ON GENERALIZED SURFACES OF FINITE TOPOLOGICAL TYPES [1]

L. C. YOUNG

1. Background and notation. In many branches of mathematics ideal elements are introduced with the object of ensuring that certain natural mathematical principles shall have universal validity. At first sight the ideal elements may perhaps appear somewhat far-removed from the more intuitive concepts which they supplement, but they are necessary for a true understanding of these concepts. In the Calculus of Variations, these ideal elements play a part which is similar to that of complex numbers in Algebra or points at infinity in Geometry: we call them generalized curves, generalized surfaces, generalized hypersurfaces. With their help, existence theorems such as the one obtained recently (and rather suddenly perhaps, after a fifteen year lull) by several mathematicians [Danskin 2, Sigalov 10, Cesari 1a] become automatic: a very very general result of this kind is given below (9.2).

There is one radical difference between the notion of generalized surface that we develop and all previous generalizations of elementary surfaces: it makes no attempt to copy the familiar features that our intuition attaches to an elementary surface, but prefers to ignore these features and to retain only the essential properties actually used in formulating problems in Analysis and in the Calculus of Variations. Yet this radical difference merely exists in order to achieve, more fully, exactly the same acknowledged aim: to attain a better understanding of surfaces *in the most elementary sense* and of the laws and principles by which they are governed.

In the present paper, and in its predecessor [Young 18], we have, perhaps, not been quite radical enough as regards the *boundary* of a surface: a more satisfying treatment of this last notion and of its implications is given in [Fleming and Young 20]; unfortunately it seems applicable only in the oriented case. We present here, not a theory which is complete or final, but one which includes all the existence theorems that workers in the field have been striving for. Naturally, we modify as little as possible their assumptions. In particular, in contrast to [Young 19], we restrict ourselves to admissible surfaces with a bounded Euler characteristic. Finally we emphasize that the surfaces admitted are situated in Euclidean spaces of unrestricted dimensions, in which the basic algebraic and topological facts (affecting the Calculus of Variations as well as other topics such as functions of

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several complex variables) are by no means cleared up. Our ignorance of these facts has forced us sometimes to adopt devious routes (and so to add to the length of this paper) but it has not materially influenced the generality of our results. We draw attention below (§ 10) to seemingly simple questions which correspond, in the classical theory, to very real difficulties.[2]

We are concerned with a generalization of the notion of surface in the *parametric* form [for the non-parametric form see Young 12; for the case of curves, see Young 11, McShane 4 d]. If one thinks of such a surface as somehow represented by a function $x(u, v)$, whose values lie in Euclidean m-space X while (u, v) ranges in the unit square $R(0 \leq u \leq 1,\ 0 \leq v \leq 1)$, the question arises as to when *two* such $x(u, v)$ are to be regarded as defining a *same* surface. To this question, conflicting answers have been given and every such answer gives rise to a different notion of parametric surface: thus, a polyhedron, as defined in combinatory topology, has representations $x(u, v)$ which are not equivalent in the sense of (say) Fréchet. The key to the present theory lies precisely in a new specification of the equivalence of two $x(u, v)$, a specification which is closest to that of combinatory topology.

We require some preliminary definitions. We say that $x(u, v)$ is a generalized Dirichlet representation, if it is defined and bounded for (u, v) in R and takes values in X, if it is absolutely continuous on the intersection of R with almost every line $u = $ const. or $v = $ const., if its restriction to the perimeter of R is continuous, and if its Dirichlet integral

$$I(x, R) = \frac{1}{2} \iint_R (x_u^2 + x_v^2)\, du dv$$

is finite. In the sequel, all the representations which occur, will, unless the contrary is expressly stated, be generalized Dirichlet representations, and we denote the class of these by D. Let us emphasize that *we do not restrict ourselves to continuous representations* except where we say so expressly. (In classical analysis, stress is laid on the continuity of the representations $x(u, v)$ which are admitted; we have dropped this requirement and replaced it by others, which also are satisfied by every elementary surface—whatever meaning we may agree to assign to this term—and which have been chosen for reasons explained elsewhere[3] [Young 17 b]; actually the removal of the continuity requirement strengthens, not

2. This is one of a number of instances in which the classical theory is confronted with the same technical difficulties as our theory. Thus, in establishing their existence theorem, Sigalov and Cesari use and refine the techniques that were planned [Young 15, 17 a] to build the present theory. Similarly some of the techniques developed here, notably in (3.2), (2.3), and in §6, have applications quite independent of our theory.

