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SEMINAR ON TRANSFORMATION GROUPS

BY

Armand Borel

WITH CONTRIBUTIONS BY

G. Bredon

E. E. Floyd

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TABLE OF CONTENTS

	Page
INTRODUCTION	1
CHAPTER I: COHOMOLOGY MANIFOLDS, by A. Borel	5
§1. Preliminaries	5
§2. Local Betti Numbers Around a Point	7
§3. The Notion of Cohomology Manifold	9
§4. Some Properties of Cohomology Manifolds	10
§5. Appendix on Cohomological Dimension	18
CHAPTER II: HOMOLOGY AND DUALITY IN GENERALIZED MANIFOLDS by A. Borel	23
§1. Homology Groups for Locally Compact Spaces	23
§2. Duality in Generalized Manifolds	25
§3. Existence of Fundamental Sheaves for Homology	27
§4. Closed Subsets of Euclidean Spaces	30
§5. Complements	31
CHAPTER III: PERIODIC MAPS VIA SMITH THEORY, by E. E. Floyd	35
§1. The Leray Spectral Sequence	35
§2. Transfer Homomorphism	37
§3. An Euler Characteristic Formula	39
§4. The Smith Sequences	40
§5. Orbit Spaces of Finite Groups	43
CHAPTER IV: THE ACTION OF Z_p OR T^1 : GLOBAL THEOREMS, by A. Borel	49
§1. Transformation Groups	49
§2. Some Remarks on the Cohomology of B_G	50
§3. The Space X_G	52
§4. The Fixed Point Set of a Prime Period Map in a Cohomology Sphere	55
§5. The Action of the Circle Group	60
§6. The Quotient of a Cohomology Sphere by Z_p or T^1	63
CHAPTER V: THE ACTION OF Z_p OR T^1 : LOCAL THEOREMS, by A. Borel	67
§1. Conservation of Cohomological Local Connectedness	67
§2. The Fixed Point Set of a Prime Period Map in a Cohomology Manifold	74
§3. The Fixed Point Set of a Toral Group in a Cohomology Manifold	80
§4. Remarks on Local Groups of the Quotient Space	83

CONTENTS

	Page
CHAPTER VI: ISOTROPY SUBGROUPS OF TORAL GROUPS, by E. E. Floyd	85
§1. Introduction	85
§2. A Regular Convergence Theorem	86
§3. Two Lemmas	89
§4. Proof of Theorem 1.2	90
CHAPTER VII: FINITENESS OF NUMBER OF ORBIT TYPES, by G. E. Bredon	93
§1. Preliminary Remarks	93
§2. Statements of the Main Results	93
§3. Proof of the Main Theorem	95
CHAPTER VIII: SLICES AND EQUIVARIANT IMBEDDINGS, by R. S. Palais	101
§1. Notation and Preliminaries	101
§2. Orbit Types	104
§3. Slices	105
§4. Equivariant Imbeddings in Euclidean Space	112
CHAPTER IX: ORBITS OF HIGHEST DIMENSION, by D. Montgomery	117
§1. Introduction	117
§2. The Set B	117
§3. The Set D	121
§4. The Set $B \cap D$	123
§5. Conditions Under Which $\dim_{\mathbb{Z}} D \leq n - 2$	125
§6. Remarks on the Differentiable Case	128
CHAPTER X: THE SPECTRAL SEQUENCE OF A BIFILTERED MODULE, by A. Borel	133
§1. Spectral Sequences	133
§2. The Notion of Bifiltration	134
§3. The Terms E_0 and E_1	136
§4. Two Further Assumptions	139
§5. The Term E_2	141
§6. Homomorphisms	142
§7. Remarks	144
CHAPTER XI: THE SPECTRAL SEQUENCE OF FARY, by A. Borel	145
§1. Sheaves	145
§2. Continuous Maps	146
§3. The Spectral Sequence of Fary	149
§4. Locally Compact Spaces	152

CONTENTS

	Page
CHAPTER XII: FIXED POINT THEOREMS FOR ELEMENTARY COMMUTATIVE GROUPS I, by A. Borel	157
§1. Some Fiberings	157
§2. The Topology of the Leray Sheaf	160
§3. Fixed Point Theorems	162
§4. Applications to Projective Spaces	167
§5. Applications to Compact Lie Groups	169
§6. Applications to Homologically Kählerian Manifolds	170
CHAPTER XIII: FIXED POINT THEOREMS FOR ELEMENTARY COMMUTATIVE GROUPS II, by A. Borel	173
§1. Notation	173
§2. Cohomology Spheres	175
§3. Some Local Concepts	180
§4. Cohomology Manifolds	181
CHAPTER XIV: ONE OR TWO CLASSES OF ORBITS, by A. Borel	185
§1. One Class of Orbits on Spheres	185
§2. A Sufficient Condition for the Existence of Fixed Points on $\mathbb{R}^n$	188
§3. Two Classes of Orbits on Euclidean Space	190
CHAPTER XV: FIXED POINT SETS AND ORBITS OF COMPLEMENTARY DIMENSION, by G. E. Bredon	195
§1. Introduction	195
§2. Preliminary Results	198
§3. A Theorem on Cross-Sections	207
§4. Case I. $\text{rank } (G) = \text{rank } (H)$	211
§5. Case II. $\text{rank } (H) < \text{rank } (G)$	214
§6. Rank One Case	215
§7. A Local Cross-Section for the Orbits of G Near F	223
§8. Conclusion of the Proof of Theorem 1.4	230
CHAPTER XVI: REMARKS ON THE SPECTRAL SEQUENCE OF A MAP, by A. Borel	233
§1. Cohomology with Compact Supports	233
§2. The General Case	235
§3. Inductive Limits	239
§4. On the Leray Sheaf	241
§5. A Case of Validity for the Künneth Rule	243

SEMINAR ON TRANSFORMATION GROUPS

INTRODUCTION

In this book, a transformation group is a compact Lie group G acting on a topological, usually locally compact, space X . Transformation groups are discussed mainly from the point of view of algebraic topology, and the problems most often center around relations between topological properties of G , X , the fixed point set, the orbits, and the space of orbits X/G . A typical topic along these lines, and chronologically one of the first ones, is P. A. Smith's theory of prime period maps of homology spheres.

Apart from some modifications and additions, this book consists of the Notes of a Seminar held at the Institute for Advanced Study in 1958-59. Familiarity with the basic notions in the theory of transformation groups and with algebraic topology is assumed. As to the latter, the results concerning sheaves, spectral sequences and fibre bundles, for which references to the expositions of Cartan, Godement, Steenrod or the undersigned can be made, are usually taken for granted. However, certain topics for which this is not possible are given an independent discussion. As a result, five chapters (I, II, X, XI, XVI) are concerned with questions of algebraic topology and make no mention of transformation groups. We first briefly summarize those.

Chapter I is devoted to cohomology manifolds. These are spaces which have the local cohomological properties of manifolds. The definition adopted here is equivalent to that of a locally orientable generalized manifold in the sense of Wilder. Chapter II introduces a homology theory for locally compact spaces and uses it in order to derive a Poincaré duality theorem for cohomology manifolds. Chapter X is purely algebraic, and is concerned with the spectral sequences attached to a differential module endowed with two filtrations. The discussion is carried only as far as is needed to supply a convenient algebraic framework for the discussion of Fary's modification of the spectral sequence of a map (XI). It is obtained by combining Leray's filtration with a filtration stemming from a decreasing sequence of closed subspaces. To Chapter XVI we have relegated some remarks about the spectral sequence of a map, including a

brief summary of its definition, a procedure for putting on it a module structure over the cohomology of the base, a discussion of families of supports, a sufficient condition under which the stalk of the Leray sheaf in a fibre bundle is the cohomology of the fibre, and an application to the cohomology of certain products.

As to Chapters I and II, the undersigned would like to point out that cohomology manifolds are emphasized for technical convenience rather than for the sake of generality. When it comes to manifolds, genuine ones are of the course the most important ones, and differentiable transformation groups already form a very interesting special case. However, if one does not wish to assume that all group actions (including those on the one point compactification of a euclidean space) are differentiable, then assumptions of a cohomological character are much more tractable with the methods used in this seminar and have a more "hereditary" character (e.g., Theorem 4.10 (b) of I, or 3.2 of V are false for manifolds).

The discussion of transformation groups starts with Chapter III, where Floyd gives a sheaf-theoretic interpretation of Smith's theory of prime period maps and applies the transfer homomorphism to the discussion of the orbit space of a finite group.

Prime period maps are also studied in Chapters IV and V, together with the action of the circle group T^1 , from a different point of view. It consists of a systematic use of the twisted product $X_G = X \times_{G, E_G} G$ of X with a universal bundle for G . This space on one hand is a fibre bundle over the classifying space B_G of G , with fibre X , and on the other hand it has a projection π_1 on X/G , such that the inverse image of a point Y of X/G can be identified with the classifying space of the stability group of any point in this orbit. It allows us to tie together the cohomology groups of X , X/G , and the fixed point set F , with those of the classifying spaces of the stability groups and of G . After a few general remarks about X_G , Chapter IV proves various, mostly known, results about prime period maps on homology spheres and their analogues for the circle group. Chapter V is devoted to the local theorems for these two cases, stating mainly that F and X/G are cohomologically locally connected if X is, and that F is a cohomology manifold if X is.

The discussion of elementary abelian p -groups (by which we mean direct products of cyclic groups Z_p of order p if p is prime, of circle groups if $p = 0$) is resumed in Chapters XII, XIII, with the help of Fary's spectral sequence. In Chapter XII, it is shown that if X is totally non-homologous to zero in X_G , then G has fixed points. In

Chapter XIII a relation is obtained between the dimensions of X , F , and the fixed point sets of the subgroups of index p if p is prime, or of the closed connected subgroups of codimension 1 if $p = 0$, when X is a cohomology sphere or a cohomology manifold. In contrast with the results of Chapters IV, V, those of Chapters XII, XIII do not seem to follow by induction from the case $G = \mathbb{Z}_p, T^1$.

Chapters V to IX are devoted to some known basic theorems: the local finiteness of the number of orbit types in a cohomology manifold, first when G is a torus (VI), and then in the general case (VII); the existence of equivariant embeddings in euclidean spaces and of a slice (VIII); various results concerning the orbits of the highest dimension in a cohomology manifold (IX).

In Chapter XIV it is shown that if a compact Lie group G acts effectively on a sphere with one class of orbits and is not transitive, then $G = T^1, SU(2)$ and G acts freely, and that if G acts on euclidean space with two classes of orbits, then it has a fixed point.

Chapter XV presents new results of Bredon pertaining to the action of a compact Lie group acting on a cohomology n -manifold when the fixed point set F has the greatest possible dimension allowed by the results of Chapter IX, namely $n - k - 1$ where k is the highest dimension of the orbits. The highest dimensional orbits are then integral cohomology spheres, the orbit space around a fixed point x is a cohomology manifold with boundary F and there is a local cross-section around x for the orbits.

In short, these Notes give an exposition of some basic theorems and of results obtained by cohomological methods in the theory of compact Lie groups of transformations. They make no claim at completeness, although it has been tried to give a fairly comprehensive discussion of the topics chosen, and do not cover all aspects of the theory of transformation groups. It should also be pointed out that purely cohomological methods, while forming a major part of the subject at present, have their limitations, as is shown by well known counter examples; and that in view of this, it would certainly be very desirable to make more effective use of differentiability assumptions than has been possible so far.

As to references, each chapter carries its own bibliography. The numbering of lemmas, theorems, remarks, etc. in each chapter is cumulative. For example 2.1 means Section 2.1 in the same chapter; (IV, 2.1) means Section 2.1 in Chapter IV; 3.2(1) refers to formula (1) or assertion (1) in 3.2, as the case may be.

Finally, I would like to thank G. Bredon for his help in checking the final version of these Notes.

I: COHOMOLOGY MANIFOLDS

A. Borel

§1. Preliminaries

1.1. L will always denote a principal ideal domain, Z the ring of integers, K_p a field of characteristic p (p prime or zero), Z_p the prime field of characteristic p . All topological spaces will be Hausdorff and, unless otherwise stated, locally compact. $\bar{A}$ denotes the closure of a subset A of a topological space.

$H_C^i(X; L)$, (respectively $H^i(X; L)$) is the i -th cohomology group of X , in the sense of [4], with coefficients in L , and compact (respectively closed) supports. $H_C^*(X; L)$ (respectively $H^*(X; L)$) is the direct sum of the groups $H_C^i(X; L)$ (respectively $H^i(X; L)$). Thus $H_C^*(X; L)$ (respectively $H^*(X; L)$) is the Alexander-Spanier cohomology ring of X , coefficients in L , with compact (respectively closed) supports (respectively when X is paracompact).

$X \sim_L Y$ will mean $H_C^*(X; L) = H_C^*(Y; L)$. If $L = K_p$, we also write $X \sim_p Y$.

Given an open subset U of X the standard homomorphism $H_C^*(U; L) \longrightarrow H_C^*(X; L)$ will be denoted by j_{XU}^* and its restriction to $H_C^1(U; L)$ by j_{XU}^1 . If V is open in U , then $j_{XV}^* = j_{XU}^* j_{UV}^*$. We have trivially

$$(1) \quad H_C^1(X; L) = \text{ind} \lim (H_C^1(U; L), j_{UV}^1) \quad (U, V \text{ open relatively compact}).$$

If A is a closed subspace of X , the natural homomorphism $H^*(X; L) \longrightarrow H^*(A; L)$ (respectively $H_C^*(X; L) \longrightarrow H_C^*(A; L)$) is denoted by r_{XA}^* (respectively r_{cXA}^*) and its restriction to elements of degree i by r_{XA}^i (respectively r_{cXA}^i). We recall the exact cohomology sequence

$$(2) \quad \longrightarrow H_C^1(U; L) \xrightarrow{j_{XU}^1} H_C^1(X; L) \xrightarrow{r_{cXA}^1} H_C^1(A; L) \longrightarrow H_C^{i+1}(U; L) \longrightarrow$$

for U open in X and $A = X - U$.

Finally the homomorphism $H^*(Y; L) \longrightarrow H^*(X; L)$ (respectively $H_C^*(Y; L) \longrightarrow H_C^*(X; L)$) induced by a continuous (respectively and proper) map $f: X \longrightarrow Y$ is denoted by f^* (respectively f_C^*).

1.2. Dimension. By $\dim_L(X)$ is meant the value on X of the dimension function over L introduced by H. Cohen [3], namely

$$(3) \quad \dim_L X \leq n \iff H_C^{n+1}(U; L) = 0 \text{ for all open } U \subset X.$$

If $L = K_p$, we write $\dim_p X$ for $\dim_L X$. The condition on the right hand side is equivalent to: r_{cXA}^n is surjective for all closed subsets A of X (cf. [3], it is proved there for X compact, but the proof extends easily to the locally compact case). It also implies that $H_C^j(U; L) = H_C^j(X - U; L) = 0$ for all $j \geq n + 1$ and U open in X . Also $\dim_L X$ is $\geq$ the dimension over L of any locally compact subspace, and is the upper limit of the dimension over L of the compact subspaces of X .

H. Cartan [2] has introduced the following notion of dimension in Φ -cohomology where Φ is a "paracompactifying family" of closed subsets (see §5):

$$(4) \quad \dim_{\Phi, L} X \leq n \iff H^{n+1}(U; \underline{A}) = 0 \text{ (} U \text{ open in } X, \underline{A} \text{ sheaf of } L\text{-modules)}.$$

This implies $H^j(U; \underline{A}) = 0$ for $j \geq n + 1$. By results of [4] and of Floyd (see §5) these two dimension functions are equal.

1.3. Cohomological local connectedness. X is k -clc over L or k - clc_L at x if given a neighborhood U of x there exists a neighborhood V of x inside U such that $\text{Im } r_{UV}^k = 0$ (if $k = 0$, one has to take reduced cohomology groups). The space is clc_L^n (at x) if it is k - clc_L for all $k \leq n$, clc_L^∞ if it is clc_L^n for all n , and clc_L if it is clc_L^∞ and if given U there exists V such that $\text{Im } r_{UV}^* = 0$. The space X is k - clc_L , clc_L^n , clc_L^∞ or clc_L if it is so at every point. The property clc_L^0 is equivalent to local connectedness. If $L = K_p$, we replace the subscript L by p .

These definitions are valid for an arbitrary space. If X is locally paracompact at x then, as is well known [4, p. 193], given $c \in \text{Ker } r_{Ux}^*$, there exists a neighborhood V of x in U such that $c \in \text{Ker } r_{UV}^*$. Therefore, in this case, X is k -clc at x if and only given a neighborhood U of x there exists a neighborhood V of x in U such that $\text{Im } r_{UV}^k$ is a finitely generated L -module. It is clc if there exists V such that $\text{Im } r_{UV}^*$ is finitely generated.

We recall that if X is compact and clc over L , then

$H^1(X; L)$ is a finitely generated L -module for each integer i . This is proved in Wilder [6]. For more recent proofs, see Floyd, Trans. A. M. S. 84(1957), 319-337, E. Dyer, Annals of Math., 67(1958), 119-149.

Let Y be another space. The projection of $X \times Y$ onto one of the factors induces an injective map in cohomology with closed supports. This and the Künneth rule applied to compact subsets of X and Y show at once that $X \times Y$ has any one of the above properties at (x, y) if and only if X and Y have this property at x and y respectively.

In the sequel we deal almost exclusively with spaces of finite cohomological dimension. In that case, the properties clc and clc^∞ are of course equivalent.

§2. Local Betti Numbers Around a Point

2.1. DEFINITION. The space X has the i -th local Betti number k around x (over L), in writing $p^i(x; L) = k$, if given an open neighborhood U of x , there exist open neighborhoods $W \subset V \subset U$ of x with $\bar{W} \subset V$, $\bar{V} \subset U$, such that for any open neighborhood $W' \subset W$ of x , $\text{Im } j_{VW'}^1 = \text{Im } j_{VW}^1$ and is a free L -module of rank k . We write $p^1(x; L) \leq \omega(p^1(x; L))$ is at most increasingly infinite) if given U there exists $V \subset U$ such that $\text{Im } j_{UV}^1$ is finitely generated.

The local Betti numbers around a point (a notion introduced by Alexandroff (Ann. of Math. 36(1935), 1-35)) are traditionally defined with respect to fields of coefficients, and in this case our definition is equivalent to the usual one. When L is not a field, 2.1 will cover our needs, but will probably not be the final definition. It is likely that one will also have to introduce local torsion coefficients, which in our case would be zero. The condition $\text{Im } j_{VW'}^1 = \text{Im } j_{VW}^1$ is not necessary to state a meaningful definition, but seems to be needed wherever one wants to put this notion to use. It would also be desirable to introduce local groups at a point. This is easily done in homology, where these groups form in fact a sheaf in a natural fashion (see [1] and Chapter II). One may of course consider the projective limit of the groups $H_C^1(U; L)$ with respect to the maps j_{VU}^1 . If L is a field, and if $p^1(x; L) \leq \omega$, the dimension of this limit is in fact equal to $p^1(x; L)$. When L is not a field, these groups have been introduced and studied by Conner and Floyd (Michigan M. J. 6(1959), 33-43) under the assumption $\text{Im } j_{VW'}^1 = \text{Im } j_{VW}^1$ made in 2.1 and one further axiom expressing an analogous stability of kernels.

2.2. THEOREM. Let X be finite dimensional over L . Then the three following conditions are equivalent.

(1) X is clc_L ; (ii) $p^1(x; L) \leq \omega$ for all $x \in X$; (iii) For any pair $P \subset Q$ of open subspaces of X with $\bar{P}$ compact, contained in Q , $\text{Im } j_{PQ}^*$ is finitely generated.

This theorem is due to Wilder [6] when L is a field. It follows from 6.2, 6.4 in [1], whose proofs are essentially a translation in another framework of those of Wilder. (iii) is Wilder's property (P, Q) .

2.3. THEOREM. Let X be finite dimensional over L . Then $\dim_L X \leq n$ if and only if $p^1(x; L) = 0$ for all $x \in X$ and $i \geq n + 1$.

The only if part follows from the definitions. To prove the converse, let $A_{k,L}$ be the property:

$(A_{k,L})$: for each open pair V, U with $\bar{V} \subset U$, $j_{UV}^k(H_C^k(V; L)) = 0$.

By 1(1), if $A_{k,L}$ is true, then $H_C^k(U; L) = 0$ for each open U . Since $\dim_L X < \infty$, $A_{k,L}$ is true for k big enough, hence the theorem will follow by descending induction from the

2.4. PROPOSITION. If $A_{k+1,L}$ is true on X and if $p^k(x, L) = 0$ for all $x \in X$, then $A_{k,L}$ is true on X .

Let $V \subset U$ be open in X , with $\bar{V} \subset U$. We have to show that $\text{Im } j_{UV}^k = 0$ and, in view of 1(1), it is enough to consider the case in which $\bar{V}$ is compact. By the assumption on p^k we can therefore find a finite number of open sets U_i ($1 \leq i \leq m$) with $\bar{U}_i \subset U$, $\text{Im } j_{U,U_i}^k = 0$, and whose union contains $\bar{V}$. We can then construct open sets U_{ij} , such that $U_{ij} = U_j$, $\bar{U}_{ij} \subset U_{i-1,j}$ and $U_{i1} \cup \dots \cup U_{im} \supset \bar{V}$ ($1 \leq i, j \leq m$); it is then clearly enough to show that $\text{Im } j_{U,U_{i1} \cup \dots \cup U_{im}}^k = 0$ for $i = 1, 2, \dots, m$. By induction, this is reduced to the following lemma:

2.5. LEMMA. Assume that $A_{k+1,L}$ is true on X . Let $V_i \subset U_i$ ($i = 1, 2$) be open subspaces with $\bar{V}_1 \subset U_1$ and let $V = V_1 \cup V_2$. If $\text{Im } j_{X,U_1}^k = 0$ ($i = 1, 2$), then $\text{Im } j_{XV}^k = 0$.

PROOF. We consider the following commutative diagram, where the horizontal homomorphisms are the natural maps j induced by inclusion and where the vertical maps are parts of exact sequences of the Mayer-Vietoris type (see [1], §8 for more details) (and the coefficients L are omitted)

$$\begin{array}{ccccc}
 H_C^k(V_1) + H_C^k(V_2) & \longrightarrow & H_C^k(U_1) + H_C^k(U_2) & \xrightarrow{g} & H_C^k(X) \\
 \downarrow & & \downarrow & \nearrow & \\
 H_C^k(V) & \longrightarrow & H_C^k(U_1 \cup U_2) & & \\
 \downarrow & & \downarrow & & \\
 H_C^{k+1}(V_1 \cap V_2) & \xrightarrow{f} & H_C^{k+1}(U_1 \cap U_2) & &
 \end{array}$$

The lemma follows then by simple diagram chasing from the nullity of $\text{Im } f$ and $\text{Im } g$.

REMARK. This proof is almost identical to that of Property 6.2 in [1]. The difference is that the condition of being zero imposed on images by $A_{k,L}$ and on $p^k(x; L)$ must be replaced by "being finitely generated". One would get similar results in the same way by replacing "zero" in those conditions by "congruent zero mod $\underline{C}$ " where $\underline{C}$ is a class of L -modules in the sense of Serre.

§3. The Notion of Cohomology Manifold

3.1. DEFINITION. Let n be a positive integer, and L a principal ideal domain. The space X is a Wilder n -manifold if $\dim_L X$ is finite and if $p^1(x; L) = 0$, $p^n(x; L) = 1$ ($1 \neq n$; $x \in X$).

Let X be a Wilder n -manifold. Then by 2.2 it is clc over L , in particular it is locally connected, and by 2.3 we have $\dim_L X = n$.

3.2. DEFINITION. A connected Wilder n -manifold is orientable if $H_C^n(X; L)$ contains a free L -module A of rank 1 such that $x \in X$ has a fundamental system of connected open neighborhoods U for which $A = \text{Im } j_{XU}^n$. A Wilder n -manifold is orientable if all of its connected components are orientable. It is locally orientable if each point has an open orientable neighborhood.

Thus, if X is orientable, and connected, the intersection of all the images $\text{Im } j_{XU}^n$, where U runs through the non-empty open subsets of X , is a free submodule of rank 1. It can be seen without difficulty and will in any case follow from 4.3 that in a locally orientable Wilder n -manifold, each point has a fundamental system of orientable open neighborhoods. It is not known whether a Wilder n -manifold is always locally orientable. In this seminar we shall always assume local orientability, and in order to abbreviate, we introduce the following:

3.3. DEFINITION. A cohomology n -manifold over L (an n - cm over L or an n - cm_L) is a locally orientable Wilder n -manifold over L .

REMARK. Wilder n -manifolds over fields are the main topic of [6], which sums up all results concerning them known at that time. Smith, in his work on periodic transformations, has introduced a notion of homology manifold over a prime field Z_p , which has turned out to be equivalent to that of an n -cm over Z_p (see Yang [7]). In [7], Yang introduces and studies homology manifolds over the reals mod 1, which correspond to our n -cm over Z . He proves there (in the language of homology) the results 4.2 to 4.5 below, as well as a lemma similar to 2.4, although stated in a more special case.

3.4. Remark on the orientability of Wilder Manifolds. Let X be a Wilder n -manifold over the principal ideal domain L , and let $x \in X$. The definition of local Betti numbers shows the existence of a subgroup $H(x) \subset H^n_c(X; L)$ and of a basic system $\underline{U}_x$ of open neighborhoods of x for which $\text{Im } j_{XU}^n = H(x)$, $\text{Im } j_{XU}^i = 0$ ($i \neq n$), for $U \in \underline{U}_x$. With this notation, we have the following assertion:

Let X be a Wilder n manifold over L . Then X is orientable if and only if the following is true:
 $H(x) \cong L$ for every $x \in X$ and if $H(y) \subset H(x)$,
 then $H(y) = H(x)$, ($y, x \in X$).

The necessity of the condition follows from the definition of orientability. Assume now this condition to be true. Let H be a subgroup of $H^n_c(X; L)$ isomorphic to L and $x \in X$ such that $H(x) = H$. If $U \in \underline{U}_x$ and $y \in U$, then $H(y) \subset H(x)$, hence $H(y) = H(x) = H$. Therefore $X(H) = \{x, H(x) = H\}$ is open in X . Orientability follows now since X is locally connected and $X(H) \cap X(H')$ is empty if $H \neq H'$.

If L is a field, then the condition may be stated slightly more simply in the following way:

Let X be a Wilder n -manifold over a field L . Then X is orientable if and only for every non-empty open subset U the homomorphism j_{XU}^n has a non-zero image.

We also remark that the condition " $\text{Im } j_{XU}^n \neq 0$ for all non-empty open U " is equivalent to " $\text{Im } j_{XU}^* \neq 0$ for all non-empty open U ", and that above, we may also define $H(x)$ as $\text{Im } j_{XU}^*(U \in \underline{U}_x)$.

§4. Some Properties of Cohomology Manifolds

4.1. We recall first the so called "minimality principle" for (Alexander-Spanier) cohomology with compact carriers: Given $c \in H_c^k(X; L)$,

$c \neq 0$, the set of closed subspaces for which $r_{cXF}^*(c) \neq 0$, ordered by inclusion, has a minimal element.

PROOF. Let $(F_i)_{i \in I}$ be a totally ordered set of closed subsets on which c induces a non-zero element, and let $F = \bigcap_{i \in I} F_i$. We want to show that $r_{XF}^*(c) \neq 0$. Assume it is not true, and let K be a fine cover (couverture fine), in the sense of Leray, for cohomology with compact carriers. Its section by a closed subset A , i.e., the quotient of K by elements with supports in $X - A$, endowed with the supports, $S(c) \cap A$, is then a fine cover for A . Therefore, if $r_{XF}^*(c) = 0$, there exists a representative cocycle c' of c in K and $k \in K$ such that $S(c' - dk)$ does not meet F . Since it is compact there exists an $i \in I$ for which $S(c' - dk) \cap F_i = \emptyset$, whence $r_{XF_i}^*(c) = 0$. The minimality principle follows from Zorn's lemma.

4.2. LEMMA. Let X be an n -cm and F be a closed subspace of X . Assume that for any relatively open non-empty subset C of F , $\text{Im } j_{FC}^n \neq 0$. Then F is open in X .

It is enough to show that $U \cap F$ is open in U when U belongs to some open covering of X ; since our assumption on F is a fortiori fulfilled for $U \cap F$, we may assume X to be orientable and connected. Let Y be an open subset of X meeting F such that $\text{Im } j_{XY}^n = \bigcap_V \text{Im } j_{XV}^n$ (V running through the non-empty open subsets of X). We want to prove that $Y \subset F$. We consider the following commutative diagram where the horizontal lines are parts of cohomology sequences 1(2), and the coefficients L are omitted.

$$\begin{array}{ccccccc}
 H_C^n(X - F) & \longrightarrow & H_C^n(X) & \longrightarrow & H_C^n(F) & \longrightarrow & 0 \\
 & & \uparrow j_{XY}^n & & \uparrow j_{F, Y \cap F}^n & & \\
 H_C^n(Y - Y \cap F) & \longrightarrow & H_C^n(Y) & \longrightarrow & H_C^n(Y \cap F) & \longrightarrow & 0
 \end{array}$$

By 2.4, $H_C^{n+1}(Y) = H_C^{n+1}(X) = 0$, hence the horizontal lines are exact. If $Y \cap F \neq Y$ then by our choice of Y , $\text{Im } j_{XY}^n = \text{Im } j_{X, Y - Y \cap F}^n$; the nullity of $\text{Im } j_{F, Y \cap F}^n$ follows then by diagram chasing, whence a contradiction.

4.3. THEOREM. Let X be a connected n -cm over L . Then

(1) For every non-empty open subset U , the

homomorphism j_{XU}^n is surjective, hence $H_G^n(A; L) = 0$ for every proper closed subset A of X .

(2) X is orientable if and only if $H_G^n(X; L) = L$.

If X is orientable and U is an open subset, then U is orientable and, if U is moreover connected, j_{XU}^n is an isomorphism. If X is non-orientable and L is a field, then $H_G^n(X; L) = 0$.

If j_{XU}^n is not surjective, there exists a $c \in H_G^n(X; L)$ and a proper closed subset A for which $r_{cXA}^*(c) \neq 0$. By 4.1 we can find a minimal proper closed subset F such that $r_{cXF}^*(c) \neq 0$. Consider the commutative diagram, where the two columns are exact by 2.3 and where L is omitted.

$$\begin{array}{ccccc}
 H_G^n(U) & \longrightarrow & H_G^n(X) & & \\
 \downarrow & & \downarrow & & \\
 H_G^n(U \cap F) & \longrightarrow & H_G^n(F) & \longrightarrow & H_G^n(F - F \cap U) \\
 \downarrow & & \downarrow & & \\
 0 & & 0 & &
 \end{array}$$

the maps being either restriction maps r_G^* or inclusion maps j^* . If $F \cap U$ is non-empty, then $r_{cX, F \cap U}^*(c) = 0$ by the minimality of F , hence $r_{cXF}^*(c)$ belongs to the image of $j_{F, F \cap U}^n$, and F is open by 4.2. Since it is closed, non-empty, proper and X is connected, this is a contradiction. The second part of (1) follows then from the first part by the cohomology sequence and 2.3, and (2) is an obvious consequence of (1) and the definitions.

4.4. DEFINITION. Let X be an n -cm $_L$, and U be an open connected orientable subspace of X . An open connected subset V of U is said to be adapted to U if j_{VU}^n is an isomorphism and $j_{VU}^1 = 0$ for $1 \neq n$.

By 4.3, each point $x \in U$ has a fundamental system of neighborhoods which are adapted to U .

4.5. COROLLARY. Let X be an (orientable) n -cm over Z . Then X is an (orientable) n -cm over any principal ideal domain L .

This follows from 2.4, 4.2 and the universal coefficient formula

$$0 \longrightarrow H_C^k(X; Z) \otimes L \longrightarrow H_C^k(X; L) \longrightarrow \text{Tor}(H_C^{k+1}(X; Z), L) \longrightarrow 0$$

4.6. COROLLARY. Let X be a connected n -cm and F be a proper closed subset. If F is an m -cm, then $m < n$.

Let x be a point of F which is not an interior point. Then $F \cap U$ is a proper closed subset of U for any open connected U containing x , and therefore $H_C^n(F \cap U; L) = 0$ by 4.3. Thus the n -th local Betti number of F around x must then a fortiori be zero, and 4.6 follows from the definitions and 2.3.

4.7. COROLLARY. Let X be a connected n -cm and A be a closed subset such that $\dim_L A \leq n - 2$. Then $X - A$ is connected.

It is enough to show that if U is a connected open neighborhood of an arbitrary point $a \in A$, then $U - U \cap A$ is connected. Hence we may assume X to be orientable. Since, by the cohomology sequence, $H_C^n(X - A; L) \cong H_C^n(X; L)$, our assertion follows from 4.3.

4.8. PROPOSITION. Let X be a connected n -cm over L . If $L = Z_2$ then X is orientable. If $L = Z$ and X is not orientable, then $H_C^n(X; Z) = Z_2$, hence X considered as an n -cm over a field of characteristic $\neq 2$ is not orientable.

Assume $L = Z_2$. Since X is locally connected, $H_C^n(X; Z_2)$ is the inductive limit of the vector spaces $H_C^n(U; Z_2)$, where U runs through the relatively compact open connected subsets of X . Each one of those is contained in a finite union of connected orientable open subsets. Therefore it will be enough to show that if U, V are open connected orientable, then $U \cup V$ is orientable, i.e., $H_C^n(U \cup V; Z_2) = Z_2$. To this end we consider the Mayer-Vietoris sequence

$$(1) \quad H_C^n(U \cap V) \xrightarrow{f} H_C^n(U) + H_C^n(V) \xrightarrow{g} H_C^n(U \cup V) \longrightarrow 0.$$

For each connected component W of $U \cap V$, j_{UW}^n and j_{VW}^n are isomorphisms by 4.3, hence the image of f is the subgroup generated by (a, b) where a and b are the generators of $H_C^n(U; Z_2)$ and $H_C^n(V; Z_2)$ respectively. Therefore $\text{Im } g$ is 1-dimensional.

Let now $L = Z$. From 4.3 we know that if X is not orientable, $H_C^n(X; Z)$ is a proper quotient of Z , and since X is an n -cm mod 2, it follows from what has just been proved that $H_C^n(X, Z)$ is not zero, of finite order divisible by 2. It remains to show that its order is ≤ 2 .

Since $H_C^n(X; Z)$ is inductive limit of the $H_C^n(U; Z)$ with U open connected relatively compact, and is not isomorphic to Z , there must exist, given an orientable connected open U , an open connected relatively compact U' containing U such that $\ker j_{U',U}^n \neq 0$. Since U' is contained in a finite union of connected orientable relatively compact sets, a trivial induction shows that there must exist a U' of the form $U \cup V$ with V connected orientable, for suitable U . We consider again 4.8(1), but for integral cohomology; let a, b be generators of $H_C^n(U; Z)$ and $H_C^n(V; Z)$, and W be a connected component of $U \cap V$. Since $j_{U,W}^n$ and $j_{V,W}^n$ are isomorphisms, $f(H_C^n(W; Z))$ is either the subgroup A_1 generated by (a, b) or the subgroup A_2 generated by $(a, -b)$. Since $\text{Im } g = H_C^n(U \cup V; Z) \neq Z$, both A_1 and A_2 must occur as images for different connected components of $U \cap V$, and then $H_C^n(U \cup V; Z) = Z_2$. But then $H_C^n(X; Z)$ has at most order 2 since $j_{X,U \cup V}^n$ is surjective by 4.3.

4.9. Let L be a principal ideal domain. By the Künneth rule and 1.2, it is clear that $\dim_Z X \leq n \longrightarrow \dim_L X \leq n$.

PROPOSITION. Let X be an n -cm over L , and F be a closed subset of X . Then:

- (a) $\dim_L F = n$ if and only if F has an interior point;
- (b) $\dim_L F = n - 1$ if and only if F has no interior point and separates X locally around one of its points.

COROLLARY. Let X be an n -cm over Z and F be a closed subset of X . Then $\dim_Z F = n$ (respectively $\dim_Z F = n - 1$, respectively $\dim_Z F \leq n - 2$) if and only if $\dim_L F = n$ (respectively $\dim_L F = n - 1$, respectively $\dim_L F \leq n - 2$).

(F is said to separate X locally around x if it has a basic system of connected neighborhoods U in X such that $U - F$ is not connected.)

PROOF of (a): If F has an interior point x , then the local Betti numbers of x in F and in X are the same, and since $p^n(x; L) = 1$, we have $\dim_L F \geq n$. If F has no interior point, then for every $x \in F$ and open connected neighborhood U of x in X , we have $H_C^n(F \cap U; L) = 0$ by 4.3(a) hence $\dim_L F \leq n - 1$ by 2.3.

PROOF of (b): Let $U \supset V$ be open connected orientable subsets of X , such that $j_{UV}^{n-1} = 0$. Consider the commutative diagram

$$\begin{array}{ccccccc}
H_C^{n-1}(U) & \longrightarrow & H_C^{n-1}(U \cap F) & \longrightarrow & H_C^n(U - F) & \longrightarrow & H_C^n(U) \longrightarrow 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_C^{n-1}(V) & \longrightarrow & H_C^{n-1}(V \cap F) & \xrightarrow{\delta} & H_C^n(V - F) & \longrightarrow & H_C^n(V) \longrightarrow 0
\end{array}$$

where the rows are parts of exact cohomology sequences, the vertical maps are j maps and cohomology is with respect to L . By 4.3, $H_C^n(U) = H_C^n(V) = L$, j_{UV}^n is an isomorphism, and $H_C^n(V - F)$ is a free L -module whose rank is equal to the number of connected components of $V - F$. Therefore, δ is not zero if and only if $V - F$ is not connected. Moreover, since $j_{UV}^{n-1} = 0$, it follows that if $j_{UF, V \cap F}^{n-1} \neq 0$, then $V - F$ is not connected. Let now $\dim_L F = n - 1$. Then F has no interior point by (a). Also, by 2.3 there is an $x \in F$ such that the $n-1$ -th local Betti number of F at x is not zero. This, together with the fact that in X , $p^{n-1}(x) = 0$, implies that we can find an open connected orientable U containing x , and a basic system of open connected orientable neighborhoods V of x in U such that $j_{UV}^{n-1} = 0$ and $j_{UF, V \cap F}^{n-1} \neq 0$. Then by the remark just made, $V - F$ is not connected.

If now F separates X locally around x , then for suitable V , δ is not zero, hence $H_C^{n-1}(V \cap F) \neq 0$ and $\dim_L F \geq n - 1$; if, moreover, F has no interior point, then $\dim_L F = n - 1$ by (a).

4.10. THEOREM. (a) Let X be an s -cm and Y be a t -cm over L . Then $X \times Y$ is an $(s + t)$ -cm over L , and is orientable if and only if X and Y are orientable.

(b) Let $L = \mathbb{Z}, K_p$. Let $X \times Y$ be a connected n -cm $_L$. Then X is an s -cm $_L$ and Y a t -cm $_L$ ($s + t = n$), both orientable if and only if $X \times Y$ is orientable.

The first part is an elementary consequence of the definitions and the Künneth rule, and is left to the reader. We use implicitly the fact that $\dim_L(X \times Y)$ is finite. More generally, it is known that $\dim_L(X \times Y) \leq \dim_L X + \dim_L Y$ for arbitrary locally compact spaces X, Y . This follows for instance from Proposition 4 in [2, XXI].

Let now $X \times Y$ be an n -cm $_L$: Then (see end of 1.3) X and Y are clc $_L$. Let first $L = K_p$. Then, for subspaces U and V of X and Y respectively, we have $H_C^*(U \times V; L) = H_C^*(U; L) \otimes H_C^*(V; L)$ and this isomorphism commutes with maps. Our assertion follows in this case easily from 4.3.

Let now $L = \mathbb{Z}$. Then $X \times Y$ is an orientable n -cm over K_p

if it is orientable over Z (4.5). Therefore, by the above, there exists for each p (prime or zero) integers s_p, t_p ($s_p + t_p = n$) such that X is an s_p -cm and Y a t_p -cm mod p , both orientable if $X \times Y$ is. Let $U \subset X, V \subset Y$ be open connected, such that $U \times V$ is orientable. By the Kunneth theorem, there is an exact sequence

$$(1) 0 \rightarrow \Sigma_a H_C^a(U; Z) \otimes H_C^{n-a}(V; Z) \rightarrow H_C^n(X \times Y; Z) \rightarrow \Sigma_a \text{Tor}(H_C^{a+1}(U; Z), H_C^{n-a}(V; Z)) \rightarrow 0.$$

Since the middle term is isomorphic to Z by 4.3, it follows that there exists an integer $s(U, V)$ such that

$$H_C^a(U; Z) \otimes H_C^{n-a}(V; Z) = 0 \quad (a \neq s(U, V)), \quad H_C^a(U; Z) \otimes H_C^{n-a}(V; Z) = Z \quad (a = s(U, V)).$$

Let $U' \subset U, V' \subset V$ be open and connected. The exact sequence (1) being natural with respect to maps, 4.3(2) shows that $s(U', V') = s(U, V)$. Since $X \times Y$ is connected, it follows at once that $s(U, V)$ is independent of U, V . Call it s and put $t = n - s$. Thus we have

$$H_C^s(U; Z) \otimes H_C^{n-s}(V; Z) = Z$$

whenever $U \subset X, V \subset Y$ are open, connected, and such that $U \times V$ is orientable. For $U' \subset U, V' \subset V$ open connected and small enough, $U' \times V'$ is adapted to $U \times V$ (4.4) and $\text{Im } j_{U', U}^*, \text{Im } j_{V', V}^*$ are finitely generated (2.2). Applying 4.3 to the case $L = K_0$ we see that these groups, modulo torsion, have rank one. Therefore, if $U'' \subset U'$ and $V'' \subset V'$ are open connected, $\text{Im } j_{U'', U''}^*$ and $\text{Im } j_{V'', V''}^*$ have finite indices in $\text{Im } j_{U', U'}^*$ and $\text{Im } j_{V', V'}^*$ respectively. However, we have already proved that the composition of the maps

$$\text{Im } j_{U'', U''}^* \otimes \text{Im } j_{V'', V''}^* \rightarrow \text{Im } j_{U', U'}^* \otimes \text{Im } j_{V', V'}^* \rightarrow H_C^s(U; Z) \otimes H_C^t(V; Z) = Z$$

is surjective. Since the four groups occurring in the two first terms, modulo torsion, have rank one, it follows that the finite indices mentioned above are bounded from above. In other words, there exist U', V' such that for any $W \subset U', T \subset V'$, open connected, $\text{Im } j_{W, W}^* = \text{Im } j_{U', U'}^*$, $\text{Im } j_{T, T}^* = \text{Im } j_{V', V'}^*$. Our contention will then be a consequence of the following lemma:

4.11. LEMMA. Let M be a connected space which is finite dimensional and oic over Z . For every p (prime or zero) there exists an integer n_p such that M is an orientable n_p -cm over K_p . For every

open connected U in M and $x \in U$, there exists a subgroup $A(x)$ of $H_c^*(U; Z)$, isomorphic to Z , which is contained in $\text{Im } j_{UV}^*$ for every open $V \subset U$ containing x . Then M is an orientable n -cm over Z and $n = n_p$ for all p .

$j_{UV,L}^*$ will denote $j_{U,V}^*$ for cohomology with coefficients in L . If $\bar{V}$ is compact in U , then by property (P, Q), (2.2), $\text{Im } j_{UV,Z}^*$ is finitely generated. If V is small enough, then V is adapted to U over K_p , therefore $\text{Im } j_{U,V,K}^*$ has dimension 1 and $\text{Im } j_{U,V,Z}^*$ is the direct sum of Z and a finite group; it contains then $A(x)$ as a subgroup of finite index; consequently, if W runs through a basic system of open neighborhoods of x in V , then the index of $\text{Im } j_{U,W,Z}^*$ in $\text{Im } j_{U,V,Z}^*$ is bounded from above, and there exists a neighborhood W of x , such that $H = \text{Im } j_{U,W,Z}^* \cong Z + S$, where S is finite, and that $\text{Im } j_{U,W,Z}^* = \text{Im } j_{U,W',Z}^*$ for every open neighborhood of x in W . We claim that $S = (e)$. If $S \neq (e)$, let p be a prime divisor of $\text{ord } S$. Then $H \otimes_{Z_p}$, considered as vector space over Z_p has at least dimension two. Since $\text{Im } j_{U,W',Z}^*$ is also equal to H (and the tensor product is right exact!), it follows that the image of the natural map

$$H_c^*(W'; Z) \otimes_{Z_p} \longrightarrow H_c^*(W; Z) \otimes_{Z_p}$$

has at least dimension two; the universal coefficient formula and naturality imply then that $\text{Im } j_{W,W',Z_p}^*$ has dimension ≥ 2 . This, being valid when W, W' describe basic systems of neighborhoods of x , shows that the sum of the local Betti numbers mod p of M at x is at least two, in contradiction with the fact that M is a cm over Z_p . Therefore $S = (e)$ and we have shown the existence of a subgroup $B(x) \cong Z$ of $H_c^*(U; L)$ which is contained in $\text{Im } j_{UV}^*$ for every open neighborhood V of x in U , and is equal to this image if V is small enough. Let $n(x)$ be the integer such that $B(x) \subset H_c^{n(x)}(U; Z)$. From the universal coefficient formula, $n(x) = n_p$, for all p , thus $n = n(x)$ is independent of x , and all n_p are equal to n . This shows already that M is a Wilder n -manifold over Z (3.1). By 3.4, local orientability will follow if we prove that $B(y) \subset B(x)$ ($y, x \in U$) implies $B(y) = B(x)$. Suppose it is not so, and let p be a divisor of the index of $B(y)$ in $B(x)$. Then we would have $\text{Im } j_{U,W,Z_p}^* = 0$ for a sufficiently small open neighborhood W of y ; however, U being orientable mod p , this contradicts 4.3.

REMARK. An extension of 4.10 to generalized manifolds with boundary will be found in a forthcoming paper of F. Raymond, "Separation and uniqueness theorems for generalized manifolds with boundary" to appear

in Michigan J. Math.

§5. Appendix on Cohomological Dimension

This section is mainly devoted to the proof of 5.3. This theorem is due to Grothendieck for the case $L = \mathbb{Z}$, X paracompact [5, §3. 6a; 4, p. 197], and to Floyd (unpublished) in the general case.

5.1. LEMMA (Floyd). Let S be a class of sheaves on the (not necessarily locally compact) space X satisfying the following conditions:

- (i) If $A \in S$ and $A' \cong A$, then $A' \in S$.
- (ii) If $0 \rightarrow A'' \rightarrow A \rightarrow A' \rightarrow 0$ is exact, then $A, A'' \in S$ implies $A' \in S$ and $A', A'' \in S$ implies $A \in S$.
- (iii) If $A = \bigcup_1 A_1$ where (A_1) is a directed upward family of subsheaves of A belonging to S , then $A \in S$.
- (iv) For each open $U \subset X$, the subsheaf $L_U = (U \times L) \cup (X \times 0)$ of the constant sheaf $L = X \times L$ is in S .

Then S consists of all sheaves on X .

PROOF. Every sheaf $\underline{A}$ is a homomorphic image of a direct sum $\bigoplus_{1 \in I} \underline{L}_{U_1}$ [4, p. 142]; here 1 ranges over all sections $U_1 \rightarrow \underline{A}$ and the homomorphism φ is defined naturally from the sections. If $\varphi(\bigoplus_{1 \in I} \underline{L}_{U_1}) \in S$ for each finite family $I' \subset I$, then $\underline{A} \in S$ by (iii). Hence it is enough to prove that $\underline{A} \in S$ when I is finite. By (ii), it is then sufficient to prove that every homomorphic image $\underline{A}$ of a sheaf $\underline{L}_U$ belongs to S ; by (ii) and (iv) it is therefore enough to show that every subsheaf of L belongs to S .

L , being a principal ideal domain, is a unique factorization domain. We denote by $N(I)$, (I proper ideal in L), the number of prime ideals of which I is the product. Thus $I' \supset I$, implies $n(I') \leq n(I)$.

The stalk of $\underline{A}$ at $x \in X$ is an ideal of L , and we have clearly $\underline{A}_y \supset \underline{A}_x$ for y sufficiently near x ; therefore if U_1 is the subspace of points x for which $n(\underline{A}_x) \leq 1$, then the U_1 form an increasing sequence of open subsets whose union is $U = \{x, \underline{A}_x \neq 0\}$, and $\underline{A}$ is the union of the increasing sequence of subsheaves $\underline{A}_1 = \underline{A}_{U_1}$. By (iii), it is enough to show that the $\underline{A}_1$ belong to S . We have $\underline{A}_0 = \underline{L}_{U_0} \in S$,

hence, using (ii) and induction on i , it suffices to show that $\underline{A}_1/\underline{A}_{i-1} \in S$. This last sheaf is concentrated on $U_i - U_{i-1}$; moreover $U_i - U_{i-1}$ is the union of disjoint relatively open subsets V_j where j runs through the ideals I_j of L which are products of exactly i prime factors and where $\underline{A}_X$ is equal to I_j on V_j . Since $I_j \cong L$ as an L -module, it follows that there exist open sets W_j, W_j' with $V_j = W_j - W_j'$ such that

$$(\underline{A}_1/\underline{A}_{i-1})_{V_j} \cong \underline{L}_{W_j}/\underline{L}_{W_j'}.$$

(iv), (ii) and (iii) imply then that $\underline{A}_1/\underline{A}_{i-1} \in S$.

5.2. A paracompactifying family Φ on a (not necessarily locally compact) space X is a family of closed subsets of X satisfying the following conditions:

- (a) The elements of Φ are closed and paracompact.
- (b) Φ is closed under finite unions.
- (c) Every closed subset of an element of Φ belongs to Φ .
- (d) Every element of Φ has a neighborhood in Φ .
- (e) The elements of Φ cover X .

Except for (e), which is added here for convenience, these are the usual conditions of H. Cartan [2, Exp. XVII; 4 p. 150]. When X is locally compact, as again will be tacitly assumed from now on, any Φ contains the family of compact subsets of X .

5.3. THEOREM. Let Φ be a paracompactifying family on X . Then, in the notation of 1.2, we have $\dim_L X = \dim_{\Phi, L} X$.

PROOF. For U open in X , we have

$$(1) \quad H_{\Phi}^n(U; L) = H_{\Phi}^n(X; \underline{L}_U)$$

where the left hand side means the cohomology of U with respect to the family of elements of Φ contained in U [4, p. 190]. If moreover U is relatively compact, then $\Phi|_U$ is the family of compact subsets of U , hence

$$(2) \quad H_{\Phi}^n(U; L) = H_{\bar{C}}^n(U; L) \quad (\bar{U} \text{ compact}).$$

Therefore, by 1(1), $\dim_L X \leq \dim_{\Phi, L} X$. We now prove that $\dim_L X \geq \dim_{\Psi, L} X$ when Ψ is the family of compact subsets of X . We have then to show that if $H_{\bar{C}}^n(U; L) = 0$ for each open U (for a given n), then the class S of sheaves for which $H_{\Psi}^j(X; A) = 0$ ($j \geq n$) consists of all sheaves, and

for this it is enough to show that S satisfies the conditions (i), (ii), (iii), (iv) of 5.1. (i) is clear, (ii) follows from the exact cohomology sequence [4, p. 174], (iii) from the equality

$$H_{\Psi}^j(X; \underline{A}) = \text{ind} \lim H_{\Psi}^j(X; \underline{A}_1)$$

[4, p. 194], and (iv) from the equality $H_C^n(U; L) = H_{\Psi}^n(X; \underline{L}_U)$, (U open in X).

In case X is paracompact and ϕ is the family of closed subsets of X , then $\dim_{\phi, L} X = \dim_{\Psi, L} X$ is proved in [4, p. 196], (for $L = \mathbb{Z}$, but the proof given there is seen also to hold for a general L).

Consider finally the general case. By a lemma of H. Cartan [2, XVII p. 6], we have

$$\dim_{\phi, L} X = \max \dim_{\phi} |M, L^M$$

where M runs through the elements of ϕ and $\phi | M$ consists of the elements of ϕ contained in M , that is, of all closed subsets of M . Hence

$$\dim_{\phi, L} X = \max \dim_{\phi} |M, L^M = \max \dim_{\Psi, L} M = \dim_{\Psi, L} X = \dim_L X.$$

5.4. COROLLARY. Let ϕ be a paracompactifying family on X . Then $\dim_L X \leq n$ if and only if $H_{\phi}^{n+1}(U; L) = 0$ for all open U in X .

1(1) and 5.3(1), (2) show that the condition

$$(A_{n+1}) : H_{\phi}^{n+1}(U; L) = 0$$

for all open U implies $\dim_L X \leq n$. The converse follows from 5.3 and 5.3(1).

A particular case: Let X be paracompact and ϕ be the family of closed subsets. Then $H_{\phi}^j(U; L)$ may be identified with the relative Čech or Alexander-Spanier cohomology group $H^j(X \bmod A; L)$, where $A = X - U$. Therefore $\dim_L X \leq n$ if and only if $H^{n+1}(X \bmod A; L) = 0$ for every closed subset A .

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II: HOMOLOGY AND DUALITY IN GENERALIZED MANIFOLDS

A. Borel

Our main purpose here is to obtain a Poincaré duality theorem in cohomology manifolds. Even if one is interested mainly in a statement involving only cohomology, one has to use in the proof groups which play the role of homology groups, and therefore this presupposes some homology theory. The approach of [2] led to satisfactory results only when the coefficients form a field. We shall give here another one, which will allow us to make use of universal coefficient formulas in the general case. The present paper, which differs considerably from the oral lecture, represents joint work of J. C. Moore and the author.

The notation is that of I. We assume familiarity with sheaf theory [3, 5]. In particular the following notion and notation of [5] will be used: A family of supports ϕ in the space X is a collection of closed subsets such that the union of two elements of ϕ belongs to ϕ , and that every closed subset of an element of ϕ is in ϕ . Given a sheaf $\underline{F}$ on X , $\Gamma(\underline{F})$, $\Gamma_\phi(\underline{F})$, $\underline{F}(A)$ ($A \subset X$) denote respectively the sections of $\underline{F}$, those with support in ϕ , and the sections over the subspace A . "Section" will always mean "continuous section". A sheaf $\underline{F}$ is called flabby (respectively soft) if the restriction map $\Gamma(\underline{F}) \rightarrow \underline{F}(A)$ is surjective for every open (respectively closed) subspace A of X . Thus flabby (respectively soft) sheaf is the translation of *faisceau flasque* (respectively *mou*) of [5]. If K is a grating on X , K_U will denote the subgrating of elements with supports in the subspace U . L is a principal ideal domain given once for all and Hom , Ext will be taken over L . Unless otherwise stated, all topological spaces are locally compact and Hausdorff.

§1. Homology groups for locally compact spaces

1.1. DEFINITION. A sheaf $\underline{C}$ of L -modules on X is a fundamental sheaf for homology with coefficients in L (and supports in a given family ϕ of supports) if

(1) $\underline{C}$ is graded, and has a boundary operator lowering degrees by 1.

(ii) $\underline{C}$ is flabby (or ϕ -soft, or ϕ -fine, if ϕ is a paracompactifying family, see I, §5).

(iii) For each open U in X we have a split exact sequence

$$(1) \quad 0 \rightarrow \text{Ext}(H_c^{i+1}(U; L), L) \rightarrow H_1(\underline{C}(U)) \rightarrow \text{Hom}(H_c^i(U; L), L) \rightarrow 0$$

and if V is open in U , the sequences (1) for U, V are compatible with the maps induced from j_{UV}^* and the restriction $\underline{C}(U) \rightarrow \underline{C}(V)$.

If L is a field, and X is paracompact, then for any fundamental grating for the cohomology with compact carriers K , for instance the Alexander-Spanier cochains with compact supports, the sheaf associated to the grating $\text{Hom}(K, L)$ is fundamental for homology [2].

1.2. Let $\underline{C}$ be a fundamental L -sheaf, and let ϕ be a family of supports on X . Then for $U \supset V$ open we denote by j_*^{UV} the homomorphism $H_*(\underline{C}(U)) \rightarrow H_*(\underline{C}(V))$ induced by the restriction, and by j_1^{UV} its restriction to $H_1(\underline{C}(U))$. We put

$$H_1(X; L) = H_1(\Gamma(\underline{C})); H_1^\phi(X; L) = H_1(\Gamma_\phi(\underline{C})); H_1^X(X; L) = H_1(\underline{C}_X) \quad (x \in X)$$

(2)

$$H_*(X; L) = \Sigma H_1(X; L); H_*^\phi(X; L) = \Sigma H_1^\phi(X; L); H_*^X(X; L) = \Sigma H_1^X(X; L).$$

Here $\underline{C}_x$ denotes of course the stalk of $\underline{C}$ at x ; the group $H_1(\underline{C}_x)$ is the 1-th local homology group at x , and, in case say it is finitely generated, the rank of $H_1(\underline{C}_x)$ modulo its torsion subgroup corresponds to Alexandroff's local Betti number in x (see [2] for some remarks on the relation between the local Betti numbers in x and those around x , as defined in I). The groups $H_1^X(X; L)$ are then the stalks of the homology sheaf $\underline{H}_1(\underline{C})$, and $\underline{H}(\underline{C})$ will be called the local homology sheaf.

By 1.1(iii), $H_*(\underline{C}(U))$ is independent of the particular fundamental sheaf considered, and this legitimizes the notation $H_*(U; L)$. We shall allow ourself to use it for the local groups, although we do not know whether the axioms are sufficient to insure uniqueness, because this will be the case anyway for cohomology manifolds. We shall come back to this in 2.5 and 5.1.

1.3. THEOREM. Any locally compact space has fundamental sheaves for homology with degrees ≥ -1 . For any such

sheaf $\underline{C}$ and any $x \in X$ we have an exact sequence

$$(3) \quad 0 \rightarrow \operatorname{Dir} \lim \operatorname{Ext}(H_C^{i+1}(U; L), L) \rightarrow H_1(\underline{C}_X) \rightarrow \operatorname{Dir} \lim \operatorname{Hom}(H_C^i(U; L), L) \rightarrow 0$$

U running through a fundamental system of neighborhoods of x , and the limit being taken with respect to the maps induced by the homomorphisms j_{UV}^* .

For the proof, see Section 3. Another construction, valid for closed subsets of euclidean spaces, will be given in Section 4.

REMARK. The homology groups defined by N. Steenrod (Annals 41) by means of regular cycles for a compact metric space X are known to be related to the Čech cohomology by a split exact sequence (1) with $U = X$) (Eilenberg-MacLane, Annals 43, p. 825). In this case our groups coincide therefore with the Steenrod groups (except however for a shifting of the degrees by 1). Note that with our convention for the degree we have by (1), $H_1(X; L) = 0$ for $i \leq -2$, but $H_{-1}(X; L) = \operatorname{Ext}(H_C^0(X; L), L)$ may be nonzero if the space is not locally connected.

§2. Duality in generalized manifolds

2.1. DEFINITION. The space X is an $L - n$ space if $\dim_L(X)$ is finite and if for any $x \in X$, we have $H_1^X(X; L) = 0$ ($i \neq n$) and $H_n^X(X; L) = L$. An $L - n$ space is locally orientable (respectively orientable) if the local homology sheaf is locally simple (respectively simple).

In the locally orientable case, the sheaf $\underline{H}_n^X(X, L)$ of the local groups will be called the orientation sheaf. A locally orientable $L - n$ space which is clc_L will also be called a homology n -manifold over L (a $n - \operatorname{hm}_L$).

2.2. THEOREM. Let X be an $L - n$ space, $\underline{C}$ a fundamental sheaf for homology over L , and ϕ a family of supports in X . Then

$$(1) \quad H_1^\phi(X; L) \cong H_\phi^{n-1}(X; \underline{H}_n(\underline{C})) .$$

Let us denote by $\underline{A}$ the sheaf $\underline{C}$ with the modified grading $\underline{A}^1 = \underline{C}_{n-1}$. Then $H^1(\underline{A}_X) = 0$ ($i \neq 0$), $H^0(\underline{A}_X) = L$, and $H^{n-1}(\Gamma_\phi(\underline{A})) = H_1(\Gamma_\phi(\underline{C}))$. By [5, Chap. II, Theor. 4.6.1] there is a spectral sequence in which

$$E_2^{p,q} = 0 \quad (q \neq 0), \quad E_2^{p,0} \simeq H_{\phi}^p(X, \underline{H}^0(\underline{A}_X)) \simeq H_{\phi}^p(X; \underline{H}_n(\underline{C}))$$

and E_{∞} is the graded module associated to some filtration of $H(r_{\phi}(\underline{A}))$; the $\dim_{\mathbb{L}} X$ being finite, this filtration may be assumed to be regular or even defined by a finite number of submodules [5, p. 193]. Since we have here $E_{\infty}^{p,q} = 0 \quad (q \neq 0)$, $E_{\infty}^{p,0} = E_2^{p,0}$, the theorem follows.

REMARK. Let us denote by ϕ the isomorphism of 2.2 as obtained in the above proof, and let $U \supset V$ be open. Then we have a commutative diagram

$$\begin{array}{ccc} H_1(r_{\phi \cap U}(\underline{C}_U)) & \xrightarrow{\phi} & H_{\phi \cap U}^{n-1}(U; \underline{H}_n(\underline{C}_U)) \\ \downarrow j_*^{UV} & & \downarrow r^* \\ H_1(r_{\phi \cap V}(\underline{C}_V)) & \xrightarrow{\phi} & H_{\phi \cap V}^{n-1}(V; \underline{H}_n(\underline{C}_V)) \end{array}$$

where r^* is the restriction map in cohomology, and $\phi \cap U$, $\phi \cap V$ denote the families of intersections of the elements of ϕ by U and V respectively. This follows from the fact that the proof of the basic result of [5] quoted above is obtained by examining the two spectral sequences of a double complex, namely the sections of a flabby resolution of $\underline{C}$ and that the restriction of such a grating on U or V has the same properties with respect to U or V .

2.3. COROLLARY. Let X be an L - n space. Then there is a split exact sequence

$$(3) \quad 0 \rightarrow \text{Ext}(H_{\mathbb{C}}^{1+1}(X; L), L) \rightarrow H^{n-1}(X; \underline{H}_n(\underline{C})) \rightarrow \text{Hom}(H_{\mathbb{C}}^1(X; L), L) \rightarrow 0$$

where $\underline{C}$ is a fundamental sheaf for homology on X .

We refer to [2] for some consequences of 2.3 when L is a field, in particular for the fact that if X is connected orientable, then the isomorphism $H^1(X; L) \rightarrow \text{Hom}(H_{\mathbb{C}}^{n-1}(X; L), L)$ is induced by the cup product pairing; and for remarks on the relation between 2.3 and the results of Wilder [6].

2.4. PROPOSITION. (a) Let L be a field. Then a Wilder n -manifold over L (respectively a n - cm over L , respectively an orientable n - cm over L) is an (respectively a locally orientable, respectively an orientable) L - n space which is clc over L , and conversely.

(b). Let X be an (orientable) n -cm over L . Then X is an (orientable) n -hm $_L$.

Let X be a Wilder n -manifold over L . It is clc over L (I, 2.2). If L is a field, then for any open U , $H_1(U; L)$ is the dual vector space to $H_0^1(U; L)$ and j_{UV}^1 (V open in U) is the transpose of j_{UV}^* . Then it follows without difficulty from 1.3 that X is an L - n space. Assume now that X is clc over L and satisfies condition L - n , L being again a field. Then for any open neighborhood U of x there exists an open neighborhood V of x in U such that $\text{Im } j_{UV}^1$ is finite dimensional. Under the latter condition it is easily seen that $\dim H_1^X(X; L)$ is equal to the i -th Betti number of X around x . Therefore X is a Wilder n -manifold. Let now U be a connected orientable open subset of an n -cm over L . Then (I, 4.3) $H_0^n(U; L) \cong L$ and j_{UV}^n is an isomorphism for every connected open subset V of U . Also, each point $x \in X$ has neighborhoods for which $\text{Im } j_{UV}^1 = 0$ for $1 \neq n$. This, together with 1.3 shows that X satisfies condition L - n at each point of U , and moreover that over U , the orientation sheaf $\underline{H}_n(\mathbb{C})$ is isomorphic to the simple sheaf $U \times H_0^n(U; L)$. This proves (b) and the remaining direct part of (a).

Let now X be an n -hm $_L$, where L is a field, and U be a connected orientable open subset. Then by 2.2, $H_n(X; L) = H^0(X, L) \cong L$. Using the diagram (2), for $i = n$, it follows that if V is open connected in U , then j_{UV}^n is an isomorphism. But here $H_n(U; L) = \text{Hom}(H_0^n(U; L), L)$ and j_{UV}^n is transpose of j_{UV}^n . Therefore $H_0^n(U; L) \cong H_0^n(V; L) \cong L$ and j_{UV}^n is an isomorphism. Thus U is an orientable Wilder n -manifold, and this ends the proof of (a).

2.5. Since it is not clear that the conditions put on a fundamental sheaf for homology insure uniqueness of the derived sheaf, there is some ambiguity in 2.1, because it is not said whether the condition L - n is required for some or for all fundamental sheaves for homology. The simplest is probably to require it just for the restricted fundamental sheaves in the sense of 5.1. We also note that the proof of 2.4 shows the following: if X is an (orientable) n -cm then it is an (orientable) L - n space for any fundamental sheaf; if L is a field, and X is an (locally orientable, orientable) L - n space for some fundamental sheaf, then it is for all fundamental sheaves.

§3. Existence of fundamental sheaves for homology

3.1. LEMMA. Let $\mathcal{K}$ be the family of compact subsets of X , K be a $\mathcal{K}$ -fine grating [3b, XVIII-9] with compact supports, and A be an L -module. Let $\underline{C}'$ be the

presheaf over X defined by $\underline{C}'(U) = \text{Hom}(K_U, A)$, the restriction map s_{UV} (V open in U) being the transpose to the inclusion $K_V \longrightarrow K_U$. Then the natural map of $\underline{C}'$ into the presheaf defined by the cross sections of the sheaf $\underline{C}$ associated to $\underline{C}'$ is an isomorphism, hence [5, p. 111], $\underline{C}'(U) = \underline{C}(U)$. If A is injective, then $\underline{C}$ is flabby.

By [5, Chap. II, 1] our first assertion is equivalent to the following one:

Let $(U_i)_{i \in I}$ be a family of open subsets of X , and U be their union. Then (a) If $t, t' \in \underline{C}'(U)$ have equal restriction on each U_i , then $t = t'$. (b) Let there be given $t_i \in \underline{C}'(U_i)$ in such a way that for all $i, j \in I$, t_i and t_j have equal restrictions on $U_i \cap U_j$. Then there exists a $t \in \underline{C}'(U)$ whose restriction to U_i is t_i for every $i \in I$.

PROOF of (a): Let $c \in K_U$ and I' be a finite subset of I such that the U_i with $i \in I'$ cover the support $S(c)$ of c . Let $\underline{V}$ be an open covering of X formed by set V_0 whose closure does not meet $S(c)$ and by elements $V_i (i \in I')$ such that $\bar{V}_i \subset U_i (i \in I')$. Let (r_j) be a partition of K subordinated to this covering. Then $c = \sum_{i \in I'} r_j c$ and $r_j c \in K_{U_i}$. By assumption $t(r_i c) = t'(r_i c)$ for all $i \in I'$, hence $t(c) = t'(c)$ by linearity.

PROOF of (b): Let c, I' and $\underline{V}$ be as before. We tentatively define t by $t(c) = \sum_{i \in I'} t_i(r_i c)$. In order to legitimize this, we have to prove that $t(c)$ is independent of the different choices made. Let then J be another finite subset of I such that the $U_j (j \in J)$ cover c and let us construct as before endomorphisms s_j of K such that $s_j(K) \subset K_{U_j}$ and that $c = \sum_{j \in J} s_j c$. We have to show that $t'(c) = \sum t_j(s_j c)$ is equal to $t(c)$.

We have $s_j t_i c \in K_{U_j \cap U_i}$ hence

$$t_j(s_j t_i c) = t_i(s_j t_i c) \quad (j \in J, i \in I')$$

and also $r_i c = \sum_j s_j r_i c$, $s_j c = \sum_i s_j r_i c$. Therefore

$$t_i(r_i c) = \sum_j t_i(s_j r_i c) = \sum_j t_j(s_j r_i c)$$

hence

$$t(c) = \sum_i t_i(r_i c) = \sum_{i,j} t_j(s_j r_i c) = \sum_j t_j(s_j c) = t'(c)$$

and (b) follows.

Let now A be injective. The restriction map $s_{UV}(U \supset V, \text{ open})$ is then surjective, hence $\underline{C}$ is flabby.

3.2. PROOF of Theorem 1.3. Let D be a fundamental L -sheaf (for cohomology) which is ψ -fine, ψ being the family of compact subsets of X , for instance the sheaf of germs of Alexander-Spanier cochains, and let $K = r_{\psi}(D)$. Then $H^*(K) = H_c^*(X; L)$. Let A_0 be an injective module containing L and $A_1 = A_0/L$. Since L is hereditary, A_1 is also injective. We view $A = A_0 + A_1$ as an augmented complex over L via the exact sequence

$$(1) \quad 0 \rightarrow L \rightarrow A_0 \rightarrow A_1 \rightarrow 0,$$

hence

$$(2) \quad H_0(A) \cong L, H_1(A) = 0 \quad (i \geq 1).$$

Let $C = \text{Hom}(K, A)$ be endowed with the grading

$$C_1 = \text{Hom}(K^1, A_0) + \text{Hom}(K^{1+1}, A_1)$$

(thus $C_1 = 0$ $1 \leq -2$, $C_{-1} = \text{Hom}(K^0, A_1)$), and the differential

$$df(k) = d(f(k)) + (-1)^{i+1} f(dk) \quad (f \in C_1, k \in K).$$

Then $U \longrightarrow \text{Hom}(K_U, A)$ is the presheaf of cross sections of a flabby sheaf $\underline{C}$ by 3.1. By [4, Cor. 5.2, p. 373] or [5, Théor. 5.4.2, p. 101] and (2) above, we have then an exact sequence

$$(3) \quad 0 \rightarrow \text{Ext}(H^{1+1}(K_U), L) \rightarrow H_1(\text{Hom}(K_U, A)) \rightarrow \text{Hom}(H^1(K_U), L) \rightarrow 0.$$

Moreover, if $F = F_1 + F_0$ is a projective resolution of K_U , the sequence (3) is canonically isomorphic to the corresponding exact sequence for F by [4, pp. 373-4], hence it splits, by [4, Theor. 3.2a, p. 114]. Also, this exact sequence is compatible with homomorphisms, therefore $\underline{C}$ has all the properties required in 1.1.

Since $\underline{C}$ is flabby, its stalk $\underline{C}_x$ at x is equal to the quotient of $r(\underline{C}) = \text{Hom}(K, A)$ by the submodule of sections vanishing at x , and also to $\text{Dir} \lim (\text{Hom}(K_U, A), s_{UV}), U$ running through the open neighborhoods of x . Therefore

$$H_1(\underline{C}_X) = \text{Dir lim } H_1(U; L)$$

and the exact sequence 1.3 follows then from 1.1(iii) and the fact that direct limits and exact sequences commute.

3.3. REMARK. Since A is an injective resolution of L , the homology of $\text{Hom}(K, A)$ represents the "hyperhomology invariants" of $\text{Hom}(K, L)$ in the sense of [4, Chapt. XVII], which may differ from $H(\text{Hom}(K, L))$. One could also compute them as $H(\text{Hom}(F, L))$, F being a projective resolution of K .

§4. Closed subsets of euclidean spaces.

In this section we describe briefly another construction of fundamental sheaves for homology valid for separable metric spaces of finite covering dimension, which makes use of basic gratings.

4.1. A basic L-grating (carapace basique, cf. [3b, XVIII-6]) is a grating, which is a free L -module and has moreover a basis $(x_i)_{i \in I}$ such that the support of a finite linear combination of the x_i is exactly the union of the supports of the x_i which have a non-zero coefficient. For instance the singular chains with coefficients in L always form a basic L -grating.

4.2. LEMMA. Let X be a separable metric space of finite covering dimension. Then

(a) X has a fundamental sheaf for cohomology which is homotopically fine and whose sections with compact carriers form a basic grating with degrees bounded from above.

