound magnetizing turns are gradually pushed round the circumference till they occupy the position shown in Fig. 96: during this process the magnetizing current I is kept unchanged. As the magnetizing coil is gradually bunched together we should expect less flux to pass through the search coil S and hence that the flux-meter needle would move continuously as the magnetizing coil is shut up on itself. Our expectations would be fulfilled, but on the other hand we should find the total change of flux was a very small fraction of the total flux through S. We thus make the very important experimental discovery that the flux through a search coil such as S is substantially independent of the position of the magnetizing winding. If the ring had had a non-magnetic core, say of wood, the change would have been enormous, but with an iron core the change is negligible. The discovery that the flux round an iron ring is almost independent of the disposition of the magnetizing winding is very important to the electrical engineer.

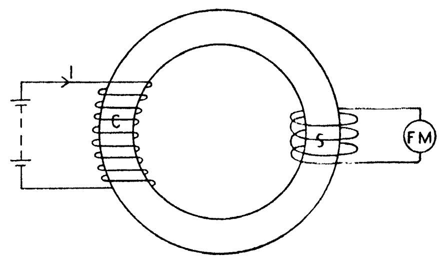


FIG. 96

We stated the change of flux would be negligible, but the thoughtful reader should inquire what we consider negligible; is it merely that the change was less than we expected? It is impossible to state a general figure for the change because this depends on the flux density and also on the shape of the ring, but in general the change of flux would be of the order of 2 per cent. and would certainly require careful measurement to detect.

This fortunate property of an iron ring has had a great effect on the development of electrotechnics: had it been otherwise it is very improbable that dynamo design would have been brought to an exact science.

In Part II of this chapter we shall attempt to reconcile this discovery with our knowledge of magnets and the magnetic qualities of iron, but at present we shall accept it without question and see how it allows us to obtain approximate solutions of some very intricate problems.

Since we realize that the flux round a ring such as that in Fig. 96 is very nearly constant, we can make a very close estimate of the number of ampere turns required to produce a given flux. So long as the radius of the cross-section is small compared with the radius of the ring, it is found that the flux density is substantially constant over any area of cross-section: in these circumstances we shall describe the total flux divided by the area as the flux density in that cross-section. Let it be required to produce a flux density B through the flux-meter coil in Fig. 96, and let reference to the reversal curve for the iron show that this requires a magnetic force H. Round any closed path threading the current we know that $\int H\,dl = \frac{4\pi IT}{10}$, but in this problem the magnitude and direction of H is known only at the point where B is specified. But since we find that the flux density is substantially constant and uniform, we realize that the lines of magnetic force must be approximately circles, as they would be if the magnetizing winding was distributed uniformly. Therefore, as a first approximation, we shall calculate the ampere turns in the same way we should do if the winding was distributed uniformly. Thus

$$IT = \frac{10}{4\pi} \times \text{mean circumference} \times H.$$

A second approximation can be made by assuming that the flux density increases uniformly with distance in proceeding round the ring from the flux-meter coil to the magnetizing winding, and that the total increment is some assigned amount, say 5 per cent. The process will be made clear by a numerical example: thus suppose it is required to obtain a flux density of 10,000 lines sq. cm. through the flux-meter coil in a ring whose radius of cross-section is 1 cm. and whose mean circumference is 30 cm. If the iron is Stalloy this will require an H of 3·5 at the place where B is 10,000: assuming that B is constant, we then find that $IT = \frac{10}{4\pi} \times 30 \times 3{\cdot}5 = 84$. Now suppose the flux density increases uniformly to a maximum of 10,500 lines sq. cm. in the magnetizing winding, then H there must be 3·8 and the mean value of H is 3·65: hence on this assumption the necessary ampere turns must be increased by 4·3 per cent. In general the necessary ampere turns will exceed that calculated from the assumption of uniform flux density by less than the difference between the maximum and minimum flux density in the ring, and by experiment this difference is small.

There is another way of proceeding which depends on the general proposition that the lines of force will dispose themselves so that for a given number of ampere turns the flux through the coil is a maximum. Thus we might postulate a certain flux density in the coil and assume this diminished round the ring according to some

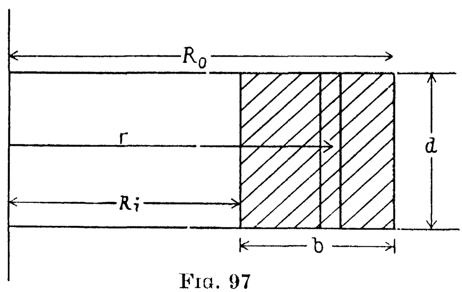


FIG. 97

assigned law. We could then find the distribution of H appropriate to this assigned law and so find $\int H\,dl$. We could then distribute the flux in some other assigned way and recalculate $\int H\,dl$: if the second value of this integral was smaller than the first, then the second approximation would be closer to the correct answer. But the problem is seldom of sufficient importance to require this elaborate procedure.

Before going farther the reader should pause and realize that the assumption of no leakage flux is impossible, but ignoring its magnitude does not make a serious error in the numerical calculation.

17. Difference between mean flux density and the density at the mean radius

In the numerical examples of magnetic testing with a uniformly wound anchor ring it was assumed that $B = \phi/A$, where ϕ is the total flux through the cross-sectional area A. This value gives the mean flux density, but B cannot be quite constant over any given cross-section because H is less along the outside circumference than along the inside. We will now estimate the degree of approximation involved in this process. Let Fig. 97 represent the cross-section of a ring specimen of constant permeability and let H be the magnetizing force at radius r, then

$$2\pi r H = 4\pi I T.$$

$$\begin{aligned}
\phi &= d\int_{R_i}^{R_0} B\,dr \\
&= \mu d\int_{R_i}^{R_0} H\,dr \\
&= \mu d\int_{R_i}^{R_0} \frac{4\pi I T}{2\pi r}\,dr \\
&= 2IT\mu d\log_e\frac{R_0}{R_i} \\
&= 2IT\mu d\log_e\left(1+\frac{b}{R_i}\right) \\
&= 2IT\mu d\frac{b}{R_i}\left(1-\frac{b}{2R_i}+\frac{b^2}{3R_i^2}\cdots\right), \text{ if } R_i > b \\
&= \frac{2IT\mu d b}{R_i+\frac{b}{2}}\left(1+\frac{b}{2R_i}\right)\left(1-\frac{b}{2R_i}+\frac{b^2}{3R_i^2}\cdots\right) \\
&= \mu H_m A\left(1-\frac{1}{12}\frac{b^2}{R_i^2}-\frac{1}{12}\frac{b^3}{R_i^3}\cdots\right),
\end{aligned}$$

where H_m is the magnetizing force along the mean circumference. So if $R_i > 5b$, then ϕ/A gives the value of B appropriate to H_m with an error of less than 0·3 per cent. Since μ is not a constant, the difference between B_i and B_0 will be less than we have supposed and ϕ/A will be an even closer approximation to the value of B along the mean circumference.

18. Magnetomotive force

The line integral $\int E\,dl$ of electric force round a closed path is called electromotive force, and similarly $\int H\,dl$ is called magnetomotive force and written m.m.f. With every m.m.f. there is associated a flux, and for a uniformly wound ring solenoid we have

$$4\pi I T = Hl = \frac{Bl}{\mu}.$$

$$\therefore\quad \text{m.m.f.} = \phi\frac{l}{\mu A}.$$

This is the same form of equation as that for current flow in a conductor of length l and cross-section A, or for heat flow due to a temperature difference, and μ corresponds to thermal or electric conductivity. But whereas these last are constant, μ varies over an enormous range, and therefore the equation is of very little practical utility. Nevertheless, the expression $\frac{l}{\mu A}$ corresponds to a magnetic resistance and it is called the Reluctance of the circuit. But the reader will be well advised if he refuses ever to use numerical values of reluctance, because the process obscures the fundamental assump-

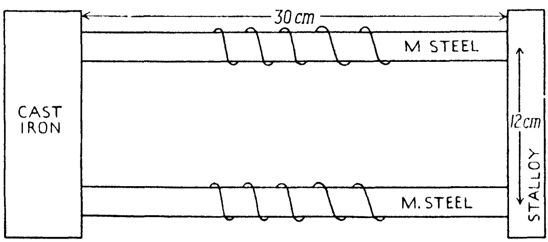


FIG. 98

tions and principles of the calculation. In designing a magnetic circuit the first thing is to choose the flux density in each portion and then make the areas of cross-section such as to carry the total flux with these assigned values of B: the range of choice for B is small because iron is used wastefully if B is less than about 8,000, and it requires an excessive H to obtain a B greater than 16,000. Having chosen B, the appropriate value of H can be found from the reversal curve and then the value of $\int H\,dl$ can be found at once. There is no advantage in working in terms of reluctance. The process is analogous to the design of a steel structure: there the first thing is to find the load which each member must carry, and this corresponds to finding the total flux required. Then the designer fixes on the tons per square inch, corresponding to B, which he thinks desirable for the particular kind of steel and for the type of load to be carried by the member in question; then he makes the area such as to carry the load with that stress intensity.

A numerical example should make clear the process of designing a magnetic circuit. Suppose the magnetic circuit, shown in Fig. 98, is to carry a flux of one megaline (10^6 lines), estimate the necessary ampere turns and cross-sectional areas. Although this magnetic circuit is not a simple ring, and although it is composed of different materials, we shall again assume the leakage flux is so small a fraction of the total that its presence makes a negligible addition to the total ampere turns. We will start by making $B = 8{,}000$ in the mild steel, 14,000 in the Stalloy, and 4,000 in the cast iron: from the magnetization curves we find the relevant values of H happen to be 10 for each material. Hence $\int H\,dl = 10\times 30+10\times 12+10\times 30+10\times 12 = 840$, and so $IT = 670$. The path length in the end pieces is very uncertain, but evidently 14 cm. would be a closer estimate, and then we should have found that $IT = 700$. The necessary areas are $A_{MS} = 125$ sq. cm., $A_{CI} = 500$ sq. cm., $A_S = 71$ sq. cm. If the mild steel pieces are to be round rods their radii must be 6·5 cm., so they cannot be placed 12 cm. apart. Increasing the flux density in them to 16,000 would allow their radii to be reduced to 4.5 cm., but this would entail about 5,000 ampere turns. So either the frame shown cannot carry a megaline of flux or else the centre distance must be increased to, say, 20 cm. If this is done, $\int H\,dl = 600+400 = 1{,}000$. Other considerations intrinsic to this particular magnet and its purpose may dictate slightly different dimensions, but those chosen suit the magnetic qualities of the materials. Throughout the process we have worked with B as the independent variable. If the areas are given, the first thing is to find the flux density in each portion of the magnetic circuit, and when this has been done we have gauged the magnetic qualities of the problem. If the reluctance method is used, B never appears explicitly and the whole calculation becomes a mere piece of arithmetic exhibiting nothing but the final answer. It is surely undesirable tacitly to ignore the degree of magnetization possible for the iron when this should be the guiding-point of the whole calculation. Furthermore, the basis of the reluctance method is the supposition that the leakage flux is a small fraction of the total, and this is substantially correct only when the flux density is within certain assigned limits.

Reluctance is a useful word to describe what otherwise would require a sentence. Thus in the previous numerical example we can say that increasing the area of mild steel will reduce the total reluctance, and thereby state briefly that the same flux can be produced by fewer ampere turns if the mild steel has a larger area. In similar contexts reluctance is a useful word in the technical vocabulary, but the reader will be well advised to restrict its use to such purposes.

19. Ring coil with an air gap

If a narrow gap is cut across an iron ring, as shown in Fig. 99, very strong poles are formed on the flat faces of the slit and the total flux through the magnetizing coil is much less than before the gap was made. The term leakage flux has been used to denote the flux due to surface polarity on the iron, and accordingly the reader may expect it to include the flux from the surfaces of the air gap; but in this problem the meaning of the term is restricted to that flux only

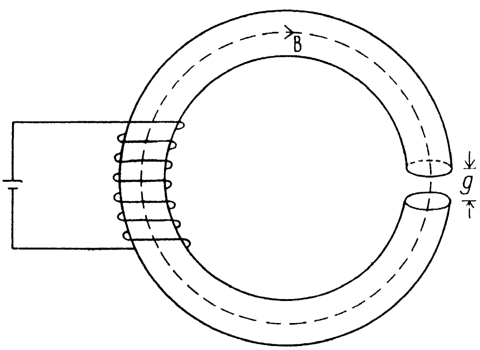


Fig. 99

which springs from the surface polarity on the curved surfaces of the ring and explicitly excludes the flux from the flat faces of the air gap. When the gap is cut the leakage flux does not increase enormously but the flux through the magnetizing coil does decrease enormously: if the leakage flux is expressed as a fraction of the total flux through the magnetizing coil, then its increase is very considerable, but this is due very much more to the decrease of total flux than to the increase of leakage. We shall now calculate approximately the flux density in the air gap and see what causes the large decrease of total flux.

If the gap length is small compared with a side of the cross-section, the flux density over most of the faces of the gap will be constant; near the edges of these areas the lines of force will bulge, and here there will be a smaller flux density, but experience and calculation agree in showing that this is appreciable only over a strip of width $\frac{1}{2}g$, running round the boundary of the gap face. In taking a unit pole round a closed path following the mean circumference, the total work done falls into two distinct categories. In crossing the gap the force is constant and very large because it is due to the polarity on two large flat opposing areas; inside the iron the force is not constant and increases slightly as we pass from the gap to the coil, but its magnitude is very small compared with that in the air gap because inside iron the H is very small compared with the B it produces. Expressing the work law in symbols for the path described, we have

$$\frac{4\pi IT}{10} = Bg + \int H\,dl,$$

where B is the flux density in the air gap and H is the magnetizing force at some point in the iron. Since H in the iron is due mainly to an unknown distribution of leakage flux, $\int H\,dl$ cannot be found, and we can say only that it is greater than Hl, where H is the magnetizing force corresponding to the air gap flux density B. However, as a first trial let us make $\int H\,dl = Hl$ so as to find the order of this quantity compared with Bg. Then

$$\frac{4\pi IT}{10} = Bg + Hl$$

$$= Bg\left(1+\frac{1}{\mu}\frac{l}{g}\right).$$

Since μ is of the order of 2000, $\frac{1}{\mu}\frac{l}{g}$ is less than 10 per cent. of unity so long as g is greater than $\frac{l}{200}$. If the length of the gap is an appreciable fraction of the length of the iron, then Hl is negligible compared with Bg. Experience shows the flux density round the iron does not increase enormously and therefore $\int H\,dl$ cannot well be more than, say, twice Hl: hence, whatever be the unknown value of $\int H\,dl$, we conclude it will have little effect on the calculated value of the flux density in the air gap. Therefore B will be calculated from the approximate formula

$$\frac{4\pi IT}{10} = Bg\left(1+\frac{1}{\mu}\frac{l}{g}\right) \doteqdot Bg,$$

but in using this we shall not fall into the error of supposing that the leakage flux is a negligible fraction of the flux through the magnetizing coil. If the magnetizing current is kept constant, the flux density in the air gap should be inversely proportional to the gap width, and experiment shows this is correct to a very high degree of accuracy.

Before a gap is cut across the iron, the flux density in the place where it will be cut is $\frac{B_1 l}{\mu} = \frac{4\pi IT}{10}$, and after the cut it is approximately $B_2 g = \frac{4\pi IT}{10}$: hence cutting a gap reduces the flux density there in the ratio $\frac{B_2}{B_1} = \frac{l}{\mu g}$, if the magnetizing current is not altered.

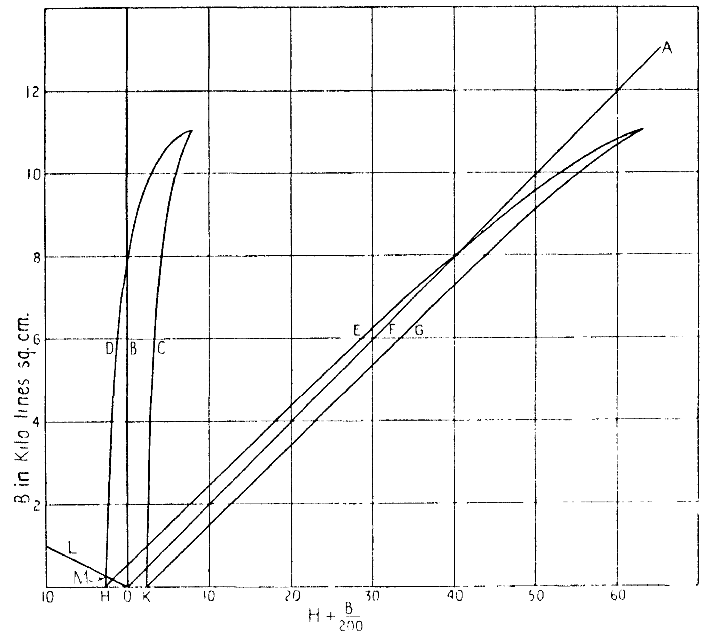


FIG. 100

In Part II of this chapter we shall try to understand how it comes about that H is substantially constant all along the iron portion of the path.

20. Hysteresis loop for an iron ring with an air gap

With the same assumption that H is sensibly uniform all along the iron part of the magnetic circuit, it is simple to derive the relation between magnetizing current and flux density when an iron ring, having a small air gap, is carried round a magnetic cycle having an assigned value of B_{max}. Thus

$$4\pi IT = Hl + Bg.$$

$$\therefore \quad \frac{4\pi IT}{l} = H + \frac{g}{l}B.$$

For any given value of B there are two values of H which can be read from the cyclic curve of the iron appropriate to the chosen value of B_{max}, and thus can be found the magnetizing current required to produce this value of B both on the ascending and on the descending side of the cycle. The process should be made clear by Fig. 100, which shows the cyclic curve for a certain kind of iron magnetized to a maximum flux density of 11,000 lines sq. cm. It also shows the cyclic curve for a ring of this iron having an air gap whose length is 0·5 per cent. of the length of the iron. The line OA is drawn with a slope such that $H = B/200$: to obtain the two points on the loop corresponding to, say, $B = 6000$, make $FG = BC$ and $EF = DB$, then E and G are two points on the loop. The loop for the circuit with an air gap is obtained by shearing over the cyclic curve so that it is plotted with OK and OA as axes. The presence of the air gap very much reduces the difference between the two values of B corresponding to any particular magnetizing current, and the larger the air gap the more nearly the two sides of the loop tend to coincide: it should be clear that the area of the loop is unaltered by the shearing process.

If the iron is in the cyclic state and if the current is brought to its maximum value and then broken, the remanent flux density will be 500 lines sq. cm. in contrast to 8,000 lines sq. cm., which it would have been if the air gap had not existed. When the magnetizing current has been broken the m.m.f. round the ring is zero, and so $Hl+Bg = 0$. It is not necessary to plot the whole cycle in order to obtain the remanent flux density, for we have only to use the cyclic curve to solve graphically the equation $-\frac{Bg}{l} = H$. Thus, to find the remanent B when $g/l = 1/100$, draw through O the line OL, such that $\tan LOH = 1/100$ and this cuts the cyclic curve at the point M where the flux density is approximately 250 lines sq. cm. Since the sides of the cyclic curve rises almost vertically, the remanent flux density may be estimated approximately from the formula

$$B_{rem} = \frac{l}{g} \times \text{coercive force.}$$

21. The magnetic potential meter

Let Fig. 101 represent a piece of iron magnetized in the direction of the arrows: at any point between A and B there is a magnetic force, and the difference of magnetic potential between these two points is the work done in carrying a unit pole between them; that is, $V_{AB} = \int_A^B H\,dl$. But the difference of potential between A and B is independent of the path followed so long as that path does not thread

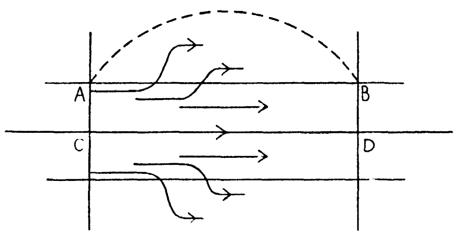


FIG. 101

a current. Thus, if $\int_A^B H\,dl$ is measured along the path shown dotted in the figure, the result will be the same as if the path had been $ACDB$. Outside the iron the H at any point is evidently due to the leakage lines, or surface polarity, whichever term is preferred.

It is possible to use a flux meter to measure the magnetic p.d. between any two points, and the arrangement is due to Professor A. P. Chattock (see *Phil. Mag.*, 1887, vol. xxiv, p. 94). Let a flat strip, about 30 cm. long, of flexible material be wound uniformly with fine wire so as to form a long solenoid of rectangular cross-section S having N turns per unit length, and let the two ends of the winding be taken to a flux meter. To measure the magnetic p.d. between the two points AB, take the flexible strip solenoid and bend it so that the ends rest on the points A and B; then move the strip right away out of the field and in this process the flux meter will give a reading which is proportional to the magnetic p.d. between A and B. The ends of the solenoid may be likened to two terminal spikes connected to a voltmeter and used to measure the electric p.d. between any two points on a conductor.

The solenoid is shown diagrammatically in Fig. 102 and the whole equipment in Fig. 103. Let H in Fig. 102 be the magnetic field, due to surface polarity, etc., at some point P inside the flat solenoid: then the flux through a turn at P is $HS\cos\theta$, and the flux turns in a short length dl are $SNH\cos\theta\,dl$. The total flux turns threading the

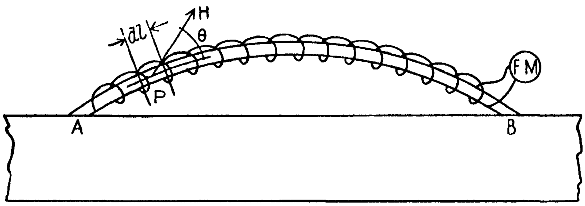


Fig. 102

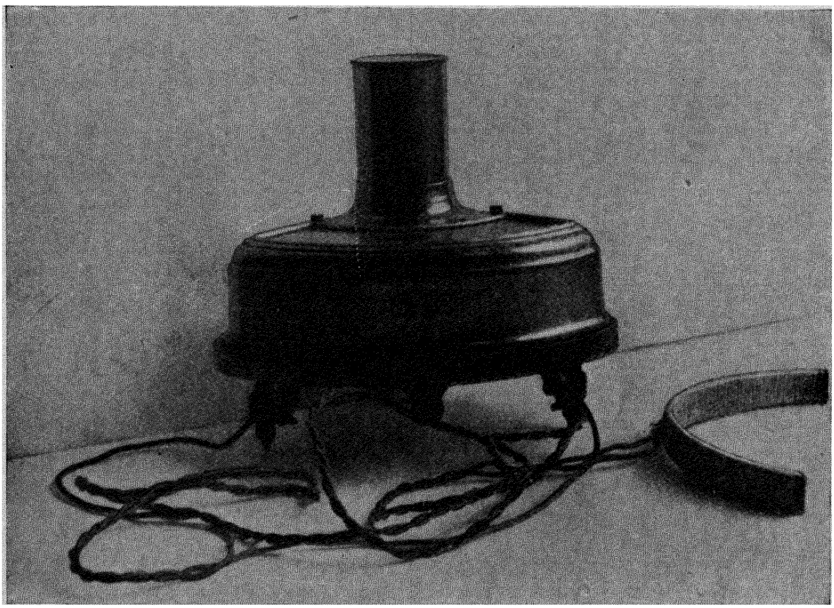


Fig. 103

solenoid from A to B are

$$\begin{aligned}\phi T &= \int SNH\cos\theta\,dl\\ &= SN\int H\cos\theta\,dl\\ &= SN \times \text{magnetic p.d.}_{AB}.\end{aligned}$$

$$\therefore\quad \text{Magnetic p.d.}_{AB} = \frac{\phi T}{SN}.$$

The cross-section of the solenoid must be small, otherwise its ends do not approximate to points: it is usually made flat so that its dimension in the direction AB is small and the larger dimension is set perpendicular to the estimated direction of the flux in the iron.

Magnetic p.d. may be measured in dyne cm. units, but there is no name for a larger unit to correspond with the volt in electricity. It is common practice to measure magnetic p.d. in hybrid units of ampere turns, because nearly all magnets are electromagnets and for every unit of magnetic p.d. there must exist somewhere in the magnetic circuit a corresponding number of ampere turns. The relation between dyne cm. units and ampere turn units is

$$\frac{4\pi IT}{10} = Hl = \text{magnetic p.d.}$$

So with the magnetic potential meter,

$$\text{Magnetic p.d. in ampere turns} = \frac{\text{flux meter reading in flux turns}}{\frac{4\pi SN}{10}}.$$

To estimate the sensitivity of such an instrument, suppose the flux-meter calibration is 15,000 flux turns per division and that the flexible strip is $4 \times 0{\cdot}3$ cm. wound with 50 turns/cm. (S.W.G. 40 D.S.C. wire). Then the magnetic p.d. in dyne cm. units $= \frac{15000}{60} \times$ flux-meter reading, or the reading of the flux meter is 250 units of p.d. per division or 200 ampere turns per division. So the solenoid and flux meter is thoroughly insensitive and is of little use except for measuring the p.d. across the air gap of a ring.

III

IRON IN A MAGNETIC FIELD

PART II

It is now proposed to amplify some of the statements made in the first part of this chapter and add some problems which are of secondary importance to the general reader: the following pages should be read with close attention by those who are particularly interested in electromagnetism but may be omitted by those who are content with mastering the broad principles of the subject. These pages have been extracted from their natural places in Part I in the hope of assisting the reader to gain first a general perspective. They are concerned mostly with reconciling alternative methods of solving a problem, and this is a very valuable exercise for those minds which delight in approaching each problem armed only with the most fundamental principles together with their mechanical model describing its logical consequences.

1. Force between poles submerged in a medium of constant permeability

So far everything has been derived from the supposition that two poles m and m' exert a force on one another equal to mm'/r^2, where the distance r between them may be occupied by any material. But the law of force is often expressed by the equation $F = mm'/\mu r^2$, where μ is the permeability of the medium in which the poles are submerged. This is a correct expression for the law of force, but it appears to broaden the initial mystery of magnetism since the force is not a unique property of nuclear poles but depends also on the medium in which these poles happen to be situated. It is possible to regard the force in this way, but it is surely a simplification to consider that every pole can exert a certain definite force which in all circumstances depends only on the distance and is quite independent of the intervening medium: when we take this standpoint we find that the presence of μ in the formula is due only to the convenient device of reckoning the effect of a myriad nuclear poles by means of the factor called permeability.

Before considering the force between two poles submerged in a medium of permeability μ, we will consider the somewhat simpler problem of a flat iron plate placed across what was previously a uniform field of strength h. The uniform field h will induce a uniform south-polar intensity I over the face AB (see Fig. 104) and a similar north-polar intensity over the face CD. The field at a point such as P in Fig. 104 is now the resultant of the original field h and the field due to the surface polarity over the faces AB and CD. By § 5 (a) of Chap. I, the surface AB contributes at P a force $-2\pi I$ and the face CD a force $2\pi I$, and hence the field at any external point is unaltered by the presence of the plate. Now consider the force inside a cavity such as that depicted by $EFGH$. The total force experienced by a unit pole inside such a cavity is that denoted by the symbol B and

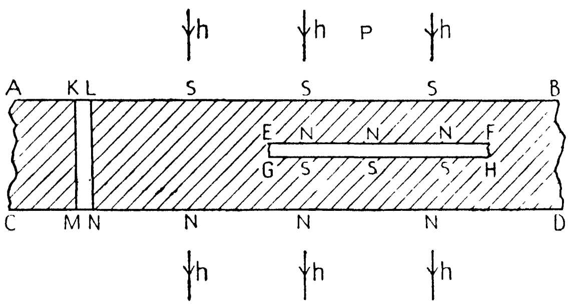


FIG. 104

has been split up (see pp. 108–11) into a contribution from the broken magnetic chains and a contribution from all other polarity in the region. The broken magnetic chains on the face EF contribute a force $-2\pi I$ and those on GH a force $-2\pi I$. The force from the other poles in the region is h from the original field, $+2\pi I$ from the surface AB and $+2\pi I$ from the surface CD. Hence

$$\begin{aligned} B &= -4\pi I + (h + 4\pi I) \\ &= h. \end{aligned}$$

And so we find the induction inside the plate is the same as it would be at the same point of space if the plate were removed.

Consider now the force inside a small hole drilled through the plate, such as that shown by $KLMN$ in Fig. 104. No magnetic chains have been broken by making this hole, and hence the force inside it is that due to the poles in the neighbourhood and is that which has always been designated by the symbol H (see p. 109).

But $B = \mu H$, and we have just shown that $B = h$, whence

$$H = \frac{h}{\mu}.$$

So the presence of the iron reduces the magnetic force inside it to $\frac{1}{\mu}$ of its previous value. But the reduction of force is not due to some fresh and unexplained property of the iron but is due to the surface polarity induced on the surfaces of the plate, and this in turn is due to the orientation of the magnetic particles which form the iron. The intensity of this surface polarity depends on the quality of the iron, as described by the parameter μ. We can find the intensity of this polarity as follows: the force, in the hole, due to the polarity on AB and CD is (see Chap. I, § 5 (b)) $-4\pi I$, whence

$$H = h-4\pi I.$$

$$\therefore \quad I = \frac{1}{4\pi}(h-H)$$

$$= \frac{h}{4\pi}\left(1-\frac{1}{\mu}\right).$$

If a unit pole is taken along a path parallel to the field, the work done will be

$$W = hx+\frac{ht}{\mu}$$

$$= h\left(x+\frac{t}{\mu}\right),$$

where x is the distance in air and t is the thickness of the plate.

Now consider an isolated north pole, spherical in form, which is submerged in an infinite medium which has a constant and uniform permeability; the magnetic particles of the medium will arrange themselves into chains which are all radial from the pole. Since μ is constant there will be no unneutralized poles in the medium (see p. 113). Let a small cavity be scooped out of the medium having two walls which are parts of spherical shells centred on the pole; the arrangement is indicated in Fig. 105. Each of these spherical walls will have a uniform intensity of magnetization due to the ends of the magnetic chains which were broken in its formation. The continuity of the magnetic chains, or in other words the uniformity of μ, demands that the intensity I at a radius r must be such that $4\pi r^2 I = M$, where M is the aggregate strength of the south poles adhering to the surface of the original north pole of strength m. In the cavity the induction will be radial, and its value B will be due to the surface magnetization on the walls of the cavity together with the force from the south poles of aggregate strength M and the force from the original pole of strength m.

Hence
$$B = 2\pi I_1 + 2\pi I_2 + \frac{m}{r^2} - \frac{M}{r^2}$$
$$= \frac{M}{2r^2} + \frac{M}{2r^2} + \frac{m}{r^2} - \frac{M}{r^2}$$
$$= \frac{m}{r^2}.$$

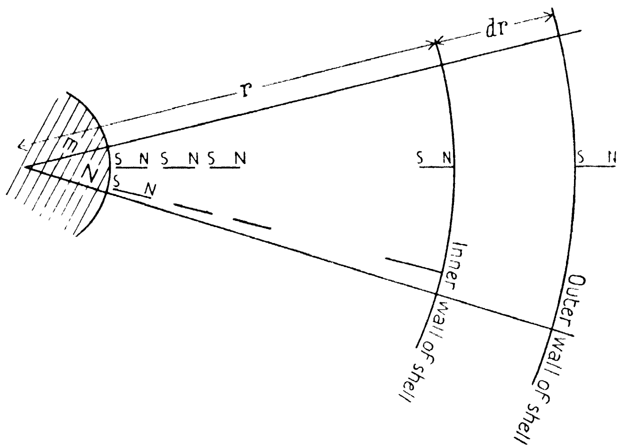


FIG. 105

Hence we find that the B in the medium is the same as if the medium had been absent, and consequently the H in the medium is $\frac{m}{\mu r^2}$.

The H in the medium is due to the difference between the force due to m and that due to M, whence

$$H = \frac{m-M}{r^2}.$$
$$\therefore \quad M = m - Hr^2$$
$$= m\left(1 - \frac{1}{\mu}\right).$$
$$\therefore \quad I = \frac{m}{4\pi r^2}\left(1 - \frac{1}{\mu}\right).$$

The effect of the ends of the magnetic chains which must be broken to insert the original north pole in the medium is to reduce the force in a worm-hole cavity to $1/\mu$ of what it would have been in the absence of the medium. If a pole of strength m' is placed in such a worm-hole cavity the force will be $F = mm'/\mu r^2$, and the force from the original pole does appear at first sight to depend on the medium. If the pole m' is simply submerged in the medium the force on it is apparently still equal to $F = mm'/\mu r^2$, and so it seems the chains which end on it can transmit no net pressure. Evidently the experiment could not be performed in a solid medium and a magnetizable liquid would have to be used. The most magnetizable liquid is a solution of iron in methyl alcohol, and for this $\mu = 1+1/1000$, hence the law of force is not capable of accurate verification.