3. See also §2 below.

only our methods, but also our results, for we shall see below (9.2) and (9.9) that the solutions of acceptable variational problems are still continuous.)

We shall use the symbol x both for a point of Euclidean m-space and for a generalized Dirichlet representation $x(u, v)$. To avoid confusion, we write in the former case $x \in X$ and in the latter $x \in D$. It is convenient to write further $x \in D(N)$ to mean that $x \in D$ and that $I(x, R) < N$. A similar notation will be used in the case of the Jacobian $j(u, v)$ of $x(u, v)$, which is defined as the vector product of the partial derivatives x_u and x_v and exists almost everywhere in R. We write $(x, j) \in D$ to mean that $x \in D$ and that j is the Jacobian of x.

The values of any $j(u, v)$ are bi-vectors. They lie in the space J whose elements can be identified with the skew-symmetric matrices $j = (j_{rs})\ (r, s = 1, 2, \cdots, m)$ of rank 2 or 0. We define the norm $|j|$ for $j \in J$ as the square root of Σj_{rs}^2 (summed for $1 \leq r < s \leq m$). We denote by J_1 the subset of J consisting of the j of norm 1, and by J^* the superspace obtained from J by deleting the condition that j be of rank 2 or 0. We write further (E, J), (E, J_1), (E, J^*) for the cartesian product of a subset E of X with J, J_1 or J^*. We note the following properties of J:

(a) Any two $j \in J$ whose sum belongs to J are expressible as the vector products of two points of X with a same point of X.

(b) Given a finite number of $j \in J$ whose sum is zero, there exist a finite number of $j' \in J$, such that, writing $j'' = -j'$, the sum $\Sigma j + j' + j''$ can be rearranged in blocks of not more than 3 terms, so that the sum of the terms in each block is zero.

(c) Given two bi-vectors j_1, j_2 there exists a bi-vector j such that $j_1 - j$ and $j - j_2$ are bi-vectors and that

$$|j_1 - j_2|^2 = |j_1 - j|^2 + |j - j_2|^2.$$

Here (a) and (c) are easily established by suitable choice of the coordinate axes of the space X. Moreover (b) is trivial both in the case in which the j all lie in a same coordinate plane, and in the case in which they consist of a given bi-vector and the negatives $-j_{rs}$ of its components j_{rs}. To deal with the general case, it is now sufficient to reduce it to these two trivial cases, by including among the j' the negatives of all the components of each j.

We write F for the space of continuous functions $f = f(x, j)$, $(x, j) \in (X, J_1)$, with the norm[4] $|f| = \sup |f(x, j)|$ for $|x| \leq B$ where B is a sufficiently large

4. Or, more properly, pseudo-norm. The topology is the same as that derived from the norm $\Sigma 2^{-n} Q^n(f)/\{1 + Q^n(f)\}$ where $Q^n(f) = \mathrm{Sup}\,|f(x, j)|$ for $|x| \leq n$. See [12] p. 87 for an analogous situation.

constant depending on the context. The extension of an $f \in F$ to $(x, j) \in (X, J)$ by the homogeneity relation $f(x, \rho j) = \rho f(x, j)$ for $\rho \geq 0$ will be denoted by the same symbol and termed parametric integrand or simply *integrand.* The values taken by each f will be real numbers or else points of some Euclidean space. We term the integrand f *symmetric* if $f(x, j) = f(x, -j)$, *real* if it takes real values, *non-negative* if it takes non-negative values, for each $(x, j) \in (X, J)$; whenever in the sequel we speak of an integrand in connection with a *non-oriented* surface, it will be understood that the integrand is symmetric.