(b) X has a fundamental L -sheaf for homology with degrees bounded from above.

We recall first the operation of section of a grating K on a space X by a subspace Y in the sense of Leray (see e.g., [1, Exp. II]): as a module it is K/K_{X-Y} , the support of $c \in K/K_{X-Y}$ being defined as the intersection of Y by the support of any element of the K_{X-Y} -coset represented by c . This section is clearly fine (respectively homotopically fine) if K is. Moreover if K is basic with basis (x_i) , then K/K_{X-Y} is also basic, with a basis represented by the x_i whose support meets Y . Therefore, if X has a homotopically fine basic grating with compact supports, so does every closed subset Y of X . Moreover, a basic grating with compact supports K can always be identified

with the sections with compact supports of its associated sheaf $\underline{F}$. Since the associated sheaf of the section of K by Y is clearly the sheaf $\underline{F}_Y$ induced by $\underline{F}$ on Y , and since $\underline{F}_Y$ is a fundamental sheaf for cohomology on Y if $\underline{F}$ is on X , we see that if (a) is true for X , it is true for every closed subspace of X .

By the Menger-Nöbeling theorem, a locally compact separable metric space of finite covering dimension is homeomorphic to a closed subset of some euclidean space R^n . It is therefore enough to prove (a) for R^n . In this case we take the sheaf $\underline{S}$ of germs of singular chains with coefficients in L , giving the degree $n - i$ to the i -dimensional chains. It is well known to be a fundamental sheaf for cohomology. Its sections with compact supports are then the singular chains with coefficients in L . This proves (a).

Let K be the grating thus obtained. Its degrees are therefore bounded from above by n . Let $\underline{C}'$ be the presheaf defined by $U \longrightarrow \text{Hom}(K_U, L)$, graded by $\underline{C}'_i(U) = \text{Hom}(K_U^i, L)$, and with differential the transpose of that of K let $\underline{C}$ be the associated sheaf, and $f : \underline{C}'(U) \longrightarrow \underline{C}(U)$ the natural map. It is always surjective; this follows from the fact that K is basic in exactly the same way as the corresponding statement for singular cochains [5, p. 160]. Since K is basic, K_V is a direct summand in $K_U(V \subset U)$ hence the restriction map $\text{Hom}(K_U, L) \longrightarrow \text{Hom}(K_V, L)$ is surjective; thus $\underline{C}$ is flabby.

Here K_U is free, therefore we have a split exact sequence

$$0 \rightarrow \text{Ext}(H^{i+1}(K_U), L) \rightarrow H_1(\text{Hom}(K_U, L)) \rightarrow \text{Hom}(H^i(K_U), L) \rightarrow 0$$

(see the reference given below 3.2(3)), which is compatible with the restriction maps. In order to obtain the property 1.1 (iii), it is enough then to show that $f : \underline{C}'(U) \longrightarrow \underline{C}(U)$ induces an isomorphism for homology, or in other words that $H(\ker f) = 0$. This can also be expressed by saying that the elements of $\text{Hom}(K_U, L)$ which vanish on all elements of K_U with sufficiently small supports have zero homology. This assertion is proved by using the fact that K_U is homotopically fine, in the same way as in the case of singular cochains [3a, Exp. VIII, p. 4].

§5. Complements

5.1. Let $\underline{C}, \underline{C}'$ be fundamental sheaves for homology. We have $H_*(\underline{C}(U)) = H_*(\underline{C}'(U))$ by 1.1 (iii) but since the splitting of the exact sequence is not natural in general, it is not certain that this isomorphism can be so chosen as to commute with restriction maps. It is therefore conceivable that the derived sheaves $\underline{H}(\underline{C})$ and $\underline{H}(\underline{C}')$ might not be

isomorphic. Nevertheless one can insure uniqueness by taking a narrower class of fundamental sheaves, which will be, however, big enough so that one can define f_* in the usual way (see below), namely essentially those which have been constructed in §§3, 4.

Let us remark first that the isomorphisms of homological algebra mentioned in §3 are sufficiently "canonical" as to insure that $H_1(\underline{C}(U))$ does not depend, up to canonical isomorphisms, on the particular injective resolution A of L , which in any case could be normalized by requiring A_0 to be the quotient field of L ; also in §4, we could as well consider $\text{Hom}(K, A)$ instead of $\text{Hom}(K, L)$. It is also clear that if K, K' fulfill the conditions of §3, or §4, and if there is a homomorphism $f : K \longrightarrow K'$ inducing an isomorphism of $H(K_U)$ onto $H(K'_U)$ for each open subset U , then $H(\text{Hom}(K'_U, A))$ and $H(\text{Hom}(K_U, A))$ will be related by an isomorphism commuting with restrictions.

Let $\underline{F}$ be a fundamental sheaf for cohomology on X . This implies in particular that $H^1(\underline{F}) = 0$ ($1 \neq 0$) and that there is an isomorphism φ of $H^0(\underline{F})$ onto the simple sheaf $L \times X$. Let us say that $\underline{F}$ has local units if given any compact subset $Y \subset X$, there exists a cross section s with compact support of $\underline{F}^0$ such that for $x \in Y$, $s(x)$ is a cocycle of the element of $H^0(\underline{F}_x)$ mapped into 1 by φ . The Alexander-Spanier sheaf, or the sheaves constructed in §4, or the canonical flabby resolution of L in [5] all have local units.

If $\underline{F}'$ is another such sheaf, and if r_c denotes sections with compact supports, then by the fundamental construction of Leray, as modified by Fary [see 1, Exp. II], there are natural embeddings

$$r_c(\underline{F}) \longrightarrow r_c(\underline{F} \otimes \underline{F}') \longleftarrow r_c(\underline{F}')$$

which induce the isomorphism of the uniqueness theorem [1, Exp. III]. Let us then define a restricted fundamental sheaf for homology as one associated to a presheaf $U \longrightarrow \text{Hom}(K_U, A)$ where A is an injective resolution of L , and $K = r_c(\underline{F})$, $\underline{F}$ being a fundamental sheaf for cohomology having local units. We have then the

PROPOSITION. Let $\underline{C}, \underline{C}'$ be two restricted fundamental sheaves for homology. Then there are canonical isomorphisms $t_U : H_1(\underline{C}(U)) \longrightarrow H_1(\underline{C}'(U))$ which commute with the restriction maps $j_{UV}^1 (U \subset V \text{ open})$. Consequently the derived sheaves $H(\underline{C})$ and $H(\underline{C}')$ are canonically isomorphic.

5.2. The map f_* and the exact homology sequence. Let $f : X \longrightarrow Y$ be a continuous proper map, $\underline{F}$, (respectively $\underline{G}$) be a fundamental sheaf with local units on X (respectively Y), and let

$f'(\underline{G})$ be the sheaf induced by $\underline{G}$ on X . Then $\underline{F} \otimes f'(\underline{G})$ is a fundamental cohomology sheaf on X with local units and there is a natural homomorphism $\bar{f} : \Gamma_{\underline{C}}(\underline{G}) \longrightarrow \Gamma_{\underline{C}}(f'(\underline{G}) \otimes \underline{F})$ which defines $f_{\underline{C}}^* : H_{\underline{C}}^*(Y; L) \longrightarrow H_{\underline{C}}^*(X; L)$. If X is a closed subset of Y , this map can be identified with the one induced by the natural map $\Gamma_{\underline{C}}(\underline{G}) \longrightarrow \Gamma_{\underline{C}}(\underline{G})/\Gamma_{\underline{C}}(\underline{G})_{Y-X}$ ([1], Exp. VII, [2] 5, p. 199). We define then $f_* : H_*(X; L) \longrightarrow H_*(Y; L)$ as the map induced by the homomorphism $\text{Hom}(\Gamma_{\underline{C}}(f'(\underline{G}) \otimes \underline{F}), A) \longrightarrow \text{Hom}(\Gamma_{\underline{C}}(\underline{G}), L)$, the transpose of $\bar{f}$

PROPOSITION. Let U be an open subset of X and $Y = X - U$. Then there is an exact sequence

$$(1) \quad \dots \longrightarrow H_1(Y; L) \xrightarrow{f_*} H_1(X; L) \xrightarrow{j_1^{XU}} H_1(U; L) \longrightarrow H_{1-1}(Y; L) \longrightarrow \dots$$

Let K be the grating of sections with compact supports of a fundamental sheaf for cohomology on X having local units. Then K_U and K/K_U have the same property with respect to U and $X - U$. Since A is injective, the exact sequence $0 \longrightarrow K_U \longrightarrow K \longrightarrow K/K_U \longrightarrow 0$ yields an exact sequence

$$(2) \quad 0 \longrightarrow \text{Hom}(K/K_U, A) \longrightarrow \text{Hom}(K, A) \longrightarrow \text{Hom}(K_U, A) \longrightarrow 0$$

and (1) will be the derived homology sequence of (2).

5.3. We finally remark that it is also possible to define homology with coefficients in a sheaf, but we shall not enter into details here, since this would require some modifications of the constructions made in §3, for instance, in order to get a torsion free fundamental sheaf for homology (a property which the sheaf obtained in §4 clearly has). If this is so, then the condition $L - n$ implies $H_1(\Gamma(\underline{C} \otimes \underline{B})) = H^{n-1}(X; H_n(\underline{C}) \otimes \underline{B})$, for any sheaf $\underline{B}$ by the same proof as for 2.2. Also, using injective sheaves, one can dispense with the local units in 5.1. For all this, see a forthcoming joint paper of J. C. Moore and the author.

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III: PERIODIC MAPS VIA SMITH THEORY

E. E. Floyd

§1. The Leray spectral sequence

It will be supposed throughout that X is a locally compact Hausdorff space and that G is a finite group acting on X . There is then the space of orbits X/G and the orbit map $\pi : X \longrightarrow X/G$. The map π is open and proper $\pi^{-1}\pi x$ is the orbit Gx for $x \in X$. The main purpose here is to put the Smith theory of periodic maps [12] in the framework of sheaf theory.

Fix a paracompactifying family Φ (see I-5.2) which also has $U_{g \in G} M \in \Phi$ whenever $M \in \Phi$. For example, Φ may be the family of compact subsets of X ; if X is paracompact, Φ may be the family of closed subsets of X . There results a family Φ^* for X/G , where $N \in \Phi^*$ if $\pi^{-1}N \in \Phi$. It may be verified that Φ^* is a paracompactifying family for X/G . To simplify notation, we use $H^*(X; \underline{A})$ for $H_\Phi^*(X; \underline{A})$ and $H^*(X/G; \underline{A})$ for $H_{\Phi^*}^*(X/G; \underline{A})$, where $\underline{A}$ is a sheaf on X or X/G .

There is the Leray spectral sequence for π [2, XXI], which has as E_2 -term

$$E_2^{p,q} = H^p(X/G, H^q(\pi^{-1}y; L))$$

and whose E_∞ -term is associated with $H^*(X; L)$. Since $E_2^{p,q} = 0$ for $q \neq 0$ (since $\pi^{-1}y$ is finite), then $E_2^{p,0} \cong E_\infty^{p,0} = H^p(X; L)$. That is, denoting by $\underline{A}$ the sheaf $U_y H^0(\pi^{-1}y; L)$, $H^p(X/G; \underline{A}) \cong H^p(X; L)$.

We need an explicit form for the isomorphism $H^p(X/G; \underline{A}) \cong H^p(X; \underline{A})$. There is $\pi : X \longrightarrow X/G$; there is also a sheaf homomorphism from $\underline{A} = U_y H^0(\pi^{-1}y; L)$ to the constant sheaf $X \times L \cong U_{x \in X} H^0(x; L)$ which is compatible with π in the sense of Cartan [2, XIV 7]. This is the collection $\varphi = \{\varphi_x\}$ of homomorphisms $H^0(\pi^{-1}\pi x; L) \longrightarrow H^0(x; L)$ induced by inclusion. The isomorphism is induced by the pair π, φ .

Each $H^0(\pi^{-1}y; L)$ contains in a natural way the ring L ; that

is, the constant sheaf $X/G \times L$ may be considered as embedded in $\underline{A}$. The inclusion $X/G \times L \subset \underline{A}$ induces a homomorphism $H^p(X/G; L) \longrightarrow H^p(X/G; \underline{A})$; commutativity holds in

$$\begin{array}{ccc} & H^p(X/G; L) & \\ \swarrow & & \searrow \pi^* \\ H^p(X/G; \underline{A}) & = & H^p(X; L) \end{array} .$$

We may consider $\underline{A}$ as given by the presheaf $U \longrightarrow \underline{A}(U) = H^0(\pi^{-1}U; L)$ for compact neighborhoods U . Since G operates on $\pi^{-1}U$, G also operates on $\underline{A}(U)$, and hence on the sheaf $\underline{A}$. On the stalk $\underline{A}_y = H^0(\pi^{-1}y; L)$, $g : \underline{A}_y \longrightarrow \underline{A}_y$ is just the homomorphism $(g \mid \pi^{-1}y)^* : H^0(\pi^{-1}y; L) \longrightarrow H^0(\pi^{-1}y; L)$. We regard G as operating on the right, so that $(\gamma g)g' = \gamma(gg')$ for $\gamma \in \underline{A}$.

There are the commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow \pi & & \downarrow \pi \\ X/G & \longrightarrow & X/G \end{array} \qquad \begin{array}{ccc} X \times L & \xleftarrow{\bar{g}} & X \times L \\ \uparrow \varphi & & \uparrow \varphi \\ \underline{A} & \xleftarrow{g} & \underline{A} \end{array}$$

of maps and compatible sheaf homomorphisms, where on given stalks, the sheaf homomorphisms are

$$\begin{array}{ccc} H^0(x; L) & \xleftarrow{g^*} & H^0(gx; L) \\ \uparrow \varphi_x & & \uparrow \varphi_{gx} \\ H^0(\pi^{-1}\pi x; L) & \xleftarrow{g_*} & H^0(\pi^{-1}\pi gx; L) \end{array} .$$

Hence there is the commutative diagram

$$(1.1) \quad \begin{array}{ccc} H^p(X; L) & \xleftarrow{g^*} & H^p(X; L) \\ \uparrow & & \uparrow \\ H^p(X/G; \underline{A}) & \xleftarrow{\quad} & H^p(X/G; \underline{A}) \end{array}$$

where the bottom homomorphism is induced by $g : \underline{A} \longrightarrow \underline{A}$.

§2. Transfer homomorphisms

We prove some results special cases of which have been given by Eckmann [4] and Liao [9], and the general case by Conner [3].

2.1. THEOREM. Suppose the finite group G operates on the locally compact Hausdorff space X . There is a homomorphism $\mu : H^p(X; L) \longrightarrow H^p(X/G; L)$ such that in the diagram

$$H^p(X; L) \begin{array}{c} \xrightarrow{\mu} \\ \xleftarrow[\pi]{*} \end{array} H^p(X/G; L)$$

we have $\pi^* \mu(\gamma) = \sum_{g \in G} g^* \gamma$, $\mu \pi^*(\gamma) = (\text{order } G) \gamma$.

PROOF. Consider the embedding of the constant sheaf $X/G \times L$ in $\underline{A}$. If we denote by $\underline{A}^G$ the subset of $\underline{A}$ consisting of elements fixed under all operations of G on $\underline{A}$, it is seen that $X/G \times L = \underline{A}^G$. Hence $H^n(X/G; \underline{A}^G) = H^n(X/G; L)$. The homomorphisms

$$\underline{A}^G \begin{array}{c} \xrightarrow{j} \\ \xleftarrow[\sigma]{} \end{array} \underline{A}$$

where j is inclusion and $\sigma = \sum g$, induce

$$H^n(X/G; L) \begin{array}{c} \xrightarrow{\pi^*} \\ \xleftarrow[\mu]{} \end{array} H^n(X; L) .$$

The conclusions follow readily.

Note also that μ is canonical. That is, suppose X_1, G, ϕ_1 and X_2, G, ϕ_2 consist of locally compact spaces X_i , actions of G on X_i , and admissible paracompactifying families ϕ_i . Suppose $f : X_1 \longrightarrow X_2$ is equivariant (that is, $fg = gf$ for all $g \in G$) and that $N \in \phi_2$ implies $f^{-1}N \in \phi_1$. There is an induced map $\bar{f} : X_1/G \longrightarrow X_2/G$ with $\bar{f}(Gx_1) = G(fx_1)$. Then commutativity holds in

$$(2.2) \quad \begin{array}{ccc} H^p(X_2; L) & \xrightarrow{\mu} & H^p(X_2/G; L) \\ \downarrow f^* & & \downarrow \bar{f}^* \\ H^p(X_1; L) & \xrightarrow{\mu} & H^p(X_1/G; L) . \end{array}$$

commutativity follows from a commutative diagram of maps and sheaf homomorphisms.

A proof of a general form of the following corollary may be found in Grothendieck [7, 202].

2.3. COROLLARY. Suppose that L is a field of characteristic 0 or of characteristic prime to order G . Then

$$H^n(X/G; L) \cong [H^n(X; L)]^G.$$

PROOF. Consider the element $r = \text{order } G \in L$. By hypothesis, $r \neq 0$. Let $\mu'' = r^{-1}\mu$. Then, in 2.1, $\pi^*\mu'' = r^{-1}\pi g^*$ and $\mu''\pi^* = \text{identity}$. Since $\mu''\pi^* = \text{id}$, π^* is an isomorphism onto. Since $g^*(\pi^*(\gamma)) = (\pi g)^*\gamma = \pi^*\gamma$, $\text{Image } \pi^* \subset [H^n(X; L)]^G$. Since $\gamma' \in [H^n(X; L)]^G$ implies $\pi^*\mu''(\gamma') = \gamma'$, $\text{Image } \pi^* \supset [H^n(X; L)]^G$. The corollary follows.

In a later section, we use a slight modification of 2.1, a form of the transfer [6]. Suppose K is a subgroup of G . Then K acts on X and there is the orbit map $\pi_1 : X \rightarrow X/K$. There is also the unique map $\pi_2 : X/K \rightarrow X/G$ with $\pi_2\pi_1 = \pi$; π_2 maps an orbit Kx of X/K into the orbit Gx of X/G .

2.4. PROPOSITION. There is a homomorphism

$\mu' : H(X/K; L) \rightarrow H(X/G; L)$ such that $\mu'\pi_2$ is multiplication by order $G/\text{order } K$.

PROOF. From the Leray sequence for the finite-to-one map $\pi_2 : X/K \rightarrow X/G$, we have $H^*(X/G, H^0(\pi_2^{-1}y; L)) \cong H^*(X/K; L)$. Note now that $\pi_2^{-1}y = \pi_1\pi_1^{-1}y = \pi_1^{-1}y/K$. There is the sheaf homomorphism $\Theta : \bigcup_y H^0(\pi_2^{-1}y; L) \rightarrow \underline{A} = \bigcup_y H^0(\pi_1^{-1}y; L)$ defined on each stalk by the homomorphism $H^0(\pi_2^{-1}y; L) \rightarrow H^0(\pi_1^{-1}y; L)$ induced by $\pi_1 : \pi_1^{-1}y \rightarrow \pi_2^{-1}y$. We must compute the image of this homomorphism. Since $\pi_2^{-1}y$ and $\pi_1^{-1}y$ are finite, the map $H^0(\pi_2^{-1}y; L) \rightarrow H^0(\pi_1^{-1}y; L)$ may be identified with the cochain map $\pi_1^* : C^0(\pi_2^{-1}y; L) \rightarrow C^0(\pi_1^{-1}y; L)$, where C^0 denotes the usual 0-dimensional cochain group. It is seen that this is an isomorphism into, and has image $[H^0(\pi_1^{-1}y; L)]^K$. Hence $\bigcup H^0(\pi_2^{-1}y; L) \cong \underline{A}^K$ and $H^*(X/G; \underline{A}^K) \cong H^*(X/K; L)$.

We define now $\sigma' : \underline{A}^K \rightarrow \underline{A}^G$. For this, it is sufficient to consider any sheaf $\underline{A}$ upon which G operates (say, on the right). Suppose $g_1, \dots, g_k$ ($k = \text{order } G/\text{order } K$) is a complete set of representatives of right cosets of K in G . Define $\sigma' : \underline{A}^K \rightarrow \underline{A}$ by $\sigma'(\gamma) = \gamma g_1 + \dots + \gamma g_k$. It is clear that σ' is independent of the choice of $g_1, \dots, g_k$. Moreover,

$$(\sigma'\gamma)g = \gamma g_1 g + \dots + \gamma g_k g.$$

Since $g_1g, \dots, g_kg$ is also a complete set of representatives of right cosets of K , $(\sigma'\gamma)g = \sigma\gamma$. Hence $\text{Image } \sigma' \subset \underline{A}^G$. There are then the diagrams

$$\underline{A}^G \xrightarrow[\sigma']{j} \underline{A}^K,$$

and the induced diagram

$$H^n(X/G; L) = H^n(X/G; \underline{A}^G) \xrightarrow[\longrightarrow]{\mu'} H^n(X/G; \underline{A}^K) = H^n(X/K; L)$$

yields the stated property.

§3. An Euler characteristic formula

Consider now a homeomorphism $g : X \longrightarrow X$ with $g^p = \text{identity}$, p prime. Then the cyclic group $Z_p = \{1, g, \dots, g^{p-1}\}$ acts on X . Since Z_p has no nontrivial subgroups, the isotropy subgroups are either 1 or Z_p . Denoting by F the set of fixed points of g , then Z_p operates freely on $X - F$. We first consider the case in which F is empty.

3.1. LEMMA. Suppose that $G = Z_p$, p prime, operates freely on the space X , and that the coefficient ring L is Z_p . Suppose that $\underline{A}$ is the sheaf on X/G of Section 1. If g is a generator of Z_p and $\tau = 1 - g$ is the operator of $\underline{A}$, then the sequence

$$\underline{A} \supset \tau \underline{A} \supset \tau^2 \underline{A} \supset \dots \supset \tau^{p-1} \underline{A} \supset 0$$

has $\underline{A}/\tau \underline{A}$, $\tau \underline{A}/\tau^2 \underline{A}$, $\dots$, $\tau^{p-1} \underline{A}$ all isomorphic to the constant sheaf $X/G \times Z_p$.

PROOF. The proof of this lemma is obtained by the techniques of Smith and Richardson [11]. Note that $\tau^{p-1} = (1 - g)^{p-1} = 1 + g + \dots + g^{p-1} = \sigma$ and $\tau^p = 0$. Now $\text{Kernel } \tau = \underline{A}^G$, since g generates Z_p . Since Z_p operates freely, this is also seen to be $\text{Image } \sigma : \underline{A} \longrightarrow \underline{A}$. Hence $\text{Kernel } \tau = \text{Image } \tau^{p-1}$. Suppose it has been shown that $\text{Kernel } \tau^i = \text{Image } \tau^{p-i}$, and consider the proposition $\text{Kernel } \tau^{i+1} = \text{Image } \tau^{p-i-1}$. Since $\tau^p = 0$, $\text{Image } \tau^{p-i-1} \subset \text{Kernel } \tau^{i+1}$. Suppose $\tau^{i+1}(r) = 0$. Then $\tau^i(\tau r) = 0$ and $\tau r = \tau^{p-1} r'$. Hence $\tau(r - \tau^{p-i-1} r') = 0$ and $r - \tau^{p-i-1} r' = \tau^{p-1} r''$. Hence $r = \tau^{p-i-1}(r' + \tau^i r'')$. Hence for $1 \leq i \leq p-1$, $\text{Kernel } \tau^i = \text{Image } \tau^{p-i}$; in particular, $\text{Kernel } \tau^{p-1} = \text{Image } \sigma$.

Consider now the homomorphism τ^{p-1-1} of $\tau^1 \underline{A}$ onto $\tau^{p-1} \underline{A}$. If $\tau^1 r$ is in the kernel, then $\tau^{p-1-1}(\tau^1 r) = \tau^{p-1} r = 0$ so that $r = \tau r^1$. Hence the kernel is $\tau^{1+1} \underline{A}$ and $\tau^1 \underline{A} / \tau^{1+1} \underline{A} \sim \tau^{p-1} \underline{A} \sim \underline{A}^G \sim X/G \times Z_p$.

A slightly different proof of the following may be found in [5].

3.2. THEOREM. Suppose the cyclic group $G = Z_p$, p prime, operates freely on X , where $\dim_{Z_p} X < \infty$ and $H^*(X; Z_p)$ is finite dimensional. Then $H^*(X/G; Z_p)$ is finite dimensional and $x(X; Z_p) = p \cdot x(X/G; Z_p)$ where $x(Y; Z_p)$ denotes $\sum (-1)^n \dim H^n(X; Z_p)$.

PROOF. Let $\underline{A}^1 = \tau^1 \underline{A}$, $0 \leq 1 \leq p-1$. There is the filtration

$$\underline{A} = \underline{A}^0 \supset \underline{A}^1 \supset \dots \supset \underline{A}^{p-1} \supset 0$$

of the sheaf $\underline{A}$. Given such a finite filtration of the sheaf $\underline{A}$, there is a spectral sequence $(E_r^{s,t})$ with $E_1^{s,t} = H^{s+t}(X/G; \underline{A}^s / \underline{A}^{s+1})$ and with E_∞ associated with a filtration of $H^*(X/G; \underline{A}) = H^*(X; Z_p)$. By 3.1, $E_1^{s,t} = H^{s+t}(X/G; Z_p)$ for $0 \leq s \leq p-1$. Of course, $E_1^{s,t} = 0$ for $s < 0$ or $s \geq p$. If $H^*(X/G; Z_p)$ is finite dimensional then we have

$$x(E_1) = x(E_2) = \dots = x(E_\infty) = x(X; Z_p)$$

and $x(E_1) = \sum (-1)^{s+t} \dim E_1^{s,t} = p \cdot x(X/G; Z_p)$.

Hence it is sufficient to prove $H^*(X/G; Z_p)$ finite dimensional. Now $\pi: X \rightarrow X/G$ is a local homeomorphism. Since $\dim_{Z_p} X < \infty$ and since dimension is a local property (see 1-2.3), $\dim_{Z_p} X/G < \infty$. Suppose now that it has been shown that $\dim H^j(X/G; Z_p) < \infty$ for all $j > n$. Then $\dim E_1^{s,t} < \infty$ for $s+t > n$ and $\dim E_r^{s,t} < \infty$ for $s+t > n$. Now $E_{r+1}^{0,n}$ is Kernel $d_r^{0,n}; E_r^{0,n} \rightarrow E_r^{r,n-r+1}$. Since $E_r^{r,n-r+1}$ is finite dimensional, then $\dim E_{r+1}^{0,n} < \infty$ if $\dim E_r^{0,n} < \infty$. Since $\dim E_\infty^{0,n} < \infty$, it follows that $\dim E_1^{0,n} = \dim H^n(X/G; Z_p) < \infty$. The induction follows.

It should be noted that the finite dimensionality of $H^n(X/G; Z_p)$ follows more generally from the Cartan-Leray spectral sequence [1]; it has been included here for completeness.

§4. The Smith sequences

Here we confine ourselves to the action of a cyclic group $G = Z_p$, p prime, on a locally compact Hausdorff space X . Denote by $\underline{A}$ the sheaf of Section 1, by g a generator of Z_p , and by τ, σ the endomorphisms

$1 - g, 1 + g + \dots + g^{p-1}$ of $\underline{A}$. If one of these is denoted by ρ , the other will be denoted by $\bar{\rho}$. The coefficient ring L is assumed to be Z_p . Denote by F the set of stationary points of Z_p ; F is also the set of fixed points of g . If $f : \underline{A} \longrightarrow X/G$ is the natural map of the sheaf, then $\underline{A}_U$ (U open in X/G) denotes $f^{-1}U \cup \{0_y : y \in X/G\}$ and $\underline{A}_M$ (M closed) denotes $\underline{A}/\underline{A}_{X-M}$.

We develop a cohomology form of the special homology groups of P. A. Smith. These were originally defined by Smith (for references, see [12]) and used by Smith, Richardson [10], Liao [9], and the author [5] to develop a theory of periodic maps. We confine ourselves here to a brief global treatment, as the topics will be dealt with later using the spectral sequences of Cartan-Leray and Borel [1].

4.1. LEMMA. The sequence

$$0 \longleftarrow \bar{\rho}\underline{A} \longrightarrow \underline{A} \xrightarrow{\rho \oplus \eta} \rho\underline{A} \oplus \underline{A}_F \longrightarrow 0$$

is exact, where $\underline{A} \longrightarrow \underline{A}$ is inclusion and where $\eta : \underline{A} \longrightarrow \underline{A}_F$ is the quotient homomorphism.

PROOF. It is sufficient to verify exactness on each stalk. For $y \in X/G - F$, this reduces to exactness of

$$0 \longrightarrow \bar{\rho}A_y \longrightarrow A_y \longrightarrow \rho A_y \longrightarrow 0$$

which has been shown in the proof of 3.1. For $y \in F$, it is seen that $\bar{\rho}A_y = \rho A_y = 0$, so that exactness reduces to exactness of

$$0 \longrightarrow 0 \longrightarrow A_y \longrightarrow A_y \longrightarrow 0.$$

4.2. DEFINITION. The groups $H^n(X/G; \rho\underline{A})$ will be called the Smith special cohomology groups. When no confusion results, denote them simply by $H^n(\rho)$. The exact cohomology sequence

$$\dots \rightarrow H^n(X/G; \bar{\rho}\underline{A}) \rightarrow H^n(X/G; \underline{A}) \rightarrow H^n(X/G; \rho\underline{A}) \oplus H^n(X/G; \underline{A}_F) \rightarrow \dots$$

of (4.1) becomes

$$\dots \rightarrow H^n(\bar{\rho}) \rightarrow H^n(X; Z_p) \rightarrow H^n(\rho) \oplus H^n(F; Z_p) \rightarrow H^{n+1}(\bar{\rho}) \rightarrow \dots$$

and will be called a Smith sequence.

Note that $\rho\underline{A}$ is the constant sheaf $(X/G \times Z_p)_{X/G-F}$ so that

$$H^n(\sigma) = H_{\phi*}^n(X/G - F; Z_p).$$

Theorems of the following type have been proved in [5], and by Heller [8].

4.3. THEOREM. If $G = Z_p$, p prime, acts on X , where $\dim_{Z_p} < \infty$, then

$$\dim H^n(X/G - F; Z_p) + \sum_{i \geq n} \dim H^i(F; Z_p) \leq \sum_{i \geq n} \dim H^i(X; Z_p)$$

for any integer n .

PROOF. Consider the Smith sequences

$$(n) \quad H^n(X; Z_p) \longrightarrow H^n(\sigma) + H^n(F; Z_p) \longrightarrow H^{n+1}(\tau)$$

$$(n+1) \quad H^{n+1}(X; Z_p) \longrightarrow H^{n+1}(\tau) + H^{n+1}(F; Z_p) \longrightarrow H^{n+2}(\sigma)$$

$$(n+2) \quad H^{n+2}(X; Z_p) \longrightarrow H^{n+2}(\sigma) + H^{n+2}(F; Z_p) \longrightarrow H^{n+3}(\tau)$$

$$(n+2i) \quad H^{n+2i}(X; Z_p) \longrightarrow H^{n+2i}(\sigma) + H^{n+2i}(F; Z_p) \longrightarrow H^{n+2i+1}(\tau) \quad .$$

Since $A \longrightarrow B \longrightarrow C$ exact implies $\dim B \geq \dim A + \dim C$, we obtain

$$[n] \quad \dim H^n(\sigma) + \dim H^n(F; Z_p) \leq \dim H^n(X; Z_p) + \dim H^{n+1}(\tau)$$

$$[n+1] \quad \dim H^{n+1}(\tau) + \dim H^{n+1}(F; Z_p) \leq \dim H^{n+1}(X; Z_p) + \dim H^{n+2}(\sigma)$$

$$\dim H^{n+2}(\sigma) + \dim H^{n+2}(F; Z_p) \leq \dim H^{n+2}(X; Z_p) + \dim H^{n+3}(\tau)$$

$$[n+2i] \quad \dim H^{n+2i}(\sigma) + \dim H^{n+2i}(F; Z_p) \leq \dim H^{n+2i}(X; Z_p) + \dim H^{n+2i+1}(\tau)$$

If $\dim H^{n+2i+1}(\tau) < \infty$, it is seen by going backwards step by step that all terms are finite. If $n+2i > \dim_{Z_p} X$, then all terms of $[n+2i]$ are 0. The terms $H^{n+2i}(\sigma)$ are 0 since $\dim_{Z_p}(X/G - F) \leq \dim(X - F)$ and since ρ_A is essentially a sheaf on $X/G - F$. The conclusion follows by addition and cancellation.

4.4 THEOREM. Suppose that Z_p , p prime, acts on X , where $\dim_{Z_p} X < \infty$ and where $\dim H^*(X; Z_p) < \infty$. Then $\dim H^*(F; Z_p) < \infty$ and $\chi(X; Z_p) = \chi(F; Z_p) \pmod{p}$.

PROOF. It follows from 4.3 that $\dim H^n(F; Z_p) < \infty$. From the

cohomology sequence

$$(1) \quad H_{\Phi}^n(F; Z_p) \longrightarrow H_{\Phi}^{n+1}(X - F; Z_p) \longrightarrow H_{\Phi}^{n+1}(X; Z_p)$$

we see that $H_{\Phi}^{n+1}(X - F; Z_p)$ is finite dimensional for all n . Hence, by 3.2, $x_{\Phi}(X - F; Z_p)$ is divisible by p .

Now from the cohomology sequence (1),

$$\begin{aligned} x_{\Phi}(X; Z_p) &= x_{\Phi}(F; Z_p) + x_{\Phi}(X - F; Z_p) \\ &= x_{\Phi}(F; Z_p) \pmod{p} . \end{aligned}$$

We obtain now the well-known Smith theorems [12], as well as a dimensional parity result [5].

4.5. COROLLARY. Suppose that Z_p, p prime, acts on X , where $\dim_{Z_p} X < \infty$ and where $H_{\Phi}^*(X; Z_p) \cong H^*(S^n; Z_p)$. There exists an $-1 \leq r \leq n$ such that $H_{\Phi}^*(F; Z_p) = H^*(S^r; Z_p)$, where the (-1) -sphere is interpreted as the empty set. If p is odd, then $n - r$ is even.

4.6. COROLLARY. Suppose that Z_p, p prime, acts on X , where $\dim_{Z_p} X < \infty$ and where $H^1(X; Z_p) = 0$ for

$1 > 0$, $H^0(X; Z_p) \cong Z_p$. Then F is not empty and $H^1(F; Z_p) = 0$ for $1 > 0$, $H^0(F; Z_p) \cong Z_p$.

Both of the corollaries follow easily from (4.3) and (4.4).

§5. Orbit spaces of finite groups

5.1 PROPOSITION. If the finite group G operates on the locally compact Hausdorff space X , then $\dim_L X/G = \dim_L X$.

PROOF. Note first that if A is a closed subset of a space X , then $\dim_L X = \max \{\dim_L A, \dim_L (X - A)\}$. For if U is open in X we have the exact sequence

$$\dots \longrightarrow H_G^n(U - A \cap U) \longrightarrow H_G^n(U) \longrightarrow H_G^n(A \cap U) \longrightarrow \dots$$

where coefficients are in L . Hence the equality holds.

Let $n = \text{order } G$, and let A_1 denote the set of all $x \in X$ with order $G_x \geq 1$, where G_x denotes the isotropy subgroup. Then

$$X = A_1 \supset A_2 \supset \dots \supset A_n \supset A_{n+1} = \emptyset$$

is a sequence of closed subsets of X ; each A_1 is G -invariant, since $Gg_x = gG_xg^{-1}$. Moreover, $A_1 - A_{1+1} = \bigcup_H B_H$ where the summation is over all subgroups H of order 1 and where $B_H = \{x : G_x = H\}$. The B_H are pairwise disjoint and open in $A_1 - A_{1+1}$; also the orbit map $\pi : X \longrightarrow X/G$ is a local homeomorphism on each B_H and hence of $A_1 - A_{1+1}$ onto $\pi A_1 - \pi A_{1+1}$. Hence, since dimension is a local property, $\dim_L(A_1 - A_{1+1}) = \dim_L(\pi A_1 - \pi A_{1+1})$. Moreover π is a homeomorphism of A_n onto πA_n . Inductively, suppose $\dim_L A_{1+1} = \dim_L \pi A_{1+1}$. Then

$$\begin{aligned} \dim_L \pi A_1 &= \max (\dim_L \pi A_{1+1}, \dim_L (\pi A_1 - \pi A_{1+1})) \\ &= \max (\dim_L A_{1+1}, \dim_L (A_1 - A_{1+1})) \\ &= \dim_L A_1. \end{aligned}$$

For $i = 1$ we receive $\dim_L X = \dim_L X/G$. References to similar theorems for compact Lie groups G of transformations may be found in Yang [13].

5.2. THEOREM. Suppose that a finite group G acts on X , where $\dim_L X < \infty$ and $H^1(X; L)$ is finitely generated (respectively trivial) for all $i \geq N$, where L is either the ring of integers $\mathbb{Z}$ or a field K_p of characteristic p . Then $H^i(X/G; L)$ is finitely generated (respectively trivial) for all $i \geq N$.

PROOF. We prove the theorem first in the case $G = \mathbb{Z}_p$ (p prime). If $L = K_q$ ($q \neq p$), then the assertions follow from 2.3. If $L = K_p$, they follow from 4.3. Consider now the case $L = \mathbb{Z}$.

From the sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow 0$, p prime, we obtain

$$\dots \longrightarrow H^n(X; \mathbb{Z}) \longrightarrow H^n(X; \mathbb{Z}_p) \longrightarrow H^{n+1}(X; \mathbb{Z}) \longrightarrow \dots$$

Hence $H^n(X; \mathbb{Z})$ and $H^{n+1}(X; \mathbb{Z})$ finitely generated [trivial] implies $H^n(X; \mathbb{Z}_p)$ is finitely generated [trivial]. By 4.3, $H^n(X/G; \mathbb{F}; \mathbb{Z}_p)$ and $H^n(\mathbb{F}; \mathbb{Z}_p)$ are finitely generated [trivial], and hence so is $H^n(X/G; \mathbb{Z}_p)$. Consider the exact sequence

$$\begin{array}{ccccc}
 \dots & \longrightarrow & H^n(X/G; Z) & \xrightarrow{p} & H^n(X/G; Z) & \xrightarrow{q} & H^n(X/G; Z_p) \\
 & & \searrow \pi^* & & \uparrow \mu & & \\
 & & & & H^n(X; Z) & &
 \end{array}$$

of cohomology $0 \longrightarrow Z \longrightarrow Z \longrightarrow Z_p \longrightarrow 0$, to which the maps of 2.1 have been added. Since $H^n(X; Z)$ is finitely generated [trivial], so is Image p . Since $H^n(X/G; Z_p)$ is finitely generated [trivial], so is Image q . Since Kernel q and Image q are finitely generated [trivial], so is $H^n(X/G; Z)$. Hence the theorem is true for $G = Z_p$, p prime.

Suppose now that G is finite solvable of non-prime order r , and that the theorem has been proved for all solvable groups of order $< r$. Then G contains a non-trivial normal subgroup K , which is solvable since G is. Moreover, G/K is solvable, and K and G/K are of order $< r$. Now K acts on X and there is the orbit map $\pi_1 : X \longrightarrow X/K$. A point Kx of X/K is acted upon by G , since $g(Kx) = Kgx \in X/K$. Under this action of G on X/K , K acts trivially on X/K . Hence G/K acts on X/K . It may be seen that we can identify X/G with $(X/K)/(G/K)$. Let $\pi_2 : X/K \longrightarrow X/G$ be the natural map. By induction hypothesis, $H^i(X/K; L)$ is finitely generated [trivial] for $i \geq N$ as then also is $H^i(X/G; L)$. Hence the theorem follows for solvable finite groups G .

Suppose now that G is an arbitrary finite group, with order $G = p_1^{n_1} \dots p_k^{n_k}$. There are then Sylow subgroups $K_1, \dots, K_k$ of G with order $K_j = p_j^{n_j}$. Since K_j is solvable, $H^i(X/K_j; Z)$ is finitely generated [trivial] for $i \geq N$. We now use a device due to C. N. Lee (unpublished). There are the maps

$$X \xrightarrow{\pi_{1,j}} X/K \xrightarrow{\pi_{2,j}} X/G$$

whose composition is π . Hence there is

$$\pi_\sigma = \pi_{2,1}^* + \dots + \pi_{2,k}^* : H^*(X/G; L) \longrightarrow H^*(X/K_1; L) + \dots + H^*(X/K_k; L).$$

Note for future reference that π_σ is canonical. That is, if $f : X \longrightarrow X'$ where X' is locally compact, where G operates on X' and f is equivariant, and where f is appropriately related to the support families for X and X' , then commutativity holds in

$$\begin{array}{ccc}
 H^*(X'/G; L) & \xrightarrow{\pi_\sigma} & \Sigma H^*(X'/K_1; L) \\
 \downarrow & & \downarrow \\
 H^*(X/G; L) & \xrightarrow{\pi_\sigma} & \Sigma H^*(X/K_1; L)
 \end{array}$$

where the vertical homomorphisms are induced by $f : X \longrightarrow X'$.

We show now that π_σ is injective. If $\gamma \in H^*(X/G; L)$ is in kernel π_σ then $\pi_{2,1}^*(\gamma) = 0$ (all 1). Hence by 2.3

$$(\text{order } G / \text{order } K_1) \gamma = \begin{pmatrix} n_1 & & n_{1-1} & n_{1+1} & & n_k \\ p_1 & \dots & p_{1-1} & p_{1+1} & \dots & p_k \end{pmatrix} \gamma = 0$$

for all 1. It follows that $\gamma = 0$. Hence π_σ is injective for an arbitrary coefficient group. Since for $i \geq N$, each $H^i(X/K_j; L)$ is finitely generated [trivial], so is $H^i(X/G; L)$.

5.3. COROLLARY. Suppose that a finite solvable group G operates freely on X , where $\dim_{\mathbb{Z}} X < \infty$ and $H^*(X; \mathbb{Z})$ is finitely generated. Then $H^*(X/G; \mathbb{Z})$ is finitely generated, and $x(X/G)$, $x(X)$ are independent of the field used to define them. Moreover $x(X) = (\text{order } G)x(X/G)$.

PROOF. That $H^*(X/G; \mathbb{Z})$ is finitely generated follows from 5.2. That $x(X)$, $x(X/G)$ is independent of the field follows from the universal coefficient theorem. For $G = \mathbb{Z}_p$, p prime, the theorem has been proved in 3.2. It follows in the general case by induction just as in 5.2.

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IV: THE ACTION OF Z_p OR T^1 : GLOBAL THEOREMS

A. Borel

§1. Transformation groups

We collect here for future reference some elementary facts and notation.

Let G be a topological group. A G -space is a space on which G operates (continuously of course). Then $G_x = \{g \in G, g \cdot x = x\}$ denotes the stability or isotropy group of $x \in X$, and for a subset A of X , $G \cdot A = \{g \cdot a, g \in G, a \in A\}$ is the union of the orbits $G \cdot a$ of the points of A . It is open if A is. The space of orbits is denoted by X/G , or when G is fixed once and for all, by X' , and $\pi_X : X \longrightarrow X'$ is the natural map assigning to $x \in X$ its orbit. X' is assumed to be endowed with the quotient topology, and π_X is open and continuous. The fixed point set of a subgroup H in an H -invariant subspace will be denoted $F(H; A)$.

Two orbits are said to be of the same type if their stability groups are conjugate (by inner automorphisms) in G . We say that the orbit structure is finite (respectively locally finite) if the stability groups belong to a finite number of conjugacy classes (respectively if each point has an invariant neighborhood with finite orbit structure).

When G is compact, the following facts are easily proved: X/G is Hausdorff (respectively locally compact) if X is. The natural map $G/G_x \longrightarrow G \cdot x$ is an isomorphism. If A is closed in X , $G \cdot A$ is closed, hence π_X is closed, and therefore also proper (since the orbits are compact).

1.2. Let N be a closed invariant subgroup of G . Then X/N is in an obvious way a (G/N) -space. Let $\alpha : G \longrightarrow G/N$ and $\beta : X \longrightarrow X/N$ be the natural maps. Then, clearly, $\alpha(G_x) = (G/N)_{\beta(x)}$ for any $x \in X$. In particular: if N contains all isotropy groups, G/N acts freely on X/N ; if G is connected and N is finite, then β is a homeomorphism of $F(G; X)$ onto $F(G/N; X/N)$.

1.3. Assume that G also operates on a space Y . We denote by $X \times_G Y$ the orbit space of $X \times Y$ under the "diagonal action" $g(x, y) = (g \cdot x, g \cdot y)$. In other words, it is the quotient of $X \times Y$ by the equivalence relation $(x, y) \sim (g \cdot x, g \cdot y)$. The canonical projections of $X \times Y$ onto its two factors induce projections π_1, π_2 of $X \times_G Y$ onto X/G and Y/G respectively. Let $y \in Y$ and y' be its orbit. Then it is easily seen that $\pi_2^{-1}(y')$ may be identified with X/G_y . Moreover, if Y is a locally trivial principal G -bundle, then π_2 is the projection of a locally trivial fibre bundle, with typical fibre X and structural group G . If G acts trivially on X , then $X \times_G Y = X \times (Y/G)$ and π_1, π_2 are the natural projections. There are of course analogous statements with X and Y interchanged. Here we have assume G to operate on the left on X, Y . If we are given a space E on which G operates on the right, we consider instead the left operations defined by $g \cdot x = x \cdot g^{-1}$ ($x \in E, g \in G$).

Let G^* be another topological group, $\lambda : G \longrightarrow G^*$ a homomorphism, and X^* be a G^* -space. A λ -equivariant map $f : X \longrightarrow X^*$ is a continuous map such that $f(g \cdot x) = \lambda(g) \cdot f(x)$. It induces a map $\bar{f} : X/G \longrightarrow X^*/G^*$. If $G = G^*$ and λ is the identity, then f is said to be equivariant. Let Y^* be another G^* -space and $h : Y \longrightarrow Y^*$ be a λ -equivariant map. Then $f \times h : X \times Y \longrightarrow X^* \times Y^*$ is a λ -equivariant; it induces a map $\mu : X \times_G Y \longrightarrow X^* \times_{G^*} Y^*$ and we have a commutative diagram

$$\begin{array}{ccccc} X/G & \xleftarrow{\pi_1} & X \times_G Y & \xrightarrow{\pi_2} & Y/G \\ \bar{f} \downarrow & & \mu \downarrow & & \downarrow \bar{g} \\ X^*/G^* & \xleftarrow{\pi_1} & X^* \times_{G^*} Y^* & \xrightarrow{\pi_2} & Y^*/G^* \end{array} .$$

§2. Some remarks on the cohomology of B_G

A module S graded by submodules S_i is said to be of finite type if each S_i is finitely generated.

2.1. Cohomology of finite groups. Let G be a finite group, A a module on which it operates. A^G denotes the set of elements fixed under G , and $H^*(G; A)$ is the cohomology group of G with coefficients in A (see, e.g., [4] Chap. XII for this and the results mentioned below).

(1) $H^0(G; A) = A^G$; for $i \geq 1$, we have $nH^1(G; A) = 0$, ($n = \text{ord. } G$).

(2) Let $G = Z_m$, generated by an element g . Let $NA, {}_N A, IA$ be respectively the subgroup generated by the norms $g \cdot a + g^2 \cdot a + \dots + g^m \cdot a$, by the elements of norm zero, and by the elements $g \cdot a - a$.

Then

$$H^1(G; A) = A^G/NA \quad (1 \text{ even, } \neq 0); \quad H^1(G; A) = N^A/IA \quad (1 \text{ odd}).$$

(3) Let p prime, $m = p^s$, and $A = Z_p$. Then, clearly, G acts trivially on A . We have

$$H^*(Z_2; Z_2) = Z_2[a] \quad (d^0 a = 1)$$

$$H^*(Z_m; Z_p) = \wedge[a] \otimes Z_p[b] \quad (d^0 a = 1, d^0 b = 2; m = p^s, m \neq 2).$$

Thus, if $m \neq 2$, the multiplication by a is injective on $H^{2i}(Z_m; Z_p)$ and annihilates $H^{2i-1}(Z_m; Z_p)$, ($i \geq 1$).

2.2. We also recall that if $G = T^k$, the k -dimensional torus, $H^*(B_G; L)$ is a ring of polynomials in k variables of degree two.

2.3. If G is a compact Lie group, $H^*(B_G; L)$ is of finite type. For G connected this follows from the fact that in the universal fibering, the cohomology of the total space and the fibre are of finite type and the sheaf of the cohomology groups $H^*(G; L)$ is constant. In the general case we use the fact that B_{G^0} (G^0 identity component of G) is a regular covering of B_G with group G/G^0 and that the cohomology of a finite group with respect to a finitely generated module is of finite type, since it can be computed by means of a chain complex of finite type.

2.4. PROPOSITION. Let $L = K_p$, A , and $G \neq (E)$.

(a) If G is finite, and $L = K_p$ with p not dividing $\text{ord } G$, then $H^*(B_G; L)$ is trivial (i.e., $H^i(B_G; L) = 0$ for $i \geq 1$).

(b) In all other cases, there are arbitrary large i such that $H^i(B_G; L) \neq 0$, and contains a non-zero free L -module when G is infinite.

PROOF (patterned after an unpublished remark of Serre about finite groups). Let M be a compact connected orientable manifold on which G acts freely, and on the cohomology of which G acts trivially. This always exists, take for instance a group $SO(n)$ in which G is embedded by means of a faithful representation. Let $n = \dim M$. Since M is a principal bundle, there is a spectral sequence leading from $E_2 = H(B_G; H^*(M; L))$ to $H^*(M/G; L)$, as will be recalled in 3.4.

Assume first $H^*(B_G; L)$ to be trivial. It follows from the spectral sequence that $\pi^*: H^*(M/G; L) \rightarrow H^*(M; L)$ is an isomorphism, hence $\dim M/G = \dim M$, and G is finite; moreover, since the Brouwer degree of π is then equal to $\text{ord } G$, π^* can be an isomorphism only if

$L = K_p$ with p not dividing $\text{ord } G$. Conversely it is known that in this case $H^*(B_G; L)$ is trivial (2.1). This proves (a) and shows that in case (b), there exists a $j \geq 1$ such that $H^j(B_G; L) \neq 0$. In the spectral sequence mentioned above, we have $E_2^{p,n} = H^p(B_G; L)$ since $H^n(M; L) = L$ and is acted upon trivially by G . Therefore if the i 's for which $H^i(B_G; L) \neq 0$ were bounded from above, the maximum cocycle argument would show the existence of some $m > n$ for which $H^m(M/G; L) \neq 0$ in contradiction with the fact that M/G is a compact manifold of dimension $\leq n$. There are therefore arbitrarily large i for which $H^i(B_G; L) \neq 0$; this ends the proof of (b) if L is a field, or if $L = \mathbb{Z}$ and G is finite. If now G is infinite, and $L = \mathbb{Z}$, the last part of (b) follows from the fact that $H^1(B_G; \mathbb{Z})$ is finitely generated and from the first part of (b) applied to the case $L = K_0$.

§3. The space X_G

3.1. Let G be a topological group. E_G and B_G denote respectively a universal G -bundle and a classifying space for G . Thus E_G is a principal G -bundle with trivial cohomology in all dimensions, and B_G is its base space (cf. for instance [1], [9]). If G is discrete, then $H^*(B_G, A)$ is the cohomology group of G in the sense mentioned in 2.1 and will accordingly be denoted by $H^*(G; A)$.

DEFINITION. Let X be a space on which G operates. Then we denote by X_G the space $X \times_G E_G$ (see 1.3) and by π_1, π_2 the natural maps of X_G on X/G and B_G respectively.

From the remarks made in 1.3, it follows that π_2 is a fibre map, in a fibration with typical fibre X and structural group G , and that if $x \in X$, $x' = \pi(x)$, then $\pi_1^{-1}(x') = E_G/G_x$, therefore $\pi_1^{-1}(x')$ may be identified with a classifying space for G_x .

In this book we are interested only in the case in which G is a compact Lie group and X is locally compact, and in order to avoid some technical and irrelevant complications we shall assume that E_G is compact but then N -universal (i.e., $H^i(E_G) = 0$ for $1 \leq i \leq N$) for some integer N . It is then understood that in all statements below, $i > \dim X$ or $i > n$ should, strictly speaking, be replaced by $\dim X \leq i \leq N$ or $n \leq i \leq N$. Since N can be chosen to be arbitrarily large, this convention will not lead to any confusion. For finite G , it could be avoided altogether by taking as E_G a G -free and acyclic complex, finitely generated in all dimensions, and by replacing X by some fundamental grating on which G operates.

3.2. Let G^* be a compact Lie group, $\lambda : G \longrightarrow G^*$ a continuou

homomorphism, Y a G^* -space, and $f : X \longrightarrow Y$ a λ -equivariant map. To λ there is associated an equivariant map μ of E_G into E_{G^*} for suitable E_G, E_{G^*} . This follows directly from Milnor's construction using joins, and can also be seen as follows: Let E_G, E_{G^*} be universal bundles. Then $E_G \times E_{G^*}$ considered as G -space under the action $g(u, v) = (g \cdot u, \lambda(g) \cdot v)$ is also a principal G -bundle and is clearly acyclic, hence may be viewed as a universal G -bundle. We then define μ as the projection on the second factor. It induces a map $\rho(\lambda) : B_G \longrightarrow B_{G^*}$. Then $f \times \mu : X \times E_G \longrightarrow Y \times E_{G^*}$ is λ -equivariant, hence defines a map $f_\lambda : X_G \longrightarrow Y_{G^*}$ and we have clearly a commutative diagram

$$\begin{array}{ccccc} B_G & \xleftarrow{\pi_2} & X_G & \xrightarrow{\pi_1} & X/G \\ \rho(\lambda) \downarrow & & \downarrow f_\lambda & & \downarrow f' \\ B_{G^*} & \xleftarrow{\pi_2} & Y_{G^*} & \xrightarrow{\pi_1} & X/G^* \end{array} .$$

If G is an invariant subgroup of G^* , λ the inclusion map, $X = Y$ and f the identity, then f and $\rho(\lambda)$ are the projection maps of principal (G^*/G) -bundles.

3.3. X_G is a bundle over B_G , with fibre X , structural group G , projection map π_2 . Assume that there is at least one fixed point a . Since it is invariant under the structural group, the fibre over $y \in B_G$ has a well defined point a_y corresponding to a under any of the allowable homomorphisms of $\pi_2^{-1}(y)$ over X , and $y \longrightarrow a_y$ is then clearly a continuous cross section. Consequently: if there is a fixed point, then $\pi_2^* : H^*(B_G; L) \longrightarrow H^*(X_G; L)$ is injective and $\text{Im } \pi_2^*$ is a direct summand for any coefficient ring L .

If a is viewed as a point of X/G , then the above cross section may be identified with $\pi_1^{-1}(a)$, therefore the natural restriction map $H^*(X_G; L) \longrightarrow H^*(\pi_1^{-1}(a); L)$ is surjective. It follows that $\text{Im } r_{X_G}^*$ contains the image of $\text{Im } r_{X/G}^* \otimes H^*(B_G; L)$ under the canonical map of $H^*(F; L) \otimes H^*(B_G; L)$ into $H^*(F \times B_G; L)$.

3.4. Applications. Assume that $H^i(B_{G_x}; L) = 0$ for $1 \leq i \leq N$ and all $x \in X$. Then by the Vietoris mapping theorem $H^i(X_G; L) = H^i(X/G; L)$ for $1 \leq i \leq N$; the same is true in cohomology with compact supports if X is locally compact. There is then a spectral sequence (that of π_2) leading to the cohomology of X/G (up to N), and in which $E_2 = H^*(B_G; H^*(X; L))$ (or $E_2 = H^*(B_G; H_c^*(X; L))$ as the case may be). We mention three cases to which this remark applies:

(a) X is a principal G -bundle. See [1] for numerous applications.

If G is finite, we get in this way the Leray-Cartan spectral sequence for regular coverings.

(b) G is a circle, without fixed points in X , and L is a field of characteristic zero. In that case G_X is finite, hence $H^*(B_{G_X}; L)$ is trivial by 2.1.

(c) G is finite, L is a field of characteristic zero or prime to $\text{ord } G$. In this case, moreover, 2.1 implies that in the above spectral sequence of π_2 we have $E_2^{p,q} = 0$ for $p \neq 0$, and $E_2^{0,q} = H^q(X; L)^G$. Hence we have $H^*(X/G; L) \cong H^*(X; L)^G$, as was already proved in III, 2.3; it would also be easy to check here that the isomorphism is induced by π_X^* .

3.5. Let Y be another space acted upon by G and $f: X \rightarrow Y$ a continuous equivariant map. Then (3.2), f induces a map $f_G: X_G \rightarrow Y_G$ compatible with the projections π_1, π_2 and inducing the identity on B_G ; it will therefore also yield homomorphisms of the corresponding spectral sequences. This applies in particular if Y is a G -invariant subspace of X , hence Y_G may then be identified with a subspace of X_G . If $Y = F$ is the set of fixed points of G in X , then we have a natural inclusion of $F_G = F \times B_G$ into X_G and the restrictions of π_1, π_2 to F_G are the canonical projections on the two factors.

3.6. PROPOSITION. Let G be a compact Lie group acting on the locally compact space X , and let F be a closed invariant subspace. Assume that for $x \notin F$, $H^1(B_G; L) = 0$ for $1 \leq i \leq M$ and that $H^1_0(X' - F'; L) = 0$ for $n < i \leq M$. Then the restriction map $H^1_c(X_G; L) \rightarrow H^1_c(F_G; L)$ is an isomorphism for $n < i < M$.

Let $U = X - F$ and $U' = X' - F'$ be its image in X' . Then we can apply 3.4 to $\pi_2: U_G \rightarrow U'$, which implies that $H^1_c(U_G; L) = H^1_c(U'; L)$ for $1 \leq i \leq M$, hence $H^1_c(U_G; L) = 0$ for $n < i \leq M$. Clearly $U_G = X_G - F_G$ and the proposition now follows by use of the cohomology sequence with compact carriers for $X_G \bmod F_G$.

3.7. Applications. The assumptions of 3.6 are fulfilled in each of the following cases:

(a) $\dim_L X = n$, G acts freely outside F (because $\dim_L X' - F' \leq \dim_L X - F$). This follows here simply from the fact that each point $y \in X' - F'$ has a closed neighborhood homeomorphic to a closed subset of X . For a more general result see [10] or (VIII, 3.16.).

(b) $G = L = \mathbb{Z}_p$ (p prime), $\dim_p X = n$, and F the fixed point set; because then G acts freely outside F , and we are in case (a).

(c) F is a circle, L a field of characteristic zero, $\dim_L X = n$, and F the fixed point set. The assumption on $H^*(B_G; L)$ follows from 3.4 (b). The other one from the fact that $\dim_L X' \leq \dim_L X$ [10].

3.8. Proposition 3.6 is also valid for a not necessarily locally compact space in $\ast$ -cohomology, where $\ast$ is a paracompactifying family containing the saturation of its elements. The proof uses the Vietoris mapping theorem for $\ast$ -cohomology and proper maps, and the cohomology sequence in $\ast$ -cohomology, but is otherwise the same. The assumption on $H^*(X' - F)$ must then be made for $\ast$ -cohomology where $\ast$ is the image of $\ast$ (which is paracompactifying under the assumption made on $\ast$).

3.9. REMARK. All our discussion will center around the space X_G , and the remarks 3.6, 3.7 will be basic. Similar arguments have been used by Conner [5] when G is a circle, in rational cohomology. For an algebraic analogue when G is discrete, see [7, Chap. V]. The space X_G and the embedding $F(X; G) \times B_G \subset X_G$ were also mentioned to the author by A. Shapiro. The proof of Smith's theorem 4.3 is also related to that of [2].

§4. The fixed point set of a prime period map in a cohomology sphere

The present section is concerned with results already proved in III, except for 4.5, 4.7, and is meant mainly as an illustration of the method underlying this lecture. In this paragraph, p is a fixed prime. Unless otherwise stated, cohomology is taken with respect to Z_p and coefficients are omitted. We also write $H^*(G; A)$ for $H^*(B_G; A)$ when G is finite. Except in 4.5, 4.6, G is a cyclic group of prime order p acting effectively on a space X , F its fixed point set and (E_r) the spectral sequence of the fibering $\pi_2 : X_G \longrightarrow B_G$.

4.1. Let $\dim_{Z_p} X = n$. Then $\dim H^*(X) \geq \dim H^*(F) = \dim H^1(X_G)$ ($1 \geq n + 1$). (The first inequality is contained in III, 4.3). By 3.7 (b) $H^1(X_G) \cong H^1(F \times B_G)$ for $1 \geq n + 1$. But, by 2.1 and the Künneth rule, $H^1(F \times B_G) = H^*(F)$ for $1 \geq n$, whence the equality of 4.1. On the other hand

$$(1) \dim H^1(X_G) = \sum_s \dim E_\infty^{1-s, s} \leq \sum_s \dim E_2^{1-s, s} = \sum_s \dim H^{1-s}(G; H^s(X))$$

and the inequality follows from 2.1. The proof also shows that $\dim H^*(X) = \dim H^*(F)$ if and only if $E_2 = E_\infty$ and G acts trivially on

$H^*(X)$.

If G acts trivially on $H^*(X)$, then $\dim H^*(X) = \dim H^*(F)$ if and only if $E_2 = E_\infty$. The if part follows from the preceding proof and 2.1. Conversely, if $E_2 \neq E_\infty$, then, using 2.1 (3), it is easily seen that there are elements of arbitrary high degrees on which some differential is non-zero; therefore $\dim E_\infty^{1-s,s} < \dim E_2^{1-s,s}$ for suitable $1 > n$ and $s \geq 0$, and we have a strict inequality in (1).

4.2. If X is acyclic over Z_p , then in (E_r) we have only the fibre degree zero; therefore $H^1(X_G) = E_\infty^{1,0} = E_2^{1,0} = H^1(G; Z_p) = Z_p$, hence $\dim H^*(F) = 1$, and F is non-empty and acyclic (III, 4.6).

4.3. Assume now that X is a homology n -sphere over Z_p . We want to show that F is a homology r -sphere ($-1 \leq r \leq n$) over Z_p , where (-1) -sphere stands for the empty set.

First let $n = 0$. Then X consists of two acyclic connected components, X_1, X_2 . If these are permuted (which is possible only if $p = 2$ of course), then F is empty. If not, 4.2 shows that F is a homology 0-sphere.

From now on assume $n \geq 1$. Then, by 2.1 (3), $E_2 = H^*(Z_p) \otimes H^*(X)$. We distinguish three cases.

(1) If $E_2 = E_\infty$, then (4.1) $\dim H^*(F) = \dim H^*(X)$, thus F is a non-empty homology sphere.

(ii) If n is even and p odd, then $E_2 = E_\infty$. This is the well known argument for sphere fiberings: Let z be a generator of $H^n(X)$. Since $H^*(X) = H^*(S_n)$, we have $E_2 = E_{n+1}$ and $E_{n+2} = E_\infty$, and $E_2 = E_\infty$ if and only if $d_{n+1}(1 \otimes z) = 0$. (This is of course for arbitrary p, n .) If now n is even and p odd, then $d_{n+1}(1 \otimes z^2) = 2 d_{n+1}z \otimes z = 0$ since $z^2 = 0$ hence $d_{n+1}(1 \otimes z) = 0$.

(iii) If $E_2 \neq E_\infty$, then F is empty. In fact, since $H^*(X)$ is generated by transgressive elements, our assumption implies that $\kappa_2^*: H^*(B_G) \rightarrow H^*(X_G)$ is not injective and our contention follows from 3.3.

This shows that F is a homology r -sphere. That $r \leq n$ was proved in III. A proof within the present framework may be found in ([2], 3.6) and will not be repeated here.

4.4. Under the assumptions of 4.3, if p is odd, then $n - r$ is even (III, 4.5). If $n = 0$, see the beginning of the proof of 4.3. If F is empty (i.e., if $r = -1$), then n must be odd by (1), (ii) above. Now let F be non-empty and $n \geq 1$. The rings $H^*(X_G)$ and $H^*(F_G)$ are modules over $H^*(Z_p)$ and these operations commute with the

restriction map $H^*(X_G) \longrightarrow H^*(F_G)$ (XVI, §1). Let α be the operation defined by the generator of degree 1 of $H^*(Z_p)$ (see 2.1), on these rings or on (E_r) . Let 1 be even. If r is odd (respectively even), then α annihilates $H^{1-r}(Z_p) \otimes H^r(F)$, (respectively is injective on $H^1(F_G)$). Since F is assumed to be non-empty, we have (4.3 (iii)) $E_2 = E_\infty$, hence on E_∞ we see similarly that if n is odd, (respectively even) then α annihilates $E_\infty^{1-n,n}$ (respectively is injective on the terms of total degree n). This implies easily that $\alpha : H^1(X_G) \longrightarrow H^{1+1}(X_G)$ is not injective (respectively is injective) for n odd (respectively even). The fact that n and r have the same parity follows now because the restriction $H^*(X_G) \longrightarrow H^*(F_G)$ commutes with α and is an isomorphism in degrees $1 \geq n+1$.

4.5. By a trivial induction, the inequality $\dim H^*(X) \geq \dim H^*(F)$ and 4.2, 4.3, 4.4 remain true if G is a finite p -group. Also, (in view of 2.1), 4.1 to 4.4 and their proofs are valid as they stand if $G = Z_m$ ($m = p^s$, $s \geq 1$) and is assumed to act freely outside F . The statement 4.4 is of course not generally true for $p = 2$. However we have the following result:

Let us keep the notation of 4.3, but assume that $G = Z_m$ with $m = 2^s$, $s \geq 2$ and that G acts freely outside F . Then $n - r$ is even.

Assume first F to be empty, $n \geq 1$. We want to show that the assumption n even leads to a contradiction. Since F is empty, we must have $E_2 \neq E_\infty$, by 4.1, hence, in the notation of 2.1 and 4.3 (ii)

$$d_{n+1}(1 \otimes z) = ab^t \otimes 1 \quad (t = n/2) .$$

It follows readily that

$$d_{n+1}(E_{n+1}^{2k,n}) = E_{n+1}^{2k+n+1,0} \quad d_{n+1}(E_{n+1}^{2k+1,n}) = 0$$

and consequently that

$$E_\infty^{2k,n} = E_\infty^{2k+1+n,0} = 0; \quad E_\infty^{2k+1,n} = E_\infty^{2k+n,0} = Z_2$$

hence

$$H^1(X_G) = Z_2 \quad (1 \geq n+1) ,$$

but this contradicts 4.1 and 4.3.

If F is not empty and $n \geq 1$, the proof of 4.5 is identical to that of 4.4. Finally if $n = 0$, then the square of a generator of G must necessarily leave each component of X invariant, hence (4.3, 4.5) its fixed point set is a homology 0-sphere. Since by assumption it is equal to F , we have that $n - r = 0$ is also even in this case.

4.6. In 4.7 the following properties of Steenrod's operation $Sq^1 : H^1(X; Z_2) \longrightarrow H^{1+1}(X; Z_2)$ will be used: it commutes with maps, it is zero on the reduction mod 2 of integral cohomology classes; if $H^{1+1}(X; Z)$ is a group of type $(2, 2, \dots, 2)$, then Sq^1 defines a homomorphism of $H^1(X; Z_2)$ onto $H^{1+1}(X; Z) \otimes Z_2$, hence in particular Sq^1 is zero on $H^1(Z_2; Z_2)$ if and only if 1 is even (see 2.1); if $X = A \times B$ and $c = a \otimes b$ with $a \in H^*(A; Z_2)$, $b \in H^*(B; Z_2)$, then $Sq^1(c) = Sq^1 a \otimes b + a \otimes Sq^1 b$.

4.7. We assume here that X is a compact cohomology n -sphere mod 2, whose integral cohomology group is finitely generated, and that $p = 2$. Let T_1 be the torsion subgroup of $H^1(X; Z)$, and T the sum of the T_1 . The universal coefficient formula implies that $\text{ord } T$ is odd, $H^1(X; Z) = T_1$ ($i \neq 0, n$), $H^n(X; Z)/T_n = Z$, $H^0(X; Z) = Z$. On $H^n(X; Z)/T_n$, the involution g acts either trivially or by $z \longrightarrow -z$. We say accordingly that g preserves or reverses the orientation. By 4.3, F is a cohomology r -sphere mod 2 ($-1 \leq r \leq n$).

THEOREM (Liao [8]). $n - r$ is even if and only if g preserves the orientation of X ($n \geq 1$).

Let F be non-empty. Then $H^*(F_G; Z_2) = H^*(Z_2; Z_2) \otimes H^*(S_r; Z_2)$. Sq^1 is zero on $H^*(S_r; Z_2)$, since it increases degree by 1, and on $H^0(S_r; Z_2)$, which is the reduction mod 2 of $H^0(S_r; Z)$. It follows (4.6) that for 1 even Sq^1 is zero on $H^1(F_G; Z_2)$ if and only if r is even. Since Sq^1 commutes with $H^*(X_G) \longrightarrow H^*(F_G)$ it will then be enough to show

(*) Let F be non-empty. Then Sq^1 is zero on $H^{21}(X_G; Z_2)$ if and only if either n is even and G preserves the orientation or n is odd and G reverses the orientation. If F is empty, then n is even if and only if G reverses the orientation.

Since T has odd order, we have $H^j(Z_2; T) = 0$ for $j \geq 1$ by 2.1, hence the natural map $H^j(Z_2; H^*(X; Z)) \longrightarrow H^j(Z_2; H^*(X; Z)/T)$ is an isomorphism for $j \geq 1$ by the exact cohomology sequence associated to $0 \longrightarrow T \longrightarrow H^*(X; Z) \longrightarrow H^*(X; Z)/T \longrightarrow 0$. Using 2.1, we easily obtain for the spectral sequence (E_r^1) of π_2 over the integers:

$$E_2^{2j,0} = Z_2, \quad E_2^{2j-1,0} = E_2^{j,k} = 0 \quad (j \geq 1; 1 \leq k \leq n-1);$$

(1)

$$E_2^{j,t} = 0 \quad (t \geq n+1);$$

if G preserves the orientation then

$$(2) \quad E_2^{2j,n} = Z_2, \quad E_2^{2j-1,n} = 0 \quad (j \geq 1);$$

if G reverses the orientation then

$$(3) \quad E_2^{2j,n} = 0; \quad E_2^{2j-1,n} = Z_2 \quad (j \geq 1).$$

Assume now that G preserves the orientation. If n is even, then $H^{2i+1}(X_G; Z) = 0$ ($2i \geq n$) since E_2' has no non-zero term of total degree $2i+1$, therefore $H^{2i+1}(X_G; Z_2) = H^{2i+1}(X_G; Z) \otimes Z_2$ ($2i \geq n+1$) and is annihilated by Sq^1 . If n is odd, then $H^1(X_G; Z)$ is Z_2 or 0 ($i \geq n+1$) because by (1) and (2) the terms of total degree 1 in E_2' form a group isomorphic to Z_2 . If moreover F is non-empty, then $H^1(X_G; Z_2) = H^1(F_G; Z_2)$ is two dimensional. The universal coefficient formula implies then that $H^1(X_G; Z) = Z_2$ for all $i \geq n+1$, consequently Sq^1 does not annihilate $H^1(X_G; Z_2)$. This proves (*) for F non-empty and G preserving the orientation. If G reverses the orientation the proof is quite similar and is left to the reader.

Assume now F to be empty. We then have $H^1(X_G; Z_2) = 0$ for $i \geq n+1$. But in E_2' all groups $E_2^{p,q}$ with $p+q \geq n+1$ are finite 2 groups, therefore, in order to have $H^1(X_G; Z_2) = 0$, we must have $E_2^{p,q} = 0$ for $p+q \geq n+1$. If G preserves the orientation then $E_2^{2j,0} = Z_2$ must be a boundary for some differential. Since for $2j \geq n+1$, the only possible non-zero elements of total degree $2j-1$ have the fibre degree n , we must then have $E_2^{2j-n-1,n} \neq 0$ and hence, by (2), $2j-n-1$ is even, that is, n is odd. If G reverses the orientation, one sees in the same way that n must be even, and this ends the proof of (*).