The expression $F = \dfrac{mm'}{\mu r^2}$ should be familiar to an engineer and he should now understand just how much is implied in the statement; but beyond this it should be dismissed and every problem in which it might be involved should be dealt with by the method just used for establishment. The reader is particularly warned against the error of supposing that if a block of iron intervenes between a pole and a point in space, then the force due to the pole is altered because it acts through the iron: the force at the given point will be altered by the presence of the iron, but this is due to the surface polarity induced on the iron.

It is difficult for the human mind to visualize how a pole can produce a force at a distant point unless there is some sort of substance, between the pole and the point, through which the force can be transmitted. Accordingly the physicist has proposed the hypothesis of the ether as a medium filling all space: it is a purely hypothetical medium invented to reduce slightly the mystery of attraction. It was an idea which seemed very necessary to Faraday, who felt bound to conceive of this all-pervading medium, which discloses itself to human beings only by the fact that attractions and repulsions exist. It will depend on individual temperament whether or not this medium must, as a matter of necessity, be visualized or whether it is regarded just as a possible explanation of what a being of superior perceptions would recognize. But there is nothing to suggest that this hypothetical medium is not perfectly continuous, and, unlike iron, we need not suppose it has a granular structure which is composed of magnetic particles. Though we may choose to say that the permeability of ether is unity, the very notion of permeability as developed in this book seems superfluous when applied to the continuous ether. If we supposed that iron was a continuous structure, then we should be forced to imagine that the force due to a pole depended on the surrounding medium. But as it is, we consider that the force due to a pole is unalterable by any means, and acts through ether or empty space, whichever it may be. If the surrounding district is scattered with myriads of magnetic particles, then we must reckon the force from each of their poles, and this is done by means of the factor called permeability. With this understanding the reader may prefer to say that $F = mm'/r^2$, and the author hopes he will prefer to say this. It will then seem undesirable to attribute physical properties to permeability, which will be regarded as a ratio representing a statistical reckoning of myriads of poles none of which can be located separately.

An exact mathematical solution of magnetic problems can be obtained only when μ is treated as a constant, and so no problem involving iron can be solved correctly. Exact solutions have been found for a few problems where the permeability is treated as a constant, and these solutions sometimes give useful qualitative results for real problems. Since the permeability of iron varies over an enormous range, solutions valid for a constant permeability may appear absurd. But the solution of any real problem is a process of successive approximations, and the approach by treating μ as a constant often yields useful qualitative results. This is specially true of the field external to the iron, and the approximation is successful largely because a line of force enters almost normally into an iron face (see next section). The whole problem of ferro-magnetism appears to be beset with insurmountable difficulties, and yet by good fortune the engineer is able to surmount most of them with a scaling-ladder whose crudity is almost ludicrous. He owes this power to the enormous permeability of iron which allows him to make very drastic simplifications without spoiling his solution appreciably.

We have had one example of this in the iron ring with a concentrated magnetizing coil: the fact that the flux round the ring is sensibly constant is due to the power of a small leakage flux to produce a sensibly constant H round the ring. In ignoring the leakage flux in calculation, we are shutting our eyes to the very factor which makes our method of solution a good approximation.

2. Condition at the boundary between two different media

Let BB in Fig. 106 be the trace made in the paper by the plane interface between two different media. Lines of induction in the lower medium are at θ_1 to the normal, and these enter the second medium at θ_2 to the normal: this is shown in Fig. 106. Let *eoc* and *abd* be small portions of two equipotential surfaces which are close to one another. We remember that inside the iron the potential is defined as derived from H only. In curling round the elementary path *ocbao* no current is enclosed, and hence by the work law

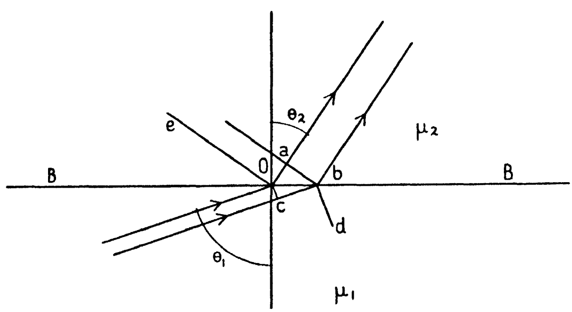


FIG. 106

$\int H\,dl = 0$. Along *oc* and *ab* no work is done because these portions of the path are equipotentials; hence

$$\int H\,dl = H_1 cb - H_2 oa = 0.$$

But $$cb = ob\sin\theta_1 \text{ and } oa = ob\sin\theta_2.$$

$$\therefore\quad H_1\sin\theta_1 = H_2\sin\theta_2.$$

Thus the component of H along the surface is continuous in passing from medium 2 to medium 1.

The lines of induction are continuous and so the flux density is the same each side of the surface, hence

$$B_1 oc = B_2 ab,$$

or $$B_1\cos\theta_1 = B_2\cos\theta_2,$$

or $$\mu_1 H_1\cos\theta_1 = \mu_2 H_2\cos\theta_2.$$

$$\therefore\quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}.$$

If the medium 2 is air, $\mu_2 = 1$, and then

$$\tan\theta_1 = \mu_1\tan\theta_2.$$

Now suppose $\mu_1 = 50$ and that $\theta_1 = 85$ degrees; then $\tan\theta_1 = 11{\cdot}4$.

Whence $$\tan\theta_2 = 0{\cdot}23$$

or $$\theta_2 = 13^\circ.$$

So lines of B inside the iron, which are nearly parallel to the surface, emerge almost perpendicular to the surface. The appropriate value of μ for iron is more nearly of the order of 1,000, and if this value is taken for μ, then $\theta_2 = 0{\cdot}6$ degrees. The reader should remember that any line of force which emerges from the iron into air must emerge

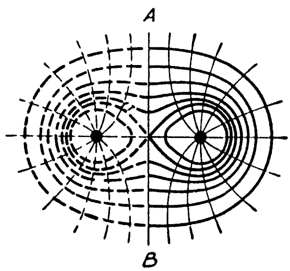


FIG. 107*a*

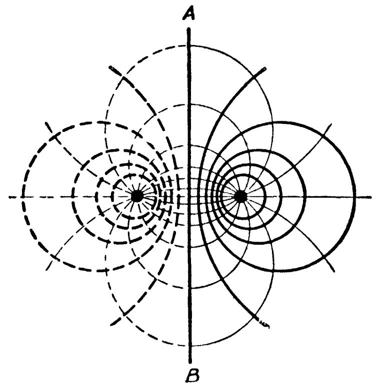


FIG. 107*b*

very nearly normally to the surface. If iron had infinite permeability, then every line of force in air would enter perpendicularly to the iron surface, and every iron surface would be an equipotential surface. In practice every iron face is very nearly an equipotential surface.

3. Linear current near an iron face of infinite permeability

We will now give two examples which may be solved by means of the last proposition. Consider Figs. 107 *a* and 107 *b* which show the lines of force and magnetic equipotentials for similarly and for oppositely directed currents. In Fig. 107 *a* the line AB is perpendicular to all the lines of force and so is an equipotential surface and might therefore be an iron face. If the left-hand wire is removed and the space to the left of AB filled in with an infinite slab of iron having its face at AB, the field to the right of AB will not be affected by the change. So the field produced by a wire placed in front of an infinite slab of iron is the same as the field which would be produced by the current and a similar image current. The magnetic force in the iron is zero everywhere, but B is finite, which is possible since μ is infinite. The right-hand conductor is in the field due to the surface polarity induced on AB, and this surface polarity produces the same field at every point to the right of AB as would be produced by the image current if the iron was absent. Hence there is a force on the conductor tending to drive it towards the iron face, and the value per unit length of this force is*

$$F = i^2/D,$$

where D is the distance from the centre of the wire to the face of the iron.

Now consider Fig. 107 *b*. Here the line AB is a line of force. Let the space to the right of AB be filled in with an infinite block of iron and let the left-hand current be removed. The field to the right of AB will not be disturbed by this process because there is no surface polarity on AB, and the field to the left of AB will be zero. If a line of force inside the iron coincides with the surface, then no line can cross the surface. So the circular lines to the right of AB are the lines of force for a linear current buried in an infinite slab of infinitely permeable iron. They are also the lines of force for a linear current inside a finite circular hole, so long as this hole coincides with a line of force.

4. Further discussion of an iron ring with a concentrated coil

In § 16, p. 142, we discussed the iron ring with a concentrated coil and were able to obtain an approximate solution by invoking the help of the experimental discovery that the flux density round the ring is nearly constant and roughly equal to what it would be if the coil were distributed uniformly. This important discovery should not be accepted as a starting-point whose explanation is beyond human knowledge, but we must think it out until we have seen that it is just what we should expect to follow from the law of force between poles.

On p. 146 we defined magnetomotive force and reluctance and saw that there was a certain similarity between a magnetic circuit and an electric circuit. We may therefore compare the coil to a battery which supplies current to a thick copper conductor which is submerged in a feebly conducting solution. From our experience of

* See Chap. I, § 11 (*b*).

electric circuits and our conception of resistance we shall then realize that very little current will leak from the surface of the wire which will have sensibly the same current all along it. But this is by no means an explanation of the magnetic problem, for it merely compares it with another problem which seems to be similar but also is unexplained. Since the B round the ring is substantially uniform, so must the H be also, and the H can be produced only by currents or poles. The H due to the current is appreciable only very close to the coil, and hence fresh poles must be formed in such a way as to maintain H almost uniform round the ring. There is the same problem in the electric circuit, for the charges on the plates of the battery produce a field which is appreciable only at near-by points, and yet we deduce from Ohm's law that there must be the same electric force acting on every moving electron in any part of the circuit. We may choose to think of this force as transmitted from the battery like a hydrostatic pressure from a pump, but we may prefer to attribute it to a surface and volume charge of electrons which must collect along the wire in order to produce the electric field which keeps the current flowing. Similarly with the magnetic problem, we may choose to describe the effect in terms of the permeability of two media, but we may prefer to attribute the substantially constant H to the action of the surface polarity which demonstrably exists. If this surface distribution could be calculated, then the application of the inverse square law would show that it produces an H which is approximately constant round the ring.

Philosophically it all depends on whether we like to regard poles as having what is commonly called real existence and the ether as having no real existence, or whether we choose to regard poles as some singularity in the ether and as one of the few means by which this all-pervading medium manifests its presence to human senses. In fact whether poles exist as isolated entities in empty space or whether they are merely peculiarities in an all-pervading unempty space, like occasional vortices in an ocean.

The continuous change of density which must occur when water-pressure is transmitted through a pipe, corresponds to the charge distributed along a wire carrying a current or to the surface polarity along the magnetized ring.

We will now attempt to show that a comparatively small leakage flux produces a surface polarity sufficient to maintain H nearly constant round a ring, such as that illustrated diagrammatically in Fig. 108. The ring is supposed to be formed of a round rod of radius a bent into a circle of radius R. The magnetizing coil consists of a few concentrated turns, and the lines of force sketched in the figure are intended to represent the field which the coil would produce in the absence of the iron. The dissymmetry of this field with respect to the iron must produce surface polarity, and this in turn orientates the magnetic particles of the iron and hands on the effect of the coil from place to place round the ring. Experiment shows that B is substantially constant all round the ring and that the leakage is small,

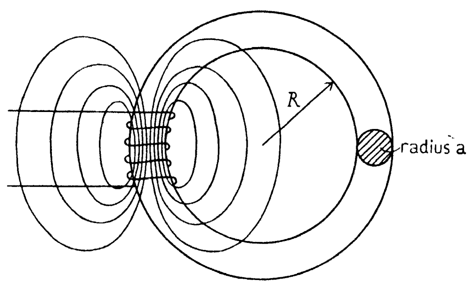


Fig. 108

yet it is to this almost negligible leakage that the constancy of B is due.

When the winding is distributed uniformly $H_1 = \frac{4\pi IT}{2\pi R}$, but when these same turns are bunched together the field at their centre due to the current is $H_2 \doteqdot \frac{2\pi IT}{a}$: whence $\frac{H_1}{H_2} = \frac{a}{\pi R}$, and this ratio is equal to, say, 1/50. But a fiftyfold increase of H due to the current produces a negligible increment of B inside the coil, and hence surface polarity must produce an opposing H which leaves conditions there almost unchanged. Consider now a point in the iron diametrically opposite to the coil; here $H_3 \doteqdot \frac{\pi a^2 IT}{(2R)^3}$, whence $H_3/H_1 = \frac{a^2}{16R^2} \doteqdot \frac{1}{2000}$. Thus at such a point the coil contributes a negligible fraction to the total H, which must be due almost entirely to surface polarity. Fig. 109 shows some of the lines due to surface polarity, and we see that these decrease the H inside the coil and increase it at all external points. We now see how it comes about that the surface polarity, or 'leakage flux' as the engineer usually terms it, tends to equalize the value of H round the ring, and shall make a rough calculation to find how much leakage flux is required to contribute the necessary H. Thus consider the force at S due to the surface polarity on two bands of width a, distant d from S (see Fig. 110). Suppose that the density

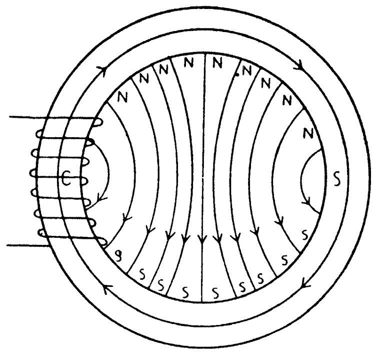


FIG. 109

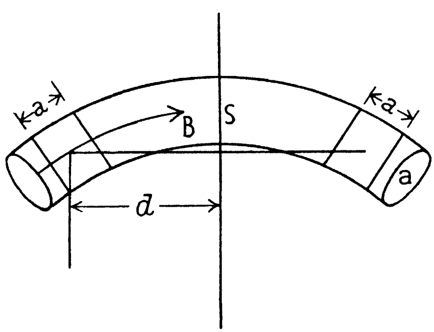


FIG. 110

of the flux leaving these bands is uniform and equal to $B/200$; then the flux leaking from one band surface is

$$\frac{2\pi a^2 B}{200} = \frac{\pi a^2 B}{100} = 0{\cdot}01\phi.$$

Expressing this leakage flux in terms of pole strength m, we have

$$\frac{\pi a^2 B}{100} = 4\pi m.$$

$$\therefore \quad m = \frac{a^2 B}{400}.$$

Hence the force at S due to these two poles is

$$F \doteqdot \frac{2a^2B}{400d^2} = \frac{10a^2}{d^2}H, \text{ if } \mu = 2000,$$
$$\doteqdot H, \text{ if } d = 3a.$$

Hence 0·5 per cent. of the total flux leaking from such a band, distant $3a$ from S, is sufficient to produce at S the requisite magnetic force if $\mu = 2000$. But the leakage from all parts of the surface adds its contribution to the H at S, and therefore a leakage of much less than 0·5 per cent. will be needed from the particular surface considered. Had the permeability been 200 instead of 2,000, the assumed amount of leakage flux would have produced a magnetizing force of only $H/10$. It is because a given B requires a very small H that a small fractional amount of leakage can suffice to produce surface polarity of the necessary strength.

Had the flux density in the ring been, say, 30,000 lines sq. cm., then the leakage flux would have to be relatively increased because so large a value of B would require a disproportionately large value of H. So, in conclusion, we realize why the flux density round a ring will be roughly constant provided its value lies between, say, 2,000 and 14,000 lines sq. cm. In such circumstances it is permissible to calculate the flux which will be produced by a given number of ampere turns in the same way that we should do if the ampere turns were distributed uniformly.

It is impossible to calculate accurately the distribution of leakage flux from a ring, and an exact mathematical solution is quite out of the question even if μ for iron were constant. The analytical difficulties are so preposterous that the problem is not worth pursuing for its own sake because its practical interest is very small. Rough estimates can be made of leakage flux from some forms of magnet, but in the rare circumstances where information is required it is usually necessary to resort to experiment.

5. Iron ring with an air gap

We will now consider the mechanism by which the flux through the coil is decreased by the process of cutting an air gap across the iron ring, such as that shown in Fig. 99. The imaginary process of carrying a unit pole round the ring and across the air gap shows that the flux must be reduced because the bulk of the work is done in crossing the gap. But this process does not describe the mechanism by which the magnetic force at a given point in the iron becomes enormously reduced when the gap is cut. To make the problem very vivid let us state it in an anthropomorphic form. Let the reader picture himself bearing a unit pole in his hand and making the circuital tour through a previously provided tunnel in the iron and then across the air gap. The struggle to get across the gap and the easy walk through the iron would demonstrate strikingly the large reduction of flux, but not how and why it occurred. Now let him picture himself as a magnetic particle somewhere in the iron and let him be conscious of the attraction of his neighbours and the forces from the surface polarity or leakage flux. When the gap is cut he finds that the forces from the leakage flux decrease very much, and the influence of his neighbours causes him to yaw away from the line of formation, and he sees his neighbours do likewise. This yawing from the formation we describe as the reduction of B in the iron. Now how does cutting the gap alter everywhere the superposed H and disorder the formation of the magnetic chains?

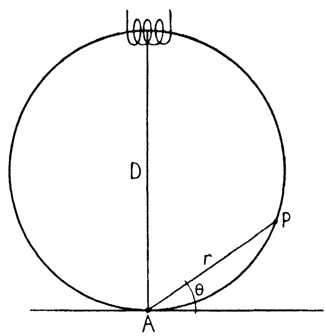


FIG. 111

Our first instinct may be to suspect that the necessary diminution of H is produced by the forces from the polarity on the faces of the air gap, but we shall now show that this is not so.*

At a distant point on the ring, these two pole faces of strength m, separated by a distance g, may be regarded as a magnetic particle (see Chap. I, § 3) of moment $M = mg$: the force at P for the particle situated at A in Fig. 111 is

$$H_r = \frac{2M\cos\theta}{r^3} = \frac{2M\cos\theta}{D^3\sin^3\theta}$$

and

$$H_t = \frac{M\sin\theta}{r^3} = \frac{M\sin\theta}{D^3\sin^3\theta}.$$

* The term leakage flux has hitherto been applied to any surface polarity on the iron and therefore might be expected to include that on the faces of the air gap. But in the problem of the ring with an air gap, the term leakage flux is tacitly assumed to exclude that flux which enters or leaves the parallel faces of the gap. This distinction may seem surprising to the general reader; but it is quite natural to the engineer, for in most of his machines the working apparatus is situated in the air gap. It is the flux which crosses the gap which is useful to him, and so he regards the leakage lines as so much wastage or even as a mild annoyance.

Resolving these two along the circumference of the circle, we have a resultant H_T, where

$$H_T = \frac{M(2-\sin^2\theta)}{D^3\sin^3\theta}$$
$$= \frac{mg(2-\sin^2\theta)}{D^3\sin^3\theta}$$
$$= \frac{Bga^2(2-\sin^2\theta)}{4D^3\sin^3\theta}$$
$$\doteqdot \frac{4\pi ITa^2}{4D^3}\frac{(2-\sin^2\theta)}{\sin^3\theta}.$$

The value of H_T varies greatly as we proceed from the gap to the coil, but we notice with interest that to this degree of approximation it is independent of the gap length and depends only on the current turns in the coil.

Hence the general big reduction of the net mean H in the iron is not due to the polarity on the gap faces, since they produce a reduction which is independent of g. Nor is their effect appreciable except at near-by points: thus the value of H_T inside the coil is

$$H_T = \frac{2\pi ITa^2}{D^3},$$

and before the gap was cut, H_1 the net magnetic force there was given by

$$4\pi IT \doteqdot H_1\pi D.$$

$$\therefore \frac{H_T}{H_1} = \frac{\pi}{2}\frac{a^2}{D^2} \doteqdot \frac{1}{100},$$

and so the pole faces decrease the magnetizing force inside the coil by a negligible amount.

But cutting the gap has somehow decreased the net magnetic force in a ratio which is of the order of $\frac{l}{\mu g+l}$, and this must be due to an increase of leakage flux from the curved surface of the iron. It might seem, therefore, that this leakage has now become enormous, yet experience shows this is not so. How, then, can we explain a comparatively small increment of leakage causing so enormous a change in the net value of H? Let us return to the ring without an air gap but with a concentrated coil. Experience has shown us that the net H everywhere is given approximately by $4\pi IT = \pi DH$, or $H = \frac{4IT}{D}$.

The magnetizing force inside the coil due to the current is approximately $H_c = \dfrac{2\pi IT}{a}$: let H_l be the magnetizing force inside the coil due to surface polarity on the uncut ring; then

$$H_c - H_l = H.$$

$$\therefore \quad H_l = 4\pi IT\left(\frac{1}{2a}-\frac{1}{D}\right)$$

$$= H\left(\frac{D}{2a}-1\right)$$

$$\doteqdot H(50-1)$$

$$= 49H.$$

Now suppose that cutting the gap causes the leakage flux to increase by such an amount that inside the coil H_l is increased by, say, 1 per cent. Then the net $H = H_c - H_l$

$$= \frac{D}{2a}H - 49\times 1{\cdot}01H$$

$$\doteqdot H(50-49{\cdot}5) = 0{\cdot}5H.$$

So if the change of leakage flux causes H_l to increase by 1 per cent., this will reduce the net magnetizing force from H to $0{\cdot}5H$: the net H is thus the small difference between two large quantities, and a small change of one will modify the resultant profoundly.

Hence, cutting the gap must cause an increase of leakage flux, but the increment is very small compared with the enormous decrease of total flux through the iron. No doubt there is a considerable change in the distribution of leakage flux when a minute gap has been cut, but subsequent increase of the gap-width seems to make little difference to its magnitude or distribution.

The total flux through the magnetizing coil may be regarded as the sum of two components: one component is roughly constant and the other varies inversely as the length of the gap. The ratio of the flux through the coil to the flux crossing the gap is called the Leakage Coefficient or Dispersive Coefficient.

Suppose that the flux through the coil is 1 per cent. greater than the flux through the iron at the place where the gap is to be cut: let a gap be cut such that $g/l = 1$ per cent. and $\mu = 2000$, then the flux which passes right round the ring will fall to 1/20th of its previous value. The supposedly constant leakage flux will now have become 20 per cent. of the total, and the leakage coefficient will be 1·2.

It is often required to calculate the inductance of an iron ring with an air gap, because such are used a great deal in alternating current practice. We may remind the reader that the inductance of a coil is defined as being the flux turns per unit current (see Chap. I, § 15). The flux turns due to the flux which passes right round the ring and across the gap is readily calculated, and if there were no leakage flux we should have

$$L = BAT \doteq \frac{4\pi AT^2}{g}.$$

Comparing this with the formula for the inductance of an air-cored ring solenoid (see Chap. I, equation (23)) we see that the iron-cored ring with a gap of length g has the same inductance as an air-cored solenoid of length g. It would be physically impossible to wind the same number of turns on a solenoid of length g (g is probably only a few millimetres), so one way of regarding the effect of the iron is as a device which permits the effective length of the solenoid to be g while providing plenty of space to accommodate the winding.

But the inductance of an iron-cored air-gap solenoid is appreciably greater than the value given above, because some or all of the turns are threaded by the leakage flux, and hence $L = x\frac{4\pi AT^2}{g}$, where x is the leakage coefficient. But since x is a function of g, this formula is not of much practical utility. We have seen reasons for supposing the leakage flux does not change very much with g, though some change is essential since it is by this means that g alters the net H inside the iron. Hence the next approximation to the inductance of a ring with an air gap may be represented by Fig. 112, which represents a hypothetical ring having no leakage flux, connected in series with a small air-cored inductance whose value is such that the total flux turns in the two coils is the same as that in the real ring. Fig. 112 is merely a way of drawing the system so as to separate clearly the leakage flux from that which passes right round the ring and across the air gap: the leakage inductance certainly is a function of g, but we are hoping to find that a large change of g makes a small change in its value. Accordingly we hope to find that

$$L = L' + \frac{4\pi AT^2}{g}.$$

One way of testing this relationship is to plot the current I against the gap g when an alternating voltage of constant value and fre-

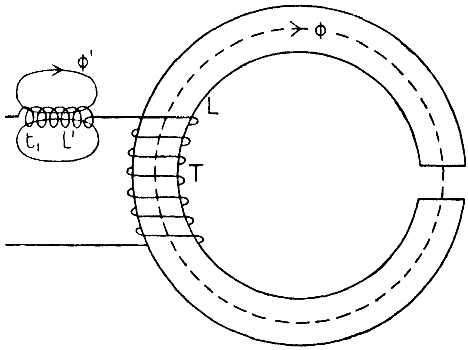


FIG. 112

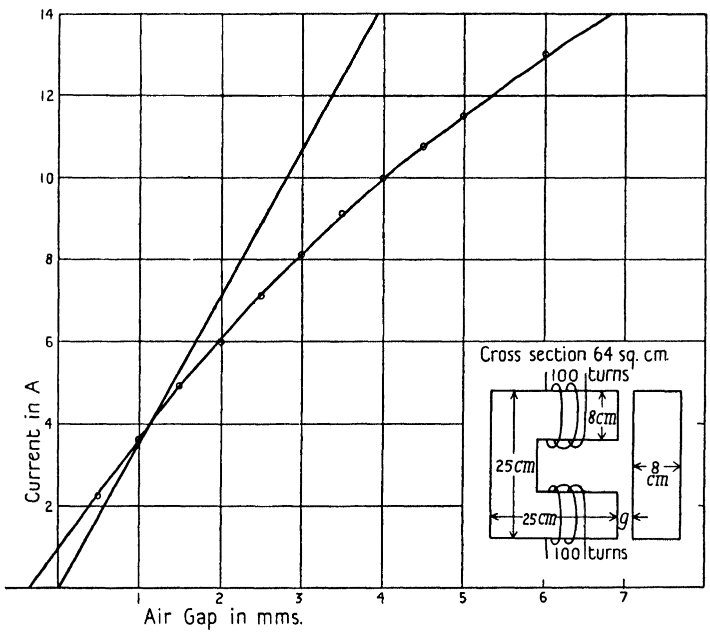


FIG. 113

quency is applied to the winding. An example of such a test is shown in Fig. 113, for the iron core having the dimensions shown in the inset sketch. It will be seen that the curve of Fig. 113 starts as a straight line, which does not go through the origin because the reluctance of the iron path is not zero: assuming that $\mu = 3000$, then it can be calculated that the effect of the iron is equivalent to lengthening the air gap by 0·3 mm. and the intercept in Fig. 113 bears out this value. The straight line through the origin is the calculated curve assuming there is no leakage and that the iron reluctance is zero. If the suspected relation between I and g is correct, the equation of the curve will be of the form

$$I = \frac{g}{ag+0{\cdot}28},$$

where a is due to the leakage inductance and 0·28 is the calculated coefficient appropriate to the area, turns, frequency, and voltage. When $g = 8$ mm., the measured value of I was 15·2 amp.: taking g as 8·3 mm., we find $a = 0{\cdot}032$. Using this figure it may be found that when $g = 3$ (or equivalent gap 3·3 mm.), $I = 8{\cdot}5$ amp., whereas the observed current is 8·1 amp., which is 5 per cent. less than the calculated value.

The equation $$I = \frac{g}{0{\cdot}029g+0{\cdot}305}$$

gives the correct current when the gap is 4 mm. and also when it is 8 mm. and fits the curve very satisfactorily at all points: thus, when the gap is 1 mm., the observed current is 5 per cent. less than the calculated. When the gap is very small the calculated current is very sensitive to the exact value used for the equivalent length of the iron, and hence an appreciable fractional error for the current with a small air gap does not mean that the assumed form of the curve is incorrect. Though a curve such as Fig. 113 does not prove the leakage flux is constant, it does show that it may be treated as such for the practical purpose of finding the inductance with a given air gap. When g becomes very large, the equation shows that the current will tend to the limiting value of 33 amp. with 182 V. and 50 cycles/sec.; this yields a value of 17·5 mH. for the leakage inductance. When the head-piece of the magnet was removed entirely, corresponding to $g = \infty$, the inductance was found to be 17 mH. The leakage inductance cannot well be calculated, but once the magnet has been made, its effective value can be found by measuring the inductance of the coil with the head-piece removed. The writer is not able to forecast any relation between the leakage inductance and the size of the magnet.

It follows readily that the leakage coefficient for any air gap can be found at once from a curve such as Fig. 113: it is the ratio of the observed current to the current there would be with no leakage. Thus for the magnet described by Fig. 113 it is

$$x = \frac{0{\cdot}28}{0{\cdot}029g+0{\cdot}305}.$$

6. Equivalent length of a joint in the iron circuit

It is not often practicable to make a magnetic circuit without a joint in it, and it is of interest to inquire how nearly a smooth joint

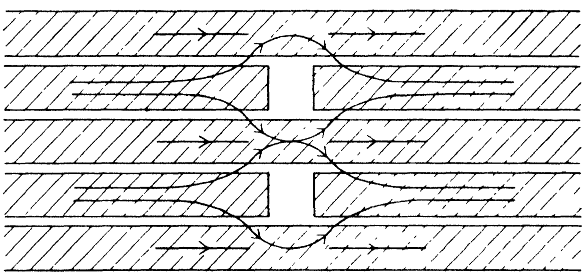


FIG. 114

approximates to a continuous iron circuit: for example, is the flux in the magnet described by Fig. 112 the same when the gap is squeezed up tight as it would be if the iron were continuous? Interesting data are to be found in Ewing's *Magnetic Induction in Iron*, pp. 285–93, where it is stated that a smooth joint which has been scraped to fit, is equivalent to an air film 0·03 mm. thick, and a smooth unscraped joint is equivalent to an air film of about 0·04 mm. The effective length of a scraped joint decreases if the joint is put in compression, and a compressive stress of about 1·5 tons/sq. inch is sufficient to remove the effect of the joint; on the other hand, the effective length of an unscraped joint cannot be reduced appreciably by any compressive stress.

7. Equivalent length of the joint between laminations

The stampings from which magnets are built up can seldom be made without a joint, which is necessary to allow them to be threaded through the coil; also, in very large magnets the stampings could not be made all in one piece. To reduce the net effect of these joints it is common practice to build up the magnet so that successive joints do not come above one another. The arrangement is illustrated by Fig. 114, which shows that each joint has a cover-plate which is continuous at that place. However, the net effect of the joints is still not zero because the flux has to cross the air space from one stamping to the next, as shown in the figure.

We will now calculate the increment of reluctance due to the joints on the assumption that the permeability is constant, that in the iron the flux runs parallel to the laminations, and that the flux crosses perpendicularly from one plate to the next. We have to express in

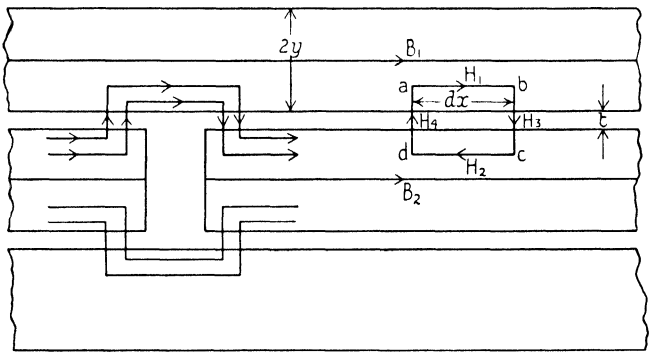


FIG. 115

symbols that round the rectangle *abcd* in Fig. 115, $\int H\,dl$ is zero, that the flux across any section of the plates is constant, and that there is no polarity in the iron.