Corresponding to any $(x, j) \in D$, we form for each integrand $f \in F$ the classical surface integral

$$L(f) = \iint_R f[x(u, v),\ j(u, v)]\, du\, dv. \tag{1.1}$$

We remark that the various definitions of equivalence of parametric representations, which normally occur in Analysis, have the following property: if two $x \in D$ are acceptable representations according to the definition in question—in particular are continuous—and represent the same oriented (or non-oriented) parametric surface, then, for each integrand f (or each symmetric f), they give the same value to $L(f)$. It is this property (which need only be extablished for a dense subset of the relevant $f \in F$) that *we take for our definition.* In the case of Fréchet equivalence of surface representations whose values lie in Euclidean 3-space, this property is a special case of very general results, obtained for instance by Reichelderfer [8, (17) p. 283 and (20) p. 284]. We say that two $x \in D$ define the same oriented parametric surface, if and only if, for each $f \in F$, they give the same value to $L(f)$; and that they define the same non-oriented parametric surface, if and only if, for each symmetric $f \in F$, they give the same value to $L(f)$.

An oriented or non-oriented parametric surface may now be defined as a maximal class consisting of $x \in D$, all of which give the same value to the functional $L(f)$ of (1.1), either for each integrand f, or for each symmetric integrand. But since the functional determines the class and vice-versa, it is simpler to say that the functional, defined for all integrands or for all symmetric integrands, *is* our oriented or non-oriented parametric surface, and that an $x \in D$ for which (1.1) holds for every integrand or for every symmetric integrand is a parametric representation of $L(f)$.

More generally, a continuous linear functional $L(f)$ will be termed a *generalized oriented, or non-oriented, surface,* if it is defined for $f \in F$, or for symmetric $f \in F$, and its value is non-negative whenever such an f is non-negative.

The generalized surface $L(f)$ is termed *limit* (or sometimes: weak limit) of the sequence of generalized surfaces $L_n(f)$, $n = 1, 2, \cdots$, or in particular, of parametric surfaces, if for each relevant integrand f (each $f \in F$ in the oriented case, each symmetric f in the non-oriented case) the values $L_n(f)$ converge (weakly) to $L(f)$. It is easy to see that every generalized surface is expressible as the limit of a sequence of uniformly bounded parametric surfaces.

The notion of limit can be associated with a distance termed *McShane distance* [Young 18 (3.3) p. 62] provided that there is a sphere $|x| \leq B$ of the space X such that all the $L_n(f)$ and $L(f)$ concerned vanish when $f(x, j)$ is zero for all x in this sphere. The McShane distance of two such generalized surfaces $L_1(f)$ and $L_2(f)$ is the maximum modulus of $L_1(f) - L_2(f)$ for the class of functions $f(x, j)$ with Lipschitz constant ≤ 1 and with maximum modulus ≤ 1 when x is restricted to the sphere $|x| \leq B$ and j to J_1.

We reserve however the term *distance* for an independent (much larger) quantity: we term distance (strong distance) of the generalized surfaces $L(f)$ and $L'(f)$, the norm of the linear functional $L - L'$, i.e., the expression

$$\operatorname{Sup} |L(f) - L'(f)|$$

where the supremum is for all $f \in F$ whose norm is ≤ 1, or all such symmetric f in the non-oriented case. The generalized surface $L(f)$ is termed *strong limit* of the sequence of generalized surfaces $L_n(f)$, $n = 1, 2, \cdots$, if the distance of $L(f)$ and $L_n(f)$ tends to 0 as $n \to \infty$, and the carriers of the $L_n(f)$ are uniformly bounded. The distance of two surfaces $L(f)$ and $L'(f)$ reduces to the norm of L in the case $L'(f) = 0$ and its value is then $L(f_0)$ where $f_0(x, j) = 1$ for $(x, j) \in (X, J_1)$; we term this value the *area* of the generalized surface $L(f)$. In the oriented case we define also the *oriented area* of $L(f)$: we mean by it the value in J^* taken by $L(f)$ when $f(x, j) = j$.