4.8. REMARK. In 4.1 to 4.4 it is not necessary to assume X to be locally compact. This is clear from the purely algebraic approach of [7], as is mentioned there. It could also be seen here using 3.8 and the fact that Künneth's rule is valid for $F \times B_G$, because B_G may be assumed to be a compact manifold (see XVI, §5).

§5. The action of the circle group

The identity component of the compact Lie group G will be denoted by G^0 .

5.1. LEMMA. If $H_G^*(X; L)$ is of finite type, then so is $H_G^*(X_G; L)$.

G^0 acts trivially on $H_G^*(X; L)$, therefore the term E_2 of the spectral sequence of $X_{G^0} \longrightarrow B_{G^0}$ is given by the Künneth rule. Since $A \otimes B$ and $\text{Tor}(A, B)$ are finitely generated if A, B are, it follows, using 2.3, that E_2 is of finite type, hence a fortiori $H_G^*(X_{G^0}; L)$ is of finite type.

Let $N = G/G^0$. It is a finite group. By 3.3, X_{G^0} is a principal N -bundle over X_G . There is then (3.4 (a)) a spectral sequence leading to $H_G^*(X_G; L)$ in which $E_2 = H^*(N; H_G^*(X_{G^0}; L))$. Since the cohomology of a finite group with coefficients in a finitely generated module is of finite type our result follows.

REMARK. Taking the results of XVI, §§4, 5 into account, one sees easily that 5.1 and its proof remain valid in cohomology with closed supports, for a non-necessarily locally compact space X .

5.2. LEMMA Let $\dim_L X$ be finite. Assume that $H_G^*(B_G; L)$ is trivial for $x \notin F$ and that $H_G^*(X; L)$ is finitely generated (respectively vanishes in degrees $\geq N$). Then

- (a) the same is true for $H_G^*(X'; L)$;
- (b) the same is true for $H_G^*(F; L)$ if G is infinite.

In the proof, the coefficients L are omitted. If G is finite see III, 5.2. From now on G will be infinite, hence (2.4), $H^*(B_G)$ contains non-zero free submodules in arbitrary high dimensions. We show first that $H_G^*(F)$ and $H_G^*(X')$ are finitely generated if $H_G^*(X)$ is. By 5.1, $H_G^*(X_G)$ is of finite type, hence (3.6) $H_G^1(F_G)$ is finitely generated for $1 > \dim_L X$. By 2.4 and the Künneth rule, each $H_G^S(F)$ is embedded in some $H_G^1(F_G)$ for suitably large 1 , therefore $H_G^*(F)$ is finitely generated; it also follows that $H_G^*(F_G)$ is of finite type, hence, by the cohomology sequence, $H_G^*((X - F)_G)$ is of finite type. Using the isomorphism $H_G^*((X - F)_G) = H_G^*(X' - F)$ of 3.4 and $\dim_L X' \leq \dim_L X$, it

follows that $H_C^*(X' - F)$ is finitely generated; finally, the cohomology sequence of $X' \bmod F$ shows that $H_C^*(X')$ is also finitely generated.

There remains to prove that if $H_C^*(X)$ vanishes in degrees $\geq N$, so do $H_C^*(F)$ and $H_C^*(X')$. We assume first that G is connected. The diagram

$$\begin{array}{ccccccc} H_C^1(X) & \longrightarrow & H_C^1(F) & \longrightarrow & H_C^{1+1}(X - F) & \longrightarrow & H_C^{1+1}(X) \longrightarrow \\ \uparrow \pi^* & & \uparrow & & \uparrow \pi^* & & \uparrow \pi^* \\ H_C^1(X') & \longrightarrow & H_C^1(F) & \longrightarrow & H_C^{1+1}(X' - F) & \longrightarrow & H_C^{1+1}(X') \longrightarrow \end{array}$$

of cohomology sequences, together with $H_C^1(X) = 0$ for $1 \geq N$, shows readily that it is enough to prove that $H_C^1(X' - F) = 0$ for $1 \geq N$; for this, we consider the spectral sequence (E_r) of $\pi : X - F \longrightarrow X' - F$. If $y = \pi(x)$, then $\pi^{-1}(y) = G/G_x$. By the assumption and 2.4, G_x is finite and L is a field of order prime to $\text{ord } G_x$. Since G acts trivially on its own cohomology, it follows (III, 2.3) that the projection $G \longrightarrow G/G_x$ yields an isomorphism $\alpha_y : H^*(\pi^{-1}(y)) \longrightarrow H^*(G)$ which is independent of the choice of x in $\pi^{-1}(y)$. Moreover, the existence of a slice at x (see VII) implies that there is an equivariant retraction of a saturated neighborhood of $\pi^{-1}(y)$ onto $\pi^{-1}(y)$. It follows immediately that for any $a \in H^*(G)$, the map $y \longrightarrow \alpha_y(a)$ is a continuous cross section of the sheaf defined over $X' - F$ by the groups $H_C^*(\pi^{-1}(y))$ (see also XII, §2); hence this sheaf is constant, and we have ordinary co-efficients in E_2 . If $k = \dim G$ and s is the highest dimension for which $H_C^1(X' - F) \neq 0$, then the maximum cocycle argument gives $H_C^s(X' - F) \cong H_C^{s+k}(X - F)$. However, the above diagram shows that π^* is surjective in dimensions $\geq N$, hence we must have $s < N - k$.

If now G is not connected, combine the result just obtained for G^0 with 2.4 and III, 3.4.

5.3. THEOREM. Let $\dim_L X$ be finite and $H_C^*(X; L)$ be finitely generated (respectively vanishing in degrees $\geq N$). Then the same is true

- (a) for X/G and for F if $L = K_0$, $G = T^k$;
- (b) for X/G if $L = K_p$, Z, G is a finite extension of a toral group, and the orbit structure is finite;
- (c) for F , either if $L = K_p$ and G is a finite p -group, or if $L = K_p$, $Z, G = T^k$, and the orbit structure is finite.

If any of these assertions, for a given L , is true for a closed invariant subgroup N of G and for G/N , then it is true for G since $F(G; X) = F(G/N; F(N; X))$ and $X/G = (X/N)/(G/N)$. It is therefore enough to prove the various assertions of 5.3 in the two cases G finite and $G = T^1$. The former one being taken care of by (III, 4.3, 5.2), we assume from now on that $T = T^1$. Then G_x is finite for $x \notin F$. If $L = K_0$, (a), (b), (c) follow from 2.1, 5.2. In the remaining cases, there are a finite number of distinct isotropy groups. Let N be a finite subgroup of G containing all G_x ($x \notin F$). By III, 5.2, $H_c^*(X/N; L)$ is finitely generated (respectively vanishes in degrees $\geq N$). Now, G/N is a circle group which operates on X/N freely outside the fixed point set (1.2), therefore the cohomology groups of $(X/N)/(G/N)$ and $F(G/N; X/N)$ are finitely generated (respectively vanish in degrees $\geq N$) by 5.2. Since these two spaces are homeomorphic to X/G and $F(G; X)$ (see (1.2)), this concludes the proof.

5.4. Let $G = T^1$ and $\dim_0 X$ be finite. Then, for $a = 0, 1$, $i \equiv a \pmod{2}$ and $i > \dim_0 X$, we have

$$(1) \sum_{s \equiv a(2)} \dim H_c^s(X; K_0) \geq \sum_{s \equiv a(2)} \dim H_c^s(F; K_0) = \dim H_c^1(X_G; K_0),$$

and there is equality if and only if the spectral sequence over K_0 of $X_G \longrightarrow B_G$ is trivial.

The first inequality is due to Conner [5]. For $i > n$, we have $H_c^1(X_G; K_0) = H_c^1(F \times B_G; K_0)$ by 3.7 (c), and the equality of the second and third terms in (1) follow from 2.2 and the Künneth rule. In the spectral sequence (E_r) of $X_G \longrightarrow B_G$ over K_0 we have

$$(2) \dim H_c^1(X_G) = \sum E_\infty^{1-s, s} \leq \sum E_2^{1-s, s} = \sum H^{1-s}(B_G; K_0) \otimes H_c^s(X; K_0)$$

whence the inequality of (1), and the equality if $E_2 = E_\infty$. Now if there is a non-zero differential in (E_r) , it is easily seen, using the multiplicative structure of $H^*(B_G; L)$, that there are elements of arbitrarily high degree on which some differential is non-zero, hence $\dim E_\infty^{1-s, s} < \dim E_2^{1-s, s}$ for suitable $i > n$ and $s \geq 0$, and therefore we have a strict inequality in (1).

5.5 Let $G = T^k$, and $\dim_0 X$ be finite. Then

$$\sum_{s \equiv a(2)} \dim H_c^s(X; K_0) \geq \sum_{s \equiv a(2)} \dim H_c^s(F; K_0) \quad (a = 0, 1).$$

This follows from 5.4 by induction on k . It will be proved in a later Chapter (XII), together with a similar result mod p , that if, under the present assumptions, the spectral sequence of $X_G \longrightarrow B_G$ is trivial, and

$\dim H_C^*(X, K_0)$ is finite, then $\dim H_C^*(X; K_0) = \dim H_C^*(F; K_0)$.

5.6. It follows in particular from 5.5 that if X is acyclic over Q , so is F .

5.7. Let X be compact and finite dimensional over K_0 , $X \sim_0 S_n$ and $G = T^k$. Then F is a cohomology r -sphere ($-1 \leq r \leq n$) over K_0 and $n - r$ is even [5].

The proof is entirely analogous to that of the Smith's theorem (4.3), with $r \leq n$ following now from 5.3, and the dimensional parity from 5.5.

5.8. LEMMA. Let $G = T^1$, $\dim_L X < \infty$, $H^*(X; Z_2) = H^*(S_n; Z_2)$, $F = F(Z_2; X)$, where Z_2 is the subgroup of order two of G , and r be the integer such that $F \sim_2 S_r$. Then $n - r$ is even.

For $x \notin F_1$, G_x is finite of odd order, therefore, by 2.1, 3.6 $r^* : H^*(X_G; Z_2) \rightarrow H^*(F_1G; Z_2)$ is an isomorphism in sufficiently high dimensions. By use of 2.2 we deduce immediately that the spectral sequences mod 2 of $X_G \rightarrow B_G$ and $F_1G \rightarrow B_G$ are simultaneously trivial or not trivial, and the lemma follows.

5.9. THEOREM. Let $G = T^k$, $L = Z$, K_p , X be compact, $X \sim_L S_n$, $\dim_L X < \infty$. Assume moreover that G has only a finite number of distinct isotropy groups on X . Then $F \sim_L S_r$ and $n - r$ is even.

By induction, it is enough to consider the case $k = 1$. For a given prime p , there exists an integer s such that F is the fixed point set of the subgroup of order p^s of G . Thus for $L = K_p$, ($p \neq 0$), our assertion follows from 4.4 if $p \neq 2$, from 5.8 if $p = 2$. For $p = 0$, we may apply 5.7.

In the case $L = Z$, the theorem is a consequence of the above, 5.3 (c) and the universal coefficient theorem.

§6. The quotient of a cohomology sphere by Z_p or T^1

The topic named in the title of this section can be studied conveniently in the present framework. We content ourselves by giving some examples, without aiming at completeness.

6.1. THEOREM [6, 8]. Let p be a prime (respectively,

$p = 0$). Let $G = Z_p$ (respectively, $G = T^1$), $X \sim_p S_n$ and $\dim_p X < \infty$. Let r be such that $F \sim_p S_r$. Then
 $H^1(X'; K_p) = H^{1-r-1}(B_G; K_p)$ ($r + 2 \leq 1 \leq n$),
 $H^1(X'; K_p)' = 0$ otherwise for $1 \neq 0$.

If F is empty we are essentially in the principal bundle case and our statement is well known. From now on F is not empty. Then $\pi_2^* : H^*(B_G) \longrightarrow H^*(X_G)$ is injective (3.3). Since $X \sim_p S_n$, the spectral sequence of this fibering is then trivial and

$$(1) \quad H^1(X_G) = H^1(B_G) + H^{1-n}(B_G) \quad .$$

Consider now the spectral sequence of the map $\pi_1 : X_G \longrightarrow X'$. There $E_2 = H^*(X'; \underline{F})$ where $\underline{F}$ is the Leray sheaf of π_1 , whose stalk at $y \in Y'$ is $H^*(\pi^{-1}(y))$. (See XVI, §§1, 2). Here $\pi^{-1}(y)$ may be identified with $B_{G_X}(x \in \pi_X^{-1}(y))$ by 3.1. For $y \notin F$, its cohomology is trivial (2.1), hence $\underline{F}^q$ ($q \neq 0$) is concentrated on $\pi_1^{-1}(F)$. But $\pi^{-1}(F) = F_G = F \times B_G$, therefore $\underline{F}^q$ is constant over F , and we get

$$(2) \quad \begin{aligned} E_2^{s,0} &= H^s(X') & (s \geq 0) \\ E_2^{r,t} &= H^r(F; H^t(B_G)) = H^t(B_G) & (t \neq 0) \\ E_2^{0,t} &= H^0(F; H^t(B_G)) = H^t(B_G) & (t \neq 0) \\ E_2^{s,t} &= 0 & (s \neq 0, r; t \neq 0) \quad . \end{aligned}$$

On the other hand, we assert that

$$(3) \quad \pi_1^*(H^1(X')) = 0 \quad (1 \geq 1)$$

In fact, if (3) is wrong, then $\text{Im } \pi_1^*$ contains some power of the polynomial generator of $H^*(B_G)$; (this is clear for $p = 0, 2$; for p odd, it follows from $H^{2i}(Z_p; Z_p) = \beta H^{2i-1}(Z_p; Z_p)$ where β is the Bockstein homomorphism) hence $H^j(X') \neq 0$ for arbitrarily high j . However it must be zero for $j \geq n$; this last assertion follows from III, 4.3 or [2] if $p \neq 0$; and the argument of [2] is also valid for $p = 0$. Using (1), (2), (3) and 3.4, it is then seen without difficulty that the differential d_1 ($2 \leq 1 \leq n - r$) induces an isomorphism of $E_1^{r,i-1} = E_2^{r,i-1}$ onto $E_1^{1+r,0} = E_2^{1+r,0}$ and that otherwise, the differentials are zero. This proves the theorem. In writing (2) we tacitly assumed $r \neq 0$. If $r = 0$ we have $E_2^{0,t} = H^t(B_G) + H^t(B_G)$, $E_2^{s,t} = 0$ ($s, t \neq 0$) and the rest of the proof is the same.

6.2. LEMMA. Let $G = T^1$, p be a prime or zero,

$\dim_p X < \infty$, X compact, $X \sim_p S_n$, and r be the integer such that $F_1 = F(Z_p; X)$ is a cohomology r -sphere mod p . Then (4.3, 5.8) $n - r$ is even and $H^1(X' - F_1'; K_p)$ is equal to K_p for $i = r + 1, r + 3, \dots, n - 1$ and is zero otherwise.

We leave the case $n = r$ to the reader and assume $n - r \geq 2$.

Then

$$(4) \quad H_C^{r+1}(X - F_1; K_p) = H_C^n(X - F_1; K_p) = K_p; \quad H_C^1(X - F_1; K_p) = 0 \quad (i \neq r + 1, n)$$

by the cohomology sequence of $X \bmod F_1$. For $x \notin F_1$, G_x is of finite order prime to p , therefore $H^*(B_{G_x}; K_p)$ is trivial (2.1) and

$$(5) \quad H^*((X - F_1)_G; K_p) = H^*(X' - F_1'; K_p)$$

by 3.4. Thus the spectral sequence of $(X - F_1)_G \longrightarrow B_G$ leads from $H^*(B_G) \otimes H_C^*(X - F_1)$ to $H^*(X' - F_1')$. Since the latter vanishes in high dimensions, our assertion follows from (4) by an easy spectral sequence argument.

6.3. THEOREM [6]. Let $G = T^1$ have a finite number of distinct isotropy groups on the compact space X . Let $L = Z, K_p$. Assume $\dim_L X < \infty$ and $X \sim_L S_n$. Let r be the integer, congruent to $n \bmod 2$, such that $F \sim_L S_r$ (see 5.9). Then $H^1(X'; L) = L$ for $i = 0$ and $i = r + 3, r + 5, \dots, n - 1$ and is zero otherwise.

If $L = K_0$, see 6.1. If $L = K_p$ (p prime) and, in the notation of 6.2, $F = F_1$, then use 6.2 and the cohomology sequence of $X' \bmod F$. If $F \neq F_1$, proceed by induction on n , apply the hypothesis to F_1 and use again 6.2 and the cohomology sequence. If $L = Z$, the result follows from this and from 5.3 (b).

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V: THE ACTION OF Z_p OR T^1 : LOCAL THEOREMS

A. Borel

§1. Conservation of cohomological local connectedness

1.1. The following elementary remark will be used in the sequel.

Let

$$\begin{array}{ccccc}
 & & A' & \xrightarrow{u'} & A'' \\
 & & \downarrow f & & \downarrow v' \\
 B & \longrightarrow & B' & \longrightarrow & B'' \\
 & & \downarrow g & & \\
 & & C & \xrightarrow{v} & C'
 \end{array}$$

be a commutative diagram of modules and homomorphisms, with exact middle row. If $v' \circ u'$ and $v \circ u$ have zero (respectively, finitely generated) images, then so does $g \circ f$.

1.2. LEMMA. Let $U = U_0 \supset U_1 \supset U_2 = V$ be closed invariant subsets of X such that $\text{Im } r_{cU_1 U_{1+1}}^*$ ($i = 0, 1$) is of finite type. Then $\text{Im } r_{cU_G V_G}^*$ is of finite type.

Let G^0 be the identity component of G . We prove first that $\text{Im } r_{cU_{G^0} V_{G^0}}^*$ is of finite type. Let $({}_1 E_r)$ be the spectral sequence over L of $U_{G^0} \longrightarrow G_{G^0}$, and $r_{1,1+1}$ the homomorphism $({}_1 E_r) \longrightarrow ({}_{1+1} E_r)$ defined by inclusion. Since G^0 is connected, we have ordinary coefficient in ${}_1 E_2$, hence there are exact sequences

$$0 \rightarrow H^r(B_{G^0}; L) \otimes H_c^s(U_1; L) \rightarrow {}_1 E_2^{r,s} \rightarrow \text{Tor}(H^{r+1}(B_{G^0}; L), H_c^s(U_1; L)) \rightarrow 0$$

compatible with the maps $r_{1,i+1}$. Since $H^*(B_{G^0}; L)$ is of finite type, the maps of the $\otimes$ and Tor terms have finitely generated images, hence ${}_1E_2^{r,s} \longrightarrow {}_{1+2}E_2^{r,s}$ has a finitely generated image (1.1). It is then true a fortiori for the E_∞ terms, and our contention follows.

U_{1G^0} is a regular covering of U_{1G} , with group $N = G/G^0$ (IV, 3.2). In the Cartan-Leray spectral sequence of this covering, ${}_1E_2 = H^*(N; H_c^*(U_{1G^0}; L))$, and there is again a natural map $({}_1E_r) \longrightarrow ({}_{1+1}E_r)$. Since the cohomology of a finite group with respect to a finitely generated module is of finite type, $H^r(N; H_c^s(U_{1G^0}; L)) \longrightarrow H^r(N; H_c^s(V_{G^0}; L))$ has a finitely generated image, and the lemma follows as above.

1.3. LEMMA. Let $U = U_0 \supset U_1 \supset U_2 = V$ be invariant open subsets of X . If $\text{Im } j_{U_1 U_{1+1}}$ is of finite type ($i = 0, 1$) then $\text{Im } j_{U_G V_G}$ is of finite type.

The proof is the same as that of 1.2, except for a reversing of the arrows, and is left to the reader.

1.4. PROPOSITION. Let X be a G -space, which is finite dimensional and clc over L . Assume that $H^*(B_G; L)$ is trivial for all x outside the fixed point * set F . Let $L = K_p, \mathbb{Z}$ and G be infinite if $L = \mathbb{Z}$. Then

(a) X/G is clc_L .

(b) if moreover $H^*(B_G; L)$ is not trivial, F is clc_L .

Coefficients L are omitted in the proof. We show first that if $H^*(B_G)$ is also trivial or if F is empty, then X/G is clc_L , using Wilder's property (P, Q) (see I, 2.2). Let Q', P' be open in X' , with $\bar{Q}'$ compact and in P' ; there clearly exists R' open in X' , with $\bar{Q}'$ in R' and $\bar{R}'$ compact in P' . Let P, Q, R be their inverse images in X . Then $\bar{Q}$ and $\bar{R}$ are compact and contained in R and P respectively. Since X has property (P, Q) , $\text{Im } j_{RQ}^*$ and $\text{Im } j_{PR}^*$ are finitely generated. By 1.3, $\text{Im } j_{P_G Q_G}^*$ is then of finite type. However, since $H^*(B_{G_x})$ is trivial for all x , $\pi_1 : P_G \longrightarrow P'$ and $Q_G \longrightarrow Q'$ induce isomorphisms for cohomology (IV, 3.4), hence $\text{Im } j_{P', Q'}^*$ is finitely generated, and X' has property (P, Q) . This proves (a) for the case in which $H^*(B_G)$ is trivial and also proves, in case (b) that $X' - F$ is clc_L . Assume now that $H^*(B_G)$ is non-trivial that F is not empty, and that G is infinite if $L = \mathbb{Z}$. It remains to show that F is clc_L and that X' is clc_L around a point of F .

Let $x \in F$. Since it has arbitrarily small invariant neighborhoods, we can find by 1.3 closed compact invariant neighborhoods $U \supset V$ of x such that $\text{Im } r_{UV}^*$ and $\text{Im } r_{U/G, V/G}^*$ are of finite type. Consider now the commutative diagram

$$\begin{array}{ccc} H^1(U_G) & \longrightarrow & H^1((U \cap F)_G) \\ \downarrow & & \downarrow r^1 \\ H^1(V_G) & \longrightarrow & H^1((V \cap F)_G) \end{array}$$

where the maps are the natural restrictions. The horizontal ones are isomorphisms for $1 > \dim_L X$ by (IV, 3.6), therefore $\text{Im } r^1$ is finitely generated. Now $(U \cap F)_G = (U \cap F) \times B_G$, $(V \cap F)_G = (V \cap F) \times B_G$ and the inclusion is the product of the identity on B_G and of the inclusion $V \cap F \longrightarrow U \cap F$. Given an integer s , there exists $1 > \dim_L X$ such that $H^{1-s}(B_G)$ contains a non-zero free L -module (IV, 2.4). It follows, using Künneth's rule, that $H^s(V \cap F)$ and $H^s(U \cap F)$ may be identified respectively with direct summands of $H_C^1((V \cap F)_G)$ and $H_C^1((U \cap F)_G)$ on which r^1 induces the map $r_{U \cap F, V \cap F}^s$. Therefore, $\text{Im } r_{U \cap F, V \cap F}^s$ is finitely generated, and F is clc_L around x . It remains to show that X' is clc_L around x . For this it is enough to prove:

(*) Given an invariant neighborhood U in X of $x \in F$, there exists an invariant neighborhood V of x in U such that $\text{Im } r_{U/G, V/G}^*$ is finitely generated.

If W is a neighborhood of x in U , then the intersection T of W and its transforms is an invariant neighborhood of x , and if r_{UW}^* is trivial or finitely generated, so is r_{UT}^* . Using the above we can then find invariant closed neighborhoods $U \supset V \supset W$ of x such that r_{UV}^* , r_{VW}^* , $r_{U \cap F, V \cap F}^*$, $r_{V \cap F, W \cap F}^*$ have finitely generated images. If we then write down the three rows diagram whose rows are the exact cohomology sequences of $U \bmod U \cap F$, $V \bmod V \cap F$ and $W \bmod W \cap F$, and whose vertical maps are the natural restriction maps we see then, by 1.1 that the image of $\text{Im } r_{U \cap F, W \cap F}^*$ is finitely generated. Similarly, there exists a closed invariant neighborhood S of x in W such that $\text{Im } r_{S \cap F, S \cap F}^*$ is finitely generated. By 1.2, $r_C^* : H_C^*(U \cap F)_G \longrightarrow H_C^*(S \cap F)_G$ has a finitely generated image. It follows then from IV, 3.4 that $\text{Im } r_{U \cap F, S \cap F}^*$ is finitely generated.

From what has been proved so far, we deduce the existence of compact neighborhoods $V, W (U \supset V \supset W)$ of x in X , invariant under G , such that $r_{U \cap F, V \cap F}^*$ and $r_{V \cap F, W \cap F}^*$ ($U' = U/G$, $V' = V/G$, $W' = W/G$) have finitely generated images. We then consider the commutative diagram

$$\begin{array}{ccccccc}
\longrightarrow & H_c^1(U' - F) & \longrightarrow & H^1(U') & \longrightarrow & H^1(F \cap U') & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_c^1(V' - F) & \longrightarrow & H^1(V') & \longrightarrow & H^1(F \cap V') & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_c^1(W' - F) & \longrightarrow & H^1(W') & \longrightarrow & H^1(F \cap W') & \longrightarrow
\end{array}$$

where the rows are exact cohomology sequences and the vertical maps are restriction maps. 1.1 then shows that $r_{U', W'}^*$ has a finitely generated image, and this proves (*).

1.5. THEOREM (Floyd [4]). Let X be a locally compact space which is finite dimensional and clc over Z , and let G be a finite group operating on X . Then X/G is clc over Z (and finite dimensional over Z by III, 5.1).

First we consider the case where $G = Z_p$. If $y \notin F$, then X' is clearly clc around y . Let now $y \in F$. By the universal coefficient formula, X is also clc over Z_p , hence there exists by 1.4 compact invariant neighborhoods $U \supset V \supset W$ of y in X such that $H^*(V; Z) \longrightarrow H^*(W; Z)$ and $H^*(U/G; Z_p) \longrightarrow H^*(V/G; Z_p)$ have finitely generated images. We form the following commutative diagram in which the rows are the exact sequences defined by means of the exact coefficient sequence

$$0 \longrightarrow Z \xrightarrow{p} Z \longrightarrow Z_p \longrightarrow 0$$

and the vertical maps are the restriction maps:

$$\begin{array}{ccccccc}
\longrightarrow & H^1(U'; Z) & \xrightarrow{\bar{p}} & H^1(U'; Z) & \longrightarrow & H^1(U'; Z_p) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H^1(V'; Z) & \xrightarrow{\bar{p}} & H^1(V'; Z) & \longrightarrow & H^1(V'; Z_p) & \longrightarrow \\
& \downarrow \nu & & \downarrow & & \downarrow & \\
\longrightarrow & H^1(W'; Z) & \xrightarrow{\bar{p}} & H^1(W'; Z) & \longrightarrow & H^1(W'; Z_p) & \longrightarrow
\end{array}$$

($U' = U/G$. etc.) We want to show that $H^1(U'; Z) \longrightarrow H^1(W'; Z)$ has a finitely generated image. Since we already know this to be true for $H^1(U'; Z_p) \longrightarrow H^1(V'; Z_p)$ it is enough (1.1) to show it for the composite

map $\alpha = \bar{p} \circ \nu : H^1(V'; Z) \longrightarrow H^1(W'; Z)$. To this effect we consider the following diagram

$$\begin{array}{ccccc}
 H^1(V'; Z) & \xrightarrow{\bar{p}} & H^1(V'; Z) & & \\
 \searrow \pi_V^* & & \nearrow \mu & & \\
 & H^1(V; Z) & & & \\
 \downarrow r_{V', W'}^1 & & \downarrow r_{VW}^1 & & \downarrow r_{V', W'}^1 \\
 & H^1(W; Z) & & & \\
 \nearrow \pi_W^* & & \searrow \mu & & \\
 H^1(W'; Z) & \xrightarrow{\bar{p}} & H^1(W'; Z) & &
 \end{array}$$

Here the maps π^* are transposes of the projections $V \longrightarrow V'$ and $W \longrightarrow W'$, and μ is the transfer map. We have $\bar{p} = \mu \circ \pi^*$ by (III, 2.1), and the commutativity in the remaining parts of the diagram is clear. Since $H^1(V; Z) \longrightarrow H^1(W; Z)$ has finitely generated image by construction, the same is then true for $\alpha : H^1(V'; Z) \longrightarrow H^1(W'; Z)$, and this ends the proof in the case $G = Z_p$. If G is a finite p -group, the theorem follows then by induction on $\text{ord } G$.

Let now G be a finite group of order $n = p_1^{n_1} \dots p_k^{n_k}$ (p_i prime, $p_i \nmid p_j$ for $i \neq j$) and let K_1 be a Sylow subgroup of order $p_1^{n_1}$ ($1 \leq i \leq k$). Let $U' \supset V'$ be open in X/G , with $\bar{V}'$ compact and contained in U' , and U, V their inverse images in X . Then V (respectively V/K_1) and U (respectively, U/K_1) are open in X (respectively X/K_1) and $\bar{V}$ (respectively $\bar{V}/K_1$) is compact and contained in U (respectively U/K_1), ($1 \leq i \leq k$). In view of I, 2.2 and what has been proved previously, $\text{Im } j_{UV}^*$ and $\text{Im } j_{U/K_1, V/K_1}^*$ ($1 \leq i \leq k$) are finitely generated and it is enough to show that $\text{Im } j_{U', V'}^*$ is finitely generated (integral cohomology). This follows from the existence of a commutative diagram

$$\begin{array}{ccc}
 H_c^*(U'; Z) & \xrightarrow{\sigma} & \Sigma H_c^*(U/K_1; Z) \\
 \uparrow j_{U', V'}^* & & \uparrow j^* \\
 H_c^*(V'; Z) & \xrightarrow{\sigma} & \Sigma H_c^*(V/K_1; Z)
 \end{array}$$

where $j^* = \Sigma j_{U/K_1, V/K_1}^*$, σ is the sum of the transposes of the projection

maps (see III, proof of 5.2), and from the fact that π_g is injective (III, 5.2).

1.6. THEOREM. Let X be clc and finite dimensional over L . Then

- (a) If $L = K_0$, and $G = T^k$, then X/G and F are clc_L .
- (b) If $L = K_p$ or Z , G is a finite extension of a toral group and the orbit structure is locally finite, then X/G is clc_L .
- (c) If $L = K_p$, G is a finite p -group, or if $L = K_0$, or Z , $G = T^k$ and the orbit structure is locally finite, then F is clc_L .

If any of these assertions is true for a given L , for a closed invariant subgroup N of G and G/N , then it is true for G , since $F(G; X) = F(G/N; F(N; X))$ and $X/G = (X/N)/(G/N)$. It is therefore enough to prove the assertions of 1.6 in the cases $G = T^1$ and G finite, and more specifically (a) for $G = T^1$, (b) for G finite or $G = T^1$ and (c) for $G = Z_p$ or $G = T^1$.

If G is finite, (b) follows from 1.5; if $G = Z_p$, (c) follows from 1.4. From now on $G = T^1$. Then G_x is finite for $x \notin F$. If $L = K_0$, (a), (b), (c) follows then from 1.4 and IV, 2.1. In the remaining cases, the orbit structure is assumed to be locally finite; since clc is a local property, we may assume that G has only a finite number of distinct isotropy groups on X . Let N be a finite subgroup of G containing all G_x for $x \notin F$. The space X/N is clc_L by 1.5. The group $G/N = T^1$ operates on X/N , freely outside its fixed point set, therefore $F(G/N; X/N)$ and $(X/N)/(G/N)$ are clc_L by 1.4. Since these spaces are homeomorphic to $F(G; X)$ and X/G respectively, this concludes the proof.

REMARK. When $L = Z_p$ and G is a p -group, the fact that F is clc_L is due to Smith [5, Theorem 6.15], and that fact that X' is clc_L is due to Floyd [3, Theorem 1'].

1.7. The following lemma will be needed in VI. It could easily be stated in a slightly more general form, so as to embody an analogous result of Conner [2] for the case $G = T^1$, $L = K_0$, and be used above to prove the cohomological local connectedness of F .

Given a space A and a subspace B the map r_{AB}^* is said to be trivial if there exists a point $b \in B$ such that $\text{Ker } r_{AB}^* = \text{Ker } r_{Ab}^*$.

LEMMA. Let X be a locally compact space which is of finite dimension n and clc over Z_p . Let $G = Z_p$ operate on X , F be its fixed point set, and $U = A_0 \supset A_1 \supset \dots \supset A_n = V$ be a decreasing sequence of invariant compact subsets, with V connected, such that $r_{j-1,j}^* = r_{A_{j-1}, A_j}^*$ is trivial for $j = 0, \dots, n-1$. Then $\dim \text{Im } r_{U \cap F, V \cap F}^* \leq 1$.

Let $({}_j E_r)$ be the spectral sequence of the projection of K_{jG} onto B_G and $r_{j-1,j}^*$ the homomorphism of $({}_{j-1} E_r)$ into $({}_j E_r)$, or of $H^*(A_{j-1,G})$ into $H^*(A_{jG})$, the transpose of the inclusion map. In particular

$$r_{j-1,j}^* : {}_{j-1} E_2 \cong H^*(Z_p; H^*(A_{j-1})) \longrightarrow H^*(Z_p; H^*(A_j)) \cong {}_j E_2$$

is the coefficient map defined by $r_{j-1,j}^*$. Since the latter is trivial, we have

$$(1) \quad r_{j-1,j}^*({}_{j-1} E_r^{s,t}) = 0 \quad (t \geq 1; s \geq 0; r \geq 2; j < n) .$$

Let ${}_j J^{i-s,s}$ be the submodules defining the filtration of $H^i(A_{jG})$ associated to the fibering over B_G . We have then (since the fibre degree is bounded by n):

$$(2) \quad \begin{aligned} H^i(A_{jG}) &= {}_j J^{0,i} = {}_j J^{i-n,n} \supset {}_j J^{i-n+1,n-1} \supset \dots \supset \\ &{}_j J^{i-1,1} \supset {}_j J^{i,0} \supset {}_j J^{i+1,-1} = 0 \end{aligned}$$

with

$$(3) \quad {}_j E_\infty^{s,t} = {}_j J^{s,t} / {}_j J^{s+1,t-1} .$$

We have of course

$$r_{j-1,j}^*({}_{j-1} J^{s,t}) \subset {}_j J^{s,t}$$

and $r_{j-1,j}^*$ is compatible with the isomorphism (3). Therefore (1) implies

$$(4) \quad r_{j-1,j}^*({}_{j-1} J^{i-s,s}) \subset {}_j J^{i-s+1,s-1} \quad (s \geq 1; j \leq n-1; i \geq 0) .$$

Since r_{U_G, V_G}^* is the composition of the maps $r_{j-1, j}^*$ it follows that

$$\text{Im } r_{U_G, V_G}^1 \subset n^{j^1, 0}.$$

But $n^{j^1, 0} = \pi_2^*(H^*(B_G))$, since we use closed supports and V is connected, hence (using IV, 2.1)

$$(5) \quad \dim \text{Im } r_{U_G, V_G}^* \leq 1.$$

We consider now the diagram

$$\begin{array}{ccc} H^1(U_G) & \longrightarrow & H^1((F \cap U) \times B_G) \\ \downarrow r_{U_G, V_G}^1 & & \downarrow r^1 \\ H^1(V_G) & \longrightarrow & H^1((F \cap V) \times B_G) \end{array},$$

where the horizontal maps are given by IV, 3.5 and $r^1 = r_{(F \cap U)_G, (F \cap V)_G}^1$. The horizontal maps are isomorphisms for $1 \geq n+1$ by IV, 3.6, hence the lemma follows readily from (5).

We leave it as an exercise to the reader to show, using the spectral sequences $(j_* E_r)$, that under the assumptions of the lemma (without requiring V to be connected), then $U \cap F$ is not empty if no K_j is empty.

§2. The fixed point set of a prime period map in a cohomology manifold

2.1. LEMMA. Let X be an n -cm over L . Let $U_0 \supset U_1 \supset \dots \supset U_{2n}$ be connected open orientable invariant subsets of X with U_1 adapted to U_{1-1} in the sense of I, 4.4 ($1 \leq i \leq 2n$), and let $U = U_0$, $V = V_n$. Then for all i , $\text{Im } j_{U_G, V_G}^i$ is a direct summand and is isomorphic to $H^{i-n}(B_G; H_G^n(U; L))$, hence to $H^{i-n}(B_G; L)$ if G preserves the orientation of U .

Here $H^k(B_G; H_G^s(U; L))$ means the k -th cohomology group of B_G with coefficients in the locally constant sheaf formed by the s -th cohomology groups of the fibres in the fibering of U_G over B_G . For $s = n$,

this sheaf is locally isomorphic to $L \times B_G$ and there are essentially two cases, depending on whether G preserves or reverses the orientation.

The proof will be analogous to that of Lemma 1.7. Let $({}_jE_r)$ be the spectral sequence of $U_{jG} \longrightarrow B_G$ and $f_{j,j-1}^*$ be the homomorphism defined by the inclusion of U_j in U_{j-1} . The inclusion $U_{jG} \longrightarrow U_{j-1,G}$ also defines a homomorphism $k_j : H_G^*(U_j; L) \longrightarrow H_G^*(U_{j-1}; L)$ and on the E_2 level $f_{j,j-1}^*$ is the cohomology map associated to this coefficient homomorphism. Since U_j is adapted to U_{j-1} , the map k_j is zero in degrees $\neq n$ and is an isomorphism in degree n . Therefore

$$(1) \quad f_{j,j-1}^*({}_jE_r^{s,t}) = 0 \quad (t \neq n, r \geq 2; j \geq 1)$$

(actually first for $r = 2$, but then a fortiori for $r \geq 2$), and

$$(2) \quad f_{j,j-1}^* \text{ is an isomorphism of } {}_jE_2^{s,n} \text{ onto } {}_{j-1}E_2^{s,n} \quad (j, s \geq 1).$$

We now want to prove:

(*) For $j \leq n$, ${}_jE_2^{s,n}$ ($s \geq 1$) consists of permanent cocycles, so that ${}_jE_2^{s,n} \cong {}_jE_\infty^{s,n}$ ($j \leq n; s \leq 0$). The map $f_{j,j-1}^*$ is zero on ${}_jE_\infty^{s,t}$ ($t \neq n$) and is an isomorphism on ${}_jE_\infty^{s,n}$ ($j \leq n; s \geq 0$).

The top fibre degree in $({}_jE_r)$ is n , hence no non-zero element of ${}_jE_r^{s,n}$ ($j \leq 2n, s \geq 0; r \geq 2$) can be a coboundary. Thus the assertions in (*) are consequences of the first one and (1) and (2). As to the former, it will follow by induction from:

(**). Assume that for some $j \geq 1$ and $k \geq 2$ the elements of ${}_jE_2^{s,n}$ are d_r -cocycles for $2 \leq r \leq k-1$. Then ${}_{j-1}E_2^{s,n}$ consists of d_r -cocycles for $2 \leq r \leq k$.

Since n is the top fibre degree, we have an isomorphism of ${}_jE_2^{s,n}$ onto ${}_jE_k^{s,n}$ (defined by the map κ_k^2 in the notation of Leray). Since $f_{j,j-1}^*$ is an isomorphism, the same will be true with j replaced by $j-1$ and we get a commutative diagram, in which all maps are isomorphisms

$$\begin{array}{ccc} {}_jE_2^{s,n} & \longrightarrow & {}_{j-1}E_2^{s,n} \\ \downarrow & & \downarrow \\ {}_jE_k^{s,n} & \longrightarrow & {}_{j-1}E_k^{s,n} \end{array}$$

Now

$$d_k(j_{-1}E_k^{s,n}) = d_k(f_{j,j-1}^*(j_{-1}E_k^{s,n})) = f_{j,j-1}^*(d_k(j_{-1}E_k^{s,n})) \subset f_{j,j-1}^*(E_k^{s+k,n-k+1})$$

and (1) implies that $d_k(j_{-1}E_k^{s,n}) = 0$. This proves (**), hence also (*).

j_{∞}^E is the "graded module associated to $H_C^*(U_{jG})$ suitably filtered". This means that we have a decreasing sequence of submodules

$$H_C^1(U_{jG}; L) = j^{J^0,1} \supset j^{J^1,1-1} \supset \dots \supset j^{J^1,0} \supset j^{J^{1+1},-1} = 0$$

with

$$(3) \quad j_{\infty}^{E^s,t} \cong j^{J^s,t} / j^{J^{s+1},t-1},$$

and $j^{J^0,1} = j^{J^{1-n},n}$ since the top fibre degree is n . The filtration considered here comes from the fibering π_2 hence $f_{j,j-1}^*(j^{J^s,t}) \subset j_{-1}^{J^s,t}$ and the homomorphism $f_{j,j-1}^*$ is compatible with (3). Therefore (1) implies

$$(4) \quad f_{j,j-1}^*(j^{J^{1-k},k}) \subset j_{-1}^{J^{1-k+1},k-1} \quad (j \geq 1; k \neq n).$$

Since $j_{U_{jG}V_G}^* = f_{1,0}^* \cdot f_{2,1}^* \cdot \dots \cdot f_{n,n-1}^*$, (3), (4), (*) yield

$$(5) \quad \ker j_{U_{jG}V_G}^1 = j^{J^{1-n+1},n-1}.$$

Since in the commutative diagram

$$(6) \quad \begin{array}{ccc} H_C^1(V_G; L) & \xrightarrow{j_{U_{jG}V_G}^1} & H_C^1(U_G; L) \\ \downarrow h_j & & \downarrow h_0 \\ o_{E_{\infty}^{1-n,n}} & \longrightarrow & o_{E_{\infty}^{1-n,n}} \end{array}.$$

the bottom map is an isomorphism by (*) and the kernel of h_j is $j^{J^{1-n+1},n-1}$ the lemma follows.

2.2. THEOREM (Smith [5, Theorem 1]). Let p be a prime number, X a cohomology manifold over Z_p , G a finite p -group acting on X , and F the fixed point set of G . Then F is a cohomology manifold over Z_p , and is orientable if X is.

If N is invariant in G , then the fixed point set of N is

invariant under G . Hence, by a trivial induction, it is enough to consider the case where $G = Z_p$. Assume F non-empty (otherwise there is nothing to prove) and let $x \in F$. Let U be an orientable connected neighborhood of x in X . Replacing it by the component of x in the intersection of U with its transforms under G , we may assume it to be invariant under G . Now if a neighborhood W of $y \in F \cap U$ is adapted to U , then every neighborhood of y in W is also adapted to U hence by the same construction, we see that there exists a sequence $U = U_0 \supset \dots \supset U_{2n}$ of invariant open connected neighborhoods of y , with U_j adapted to U_{j-1} . Putting $V = U_n$ we have a commutative diagram

$$(7) \quad \begin{array}{ccc} H_C^1(U_G) & \longrightarrow & H_C^1((F \cap U)_G) = H_C^1((F \cap U) \times B_G) \\ \uparrow j_{U_G, V_G}^1 & & \uparrow j^1 = j_{(F \cap V)_G, (F \cap U)_G}^1 \\ H_C^1(V_G) & \longrightarrow & H_C^1((F \cap V)_G) = H_C^1((F \cap V) \times B_G) \end{array}$$

where the horizontal maps are given by IV, 3.4. From IV, 2.1 and the Künneth rule we have for $1 \geq n$, $H_C^1((F \cap U)_G) = H_C^*(F \cap U)$ and clearly, this isomorphism carries j^1 over to $j_{F \cap U, F \cap V}^*$. For $1 \geq n+1$, the horizontal maps are isomorphisms by IV, 3.5 and j_{U_G, V_G}^1 has a 1-dimensional image by 2.1. Hence $\text{Im } j_{F \cap U, F \cap V}^*$ is 1-dimensional. This shows first (for $x = y$) that the sum of the local Betti numbers of F at x is 1. Therefore (I, 2.2), F is clc; let $k(x)$ be the dimension in which F has local Betti number 1 at x . Then it is easily seen that the set of $x \in F$ for which $k(x) = k$ is open. Since F is locally connected, it follows that it is a Wilder manifold. Also, since the above discussion is valid for any $y \in F \cap U$, we see by I, 3.4 that F is locally orientable. If now a given connected component F_0 of F is contained in an orientable component of X , we may assume $U \cap F = F_0$, hence, again by I, 3.4, F_0 is orientable.

2.3. THEOREM. We keep the assumptions of 2.2. Let $k = \dim_p F_0$, $n = \dim_p X_0$, where F_0 is a connected component F_0 of F , and X_0 the connected component X_0 of X containing F_0 . Then $n - k$ is even in each of the following cases:

- (a) (Bredon [1]), p is odd.
- (b) $p = 2$, $G = Z_4$ and G acts freely outside F_0 in X_0 .

By induction again, it is enough to prove (a) for $G = \mathbb{Z}_p$. In the following we discuss simultaneously the case (b) and the case $G = \mathbb{Z}_p$, p odd. We have then (IV, 2.1):

$$(8) \quad H^*(G; \mathbb{Z}_p) = \Lambda a \otimes \mathbb{Z}_p[b] \quad (d^0 a = 1, d^0 b = 2) \quad .$$

The operator defined by a on the cohomology of a fibering over B_G (see XVI, §1) will be denoted by α .

Let $x \in F_0$, U an open connected orientable neighborhood of x in X such that $U \cap F$ is connected, hence equal to $U \cap F_0$, and orientable. We then can easily find a sequence (U_j) of neighborhoods of x satisfying the assumptions of 2.1, such that $V \cap F$ is connected. As was seen in the proof of 2.2 $V \cap F$ is then adapted to $U \cap F$, therefore the image of $j^1 : H_G^1((F \cap V)_G) \longrightarrow H_G^1((F \cap U)_G)$ is equal to $H_G^r(F \cap U) \otimes H^{1-r}(G)$. Then (8) implies that α annihilates $\text{Im } j^1$ if and only if $1 - r$ is odd. On the other hand, in the notation of the proof of 2.1

$${}_n E_\infty^{1-n, n} = {}_n E_2^{1-n, n} = H^{1-n}(G) \otimes H_G^n(V)$$

hence $\alpha({}_n E_\infty^{1-n, n}) = 0$ if and only if $1 - n$ is odd. 2.3 follows now from 2.2 (7) and from the fact that in the diagram (7), all maps commute with α , and the horizontal homomorphisms are bijective.

2.4. Theorem 2.3 corresponds to (IV, 4.4). We now want to discuss an analogue for cohomology manifolds of IV, 4.7 when F is not empty. The proof parallels that of 2.3, with α replaced by Sq^1 . Since it does not involve new ideas, it will be dealt with more briefly.

Let X be an n -cm over Z , operated upon effectively by $G = \mathbb{Z}_2$, with a non-empty set of fixed points F . Let $x \in F$ and $U \supset V$ be invariant orientable connected neighborhoods of x in X . Then if G acts trivially (non-trivially) on $H_G^n(U; Z) = Z$, G acts in the same way on $H_G^n(V; Z)$ since the isomorphism j_{UV}^n commutes with G . We say that G preserves (respectively reverses) the local orientation of X around x if it acts trivially (respectively non-trivially) on $H_G^n(U; Z) = Z$. Here X is also a n -cm over $\mathbb{Z}_2$, hence by 2.2 F is a generalized manifold over $\mathbb{Z}_2$, in particular it is locally connected. In the following, $j_{..L}^*$ will stand for the map $j_{..}^*$ in cohomology with coefficients in L .

LEMMA. Let $U_0 \supset U_1 \supset \dots \supset U_{2n}$ be a sequence of invariant connected open subsets of X , orientable over Z , with U_{j+1} adapted to U_j ($j \geq 1$), and

let $U = U_0$, $V = U_n$. Then $\text{Im } j_{U_G V_G, Z}^{2k+n} = Z_2$,
 $\text{Im } j_{U_G V_G, Z}^{2k+1+n} = 0$ (respectively $\text{Im } j_{U_G V_G, Z}^{2k+n} = 0$,
 $\text{Im } j_{U_G V_G, Z}^{2k+1+n} = Z_2$), if G preserves (respectively
 reverses) the orientation of U . The operation
 Sq^1 is zero on $\text{Im } j_{U_G V_G, Z_2}^{2k+n}$ (respectively
 $\text{Im } j_{U_G V_G, Z_2}^{2k+1+n}$) if G preserves (respectively re-
 verses) the orientation.

The first assertion follows from 2.1 and IV, 2.1.

To prove the second assertion, let us consider the diagram

$$\begin{array}{ccccccc}
 0 \longrightarrow & H_c^1(V_G; Z) \otimes Z_2 & \longrightarrow & H_c^1(V_G; Z_2) & \longrightarrow & \text{Tor}(H_c^{1+1}(V_G; Z), Z_2) & \longrightarrow 0 \\
 & \downarrow j_1 & & \downarrow j_2 & & \downarrow j_3 & \\
 0 \longrightarrow & H_c^1(U_G; Z) \otimes Z_2 & \longrightarrow & H_c^1(U_G; Z_2) & \longrightarrow & \text{Tor}(H_c^{1+1}(U_G; Z), Z_2) & \longrightarrow 0
 \end{array}$$

in which the rows are the exact universal coefficient sequences and the j_i 's are defined by $j_{U_G V_G}^*$. Assume the G preserves the orientation

and let $1 = 2k + n$. Then the first assertion implies that $j_3 = 0$, hence $\text{Im } j_2$ consists of reduction mod 2 of integral classes and is therefore annihilated by Sq^1 . One argues in the same way if G reverses the orientation.

2.5. THEOREM (Bredon [1]). Let G be a group of order 2 operating effectively on an n -cm over Z , with fixed points. Let F_0 be a connected component of the fixed point set of G , and $r = \dim_2 F_0$. Then $n - r$ is even (respectively odd) if and only if G preserves (respectively reverses) the local orientation around some point of F_0 .

COROLLARY. If G preserves the local orientation around one point of F_0 , then it does so around every point of F_0 .

Let $x \in F_0$. Using 2.1, 2.2 and the lemma, we first find orientable connected neighborhoods $U \supset V$ of x in X with V adapted to U over Z , hence also over Z_2 , and $U \cap F$, $V \cap F$ connected,

orientable mod Z_2 , $V \cap F$ adapted to $U \cap F$ over Z_2 . Then the image of $j^1 : H_G^1((V \cap F)_G; Z_2) \longrightarrow H_G^1((U \cap F)_G; Z_2)$ is $H^{1-r}(Z_2; Z_2) \otimes H_G^r(F \cap U; Z_2)$. Therefore Sq^1 is zero on $\text{Im } j^1$ if and only if $1 - r$ is odd. The theorem follows now from the lemma by using the diagram (7) in cohomology mod 2, noting that all maps commute with Sq^1 and that the horizontal ones are isomorphisms for $1 \geq n + 1$, by IV, 3.7.

2.6. The following result is due to C. T. Yang [6, Theorem 1] and will be used in a later lecture.

THEOREM. Let X be a connected n -cm over Z , G be a finite group acting effectively on X and F the fixed point set. If $G \nmid (e)$, then $\dim_Z F \leq n - 1$, and if the equality holds, then $G = Z_2$.

In this proof, $F(H)$ denotes the fixed point set of a subgroup H of G . If H is a cyclic group of prime order q , then $\dim_P F(H) \leq n - 1$ (by 2.2 and I, 4.6) hence $\dim_Z F(H) \leq n - 1$ by I, 4.9 and $\dim_Z F \leq n - 1$ since F is closed in $F(H)$. In order to prove the second assertion, it suffices by I, 4.9 to show that if $G \nmid Z_2$, there exists a prime q and a subgroup H for which $\dim_Q F(H) \leq n - 2$. If the order of G is divisible by an odd prime, this follows from 2.4. Let now G be of order 2^s ($s \geq 2$), and let $x \in F$. The subgroup N of G which leaves the local orientation around x invariant is clearly of index 1 or 2, hence of order ≥ 2 , and contains therefore a subgroup of G of order 2; for the latter we have then $\dim_Z F(H) \leq n - 2$ by 2.5.

§3. The fixed point set of a toral group in a cohomology manifold

3.1. **PROPOSITION.** Let $G = T^1$, X be an n -cm over Z_2 , Y a connected component of the fixed point set of the subgroup H of order 2 of G , and r the dimension of Y over Z_2 . If Y contains a point fixed under G , then $n - r$ is even.

By Smith's theorem (2.2) Y is an r -cm over Z_2 . Let $y \in Y$, and (U_i) ($0 \leq i \leq 2n$) a decreasing sequence of open connected H -invariant neighborhoods of y , with U_i adapted to U_{i-1} ($1 \leq i \leq 2n$). It follows from the proof of 2.2, that $U_n \cap Y$ is adapted to $U_0 \cap Y$. Assume now that y also belongs to $F(G; X)$. It has arbitrarily small neighborhoods invariant under G , therefore, by the remark just made, there exists a sequence (U_i) ($0 \leq i \leq 2n$) of open connected G -invariant neighborhoods of y such that U_i is adapted to U_{i-1} and $U_i \cap Y$ is adapted to $U_{i-1} \cap Y$.

for $i = 1, \dots, 2n$; these sets are clearly invariant under G . Let then $U_n = V$, $U_0 = U$ and consider the diagram, in cohomology mod 2

$$\begin{array}{ccc} H_G^1(U_G) & \xrightarrow{\lambda} & H_G^1((U \cap Y)_G) \\ \uparrow j_{U_G, V_G}^1 & & \uparrow j^1 \\ H_G^1(V_G) & \xrightarrow{\mu} & H_G^1((V \cap Y)_G) \end{array}$$

where $j^1 = j_{(U \cap Y)_G, (V \cap Y)_G}^1$ and the horizontal maps are restrictions. By 2.1 and our assumption on the U_i .

$$(1) \quad \text{Im } j_{U_G, V_G}^1 \cong H^{1-n}(B_G), \quad \text{Im } j^1 \cong H^{1-r}(B_G).$$

On the other hand, $F(H_2; X)$ is locally connected, so that it may be assumed that $U \cap F(H_2; X) = U \cap Y$. Then for $x \in U - Y$, G_x is a finite group of odd order, and $H^*(B_{G_x}; Z_2)$ is trivial and therefore (IV, 3.6) λ and μ are isomorphisms for $i > n$. Since $H^k(B_G; Z_2)$ is zero for k odd and isomorphic to Z_2 for k even, our assertion now follows from (1).

REMARK. In the usual action of G on the projective plane, $F(G; X)$ is a point and $F(H; X)$ is a point and a circle. Thus the assumption $Y \cap F(G; X)$ non-empty is essential.

3.2. THEOREM (Conner-Floyd). Let $G = T^k$, $L = K_p$, Z and X be a n -cm over L . Then F is a cm over L . The dimension of each connected component of F has the same parity as n . F is orientable over L if X is.

By induction, it is enough to consider the case $k = 1$. If $L = K_0$, the proof is quite the same as that of 2.2, the dimensional parity coming now from the fact that $H^*(B_G; L)$ vanishes in odd dimensions, and will not be given.

Let now $L = K_p$ (p prime). Let F_0 be a component of F and let F_1 ($i \geq 1$) be the component of $F(Z_{p^i}; X)$ containing F_0 , where Z_{p^i} denotes the subgroup of order p^i contained in G . The F_1 are cohomology manifolds over Z_p by 2.2, and $F_{i+1} \subset F_i$. Therefore (I, 4.6) $F_1 = F_j$ for $i, j \geq i_0$, hence $F_1 = F_0$ for i large enough. Our assertion follows then from 2.2, 2.3 and 3.1.

From now on, $L = Z$. An r -cm over Z is a r -cm over any field, and it is orientable over Z if and only if it is orientable over some field of characteristic different from two, therefore it will be enough to show that if U is an invariant open connected orientable set such that $U \cap F$ is connected, then $U \cap F$ is an r -cm over Z , for some r . To this effect, let V be as in 2.1 and consider the diagram (where cohomology is over Z)

$$\begin{array}{ccccccc} H_c^1(U_G - F_G) & \longrightarrow & H_c^1(U_G) & \xrightarrow{\lambda} & H_c^1((F \cap U)_G) & \longrightarrow & H_c^{1+1}(U_G - F_G) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H_c^1(V_G - F_G) & \longrightarrow & H_c^1(V_G) & \xrightarrow{\mu} & H_c^1((F \cap V)_G) & \longrightarrow & H_c^{1+1}(V_G - F_G) \end{array}$$

For $x \notin F$, G_x is finite, hence $H^*(B_{G_x}; Q)$ is trivial and (IV, 3.4)

$H_c^1(U_G - F_G; Q) = 0$ for $1 > n$, Q being the field of rational numbers, therefore $H_c^1(U_G - F_G; Z)$ is a torsion group. Let $A_1 = \text{Im } j_{U_G, V_G}^1$. By 2.1, $A_1 = H^{1-n}(B_G; Z)$, hence $A_1 = Z$ for $1 - n$ even, $A_1 = 0$ for $1 - n$ odd. Moreover, since $H_c^1((U_G - F_G); Z)$ is a torsion group, $A_1 = \lambda(A_1)$ is isomorphic to A_1 for $1 > n$.

Now $(U \cap F)_G = (U \cap F) \times B_G$, hence $H_c^1((U \cap F)_G; Z) = \sum_{s=1}^n H_c^s(U \cap F; Z)$; the same is true of course with U replaced by V ; it is clear that these isomorphisms carry j^1 into the sum of the $j_{U \cap F, V \cap F}^s$; therefore, $\text{Im } j_{U \cap F, V \cap F}^*$ contains a subgroup $A' = Z$ (namely A_1 for $1 = n(2)$). If W is open invariant in V , then $\text{Im } j_{U_G, W_G}^1$, which is a priori contained in $\text{Im } j_{U_G, V_G}^1$, is in fact equal to it, since it is a direct summand by 2.1. As a consequence $A' \subset \text{Im } j_{U \cap F, W \cap F}^*$. Clearly, if W runs through a fundamental system of invariant neighborhoods of a point $x \in U \cap F$, then $W \cap F$ runs through a basic system of neighborhoods of x in F . Therefore we have shown that for each $x \in U \cap F$, there exists a subgroup $A(x)$ of $H_c^*(U \cap F; Z)$, isomorphic to Z , and contained in $\text{Im } j_{U \cap F, W}^*$ for every open neighborhood W of x in $U \cap F$. On the other hand, we already know that $U \cap F$ is an orientable cm over any field. Moreover, it will be shown in VI that the orbit structure of a torus operating on a cm_Z is locally finite. Assuming this, F is clc_Z by 1.6. The fact that $U \cap F$ is an r -cm over Z now follows from (I, 4.11).

REMARK. The local finiteness of the orbit structure was used only in the case $L = Z$, at the end of the preceding proof. In particular, the argument for $L = K_p$, p prime, is a simple reduction to the case of $G = Z_p$, based on 2.2 and I, 4.6 which did not need any assumption about

the orbit type. We point this out because in VI, the proof of the local finiteness of the orbit type will make use of the above Theorem 3.2, in the case $L = K_p$, p prime

§4. Remarks on local groups of the quotient space

We make here some remarks about the local cohomology groups of X/G around a fixed point x , in the cases considered in this chapter. They are merely meant to point out further applications of the method used here, and make no pretense at giving full proofs. These groups are computed in [1]. Roughly speaking, if n and r are respectively the dimension of the components of X and F containing x , they are the same as the cohomology groups of the quotient S_n/G , when $F(G; S_n)$ is a cohomology r -sphere. Such results, and the analogous ones for the circle group, can be obtained by combining the methods used in IV, §6 and in the proof of 2.1. More specifically, one takes a decreasing sequence of adapted neighborhoods (U_i) of x such that the $U_i \cap F$ are also adapted and considers the homomorphisms of the spectral sequences of $U_i G \longrightarrow B_G$ and of $U_i G \longrightarrow U_i/G$.

This is however a somewhat clumsy procedure, chiefly because the homomorphisms go in the "wrong" direction, namely, the one which leads to projective limits. Therefore it is better to try to avail oneself of the localization device of XIII, which uses inductive limits. Using arguments similar to those of XIII, it is seen without difficulty that the limit spectral sequence at $x \times B_G$ of $X_G \longrightarrow X'$ is the same as the spectral sequence of $Y \longrightarrow Y/G$ when Y is a cohomology $(n-1)$ -sphere, and $F(G; Y)$ a cohomology $(r-1)$ -sphere, the groups $H^1(Y')$ taking the place of the local groups $I_X^1(X')$. Thus in particular the results of IV, §6 yield the local groups $I_X^1(X)$. Of course, this result can be used only if one knows how to relate the groups $I_X^1(X')$ to the usual local cohomology groups. According to some work now in progress by Raymond, it seems that, under suitable assumptions for the underlying space M , (for instance, metrizable, clc), $I_X^1(M)$ may be identified with the $(i+1)$ -st local cohomology group of M at x , (using reduced cohomology for $i = 0$). This of course checks with, and in fact would yield, the results of Bredon mentioned above.

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VI: ISOTROPY SUBGROUPS OF TORAL GROUPS

E. E. Floyd

§1. Introduction

The purpose here is to prove the following:

1.1. THEOREM. Suppose a toral group G acts on an n - cm X over Z . If C is a compact subset of X , then there are only a finite number of isotropy subgroups G_x , $x \in C$.

For differentiable actions, this is a special case of a theorem of Yang [8] (see also VII and VIII). For general actions, this has been proved for compact orientable manifolds by the author [3]; Mostow extended it to compact manifolds [6], as well as to compact Lie groups (see VII). In the above form, the theorem has been proved by Larry Mann [4].

Theorem 1.1 will be obtained as a corollary of (1.2) below, which will be proved in Section 4.

1.2. THEOREM. Suppose the compact abelian Lie group G acts on the n - cm X over Z , and that $\{G_i\}$ is a sequence of closed subgroups of G with $\lim G_i = G$. If C is a compact subset of X and if $F(G_i)$, $F(G)$ denote the stationary points of G_i , G respectively then $F(G_i) \cap C = F(G) \cap C$ for i sufficiently large.

We show now that (1.2) implies (1.1). Suppose (1.2) holds but that C is a compact subset of X with $\{G_x : x \in C\}$ infinite. There is a sequence $\{G_{x_1}\}$ of distinct isotropy subgroups, $x_1 \in C$. The collection of all closed subsets of the compact metric space G is itself a compact metric space, using the Hausdorff metric. Hence we may as well suppose, taking a subsequence, that the sequence $\{G_{x_1}\}$ converges in the

Hausdorff metric to a closed subgroup H of G . Since almost all G_{x_1} are contained in each neighborhood of H , it follows from a theorem of Montgomery and Zippin [5] that almost all G_{x_1} are conjugate under inner automorphisms to subgroups of H . Since G is abelian, then almost all $G_{x_1} \subset H$. Hence $\{G_{x_1}\}$ is a sequence of closed subgroups of H with $\lim G_{x_1} = H$. Now $F(G_{x_1}) \cap C = F(H) \cap C$ for i large by (1.2). Hence $F(G_{x_1}) \cap C = F(G_{x_j}) \cap C$ for i, j large. Since $x_1 \in F(G_{x_1}) \cap C$ then $x_1 \in F(G_{x_j})$ and $G_{x_j} \subset G_{x_1}$. Similarly $G_{x_1} \subset G_{x_j}$ and we get a contradiction.

§2. A regular convergence theorem

We prove in this section a lemma which is the purely topological part of the proof of (1.2). The reader may well decide to skip it temporarily, and return to it after reading the later sections.

2.1. Suppose that X is a connected, orientable n -cm over L . Suppose also that $A \supset A'$ are closed subsets of X such that there is a separation $X - A' = U' \cup V'$ (U' and V' open, non-empty and disjoint) with neither U' nor V' contained in A . Then $r_{CAA'}^{n-1} \neq 0$.

PROOF. There is the separation $X - A = U \cup V$ with $U = U' \cap (X - A)$, $V = V' \cap (X - A)$. Consider the following diagram, with coefficients in L and each row the cohomology sequence of a pair:

$$\begin{array}{ccccccc}
 H_c^{n-1}(A) & \xrightarrow{\delta} & H_c^n(U) \oplus H_c^n(V) & \xrightarrow{j} & H_c^n(X) & \xrightarrow{\quad} & \\
 \downarrow r & & \downarrow j'' & & \downarrow & & \\
 H_c^{n-1}(A') & \xrightarrow{\delta'} & H_c^n(U') \oplus H_c^n(V') & \xrightarrow{j'} & H_c^n(X) & \xrightarrow{\quad} &
 \end{array}$$

Now j_{XU}^n and j_{XV}^n are onto. If c is a non-zero element of $H_c^n(X)$, there are elements c_1, c_2 in $H_c^n(U), H_c^n(V)$ respectively with $j_{XU}^n(c_1) = j_{XV}^n(c_2) = c$. Then $j(c_1 \oplus c_2) = c$, and $c_1 \oplus c_2 = \delta(c')$ for some $c' \in H_c^{n-1}(A)$. Then $r(c') \neq 0$, since $j''(c_1 \oplus c_2) \neq 0$. The remark follows.

2.2. NOTATION. Suppose that $\{Y_i\}$ is a sequence of closed subsets of the locally compact Hausdorff space X . Say that $\{Y_i\}$ converges to the subset Y of X , written $\lim Y_i = Y$, if every neighborhood

of a point of Y intersects all but a finite number of Y_i , while every $x \notin Y$ has a neighborhood intersecting Y_i for only finitely many i . If $\lim Y_i = Y$ then $Y \cup \bigcup Y_i$ is closed in X . We need now a variant of the regular convergence concept; for an exposition of regular convergence, see White [7]. Suppose $\lim Y_i = Y$, and that $\{L_i\}$ is a sequence of coefficient rings (all principal ideal domains). Say that $\{Y_i\}$ converges regularly to Y over $\{L_i\}$ if given $y \in Y$ and a compact neighborhood A of y , there is a compact neighborhood $B \subset A$ of y and an integer I with the restrictions

$$H^*(A \cap Y_i; L_i) \longrightarrow H^*(B \cap Y_i; L_i)$$

$$H^*(A \cap Y; L_i) \longrightarrow H^*(B \cap Y; L_i)$$

trivial for all $i \geq I$ (using reduced cohomology).

We use Dyer's version [1, p. 129] of a theorem of the author in the following. By a closed covering α of a space X we mean a finite collection of closed subsets of X which covers X . If $A \supset B$ are compact subsets of a space X , and if α, β are closed coverings of A, B respectively with $\beta > \alpha$, write $\beta^n \gg \alpha$ (over L) if there is a projection $\pi: \beta \longrightarrow \alpha$ such that for arbitrary elements $B_0, \dots, B_k$ of β , $r_{\pi B_0}^1 \dots \pi B_k, B_0 \cap \dots \cap B_k = 0$ for all $i \leq n$ (using reduced cohomology). The theorem in question yields the following: if $A_{-1} \supset A_0 \supset \dots \supset A_{2n}$ are compact subsets of a space and if $\alpha_{-1} \ll^n \alpha_0 \ll^n \dots \ll^n \alpha_{2n}$ (over L), where α_1 is a closed covering of A_1 , then in the diagram (coefficients in L)

$$\begin{array}{ccccc} H^k(\alpha_{-1}) & \xrightarrow{\pi} & H^k(\alpha_n) & \xrightarrow{\pi'} & H^k(\alpha_{2n}) \\ \downarrow \lambda_1 & & \downarrow \lambda_2 & & \downarrow \lambda_3 \\ H^k(A_{-1}) & \xrightarrow{r} & H^k(A_n) & \xrightarrow{r'} & H^k(A_{2n}) \end{array}$$

where the π 's are projections, the r 's are restrictions, and the λ 's are the natural maps, we have $\text{Image } r \subset \text{Image } \lambda_2$ and $\text{Kernel } \pi' \supset \text{Kernel } \lambda_2$ for all $k \leq n$.

2.3. THEOREM. Suppose that X is a locally compact Hausdorff space with $n = \dim_{\mathbb{Z}} X < \infty$, and that $\{L_i\}$ is a sequence of coefficient rings. Suppose that $\{Y_i\}$ is a sequence of closed subsets of X which converges regularly over $\{L_i\}$ to the non-empty subset Y ,

where Y_i is a connected orientable m_i -cm over L_i and Y is a connected orientable n_1 -cm over L_1 for every i . Then $m_i = n_1$ for i sufficiently large. In particular, if $Y_i \supset Y$ for all i then $Y_i = Y$ for i sufficiently large.

PROOF. Consider first the case in which Y consists of a single point y , so that $n_1 = 0$ for all i . Suppose the conclusion false; we may as well assume $m_i > 0$ for all i . If U is a neighborhood of y with $\bar{U}$ compact, then $Y_i \subset U$ for i large. For otherwise $Y_i \cap (\bar{U} - U) \neq \emptyset$ for arbitrarily large i since Y_i is connected, so that we would have $Y \cap (\bar{U} - U) \neq \emptyset$. If now A and B are compact neighborhoods of y then $A \cap Y_i = B \cap Y_i = Y_i$ for i large. Then

$$H^{m_i}(A \cap Y_i; L_i) \longrightarrow H^{m_i}(B \cap Y_i; L_i)$$

is the identity map of $H^{m_i}(Y_i; L_i)$, which is non-trivial since Y_i is orientable. This contradicts regular convergence, so that $m_i = 0$ for i large.

Consider hereafter the case in which Y consists of more than one point. Let $y \in Y$, and let U be an open set (in X) containing y with $\bar{U}$ compact and with $Y \not\subset \bar{U}$. Denote by C the set $Y \cap (\bar{U} - U)$. Cover C with a finite collection α_{-1} of compact sets in X whose interiors cover C , and denote by A_{-1} the union of the elements of α_{-1} . Suppose α_{-1} so small that $y \notin A_{-1}$ and $Y - \bar{U} \not\subset A_{-1}$. Suppose also that

$$(2.4) \quad \text{if } B_0, \dots, B_k \in \alpha_{-1} \text{ and } B_0 \cap \dots \cap B_k \neq \emptyset$$

then $\text{int } B_0 \cap \dots \cap \text{int } B_k \cap C \neq \emptyset$. This is easily satisfied (see [2, (4.5)]). Then $\text{nerve } \alpha_{-1} = \text{nerve } \alpha_{-1} \cap Y = \text{nerve } \alpha_{-1} \cap Y_i$ for i large.

For each $y' \in C$ select an element $B \in \alpha_{-1}$ containing y' in its interior. There is a neighborhood $B_{y'} \subset B$ of y' with $H^*(B \cap Y; L_1) \longrightarrow H^*(B_{y'} \cap Y; L_1)$, $H^*(B \cap Y_i; L_1) \longrightarrow H^*(B_{y'} \cap Y_i; L_1)$ trivial for i large. There is a finite collection β of $B_{y'}$'s whose interiors cover C . There is then a star refinement α_0 of β whose interiors cover C . We may also suppose α_0 so small that (2.4) holds for it. Define A_0 to be the union of the elements of α_0 . It is then seen that $\alpha_0 \cap Y_1 \xrightarrow{n} \alpha_{-1} \cap Y_1$ over L_1 and $\alpha_0 \cap Y \xrightarrow{n} \alpha_{-1} \cap Y$ over L_1 for i large (see [1, p. 132]). Moreover $\text{nerve } \alpha_0 = \text{nerve } \alpha_0 \cap Y = \text{nerve } \alpha_0 \cap Y_i$ for i large.

Continue to get $\alpha_{-1}, \alpha_0, \dots, \alpha_{2n}$; let A_1 denote the union of the elements of α_1 . Suppose the construction made so that

$\alpha_{k+1} \cap Y_1 \xrightarrow{n} \alpha_k \cap Y_1, \alpha_{k+1} \cap Y \xrightarrow{n} \alpha_k \cap Y$ over L_1 for i large.