Let the flux density crossing from one plate to the next be B at a distance x from the joint. Accordingly we have

$$H_1 dx + H_3 t - H_2 dx - H_4 t = 0.$$

$$\therefore \quad (H_1 - H_2)dx = (H_4 - H_3)t$$

$$= \left\{B - \left(B + \frac{\partial B}{\partial x}\,dx\right)\right\}t.$$

$$\therefore \quad (B_1 - B_2) = -\mu t \frac{dB}{dx}.$$

$$B_1 + B_2 = \text{constant} = 2b.$$

$$\therefore \quad -\mu t \frac{dB}{dx} = 2(B_1 - b),$$

also $$B_1 y + B\,dx = y\left(B_1 - \frac{\partial B_1}{\partial x}\,dx\right).$$

$$\therefore \quad B = -y\frac{dB_1}{dx}.$$

$$\therefore \quad -\mu t\frac{dB}{dx} = \mu ty\frac{d^2B_1}{dx^2}.$$

$$\therefore \quad \frac{d^2B_1}{dx^2} = \frac{2}{\mu ty}(B_1-b)$$

$$\doteqdot \alpha^2(B_1-b).$$

$$\therefore \quad B_1 = b+Ae^{\alpha x}+Ce^{-\alpha x}.$$

When $x = \infty$, $\quad B_1 = b. \quad \therefore \quad A = 0.$

$$\therefore \quad B_1 = b+Ce^{-\alpha x}.$$

We shall now assume that an inappreciable amount of flux crosses the butt joint between two successive stampings in one plane: this is a reasonable assumption because the separation between two plates is only the thickness of the paper insulation which is 0·0013 inches, whereas the width across the joint may well be about 0·02 inches. On this assumption the flux density in the lower plate is zero at the origin and that in the upper plate is then $2b$. Accordingly we have $C = b$, and so

$$B_1 = b(1+e^{-\alpha x}).$$

$$\therefore \quad H_1 = \frac{b}{\mu}(1+e^{-\alpha x}).$$

$$\int_0^x H_1\,dx = \frac{b}{\mu}\left[x-\frac{1}{\alpha}e^{-\alpha x}\right]_0^x \doteqdot \frac{b}{\mu}\left[x+\frac{1}{\alpha}\right], \text{ when } \alpha x \gg 1.$$

This shows that the work done in taking a unit pole along the inside of the upper plate is increased by the joint in the lower plate, and the work done over a given distance x is the same as if the joint had been absent and the distance had been increased by a length $\frac{1}{\alpha} = \sqrt{\frac{\mu ty}{2}}$. Giving the values $\mu = 2000$, $y = 0{\cdot}055$ cm. (20 mil plates), and $t = \frac{3{\cdot}3}{10^3}$ cm. (paper 1·3 mils thick), we find that $\frac{1}{\alpha} = 0{\cdot}3$ cm. The flux density in the plates will have settled down sensibly to its undisturbed value in a distance such that, say, $4\alpha x = 1$ or $x = 1{\cdot}2$ cm. The flux density through the paper insulation is found to be

$$B = \sqrt{\frac{2y}{\mu t}}\,be^{-\alpha x}.$$

8. Energy of the magnetic field

Every magnetic field can be considered to be the resultant effect of a large number of elementary magnets, and work would have to be done to bring these elementary magnets from infinity into their final relative positions, because they are mutually repellant. This energy is stored and could be recovered if they were again dispersed to infinity. We cannot say where this energy is stored, any more than we can say where the energy of a steel spring is stored. We commonly think of the magnets being surrounded by the ether, and so we may suppose the energy is stored in the state of stress of this hypothetical continuous medium. Because steel is obvious to our senses we commonly think of the energy of a steel spring as being stored in the steel, which we then think of as a continuous medium. But when we think of steel as a molecular structure we are no longer clear about the meaning of stress, and we have to think of molecular bond which again we do not understand. Hence even in a steel spring we are presumably forced to think of the energy as an ether stress. Or in other words, baffled by finding all media discontinuous, we invent a continuous ether to support the continuous stresses of our supposition.* The energy of a magnetic field could be reckoned in terms of the strength and position of its component magnets, but this process leads to an alternative expression in terms of the flux density at any point of the field. It was stated in Chapter I that the energy of a magnetic field could be expressed by the equation

$$W = \iiint \frac{H^2}{8\pi}\, dxdydz,$$

where $\iiint$ represents integration through all space and H is the strength of the magnetic field at the point xyz. This relation may be interpreted as stating that the energy is distributed through the field at the rate $\frac{H^2}{8\pi}$ per unit volume, but of course it cannot be shown that this amount of energy is situated in any particular unit volume since nothing is known about the mechanism by which energy is stored. The field may be filled with magnetic particles and then their energy will have to be reckoned by means of the permeability of the so-called surrounding medium. If the permeability at the point (x, y, z) is μ, then it can be shown that

* Compare p. 160.

$$W = \iiint \frac{\mu H^2}{8\pi}\,dxdydz$$
$$= \iiint \frac{HB}{8\pi}\,dxdydz.$$

In this equation μ must be a constant but need not be uniform: it must not be a function of H, but it may be a function of position. It must be remembered that these formulae neglect the existence of hysteresis.

The energy in the field is often a useful analytical weapon in the solution of problems, and it is also a fruitful source for examination questions: for these two reasons the reader must be familiar with the two previous expressions for W, but otherwise they are of very little value or importance.

We will now give an example of the application of these formulae to a field whose distribution has been postulated. Consider an iron ring with an air gap and postulate that H is uniform in all parts of the iron and that there is a uniform flux density B in the air gap. According to these suppositions the field is zero everywhere except in the iron and between the faces of the gap, and hence the summation through all space is restricted to these two regions. Accordingly

$$W = \frac{\mu H^2}{8\pi}Al + \frac{B^2}{8\pi}Ag$$
$$= \frac{AB^2g}{8\pi}\left(1+\frac{l}{\mu g}\right).$$

This result could have been obtained from the energy associated with a self-inductance L, for

$$W = \tfrac{1}{2}LI^2 \quad \text{(see Chap. II, § 7)}$$
$$= \tfrac{1}{2}BATI$$
$$= \frac{B^2Ag}{8\pi}\left(1+\frac{l}{\mu g}\right),$$

since
$$4\pi IT = Bg\left(1+\frac{l}{\mu g}\right).$$

Or, expressing W in terms of B and current turns, we have

$$W = \frac{AB}{8\pi}(Hl+Bg)$$
$$= \frac{ABIT}{2}.$$

9. The pull between two magnetized surfaces

To calculate the pull between two magnets it is first necessary to discover the distribution of polarity over their surfaces and then apply the inverse square law to calculate the force between any element of polarity and all other elements. But here again we can transform the integration and express the force in terms of an integral involving the field strength at points of space between the magnets.

We will begin the discussion by considering the force pulling two magnetized surfaces together, for example, the force tending to close the air gap of the iron rings we have been discussing. Let the two gap faces have a uniform intensity I' of surface polarity. At a point of space between the two surfaces the force on a unit pole would be

$$B = H+2\pi I'+2\pi I'.$$

But the force on a unit pole in one of the surfaces is less than B by the contribution from one surface. There is polarity I' per unit area and therefore the pull on the face is

$$\begin{aligned} P &= (H+2\pi I')I'A \\ &= \left\{H+2\pi\frac{(B-H)}{4\pi}\right\}\frac{(B-H)}{4\pi}A \\ &= \frac{B^2-H^2}{8\pi}A \\ &= \frac{B^2}{8\pi}\left(1-\frac{1}{\mu^2}\right)A. \end{aligned}$$

This has expressed the pull in terms of the field strength which the surface polarity produces in the air gap. But though this is the contribution of one face to the pull, it is not necessarily the force tending to close the gap because we have ignored the pull between the surface polarity which we have described as leakage flux.

We will now approach the problem by an energy method and use the expression for W found in § 8 above. Let the gap be opened a very small amount x by means of a force P, then external work is supplied of amount Px. The result of opening the gap is to make a very small decrease in the flux density, and during the process an e.m.f. will be induced in the winding in a direction which would cause a current to flow which would tend to maintain the flux. Therefore the induced e.m.f. is in the same direction as the current, and hence it does work on the current and this work is returned to the battery. We shall suppose that during the change the resistance is being adjusted so as to keep the current unaltered by the induced e.m.f. Opening the air gap delivers energy to the battery and it also reduces the total store of energy in the magnetic field, since the flux density decreases by an amount dB. In order to make the evaluation of W possible, we must assume that H is uniform everywhere in the iron so as to avoid the complication of leakage field: then, from the end of § 8 above, the decrease of energy in the field is $\frac{1}{2}AITdB$. The equation of energy balance is accordingly

$$Px+\frac{AIT}{2}dB = I\int_0^t \frac{d\phi}{dt}\,dt$$

$$= I\int_0^t AT\frac{db}{dt}\,dt$$

$$= AITdB.$$

$$\therefore\quad Px = \frac{AIT}{2}dB.$$

But $$4\pi IT = B\left(g+\frac{l}{\mu}\right) = (B-dB)\left(g+x+\frac{l}{\mu}\right).$$

$$\therefore\quad dB = \frac{Bx}{g+x+l/\mu}$$

$$= \frac{B(B-dB)x}{4\pi IT}.$$

$$\therefore\quad \frac{P}{A} = \frac{B(B-dB)}{8\pi} = \frac{B^2}{8\pi}, \text{ in the limit.}$$

This expression differs from that found previously by the term $1/\mu^2$, which in practice is less than one in a million. The first expression was evidently incomplete because it ignored the pull contributed from the surface poles of the leakage flux, but the second expression has been derived from the supposition that there is no leakage field: so there appears to be complete confusion.

9 a. The Maxwell stress

In order to find the correct answer we must use a method developed by Maxwell. Faraday and Maxwell wished to describe and attribute the pull between poles and currents to the terminal effect of stresses transmitted through a continuous isotropic medium, the ether. Maxwell shows that the forces we observe could be attributed to the following stresses propagated through the ether.

(1) A hydrostatic pressure of amount $\dfrac{H^2}{8\pi}$.

(2) A tension along the lines of force of amount $\dfrac{BH}{4\pi}$, where B and H are the values of the induction and the magnetic force at the point considered.

These are equivalent to

(1) A compressive stress $\dfrac{H^2}{8\pi}$ perpendicular to the lines of force.

(2) A tension stress $\left(\dfrac{BH}{4\pi}-\dfrac{H^2}{8\pi}\right)$ along the lines of force.

If we are considering the stress outside the iron, then $B = H$ and the tension along the lines reduces to $\dfrac{B^2}{8\pi}$.

To calculate the pull tending to close the air gap in an iron ring, we must integrate this hypothetical stress over some plane between the air faces. The plane midway between the faces is a plane of symmetry and every line will cut through it perpendicularly, and therefore $P = \iint \dfrac{B^2}{8\pi}\,dxdy$, where $\iint$ denotes integration over the particular plane. So it appears that our second derivation of the force closing the air gap was correct, though it was derived from an impossible distribution of flux.

It must be understood that the tension $\dfrac{B^2}{8\pi}$ is a device which provides an alternative method of performing the integration of the force from the component poles. The distribution of the field must first be calculated, and this is equivalent to finding the distribution of polarity. It is difficult to be certain that the Maxwell stress is exactly correct in real problems, all of which involve hysteresis effects.

Perhaps the wisest method for the student to calculate the force closing the air gap is that used in the beginning of § 9 above, and then to realize that for iron the term $1/\mu^2$ is negligible. This is perhaps sounder and more convincing than energy methods. However, the energy in the medium and the Maxwell stress have been explained, and the reader will meet both of these in any advanced text-book of magnetism.

We have now a simple means of calculating the lifting force of a magnet. Since it is impracticable to produce a field strength greater than, say, 20,000 lines per sq. cm., it follows that the pull from an electromagnet will be less than 10^7 dynes sq. cm., which is equivalent to 16·2 kg./sq. cm. or 3 tons/sq. ft. of lifting surface.

10. An example of calculating the force by the energy method

The energy method may be used to calculate approximately the force tending to pull the two electromagnets, shown in Fig. 116, into

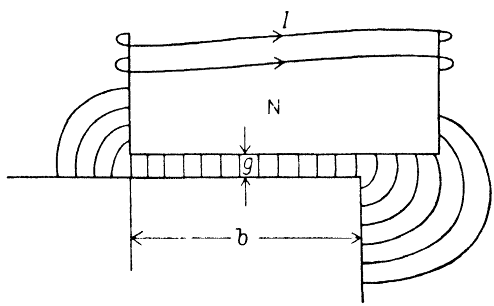


FIG. 116

line with one another end to end. If the overlap b is large compared with the air gap g, experience shows that the fringing lines will be sensibly independent of the length b. If the north pole is moved a small distance x to the left, the main effect on the distribution of field is an increase in the volume of the zone where the lines of force are sensibly straight and parallel: this zone will extend over most of the length b so long as b is many times greater than g. If the field strength B in this zone is uniform, the volume of uniform field will be increased an amount gx by the movement of the north pole. Allowing the magnet to move will increase the total flux through the winding and generate therein an opposing e.m.f. against which the battery has to do work: the work from the battery will move the magnet and increase the energy stored in the field.

$$\therefore \quad Px+\frac{B^2}{8\pi}gx = I\int T\frac{db}{dt}\,dt = ITBx$$

$$\doteqdot \frac{gB^2x}{4\pi}, \text{ since } 4\pi IT \doteqdot Bg.$$

$$\therefore \quad P = \frac{B^2g}{8\pi}.$$

Such an effect occurs in electric motors, because the surface of the armature is furrowed by slots to carry the winding. Each time a slot passes away from the edge of a pole there is a force on the armature tending to keep the slots placed symmetrically with respect to the poles. If the armature is turned round slowly by hand this cogging effect, as it is called, may readily be felt.

11. Force between two long parallel magnetized rods

Suppose a piece of thin iron wire is bent into the form shown in Fig. 117 and magnetized by a winding round the short side of the

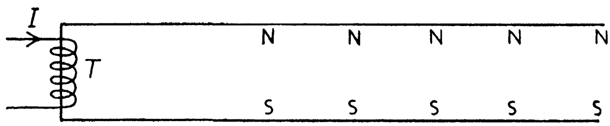


FIG. 117

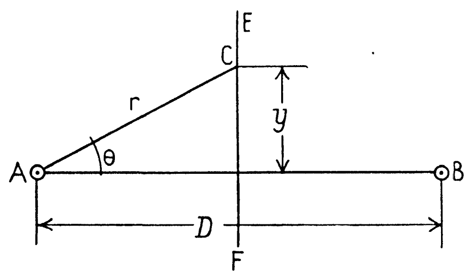


FIG. 118

rectangle, then there is sensibly the same magnetic p.d. between the rods at any point in their length. The whole of one rod will be north polar and the whole of the other south polar: let the pole strength per unit length be m. Then it follows (by the same piece of integration that was used in Chap. I, § 10, to find the force outside a long straight current) that the force due to either rod is $2m/r$ at a distance r away from it: the direction of the force is radial. A cross-section of the two wires is shown in Fig. 118, and the force between them is

$$F = \frac{2m^2}{D}.$$

We will now use the Maxwell stress method to obtain the same result, by calculating the total tensile stress which may be supposed to exist across the plane EF. The flux density at the point C in

Fig. 118 is $\frac{4m}{r}\cos\theta$, and its direction is perpendicular to the plane EF, whence

$$F = \int_{-\infty}^{+\infty} \frac{B^2}{8\pi}\,dy$$

$$= \frac{2m^2}{\pi}\int_{-\infty}^{+\infty} \frac{\cos^2\theta}{r^2}\,dy$$

$$= \frac{8m^2}{\pi D}\int_{0}^{\pi/2} \cos^2\theta\,d\theta$$

$$= \frac{2m^2}{D}.$$

Now $$4\pi IT = V_1 - V_2 = 4m\log\frac{D}{r}.$$

$$\therefore\quad F = \frac{2\pi^2 I^2 T^2}{D\left(\log\frac{D}{r}\right)^2}.$$

In this expression the radius r of the rods is supposed to be very small compared with D. But by an extension of the process we can find the force between two rods of finite radius r, and in these the density of leakage flux will not be uniform round the circumference and will be a maximum at the two points which are closest to one another. The magnetic potential at a distance r_1 from A and r_2 from B is $V = 2m\log r_1/r_2$, and therefore if a point P moves so that r_1/r_2 is constant it will always be on an equipotential surface: but the surface of a rod must be an equipotential surface since there is no magnetic p.d. between different parts of it. By a well-known geometrical proposition the path traced out by a point which moves so that the ratio of its distances from two fixed points is constant, is a circle of which the fixed points are called inverse to one another. The surfaces of the rods are thus equipotentials of line magnets placed, not at their centres but at the points which are mutually inverse to both. This is illustrated by Fig. 119 in which the lines of force are shown as though radiating from the inverse points: the flux density at the surface is clearly a maximum at the point where the cylinders are nearest to one another.

The force of attraction between the two cylinders is the same as

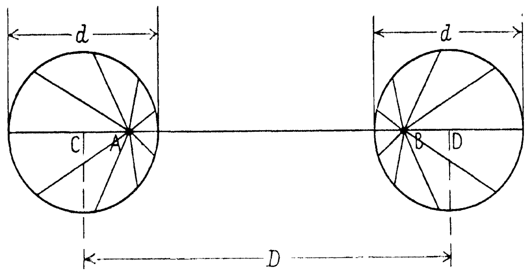
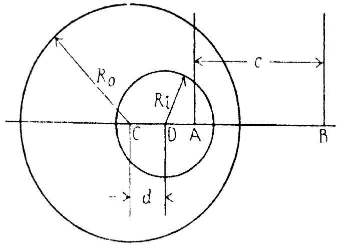


FIG. 119



$$R_0 - R_i = g$$
$$R_i \equiv R$$

FIG. 120

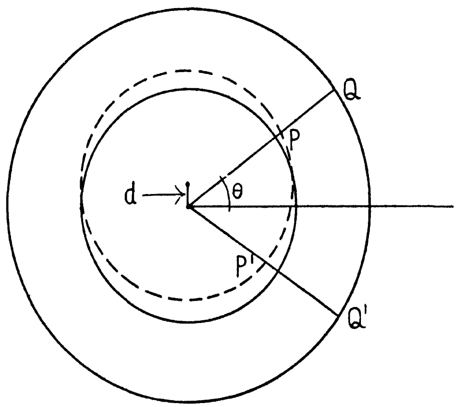


FIG. 121

the force between two line magnets at a distance apart AB. This follows because A and B can be the inverse points of an infinite series of circles with their centres on the line AB: one such circle may be that shown in the figure with its centre at C, while the other may have a very small radius and so have its centre very near B. The polarity on the cylinder centred at C produces at B the field appropriate to a line magnet at A, and so the cylinder attracts the line magnet at B with a force $\frac{2m^2}{AB}$. But the line magnet at B must attract the cylinder with an equal force, and hence the reaction between the field of B and the non-uniform polarity round the cylinder centred at C is the same as if all C's polarity were concentrated at A. It can be shown from the property of inverse points that $AB = \sqrt{D^2-d^2}$.

$$\therefore \quad F = \frac{2m^2}{\sqrt{D^2-d^2}}.$$

The inverse points can be used to calculate the force between two magnetized cylinders one of which wholly encloses the other, as shown by Fig. 120: in practice such a system occurs in the magnet of a moving-coil loud speaker and is illustrated in Chap. II, Figs. 55 and 60. If the core is truly concentric with the outer cylinder the force is zero, but any displacement produces a force tending to increase the eccentricity. It follows from the previous argument that the field between the cylinders is the same as there would be in this space if there were line magnets placed at the common inverse points A and B in Fig. 120. Whence

$$F = \frac{2m^2}{AB}.$$

It follows from the property of inverse points that

$$\begin{aligned} c^2d^2 &= \{(R_0+R_i)^2-d^2\}\{(R_0-R_i)^2-d^2\} \\ &\equiv \{(2R+g)^2-d^2\}(g^2-d^2). \end{aligned}$$

$$\therefore \quad F = \frac{2m^2d}{\sqrt{g^2-d^2}\sqrt{(2R+g)^2-d^2}}$$

$$\doteq \frac{2m^2d}{(2R+g)\sqrt{g^2-d^2}},$$

when the eccentricity is small.

If B is the mean flux density at the mean radius, then it follows from Gauss's theorem that

$$2\pi\left(R+\frac{g}{2}\right)B = 4\pi m.$$

$$\therefore \quad F = \frac{B^2 d(R+g/2)}{4\sqrt{g^2-d^2}}.$$

The exact distribution of flux happens to be soluble for this problem, but it is interesting to see how near an approximation would have been obtained by assuming that the flux density at any point was inversely proportional to the distance between the cylinders at that point. Thus consider Fig. 121 which shows two cylinders whose radii are not very different and whose axes are displaced a distance d. Then, if B_P is the flux density at the point P, we shall suppose that $B_P . PQ = Bg$, and to the first order $PQ = (g-d\sin\theta)$. The pull per unit area at P is $\frac{B_P^2}{8\pi}$ and at P' it is $\frac{B_{P'}^2}{8\pi}$. The net upward pull due to elements at P and at P' is

$$\begin{aligned} dF &= \frac{(B_P^2-B_{P'}^2)}{8\pi}\sin\theta R d\theta \\ &= \frac{B^2 g^2}{8\pi}\left\{\frac{1}{(g-d\sin\theta)^2}-\frac{1}{(g+d\sin\theta)^2}\right\}R\sin\theta\, d\theta \\ &= \frac{B^2 g^2}{8\pi}\times\frac{4dgR\sin^2\theta\, d\theta}{(g^2-d^2\sin^2\theta)^2} \\ &\doteqdot \frac{B^2}{2\pi}\frac{dR}{g}\sin^2\theta\, d\theta, \text{ if } g^2 \gg d^2. \end{aligned}$$

$$\begin{aligned} \therefore \quad F &\doteqdot 2\frac{B^2 dR}{2\pi g}\int_0^{\frac{1}{2}\pi}\sin^2\theta\, d\theta \\ &= \frac{B^2 Rd}{4g} \\ &= \frac{B^2 d(R+g/2)}{4g}\left(1+\frac{g}{2R}\right). \end{aligned}$$

The approximate solution is thus seen to be substantially in agreement with the exact solution so long as g is small compared with R, and d is small compared with g. But the total pull on the surface of the core is $\frac{B^2}{8\pi}\times 2\pi(R+g/2) = \frac{B^2(R+g/2)}{4}$, and so we find that the pull due to a small eccentricity d, is d/g of the total pull: this is a simple way of remembering the result.

12. Magnetic p.d. across a uniform cylindrical air gap

The same flux crosses every cylindrical shell between the two iron surfaces, and therefore $Br = B_i R_i$.

$$\therefore \quad \text{Magnetic p.d.} = \int_{R_i}^{R_0} B\,dr$$

$$= B_i R_i \int_{R_i}^{R_0} \frac{dr}{r}$$

$$= B_i R_i \log_\epsilon\left(1+\frac{g}{R_i}\right).$$

$$\therefore \quad 4\pi IT \doteqdot B_i g\left(1-\frac{g}{2R}+\frac{g^2}{3R^2}\right), \text{ if } \frac{g}{R} \ll 1.$$

Numerical example. Ignoring the reluctance of the iron, find the number of ampere turns required to produce a flux density of 5,000 lines per sq. cm. on the surface of the inner cylinder in Fig. 55 of Chap. II if the air gap is 1 cm. and the radius of the core is 4 cm.

$$\frac{4\pi IT}{10} = 5000\times 1\left(1-\frac{1}{8}+\frac{1}{48}\cdots\right).$$

$$\therefore \quad IT = 4000\times\frac{43}{48}$$

$$= 3580. \quad \text{ANS.}$$

EXAMPLES TO CHAPTER III

1. A laminated iron ring has a cross-section of 100 sq. cm. and a mean length of 100 cm. It has two separate windings, one of 1,000 turns and one of 40 turns. The 1,000-turn coil is connected to a high-resistance voltmeter: current through the 40-turn coil is varied, from a positive maximum to a

Time in secs.	0	10	20	30	40	50	60	70	80	90	100	110
Current in A.	13·6	3·0	0·4	−1·0	−1·7	−2·4	−2·8	−3·0	−3·2	−3·4	−3·5	−3·6

Time in secs.	120	130	140	150	160	170	180	190	200	210	220
Current in A.	−3·7	−3·8	−4·0	−4·4	−5·0	−5·5	−6·4	−7·4	−8·6	−10·4	−13·6

negative maximum, in such a way that the voltage induced in the other winding has a constant value of 0·1 V. The relation between current and time is given in the table. Plot the hysteresis cycle for the iron and find the energy loss in joules per cycle. (M.S.T. 1928.)

See p. 116. $V = A\dfrac{dB}{dt} \times 10^{-8}$. $\therefore \quad \dfrac{db}{dt} = 100$ lines sq. cm./sec.

$\therefore \quad B_{max} = \dfrac{22000}{2} = 11$ kilolines sq. cm.

$Hl = \dfrac{4\pi IT}{10}$. $\therefore \quad H = 0{\cdot}5I$. Hence plot cyclic curve.

Voltage induced in magnetizing winding $= \dfrac{40}{1000} \times 0{\cdot}1$ V. $= 4$ mV.

Energy loss per cycle $= \dfrac{4}{10^3} \times$ mean current $\times\, 220$.

Or, alternatively, find area of cyclic curve and multiply by volume of iron.

ANS. 6·5 joules.

2. The uniform radial magnetic field in the air gap of a flux meter has a strength of 2,000 lines sq. cm.: the core is 1 cm. radius and 2 cm. long, and the coil has 250 turns. Find the calibration in flux turns/radian. See p. 122.

ANS. 2×10^6.

3. Show that if friction be neglected, the damping torque of a ballistic galvanometer of usual type is proportional to $\dfrac{1}{R}\dfrac{d\theta}{dt}$, where R is the resistance of the galvanometer circuit and θ is the deflexion at any time t.

A ballistic galvanometer is connected in series with the secondary coil of a standard field whose constant is 800,000 flux-turns linked per ampere flowing, the total resistance of the galvanometer circuit being 10,000 ohms. When 2 amp. are reversed in the primary circuit of the standard field the first resulting fling of the galvanometer is 100 divisions and the sixth to the same side is 20 divisions. The standard field is now replaced by a condenser charged to a p.d. of 10 volts. If the fling on discharging the condenser through the galvanometer be 120 divisions, deduce the capacity of the condenser. (M.S.T. 1923.)

See p. 124. Quantity of electricity through galvanometer $= 3{\cdot}2\ \mu$C: damping factor $\Delta = 1{\cdot}08$. If undamped, $q = \dfrac{3{\cdot}2}{108}\mu$C/division. For condenser discharge $\Delta = 1$, because current flow is negligible.

ANS. $C = 0{\cdot}355\ \mu$F.

4. The particulars of a 'standard field' are as follows:

Primary coil: 800 turns, mean diam. 10 cm., length 50 cm.

Secondary coil: 200 turns, mean diam. 6 cm.

The secondary coil of the standard field is put in series with a slow-period ballistic galvanometer and with a single turn wound round the root of one pole of a 4-pole wave-wound generator. Reversal of 4 amp. in the primary of the standard field gives a fling of 125 divisions of the galvanometer, and when the full excitation of the generator is rapidly reduced to zero a fling of 197 divisions results. If the flux entering the armature is $1{\cdot}13 \times 10^6$ lines, deduce the value of the dispersive coefficient of the magnetic circuit. (M.S.T. 1924.) ANS. 1·24: constant of standard field = $1{\cdot}11 \times 10^5$ flux turns/amp.

5. The mean diameter of an iron ring is 15 cm., and it is wound uniformly with a magnetizing winding of 300 turns. The area of cross-section is 2 sq. cm., and there is a concentrated search coil of 10 turns, connected to a flux meter whose calibration is 15×10^3 flux-turns/division. On reversing a current of 2 amp. in the magnetizing winding, the change of flux-meter reading is 40 divisions. Find the flux density in the iron and the permeability.

ANS. $H = 8I$. $2B \times 2 \times 10 = 15 \times 10^3 \theta$. $\therefore B = 15{,}000$ lines sq. cm. $\mu = \dfrac{15000}{16}$.

6. A cast-iron ring is 10 cm. in mean diameter, 1·5 sq. cm. in cross-sectional area, and is wound uniformly with a coil of 100 turns. Use the curve of Fig. 82 to find the suitable number of turns for the search coil so that a flux density of 8,000 lines sq. cm. may be measured with a flux meter whose whole range is a change of 15×10^5 flux-turns: find also the necessary magnetizing current.

ANS. About 60 turns and 13 amp.

7. Why is it necessary to know the weight of iron in order to calculate the cross-sectional area of a ring specimen which is built up of laminations?

8. On what basis does the 'Steinmetz law' (hysteresis loss = $\eta B^{1{\cdot}6}$ ergs per c.c. per cycle) rest? How do the hysteresis and eddy-current losses vary with the speed and the temperature?

An iron specimen has a volume of 200 c.c.: when rotated 50 times a second in a field giving an induction of 5,000 the total loss in hysteresis and eddy currents is 5 watts. Find the total loss with an induction of 10,000 and a speed of 25 revs. per sec., if the Steinmetz coefficient for the iron be 0·0015. (I.C.E. 1920.)

ANS. Eddy loss, 3·71 W.; hysteresis loss, 1·93 W. See pp. 141 and 134.

9. Describe a method of finding a (B, H) cyclic curve for a sample of sheet steel.

The table below gives the loss (W) in a specimen of sheet steel for various values of the maximum flux-density (B) and the frequency (n).

B	n	W
5,000	20	6·2
	40	15·6
	60	28·0
9,000	20	16·8
	40	43·4
	60	80·0

Show that these figures indicate that the hysteresis loss in the specimen is proportional to nB^x, and deduce the value of x. (M.S.T. 1931.) ANS. 1·63.

10. Show that the hysteresis loss in a magnetic cycle is proportional to the area of the (B, H) curve.

An iron-cored choking coil has a core loss of 70 watts with a given applied p.d. at a frequency of 70 cycles per second: the eddy-current loss is then one-quarter of the hysteresis loss. Find the core loss with the same applied p.d. at a frequency of 50. (I.C.E. 1927.) ANS. 82·5 W.

11. Give a complete diagram of the apparatus and connexions for determining the (B, H) curve for an iron ring specimen.

In a test the specimen consisted of 25 rings of thickness 0·01 in., internal diameter 4 in., and external diameter 5 in. Primary turns 80; secondary turns 200. Constant of standard field 6×10^5 lines per ampere. Galvanometer fling from standard field for 1·5 amp. reversed, 14 divisions. Fling from specimen for 1·2 amp. reversed, 20 divisions. Calculate from these data the mean values of H and B. (I.C.E. 1926.) ANS. $B = 8{,}000$ lines sq. cm.; $H = 3·36$.

12. A specimen of iron transformer sheet is tested, and the hysteresis loss per c.c. per cycle is 700 ergs when the maximum B in the cycle is 4,000. A transformer is built containing 60 lb. of this iron, and the total loss in hysteresis and eddy currents at 50 cycles per second is found to be 16 watts, with the same maximum B. What will be the total loss for a maximum B of 5,500 at a frequency of 60 cycles per second? (Specific gravity of iron = 7·7.) (I.C.E. 1926.)

ANS. $B = 4{,}000$. Hysteresis loss = 12·4 W.; eddy loss = 3·6 W.; total loss at $B = 5{,}500$ is 35 W.

13. Show that the eddy-current loss in sheet iron varies inversely as the square of the thickness of the sheet.

The total iron loss in a transformer core when working at a B of 5,000 and a frequency of 50 cycles per second is 30 watts. The weight of iron is 20 kg., the specific gravity 7·7, and the coefficient η in the Steinmetz formula $h = \eta B^{1·6}$ is 0·0015. Calculate the eddy-current loss. (I.C.E. 1921.) ANS. 10·5 W.

14. Calculate the eddy-current loss in W./lb. in Stalloy plates 0·5 mm. thick at a frequency of 100 cycles/sec. and at a flux density of 12,000 lines sq. cm.

ANS. 0·576 W./lb. See p. 139.

15. Calculate the maximum thickness of Stalloy plates for which it is permissible to assume that the eddy-current loss varies as the square of the frequency, for a frequency of 25 cycles/sec. ANS. 1·4 mm. See p. 142.

16. A ring specimen has an inner radius of 6 cm. and an outer radius of 8 cm. Assuming a constant permeability, estimate the difference between the mean flux density and the flux density at the mean radius.

ANS. 1·24 %. See p. 146.

17. A ring specimen of Stalloy has a mean diameter of 15 cm. and there is an air-gap of 0·3 mm. width cut radially across the ring.

How many ampere turns are required to produce a flux density of 11,000 lines across the air-gap? The coercive force for this maximum flux density is $H_c = 1·75$; estimate the residual flux density when the current is broken.

ANS. 430 ampere turns: about 2,750 lines sq. cm. See pp. 150 and 152.

18. What do you understand by *magnetic force, magnetomotive force*?

The table below gives figures for a complete magnetic circuit through the armature and yoke of a dynamo and along the centre-lines of two adjacent poles.

Part of circuit.	Length in cm.	B in c.g.s. units.
Each air-gap . . .	0·3	7,000
Each armature tooth . .	3	15,800
Armature core . . .	15	13,000
Each pole body . . .	11	16,000
Yoke	60	10,000

The poles and yoke are of cast steel and the armature stampings are of Stalloy. Calculate the ampere turns per pole. Ans. 2,750.

If the effective area of a pole face is 600 sq. cm. and the dynamo has 4 poles and is wave-wound with 418 conductors, estimate the e.m.f. generated at 750 r.p.m. (M.S.T. 1931.) Ans. 435 V.

19. In a closed solenoid with an air core, show that the energy stored per cubic centimetre of the core is $\frac{H^2}{8\pi}$ ergs, where H is the strength of the magnetizing force in the core in c.g.s. units.