One consequence of our definitions is that *addition* of parametric or generalized surfaces now simply means addition of the corresponding linear functionals, an algebraic instead of a somewhat constricted geometrical operation. It is important to note, and easy to verify, that the sum of two oriented, or non-oriented, parametric surfaces is itself an oriented, or non-oriented, parametric surface. This shows incidentally that the notion of parametric surface would not be widened if we allowed (u, v) to range on a two-dimensional manifold instead of a square.

On the other hand, the intuitive process of sticking together two surfaces "along a common edge" will be replaced here by a much more complicated "sewing process" (3.2).

We shall require a few more definitions. We say that the parametric, or generalized, surface $L(f)$, oriented or non-oriented, is *of the type of the sphere*, if

$L(f) = \lim L_n(f)$ where $L_n(f)$ is a parametric surface with at least one representation $x_n \in D$ such that the restriction of x_n to the perimeter of R is constant. (We have altered the terminology of Young [18], where such an $L(f)$ was termed "closed".) Similarly, we say that $L(f)$ is *of the type of a disc*, if $L(f) = \lim L_n(f)$, where $L_n(f)$ is a parametric surface with at least one representation $x_n \in D$ whose restriction to the perimeter of R describes a fixed simple closed oriented Fréchet curve C independent of n, termed *rim* of the disc.

We term piecewise linear a continuous representation $x \in D$ which is linear in each of the parts into which R is subdivided by some finite triangulation. A parametric surface with a piecewise linear representation is termed a *polyhedron* and the latter will be said to be a *polyhedral sphere*, or a *polyhedral disc*, if it is of the type of a sphere, or of the type of a disc. A polyhedral disc will be termed *flat-rimmed* if its rim is situated in a plane of the space X.

2. Approximation to a micro-representation. We term *carrier* of a generalized surface $L(f)$ the smallest closed set of points $E \subset X$ such that $L(f)$ assumes the value zero whenever the restriction of its integrand $f(x, j)$ to $(x, j) \in (E, J)$ is zero. A generalized surface whose carrier consists of a single point is termed *micro-surface*.

Given a generalized surface $L(f)$, or in particular a micro-surface, and given a point $x_0 \in X$, the micro-surface $M(f)$ defined by writing $M(f) = L(f_0)$ where $f_0(x, j) = f(x_0, j)$, is said to be obtained by *concentrating* $L(f)$ at the point x_0; when x_0 is the origin, we term $M(f)$ the *shape* of $L(f)$; in the case in which $L(f)$ is a micro-surface, we say also that $M(f)$ is obtained by *translating* $L(f)$ to the point x_0. In the case in which $L(f)$ is a polyhedron, or a polyhedral sphere, or a flat-rimmed polyhedral disc, and is moreover *oriented*, then the micro-surface $M(f)$ obtained by concentrating it at some point $x_0 \in X$, will be termed a *micro-polyhedron*, or a *polyhedral micro-sphere*, or a *polyhedral micro-disc*. The limit of a sequence of polyhedral micro-spheres, or of a sequence of polyhedral micro-discs, will be termed a *micro-sphere*, or a *micro-disc*. We observe that a micro-sphere is at the same time a micro-disc (but not vice-versa), since a polyhedral sphere has an arbitrarily small distance from a flat-rimmed polyhedral disc. We observe further that a micro-sphere, or micro-disc, $M(f)$, is expressible as the limit of a polyhedral micro-sphere or flat-rimmed micro-disc $M_n(f)$ with the same carrier (consisting of a point $x_0 \in X$); it is important to observe, in the case of a micro-disc, that we can stipulate further that $M_n(f)$ possess the same oriented area j_0 as $M(f)$, and that the shape of $M_n(f)$ be that of a flat-rimmed polyhedral disc $L_n(f)$ whose rim is the oriented perimeter of an *arbitrarily given* parallelogram π of oriented area j_0: to this effect it is sufficient to modify $M_n(f)$ by the following construction.