Since A_k contains $C = Y \cap (\bar{U} - U)$ in its interior, it follows from $\lim Y_1 = Y$ that A_k contains $Y_1 \cap (\bar{U} - U)$ for i large. Hence for i large there is the separation of $Y_1 - A_k \cap Y_1$, separating it into its intersection with U and with $X - \bar{U}$. Hence by (2.1),

$$H^{m_1-1}(A_{-1} \cap Y_1; L_1) \longrightarrow H^{m_1-1}(A_{2n} \cap Y_1; L_1)$$

is non-trivial. In the diagram

$$(2.5) \quad \begin{array}{ccccc} H^k(\alpha_{-1} \cap Y_1; L_1) & \xrightarrow{\pi} & H^k(\alpha_n \cap Y_1; L_1) & \xrightarrow{\pi'} & H^k(\alpha_{2n} \cap Y_1; L_1) \\ \downarrow \lambda_1 & & \downarrow \lambda_2 & & \downarrow \lambda_3 \\ H^k(A_{-1} \cap Y_1; L_1) & \xrightarrow{r} & H^k(A_n \cap Y_1; L_1) & \xrightarrow{r'} & H^k(A_{2n} \cap Y_1; L_1) \end{array}$$

$$(2.6) \quad \text{Image } r(\text{Image } \lambda_2, \text{Kernel } \pi') \text{Kernel } \lambda_2$$

for i large and $k \leq n$. Since $r'r \neq 0$ for $k = m_1 - 1$, the equations (2.6) imply that $\pi' \neq 0$ for $k = m_1 - 1$ and i large. Since $\text{nerve } \alpha_k = \text{nerve } \alpha_k \cap Y = \text{nerve } \alpha_k \cap Y_1$ for i large then π' coincides with $\pi' : H^k(\alpha_n \cap Y; L_1) \longrightarrow H^k(\alpha_{2n} \cap Y; L_1)$. But there is a similar diagram to (2.5) with Y_1 replaced by Y and with (2.6) holding. Since $\pi' \neq 0$, then $\lambda_2 \neq 0$ and $H^{m_1-1}(A_n \cap Y; L_1) \neq 0$, i large. Since $A_n \cap Y$ is a proper compact subset of Y , then $n_1 = \dim_{L_1} Y \geq m_1$ (i large) by I, Theorem 4.3. A symmetric argument shows $m_1 \geq n_1$, so that $m_1 = n_1$ for i large.

§3. Two lemmas

3.1. Suppose that G , either a toral group or a p -group (p prime), acts on an (orientable) cm X over Z_p . Then each component of $F(G)$ is an (orientable) cm over Z_p .

This was proved in V, 3.2.

3.2. Suppose that G , either a p -group (p prime) or a toral group, acts on X , where $\dim_p X \leq n < \infty$ and each component of X is a cm over Z_p . Suppose that $A_0 \supset A_1 \supset \dots \supset A_{(n+1)}$ is a sequence of compact invariant subsets of X with $r_{A_i A_{i+1}}^* = 0$ over Z_p for each i . Then

$$r_{F(G) \cap A_0, F(G) \cap A_{(n+1)}}^* = 0 \text{ over } Z_p.$$

PROOF. Since $r_{A_0 A_1}^*$ is trivial in dimension 0, A_1 is contained in a single component of A_0 . [This uses the fact that every neighborhood of a component of a compact Hausdorff space contains an open and closed neighborhood.] Since A_1 is invariant, the component of X containing A_1 is also invariant. It follows that it is sufficient to prove the assertion in case X is connected; it is trivially true in case $n = 0$. Suppose it has been proved in dimensions $< n$, and that G operates effectively.

Now $G \supset Z_p$, and each component of $F(Z_p)$ is a cm over Z_p of dimension $< n$. Note also that A_{n+1} is contained in a single component A'_n of A_n , and that A'_n is invariant. Then $A_0 \supset A_1 \supset \dots \supset A_{n-1} \supset A'_n$ are invariant under Z_p , and $r_{A_0 A_1}^*, \dots, r_{A_{n-1} A'_n}^*$ are trivial. By V (1.7), $r_{F(Z_p) \cap A_0, F(Z_p) \cap A'_n}^* = 0$. Hence $r_{F(Z_p) \cap A_0, F(Z_p) \cap A_{n+1}}^* = 0$. Considering $F(Z_p) \cap A_0 \supset F(Z_p) \cap A_{n+1} \supset F(Z_p) \cap A_{2(n+1)} \supset \dots \supset F(Z_p) \cap A_{n!(n+1)}$, we see that each set is invariant under G , and that $r_{F(Z_p) \cap A_{k(n+1)}, F(Z_p) \cap A_{(k+1)(n+1)}}^* = 0$. Moreover $\dim_p F(Z_p) \leq n - 1$. By induction, $r_{F(G) \cap A_0, F(G) \cap A_{(n+1)}}^* = 0$.

§4. Proof of Theorem 1.2

Suppose now that X is a cm over Z , that the compact abelian Lie group G acts on X and that $\{G_i\}$ is a sequence of closed subgroups of G with $\lim G_i = G$. Let C be a compact subset of X . We must show that $F(G_i) \cap C = F(G) \cap C$ for i large.

Clearly $F(G_i) \supset F(G)$. It may be seen that $\lim F(G_i) = F(G)$. It is sufficient to prove that every $x \in F(G)$ has a neighborhood C with $F(G_i) \cap C = F(G) \cap C$ for i large. Every $x \in F(G)$ can be seen to have arbitrarily small connected invariant neighborhoods. In particular we may choose an orientable such neighborhood. Hence it is sufficient to prove the theorem for X orientable.

For purposes of induction, we replace the condition that X be an orientable cm over Z by the following: (1) $\dim_Z X < \infty$, (2) every component of X is an orientable cm over Z_p for every prime p , (3) given $x \in X$ and a compact neighborhood A of x there is a compact neighborhood $B \subset A$ of x with $r_{AB}^* = 0$ for all coefficient groups Z_p , p prime. Note that if X satisfies (1), (2), (3) and a toral group G acts on X , then $F(G)$ has (1), (2), (3).

For (1) is obvious, (2) follows from 3.1 and (3) from 3.2.

An orientable cm X over Z satisfies (2) by I 4.5. An easy argument using the universal coefficient theorem shows that (3) holds.

We shall now prove the theorem in case G is a toral group and G_1 is a subgroup of prime power order $p_1^{n_1}$. Suppose $x \in F(G)$ and that A is a compact neighborhood of x . There is a sequence $A \supset A_0 \supset \dots \supset A_{(n+1)}$ $n = \dim_Z X$, of compact neighborhoods of x with A_1 invariant and $r_{A_1 A_{1+1}}^* = 0$ for all coefficient groups Z_p , p prime. From 3.2,

$$(4.1) \quad r_{A_0 \cap F(G_1), A_{(n+1)} \cap F(G_1)}^* = 0, \quad r_{A_0 \cap F(G), A_{(n+1)} \cap F(G)}^* = 0$$

with coefficients in Z_{p_1} for all i . This being true in dimension 0, $A_{(n+1)} \cap F(G_1) \cap F(G)$ is contained in a single component of $F(G_1)(F(G))$, namely the component containing x . If $M_1(M)$ is the component of $F(G_1)(F(G))$ containing x , it is then clear that it is sufficient to show $M_1 = M$ for i large.

Let now $E_1(E)$ be the union of all components of $F(G_1)(F(G))$ not containing x . Since $F(G_1)$ and $F(G)$ are clc and hence locally connected, E_1 and E are closed. Since $F(G) \cup UF(G_1)$ is closed and since E_1 is closed in $F(G_1)$ then $F(G) \cup UE_1$ is closed. From the preceding paragraph, every point of M has a neighborhood not intersecting E_1 , i large. Hence $E \cup UE_1$ is closed. Since $F(G_1)$, $F(G)$, M , M_1 are all invariant, it follows that $E \cup U_1 E_1$ is invariant. Hence we may replace X by its open invariant subset $X - (E \cup U_1 E_1)$. That is, we are reduced to the case in which $F(G_1)$ and $F(G)$ have a single component, M_1 and M respectively. Now from (4.1) we get that $\{F(G_1)\}$ converges regularly to $F(G)$ over $\{Z_{p_1}\}$. Also $F(G_1)$ and $F(G)$ are orientable over Z_{p_1} . Hence by (2.3), $M_1 = M$ for i large. Hence the theorem is proved in case G is toral and each G_1 is of prime power order.

We now prove the theorem in the general case by induction on $\dim G$, noting first that it is obvious for $\dim G = 0$. Suppose $\dim G = n > 0$ and that the theorem holds for dimensions $< n$.

Since $\lim G_1 = G$, G_1 has a subgroup H_1 of prime power order with order $H_1 \rightarrow \infty$. Select a subsequence of $\{H_1\}$ which converges to a subgroup H of G . We may as well suppose this the original sequence. By the argument of Section 1, $H_1 \subset H$ for i large. Now $\dim H > 0$ since $\text{ord } H_1 \rightarrow \infty$. Replacing H by its identity component H^C and H_1 by H_1 by $H_1 \cap H^C$, we may as well suppose H a toral group.

Now $H_1 \subset H$, $\lim H_1 = H$, H_1 is of prime power order, and H is

a toral group of $\dim > 0$. By our earlier case, $F(H_1) \cap C = F(H) \cap C$ for i large. Also $F(H)$ satisfies (1), (2), (3). We have that G/H acts on $F(H)$ and that the set of stationary points of G/H acting on $F(H)$ is precisely $F(G)$. Let $\lambda : G \longrightarrow G/H$ denote the quotient map. Then $\{\lambda(G_1)\}$ is a sequence of subgroups of G/H with $\lim \lambda(G_1) = G/H$. Hence, by induction, $F(G/H, F(H) \cap C) = F(\lambda(G_1), F(H) \cap C)$ for i large. But $F(G_1) \cap C \subset F(H_1) \cap C = F(H) \cap C$. Hence $F(G_1) \cap C \subset F(\lambda(G_1), F(H) \cap C)$ for i large. That is, $F(G_1) \cap C \subset F(G) \cap C$ for i large. The opposite inclusion is obvious, and the theorem is proved.

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VII: FINITENESS OF NUMBER OF ORBIT TYPES

G. E. Bredon

§1. Preliminary Remarks

In Chapter V E. E. Floyd proved the following result:

1.1. THEOREM. If M is a cm over Z and if T is a toral group acting on M , then, for any compact set $A \subset M$, there are at most a finite number of distinct isotropy subgroups T_p of T for $p \in A$.

For M compact and orientable this result was originally proved by Floyd in [1]. The above generalization is due to L. Mann [2]. Floyd's original result was generalized by G. D. Mostow [5] to apply to compact manifolds M with a compact Lie group G of transformations, the result giving in this case that there are at most a finite number of non-conjugate isotropy subgroups of G . An examination of Mostow's proof reveals that there is essentially no use of the theory of cohomology manifolds or of the theory of transformation groups which is not already contained in the statement of Floyd's theorem. That is, the transition between these two theorems is entirely group theoretical, and therefore it should be possible to extend the above (local) theorem in exactly the same fashion. The purpose of the present chapter is to give this more general group theoretical formulation of Mostow's result and its applications to the theory of transformation groups. Most of the proofs given here are contained in Mostow's papers [4] and [5] and are only slightly altered to suit our present purposes.

By the term subgroup (as distinct from analytic subgroup) we will always mean a closed subgroup.

§2. Statements of the main results

Our main group theoretical result is

2.1. THEOREM. Let G be a compact Lie group and let $\mathcal{S}$ be a class of subgroups of G which is closed with respect to conjugation in G and is such that the set $\{S \cap T; S \in \mathcal{S}\}$ is finite, where T is a maximal torus of G . Then there are at most a finite number of non-conjugate members S of $\mathcal{S}$.

(Note that the hypothesis of 2.1 implies immediately that the set $\{S \cap T; S \in \mathcal{S}\}$ is finite for any toral subgroup T of G , since any torus is contained in a maximal torus and the maximal tori are all mutually conjugate in G .)

2.2. THEOREM. If G is a compact Lie group acting on a cm M over Z , then, for any compact set $A \subset M$, there are at most a finite number of non-conjugate isotropy subgroups G_p of G for $p \in A$.

PROOF. Since $G(A) \supset A$ is compact and invariant, we may assume that A is invariant. Since $gG_p g^{-1} = G_{g(p)}$, the collection $\{G_p; p \in A\}$ is closed under conjugation and, since $G_p \cap T = T_p$, the collection $\{G_p \cap T; p \in A\}$ is finite by 1.1. Thus the result follows immediately by applying 2.1 to the collection $\mathcal{S} = \{G_p; p \in A\}$.

It is desirable, in view of possible further generalizations of 1.1, to formulate a theorem for transformation groups which is in between 2.1 and 2.2 with regard to generality, that is, a result phrased in the language of transformation groups, but applying to spaces more general than cohomology manifolds. For this we need the following definition.

2.3. DEFINITION. A space X is said to have property F if, for every toral group T acting on X and every compact subset $A \subset X$, there are at most a finite number of isotropy subgroups T_p of T for $p \in A$.

Now Theorem 1.1 merely asserts that every cm over Z has property F .

2.4. THEOREM. If G is a compact Lie group acting on a space X having property F , then for any compact subset $A \subset X$ there are at most a finite number of non-conjugate isotropy subgroups G_p of G for $p \in A$.

The proof is exactly the same as that of 2.2.

PROBLEM (Floyd). Does every finite complex have property F?

§3. Proof of the main theorem

If H is a Lie group, we denote the connected component of the identity in H by H^0 . We shall first show, under the hypothesis of 2.1, that there are at most a finite number of non-conjugate S^0 for $S \in \mathcal{S}$, and then will consider the resulting finite extension problem $S^0 \longrightarrow S$. Thus we first consider

3.1. THEOREM. Under the hypothesis of 2.1, there are at most a finite number of non-conjugate S^0 for $S \in \mathcal{S}$.

PROOF. We wish to reduce this problem to the case in which all the S^0 are normal in G . This reduction is accomplished by the following important lemma.

3.2. LEMMA. Let G be a compact Lie group. There are at most a finite number of non-conjugate subgroups of G which are normalizers of (connected) analytic subgroups.

PROOF. Let $\mathfrak{g}$ be the Lie algebra of G and let E be the exterior algebra of $\mathfrak{g}$. Let P be the projective space of one-dimensional subspaces of E . Every linear subspace $\underline{h} \subset \mathfrak{g}$ determines a point of P as follows. Say that $\underline{h}$ is spanned by the independent vectors $v_1, v_2, \dots, v_n$, then $v_1 \wedge v_2 \wedge \dots \wedge v_n$ determines a point of P . This point does not depend on the choice of basis for $\underline{h}$ since if the vectors $w_1 = \sum a_{1j} v_j$ also span $\underline{h}$ then $\det(a_{1j}) \neq 0$ and hence $w_1 \wedge \dots \wedge w_n = \det(a_{1j}) v_1 \wedge \dots \wedge v_n$ determines the same point in P as does $v_1 \wedge \dots \wedge v_n$. We denote this point by $\underline{h}^*$.

The adjoint representation of G on $\mathfrak{g}$ induces a projective representation of G on P by $g(v_1 \wedge \dots \wedge v_n) = \text{ad}(g)(v_1) \wedge \dots \wedge \text{ad}(g)(v_n)$. Clearly a subgroup $N \subset G$ leaves $\underline{h}$ invariant if and only if N leaves $\underline{h}^*$ stationary in this representation. If H is an analytic subgroup of G with Lie algebra $\underline{h}$ then we have that, for any given $g \in G$, $gHg^{-1} = H$ if and only if $\text{ad}(g)(\underline{h}) = \underline{h}$ and this holds if and only if $g(\underline{h}^*) = \underline{h}^*$. Thus K is the normalizer of H if and only if $K = G_{\underline{h}^*}$ in this representation.

We have thus identified the normalizers in G with certain isotropy subgroups of G in a differentiable action of G on a compact manifold. Therefore we will be done if we can show that such an action admits at most a finite number of non-conjugate isotropy subgroups. This was first proved by Yang and may be shown as follows. Another proof will be given in Chapter VIII.

We label three different propositions as follows:

$L(n)$: Every differentiable (C^1) action of any compact Lie group on an n -manifold M^n has at most a finite number of non-conjugate isotropy subgroups locally on M^n .

$C(n)$ Every differentiable action of any compact Lie group on a compact n -manifold M^n has at most a finite number of non-conjugate isotropy subgroups on M^n .

$S(n)$: Every differentiable action of any compact Lie group on an n -sphere S^n has at most a finite number of non-conjugate isotropy subgroups on S^n .

Obviously, $L(n)$ implies $C(n)$ and $C(n)$ implies $S(n)$. We shall prove that $S(n)$ implies $L(n+1)$, which, together with the obvious fact $L(0)$, will prove the validity of these propositions for all n .

Let G be a compact Lie group acting on M^{n+1} and let p be an arbitrary point of M^{n+1} . By a theorem of Bochner (see [3, p. 206]) there is an open neighborhood U of p in which coordinates may be chosen so that G_p acts linearly in these coordinates. Thus G_p is equivalent in U to the cone over an orthogonal action of G_p on an n -sphere S^n . As such, G_p has at most one more isotropy subgroup in U than it has in this action on S^n and thus, by property $S(n)$, G_p has at most a finite number of non-conjugate isotropy subgroups in U . Let K be a compact neighborhood of the identity in G such that $K(p) \subset U$. Then by a theorem of Montgomery and Zippin [3, pp. 215-216], there is a neighborhood $V \subset U$ of p such that for any $q \in V$ there is a $g \in K$ such that $G_{g(q)} = gG_qg^{-1} \subset G_p$. We may assume that V is so small that $K(V) \subset U$. Thus since $gG_qg^{-1} = (G_p)_{g(q)}$ and $g(q) \in U$, it follows from the above remarks about G_p that there are at most a finite number of non-conjugate G_q for $q \in V$. This completes the proof that $S(n)$ implies $L(n+1)$, and hence also of Lemma 3.2.

We now return to the proof of Theorem 3.1. By replacing G in turn by the finite number of non-conjugate normalizers $N_1, \dots, N_k$ of the subgroups S° for $S \in \mathcal{S}$ and replacing $\mathcal{S}$ by the subcollections $\mathcal{S}_1 = \{S \in \mathcal{S}; N_1 \text{ is the normalizer of } S^\circ\}$ it is seen that the group N_1 and the collection $\mathcal{S}_1$ satisfy the hypothesis of Theorem 3.1. If $S \in \mathcal{S}$ and N is the normalizer of S° , then there is a $g \in G$ such that $gNg^{-1} = N_1$ for some i and it follows that $gSg^{-1} \in \mathcal{S}_1$. Thus it clearly suffices to assume that all the groups S° for $S \in \mathcal{S}$ are normal in G . Moreover, since $S \cap T = (S \cap G^\circ) \cap T$, we may assume that G is connected.

It is well-known that every compact connected Lie group S° is a product $S^\circ = CZ$, where C is the commutator subgroup of S° and is semi-simple and where Z is the connected component of the identity in the center of S° (see, for instance, [6, p. 282]). Clearly, by their descriptions, C and Z are normal in G . Since G is connected, the automorphisms of Z induced by inner automorphisms of G form a connected subgroup of the discrete group of all automorphisms of Z and thus are trivial. That is, Z is central in G .

Let T be the connected component of the identity in the center of G . Then clearly $Z \subset (T \cap S)^\circ$. Also $T \cap S$ is in the center of S° and thus $(T \cap S)^\circ \subset Z$. Therefore, $Z = (T \cap S)^\circ$ and by hypothesis there are only a finite number of $T \cap S$. Hence there are only a finite number of possible candidates for Z .

There are also only a finite number of candidates for C , since any compact Lie group contains at most a finite number of normal semi-simple subgroups. To prove this we note that, since a semi-simple group is its own commutator subgroup, such a group C must be contained in the commutator subgroup of G which is semi-simple. Thus G may be assumed to be semi-simple. Letting $\mathfrak{g}$ be the Lie algebra of G , we have that $\mathfrak{g} = \mathfrak{g}_1 + \dots + \mathfrak{g}_n$, where the $\mathfrak{g}_i$ are the minimal ideals of $\mathfrak{g}$. Now let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. $\mathfrak{h}$ must contain a minimal ideal of $\mathfrak{g}$, say $\mathfrak{g}_1$, and, by the fact that the minimal ideals are simple, $\mathfrak{h}$ must split $\mathfrak{h} = \mathfrak{g}_1 + \mathfrak{k}$ and clearly $\mathfrak{k} \subset \mathfrak{g}_2 + \dots + \mathfrak{g}_n$. It follows easily that $\mathfrak{k}$ is an ideal of $\mathfrak{g}$ and by induction we may assume that $\mathfrak{k} = \mathfrak{g}_{1_1} + \dots + \mathfrak{g}_{1_m}$. Thus it follows immediately that every ideal $\mathfrak{h}$ of $\mathfrak{g}$ is of the form $\mathfrak{h} = \mathfrak{g}_{1_1} + \dots + \mathfrak{g}_{1_r}$ which are only finite in number.

We have shown that there are only a finite number of candidates for C and Z and hence there are only a finite number of possible S° , which completes the proof of Theorem 3.1.

Let us now consider the finite extension problem $S^\circ \longrightarrow S$ and thereby complete the proof of Theorem 2.1. By Theorem 3.1 and Lemma

3.2 we may assume that all $S \in \mathcal{S}$ have the same connected component $S^0 = K$ and that K is normal in G . Let T be a maximal torus of G .

Given $s \in S \cap G^0$ for some $S \in \mathcal{S}$, there is an element $g \in G$ such that $gsg^{-1} \in T$. We have that $s^n \in K$ if and only if $(gsg^{-1})^n = gs^n g^{-1} \in gKg^{-1} = K$ which means that

$$(1) \quad \text{ord}(sK) \text{ in } G/K = \text{ord}(gsg^{-1}K) \text{ in } G/K.$$

Moreover $gsg^{-1} \in gSg^{-1} \cap T = S' \cap T$ for some $S' \in \mathcal{S}$, and there are only a finite number of $S' \cap T$ for $S' \in \mathcal{S}$ by hypothesis. We also have that $(S' \cap T)K/K \subset S'/K$ which is a finite group. Thus, by (1), the order of sK in G/K is the same as the order of some element in one of a finite number of finite groups and therefore is bounded. Moreover, if $s \in S \in \mathcal{S}$, and $r = \text{ord}(G/G^0)$, then $s^r \in S \cap G^0$, and therefore the order of s is bounded since the order of s^r is bounded. That is,

$$(2) \quad \text{There is an integer } n \text{ such that } \text{ord}(sK) \leq n \text{ in } G/K \text{ for all } s \in S \text{ and all } S \in \mathcal{S}.$$

Since $S_1/K \sim S_2/K$ in G/K if and only if $S_1 \sim S_2$ in G , it follows from (2), upon letting $G/K = H$ and $S/K = F$, that it suffices to prove the following lemma.

3.3. LEMMA. Let H be a compact Lie group. If $\mathcal{F}$ is a collection of finite subgroups of H and n is a fixed integer such that $\text{ord}(g) \leq n$ for all $g \in F \in \mathcal{F}$, then there are at most a finite number of non-conjugate members F of $\mathcal{F}$.

PROOF. We shall first establish

$$(I) \quad \text{There is a number } m \text{ such that } \text{ord}(F) \leq m \text{ for all } F \in \mathcal{F}.$$

PROOF. Let μ be Haar measure and let U be a symmetric neighborhood of the identity in H so small that U^2 contains no element of order less than $n + 1$. Then if $F \in \mathcal{F}$ and $g \nmid g' \in F$ we have that $gU \cap g'U = \emptyset$ since otherwise U^2 contains $g'g^{-1} \in F$ which is of order at most n . Thus $\mu(H) \geq \mu(FU) = \text{ord}(F)\mu(U)$ which gives us the bound $\mu(H)/\mu(U)$ for $\text{ord}(F)$.

By (I) it now suffices to prove

$$(II) \quad \text{There are at most a finite number of non-conjugate members}$$

$F \in \mathcal{F}$ of order k , where k is any fixed integer.

PROOF. Suppose this is false and let $F_1, F_2, \dots$ be a sequence of mutually non-conjugate members of $\mathcal{F}$ all of order k . Since there are only a finite number of distinct abstract groups of order k we may assume that the F_i are isomorphic to one another. Let $g_1^{(1)}, g_1^{(2)}, \dots, g_1^{(k)}$ be the elements of F_1 in such an order that the correspondence $g_1^{(p)} \longleftrightarrow g_j^{(p)}$ gives an isomorphism between F_1 and F_j for all i and j . By passing to a subsequence we may assume that $\lim g_i^{(p)} = g^{(p)}$ exists for all $p = 1, 2, \dots, k$. If $g_1^{(p)} g_1^{(q)} = g_1^{(r)}$ then it follows upon passing to the limit that $g^{(p)} g^{(q)} = g^{(r)}$. Thus $F = \lim F_i$ is a finite subset of H closed under multiplication and, as such, it is a subgroup of H . By a theorem of Montgomery and Zippin [3, p. 216] it follows that F_i is conjugate to a subgroup of F (and hence conjugate to F) for all large i . Thus F_i and F_j are conjugate for all large i and j , and this contradiction completes the proof.

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VIII: SLICES AND EQUIVARIANT IMBEDDINGS

R. S. Palais

§1. Notation and Preliminaries

G will always denote a compact Lie group. Unless otherwise stated, the dimension of a separable metric space is the classical covering dimension.

1.1. DEFINITION. A G -space is a completely regular space X together with an action of G on X . We denote by X/G the orbit space of X and by π_X the natural map of X on X/G .

We recall that the topology for X/G is the strongest making π_X continuous, i.e., $S \subseteq X/G$ is open (closed) if and only if $\pi_X^{-1}(S)$ is open (closed).

1.2. PROPOSITION. X/G is completely regular.

PROOF. Given $\tilde{x} \in X/G$ and a closed subset $\tilde{F}$ of X/G not containing $\tilde{x}$, $F = \pi_X^{-1}(\tilde{F})$ is a closed invariant subset of X disjoint from the compact set $\tilde{x}$ and by Urysohn's lemma (applied to $\tilde{x}$ and the closure of F in the Čech compactification of X) there exists $f : X \rightarrow I$ with $f|_{\tilde{x}} \equiv 0$ and $f|_F \equiv 1$. Let $f^*(x) = \int_G f(gx) dg$ and $\tilde{f} = f^* \circ \pi_X^{-1}$. Then $\tilde{f} : X/G \rightarrow I$ and $\tilde{f}(\tilde{x}) = 0$, $\tilde{f}|_{\tilde{F}} \equiv 1$. q.e.d.

1.3. PROPOSITION. π_X is open.

PROOF. If O is open in X then $\pi_X^{-1}(\pi_X(O)) = GO = \bigcup_{g \in G} gO$ is open, hence $\pi_X(O)$ is open.

Since X/G can clearly have only one topology making π_X both continuous and open:

1.4. COROLLARY. The topology of X/G is uniquely determined by the conditions that π_X be open and continuous.

1.5. PROPOSITION. If X is a metrizable G -space there is a metric ρ for X relative to which each operation of G is isometric. If we define $\tilde{\rho}(\tilde{x}, \tilde{y}) = \inf \{\rho(x, y) \mid x \in \tilde{x}, y \in \tilde{y}\}$ then $\tilde{\rho}$ is a metric for X/G . If X is separable so is X/G .

PROOF. If ρ^* is any metric for X then it is easily checked that $\rho(x, y) = \int_G \rho(gx, gy) dg$ is a metric for X and $\rho(gx, gy) = \rho(x, y)$ by the invariance of Haar measure. It is clear that π_X is distance decreasing and hence continuous relative to ρ and $\tilde{\rho}$. Moreover π_X maps the ρ ϵ -ball about x onto the $\tilde{\rho}$ ϵ -ball about $\pi_X(x)$ and hence is open relative to ρ and $\tilde{\rho}$ so, by 1.4, $\tilde{\rho}$ induces the correct topology. If X is separable it is Lindelöf, hence $X/G = \pi_X(X)$ is Lindelöf and, being metrizable, separable.

1.6. PROPOSITION. π_X is a closed mapping.

PROOF. Let F be closed in X and let x be adherent to $\pi_X^{-1}(\pi_X(F)) = GF$. Choose nets $\{g_\alpha\}$ in G and $\{f_\alpha\}$ on F so that $g_\alpha f_\alpha \rightarrow x$. Since G is compact we can suppose $g_\alpha \rightarrow g$. Then $f_\alpha \rightarrow g^{-1}x$ so $g^{-1}x \in F$ and $x \in GF$. Hence $\pi_X^{-1}(\pi_X(F))$ is closed so $\pi_X(F)$ is closed.

1.7. PROPOSITION. π_X is a proper mapping.

PROOF. It is a general fact that a map of a Hausdorff space into a Hausdorff space is proper if it is closed and the pre-image of every point is compact.

1.8. PROPOSITION. If J is an invariant subset of the G -space X then every neighborhood of J includes an invariant neighborhood of J .

PROOF. If V is an open set including J then $O = X - \pi_X^{-1}(\pi_X(X - V))$ is clearly invariant and included in V . It is open by 1.6 and $J \subset O$ follows easily from the invariance of J .

1.9. DEFINITION. A G -space X is:

(1) Euclidean: if X is a finite dimensional real vector space with an orthogonal structure and the action of G on X is an orthogonal representation.

(2) Riemannian: if X is a Riemannian manifold and each operation of G is an isometry.

(3) Differentiable: if X is a differentiable ($= C^\infty$) manifold and each operation of G is differentiable (it then follows by a theorem of Bochner and Montgomery [1] that the natural map of $G \times X \rightarrow X$ is differentiable).

REMARK. Since an isometry of a Riemannian manifold is differentiable (see e.g., [2]) a Riemannian G -space is differentiable. Conversely given a differentiable G -space, X , by averaging any Riemannian tensor for X relative to Haar measure in an obvious way we get a Riemannian tensor for X which is invariant under the operations of G , so a differentiable G -space can always be considered Riemannian.

1.10. DEFINITION. Let X and Y be G -spaces. A mapping $f : X \rightarrow Y$ is equivariant if $f(gx) = gf(x)$ for all $(g, x) \in G \times X$. If in addition f is one-to-one on each orbit of X we call f isovariant.

1.11. PROPOSITION. If $f : X \rightarrow Y$ is equivariant then $G_x \subseteq G_{f(x)}$ for all $x \in X$ and f is one-to-one on $G(x)$ if and only if equality holds. Hence if f is isovariant $G_x = G_{f(x)}$ for all $x \in X$.

PROOF. $G_x \subseteq G_{f(x)}$ is trivial. If f is one-to-one on $G(x)$ then $(f|G(x))^{-1}$ is equivariant and we get the reverse inclusion. Conversely if $G_x = G_{f(x)}$ and $f(g_1x) = f(g_2x)$ then $g_2^{-1}g_1f(x) = f(x)$ so $g_2^{-1}g_1x = x$ so $g_1x = g_2x$ and f is one-to-one on $G(x)$.

1.12. TIETZE-GLEASON LEMMA. If K is a compact (respectively, closed) invariant subspace of the G -space (respectively, normal G -space) X and if $f : K \rightarrow E$ is an equivariant map of K into a Euclidean G -space, then f admits an equivariant extension $\tilde{f} : X \rightarrow E$.

PROOF. By Tietze's extension theorem (applied, in the first case, to the Čech compactification of X) we can find a continuous extension $f^* : X \rightarrow E$. Let $\tilde{f}(x) = \int_G g^{-1}f^*(gx)dg$. For $x \in K$ $\tilde{f}(x) = \int g^{-1}gf^*(x)dg = f(x)$ so $\tilde{f}$ extends f . By the invariance of Haar measure

$$\tilde{f}(\gamma x) = \int g^{-1}f^*(g\gamma x)dg = \int \gamma g^{-1}f^*(gx)dg = \gamma \tilde{f}(x)$$

so $\tilde{f}$ is equivariant.

§2. Orbit types

If H is a closed subgroup of G we denote by (H) the set of conjugate subgroups $\{gHg^{-1} \mid g \in G\}$. We call sets of the form (H) G-orbit types. If X is a G -space and $\omega \in X/G$ an orbit in X , say $\omega = G(x)$ then since $G_{gx} = gG_xg^{-1}$ it follows that $\{G_\omega \mid \omega \in \Omega\} = \{G_x\}$ is a G -orbit type which we call the orbit type of Ω and denote by $[\Omega]$ (note $[G(x)] = (G_x)$). Two orbits, in perhaps different G -spaces are called equivalent if there exists an equivariant homeomorphism of one onto the other.

2.1. PROPOSITION. Two orbits are equivalent if and only if they are of the same type.

PROOF. If Ω and Ω' are equivalent then by 1.11 they are of the same type. Conversely if Ω and Ω' are of the same type then we can find $\omega \in \Omega$ and $\omega' \in \Omega'$ with $G_\omega = G_{\omega'}$. Then $g\omega \longrightarrow g\omega'$ is clearly a well defined, continuous, one-to-one equivariant map of Ω onto Ω' . Since Ω is compact this map is a homeomorphism.

If X is a G -space then those subsets of X which are unions of all orbits of a fixed type form a partitioning of X into invariant subsets. We call this partitioning of X (or equivalently of X/G), each subset labeled by the corresponding orbit type, the orbit structure of X . We recall from Chapters VI and VII of Floyd and Bredon that if X is a cohomology manifold the orbit structure is locally finite and hence, if X is compact, finite (we shall give a very simple proof, based on the existence of slices, when X is a differentiable G -space later). The significance of this is apparent from the following theorem of Mostow [3], the proof of which is one of our main goals.

THEOREM. The following conditions are necessary and sufficient for a G -space X to admit an equivariant imbedding in some Euclidean G -space:

- (1) X is metrizable, separable, and of finite dimension.
- (2) X has finite orbit structure.

The first step towards proving this is the fact, noted independently by Mostow [3] and Palais [4], that a single orbit can always be isovariantly imbedded in a Euclidean G -space.

2.2. PROPOSITION. If H is a closed subgroup of G there exists a Euclidean G -space E and a $v \in E$

such that $G_v = H$.

PROOF. Let R be right regular representation of G in the real $L^2(G)$ and let $\bar{f}$ be a continuous real valued function on the left coset space G/H which assumes the value 1 only at H . Define $f \in L^2(G)$ by $f(g) = \bar{f}(gH)$. It is clear that $H = \{g \in G \mid R_g f = f\}$. By Peter-Weyl $L^2(G) = \bigoplus_{i=1}^{\infty} E_i$ where the E_i are finite dimensional subspaces invariant under R . Let f_i be the projection of f on E_i , and $H_i = \{g \in G \mid R_g f_i = f_i\}$. Then the H_i are closed subgroups of G such that $H = \bigcap_i H_i$. Now the closed subgroups of a compact Lie group satisfy the descending chain condition (at each step in a properly descending chain either the dimension or number of components must drop), hence we can find $i_1, \dots, i_n$ such that $H = \bigcap_{j=1}^n H_{i_j}$. We put $E = \bigoplus_{j=1}^n E_{i_j}$ and $v = f_{i_1} + \dots + f_{i_n}$.

2.3. THEOREM. If X is a G -space and $x \in X$ there exists an equivariant map of X into a Euclidean G -space which is one-to-one on $G(x)$.

PROOF. By 2.2 we can find a Euclidean G -space E and $v \in E$ with $G_v = G_x$. Then $G(v)$ and $G(x)$ are orbits of the same type so 2.1 they are equivalent, i.e., there exists an equivariant homeomorphism of $G(x)$ onto $G(v)$. By 1.12 this can be extended to an equivariant map of X into E .

§3. Slices

3.1. DEFINITION. Let H be a closed subgroup of G . A local cross-section in G/H is a non-singular differentiable map $x : U \longrightarrow G$ such that U is an open neighborhood of H in G/H , $x(H) = e$, and $\Pi \circ x = \text{identity}$ where $\Pi : G \longrightarrow G/H$ is canonical.

REMARK. The existence of local cross-sections follows easily from Chevalley [5] pp. 109, 110.

3.2. DEFINITION. Let X be a G -space and H a closed subgroup of G . An H -slice in X is a subset S of X such that

(1) S is invariant under H .

(2) $gS \cap S \neq \emptyset \implies g \in H$.

(3) If $x : U \longrightarrow G$ is a local cross section in G/H then the map $F : U \times S \longrightarrow X$ defined by $F(u, s) = x(u)s$ is a homeomorphism of $U \times S$ onto an open set in X .

If $x \in X$ then a slice at x is a G_x -slice in X which contains x .

3.3. PROPOSITION. If S is an H -slice in X then $\pi_X|_S$ is an open map of S into X/G and in particular $\pi_X(S)$ is open.

PROOF. Let V be open in S . Using the notation of 3.2, $F(U \times V)$ is open in X and hence $\pi_X(F(U \times V))$ is open in X/G . Clearly $\pi_X(F(U \times V)) \subseteq \pi_X(V)$ and since $F(U \times V) \supseteq F(\{H\} \times V) = x(H)V = eV = V$ we in fact have equality.

3.4. PROPOSITION. If S is an H -slice in X then $G_s \subseteq H$ for all $s \in S$.

PROOF. Immediate from (2) of 3.2.

3.5. COROLLARY. If there exists a slice at x then there is a neighborhood of x at each point of which the isotropy group is conjugate to a subgroup of G_x .

PROOF. Using the notation of 3.2 with $G_x = H$, $F(U \times S)$ is a neighborhood of x and $G_{x(u)s} = x(u)G_s x(u)^{-1}$ while $G_s \subseteq G_x$ by 3.4. We note that an H -slice S in X is an H -space. The next result says that S/H can be naturally identified with GS/G .

3.6. PROPOSITION. If S is an H -slice in X then $h: Hs \rightarrow Gs$ is a homeomorphism of S/H onto $\pi_X(S) = GS/G$.

PROOF. If $Gs = Gs'$ with $s, s' \in S$ then $gs = s'$ for some $g \in G$ and, by (2) of 3.2, $g \in H$ so $Hs = Hs'$ and h is one-to-one. Now $\pi_S: S \rightarrow S/H$ is open and continuous and by 3.3 so also is $\pi_X|_S$. Since $\pi_X|_S = h \circ \pi_S$ it follows that h is continuous and open, hence a homeomorphism.

The following powerful and elegant result is essentially due to Mostow [3] although the basic idea goes back to Gleason [6]. While easily patched up, Mostow's proof contains several errors and in fact the statement of the theorem given in [3] is easily seen to be incorrect.

3.7. THEOREM. If S' is an H -slice in the G -space Y and if f is an equivariant map of the G -space X into Y , then $S = f^{-1}(S')$ is an H -slice in X .

PROOF. Since $f(HS) = Hf(S)$ $HS' = S'$ it follows that $HS \subseteq f^{-1}(S') = S$ so S is invariant under H . Since $gS' \cap S' \supseteq gf(S) \cap f(S) = f(gS) \cap f(S) \supseteq f(gS \cap S)$ it follows that $gS \cap S \neq \emptyset \implies gS' \cap S' \neq \emptyset \implies g \in H$. Thus S satisfies (1) and (2) of 3.2 and it remains to verify (3). Let $x : U \longrightarrow G$ be a local cross-section in G/H and let $F' : (u, s') \longrightarrow x(u)s'$ be the corresponding homeomorphism of $U \times S$ onto an open set W' of Y . We will complete the proof by showing that $F : (u, s) \longrightarrow x(u)s$ is a homeomorphism of $U \times S$ onto $f^{-1}(W') = W$.

(1) F is continuous. Obvious.

(2) F is onto W .

PROOF. Let $w \in W$ so $f(w) = x(u)s'$. Then $s' = x(u)^{-1}f(w) = f(x(u)^{-1}w) = f(s)$ where $s = x(u)^{-1}w$ is in S because $f(s) = s' \in S'$. Then $w = x(u)s = F(u, s) \in F(U \times S)$, so $W \subseteq F(U \times S)$. Conversely $f(F(U \times S)) = F'(U \times f(S)) \subseteq F'(U \times S') = W'$ so $F(U \times S) \subseteq f^{-1}(W') = W$.

(3) F is one-to-one and F^{-1} is continuous.

PROOF. We prove both facts at once by showing that if $F(u_\alpha, s_\alpha) \longrightarrow F(u, s)$ then $u_\alpha \longrightarrow u$ and $s_\alpha \longrightarrow s$. In fact applying f to $F(u_\alpha, s_\alpha) \longrightarrow F(u, s)$ we see that $F'(u_\alpha, f(s_\alpha)) \longrightarrow F'(u, f(s))$ and since F' is bicontinuous $u_\alpha \longrightarrow u$. But then $x(u_\alpha)^{-1} \longrightarrow x(u)^{-1}$ which together with $x(u_\alpha)s_\alpha = F(u_\alpha, s_\alpha) \longrightarrow F(u, s) = x(u)s$ implies that $s_\alpha \longrightarrow s$. q.e.d.

REMARK. If $x \in X$ and S is a slice of x then define $f : GS \longrightarrow Gx$ by $f(gs) = gx$; $g \in G$, $s \in S$. It is easily checked that f is a well-defined equivariant retraction of GS onto Gx . What is more it can be shown that the triple (GS, G_x, f) has the structure of a fibre bundle, the fibre being S , the structural group G_x/K (where $K = \cap_{s \in S} G_s$) and the associated principal bundle being G/K .

Conversely it follows from 3.7 that if O is an open invariant neighborhood of Gx and f is an equivariant retraction of O onto Gx then $f^{-1}(x)$ is a slice at x (for clearly $\{x\}$ is a G_x -slice in Gx) and we recover f from this slice by the above process. In other words there is a natural one-to-one correspondence between slices at x and equivariant retractions of open invariant neighborhoods of Gx onto Gx .

The following result is due to Koszul [6]. A shorter proof using Bochner's theorem on linearity at stationary points (which, by the way, is a consequence of the theorem) will be found in Mostow [3] page 444.

3.8. THEOREM. If X is a differentiable G -space and $x \in X$ then there exists a slice S at x . Moreover S can be so chosen that, in a suitable coordinate system about x , G_x act linearly and S is an open disc with center x in an invariant subspace.

PROOF. As noted in the remark following 1.9 we can suppose that X is a Riemannian G -space, in which case it follows that if v is a tangent vector to X for which $\text{Exp}(v)$ is defined then $\text{Exp}(\delta g(v))$ is defined and equals $g\text{Exp}(v)$ for all $g \in G$ (here δg is the differential of g when g is considered a diffeomorphism of X). Let $\Sigma = G(x)$ and let $N(\Sigma)$ be the normal bundle of Σ in X . Since Σ is a compact submanifold of X it is well known that for ε sufficiently small Exp maps $N(\Sigma, \varepsilon) = \{v \in N(\Sigma) \mid \|v\| < \varepsilon\}$ diffeomorphically onto $B(\Sigma, \varepsilon) = \{x' \in X \mid \rho(x', \Sigma) < \varepsilon\}$. Now $\Sigma_x^\perp(\varepsilon) = \{v \in \Sigma_x^\perp \mid \|v\| < \varepsilon\}$ is an open disc in $\Sigma_x^\perp$ and $\Sigma_x^\perp$ is invariant under δg for $g \in G_x$, hence because $\text{Exp} \delta g = g \text{Exp}$ it follows that in any Riemannian normal coordinate system about x G_x acts linearly (in fact orthogonally) and $S = \text{Exp}(\Sigma_x^\perp(\varepsilon))$ is an open disc centered at x in an invariant subspace. We now verify the three conditions of (3.2) that will show that S is a G_x -slice.

(1) $G_x S = S$. Already proved.

(2) $gS \cap S \neq \emptyset \implies g \in G_x$.

PROOF. Suppose $g(\text{Exp } v) = \text{Exp } w$ with $v, w \in \Sigma_x^\perp(\varepsilon)$. Then $\text{Exp}(\delta g(v)) = \text{Exp}(w)$ and $\delta g(v) \in N(\Sigma, \varepsilon)$. Since Exp is a diffeomorphism on $N(\Sigma, \varepsilon)$ it follows that $\delta g(v) = w$. But $w \in \Sigma_x^\perp$ and $\delta g(v) \in \Sigma_g^\perp(x)$ and so we must have $g(x) = x$.

(3) Let $\chi : U \longrightarrow G$ be a local cross-section in $\dot{G}/G_x$ and define $K : (u, v) \longrightarrow \delta\chi(u)v$ on $U \times \Sigma_x^\perp(\varepsilon)$. Then K is a diffeomorphism of $U \times \Sigma_x^\perp(\varepsilon)$ onto an open set W in $N(\Sigma, \varepsilon)$, its inverse in fact is $\omega \longrightarrow \delta\chi(h^{-1}\Pi(\omega))^{-1}\omega$ where Π is the fibre projection of $N(\Sigma) \longrightarrow \Sigma$ and h is the diffeomorphism $gG_x \longrightarrow gx$ of G/G_x onto Σ . Then $\tilde{F} = \text{Exp} \circ K$ is a diffeomorphism of $U \times \Sigma_x^\perp(\varepsilon)$ onto an open set $V = \text{Exp}(W)$ in X and since $\text{Exp}(\delta\chi(u)v) = \chi(u) \text{Exp}(v)$, $F : (u, s) \longrightarrow \chi(u)s$ is a diffeomorphism of $U \times S$ onto V .

3.9. THEOREM (Mostow [3]). If X is a G -space and $x \in X$ there exists a slice at x .

PROOF. By 2.3 there is an equivariant map f of X onto a Euclidean G -space E which is one-to-one on $G(x)$. By 3.8 there exists a $G_{f(x)}$ -slice S' in E containing x . By 3.7 $S = f^{-1}(S')$ is a

$G_f(x)$ -slice in X containing x . But by 1.11 $G_f(x) = G_x$ so S is a slice at x .

3.10. COROLLARY (Montgomery-Zippin [7]). If X is a G -space and $x \in X$ there is a neighborhood W of x such that G_w is conjugate to a subgroup of G_x for all $w \in W$.

PROOF. 3.5 and 3.9.

3.11. PROPOSITION. Let X be a G -space and $\{S_\alpha\}$ a collection of H -slices in X such that the sets $\{GS_\alpha\}$ are pairwise disjoint. Then $S = \bigcup_\alpha S_\alpha$ is an H slice in X .

PROOF. Clearly S is invariant under H . If $gx_\alpha = x_\beta$ with $x_\alpha \in S_\alpha$ and $x_\beta \in S_\beta$ then since $GS_\alpha \cap GS_\beta = \emptyset$ we must have $\alpha = \beta$ and hence $g \in H$. Thus $gS \cap S \neq \emptyset \implies g \in H$. Now by 3.3 $GS_\alpha = \pi_X^{-1}(\pi_X(S_\alpha))$ is open in X hence $S_\alpha = S \cap GS_\alpha$ is open in S . If $x : U \longrightarrow G$ is a local cross-section in G/H then $F : (u, s) \longrightarrow x(u)s$ maps each $U \times S_\alpha$ homeomorphically onto an open set W_α of X . Since the $U \times S_\alpha$ form a disjoint open covering of $U \times S$ and the W_α a disjoint open covering of $W = \bigcup_\alpha W_\alpha$, F maps $U \times S$ homeomorphically onto W .

The following metatheorem will have several important applications in the sequel.

3.12. METATHEOREM. Let S be a statement valued function defined for all compact Lie groups. If the truth of $S(H)$ for every proper closed subgroup of an arbitrary compact Lie group G entails the truth of $S(G)$, then $S(G)$ is true for every compact Lie group G .

PROOF. Assume $S(G)$ is false for some compact Lie group G and let n be the least integer which is the dimension of a compact Lie group G for which $S(G)$ is false. Among all compact Lie groups of dimension n for which $S(G)$ is false choose one with fewest connected components and note that $S(H)$ is true for all proper closed subgroups of G .

As a first application we prove a special case of the result proved earlier in the book by Floyd and Bredon.

3.13. THEOREM. A differentiable G -space has locally finite orbit structure.

PROOF. By 3.12 we can assume that the theorem holds for all proper closed subgroups H of G . If $\dim X = 0$ the theorem is obvious and we can therefore also assume that the theorem holds for differentiable G -spaces of dimension less than $\dim X$. Let $x \in X$. If x is not a fixed point let S be a slice at x , which by 3.8 can be chosen to be a differentiable G_x -space. Since G_x is a proper closed subgroup of G we can find a neighborhood V of x in S in which only a finite number of G_x -orbit types occur. Then for $v \in V$ G_v is conjugate in G_x to one of a finite number of subgroups of G . Then by 3.3 GV is a neighborhood of x and clearly only a finite number of orbit types occur in GV .

If x is a fixed point then by Bochner's theorem (or by 3.8 which generalizes Bochner's theorem) we can find a coordinate system about x in which the action of G is linear and in fact orthogonal. Then if S is a sphere with center x in the coordinate system S is a differentiable G -space of dimension less than $\dim X$ so S has locally finite orbit structure and, being compact, actually has finite orbit structure. Since the isotropy group of a point is clearly constant on open rays only a finite number of orbit types occur within the coordinate system. Thus for any x in X we can find a neighborhood in which only a finite number of orbit types occur.

3.14. PROPOSITION. If X is a G -space with (locally) finite orbit structure and if H is a closed subgroup of G then X has (locally) finite orbit structure when considered as an H -space.

PROOF. It clearly suffices to show that if Σ is an orbit of type (K) then Σ has finite orbit structure as an H -space. But since Σ is equivalent to G/K we see that Σ is a compact differentiable H -space and the proposition follows from 3.13.

3.15. COROLLARY. If X is a G -space with (locally) finite orbit structure then any H -slice has (locally) finite orbit structure as an H -space.

We now make a second application of 3.12.

3.16. THEOREM. If X is a separable metric G -space of dimension n then $\dim X/G \leq n$.

PROOF. We recall that by 1.5 X/G is separable metric. Let F be the set of stationary points of X . Then F is a closed subspace of X/G of dimension $\leq n$. Let $U = X/G - F$. Since U is open and hence an F_σ , it follows from the sum theorem that it suffices to prove that X/G has dimension $\leq n$ at points of U . Let $\tilde{x} \in U$ and let S be a slice at a point x of $\tilde{x}$. We can suppose by 3.12 that the theorem holds for the proper closed subgroup G_x of G and hence that $\dim S/G_x \leq \dim S \leq n$. By 3.6 $\dim \pi_x(S) = \dim GS/G = \dim S/G_x \leq n$ and since by 3.7 $\pi_x(S)$ is a neighborhood of $\tilde{x}$, U has dimension $\leq n$ at $\tilde{x}$.

REMARK. The formal properties of $\dim$ used in the above proof include only topological invariance, the sum and monotonicity theorems, and the fact that $\dim X \leq n$ provided each point x of X has a neighborhood with $\dim U \leq n$. Thus if we have a G -space X with such a dimension function defined for X and X/G , then $\dim X/G \leq \dim X$. In particular if L is a principal ideal domain and $\dim_L$ is the cohomological dimension of H . Cohen, defined for locally compact spaces (cf. reference in I), then if X is a locally compact G -space, $\dim_L X/G \leq \dim_L X$.

The following is apparently due to Milnor, at least in its present form. It replaces the much more complicated Theorem 4.1 of Mostow [3].

3.6. THEOREM. Let X be a paracompact space with covering dimension n and let $\{U_\alpha\}$ be an open covering of X . Then there is an open covering of X refining $\{U_\alpha\}$, $\{G_{1\beta}\}_{\beta \in \beta_1}$ $i = 0, \dots, n$ such that $G_{1\beta} \cap G_{1\beta'} = \emptyset$ if $\beta \neq \beta'$.

PROOF. By making an initial refinement of $\{U_\alpha\}$ we can suppose that the order of $\{U_\alpha\}$ is at most n , i.e., no $x \in X$ is contained in more than $n + 1$ U_α . Let $\{\varphi_\alpha\}$ be a locally finite partition of unity with support $\varphi_\alpha \subseteq U_\alpha$. Given $i = 0, \dots, n$ let $B_1 =$ the set of unordered $i+1$ -tuples from the indexing set of the $\{U_\alpha\}$. Given $\beta = (\alpha_0, \dots, \alpha_i) \in B_1$ we set $G_{1\beta} = \{x \in X \mid \varphi_{\alpha_j}(x) > 0 \text{ and } \alpha \neq \beta \implies \varphi_\alpha(x) < \varphi_{\alpha_j}(x) \text{ } j = 0, \dots, i\}$. Since in a neighborhood of any point of X only a finite number of φ_α are not identically zero each $G_{1\beta}$ is open. Clearly $G_{1\beta} \cap G_{1\beta'} = \emptyset$ if $\beta \neq \beta'$ and $G_{1\beta} \subseteq \cap_{\alpha \in \beta} U_\alpha$ so support $\varphi_\alpha \subseteq \cap_{\alpha \in \beta} U_\alpha$ so $\{G_{1\beta}\}$ refines $\{U_\alpha\}$. It remains to show that $\{G_{1\beta}\}$ covers X . Given $x \in X$ let $(\alpha_0, \dots, \alpha_m)$ be the indices such that $\varphi_\alpha(x) > 0$. Since $x \in \cap_{i=0}^n \text{support } \varphi_\alpha \subseteq \cap_{i=0}^n U_{\alpha_i}$, $m \leq n$. We can suppose $\varphi_\alpha(x) = \dots = \varphi_{\alpha_1}(x) > \varphi_{\alpha_{1+1}}(x) \geq \dots \geq \varphi_{\alpha_m}$ in which case clearly $x \in G_{1\beta}$ where $\beta = (\alpha_1, \dots, \alpha_m)$.

3.17. DEFINITION. Let X be a G -space and H a closed subgroup of G . A subset of X/G is called H -liftable if it is the image under π_X of an H -slice in X .

3.18. LEMMA. An open subset of an H -liftable set is H -liftable.

PROOF. If S is an H -slice and V is open in $\pi_X(S)$ then it is immediate from the definition that $S \cap \pi_X^{-1}(V)$ is an H -slice.

The following is one of the crucial steps in getting equivariant imbeddings in Euclidean space.

3.19. THEOREM. Let X be a separable metric G -space of dimension h , H a closed subgroup of G and $X_{(H)}$ the union of all orbits in X of type (H) . Then there exist $n + 1$ H -slices in X $S_0, \dots, S_n$ such that $X_{(H)} \subseteq GS_0 \cup \dots \cup GS_n$.

PROOF. Let $\tilde{X}_{(H)} = \pi_X(X_{(H)})$. We must show that $\tilde{X}_{(H)}$ is included in the union of $n + 1$ H -liftable sets. Let $\{U_\alpha\}$ be the collection of all H -liftable subsets of X/G and let $\tilde{Y} = \bigcup_\alpha U_\alpha$. By 3.9 $\tilde{X}_{(H)} \subseteq Y$. Since Y is metrizable (and hence paracompact) by 1.5 and of dimension $\leq n$ by 3.15 and since the U_α are open by 3.3 it follows from 3.16 that we can find an open covering of Y refining $\{U_\alpha\}$, $\{G_{1\beta}\}_{\beta \in B}$ $1 = 0, \dots, n$ such that $G_{1\beta} \cap G_{1\beta'} = \emptyset$ if $\beta \neq \beta'$. By 3.18 each $G_{1\beta}$ is H -liftable and it follows from 3.11 that $\tilde{S}_1 = \bigcup_{\beta \in B_1} G_{1\beta}$ is also H -liftable. Clearly $Y = \tilde{S}_0 \cup \dots \cup \tilde{S}_n$ and since $\tilde{X}_{(H)} \subseteq Y$ we are through.

§4. Equivariant imbeddings in Euclidean space

4.1. LEMMA. If a G -space admits an equivariant imbedding in a Euclidean G -space then it admits an equivariant imbedding in the unit sphere of a Euclidean G -space.

PROOF. Let f_1 be an equivariant imbedding of X in a Euclidean G -space V , W a one-dimensional Euclidean G -space on which G acts trivially, w a non-zero vector in W and define $f_2 : X \longrightarrow V \oplus W$ by $f_2(x) = (f_1(x), w)$. Then $f(x) = f_2(x)/\|f_2(x)\|$ defines an equivariant imbedding of X in the unit sphere of $V \oplus W$.

4.2. THEOREM (Mostow [3]). Let X be a metrizable

G -space and F a closed invariant subspace of X .

If both F and $X - F$ admit equivariant imbeddings in Euclidean G -spaces so does X .

PROOF. Let f_1 be an equivariant imbedding of F in a Euclidean G -space V , extended by 1.12 to an equivariant map of X into V . Let ρ be an invariant metric for X (see 1.5) and define $h(x) = \inf \{(\rho(x, z) + \|f_1(x) - f_1(z)\| \mid z \in F)\}$. Then h is continuous on X , positive on $X - F$, invariant under the action of G , and $x_n \rightarrow x \in F \implies h(x_n) \rightarrow 0$.

Now let f_2^* be an equivariant imbedding of $X - F$ in a Euclidean G -space W which, by 4.1, we can suppose satisfies $\|f_2^*(x)\| = 1$. Define $f_2(x) = h(x)f_2^*(x)$. That f_2 is continuous, one-to-one and equivariant is clear. If $f_2(x_n) \rightarrow f_2(x)$ then $h(x_n) = \|f_2(x_n)\| \rightarrow \|f_2(x)\| = h(x)$ so $f_2^*(x_n) \rightarrow f_2^*(x)$ and hence $x_n \rightarrow x$, so f_2 is an imbedding. Moreover we can extend f_2 to a continuous equivariant map of X into W by defining $f_2(x) = 0$ for $x \in F$. We now define $f : X \rightarrow V \oplus W$ by $f(x) = (f_1(x), f_2(x))$. It is clear that f is continuous, equivariant and a homeomorphism on each of F and $X - F$. Since $f_2(x) = 0$ $x \in F$ and $f_2(x) \neq 0$ $x \in X - F$ it follows that f is one-to-one. Now suppose $f(x_n) \rightarrow f(x)$. The proof will be complete if we can show that $x_n \rightarrow x$. If $x \in X - F$ then $f_2(x) \neq 0$ hence $f_2(x_n) \neq 0$ for large n , hence $x_n \in X - F$ for large n and, since f is a homeomorphism on $X - F$, $x_n \rightarrow x$. Now suppose $x \in F$. Then by definition of h we can choose $z_n \in F$ so that $\rho(x_n, z_n) \leq 2h(x_n)$ and $\|f_1(x_n) - f_1(z_n)\| \leq 2h(x_n)$. Since $h(x_n) = \|f_2(x_n)\| \rightarrow \|f_2(x)\| = 0$ it follows that $\lim f_1(z_n) = \lim f_1(x_n) = f_1(x)$, and since f_1 is a homeomorphism on F $z_n \rightarrow x$. But then since $\rho(x_n, z_n) = 2h(x_n) \rightarrow 0$, $x_n \rightarrow x$. q.e.d.

4.3. COROLLARY. Let X be a separable, metric, finite dimensional G -space and F the set of stationary points of X . If $X - F$ admits an equivariant imbedding in a Euclidean G -space then so does X .

PROOF. F is closed in X and being a finite dimensional separable metric space admits an imbedding in a Euclidean space V which is automatically equivariant if we let G act trivially on V .

4.4. COROLLARY. Let X be a metrizable, separable G -space and $U_1, \dots, U_n$ a covering of

X by open invariant subsets. If each U_1 admits an equivariant imbedding in a Euclidean G -space then so does X .

PROOF. If $n = 1$ the theorem is trivial so we can assume inductively that $U_1 \cup \dots \cup U_{n-1}$ admits an equivariant imbedding in a Euclidean G -space. Then $F = X - U_n \subset U_1 \cup \dots \cup U_{n-1}$ admits an equivariant imbedding in a Euclidean G -space, and since $X - F = U_n$ does also, so does X by 4.2.

4.5. THEOREM. Let X be a G -space and S an H -slice in X . If S admits an H -equivariant imbedding in a Euclidean H -space then GS admits a G -equivariant imbedding in a Euclidean G -space.

PROOF. Let f_1 be an H -equivariant imbedding of S in a Euclidean H -space V . It is immediate from the Frobenius reciprocity theorem that there is a Euclidean G -space W which considered as an H -space contains V as an invariant linear subspace (see [8], italicized remark bottom of page 83). Let U be a Euclidean G -space with a vector $u \in U$ such that $G_u = H$ (see 2.2). Define $f : GS \longrightarrow W \oplus U$ by $f(gs) = (gf_1(s), gu)$. We note first that f is well defined, for if $g_1s_1 = g_2s_2$ then $g_2^{-1}g_1s_1 = s_2$ and by (2) of 3.2 $g_2^{-1}g_1 \in H$. Since f_1 is H -equivariant $g_2^{-1}g_1f_1(s_1) = f_1(s_2)$ so $g_1f(s_1) = g_2f(s_2)$ and since $G_u = H$ $g_1u = g_2u$. It is clear that f is continuous and equivariant so it remains to show that f is one-to-one and that f^{-1} is continuous. We can kill both birds with one stone by showing that $f(g_ns_n) \longrightarrow f(gs)$ implies $g_ns_n \longrightarrow gs$. Now $f(g_ns_n) \longrightarrow f(gs)$ gives $g_nu \longrightarrow gu$ and since $G_u = H$ it follows that $g_n = \gamma_n h_n$ where $h_n \in H$ and $\gamma_n \longrightarrow g$. But $f(g_ns_n) \longrightarrow f(gs)$ also gives $g_nf_1(s_n) \longrightarrow gf_1(s)$ or $\gamma_nf_1(h_ns_n) = \gamma_nh_nf_1(s_n) \longrightarrow gf_1(s)$ and since $\gamma_n \longrightarrow g$ we get $f_1(h_ns_n) \longrightarrow f_1(s)$. Since f_1 is a homeomorphism on S we see that $h_ns_n \longrightarrow s$ and hence $g_ns_n = \gamma_nh_ns_n \longrightarrow gs$. q.e.d.

4.6. THEOREM [Mostow]. If G is a compact Lie group and X is a separable, metrizable G -space of finite dimension and with finite orbit structure then S admits an equivariant imbedding in a Euclidean G -space.

PROOF. By 3.12 we can assume that the theorem holds if in the statement we replace G by any of its proper closed subgroups, and by 4.3 we can suppose that X has no stationary points. Let $(H_1), \dots, (H_k)$

be the orbit types occurring in X and note that each H_j is a proper closed subgroup of G . By 3.19 we can find subsets $S_j^1, j = 1, \dots, k$, $j = 0, \dots, \dim X$, of X such that S_j^1 is an H_j -slice in X and $X = \bigcup_{j=1}^k GS_j^1$. Now by 3.15 each S_j^1 has finite orbit structure as an H_j -space and of course each S_j^1 is separable, metric, and of finite dimension. Thus each S_j^1 admits an H_j -equivariant imbedding in a Euclidean H_j -space. By 5.4 each GS_j^1 admits a G -equivariant imbedding in a Euclidean G -space. Since each GS_j^1 is open in X (3.3) and $X = \bigcup_{j=1}^k GS_j^1$ it follows by 4.4 that X admits an equivariant imbedding in a Euclidean G -space. q.e.d.

4.7. COROLLARY. If in 4.6 we assume in addition that X is locally compact, then the equivariant imbedding of the conclusion can be chosen so as to be a proper map.

PROOF. By 4.1 and 4.6 we can find an equivariant imbedding f of X into the unit sphere in some Euclidean G -space V . Let ϕ be a positive real valued invariant function on X which vanishes at infinity (e.g., $\phi(x) = \sum_n 2^{-n} \psi_n(\|x\|)$ where $\{\psi_n\}$ is a locally finite partition of unity for X/G with support ψ_n compact) and define $f^*(x) = (1/\phi(x))f(x)$. It is clear that f^* is an equivariant imbedding of X in V (cf. proof that f_2^* is an imbedding in 4.2). Let K be any compact subset of V and put $M = \sup\{\|k\| \mid k \in K\}$. Then $f^{*-1}(K) \subset \{x \in X \mid \phi(x) \geq M\}$ which is a compact subset of X , hence f^* is proper.

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IX: ORBITS OF HIGHEST DIMENSION

Deane Montgomery

§1. Introduction

Throughout this lecture G will be a compact Lie group acting on a space X which is an n -manifold over Z . For any $x \in X$, the isotropy group, G_x , is the set of elements $g \in G$ such that $g(x) = x$. The number of components in G_x is denoted by $m(x)$ which is, therefore, an integral valued and upper semi-continuous function on X . There is a highest dimension for the orbits in X and this highest dimension will be denoted by r ; the set of points $x \in X$ on orbits of dimension less than r will be denoted by B . The set B is closed and more generally the set of points on orbits of dimension at most k , $k \geq 0$, is a closed set.

For any integers $u \geq 0$, $v \geq 1$, let

$$X_{u,v} = \{x : x \in X, \dim G(x) = u, m(x) = v\}$$

$$X_u = \bigcup_{v=1}^{\infty} X_{u,v}$$

$$Z_{u,v} = X_r \cup X_{r-1} \cup \dots \cup X_{u+1} \cup X_{u,1} \cup \dots \cup X_{u,v}.$$

For every $x \in X$, it is true that for y sufficiently near to x , G_y is conjugate to a subgroup of G_x so that $Z_{u,v}$ is open. Each set $X_{u,v}$ is locally a product of an orbit and a cross-section; any component of $X_{u,v}$ contains orbits of only one type, that is if x and y are in a component of $X_{u,v}$, then G_x is conjugate to G_y .

Cohomology is usually taken with compact supports and with coefficients in Z . There is a natural map f from X to the orbit space $X' = X/G$ and f takes B to $B' = B/G$. This lecture contains some of the results of [1, 2, 3, 4, 5].

§2. The set B

THEOREM 2.1. Let G be a compact Lie group acting

on a space X which is an n -cm over Z . Then $\dim_Z B \leq n - 2$. Hence if X is connected, B does not separate X ; thus if X is connected, then X_r is connected.

It will be sufficient to prove that each $X_{u,v}$ intersects $\bar{X}_r - X_r$ in a set of dimension at most $n - 2$. In order to prove this, choose p ,

$$p \in X_{u,v} \cap (\bar{X}_r - X_r) ,$$

and let Y be a slice at p . Then Y is an $(n-u)$ -cm over Z and the topological product of Y and an open cell of dimension u is an open neighborhood of p . Since $p \in \bar{X}_r$, there is a point $y, y \in Y, y \in X_r$ and

$$\dim G_p > \dim G_y .$$

The group G_p acts on Y as a transformation group and because $\dim G_p > \dim G_y$ there exists in G_p a cyclic group H , such that $H/(H \cap G_y)$ has order > 2 . Hence in the operation of H on Y , $F(H)$, the set of points stationary under H , satisfies (see V, 2.6)

$$\dim_Z F(H) \leq n - u - 2 .$$

But $X_{u,v} \cap Y \subset F(H)$ and hence

$$\dim_Z (X_{u,v} \cap Y) \leq n - u - 2$$

and

$$\dim_Z X_{u,v} \leq n - 2 .$$

This completes the proof of the theorem.

THEOREM 2.2. Let G be a compact connected Lie group acting on a space X which is an n -cm over Z , and let t be any integer $0 \leq t < r$. Then the union of all orbits of dimension $\leq t$, is a closed set of dimension $\leq n - r + t - 1$.

As above let

$$Z_{u,v} = X_r \cup X_{r-1} \cup \dots \cup X_{u+1} \cup X_{u,1} \cup X_{u,2} \cup \dots \cup X_{u,v} .$$

Then $Z_{u,v}$ is open and $Z'_{u,v} = f(Z_{u,v})$ is also open. Let

$$k_{u,v} = \dim_Z X'_{u,v} .$$

Then

$$\dim_Z X_{u,v} = k_{u,v} + u$$

and in order to prove the theorem it will be sufficient to prove

$$k_{u,v} \leq n - r - 1, \quad 0 \leq u < r .$$

The proof will be made by contradiction, so assume the above inequality is not always true; it will then be false for a largest u . Hence there is an $X_{\alpha,\beta}$, $0 \leq \alpha < r$, $k = k_{\alpha,\beta} > n - r - 1$, while if $u > \alpha$ or $u = \alpha$ and $v < \beta$, then $k_{u,v} \leq n - r - 1$.

Let p be a point of $X_{\alpha,\beta}$ in every neighborhood of which $\dim_Z X_{\alpha,\beta} = k + \alpha$, and choose a slice Y at p , $\bar{Y} \subset Z_{\alpha,\beta}$; let F be the set of points of the slice which are left fixed by G_p . Note that Y is an $(n-\alpha)$ -gm over Z and the choice of α, β guarantees that

$$\dim_Z Z'_{\alpha,\beta} = k .$$

There exist compact sets M, N in F , $M \supset N$, such that there is an element $a \in H_C^{k-1}(N, Z)$ which cannot be extended to M but which can be extended to any proper compact subset of M which includes N .

Now let M_1, M_2 be proper compact subsets of M such that

$$M \supset M_1 \supset M_2 \supset N .$$

The following diagram is commutative and the rows are exact:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_C^{k-1}(M; Z) & \longrightarrow & H_C^{k-1}(M_1; Z) & \longrightarrow & H_C^k(M-M_1; Z) \longrightarrow H_C^k(M; Z) \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & H_C^{k-1}(M; Z) & \longrightarrow & H_C^{k-1}(M_2; Z) & \longrightarrow & H_C^k(M-M_2; Z) \longrightarrow H_C^k(M; Z) \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & H_C^{k-1}(M; Z) & \longrightarrow & H_C^{k-1}(N; Z) & \longrightarrow & H_C^k(M-N; Z) \longrightarrow H_C^k(M; Z) \longrightarrow \dots \end{array}$$

The diagram and the facts above show there is an element b_1 in

$H_C^k(M - M_1; Z)$ carried to an element b_2 in $H_C^k(M - M_2; Z)$ where $b_1 \neq 0$ and $b_2 \neq 0$.

As before f will denote the map from X to $X' = X/G$; then M is carried to M' , M_1 to M'_1 , and so on. Let m be a point of $M - N$. Choose $U_1 \subset U_2$ to be open sets in $Y - N$, which are invariant under G_p . Now let $M_1 = M - U_1$ so that

$$M - M_1 = M \cap U_1, \quad i = 1, 2$$

and similarly

$$M' - M'_1 = M' \cap U'_1, \quad i = 1, 2.$$

We shall, for the remainder of this proof, consider f as mapping Y onto Y' .

The map f is a homeomorphism of M onto M' so b_1 , $i = 1, 2$ has a corresponding element $b'_1 \in H_C^k(M' - M'_1; Z)$. Now $\dim Y' \leq k$ and hence b'_1 can be extended to U'_1 . The following diagram is commutative ($M - M_1$ is closed in U_1 and $M' - M'_1$ is closed in U'_1):

$$\begin{array}{ccc} H_C^k(U'_1; Z) & \longrightarrow & H_C^k(M' - M'_1; Z) \\ \downarrow & & \downarrow \\ H_C^k(U_1; Z) & \longrightarrow & H_C^k(M - M_1; Z) \end{array}.$$

This commutative diagram shows that b_1 can be extended to U_1 . The following diagram is also commutative:

$$\begin{array}{ccc} H_C^k(U_1; Z) & \longrightarrow & H_C^k(M - M_1; Z) \\ \downarrow & & \downarrow \\ H_C^k(U_2; Z) & \longrightarrow & H_C^k(M - M_2; Z) \end{array}.$$

Since U_1 may be chosen as small as desired and since $k \leq n - \alpha - 2$, we may assume the map

$$H_C^k(U_1; Z) \longrightarrow H_C^k(U_2; Z)$$

is trivial. But this is inconsistent with the fact that b_1 and b_2 are covered, $b_1 \neq 0$, $b_2 \neq 0$, and $b_1 \longrightarrow b_2$. This contradiction proves the theorem.

COROLLARY. Let G and X be as above. Then $\dim X' \leq n - r$ and $\dim B' \leq n - r - 1$.

§3. The Set D

As above, G is to be a compact Lie group acting on X , an n -cm over Z , and the highest dimension of any orbit is r . Let

$$s = \min m(x), x \in X_r$$

$$D = \bigcup_{v \geq s} X_{r,v}.$$

For any point $p \in X_r$, choose a connected slice K at p , and let G_K be the subgroup of G_p which leaves every point of K fixed. Then G_K is an invariant subgroup of G_p and $G_K \supset G_p^*$ where G_p^* is the identity component of G_p ; $H = G_p/G_K$ is a finite group which acts effectively on K .

LEMMA 3.1. If $p \in X_{r,s} \cap D$, then for every $x \in K$, s is a factor of $m(x)$ and $m(x)/s$ is the order of H_x . Therefore a point $x \in K$ belongs to D if and only if H_x is not trivial.

Let y be a point in $X_{r,s} \cap K$. Since $X_{r,s}$ is open, some neighborhood of y (relative to K) is in $X_{r,s}$ and hence G_y leaves all of K fixed so that for any $x \in K$

$$G_y \subset G_K \subset G_x.$$

This proves that if $x \in K$ then s is a factor of $m(x)$. Since $G_K \subset G_y$, it follows that

$$G_K = G_y.$$

This proves the Lemma.

LEMMA 3.2. $\dim_Z D \leq n-1$ and if X is connected then $\bar{X}_{r,s} = X$.