An iron ring, made of iron whose magnetic properties are given below, has a mean diameter of 15 cm. and a cross-section of 8 sq. cm. A slit 2 mm. wide is made in the ring at one point. Calculate the ampere turns required on the ring to give a pull of 40 kilogrammes tending to close up the slit. (M.S.T. 1921.)

B	8,000	10,000	12,000
H	3·15	4·45	6·60

Ans. $B = 11,100$ lines sq. cm., 2,000 ampere turns.

20. Define the terms *magnetic force, induction, reluctance.*

A ring of iron measures 40 cm. along its axis: it is magnetized with $H = 8{\cdot}25$. Find the ampere turns required. A slot of 1 mm. is cut in it: if the relation between B and H is given by a straight line from $H = 1{\cdot}27$ ($B = 2,460$) to $H = 1{\cdot}45$ ($B = 3,080$), and if the magnetizing current is the same, find the new induction and magnetizing force. (M.S.T. 1919.)

Ans. 261 ampere turns: $B = 2,640$, for which $H = 1{\cdot}36$.

21. Define *magnetomotive force* and *reluctance.*

A soft iron ring, 20 sq. cm. in section and 10 cm. in mean radius, has an air-gap 2 mm. long cut at one point. It is wound with a coil of 1,000 turns. What current will be required to produce a pull of 100 kg. tending to close up the gap if the (B,H) curve of the iron be such that when $H = 5$, $B = 10,000$, and when $H = 7$, $B = 12,000$, the curve being practically straight over this range? (M.S.T. 1916.) Ans. $B = 11,100$ lines sq. cm.; 2·06 A.

22. If the flux density in mumetal is vanishingly small, find the angle at which a line of induction emerges from the iron if it is inclined at 85 degrees to the normal inside the iron. See p. 162. Ans. 4·4′ from normal.

23. A long straight wire, carrying a current of 100 amp., runs parallel to an iron plate and 10 cm. above the surface. Estimate the force of attraction between the plate and the current. See p. 164. Ans. 10 dynes/cm.

IV

EQUATIONS OF ELECTROMAGNETISM AND SOME SPECIAL PROBLEMS

THE student should be able to express the fundamental laws of magnetism in mathematical symbols. This chapter is restricted to two-dimensional fields, whose variation occurs only in the xy-plane and not in the direction of the axis of z: the extension to a three-dimensional field follows readily and is seldom required by engineers.

1. Gauss's theorem expressed in Cartesian coordinates

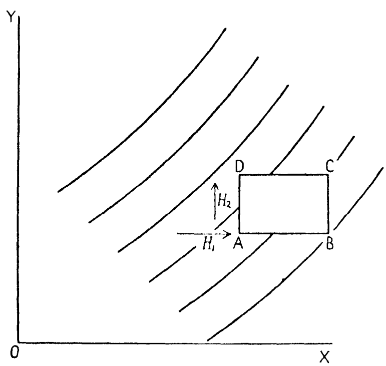


FIG. 122

Let Fig. 122 represent some lines of force of any two-dimensional field. Gauss's theorem states that the net flux passing out of any closed surface in the field is 4π times the enclosed pole strength. To express this in Cartesian coordinates, the suitable Gauss surface is a rectangular parallelepiped of sides dx and dy and of unit length into the paper, see Fig. 122. Let H_1 and H_2 be the X and Y components respectively of the magnetic force at some point A in the field. Then at B and D it is $H_1+\frac{\partial H_1}{\partial x}dx$ and $H_2+\frac{\partial H_2}{\partial y}dy$ respectively. No flux passes in or out of the face $ABCD$ because the field is two-dimensional, and has no component in this direction. The flux passing into the face whose sides are AD, and unity into the paper, is $H_1 dy 1$, and that passing out of the face CB is $\left(H_1+\frac{\partial H_1}{\partial x}dx\right)dy1$. Hence the net flux passing out of the rectangular block per unit length is

$$\left(H_1+\frac{\partial H_1}{\partial x}dx\right)dy-H_1dy+\left(H_2+\frac{\partial H_2}{\partial y}dy\right)dx-H_2dx$$
$$=\left(\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}\right)dxdy.$$

If the density of pole strength in the block is ρ, the pole strength enclosed is $\rho dxdy$. Hence Gauss's theorem expressed in Cartesian coordinates is

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y} = 4\pi\rho. \tag{1}$$

For every point in free space, that is to say, for every point not occupied by a pole,

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y} = 0. \tag{2}$$

When the reader sees this differential equation he should not regard it necessarily as an invitation to perform some recondite mathematical juggling, but should recognize it as a way of saying that the field results from an inverse square law. Equation (2) is simply a way of stating that lines of force end on poles. Or alternatively, if H_1 and H_2 are the components of velocity at a point in a fluid, it states that the fluid is incompressible; or, if H_1 and H_2 are components of electric current, the equation states that the current is steady and no charge is accumulating at any point.

There is another notation for writing (2) which is

$$\text{div}\, H = 0. \tag{3}$$

In this statement div is short for the word divergence and the equation states that H is a vector whose divergence is zero at every point; the word divergence here implies increase or decrease. A directed quantity, or vector, whose flux does not increase or decrease is called a 'solenoidal vector'.* In one mathematical notation it would be said that H is a solenoidal vector provided that div $H = 0$. We repeat once more that this seemingly strange and abstract statement is merely an alternative description of the familiar inverse square law.

Specific problems are often solved more elegantly by using the potential† V rather than the force H. The potential and the force are related by the equations

$$-\frac{\partial V}{\partial x} = H_1 \text{ and } -\frac{\partial V}{\partial y} = H_2. \tag{4}$$

An alternative and more comprehensive method of stating equations (4) is

$$H = -\text{grad}\, V, \tag{5}$$

where grad stands for the word gradient; equation (5) merely states

* See also Chap. III, p. 112. † See Chap. I, § 2.

that the force in any direction is equal to the gradient of the potential in the same direction.

Combining (3) and (4), the inverse square law can be expressed in terms of potential, and accordingly

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) = 0,$$

or

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2} = 0. \tag{6}$$

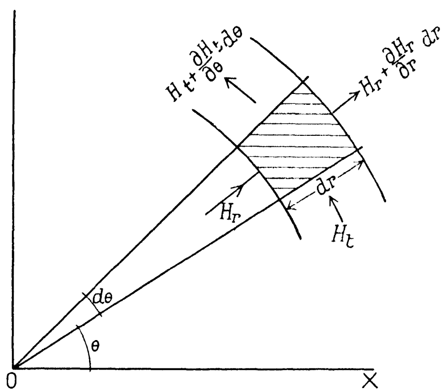


FIG. 123

This equation is called Laplace's equation and is often written more shortly as

$$\nabla^2 V = 0. \tag{7}$$

Any magnetic field must be derived from an inverse square law, and this is expressed in Cartesians by stating that the potential at every point of free space must be such that

$$\nabla^2 V = 0.$$

The reader must not regard Laplace's equation as a differential equation which can be solved by a standard process and made to turn out the answer to any problem. It is little more than a bare statement that the problem arises from the inverse square law and every solution has to be found on its own merits, by sheer hard thinking combined with guess-work: very few solutions have been found. The equation, considered as such, is the happy hunting-ground for the professional mathematician and the bane and irritation of the engineer, who regards it as little more than an obvious statement of the inverse square law.

Writing Laplace's equation in vector notation, we have

$$\text{grad div } H = 0. \tag{8}$$

2. Polar expression for Gauss's and Laplace's equations

Sometimes it is more convenient to split the field into radial and tangential components. Then the flux passing out of unit length of a prism having the cross-section shown shaded in Fig. 123 is

$$\left(H_r + \frac{\partial H_r}{\partial r}dr\right)(r+dr)d\theta - H_r r d\theta + \left(H_t + \frac{\partial H_t}{\partial \theta}d\theta\right)dr - H_t dr$$

$$= r\frac{\partial H_r}{\partial r}drd\theta + H_r drd\theta + \frac{\partial H_t}{\partial \theta}drd\theta.$$

But $$-\frac{\partial V}{\partial r} = H_r \quad \text{and} \quad -\frac{1}{r}\frac{\partial V}{\partial \theta} = H_t.$$

Hence $$\frac{\partial V}{\partial r} + r\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial^2 V}{\partial \theta^2} = 0,$$

or $$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0. \tag{9}$$

Equation (9) is a third way of expressing the inverse square law.

A solution of equation (9) is

$$V = Ar^n \cos n\theta,$$

for $$\frac{\partial V}{\partial r} = nAr^{n-1}\cos n\theta,$$

and $$\frac{\partial^2 V}{\partial r^2} = n(n-1)Ar^{n-2}\cos n\theta,$$

and $$\frac{\partial^2 V}{\partial \theta^2} = -n^2 Ar^n \cos n\theta.$$

Whence, substituting in (9), we have

$$n(n-1)+n-n^2 = 0.$$

So $V = Ar^n \cos n\theta$ is a possible solution of (9). It can be shown in the same way that $V = Br^n \sin n\theta$ is also a solution and that $V = Cr^{-n}\cos n\theta$ and $V = Dr^{-n}\sin n\theta$ are also possible solutions.

So a complete series solution is

$$V = \sum (A_n r^n + B_n r^{-n})(\cos n\theta + \sin n\theta).$$

But the field at a given point has a definite value, and so every time

θ increases by 2π, the value of V must repeat itself; this means that n must be an integer.

So $$V = \sum_{1}^{\infty} (Ar^n + Br^{-n})(\cos n\theta + \sin n\theta).$$

If it is obvious that the field is symmetrical about the line $\theta = 0$, then V at the point (r, θ) must be equal to V at the point $(r, -\theta)$; then the $\sin n\theta$ terms will not suit, and the solution simplifies to

$$V = \sum_{1}^{\infty} (Ar^n + Br^{-n})\cos n\theta.$$

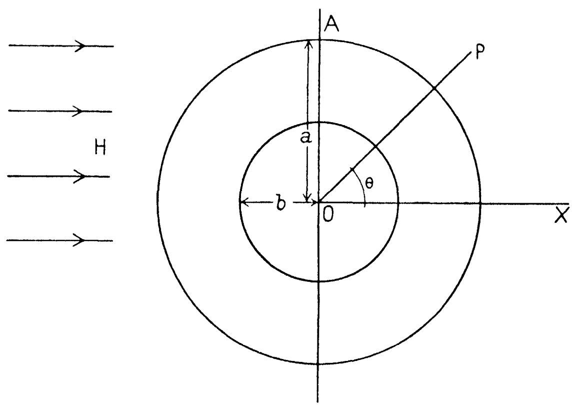


FIG. 124

3. Iron cylinder in uniform field

A uniform iron cylinder of constant permeability μ and having external and internal radii a and b respectively is placed with its axis perpendicular to a uniform field H. Find the field inside the cylinder, inside the iron, and outside the iron. Let V_1, V_2, and V_3 be the potentials outside the iron, in the iron, and inside the cylinder respectively.

At an infinite distance from the cylinder the original field will be undisturbed, and there

$$H_r = H\cos\theta$$
$$= -\frac{\partial V}{\partial r}.$$

But $$-\frac{\partial V}{\partial r} = -\sum_{1}^{\infty} \{nAr^{n-1} - nBr^{-(n+1)}\}\cos n\theta.$$

So in this problem n can have the value unity only, since at infinity

$$H\cos\theta = -\left(A - \frac{B}{r^2}\right)\cos\theta.$$

But A must equal $-H$, since at infinity the term $\frac{B}{r^2}$ is zero.

So $$V_1 = -\left(Hr - \frac{B}{r}\right)\cos\theta.$$

At each interface the tangential component of H and the normal component of B must be continuous. (NOTE.—Remember that in the iron V refers to H only, see p. 111.)

Therefore $$\left(\frac{\partial V_1}{\partial r}\right)_{r=a} = \mu\left(\frac{\partial V_2}{\partial r}\right)_{r=a}$$

and $$\frac{1}{a}\left(\frac{\partial V_1}{\partial \theta}\right)_{r=a} = \frac{1}{a}\left(\frac{\partial V_2}{\partial \theta}\right)_{r=a}$$

and similar relationships between V_2 and V_3.

$$\left(\frac{\partial V_1}{\partial r}\right)_{r=a} = -\left(H + \frac{B}{a^2}\right)\cos\theta.$$

Since this must be equal to $\left(\frac{\partial V_2}{\partial r}\right)_{r=a}$ for all values of θ, we see that the only permissible value for n is unity in the expression for V_2, and likewise in V_3.

So $$V_2 = -\left(Cr - \frac{D}{r}\right)\cos\theta$$

and $$V_3 = -\left(Er - \frac{F}{r}\right)\cos\theta.$$

When $r = 0$, V_3 must be finite, so F must be zero; whence

$$V_3 = -Er\cos\theta.$$

Applying the four boundary conditions, it follows that

$$H + \frac{B}{a^2} = \mu\left(C + \frac{D}{a^2}\right),$$

$$H - \frac{B}{a^2} = C - \frac{D}{a^2},$$

$$\mu\left(C + \frac{D}{b^2}\right) = E,$$

$$C - \frac{D}{b^2} = E.$$

Hence it follows that

$$C = -\frac{D}{b^2}\frac{(\mu+1)}{(\mu-1)},$$

$$E = -\frac{2\mu}{\mu-1}\frac{D}{b^2},$$

$$E = \frac{4\mu H}{(\mu+1)^2-(\mu-1)^2\frac{b^2}{a^2}},$$

and

$$B = \frac{\mu-1}{\mu+1}\frac{1-\frac{b^2}{a^2}}{1-\frac{(\mu-1)^2}{(\mu+1)^2}\frac{b^2}{a^2}}a^2H.$$

The quantity E is the strength of the uniform field in the hollow inside of the cylinder; since $\mu \gg 1$, we have

$$E \doteqdot \frac{4}{\mu}\frac{H}{1-\frac{b^2}{a^2}}. \tag{10}$$

When the inside radius is small compared with the outside radius the internal field tends to the value $\frac{4}{\mu}$ of the undisturbed field outside the cylinder. The inside of the cylinder is screened magnetically from the outside field; it should be noticed that with finite permeability the inside field cannot be reduced to zero however thick the cylinder. A hollow cylinder of iron is often used to screen a galvanometer from the magnetic field of the earth. In very weak fields the permeability of soft iron is constant and has a limiting value of round about 200,* so a single shield of soft iron cannot reduce the internal field to less than about 2 per cent. of the external value. The new magnetic alloys called permaloy and mumetal (see p. 127) have a very large value of μ in very weak fields, and consequently these make much more effective galvanometer shields than ordinary iron. More effective shields can be made by using several concentric shields than by using one very thick shield; at first sight this seems surprising, but it may be seen as follows. The field inside the cylinder is uniform, and so if another much smaller cylinder is placed inside the first, the field inside it will be approximately equal to $\frac{4}{\mu}$ of the field

* But compare this with the value for mumetal; see Chap. III, Fig. 86.

inside the outer cylinder. The final internal field will then be approximately equal to $\left(\frac{4}{\mu}\right)^2$ of the value outside the outer cylinder.

The applied external field induces a polarity over the outside surface and over the inside surface of the cylinder, and the intensity of this surface magnetization can be found as follows. Inside the iron the radial component of H is

$$H = -\frac{\partial V_2}{\partial r}$$

$$= \left(C+\frac{D}{r^2}\right)\cos\theta$$

$$= D\left(-\frac{\mu+1}{\mu-1}\frac{1}{b^2}+\frac{1}{r^2}\right)\cos\theta.$$

$$\therefore \quad H_a = D\left(\frac{1}{a^2}-\frac{\mu+1}{\mu-1}\frac{1}{b^2}\right)\cos\theta,$$

and

$$H_b = -\frac{2D}{b^2}\frac{1}{\mu-1}\cos\theta.$$

Now

$$B = H+4\pi I.$$

$$\therefore \quad I = \frac{\mu-1}{4\pi}H.$$

So the surface density of magnetization has the values

$$I_a = \frac{\mu-1}{4\pi}D\left(\frac{1}{a^2}-\frac{\mu+1}{\mu-1}\frac{1}{b^2}\right)\cos\theta,$$

and

$$I_b = -\frac{2D}{4\pi}\frac{\cos\theta}{b^2}.$$

Thus the surface density of magnetization varies as $\cos\theta$. It should be remembered that a surface density of magnetization which varies as $\cos\theta$ produces a uniform field inside the cylinder; this has applications in many engineering problems. (See also Chap. II, § 10, p. 93.)

3 a. Linear current along the axis of the cylinder

Let a current i be placed along the axis of the cylinder. If the current was in the undisturbed field outside the cylinder there would be a force $F = Hi$ per unit length tending to push the current perpendicular to the field. But when the current is surrounded by the iron cylinder the current is situated in a very weak magnetic field, and then the force on the conductor tends to the value $F = \frac{4Hi}{\mu}$:

so the iron tube screens the force from the wire. In Chapter I we calculated the forces on currents placed in magnetic fields, and now it would seem as if these forces could be reduced to very small values by enclosing the current in an iron tube. This is true; but there will then be a force tending to move the tube perpendicular to the field, and it can be shown that the force on the tube plus the force on the wire is equal to the force which would be on the wire if it was in

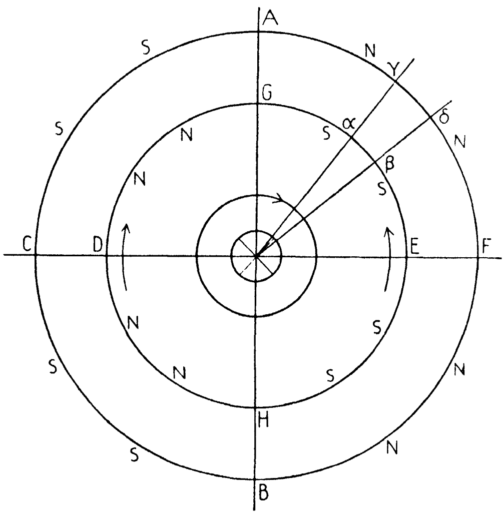


FIG. 125

the undisturbed field. This proposition is of great practical importance in the construction of dynamos, where the conductors are buried in slots or in tunnels in the iron of the armature. It is a great practical convenience to place the conductors in slots, but in doing so it places them in a much weaker magnetic field. The purpose of the conductor is to produce a force on the armature tending to rotate it. If burying the conductor reduced the force on the armature the purpose of the machine would be defeated. But in fact the force on the armature is not reduced, but most of the force which would have been on the conductor is now transferred to the iron. This is very desirable because the driving force does not come on the insulation, which would readily be damaged thereby.

It will now be shown that the force on the tube plus the force on the wire is equal to the force which would be on the wire in the absence of the tube. Fig. 125 shows the wire at the centre of the tube. The semi-circumference ACB carries a south induced polarity and the semi-circumference GDH carries a north polarity; there is S on GEH and N on AFB. This surface polarity is acted on by the field of the wire, whose lines of force are concentric circles. The action of the field on the internal polarity gives a general upward tending force, as shown by the arrows; the action of the field on the external polarity gives a downward force. The downward component of force due to elements of surface such as $\alpha\beta$ and $\gamma\delta$ is

$$dF_V = 2i\left(-\frac{I_b}{b}bd\theta+\frac{I_a}{a}ad\theta\right)\cos\theta$$

$$= 2i(I_a-I_b)\cos\theta\,d\theta.$$

$$= 2i\frac{(\mu-1)}{4\pi}D\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos^2\theta\,d\theta.^*$$

$$\therefore\quad F_V = \frac{4i(\mu-1)(b^2-a^2)}{4\pi a^2b^2}D\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\cos^2\theta\,d\theta$$

$$= \frac{i(\mu-1)(b^2-a^2)}{2a^2b^2}D$$

$$= \frac{i(\mu-1)^2(a^2-b^2)}{4\mu a^2}E$$

$$= \frac{(\mu-1)^2\left(1-\frac{b^2}{a^2}\right)Hi}{(\mu+1)^2-(\mu-1)^2b^2/a^2}$$

$$= \left\{1-\frac{4\mu}{(\mu+1)^2-(\mu-1)^2b^2/a^2}\right\}Hi.$$

But the force on the wire is

$$F'_V = Ei = \frac{4\mu}{(\mu+1)^2-(\mu-1)^2\frac{b^2}{a^2}}Hi.$$

$$\therefore\quad F_V+F'_V = Hi.$$

So we arrive at the remarkable discovery that the force on the tube and the force on the wire together equal the force which would have

* See p. 201.

been on the wire in the absence of the tube. The upward force on the inner surface of the iron is

$$F''_V = 4i \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} I_b \cos\theta \, d\theta,$$

$$= \frac{8Di}{4\pi b^2} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \cos^2\theta \, d\theta$$

$$= -\frac{iD}{b^2}$$

$$= \frac{\mu-1}{2\mu} Ei$$

$$\doteq \frac{Ei}{2}.$$

3 b. E.m.f. induced in a circuit shielded by iron tubes

Now suppose the shielded wire shown in Fig. 125 is part of a circuit which is being moved across a field, whose strength changes slowly with distance so that each side of the circuit can be considered to be in a sensibly constant field at any given moment. The conductor is cutting across a very weak field, so by applying the cutting rule it would seem as if the circuit was shielded from the induced e.m.f. But when the cutting rule was deduced, in Chapter II, § 2, it was assumed that the field strength at any given point of space had no time rate of change due to the movement of the circuit; on this supposition only is the cutting rule correct. But when the conductor is shielded in an iron tube, the field strength at a given point of space, within the circuit, is changing because the tube is approaching it. So the e.m.f. calculated by the cutting rule will not be the whole e.m.f. round the circuit. The e.m.f. induced by the time rate of change of the flux in space must also be reckoned and added to that found from the cutting rule. If this were done the result would be the same as if the cutting rule had been applied to the unshielded conductor moving across the undisturbed field.

So the e.m.f. generated by a dynamo is unaffected by placing the conductors in slots, and can be calculated by applying the cutting rule to the field which would exist in the absence of the slots.

3 c. Field inside a spherical iron shell

The solution of this problem is very similar to that for the tube, and will not be given in detail. It can be shown that the field inside the shell is uniform and its strength is

$$E = \frac{9\mu H}{2\left\{(1+\mu)^2-(\mu-1)^2\frac{b^3}{a^3}+\frac{\mu}{2}\right\}}. \tag{11}$$

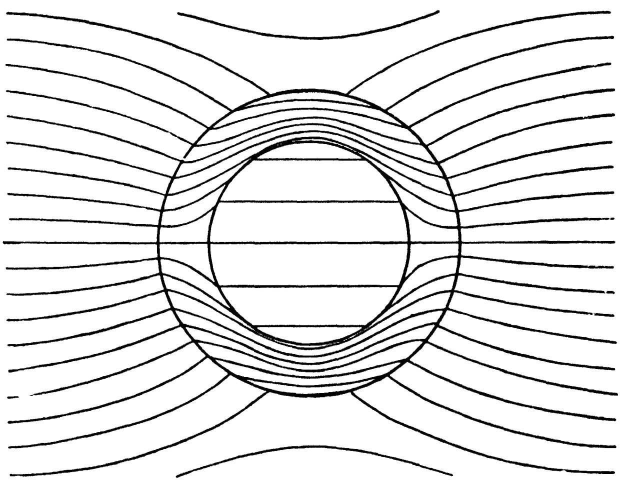


FIG. 126

When μ is large and the shell is thick, this tends to the value

$$E = 4{\cdot}5\frac{H}{\mu}.$$

Fig. 126 shows the field round and in a spherical shell in which $a/b = 3/2$ and in which $\mu = 14$.

If the shell is solid, it may be shown that the external potential is

$$V = -Hr\cos\theta+\frac{\mu-1}{\mu+2}H\frac{a^3}{r^2}\cos\theta.$$

3 d. The field outside a cylinder and a sphere

The magnetic potential V_1 outside a cylinder is

$$V_1 = -Hr\cos\theta + B\cos\theta/r.$$

The second term of this expression for V is the potential of two oppositely directed equal currents i, separated by a small distance d, such that $B = 2id$. This can be seen as follows. Consider first the magnetic potential due to a long straight current; then

$$-\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{2i}{r}$$

and

$$-\frac{\partial V}{\partial \theta} = 0.$$

$$\therefore \quad V = -2i\theta + K.$$

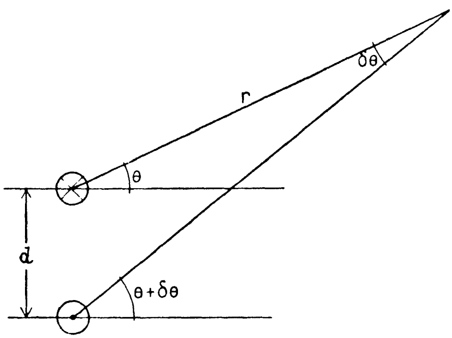


FIG. 127

Now consider the potential due to equal and oppositely directed currents as shown in Fig. 127.

$$\begin{aligned} V &= 2i(\theta+d\theta)-2i\theta \\ &= 2id\theta \\ &= \frac{2id\cos\theta}{r}, \text{ since } d\cos\theta \doteqdot rd\theta. \end{aligned}$$

So the field outside the cylinder is the same as would be produced by placing two long equal and oppositely directed currents in the field, each of strength i and separated by an infinitesimal distance d, such that $2id = B$; the distance d being perpendicular to the uniform field.

When the cylinder is solid,

$$B = \frac{\mu-1}{\mu+1}a^2H,$$

whence

$$id = \frac{\mu-1}{\mu+1}\frac{a^2H}{2}.$$

It follows in a similar manner that the field outside a sphere could be reproduced by removing the sphere and placing at its centre a small coil of vanishingly small area A, carrying a current i such that

$$Ai = \frac{\mu-1}{\mu+1}a^3H.$$

The plane of the small coil is placed perpendicular to the plane of the field.

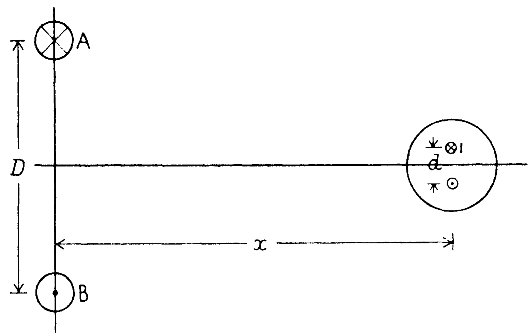


FIG. 128

4. The increase of self-inductance of a circuit due to the presence of iron in the field

If a lump of iron is placed in the field of a current, the induced polarity will increase the flux through the circuit and thereby increase the self-inductance. An approximate expression for the increment due to a long iron cylinder of small cross-section may be found as follows. Consider Fig. 128, which shows a small cylinder of radius a in the field of two long straight currents, placed at A and B. If the cylinder is small compared with D and x, it will be in a substantially uniform field H and so the external field may be imitated by replacing the cylinder by two long parallel wires carrying a current i and separated by a distance d such that

$$id = \frac{\mu-1}{\mu+1}\frac{a^2H}{2}.$$

The flux from circuit (1) which threads the equivalent circuit (2) is Hd per unit length.

$$\therefore \quad M_{12} = Hd.$$

The flux through circuit (1) in the presence of circuit (2) is

$$\begin{aligned}\phi &= L+M_{21}i \\ &= L+M_{12}i, \text{ since } M_{12} = M_{21} \text{ (see p. 44)}, \\ &= L+Hdi \\ &= L+H\frac{\mu-1}{\mu+1}\frac{a^2H}{2} \\ &= L+\frac{\mu-1}{\mu+1}\frac{a^2H^2}{2}.\end{aligned}$$

So the iron increases the inductance by an amount which is approximately equal to $\frac{a^2H^2}{2}$ per unit length.

For the arrangement shown in Fig. 128,

$$H = \frac{2D}{x^2+\frac{D^2}{4}}.$$

$$\therefore \quad \phi \doteqdot L+\frac{2D^2a^2}{x^4}. \tag{12}$$

The attractive force between the circuit and the cylinder may be found by calculating the attraction between the circuit, and the circuit equivalent to the iron.

Following the same process for a small sphere placed in the vicinity of a circuit, we have

$$\begin{aligned}\phi &= L+M_{21}i \\ &= L+HAi \\ &= L+\frac{\mu-1}{\mu+2}a^3H^2.\end{aligned} \tag{13}$$

5. The potential of a long straight current expressed as a Fourier series

Let O' in Fig. 129 represent the cross-section of a current flowing into the paper, then V, its potential at P, is $V = i\psi$. Sometimes it is inconvenient to take the origin at the current; let the origin be taken at O. It is possible to express ψ as a Fourier series of θ, and it can be shown that

$$\left.\begin{aligned}\psi &= -\sum_1^\infty \frac{1}{n}\frac{r^n}{c^n}\sin n\theta \\ \text{when} \quad & r < c,\end{aligned}\right\} \tag{14a}$$

and
$$\left.\begin{aligned} \psi &= \theta+\sum_{1}^{\infty}\frac{1}{n}\frac{c^n}{r^n}\sin n\theta \\ r &> c. \end{aligned}\right\} \tag{14 b}$$
when

5 a. Two-core cable inside an iron tube

We can use the previous expression to determine the field of a two-core cable which is shielded in an iron tube. Let each core of the cable be placed at a distance c from the axis of the tube whose external and internal radii are a and b respectively. Outside the

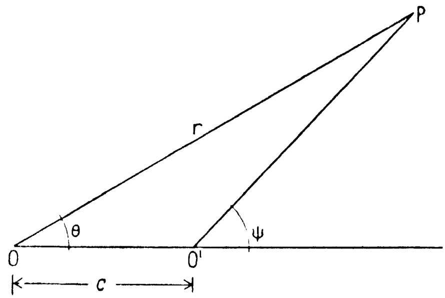


FIG. 129

cylinder V must be of the form $V = \sum_{1}^{\infty}(Ar^n+Br^{-n})\cos n\theta$. By considering the symmetry of the field we see that the potential at (r,θ) is equal and opposite to the potential at the point $(r,-\theta)$; so the cosine terms are inadmissible. Also V at the point (r,θ) must equal V at the point $(r,-\theta)$, so it follows that only odd multiples of θ are admissible. Hence

$$V = \frac{B_1}{r}\sin\theta+\frac{B_3}{r^3}\sin 3\theta+\frac{B_5}{r^5}\sin 5\theta+\dots.$$

By expressing the condition for the continuity of the normal component of flux density and the tangential component of magnetic force at the inner and outer surface of the tube, it can be shown that

$$B_{2n+1} = \frac{16\mu Ic^{2n+1}}{2n+1}\Big/\left\{(\mu+1)^2-\left(\frac{b}{a}\right)^{2n+2}(\mu-1)^2\right\}$$

$$\doteqdot \frac{16Ic^{2n+1}}{\mu(2n+1)\left\{1-\left(\frac{b}{a}\right)^{2n+2}\right\}}, \text{ when } \mu \gg 1.$$

If the tube is absent $\mu = 1$, and then

$$B_{2n+1} = \frac{4Ic^{2n+1}}{2n+1}.$$

For a thin tube of thickness t

$$B_{2n+1} \doteqdot \frac{8Ic^{2n+1}a}{\mu(2n+1)(n+1)t},$$

and then

$$V = \frac{8Ia}{\mu t}\left(\frac{c}{r}\sin\theta + \frac{c^3}{6r^3}\sin 3\theta + \frac{c^5}{15r^5}\sin 5\theta + \ldots\right). \tag{15}$$

So at a distance such that c^2/r^2, etc. can be ignored, the tube reduces the external field in the ratio

$$\frac{V}{V'} = \frac{2a}{\mu t}.$$

Thus suppose $\dfrac{t}{a} = \dfrac{1}{10}$ and $\mu = 1000$; then the tube reduces the external field to 2 per cent. of the value it would have without a screen.

6. Equations of current flow in a conductor

It is worth while to point out that the equations of current flow in a solid conductor of specific resistance ρ are the same as the equations for the magnetic field in a medium of permeability μ. The equations of current flow will now be developed so that the reader can turn readily from one problem to the other.

Let the current density have components i_1 and i_2 at a point P in a conductor of specific resistance ρ. The terms 'lines and tubes of current flow' are self-explanatory, and likewise the term 'current density'. Applying Ohm's law to the flow along an infinitesimal length of a tube of flow, we have

$$V - \left(V + \frac{\partial V}{\partial x}dx\right) = \rho\,dx\,i_1$$

or

$$-\frac{\partial V}{\partial x} = \rho i_1.$$

Similarly,

$$-\frac{\partial V}{\partial y} = \rho i_2.$$

In a steady flow of current, as much current must flow out of a closed surface as flows into it, and so (compare equation (1) of this chapter)

$$\left.\begin{aligned}\frac{\partial i_1}{\partial x}+\frac{\partial i_2}{\partial y} &= 0,\\ \text{i.e.}\qquad \operatorname{div} i &= 0.\end{aligned}\right\} \tag{16}$$

Hence the electric potential produced by the current flow obeys the same equation as the magnetic potential of a field, namely, $\nabla^2 V = 0$.