By making use of property (c) of the space J, we first construct a parallelogram π_n, whose oriented area equals the oriented area j_n of $M_n(f)$, and whose corners are close to those of π. This is possible since $j_n - j_0$ is small for large n. We can then add to π_n not more than eight triangular faces of small area, so as to form a flat-rimmed polyhedral disc $L'_n(f)$ whose rim is the same as that of π. We now cover, by Vitali's theorem, an arbitrarily large proportion of π_n with plane figures T whose rims are polygons derived from the rim of our original $L_n(f)$ by magnification and translation. By fitting onto these rims, flat-rimmed polyhedral discs derived from $L_n(f)$ itself by similar magnification and translation, and substituting them for the corresponding figures T, we derive from $L'_n(f)$ a flat-rimmed polyhedral disc $L^*_n(f)$ whose rim is that of π and whose concentration at x_0 is a polyhedral micro-disc $M^*_n(f)$ at a small distance ρ_n from $M_n(f)$. If we arrange for ρ_n to tend to zero as $n \to \infty$, the micro-disc $M^*_n(f)$ is the desired modification of $M_n(f)$.

A micro-disc $M_{u,v}(f)$, which depends on the parameters u and v, and is defined for almost every $(u, v) \in R$, will be termed parametric micro-representation, or simply, *micro-representation,* if its carrier is the value at (u, v) of an $x \in D$ whose Jacobian $j(u, v)$ equals the oriented area of $M_{u,v}(f)$ (condition of tangency) and if its value for each f constitutes a function of (u, v) integrable in R (condition of integrability). A function $x \in D$ which coincides at almost every (u, v) of R with the carrier of the micro-disc $M_{u,v}(f)$, will be termed *the carrier at* (u, v) of the given micro-representation. Two carriers $x \in D$ of a same micro-representation agree almost everywhere in R and also at every point of the perimeter, and we shall not distinguish between them.

The generalized, oriented or non-oriented, surface $L(f)$ will be said to possess a micro-representation $M_{u,v}(f) \in D^*$, if for all relevant integrands f we have

$$L(f) = \iint_R M_{u,v}(f)\,du\,dv. \tag{2.1}$$

We observe that (2.1) constitutes a generalization of (1.1): parametric surfaces constitute a subclass of the generalized surfaces which possess micro-representations. In (2.2) below, we state that in the class of generalized surfaces which possess micro-representations whose carriers $x(u, v)$ have a fixed restriction to the perimeter of R, the subclass consisting of parametric surfaces with the same property is dense.

We write Pol, Pol-int for the classes of continuous $x(u, v) \in D$ which are, respectively, piecewise linear in R and piecewise linear in the interior of R, i.e., in every smaller concentric square. We write Lip, Lip-int, and quasi-Lip, for the classes of continuous $x(u, v) \in D$ which are, respectively, Lipschitzian in R, Lipschitzian in the interior of R, i.e., in every smaller concentric square, and quasi-Lipschitzian: we term quasi-Lipschitzian a continuous $x(u, v) \in D$ such that the interior of R is the sum of two sequences A_n and B_n, where each A_n is a closed set on which $x(u, v)$ is Lipschitzian and each B_n is a segment parallel to one of the axes on which the restriction of $x(u, v)$ is an absolutely continuous function of one variable. All such representations are special in the sense of [17 (b) §3], i.e., their classical areas coincide with their intrinsic areas on the interior of R; actually the special representations used in [17(b)] can be replaced by quasi-Lipschitzian representations without affecting the proofs. (Special representations are introduced mainly because some mathematicians do not consider continuous $x(u, v) \in D$ admissible without a further restriction: the following pair of theorems, applied in the case of a parametric surface $L(f)$, suggest that even a restriction to continuous $x(u, v) \in D$ is unnatural.)

(2.2) *Let the generalized surface* $L(f)$ *possess a micro-representation* $M_{u,v}(f) \in D^*$ *whose carrier is* $x(u, v) \in D$. *Then* $L(f)$ *is the limit of a sequence of parametric surfaces* $L_n(f)$, *such that each* $L_n(f)$ *has a continuous special representation* $x_n(u, v) \in$ Pol-int *whose restriction to the perimeter of* R *coincides with the corresponding restriction of* $x(u, v)$. *Moreover if this restriction is Lipschitzian, then* $x_n(u, v) \in$ Lip, *if it is piecewise linear then* $x_n(u, v) \in$ Pol.