If $\bar{X}_{r,s} \neq X$, then $\bar{X}_{r,s} - \text{Int } \bar{X}_{r,s}$ is not null and is $(n-1)$ -dimensional so that it is not contained in B . Choose p not in B and in $\bar{X}_{r,s} - \text{Int } \bar{X}_{r,s}$ and let K and H be as above with $K \cap B = \emptyset$. Then $K - \bar{X}_{r,s}$ is non-null, open in K , and is contained in D . Hence

$$K \cap D = K - X_{r,s}$$

contains an inner point of K . By Lemma 3.1 a point is in $K \cap D$ if and only if H_x is not trivial. But the fixed point set of any non trivial

H_x is nowhere dense in K , and of dimension at most $(n - r - 1)$ over Z . This contradiction proves that $\bar{X}_{r,s} = X$. Since

$$B \cup D \subset \bar{X}_{r,s} - X_{r,s},$$

$B \cup D$ is nowhere dense and hence

$$\dim_{\mathbb{Z}} D \leq \dim_{\mathbb{Z}} (B \cup D) \leq n - 1.$$

This completes the proof.

For any integer $k > 1$ let

$$D_k = \{x : x \in D, ks \mid m(x)\}$$

where $ks \mid m(x)$ means that ks is a factor of $m(x)$.

LEMMA 3.3. $D = \bigcup D_k$, k prime; the set $D^* = \bigcup D_k$, k an odd prime or 4, is of dimension $\leq n - 2$.

This follows from results on fixed points, and slices. In this connection see V 2.6.

LEMMA 3.4. If X is connected, the set $X'_{r,s}$ is connected, and if furthermore G is connected, then $X_{r,s}$ is connected.

Let M be an open invariant subset of $X_{r,s}$ such that M' is a component of $X'_{r,s}$ and suppose $M \not\subset X_{r,s}$, so that

$$\bar{M} - \text{Int } \bar{M} \neq \emptyset.$$

Now $\bar{M} - \text{Int } \bar{M}$ is an $(n-1)$ -dimensional subset of $B \cup D$; since $B \cup D^*$ is $(n-2)$ -dimensional it follows that there is a point p of $\bar{M} - \text{Int } \bar{M}$ belonging to $D - D^* = X_{r,2s}$.

Take K and H as before; by Lemma 3.1 H is of order 2. Let T be the non-zero element of H . It can be proved (see for example [5] Lemma 16) that there exists a neighborhood A of p in K such that $A - D$ has exactly two components C and $T(C)$. Since $p \in \bar{M}$, M includes some points of either C or $T(C)$, say C . It follows that $C \subset M$, and because M is invariant it also follows that $C \cup T(C) \subset M$. Hence $p \in \text{Int } \bar{M}$ and this contradiction proves M' connected which proves the Lemma.

Orbits in D are called exceptional orbits of highest dimension and orbits in $X_{r,s}$ are called principal orbits.

§4. The Set $B \cap D$

THEOREM 4.1. Let G be a compact Lie group acting on X , an n -cm over Z . Let $u, v, u < r$, be fixed integers; let $p \in X_{u,v}$ be such that $\dim_Z(X_{u,v} \cap W) = n - 2$ for any neighborhood W of p . Then some neighborhood of p contains no point of D . Also W can be chosen as to contain only points of $X_r \cup X_{u,v}$.

We note first that $u = r - 1$, as, otherwise, $\dim_Z X_{u,v} < n - 2$. Let K be a connected slice at p ; then K is an $(n - r + 1)$ -cm over Z and

$$\dim_Z[K \cap F(G_p) \cap W] = n - r - 1$$

for any neighborhood W of p . Take $y \in K \cap X_r$. Then

$$\dim_Z G_p - \dim_Z G_y = 1$$

and G_y^* is an invariant subgroup of G_p^* . Then set $K \cap X_r$ is connected and hence for y and z in $K \cap X_r$, we have

$$G_y^* = G_z^* = G_K^* .$$

The group G_p/G_K has dimension one and from now on we discuss the action of this group (or of G_p) on K ; it has the same orbits as G_p on K . The group $(G_p/G_K)^*$ is a circle and its fixed point set, $F(G_p/G_K)^*$ is a cm over Z which includes $F(G_p)$ so that it must be an $(n - r - 1)$ -cm over Z . To be precise, this remark applies to that component of $F(G_p/G_K)^*$ which includes p ; but by choosing K small we may assume $F(G_p/G_K)^*$ is as stated.

For convenience let $A = G_p/G_K$. Then A/A^* is a finite transformation group acting on $F(A^*)$, an $(n - r - 1)$ -cm over Z . Let a_1 be an element of A/A^* so that a_1 has finite period. Now $F(a_1)$ has dimension $(n - r - 1)$ over Z and hence a_1 leaves all of $F(A^*)$ fixed. It has thus been shown that

$$F(A) = F(A^*) .$$

The group A^* has only a finite number of isotropy groups in

some neighborhood of p . Moreover every element of A^* preserves orientation so we may assume, by rechoosing K , that A^* acts freely in $K - F(A)$.

If $x \in K - F(A)$, then A_x is finite, and $A_x \cap A^* = e$.

The group A has only a finite number of classes of isotropy groups and, again by rechoosing K , we may assume that if $x \in K - F(A)$, $ax = x$, a of prime order then

$$a^2 = e, \dim F(a) = n - r,$$

and a reverses the orientation of K . Thus A acts freely except for some elements of order 2 of the kind described.

Let A_1 be the subgroup of A consisting of all elements a which commute with all of A^* ; thus for $a \in A_1$, $g \in A^*$ we have

$$aga^{-1} = g.$$

Consider the action of A_1 on K , and let $a \in A_1$ be such that

$$\dim_Z F(a) = n - r.$$

Since $a \in A_1$, A^* leaves $F(a)$ invariant so the fixed points of A^* for this action have dimension at most $(n - r - 2)$ over Z whereas we know $\dim_Z F(A^*) = n - r - 1$. This contradiction proves that A_1 acts freely in $K - F(A)$. This proves the theorem if $A = A_1$.

Assume now $A \not\subset A_1$. If A acts freely in $K - F(A^*)$ there is nothing to prove. If not take $a \in A$, $a \neq e$, and x in $K - F(A)$ such that

$$ax = x.$$

By the choices made, a reverses orientation, and

$$a^2 = e, \dim F(a) = n - r.$$

For $g \in A^*$, we have, since a is not in A_1 ,

$$aga^{-1} = g^{-1}.$$

Hence if $y \in F(a) \cap [K - F(A)]$, $g \in A^*$, then

$$agy = g^{-1}y.$$

This proves that for y as above there is one and only one other point in $A^*(y)$ which is left fixed by a . Thus, locally, $F(a)$ is a cross-section of the orbits of $K - F(A)$ that it touches.

The group A/A^* acts on the orbit space

$$[K - F(A)]/A^*.$$

The element a leaves an open subset of this orbit space fixed, as follows from the above. Hence a leaves the entire space $[K - F(A)]/A^*$ fixed; then $a A^*(y) = A^*(y)$ for all $y \in K - F(A)$. Since

$$agx = g^{-1}x,$$

a reverses the orientation of $A^*(x)$ and hence it reverses the orientation of every orbit of A^* in $K - F(A)$. Thus a has precisely two fixed points on each of these orbits.

Suppose now that $b \in A$, $b \neq e$, $bx = x$. Then b has the same properties as those just found for a .

But $ab \in A$, so $abx = x$ implies $ab = e$. Therefore $b = a^{-1} = a$ so $A_x = Z_2$ and the same is true for any A_y , $y \in K - F(A)$. This proves the theorem.

§5. Conditions under which $\dim_Z D \leq n - 2$.

It has been seen that $\dim_Z D \leq n - 1$ and it will be shown here that for E^n or S^n and certain more general spaces that $\dim_Z D \leq n - 2$. For this purpose the following is convenient.

LEMMA 5.1. If $\dim_Z D = n - 1$, then $H_C^{n-1}(\bar{D}, Z_2) \neq 0$.

Let p be a point of D such that for every neighborhood W of p

$$\dim_Z(D \cap W) = n - 1.$$

Let K be a slice at p and let H be the group G_p/G_K . If T is an element in H of order 2, then $F(T)$ is a cm over Z_2 . We need consider only elements T of order 2 such that the component of $F(T)$ which includes p is of dimension $n - r - 1$ over Z (there must be such by hypothesis); such a component is of dimension $n - r - 1$ over Z_2 and must be an $(n - r - 1)$ -cm over Z_2 . Therefore, (multiplying by an r -cell Q in G), p is in an $(n - 1)$ -cm over Z_2 which is in D .

Now let A be the set of all points p in D such that p is in an $(n-1)$ -cm over Z_2 which is in D . The following fact will be needed.

LEMMA 5.2. $\bar{A} - A$ is in B .

The proof follows by studying the section K and the group H and is omitted.

From A we now form a new space R as follows. The points of R are of the following kinds:

(1) The points p in A with the property that if any two $(n-1)$ -cms over Z_2 include p and are in D then these two coincide in some neighborhood of p ;

(2) the points p in A not in class (1); a point p in class (2) is included in a number (finite) of sets of the form $QF(T)$. Denote these sets by $Y_1, \dots, Y_k$. In this case p is to be replaced by k points $p_1, \dots, p_k$. The set R is now topologized by neighborhoods as follows:

(a) a point of class (1) has as a neighborhood any connected open set in A which is an $(n-1)$ -cm over Z_2 ; that is it has the same local neighborhoods as before.

(b) for a point p of type 2 we choose as a neighborhood of p_1 any open set in the set Y_1 above which includes p_1 .

As thus topologized, R becomes an $(n-1)$ -cm over Z_2 . There is a natural map

$$f : R \longrightarrow A$$

and any p in R has a neighborhood U in R which is mapped homeomorphically onto $f(U)$. The map f induces the homomorphism

$$f^* : H_C^{n-1}(A, Z_2) \longrightarrow H_C^{n-1}(R, Z_2) .$$

The latter group is not trivial. The local property of f shows that any cocycle of R is the image of one in A . Hence $H_C^{n-1}(A, Z_2)$ is not trivial.

Now A is open in D and in $\bar{D}$; moreover $\bar{A} - A \subset \bar{D} - D$, so that by Theorem 4.1

$$\dim_2(\bar{A} - A) \leq n - 3 .$$

The following cohomology sequence over Z_2 is exact:

$$\begin{aligned} \dots \rightarrow H^{n-2}(\bar{A}-A; Z_2) \rightarrow H^{n-1}_C(A; Z_2) \rightarrow H^{n-1}_C(\bar{A}; Z_2) \rightarrow H^{n-1}_C(\bar{A}-A; Z_2) \\ \rightarrow H^n_C(A; Z_2) \rightarrow \dots \end{aligned}$$

Since $H^{n-2}_C(\bar{A}-A; Z_2) = 0$ we see that $H^{n-1}_C(\bar{A}; Z_2)$ is non-trivial. The following is also exact:

$$\rightarrow H^{n-1}_C(\bar{D}, Z_2) \rightarrow H^{n-1}_C(\bar{A}; Z_2) \rightarrow H^n_C(\bar{D} - \bar{A}; Z_2) \rightarrow \dots$$

But $H^n_C(\bar{D} - \bar{A}; Z_2) = 0$ so that $H^{n-1}_C(\bar{D}; Z_2)$ is non trivial.

THEOREM 5.3. Let G be a compact Lie group acting on X , an n -cm over Z , and assume $H^{n-1}_C(X, Z_2) = 0$. Then $\dim_{Z_2} D \leq n-2$, for G connected.

Assume the theorem false so that $\dim D = n-1$. Then $H^{n-1}_C(\bar{D}; Z_2) \neq 0$. Consider the following sequence over Z_2 .

$$\rightarrow H^{n-1}_C(X; Z_2) \rightarrow H^{n-1}_C(\bar{D}; Z_2) \rightarrow H^n_C(X-\bar{D}; Z_2) \rightarrow H^n_C(X; Z_2) \rightarrow H^n_C(\bar{D}; Z_2) \rightarrow \dots$$

Then $X - \bar{D}$ is not connected but this is a contradiction and proves the theorem.

THEOREM 5.4. Let G be a compact connected Lie group acting on X , an n -cm over Z and let $p \in X$ be a stationary point, that is $G(p) = p$. Then there is an open set W' including p such that $\dim(W' \cap D) \leq n-2$.

Let U be open invariant neighborhood of p . There exist open neighborhoods $W \subset V \subset U$ of p with $\bar{W} \subset V$, $\bar{V} \subset U$ such that for every open neighborhood $W' \subset W$ of p and for coefficients in Z_2

$$\begin{aligned} \text{Im } j_{W'}^k &= \text{Im } j_W^k = 0, \quad k \neq n \\ &= Z_2, \quad k = n \end{aligned}$$

We may assume W' is invariant.

Assume W' does not satisfy the requirement of the theorem.

Let $R_{W'}$ and R_V be the parts of R associated with W' and V respectively. The map $H^{n-1}_C(R_{W'}; Z_2) \rightarrow H^{n-1}_C(R_V; Z_2)$ is not trivial

by I, 4.3. The following diagram is commutative:

$$\begin{array}{ccc} H_C^{n-1}(W' \cap A; Z_2) & \longrightarrow & H_C^{n-1}(R_{W'}; Z_2) \\ \downarrow & & \downarrow \\ H_C^{n-1}(V \cap A; Z_2) & \longrightarrow & H_C^{n-1}(R_V; Z_2) \end{array} .$$

Hence $H_C^{n-1}(W' \cap A; Z_2) \longrightarrow H_C^{n-1}(V \cap A; Z_2)$ is not trivial. The two diagrams below are commutative:

$$\begin{array}{ccccccc} H_C^{n-2}(W' \cap (\bar{A}-A); Z_2) & \longrightarrow & H_C^{n-1}(W' \cap A; Z_2) & \longrightarrow & H_C^{n-1}(W' \cap \bar{A}; Z_2) & \longrightarrow & H_C^{n-1}(W' \cap (\bar{A}-A); Z_2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_C^{n-2}(V \cap (\bar{A}-A); Z_2) & \longrightarrow & H_C^{n-1}(V \cap A; Z_2) & \longrightarrow & H_C^{n-1}(V \cap \bar{A}; Z_2) & \longrightarrow & H_C^{n-1}(V \cap (\bar{A}-A); Z_2) \end{array}$$

and

$$\begin{array}{ccc} H_C^{n-1}(W' \cap \bar{D}; Z_2) & \longrightarrow & H_C^{n-1}(W' \cap \bar{A}; Z_2) \\ \downarrow & & \downarrow \\ H_C^{n-1}(V \cap \bar{D}; Z_2) & \longrightarrow & H_C^{n-1}(V \cap \bar{A}; Z_2) \end{array} .$$

Since $H_C^{n-2}(W' \cap (\bar{A}-A); Z_2) = H_C^{n-2}(V \cap (\bar{A}-A); Z_2) = 0$, and similarly for $n-1$, the map in the left column of the second diagram is not trivial. Consider now the following commutative diagram where coefficients in Z_2 are understood:

$$\begin{array}{ccccccccccc} \longrightarrow & H_C^{n-1}(W') & \longrightarrow & H_C^{n-1}(W' \cap \bar{D}) & \longrightarrow & H_C^n(W' - \bar{D}) & \longrightarrow & H_C^n(W') & \longrightarrow & H^n(W' \cap \bar{D}) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & H_C^{n-1}(V) & \longrightarrow & H_C^{n-1}(V \cap \bar{D}) & \longrightarrow & H_C^n(V - \bar{D}) & \longrightarrow & H_C^n(V) & \longrightarrow & H^n(V \cap \bar{D}) & \longrightarrow \end{array}$$

Now $\text{Im}[H_C^{n-1}(W' \cap \bar{D}) \longrightarrow H_C^{n-1}(V \cap \bar{D})]$ can not be trivial. But this leads to a contradiction as in the global case of Theorem 5.3.

§6. Remarks on the Differentiable Case

In this section, G is to be a compact connected Lie group acting on a space X which will be either R^n or S^n , and G acts differentiably, by which we mean that each homeomorphism of G is of class C^1 in the ordinary differentiable structure of X .

Let U be defined as all points on principal orbits, that is

$$U = X_{\Gamma} - D .$$

Thus B, D, U are invariant and

$$X = B \cup D \cup U .$$

For the differentiable case, any component of $X_{u,v}$ is a manifold. We shall prove the following:

THEOREM 6.1. Let a compact connected group act differentiably on $X = \mathbb{R}^n$ or S^n . Then $U' \cup D'$ is simply connected. Furthermore U' is orientable.

For the proof it is convenient to use the following:

LEMMA 6.2. Under the hypothesis of Theorem 6.1, let Q be an $(n-2)$ -dimensional component of an $X_{u,v}$, and take $b' \in Q'$. Then X' contains a closed $(n-r)$ -cell which is a neighborhood of b' ; the boundary of the cell contains b' and also contains a neighborhood of b' relative to B' . Furthermore the cell may be so chosen as to contain no point of D' .

Let b be such that $f(b) = b'$, and let K be a slice at b . The situation is now similar to that in Theorem 4.1 and in addition we now have differentiability so that, by Bochner's theorem, G_b/G_K is orthogonal in properly chosen coordinates around b . In this case we can see that each orbit of G_b/G_K in $K - F(G_b)$ is a circle and the action can actually be obtained from the action in a plane multiplied by an $(n-r-1)$ -cell. In the plane action there exists a closed arc, with one end point at the origin, which is a cross-section of all orbits in a neighborhood of the fixed point. This arc, multiplied by the $(n-r-1)$ -cell, gives an $(n-r)$ -cell cross-section and this cell projects homeomorphically under f and the image is the cell required. This proves the Lemma.

LEMMA 6.2. Let $X = \mathbb{R}^n$ or S^n and let K be a finite C^1 complex in X , $\dim K = 1$. Then X may be triangulated so that no $(n-1-1)$ -cell of X touches K .

The proof of this Lemma will be omitted.

The proof that $U' \cup D'$ is simply connected will now be sketched.

Let α' be a closed path in $U' \cup D'$ with end points at p' . Choose p in $f^{-1}(p')$. Since G is connected there exists a closed path α which covers α' ,

$$f\alpha = \alpha'.$$

The closed path α bounds a 2-cell β in X , and for the proof it must be shown that β can be deformed so that it does not intersect B . We know that

$$\dim(X_0 \cup \dots \cup X_{r-2}) \leq n - 3.$$

Using this and Lemma 6.2, it can be shown that β can be deformed so that

$$B \subset X_{r-1} \cup U \cup D.$$

In the same way, β can be deformed so that if it touches a component of X_{r-1} , then that component has dimension $n - 2$.

Next consider $\beta' = f(\beta)$. If $b' \in \beta' \cap B'$ then b' is on the boundary of an $(n-r)$ -cell as shown in Lemma 6.1. Hence we may deform β' , by pushing in the face of the cell so that b' is not in β' . By a finite number of such steps, we obtain a β' which is in $U' \cup D'$ and this completes the proof that $U' \cup D'$ is simply connected.

The following corollary may be proved in a similar way.

COROLLARY 6.3. Under the hypothesis of Theorem 6.1, assume in addition that $\dim D \leq n - 3$. Then U' is simply connected.

The proof of the corollary is omitted. In order to complete the proof of Theorem 6.1, it must be shown that U' is orientable. For this it is convenient to use the set E defined as follows:

$$E = \{x : x \in D, m(x|D) \text{ continuous at } x, \\ \dim_{\mathbb{Z}} D = n - 2 \text{ at } x\}.$$

Note that $\dim_{\mathbb{Z}} D \leq n - 2$ by Theorem 5.3.

LEMMA 6.4. Under the hypothesis of Theorem 6.1, $U' \cup E'$ is simply connected.

The proof of this lemma is based on the fact that $\dim_{\mathbb{Z}}(E - D) \leq n - 3$, and will be omitted.

LEMMA 6.5. $U' \cup E'$ is a manifold.

Of course $U' \cup E'$ is a manifold at a point of U' so for the proof of the Lemma it is only necessary to examine a point x' in E' . Choose x so that $f(x) = x'$ and let K be a slice at x . Then G_x/G_K is a finite group acting on an $(n-r)$ -cell and leaving fixed an $(n-r-2)$ -cell. Then the action of G_x/G_K is really given by the action of a finite group in the plane multiplied by the $(n-r-2)$ -cell. The group acting on the 2-cell must be a cyclic group, as we see from the definition of E , and the base space of such an action on a 2-cell is again a 2-cell. Hence $U' \cup E'$ is locally an $(n-r)$ -cell at b' . This proves the Lemma.

We now know that $U' \cup E'$ is a simply connected manifold and hence $U' \cup E'$ is orientable. It follows that U' , which is an open submanifold, is orientable. This completes the proof of Theorem 6.1.

COROLLARY 6.6. Under the hypothesis of 6.1, no closed path in U' (or in U) can reverse the orientation of an orbit in U . Therefore the r -dimensional homology of an orbit in U determines a constant sheaf over U' .

A principal orbit in X (or any orientable space) is orientable. The following stronger fact follows also.

COROLLARY 6.7. Under the hypothesis of 6.1, every orbit of highest dimension is orientable.

COROLLARY 6.8. Under the hypothesis of 6.1, let $n - r$ be odd. Then there are no isolated orbits in D .

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X: THE SPECTRAL SEQUENCE OF A BIFILTERED MODULE

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The purpose of this lecture is to supply the algebraic framework for a discussion of Fary's modification of the spectral sequence of a continuous map.

§1. Spectral Sequences

We review here some standard notation.

1.1. Modules are always over some fixed groundring L . Let A be filtered by submodules $F^p A$ ($p \in \mathbb{Z}$) (i.e., $F^p A \supset F^{p+1} A$, $\bigcup_p F^p A = A$). Then the graded module GA is the direct sum of the $F^p A / F^{p+1} A$.

Let moreover A be endowed with a differential d . Then the spectral sequence (E_r) or more precisely (E_r, d_r) of A is defined by

$$E_r^p = \Sigma_p E_r^p, \quad E_r^p = C_r^p / C_{r-1}^{p+1} + D_{r-1}^p$$

where

$$C_r^p = \{x \in F^p A, dx \in F^{p+r} A\}, \quad D_r^p = dC_r^{p-r}.$$

Let λ_r^p be the projection of C_r^p onto E_r^p . Then the restriction d_r^p of d_r to E_r^p is characterized by

$$(1) \quad d_r^p \lambda_r^p x = \lambda_r^{p+r} dx \quad (x \in C_r^p)$$

and we have $E_{r+1} = H(E_r)$.

Let κ_{r+1}^r be the projection of the module of d_r -cycles in E_r onto E_{r+1} , and for $r < s$, $\kappa_s^r = \kappa_s^{s-1} \circ \dots \circ \kappa_{r+1}^r$ (with a suitable domain of definition). Then

$$(2) \quad \kappa_s^r \circ \lambda_r^p(x) = \lambda_s^p(x) \quad (x \in C_s^p).$$

The filtration of A induces a filtration in $H(A)$ and E_∞ is defined as $GH(A)$. If $dA^p \subset A^p$, the only case of interest in the sequel, then $E_r = GA$ for $r \leq 0$, and $d_r = 0$ for $r \leq -1$.

1.2. LEMMA. Assume that $H(F^p A) \longrightarrow H(A)$ is injective for all p . Then $d_r = 0$ ($r \geq 1$).

We have to show that for any $x \in C_r^p$, $d_r \lambda_r^p x = 0$. The element dx is a cycle of $F^{p+r} A$ which is a boundary in $F^p A$, hence in A , therefore, by our assumption, there exists $y \in F^{p+r} A$ such that $dy = dx$. If $r \geq 1$, this implies $y \in C_{r-1}^{p+1}$ hence $\lambda_{r-1}^{p+1} y = 0$, and

$$d_r \lambda_r^p(x) = d_r \lambda_r^p(x - y) = \lambda_r^{p+r} d(x - y) = 0.$$

§2. The notion of bifiltration

On $Z \times Z$ we consider the partial ordering defined by $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$.

2.1. DEFINITION. Let A be a module. A bifiltration of A is a family of submodules $F^{s,t} A$ indexed by $Z \times Z$, such that

$$(F) \quad F^{s,t} A \cap F^{s',t'} A = F^{s^*,t^*} A \quad (s^* = \max(s, s'), t^* = \max(t, t')), \quad U F^{s,t} A = A.$$

To this bifiltration are associated three filtrations: the left filtration defined by the modules $F^{s,*} A = \sum_t F^{s,t} A$, the right filtration, defined by the modules $F^{*,t} A = \sum_s F^{s,t} A$ and the total filtration defined by $F^p A = \sum_{s+t=p} F^{s,t} A$. Clearly $F^{s,*} A = \sum_{s' \geq s} F^{s',t} A$, $F^{*,t} A = \sum_{t' \geq t} F^{s,t'} A$, $F^p A = \sum_{s+t \geq p} F^{s,t} A$. Also, (F) implies

$$(F)' \quad F^{s,t} A \supset F^{s',t'} A \quad \text{if} \quad (s, t) \leq (s', t') \quad U_{s,t} F^{s,t} A = A.$$

If A is differential, we assume moreover that $d(F^{s,t} A) \subset F^{s,t} A$, for all $s, t \in Z$.

In §§2 to 5, the module A and its bifiltration are chosen once for all, and we shall write $A^{s,t}$, U^s , V^t , A^p for $F^{s,t} A$, $F^{s,*} A$, $F^{*,t} A$ and $F^p A$ respectively.

2.2. Strictly speaking, this should be called decreasing bifiltration. Other types of bifiltration can be reduced to it by the familiar device of replacing an integer by its opposite. For instance assume that in (F) we replace $t^* = \max(t, t')$ by $t^* = \min(t, t')$. Then the left filtration is decreasing and the right filtration increasing. We

can go back to our case by substituting $(s, -t)$ for (s, t) , or we can carry out the discussion directly in this notation. This entails of course defining the total filtration by $F^p A = \sum_{s-t \geq p} F^{s,t} A$ and brings some obvious changes of signs in the statements of the theorems below. In the next lecture the bifiltration will be precisely of this type, and we shall use without further comment the slightly modified forms of 3.2, 5.1.

2.3. If one wishes to, one may consider 2.1 as a particular case of the following concept. Let M be a partially ordered set by a relation $a \leq b$ in which any two elements have a sup (i.e., $\sup(a, b) \geq a, b$ and is the unique element which is smaller than any element $c \geq a, b$). A M-filtration of a module A is a family of submodules $F^m A$ indexed by M such that

$$\bigcup_m F^m A = A \quad \text{and} \quad F^m A \cap F^{m'} A = F^{m^*} A \quad \text{with} \quad m^* = \sup(m, m') .$$

A filtration is thus a Z -filtration, Z being ordered in the usual way. Let $f : M \longrightarrow Z$ be an order preserving map. Then to any M -filtration there is associated a filtration defined by $F^p A = \sum_{m, f(m) \geq p} F^m A$. In 2.1 $M = Z \times Z$ and the three filtrations correspond to the maps $(s, t) \longrightarrow s$, $(s, t) \longrightarrow t$, $(s, t) \longrightarrow s + t$.

We proceed now to list some elementary properties of a bifiltration. $a^{p,q}$ will denote an element of $A^{p,q}$.

$$2.4. \quad U^s \cap V^t = A^{s,t}$$

For s fixed (respectively t fixed) the modules $A^{s,q}$ (respectively $A^{p,t}$) form a decreasing sequence, therefore

$$U^s \cap V^t = \sum_q (A^{s,q} \cap V^t) = \sum_{p,q} (A^{s,q} \cap A^{p,t})$$

$$U^s \cap V^t = \sum_{p \leq s, q \leq t} (A^{s,q} \cap A^{p,t}) = A^{s,t} \quad (\text{by } F)$$

REMARK. Given two filtrations by submodules A^s, B^t on a module C , the submodules $F^{s,t} C = A^s \cap B^t$ clearly define a bifiltration. 2.3 shows that in fact this definition is equivalent to 2.1 (however, if (F) were replaced by $(F)'$ 2.1 would define a broader class of bifiltrations).

2.5. $A^{s,t} \cap V^{t'} = A^{s,t^*}$ ($t^* = \max(t, t')$), $A^{s,t} \cap U^{s'} = A^{s^*,t}$ ($s^* = \max(s, s')$). $A^{s,t} \cap V^{t'} = \sum_s (A^{s,t} \cap A^{s',t'}) = \sum_{s' \leq s} (A^{s,t} \cap A^{s',t'}) = A^{s,t^*}$ by F and the fact that the $A^{s',t'}$ (t' fixed) form a decreasing sequence. Same proof for the second equality.

$$2.6. \quad U^s \cap \left(\sum_{k+s \geq p, l \geq t} A^{k,l} \right) = \sum_{\substack{k+l \geq p \\ l \geq t, k \geq s}} A^{k,l} .$$

Let $A = \sum_{k+s \geq p; k \geq k_0; s \geq t} a^{k,s}$ be in U^s and let a_1 be the sum of elements in the right hand term whose first index k is equal to k_0 . If $k_0 < s$, then, $a_1 \in U^{k_0+1} \cap V^m$, where $m = \max(p - k_0, t)$. Let $m' = \max(p - k_0 - 1, t)$. Then $m' \leq m$, hence $a_1 \in U^{k_0+1} \cap V^{m'}$, that is, by 2.4, $a_1 \in A^{k_0+1, m'}$, with $k_0 + 1 + m' \geq p$, and $m \geq t$. Our assertion follows then by induction on k_0 . In the same way, one proves

$$(1) \quad V^t \cap \left(\sum_{\substack{k+s \geq p \\ k \geq s}} A^{k,s} \right) = \sum_{\substack{k+s \geq p \\ k \geq s, s \geq t}} A^{k,s}$$

$$2.7. \quad A^{s,t} \cap A^p = \sum_{\substack{k+s \geq p \\ k \geq s, s \geq t}} A^{k,s}$$

By 2.4, 2.6 (1),

$$A^{s,t} \cap A^p = U^s \cap (V^t \cap A^p) = U^s \cap (V^t \cap \sum_{k+s \geq p} A^{k,s}) = U^s \cap \left(\sum_{s \geq t} \sum_{k+s \geq p} A^{k,s} \right)$$

and 2.7 follows from 2.6.

2.8. $(\sum_{s+t=p} A^{s,t}) \cap A^{p+1} = \sum_{s+t=p} (A^{s,t} \cap A^{p+1})$. Let $a = \sum_{s \geq s_0} a^{s, p-s}$ be in A^{p+1} . By 2.6 we may write

$$a = \sum_{k \geq s_0} b^{k, p+1-k}.$$

Then $a^{s_0, p-s_0} - b^{s_0, p+1-s_0} \in U^{s_0+1} \cap V^{p-s_0} = A^{s_0+1, p-s_0} \subset A^{p+1}$; therefore $a^{s_0, p-s_0} \in A^{p+1}$, and $a - a^{s_0, p-s_0} \in A^{p+1}$. 2.8 follows then by an induction.

§3. The terms E_0 and E_1

3.1. NOTATION. E_r will be the spectral sequence of the bi-filtered differential module A with respect to the total filtration.

For s (respectively t) fixed, the modules $A^{s,t}$ define a filtration of U^s (respectively V^t), and hence also of the quotients $U^s/U^{s'}$ ($s < s'$), $V^t/V^{t'}$ ($t < t'$). The corresponding spectral sequences are denoted $E_r(U^s/U^{s'})$, $E_r(V^t/V^{t'})$. We write $F_r(s)$ for $E_r(U^s/U^{s+1})$ and $G_r(t)$ for $E_r(V^t/V^{t+1})$. The r -th differential in $F_r(s)$ (respectively $G_r(t)$) is denoted d_r^U (respectively d_r^V).

3.2. THEOREM. In the preceding notation, we have

- (1) $F_0^t(s) = G_0^s(t) = A^{s,t}/(A^{s+1,t} + A^{s,t+1})$.
- (2) E_0^p is the direct sum of the modules $F_0^s(t) = G_0^t(s)$ for $s + t = p$, and $d = d_0^U = d_0^V$.
- (3) $F_1^t(s) = G_1^s(t)$ and $E_1^p = \sum_{s+t=p} G_1^s(t)$ (direct)
- (4) $d_1 = d_1^U + d_1^V$
- (5) A d_1 -cocycle of $G_1^s(t)$ has a representative element in $A^{s,t} \cap C_2^p$.

By definition and 2.5,

$$(V^t/V^{t+1})^s = A^{s,t}/A^{s,t} \cap V^{t+1} = A^{s,t}/A^{s,t+1}$$

hence

$$G_0^s(t) = (A^{s,t}/A^{s,t+1})/(A^{s+1,t}/A^{s+1,t+1}) = A^{s,t}/A^{s+1,t} + A^{s,t+1}.$$

In the same way, the last term is seen to be equal to $F_0^t(s)$.

By definition, E_0^p is the sum of the submodules $A^{s,t}/A^{s,t} \cap A^{p+1}$. By 2.7, $A^{s,t} \cap A^{p+1} = A^{s+1,t} + A^{s,t+1}$, and by 2.8 the sum is direct. This proves the first part of (2). The equality $d_0 = d_0^U = d_0^V$ then follows directly from the definition of the 0-th differential. (3) follows obviously from (2).

Let $x \in A^{s,t} \cap C_1^p$ ($p = s + t$). Since $dx \in A^{s,t} \cap A^{p+1}$ we may write, by 2.7

$$dx = a + b \quad (a \in A^{s+1,t}, b \in A^{s,t+1}).$$

$ddx = 0$ then shows that $da = -db$, and hence $da, db \in A^{s+1,t+1}$ by (F), which means that $a, b \in C_1^{p+1}$. We have

$$d_1 \lambda_1^p x = \lambda_1^{p+1}(dx) = \lambda_1^{p+1}(a + b) = \lambda_1^{p+1}a + \lambda_1^{p+1}b.$$

But $a = dx \bmod V^{t+1}$, $b = dx \bmod U^{s+1}$, and therefore the image of a in $G_1^{s+t}(t)$ represents $d_1^V x$ and its image in $F_1^{t+1}(s)$ represents $d_1^U x$, proving (4).

To prove (5) let $f_{s,t}$ be the inclusion map $A^{s,t} \longrightarrow A$. Consider $A^{s,t}$ as bifiltered by the submodules $A^{s,t} \cap A^{s',t'}$. Then $f_{s,t}$

is compatible with the bifiltration and induces a map of the spectral sequence $(E_r(A^{s,t}))$ of $A^{s,t}$, with respect to the total filtration, into E_r . It follows immediately from (1) to (4) applied to $A^{s,t}$ that this map is an isomorphism of $E_r(A^{s,t})$ onto $\Sigma_{s' \geq s, t' \geq t} G_r^{s'}(t')$ for $r = 0, 1$.

Therefore any d_1 -cycle of $G_1^s(t)$ is the image of a d_1 -cycle of the corresponding part in $E_1(A^{s,t})$; consequently it has a representative in the image of $f_{s,t}$, that is in $A^{s,t}$.

3.3. An interpretation of d_1^U . Consider the exact sequence

$$0 \longrightarrow V^{t+1}/V^{t+2} \xrightarrow{f_t} V^t/V^{t+2} \xrightarrow{g_t} V^t/V^{t+1} \longrightarrow 0.$$

f_t, g_t are clearly compatible with the filtrations and hence define homomorphisms of the spectral sequences. It follows from 2.5 that the sequence

$$0 \longrightarrow (V^{t+1}/V^{t+2})^s \longrightarrow (V^t/V^{t+2})^s \longrightarrow V^t/V^{t+1})^s \longrightarrow 0$$

may be written

$$0 \longrightarrow A^{s,t+1}/A^{s,t+2} \longrightarrow A^{s,t}/A^{s,t+2} \longrightarrow A^{s,t}/A^{s,t+1} \longrightarrow 0$$

and hence it is also exact. By a standard argument,

$$(1) \quad 0 \longrightarrow G_0(t+1) \longrightarrow E_0(V^t/V^{t+2}) \longrightarrow G_0(t) \longrightarrow 0$$

is then an exact sequence of complexes.

LEMMA. The diagram

$$(2) \quad \begin{array}{ccc} G_1^s(t) & \xrightarrow{\delta_1} & G_1^s(t+1) \\ \downarrow & & \downarrow \\ F_1^t(s) & \xrightarrow{d_1^U} & F_1^{t+1}(s) \end{array},$$

in which the vertical maps are the isomorphisms of 3.2 (3) and δ_1 is the coboundary homomorphism attached to the exact sequence (1), is commutative.

This is a straightforward consequence of the definitions. Details are left to the reader.

§4. Two Further Assumptions

4.1. The discussion of E_2 will be carried out only under the following assumptions

- (A) $d_r^U = 0$ for $r \geq 1$, that is $F_1^t(s) = F_r^t(s)$ for $s, t \in \mathbb{Z}, r \geq 1$.
- (B) A has a grading by submodules ${}^n A$, which induces gradings of the $A^{s,t}$, with respect to which d is of degree 1, and with respect to which the total filtration is regular (that is, for each n , there exists $p(n)$ such that ${}^n A \cap A^p = 0$ for $p > p(n)$).

We have $(U^s/U^{s+1})^t = A^{s,t}/U^{s+1} \cap A^{s,t} = A^{s,t}/A^{s+1,t}$. Therefore, by 1.2, condition (A) is implied by

- (A') $H^*(A^{s,t}/A^{s+1,t}) \longrightarrow H(U^s/U^{s+1})$ is injective for all $s, t \in \mathbb{Z}$.

Moreover, using the description of the spectral sequence given in [1, pp. 317-18], it may be seen without difficulty that (A) + (B) is equivalent to (A') + (B). We omit the proof of this fact, which will not be needed.

In this section, we derive some consequences from (A) (4.2 to 4.4), and one from (A) and (B).

4.2. If $d_1^U = 0$ then $d_1 = d_1^V$, and $E_2^p = \Sigma G_2^s(t)(s + t = p)$. This follows from 3.2.

4.3. If $d_1^U = 0$, then (3.3 (1)) gives rise to an exact sequence

$$(3) \quad 0 \longrightarrow G_1^s(t+1) \xrightarrow{f_t^*} E_1(V^t/V^{t+2}) \xrightarrow{g_t^*} G_1(t) \longrightarrow 0$$

of graded complexes with differentials of degree + 1. Therefore there exists an exact triangle

$$(4) \quad \begin{array}{ccc} G_2(t+1) & \xrightarrow{f_t^*} & E_2(V^t/V^{t+2}) \\ & \searrow \delta_2 & \swarrow g_t^* \\ & G_2(t) & \end{array}$$

and in particular a coboundary operator

$$\delta_2 : G_2^s(t) \longrightarrow G_2^{s+1}(t+1) \quad .$$

4.4. LEMMA. Assume (A) is fulfilled. For $r = 1, 2$, let x be an element of $A^{s,t} \cap C_r^p(s+t=p)$ such that $dx \in A^{s,t'} \bmod U^{s+1}$ for some $t' \geq t+r$. Then there exists $y \in A^{s,t+1} \cap C_r^p$ such that $d(x+y) \in A^{s,t'+1} \bmod U^{s+1}$.

Let x^* denote the element of U^s/U^{s+1} represented by x . By assumption, $dx^* \in (U^s/U^{s+1})^{t'}$, therefore, by definition $x^* \in C^t(U^s/U^{s+1})$ where $k = t' - t$ is $\geq r$. In the spectral sequence of U^s/U^{s+1} we have, using (A),

$$d_k^U \lambda_k^t x^* = \lambda_k^{t'} dx^* = 0$$

which means that

$$dx^* \in C_{k-1}^{t'+1} + D_{k-1}^{t'} = C_{k-1}^{t'+1} + dC_{k-1}^{t+1}$$

(the notation referring of course to the spectral sequence of (U^s/U^{s+1})), and this is equivalent to

$$dx = a^{s,t'+1} + da^{s,t+1} \bmod U^{s+1} \quad (a^i, j \in A^i, j) .$$

We want to show that $y = -a^{s,t+1}$ satisfies our conditions. Clearly $d(x+y) \in A^{s,t'+1} \bmod U^{s+1}$, and $y \in A^{s,t+1}$.

By assumption $dx = a^{s,t'} \in A^{s,t'} \bmod U^{s+1}$, hence

$$dy = a^{s,t'+1} - a^{s,t'} + b \quad (b \in U^{s+1}) .$$

We have $t' \geq t+r \geq t+1$, therefore $b \in U^{s+1} \cap V^{t+1}$, that is (2.4), $b \in A^{s+1,t+1}$, hence $b \in A^{p+2}$. If now $r = 1$, this shows that $dy \in A^{p+1}$, hence $y \in C_1^p$. If $r = 2$, then $dy \in A^{p+2}$, hence $y \in C_2^p$, which ends the proof.

4.5. LEMMA. Assume that A satisfies (A) and (B).

Let $r = 1, 2$ and let $x \in A^{s,t} \cap C_r^p \cap {}^nA$. Then there exists $y \in A^{s,t+1} \cap C_r^p \cap {}^nA$ such that $d(x+y)$ is in $A^{s+1,t}$ if $r = 1$, and in $A^{s+1,t+1} + A^{s+2,t}$ if $r = 2$, ($s+t=p$).

Since $A^{s,t} \cap A^{p+r} = A^{s,t+r} \bmod U^{s+1}$ by 2.7, we have $dx \in A^{s,t+r} \bmod U^{s+1}$. Using induction on t' , and 4.4, we can then find for any $q \geq t+r$ an element $y_q \in A^{s,t+1} \cap C_r^p$ such that

$d(x + y_q) \in A^{s,q} \bmod U^{s+1}$, and it is clear that y_q may be taken in ${}^n A$. If q is big enough, then $A^{s,q} \cap {}^{n+1} A = 0$ by condition (B), hence there exists $y \in A^{s,t+1} \cap C_2^p \cap {}^n A$ such that $d(x + y) \in U^{s+1}$, hence $d(x + y) \in U^{s+1} \cap A^{s,t} \cap A^{p+r}$. Our conclusion follows then from 2.5, 2.7.

§5. The term E_2

5.1. THEOREM. Assume that A satisfies conditions (A) and (B) of 4.1. Then $E_2^p = \Sigma G_2^s(t)$, ($s + t = p$), (direct), and $d_2 = d_2^V + \delta_2$; that is, the restriction of d_2 to $G_2^s(t)$ is the sum of the second differential $d_2^V : G_2^s(t) \longrightarrow G_2^{s+2}(t)$ of the spectral sequence of V^t/V^{t+1} and of the coboundary operator $\delta_2 : G_2^s(t) \longrightarrow G_2^{s+1}(t+1)$ defined in 4.3.

In the proofs, all elements are assumed to be homogeneous with respect to the grading by the modules ${}^n A$, which will not be mentioned any more.

We have already seen (4.2) that E_2 may be identified in a natural way with the direct sum of the $G_2^s(t)$. Let $x \in G_2^s(t)$, or, strictly speaking, in the submodule of E_2 corresponding to $G_2^s(t)$ under that identification. By 3.2 (5), g has a representative $x_0 \in A^{s,t} \cap C_2^p$. By 4.5, there exists $y \in A^{s,t+1} \cap C_2^p$ such that $d(x_0 + y) \in A^{s+1,t+1} + A^{s+2,t}$. The element y belongs to $C_1^{p+1}(p = s + t)$, hence $\lambda_2^p y = 0$ and $\lambda_2^p(x_0 + y) = g$. Therefore $x = x_0 + y$ is a representative of g in $A^{s,t} \cap C_2^p$ for which

$$(1) \quad dx = a + b \quad (a \in A^{s+1,t+1}, b \in A^{s+2,t}) .$$

We have $da = -db$, hence $da, db \in A^{s+1,t+1} \cap A^{s+2,t} = A^{s+2,t+1}$ (see 2.1), and $a, b \in C_1^{p+2}$. They belong to the domain of definition of λ_1^{p+2} and we have $\lambda_1^{p+2} dx = \lambda_1^{p+2} a + \lambda_1^{p+2} b$. Of course $d_1 \lambda_1^{p+2} dx = 0$. Now (3.2), E_1 is bigraded by s, t and d_1 is homogeneous (of bidegree $(1, 0)$) with respect to this bigrading, therefore $d_1 \lambda_1^{p+2} a = d_1 \lambda_1^{p+2} b = 0$, the elements $\lambda_1^{p+2} a, \lambda_1^{p+2} b$ are then in the domain of definition of κ_2^1 and we have

$$d_2 g = \lambda_2^{p+2} dx = \kappa_2^1 \lambda_1^{p+2} dx = \kappa_2^1 \lambda_1^{p+2} a + \kappa_2^1 \lambda_1^{p+2} b$$

and it will be sufficient to prove that

$$(2) \quad \kappa_2^1 \lambda_1^{p+2} a = d_2^V g, \quad \kappa_2^1 \lambda_1^{p+2} b = \delta_2 g .$$

This is just a matter of comparing definitions. Namely, let x^*, a^* be the elements of V^t/V^{t+1} defined by x and a . Since $b \in A^{s+1, t+1}$, we have $dx^* = a^* \in F^{s+2}(V^t/V^{t+1})$, hence $x^* \in C_2^p(V^t/V^{t+1})$ is a representative of g viewed now as element of the spectral sequence of V^t/V^{t+1} . In the latter, we have then

$$d_2^V g = \lambda_2^{p+2} dx^* = \kappa_2^1 \lambda_1^{p+2} dx^* = \kappa_2^1 \lambda_1^{p+2} a^* \in G_2^{s+2}(t) .$$

Since the identification of E_2 with the sum of the $G_2^s(t)$ clearly maps $\kappa_2^1 \lambda_1^{p+2} a^*$ onto $\kappa_2^1 \lambda_1^{p+2} a$, this proves the first part of (2).

Let x', b' be the elements of V^t/V^{t+2} defined by x and b . Since $a \in V^{t+2}$, we have $dx' = b'$, hence $x' \in C_1^s(V^t/V^{t+2})$. In the notation of 4.3, we have

$$g_t^* \lambda_1^s(x') = \lambda_1^s x$$

which means that $\lambda_1^s x'$ is sent onto a d_1 -cycle of $G_1^s(t)$ representing the element $g \in G_2^s(t)$. Now $b \in A^{s+1, t+1}$ defines an element $b^* \in V^{t+1}/V^{t+2}$ of filtration $s+1$. As we have seen, $db \in A^{s+2, t+1}$, hence $b^* \in C_1^{p+2}(V^{t+1}/V^{t+2})$. We have $f_t^*(\lambda_1^{p+2} b^*) = \lambda_1^{p+2} b'$ in the notation of 4.3. Since $b' = dx'$, this implies, by the definition of a coboundary operator, that $\lambda_1^{p+2} b^*$ is a d_1^V -cycle of $G_1^{s+1}(t+1)$ whose homology class in $G_2^{s+1}(t+1)$ is $\delta_2 g$. Modulo obvious identifications again, this proves the second part of (2).

REMARK. In this proof, only the part of 4.5 corresponding to $r=2$ has been used. The assertion for $r=1$ allows one to prove directly that in the decomposition (1), a and b can be chosen so as to be in C_2^{p+2} , so that one may write directly $d_2 g = \lambda_2^{p+2} a + \lambda_2^{p+2} b$; however, this would not shorten appreciably the above proof.

§6. Homomorphisms

In the proof of 3.2, mention was made of a homomorphism of spectral sequences induced by the inclusion $A^{s,t} \longrightarrow A$. If we consider $U^k, A/U^k$ to be filtered by $F^{s,t} U^k = A^{s,t} \cap U^k$, and by $F^{s,t}(A/U^k) = A^{s,t}/A^{s,t} \cap U^k$ then the inclusion $u_k : U^k \longrightarrow A$ and the projection $u_k' : A \longrightarrow A/U^k$ are compatible with the filtrations and induce, in particular, homomorphisms of the spectral sequences with respect to the total filtration. Similar remarks apply to $v_s : V^s \longrightarrow A$ and $v_s' : A \longrightarrow A/V^s$. These bifiltrations satisfy (B) if A does.

6.1. LEMMA. If A satisfies (A') , then so do

U^k , V^l , A/U^k , A/V^l , and $A^{s,t}$.

We put $s' = \max(s, k)$, $t' = \max(t, l)$. Then $F^{s,t}V^l = A^{s,t} \cap V^l = A^{s,t'}$ and $F^{s,*}V^l = A^{s,l}$ by 2.5. Therefore

$$F^{s,t}V^l/F^{s+1,t}V^l = A^{s,t'}/A^{s+1,t'}; \quad F^{s,*}V^l/F^{s+1,*}V^l = A^{s,l}/A^{s+1,l} \subset U^s/U^{s+1}.$$

Hence condition (A') for A implies condition (A') for V^l .

$F^{s,t}(A/V^l) = A^{s,t}/A^{s,t'}$, and $F^{s,*}(A/V^l) = U^s/(U^s \cap V^l) = U^s/A^{s,l}$, therefore

$$F^{s,t}(A/V^l)/F^{s+1,t}(A/V^l) = A^{s,t}/(A^{s,t'} + A^{s+1,t'}) ,$$

$$F^{s,*}(A/V^l)/F^{s+1,*}(A/V^l) = U^s/(U^{s+1} + A^{s,l}) .$$

If $t \geq l$, then $F^{s,t}(A/V^l)/F^{s+1,t}(A/V^l) = 0$, and there is nothing to prove. If $t \leq l - 1$ then $t' = l$, and we have the following commutative diagram, whose rows are exact by 2.5.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{s,l}/A^{s+1,l} & \longrightarrow & A^{s,t}/A^{s+1,t} & \longrightarrow & F^{s,t}(A/V^l)/F^{s+1,t}(A/V^l) \longrightarrow 0 \\ & & \downarrow j_1 & & \downarrow j_2 & & \downarrow j_3 \\ 0 & \longrightarrow & A^{s,l}/A^{s+1,l} & \longrightarrow & U^s/U^{s+1} & \longrightarrow & F^{s,*}(A/V^l)/F^{s+1,*}(A/V^l) \longrightarrow 0 . \end{array}$$

In the corresponding diagram for the cohomology sequences, j_1^* is the identity, and, by assumption, j_2^* is injective. Then j_3^* is injective by diagram chasing.

The proofs for U^k , A/U^k , $A^{s,t}$ are as elementary as for V^l and left to the reader.

6.2. PROPOSITION.

a) the map u_k induces an isomorphism of $E_r(U^k)$ onto $\Sigma_{r \geq k}^s G_r^s(t)$ ($s \geq k$) for $r = 0, 1$, and if A satisfies (A'), (B), also for $r = 2$. The map u_k^1 induces a surjective homomorphism of $E_r(A)$ onto $E_r(A/U^k)$ whose kernel is $\text{Im } u_k$ for $r = 0, 1$.

b) v_l induces an isomorphism of $E_r(V^l)$ onto $\Sigma_{t \geq l}^s G_r^s(t)$ for $r = 0, 1$, and also for $r = 2$ if A satisfies (A'), (B). The map v_l^1 induces a surjective homomorphism of $E_r(A)$ onto $E_r(A/V^l)$ whose kernel is $\text{Im } v_l$ for $r = 0, 1$.

c) $f_{s,t} : A^{s,t} \longrightarrow A$ induces an isomorphism of $E_r(A^{s,t})$ onto $\Sigma G_r^{s'}(t')(s' \geq s, t' \geq t)$ for $r = 0, 1$, and also for $r = 2$ if A satisfies (A), (B).

This follows easily from 3.2, 5.1 and 6.1.

§7. Remarks

The simplest case of a bifiltration is of course the bifiltration associated with a double complex $A = \Sigma A^{m,n}$ by $F^{s,t}A = \Sigma A^{m,n}$ ($m \geq s; n \geq t$). But then, the total filtration is induced from the total degree, and is not of interest. On the other hand, we have not introduced here the analogues of the spectral sequences of a double complex with respect to the left and right filtrations.

However, a non-trivial case of the spectral sequence of a bifiltered complex appears in the earlier and basic paper of Leray, [2]: namely when one of the filtrations is induced by a grading. In particular, consider $A = B \otimes C$ where B is graded by submodules B^i , has a differential of degree 1, and C is filtered by submodules $F^j C$. Then define $F^p A = \Sigma_{i+j \geq p} B^i \otimes F^j C$. [2, Definition 14.2]. Then [2, p. 37], if B has no torsion, we have

$$E_2 = H(B \otimes E_1(C).)$$

(with respect to the total differential), as also follows from 3.2. Also, if C satisfies (A'), so does A . We omit the details since the topological analogue of this example will be the main concern of the next lecture.

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XI: THE SPECTRAL SEQUENCE OF FARY

A. Borel

The spectral sequence of Fary combines the spectral sequences attached to a continuous map and to a single space filtered by closed subspaces. It is obtained by changing the filtration in Leray's construction. Fary's treatment [4] is valid for cohomology with compact supports of locally compact spaces. We shall discuss it in a somewhat more general case, in the framework of sheaf theory, without, however, striving for the greatest generality; in fact, in order to arrive easily at an interesting interpretation of the terms occurring in E_2 , we shall make rather strong restrictions on the families of supports and on the map. As a consequence, when applied to the case of Fary, the "general" theorem contains the results of [4] only when either the map is proper or the total space is metrizable. As far as applications go, this is harmless enough, but nevertheless, in order to free us from any such assumption in this case, and also for the benefit of the reader interested mainly in Fary's case, we shall devote Section 4 to it, making use only of the theory of Leray proper, as described in [1].

§1. Sheaves

1.1. Let $\underline{M}$ be a sheaf on a space X , and A be a subspace of X . The sheaf induced by $\underline{M}$ on A is denoted $\underline{M} \mid A$. If A is locally closed (i.e., $A = U \cap F$, U open, F closed) then, given $\underline{M}$, there is a unique sheaf $\underline{M}_A$ on X which induces $\underline{M} \mid A$ on A and zero on $X - A$. If A is open (respectively closed), then $\underline{M}_A$ may be identified with a subsheaf (respectively quotient sheaf) of $\underline{M}$. For A closed there is an exact sequence

$$(1) \quad 0 \longrightarrow \underline{M}_{X-A} \longrightarrow \underline{M}_X \longrightarrow \underline{M}_A \longrightarrow 0.$$

If A and B are both locally closed, then

$$(2) \quad \underline{M}_{A \cap B} = (\underline{M}_A)_B = (\underline{M}_B)_A .$$

For all this, see [5, Chap. II, 2.9]. This characterization of $\underline{M}_A$ also shows that if $\underline{N}$ is another sheaf on X , and A is locally closed, then

$$\underline{M} \otimes \underline{N}_A = \underline{M}_A \otimes \underline{N} = (\underline{M} \otimes \underline{N})_A .$$

1.2. Let Φ be a family of supports in X , and A a subspace of X . Then [5]:

$$\Phi|A = \{S \in \Phi, S \subset A\} \quad \Phi \cap A = \{S \cap A, S \in \Phi\} .$$

They are both families of supports in A , and $\Phi|A$ is also, of course, a family of supports in X . If A is closed, $\Phi|A = \Phi \cap A$. If A is locally closed

$$(3) \quad \Gamma_{\Phi} \underline{M}_A = \Gamma_{\Phi|A} \underline{M}_A \cong \Gamma_{\Phi|A} (\underline{M}|A) .$$

The first equality of 1.1 (3) is obvious. By 1.1 (2), it is enough to prove the second one for A open and for A closed. If A is open, this means that a section of $\underline{M}$ over A , with support closed in X , extended by zero outside A is a section of $\underline{M}_A$, which is clear. If A is closed, see [5, Lemme 4.9.2, p. 188].

§2. Continuous maps

X, Y are topological spaces, $f : X \longrightarrow Y$ a continuous map, ψ, Φ families of supports on X and Y respectively. We shall lean heavily on XVI, Section 2.

2.1. Very well adapted families. ψ, Φ will be said to be very well adapted if they are well adapted and if $f^{-1}(\Phi) \subset \psi$.

If f is closed and surjective, then the families of all closed subsets are very well adapted. If X, Y are locally compact, and f is proper and surjective, the families of compact subsets are very well adapted.

LEMMA. Let ψ and Φ be very well adapted, and Y regular.

(a) $\psi|f^{-1}A$ and $\Phi|A$ are very well adapted for A locally closed in Y .

(b) If ψ, Φ are paracompactifying and $\underline{M}$ is a ψ -soft sheaf on X , $f_{\psi} \underline{M}$ is Φ -soft.

The assertion (a) follows from XVI, 2.3.

Let $S \in \Phi$ and c be a section of $f_{\Phi} \underline{M}$ over S . There exists [5, p. 150] an open neighborhood U of S and $c' \in f_{\Phi} \underline{M}(U)$ whose restriction to S is c . Since Φ is paracompactifying and since a paracompact space is normal, there exists an open neighborhood V of S in U whose closure $\bar{V}$ is contained in U and belongs to Φ . Let us now consider c' as an element of $\Gamma_{\Psi}(U) \underline{M}|f^{-1}U$. By our assumption $f^{-1}\bar{V} \in \Psi$, therefore, $\underline{M}$ being Ψ -soft, there exists $d \in \Gamma_{\Psi} \underline{M}$ whose restriction to $f^{-1}\bar{V}$ is equal to $c'|f^{-1}\bar{V}$. It follows then that d , viewed as an element of $f_{\Phi} \underline{M}(Y)$ has the same restriction as c' to V , hence also to S . Thus $d|S = c$, which proves (b).

2.2. In the sequel Φ will be paracompactifying. Let then $\underline{C}$ be a fine torsion-free resolution of the groundring L on Y (e.g., the Alexander-Spanier sheaf, or also the canonical resolution of [5]). We recall then that $\underline{C} \otimes \underline{N}$ is a fine, hence Φ -soft, resolution of the sheaf $\underline{N}$ on Y , and [2, XVI, 6, 3.6] that the functor $\underline{N} \longrightarrow \Gamma_{\Phi}(\underline{C} \otimes \underline{N})$ is exact. Of course, it is not necessary to use this particular type of resolution, but it seems slightly more convenient, at least from the typographical point of view, when dealing with restrictions to subspaces.

2.3. PROPOSITION. Let Y be regular, Ψ and Φ be paracompactifying and well adapted. Let $\underline{M}$ be a sheaf on X , A a locally closed subset of Y and $A^* = f^{-1}A$. Let $\underline{C}$ be a torsion free Φ -fine resolution of the groundring L on Y . If Ψ and Φ are very well adapted (respectively X is metrizable) and $\underline{M}^*$ is a Ψ -soft (respectively flabby) resolution of $\underline{M}$, the spectral sequence of $\Gamma_{\Phi}(\underline{C} \otimes (f_{\Psi} \underline{M}^*)_A)$, filtered by the degree of $\underline{C}$, may be canonically identified with $(E_r, f|A^*)$ and $H(f_{\Psi}|A^*(\underline{M}^*|A^*))$ is the restriction to A of $H(f_{\Psi} \underline{M}^*)$.

The last assertion follows already from XVI, 2.5 (under slightly weaker assumptions).

It is clear from the definitions that $\underline{C} \otimes (f_{\Psi} \underline{M})_A = (\underline{C} \otimes f_{\Psi} \underline{M})_A$. Hence $\Gamma_{\Phi}(\underline{C} \otimes (f_{\Psi} \underline{M})_A) = \Gamma_{\Phi}|_A(\underline{C} \otimes f_{\Psi} \underline{M})$ and this, by 1.2 (3) is equal to $\Gamma_{\Phi}|_A((\underline{C} \otimes f_{\Psi} \underline{M})|A) = \Gamma_{\Phi}|_A(\underline{C}|A \otimes (f_{\Psi} \underline{M})|A)$. Using XVI, 2.5, we get

$$(1) \quad \Gamma_{\Phi}(\underline{C} \otimes (f_{\Psi} \underline{M}^*)_A) \cong \Gamma_{\Phi}|_A(\underline{C}|A \otimes f_{\Psi}|A^*(\underline{M}^*|A^*)).$$

$\underline{C}|A$ is a $(\Phi|A)$ -fine torsion free resolution of the groundring. If X is metrizable and $\underline{M}^*$ is flabby, then $\underline{M}^*|A^*$ is flabby [5, Chap. II, 3.3,

Cor. 2], and $f_{\psi|A}(\underline{M}^*|A^*)$ is $(\phi|A)$ -soft [6, 3.7]. If ψ and ϕ are very well adapted, then so are $\psi|A^*$ and $\phi|A$ (2.1); moreover $\underline{M}^*|A^*$ is $(\psi|A^*)$ -soft [5, Théor. 3.5.5], and $f_{\psi|A^*}(\underline{M}^*|A^*)$ is $(\phi|A)$ -soft (2.1). Thus, in both cases, the right hand side of (1), filtered by the degree in $\underline{C}|A$, yields the spectral sequence of $f|A^*$ (with respect to $\psi|A^*$, $\phi|A$ and $\underline{M}|A^*$ of course).

2.4. COROLLARY. Let B be a relatively closed subspace of A . Then the homomorphisms μ, ν of the sequence

$$(2) \quad 0 \longrightarrow r_{\phi}(\underline{C} \otimes (f_{\psi} \underline{M}^*)_{A-B}) \xrightarrow{\mu} r_{\phi}(\underline{C} \otimes (f_{\psi} \underline{M}^*)_A) \xrightarrow{\nu} r_{\phi}(\underline{C} \otimes (f_{\psi} \underline{M}^*)_B) \longrightarrow 0$$

which is exact by 2.2, yield, via the identification

2.3 (1), the natural homomorphisms

$$(E_r, f|A - B) \longrightarrow (E_r, f|A) \longrightarrow (E_r, f|B).$$

Note also that since (2) is also exact for the modules F^D defining the filtrations, the E_0 terms form an exact sequence, and the E_i terms a complex.

2.5. REMARKS. In Sections 2 and 3, use is made of several kinds of resolutions of sheaves (injective, flabby, ϕ -soft, fine), and a few words about the motivations might not be out of order, although these are of course implicit in the proofs. As already mentioned in 2.2 there is no need to use $\underline{C} \otimes \underline{N}$ ($\underline{C}$ fine, torsion free) as a ϕ -soft resolution of $\underline{N}$ on Y , and any one of the three other types would do. As to the resolution $\underline{M}^*$ of $\underline{M}$ ($\underline{M}$ sheaf on X) the requirements are that (a) it should behave nicely with respect to f_{ψ} , that is, $f_{\psi} \underline{M}^*$ should consist of ϕ -acyclic sheaves (ϕ paracompactifying); (b) it should restrict nicely to a locally closed subspace, that is $\underline{M}^*|A^*$ should be a resolution of $\underline{M}|A^*$ by $(\psi|A^*)$ -acyclic sheaves having $(\phi|fA^*)$ -acyclic images under $f_{\psi|A^*}$.

Now the restriction of a ψ -soft sheaf to A^* is $(\psi|A^*)$ -soft, but, unless ψ and ϕ are very well adapted, it is not clear that the f_{ψ} -image of a ψ -soft sheaf is ϕ -soft. On the other hand, if $\underline{N}$ is flabby or injective, then $f_{\psi} \underline{N}$ is ϕ -soft, and $\underline{N}|A^*$ is flabby (injective) for A^* open, but not necessarily for A^* closed, unless X is metrizable. This is why in 2.3, 3.2 we assume either X metrizable, ψ, ϕ well adapted or ψ and ϕ very well adapted. Then we may take as $\underline{M}^*$ a ψ -soft resolution in the latter case and a flabby or injective resolution in either case.

§3. The spectral sequence of Fary

3.1. $f : X \longrightarrow Y$ is a continuous map, $Y = Y_0 \supset Y_1 \supset \dots \supset Y_n \supset \dots$ a decreasing sequence of closed subspaces of Y with empty intersection, and $X_1 = f^{-1}Y_1$. Given a sheaf $\underline{N}$ on Y , we write $\underline{N}_t$ for $\underline{N}_{Y-Y_{t+1}}$ and $\underline{N}_{t,t'}$ for $\underline{N}_{Y_t-Y_{t'}}$ ($t' < t$). We also set $Y_1 = Y$ for $1 \leq 0$ so that $\underline{N}_t = 0$ for $t < 0$. The subsheaves $\underline{N}_t$ form then an increasing filtration of $\underline{N}$, which is bounded from below by zero, and is bounded from above by m when Y_{m+1} is empty.

ψ, ϕ are paracompactifying families of supports on X and Y , and $\underline{M}$ is a sheaf on X . Let $\underline{M}^*$ be a flabby resolution of $\underline{M}$, $\underline{C}$ a torsion free fine resolution of the groundring on Y , and $A = r_\phi(\underline{C} \otimes f_{\psi} \underline{M}^*)$. The sheaf $f_{\psi} \underline{M}^*$ will be considered to be filtered by the subsheaves $(f_{\psi} \underline{M}^*)_t$, and we define

$$F^{s,t}A = \sum_{1 \geq s} r_\phi(\underline{C}^1 \otimes (f_{\psi} \underline{M}^*)_t)$$

$$\underline{U}^s = F^{s,*}A = \sum_{1 \geq s} r_\phi(\underline{C}^1 \otimes f_{\psi} \underline{M}^*)$$

$$V^t = F^{*,t} = r_\phi(\underline{C} \otimes (f_{\psi} \underline{M}^*)_t) .$$

By 1.1, 1.2, we have

$$F^{s,t}A = \sum_{1 \geq s} r_\phi((\underline{C}^1 \otimes f_{\psi} \underline{M}^*)_t) = \sum_{1 \geq s} r_\phi|_{Y-Y_{t+1}}(\underline{C}^1 \otimes f_{\psi} \underline{M}^*)$$

$$V^t = r_\phi|_{Y-Y_{t+1}}(\underline{C} \otimes f_{\psi} \underline{M}^*) .$$

Clearly $F^{s,t}A = U^s \cap V^t$, hence the $F^{s,t}A$ define a bifiltration of A (X, 2.1, 2.4). More accurately this bifiltration is decreasing in s , increasing in t , so that the condition F of X, 2.1 is replaced by $F_{s,t}A \cap F^{s',t'}A = F^{s^*,t^*}A$ with $s^* = \max(s, s')$, $t^* = \min(t, t')$, (see X, 2.2). We put then (loc. cit.)

$$F^pA = \sum_{s-t \geq p} r_\phi(\underline{C}^s \otimes (f_{\psi} \underline{M}^*)_t) .$$

Fary's spectral sequence is the spectral sequence of A with respect to the total filtration (F^pA) . The module A is the same as the one used to define the Leray spectral sequence of f (see XVI, 2.6), so that $H(A) = H_{\psi}^*(X; \underline{M})$.

A has a total grading defined by

$$n_A = \sum_{a+b=n} r_\phi(\underline{C}^a \otimes (f_{\psi} M^b)_t)$$

(where $\underline{M}^b$ denotes the sheaf of elements of degree b in $\underline{M}^*$), and the total differential of A increases it by 1.

3.2. LEMMA. We keep the preceding notation. Then

(a) A satisfies the conditions (A'), (B) of X, 4.1.

(b) If either ψ and ϕ are very well adapted or X is metrizable and ψ and ϕ well adapted, the spectral sequence of V^t/V^{t-1} ($t' < t$) with respect to the filtration defined by s , may be canonically identified with $(E_r(f|X_{t'+1} - X_{t+1}))$. In particular the homomorphisms $V^{t-1}/V^{t-2} \longrightarrow V^t/V^{t-2} \longrightarrow V^t/V^{t-1}$ yield the natural homomorphisms $(E_r(f|X_{t-1} - X_t)) \longrightarrow (E_r(f|X_{t-1} - X_{t+1})) \longrightarrow (E_r(f|X_t - X_{t+1}))$.

Since the grading of $\underline{M}^*$ is positive, we have $n_A \cap F^p A = 0$ for $p > n$, hence (B) is fulfilled.

We have $U^s/U^{s+1} = r_\phi(\underline{C}^s \otimes f_{\psi} \underline{M}^*)$ and $F^{s,t} A / F^{s+1,t} A = r_\phi(\underline{C}^s \otimes (f_{\psi} \underline{M}^*)_t)$, both modules being endowed with the differential of $\underline{M}^*$. Since $\underline{N} \longrightarrow r_\phi(\underline{C} \otimes \underline{N})$ is an exact functor, we have then

$$H(U^s/U^{s+1}) = r_\phi(\underline{C}^s \otimes H(f_{\psi} \underline{M}^*)), \quad H(F^{s,t} A / F^{s+1,t} A) = r_\phi(\underline{C} \otimes H((f_{\psi} \underline{M}^*)_t)) .$$

But $H((f_{\psi} \underline{M}^*)_t) = (H(f_{\psi} \underline{M}^*))_t$ is a subsheaf of $H(f_{\psi} \underline{M}^*)$. Since $H(F^{s,t} A / F^{s+1,t} A) \longrightarrow H(U^s/U^{s+1})$ is induced from the inclusion $(H(f_{\psi} \underline{M}^*))_t \longrightarrow H(f_{\psi} \underline{M}^*)$, it is injective (2.2). This proves the condition (A').

By definition

$$V^t/V^{t-1} = r_\phi(\underline{C} \otimes (f_{\psi} \underline{M}^*)_t) / r_\phi(\underline{C} \otimes (f_{\psi} \underline{M}^*)_{t-1}) \quad (t' < t)$$

By 2.2 again, this is equal to $r_\phi(\underline{C} \otimes (f_{\psi} \underline{M}^*)_{t'+1,t+1})$ so that the remaining assertions follow from 2.4.

3.3. In the following we write $\phi(t, t+1)$ for $\phi|Y_t - Y_{t+1}$, and $\underline{F}$ for $H(f_{\psi} \underline{M}^*)$, where $\underline{M}^*$ is a flabby (or also ψ -soft if ϕ, ψ are very well adapted) resolution of $\underline{M}$. By (XVI, 2.5), we have

$$(4) \quad \underline{F}|Y_t - Y_{t+1} = \underline{H}(f_{\psi}(t, t') \underline{M}^* | X_t - X_{t+1})$$

and

$$(5) \quad \underline{F}_Y = \underline{H}^*(f^{-1}Y; \underline{M})$$

under suitable assumptions, fulfilled in the following theorem, and this will be used implicitly in its statement.

THEOREM. We keep the previous notation. We assume either that ψ and ϕ very well adapted or that ψ and ϕ are well adapted and X is metrizable. Then the spectral sequence of Fary has the following properties:

- (a) E_r converges to E_∞ , and E_∞ is the graded module associated to $H_\psi^*(X; \underline{M})$ suitably filtered;
- (b) for $r = 0, 1, 2$, $E_r^p = \Sigma_{s-t=p} E_r^{s,2}(f|X_t - X_{t+1})$. In particular $E_2^p = \Sigma_{s-t=p} H_\phi^s(t, t+1)(Y_t - Y_{t+1}; \underline{F})$;
- (c) $d_2 = d_2^V + \delta_2$ where $d_2 : E_2^s(f|X_t - X_{t+1}) \longrightarrow E_2^{s+2}(f|X_t - X_{t+1})$ is the second differential of $(E_r, f|X_t - X_{t+1})$ and $\delta_2 : H_\phi^s(t, t+1)(Y_t - Y_{t+1}, \underline{F}) \longrightarrow H_\phi^{s+1}(t-1, t)(Y_{t-1} - Y_t; \underline{F})$ is the coboundary operator in the cohomology sequence with coefficients in $\underline{F}$ of $Y_{t-1} - Y_{t+1}$ modulo $Y_t - Y_{t+1}$;
- (d) let $E_r^{p,q}$ be the set of elements of filtration degree p , total degree $n = p + q$ in E_r . Then d_r increases p by r , decreases q by $r - 1$ and we have

$$E_2^{p,q} = \Sigma_{t \geq 0} H_\phi^{p+t}(t, t+1)(Y_t - Y_{t+1}; \underline{F}^{q-t}) .$$

This follows from 3.2 and X, 3.2, 5.1. Of course, the spectral sequence of $f|X_t - X_{t+1}$ is meant to be with respect to the families $\psi(X_t - X_{t+1})$, $\phi(Y_t - Y_{t+1})$ and with coefficients in $\underline{M}|X_t - X_{t+1}$.

We remark here that $E_r^{p,q}$ can be non-zero only if $p + q \geq 0$, $q \geq 0$, but there is a priori no restriction on p . However, if Y_m is empty, we have $E_r^{p,q} = 0$ for $p \leq -m$. As noted above ${}^nA \cap F^pA = 0$ for $p > n$, therefore the convergence mentioned in (a) has the following meaning: for $r > q + 1$, $E_r^{p,q}$ consists of permanent cocycles, we have surjective homomorphisms $E_r^{p,q} \longrightarrow E_{r+1}^{p,q} \longrightarrow \dots$; and the corresponding limit is isomorphic to $F^{p+q}H_\psi^*(X; \underline{M})/F^{p+1}H_\psi^{p+q}(X; \underline{M})$ [2, p. 235].