If a current flows across a boundary from a medium of specific resistance ρ_1 to a medium of specific resistance ρ_2, the normal component of current density must be continuous across the boundary, for otherwise electricity would continue to accumulate on the surface of separation. Hence,

$$\frac{1}{\rho_1}\frac{\partial V_1}{\partial n} = \frac{1}{\rho_2}\frac{\partial V_2}{\partial n},$$

where n stands for differentiation along the normal. The tangential component of electric force must also be continuous, so

$$\frac{\partial V_1}{\partial s} = \frac{\partial V_2}{\partial s},$$

where s stands for differentiation along the surface.

These are the same boundary conditions as those for the magnetic field and the lines of current flow will be refracted across the boundary, so that

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\rho_2}{\rho_1}. \tag{17}$$

These equations show that the lines of current flow coincide with the lines of magnetic force in a corresponding magnetic problem, and the solution for the current flow can be obtained from the solution of the corresponding magnetic problem by replacing μ by $1/\rho$.

For example, if a uniform current flow in one medium is disturbed by inserting a hollow cylinder of different conductivity, the lines of flow will alter till they take the same form as the lines of magnetic force in the problem described by Fig. 124. Thus suppose there is a uniform current flow through an ocean of salt water and that this flow is disturbed by inserting a thick copper pipe perpendicular to the flow. The current density in the salt water inside the pipe will be uniform, but very small compared with that in the undisturbed flow. The specific resistance of copper is $1{\cdot}68\ \mu\Omega/\text{cm.}^3$ and that of sea-water is $100\ \Omega/\text{cm.}^3$ So the current density inside a thick tube will tend to be reduced in the ratio $\dfrac{4\times 1{\cdot}68\times 10^{-6}}{10^2} = \dfrac{6{\cdot}72}{10^8}$.

In general there is a charge of electricity on the surface of separation, and if ρ varies through the medium there is in general a volume distribution of charge. (Compare Chap. III, p. 113.)

Reverting to the problem of the cylinder in the current flow: suppose that the cylinder is of copper and that the flow is in a solution of copper sulphate, then the conductivity of the copper may be treated as infinite compared with that of the solution. So then

$$V = -\rho I_1 r \cos\theta\left(1-\frac{a^2}{r^2}\right)$$

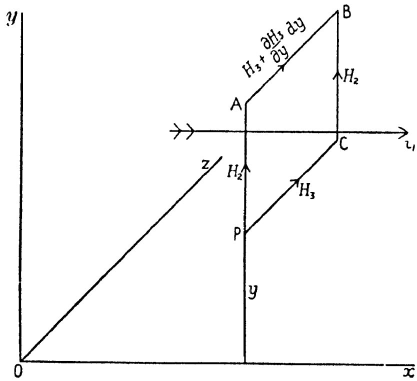


FIG. 130

and the presence of the cylinder will reduce the p.d. between two points situated at a distance r on either side of the centre of the cylinder and in a diametrical plane parallel to the current flow, in the ratio $\left(1-\frac{a^2}{r^2}\right):1$.

7. Cartesian expression for the work law

In deriving the work law, $\int H\,dl = 4\pi i$ (see Chap. I, § 7), from the equivalent magnetic shell there was a tacit assumption that the current was concentrated in a thin wire and that the threading path did not pass through the current flow. But a current in a wire of finite cross-section can be split up into streamlets each of which is the boundary of a shell, so there is no reason why the threading path should not pass through some shells and not through others, and there is no need to restrict the work law to paths outside the current: it has already been applied to a path inside a current flow in order to find the magnetic field inside a wire (see p. 52) and the magnetic field of a rational current element (see p. 23). To express the work law in Cartesian coordinates, consider a small rectangle, Fig. 130, of sides dz and dy with its corner at a point $P(x, y)$ in a two-dimensional current flow. Let i_1 be the x-component of current density through the rectangle; then the current enclosed by the area $PABC$ is $i_1 dydz$. Let H_1, H_2, and H_3 be the components of magnetic force at the point P; since the current flow is two-dimensional, neither of these components has any variation in the direction of the axis of z. Taking a unit pole round the rectangle $PCBAP$, and passing right-handedly round the current, the work done is

$$H_3 dz + H_2 dy - \left(H_3 + \frac{\partial H_3}{\partial y} dy\right) dz - H_2 dy = -\frac{\partial H_3}{\partial y} dydz.$$

So by the work law

$$-\frac{\partial H_3}{\partial y} dydz = 4\pi i_1 \, dydz.$$

$$\left.\begin{aligned} \therefore \quad -\frac{\partial H_3}{\partial y} &= 4\pi i_1. \\ \text{Similarly,} \quad \frac{\partial H_3}{\partial x} &= 4\pi i_2 \\ \text{and} \quad \frac{\partial H_1}{\partial y} - \frac{\partial H_2}{\partial x} &= 0. \end{aligned}\right\} \tag{18}$$

Equations (18) are expressed in vector notation by

$$\text{curl}\, H = 4\pi i, \tag{19}$$

where the word curl is descriptive of the line integral round a circuit.

In this process the current flowing through an area has been measured by performing an operation round the perimeter of the surface. But any number of non-planar surfaces can have the same perimeter, and so the process will not be possible unless the same current flows across each one of these areas having the same perimeter. Thus consider Fig. 131 which shows a section across two surfaces bounded by the same perimeter ACB. From the essential nature of a steady current flow, the same current flows across each of these two areas. Or, applying Gauss's theorem to the closed surface formed by the two areas which join round the perimeter ACB, we have

$$\frac{\partial i_1}{\partial x} + \frac{\partial i_2}{\partial y} = 0.$$

It can be shown by Stokes's theorem that equations of the form of

(18) are possible for any vector quantity F, provided only that $\operatorname{div} F = 0$.

If H_1, H_2, and H_3 are given at every point of space it is possible to calculate i_1 and i_2 at every point, but the converse is not possible; the H's cannot be calculated when the i's are specified. This may seem surprising since equations (18) seem to be three independent equations to find three unknowns.

But the three equations (18) are not really independent because

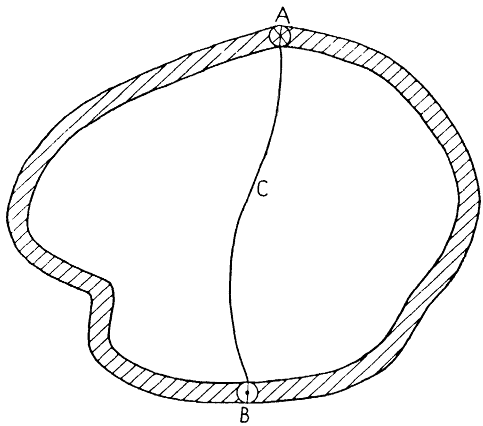


Fig. 131

they are interlinked by the necessary relation that $\operatorname{div} i = 0$. Thus

$$4\pi\frac{\partial i_1}{\partial x} = -\frac{\partial^2 H_3}{\partial x \partial y} = -4\pi\frac{\partial i_2}{\partial y}.$$

$$\therefore \quad \frac{\partial H_3}{\partial x} = 4\pi i_2,$$

since

$$\frac{\partial^2 H_3}{\partial x \partial y} = \frac{\partial^2 H_3}{\partial y \partial x}.$$

So the first two of equations (18) are not independent, and there are in reality only two equations to find three unknowns. Having specified the currents it is not possible to do more than find the difference of the H's between two points. The physical meaning of this is quite clear, for we can determine from equations (18) only the H's which are due to the currents which are recognized as such and have been curled round. A knowledge of the currents can say nothing about a superposed magnetic field which might exist from some external permanent magnets. Consider, for example, some specified current flow in a block of metal; though we can calculate the magnetic field which these currents would produce, we cannot calculate therefrom the strength of the superposed magnetic field of the earth.

8. The work law expressed in cylindrical polar coordinates

Following an analogous process round the elementary circuits shown in Fig. 132 it follows that

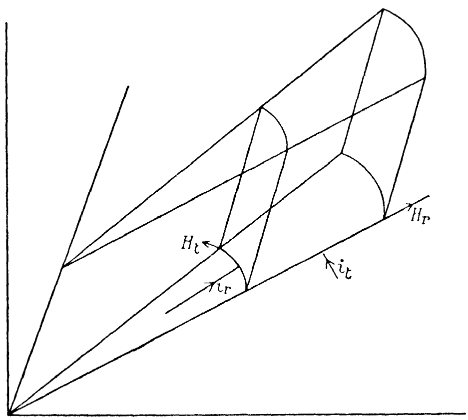


FIG. 132

$$\left.\begin{aligned} -\frac{1}{r}\frac{\partial H_3}{\partial \theta} &= 4\pi i_r \\ \frac{\partial H_3}{\partial r} &= 4\pi i_t \\ \frac{\partial}{\partial \theta}(rH_t)-\partial H_r &= 0 \end{aligned}\right\}. \tag{20}$$

For a two-dimensional current flow H_t and H_r must both be zero in so far as they arise from the current. Applying these equations to find the magnetic field due to a uniform current stream disturbed by an infinitely conducting cylinder, we have

$$-\frac{1}{r}\frac{\partial H_3}{\partial \theta} = 4\pi I\cos\theta\left(1+\frac{a^2}{r^2}\right)$$

and
$$\frac{\partial H_3}{\partial r} = -4\pi I\sin\theta\left(1-\frac{a^2}{r^2}\right).$$

Hence
$$H_3 = -4\pi Ir\sin\theta\left(1+\frac{a^2}{r^2}\right)+f(r).$$

The constant of integration, $f(r)$, cannot be determined, because at infinity we can say only that

$$\left(\frac{\partial H_3}{\partial r}\right)_{\theta=\frac{1}{2}\pi} = -4\pi I.$$

9. Law of induced e.m.f. expressed in Cartesian coordinates

The law of induced e.m.f., namely, $\int E\,dl = -\frac{d\phi}{dt}$, can also be expressed in Cartesian coordinates. Thus let H_1, H_2, and H_3 be the components of magnetic force at a point in a field, and E_1, E_2, and E_3 be the components of electric force at the same point.

Then by the same process of curling round small rectangular paths, it follows that

$$\left.\begin{aligned} \frac{\partial E_3}{\partial y}-\frac{\partial E_2}{\partial z} &= -\mu\frac{\partial H_1}{\partial t} \\ \frac{\partial E_1}{\partial z}-\frac{\partial E_3}{\partial x} &= -\mu\frac{\partial H_2}{\partial t} \\ \frac{\partial E_2}{\partial x}-\frac{\partial E_1}{\partial y} &= -\mu\frac{\partial H_3}{\partial t} \end{aligned}\right\}. \tag{21}$$

These three equations are interconnected by the necessary relation that

$$\frac{\partial E_1}{\partial x}+\frac{\partial E_2}{\partial y}+\frac{\partial E_3}{\partial z} = 0.$$

Expressed in vector notation these equations are

$$\left.\begin{aligned} \text{curl } E &= -\mu\dot{H} \\ \text{div } E &= 0 \end{aligned}\right\}. \tag{22}$$

(NOTE.—Previously equations have been derived for a two-dimensional field only; but the equations of a three-dimensional field are perfectly symmetrical and therefore are more readily written down than those for two dimensions. The reader should now be quite familiar with the process of deriving the curl or the divergence for any vector quantity, and therefore the three-dimensional equations will be used in the next few sections of this chapter.)

Here again if $\dot{H}_1$, etc. are given, E_1, etc. cannot be found because there are only two independent equations for the three components of electric force. It is possible to find only that portion of the electric force which is due to the rate of change of flux, and further information is required about the charges in the field before the electric force can be determined. It should be noticed that equations (21) do not necessarily lead to a cutting rule of the form $E = Bv$: it has been pointed out (see pp. 65 and 76) that the cutting rule for the e.m.f. round a circuit is true only in particular cases and the electric force at a point cannot be calculated from it.

10. An example which requires the use of equations (18) and (21). Current distribution in a long circular conductor

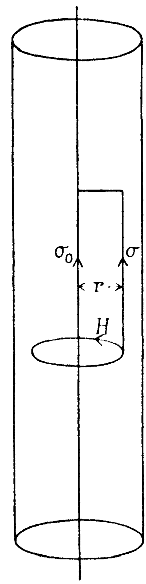


FIG. 133

The first approximation to the high-frequency resistance of a conductor was found in Chapter II (see p. 98). The process used there was to suppose that the internal magnetic field of the eddy current was negligible compared with that due to the main current flow. Consider a unit length of a straight conductor (see Fig. 133) and let the current density be σ_0 along the axis and σ at radius r. Let the magnetic field be H at radius r; it must be zero at the axis. Then applying the work law to the difference of H round two circles of radius r and $(r+dr)$,

$$\left(H+\frac{\partial H}{\partial r}\,dr\right)(r+dr)d\theta - Hrd\theta = 4\pi\sigma\, rdrd\theta.$$

$$\therefore \quad -\frac{\partial H}{\partial r}+\frac{H}{r} = 4\pi\sigma.$$

Applying the e.m.f. law to the e.m.f. round a rectangular circuit of unit axial length and radial width dr,

$$\rho\left(\sigma+\frac{\partial\sigma}{\partial r}dr\right)-\rho\sigma = -\mu\frac{\partial H}{\partial t}\,dr.$$

$$\therefore \quad \rho\frac{\partial\sigma}{\partial r} = -\mu\frac{\partial H}{\partial t}.$$

So eliminating H,

$$\frac{\partial^2\sigma}{\partial r^2}+\frac{1}{r}\frac{\partial\sigma}{\partial r} = -\frac{4\pi\mu}{\rho}\frac{\partial\sigma}{\partial t}.$$

This is the general equation for the current density. If the current is simple harmonic, the current density will be simple harmonic but its magnitude and phase will be a function of the radius.

It can be shown that

$$\sigma = A(\operatorname{ber} x\cos pt - \operatorname{bei} x\sin pt),$$

where $$\operatorname{ber} x \equiv 1-\frac{x^4}{2^2.4^2}+\frac{x^8}{2^2.4^2.6^2.8^2}-\dots$$

and $$\operatorname{bei} x \equiv \frac{x^2}{2^2}-\frac{x^6}{2^2.4^2.6^2}+\frac{x^{10}}{2^2.4^2.6^2.8^2.10^2}-\dots$$

11. The vector potential of a magnetic field

Current has been measured by the work performed in curling round it with a unit pole and changing flux density has been measured by the work performed in curling round it with an electric charge. Circuital quantities seem intrinsic to electric and magnetic measurements, and so it seems possible that a desirable method of measuring a steady flux density is to curl some vector quantity round it. No physically recognizable quantity is available as the vector to use, such as H was obvious for measuring i. But nevertheless it may be worth while to invent such a vector, as a convenient mathematical tool for calculation. So we invent a new vector, described by the symbol A, such that the line integral of A round any closed path is equal to the flux through that path. Thus let the circuit in Fig. 134 be the edge of any surface through which a flux is passing. Let θ be the inclination to B of the normal at some point in the surface and let ϵ be the inclination of A to the tangent at some point of the path. Then A is defined so that the line integral of A round the closed path is equal to the flux enclosed by the path. Written in symbols this is $\int A\cos\epsilon\, ds = \iint B\cos\theta\, dS$, where the double sign of integration stands for integration over the area and the single sign of integration stands for integration round the closed path. Expressed in vector notation this is

$$\operatorname{curl} A = B. \tag{23}$$

Expressed in Cartesian coordinates this is

$$\left.\begin{aligned} B_1 &= \frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z} \\ B_2 &= \frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x} \\ B_3 &= \frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y} \end{aligned}\right\}. \tag{24}$$

On p. 214 it was stated that such equations are possible provided only that $\operatorname{div} B = 0$. But it is essential to the meaning of B that $\operatorname{div} B$ should be zero. So it is possible to invent a vector quantity A such that $\operatorname{curl} A = B$.

The same remark applies to equations (24) as applied to equations (18) and (21), namely, that A_1, etc., cannot be found when B_1, etc., are given, because the three equations of (24) are not independent but are interconnected by the necessary relation that $\operatorname{div} B = 0$. For equations (18) and (21) the physical explanation was given that there might be a superposed field of H or E due to magnets or charges and this superposed field could be supposed to be found from a separate measurement. Since A is not recognizable by any physical

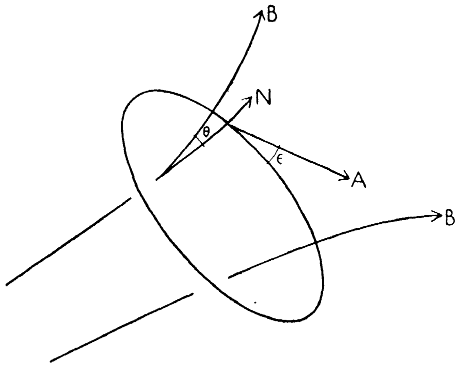


FIG. 134

measurement, a superposed field of A cannot be observed. But A is a mathematical tool invented because it is convenient in calculations, and if equations (24) are insufficient to specify A uniquely, we are free to impose any additional specification which is convenient; this will be pursued again shortly.

Now suppose that B_1, etc., arise only from currents i_1, etc., distributed through space. Combining (18) and (24), we have

$$\begin{aligned}
4\pi i_1 &= \frac{\partial H_3}{\partial y}-\frac{\partial H_2}{\partial z} \\
&= \left(\frac{\partial^2 A_2}{\partial y\partial x}-\frac{\partial^2 A_1}{\partial y^2}-\frac{\partial^2 A_1}{\partial z^2}+\frac{\partial^2 A_3}{\partial z\partial x}\right) \\
&= \left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_2}{\partial y\partial x}+\frac{\partial^2 A_3}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial x^2}-\frac{\partial^2 A_1}{\partial y^2}-\frac{\partial^2 A_1}{\partial z^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\nabla^2 A_1 \\
&= \frac{\partial}{\partial x}(\operatorname{div} A)-\nabla^2 A_1. \qquad (25)
\end{aligned}$$

This equation, and the other two similar ones for i_2 and i_3, would be much more simple if div A were zero. But there is no reason why div A should not be zero, because we can decide to make div A zero if we choose to do so. Such a procedure seems a little arbitrary, but it must not be forgotten that A is not a physically recognizable quantity but is a convenient mathematical tool, which is to be forged in the form in which it will give the best service. So far A has had no more definition than curl $A = B$; this definition is insufficient to determine A, for it determines only the gradients of A. A will be a very convenient tool if it is defined so that

$$\operatorname{curl} A = B$$

and

$$\operatorname{div} A = 0.$$

This defines A completely and the weapon is forged in a form which is convenient to use. Now the relation between A_1 and i_1, etc., is

$$\left.\begin{aligned} \nabla^2 A_1 &= -4\pi\mu i_1 \\ \nabla^2 A_2 &= -4\pi\mu i_2 \\ \nabla^2 A_3 &= -4\pi\mu i_3 \end{aligned}\right\}. \tag{26}$$

Comparing equations (26) with the potential expression for Gauss's theorem, namely,

$$\nabla^2 V = 4\pi\rho,$$

where V is defined by

$$H_1 = -\frac{\partial V}{\partial x}, \text{ etc.,}$$

we know that the potential of a unit pole is

$$V = 1/r,$$

and the potential of a system of poles is

$$V = \int \frac{\rho\, dv}{r},$$

where ρ is the pole strength in a volume element dv.

Hence, by comparison,

$$\left.\begin{aligned} A_1 &= \mu \int \frac{i_1}{r}\, dv \\ A_2 &= \mu \int \frac{i_2}{r}\, dv \\ A_3 &= \mu \int \frac{i_3}{r}\, dv \end{aligned}\right\}, \tag{27}$$

where i_1, etc., is the current density in a volume element dv situated at a point P distant r from the point Q where A is being evaluated.

So the x-component of A is calculated by adding algebraically a series of vectors which are parallel to i_1 at every point of the current field and whose magnitudes are inversely proportional to the distance from that particular component of current density. The process of evaluation is the same as that for finding the magnetic potential of a system of charges, but whereas V is a scalar quantity, A is a vector quantity and has three components. Because of the similarity in the process of evaluation, the vector A has received the name 'vector potential'. Perhaps the name is not very desirable because the idea of potential as a scalar quantity is probably firmly implanted in the reader's mind. Here is something called a vector potential, apparently a contradiction in terms. However, the undesirability of the name has the counteracting advantage of suggesting the process of evaluation.

There is nothing mysterious about vector potential; it is just a convenient tool which often simplifies calculations, just as scalar potential often simplifies calculations. Though scalar potential is defined as the work done in bringing up a pole to that point, yet in reality it is not a physically recognizable quantity. It can be measured only by tracing a path from infinity and measuring the force at every point and evaluating the work done. The potential at a given point cannot be found without moving from that point to infinity. Scalar potential is not really a more concrete thing than vector potential, but it is much more familiar to most readers, especially engineers, and its abstract character has perhaps been overlooked.

If V, the scalar potential, is defined from the equation $H = -\dfrac{\partial V}{\partial r}$, then for a unit pole $V = 1/r+K$, where K is the necessary constant of integration. But V is more convenient to use if K is zero, and consequently the mathematical tool V is formed by stating dogmatically that $K = 0$. So V is defined as being the work done in bringing up a pole from infinity, together with the extra specification that V is zero at infinity. This extra specification, which adds so much to the convenience of V, corresponds to the additional specification that $\operatorname{div} A = 0$.

The vector potential is a tool which engineers seldom use, and in general it seems very unfamiliar to them; this is a pity, because the tool is a very convenient and simple one, and its concept as a line integral measuring the flux through an area should surely appeal to them. Every dynamo designer will state that a machine has a flux of so many lines per pole; most designers would be very mystified if they were told that each pole carried so many units of vector potential. By the first system of specification he is thinking of adding up flux density over an area; in the second specification he is thinking of measuring the flux by means of a particular form of tape measure wrapped round its girth. If he uses a search coil and flux meter, which he must do for a measurement, he is putting a tape measure round the girth; so after all the idea is familiar enough.

The mutual inductance M is defined as the flux through circuit 2 due to unit current through circuit 1. Accordingly, in terms of vector potential, $M = \int A\,ds$ where the path curled round is circuit 2. But $A = \int \frac{\mu\,ds'}{r}$ where r is the distance of a point on circuit 2 from an element ds' on circuit 1. So $M = \mu \int\int \frac{ds ds'}{r}$. Similarly L, the self-inductance, defined as the flux through the circuit is

$$L = \int A\,ds.$$

But $A = \mu \int \frac{ds'}{r}$ where r is the distance to the element ds' on the circuit from some other element ds on the same circuit. So

$$L = \mu \int\int \frac{ds ds'}{r}. \tag{28}$$

12 a. Vector potential of a straight current filament

Consider a point P, distant x from a straight current running along the axis of y. Then

$$A = i \int_{-\infty}^{+\infty} \frac{dy}{r},$$

or

$$\frac{\partial A}{\partial x} = \mu H = \frac{2\mu i}{x}.$$

$$\therefore \quad A = K + 2\mu i \log x,$$

where K is a constant of integration whose value depends on the form of the remainder of the circuit. Inside the wire

$$\frac{\partial A}{\partial x} = \frac{2\mu i x}{a^2}.$$

$$\therefore \quad A = \frac{\mu i x^2}{a^2} + K'.$$

12 b. Vector potential of an infinite solenoid

Inside the solenoid the flux density is uniform and so the lines of A are circles centred on the axis of the solenoid. So

$$A \times 2\pi r = 4\pi i T \times \pi r^2.$$

$$\therefore \quad A = 2\pi i T r.$$

Outside the solenoid

$$A \times 2\pi r = 4\pi i T \pi R^2.$$

$$\therefore \quad A = \frac{2\pi R^2 i T}{r},$$

where R is the radius of the solenoid.

13. Relation between vector potential and electric force

The flux through an area can be measured by the curl of the vector potential, and the rate of change of flux through an area can be measured by the curl of the electric force induced by the change. In vector notation these two methods are expressed by the equations

$$\operatorname{curl} E = -\dot{B}$$

and

$$\operatorname{curl} A = B.$$

So, by eliminating B, we have

$$\operatorname{curl} E = -\frac{\partial}{\partial t}(\operatorname{curl} A).$$

$$\left.\begin{aligned} \therefore \quad E_1 &= -\frac{\partial A_1}{\partial t} - \frac{\partial \phi}{\partial x}, \\ E_2 &= -\frac{\partial A_2}{\partial t} - \frac{\partial \phi}{\partial y}, \\ E_3 &= -\frac{\partial A_3}{\partial t} - \frac{\partial \phi}{\partial z}, \end{aligned}\right\} \tag{29}$$

where ϕ is any function whatever which has the property that $\operatorname{div} \phi = 0$. But $\operatorname{grad} \phi$ must be an electric force and therefore ϕ must be the scalar potential of any electric charges in the field. If there are no charges, then $E_1 = -\dfrac{\partial A_1}{\partial t}$, etc.

So in a sense it may be said that A is a physically recognizable quantity, for the time rate of change of A is the electric force at a point due to the changing currents in the field; this electric force can be recognized by a test charge. In Chapter II it was stated that the cutting rule may be expressed as $E = bv$, so long as the operation is performed for a whole circuit, but it cannot be shown that the electric force at any given point of space is $E = bv$. Now it is found that $E_1 = -\frac{\partial A_1}{\partial t}$, etc., and this does not suggest a cutting rule. But even now the problem is not quite solved because any varying currents must entail charges whose position is fixed but whose magnitude varies with time (capacity effects), and hence it is impossible for $\frac{\partial \phi}{\partial x}$, etc., to be zero, though these terms may possibly be very small compared with $\dot{A}_1$, etc.

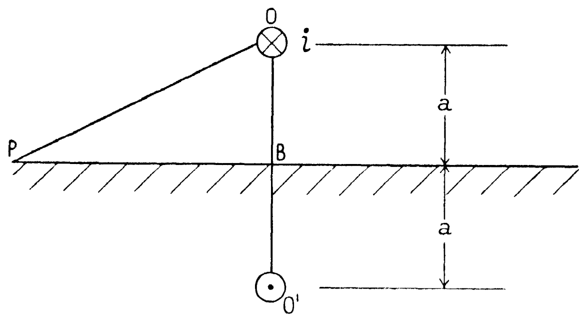


FIG. 135

14. Currents near plane faces of iron

The problem of a current which runs near one or more plane faces of iron is very important to the engineer, because it approximates to an element of many of the machines which he constructs. The problem of a current between two parallel iron faces is best solved by the help of the vector potential, and hence this class of problem has been reserved to this stage of the present chapter.

We will consider first the relatively simple problem of a long straight current running parallel to an infinite face of iron of permeability μ and at a distance a above it, as shown in Fig. 135.

We will first consider the form which the solution is likely to take and remember that μ is a very large number. The current produces an H inside the iron which is independent of μ, and this will tend to form magnetic chains which are circles centred at O. These chains must end in the face of the iron and produce there a surface polarity. It may readily be seen that this polarity produces an H which, on the whole, adds to the field above the iron and opposes the H due to the current in the iron. Since the H due to the surface polarity must be a little less than that due to the current, so as to leave a small net H to form the chains, it seems probable that the B in the iron is not very great, for otherwise it might lead to an infinite surface polarity. If this be so, then the net H in the iron will be excessively small since μ is very large. It seems possible that the surface polarity must establish an H in the iron which tends everywhere to be equal and opposite to that of the current. If this be so, then the surface polarity must produce a field in the iron which is equivalent to $-i$ at O. But the field of this polarity must be symmetrical about the iron face, and hence above the iron it is equivalent to $-i$ at O', the image point of O. The boundary conditions to be satisfied are continuity of tangential H and of normal B. Accordingly we will see if these conditions can be satisfied by supposing the H in the air is that appropriate to a current i at O and a current u at O', and that the H in the iron is that appropriate to a current v at O. The normal component of magnetic force just outside the iron will be

$$2i\frac{BP}{OP^2}+2u\frac{BP}{OP^2},$$

and just inside the iron it will be $2v\dfrac{BP}{OP^2}$.

Hence the continuity of the normal component of flux density gives

$$i+u = \mu v,$$

and the continuity of tangential magnetic force gives

$$i-u = v,$$

whence $$u = \frac{\mu-1}{\mu+1}i \quad \text{and} \quad v = \frac{2i}{\mu+1}.$$

The flux lines in the iron are therefore circles centred at O and the flux density at a distance r from O is

$$B = \frac{2\mu v}{r} = \frac{4\mu i}{(\mu+1)r},$$

and this tends to the limiting value $B = 4i/r$, when the permeability is very large.

The force driving the wire towards the iron face is

$$F = \frac{2ui}{2a} = \frac{2(\mu-1)}{(\mu+1)}\frac{i^2}{2a} \doteqdot 2\left(1-\frac{2}{\mu^2}\right)\frac{i^2}{2a}. \tag{30}$$

Fig. 136 shows the magnetic field of a wire outside a block of iron for which $\mu = 9$.

Since the flux density in the iron does not tend to a large value, but is only about twice what it would be at the same point if the iron were absent, this solution is probably a very close-approximation for real iron, where μ is by no means constant.

If the current is at a distance a below the surface of the iron, the current will arrange magnetic chains which end on the face and produce there a surface polarity. It may readily be seen that this polarity increases the H between the current and the face and de-

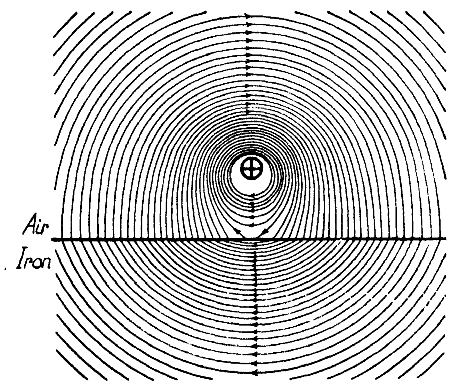


FIG. 136

creases it on the side of the current remote from the face: above the iron face the surface polarity increases the field which would exist if the iron were not present. It follows as in the preceding problem that the H in the iron is that appropriate to a current i at O' and a current u at O, where

$$u = -\frac{\mu-1}{\mu+1}i.$$

Similarly the field in air is the same as if the iron were removed and a current placed at O' of value

$$v = \frac{2\mu}{\mu+1}i.$$

Hence the lines of force in air are circles centred at O'. When μ is large, v tends to the value $2i$. Also the H in the iron tends to the value it would have if the iron were absent and a current $-i$ was placed at O'. It is quite incorrect to suppose that the iron slab screens the external space from the field of the current, for, on the contrary, it doubles this field. But the flux density in the iron is very great compared with what it would be if the iron were absent. If this solution is applied to real iron we must be sure that the greatest value of H, which occurs at the face immediately above the current, is much less than that corresponding to maximum permeability.

14 a. Field of a linear current between two parallel iron faces

We have seen that when a current is above an iron face the lines of force meet the iron almost normally, because the permeability is very great. The limiting condition of infinite permeability is approached very closely indeed for values of μ possessed by ordinary iron. The approximation is so close that we may consider the field above the iron is that corresponding to the real current together with its image. With this simplification it is possible to find the field in the air space between two parallel iron faces. For consider the lines of force of an infinite number of equal currents spaced the same distance apart: these are shown in Fig. 137. From symmetry the lines of force must be normal to planes which are perpendicular to the array of currents and midway between each pair: two such planes are shown in Fig. 137. But such planes satisfy the conditions required by two iron faces of high permeability, and hence the field between two such iron faces must be the same as that shown between the two cross-planes drawn in Fig. 137. If we can calculate the field due to an infinitely wide array of currents, we shall have found the field between two parallel iron faces.

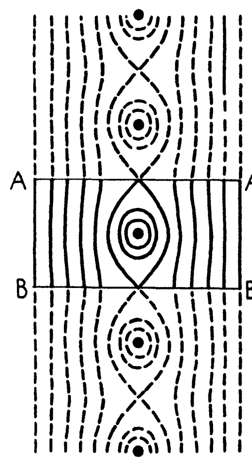


FIG. 137

This solution is of great interest in the problem of the dynamo, for there currents run parallel to two iron faces. It is true that the iron faces are there parts of concentric cylinders, but in practice the space between the cylinders, called the air gap, is so small compared with the radius of either that the surfaces may be considered plane.