(2.3) *Let the generalized surface* $L(f)$ *possess a micro-representation* $M_{u,v}(f) \in D^*$ *whose carrier is* $x(u, v) \in D$. *Then, given* $\epsilon > 0$, $L(f)$ *is distant* $< \epsilon$ *from a generalized surface* $L_\epsilon(f)$ *which possesses a micro-representation* $\in D^*$ *with a carrier* $x_\epsilon(u, v) \in$ Lip-int *whose restriction to the perimeter of* R *coincides with that of* $x(u, v)$. *Moreover, if this restriction is Lipschitzian then* $x_\epsilon(u, v) \in$ Lip; *and if* $L(f)$ *is a parametric surface then* $L_\epsilon(f)$ *is a parametric surface with the representation* $x_\epsilon(u, v)$.[5]

NOTE. In the above statements, we can further arrange that the diameter of the approximating surface $L_n(f)$ or $L_\epsilon(f)$ exceeds by at most $1/n$, or by at most ϵ, that of $L(f)$.

The proofs will be in several stages. We establish (2.3) first as this will enable us to suppose $x(u, v) \in$ Lip-int in (2.2). Part of the proof of (2.2) will be similar to that of a corresponding statement for Lipschitzian non-parametric surfaces [Young 12 (8.1) p. 93].

PROOF OF (2.3). (i) First suppose that $x(u, v)$ has a Lipschitzian restriction to the perimeter. We determine $\delta > 0$ where $\delta < \epsilon$, so that the integral of the

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5. As pointed out by a referee, the corresponding result, with distance replaced by McShane distance, would be much easier to prove.

area of $M_{u,v}(f)$ over any set of (u, v) of measure $< \delta$, and the Dirichlet integral of $x(u, v)$ over such a set, are $< \epsilon$. We determine further [cf. 17 (a) (15.3) p. 329] an open set $W \subset R$ of measure $< \delta$ such that $x(u, v)$ is Lipschitzian in $R-W$, and we denote by k a number $> 1/\delta$ which exceeds the Lipschitz constant of $x(u, v)$ in $R-W$. Since $R-W$ has density one at almost all its points, we can construct, by Vitali's theorem, a finite sum of disjoint squares $Q_\nu \subset R$ whose complement has measure $< \delta$, such that W occupies a proportion $< k^{-3}$ of the measure of each Q_ν. We denote by I_ν the Dirichlet integral of $x(u, v)$ over Q_ν, and by Π_ν the perimeter of a square R_ν, concentric to Q_ν and of side r_ν (say), such that the linear measure of $W\Pi_\nu$ is $< 4r_\nu/k^2$ and that the line integral of $x_u^2 + x_v^2$ on Π_ν is $< 2kI_\nu/r_\nu$. For each of these two inequalities, the perimeters of squares, concentric and parallel to Q_ν, which violate it, can occupy only a proportion $< k^{-1}$ of the measure of Q_ν. We can therefore select Π_ν so that R_ν occupies a proportion $> 1 - 2k^{-1} > 1 - 2\delta$ of the measure of Q_ν. The measure of $R - \Sigma R_\nu$ is then $< 3\delta$.

We can now divide each Π_ν into a finite number of arcs $\Pi_{\mu\nu}$, each of length $< 4r_\nu k^{-2}$, whose extremities do not belong to W. Hence if $x_\nu(u, v)$ is a Lipschitzian continuation into R_ν, in accordance with Mickle [5] [6], with Lipschitz constant $< k$, of the restriction of $x(u, v)$ to $R_\nu - W$, and if $l_{\mu\nu}$, $l^*_{\mu\nu}$ are the lengths of the two curves described for $(u, v) \in \Pi_{\mu\nu}$, by $x(u, v)$ and by $x_\nu(u, v)$ respectively, we have, on the one hand by Schwarz's inequality

$$\Sigma_\mu (l_{\mu\nu})^2 < (4r_\nu k^{-2})(2kI_\nu/r_\nu) = 8I_\nu k^{-1} < 8\epsilon I_\nu,$$

and on the other hand

$$\Sigma_\mu (l^*_{\mu\nu})^2 \leq (\Sigma_\mu l^*_{\mu\nu})^2 < (4r_\nu k^{-1})^2 < 16\epsilon^2 r_\nu^2 < 16\epsilon r_\nu^2,$$

whence

$$\Sigma_{\mu,\nu} (l_{\mu\nu})^2 + (l^*_{\mu\nu})^2 < 24\epsilon,$$

an equality that we shall need for a "sewing up" process.