3.4. Particular cases. If Y_1 is empty, then we get the Leray

spectral sequence of f . If $X = Y$, $\psi = \phi$ and f is the identity, then $H(f\mathcal{M}^*) = \underline{M}$, the spectral sequence of $f|X_t - X_{t+1}$ degenerates, we get

$$E_2^{p,q} = \sum_{q \geq 0} H_{\phi}^{p+q}(q, q+1)(X_q - X_{q+1}; \underline{M})$$

and $d_2 : E_2^{p,q} \longrightarrow E_2^{p+2, q-1}$ is the coboundary operator

$$H_{\phi}^{p+q}(X_q - X_{q+1}; \underline{M}) \longrightarrow H_{\phi}^{p+q+1}(X_{q-1} - X_q; \underline{M})$$

in the cohomology sequence of $X_{q-1} - X_{q+1}$ modulo $X_q - X_{q+1}$.

3.5. Homomorphisms. $V^t = \Gamma_{\phi}(\underline{C}) \otimes (f_{\psi} \mathcal{M}^*)_t$ is the module which our general construction attaches to $f|X - X_{t+1}$, the space $X - X_{t+1}$ being of course filtered by the $X_1 - X_{t+1}$. Thus $v^t : V^t \longrightarrow A$ induces a homomorphism of $E_r(f|X - X_{t+1})$ into $E_r(f)$ corresponding to the inclusion $X - X_{t+1} \longrightarrow X$.

Similarly, $A/V^t \cong \Gamma_{\phi}(\underline{C} \otimes (f_{\psi} \mathcal{M}^*)_{Y_{t+1}})$ (by exactness) can be used to construct the spectral sequence of $f|Y_{t+1}$ (see 2.3). Thus $v_t : A \longrightarrow A/V^t$ induces a map of $E_r(f) \longrightarrow E_r(f|Y_{t+1})$ corresponding to the restriction map. These homomorphisms have the naturality properties stemming from X, 6.2, and their precise statements, using 3.2, 3.3, are left to the reader.

3.6. The usual spectral sequence of a map has a cup product when $\underline{M}$ is a sheaf of rings, but this is not so for the Fary spectral sequence. However, some use of the cup product can be made at times, using the identifications or homomorphisms of 3.3, 3.5. For instance, the d_2^V are differentials with respect to the cup product. On the image of $E_r(f|X - X_1)$, there is a cup product. Also, there is similarly a cup product on the image of $E_r(f|X_{t+1} - X_{t+2})$ in $E_r(f|X_{t+1})$, which via v_t^* can sometimes yield information in $E_r(f)$, etc.

The subspaces Y_t are arbitrary so far. In the applications they will be chosen so as to make $E_2(f|X_t - X_{t+1})$ as simple as possible. This will be achieved to a certain extent if the Y_t are such that the restriction of the sheaf $H^*(f_{\psi} \mathcal{M}^*)$ to $Y_t - Y_{t+1}$ is locally constant; when this is possible, Fary [4] says that f has a simple homology type. This is the case for instance when $f|X_t - X_{t+1}$ is the projection in a locally trivial fibering.

§4. Locally compact spaces

In this section all spaces are locally compact, and we consider

only compact supports. $c(X)$ is the family of compact subsets of X , and we write Γ_c for $\Gamma_c(X)$. The discussion will be carried out in the framework of [1].

4.1. Let K be a grating over X [1, Exp. II], A a subspace of X . Then K_A is the subgrating of elements of K with supports in A . For A closed the "section" $AK = K/K_{X-A}$ of K by A will also be considered as a grating on X , with the same supports. The sheaf $\underline{F}(K)$ associated to K has $K/K_{X-x} = xK$ as stalk at the point $x \in X$, and the obvious topology [1, Exp. V]. Clearly $\underline{F}(K)_A = \underline{F}(AK)$. If K is a fine cover, then $\underline{F}(K)$ will be called a fundamental sheaf. If K is a fine cover [1, Exp. II], then K_U for U open and FK for F closed are fine covers [1, Exp. II] of U and F respectively. Therefore, if $A = U \cap F$, AK_U is a fine cover of A . It is clear that $\underline{F}(K)_A = \underline{F}(FK_U)$, so that $\underline{F}(K)|_A$ is a fundamental sheaf for A . Let $f: X \rightarrow Y$ be a continuous map. If K has compact supports, then the image grating fK of K is defined [1, Exp. VII] and we have $AfK = f(A^*K)$ if $A \subset Y$, $A^* = f^{-1}A$.

4.2. The spectral sequence of a map. Let $f: X \rightarrow Y$ be a continuous map. Given a sheaf $\underline{M}$ on X , let $\underline{M}^*$ denote the tensor product of $\underline{M}$ with the sheaf associated with a fine cover [1, Exp. II] on X . Let $\underline{K}$ be the sheaf associated with a fine cover K on Y , e.g., the Alexander-Spanier sheaf. As in [1], let us abbreviate $\Gamma_c(\underline{K} \otimes \underline{N})$ ($\underline{N}$ sheaf on Y) by $K \circ \underline{N}$. Consider then $A = K \circ f\underline{M}$ filtered by $F^p A = \sum_{i \geq p} K^i \circ f\underline{M}^*$. Its spectral sequence is the spectral sequence $(E_r(f))$ of f . In it, E_∞ is the graded module associated to $H_c^*(X; \underline{M})$ suitably filtered and $E_2^{p,q} = H_c^p(f\underline{M}^*)$, where $H(f\underline{M}^*)$ is the homology sheaf of $f\underline{M}^*$. As is known, and incidentally will follow from 4.3, its stalk at $y \in Y$ is $H_c^*(f^{-1}y; \underline{M})$. For this see [7] (or [1], where only the case of a constant sheaf $\underline{M}$ is considered, but the extension to the case of a general sheaf of coefficients is obvious).

4.3. LEMMA. Let $f: X \rightarrow Y$ be a continuous map, $\underline{M}$ be a fine sheaf on X , A a locally closed subspace of Y and $A^* = f^{-1}A$. Then $f(\underline{M}_{A^*}) = (f(\underline{M}))_A$ and $f(\underline{M}|_{A^*}) = f(\underline{M})|_A$.

(Fine stands for fine in the sense of [1], or equivalently for $c(X)$ -fine in the sense of [2, 5].)

For A open these equalities follow readily from the following fact: given $u \in \Gamma_c \underline{M}$, and K compact in A , there exist $u_1, u_2 \in \Gamma_c \underline{M}$ such that $u = u_1 + u_2$, the support $S(u_1)$ of u_1 is in a neighborhood of K in A and $S(u_2)$ does not meet K , which is seen by use of a

suitable partition of $\Gamma_{\underline{C}} \underline{M}$.

Let A be closed. The exact sequence $0 \longrightarrow \underline{M}_{X-A^*} \longrightarrow \underline{M} \longrightarrow \underline{M}_{A^*} \longrightarrow 0$ yields a sequence

$$0 \longrightarrow \Gamma_{\underline{C}} \underline{M}_{X-A^*} \longrightarrow \Gamma_{\underline{C}} \underline{M} \longrightarrow \Gamma_{\underline{C}} \underline{M}_{A^*} \longrightarrow 0$$

which is exact since $\underline{M}$ is fine [1, III-Lemma 1], hence

$$0 \longrightarrow f\Gamma_{\underline{C}} \underline{M}_{X-A^*} \longrightarrow f\Gamma_{\underline{C}} \underline{M} \longrightarrow f\Gamma_{\underline{C}} \underline{M}_{A^*} \longrightarrow 0$$

is also exact. Our assertion follows then from 1.2 (3). If A is locally closed, use 1.1 (2).

4.4. PROPOSITION. We keep the notation of 2.2. Let A be a locally closed subspace of Y and $A^* = f^{-1}A$. Then $\underline{K} \circ (f\underline{M}^*)_A \cong \Gamma_{\underline{C}(A)}(\underline{K}|_A \otimes f(\underline{M}^*|_A))$. Hence the spectral sequence of $\underline{K} \circ (f\underline{M}^*)_A$, filtered by the degree of K , may be identified with $(E_r(f|_A^*))$.

From the definition, it follows that

$$\underline{K} \circ (f\underline{M}^*)_A = \Gamma_{\underline{C}}(\underline{K} \otimes (f\underline{M}^*)_A) \cong \Gamma_{\underline{C}}((\underline{K} \otimes (f\underline{M}^*))_A) .$$

The equality follows then from 1.2 (3) (and the compatibility of $\otimes$ with restriction). As noticed in 2.1, $\underline{K}|_A$ is a fundamental sheaf for A , and consequently $\underline{M}^*|_A^*$ is the tensor product of $\underline{M}|_A^*$ with a fundamental sheaf on A^* , whence our contention.

4.5. COROLLARY. Let A be open (respectively closed). The the homomorphism of spectral sequences induced by the natural map $\underline{K} \circ (f\underline{M}^*)_A \longrightarrow \underline{K} \circ f\underline{M}^*$ (respectively $\underline{K} \circ f\underline{M}^* \longrightarrow \underline{K} \circ (f\underline{M}^*)_A$) is the homomorphism $(E_r(f|_A)) \longrightarrow (E_r(f))$ (respectively $(E_r(f)) \longrightarrow (E_r(f|_A^*))$) induced by the inclusion map.

4.6. Let now (Y_1) be a decreasing sequence of closed subspaces of Y , with empty intersection and $Y_1 = Y$ for $1 \leq 0$. As in 3.1 we put $X_1 = f^{-1}Y_1$ and write $\underline{N}_t$ for $\underline{N}_{Y-Y_{t+1}}$, where $\underline{N}$ is a sheaf on $\underline{Y}$.

Let $\underline{M}$ be a sheaf on X , $\underline{M}^*$ its tensor product with a fundamental sheaf on X , K a fine cover on Y and $A = \underline{K} \circ f\underline{M}^*$. We put

$$F^{s,t}A = \sum_{i \geq s} K^i \circ (fM^*)_t$$

$$U^s = F^{s,*}A = \sum_{i \geq s} K^i \circ fM^*$$

$$V^t = F^{*,t} = K \circ (fM^*)_t .$$

We have $F^{s,t}A = U^s \cap V^t$. The $F^{s,t}A$ define a bifiltration of A , decreasing in s , increasing in t (see X, 2.2), and we put

$$F^p A = \sum_{s-t \geq p} K^s \circ (fM^*)_t .$$

Fary's spectral sequence is the spectral sequence of A with respect to the filtration $(F^p A)$. From then on the discussion is the same as before; f is any continuous map, the conditions ϕ, ψ very well adapted or X metrizable drop, the subscripts $\psi, \phi, \psi(t, t+1)$ are replaced by c , and the references 2.3, 2.4 by 4.4, 4.5. Apart from that, Section 3 from 3.2 on goes over without change to the present case and will not be repeated.

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XII: FIXED POINT THEOREMS FOR ELEMENTARY COMMUTATIVE GROUPS I

A. Borel

The theorems referred to in the title are proved in Section 3. Applications are made in Sections 4, 5, 6. Sections 1, 2 discuss some general properties of G -spaces.

As usual, G is a compact Lie group, G_0 its identity component. (H) is the conjugacy class of a subgroup H , N_H the normalizer of H . The space of double cosets MgH (M subgroup) is denoted by $G/(M, H)$. If X is a G -space, $X_{(H)} = \{x \in X, G_x \in (H)\}$ is the subspace made up by the orbits of type G/H . As in IV and V, X_G stands for $X \times_{G_0} E_G$.

π_X is the projection of X onto the orbit space X' . We also write x', A' for $\pi_X(x), \pi_X(A)$, etc.

§1. Some fiberings

1.1. Let U be a closed subgroup of G . We recall once for all that if S is an U -slice (VIII, 3.2) then the inclusion $S \rightarrow GS$ induces a homeomorphism of S/U onto $\pi_X(GS)$, both spaces being of course endowed with the quotient topology [VIII, 3.6]. In particular $F(U; X_{(U)})$ is an N_U -slice and under this action, it is a principal (N_U/U) -bundle over $X'(U)$.

In the sequel we write F for $F(U, X_{(U)})$ and N for N_U . Let Y be a G -space, then Y/U is in a natural way a N/U -space. In particular G/U is, via right translations, a principal N/U -bundle over G/N ; this action commutes with G so that G/U is also a $G \times N/U$ space. These actions of N/U or $G \times N/U$ are always understood below.

1.2. LEMMA. Let H be a closed subgroup of G , S be an H -slice in X , and K be a subgroup of $\bigcap_{s \in S} G_s$ invariant in H . Then there is an equivariant

homeomorphism of $S \times_{H/K} G/K$ onto GS . More generally, if Y is another G -space, there is a homeomorphism of $S \times_{H/K} (Y/K)$ onto $GS \times_G Y$.

To define $S \times_H G$, we consider H as acting on the right on G . Then the left translations of G define a G -space structure on $S \times_H G$. Consider the map $(s, g) \longrightarrow g \cdot s$ of $S \times G$ into GS . It is readily seen to induce a map $\alpha : S \times_H G \longrightarrow GS$ which is continuous, bijective and equivariant. Moreover, since S is a closed subset of GS , and G, H are compact, α is a closed map, hence α^{-1} is continuous. Finally, since K acts trivially on S , we have a natural identification $S \times_H G \approx S \times_{H/K} G/K$.

The second assertion is proved analogously. Consider the inclusion α' of $S \times Y$ into $GS \times Y$. It induces a map $\alpha : S \times_H Y \longrightarrow GS \times_G Y$, which is continuous and bijective. Since α' , and the projections $S \times Y \longrightarrow S \times_H Y$, $GS \times Y \longrightarrow GS \times_G Y$ are closed, α is closed, hence is a homeomorphism. Finally $S \times_H Y = S \times_{H/K} Y/K$ because K acts trivially on S .

1.3. We recall that if X or Y is a principal bundle, $X \times_G Y$ admits a locally trivial fibering (see [2] for instance). Thus, under suitable assumptions, Lemma 1.2 will show the existence of fiberings. We now proceed to list some such cases. (E, B, F, H) denotes a locally trivial fibering with total space E , base B , typical fibre F , structural group H . The notation of 1.2 is kept.

(1) We have a fibering $(GS, G/H, S, H/K)$, since G/K is a principal H/K -bundle over G/H . Its projection may be identified with the retraction $r : GS \longrightarrow Gx$ (see Appendix).

(2) There are two fiberings of $X_{(U)}$: $(X_{(U)}, G/N, F, N/U)$ which has an equivariant projection map, and $(X_{(U)}, X'_{(U)}, G/U, N/U)$, whose projection is π_X .

By 1.2, $X_{(U)} = F \times_{N/U} G/U$, and now both factors are principal (N/U) -bundles (1.1), whence (2).

(3) If Y is a principal G -bundle, there is a fibering $(GS \times_G Y, Y/H, S, H/K)$.

(4) There is a fibering $(X_{(U)} \times_G Y, X'_{(U)}, Y/U, N/U)$. If Y is a principal G -bundle, there is moreover the fibering $(X_{(U)} \times_G Y, Y/N, F, N/U)$.

This follows from $X_{(U)} \times_G Y = F \times_{N/U} Y/U$.

If $Y = E_G$, then $Y/A = B_A$ for any closed subgroup A of G . Therefore, putting $Y = E_G$ in (3), (4) we see that

(5) There are fiberings $((GS)_G, B_H, S_K, H/K), (X_{(U)},_G, B_N, F, N/U)$ and $(X_{(U)},_G, X'_{(U)}, B_U, N/U)$.

(6) Let M be a closed subgroup of G . Then there is a fibering $(X_{(U)}/M, X'_{(U)}, G/(M, U), N/U)$.

In fact, we have $X_{(U)} = F \times_{N/U} G/U$ by 1.2, the G -space structure being induced by the left translations on G/U . Since these commute with N/U , it follows immediately that $G/(M, U)$ is again a (N/U) -space and that

$$X_{(U)}/M = F \times_{N/U} G/(M, U) .$$

1.4. Again let X, Y be G -spaces. Then $X \times Y$ is also a $G \times G$ space, and $X \times_G Y$ is the quotient of $X \times Y$ under the "diagonal" subgroup $G = (g, g)$. Therefore we have natural projections

$$X \times Y \longrightarrow X \times_G Y \xrightarrow{\beta} X' \times Y' .$$

Let $x \in X, y \in Y$, then $g_{x,y} : (g, g') \longrightarrow (gx, g'y)$ yields a homeomorphism of $(Gx) \times_G (Gy)$ with $\beta^{-1}(x', y')$. It follows that $\beta^{-1}(x', y')$ may be identified with the space $G/(G_x, G_y)$ of double cosets mod G_x, G_y . In particular, if Y is principal, then the map $g_{x,y} : z \longrightarrow (zx, y), (z \in G)$ provides an identification of G/G_x with $\beta^{-1}(x', y')$.

If both X and Y are principal bundles, then $\beta^{-1}(x', y') = G \times_G G$ may be identified with G , acted upon by $G \times G$ by means of right and left translations, and we have, as usual, a fibering

$$(1) \quad (X \times_G Y, X' \times Y', G, G \times G) .$$

This can be applied to a variety of situations, using 1.2. We give some examples.

Consider the projection $\gamma : GS \times_G Y \longrightarrow \pi_X(S) \times Y'$. By 1.2 the left hand side is $(S \times_{H/K} (G/K)) \times_G Y$. The spaces $S \times Y$ and G/K are both $((H/K) \times G)$ -spaces in the natural way, and it follows from the definition of the twisted product that

$$S \times_{H/K} (G/K) \times_G Y \cong (S \times Y) \times_{(H/K) \times G} G/K$$

therefore

$$(2) \quad GS \times_G Y \cong (S \times Y) \times_{(H/K) \times G} G/K$$

and this identification carries γ over to the projection on the first factor. If now $S \times Y$ is a principal $((H/K) \times G)$ -bundle, then we get a fibering

$$(3) \quad ((GS)_{X_G} Y, S' \times Y', G/K, (H/K) \times G)$$

Taking $S = F$, $GS = X(U)$, $H = N$, $K = U$, we see in particular that if Y is a principal G -bundle, then there is a fibering

$$(4) \quad (X_{(U)} X_G Y, X'(U) \times Y', G/U, (N/U) \times G)$$

Specializing Y to E_G in (3), (4) we also get fibre bundles

$$(5) \quad ((GS)_G, S' \times B_G, G/K, (H/K) \times G)$$

$$(6) \quad ((X_{(U)})_G, X'(U) \times B_G, G/U, (N/U) \times G)$$

§2. The topology of the Leray sheaf

2.1. Let $\pi : X \longrightarrow Y$ be a continuous map. Assume first X , Y to be locally compact, and consider cohomology with compact supports. The Leray sheaf $\underline{F}(\pi)$ or $\underline{F}$ of π is the sheaf occurring in the E_2 -term of the spectral sequence of π (cohomology with coefficients in some fixed groundring). Thus $\underline{F}_Y = H^*(\pi^{-1}Y)$; the sheaf $\underline{F}$ is associated with the presheaf $U \longrightarrow H^*(\pi^{-1}U)$ (U open in Y). Here cohomology is meant with respect to the family of relatively closed supports. An equivalent description is obtained by assigning to U open and relatively compact the cohomology with compact supports of the closure of $\pi^{-1}U$.

Apart from this, we shall be concerned only with cases in which $\pi^{-1}(y)$ is compact for every $y \in Y$, π is closed, ϕ, ψ are all closed subsets of X, Y . Then $\underline{F}(\pi)_y = H^*(\pi^{-1}y)$ and $\underline{F}$ is associated with the presheaf $U \longrightarrow H^*(\pi^{-1}U)$ (see XVI, 2.5).

2.2. PROPOSITION. Let X be a G -space, S a slice at $x \in X$, r the equivariant retraction of GS onto Gx associated with S (see VIII, 3.7), r_y its restriction to $\pi_X^{-1}(y)$. Let $c \in H^*(\pi_X^{-1}y)$. Then $y \longrightarrow r_y^*(c)$ is a continuous cross section of $\underline{F}$ over S' . Let $s \in S$, and $p_s : G/G_s \longrightarrow \pi_X^{-1}(s')$ be the natural identification. Then $p_X^{-1} \circ r_s \circ p_s : G/G_s \longrightarrow G/G_x$ is the natural projection, deriving from the inclusion $G_s \subset G_x$.

Let $c' = r^*(c)$, and let i_y be the inclusion of $f^{-1}(y)$ in GS , ($y \in S'$). Then $y \longrightarrow i_y^* c'$ is a continuous cross section (2.1). Since $r_y = r \circ i_y$, this proves the first assertion. The second one follows from the definition of r .

2.3. Let Y be another G -space, which we assume to be compact for simplicity. Let π_1 be the projection of $X \times_G Y$ onto X' . As we know (IV, 1.3) $\pi_1^{-1}(x') \simeq Y/G_x$ ($\pi_X(x) = x'$). More precisely, the map $y \longrightarrow (x, y)$ induces a homeomorphism $q_x : Y/G_x \longrightarrow \pi_1^{-1}(x')$.

Let S be a slice at x , and r be its retraction onto Gx . Then r and the identity on Y provide a retraction r_1 of $GS \times_G Y$ onto $\pi_1^{-1}(x')$.

PROPOSITION. We keep the notation of 2.3. Let r_{1y} be the restriction of r_1 to $\pi_1^{-1}(y)$. Let $c \in H^*(\pi_1^{-1}(y))$. Then $y \longrightarrow r_{1y}^*(c)$ is a continuous cross section of $\underline{F}(\pi_1)$ over $\pi_X(S)$. Moreover, if $s \in S$, then $q_x^{-1} \circ r \circ q_s : Y/G_s \longrightarrow Y/G_x$ is the natural projection deriving from the inclusion $G_s \subset G_x$.

Same proof as for 2.2.

2.4. Let X, Y be two G -spaces. As in 1.5 we consider the projection: $X \times_G Y \longrightarrow X' \times Y'$, the canonical homeomorphism $q_{x,y} : Gx \times_G Gy \longrightarrow \beta^{-1}(x', y')$, or also if Y is principal, the homeomorphism $q_{x,y} : Gx \longrightarrow \beta^{-1}(x', y')$.

Let S, T be slices at $x \in X$ and $y \in Y$ respectively. There is a retraction $GS \times_G GT \longrightarrow \beta^{-1}(x', y')$ associated with $GS \longrightarrow Gx$, $GT \longrightarrow Gy$. We denote it again by r , and $r_{a,b}$ ($a \in \pi_X(S)$, $b \in \pi_Y(T)$) will be its restriction to $\beta^{-1}(a, b)$.

PROPOSITION. We keep the notation of 2.4. Let $c \in H^*(\beta^{-1}(x', y'))$. Then $(a, b) \longrightarrow r_{a,b}^*(c)$ is a continuous cross-section of $\underline{F}(\beta)$ over $\pi_X(S) \times \pi_Y(T)$. For $s \in S$, $t \in T$ the map $q_{x,y}^{-1} \circ r_{s',t'} \circ q_{s,t} : G/G_s \times_G G/G_t \longrightarrow G/G_x \times_G G/G_y$ (respectively $G/G_s \longrightarrow G/G_x$, when Y is a principal bundle) is the natural map associated with the projections $G/G_s \longrightarrow G/G_x$, $G/G_t \longrightarrow G/G_y$.

Same proof as for 2.2.

REMARK. In Sections 1, 2, we have tried to systematize various known remarks of Palais, Floyd, Conner and the author. 1.4 and 2.4 were suggested by a proof of Conner (Mich. J. M. 6 (1959), pp. 25-32, Theorem 1).

§ 3. Fixed point theorems

3.1. An elementary commutative p-group (p prime) is a direct product of Z_p 's; the number of factors is the rank or p -rank of the group. For unification purposes we also allow p to be zero and consider an n -dimensional torus as an elementary abelian 0-group of rank n .

$P(A, t) = \sum (\text{rk } A^i) t^i$ is the Poincaré polynomial or series of a free module graded by submodules A^i of finite ranks. For a space X , $P_p(X, t) = P(H_G^*(X; Z_p))$. We write $P(A, t) \leq P(B, t)$ if $\text{rk } A^i \leq \text{rk } B^i$ for all i .

K_p is a field of characteristic p (p prime or zero).

3.2. LEMMA. Let G be an elementary commutative p -group, and X be a compact G -space. Assume $\dim_p X$, $\dim H^*(X; K_p)$ and the number of orbit types to be finite. Then there is a decreasing sequence $X = X_0 \supset X_1 \supset \dots \supset X_m \supset X_{m+1} = \emptyset$ of closed G -invariant subspaces $(X_m \neq \emptyset)$, with the properties:

- (a) $H^*(X_i; K_p)$, $H^*(X'_i; K_p)$ are finite dimensional.
- (b) For each $i \leq m$ there is a closed subgroup G_i of G such that $X_i - X_{i+1} = \{x \in X, G_x = G_i\}$ if $p \neq 0$, or $X_i - X_{i+1} = \{x \in X, G_{x_0} = G_i\}$ if $p = 0$. Moreover $\text{rk } G_i \leq \text{rk } G_j$ for $i \leq j$.

Let s_1 be the minimum of $\text{rk } G_x$, ($x \in X$), and $A_2 = \{x \in X, \text{rk } G_x \geq s_1 + 1\}$. Then A_2 is closed, and $X - A_2$ is a disjoint union of open subsets $U_{11}, \dots, U_{1k_1}$ in which G_x (or G_{x_0} if $p = 0$) is constant. We then put $X_1 = X - U_{11} - \dots - U_{1k_1}$ for $i \leq k_1$, hence $X_{k_1} = A_2$. On A_2 , $\min \text{rk } G_x \geq s_1 + 1$, so that we may proceed similarly. Since G is commutative all these subspaces are G -invariant, and (b) is proved. (a) is proved by induction on m . By definition, $X_m = F(G_m; X)$ if $p \neq 0$, $X_m = F(G_{m0}; X)$ if $p = 0$. By IV, 5.3 $H^*(X'_i; K_p)$, $H^*(X_m; K_p)$ and $H^*(X'_m; K_p)$ are finite dimensional, hence $X - X_m$, $X' - X'_m$ have also finitely generated cohomology groups over K_p . The subspaces $Y_i = X_i - X_m$ form a decreasing sequence in $X - X_m$ having the property (b), with $Y_m = \emptyset$. By induction $H_G^*(X_i - X_m; K_p)$ is finite dimensional and (a)

follows from (IV, 5.3) and the cohomology sequence.

3.3. LEMMA. Let G be an elementary commutative p -group, X be a G -space. Assume that there is a subgroup H of G such that for all $x \in X$, $G_x = H$ if $p \nmid 0$, $G_{x0} = H$ if $p = 0$. Then the Leray sheaf of the map $\pi_1 : X_G \longrightarrow X'$, with coefficients in K_p , is constant.

Let $p \nmid 0$. Then (1.3 (5)), X_G is fibered over X' , with fibre B_H , structural group $N/H = G/H$. But here H is a direct factor in G , therefore $\rho^*(H, G) : H^*(B_G; K_p) \longrightarrow H^*(B_H; K_p)$ is surjective. Since N/H acts trivially on $\text{Im } \rho^*(H, G)$ this proves the lemma for $p \nmid 0$.

Let $p = 0$. Consider the map

$$B_H \xrightarrow{t_x} B_{G_x} \xrightarrow{q_x} \pi_1^{-1}(y) \quad (y \in X', x \in \pi_1^{-1}(y))$$

induced from the map $e \longrightarrow (x, e)$ of E_G into $X \times E_G$. Here q_x is the homeomorphism considered in 2.3, and t_x is the natural map associated to $H \longrightarrow G_x$. In our case, G_x is the direct product of H with a finite group; since the latter has trivial cohomology over K_0 , t_x^* is an isomorphism. Hence $t_x^* \cdot q_x^* : H^*(\pi_1^{-1}(y); K_0) \longrightarrow H^*(B_H; K_0)$ is an isomorphism. Replacing x by $g \cdot x$ changes $t_x \cdot q_x$ by the translation of B_H by g . Since G is connected, this has no effect on cohomology, hence $t_x^* \cdot q_x^*$ depends in fact only on $y \in X'$. Let us denote it by s_y^* . The map $(y, c) \longrightarrow (y, s_y^*(c))$ is then a bijective map of the simple sheaf $Y \times H^*(B_H; K_0)$ on the Leray sheaf, and it follows from 2.3 that it is continuous. It is therefore an isomorphism of the constant sheaf onto the Leray sheaf.

REMARK. In the foregoing two particular cases of the following statement occur. Let G be a topological group, N a closed central subgroup. Then in the fibering $(B_N, B_G, G/N)$, G/N acts trivially on the cohomology of the total space. This is of course obvious when G is connected. A general proof in singular cohomology was pointed out to the author by A. Heller, using his construction of a universal bundle (Trans. Am. M. S. 80 (1955), pp. 299-344): namely, one can take for E_G a semi-simplicial group, in which the singular complex of G is embedded as a subgroup (loc. cit. p. 329). It is clear from the construction that if N is central, then N , or rather its singular complex, is a central, hence invariant, subgroup of E_G . Then G/N may be considered as a subgroup of the connected group $E_G/N = B_N$, hence it acts trivially on the cohomology of B_N .

3.4. THEOREM. Let p be a prime or zero. Let G be an elementary commutative p -group (3.1) and X be a compact G -space, for which $\dim_p X$, $\dim H^*(X; K_p)$ and the number of orbit types are finite. If X is totally non-homologous to zero over K_p in the fibering (X_G, B_G, X, π_2) , then $\dim H^*(F(G; X); K_p) = \dim H^*(X; K_p)$. In particular, F is not empty.

In the proof, cohomology is with respect to K_p , $F = F(G; X)$; and we drop the subscript p in the Poincaré series.

" X totally non-homologous to zero in X_G " is equivalent to the two conditions: G acts trivially on $H^*(X)$ and the spectral sequence of π_2 is trivial [2, §4]. We already know that under the former condition $\dim H^*(X) \geq \dim H^*(F)$ (IV, 4.1, 5.4) so that it will be enough to establish the reverse inequality. In the notation of 3.1, we have to show

$$(1) \quad P(X, 1) \leq P(F, 1) \quad .$$

Since X is totally non-homologous to zero in X_G we have

$$(2) \quad P(X_G, t) = P(X \times B_G, t) = P(B_G, t)P(X, t) \quad .$$

Choose now subspaces X_1 of X satisfying the conditions of 3.2. Then $X_{1G} = \pi_1^{-1}(X_1')$. We consider the Fary spectral sequence (XI) of $\pi_1 : X_G \longrightarrow X'$, with respect to the subspaces X_{1G} . Then (XI, 3.3)

$$E_2 = \sum_{i=0}^{1=\infty} H_G^*(X_1' - X_{1+1}'; \underline{F})$$

where $\underline{F}$ is the Leray sheaf. By 3.3, $\underline{F}$ is constant over $X_1' - X_{1+1}'$, with fibre $H^*(B_{G_1})$, hence

$$(3) \quad E_2 = \sum_1 H_G^*(X_1' - X_{1+1}'; H^*(B_{G_1}))$$

$$P(E_2; t) = \sum_1 P(X_1' - X_{1+1}', t) \cdot P(B_{G_1}, t) \quad .$$

(the grading in E_2 is given by the total degree). Now $E_\infty = GH^*(X_G)$, hence

$$P(X_G, t) = P(E_\infty, t) \leq P(E_2, t)$$

and (2), (3) yield

$$(4) \quad P(X, t) \cdot P(B_G, t) \leq \sum_1 P(B_{G_1}, t) \cdot P(X_1^i - X_{1+1}^i, t) .$$

Let $\ell_1 = \text{rk } G_1$, $\ell = \text{rk } G$, and let us denote by a an integer which is equal to 1 if $p \neq 0$ and to two if $p = 0$. Then

$$P(B_G, t) = (1 - t^a)^{-\ell}, \quad P(B_{G_1}, t) = (1 - t^a)^{-\ell_1},$$

and (4) may be written

$$P(X, t)(1 - t^a)^{-\ell} \leq \sum_1 P(X_1^i - X_{1+1}^i, t)(1 - t^a)^{-\ell_1} .$$

By assumption, 3.2, and the cohomology sequence, the left hand side and each term on the right hand side is the product of a polynomial with positive coefficients by a negative power of $1 - t^a$. For $0 < t < 1$, each represents an analytic function, whose Taylor development has positive coefficients. Therefore the formal inequality in (4) is also meaningful and true if t is interpreted as a real variable in $[0, 1)$, and the different terms as values of functions. We have then also

$$(5) \quad P(X, t) \leq \sum_1 P(X_1^i - X_{1+1}^i, t)(1 - t^a)^{\ell - \ell_1} .$$

Now the only subgroup of rank ℓ of G is G itself. Therefore $\ell_1 < \ell$ for $1 < m$, (and also for m if and only if F is empty). As a consequence, we also have, for $t \in [0, 1)$,

$$(6) \quad P(X, t) \leq P(X_m^i, t)(1 - t^a)^{\ell - \ell_m} + (1 - t^a)Q(t)$$

where $Q(t)$ is a polynomial. If now $\ell_m \neq \ell$, the right hand side tends to zero when $t \rightarrow 1$, in contradiction with the fact that $P(X, 1)$ is a strictly positive integer. Therefore $\ell_m = \ell$, hence $G_m = G$, $F = X_m = X_m^i$. Then (6) yields

$$P(X, t) \leq P(F, t) + (1 - t^a)Q(t) .$$

Since the second term in the right hand side tends to zero as $t \rightarrow 1$, this proves (1).

3.5. COROLLARY. Let G be an elementary abelian p -group (p prime or zero), X a compact connected G -space for which $\dim_p X$, $\dim H^*(X; K_p)$ and the number of orbit types are finite. Assume that G acts trivially on $H^*(X; K_p)$, and that $H^*(X; K_p)$ is generated by 1 and by its elements of lowest

strictly positive degree k , or more generally by elements which are transgressive in (X_G, B_G, X, π_2) . Then $F(G; X)$ is non-empty if and only if $E_2(\pi_2) = E_\infty(\pi_2)$.

If $E_2 = E_\infty$, then X is totally non-homologous to zero in X_G , hence $F \neq \emptyset$ by 3.4. If now $F(G; X) \neq \emptyset$, then π_2^* is injective (IV, 3.3). Since $H^*(X)$ is assumed to be generated by transgressive elements, this implies that $E_2(\pi_2) = E_\infty(\pi_2)$, in view of known facts about $\ker \pi_2^*$ and the image of the transgression [2, §5].

3.6. THEOREM. Let G be an elementary abelian p -group (p prime or zero), and X a compact connected G -space. Assume that $\dim_p X$, $\dim H^*(X; K_p)$ and the number of orbit types are finite, and that $H^*(X; K_p)$ is generated by 1 and its two-dimensional elements. If G acts trivially on $H^*(X; K_p)$ and if all subgroups of rank 2 of G have fixed points, then G has fixed points and $\dim H^*(F(G, X), K_p) = \dim H^*(X; K_p)$.

Let (E_r) be the spectral sequence of the fibering (X_G, B_G, X, π_2) with coefficients in K_p (which will be omitted). By 3.4, it will be enough to show that $E_2 = E_\infty$. If $p = 0$, this is obvious, since we have then only even degrees in E_2 , so that from now on we assume $p \neq 0$. Let $G = H_1 \times \dots \times H_s$ with $H_1 = Z_p$. Let first p be odd. Then

$$H^*(B_G) = H^*(B_{H_1}) \otimes \dots \otimes H^*(B_{H_s})$$

$$H^*(B_{H_1}) = \wedge(a_1) \otimes_{K_p}[b_1] \quad (d^0 a_1 = 1, d^0 b_1 = 2).$$

$H^*(X)$ being generated by $H^2(X)$, we have $d_2 = 0$ and (E_r) will be trivial if and only if $d_3(H^2(X)) = 0$. Let $c \in H^2(X)$. Since $H^3(X) = 0$, the first Bockstein operator β is zero on X , hence also on $d_3(c) \in H^3(B_G)$. Since $\beta a_1 = b_1 (1 \leq i \leq s)$, this implies immediately that $d_3(c)$ is a sum of monomials of the type $a_1 b_j$. Therefore, in order to show that $E_2 = E_\infty$, it is enough to prove that the image of $d_3(c)$ in $H^*(B_{H_1} \times H_j) (i \neq j)$ and $H^*(B_{H_1})$ under $\rho^*(H_1 \times H_j, G)$ and $\rho^*(H_1, G)$ is zero for $1 \leq i, j \leq s$.

Let H be a subgroup of G . There is a map $X_H \longrightarrow X_G$ commuting with π_2 and inducing $\rho(H, G): B_H \longrightarrow B_G$ on base spaces, see

(IV, 3.2). In view of the standard properties of homomorphisms of spectral sequences, it is clear that $\rho^*(H, G)(d_3(c))$ is the image of c under d_3 in the spectral sequence of (X_H, B_H, X) . By our assumption and 3.5, this image must be zero if $\text{rk } H \leq 2$. As noted above, this implies $d_3(c) = 0$ and $E_2 = E_\infty$. For $p = 2$, the proof is similar.

3.7. REMARK. In the proof of 3.4, B_G has been considered to be classifying for all dimensions. However, from the topological point of view, we have considered the spectral sequences to be taken with respect to compact supports, that is as if B_G was compact, hence classifying up to some dimension N . This can be justified in the same way as in [2, p. 181], as follows: let E_1, E_2 be two N -universal G bundles for which there exists an equivariant map $E_1 \rightarrow E_2$. This map then induces a map $f: X \times_G E_1 \rightarrow X \times_G E_2$ commuting with π_1 and π_2 ; it yields therefore also homomorphisms of the Fary spectral sequence of π_2 . It is an isomorphism on the cohomology of the fibres up to N , and an isomorphism on the base for all degrees. Since the base degree is bounded by $\dim X'$ it follows easily, as in [2, §4 (d)], that f^* is an isomorphism up to the total degree $N - \dim X$. Therefore we may say that the Fary spectral sequence considered in the proof of 3.4 is characterized and may be defined by the following property: for each N , it is isomorphic up to $N - \dim X$ to the Fary spectral sequence of $X \times_G E_1 \rightarrow X'$ where E_1 is N -universal. One argues of course similarly for the spectral sequence of π_2 , a case which was already explicitly covered in [2]. Thus defined, the spectral sequences have then the properties claimed for in the proof of 3.4.

A similar "passage to the limit" will be made implicitly in XIII, and can be justified in the same way.

§4. Applications to projective spaces

In this section we consider spaces whose cohomology mod p , for a fixed prime p , is a truncated polynomial ring $K_p[x]/(x^{s+1})$, with $d_0 x = k$ even if $p \nmid 2$. Examples are the projective spaces over the real numbers ($k = 1, p = 2$), over the complex numbers ($k = 2$), the quaternionic numbers ($k = 4$), and the projective plane over the Cayley numbers ($k = 8, s = 2$).

4.1. LEMMA. Let p be a prime and (E, B, F, π) be a fibering in which $H^*(F; K_p) = K_p[x]/(x^{s+1})$, ($d_0 x = k$, even if $p \nmid 2$) and the Leray sheaf is constant. Then F is totally non-homologous to zero in any of the following cases:

- (a) $(p, s + 1) = 1$,
- (b) $k = 4, s + 1 \nmid p, p$ odd,
- (c) F is the quaternionic projective space of real dimension $4s, (s \geq 1)$ and p is odd.

Let (E_r) be the spectral sequence of π over K_p . Then $E_{k+1} = E_2, E_{k+2} = E_\infty$, and $E_2 = E_\infty$ if and only if $d_{k+1}x = 0$. Moreover, since d_{k+1} is a differential with respect to cup product, and k is even for $p \nmid 2$, we have

$$(1) \quad d_{k+1}x^m = m d_{k+1}x \otimes x^{m-1} \quad (d_{k+1}x \in H^{k+1}(B; K_p)) .$$

In particular, this implies $(s + 1)d_{k+1}x = 0$. If $(p, s + 1) = 1$, we get $d_{k+1}x = 0$. Let now $k = 4, p \nmid 2, s + 1 \nmid p$. If $s + 1 < p$, we are in case (a). If not $x^p \neq 0$. But $x^p = \underline{p}_p^2 x$, where $\underline{p}_p^1$ denotes the i -th reduced Steenrod power, and $\underline{p}_p^2 = 2 \cdot \underline{p}_1^1$ (see [5]). Hence

$\underline{p}_p^1 x = c \cdot x^{(p+1)/2}$ with $c \neq 0$. However, the reduced powers commute with transgression hence $x^{(p+1)/2}$ must be transgressive. Since $p + 1 \geq 4$, and is prime to p this implies that $d_{k+1}x^{(p+1)/2} = 0$ and therefore, using (1), that $d_{k+1}x = 0$.

If now p is odd, F is the quaternionic projective space and $s \geq 2$, then [3, §8] $\underline{p}_p^1 x = 2 \cdot x^{(p+1)/2}$ for a suitable x , and this implies $d_{k+1}x = 0$ as before.

4.2. THEOREM. Let p be prime, G be an elementary commutative p -group, X be a compact connected G -space of finite dimension mod p and for which $H^*(X; K_p) = K_p[x]/(x^{s+1})$ ($d^0 x = k$, even if $p \nmid 2$). Then $\dim H^*(F(G, X); K_p) = s + 1$ in any of the following cases:

- (a) $(s + 1, p) = 1$,
- (b) $p \nmid 2, k = 4, s + 1 \nmid p$,
- (c) $p \nmid 2, X$ is the quaternionic projective space of real dimension $4s, (s \geq 1)$.

For any i , $H^1(X; Z_p)$ is either zero or Z_p . Since a p -group acts always trivially on Z_p , and since G is the structural group of the fibering (X_G, B_G, X, π_2) the Leray sheaf on this fibering is necessarily constant. The result follows then from 4.1 and 3.4.

REMARK. On the complex projective space $P_s(C)$, there exists for each p dividing $s + 1$ a group of projective transformations, isomorphic to $(Z_p)^2$, which has no fixed points. Also the quaternionic

projective space of odd quaternionic dimension has a group $Z_2 + Z_2$ of projective transformations without fixed points, and the Cayley plane has a group Z_3^3 of projective transformations without fixed points.

§5. Applications to compact Lie groups

5.1. Let K be a compact Lie group, U a closed subgroup, A a group of coefficients. $H^*(K/U; A)$ is said to be equal to its characteristic algebra if the characteristic homomorphism of the fibering $(K, K/U, U)$ is surjective or equivalently, if K/U is totally non-homologous to zero in the bundle $(B_U, B_K, K/U)$ [2, Théor. 22.2] (coefficients in A of course). When this is the case, K/U is totally non-homologous to zero in any fibre bundle of the form $(E/U, B, K/U)$, where (E, B, K) is a principal K -bundle [2, Cor. to Prop. 18.3].

5.2. THEOREM. Let K be a compact connected Lie group, U a closed subgroup and p be a prime or zero. If $H^*(K/U; K_p)$ is equal to its characteristic algebra, then every elementary commutative p -subgroup of K is conjugate to a subgroup of U .

Let G be an elementary commutative p -subgroup of K . We have to show that G , acting on K/U by left translations, has a non-empty fixed point set.

The projection $K \rightarrow K/U$ is G equivariant, G acting on the left on K . Hence, (IV, 3.2), there is a commutative diagram

$$\begin{array}{ccc} K_G & \longrightarrow & (K/U)_G \\ \downarrow & & \downarrow \\ B_G & \xrightarrow{\text{Id.}} & B_G \end{array} \quad .$$

Here $K_G = K \times_G E_G$ is defined by letting G act on the left on K . This commutes with the right translations; and it is easy to see that, under that action, (K_G, B_G, K) is a principal K -bundle and that $((K/U)_G, B_G, K/U)$ may be written as $(K_G/U, B_G, K/U)$. Thus, 5.2 follows from 3.4 and the result recalled in 5.1.

REMARK. For $p = 0$, the theorem means simply that if $H^*(K/U; K_0)$ is equal to its characteristic algebra, then U and K have the same rank, which is not really new. The converse is also known [2, §26].

5.3. COROLLARY. If K has no p -torsion, then every elementary commutative p -subgroup of K is contained in a torus of K .

Let T be a maximal torus. By 5.2, it is enough to show that $H^*(K/T; K_p)$ is equal to its characteristic algebra when K is p -torsion free. This follows from [2, Prop. 29.3], if we take into account the fact that K/T is always torsion free (see R. Bott, Bull. Soc. Math. France 84 (1956), pp. 251-281 or A. Borel, Proc. Nat. Ac. Sc. USA 40 (1954), pp. 1147-1151).

5.4. REMARK. It is known [3] that if K has no p -torsion, then B_K has no p -torsion and that if B_K has no p -torsion, K/T is equal to its characteristic algebra ([3, Prop. 5.3], for this part of the proposition the assumption K simply connected is not used), so that the preceding proof also works under the assumption that B_K has no p -torsion. In fact it can be checked that this is equivalent to K being p -torsion-free, and that the three following conditions are equivalent:

- (a) K has no p -torsion;
- (b) B_K has no p -torsion;
- (c) every elementary abelian p -subgroup is contained in a torus.

Also, 5.1 can be extended to the following theorem: Let G be a compact abelian subgroup of the compact connected Lie group K , and G_0 be the identity component of G . Assume that K has no p -torsion for every prime p dividing the order of G/G_0 . Then G is contained in a torus. For all this, we refer to a forthcoming paper of the author "Abelian subgroups and torsion in compact Lie groups". For earlier results pertaining to p -torsion and elementary abelian p -subgroups, see A. Borel, Bull. A.M.S. 61 (1955), pp. 397-432, §12.

§6. Applications to homologically kählerian manifolds

6.1. A compact connected manifold X will be said to be homologically kählerian if there exists an element $Q \in H^2(X; K_0)$ such that multiplication by Q^{n-s} ($2n = \dim X$, $s = 0, \dots, n$) is an isomorphism of $H^s(X; K_0)$ onto $H^{2n-s}(X; K_0)$. In particular, it is orientable. In a kählerian manifold, the cohomology class represented by the imaginary part of the metric is well known to have the property assigned here to Q .

The following theorem is due to Blanchard. [1, Théor. II. 1.2].

THEOREM. Let (E, B, F) be a fibering whose total

space is compact and typical fibre is a connected compact homologically kählerian manifold. Let (E_r) be its spectral sequence over K_0 . Assume that $E_2 = H^*(B; K_0) \otimes H^*(F; K_0)$ and that $d^2(H^1(F; K_0)) = 0$. Then F is totally non-homologous to zero in E .

In fact, E need not be compact, if one uses closed supports. The main point in the purely algebraic proof is to have, besides the assumptions made on E_2 and F , a unit in $H^*(B; K_0)$. Also, it is not really necessary to assume that F is a manifold; this part of the assumption could be replaced by the assumption that F is a cohomology manifold, or even that F is a compact connected space whose cohomology ring over K_0 obeys Poincaré duality. We note also that Blanchard proves his theorem under a slightly weaker assumption on E_2 .

6.2. THEOREM. Let G be a torus acting on a homologically kählerian compact connected manifold and F be its fixed point set.

(a) If $H^1(X; K_0) = 0$, then F is not empty and $\dim H^*(X; K_0) = \dim H^*(F; K_0)$.

(b) If $F \neq \emptyset$, then $\dim H^*(X; K_0) = \dim H^*(F; K_0)$.

G has a finite number of orbit types on X (see VI) and, being connected, acts trivially on $H^*(X; K_0)$. Hence (a) follows from 3.4, 6.1.

Now let F be non-empty. By IV, 3.3, $\pi_2^* : H^*(B_G; K_0) \rightarrow H^*(X_G; K_0)$ is injective. Since its kernel contains $d_2(H^1(X; K_0))$, (see e.g., [2, §5]), the latter group is zero, and (b) is a consequence of 3.4, 6.1.

6.3. REMARK. We have $\dim H^*(X; K_p) \geq \dim H^*(F; K_p)$ for arbitrary p (prime or zero) (IV, 5.5), so that the reverse inequality is in fact equivalent to equality. When G is a group of isometries of a kählerian manifold, Frankel [6] has proved that $\dim H^*(F; K_p) \geq \dim H^*(X; K_p)$ for any p . As a consequence, X has no p -torsion if and only if F has no p -torsion. Thus 6.2 and 6.4 are weaker than these results in the properly kählerian case; on the other hand, they are proved without differentiability assumptions. In fact, as in 6.1, the assumption " X is a manifold" is not fully used. It could be replaced for instance by " X is a cohomology manifold".

6.4. PROPOSITION. Let G be a torus acting on a compact connected homologically kählerian manifold X .

Let p be a prime. If X has no p -torsion, then $F = F(G; X)$ has no p -torsion.

We assume F to be non-empty, since there is nothing to prove otherwise. By 6.2, we have $\dim H^*(X; K_0) = \dim H^*(F; K_0)$. Since G has only a finite number of distinct isotropy groups on X (see VI), there is a p -subgroup U of G such that $F(H; X) = F(G; X)$, therefore $\dim H^*(X, K_p) \geq \dim H^*(F; K_p)$, by (IV, 4.5). The space X having no p -torsion, we have $\dim H^*(X; K_0) = \dim H^*(X; K_p)$, therefore we see that $\dim H^*(F; K_0) \geq \dim H^*(F; K_p)$. But, X being a compact manifold, $H^*(X; \mathbb{Z})$ is finitely generated, hence so is $H^*(F; \mathbb{Z})$ (IV, 5.3). The proposition follows then from the previous inequality and the universal coefficient formula.

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XIII: FIXED POINT THEOREMS FOR ELEMENTARY COMMUTATIVE GROUPS II

A. Borel

This lecture is devoted to a theorem about elementary commutative groups acting on a cohomology sphere (2.3) and to a local analogue (4.3). The latter will in fact be reduced formally to the former by use of a method of Conner [2]. It relies on certain local notions introduced in §3 and some remarks on inductive limits made in §5.

§1. Notation

1.1. In §§1, 2, 3, the notation and conventions of XII, §3, are retained. In particular, G is an elementary commutative p -group (p prime or zero), l its rank, X is a compact connected G -space for which $\dim_p X$ and the number of orbit types are finite. The subspaces X_i of X are as in XII, 3.2, except that instead of requiring $X_m \neq \emptyset$, we put $X_m = F(G; X) = F$. Thus $G_m = G$, and $G_i \neq G$ for $1 \leq m-1$.

Let $Y = X - F$, $Y_1 = X_1 - F$, and s be the smallest integer such that $\text{rank } G_1 \geq l-1$ for $1 \geq s$. Then $Y_m = \emptyset$, $F(G; Y) = \emptyset$; we have

$$F(G_1; Y) = Y_1 - Y_{1+1}, \quad F(G_1; X) = X_1 - X_{1+1} + F \quad (1 = s, \dots, m-1)$$

for $p \neq 0$ and the same relations with G_1 replaced by G_{10} for $p = 0$. Thus Y_s is the disjoint union of the closed subspaces $Y_1 - Y_{1+1}$ ($s \leq 1 < m$).

The cohomology is always understood with respect to a field K_p of characteristic p . The subscript p is dropped in the notation of the Poincaré series. Finally the constant a is equal to 1 if $p \neq 0$, to 2 if $p = 0$.

The following is a sharpening of XII, 3.1.

1.2. LEMMA. Let M be a G -space and assume that there is a subgroup H of G such that $G_y = H$ if $p \neq 0$ (respectively $G_{y_0} = H$ if $p = 0$) for all $y \in M$. Then there is a continuous map $\lambda : M_G \longrightarrow M' \times B_H$ which carries the canonical projection $\pi_1 : M_G \longrightarrow M'$ over onto the projection pr_1 of $M' \times B_H$ on its first factor, and the spectral sequence mod p of π_1 onto that of pr_1 . In particular, the former is trivial.

In both cases $p = 0$ and $p \neq 0$, H is a direct factor in G , so let K be an elementary abelian subgroup of G such that $G = H \times K$. Let us take as E_G a product $E_H \times E_K$. It follows from the definitions that

$$M_G = (M \times_K E_K) \times_H E_H = M_K \times_H E_H.$$

Since H acts trivially on M_K , the last term is equal to $M_K \times B_H$. The map λ is then defined to be the product of the canonical map $\pi_1' : M_K \longrightarrow M/K = M/G = M'$ by the identity. The proof of XII, 3.1 shows in fact that λ induces an isomorphism of the Leray sheaf of π_1 onto the Leray sheaf of pr_1 , and the second assertion follows.

Finally, for reference, we reformulate the main point in the proof of XII, 3.4 in the following:

1.3. LEMMA. Let $A_i(t), B_i(t), (1 \leq i \leq k)$ be polynomials with positive coefficients, and let b, c be strictly positive integers. Assume that we have

$$\sum_{i=0}^{1=k} A_i(t) (1 - t^b)^{-1} \leq \sum_{i=0}^{1=k} B_i(t) (1 - t^c)^{-1}$$

in the sense that for each $n \geq 0$, the coefficient of t^n in the left hand side, considered as a power series in t , is smaller than or equal to the coefficient of t^n in the right hand side. Then $b^{-k} A_k(1) \leq c^{-k} B_k(1)$.

PROOF. Both sides are power series with positive coefficients, which converge for $0 \leq t < 1$. Therefore, for $t \in [0, 1)$, the given relation is also an inequality between values of functions. We multiply then both sides by $(1 - t)^k$ and let $t \longrightarrow 1$.

§2. Cohomology spheres

2.1. In this section, X is a cohomology n -sphere mod p . By the Smith theorem, F is a cohomology r -sphere and $X_1 - X_{1+1} + F$ ($1 \geq s$) a cohomology n_1 -sphere mod p ($r \leq n_1 \leq n$). Moreover, by the dimensional parity theorem, $n - r$ and $n - n_1$ are even if $p \neq 2$.

2.2. LEMMA. In the notation of 1.1, 2.1 we have

$$P(Y'_1 - Y'_{1+1}, t) = t^{r+1}(1 + t^a + \dots + t^{n_1-r-a}) \quad (s \leq 1 < m)$$

with the convention $P(Y'_1 - Y'_{1+1}; t) = 0$ if $n_1 = r$.

On $X_1 - X_{1+1} + F$ the effective group is Z_p if $p \neq 0$, a circle group if $p = 0$; it also has F as fixed point set, so that $Y'_1 - Y'_{1+1}$ may be viewed as the quotient space of $(X_1 - X_{1+1} + F) - F$. In the case $p \neq 0$, we are then in the Smith situation and the lemma is known (see e.g., [1, Lemme 5.6]). If $p = 0$, then all stability groups are finite, and for cohomology mod p , the situation is the same as for a principal bundle [XII, 2.2]. In particular, there is a spectral sequence going from $H^*(B_G) \otimes H_G^*(X_1 - X_{1+1})$ to $H_G^*(Y'_1 - Y'_{1+1})$. The proof is then the same as in the Smith case.

2.3. THEOREM. Let G, X be as in 1.1, 2.1. Let $n(H)$ be the integer such that $F(H; X)$ is a cohomology $n(H)$ -sphere mod p , where H is a subgroup of G (closed connected if $p = 0$), and let $r = n(G)$. Then

$$(1) \quad n - r = \sum_H (n(H) - r)$$

where H runs through the subgroups of index p if $p \neq 0$, through the closed connected subgroups of co-dimension 1 if $p = 0$.

In the notation of 1.1, 2.1, our assertion is clearly equivalent to

$$(1') \quad n - r = \sum_{s \leq i \leq m-1} (n_1 - r) \quad .$$

The proof will be similar to that of XII, 3.4. We shall consider the Fary spectral sequence (E_r) mod p of $\pi_1: Y_G \longrightarrow Y'$, with respect

to the subspaces Y_{1G} , and the spectral sequence $(E_r^1) \bmod p$ of the fibre map $\pi_2 : Y_G \longrightarrow B_G$. We have of course

$$(2) \quad P(E_2^1, t) \geq P(E_\infty^1, t) = P(Y_G, t) = P(E_\infty, t) \leq P(E_2, t) .$$

By 1.2, 2.1 and XI:

$$E_2 = \sum_{i=0}^{l=m-1} H_G^*(Y_1^i - Y_{i+1}^i) \otimes H^*(B_{G_1})$$

hence

$$(3) \quad P(E_2, t) = (1 - t^a)^{l-s} \sum_{i=s}^{l=m-1} t^{r+1} (1 + t^a + \dots + t^{n_1-r-a}) + \\ + \sum_{i=0}^{l=s-1} P(Y_1^i - Y_{i+1}^i, t) (1 - t^a)^{-l_1}$$

where $l_1 = \text{rank } G_1$ is $< l$ for $i < s$.

If $n = r$, then $n = n_1$ and (1) is obvious, so assume from now on $n > r$. We shall first prove

$$(4) \quad n - r \leq \sum_{i=s}^{l=m-1} (n_1 - r) .$$

Let $r \leq n - 2$. Then, by the cohomology sequence

$$H_G^{r+1}(Y) = H_G^n(Y) = K_p; H_G^1(Y) = 0 \quad (1 \neq r+1, n)$$

and G acts trivially on $H_G^*(Y)$ (for $p \neq 0$, use the fact that a p -group always acts trivially on Z_p). Therefore

$$(5) \quad E_2^1 = H^*(B_G) \otimes H_G^*(Y)$$

$$P(E_2^1, t) = (1 - t^a)^{-l} (t^{r+1} + t^n) .$$

In (E_r^1) there are only the fibre degrees $r+1, n$, therefore $E_2^1 = E_{n-r}^1, E_\infty^1 = E_{n-r+1}^1$. Let u, v be generators of $H_G^{r+1}(Y)$ and $H_G^n(Y)$ respectively. Then d_{n-r} is determined by

$$d_{n-r}(1 \otimes v) = y \otimes u, \quad (y \in H^{n-r}(B_G)) ,$$

and we get easily

$$(6) \quad E_\infty^1 = H^*(B_G)/(y) \otimes H_G^{r+1}(Y) + \text{Ann } y \otimes H_G^n(Y)$$

where $\text{Ann } y$ is the annihilator of y , and (y) the ideal generated by y in $H^*(B_G)$. We want to prove that $\text{Ann } y = 0$. Assume the contrary. If $p = 0, 2$, $H^*(B_G)$ is a ring of polynomials in l variables of degree a , so that our assumption implies $y = 0$, $\text{Ann } y = H^*(B_G)$. Let now $p \neq 0, 2$. Then

$$H^*(B_G) = \Lambda(a_1, \dots, a_g) \otimes_{K_p} [b_1, \dots, b_g] \quad (d^0 a_1 = 1, d^0 b_1 = 2).$$

If $\text{Ann } y \neq 0$, then y must be a sum of monomials each of which contains at least one of the variables a_1 . Consequently, $\text{Ann } y$ contains the product $a_1 \dots a_g$ and also $a_1 \dots a_g \otimes_{K_p} [b_1, \dots, b_g]$. Thus, for any p , we see that

$$P(\text{Ann } y, t) \geq (1 - t^2)^{-l} t^q$$

for some $q \geq 0$. By (2), (6) we have then

$$t^{q+n}(1 - t^2)^{-l} \leq P(E_2, t),$$

and this, by 1.3, (3) implies $2^{-l} \leq 0$. This contradiction proves that $\text{Ann } y = 0$. As a consequence

$$P((y), t) = t^{n-r} P(B_G, t) = t^{n-r} (1 - t^a)^{-l}$$

and (6) yields

$$\begin{aligned} P(E_\infty^1, t) &= t^{r+1} (P(B_G, t) - P((y), t)) = t^{r+1} (1 - t^a)^{-l} (1 - t^{n-r}) \\ (7) \quad P(E_\infty^1, t) &= (1 - t^a)^{1-l} \cdot t^{r+1} \cdot (1 + t^a + t^{2a} + \dots + t^{n-r-a}). \end{aligned}$$

(Recall that if $a = 2$, then $p = 0$ and $n - r$ is even.) Since $1 + t^a + \dots + t^k$ tends to $(k + a)/a$ when $t \rightarrow 1$, (4) follows from (3), (7) and 1.3. Let now $n - r = 1$. By the parity theorem, $p = 2$. We take Z_2 for a field of coefficients. By the cohomology sequence

$$H_C^n(Y) = Z_2 + Z_2, \quad H_C^1(Y) = 0 \quad (1 \neq n),$$

and there is an exact sequence

$$0 \longrightarrow H^{n-1}(F) \longrightarrow H_C^n(Y) \longrightarrow H^n(X) \longrightarrow 0.$$

In E_2^1 there is only one fibre degree, hence $E_2^1 = E_\infty^1$. If G acts

trivially on $H_C^*(Y)$, then we should have

$$P(E_\infty^1, t) = (1 - t)^{-l} 2t^n \leq P(E_2, t)$$

and (3), 1.3 would yield a contradiction. Thus G does not act trivially on $H_C^n(Y)$. Since G acts trivially on $H^{n-1}(F)$, $H_C^n(X)$, there exists by the above exact sequence a basis u, v of $H_C^n(Y)$ such that $g \cdot u = u$ for $g \in G$ and $g \cdot v$ is either v or $u + v$, the latter value being taken for at least on $g \in G$. Therefore the subgroup H of G which acts trivially on $H_C^*(Y)$ has index 2. Let $K \cong Z_2$ be such that $G = H \times K$. Then

$$H^*(B_G; H_C^n(Y)) \cong H^*(B_H) \otimes H^*(B_K; H_C^n(Y)) ,$$

and the non-zero element k of K operates by $k \cdot u = u$, $k \cdot v = u + v$. The known results on the cohomology of cyclic groups (recalled in IV, 2.1, or [1]) imply that $H^1(B_K; H_C^n(Y))$ is zero for $1 \neq 0$ and 1-dimensional for $1 = 0$. Hence we get

$$(7') \quad P(E_2^1, t) = P(E_\infty^1, t) = (1 - t)^{1-l} t^n$$

so that, in the case $n - r = 1$, (4) follows from (2), (3), (7'), 1.3.

It remains to prove the reverse inequality. For this, we shall minorize $P(E_\infty, t)$. Let (F_r) be the Fary spectral sequence of Y_{SG} filtered by the subspaces Y_{1G} ($1 \geq s$). Y_s is a disjoint union of the subspaces $Y_1 - Y_{1+1}$ (see 1.1), therefore Y_{SG} is the disjoint union of the subspaces $(Y_{1G} - Y_{1+1,G}) = (Y_1 - Y_{1+1})_G$. For each of these the spectral sequence of π_1 is trivial (1.2), hence $F_2 = F_\infty$. The inclusion $Y_S \longrightarrow Y$ induces a homomorphism $(\alpha_r) : (E_r) \longrightarrow (F_r)$. Let us put $E_2 = A + B$ with

$$A = \sum_{i=s}^{1=m-1} H_C^*(Y_1' - Y_{1+1}') \otimes H^*(B_{G_1})$$

$$B = \sum_{i=0}^{1=s-1} H_C^*(Y_1' - Y_{1+1}') \otimes H^*(B_{G_1}) .$$

$P(A, t)$ and $P(B, t)$ are respectively the first and second terms in the right hand side of (3). The map α_2 is an isomorphism of A onto F_2 , with kernel B (see XI). Let V_r (respectively W_r) ($r \geq 2$) be the subspace of elements of A (respectively B) which are d_s -cocycles for $2 \leq s \leq r$ and V_∞ be the subspace of permanent cocycles of A . Thus $V_\infty \subset V_{r+1} \subset V_r$, $W_{r+1} \subset W_r$. Put moreover $V_1 = A$, $W_1 = B$. We claim that

$$(8) \quad d_r \kappa_r^2(V_{r-1}) \subset \kappa_r^2(W_{r-1}) .$$

Let $v \in V_{r-1}$ and let us write

$$d_r v = \kappa_r^2(x + y) \quad (x \in A, y \in B)$$

where $x + y$ is of necessity a d_t -cocycle for $2 \leq t \leq r - 1$. We have

$$\alpha_r \kappa_r^2(x + y) = \kappa_r^2 \alpha_2(x + y) = \kappa_r^2 \alpha_2 x$$

by the property of α_2 recalled above. On the other hand, since $F_2 = F_\infty$, we have $\alpha_r d_r v = d_r \alpha_r v = 0$, hence $\kappa_r^2 \alpha_2 x = 0$, and $x = 0$. Thus $d_r v = \kappa_r^2 y$, and y is then, by definition, contained in W_r .

Next we claim that κ_r^2 is injective on V_{r-1} . This is clear for $r = 2$. Let now $r \geq 3$, and $v \in V_{r-1}$ be annihilated by κ_r^2 . This means that there exists an integer t ($2 \leq t \leq r - 1$) and $x \in E_2$ such that x is a cocycle for $d_2, \dots, d_{t-1}$ and that

$$\kappa_t^2 v = d_t \kappa_t^2(x) .$$

Now

$$\kappa_t^2 \alpha_2 v = \alpha_t \kappa_t^2 v = \alpha_t d_t \kappa_t^2(x) = d_t \alpha_t \kappa_t^2(x) .$$

Since $F_2 = F_\infty$, this gives $\kappa_t^2 \alpha_2 v = 0$, hence $\alpha_2 v = 0$ and finally $v = 0$ since α_2 is injective on A . Therefore, κ_r^2 is injective on V_{r-1} , and V_r may be identified with the kernel of d_r on $V_{r-1} \cong \kappa_r^2 V_{r-1}$. This implies

$$P(V_r, t) + P(\text{Im } d_r(V_{r-1}), t) = P(V_{r-1}, t) ,$$

and, together with (8), shows by induction on r that

$$P(V_r, t) \geq P(A, t) - rP(B, t) .$$

Now the number of subspaces Y_1 and the fibre degree are both finite, so that for r big enough we have $V_\infty = V_r$, $E_\infty = E_r$. The foregoing implies then the existence of an integer N such that

$$(9) \quad P(E_\infty, t) + N P(B, t) \geq P(A, t) .$$

The reverse inequality of (4) follows now from (2), (3), (7), 1.3.

§3. Some local concepts

3.1. Let M be a space. For subspaces $A \subset B$, we denote as in I by r_{BA}^* the restriction map $H^*(B; L) \longrightarrow H^*(A; L)$ (cohomology with closed supports). Let P be a closed subspace of M . The subspaces $V - P$ where V runs through the neighborhoods of P form under inclusion a projective system, hence we may define

$$I_P^1(M; L) = \text{dir lim}(H^1(V - P; L), r_{V-P, U-P}^*) .$$

Assuming $M, M - P$ to be paracompact, and N being a closed subspace containing P , we also introduce

$$I_P^1(M - N; L) = \text{dir lim}(H_\emptyset^1(V \cap M - V \cap N; L)$$

where $\emptyset$ is the family of subsets of $V \cap M - V \cap N$ which are closed in $V \cap M$. I^* will be the sum of the I^1 , and L will often be omitted when the coefficients have been specified. When P is a point, we get a kind of local group, which was introduced by Conner [2]. It is of course enough to consider a cofinal set of V 's, and in the cases to be considered below, it will always be possible to take either closed or open neighborhoods.

Since exact sequences and inductive limits "commute", we have an exact sequence

$$(1) \quad \longrightarrow I_P^1(M - N; L) \longrightarrow I_P^1(M; L) \longrightarrow I_P^1(N; L) \longrightarrow I_P^{1+1}(M - N; L) \longrightarrow .$$

3.2. LEMMA Let M be a cohomology n -manifold over L . Then $I_m^*(M; L) = H^*(S_{n-1}; L)$ for any $m \in M$.

Here we let V run through the open connected orientable neighborhoods of m . The lemma follows from the Poincare duality (II, 2.3) and from the fact that to each V we can find a $U \subset V$ which is adapted to V in the sense of I, 4.4. (Thus $I_x^0(M; L)$ (respectively $I_x^{n-1}(M; L)$) may be canonically identified with $\text{Hom}(H_c^n(V; L), L)$ (respectively $H^0(x; L)$.) As was mentioned at the end of Chapter V, more general results have been obtained by F. Raymond.

3.3. Let $f : M \longrightarrow M'$ be a continuous map. Then the spectral sequences of $f|V - P$ form an inductive system. Their limit (E_r) will be called the local spectral sequence at P of f . We have then (see XVI, §3):

$$E_\infty = \text{Gr} I_P^*(M; L)$$

$$E_2 = \text{lim } E_2(f|V - P) ,$$

at any rate when there is a countable fundamental system of neighborhoods of P . Analogously we may define the local Fary spectral sequence at P .

§4. Cohomology manifolds

4.1. LEMMA. Let M be a G -space, $x \in M$ be a fixed point of G with a countable fundamental system of neighborhoods. Assume that there exists a subgroup H of G (closed connected if $p = 0$) such that $G_m = H$ if $p \neq 0$ (respectively $G_{m0} = H$ if $p = 0$) for every $m \neq x$. Then the local spectral sequence of $M_G \longrightarrow M'$ at x_G is trivial and we have

$$E_2 = I_X^*(M') \otimes H^*(B_H) = E_\infty = \text{Gr } I_{x_G}^*(M_G) .$$

Let V be an invariant closed neighborhood of x . Then the spectral sequence with closed supports of $(V - x)_G \longrightarrow V' - x$ satisfies 1.2. The lemma follows then by going over to the limit on the invariant neighborhoods of x , and applying XVI, 3.2.

4.2. We now come back to the notation of 1.1, and add the assumption that X is a cohomology n -manifold over K_p with a countable basis for the open sets. By (V, 2.2, 3.2) $F(H; X)$ is then a cohomology manifold mod p for any subgroup H of G (closed connected if $p = 0$). We shall assume that $F(G; X)$ is not empty and denote by x a fixed point of G . Let n_1 (respectively r) be the dimension mod p of the component of $F(G_1; X)$ (respectively F) containing x .

LEMMA. We keep the previous notation. Then for

$$i \geq s$$

$$P(I_X^*(Y_1' - Y_{1+1}'), t) = t^r (1 + t^a + t^{2a} + \dots + t^{n_1 - r - a - 1})$$

if $n_1 \neq r$, and equals zero otherwise.

Let K_1 be a subgroup of G such that $G = G_1 \times K_1$. Then as in 2.2, $(Y_1 - Y_{1+1})/K_1 = (Y_1 - Y_{1+1})/G = Y_1' - Y_{1+1}'$. We consider the local spectral sequence (I_r) mod p of $(Y_1 - Y_{1+1})_{K_1} \longrightarrow Y_1' - Y_{1+1}'$ at x_{K_1} . Since we deal with a principal bundle (or, if $p = 0$, with a principal-bundle-like situation, see 2.2) and since for any invariant neighborhood V of x , $(V \cap (Y_1 - Y_{1+1}))_K$ maps onto the base, we get

$$I_2 = H^*(B_K; I_X^*(Y_1 - Y_{1+1}))$$

$$I_\infty = \text{Gr} I_X^*(Y_1' - Y_{1+1}') \quad .$$

Now it follows from 3.1 (1), 3.2 that we have an exact sequence

$$0 \longrightarrow H^{j-1}(S_{r-1}) \longrightarrow I_X^j(Y_1 - Y_{1+1}) \longrightarrow H^j(S_{n_1-1}) \longrightarrow H^j(S_{r-1}) \longrightarrow$$

Thus we are formally in the same situation as in the proof of 2.2 (with n_1 and r decreased by 1), so that the references given there also apply now.

4.3. THEOREM. Let G be an elementary abelian p -group operating on a cohomology n -manifold mod p X . Let $x \in X$ be a fixed point and assume that x has a countable fundamental system of neighborhoods. Let $n(H)$ be the dimension mod p of the components of x in the fixed point set of a subgroup H of G (closed connected if $p = 0$), and $r = n(G)$. Then we have

$$n - r = \sum_H (n(H) - r)$$

where H runs through the subgroups of index p if $p \neq 0$, through the closed connected subgroups of co-dimension 1 if $p = 0$.