To calculate the field between AA' and BB' it is necessary to calculate the field in this region due to the infinite number of linear currents, each separated from its neighbour by a distance a. This field may be calculated most conveniently in terms of the vector potential of the currents. Consider a point $P(x,y)$ in Fig. 138. Then

$$\begin{aligned} A &= K+2i(\log r_0+\log r_1+\log r_1'+\log r_2+...) \\ &= K+i\log(r_0^2 r_1^2 r_1'^2...) \\ &= K+i\log\{[x^2+y^2][x^2+(y-g)^2][x^2+(y+g)^2]...\} \\ &= K+i\log\left\{\left[1+\frac{x^2}{y^2}\right]\left[1+\frac{x^2}{(y-g)^2}\right]...\times y^2(y-g)^2...\right\}. \end{aligned}$$

In books on trigonometry it is proved that the first set of factors are the factors of $\dfrac{\cosh\frac{2\pi x}{g}-\cos\frac{2\pi y}{g}}{2\sin^2\pi y/g}$ (see, for example, J. B. Lock, *Higher Trigonometry*, p. 97), and the second set of factors are the factors of $\dfrac{1}{\pi^2}\sin^2\dfrac{\pi y}{g}$ (see Lock, ibid., p. 92). On choosing the arbitrary constant K so that $K = -i\log\left(\dfrac{1}{2\pi^2}g^2.g^4.2^4.g^4...\right)$, it follows that

$$A = i\log\left(\cosh\frac{2\pi x}{g}-\cos\frac{2\pi y}{g}\right). \tag{31}$$

Now
$$H_x = \frac{\partial A}{\partial y} = \frac{2\pi i}{g}\frac{\sin\frac{2\pi y}{g}}{\cosh\frac{2\pi x}{g}-\cos\frac{2\pi y}{g}} \tag{32}$$

and
$$H_y = -\frac{\partial A}{\partial x} = -\frac{2\pi i}{g}\cdot\frac{\sinh\frac{2\pi x}{g}}{\cosh\frac{2\pi x}{g}-\cos\frac{2\pi y}{g}}. \tag{33}$$

The lines of force, drawn from these equations, are shown in Fig. 139. It should be noticed that the lines become sensibly straight at a distance from the wire about equal to the width of the gap. From symmetry all the lines must be perpendicular to the mid plane through the wire, so the same solution will give the field of a conductor which is on one face of the iron. The lines of force for this are shown in Fig. 140, which, it should be noticed, is a replica of the upper or lower half of Fig. 139.

The most interesting value of H_y is when $y = 0$ or $y = g/2$: this gives the flux density on the iron face and across the mid plane

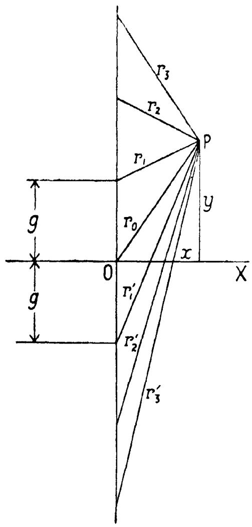


Fig. 138

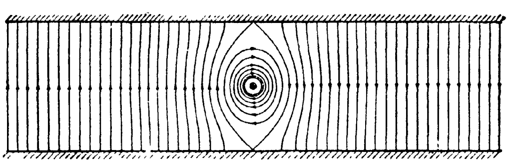


Fig. 139

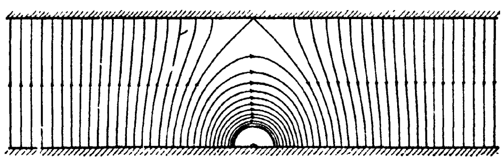


Fig. 140

respectively for Fig. 139. Or it gives the flux density on either iron face of Fig. 140. These values are

$$H_i = -\frac{2\pi i}{g}\frac{\sinh\dfrac{2\pi x}{g}}{\cosh\dfrac{2\pi x}{g}+1} = -\frac{2\pi i}{g}\tanh\frac{\pi x}{g}, \tag{34}$$

and $$H_m = -\frac{2\pi i}{g}\coth\frac{\pi x}{g}. \tag{35}$$

The values of H_i and H_m are plotted as a function of x in Fig. 141.

It should be noticed from Fig. 141 that H_i and H_m have become sensibly constant when $x > g/2$.

When $x = 0$, H_m appears to become infinite, because the finite diameter of the wire has not been allowed for. But,

$$\begin{aligned} H_m &= -\frac{2\pi i}{g}\frac{e^{\frac{\pi x}{g}}+e^{-\frac{\pi x}{g}}}{e^{\frac{\pi x}{g}}-e^{-\frac{\pi x}{g}}} \\ &\doteqdot -\frac{2\pi i}{g}\frac{\left(1+\dfrac{\pi^2x^2}{g^2}\right)}{\dfrac{\pi x}{g}}, \text{ when } \frac{\pi x}{g} \ll 1, \\ &= -\frac{2i}{x}\left(1+\frac{\pi^2x^2}{g^2}\right). \end{aligned}$$

Hence, close to the wire H_m approaches very near to the value it would have if the iron were absent. If the radius of the wire is r, the limiting value of H_m is not infinite but is $H_m = 2i/r$.

An approximate solution can be obtained by using the work law only. Experience tells us that at an appreciable distance from the wire the lines of force will be sensibly straight. Take a unit pole round a path which is straight across the gap, through the iron parallel to a face, back across the gap at the same distance on the other side of the wire, and back to the starting-point through the iron. The work done in the iron is zero because the permeability is infinite and the work done at each crossing of the gap is Hg where H is the mean value of H_y. Hence

$$2Hg = 4\pi i.$$

$$\therefore \quad H = \frac{2\pi i}{g}.$$

This is correct for the mean value of H_y across the air gap at any distance from the wire. Exact analysis shows that

$$\frac{H_m - H_i}{H_{\text{mean}}} = 2\,\text{cosech}\,\frac{2\pi x}{g}.$$

So the fractional difference is only 0·07 per cent., when $x = g$.

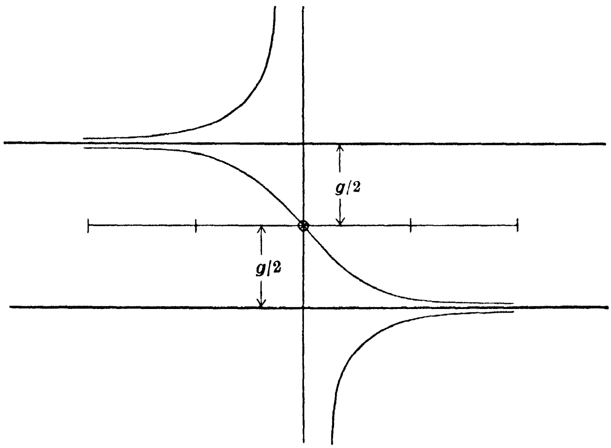


FIG. 141

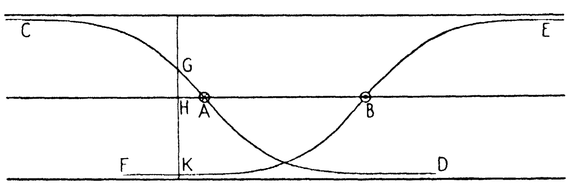


FIG. 142

14 b. Two oppositely directed currents in a uniform air gap

Let there be equal currents oppositely directed and separated by a distance s, each wire midway between two iron faces separated by a distance g. The resulting field beyond and between the wires can be visualized by the help of Fig. 142. The currents are placed at A and at B. The field due to A is represented by the curve CAD and that due to B by the curve EBF. The resultant field on the iron face is obtained by taking the algebraic sum of these two curves. Consider some ordinate such as GHK, then HK is always a little greater than GH, so there is no point on the iron face at which the field reverses, and so there is no neutral line. Between the wires the flux is the sum of the two curves; if s is much greater than g, the field strength will be sensibly constant over the greater part of the span between the wires and of value $\frac{4\pi i}{g}$. The field form on the iron face, that is to say, the curve of H_y and x, will have the shape typified by Fig. 143 (a) and (b).

The maximum flux density on the iron face is

$$\begin{aligned} H &= \frac{4\pi i}{g}\tanh\frac{\pi s}{2g} \\ &= 0{\cdot}762\times\frac{4\pi i}{g}, \text{ when } s = \frac{2}{\pi}g, \\ &= 0{\cdot}964\times\frac{4\pi i}{g}, \text{ when } s = \frac{1}{\pi}g. \end{aligned}$$

The maximum flux density on the mid plane is

$$H = \frac{4\pi i}{g}\coth\frac{\pi s}{2g}.$$

Since the mid plane of the coil cuts perpendicularly across all the lines of force, this plane may be replaced by an iron surface. So the same solution gives the field for conductors lying on one iron face. The magnetic field of a coil whose span is 2·68 times the gap is shown in Fig. 144, and this is also the upper or lower half of the field of two wires placed midway between iron faces separated by twice the distance.

14 c. Inductance of a coil side midway between two iron faces

The inductance of a circuit is the flux threading it per unit current. Since $\mu H = \text{curl } A$, the flux is readily calculated from the vector potential: thus

$$\begin{aligned} L &= -2(A_r - A_s) \\ &= -2\left\{\log\left(\cosh\frac{2\pi r}{g}-1\right)-\log\left(\cosh\frac{2\pi s}{g}+1\right)\right\} \\ &= 2\left\{\log 2\sinh^2\frac{\pi s}{g}-\log 2\sinh^2\frac{\pi r}{g}\right\} \\ &= 4\left\{\log\sinh\frac{\pi s}{g}-\log\frac{\pi r}{g}\right\} \\ &= 4\left\{\log e^{\frac{\pi s}{g}}+\log\left(1-e^{-\frac{2\pi s}{g}}\right)-\log 2-\log\frac{\pi r}{g}\right\} \end{aligned}$$

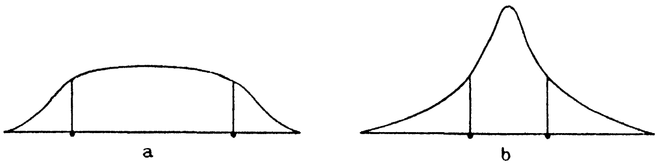


FIG. 143

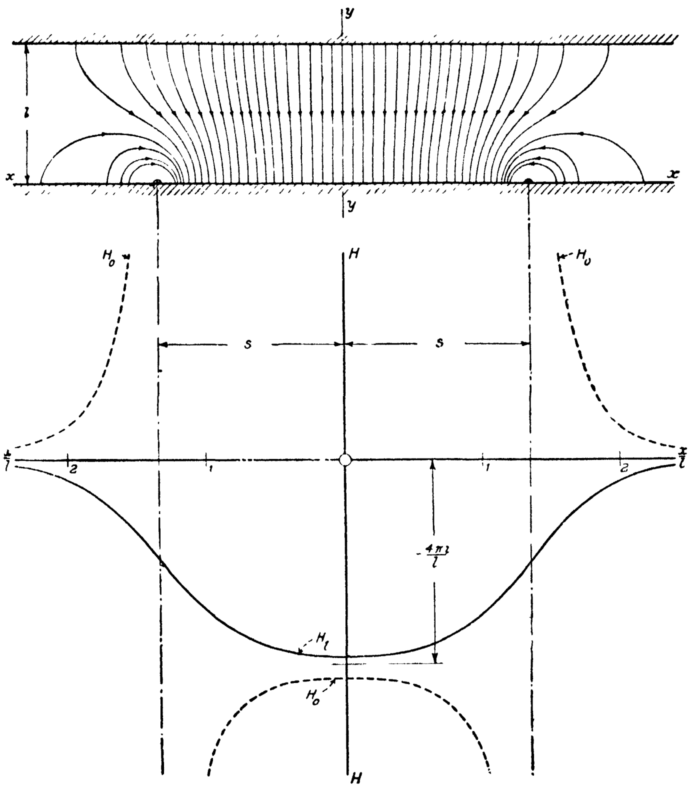


FIG. 144

$$\doteqdot \frac{4\pi s}{g}\left\{1-\frac{g}{\pi s}\log\frac{2\pi r}{g}\right\}.$$

So, adding the term for the flux within the wire (see p. 52),

$$L \doteqdot \frac{4\pi s}{g}\left\{1+\frac{g}{\pi s}\left(\log\frac{g}{2\pi r}+\frac{1}{4}\right)\right\}. \tag{36}$$

Flux through the iron face. The flux entering either iron face is less than the flux passing between the two wires, because some lines of force are closed curves round the wire. The difference between the flux passing between the wires and that entering the iron will be termed the leakage inductance. On the top iron face

$$\begin{aligned} A &= i\log\left(\cosh\frac{2\pi x}{g}+1\right) \\ &= i\left(2\log\cosh\frac{\pi x}{g}+\log 2\right). \end{aligned}$$

$$\begin{aligned} \therefore\quad 2\phi_{0-x} &= 4i\left\{\log\cosh\frac{\pi x}{g}-\log\cosh\frac{\pi(x-s)}{g}\right\} \\ &\doteq 4i\left\{\frac{\pi x}{g}-\log 2-\frac{\pi(x-s)}{g}+\log 2\right\}, \text{when } \frac{x}{g}\to\infty, \\ &= \frac{4\pi i s}{g}. \end{aligned}$$

$$\begin{aligned} \therefore\quad L_{\text{leakage}} &= \frac{4\pi s}{g}\left\{1+\frac{g}{\pi s}\left(\log\frac{g}{2\pi r}+\frac{1}{4}\right)\right\}-\frac{4\pi s}{g} \\ &= 4\left(\log\frac{g}{2\pi r}+\frac{1}{4}\right). \qquad (37) \end{aligned}$$

So the leakage inductance per unit length is equal to the self-inductance of two wires in air placed at a distance apart $g/2\pi$.

14 d. Force between two wires in the mid plane of an air gap

Each current lies in a field due to the other current of magnitude $H_y = \frac{2\pi i}{g}\coth\frac{\pi s}{g}$, so the force pressing the wires apart or attracting them together is

$$\begin{aligned} F &= \frac{2\pi i^2}{g}\coth\frac{\pi s}{g} \\ &\doteq \frac{2\pi i^2}{g}\left(1+2e^{-\frac{\pi s}{g}}\right). \qquad (38) \end{aligned}$$

14 e. Wire near the bottom of a rectangular slot

If two similarly directed currents are placed in the mid plane of a uniform air gap, it is clear that the plane midway between the wires and perpendicular to the faces of the air gap must be a magnetic equipotential. For from symmetry any line of force which crosses this plane must cut it perpendicularly. Accordingly, this plane may be replaced by an iron interface, and then the conductor is near the bottom of an infinitely deep slot. The lines of force are shown in Fig. 145; the force attracting the wire to the bottom of the slot may be calculated from (38) above.

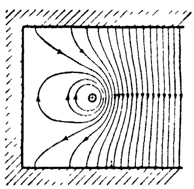


FIG. 145

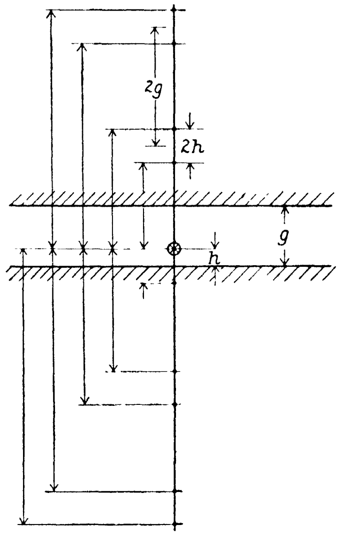


FIG. 146

14 f. Field of a current placed anywhere between two iron faces

Let the wire be placed at a height h above the lower iron surface (see Fig. 146). Then it follows that the field in the air gap can be calculated from the field of a doubly infinite series of image currents. The spacing between neighbouring images is $2h$ and the spacing between neighbouring pairs is $2g$. The vector potential for one set can be obtained from equation (31) by shifting the origin to the point $(0, -h)$ and for the second pair by shifting it to the point $(0, h)$. Accordingly we have

$$A = i\log\left\{\cosh\frac{\pi x}{g} - \cos\frac{\pi(y-h)}{g}\right\} + \\ + i\log\left\{\cosh\frac{\pi x}{g} - \cos\frac{\pi(y+h)}{g}\right\}. \tag{39}$$

Equation (39) can be used to find the field due to any system of currents placed at any points between the iron faces. We shall use it to find the inductance per unit length of two wires placed as shown in Fig. 147. Let each wire have a radius r: then the vector potential at the point $(0, h+r)$ due to one wire is

$$\begin{aligned} A &= i\log\left(1-\cos\frac{\pi r}{g}\right)+i\log\left\{1-\cos\frac{\pi(2h+r)}{g}\right\} \\ &= i\log 2\sin^2\frac{\pi r}{2g}+i\log 2\sin^2\frac{\pi(2h+r)}{2g} \\ &= 2i\left\{\log\sin\frac{\pi r}{2g}+\log\sin\frac{\pi(2h+r)}{2g}+\log 2\right\} \\ &\doteqdot 2i\left\{\log\frac{\pi r}{2g}+\log\sin\frac{\pi h}{g}+\log 2\right\}. \end{aligned}$$

The vector potential at the point $(0, h+s)$ is

$$\begin{aligned} A &= i\left\{\log\left(1-\cos\frac{\pi s}{g}\right)+\log 2\right\} \\ &= 2i\left(\log\sin\frac{\pi s}{2g}+\log 2\right). \end{aligned}$$

$$\therefore\quad L = 4\left(\log\sin\frac{\pi s}{2g}-\log\sin\frac{\pi h}{g}-\log\frac{\pi r}{2g}\right);$$

including the term for the flux within the wire

$$L = 4\left(\log\sin\frac{\pi s}{2g}-\log\sin\frac{\pi h}{g}-\log\frac{\pi r}{2g}+\frac{1}{4}\right);$$

if $s = \frac{1}{2}g$,

$$L = 4\left(\log\frac{g}{\pi r}+\frac{1}{4}+\log 2\right). \tag{40}$$

14 g. Force on a conductor placed not in the middle of the gap

The components of magnetic force can be found by differentiating equation (39): thus

$$H_x = \frac{\pi i}{g}\left[\frac{\sin\frac{\pi}{g}(y+h)}{\cosh\frac{\pi x}{g}-\cos\frac{\pi}{g}(y+h)}+\frac{\sin\frac{\pi}{g}(y-h)}{\cosh\frac{\pi x}{g}-\cos\frac{\pi(y-h)}{g}}\right] \tag{41}$$

and

$$H_y = -\frac{\pi i}{g}\left[\frac{\sinh\frac{\pi x}{g}}{\cosh\frac{\pi x}{g}-\cos\frac{\pi(y+h)}{g}}+\frac{\sinh\frac{\pi x}{g}}{\cosh\frac{\pi x}{g}-\cos\frac{\pi(y-h)}{g}}\right]. \tag{42}$$

Now examine H_x when $x = 0$ and $y = h+r$, where $r \to 0$:

$$H_x = \frac{\pi i}{g}\left[\frac{\sin\frac{2\pi h}{g}}{1-\cos\frac{2\pi h}{g}}+\frac{2}{\pi r/g}\right]$$

$$= \frac{\pi i}{g}\cot\frac{\pi h}{g}+\frac{2i}{r}.$$

The second term of this expression is the force at a distance r from a straight wire which is distant from iron. So the effect of the iron

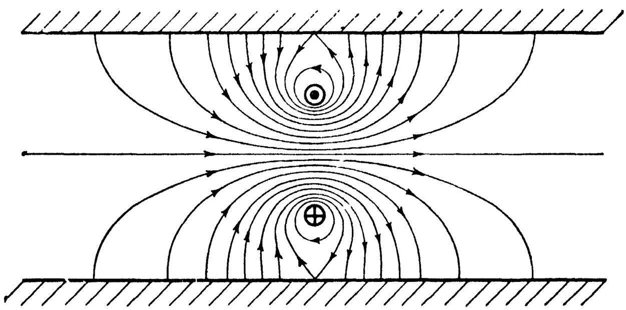


FIG. 147

is to place the wire in a horizontal field of value $\frac{\pi i}{g}\cot\frac{\pi h}{g}$, thus showing that the wire is attracted towards the iron with a force

$$F = \frac{\pi i^2}{g}\cot\frac{\pi h}{g}.$$

Now consider the force repelling two wires arranged as in Fig. 147. To calculate this force we require to find the component of H_x at one wire due to the current in the other wire: this we obtain by using equation (41) to calculate H_x at the point $(0, g-h)$. On making the substitution we find

$$H_x = \frac{\pi i}{g}\frac{\sin\frac{\pi}{g}(g-2h)}{1-\cos\pi\frac{(g-2h)}{g}}$$

$$= \frac{\pi i}{g}\frac{\sin\frac{2\pi h}{g}}{1+\cos\frac{2\pi h}{g}} = \frac{\pi i}{g}\tan\frac{\pi h}{g}.$$

So there is a force on each wire of value $\frac{\pi i^2}{g}\cot\frac{\pi h}{g}$ due to the attraction of the iron surface and a force $\frac{\pi i^2}{g}\tan\frac{\pi h}{g}$ due to the repulsion from the other wire.

$$\therefore \quad F = \frac{\pi i^2}{g}\left(\tan\frac{\pi h}{g}+\cot\frac{\pi h}{g}\right)$$

$$= \frac{2\pi i^2}{g}\operatorname{cosec}\frac{2\pi h}{g}$$

$$\doteqdot \frac{i^2}{h}\left(1+\frac{2}{3}\frac{\pi^2 h^2}{g^2}\right) \quad \text{if} \quad \frac{h}{g} \ll \frac{1}{2\pi}.$$

If the wires are placed a distance $2D$ apart,

$$2h+2D = g.$$

$$\therefore \quad F = \frac{2\pi i^2}{g}\operatorname{cosec}\frac{2D\pi}{g}$$

$$\doteqdot \frac{2i^2}{2D}\left(1+\frac{2}{3}\frac{\pi^2 D^2}{g^2}\right) \quad \text{if} \quad \frac{D}{g} \ll \frac{1}{2\pi}.$$

15. Skin effect of alternating currents

We considered in Chapter II the eddy-current system, in a round wire, produced by the magnetic field of a uniformly distributed alternating current, and found that the total effect could be expressed as an apparent increase of resistance. In deriving the expression for the resistance, the magnetic field of the eddy-current system was ignored, with the result that the equation on p. 99 is substantially correct only if the increase of resistance is small: the criterion of validity was stated to be $\frac{\pi p^2 a}{\rho} \nless 1$. A similar process was used for calculating the eddy-current loss in transformer plates, and on p. 142 will be found a discussion of the maximum thickness of iron plates to which the equation on p. 138, § 15 is applicable.

In § 10 of this chapter we have derived the correct equation for the current density in a round wire and stated the solution in terms of Bessel functions of imaginary argument. But to avoid a long digression on Bessel functions, no general expression was found for the resistance of a wire at any frequency.

We can now approach the problem by the much simpler process of considering the current flow in an infinite flat plane. This process avoids troublesome computation of Bessel functions and also brings out more clearly the physical mechanism of the effect.

Suppose an alternating current is flowing in an infinite block of material, of specific resistance ρ and permeability μ: this is illustrated in Fig. 148. Of course the specification of the problem is fictitious and the reader should resent the idea of infinite slabs of infinite

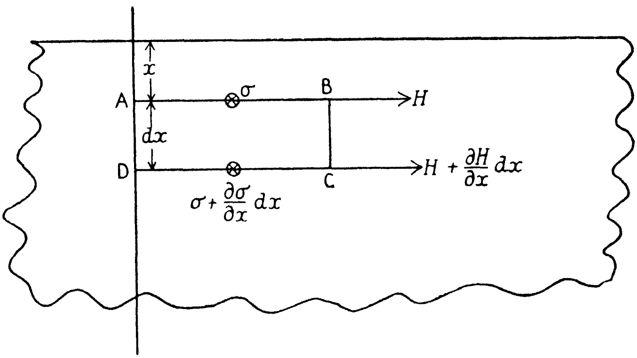


FIG. 148

thickness. What we really mean is this: we realize that the current distribution must tend to that which produces no internal magnetic field (see Chap. II, p. 82, etc.), and this distribution must be one in which the current density decreases rapidly from the surface inwards. Realizing that the current must be confined sensibly to a surface skin, it is reasonable to suppose that the curvature of the surface cannot appreciably affect the inward rate of decrease so long as the radius of curvature is very great compared with the depth of the surface skin. Hence we presume that the inward rate of decrease of current from the surface of a cylindrical or elliptical wire must tend in the limit to that which would obtain from the surface of an infinite flat slab. Likewise, if the conductor is a plate of finite thickness we suppose that the current distribution inwards from either face will tend in the limit to be the same as if the other face was infinitely far removed. We now understand what is meant by the hypothetical system illustrated on Fig. 148 and may proceed to calculate the current density σ and magnetic force H at a depth x below the surface.

Applying the work law round the rectangle $ABCD$, we have

$$4\pi\sigma\,dx = \frac{\partial H}{\partial x}\,dx,$$

and applying the e.m.f. law to the flux through a rectangle of side dx and unit depth into the paper, we have

$$-\mu\frac{\partial H}{\partial t} = \rho\frac{\partial \sigma}{\partial x}.$$

Hence, combining these two equations, we obtain

$$\left.\begin{aligned} \frac{4\pi\mu}{\rho}\frac{\partial \sigma}{\partial t} &= \frac{\partial^2 \sigma}{\partial x^2} \\ \text{and}\qquad \frac{4\pi\mu}{\rho}\frac{\partial H}{\partial t} &= \frac{\partial^2 H}{\partial x^2} \end{aligned}\right\}. \tag{43}$$

If the current is simple harmonic, then σ and H will retain the same frequency at any depth but their magnitude and phase will be a function of x. The solution of (43) is

$$\sigma = \sigma_s e^{-mx}\cos(pt-mx), \tag{44}$$

where σ_s is the current density at the surface and

$$m = \sqrt{\frac{2\pi p\mu}{\rho}}.$$

That this is a solution, may be shown as follows:

$$\frac{\partial \sigma}{\partial t} = -p\sigma_s e^{-mx}\sin(pt-mx),$$

and

$$\frac{\partial \sigma}{\partial x} = \sigma_s m e^{-mx}\{\sin(pt-mx)-\cos(pt-mx)\},$$

$$\frac{\partial^2 \sigma}{\partial x^2} = -2\sigma_s m^2 e^{-mx}\sin(pt-mx).$$

Hence substituting in (43) we find that

$$\frac{4\pi\mu p}{\rho} = 2m^2$$

or

$$m = \sqrt{\frac{2\pi\mu p}{\rho}}.$$

Similarly $H = H_0 e^{-mx}\cos(pt-mx)$.

So far we have obtained only the current density, but we require to know i, the total current flowing into the paper, per unit width of the slab. Accordingly

$$
\begin{aligned}
i &= \int\limits_0^\infty \sigma\,dx \\
&= \sigma_s \int\limits_0^\infty e^{-mx}\cos(pt-mx) \\
&= \sigma_s\left[e^{-mx}\left\{\frac{\cos(pt-mx)}{D-m}\right\}\right]_0^\infty \\
&= \sigma_s\left[e^{-mx}\frac{\{\sin(pt-mx)+\cos(pt-mx)\}}{-2m}\right]_0^\infty \\
&= \frac{\sigma_s}{\sqrt{2}m}\cos\left(pt-\frac{\pi}{4}\right) \equiv I\cos\left(pt-\frac{\pi}{4}\right).
\end{aligned} \tag{45}
$$

The power loss per unit volume is

$$
\begin{aligned}
dw &= \sigma^2\rho\,dx \\
&= \rho\sigma_s^2\,e^{-2mx}\cos^2(pt-mx) \\
&= \frac{\rho\sigma_s^2}{2}e^{-2mx}\{1+\cos 2(pt-mx)\}.
\end{aligned}
$$

Hence the mean power loss per unit volume is

$$dw = \frac{\rho\sigma_s^2}{2}e^{-2mx}.$$

Hence the total mean power loss per unit width of the slab is

$$
\begin{aligned}
w &= \frac{\rho\sigma_s^2}{2}\int\limits_0^\infty e^{-2mx}\,dx \\
&= -\frac{\rho\sigma_s^2}{4m}\left[e^{-2mx}\right]_0^\infty \\
&= \frac{\rho\sigma_s^2}{4m} \equiv \rho m\frac{I^2}{2}.
\end{aligned}
$$

But if the current $i = I\cos(pt-\frac{1}{4}\pi)$ flows in a conductor of resistance R, we should have

$$w = R\frac{I^2}{2}.$$

$$\therefore \quad R = \rho m. \tag{46}$$

Hence the energy loss is the same as if a uniformly distributed current flowed in a surface strip of thickness $1/m$: we may say that the resistance of an infinitely thick slab is such that the slab has an effective thickness $1/m$. This thickness is sometimes termed the depth of penetration, but the reader must not suppose the current is concentrated in a surface layer of this thickness. The current penetrates to an infinite depth, but the amplitude of σ decreases so rapidly that it has become very small indeed at a finite depth. At a depth $4/m$, the current density is less than 2 per cent. of the value at the surface, and hence the depth of penetration is, say, $4/m$.

We must now obtain an idea of the numerical value of m, and will calculate it for copper, stalloy, and mumetal: for these materials the relevant values of ρ and μ will be taken as 1·68 $\mu\Omega/\text{cm.}^3$ and 1, 60 $\mu\Omega/\text{cm.}^3$ and 3,000, 40 $\mu\Omega/\text{cm.}^3$ and 8×10^4, respectively.

In the formula for m, ρ must be expressed in c.g.s. units of resistance, which are such that 1 $\mu\Omega$ = 1,000 c.g.s. units. Accordingly it may be found that m has the respective values $0{\cdot}153\sqrt{n}$, $1{\cdot}4\sqrt{n}$, and $8{\cdot}8\sqrt{n}$. Accordingly the current density will have fallen to 2 per cent. of its surface value at a depth $\frac{26{\cdot}1}{\sqrt{n}}$, $\frac{2{\cdot}86}{\sqrt{n}}$, and $\frac{0{\cdot}45}{\sqrt{n}}$ cm. respectively.

Thus for copper at 10^6 cycles/sec. this depth is 0·261 mm., and for mumetal at 100 cycles/sec. it is 0·45 mm. Hence we now see why the simple expression for eddy-current loss (p. 138), is applicable to mumetal plates of ordinary thickness only if the frequency is very small.

Since the phase of σ is a function of x it follows that the direction of flow will reverse periodically with respect to the direction of flow at the surface. Thus at a depth $x = \frac{2\pi}{m}$ the current flow at every instant of time will be exactly opposite to σ_s; but at this depth the amplitude is $e^{-2\pi} = \frac{1}{500}$ of σ_s and so the reverse currents are of little practical importance. Also the energy loss in the first wave-length is less than the total loss by $\frac{1}{(500)^2}$.

15 a. Plate of finite thickness

We are often concerned with transformer plates of finite thickness t and may need to estimate the effect on the foregoing analysis of this finite thickness. If the value of m is such that $4/m$ is less than

$\frac{1}{2}t$, then clearly we may ignore the finite thickness: on the other hand, if $4/m$ is much greater than $\frac{1}{2}t$, then we may presume the current density increases linearly from the central value to a maximum at the surface. The solution for a plate of finite thickness is quite straightforward but rather laborious. If σ_0 is the current density at the middle of the plate, it can be shown that

$$\frac{\sigma_0}{\sigma_s} = \frac{\sqrt{2}}{\sqrt{\cosh mt + \cos mt}}$$

$$\doteqdot 1 - \frac{m^4t^4}{48} \text{ if } mt < 2,$$

or

$$\doteqdot 2e^{-mt/2}(1 - e^{-mt}\cos mt) \text{ if } mt \gg 1.$$

Thus we see that when mt is large, the current density at the middle of the plate differs insensibly from the sum of the current densities at this depth which would exist if penetration from each face was independent of the existence of the other face.

The same equations can be used to determine the magnetic force at a given depth in a plate which is situated in a uniform field $H\cos pt$: we have only to substitute H for σ in equation (44) above. We are now in a position to amend the equation for eddy-current loss in transformer plates and are better able to understand why the apparent value of μ decreases as the frequency rises. Thus for mu-metal plates at 50 cycles/sec., $m = 60$: if $t = 0{\cdot}5$ mm. (20 mil. plates), then $mt = 3$, and hence

$$H_0 = 0{\cdot}47\, H_s.$$

Therefore the eddy currents then produce so great a demagnetizing effect that the flux density at the middle of the plate has only about half the static value.