We denote by T the mapping of R_ν onto a parallel concentric square of half the side, obtained by halving the scale, and we write $H_{\mu\nu}$ for the subdomain of the annulus $R_\nu - TR_\nu$ bounded by $\Pi_{\mu\nu}$, $T\Pi_{\mu\nu}$ and two radial sides, i.e., two segments situated on half-lines issuing from the centre of R_ν. Further, we divide $R - \Sigma R_\nu$ into a finite number of simply connected domains G_λ by parallels to the axes on which the restriction of $x(u, v)$ is absolutely continuous. We define a vec-

6. If the continuation is in accordance with McShane [Young 12 (9.2) p. 93] we need only replace k by a constant multiple.

tor-function $x^*(u,v)$ in R by equating it to $x(u,v)$ on the boundaries of the G_λ and to $x\{T^{-1}(u,v)\}$ in each TR_ν, and then by interpolating linearly on each radial side of the $H_{\mu\nu}$ and harmonically in the interior of each $H_{\mu\nu}$ and G_λ. Since the extremity on Π_ν of a radial side does not belong to W, and since T transforms it into the extremity on $T\Pi_\nu$, we see that $x^*(u,v)$ takes the same value at both extremities and is thus constant on each radial side of the $H_{\mu\nu}$. Since $x_u^{*2}+x_v^{*2}$ has a finite line integral on the sides of $H_{\mu\nu}$, it is easy to construct a function of the class D whose restriction to the sides of $H_{\mu\nu}$ coincides with $x^*(u,v)$, and from this it follows, by the minimum property of the harmonic function that $x^*(u,v)$ has a finite Dirichlet integral in each $H_{\mu\nu}$. Since its Dirichlet integral in ΣG_λ is similarly $< 3\epsilon$ it follows that $x^*(u,v) \in D$. We define further a micro-disc $M^*_{u,v}(f)$ equal to $4M_{u',v'}(f)$ in each of the sets $T(R_\nu - W)$, where $(u',v') = T^{-1}(u,v)$ and equal elsewhere in R to $f\{x^*(u,v), j^*(u,v)\}$ where $j^*(u,v)$ is the Jacobian of $x^*(u,v)$.

On $H_{\mu\nu}$ the function $x^*(u,v)$ defines a harmonic surface whose area, according to the isoperimetric relation [Young 16], is $<(l_{\mu\nu} + l^*_{\mu\nu})^2/4 < \{(l_{\mu\nu})^2 + (l^*_{\mu\nu})^2\}/2$. The sum of all such harmonic areas is thus $< 12\epsilon$. Moreover the area integral of $x^*(u,v)$ on $\Sigma\, T(R_\nu W)$ is clearly $< k^{-1} < \epsilon$, and its area integral on $R - \Sigma R_\nu$ does not exceed the corresponding Dirichlet integral which is $< 3\epsilon$. Since the integral over $\Sigma R_\nu W$ of the area of $M_{u,v}(f)$ is clearly $<\epsilon$, we see that the sum of the areas of the generalized surfaces

$$\int\int_{R-\Sigma(R_\nu - W)} M_{u,v}(f)\,du\,dv \quad\text{and}\quad \int\int_{R-\Sigma T(R_\nu - W)} M^*_{u,v}(f)\,du\,dv$$

is less than 19ϵ. Since, moreover, for each ν the integral of $M_{u,v}(f)$ over $R_\nu - W$ equals that of $M^*_{u,v}(f)$ over $T(R_\nu - W)$, it follows that the distance of $L(f)$ from the generalized surface

$$L^*(f) = \iint_R M^*_{u,v}(f)\,du\,dv$$

is $< 19\epsilon$.