In the notation of 1.1, 4.2, the theorem is equivalent to

$$n - r = \sum_{s \leq i < m} (n_1 - r) \quad .$$

Let (E_T) be the local Fary spectral sequence mod p , at x_G , of $\pi_1 : Y_G \longrightarrow Y'$ and let (E_T') be the local spectral sequence mod p , at x_G of the fibre map $\pi_2 : Y_G \longrightarrow B_G$. Then we have again 2.3 (2). Moreover

$$P(E_\infty, t) = P(E_\infty', t) = P(I_{x_G}^*(Y_G), t) \quad .$$

(Since E_G is assumed to be compact, the subspaces V_G form a fundamental system of neighborhoods of x_G in X_G if V runs through a fundamental system of neighborhoods of $x \in X$. Thus the countability condition of 3.3 is fulfilled.) Using 4.1 and XVI, §3, we get for E_2 the same form as in 2.3 with H_G^* replaced by I_X^* . Together with 4.2 this gives again the formula 2.3 (3), for $P(E_2, t)$, except that $r + 1$ is replaced by r ,

and n_1 by $n_1 - 1$. On the other hand, by 3.1, 3.2, we have an exact sequence

$$\longrightarrow H^j(S_{r-1}) \longrightarrow I_X^{j+1}(Y) \longrightarrow H^{j+1}(S_{n-1}) \longrightarrow H^{j+1}(S_{r-1}) \longrightarrow$$

and, by an easy application of XVI, §3,

$$E_2' = H^*(B_G, I_X^*(Y)) .$$

From this we see that we are formally in the case of a cohomology $(n-1)$ -sphere, with n_1, r replaced by $n_1 - 1, r - 1$. Therefore the proof of 2.3 also applies here (with, of course, 4.1 replacing 1.2, and (α_p) being defined as a homomorphism of local spectral sequences). It shows then that $(n - 1) - (r - 1)$ is the sum of the differences $(n_1 - 1) - (r - 1)$, $(i = s, \dots, m - 1)$, as was to be proved.

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XIV: ONE OR TWO CLASSES OF ORBITS

A. Borel

This lecture is mainly devoted to the discussion of groups acting on spheres with one class of orbits or on euclidean spaces with two classes of orbits, and gives the proof of results announced in [3]. G is always a compact connected Lie group, $\neq (e)$, and X is a locally compact G -space. $A \sim_L B$ means that $H_C^*(A; L) \cong H_C^*(B; L)$; if L is a field of characteristic p , we also write $A \sim_p B$.

§1. One class of orbits on spheres

The Hopf fiberings of S_{2n-1} in circles and S_{4n-1} in 3-spheres are examples of groups acting non-transitively on spheres with one class of orbits. We do not know to what extent these are unique, but the following theorem shows that any rate the groups occurring there are the only possible ones.

1.1. THEOREM. We assume that G is effective on X , has one class of orbits, X is compact and that $\pi_1(X) = \pi_0(X) = 0$. Then if $\dim_0 X < \infty$ and $X \sim_0 S_n$, either X' is acyclic over K_0 and $G(x) \sim_0 S_n$ ($x \in X$) or G has rank 1 and $G_x = (e)$. If moreover $\dim_2 X < \infty$, and $X \sim_2 S_n$, then $G \neq SO(3)$.

Let $x \in X$ be chosen once for all. Let N be the normalizer of G_x , $M = N/G_x$ and $F = F(G_x; X)$. Then F is a principal M -bundle over X' . Since $\pi_1(X) = 0$, we also have $\pi_1(X') = 0$ and therefore, by the homotopy sequence, $\pi_0(M) = \pi_0(F)$. This implies that a connected component F_0 of F is a principal M_0 -bundle, over X' . The inverse image N_1 of M_0 in N is the subgroup of G which leaves F_0 invariant. It is easily seen that F_0 is an N_1 -slice (see VIII, 3.2 for the definition) hence (XII, 1.2)

$$(1) \quad X = F_0 \times_{M_0} (G/G_X) \quad .$$

It follows in particular that X is fibered over G/N_1 , with connected typical fibre F_0 , hence

$$(2) \quad \pi_1(G/N_1) = 0, \quad \pi_0(N_1) = 0$$

(the second equality follows from the first one by the homotopy sequence of $G \bmod N_1$). If $M_0 = (e)$, then $N_1 = G_X$, hence G_X is connected and G/G_X is orientable; moreover, $F_0 = X'$, and $X = X' \times (G/G_X)$. Since G/G_X is orientable, X' is acyclic and $n = \dim G/G_X$. Let now $M_0 \neq (e)$, and $\beta : X \longrightarrow X' \times (G/N_1)$ be the natural projection (XII, 1.4). Both factors in (1) are principal bundles, therefore β is the projection in a fibering with typical fibre M_0 and structural group $M_0 \times M_0$ (XII, 1.4). In the spectral sequence of β over K_0 we have then

$$E_2 = H^*(X' \times (G/N_1); K_0) \otimes H^*(M_0; K_0) \quad .$$

It follows then from ([1], [2, Chap. IV]) that M_0 is a rational odd dimensional cohomology sphere. Being a group, it must then have rank 1, and dimension s equal to 1 or 3. The Gysin sequence yields

$$H^*(X' \times (G/N_1); K_0) = K_0[z]/(z^{q+1}) \quad (d^0 z = s + 1, q(s + 1) + s = n).$$

If X' is acyclic, then $\dim G/N_1 = n - s$, $\dim G/G_X = n$ and the spectral sequence of $(G/G_X, G/N_1, M_0)$ shows that $G/G_X \sim_0 S_n$. If X' is not acyclic, then G/N_1 must be acyclic; since G/N_1 is orientable by (2), we get $G = N_1$; thus G_X is invariant in G . But G is effective; therefore, $G_X = (e)$, and $G = N_1 = M_0$ has rank 1. This proves the first assertion.

Assume now that $\dim_2 X < \infty$, $X \sim_2 S_n$ and $n \neq \dim G(x)$. The last condition insures that X' is not acyclic over the rationals. Thus G has rank 1, and X is a principal G -bundle. If now $G = SO(3)$, then it contains a subgroup $H = \mathbb{Z}_2 + \mathbb{Z}_2$ and X is also a principal H -bundle. As is easily seen by use of the spectral sequence of (X_H, B_H, X) this contradicts the condition $\dim_2 X < \infty$. We do not reproduce this known argument, see e.g., [4, Exp. XIII, p. 7].

1.2. REMARK. The theorem is also valid in cohomology with closed supports when X is locally compact, but not necessarily compact. The proof is essentially the same; the only point which requires some care is the application of the Künneth formula to $X' \times (G/N_1)$ or $X' \times (G/G_X)$. It can be legitimized because one of the two factors is compact and c.l.c.

1.3. The proof of 1.1 given here is not quite the one sketched in [3]. Notably the use of β has been borrowed from [5], where the following generalization of the first assertion of 1.1 is proved.

THEOREM (Conner). Assume $\dim_0 X < \infty$, $H^*(X; K_0) = H^*(S_n; K_0)$, X to be simply connected, globally and locally arcwise connected, and G to be effective on X , with all orbits of the same dimension. Then either X' is acyclic or G has rank 1 and all isotropy groups are finite.

The proof differs from that of 1.1 only by some minor technical complications, which we proceed to indicate briefly.

One remarks first that the identity components of the isotropy groups are conjugate to each other (this follows from the fact that G_y is conjugate to a subgroup of G_x if y is sufficiently near x). Define then $F = F(G_{x_0}; X)$, N to be the normalizer of G_{x_0} , and $M = N/G_{x_0}$. Again $F/N = X'$; F is a N -slice and we have (1), hence a fibering $(X, G/N, F)$ with structural group M . Thus F is also locally arcwise connected. Let F_0 be an arc component. It is then closed in X , meets every fibre. Let M be the subgroup of M leaving F_0 invariant, and N_1 its inverse image in G . Then F_0 is a N_1 -slice and we also have $X = F_0 x_{M_1}(G/G_x)$. This yields a fibering $(X, G/N_1, F_0)$, with connected fibres, and therefore we get again 1.1 (2). Since locally isomorphic compact connected Lie groups have canonically isomorphic cohomology rings over a field of characteristic zero, it follows from XII, 2.3 that the Leray sheaf of β is locally constant. Moreover the base is locally arcwise connected and simply connected ($\pi_1(X') = 0$ follows from [6], Corollary 2 to Theorem 2). Therefore the Leray sheaf of β is constant, and we conclude as in 1.1 that either X' is acyclic, or $G_{x_0} = (e)$ and rank $G = 1$.

1.4. **THEOREM.** Let G act differentiably on a differentiable connected manifold M . Assume that the fixed point set F is not empty and that there is one class of orbits besides fixed points. Then the non-trivial orbits are spheres and either their dimension is equal to $(\dim M - \dim F - 1)$ or $G = T^1, Sp(1)$, and $G_x = (e)$ for $x \notin F$.

Let $x \in F$. Assume M to be endowed with some invariant Riemannian metric and consider a sphere S_{n-1-k} ($n = \dim M$, $k = \dim F$ at

x) formed by the end points of geodesic segments starting at x , orthogonal to F , of a common small length. Then G acts effectively on S_{n-k-1} with one class of orbits, and we apply 1.1.

1.5. If we add $M \sim_{\mathbb{Z}} \mathbb{R}^n$ to the assumptions of 1.4, then $F \sim_{\mathbb{Z}} \mathbb{R}^k$. This is not difficult to show. However, since much more will be proved in XV, 1.4 and 1.5 we do not give a direct argument for it.

§2. A sufficient condition for the existence of fixed points on $\mathbb{R}^n$

This section contains some general remarks aimed at an elementary criterion for the existence of fixed points, which will be used in §3.

2.1. As usual, N_U is the normalizer of a subgroup U of G , and T a maximal torus of G . We recall that $W(G) = N_T/T$ is the Weyl group of G .

The conjugacy class of $U \subset G$ is denoted (U) . We write $(U) \leq (V)$ if every element of (U) is contained in some element of (V) , and say then that (U) is smaller than (V) . As in VIII, XII, $X_{(U)} = \{x \in X, G_x \in (U)\}$.

2.2. PROPOSITION. Let V be a closed subgroup of G containing N_T . Then $V = N_V$ and $V/V_0 = N_T/N_T \cap V_0$, V_0 being the identity component of V .

It is known that every automorphism of V_0 leaves some maximal torus invariant (see e.g., A. Borel and G. Mostow, *Annals of Math.* 61 (1955), pp. 389-405). Therefore any element of V or N_V is conjugate under V_0 to an element of N_T . The proposition follows immediately.

2.3. PROPOSITION. Let V_1, V_2 be closed, conjugate subgroups of G . Assume that each contains the normalizer in G of a maximal torus of G and that $V_1 \cap V_2$ contains a maximal torus T_0 of G . Then $V_1 = V_2$.

By the conjugacy of maximal tori, V_1 contains the normalizer in G of any one of its maximal tori ($i = 1, 2$), therefore $V_1 \cap V_2$ contains the normalizer of T_0 . By assumption there exists $g \in G$ such that $\text{Ad } g(V_1) = gV_1g^{-1} = V_2$. Then $\text{Ad } g(T_0) = T'$ is a maximal torus of V_2 . Using the conjugacy of maximal tori in V_2 we may find therefore $h \in N_{T_0}$ such that $\text{Ad } h(V_1) = V_2$. By our initial remark, $h \in V_1 \cap V_2$, hence $V_1 = V_2$.

2.4. LEMMA. Assume that on X there is a smallest class (U) and a biggest class (V) of isotropy groups, in the sense of 2.1, and that an element of (U) is contained in exactly one element of (V) .

Then there is a continuous equivariant map

$\alpha : X \longrightarrow G/N_V$ whose restriction to $X_{(V)}$ is the projection in the fibering $(X_{(V)}, G/N_V, F(V; X_{(V)}))$.

(U) being the smallest class of isotropy groups, the subspace $F = F(U; X)$ meets every orbit. Let $x \in X$, $x_0 \in G(x) \cap F$ and g be such that $g \cdot x_0 = x$. We put tentatively $\alpha(x) = g \cdot o$ where o denotes the coset N_V in G/N_V . Now, if $y, z \in F$ and $g \cdot y = z$, then $gG_y \cdot g^{-1} = G_z$ hence $gVg^{-1} \supset U$, and $g \in N_V$. This implies that α is a well defined map. The verification of continuity and equivariance is straightforward and left to the reader. The last assertion follows then from the definitions.

2.5. Let the one-point compactification of X be an n -cm over Z and $X \sim_Z R^n$. Assume that there is a biggest class (V) of isotropy groups. Then $(N_T) \leq (V)$.

By IV, 5.9, $F(T; X) \sim_Z R^k$ with $n - k$ even. Thus $(T) \leq (V)$ and $W(V)$ is a subgroup of $W(G)$. Let p be a prime, N_p be the inverse image in N_T of a Sylow p -group of $W(G)$. By the Smith theorems, N_p has a fixed point in $F(T; X)$, hence $(N_p) \leq V$. Therefore for each p , a Sylow p -group of $W(V)$ is also a Sylow p -group of $W(G)$. As a consequence $W(V) = W(G)$, and $(N_T) \leq (V)$.

2.6. THEOREM. Assume that $X \sim_Z R^n$ and the one point compactification of X is a n -cm $_Z$. Let (U) be the smallest class of isotropy groups (IX) . Assume that there is a biggest class (V) of isotropy groups and that U is contained in exactly one element of (V) . Then G has a fixed point, that is, $G = V$.

By 2.5, V contains N_T for a suitable T , therefore (2.2) $V = N_V$ and (XII, 1.3), $X_{(V)} = F(V; X_{(V)}) \times G/V$. Thus the restriction to $X_{(V)}$ of the map $\alpha : X \longrightarrow G/V$ of 2.4 is the projection on one factor and α has a cross section. As a consequence, α^* is injective.

Let $k = \dim G/V$. Since $H^k(G/V; Z_2) \neq 0$ we also have $H^k(X; Z_2) \neq 0$. By Poincaré duality (II, 2.3), $H_c^{n-k}(X; Z_2) \neq 0$. Therefore $n - k = n$, $k = 0$, and $G = V$.

2.7. COROLLARY. If G has only one class of orbits on X , then it operates trivially.

§3. Two classes of orbits on euclidean space

3.1. THEOREM. Let $X \sim_{\mathbb{Z}} \mathbb{R}^n$ and the one-point compactification of X be a n -cm $_{\mathbb{Z}}$. If G has two classes of orbits on X , then it has a fixed point.

If G is a torus, the existence of fixed points follows from IV, 5.9 (without any assumption on the number of orbit types), so assume G to be non-commutative. If $n = 1$, the orbits are zero dimensional, hence points, and the action is trivial. Proceeding by induction, we assume the theorem to be true in dimensions $< n$.

Let (U) and (V) be the smaller and bigger class of isotropy groups. We have to prove that $G = V$. By 2.2, 2.5

$$(1) \quad (N_T) \leq (V) \quad N_V = V,$$

and therefore (XII, 1.3)

$$(2) \quad X_{(V)} = F(V; X_{(V)}) \times G/V.$$

Let first U be infinite and let S be a maximal torus of U . We distinguish two cases.

(i) S contains an element regular in G . This means that S belongs to exactly one maximal torus T_0 of G . By 2.3 and (1), U is contained in one and only one element of (V) , and we may apply 2.6.

(ii) S is singular in G . Its centralizer H , which is known to be connected, is then not commutative. It leaves $Y = F(S; X)$ invariant, hence Y is an H - or an (H/S) -space. By IV, 5.9 and V, 3.2, $Y \sim_{\mathbb{Z}} \mathbb{R}^k$ ($n - k \neq 0$, even) and the one-point compactification of Y is a k -cm over $\mathbb{Z}$. The isotropy groups G_y ($y \in Y$) are the elements of (U) , (V) which contain S , and of course $H_y = G_y \cap H$. We want to show that the isotropy groups H_y form two classes in H . This will be a consequence of the following assertion:

(*) Let A_1, A_2 be subgroups of G containing S and both belonging either to (U) or to (V) . Then $A_1 \cap H$ and $A_2 \cap H$ are conjugate in H .

Let first $A_1, A_2 \in (V)$, $S \subset A_1, A_2$. Let T_1 be a maximal

torus of A_1 containing S ($i = 1, 2$); by definition it belongs to H and by (1) it is maximal in G , hence in H . Therefore there exists $h \in H$ such that $\text{Ad } h(T_1) = T_2$. Then $\text{Ad } h(A_1)$ and A_2 both contain T_2 ; by (1) and 2.3 they coincide, and $\text{Ad } h(A_1 \cap H) = A_2 \cap H$.

Let now $A_1, A_2 \in (U)$, containing S . The torus S being maximal in A_1, A_2 , there exists $g \in N_S$ such that $\text{Ad } g(A_1) = A_2$ (same proof as in 2.3). Then $g \in N_H$, which means that $(A_1 \cap H)/S$ and $(A_2 \cap H)/S$ are conjugate by an automorphism of H/S . Since S is invariant, of maximal rank in $A_1 \cap H$, the group $(A_1 \cap H)/S$ is finite, whereas it is infinite if $A_1 \in (V)$. But Y being a connected k -cm over Z , H/S has only one minimal class of isotropy groups on Y . Thus $(A_1 \cap H)/S$ and $(A_2 \cap H)/S$ must belong to one conjugacy class of subgroups in H/S ; this ends the proof of (*).

The induction assumption, applied to H/S operating on Y shows now that if $V' \in (V)$ contains S then it contains H , and therefore any maximal torus T_0 of H . Applying (1) and 2.3, we see that U is contained in exactly one element of (V) , and $G = V$ follows again from 2.6.

From now on, U is finite. Together with (2), 2.2, this implies

$$(3) \quad (G/V) \times X'_{(V)} = F(V; X_{(V)}) = F(V; X) = F(T; X) \quad (T \subset V) .$$

Also, every infinite subgroup of T has the same fixed point set as T . But then (XIII, 2.3), applied to the one point compactification of X , for $p = 0$, shows that $\dim T = 1$. Thus G has rank 1, $G = \text{Sp}(1)$, $\text{SO}(3)$.

Assume now that $G = \text{Sp}(1)$ and acts effectively. Its center C has order 2 and does not belong to U , since otherwise it would be contained in all isotropy groups. On the other hand, it belongs to the maximal tori of G , hence to V . Therefore $F(C; X) = X_{(V)}$. By the Smith theorems $X_{(V)} \sim_2 \mathbb{R}^p$, $F(T; X) \sim_2 \mathbb{R}^q$. This is compatible with (2), (3), only if G/V is acyclic mod 2, that is if $G = V$.

There remains the case $G = \text{SO}(3)$. The following facts about this group will be used:

(**) Every element $\neq e$ is contained in one and only one maximal torus. The only closed infinite proper subgroups are T , N_T and their conjugates. The group N_T is the union of two cosets $T \cup aT$ of T and aT consists of elements of order 2. We have $btb^{-1} = t^{-1}$ for $t \in T$, $b \in aT$; in particular N_T is a split extension of Z_2 by T . Let $H = Z_2 + Z_2$ be the subgroup of diagonal matrices, T_1, T_2, T_3 the

maximal tori containing the three non-zero elements of H , and N_1 be the normalizer of T_1 . The group N_H/H is isomorphic to the group σ_3 of permutations of 3 objects and its natural action on the T_1 's or the N_1 's is that of the full permutation group. The N_1 's are the only elements of (N_T) which contain H .

The proof in the case $G = SO(3)$ will be by contradiction, so we assume $G \neq V$. By (1), (**), $V = N_T$, hence $G/V = P_2$, the real projective plane, and

$$(4) \quad \begin{aligned} F(H; X_{(V)}) &= F(N_1; X) \cup F(N_2; X) \cup F(N_3; X) \\ (F(N_1; N) \cap F(N_j; X) &= \emptyset, 1 \neq j) \end{aligned}$$

Moreover, by the Smith theorems,

$$(5) \quad F(N_1; X) \sim_2 R^s, \quad F(H; X) \sim_2 R^t$$

and $F(N_1; X), F(H; X)$ are cohomology manifolds mod 2.

It follows first from (4), (5), that $F(H; X) \neq F(H; X_{(V)})$; this means that H has fixed points in $X_{(U)}$, that is $(H) \subset (U)$. Next we want to prove that $(H) = (U)$. For suitable U, T we have inclusions $H \subset U \subset N_T$. Therefore $U \cap T = Z_m$ is of index 2 in U and (see (**)) all elements of U not in $U \cap T$ have order 2. Assume now $U \neq H$. Then $m \neq 2$, and $U \cap T$ contains all elements of order m of U . It is therefore a characteristic subgroup of U , and is the intersection of U with any torus T' whose normalizer contains U . As a consequence, U is contained in only one element of (V) , and $G = V$ by 2.6, a contradiction with our present assumption. Thus $(U) = (H)$, and

$$(6) \quad F(H; X_{(H)}) = F(H; X) - U_1 F(N_1; X) \quad .$$

$F(H; X_{(H)})$ is a t -cm mod 2, and also a principal $N_H/H = \sigma_3$ bundle over $X_{(H)}$; the latter is the base space of the fibering of $X_{(H)}$ by the orbits G/H , hence is a $(n-3)$ -cm over Z_2 or even over Z . Therefore $t = n-3$.

$F(N_1; X)$ is a closed proper s -cm mod 2 of $F(H; X)$, hence (I, 4.6) $s < t$. Assume first $s = t-1 = n-4$. By (5), (6) and the cohomology sequence of $F(H; X)$ mod $F(H; X_{(V)})$

$$H^1_0(F(H; X_{(H)}); Z_2) = 0 \quad (1 \neq n-3) \quad ,$$

and there is an exact sequence

$$0 \longrightarrow H_C^{n-4}(F(H; X_{(V)}); Z_2) \longrightarrow H_C^{n-3}(F(H; X_{(H)}); Z_2) \longrightarrow H_C^{n-3}(F(H; X); Z_2) \longrightarrow 0.$$

By (I, 4.3, 4.8), $F(H; X_{(H)})$ has four connected components, each of which is $\sim_2 R^{n-3}$. The group N_H/H permutes the N_1 transitively (see (**)), hence it also operates transitively on the subspaces $F(N_1; X)$. The above exact sequence shows then that N_H/H operates faithfully on $H_C^{n-3}(F(H; X_{(H)}); Z_2)$, therefore its natural representation as group of permutations of the four components of $F(H; X_{(H)})$ is faithful. Now every subgroup isomorphic to σ_3 of the permutation group of four objects σ_4 has a fixed point. (In fact the normal subgroup K of order 3 of σ_3 must operate cyclically on three of the four objects and leave the fourth one, say P , fixed; since K is invariant in σ_3 , the latter must also leave P fixed.) Therefore one of the components of $F(H; X_{(H)})$, say Y , is invariant under $N_H/H = \sigma_3$; by the above $Y \sim_2 R^{n-3}$ and Y is a principal σ_3 -bundle. We have then $(Y/K) \sim_2 R^{n-3}$, where K is the subgroup of order three of σ_3 (III, 2.3). Since $Z_2 = \sigma_3/K$ acts freely on Y/K , we get a contradiction with the Smith theorem.

This contradiction shows that $s < n - 4$. Then $H_C^{n-3}(F(H; X_{(H)}))$ Z_2 and $F(H; X_{(H)})$ is connected. Moreover, since $\dim G/V = 2$, we have $H_C^{n-2}(X_{(V)}; Z_2) = 0$ by (2), hence $H_C^{n-1}(X_{(H)}; Z_2) = 0$ by the cohomology sequence of $X \bmod X_{(V)}$. The Poincaré duality (II, 2.3) gives then $H^1(X_{(H)}; Z_2) = 0$. The fibres of $\pi : X_{(H)} \longrightarrow X'_{(H)}$ are connected, therefore π^* is injective in dimensions 0, 1 (this follows trivially from the spectral sequence), and we also have $H^1(X'_{(H)}; Z_2) = 0$.

Now, by the above, $F(H; X_{(H)})/K$ is a principal Z_2 -bundle over $X'_{(H)}$. Since the total space is connected, $H^1(X'_{(H)}; Z_2) \neq 0$ (to see this consider for instance the Cartan-Leray spectral sequence of this covering; we have $Z_2 = H^1(Z_2; Z_2) = E_2^{1,0} = E_\infty^{1,0} \subset H^1(X'_{(H)}; Z_2)$). Thus we get a contradiction, and this ends the proof.

REMARK. According to F. Raymond, "The end point compactification of manifolds", to appear in Pacific M. J., if X is a $n - \text{cm}_Z$ with the integral cohomology of R^n , then the one-point compactification of X is an $n - \text{cm}_Z$; since it has clearly the integral cohomology of S_n , we could also assume in 3.1 that X is an $n - \text{cm}_Z$ satisfying $X \sim_Z R^n$.

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XV: FIXED POINT SETS AND ORBITS OF
COMPLEMENTARY DIMENSION

G. E. Bredon

§1. Introduction

Let G be a compact Lie group operating on an n -cm M over Z , and say that the maximal dimension of any orbit is k . Then, according to IX, 2.2, we have that $\dim F(G, M) \leq n - k - 1$. We shall consider here the extreme case in which $\dim F(G, M) = n - k - 1$, and shall obtain fairly complete results. In fact, these actions, for separable M of finite covering dimension, are characterized locally about points in F up to knowledge of cohomology manifolds with boundary and of homogeneous spaces which are integral homology spheres. The main theorem (1.4) shows that the set of points of $F(G, M)$ about which the dimension of $F(G, M)$ is $n - k - 1$, is open and closed in $F(G, M)$ and is an $(n - k - 1)$ -cm over Z . Moreover, the action of G near any such point of F is equivalent locally to the action obtained by multiplying a cohomology manifold with boundary by a homogeneous homology sphere, identifying the spheres to points on the boundary, and letting the group of the sphere act in the natural way. If M is itself a cohomology sphere, then these local results go over into global results (1.6).

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In this section we shall state the main theorem and prove two corollaries from it. In Sections 2 and 3, some preliminary results needed for the proof of the main theorem are given. In Section 4, the case k even of the main theorem is proved and the k odd part is proved in the remaining sections.

Before stating the main theorem we shall need some definitions.

By a principal orbit we mean an orbit with minimal isotropy group. By an exceptional orbit we mean an orbit of maximal dimension k which is not a principal orbit. By a singular orbit we mean an orbit of dimension less than k .

The orbit space of a space X by a group K will be denoted by X/K or by X' when no confusion results. We denote by π the orbit map $X \longrightarrow X/K$.

DEFINITION 1.1. We say that $\dim_L(X, x) = n$ if n is the infimum of $\dim_L U$, where U ranges over the neighborhoods of x in X . We use $\dim$ for $\dim_Z$.

Clearly, if $\dim X = n$, then the set of points x , for which $\dim(X, x) = n$, is closed and, by I, 2.3, this set is non-empty.

DEFINITION 1.2. If F is a closed subset of a space X , then we denote by X^{dF} the space obtained by doubling X on F . That is, X^{dF} is obtained from two copies of X by identifying corresponding points of F .

DEFINITION 1.3. A space X is said to be an n -cm over L with boundary F if X^{dF} is an n -cm over L .

We can now state our main theorem and its corollaries. The hypotheses below of finite covering dimension and of local separability are necessary only for k odd, and are used only in Section 3.

THEOREM 1.4. Let G be a compact Lie group operating on a locally separable n -cm M over Z of finite covering dimension. Say that (H) is the type of the principal orbits and let $k = \dim G/H$. Then if $\dim(F(G, M), x) = n - k - 1$ we have that

- (1) $H^1(G/H) = H^1(S^k)$.
- (2) M/G is an $(n - k)$ -cm over Z with boundary $F(G, M)$ locally at $\pi(x)$.
- (3) There is a local cross-section at x contained in $F(H, M)$ for the orbits of G on M .
- (4) There are exactly two types of orbits near x (principal and fixed).
- (5) $\text{rank}(G) = \begin{cases} \text{rank}(H) & \text{if } k \text{ is even} \\ \text{rank}(H) + 1 & \text{if } k \text{ is odd.} \end{cases}$

REMARK. Note that the condition on dimension in 1.4 is equivalent to the condition $\dim(F(G, M), x) = \dim M/G - 1$.

Note that, since 1.4 is a local theorem, we may, for the proof, replace M by any invariant open neighborhood of x . This will be done several times in the proof of 1.4 without specific mention, in order not to overly complicate the notation.

COROLLARY 1.5. With the notation of Theorem 1.4 and of IX, Section 1, say that $z \in X_{uv}$, $H \subset G_z$ and $\dim(X'_{uv}, \pi(z)) = n - k - 1$, then the hypotheses of Theorem 1.4 hold for the action of G_z on a slice at z , so that

- (1) $H^1(G_z/H) = H^1(S^{k-u})$.
- (2) M/G is an $(n - k) - \text{cm}$ over Z with boundary X'_{uv} locally at $\pi(z)$.
- (3) There is a local cross-section at z contained in $F(H, M)$ for the orbits of G on M .
- (4) There are exactly two types of orbits of G near z (principal and orbits in X_{uv}).
- (5) $\text{rank}(G_z) = \begin{cases} \text{rank}(H) & \text{if } k - u \text{ is even} \\ \text{rank}(H) + 1 & \text{if } k - u \text{ is odd.} \end{cases}$

PROOF. Let S be a slice at z . Then $\dim S = n - u$, $\dim G_z/H = k - u$ and $\dim(F(G_z, S), z) = \dim(X'_{uv}, \pi(z)) = n - k - 1$, since

$$X'_{uv} \approx \frac{X_{uv} \cap S}{G_z}$$

locally at z and

$$\frac{X_{uv} \cap S}{G_z} \approx F(G_z, S)$$

since $X_{uv} \cap S = F(G_z, S)$. This proves the first statement. (1) - (5) follow easily from the corresponding statements in Theorem 1.4.

COROLLARY 1.6. With the notation of Theorem 1.4 say that $\dim F(G, M) = n - k - 1$ and that $H^1(M) = H^1(S^n)$. Then,

- (1) $H^1(G/H) = H^1(S^k)$.
- (2) M/G is a cohomology $(n-k)$ -cell. That is, $F(G, M)$ is an $(n - k - 1) - \text{cm}$ over Z with the cohomology of S^{n-k-1} and $(M/G)^{dF}$ is an $(n - k) - \text{cm}$ over Z with the cohomology of S^{n-k} .

- (3) There is a global cross-section in $F(H, M)$ for the orbits of G on M .
- (4) There are exactly two types of orbits of G on M .
- (5) $\text{rank}(G) = \begin{cases} \text{rank}(H) & \text{if } k \text{ is even.} \\ \text{rank}(H) + 1 & \text{if } k \text{ is odd.} \end{cases}$

PROOF. For any space X let cX denote the open cone over X . From the exact cohomology sequence with compact supports of the pair $(cX, \text{vertex of } cX)$ and from the fact that $H_c^1(X \times (0, 1)) \approx H_c^{1-1}(X)$ we see easily that cX is an $(m+1)$ -cm over Z if and only if X is an m -cm over Z and $H^1(X) = H^1(S^m)$.

In the present situation, G acts in a natural way on cM which is an $(n+1)$ -cm over Z . Moreover, $F(G, cM) = cF(G, M)$ so that $\dim(F(G, cM), \text{vertex of } cM) = n - k$. Thus the conditions of Theorem 1.4 apply for the action of G on cM and for $x = \text{vertex of } cM$.

Conclusions (1) and (5) are immediate. Moreover, $F(G, cM) = cF(G, M)$ is an $(n-k)$ -cm over Z , so that $F(G, M)$ is an $(n-k-1)$ -cm over Z with the cohomology of S^{n-k-1} . Also

$$(cM/G)^d(cF) = (c(M/G))^d(cF) = c(M/G)^dF$$

is an $(n-k+1)$ -cm over Z , so that $(M/G)^{dF}$ is an $(n-k)$ -cm over Z with the cohomology of S^{n-k} . This proves (2).

We also have that there is a local cross-section near x : $f: U \rightarrow cM$ where U is a neighborhood of $\pi(x)$ in $\frac{cM}{G}$. But $\frac{cM}{G} = c\frac{M}{G}$ and we can assume that M/G is imbedded naturally in $\frac{cM}{G}$ in such a way that $M/G \subset U = \text{dom } f$. Thus $f|_{M/G}$ will be a global cross-section for the orbits of G on M . This proves (3) and (4) follows in a similar way.

REMARK. Although the differentiable version of Theorem 1.4 is practically trivial, the differentiable version of Corollary 1.6 is by no means obvious and does not follow from the "differentiable 1.4", even if we assume that $M \approx S^n$ in 1.6.

§2. Preliminary results

We shall need the following well-known result. This is just Theorem 1.4 for $k = 0$.

LEMMA 2.1. If G is a finite group operating on an n -cm M over Z and if $\dim(F(G, M), x) = n - 1$,

then the component of $F(G, M)$ containing x is an $(n - 1)$ - cm over Z , $G \approx Z_2$ effectively, and M/G is an n - cm over Z with boundary F locally at x . Moreover, F separates M locally at x into two components, the closure of either of which is a local cross-section near x for the orbits of G .

PROOF. If G is effective then $G \approx Z_2$ by V, 2.6. The proofs of the other statements can be found in both [8], Lemma 16 and in [1] Theorem 7.8, and will not be repeated here.

REMARK. Note that for $\dim G > 0$, the conditions of Theorem 1.4 still hold for the identity component G^* of G acting on M , if they hold for G on M . Using this fact, we claim that if Theorem 1.4 holds for (G^*, M) , for some particular transformation group (G, M) , then Theorem 1.4 is also true for (G, M) , with the possible exception of the condition "contained in $F(H, M)$ " in (3). To see this, note that G/G^* acts on M/G^* with orbit space M/G . Moreover, $F(G^*, M)$ is an $(n - k - 1)$ - cm over Z and G/G^* acts on it with a fixed point set $F(G, F(G^*, M)) = F(G, M)$ of dimension $n - k - 1$. Thus $F(G, M) = F(G^*, M) = F$ near x . We also have that, near x , G/G^* acts on $(M/G^*)^{dF}$ leaving F stationary. This action does not interchange components of $(M/G^*)^{dF} - F$, so that G/G^* must act trivially on (M/G^*) by Lemma 2.1. This means that the orbits of G are connected and that

$$\frac{M}{G} \approx \frac{M}{G^*}.$$

The conclusions of 1.4 for G now follow from those of G^* , except possibly for the condition "contained in $F(H, M)$ " of (3). Thus, except for this one point, we may assume that G is connected in the proof of 1.4. That the full statement of (3) is also valid will be shown at the end of Section 7 for the case $\text{rank}(G) > \text{rank}(H)$. This fact for equal ranks is almost trivial and will be left to the reader.

We shall now introduce some basic notation and use it to obtain some characterizations of n - cms with boundary.

DEFINITION 2.2. Let X be a space and F a closed subspace. Let Y be an open or closed subset of X . (We will usually use $Y = X$, F , or $X - F$.) We say that

$$H_G^1(Y) =_{LF} K$$

(read "equals locally near F " for $=_{LF}$) if and only if every point in F has a neighborhood $N \subset X$ such that for any open set $U \subset N$ and any

point $x \in U \cap F$, there is an open neighborhood $V \subset U$ of x such that if $W \subset V$ is open and $W \cap F \neq \emptyset$ then

$$\operatorname{Im} j_{UNY, WNY}^1 = \operatorname{Im} j_{UNY, VNY}^1 \approx K.$$

It is easy to show that if $\dim X < \infty$ then X is an n -cm over Z if and only if

$$H_C^1(X, Z) =_{LX} \begin{cases} Z & \text{if } n \\ 0 & \text{if } n \neq n \end{cases}.$$

Note also that the important case $H_C^1(Y) =_{LF} 0$ can be formulated more easily: $H_C^1(Y) =_{LF} 0$ if and only if given U open and $x \in U \cap F$, there is an open $V \subset U$, $x \in V$, such that $j_{UNY, VNY}^1 = 0$.

For convenience we will use the notation $H_C^1(Y) \neq_{LF} 0$ to mean that it is not true that $H_C^1(Y) =_{LF} 0$.

LEMMA 2.3. X is an n -cm over Z with boundary $F \subset X$, $F \neq \emptyset$ if and only if

- (i) F is an $(n-1)$ -cm over Z .
- (ii) $X - F$ is an n -cm over Z .
- (iii) $H_C^1(X, Z) =_{LF} 0$ for all i .

PROOF. If X is an n -cm with boundary F , then (i) follows from Lemma 2.1 applied to the involution induced by the doubling operation. (ii) is trivial and (iii) follows easily from the Mayer-Vietoris sequence.

Conversely, assume (i), (ii), and (iii) and let $M = X^{dF}$ given by $M = X_1 \cup X_2$, $F = X_1 \cap X_2$, $X_1 \approx X$. The Mayer-Vietoris sequence gives immediately that

$$H_C^1(M) =_{LF} \begin{cases} Z & \text{if } n \\ 0 & \text{if } n \neq n \end{cases}$$

and the exact cohomology sequences also give that, for $j = 1, 2$,

$$H_C^1(M - X_j) \approx H_C^1(X - F) =_{LF} \begin{cases} Z & \text{if } n \\ 0 & \text{if } n \neq n \end{cases}.$$

Given any suitably small open set U in M and any point $x \in U \cap F$, there exists, by Definition 2.2, a neighborhood V of x such that for any $W \subset V$, $W \cap F \neq \emptyset$ we have that (the first equality being a definition for arbitrary $W \subset V$)

$$J_W = \text{Im } j_{(M-X_1) \cap U, (M-X_1) \cap W}^n = \text{Im } j_{(M-X_1) \cap U, (M-X_1) \cap V}^n \approx \mathbb{Z} \quad ,$$

$$j_{X_1 \cap U, X_1 \cap V}^{n-1} = 0, \quad \text{and}$$

$$\text{Im } j_{UV}^n = \text{Im } j_{UV}^n \approx \mathbb{Z} \quad .$$

Consider the diagram

$$\begin{array}{ccccccc} H_C^{n-1}(X_1 \cap W) & \longrightarrow & H_C^n((M - X_1) \cap W) & \longrightarrow & H_C^n(W) & \longrightarrow & H_C^n(X_1 \cap W) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_C^{n-1}(X_1 \cap V) & \longrightarrow & H_C^n((M - X_1) \cap V) & \longrightarrow & H_C^n(V) & \longrightarrow & H_C^n(X_1 \cap V) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_C^{n-1}(X_1 \cap U) & \longrightarrow & H_C^n((M - X_1) \cap U) & \longrightarrow & H_C^n(U) & \longrightarrow & H_C^n(X_1 \cap U) \quad . \end{array}$$

If W is sufficiently small and $W \cap F \neq \emptyset$ then $j_{X_1 \cap V, X_1 \cap W}^n = 0$ and $\text{Im } j_{VW}^n \approx \mathbb{Z}$. Diagram chasing yields that $J_W = J_V$ maps isomorphically onto $\text{Im } j_{UV}^n = \text{Im } j_{UV}^n \approx \mathbb{Z}$.

Now for general $W \subset V$, either $W \cap F \neq \emptyset$ in which case $\text{Im } j_{UV}^n = \text{Im } j_{UV}^n$ by definition, or $W \cap F = \emptyset$ (say that $W \cap X_2 \neq \emptyset$) in which case we still know that $J_W = J_V$, since $M - X_1$ is an n -cm, and hence J_W maps isomorphically onto $\text{Im } j_{UV}^n$ as before (since J_V does). Thus $\text{Im } j_{UV}^n \supset \text{Im } j_{U, (M-X_1) \cap W}^n = \text{Im } j_{UV}^n \supset \text{Im } j_{UV}^n$ so that $\text{Im } j_{UV}^n = \text{Im } j_{UV}^n$ as before.

Thus $H_C^n(M) = {}_{\text{LM}}\mathbb{Z}$. Dimensions $i \neq n$ are much easier and will be left to the reader.

LEMMA 2.4. Say that the compact Lie group G operates on the n -cm M over Z with $F = F(G, M)$ and such that all orbits outside F are of type (H) with $\dim G/H = k$. Then M/G is an $(n - k)$ -cm over Z with boundary F if and only if

$$(i) \quad F \text{ is an } (n - k - 1) \text{-cm over } Z.$$

$$(ii) \quad H_C^1(M/G - F, Z) = {}_{\text{LF}} \left\{ \begin{array}{l} Z \text{ if } n - k \\ 0 \text{ if } n - k \end{array} \right.$$

PROOF. The necessity of (i) and (ii) is clear from Lemma 2.3 and the exact cohomology sequence. We shall prove the sufficiency by showing that conditions (i), (ii), and (iii) of Lemma 2.3 hold with $(n - k)$ replacing n . Thus it clearly suffices to show that $H_C^1(M/G) = {}_{\text{LF}}0$. Since

the case $k = 0$ is part of Lemma 2.1 we shall assume that $k > 0$ and hence that F does not separate M locally near any point.

Given U open in M and $x \in U \cap F$ let $U_1 \subset U_2 \subset U_3 \subset U_4 \subset U$ be open connected invariant neighborhoods of x in M , and put $V_1 = U_1/G$. We may assume that for $a = 1, 2, 3$,

$$(1) \quad \text{Im } j_{U_{a+1}, U_a}^1 \approx \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

$$(2) \quad \text{Im } j_{U_{a+1} \cap F, U_a \cap F}^1 \approx \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & i \neq n - k - 1 \end{cases}$$

$$(3) \quad \text{Im } j_{V_{a+1} - F, V_a - F}^1 \approx \begin{cases} \mathbb{Z} & i = n - k \\ 0 & i \neq n - k \end{cases}.$$

By chasing the diagram

$$\begin{array}{ccccccc} & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_c^{n-k-1}(U_a) & \longrightarrow & H_c^{n-k-1}(U_a \cap F) & \longrightarrow & H_c^{n-k}(U_a - F) & \longrightarrow & H_c^{n-k}(U_a) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \end{array}$$

is is easy to see that $\text{Im } j_{U_3 \cap F, U_1 \cap F}^{n-k-1} \approx \mathbb{Z}$ maps isomorphically onto $\text{Im } j_{U_3 - F, U_1 - F}^{n-k}$.

However, the diagram

$$\begin{array}{ccc} & & H_c^{n-k}(V_a - F) \\ & \nearrow & \downarrow \pi^* \\ H_c^{n-k-1}(U_a \cap F) & & H_c^{n-k}(U_a - F) \\ & \searrow & \end{array}$$

is commutative and, since $V_a - F$ is a connected $(n - k) - \text{cm}$ over $\mathbb{Z}$, it follows easily that $\text{Im } j_{V_3 \cap F, V_1 \cap F}^{n-k-1}$ maps isomorphically onto $\text{Im } j_{V_3 - F, V_1 - F}^{n-k} = \text{Im } j_{V_3 - F, V_2 - F}^{n-k} \approx \mathbb{Z}$.

The lemma will now follow easily by use of the diagram

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & H_G^j(V_a - F) & \longrightarrow & H_G^j(V_a) & \longrightarrow & H_G^j(V_a \cap F) \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow
 \end{array}$$

Our next lemma will be useful in showing the non-existence of exceptional orbits in the situation of 1.4.

LEMMA 2.5. Let G be a compact Lie group acting on an n -cm M over Z with principal orbit type (H) . Let T be a maximal torus of H and $N(T)$ the normalizer of T in G . If $z \in F(T, M)$ is on an exceptional orbit of G in M , then z is also on an exceptional orbit of $N(T)$ acting on the cm $F(T, M)$.

PROOF. We may assume that $H \subset G_z$. Then z is exceptional if and only if $1 < \text{ord } G_z/H < \infty$. Since G_z/H is a finite set, it is left pointwise fixed by the action of T on the left. But a simple calculation gives that

$$F(T, G_z/H) = \frac{(N(T) \cap G_z)H}{H} \approx \frac{N(T) \cap G_z}{N(T) \cap H}.$$

Thus

$$\text{ord } G_z/H = \text{ord } \frac{N(T) \cap G_z}{N(T) \cap H}.$$

Since the principal orbit type of $N(T)$ on $F(T, M)$ is $(N(T) \cap H)$ and since $N(T)_z = N(T) \cap G_z$, the lemma follows.

LEMMA 2.6. If G is a compact Lie group acting on an n -cm M over Z in such a way that M/G is an $(n-k)$ -cm over Z and has a local cross-section contained in $F(H, S)$ near each point $x \in F(H, M)$, where (H) is the type of a principal orbit and S is some slice at x , then all orbits of G are of type (H) . That is, $M \longrightarrow M/G$ is a bundle with fibre G/H .

PROOF. Note that $k = \dim G/H$. Let x be a point on a non-principal orbit. We may assume that if y is on a non-principal orbit and is sufficiently close to x , then $G_y \sim G_x$. With this choice of x we see easily that the action of G_x on a slice S at x has the properties assumed by the lemma (assuming $H \subset G_x$ as we may), and also possesses only principal orbits and fixed point orbits. Thus we may assume that the only orbits of G are either principal or fixed points and that $F = F(G, M) \neq \emptyset$.

Assume first that F consists of singular orbits, that is, that $k > 0$. Then F cannot separate M locally near any point and hence F cannot separate M/G locally near any point. Thus $s = \dim F \leq n - k - 2$.

By I, 2.3 we can find a point $z \in F$ and open invariant neighborhoods $V_0 \supset V_1 \supset V_2$ of z such that

$$(1) \quad j_{F \cap V_0, F \cap V}^S \neq 0 \text{ for all open neighborhoods } V \subset V_0 \text{ of } z$$

$$(2) \quad j_{V_{a-1}, V_a}^1 = 0 \text{ for } 1 \neq n$$

$$(3) \quad j_{V'_{a-1}, V'_a}^1 = 0 \text{ for } 1 \neq n - k, \text{ where } V'_j = V_j/G.$$

By the diagram

$$\begin{array}{ccccc} & & H_c^S(F \cap V_2') & \longrightarrow & H_c^{S+1}(V_2' - F) \\ & & \downarrow & & \downarrow \\ H_c^S(V_1') & \longrightarrow & H_c^S(F \cap V_1') & \longrightarrow & H_c^{S+1}(V_1' - F) \\ \downarrow & & \downarrow & & \downarrow \\ H_c^S(V_0') & \longrightarrow & H_c^S(F \cap V_0') & & \end{array}$$

we see that $j_{V_1' - F, V_2' - F}^{S+1} \neq 0$, for otherwise (1) would be false for $V = V_2$.

However, since there is a cross-section, we have that

$$M - F \approx \frac{M - F}{G} \times G/H$$

so that $j_{V_1' - F, V_2' - F}^{S+k+1} \neq 0$ and since $s \leq (s + k + 1) - 2$ we obtain $j_{V_1', V_2'}^{S+k+1} \neq 0$, which implies by (2) that $s + k + 1 = n$. However, as already remarked $s \leq n - k - 2$, which is a contradiction and shows that our assumption $k > 0$ is untenable. Thus $k = 0$ and G is effectively finite. But then

$$M - F \approx \frac{M - F}{G} \times G/H ,$$

with G/H a finite set of points. Thus $M - F$ is not connected and hence $s = n - 1$. This implies, by Lemma 2.1, that $G \approx Z_2$ effectively and M/G is an n -cm over Z with non-empty boundary, contrary to our assumption that M/G is a cm (without boundary).

LEMMA 2.7. Let G be a Lie group acting on an n -cm M over Z with (H) the type of the principal orbits. Let T be a maximal torus of H . For any point $z \in M$ with $G_z \supset H$, if S is a slice in M at z , then locally near z , $F(T, S)$ is a slice at z for the orbits of $N(T)$ on $F(T, M)$. Furthermore, the natural map

$$\frac{F(T, M)}{N(T)} \xrightarrow{f} M/G$$

is a homeomorphism locally near any fixed point of G .

PROOF. To show that $F(T, S)$ is a slice, it clearly suffices to show that $N(T)F(T, S)$ is open in $F(T, M)$ or equivalently that

$$\dim F(T, S) = \dim F(T, M) - \dim \frac{N(T)}{N(T) \cap G_z} .$$

But since $F(T, S)$ is a cm and may be assumed to be connected, we may compute its dimension by restricting our attention to the neighborhood of a point $z' \in S$ with $G_{z'} = H$. But near z' we know that $S \longrightarrow S/G_z \approx M/G$ is a product bundle. Since

$$F(T, G_z/H) = \frac{(N(T) \cap G_z)H}{H} \approx \frac{N(T) \cap G_z}{N(T) \cap H}$$

we have that

$$\dim F(T, S) = n - k + \dim \frac{N(T) \cap G_z}{N(T) \cap H} .$$

Similarly

$$\dim F(T, M) = n - k + \dim \frac{N(T)}{N(T) \cap H}$$

and hence $F(T, S)$ has the proper dimension, as claimed.

To prove the second part of the lemma we may assume that $F(T, M)$ is connected and of dimension $n - k + \dim(N(T)/T)$.

Let $B = \{x \in M; \text{rank}(G_x) > \text{rank}(H)\}$. Then B is closed and is contained in the singular set. Also $B \cap F(T, M)$ is contained in the singular set for $N(T)$ on $F(T, M)$. Thus by IX, 2.1, B cannot disconnect M locally near any point and $B \cap F(T, M)$ cannot disconnect $F(T, M)$ locally near any point.

For $x \in F(T, M)$ we now consider $F(T, G(x)) \sim F(T, G/G_x)$. If $T' \supset T$ is a maximal torus of G_x and $gG_x \in F(T, G/G_x)$ we see that $g^{-1}Tg \subset G_x$ and by changing g by an element of G_x we may assume that $g^{-1}Tg \subset T'$. Then, as is well-known ([7], Théorème 2, pp. 23-05), there is an element $n \in N(T_0)$, where $T_0 \supset T'$ is a maximal torus of G , such that $ng^{-1}Tgn^{-1} = T$. Thus $g = (gn^{-1})n \in N(T)N(T_0)$. Since $T_0 \subset N(T)$ we may take for n one of a system $\{n_i\}$ of representatives for $N(T_0)/T_0$. Thus we see that $F(T, G(x)) \subset U_1 N(T) n_1(x)$.

This shows that the natural map $f : F(T, M)/N(T) \longrightarrow M/G$ is finite to one. Note also that if $T' = T$, then $F(T, G(x)) = N(T)(x)$ so that on $(F(T, M) - B)/N(T)$, f is one-one.

Now let $z^* \in \pi(B) = B^* \subset M/G$ and assume that $f^{-1}(z^*) = \{z_1, z_2, \dots, z_j\}$ with $j > 1$. Let U_1 be pairwise disjoint open neighborhoods of z_1 in $F(T, M)/N(T)$. Let K^* be a compact neighborhood of z^* in M/G and let $K = \pi^{-1}(K^*)$. We claim that $U^* = U_1 f(U_1)$ is a neighborhood of z^* in M/G . If not, then there is a directed system $y_\alpha^* \longrightarrow z^*$, $y_\alpha^* \in K^* - U^*$. For each y_α^* let $y_\alpha \in f^{-1}(y_\alpha^*)$. Then y_α is in the compact set

$$\frac{F(T, M) \cap K}{N(T)} = U_1 U_1.$$

Thus there is a subsystem y_β converging to a point y of this set. But $f(y) = f(\lim y_\beta) = \lim f(y_\beta) = \lim y_\beta^* = z^*$ so that $y = z_1 \in U_1$ for some 1 , which is a contradiction.

Thus let $V^* \subset U^*$ be an open neighborhood of z^* in M/G , let $\bar{v} \subset M$ be a component of $\pi^{-1}(V^*)$ which contains a point projecting onto z^* , and put $W^* = \pi(W) \subset V^*$. Let $W_1 = f^{-1}(W^*) \cap U_1$. Since f is a homeomorphism outside of B , we have that $W^* - B^* \sim f^{-1}(W^* - B^*) = j_1(W_1 - B/N(T))$ is not connected, contrary to the fact that $W^* - B^* = f(W - B)$ is the continuous image of a connected set. This contradiction finishes the proof.

The following lemma is one of Gysin's results [3] and is well-known, but for sake of completeness we give a short proof here.

LEMMA 2.8. Let $E \xrightarrow{F} B$ be a fibre bundle with E , F and B manifolds. Say that $H^1(E) = H^1(S^k)$ and $H^1(F) = H^1(S^s)$, $0 < s < k$. Then s is odd, $\frac{k+1}{s+1} = m \in \mathbb{Z}$, and $H^1(B) \sim \mathbb{Z}[a]/a^m$ (as a ring) where degree $a = s + 1$. Thus $\chi(B) = \frac{k+1}{s+1} > 1$.

PROOF. Let E_2^{pq} be the spectral sequence of this fibering. Then with coefficients in $\mathbb{Z}_2$ we have $E_2^{pq} \sim H^p(B, \mathbb{Z}_2) \otimes H^q(F, \mathbb{Z}_2)$. Thus $0 = H^{k-1}(E) = E_{\infty}^{k-s-1, s} = E_2^{k-s-1, s} \sim H^{k-s-1}(B, \mathbb{Z}_2)$ so that B is orientable. This implies that the structure group acts trivially on $H^1(F, \mathbb{Z})$ and consequently for coefficients in $\mathbb{Z}$ we also have $E_2^{pq} \sim H^p(B) \otimes H^q(F)$.

We have that $E_2^{pq} = 0$ for $q \neq 0, s$, so that $d_{s+1}^{O, s} : E_{s+1}^{O, s} \rightarrow E_{s+1}^{s+1, 0}$ must be non-trivial. Let z be a generator of $H^s(F)$. Then $0 \neq d_{s+1}^{O, s}(1 \otimes z) = u \otimes 1$ say. But $0 = d_{s+1}^{O, s}(1 \otimes z^2) = (-1)^s(1 \otimes z)d_{s+1}^{O, s}(1 \otimes z) = (-1)^s(1 \otimes z)(u \otimes 1) + u \otimes z = (1 + (-1)^s)(u \otimes z)$. Thus, since $u \otimes z \neq 0$ we have that s is odd.

The cohomology of B follows easily from the Gysin sequence

$$\dots \rightarrow H^{1-1}(E) \rightarrow H^{1-s-1}(B) \rightarrow H^1(B) \rightarrow H^1(E) \rightarrow \dots$$

in which it is well-known that the map $H^{1-s-1}(B) \rightarrow H^1(B)$ is the cup product with an element $a \in H^{s+1}(B)$.

§3. A theorem on cross-sections

For the proof of Theorem 1.4 in the case k odd we must prove a theorem on cross-sections of principal bundles. This theorem and its proof are slight extensions of a similar theorem of Montgomery and Mostow [5]. Since the main argument of their proof is unchanged we shall not reproduce it here, but shall only show how their argument can be adapted to the present situation.

THEOREM 3.1. Let X be a locally compact, paracompact space which is clo over Z . Let $n < \infty$ be the classical covering dimension $\dim_{\text{cov}} X$. Let $X = X_0 \supset X_1 \supset X_2 \supset \dots \supset X_n$ be open paracompact subsets of X such that

$$\text{Im}(H^j(X_1, \mathbb{Z}_q) \rightarrow H^j(X_{1+1}, \mathbb{Z}_q)) \sim \begin{cases} \mathbb{Z}_q & j = 0 \\ 0 & j \neq 0 \end{cases}.$$

for $0 \leq i < n$ and all $q \geq 0$ (where, by Z_0 we mean Z). Let Y be a k -simple, for all k , finite polyhedron and let f be a continuous map of X into Y . Then $f|_{X_n}$ is homotopic to a constant.

PROOF. We first point out some facts about covering dimension. It is known [2], Corollary 3.6, that, since a paracompact locally compact space V is normal, its covering dimension is the least integer n such that every continuous map of a closed subset of V into S^n can be extended to a map of V into S^n .

Using this characterization and the fact that each point of a paracompact locally compact space is contained in an open and closed σ -compact subspace, it follows that

$$\dim_{\text{cov}} V = \sup \{ \dim_{\text{cov}} K; K \text{ compact, } K \subset V \}$$

for all paracompact locally compact spaces V . It also follows from this that if $U \subset V$ are both paracompact and locally compact then

$$\dim_{\text{cov}} U \leq \dim_{\text{cov}} V.$$

Returning to the situation at hand, we have that $\dim_{\text{cov}} X_i \leq n$ for all i . We find upon examination that the proof of Theorem 5.2 of [5] will prove our theorem provided that we can find arbitrarily small, n -dimensional (at most) closed covers α_i of X_i such that if K_i is the nerve of α_i and $\pi_i : K_i \rightarrow K_{i-1}$ is a refinement projection, then $\pi_i^* : H^1(K_{i-1}, Z_q) \rightarrow H^1(K_i, Z_q)$ is trivial for all $i \geq 1$ and all $q \geq 0$. We now show that this can be done. If α is a cover with nerve K then we denote $H^1(\alpha) = H^1(K)$.

Let α_0 be an arbitrary n -dimensional closed covering of X_0 and let $\beta_1 = \alpha_0|_{X_1}$. Let α_1 be an n -dimensional refinement of β_1 such that (β_1, α_1) is a Z_q -determining pair for $H^*(X_1, Z_q)$, for all $q \geq 0$ (see [5] for definitions). Consider the following commutative diagram ($j \geq 1$).

$$\begin{array}{ccccc}
 & & H^j(\alpha_0) & & \\
 & \swarrow & \downarrow \lambda_0 & \searrow & \\
 g & & H^j(X_0) & & \pi_1^* \\
 & \swarrow & \downarrow i^* & \searrow & \\
 & & H^j(X_1) & & \\
 & \swarrow & & \nwarrow \lambda_1 & \\
 H^j(\beta_1) & \xrightarrow{h} & & & H^j(\alpha_1)
 \end{array}$$

We have by definition that λ_1 maps $\text{Im } h$ isomorphically onto $H^1(X_1)$. However $\lambda_1 \pi_1^* = i^* \lambda_0 = 0$, since $i^* = 0$, and $0 = \lambda_1 \pi_1^* = \lambda_1 h g$. Hence $\pi_1^* = h g = 0$ since λ_1 is an isomorphism on $\text{Im } h$. Thus α_0 and α_1 satisfy our condition and the rest of the α_1 can be constructed in the same manner. This completes the proof of the theorem.

COROLLARY 3.2. Let X and X_1 be as in Theorem 3.1, and let G be a compact connected Lie group. Then any principal fibering with fibre G and with X as the base space is trivial over X_1 (that is, admits a cross-section over X_1).

PROOF. This follows exactly as does Corollary 5.3 of [5].

COROLLARY 3.3. Let G be a compact connected Lie group operating on a locally separable, locally compact space M of finite covering dimension. Assume that G is free outside of $F = F(G, M) \neq \emptyset$, $\frac{M-F}{G}$ is an $m - \text{cm}$ over Z , and that

$$H_c^1 \left(\frac{M-F}{G}, Z \right) =_{LF} \left\{ \begin{array}{l} Z \text{ } 1 = m \\ 0 \text{ } 1 \neq m \end{array} \right.$$

Then there is a local cross-section for the orbits of G near any point $x \in F$.

PROOF. We may assume that M is separable. Then $\frac{M-F}{G}$ is also separable and hence paracompact. It follows easily that $\frac{M-F}{G}$ is of finite covering dimension. Also from the hypothesis and by the Poincaré duality, it is clear that given a neighborhood U of x in M/G we can find a neighborhood $V \subset U$ of x in M/G such that

$$\text{Im}(H^1(U-F) \longrightarrow H^1(V-F)) \approx \left\{ \begin{array}{l} Z \text{ } 1 = 0 \\ 0 \text{ } 1 \neq 0 \end{array} \right.$$

By the exact cohomology sequence associated with the coefficient sequence $0 \longrightarrow Z \longrightarrow Z \longrightarrow Z_q \longrightarrow 0$ we can also do this for coefficients in Z_q , simultaneously for all $q \geq 0$.

Also $U-F$ and $V-F$ are paracompact, since they are separable. Thus the conditions of Corollary 3.2 may be satisfied with $X_1 = \frac{W}{G} - F$ for some invariant open neighborhood W of x in M . Let f be a cross-section

$$f : \frac{W}{G} - F \longrightarrow W - F,$$

which exists by Corollary 3.2. Then if we define

$$g(y) = \begin{cases} y & y \in F \cap W/G \\ f(y) & y \in \frac{W}{G} - F \end{cases}$$

we have that $g : \frac{W}{G} \longrightarrow W$ is a cross-section for the orbits of G in W .

The following lemma will be used several times in later sections.

LEMMA 3.4. Let G be a finite solvable group and $L = Z$ (respectively $G \approx Z_p$, $L = Z_p$) acting on a space X freely outside $F(G, X)$. Say that $x \in F$ has a fundamental system of invariant open neighborhoods Y_α such that each

$$\frac{Y_\alpha - F}{G}$$

is connected. Assume further that $\frac{X - F}{G}$ is an orientable n -cm over L and that

$$H_c^{n-1} \left(\frac{X - F}{G}, L \right) = Lx^0.$$

Then there is a local cross-section at x for the orbits of G .

PROOF. Since any solvable group G has a normal subgroup K such that $G/K \approx Z_p$ for some p , the first case follows from the second by an induction on $\text{ord } G$, since G/K acts on $\frac{X - F}{K}$ with orbit space $\frac{X - F}{K}$, and since, if $\frac{X - F}{K}$ breaks into p components, then correspondingly $X - F$ breaks into p subsets (disjoint and open) upon each of which K acts with orbit space $\frac{X - F}{G}$. Thus we shall assume that $G \approx Z_p$ and $L = Z_p$.

Assume that $V = X - F$ is connected and let W be the complement of F in some invariant open neighborhood of x , such that $j_{V/G, W/G}^{n-1} = 0$. Assuming the lemma to be false we may take W to be connected. Consider the diagram of Smith sequences:

$$\begin{array}{ccccccc} H_\sigma^{n-1}(W, Z_p) & \longrightarrow & H_\tau^n(W, Z_p) & \longrightarrow & H_c^n(W, Z_p) & \longrightarrow & H_\sigma^n(W, Z_p) \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_\sigma^{n-1}(V, Z_p) & \longrightarrow & H_\tau^n(V, Z_p) & \longrightarrow & H_c^n(V, Z_p) & \longrightarrow & H_\sigma^n(V, Z_p) \longrightarrow 0 \end{array}$$

It is well-known that the last three vertical maps must be isomorphisms and that all the groups in dimension n are isomorphic to Z_p (this follows, for instance, from [1], Lemma 7.1). These facts clearly imply that the map $H_G^{n-1}(W, Z_p) \longrightarrow H_G^{n-1}(V, Z_p)$ is non-trivial.

This is inconsistent with the fact that there is a natural commutative diagram:

$$\begin{array}{ccc} H_G^{n-1}(W, Z_p) & \longrightarrow & H_G^{n-1}(V, Z_p) \\ \downarrow \sim & & \downarrow \sim \\ H_G^{n-1}(W/G, Z_p) & \longrightarrow & H_G^{n-1}(V/G, Z_p) \end{array}$$

and the assumption $j_{V/G, W/G}^{n-1} = 0$, which finishes the proof of the lemma.

§4. Case I. rank (G) = rank (H)

Let T be a maximal torus of H . Then, since $N(T)$ is effectively finite on $F(T, M)$, we have, by Lemma 2.7, that $\dim F(T, M) = n - k$. Thus $F(T, M)$ is an $(n - k)$ -cm over Z .

Let $y \in F(T, M)$ be a point with isotropy group $G_y = H$. Then a slice S at y for the orbits of G is a local cross-section each point of which has isotropy group H . Moreover S may be taken to be an open subset of $F(T, M)$, since $\dim S = \dim F(T, M) = n - k$. Since $N(T) \cap H$ leaves the open subset S of $F(T, M)$ stationary, it acts trivially on $F(T, M)$. Moreover, if $g \in N(T) - H$ then $g(y) \neq y$, so that we have that $\frac{N(T)}{N(T) \cap H}$ acts effectively on $F(T, M)$. Moreover, $N(T) \not\subset H$ since if so the principal orbit G/H would be non-orientable, which is impossible in a cohomology manifold whenever there are fixed points. (The existence of a fixed point implies that there is a principal orbit contained in an orientable subset of the cm, but this orbit is contained in an open set of the form $U \times G/H$ where U is a cm, and thus G/H is orientable.)

By Lemma 2.1 applied to the action of $\frac{N(T)}{N(T) \cap H}$ on $F(T, M)$ we have that $\frac{N(T)}{N(T) \cap H} \sim Z_2$ and that $F(N(T), M) = F(N(T), F(T, M))$ is an $(n - k - 1)$ -cm over Z . Moreover $\frac{M}{G} \sim \frac{F(T, M)}{N(T)}$, by 2.7, is an $(n - k)$ -cm over Z with boundary $F(N(T), M)$ near x .

By dimension we have that for some $y \in F(G, M)$ near x , $F(G, M) = F(N(T), M)$ locally near y .

DEFINITION 4.1. We say that property L_y holds if $y \in F(G, M)$

and $F(G, M) = F(N(T), M)$ locally at y .

Thus if L_y holds we have that M/G is an $(n - k) - \text{cm}$ over Z with boundary $F(G, M)$ near y .

LEMMA 4.2. There are no exceptional orbits near x .

PROOF. Let $G(z)$ be an exceptional orbit. We may assume that $z \in F(T, M)$. Thus by Lemma 2.5, z is on an exceptional orbit of $N(T)$ on $F(T, M)$, that is, $z \in F(N(T), M)$. Let $E \neq \emptyset$ be the exceptional set. Then $E \cap F(N(T), M)$ is open in $F(N(T), M)$ and non-empty. Thus, since $F(N(T), M)$ is an $(n - k - 1) - \text{cm}$ over Z and touches an orbit at most in one point we have that $\dim E \geq (n - k - 1) + k = n - 1$. This is impossible near a fixed point for G connected, by a theorem of Montgomery and Yang [6], Theorem 4.

Since there are no exceptional orbits near x and since the singular set cannot disconnect M even locally we have that the set of points of a component of $F(T, M) - F(N(T), M)$, near x , which are on principal orbits, is a connected set dense in this component. It follows easily that $F(H, M) \subset F(T, M)$ is either all of $F(T, M)$ or is the closure of one component of $F(T, M) - F(N(T), M)$. In either case, since the closure of a component of $F(T, M) - F(N(T), M)$ touches each orbit of G on M in exactly one point, we have that there is a cross-section near x contained in $F(H, M)$.

LEMMA 4.3. If L_y holds, then there are no singular orbits near y other than fixed points.

PROOF. Let $z \in F(H, M) - F(G, M)$ be on a singular orbit. L_y implies that if z is sufficiently close to y , then locally at z we have that $F(T, M) = F(H, M)$ forms a local cross-section for the orbits of G near z . Let S be a slice at z . Then, locally at z , $F(T, S) = F(H, S) \subset F(H, M)$ touches every orbit of G on M . But $F(H, M)$ is a local cross-section near z and therefore $F(H, S) = F(H, M) = F(T, M)$ (near z) is a local cross-section for the orbits of G near z . Since $F(T, M)$ is a cm over Z , the lemma now follows from Lemma 2.6.

LEMMA 4.4. $H^1(G/H) = H^1(S^k)$.

PROOF. Since L_y holds near some point y we may restrict our attention to the neighborhood of such a point. The cohomology sequence

$$\dots \longrightarrow H^1_0(M - F) \longrightarrow H^1_0(M) \longrightarrow H^1_0(F) \longrightarrow \dots$$

then yields easily that

$$H_G^1(M - F, L) \cong_{LF} \begin{cases} L & \text{if } i = n - k, k \\ 0 & \text{otherwise} \end{cases},$$

where L is either $\mathbb{Z}$ or a field.

But, near y , $M - F$ consists entirely of points on principal orbits and has a cross-section in $F(H, M)$ for these orbits, so that

$$M - F \approx \frac{M - F}{G} \times G/H.$$

We also know that

$$H_G^1\left(\frac{M - F}{G}, L\right) \cong_{LF} \begin{cases} L & \text{if } i = n - k \\ 0 & \text{otherwise} \end{cases}.$$

If L is a field, then the Künneth relations give easily that

$$\begin{aligned} H_G^1(M - F, L) &\cong_{LF} H_G^{n-k}\left(\frac{M - F}{G}, L\right) \otimes H^{1-n+k}(G/H, L) \\ &\approx H^{1-n+k}(G/H, L). \end{aligned}$$

Thus $H^j(G/H, L) = H^j(S^k, L)$ for any field L , but since G/H is a finite complex, this implies that the same is true for $L = \mathbb{Z}$, as was to be shown.

Note that the assumption that G is connected is not needed for Lemma 4.4. That is, if U is a sufficiently small invariant neighborhood of a point $y \in F(G, M)$ where L_y holds then Theorem 1.4 has been proved for the particular transformation group (G^*, U) , and hence it is also true for (G, U) by the remark below 2.1.

It now suffices to show that L_X holds.

LEMMA 4.5. L_X holds.

PROOF. Let $z \in F(N(T), M) - F(G, M)$. We may assume that $H \subset G_z$ and that z has a neighborhood Y such that $z' \in Y \cap F(N(T), M)$ implies that $G_{z'} \sim G_z$. Then if $z \in X_{UV}$ we have that $\dim(X_{UV}^1, \pi(z)) = \dim(F(N(T), M), z) = n - k - 1$, so that, by Corollary 1.5, the hypotheses of Theorem 1.4 hold for the action of G_z on a slice S at z . Thus if $\dim(G_z/H) = s$ we have by Lemma 4.4 that $H^1(G_z/H) = H^1(S^s)$ and that $0 < s < k$ by Lemma 4.2 and the assumption on z that $k - s = \dim G(z) >$

since $z \notin F(G, M)$ and G is connected. Moreover, k is even, since H is of maximal rank. This contradicts Lemma 2.8 and finishes Case I.

§5. Case II. $\text{rank}(H) < \text{rank}(G)$

Let T be a maximal torus of H and let $T_0 \supset T$ be a maximal torus of G .

Then, of course, $F(T_0, M)$ meets only singular orbits (and possibly not all of these). $N(T_0)$ acts on $F(T_0, M)$ and, since

$$F(T_0, G/G_y) = \frac{N(T_0)G_y}{G_y}$$

when $T_0 \subset G_y$ and is empty if $\text{rank}(G_y) < \text{rank}(G)$, the natural map

$$\frac{F(T_0, M)}{N(T_0)} \longrightarrow M/G$$

is an imbedding. Since $F(T_0, M)$ touches only singular orbits of G we have that

$$\dim \frac{F(T_0, M)}{N(T_0)} \leq n - k - 1$$

by IX, Corollary to 2.2. However, $N(T_0)$ is effectively finite on the cm $F(T_0, M)$, so that $\dim F(T_0, M) \leq n - k - 1$ and since $F(G, M) \subset F(T_0, M)$ is of dimension $n - k - 1$ it follows that we may assume $F(T_0, M)$ to be a connected $(n - k - 1)$ -cm over Z . Also $F(G, M)$ must contain an open subset of $F(T_0, M)$ by dimension. Since $N(T_0)$ leaves this open set stationary it must leave all of $F(T_0, M)$ stationary. That is, $y \in F(T_0, M)$ implies that $N(T_0) \subset G_y$.

DEFINITION 5.1. We say that property L_y^1 holds if and only if $y \in F(G, M)$ and $F(G, M) = F(T_0, M)$ locally at y .

As we have seen above, there are points y , arbitrarily close to x , at which L_y^1 holds.

We shall now devote our attention to $F(T, M)$ which may also be assumed to be connected. We have, by the remark below 1.4 and by Lemma 2.7, that the action of $N(T)$ on $F(T, M)$ satisfies the hypotheses of Theorem 1.4. We will first prove the theorem for this action.

LEMMA 5.2. $\text{rank}(G) = \text{rank}(H) + 1$, k is odd.

PROOF. That k is odd follows from the statement on the ranks, since the dimension of a compact Lie group is congruent modulo two to its rank.

Let K be the subgroup of $N(T)$ stationary on $F(T, M)$. Then $T \subset K \subset H$ so that $\text{rank}(K) = \text{rank}(H)$ and it suffices to show that $\text{rank}\left(\frac{N(T)}{K}\right) = 1$. Thus, equivalently, it suffices to show that $\text{rank}(G) = 1$ whenever H is finite. We may also assume for this proof that $F(T_0, M) = F(G, M)$, that is, we may confine our attention to the neighborhood of a point y for which L_y^1 holds.