15 b. High-frequency resistance of round wires

If the frequency is such that $1/m$ is much less than the radius a of the wire, we may ignore the curvature and consider the wire has a resistance ρm per unit axial length and per unit length of circumference. Accordingly the resistance R_n at frequency n is

$$R_n = \frac{\rho m}{2\pi a}$$

$$= \frac{\sqrt{n\rho\mu}}{a}.$$

But
$$R_0 = \frac{\rho}{\pi a^2}.$$

$$\therefore \quad \frac{R_n}{R_0} = \frac{z\sqrt{2}}{4}, \text{ where } z^2 \equiv \frac{4\pi p\mu a^2}{\rho} = \frac{4p\mu}{R_0}.$$

More exact calculation by the ber, bei formula (see p. 218) shows that
$$\frac{R_n}{R_0} = \frac{z\sqrt{2}+1}{4}. \tag{47}$$

This differs insensibly from the value we have just derived if z is greater than, say, 10: this occurs when $ma = 7$. We have seen that the current density has fallen to 2 per cent. of its surface value at a depth $4/m$: if $ma = 7$, this occurs at a depth $\frac{4}{7}a$. So it seems that if $4/m$ is less than about half the radius of curvature, then an element of any surface may be treated as part of an infinite slab.

On p. 99 we calculated the value of $\frac{R_n}{R_0}$ for small values of z; now we have calculated it for large values of z. For intermediate values of z it must be calculated from tables of ber, bei functions. If this is done it will be found that
$$\frac{R_n}{R_0} = 1+F(z),$$
where the relation between $F(z)$ and z is shown in the following table:

z	1·5	2	2·5	3	3·5
$F(z)$	0·0258	0·0782	0·1756	0·318	0·492

When $z = 3, F(z) = \frac{z\sqrt{2}-3}{4}$ to an accuracy of closer than 2 per cent.

15 c. Effective resistance of a two-core cable

In § 12 of Chapter II we discussed the energy loss due to eddy currents in a circular cylinder placed in a uniform magnetic field, and derived the expression $w = \frac{\pi p^2 H^2 R^4}{8\rho}$, on the supposition that the magnetic field of the eddy currents was negligible compared with the penetrating field. The uniform field H may possibly arise from the proximity of a neighbouring parallel current at a distance D from the centre of the wire in which the eddy currents are generated: such a field will be sensibly uniform if $D \gg R$.

Then $$w = \frac{\pi p^2 R^2}{8\rho}\frac{R^2}{D^2}4I^2.$$

If the wires are the cores of a two-core cable, then it follows that the proximity of the two cores increases the effective resistance of either by an amount

$$R = \frac{\pi p^2 R^2}{\rho}\frac{R^2}{D^2} = \frac{\pi p^2 R^2}{4\rho}\frac{d^2}{D^2}.$$

Hence $$\frac{R_n}{R_0} = \left\{1+\frac{z^4}{192}\left(1+\frac{3d^2}{D^2}\right)\right\}. \tag{48}$$

This is valid for values of z not much greater than unity. For large values of z, the current density will tend to a surface layer, distributed in such a way that there is no internal magnetic field. The distribution of surface density may be calculated from arguments similar to those relating to Fig. 119. It is unnecessary to develop the details here; they are quite straightforward but involve the derivation of a Fourier expansion. When this has been done, it may be shown that

$$\sigma = \frac{I}{\pi d}\left\{1+2\sum_1^\infty x^n \cos n\theta\right\}\sin pt,$$

where θ is the angle included between the radius to any point P on the circumference of one wire and the line joining the two centres, and where

$$x^2 = \frac{2D^2-d^2-2D\sqrt{D^2-d^2}}{d^2}.$$

To calculate the increase of effective resistance due to proximity it is necessary to find the mean value of σ^2 taken round the circumference.

$$\int_0^{2\pi} \sigma^2\,d\theta = \frac{I^2}{\pi^2 d^2}\int_0^{2\pi}\{1+4(x\cos\theta+x^2\cos 2\theta+...)+$$
$$+4(x\cos\theta+x^2\cos 2\theta+...)^2\}\,d\theta$$
$$= \frac{I^2}{\pi^2 d^2}\int_0^{2\pi}\left\{1+\frac{4x^2}{2}(1+\cos 2\theta)+\frac{4x^4}{2}(1+\cos 4\theta)+...\right\}d\theta$$
$$= \frac{I^2}{\pi^2 d^2}\{2\pi+4\pi(x^2+x^4+x^6+...)\}$$
$$= \frac{2\pi I^2}{\pi^2 d^2}\left\{1+\frac{2x^2}{1-x^2}\right\}. \tag{49}$$

The effective resistance per unit length of each wire is

$$R_n = \frac{\rho m}{2I^2}\int_0^{2\pi} \sigma^2\,d\theta.$$

$$\therefore\quad R_n = 2\pi\frac{\rho m}{\pi^2 d^2}\left(1+\frac{2x^2}{1-x^2}\right).$$

Hence the proximity of the two wires increases the resistance of each by an amount $\frac{2x^2}{1-x^2}$ of itself, or increases the resistance of each in the ratio

$$\frac{1+x^2}{1-x^2} = \frac{D}{\sqrt{D^2-d^2}}.$$

This expression gives an upper limit to the possible increment of resistance.

16. Eddy-current losses in the sheath of a cable

Every paper-insulated cable must be covered with a seamless lead sheath: an alternating current in the cable will cause eddy currents in the surrounding sheath. An elegant solution of this problem is due to Dr. F. W. Carter,* and will be outlined here.

The eddy currents in the sheath will produce a magnetic field both inside and outside, and the net resultant field at any point is the sum of the field due to the current in the cable and that due to the eddy current in the sheath. Both fields are expressed most conveniently by means of a potential function. Let a be the radius of the thin cylindrical sheath of thickness τ and specific resistance ρ. Let V_1 and V_0 be the potential of the eddy current field inside and outside the cylinder and V' be the potential of the field of the current in the cable. At the cylindrical surface, $r = a$, the normal magnetic field must be continuous, and so there

$$\frac{\partial V_1}{\partial r} = \frac{\partial V_0}{\partial r}.$$

Applying the work law and the e.m.f. law, we have

$$4\pi\sigma = \frac{1}{a}\left(\frac{\partial V_1}{\partial\theta}-\frac{\partial V_0}{\partial\theta}\right)$$

and

$$\frac{\rho}{a}\frac{\partial\sigma}{\partial\theta} = \tau\frac{\partial}{\partial t}\left(\frac{\partial V_1}{\partial r}+\frac{\partial V'}{\partial r}\right).$$

* *Proc. Camb. Phil. Soc.*, 1927, vol. 23, p. 901.

Hence $$\frac{\partial^2 V_1}{\partial\theta^2}+\frac{\partial^2 V_0}{\partial\theta^2} = 4\pi ka^2\frac{\partial}{\partial t}\left(\frac{\partial V_1}{\partial r}+\frac{\partial V'}{\partial r}\right), \tag{50}$$

where $k = \tau/\rho$.

A solution of this equation is:

$$V_1 = \sum\left(\frac{r}{a}\right)^n\{[A_n\cos pt+B_n\sin pt]\cos n\theta+$$
$$+[A_n'\cos pt+B_n'\sin pt]\sin n\theta\}$$

and

$$V_0 = -\sum\left(\frac{a}{r}\right)^n\{[A_n\cos pt+B_n\sin pt]\cos n\theta+$$
$$+[A_n'\cos pt+B_n'\sin pt]\sin n\theta\}.$$

Let a current $I\sin pt$ be situated at $(b, 0)$. Then

$$V' = -2I\sin^{-1}\frac{r\sin\theta}{(r^2+b^2-2rb\cos\theta)^{\frac{1}{2}}}\sin pt.$$

On substituting in (50) and equating coefficients, we obtain

$$V_1 = -2mI\sum_1^\infty\left(\frac{br}{a^2}\right)^n\frac{n\cos pt+m\sin pt}{n(m^2+n^2)}\sin n\theta$$

and $$V_0 = -2mI\sum_1^\infty\left(\frac{b}{r}\right)^n\frac{n\cos pt+m\sin pt}{n(m^2+n^2)}\sin n\theta,$$

where $m = 2\pi pa\tau/\rho = 2\pi pak$. Hence

$$\sigma = -\frac{mI}{\pi a}\sum_1^\infty\left(\frac{b}{a}\right)^n\frac{n\cos pt+m\sin pt}{m^2+n^2}\cos n\theta$$

$$= -\frac{mI}{\pi a}\sum_1^\infty\left(\frac{b}{a}\right)^n\frac{\cos n\theta}{\sqrt{m^2+n^2}}\sin(pt+\lambda).$$

The mean eddy loss per unit length is accordingly

$$w = \frac{1}{2}\int_0^{2\pi}\frac{\sigma^2}{k}a\,d\theta$$

$$= \frac{mpI^2}{\pi}\int_0^{2\pi}\sum_1^\infty\left(\frac{b}{a}\right)^n\frac{\cos^2 n\theta}{m^2+n^2}$$

$$= mpI^2\sum_1^\infty\left(\frac{b}{a}\right)^n\frac{1}{m^2+n^2}.$$

If the cable has two cores, there is a current I at $(b, 0)$ and a current

$-I$ at (b, π); the net effect will be to remove the even terms in the expansion for σ and double the odd terms. Hence, then,

$$w = \frac{2\pi p^2 a\tau}{\rho} I^2 \left\{\frac{b}{a}\frac{1}{1+m^2} + \left(\frac{b}{a}\right)^3 \frac{1}{9+m^2} + \ldots\right\}$$

$$= \frac{2\pi p^2 \tau}{\rho}\frac{b}{1+m^2}\left\{1 + \left(\frac{b}{a}\right)^2 \frac{1+m^2}{9+m^2} + \ldots\right\} I^2,$$

and thus we find the eddy loss is directly proportional to the distance between the cores of the cable.

EXAMPLES ON CHAPTER IV

1. A hollow cylinder of mumetal has an internal radius of 4 cm. and an external radius of 5 cm. and is placed with its axis perpendicular to a feeble magnetic field. Find the fractional strength of the field inside the cylinder.
ANS. 0·123 per cent. See Fig. 86 and p. 200.

2. A solid iron cylinder, 1 cm. radius, is placed mid-way between two parallel wires, each 1 cm. in radius, whose centre distance is 8 cm. Calculate the inductance per unit length of the system.
ANS. $12{\cdot}3 \times 10^{-3}$ μH/cm, increase due to iron is 4 per cent. See p. 207, § 4.

3. Wire of 4 mm. radius is bent into a circle of 4 cm. radius, and an iron sphere of 1 cm. radius is placed at its centre; find the amount by which the inductance is increased by the presence of the iron.
ANS. 0·93 per cent. See equation (29), p. 55, and p. 208.

4. A two-core cable in which the distance between the cores is 4 cm. is placed inside a mumetal tube of 6 cm. radius and 1 cm. thick. Find by how much the external field of a small current is reduced by the iron tube.
ANS. 0·066 per cent.

5. A circuit consists of two long parallel wires, each 2 mm. diameter and placed 20 cm. apart centre to centre. If these wires are placed midway between iron faces which are 1 cm. apart, show that the inductance is thereby increased about eleven times and is then about 0·25 μH/cm. See p. 233.

6. Find the leakage inductance of the system of Question 5, expressed as a fraction of the self-inductance. ANS. 1·1 per cent. See p. 234.

7. If the wires in Question 5 carry a current of 100 amp., find the repulsive force between them, (*a*) when they are between the iron faces, (*b*) when the iron faces have been removed. ANS. (*a*) 630 dynes/cm; (*b*) 10 dynes/cm.

8. A long straight wire runs between, and parallel to, two parallel faces of iron which are 2 cm. apart: if there is a current of 100 amp. in the wire, calculate the force attracting it towards one face of the iron, (*a*) when the centre of the wire is midway between the iron faces, (*b*) when the centre of the wire is 0·5 cm. from one iron face.
ANS. (*a*) zero; (*b*) 157 dynes/cm. See p. 237.

9. Two wires, each carrying a current of 100 amp., are placed as shown in Fig. 147, and are 2 cm. apart. Calculate the repulsive force between them, (*a*) when the air-gap is 4 cm., (*b*) when the air-gap is 3 cm., (*c*) when the air-gap is 12 cm. ANS. (*a*) 157 dynes/cm.; (*b*) 242 dynes/cm.; (*c*) 104·6 dynes/cm.

10. A long, rectangular circuit is placed between parallel iron faces and the span between the wires is half the air-gap: the circuit can turn about an axle, such that it can be placed as shown in Fig. 147 or as shown in Fig. 142. Show that when the plane of the rectangle is perpendicular to the plane of the air-gap, the inductance is 7×10^{-10} H./cm. less than when the plane of the coil is parallel to the plane of the gap.

11. Calculate the resistance per unit length of a round copper rod, 1 cm. radius, to a simple harmonic current of frequency 10^6 cycles/sec., and express this as a multiple of the resistance to steady currents.

ANS. 41·5 $\mu\Omega$/cm., 77·5. See p. 244.

12. Calculate the resistance per unit length of a round iron rod of 1 cm. radius, specific resistance 50 $\mu\Omega/\text{cm.}^3$, and initial permeability 500, to a simple harmonic current of frequency 50 cycles/sec. ANS. 39 $\mu\Omega$/cm.

13. The copper conductors of a two-core cable are each 5 mm. diameter and are placed 2 cm. apart, centre to centre. Calculate the increase of resistance to simple harmonic currents of frequency 50 cycles/sec.

ANS. 0·0138 per cent.

14. Show that for the cable described in Question 13, the increment of resistance due to the proximity of the return conductor cannot exceed 3 per cent. (See p. 246.)

V

MAXWELL'S EQUATIONS AND THE ELECTRO-MAGNETIC FIELD

1. The Maxwell hypothesis

In this book electric current has been recognized and measured by the magnetic field it produces. When a wire is connected to the terminals of a battery, a magnetic field is created which did not exist previously. How the magnetic field is created is a complete mystery which is not penetrated by inventing an electric current in the wire; because certain magnetic effects are observable, an electric current is said to be flowing in the wire.

But there is a distinct set of phenomena described by the term electric attractions, and it is customary to describe these attractions in terms of an invented and hypothetical substance called electricity; an imponderable substance which can be subdivided indefinitely. On investigating the phenomena of electric attractions, described in terms of the hypothetical substance electricity, it appears that the electricity must flow along the wires joining the terminals of the battery. It was a distinct and separate discovery, due to Oersted in 1805, that a magnetic field accompanies the flow of electricity. A current of electricity is necessarily the rate at which the substance electricity is passing a given point in a conductor; the flow of the hypothetical substance may be likened to water flowing along a pipe. Atomic views of matter regard the water-flow as a flow of discreet particles, and so it is natural to think of the substance electricity as having an atomic structure.

This model makes current flow very concrete and the reader should imagine himself watching particles of electricity threading their way through the molecular interspaces of a conductor; he observes their velocity and concentration and so measures the current.

But all this is just a model which must not be allowed to restrict thought. Faraday's conception of current was very wide, for he says: 'By a current I mean anything progressive, whether it be a fluid of electricity, or two fluids moving in opposite directions, or merely vibrations, or, speaking more generally, progressive forces.'*

* See *Faraday's Researches* vol. 1, § 19.

Being committed to the model just described, it is necessary to accept as a current the flow of a single charge along a wire. The position of such a charge could be charted by observing the change of force on a stationary test charge placed anywhere outside the wire. Such a current is never observable by human eye, but its existence could be detected by a stationary test charge placed anywhere. Since charge is only a convenient invention, is it necessary to take so narrow a view of current, and cannot it be said to exist at any point where the movement of the charge can be detected? In fact, cannot a changing electric force be regarded as a current just as much as the passage of a charge past a given point in a wire? Such an enlarged conception of current will have many and far-reaching results. Consider the familiar experiment of charging a condenser. During the charging process charges flow along the wires on to the plates, but they do not flow across the gap between the plates: a charging current is not circuital and div i is not zero everywhere but only in the conductors leading to the plates. But between the plates there is a changing electric force, and if this is to be regarded as a current, then current flows only in closed circuits and is therefore a vector such that div i is zero everywhere. Let the condenser consist of parallel plates of area A separated by a distance d, and let the dielectric have a constant K: the condenser holds a charge q at time t at a voltage v.

Then
$$q = Cv$$
$$= \frac{KA}{4\pi}\frac{v}{d} = \frac{KA}{4\pi}E.$$
$$\therefore \quad i = \frac{dq}{dt} = \frac{KA}{4\pi}\frac{dE}{dt}.$$

There is thus a constant relation between the current i in the connecting wires and the rate of change of the electric force E between the plates. If a changing electric force is to be regarded as a current, then a conduction current i is accompanied by a current density $\frac{K}{4\pi}\frac{dE}{dt}$ of changing electric force between the plates. This changing flux of electric force is called a displacement current, to distinguish it from the conduction current i in the wires.

But the capacity C has been measured in absolute electrostatic units, and hence i has also been measured in electrostatic units. A unit of current in electrostatic measure is $1/c$ of the unit in electromagnetic measure.* So if i is to be measured in electromagnetic units,

$$i = \frac{KA}{4\pi c}\frac{dE}{dt}.$$

This enlarged conception of current is useful because it has made current flow only in closed circuits, so that $\operatorname{div} i = 0$ everywhere. But is it necessary to restrict the work law $\int H\, dl = 4\pi i$ or

$$\operatorname{curl} H = 4\pi i,$$

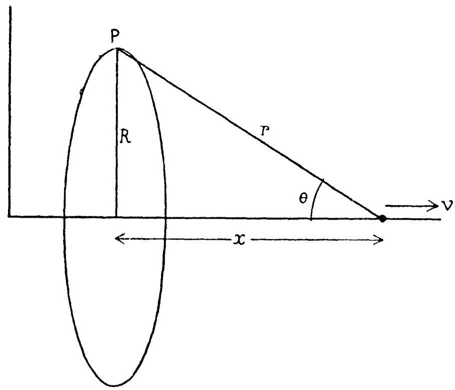


FIG. 149

to conduction currents only? Suppose it applies equally to displacement currents and let us see what results we are led to. This great suggestion is due to Clerk Maxwell, in 1863.

If at a point in space the electric force is changing at the rate $\frac{dE}{dt}$, the density of displacement current is $\frac{K}{4\pi}\frac{dE}{dt}$: applying the work law to a point in space where the electric force is changing, we have

$$\operatorname{curl} H = 4\pi \frac{K}{4\pi c}\frac{dE}{dt}$$

$$= \frac{K}{c}\frac{dE}{dt}.$$

2. Some applications of the Maxwell hypothesis

Suppose a charge q is moving with velocity v along the positive direction of the axis of x. Then at a point P, distant r from the charge, the electric field strength is $E = q/r^2$ and this is changing

* The numerical value of c is $2{\cdot}999 \times 10^{10}$.

because the charge is moving: hence at every point of space there is a displacement current and consequently a magnetic field. Symmetry shows that the magnetic field must be disposed in circles centred on the axis along which the charge is moving. Consider the magnetic field round a circle of radius R (see Fig. 149). The flux of electric force through the ring is

$$\frac{q}{r^2}2\pi r^2(1-\cos\theta) = 2\pi q(1-\cos\theta).$$

So the displacement current through the ring is

$$\frac{1}{4\pi c}\frac{d}{dt}\{2\pi q(1-\cos\theta)\}.$$

$$\therefore \quad H\times 2\pi R = 4\pi\frac{1}{4\pi c}\frac{d}{dt}\{2\pi q(1-\cos\theta)\}.$$

$$\therefore \quad H = -\frac{q}{Rc}\frac{d}{dt}(\cos\theta)$$

$$= -\frac{q}{Rc}\frac{d}{dt}\frac{x}{\sqrt{x^2+R^2}}$$

$$= -\frac{q}{Rc}\frac{R^2}{(x^2+R^2)^{\frac{3}{2}}}\frac{dx}{dt} = -\frac{qv}{c}\frac{\sin\theta}{r^2}.$$

But this is the magnetic field which can be ascribed to a current element such that $idl = qv$. (See Chap. I, p. 25.)

An electric current in a wire must be regarded as a stream of charges all moving with the same average velocity. The magnetic field of the current can be ascribed to the displacement currents due to each component moving charge; if this is done, then the foregoing analysis shows that the Maxwell hypothesis is not inconsistent with Ampère's experiments and the device of the magnetic shell. Such a method of calculation would be very artificial, because at every point the net time rate of change of electric force is zero, and therefore the net displacement current is zero everywhere. That the net magnetic field is not zero can be understood by considering two charges moving in a straight line: thus in Fig. 149 let there be a charge q moving with speed v at a distance x to the left of the circle of radius R. Then the electric flux through the ring is increasing from the left-hand charge at the same rate as it is decreasing from the right-hand charge, and the net displacement current through the ring is zero. But if the right-hand screw rule requires a decreasing electric field to be regarded as a positive current, then it requires an increasing electric field to be regarded as a negative current; so the magnetic force round the ring is twice that due to one charge although the net displacement current through the ring is zero. The student must remember that there are no displacement currents associated with a steady current flowing in a wire: it is only changing currents which give rise to displacement currents. However, the Maxwell hypothesis allows us to calculate the magnetic effect of any current flow by summing up the contributions from each component particle charge of electricity flowing somewhere along a conductor or projected through space.

We can now see whether any force is required to maintain an electric charge in uniform motion. The moving charge produces a magnetic field, and this field is changing at every point of space. So by Faraday's law of induced e.m.f. the changing magnetic field produces an electric field which is not due to the charge, *per se*, but is due to its motion. Hence, if the electric field round a moving charge is explored by a test charge, the field will not be found to obey the inverse square law, but there will be additional terms due to the motion. If there is any net axial electric force acting on the moving charge, then work will be required to maintain the motion. In Chapter IV we saw how the electric field could be calculated from the time rate of change of A, the vector potential of the changing currents. Consider the point P in Fig. 149. Then

$$E = q/r^2.$$

$$\therefore \quad \frac{dE}{dt} = -\frac{2q}{r^3}\frac{dr}{dt} = \frac{2q}{r^3}\, v\cos\theta,$$

since
$$\frac{dr}{dt} = -v\cos\theta.$$

Now from equation (27) of Chapter IV, $A = \int \frac{i\,dv}{r}$, hence the vector potential at the charge due to the displacement current through an annular ring of radius $r\sin\theta$ and width $r\,d\theta$ is

$$dA = \frac{1}{4\pi c}\frac{2qv}{r^3}\cos\theta\frac{2\pi r\sin\theta\; rd\theta dr}{r}$$

$$= \frac{qv}{2c}\frac{\sin 2\theta}{r^2}\, drd\theta.$$

Let the moving charge have a radius a, then integrating from a to infinity

$$dA = -\frac{qv\sin 2\theta\, d\theta}{2ac}.$$

The axial component is $dA\cos\theta$

$$= -\frac{qv\sin 2\theta\cos\theta\, d\theta}{2ac}$$

$$= -\frac{qv}{4ac}(\sin 3\theta - \sin\theta)\, d\theta.$$

$$\therefore \quad \text{Axial component of } A = -\frac{qv}{ac}\left[-\frac{\cos 3\theta}{3}+\cos\theta\right]_0^{\frac{1}{2}\pi}$$

$$= \frac{2}{3}\frac{qv}{ac}.$$

$$\therefore \quad \frac{dA}{dt} = \frac{2}{3}\frac{q}{ac}\frac{dv}{dt}.$$

Hence, if the charge is accelerated, the acceleration produces, by means of the displacement currents and magnetic field, an axial electric force E, such that

$$E = \frac{2}{3}\frac{q}{ac^2}\frac{dv}{dt}.$$

Hence the mechanical force F required to produce the acceleration is

$$F = Eq$$

$$= \frac{2}{3}\frac{q^2}{c^2}\frac{1}{a}\frac{dv}{dt}.$$

Hence, if a uniformly charged sphere of radius a is accelerated, its mass will appear to be greater than when it is uncharged, by an amount $\frac{2}{3}\frac{q^2}{c^2}\frac{1}{a}$.

There is a precisely analogous phenomenon for a sphere of radius a which is accelerated through a liquid of density ρ: the liquid pressure in the direction of motion is increased when the motion is accelerated, and this gives the sphere an apparent increment of mass, amounting to

$$m' = \tfrac{1}{2}.\tfrac{4}{3}\pi a^3\rho$$

$$= \tfrac{2}{3}\pi a^3\rho.$$

The force required to accelerate a charge is usually derived from the energy stored in the magnetic field, thus

$$\frac{H^2}{8\pi}\,d\tau = \frac{q^2v^2\sin^2\theta}{8\pi c^2}\,\frac{2\pi r\sin\theta\, rd\theta dr}{r^4}$$

$$= \frac{q^2v^2}{4c^2}\,\frac{\sin^3\theta\, d\theta dr}{r^2}.$$

$$\therefore \quad \iiint \frac{H^2}{8\pi}\,d\tau = \frac{q^2v^2}{c^2}\int\limits_0^\pi\int\limits_a^\infty \frac{\sin^3\theta\, d\theta dr}{r^2}$$

$$= \frac{4}{3}\,\frac{q^2v^2}{c^2}\int\limits_a^\infty \frac{1}{r^2}\,dr$$

$$= \frac{4}{3}\,\frac{q^2v^2}{ac^2}.$$

This expression shows that the energy associated with the charge varies as the square of its velocity and so is equivalent to kinetic energy of mass. It shows that the apparent mass of the body is increased by the charge by an amount $\frac{2}{3}\frac{q^2}{c^2}\frac{1}{a}$.

It should be noticed that a uniform charge on a massless sphere would still appear to have mass, and this mass is due to the displacement currents which its motion creates. Therefore it is possible that there is no mass other than that due to charge, and that what we recognize as mass is due really to the charges which exist in all bodies. Thus there is no need to assume an electron has any mass other than that which results from displacement currents. It should be noticed that this mass is not a function of the velocity, even though it originates from the velocity.

3. Electric charge moving in a magnetic field

Since the moving charge produces circular lines of magnetic force centred on the path of the charge, the motion would cause a free magnetic pole to spiral round the path and the force on a pole of strength m situated at (r, θ) with respect to the charge, will be

$$F = \frac{mq}{c}\,\frac{\sin\theta}{r^2}\,v,$$

and will be in a direction perpendicular to the motion of the charge. But the pole must produce an equal and opposite force on the moving charge. A charge can be acted on only by an electric field, and so the fixed pole must appear to the moving charge as an electric force of intensity $E = \frac{m \sin\theta\, v}{cr^2}$. But this is in agreement with the cutting-rule statement of the law of induced e.m.f., for the charge is cutting with speed v across a magnetic field of intensity $H = \frac{m \sin\theta}{r^2}$.

If we now suppose a conductor contains a large number of free electrons, all moving with the same average velocity, but as many in any one direction as in any other, we shall expect an e.m.f. to be induced when a conductor is moved across a magnetic field. If a conductor is translated as a whole, every free electron will have a general velocity in one direction superposed on the general chaotic motion. At a fixed point in space the moving conductor will produce a magnetic field. Similarly a pole fixed in space will produce, at each electron, an electric force tending to drive it perpendicular both to the magnetic field and the motion. Electrons will accumulate at one end of the conductor, and so will produce an electric field along the conductor which is equal and opposite to that produced by the motion of the conductor.

Now consider a current flowing in a circular turn of wire and let a magnetic pole be placed on the axis of the circle. The current may be regarded as a stream of charges each moving with speed v, and these charges will produce a magnetic field at the pole equal to that calculated by the Ampère shell method. The pole will produce a force F on each moving charge such that $F = \frac{m}{r^2}\frac{qv}{c}$, perpendicular to the radius vector to the pole. Therefore the charges will tend to move perpendicular to the radius vector as well as along the wire. Collisions with the atoms of the material of the wire will produce a force tending to move the circle away from or towards the pole. We now have a mechanism with which to describe the action of a pole on a circuit and also a mechanism with which to describe the voltage generated in a wire by its motion across a magnetic field.

4. Charged particle moving in a uniform magnetic field

Let a charge q move freely in a uniform magnetic field of intensity H. The velocity at any moment can be resolved into a component u in the direction of the magnetic field and a component v perpendicular to the field. If there are no impressed external forces acting on the charge, u and v will remain constant, because the force on the charge, due to the field and the motion, is always perpendicular to the motion, and so does no work. There will be a force Hqv/c on the charge, perpendicular to the field; if the charge has a mass m, this will produce an acceleration $\frac{Hqv}{cm}$ perpendicular to the field. Consider the motion in a plane perpendicular to the field: there is a velocity v and an acceleration $\frac{Hqv}{cm}$ perpendicular to it. This acceleration must be equal to $\frac{v^2}{\rho}$ where ρ is the radius of curvature of the path. Hence

$$\frac{v^2}{\rho} = \frac{Hqv}{mc}. \qquad \therefore \quad \rho = \frac{mcv}{qH}.$$

Now H is constant by hypothesis, and v is constant because the force is perpendicular to the direction of motion: therefore ρ is constant and the charge moves in a circle.

If a charge is moving in a uniform magnetic field H and has a velocity u in the direction of H and v perpendicular to H, then the complete orbit is a circular helix of radius $\frac{mcv}{qH}$ described round a line of force as axis. The charge will make one complete revolution in a time T, such that

$$2\pi\rho = vT$$

$$\therefore \quad T = \frac{2\pi mc}{qH},$$

and in this time it advances a distance uT and the pitch of the spiral is $\frac{2\pi mcu}{qH}$.

If the charge is spread uniformly on a sphere of radius a and has no mass other than that due to the motion of the charge, then

$$\rho = \frac{2}{3}\frac{q}{a}\frac{v}{cH}.$$

5. Charge moving in a uniform magnetic and electric field

Suppose that in a given region there is a uniform magnetic field and a uniform electric field, the two fields being mutually perpendicular. Let a particle be projected with speed v perpendicular to both fields.

Then it will experience a force $F_1 = Hqv/c$ from the magnetic field and a force $F_2 = Eq$ from the electric field. If F_1 and F_2 are made equal and opposite, the moving charge will not suffer any deflexion, and in this condition $v = cE/H$. So, by applying mutually perpendicular fields of appropriate strengths, the velocity of the charge can be measured. If the electric field is then removed, the charge will travel in a helix of radius $\rho = mcv/qH$. If v is supposed to have been measured by the process just described, the ratio m/q can be determined from a measurement of ρ.

This method of measurement was used by Sir J. J. Thompson to determine the ratio m/q for an electron. A stream of electrons was generated in a discharge tube and a narrow beam was passed between two parallel plates charged to a known voltage; a uniform magnetic field was applied parallel to the plates and perpendicular to the direction of the beam. The relative magnitude of the two fields was adjusted until the cathode stream of electrons suffered no net deflexion, and in this manner the velocity of the flying charges was found to be about 3×10^9 cm./sec., which is one-tenth the velocity of light.

The magnetic field was then removed and the curvature of the orbit was measured, and by means of these two measurements the ratio q/m was found to be $5{\cdot}6\times 10^7$ e.s. units of charge per gramme.

Electrons are also given off from an incandescent metal surface, as, for example, the filament of a thermionic valve. If the anode is at a positive potential V with respect to the filament, the electrons will be drawn to it by the electric field and will gain kinetic energy in the transit. If the velocity of emission is ignored, they will attain a velocity v such that

$$\tfrac{1}{2}mv^2 = Vq.$$

If a magnetic field is applied perpendicular to the electric field, then

$$\rho = \frac{mcv}{qH} = \frac{mc}{qH}\sqrt{\frac{2Vq}{m}}.$$

$$\therefore \quad \rho^2 = \frac{2c^2V}{H^2}\frac{m}{q}.$$

This gives a method of determining q/m by measuring ρ, V, and H.

The complete motion of a particle placed in mutually perpendicular magnetic and electric fields is interesting to study. Thus consider Fig. 150 in which the plate AB is maintained at a positive potential

V with respect to the earth plate. Let a negative charge be placed at the origin O and then released: from the action of the uniform electric field it will start to move towards AB. If the uniform magnetic field is directed into the paper, then the velocity of the negative charge will produce a force on the particle acting towards the right and give it a horizontal velocity u. The negative charge moving towards the right is equivalent to a positive current flowing from right to left: such a current in the magnetic field would experience a vertical force pushing it down towards OX. So the net upward force is the resultant of an upward pull due to the electric field and

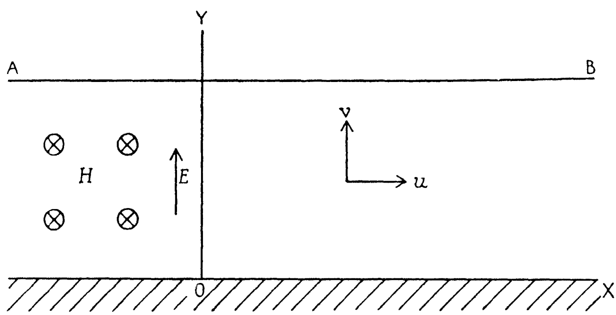


Fig. 150

a downward push which results from the horizontal velocity u which has been generated by means of the upward velocity v. With suitable values of E and H it is possible the charge might never reach the upper plate and would perform a succession of hops which carry it along OX. The equations of motion are:

$$m\frac{d^2y}{dt^2} = q\left(E-\frac{H}{c}\frac{dx}{dt}\right)$$

$$m\frac{d^2x}{dt^2} = \frac{qH}{c}\frac{dy}{dt}.$$

$$\therefore \quad \frac{d^3y}{dt^3} = -\left(\frac{qH}{mc}\right)^2\frac{dy}{dt}.$$

$$\therefore \quad \frac{d^2y}{dt^2} = -\left(\frac{qH}{mc}\right)^2 y+\beta.$$

This is the equation of a point moving in a cycloid: that is, the path of a point on the circumference of a circle of radius r which rolls with uniform angular velocity p along the axis of x. For it follows readily that the equation of such a motion is

$$y = r(1-\cos pt)$$
$$x = r(pt-\sin pt).$$

Hence, differentiating twice, we have

$$\ddot{y} = p^2r-p\dot{x}$$

and $$\ddot{x} = p\dot{y}.$$

Hence, on comparison, we find that

$$p = \frac{qH}{mc} \quad \text{and} \quad r = \frac{Emc^2}{qH^2}.$$

If the distance between the plates is greater than $2r$, the charge will not reach the upper plate and will travel along the x-axis in a series of cycloidal hops. If V is the voltage between the plates, then the condition for no current to pass is that $V = \dfrac{qH^2}{2mc^2}$.