We first show that $z \in M - F$ implies $\dim(G_z) = 0$. Say not, and assume that z is such that

$$\text{rank}(G_z) = \min \{ \max \{ \text{rank}(G_{z'}; z' \in U - F) \}; U \text{ open, } y \in U \}.$$

Also assume that T_0 contains a maximal torus T' of G_y . We then see easily that the natural map

$$\frac{F(T', M)}{N(T')} \longrightarrow M/G$$

is an imbedding, and since $F(T', M)$ touches only singular orbits we have that

$$\frac{F(T', M)}{N(T')} \leq n - k - 1.$$

Since $F(G, M) \subset F(T', M)$ is also of dimension $n - k - 1$, it must consist of principal orbits of $N(T')$ on the component of $F(T', M)$ containing y . Thus $N(T')$ acts trivially on this component. Also by the choice of T' it is clear that z may be assumed to belong to this component. But we have then that $z \in F(N(T'), M) \subset F(T_0, M) = F(G, M)$ which contradicts the fact that $z \notin F(G, M)$.

Thus $z \in M - F$ and close to y implies that $\dim G_z = 0$, so that if $T_1 \subset T_0$ and $\dim T_1 > 0$ then $F(T_1, M) = F(T_0, M)$ locally at y . By XIII, 4.3 this is clearly possible only if $\text{rank}(G) = \text{rank}(T_0) = 1$, which finishes the proof of the lemma.

§ 6. Rank one case

We must study in detail the action of $N(T)$ on $F(T, M)$. Thus we study, in this section, the case in which $\text{rank}(G) = 1$. Hence for G connected, as shall be assumed, G is effectively isomorphic to S^1 , $SO(3)$ or $Sp(1)$.

LEMMA 6.1. If L_y^1 holds then there are no exceptional orbits near y .

PROOF. Let z be on an exception orbit with $G_z \supset H$. We may assume that if z' is on an exceptional orbit and is sufficiently close to z , then $G_{z'} \sim G_z$. Let P be a p -Sylow subgroup of G_z such that $\text{ord}(P) \nmid \text{ord}(H)$. Since there are only a finite number of non-conjugate such subgroups P for z close to y , it follows easily that the lemma is true if, for each such P , the component of the cm $F(P, M)$ over Z_p which contains y is contained in $F(G, M) = F(T_0, M)$. Thus we may assume that $F(P, M)$ is connected and that $F(P, M) \supset F(G, M) = F(T_0, M)$.

Thus if $G = S^1$, then $\dim_{Z_p} F(P, M) = n - 1$ which implies that $\dim F(P, M) = n - 1$ by I 4.9, Corollary. But this is impossible by V 2.3 (a) and V 2.5 since every element of G must preserve orientation. This also follows from [6], Theorem 4.

If $\dim G = 3$, then $\dim_{Z_p} F(P, M) \geq n - 3$ since $\dim F(G, M) = n - 4$. Let U be a small neighborhood of z in $F(P, M)$ and put $Y = G(U) \subset \text{Exceptional set}$. If U is small enough then Y is a fibre bundle over Y/G with fibre G/G_z , since z was chosen to be "principal" in the exceptional set.

Thus $\dim_{Z_p} Y = \dim_{Z_p} Y/G + 3$. Also

$$\frac{Y}{G} \sim \frac{F(P, M)}{N(P)}$$

locally near z , since

$$F(P, G/G_z) = \frac{N(P)G_z}{G_z}.$$

Thus, unless $G = \text{Sp}(1)$ and $P = \text{center } G \sim Z_2$, we have that $\dim N(P) \leq 1$ so that $\dim_{Z_p} Y/G \geq \dim_{Z_p} F(P, M) - 1 \geq n - 4$. Thus $\dim_{Z_p} Y \geq n - 1$. Hence the dimension of the exceptional set is $n - 1$, which is impossible near a fixed point by [6], Theorem 4.

Thus $G \sim \text{Sp}(1)$ and $P = \text{center } G \sim Z_2$. Here $\dim_{Z_2} F(P, M) = n - 2$ by V 2.5 since an element of a connected group cannot reverse orientation. $G/P \sim \text{SO}(3)$ then acts effectively on the $(n - 2)$ -cm $F(P, M)$ over Z_2 with a fixed point set of dimension $n - 4$. Consider $K \subset G/P$, $K \sim Z_2 \oplus Z_2$. Then K acts on $F(P, M)$ and if $e \neq g \in K$, then $\dim_{Z_2} F(g, F(P, M)) = n - 4$ by V 3.1. Since $F(G, M) \subset F(g, F(P, M))$ is

an $(n - 4) - \text{cm}$ over Z and hence also over Z_2 , we must have that $F(G, M) = F(g, F(P, M))$ locally at y . That is, $Z_2 \oplus Z_2$ acts freely on $F(P, M) - F(G, M)$ locally at y , which is impossible by XIII, 4.3. This finishes the proof of the lemma.

If L_y^1 holds, then there can be no singular orbits of G near y except for fixed points. Thus by Lemma 6.1 we have that $M - F$ is a fibre bundle over $\frac{M - F}{G}$ with fibre G/H , near y . In particular, it follows that $\frac{M - F}{G}$ is an $(n - k) - \text{cm}$ over Z .

Now if $G \approx S^1$, then L_x^1 holds trivially. In this case $k = 1$ and $n - k - 1 = n - 2$. From the cohomology diagram in which the rows are

$$\dots \longrightarrow H_C^1(U - F) \longrightarrow H_C^1(U) \longrightarrow H_C^1(F \cap U) \longrightarrow \dots$$

for neighborhoods U of points in F , one can see easily that

$$H_C^1(M - F) =_{LF} \begin{cases} Z & 1 = n - 1, n \\ 0 & \text{otherwise} \end{cases}.$$

The Gysin sequences

$$\dots \longrightarrow H_C^{1-1}(U-F) \longrightarrow H_C^{1-2}\left(\frac{U-F}{G}\right) \longrightarrow H_C^1\left(\frac{U-F}{G}\right) \longrightarrow H_C^1(U-F) \longrightarrow \dots$$

show easily that

$$H_C^1\left(\frac{M - F}{G}\right) =_{LF} \begin{cases} Z & 1 = n - 1 \\ 0 & 1 \neq n - 1 \end{cases}.$$

Thus, by Lemma 2.4, we have that M/G is an $(n - 1) - \text{cm}$ over Z with boundary $F(G, M)$. Also, Corollary 3.3 implies that there is a local cross-section near x for the orbits of G .

This completes the proof of the theorem for $G \approx S^1$ and hence also for all G for which $\dim G = 1$, except as noted in the remark below Lemma 2.1.

We now turn our attention to the case in which $G \approx SO(3)$ or $Sp(1)$ and shall consider the neighborhood of a point y at which L_y^1 holds.

Consider the following table, to be explained below.

Case	G	H	$\frac{N(H)}{H}$	coefficients = L	$H^1(G/H, L)$	$H^2(G/H,$
1	Sp(1)	cyclic, odd order > 1	$N(T_0)/H$	Z_2	0	0
2	SO(3)	$Z_2 \oplus Z_2$	σ_3	Z_3	0	0
3	SO(3)	dyhedral, order $2a$, a even, $a > 2$	Z_2	Z_2	$Z_2 \oplus Z_2$	$Z_2 \oplus Z_2$
4	SO(3)	dyhedral, order $2a$, a odd	Z_2	Z_2	Z_2	Z_2
5	SO(3)	cyclic or trivial	$\frac{N(T_0)}{H}$ or G	Z_2	Z_2	Z_2
6	SO(3)	tetrahedral	Z_2	Z	0	Z_3
7	SO(3)	octahedral	(e)	Z	0	Z_2
8	SO(3)	icosahedral	(e)	Z	0	0
9	Sp(1)	(e)	G	Z	0	0

We wish to show that the first seven cases cannot occur. This will be done by considering the spectral sequences E_r^{pq} of the fiberings

$$U - F \xrightarrow{G/H} \frac{U - F}{G},$$

where U is a connected invariant open neighborhood in M of some point of F , and coefficients are understood to be in L as shown in the table. We will show first that the coefficients $H^q(G/H, L)$ of

$$U E_2^{pq} \approx H_c^p\left(\frac{U - F}{G}, H^q(G/H, L)\right)$$

are constant, for U sufficiently small. This has the immediate consequence that it may be assumed that the $(n - 3)$ -cm over Z , $\frac{M - F}{G}$ is orientable with respect to L , since

$$H_c^{n-3}\left(\frac{U - F}{G}, L\right) \approx U E_2^{n-3,3} \approx U E_\infty^{n-3,3} \approx H_c^n(U - F, L) \approx L.$$

We then show that

$$H_G^1\left(\frac{M-F}{G}, L\right) =_{LF} \begin{cases} L, & i = n-3 \\ 0 & \text{otherwise} \end{cases},$$

and that $H^2(G/H, L) = 0$. The second relation eliminates cases (3) through (7), and cases (1) and (2) will be shown to be inconsistent with Lemma 3.4, using the first relation above.

We must first show that the coefficients are constant. For the top fibre dimension, this follows from

LEMMA 6.2. $\frac{N(H)}{H}$ acts trivially on $H^k(G/H)$ whenever G is connected, H is finite and $k = \dim G/H = \dim G$.

PROOF. $N(H)/H$ acts on G/H as follows: if $\alpha \in N(H)$ then $\alpha : gH \longrightarrow \alpha g \alpha^{-1}H$. G also acts on G/H by left translation and, since G is connected, this action induces the trivial action of $H^*(G/H)$. Thus the action induced on $H^*(G/H)$ by the above action of $N(H)$ on G/H is the same as that induced by the action $\alpha : gH \longrightarrow \alpha g \alpha^{-1}H$ of $N(H)$ on G/H . It is clear that the latter action cannot reverse orientation on G/H , since locally at the origin it is the same as the adjoint representation of $N(H)$ on G and this can be extended to an action (adjoint) of G on G which must preserve orientation because G is connected.

To show that the coefficients $H^1(G/H, L)$ and $H^2(G/H, L)$ are constant, it suffices to consider only cases (3) and (6), since this is clearly trivial in the other cases.

In cases (3) and (6) let K be a Sylow 2-group in H . In case (3), K is dihedral of order ≥ 4 and in case (6) $K \approx Z_2 \oplus Z_2$. Clearly it is possible to choose $g \in N(H) - H$ (hence generating $N(H)/H$) such that $g \in N(K)$ and $g^2 \in K$. Now

$$F(K, G/H) = \frac{N(K)H}{H} \approx \frac{N(K)}{N(K) \cap H}$$

which is finite. Thus $F(K, M)$ is an $(n-3)$ -cm over Z_2 . $g \notin H$ and thus g acts non-trivially on $F(K, M)$ with fixed point set $F(G, M)$ at least. Moreover g^2 leaves $F(K, M)$ stationary, since $g^2 \in K$. Thus, locally at y , $F(g, F(K, M)) = F(G, M)$ disconnects $F(K, M)$ into two components which are permuted by g (see [8], Lemma 16). Since $F(H, M) \subset F(K, M)$ it follows that g cannot preserve a component of $F(H, M - F)$. However, the structural group can be described as

$$\frac{N_z(H)}{H},$$

where $G_z = H$ and we define

$N_z(H) = \{g \in N(H); g \text{ preserves the component of } F(H, M - F) \text{ containing } z\}$.

In our case, since $\frac{N(H)}{H} \approx Z_2$, we have shown that $N_z(H) = H$ and this completes the proof of the constancy of the coefficients. (It is of interest to note that the above proof actually shows that a component of $F(H, M - F)$ is a cross-section in these cases.)

We now prove

LEMMA 6.3.

$$H_G^1\left(\frac{M-F}{G}, L\right) =_{LF} \begin{cases} L & 1 = n - 3 \\ 0 & 1 \neq n - 3 \end{cases}.$$

PROOF. As noted before, $\frac{M-F}{G}$ is an $(n-3)$ -cm over Z and is orientable over L (at least near any point of F). Thus it suffices to show that

$$H_G^1\left(\frac{M-F}{G}, L\right) =_{LF} 0$$

for $1 \leq n - 4$.

We know that $H_G^1(M-F, L) =_{LF} 0$ for $1 \leq n - 4$ and also $U_\infty^{E^{1,0}} = U_5^{E^{1,0}}$ which implies that $M_5^{E^{1,0}} =_{LF} 0$ for $1 \leq n - 4$, that is, for every U and $z \in U \cap F$ there is a $z \in V \subset U$ such that the natural map $U, V_5^{E^{1,0}} : V_5^{E^{1,0}} \longrightarrow U_5^{E^{1,0}}$ is trivial. Assume the lemma is false and let $p \leq n - 4$ be the least integer for which

$$H_G^p\left(\frac{M-F}{G}, L\right) \neq_{LF} 0$$

and let r be the largest integer ($2 \leq r < 5$) for which $M_r^{E^{p,0}} \neq_{LF} 0$ (which exists since

$$M_2^{E^{p,0}} = H_G^p\left(\frac{M-F}{G}, L\right) \neq_{LF} 0).$$

By the universal coefficient sequences

$$0 \longrightarrow H_C^{p-r}(\frac{U-F}{G}, L) \otimes H^{r-1}(G/H, L) \longrightarrow H_C^{p-r}(\frac{U-F}{G}, H^{r-1}(G/H, L)) \longrightarrow$$

$$\text{Tor}(H_C^{p-r+1}(\frac{U-F}{G}, L), H^{r-1}(G/H, L)) \longrightarrow 0$$

it follows that $M_2^{p-r, r-1} =_{LF} 0$, since $p - r + 1 \leq p - 1$. Thus also $M_r^{p-r, r-1} =_{LF} 0$.

Given U and $z \in U \cap F$ we can choose $z \in V \subset U$ such that $U, V_j^{p-r, r-1} = 0$ and also, since $M_{r+1}^{p, 0} =_{LF} 0$ we may choose $z \in W \subset V$ such that $\text{Im } V, W_j^{p, 0} \subset \text{Im } d_r$. But then the diagram

$$\begin{array}{ccccc} & & & & W_r^{p, 0} \\ & & & & \downarrow \\ V_r^{p-r, r-1} & \xrightarrow{d_r} & V_r^{p, 0} & \xrightarrow{d_r} & 0 \\ \downarrow 0 & & \downarrow & & \\ U_r^{p-r, r-1} & \xrightarrow{d_r} & U_r^{p, 0} & & \end{array}$$

shows that $U, W_j^{p, 0} = 0$. Thus $M_r^{p, 0} =_{LF} 0$ contrary to the choice of p and r .

LEMMA 6.4. $H^2(G/H, L) = 0$.

PROOF. Since $H_C^{n-1}(M-F) =_{LF} 0$ we must have that $M_3^{n-3, 2} = M_\infty^{n-3, 2} =_{LF} 0$. Given U connected and $z \in U$ we can find a $z \in V \subset U$ such that $U, V_j^{n-5, 3} = 0$, since

$$M_2^{n-5, 3} \approx H_C^{n-5}(\frac{M-F}{G}, L) =_{LF} 0$$

by Lemma 6.3. Moreover, since $M_3^{n-3, 2} =_{LF} 0$ we can find $z \in W \subset V$ such that $\text{Im } V, W_j^{n-3, 2} \subset \text{Im } d_2$. But then the diagram

$$\begin{array}{ccccc} & & & & W_2^{n-3, 2} \\ & & & & \downarrow \\ V_2^{n-5, 3} & \xrightarrow{d_2} & V_2^{n-3, 2} & \xrightarrow{d_2} & 0 \\ \downarrow 0 & & \downarrow & & \\ U_2^{n-5, 3} & \xrightarrow{d_2} & U_2^{n-3, 2} & & \end{array}$$

shows that $U, Wj_2^{n-3,2} = 0$. However, this map becomes the natural map

$$H_G^{n-3}(\frac{W-F}{G}, H^2(G/H, L)) \longrightarrow H_G^{n-3}(\frac{U-F}{G}, H^2(G/H, L)) \approx H^2(G/H, L)$$

and, since $\frac{U-F}{G}$ is an $(n-3)$ -cm over Z , this map must be onto. Hence $H^2(G/H, L) = 0$ as claimed.

We have, so far, eliminated cases (3) through (7). To eliminate cases (1) and (2) we will show that they are inconsistent with Lemma 3.4. In case (1) note that

$$\frac{M-F}{G} \approx \frac{F(H, M-F)}{N(H)}$$

and that $F(H, M) = F(K, M)$, where K is a p -Sylow subgroup of H , p odd, $p \mid \text{ord } H$. But then $F(H, M)$ is an $(n-2)$ -cm over Z_p and cannot be disconnected by the $(n-4)$ -cm $F(G, M)$. Hence $F(H, M-F)$ may be assumed to be connected, even locally about y . Thus

$$\frac{F(H, M-F)}{T_0}$$

is connected, even locally at y , and covers

$$\frac{F(H, M-F)}{N(H)} \approx \frac{M-F}{G}$$

twice. This is impossible by Lemma 3.4 and Lemma 6.3.

In case (2),

$$F(H, G/H) \approx \frac{N(H)}{H} \approx \sigma_3 \text{ (of order 6).}$$

$\frac{N(H)}{H}$ acts on $F(H, M)$ with

$$\frac{F(H, M)}{N(H)} \approx M/G.$$

Moreover, $F(H, M)$ is an $(n-3)$ -cm over Z_2 . Let

$$K \subset \frac{N(H)}{H}, K \approx Z_2.$$

Then $F(K, F(H, M)) = F(G, M)$ which has dimension $n-4$. By [8], Lemma 16, it follows that $F(G, M)$ splits $F(H, M)$ into two pieces locally

which are permuted by K . Let V be one of these pieces of $F(H, M) - F(G, M)$. V is then connected, even locally about y . Clearly, the subgroup $J \approx Z_3$ of $\frac{N(H)}{H}$ acts on V with

$$V/J \approx \frac{M - F}{G}.$$

Since $\frac{M - F}{G}$ is an orientable $(n - 3) - \text{cm}$ over Z_3 , this contradicts Lemma 3.4 and Lemma 6.3.

We have now eliminated cases (1) through (7). For cases (8) and (9), Lemma 6.3 implies, by Lemma 2.4, that M/G is an $(n - 3) - \text{cm}$ over Z with boundary $F(G, M)$. In case (8) $F(H, M)$ is clearly a cross-section for the orbits of G , and in case (9) there is a local cross-section by Corollary 3.3.

To complete the investigation of the rank one case we must show that L'_x holds. This is equivalent, in this case, to the non-existence of singular orbits other than fixed points near x . If $G = S^1$, this is trivial as has been remarked, and if $G \approx SO(3)$ it is also trivial since there are no proper subgroups of $SO(3)$ containing the icosahedral group. Thus we may assume that $G \approx Sp(1)$ and $H = (e)$. Let z be on a singular orbit with $T_0 \subset G_z$. Then as shown at the beginning of Section 5, $N(T_0) \subset G_z$. Thus $G_z = N(T_0)$. Let S be a slice at z . Then $\dim S = n - 2$, $\dim G_z = 1$ and $\dim F(G_z, S) = n - 4$, by the same argument as used in the proof of Lemma 4.5. Hence from Theorem 1.4 for one-dimensional groups (which may be considered as proved now, except as noted in the remark below 2.1), we have the the principle orbit $G_z/H = G_z = N(T_0)$ is connected. This contradiction finishes the rank one case.

§ 7. A local cross-section for the orbits of G near F

We have shown in Section 6 that the action of $N(T)$ on $F(T, M)$ has only two orbit types near x , fixed points and principal orbits. Thus by Lemma 2.5 there are no exceptional orbits near x . Also if $G_z \supset H$, for z near x , then we have that $N(T) \cap G_z$ is either $N(T)$ or $N(T) \cap H$. We also know by Lemma 2.7 and the results of Section 6, that M/G is an $(n - k) - \text{cm}$ over Z with boundary $F(N(T), M) \subset F(T_0, M)$. Thus, near a point y at which L'_y holds, M/G has boundary $F = F(G, M)$.

Let y be a point where L'_y holds and let $z \in M - F$ be near y and such that $G_z \supset H$. Since $z \notin F$ and L'_y holds we have that $N(T) \cap G_z = N(T) \cap H$. Let S be a slice at z . Then for any $z' \in S$,

$$F(H, G_Z/G_{Z_1}) \subset F(T, G_Z/G_{Z_1}) \approx \frac{N(T) \cap G_Z}{N(T) \cap G_{Z_1}} = \frac{N(T) \cap H}{N(T) \cap H} = \text{a point},$$

we see that $F(H, S)$ is a local cross-section at z . Thus, by Lemma 2.6, there are no singular orbits near y .

We know that $N(T)$ has a cross-section for its action on $F(T, M)$, in the sense of a closed set in $F(T, M)$ touching every orbit exactly once. This yields, by Lemma 2.7, a cross-section in this sense for G on M . Moreover this cross-section is contained in $F(N(T) \cap H, M)$. We must show that it can be taken to be in $F(H, M)$.

Although $N(T)$ need not be connected, we know, by the results of Section 6 and the remark below 2.1, that its orbits on $F(T, M)$ are connected. That is,

$$\frac{N(T)}{N(T) \cap H} \approx \frac{Z(T)}{Z(T) \cap H},$$

where $Z(T)$ denotes the centralizer of T , or, in other words, the identity component of $N(T)$. Thus $N(H) = (N(H) \cap N(T))H = (N(H) \cap Z(T))H$, so that

$$\frac{N(H)}{H} \approx \frac{N(H) \cap Z(T)}{H \cap Z(T)} \subset \frac{Z(T)}{H \cap Z(T)}.$$

Now, in the case

$$\frac{Z(T)}{H \cap Z(T)} \approx \frac{SO(3)}{I}$$

we know that $N(H) = H$, since $\frac{N(H)}{H}$ is a group and since there are no proper subgroups of $SO(3)$ containing I except I itself. Thus in this case $F(H, M)$ is a cross-section for the orbits of G on M .

In the other cases we have that $H \cap Z(T)$ is normal in $Z(T)$ and

$$\frac{Z(T)}{H \cap Z(T)} \approx S^1 \text{ or } Sp(1).$$

Thus $\frac{N(H)}{H}$ may be regarded as a subgroup of $Sp(1)$. (Note that if

$$\frac{Z(T)}{H \cap Z(T)} \approx Sp(1)$$

then in fact $H \cap Z(T) = T$.)

If $\frac{N(H)}{H}$ is a finite solvable group then the cross-section is given by Lemma 3.4. If

$$\frac{N(H)}{H} \approx S^1 \text{ or } Sp(1)$$

then it is given by Corollary 3.3. If $\frac{N(H)}{H}$ is an extension of S^1 by Z_2 then

$$\frac{F(H, M - F)}{(N(H)/H)^*}$$

is a double covering of $\frac{M - F}{G}$ and hence must split (locally) by Lemma 3.4, which implies that a component of $\Gamma(H, M - F)$ is a principal bundle with fibre

$$S^1 \approx (\frac{N(H)}{H})^*$$

and base $\frac{M - F}{G}$, so that the cross-section exists by Corollary 3.3.

Thus we have to consider only the case in which

$$\frac{Z(T)}{Z(T) \cap H} = Sp(1)$$

and $\frac{N(H)}{H}$ is the icosahedral group. The remainder of this section is devoted to proving that this situation cannot arise and to extending this result to the case in which G is not connected, as promised in the remark below 2.1. We denote $N(T) \cap H$ by $N_H(T)$ and so on.

Consider the fibre maps

$$\frac{M - F}{G} \times G/H \xleftarrow{\pi_1} \frac{M - F}{G} \times G/N_H(T) \xrightarrow{\pi_2} M - F.$$

The second map π_2 is given by $\pi_2(w, gN_H(T)) = g(f(w))$, where

$$f : \frac{M - F}{G} \longrightarrow M - F$$

is a cross-section map from

$$\frac{M - F}{G} \approx \frac{F(T, M - F)}{N(T)}$$

to $F(N_H(T), M - F) \subset M - F$. π_2 is well-defined, since for $z \in F(N_H(T), M - F)$ we have $N_{G_z}(T) = N_H(T)$. Typical fibres on both sides are $H/N_H(T)$, which is acyclic over the rationals. Hence in rational cohomology π_1^* and π_2^* are isomorphisms. It then follows easily, just as in the proof of Lemma 4.4, that $H^1(G/H, R) = H^1(S^k, R)$, where R is the field of rational numbers.

The following lemma and its proof are due to A. Borel.

LEMMA 7.1. Let G be a compact connected Lie group and H a subgroup of G . Let T be a maximal torus of H and $T_0 \supset T$ a maximal torus of G . Assume that $\ell = \dim T = \dim T_0 - 1$, and that T is singular (i.e., that $\dim Z(T) = \ell + 3$). If $H^1(G/H, R) = H^1(S^k, R)$, where $k = \dim G/H$, then also $H^1(G/H^*, R) = H^1(S^k, R)$.

PROOF. If $K \subset G$ is any subgroup with maximal torus T , then, since $K/N_K(T)$ is acyclic over R , we have that

$$H^1(G/K, R) \approx H^1(G/N_K(T), R) \approx I_{W(K)} H^1(G/T, R),$$

where the latter group denotes the invariants of $H^1(G/T, R)$ with respect to the operation of the Weyl group $W(K)$. $W(K)$ may be interpreted either as

$$\frac{N_K(T)}{T}$$

or as $\text{Aut}_K(T)$ (the subgroup of $\text{Aut}(T)$ corresponding to conjugations by elements of K), since the kernel of the map

$$\frac{N_K(T)}{T} \longrightarrow \text{Aut}_K(T)$$

consists of cosets nT such that $n \in Z(T)$, and the latter group, being connected since G is connected, must act trivially on $H^1(G/T, R)$. We choose to define $W(K) = \text{Aut}_K(T)$.

Now $Z(T)$ may be decomposed $Z(T) = T \cdot Q$ where $\dim Q = 3$ ($Q \approx \text{Sp}(1)$ or $\text{SO}(3)$). Let S be some circle subgroup of Q . Then $Z(S)$ decomposes into $Z(S) = L \cdot S$ and T is a maximal torus of L . We claim that $W(K) \subset W(L)$ both in the sense of its action on T and of its action on $H^1(G/T, R)$. To see this, let $g \in N_K(T)$. Then $gZ(T)g^{-1} = Z(T)$ so that $gQg^{-1} = Q$. Thus there is an element $g_1 \in Q$ such that $g_1 S g_1^{-1} = S$

and since there are elements of $N_Q(S)$ which reverse the orientation of S we may assume that $g_1, g \in L$. The actions of g_1, g and g are identical on both T and $H^1(G/T, R)$ which proves our assertion that $W(K) \subset W(L)$.

Thus we have the natural inclusions $W(H^*) \subset W(H) \subset W(L)$. Now since G/H is an orientable cohomology k -sphere over R , we have that $W(H)$ has exactly two invariants in $H^*(G/T, R)$. Thus $W(L)$ has at most two invariants which are in dimensions 0 and k . But G/L is orientable and hence $W(L)$ has exactly two invariants. Hence $H^1(G/L, R) \approx H^1(S^k, R)$ and in particular $\dim G/L = k$. Thus $\dim L = \dim H^*$. However, as is well-known from the structure theory we have that

$$\ell + 2n(L) = \dim L = \dim H^* = \ell + 2n(H^*)$$

where $n(K)$ denotes the number of symmetries in $W(K)$ (operations in $W(K)$ which leave fixed an $\ell - 1$ dimensional subgroup). Thus $n(L) = n(H^*)$ and since the Weyl group is generated by its symmetries we conclude that $W(H^*) = W(L)$. Thus $W(H^*)$ has exactly two invariants in $H^*(G/T, R)$ and it follows that $H^1(G/H^*, R) = H^1(S^k, R)$, as was to be shown.

Returning to the case at hand we have that

$$F(T, G/H^*) = \frac{N(T)H^*}{H^*} \approx \frac{N(T)}{N(T) \cap H^*}$$

covers

$$\frac{N(T)}{N(T) \cap H} \approx \text{Sp}(1) \approx S^3$$

with fibre H/H^* . Thus $F(T, G/H^*)$ consists of $(\text{ord } H/H^*)$ 3-spheres. But $F(T, G/H^*)$ is a rational cohomology sphere by IV 5.6, and therefore we have that $H = H^*$ is connected.

Consider $Z(T) = T \cdot Q$, $Q \approx \text{Sp}(1)$. Since

$$\frac{N(H)}{H} \approx \frac{N(H) \cap Z(T)}{H \cap Z(T)} \approx \frac{N(H) \cap Q}{H \cap Q} = N(H) \cap Q$$

we have that $I = N(H) \cap Q$ is an isosahedral group. Also $N(H) = IH$. Since $I \subset N(H) \cap Z(T)$ and since any automorphism of a connected Lie group which leaves the maximal torus stationary is an inner automorphism by an element of the torus, we have that the homomorphism $I \longrightarrow \text{Aut } H$ can be

factored $I \xrightarrow{f} T/\text{center } H \longrightarrow \text{Aut } H$. Clearly f is a homomorphism and hence $I/\ker f$ is abelian. Thus $I = \ker f$, which implies that I centralizes H .

Let J be a subgroup of I of order 5, and let $K = (Z(J))^*$. Then clearly K is of maximal rank and contains H . Since H is totally non-homologous to zero in G , it is also totally non-homologous to zero in K . Thus we must have that $H^1(K/H, R) = H^1(S^r, R)$ for some $r \leq k$. Now it is known [4] that either G and H are both simple or the decomposition into simple groups is of the form $G = G' \cdot G''$, $H = H' \cdot H''$, where $G'/H' \sim G/H$ and $G'' \sim H''$ is a group of rank one. Moreover, since G is effective we have that $G'' \not\subset H$. Thus in our case G'' is not a circle group, since if it were, we would have $G'' \subset N(H) = H$, which would imply that

$$\dim \frac{N(H)}{H} \geq 1.$$

Thus H is either simple or of the form $H = H' \cdot A_1$.

Now K need not be effective on K/H , however, from the fact that $H^1(K, R) = H^1(H \times S^r, R)$ for some r , we have easily that K is either simple (but only if H is simple) or $K \sim K' \cdot A_1$ or $K \sim K' \cdot A_1 \cdot A_1$ where K' is simple. (No circle groups can appear in this decomposition of K since such a factor of K would necessarily normalize H but not lie in H , implying that

$$\dim \frac{N(H)}{H} \geq 1.)'$$

However $J \subset \text{center } K$ and hence $J \subset K'$ since $\text{center } A_1 \sim Z_2$. But the only simple groups having a center which can contain Z_5 are the groups locally isomorphic to A_n for $5|(n+1)$ (see for instance [7], p. 23-08). Thus $K' \sim A_m$, for some $m \geq 4$ (perhaps modulo a finite normal subgroup of A_m). Moreover K' is effective on K/H , since if not we would have $J \subset H$. Thus from general facts about homogeneous homology spheres [4] we conclude that $H' \subset K'$ and $K'/H' \sim K/H$ is a cohomology r -sphere over R . Also since $K' \sim A_m$, $m \geq 4$ we obtain $H' = A_{m-1}$, since A_{m-1} is the only simple group H' such that $H^1(H' \times S^r, R) = H^1(A_m, R)$ for any $m \geq 4$ and any r . For a similar reason it also follows that $G' \sim A_m$. Thus if H is simple, then $G \sim A_m$ and $K' \sim A_m$, whence $G = K' = K$ by dimension, and if H is not simple then $G \sim A_m \cdot A_1$ and $K \sim A_m \cdot A_1$ or $A_m \cdot A_1 \cdot A_1$ and again we must have $G = K$. This shows that J is central in G . But J is not central in G since it is not even central in $I \subset G$. This contradiction completes the proof that $\frac{N(H)}{H}$ cannot be an isosahedral group and hence also the proof that there

is a cross-section contained in $F(H, M)$ for the orbits of G on M near any point y for which L_y^1 holds.

We shall now show that the cross-section may be assumed to be in $F(H, M)$ even in the case in which G is not connected. Since this result is not needed to complete the proof of 1.4 for G connected and by the remark below 2.1, we may assume for (G, M) that all the other conclusions of 1.4 are known.

Let $H_1 = G^* \cap H$. Then $G/H \sim G^*/H_1$. Also

$$F(H_1, G/H) \sim F(H_1, G^*/H_1) = \frac{N(H_1) \cap G^*}{H_1}$$

and

$$F(H, G/H) = \frac{N(H)}{H} \sim \frac{N(H) \cap G^*}{H_1} \subset \frac{N(H_1) \cap G^*}{H_1}.$$

By the first part of this section we know that

$$\frac{N(H_1) \cap G^*}{H_1}$$

is either finite solvable, or isomorphic to $SO(2)$, $O(2)$, or $Sp(1)$. We also know by the proof in the first part of this section that if

$$\frac{N(H)}{H} \sim \frac{N(H) \cap G^*}{H_1} \subset \frac{N(H_1) \cap G^*}{H_1} \subset Sp(1)$$

is not the icosahedral group then there is a cross-section in $F(H, M)$ for the orbits of G on M . Thus the only case to be considered is that in which $\frac{N(H)}{H}$ is the icosahedral group (and hence for which

$$\frac{N(H_1) \cap G^*}{H_1} \sim Sp(1)).$$

However

$$\frac{N(H_1)}{H} \sim \frac{N(H_1) \cap G^*}{H_1} \sim S^3$$

and

$$\frac{N(H)}{H} = F(H/H_1, \frac{N(H_1)}{H}).$$

Thus the finite group H/H_1 acts differentiably on S^3 with a fixed point set consisting of 120 points. Let K be the effective (factor) group of $\frac{H}{H_1}$ in this action. Then, by differentiability around fixed points, we see that K is isomorphic to a subgroup of $O(3)$. Moreover, K cannot be solvable, since any solvable group acting differentiably on S^3 must have a sphere of fixed points (of dimension $-1, 0, 1$, or 2) by the Smith theorems. Thus K must contain the icosahedral group. But then the subgroup $J \approx Z_2 \oplus Z_2$ of the orientation preserving part of K must have $F(K, S^3) \subset F(J, S^3)$ as an open subset, contrary to the Smith theorems.

§8. Conclusion of the proof of Theorem 1.4.

We have seen in Section 7 that near a point y for which L_y^1 holds, there are only two types of orbits and there is a cross-section near y contained in $F(H, M)$. Thus, near y ,

$$M - F \approx \frac{M - F}{G} \times G/H$$

and the proof of Lemma 4.4 applies to show that $H^1(G/H) \approx H^1(S^k)$.

Thus it suffices to show that L_x^1 holds. Just as in the proof of Lemma 4.5, if L_x^1 does not hold then we can find a point $z \in F(T_0, M) - F(G, M)$ such that $H^1(G_z/H) = H^1(S^l)$, with l odd, since $\text{rank}(G_z) = \text{rank}(G)$. However, $N(T_0) \subset G_z$ as shown at the beginning of Section 5, so that $\chi(G/G_z) = 1$. Hence by Lemma 2.8 we must have that $l = k$. That is, $\dim G(z) = 0$. But this implies that $z \in F(G, M)$ since G is connected and this is a contradiction.

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XVI: REMARKS ON THE SPECTRAL SEQUENCE OF A MAP

A. Borel

This is an appendix to the previous chapters, in which we have put together some remarks which belong to the theory of sheaves and of spectral sequences. Our first aim is to give to the spectral sequence of a map $f : X \longrightarrow Y$ a module structure over $H^*(Y; L)$. This is done in §1 for cohomology with compact supports. §2 is a brief discussion of families of supports and of the spectral sequence of a map, used to extend §1 to a more general case. §3 contains some obvious remarks on inductive limits, used in XIII. §4 gives a sufficient condition under which the sheaf of coefficients in the spectral sequence of a locally trivial bundle has as stalks the cohomology groups of the fibres and §5 is an application to the cohomology of certain products.

§1. Cohomology with compact supports

1.1. Let L be a ring, which from 1.2 on will be as usual a principal ideal domain, and let A (respectively B) be a L -algebra (respectively L -module). Then $A \otimes B$ has a A -algebra structure, characterized by $a \cdot (a' \otimes b) = (a \cdot a') \otimes b$. If A is a graded differential algebra and B is a differential module, then

$$(1) \quad d(a \cdot c) = da \cdot c + (-1)^i a \cdot dc \quad (a \in A^i; c \in A \otimes B) ,$$

$A \otimes B$ being endowed with the usual total differential.

Let now $\underline{F}$ (respectively $\underline{G}$) be a sheaf of L -algebras (respectively L -modules) over a space Y . Then, for every U open in Y , $(\underline{F} \otimes \underline{G})(U)$ is a $\underline{F}(U)$ -algebra, hence $\underline{F} \otimes \underline{G}$ is a $\underline{F}$ -module, and, for any family of supports σ , $\Gamma_\sigma(\underline{F} \otimes \underline{G})$ is a $\Gamma_\sigma(\underline{F})$ -module, or also a $\Gamma(\underline{F})$ -module, and we have a rule analogous to (1), under similar assumptions, to be referred to also by (1).

1.2. Let now X, Y be locally compact spaces $f : X \longrightarrow Y$ be a continuous map, K (respectively M) be a fine L -cover on Y

(respectively X) and fM the image of M under f . Let $\underline{K}$, $\underline{fM}$ be the associated sheaves and $C = r_0(\underline{K} \otimes \underline{fM})$, where r_0 denotes the sections with compact supports. Then $H(C) = H_0^*(X; L)$, $r_0 \underline{K} = K$, and $H(K) = H_0^*(Y; L)$. The spectral sequence (E_r) of f is by definition the spectral sequence of C filtered by the sub-modules $C^p = \sum_{i \geq p} r_0(\underline{K}^i \otimes \underline{fM})$. The modules defining the induced filtration in $H(C)$ are denoted by J^p , and we put as usual $J^{p,q} = H^{p+q}(C) \cap J^p$, and similarly for (E_r) .

Let A be an algebra. We say that (E_r) is an A -algebra from r_0 only if each $E_r(r \geq r_0)$ is an A -module, and the action of A commutes with d_r and with the natural projection of $d_r^{-1}(0)$ on E_{r+1} .

THEOREM. In the previous notation, $H_0^*(X; L)$ and (E_r) from $r = 2$ on are $H_0^*(Y; L)$ -modules, with the following properties:

- (1) Let $y \in H_0^1(Y; L)$. Then $y \cdot H_0^j(X; L) \subset H_0^{1+j}(X; L)$;
 $y \cdot J^{p,q} \subset J^{p+1,q}$, and $y \cdot E_r^{p,q} \subset E_r^{p+1,q} (r \geq 2)$.
 The action of y commutes with the isomorphisms
 $J^{p,q}/J^{p+1,q-1} = E_{\infty}^{p,q}$.
- (2) If f is proper, then, for $c \in H_0^*(X; L)$,
 $c \longrightarrow y \cdot c$ is the cup-product by $f_c^*(y)$.
- (3) If $\underline{H}(\underline{fM})$ is constant, isomorphic to $Y \times G$,
 where G is a free L -module, then the
 $H_0^*(Y; L)$ -module structure of $E_2 = H_0^*(Y; L) \otimes G$
 is the natural one, as described in 1.1.

1.3. **PROOF.** $Z(\cdot)$ will denote cycles. First let $k \in K^1$. Then clearly $k \cdot C^p \subset C^{p+1}$. Moreover, if $k \in Z(K^1)$, then it follows readily from (1) that multiplication by k maps $Z(C^{p,q})$, $C_r^{p,q}$, $D_r^{p,q}$ into $Z(C^{p+1,q})$, $C_r^{p+1,q}$, $D_r^{p+1,q}$ respectively, therefore $H_0^*(X; L)$ and (E_r) from $r \geq 0$ on are $Z(K)$ -modules, and the action of $Z(K^1)$ increases the filtration degree p and the total degree by 1. The term E_1 is equal to $r_0(\underline{K} \otimes \underline{H}(\underline{fM}))$, where $\underline{H}(\underline{fM})$ is the homology sheaf of $\underline{fM}$ and d_1 is defined by the differential of $\underline{K}$; also the d_0 -cycles are just $r_0(\underline{K} \otimes \underline{Z}(\underline{fM}))$. Thus the $Z(K)$ -module structure of $r_0(\underline{K} \otimes \underline{H}(\underline{fM}))$ derives from the K -module structure mentioned in 1.1; if $k = du$, ($u \in K$) and c is a d_1 -cycle, then by (1), $du \cdot c = d(u \cdot c)$ hence dK maps E_r into zero from $r = 2$ on, which means that the $Z(K)$ -module E_r is in fact a $H(K)$ -module from $r = 2$ on. By the same argument, the same is true for $H(C)$, and 1.2 (1) follows easily. 1.2 (2) follows directly by comparison of the definitions of the $H_0^*(Y; L)$ -module structure and of f_c^* [1, VII-2]. Finally, if $\underline{H}(\underline{fM}) = Y \times G$, then $r_0(\underline{K} \otimes \underline{H}(\underline{fM})) \cong K \otimes G$

[1, V, Lemme 3 (e)], and if G is torsion free, $Z(K \otimes G) = Z(K) \otimes G$, $d(K \otimes G) = dK \otimes G$, whence 1.2 (3).

1.4. Let U be an open subset of X such that $V = f(U)$ is open in Y , and let K' (respectively M') the set of elements of K (respectively M) having supports in V (respectively U). Then $\underline{K}' \otimes \underline{f}M'$ is the subsheaf of $\underline{K} \otimes \underline{f}M$ whose stalk at y is equal to that of $\underline{K} \otimes \underline{f}M$ if $y \in V$ and is zero otherwise, and $r_c(\underline{K}' \otimes \underline{f}M')$ may be identified with the sections of $\underline{K} \otimes \underline{f}M$ having compact supports in V . The inclusions $K' \longrightarrow K$ and $C' \longrightarrow C$ define then the maps j_{YV}^* , j_{XU}^* and a homomorphism (j_r^*) of the spectral sequence (E_r^1) of $f|_U$ into (E_r^1) . It is obvious from the definitions that the module structures of 1.2 for f and $f|_U$ are compatible with j_{YV}^* , j_{XU}^* , (j_r^*) .

1.5. Let now $g : S \longrightarrow T$ be a continuous map, and assume that there are proper maps $\alpha : S \longrightarrow X$, $\beta : T \longrightarrow Y$ such that $\beta \circ g = f \circ \alpha$. Then there are fine covers K', M' on T, S for which there exists maps $\beta' : K \longrightarrow K'$, $\alpha' : M \longrightarrow M'$, defining β_c^* and α_c^* respectively; the induced homomorphism $\alpha' : C \longrightarrow C' = r_c(\underline{K}' \otimes \underline{g}M')$ defines then a homomorphism (α_r^*) of (E_r^1) into the spectral sequence of g . Again, it follows directly from the definitions that the module-structures of 1.2 for f and g are compatible with α_c^* , β_c^* and (α_r^*) .

§2. The general case

In the sequel, topological spaces are not necessarily locally compact, $f : X \longrightarrow Y$ is a continuous map, ψ and ϕ are families of supports on X and Y respectively.

2.1. Cohomomorphisms of sheaves. Let $\underline{B}$ be a sheaf on Y . Its inverse image in X (with respect to f) is denoted $f^*\underline{B}$, [2, p. 120], and $\bar{f}$ is the canonical homomorphism of $f^*\underline{B}$ into $\underline{B}$. Its restriction $\bar{f}_x$ to the stalk at x is an isomorphism of $(f^*\underline{B})_x$ onto $\underline{B}_f(x)$. Given a sheaf $\underline{A}$ and a f -homomorphism $g : \underline{A} \longrightarrow \underline{B}$ there is a unique homomorphism $g' : \underline{A} \longrightarrow f^*\underline{B}$ such that $g = \bar{f} \circ g'$ (loc. cit.). f^* is an exact functor, therefore if $\underline{M}$ is a resolution of $\underline{A}$, then $f^*\underline{M}$ is a resolution of $f^*\underline{M}$. A f -cohomomorphism $h : \underline{B} \longrightarrow \underline{A}$ will be a collection of homomorphisms $h_x : \underline{B}_f(x) \longrightarrow \underline{A}_x$ such that if $y \longrightarrow s(y)$ is a continuous cross-section of $\underline{B}$ over a subset K of Y , then $x \longrightarrow h_x(s(f(x)))$ is a continuous cross-section of $\underline{A}$ over $f^{-1}K$. Therefore, if $f^{-1}(\psi) \subset \psi$, then h induces a homomorphism $r_\psi(\underline{B}) \longrightarrow r_\psi(\underline{A})$, to be denoted also by h . (This definition is introduced in [2, XIV].) The collection of maps $\bar{f}_x^{-1}$ defines a cohomomorphism $f' : \underline{B} \longrightarrow f^*\underline{B}$ and there is clearly a unique homomorphism $h' : f^*\underline{B} \longrightarrow \underline{A}$ such that $h = h' \circ f'$.

2.2. The homomorphism f^* in cohomology. Assume that $f^{-1}(\emptyset) \subset \psi$ and let $\underline{A}$ and $\underline{B}$ be sheaves on X and Y respectively. Then to every f -cohomomorphism $\underline{B} \rightarrow \underline{A}$ there is associated a homomorphism f^* or $f_{\psi, \emptyset}^*$ of $H_{\emptyset}^*(Y; \underline{B})$ into $H_{\psi}^*(X; \underline{A})$. It may be viewed as the composition of the standard homomorphism $H_{\emptyset}^*(Y; \underline{B}) \rightarrow H^*(X; f^*\underline{B})$ (see [3, Chap. II, 4.16]) and of the homomorphism $H_{\emptyset}^*(X; f^*\underline{B}) \rightarrow H_{\psi}^*(X; \underline{A})$ associated to $h' : f^*\underline{B} \rightarrow \underline{A}$, or may be defined directly as follows: Let $\underline{CB}$ be a resolution of $\underline{B}$ by $\emptyset$ -acyclic sheaves, and $\underline{CA}$ be a resolution of $\underline{A}$ by ψ -acyclic sheaves for which there is a cohomomorphism $h : \underline{CB} \rightarrow \underline{CA}$ over h . This always exists: take an injective resolution of $\underline{A}$; then since $f^*\underline{CB}$ is a resolution of $f^*\underline{B}$, the map $f^*\underline{B} \rightarrow \underline{A}$ extends to $f^*\underline{CB} \rightarrow \underline{CA}$ [3, Chap. II, 7.1], and h' is the composition of this map with the standard cohomomorphism from $\underline{CB}$ to $f^*\underline{CB}$. Then h' induces a homomorphism $r_{\emptyset, \underline{CB}} \rightarrow r_{\psi, \underline{CA}}$ and the resulting map in cohomology is $f_{\psi, \emptyset}^*$. Its uniqueness follows from standard properties of injective resolutions [3, Chap. II, 7.1].

2.3. Adapted families [4, §3.7]. We recall a notation of [3]; for a subspace A of X

$$\psi|A = \{K \in \psi, K \subset A\} \quad \psi \cap A = \{S \cap K, K \in \psi\}.$$

These are families of supports in A , and $\psi|A$ is also a family of supports in X . For each U open in Y , define $\psi(U)$ as the family of relatively closed subsets A of $f^{-1}U$ having the following property: every $y \in U$ has a neighborhood V such that the closure of $f^{-1}V \cap A$ belongs to ψ . This is a family of supports in $f^{-1}U$ containing $\psi|f^{-1}U$, and equal to $\psi|f^{-1}U$ when Y is a point. If V is open in U , $\psi(U) \cap f^{-1}V \subset \psi(V)$. If A is closed in X , $\psi(U) \cap f^{-1}A \subset (\psi|f^{-1}A)(A \cap U)$. Let $\psi' = \{K \in \psi(Y), \overline{fK} \in \emptyset\}$. The families $\psi, \emptyset$ are said to be adapted if $\psi = \psi'$, to be well adapted if moreover f is ψ -closed (i.e., $f(K)$ is closed for $K \in \psi$). The families of all closed subsets are adapted, those of compact subsets in locally compact spaces are well adapted.

LEMMA. Let A be a locally closed subset of Y and $A^* = f^{-1}A$. If $\psi, \emptyset$ are well adapted and Y is regular, then $\psi|A^*$ and $\emptyset|A$ are well adapted, with respect to $f|A^*$.

Since $A = U \cap F$ (U open, F closed) it is enough to carry the proof for A open and A closed. This is elementary and left to the reader.

2.4. Image sheaf [3, 4]. Let $\underline{M}$ be a sheaf on X . Then the

presheaf $U \longrightarrow \Gamma_{\psi}(U)(\underline{M}|f^{-1}U)$, the maps being the restrictions, is a sheaf, the image sheaf $f_{\psi}\underline{M}$ of $\underline{M}$, also denoted $f_{\psi}\underline{M}$ when ψ consists of all closed subsets of X . Since, for A closed in U , we have $\psi(U) \cap A^* \subset (\psi|A^*)(A)$, the restrictions provide a natural homomorphism

$$\lim_{V \supset A} f_{\psi}\underline{M}(V) \longrightarrow \Gamma_{(\psi|A^*)(A)}(\underline{M}|A^*)$$

LEMMA If f is ψ -closed and Y regular, then $f|f^{-1}U$ (U open in Y) is $\psi(U)$ closed, and for A closed in Y , the map $\lim_{V \supset A} f_{\psi}\underline{M}(V) \longrightarrow \Gamma_{(\psi|f^{-1}Y)(A)}(\underline{M}|f^{-1}Y)$ is injective. It is an isomorphism if moreover ψ is paracompactifying, $\underline{M}$ is ψ -soft, and A is a point.

The proof of the first assertion is left to the reader. If now $\underline{M}$ is ψ -soft and ψ is paracompactifying, then $f_{\psi}\underline{M}(Y) = \Gamma_{\psi(X)}\underline{M} \longrightarrow \Gamma_{\psi|A^*}(\underline{M}|A^*)(A^* = f^{-1}A)$, is surjective; if A is a point, then $(\psi|A^*)(A) = \psi|A^*$, whence the second assertion.

2.5. PROPOSITION. Let f be ψ -closed, ψ be paracompactifying, Y be regular, A be locally closed in Y and $A^* = f^{-1}A$. Let N be a ψ -soft sheaf on X . Then $(f_{\psi}N)|A^* = f_{\psi|A^*}(N|A^*)$.

It is enough to prove this when A is open and when A is closed. In the former case, the equality follows from the definitions (for an arbitrary sheaf $\underline{M}^*$). In the latter case, the standard homomorphism $(f_{\psi}N)|A \longrightarrow f_{\psi|A^*}(N|A^*)$ is an isomorphism since by 2.4 both sheaves have the same stalk, namely $\Gamma_{\psi|f^{-1}y}(\underline{M}^*|f^{-1}y)$, at $y \in A$.

2.6. The spectral sequence of a map. The Leray sheaf. Let ψ and ϕ be adapted families, and $\underline{M}$ be a sheaf on X . We recall that if $\underline{M}^*$ is an injective (respectively flabby) resolution of $\underline{M}$, then its restriction to an open set V is an injective (respectively flabby) resolution of $\underline{M}|V$ [4, 3.13, 3, p. 148]. If X is metrizable and $\underline{M}^*$ is soft, then $\underline{M}^*|V$ is soft [3, Chap. II, 3.4.2]. In those cases therefore $H(\Gamma_{\psi(U)}\underline{M}^*)$ is canonically identified with $H_{\psi(U)}^*(f^{-1}U; \underline{M})$ and the homology sheaf of $f|\underline{M}^*$ does not depend, up to canonical isomorphisms, on the resolution of $\underline{M}$. Assume that $f_{\psi}\underline{M}^*$ consists of ϕ -acyclic sheaves, let $Cf_{\psi}\underline{M}^*$ be a resolution of $f_{\psi}\underline{M}^*$ by ϕ -acyclic sheaves and $A = \Gamma_{\phi}Cf_{\psi}\underline{M}^*$. Then [3, p. 178], $H(A) = H(\Gamma_{\phi}f_{\psi}\underline{M}^*)$, hence $H(A) = H(\Gamma_{\psi}\underline{M}^*) = H_{\psi}^*(X; \underline{M})$. The spectral sequence of A , filtered by the grading

of $\underline{C}$, is by definition the spectral sequence (E_r, f) of f . $E_\infty = \text{GH}(X; \underline{M})$ and $E_2^{p,q} = H_\phi^p(Y; H(f_* \underline{M}^*))$. As to the assumption made on $f_* \underline{M}^*$ we recall that if a sheaf $\underline{A}$ on X is injective (flabby) and ϕ, ψ are all closed subsets, then $f_* \underline{A}$ is injective (flabby) [4, 3.7.1, 3, p. 148]. If $\underline{A}$ is flabby and ϕ paracompactifying, then $f_* \underline{A}$ is ϕ -soft [4, 3.7.1]. Thus the spectral sequence is defined at any rate in those two cases, in which we may take for $\underline{M}^*$ a flabby resolution of $\underline{M}$. Its uniqueness from E_2 on follows by standard arguments using mappings into injective resolutions. $H^*(f_* \underline{M}^*)$ will be called the Leray sheaf of f .

By definition, $H(f_* \underline{M}^*)_y = \lim H_{\psi(U)}(f^{-1}U; \underline{M})$, where U runs through the open neighborhoods of y , the maps being the restrictions (they are defined since $\psi(U) \cap f^{-1}V \subset \psi(V)$). Since $\psi(U) \cap f^{-1}y = \psi|f^{-1}y$, the restriction $H_{\psi(U)}^*(f^{-1}U; \underline{M}) \longrightarrow H_{\psi|f^{-1}y}^*(f^{-1}y; \underline{M})$ is also defined, whence a natural homomorphism

$$(1) \quad g_y^* : H(f_* \underline{M}^*)_y \longrightarrow H_{\psi|f^{-1}y}(f^{-1}y; \underline{M}) .$$

If ψ is paracompactifying and $\underline{M}^*$ is ψ -soft, then $\underline{M}^*|f^{-1}y$ is $(\psi|f^{-1}y)$ -soft, therefore g_y is induced by the map

$$g_y : (f_* \underline{M}^*)_y \longrightarrow \Gamma_{\psi|f^{-1}y}(\underline{M}^*|f^{-1}y)$$

defined by the restrictions $\Gamma_{\psi(U)} \underline{M}^* \longrightarrow \Gamma_{\psi|f^{-1}y}(\underline{M}^*|f^{-1}y)$. If f is ψ -closed and Y regular, then by 2.5 g_y is an isomorphism, hence a fortiori g_y^* is an isomorphism.

2.7. Homomorphisms. Let U be an open subset of U , $U^* = f^{-1}U$ and assume ϕ to be paracompactifying. Let us take as resolution of $f_* \underline{M}^*$ its tensor product with a torsion-free fine resolution $\underline{C}$ of the groundring (e.g., the Alexander-Spanier sheaf). Then $(\underline{C} \otimes f_* \underline{M}^*)|U = \underline{C}|U \otimes (f_* \underline{M}^*)|U$ hence by 2.5 $(\underline{C} \otimes f_* \underline{M}^*)|U = \underline{C}|U \otimes f_{\psi|U^*}(\underline{M}^*|U^*)$. Since $\Gamma_\phi((\underline{C} \otimes f_* \underline{M}^*)_U) = \Gamma_\phi|U((\underline{C} \otimes f_* \underline{M}^*)|U)$ [3, p. 188] it follows that

$$\Gamma_\phi((\underline{C} \otimes f_* \underline{M}^*)_U) \longrightarrow \Gamma_\phi(\underline{C} \otimes f_* \underline{M}^*)$$

induces a homomorphism of $(E_r, f|U^*) \longrightarrow (E_r, f)$, which on the E_2 terms is the natural map $H_{\phi|U}^*(U; H(f_* \underline{M}^*)) \longrightarrow H_\phi^*(Y; H(f_* \underline{M}^*))$.

Let $f' : X' \longrightarrow Y'$ be a continuous map, ψ', ϕ' be adapted families on X', Y' and let $g : X \longrightarrow X', \bar{g} : Y \longrightarrow Y'$, be such that $f' \circ g = \bar{g} \circ f$. Assume that $g^{-1}(\psi') \subset \psi$ and $\bar{g}^{-1}(\phi') \subset \phi$. Let $\underline{A}$ and

$\underline{B}$ be sheaves on X and X' and $h : \underline{B} \longrightarrow \underline{A}$ be a g -cohomomorphism. Then there is a natural homomorphism $g^* : (E_X, f') \longrightarrow (E_X, f)$. On E_∞ it is compatible with $g_{\psi, \psi'}^*$. On the E_2 terms, it is the map $\bar{g}_X : H_\phi^*(Y'; \underline{H}(f'_\psi \underline{B}^*)) \longrightarrow H^*(Y; \underline{H}(f_\psi \underline{A}^*))$ associated with a natural $\bar{g}$ -cohomomorphism $h' : \underline{H}(f'_\psi \underline{B}^*) \longrightarrow \underline{H}(f_\psi \underline{A}^*)$.

Let $\underline{B}^*$ and $\underline{A}^*$ be flabby (or injective) resolutions of $\underline{B}$ and $\underline{A}$ for which there is a cohomomorphism $\underline{B}^* \longrightarrow \underline{A}^*$ extending h (2.2). Since $g^{-1}(\psi')(\psi)$ it is immediately seen that for U' open in Y' , $g^{-1}(\psi'(U'))(\psi(g^{-1}(U')))$. Then it follows from the definitions that we have a natural cohomomorphism $h' : f_{\psi'} \underline{B}^* \longrightarrow f_\psi \underline{A}^*$. We then take resolutions $\underline{Cf}_{\psi'} \underline{B}^*$ and $\underline{Cf}_\psi \underline{A}^*$ for which there is a map extending h' . The corresponding map $r_\phi(\underline{Cf}_{\psi'} \underline{B}^*) \longrightarrow r_\phi(\underline{Cf}_\psi \underline{A}^*)$ then yields the desired homomorphism.

2.8. Action of the cohomology of the base. Let ϕ be para-compactifying, and $\underline{C}$ be a torsion free and ϕ -acyclic resolution of the groundring. Then $\underline{C} \otimes f\mathcal{M}^*$ is a $\underline{C}$ -module, and $r_\phi(\underline{C} \otimes f\mathcal{M}^*)$ a $r_\phi(\underline{C})$ -module. Then it is clear that the analogues, of 1.2 (1), 1.2 (3), 1.4, and 1.5 hold.

§3. Inductive limits

3.1. As is well known, exact sequences and tensor product commute with direct limits. We also omit the equally straightforward verification of the fact that direct limits and spectral sequences commute. More precisely, let $(C_\alpha, f_{\alpha\beta})$ a direct system of differential filtered modules; assume that for $\alpha < \beta$, $f_{\alpha\beta}$ commutes with the differentials and maps $F^p C_\alpha$ into $F^p C_\beta$. Let then

$$C = \text{dir lim } (C_\alpha, f_{\alpha\beta})$$

be endowed with the usual limit differential and let $F^p C = \text{dir lim } (F^p C_\alpha, f_{\alpha\beta})$. Then the $F^p C$ define a filtration of C and we have

$$\text{Gr } C = \text{dir lim } (\text{Gr } C_\alpha, f_{\alpha\beta})$$

$$\text{Gr } H(C) = \text{dir lim } (\text{Gr } H(C_\alpha), f_{\alpha\beta}^*) .$$

(Gr means graded module associated to a certain filtration; on the homology we take of course the filtration induced from the given filtration.) Also, (E_X) , $({}_\alpha E_X)$ being the spectral sequences of C and C_α respectively, we have

$$E_X = \text{dir lim } ({}_\alpha E_X, f_{\alpha\beta}^*) .$$

If there exists an integer N such that ${}_{\alpha}E_r = {}_{\alpha}E_{\infty} = \text{Gr } H^*(C_{\alpha})$ for $r \geq N$ and a cofinal set of α 's, then $E_r = E_{\infty} = \text{Gr } H(C)$ for $r \geq N$.

Similarly, assume that C_{α} is graded by submodules ${}^nC_{\alpha}$, that $f_{\alpha\beta}({}^nC_{\alpha}) \subset {}^nC_{\beta}$ and that for each n there exists an integer $p(n)$ such that ${}^nC_{\alpha} \cap F^p C_{\alpha} = 0$ for all α and $p > p(n)$. Then we have the previous equality for each total degree, that is

$${}^nE_r = {}^nE_{\infty} = \text{Gr } H^n(C) \quad (r > n(p)) .$$

A homomorphism of such a system into another one yields clearly a homomorphism of the limit spectral sequences.

3.2. Let $(g_1 : X_1 \longrightarrow Y_1, f_{1j})$ ($1, j = 1, 2, \dots$) be an inverse sequence of continuous maps. For $i < j$, there is then a commutative diagram

$$\begin{array}{ccc} X_1 & \xleftarrow{f_{1j}} & X_j \\ \downarrow g_1 & & \downarrow g_j \\ Y_1 & \xleftarrow{\quad} & Y_j \end{array} .$$

Moreover $f_{1j} \circ f_{jk} = f_{1k}$ ($1 < j < k$). Let $({}_1E_r)$ be the spectral sequence of g_1 with respect to some ring L of coefficients and families ϕ_1, ψ_1 of supports on X_1 and Y_1 respectively. These families are assumed to be such that $f_{1j}^* : ({}_1E_r) \longrightarrow ({}_jE_r)$ is defined (see 2.7). Then $({}_1E_r)$ is the spectral sequence of a certain module C_1 and, by induction, these modules may be chosen so that, for $i < j$, there exists a map $f'_{1j} : C_1 \longrightarrow C_j$ giving rise to f_{1j}^* and that $f'_{jk} \circ f'_{1j} = f'_{1k}$ for $1 < j < k$. Then $\text{dir lim } (({}_iE_r), f'_{ij})$ will be the spectral sequence of $C = \text{dir lim } (C_i, f'_{ij})$. In particular, if ${}_iE_r = \text{Gr } H^*(X_i; L)$ for $r \geq N$ and all $i \geq N_0$, then

$$(1) \quad E_r = E_{\infty} = \text{Gr } \text{dir lim } (H_{\phi_1}^*(X_i; L), f_{1j}^*) \quad (r \geq N) .$$

It is of course possible to define the limit spectral sequence of an arbitrary projective system of continuous maps. However, if the system is not equivalent to a sequence, I do not know how to define C or whether (1) is still valid.

§4. On the Leray sheaf

4.1. In §§4, 5, a space is said to have property (C) if every point has a countable fundamental system of neighborhoods. The families ψ, ϕ of supports will always be the families of all closed subsets and will not be mentioned explicitly any more. A restriction map r_{AB}^* ($A \supset B$) is said to be trivial at $b \in B$ if $\text{Ker } r_{AB}^* = \text{Ker } r_{Ab}^*$.

4.2. Let $\pi : E \longrightarrow B$ be the projection in a locally trivial fibre bundle with typical fibre F . Let $b \in B$, U be a neighborhood of b over which the bundle is trivial, $\varphi : \pi^{-1}U \longrightarrow U \times F$ be an allowable homeomorphism and $h = \text{pr}_F \circ \varphi$ be the composition of φ with the projection of $U \times F$ onto F . For every V inside U , h yields a map $\pi^{-1}V \longrightarrow F$; these maps are compatible with restrictions, hence define a homomorphism

$$h_b^* : H^*(F_b; L) \longrightarrow \text{dir lim } H^*(\pi^{-1}V; L) = \underline{F}_b \quad (F_b = \pi^{-1}b) ,$$

where $\underline{F}$ denotes the Leray sheaf of π , (which may depend on φ), such that, in the notation of 2.6

$$g_b^* \cdot h_b^* = \text{Id}.$$

In particular, g_b^* is always surjective. It is injective if E is locally paracompact and if the subspaces $\pi^{-1}V$ form a fundamental system of neighborhoods of F_b [3, p. 202, remarque 4.17.1]; the latter condition is fulfilled when F is compact (without any assumption on B).

We also note that if g_b^* is an isomorphism for all $b \in B$, the bundle is trivial and B is locally connected, then the Leray sheaf is constant, isomorphic to $B \times H^*(F; L)$. It is enough to show this when B is connected. Let then E be identified with $B \times F$, and i_b be the inclusion map $f \longrightarrow b \times F$. As usual, it is enough to show that i_b^* is independent of b . Let $c \in H^*(E; L)$ and $d \in H^*(F; L)$ such that $i_b^*(c) = d$ for some $b \in B$. Then $i_b^*(c) = i_b^* \cdot h^*(d)$ hence $i_b^*(c - h^*(d)) = 0$. Since g_b^* is an isomorphism there exists a neighborhood U of b such that $r_{E, U \times F}^*$ annihilates $c - h^*(d)$. As a consequence, the points of B for which $i_b^*(c)$ has a given value is open, and our assertion follows. In other words, under the present assumptions, we have the homotopy axiom in the form of [5, §67].

We see also that if we assign to $b \times d$ ($b \in B, d \in H^*(F; L)$) the value at b of the section of the Leray sheaf defined by $h^*(d)$, then we get an isomorphism of $B \times H^*(F; L)$ onto the Leray sheaf.

4.3. THEOREM. Let (E, B, F, π) be a locally trivial fibre bundle, and F be the Leray sheaf of π . Then the map $g_b : E_b \longrightarrow H^*(F_b; L)$ ($b \in B$) of 2.6 is an isomorphism if one of the two following sets of conditions is fulfilled:

- (a) B is locally compact, clc_L , satisfies condition (C) of 4.1; F is locally connected; either F is locally compact or E is paracompact.
- (b) E is paracompact, clc_L and satisfies (C); F is locally compact.

We first assume (a). There exists therefore a decreasing sequence (U_i) ($i \geq 1$) of compact connected neighborhoods of b , whose intersection is b , over which the bundle is trivial, and such that the restriction maps $r_{i,i+1}^* : H^*(U_i; L) \longrightarrow H^*(U_{i+1}; L)$ are trivial. Let $\varphi : \pi^{-1}U_1 \longrightarrow U_1 \times F$ be an allowable homeomorphism, $\sigma_1 = \text{pr}_F \circ \varphi|_{\pi^{-1}U_1}$. Let $({}_1E_r^i)$ be the spectral sequence of σ_1 and $(E_r^i) = \text{dir lim } ({}_1E_r^i)$, the limit being taken with respect to the restriction maps. It will be enough to show that the map g_b^* of 4.2 is an isomorphism.

Under our assumptions, $\pi^{-1}U_1$ is locally paracompact, and F is locally connected. Therefore, by 4.2,

$$(1) \quad {}_1E_2^{s,t} = H^s(F; H^t(U_1; L))$$

and the map $r_{i,i+1}^* : {}_1E_2^{s,t} \longrightarrow {}_{i+1}E_2^{s,t}$ is induced by the coefficient map $H^t(U_i; L) \longrightarrow H^t(U_{i+1}; L)$. The latter being trivial, we have

$$\text{dir lim } {}_1E_2^{s,t} = 0 \quad (t \neq 0)$$

$$\text{dir lim } {}_1E_2^{s,0} = H^s(F; L) \quad .$$

This and 3.1 yield:

$$(2) \quad \text{dir lim } H^s(\pi^{-1}U_i; L) = E_\infty^{s,0} = E_2^{s,0} = H^s(F; L) \quad .$$

Since the isomorphism $H^s(F; L) \longrightarrow {}_1E_\infty^{s,0}$ is induced by the projection, the isomorphism between the extreme terms of (2) is indeed defined by h_b , and this proves our contention.

Let us now assume (b). Then both F and B are clc over L , and satisfy (C). Let $b \in B$. There exists a decreasing sequence of closed

connected neighborhoods (U_i) of b , $(i \geq 0)$, whose intersection is b , such that the restriction maps $r_{ij}^* : H^*(U_i; L) \longrightarrow H^*(U_j; L)$ are trivial $(i < j)$. Again, we assume the bundle to be trivial over U_0 and use the above notation. The preceding proof and 4.2 show that the Leray sheaf of $\text{pr}_F \circ \varphi : \pi^{-1}U_1 \longrightarrow F$ is constant, with stalks isomorphic to $H^*(U_1; L)$. Therefore we have again (1). From there on the proof is the same as in the previous case.

4.4 COROLLARY. Let (E, B, F, π) be a locally trivial bundle. Assume that $H^0(F; L) = L$, $H^1(F; L) = 0$ $(i \neq 0)$, and that one of the conditions (a), (b) of 4.3 is fulfilled. Then $\pi^* : H^*(B; L) \longrightarrow H^*(E; L)$ is an isomorphism.

4.5. THEOREM. Let T, X be topological spaces, with T connected. Let i_t be the map $x \longrightarrow t \times x$ of X in $T \times X$. Then $i_t^* : H^*(T \times X; L) \longrightarrow H^*(X; L)$ is independent of t if one of the two following sets of conditions is fulfilled.

- (a) T is locally compact, satisfies (C) of 4.1, and is clc_L ; either X is locally compact or $T \times X$ is paracompact.
- (b) $T \times X$ is paracompact, satisfies (C) and is clc_L ; X is locally compact.

§5. A case of validity for the Künneth rule

5.1. THEOREM. Let X be a compact metric space, of finite covering dimension, which is clc_L and Y be a space which either is locally compact or is such that $X \times Y$ is paracompact. Then $H^*(X \times Y; L)$ is given by the Künneth rule. Otherwise stated, there is an exact sequence

$$0 \longrightarrow \sum_a H^a(X; L) \otimes H^{1-a}(Y; L) \longrightarrow H^1(X \times Y; L) \longrightarrow \\ \longrightarrow \sum_a \text{Tor}(H^{a+1}(X; L), H^{1-a}(Y; L)) \longrightarrow 0 .$$

Let $\underline{N}$ be a fundamental sheaf on Y , for instance the canonical resolution of L , and $\underline{M}$ a fundamental sheaf on X which is torsion free and has degree bounded from above (for instance as constructed in [5, No. 41]). Furthermore, let $\underline{P}$ be a fundamental sheaf on $X \times Y$ for which

there exists a cohomomorphism $\underline{N} \longrightarrow \underline{P}$ inducing the homomorphism $H^*(Y; L) \longrightarrow H^*(X \times Y; L)$ associated with the projection on Y , (see 2.2). Let π be the projection of $X \times Y$ on X and $A = \Gamma(\underline{M} \otimes \pi \underline{P})$. Then, as was recalled in 2.6, $H^*(A) = H^*(X \times Y; L)$ and the spectral sequence of A , filtered by the degree of $\underline{M}$, is the spectral sequence (E_r) of π . In view of 4.3

$$E_2 = H^*(X; H^*(Y; L)) .$$

By definition of the image sheaf, there is a homomorphism $\Gamma(\underline{P}) \longrightarrow \Gamma(\pi \underline{P})$; composed with $\Gamma(\underline{N}) \longrightarrow \Gamma(\underline{P})$, it yields a homomorphism $\alpha : \Gamma(\underline{N}) \longrightarrow \Gamma(\pi \underline{P})$, whence a homomorphism $\alpha_x : H^*(\Gamma(\underline{N})) \longrightarrow \underline{F}_x$ ($x \in X$) where $\underline{F}$ is the Leray sheaf of π . It follows from 4.2, 4.3 that α_x is an isomorphism and that $\alpha : x \times c \longrightarrow \alpha_x(c)$ is an isomorphism of $X \times H^*(\Gamma(\underline{N}))$ onto $\underline{F}$.

Let now $B = \Gamma(\underline{M}) \otimes \Gamma(\underline{N})$. Let $k : B \longrightarrow A$ be defined as the composition $B \longrightarrow \Gamma(\underline{M}) \otimes \Gamma(\pi \underline{P}) \longrightarrow \Gamma(\underline{M} \otimes \pi \underline{P})$ where the first homomorphism is the product of the identity and of α and the second one is the obvious one. It defines a homomorphism (k_r^*) of the spectral sequence (E_r') of B , filtered by the degree of $\underline{M}$ into (E_r) . By standard facts

$$E_2' = H^*(\Gamma(\underline{M}) \otimes \underline{H}(\Gamma(\underline{N})) = H^*(X; H^*(Y; L)) .$$

Moreover it follows from the previous remarks and the definitions that k_2^* is associated with the sheaf isomorphism $X \times H^*(Y; L) \longrightarrow \underline{F}$, hence, is an isomorphism. Since the degree of $\underline{M}$ is bounded from above, it follows that $k^* : H^*(B) \longrightarrow H^*(A)$ is an isomorphism, and this proves the theorem.

REMARK. The theorem is also valid if the assumptions on X are replaced by: X is compact, clc_L of finite cohomological dimension over L , and satisfies (C). The proof is the same except that one takes for $\underline{M}$ the resolution (1) of [3, p. 195], noting that it is also torsion free, as is easily seen from its definition. Examples show that an assumption of the type "compact clc_L " on X is essential. We do not know whether the condition (C) and finite dimensionality can be dispensed with. The latter assumption was actually only used to insure regular convergence of (E_r) and (E_r') .

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