We have in general that

$$y = \frac{Emc^2}{qH^2}(1-\cos pt)$$

and $$x = \frac{Emc^2}{qH^2}\left(\frac{qH}{mc}t-\sin\frac{qH}{mc}t\right).$$

If the charge does not reach the upper plate, the time of a hop is

$$T = \frac{2\pi mc}{qH}$$

and the length of each hop is

$$x = 2\pi r = \frac{2\pi Emc^2}{qH^2}.$$

6. Maxwell's equations

Maxwell's hypothesis regards a changing electric field as equivalent to a current, and the work done in taking a unit pole round any closed path will be 4π times the conduction current enclosed plus 4π times the displacement current enclosed. The Cartesian expression of the work law now becomes

$$4\pi\left(\dot{i}_1+\frac{K}{4\pi c}\dot{E}_1\right) = \frac{\partial H_3}{\partial y}-\frac{\partial H_2}{\partial z}, \text{ etc.,}$$

where i_1 is the density of conduction current at the point x, y, z, and E_1 is the electric intensity at the same point in the medium of dielectric constant K. Since i_1 must be reckoned in electromagnetic units it is necessary to change $\dot{E}_1$, etc., into these units by dividing by the factor c whose numerical value is 3×10^{10}. The Cartesian expression for the induced e.m.f. remains unchanged, and this is

$$-\frac{\mu}{c}\dot{H}_1 = \frac{\partial E_3}{\partial y}-\frac{\partial E_2}{\partial z}, \text{ etc.}$$

In free space there are no charges or conduction currents, so that i_1, etc., are zero; then these equations written in vectorial form are

$$\left.\begin{aligned} \frac{K}{c}\dot{E} &= \operatorname{curl} H \\ -\frac{\mu}{c}\dot{H} &= \operatorname{curl} E \\ \operatorname{div} E &= 0 \\ \operatorname{div} B &= 0 \end{aligned}\right\}. \tag{1}$$

Equations (1) are generally called Maxwell's equations, since they express in symbols the Maxwell hypothesis.

Either E or H can be eliminated from equations (1) as follows:

$$\begin{aligned} \frac{K}{c}\frac{\partial^2 E_1}{\partial t^2} &= \frac{\partial}{\partial y}\left(\frac{\partial H_3}{\partial t}\right)-\frac{\partial}{\partial z}\left(\frac{\partial H_2}{\partial t}\right) \\ &= \frac{c}{\mu}\left(-\frac{\partial^2 E_2}{\partial x\,\partial y}+\frac{\partial^2 E_1}{\partial y^2}+\frac{\partial^2 E_1}{\partial z^2}-\frac{\partial^2 E_3}{\partial x\,\partial z}\right) \\ &= \frac{c}{\mu}\left\{\frac{\partial^2 E_1}{\partial y^2}+\frac{\partial^2 E_1}{\partial z^2}+\frac{\partial^2 E_1}{\partial x^2}-\frac{\partial}{\partial x}\left(\frac{\partial E_1}{\partial x}+\frac{\partial E_2}{\partial y}+\frac{\partial E_3}{\partial z}\right)\right\} \\ &= \frac{c}{\mu}\nabla^2 E_1, \text{ since } \operatorname{div} E = 0. \end{aligned} \tag{2}$$

There are similar equations for E_2 and E_3 and the H's. Equation (2) is called 'the wave-motion equation' and may be interpreted as expressing that a given state, such as E_1, travels out with a speed $\frac{c}{\sqrt{\mu K}}$. It means that if a disturbance begins to be felt at a given point at time $t = 0$, there will be some other point distant d from the given point at which the disturbance will not begin to be felt until a time $t = d\frac{\sqrt{\mu K}}{c}$ has elapsed.

7. Lorentz's equations and retarded potentials

According to the Maxwell hypothesis a periodic disturbance will fill all space with displacement currents, and the Maxwell equations relate the spatial changes of the field with the temporal changes at a given point.

But we want to know what displacement currents will be created by a given disturbance at a given point. Something equivalent to an Ohm's law for the medium is required, by which the currents can be calculated which arise from applying a voltage at a given point. Maxwell's equations are not expressed in a form which is convenient for relating the disturbance to the origin, and it is necessary to resort to the mathematical tools called scalar and vector potentials.

By applying the Maxwell hypothesis to a moving charge it has been seen that a moving charge is equivalent in magnetic effect to a current. So if there is a charge ρ in a given volume element $d\tau$, moving with component velocities v_1, v_2, and v_3, then $\rho v_1 = i_1$, etc., and so

$$\frac{4\pi}{c}\left(\rho v_1+\frac{K\dot{E}_1}{4\pi}\right)=\frac{\partial H_3}{\partial y}-\frac{\partial H_2}{\partial z}, \text{ etc.}$$

But
$$\mu H_2=\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}$$

and
$$\mu H_3=\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}.$$

Whence

$$\begin{aligned}\frac{4\pi\mu}{c}\left(\rho v_1+\frac{K\dot{E}_1}{4\pi}\right) &= \frac{\partial^2 A_2}{\partial y\,\partial x}-\frac{\partial^2 A_1}{\partial y^2}-\frac{\partial^2 A_1}{\partial z^2}+\frac{\partial^2 A_3}{\partial z\,\partial x}\\ &= \frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_2}{\partial x\,\partial y}+\frac{\partial^2 A_3}{\partial x\,\partial z}-\frac{\partial^2 A_1}{\partial x^2}-\frac{\partial^2 A_1}{\partial y^2}-\frac{\partial^2 A_1}{\partial z^2}\\ &= \frac{\partial}{\partial x}(\operatorname{div} A)-\nabla^2 A_1. \qquad (3)\end{aligned}$$

where A, the vector potential, is defined by $\mu H = \operatorname{curl} A$. But the Faraday law of induced e.m.f. is

$$-\frac{\mu}{c}\dot{H}_1=\frac{\partial E_3}{\partial y}-\frac{\partial E_2}{\partial z}, \text{ etc.,}$$

from which it has already been deduced that (see p. 223)

$$E_1=-\frac{1}{c}\frac{\partial A_1}{\partial t}-\frac{\partial V}{\partial x}, \text{ etc.,}$$

where V is any function having zero curl in the field.

$$\therefore\quad \dot{E}_1=-\frac{1}{c}\frac{\partial^2 A_1}{\partial t^2}-\frac{\partial^2 V}{\partial t\,\partial x}.$$

So, on substituting in (3),

$$\frac{4\pi\mu\rho}{c}v_1-\frac{\mu K}{c^2}\frac{\partial^2 A_1}{\partial t^2}-\frac{\mu K}{c}\frac{\partial^2 V}{\partial t\,\partial x}=\frac{\partial}{\partial x}\operatorname{div}A-\nabla^2 A_1.$$

$$\therefore\quad \nabla^2 A_1-\frac{\mu K}{c^2}\frac{\partial^2 A_1}{\partial t^2}=-\frac{4\pi\mu\rho}{c}v_1+\frac{\partial}{\partial x}\left(\operatorname{div}A+\frac{\mu K}{c}\frac{\partial V}{\partial t}\right).$$

Now A and V are defined by the equation $H = \operatorname{curl} A$ and $\operatorname{curl} V = 0$, and these equations are insufficient to determine A and V uniquely (see Chap. IV, p. 220). Maxwell's equations show that E and H satisfy a wave-motion equation, and this suggests that the derived mathematical tools A and V will be most serviceable if they are moulded so as to satisfy a wave-motion equation also. This can be done if it is decided to impose on A and V the extra condition that $\operatorname{div}A+\frac{\mu K}{c}\frac{\partial V}{\partial t}=0$.

Then $$\nabla^2 A_1=\frac{\mu K}{c}\frac{\partial^2 A_1}{\partial t^2}-\frac{4\pi\rho v_1}{c}, \text{ etc.} \tag{4}$$

But by Gauss's theorem

$$4\pi\rho=\operatorname{div}E.$$

$$\therefore\quad -4\pi\rho=\frac{1}{c}\frac{\partial}{\partial t}(\operatorname{div}A)+\nabla^2 V$$

$$=\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{\mu K}{c}\frac{\partial V}{\partial t}\right)+\nabla^2 V.$$

$$\therefore\quad \nabla^2 V=\frac{\mu K}{c^2}\frac{\partial^2 V}{\partial t^2}-4\pi\rho. \tag{5}$$

So the decision to make $\operatorname{div}A+\frac{\mu K}{c}\frac{\partial V}{\partial t}=0$ forms both A and V in such a way that both satisfy a wave equation. It may be found that the solution of this equation is

$$\left.\begin{aligned} A_1&=\frac{1}{c}\int\frac{[\rho v_1]\,d\tau}{r},\\ A_2&=\frac{1}{c}\int\frac{[\rho v_2]\,d\tau}{r},\\ A_3&=\frac{1}{c}\int\frac{[\rho v_3]\,d\tau}{r},\\ V&=\frac{1}{c}\int\frac{[\rho]\,d\tau}{r},\end{aligned}\right\} \tag{6}$$

where $[\rho]$ and $[\rho v_1]$, etc., denote that the charge and current density supposed to be in a given volume element $d\tau$ (distant r from the point where A_1, etc., are being calculated) at a given time t is not that perceived by an observer in the element at that time, but is to be taken as that which was perceived there at an earlier epoch $\left(t - r\frac{\sqrt{\mu K}}{c}\right)$. Such quantities are often called 'retarded functions', and A and V are called the vector and scalar electromagnetic potentials respectively.

So if ρ and ρv are postulated everywhere, then A and V can be calculated and from them the fundamental quantities E and H by means of the equations

$$E_1 = -\frac{1}{c}\frac{\partial A_1}{\partial t} - \frac{\partial V}{\partial x} \tag{7}$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\int \frac{[\rho v_1]}{cr}\,d\tau - \frac{\partial}{\partial x}\int \frac{[\rho]}{r}\,d\tau$$

and

$$H_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}. \tag{8}$$

This gives a process of calculating the E's and the H's which results from any assigned distribution of charge and current; the process may lead to intractable analysis, and then the difficulty must be surmounted by ingenuity and approximations.

These convenient equations have arisen by choosing to make $\operatorname{div} A + \frac{\mu K}{c}\frac{\partial V}{\partial t} = 0$. The reader might suppose this arbitrary decision involved some assumption about physically observable quantities. This is not so because A and V are derived tools which can be shaped to the most convenient form. In Chapter IV it was found convenient to define A from the equations $\mu H = \operatorname{curl} A$ and $\operatorname{div} A = 0$. When the Maxwell hypothesis is included the vector A is more serviceable when it is arranged so that $\operatorname{div} A + \frac{\mu K}{c}\frac{\partial V}{\partial t} = 0$.

8. Applications of retarded functions

These equations will now be used to calculate the fields close to an alternating current which is flowing in an open-ended wire. Consider a long thin wire of length l, terminated by conducting spheres whose radii are large compared with that of the wire. We will postulate the current and charge distribution as follows. A current $i = I\sin pt$ flows along the whole wire of length l and does not change in magnitude or phase along this length. The charge on the spheres is distributed uniformly and is $Q\cos pt$ on the upper sphere and $-Q\cos pt$ on the lower sphere. Since $i = \dfrac{dq}{dt}$, therefore $I = -pQ$. The system is shown diagrammatically in Fig. 151. This current distribution is not realizable in practice because there would be some accumulation of charge on the surface of the wire. Provided the spheres are of large diameter and l is short compared with the wave-length λ of the alternation, the assumed current distribution will be very closely correct because the charge on the surface of the wire will be negligible compared with that on the spheres. So in a volume element of wire there is a charge ρ moving with speed v_3, but the net charge in that volume does not change with time: so we have $\rho v_3\,d\tau = i\,dz$.

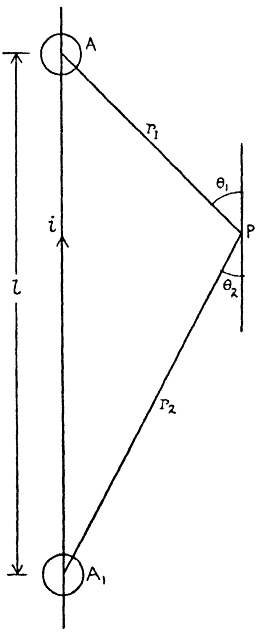


FIG. 151

Consider a point distant r from an element of charge on the lower sphere, then:

$$V = -\frac{Q}{r}\cos p\left(t-\frac{r}{c}\right)$$
$$= -\frac{Q}{r}\left(\cos\frac{pr}{c}\cos pt+\sin\frac{pr}{c}\sin pt\right).$$

The first term of this expression is in phase with the charge, and the second term is in phase quadrature with it and consequently in phase with the current. Consider the two terms separately, starting with the second; expanding $\sin pr/c$, this becomes

$$V = -\frac{Qp}{c}\left(1-\frac{p^2r^2}{6c^2}+\frac{p^4r^4}{120c^4}\cdots\right)\sin pt$$

$$\therefore\quad -\frac{\partial V}{\partial r} = -\frac{p^3Qr}{3c^3}\left(1-\frac{p^2r^2}{10c^2}\cdots\right)\sin pt.$$

Now $2\pi n = p$ and $c = n\lambda$. $\quad\therefore\quad \dfrac{p}{c} = \dfrac{2\pi}{\lambda}$.

$$\therefore\quad -\frac{\partial V}{\partial r} = -\frac{p^3Qr}{3c^3}\left(1-\frac{4\pi^2}{10}\frac{r^2}{\lambda^2}\cdots\right)\sin pt.$$

We will concern ourselves with points near the wire and for such points $r/\lambda \ll 1$.

$$\therefore \quad -\frac{\partial V}{\partial r} \doteqdot -\frac{p^3 Q r}{3c^3}\sin pt.$$

Now apply the same process to the other component of V.

$$V = -\frac{Q}{r}\left(1-\frac{p^2r^2}{2c^2}+\frac{p^4r^4}{24c^4}...\right)\cos pt.$$

$$\therefore \quad -\frac{\partial V}{\partial r} = -\left(\frac{Q}{r^2}+\frac{p^2Q}{2c^2}+...\right)\cos pt.$$

Hence
$$-\frac{\partial V}{\partial r} \doteqdot -Q\left(\frac{1}{r^2}+\frac{p^2}{2c^2}\right)\cos pt-\frac{Qp^3r}{3c^3}\sin pt.$$

When p becomes vanishingly small, this reduces to

$$-\frac{\partial V}{\partial r} = -\frac{Q}{r^2}\cos pt,$$

which is in agreement with the inverse square law of electrostatics. The additional terms are due to displacement currents induced in the ether. Now consider similarly situated elements dq of charge on the upper and lower sphere, distant respectively r_1 and r_2 from the point P in Fig. 151. The upper element contributes a repulsion $\frac{dq\,p^3r_1}{3c^3}\sin pt$ along PA, and the lower element an attraction $\frac{dq\,p^3r_2}{3c^3}\sin pt$ along $A'P$. The vertical component of the resultant force at P is

$$-\frac{dq\,p^3}{3c^3}(r_1\cos\theta_1+r_2\cos\theta_2)\sin pt = -\frac{dq\,p^3l}{3c^3}\sin pt.$$

So in the vicinity of the system there is a vertical component of electric force which is independent of the distance and at a given frequency it depends only on the product $dq.l$. So the field of all the charge on the sphere can be calculated correctly by supposing it is concentrated at the centre.

Now consider the component of electric force which arises from the vector potential

$$dA_3 = \frac{[idz]}{cr} = \frac{Idz}{cr}\sin p(t-r/c)$$

$$= \frac{Idz}{cr}\left\{\cos\frac{pr}{c}\sin pt - \sin\frac{pr}{c}\cos pt\right\}$$

$$= \frac{Idz}{cr}\left(1-\frac{p^2r^2}{2c^2}+...\right)\sin pt-\frac{Idz\,p}{c^2}\left(1-\frac{p^2r^2}{6c^2}+...\right)\cos pt.$$

When p is vanishingly small this reduces to

$$dA_3 = \frac{Idz}{cr}\sin pt,$$

which is in agreement with equation (27) of Chapter IV:

$$-\frac{1}{c}\frac{\partial}{\partial t}(dA_3) \doteq \frac{pIdz}{c^2r}\cos pt + \frac{p^2Idz}{c^3}\sin pt$$

$$= \frac{p^2Qdz}{c^2r}\cos pt + \frac{p^3Qdz}{c^3}\sin pt.$$

Now

$$E_3 = -\frac{1}{c}\frac{\partial A_1}{\partial t} - \frac{\partial V}{\partial z}$$

$$= \left\{\frac{p^2Q}{c^2}\int\frac{dz}{r} - \frac{Qp^2}{c^2}\right\}\cos pt - \left\{\frac{Q}{r^2}\right\}\cos pt + \left(-\frac{Qp^3l}{3c^3} + \frac{p^3Q}{c^3}\int dz\right)\sin pt$$

$$= \frac{p^2Q}{c^2}\left(\int\frac{dz}{r} - 1\right)\cos pt - \left\{\frac{Q}{r^2}\right\}\cos pt + \frac{2}{3}\frac{Qp^3l}{c^3}\sin pt$$

$$= -\frac{Ip}{c^2}\left(\int\frac{dz}{r} - 1\right)\cos pt - \left\{\frac{Q}{r^2}\right\}\cos pt - \frac{2}{3}\frac{Ip^2l}{c^3}\sin pt.$$

Consider each of these terms separately. The first term arises from what is usually called the inductance of the system, and, since $L = \int A_3\,dz$, we have

$$L = \iint\frac{dz\,dz'}{r} - l.$$

So the inductance is to be calculated by performing the Neumann integration (see Chap. IV, p. 222) along the length of the wire and then subtracting from this the length l.

The second term arises from what is usually called the capacity of the system and needs no comment.

The third term is in phase with the current and so represents an expenditure of work and means that the system is *radiating energy*. There is a component of electric force which is in phase with the current and is constant in value at all points along the length l. So the rate of working is

$$lE_3 i = \frac{2}{3}\frac{p^2l^2I^2\sin^2 pt}{c^3}.$$

The mean rate of working is

$$W = \frac{p^2l^2I^2}{3c^3} = \frac{4\pi^2I^2}{3c}\frac{l^2}{\lambda^2}.$$

Since the system is continuously expending work it appears to have a resistance, called the radiation resistance, and is such that

$$R_r = \frac{2}{3}\frac{p^2}{c^3}l^2$$

$$= \frac{8}{3}\frac{\pi^2}{c}\frac{l^2}{\lambda^2}.$$

Now consider a current $i = I\sin pt$ flowing round a circle of radius R. We postulate that the current has the same magnitude and phase

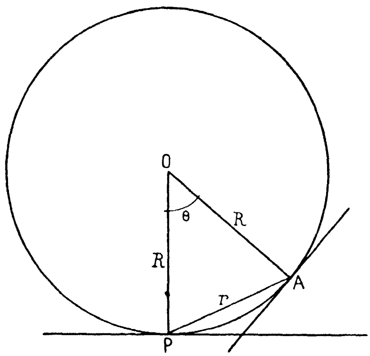


FIG. 152

at every point of the circumference, and so there is no surface charge at any point of the wire. Consequently the electric force arises only from the rate of change of vector potential, so

$$E = -\frac{\partial A}{\partial t}.$$

$$\therefore\quad dE = -\frac{pIds}{c^2r}\left(\cos\frac{pr}{c}\cos pt + \sin\frac{pr}{c}\sin pt\right).$$

Again, there is a component of electric force in phase quadrature with the current, and this represents the self-inductance effect: there is also a component in phase with the current, and this represents work done or radiation resistance. Consider the inphase component first:

$$dE \doteqdot -\frac{p^2Ids}{c^3}\left(1-\frac{p^2r^2}{6c^2}\right)\sin pt.$$

The first term of this expression is independent of the distance from the element ds of the circuit. So at any point of space the resultant due to this term must be zero for a complete circuit because any element ds will have a corresponding element pointing in the opposite direction. To find the resultant tangential electric force at any point P (see Fig. 152) on the circumference of the circle consider an element ds situated at the point A: this element contributes a tangential electric force at P of value

$$dE = \frac{p^4Ir^2}{6c^5} R\,d\theta \cos\theta \sin pt$$

$$= \frac{4p^4IR^3}{6c^5} \sin^2\frac{\theta}{2} \cos\theta\,d\theta \sin pt.$$

$$\therefore \quad E = \frac{2p^4IR^3}{3c^5} \int_0^{2\pi} \sin^2\frac{\theta}{2} \cos\theta\,d\theta \sin pt$$

$$= \frac{p^4IR^3}{3c^5} \int_0^{2\pi} (\cos\theta - \cos^2\theta)\,d\theta \sin pt$$

$$= \frac{\pi R^3p^4I}{3c^5} \sin pt.$$

So the e.m.f. round the circle

$$= \frac{2\pi^2R^4p^4I}{3c^5} \sin pt,$$

and the radiation resistance is

$$R_r = \frac{\pi^2R^4p^4}{3c^5}.$$

Now consider the inphase component of A or the quadrature component of E:

$$\delta A \doteqdot \frac{Ids}{r}\left(1 - \frac{p^2r^2}{2c^2}\right)\sin pt$$

$$= Ids\left(\frac{1}{r} - \frac{p^2r}{2c^2}\right)\sin pt.$$

The first term of this agrees with equation (28) of Chapter IV, but the second term is a function of frequency: the line integral of the first term gives the inductance for steady currents, and the line integral for the whole expression gives the inductance for currents of frequency $n = p/2\pi$. Consider the second term only:

$$A = -\frac{p^2I}{2c^2} \int_0^{2\pi} 2R^2 \sin\frac{\theta}{2} \cos\theta\,d\theta \sin pt$$

$$= \frac{4}{3}\frac{p^2R^2}{c^2} I \sin pt.$$

$$\therefore \quad \int A\,ds = \frac{8}{3}\frac{\pi p^2 R^3}{c^2} I \sin pt$$

$$= \frac{32}{3}\frac{\pi^3 R^3}{\lambda^2} I \sin pt.$$

Hence*

$$L = 4\pi R\left(\log_e \frac{R}{r} + 0{\cdot}56 + \frac{8}{3}\frac{\pi^2 R^2}{\lambda^2}\right).$$

We have now taken two simple examples to illustrate how the Maxwell hypothesis modifies the methods of calculation developed in Chapters I and II. For steady currents there is no modification, and for alternating currents the numerical value of the additional term is very small so long as the wave-length is very great compared with the dimensions of the system. However, it is on this small numerical difference that radio communication depends.

Having examined the field very close up to a current we will now examine it at a very great distance and discover the field which provides radio communication. Thus consider Fig. 151 and let the point P be now so far distant, that r_1 and r_2 are very nearly equal to one another and differ by an amount dr such that $dr = l\sin\theta$, where θ is the angle to the horizontal of the radius vector from the mid-point of l to the point P.

Then

$$V = \frac{Q}{r_1}\cos p\left(t-\frac{r_1}{c}\right) - \frac{Q}{r_1+dr}\cos p\left(t-\frac{r_1}{c}-\frac{dr}{c}\right)$$

$$= Q\cos p\left(t-\frac{r_1}{c}\right)\left\{\frac{1}{r_1}-\frac{1}{r_1+dr}\cos p\frac{dr}{c}\right\} - \frac{Q}{r_1+dr}\sin\frac{pdr}{c}\sin p\left(t-\frac{r_1}{c}\right)$$

$$\doteqdot \frac{Qdr}{r^2}\cos p\left(t-\frac{r}{c}\right) - \frac{Qpdr}{cr}\sin p\left(t-\frac{r}{c}\right)$$

$$= Ql\sin\theta\left\{\frac{1}{r^2}\cos p\left(t-\frac{r}{c}\right) - \frac{p}{cr}\sin p\left(t-\frac{r}{c}\right)\right\}.$$

$$\therefore \quad \frac{1}{r}\frac{\partial V}{\partial\theta} = Ql\cos\theta\left\{\frac{1}{r^3}\cos p\left(t-\frac{r}{c}\right) - \frac{p}{cr^2}\sin p\left(t-\frac{r}{c}\right)\right\},$$

and

$$\frac{\partial V}{\partial r} = -Ql\sin\theta\left\{\frac{2}{r^3}\cos p\left(t-\frac{r}{c}\right) - \frac{2p}{cr^2}\sin p\left(t-\frac{r}{c}\right) - \frac{p^2}{c^2 r}\cos p\left(t-\frac{r}{c}\right)\right\}.$$

* See equation (29) p. 55.

$$A_3 = \frac{1}{c}\int \frac{[idl]}{r}$$

$$= \frac{l}{cr}[i] = -\frac{plQ}{cr}\sin p\left(t-\frac{r}{c}\right).$$

$$\therefore \quad -\frac{1}{c}\frac{\partial A_3}{\partial t} = \frac{p^2 lQ}{c^2 r}\cos p\left(t-\frac{r}{c}\right).$$

$$E_r = -\frac{1}{c}\dot{A}_3\sin\theta - \frac{\partial V}{\partial r}$$

$$= 2lQ\sin\theta\left\{\frac{1}{r^3}\cos p\left(t-\frac{r}{c}\right)-\frac{p}{cr^2}\sin p\left(t-\frac{r}{c}\right)\right\}.$$

$$E_t = -\frac{1}{c}\dot{A}_3\cos\theta - \frac{1}{r}\frac{\partial V}{\partial \theta}$$

$$= -Ql\cos\theta\left\{\frac{1}{r^3}\cos p\left(t-\frac{r}{c}\right)-\frac{p}{cr^2}\sin p\left(t-\frac{r}{c}\right)-\frac{p^2}{c^2 r}\cos p\left(t-\frac{r}{c}\right)\right\}.$$

Hence we find that both the radial and tangential field contain terms whose magnitude is calculated correctly from electrostatics (compare equations (4) and (5) of p. 5) but whose phase depends on r. The tangential component contains a term

$$E_t = \frac{Ql\cos\theta}{r}\frac{p^2}{c^2}\cos p\left(t-\frac{r}{c}\right).$$

Since this varies inversely as r it is appreciable long after the terms in r^2 and r^3 are negligible: the phase of this term also depends on r.

We can now calculate the magnetic field, which from symmetry must consist of circular lines of force centred on the wire.

To do this, we use the relation

$$H_3 = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}$$

$$= -\frac{Ql}{c}\frac{\partial}{\partial x}\left\{-\frac{p}{r}\sin p\left(t-\frac{r}{c}\right)\right\}$$

$$= -\frac{Ql}{c}\left\{\frac{p}{r^2}\sin p\left(t-\frac{r}{c}\right)+\frac{p^2}{cr}\cos p\left(t-\frac{r}{c}\right)\right\}\frac{\partial r}{\partial x}$$

$$= -Ql\cos\theta\left\{\frac{p}{cr^2}\sin p\left(t-\frac{r}{c}\right)+\frac{p^2}{c^2 r}\cos p\left(t-\frac{r}{c}\right)\right\}.$$

This equation shows there is a component of H which varies inversely as r, and that this term is numerically equal in magnitude and phase to the corresponding term of E_t.

Since, at a given instant of time, both these fields will have the same phase at any two points on the same radius vector, separated by a distance $\lambda = \frac{2\pi c}{p}$, it is appropriate to describe the field as a wave propagated with a uniform velocity c.

In these expressions E is the force in dynes per electrostatic absolute unit charge, and H is the force in dynes per electromagnetic absolute unit pole. If we express the electric field in electromagnetic units, then

$$\frac{E}{c} = H$$

or

$$E = cH.$$

This is reminiscent of the cutting rule of e.m.f. since we have a field H moving with speed c.

APPENDIX

NOTE ON DIMENSIONS

SOMETIMES it is convenient to try to express magnetic and electric quantities in terms of mass, length, and time. Poles are defined to be of unit strength if there is unit force at unit distance. This leads to the equation

$$F = \frac{\alpha m m'}{r^2},$$

where α is a constant which the definition has made of unit value: but α may possibly have physical dimensions of which we know nothing. According to established convention, force has the dimensions $F = \frac{ML}{T^2}$. Accordingly, on comparing the two expressions for F and ignoring α, we find that

$$m = \frac{M^{\frac{1}{2}} L^{\frac{3}{2}}}{T}.$$

But in stating that pole strength has these dimensions we should never forget that the unknown dimensions of α have been ignored. If the law of force is stated as

$$F = \frac{m m'}{\mu r^2},$$

then we find that

$$m = \frac{M^{\frac{1}{2}} L^{\frac{3}{2}} \mu^{\frac{1}{2}}}{T}.$$

This expression appears to admit dimensions for μ. But according to the discussion on p. 157 and elsewhere, μ is a mere number, and there seems no very clear reason why we should identify $1/\alpha$ with a dimensional μ_0, the permeability of free space.*

The dimensions of H are derived from the equation

$$F = mH,$$

and hence

$$H = \frac{M^{\frac{1}{2}}}{L^{\frac{1}{2}} T}.$$

According to the argument of this book, B and H are essentially quantities of the same character, and therefore

$$B = \frac{M^{\frac{1}{2}}}{L^{\frac{1}{2}} T}.$$

* For a contrary view see Heaviside, *Electromagnetic Theory*, vol. i, p. 349, § 192.

If μ is supposed to have dimensions, then H contains $\mu^{-\frac{1}{2}}$ and B contains $\mu^{\frac{1}{2}}$.

Since current multiplied by area is equivalent to magnetic moment,

$$IL^2 = \frac{M^{\frac{1}{2}}L^{\frac{3}{2}}L}{T}.$$

$$\therefore \quad I = \frac{M^{\frac{1}{2}}L^{\frac{1}{2}}}{T}$$

and

$$Q = IT = M^{\frac{1}{2}}L^{\frac{1}{2}}.$$

The dimensions of electromotive force are obtained from equating EI to the dimensions of power, and so

$$E\frac{M^{\frac{1}{2}}L^{\frac{1}{2}}}{T} = \frac{ML^2}{T^3}.$$

$$\therefore \quad E = \frac{M^{\frac{1}{2}}L^{\frac{3}{2}}}{T^2}.$$

The dimensions of resistance are obtained from

$$R = \frac{E}{I} = \frac{L}{T}.$$

Hence, in a system where μ is not supposed to have dimensions, resistance has the dimensions of a velocity. It is interesting to notice that the dimensions of equation (18) on p. 78, which is the expression for R by the Lorentz determination, are those of a velocity.

The dimensions of inductance are obtained from the equation

$$LI = \phi.$$

$$\therefore \quad L = \frac{M^{\frac{1}{2}}L^2}{L^{\frac{1}{2}}T} \times \frac{T}{M^{\frac{1}{2}}L^{\frac{1}{2}}}$$

$$= L,$$

and thus the dimensions of inductance are found to be a length.

The dimensions of capacity are to be found from

$$C = \frac{Q}{E}$$

$$= M^{\frac{1}{2}}L^{\frac{1}{2}} \times \frac{T^2}{M^{\frac{1}{2}}L^{\frac{3}{2}}}$$

$$= \frac{T^2}{L}.$$

The expressions of the foregoing quantities in M, L, and T are perhaps not very important or significant: they are sometimes useful for checking the correctness of cumbersome algebraic expressions.

An entirely different system is obtained if we start from the equation

$$F = \frac{\beta QQ'}{r^2}$$

and ignore the dimensions of the quantity β, which the definition of charge makes numerically equal to unity. This system is called the electrostatic system, whereas the system which begins from the equation

$$F = \frac{mm'}{r^2}$$

is called the electromagnetic system.

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