We now modify $x^*(u,v)$ by making it piecewise linear (and so Lipschitzian) on the sides of the polygonal domains G_λ and $H_{\mu\nu}$ other than sides situated on the perimeters of the TR_ν or of R (where it is Lipschitzian already). The resulting function $x'(u,v)$ is defined so as to be harmonic in each G_λ or $H_{\mu\nu}$, to coincide with $x^*(u,v)$ on the perimeter of R and each TR_ν, and to be sufficiently close to $x^*(u,v)$ on the remaining sides of the G_λ and $H_{\mu\nu}$. According to [16 (b)], the area integral of $x'(u,v)$ is then as close as we please to that of $x^*(u,v)$

on the sum of the G_λ and $H_{\mu\nu}$ and we can arrange for it to be $< 9\epsilon$. Moreover $x'(u,v) \in$ Lip by construction.

We define $L'(f) = \iint_R M'_{u,v}(f)\,du\,dv$ where $M'_{u,v}(f) = M^*_{u,v}(f)$ for (u,v) in $R - \Sigma G_\lambda - \Sigma H_{\mu\nu}$ and where $M'_{u,v}(f) = f[x'(u,v),\ j'(u,v)]$ in each G_λ and $H_{\mu\nu}$, $j'(u,v)$ being the Jacobian of $x'(u,v)$. By the same calculation as before we find that the distance of $L(f)$ from $L'(f)$ is $< 19\epsilon$. Writing $L'(f) = L'(f,\epsilon)$ in its dependance on ϵ, the assertion of (2.3) in case (i) follows with $L_\epsilon(f) = L'(f, \epsilon/19)$. Evidently if $L(f)$ is a parametric surface, i.e., if $M_{u,v}(f) = f[x(u,v),\ j(u,v)]$ then $L_\epsilon(f)$ is a parametric surface with the representation $x_\epsilon(u,v)$.

(ii) We now deal with the general case, which we shall reduce simply to case (i). To this effect we denote by R_0 a rectangle interior to R on whose perimeter $x(u,v)$ is absolutely continuous, such that the Dirichlet integral of $x(u,v)$ on $R-R_0$ and also the corresponding integral of the area of $M_{u,v}(f)$ are $< \epsilon/3$. We denote by s a segment on which $x(u,v)$ is absolutely continuous and which joins the two rims of the annulus $R-R_0$, and by K the interior of a semi-circle in $R-R_0$ whose centre is the intersection of s with the perimeter of R. We write $x_0(u,v)$ for the harmonic interpolation of $x(u,v)$ in the interior of $R-R_0-s$, and $x_1(u,v)$ for the function equal to $x(u,v)$ in R_0, to $x_1(u,v)$ in $R-R_0-K$, and to the harmonic interpolation of $x_0(u,v)$ in K. We denote by R_1 a square concentric to R whose perimeter meets K and lies in $R-R_0$, and by $j_1(u,v)$ the Jacobian of $x_1(u,v)$. Clearly the generalized surface

$$L_1(f) = \iint_{R_0} M_{u,v}(f)\,du\,dv + \int\int_{R-R_0} f[x_1(u,v),\ j_1(u,v)]\,du\,dv$$

is distant $< 2\epsilon/3$ from $L(f)$ and has a micro-representation whose carrier is $x_1(u,v)$. Moreover $x_1(u,v)$ is Lipschitzian on the perimeter of R_1 and any continuous $x_\epsilon(u,v)$ Lipschitzian in R_1 and coinciding with $x_1(u,v)$ in the rest of R clearly belongs to Lip-int. We now apply the trivial modification of case (i), in which ϵ is replaced by $\epsilon/3$ and the role of R is taken by R_1, to the generalized surface derived from $L_1(f)$ by integrating its micro-representation over R_1. By adding to the approximating surface the integral on $R-R_1$ of $f[x_1(u,v),\ j_1(u,v)]$, we obtain, at a distance $< \epsilon/3$ from $L_1(f)$, a generalized surface possessing a micro-representation which coincides with that of $L_1(f)$ in $R-R_1$ and has a carrier, Lipschitzian in R_1 and equal to $x_1(u,v)$ in $R-R_1$, which therefore $\in$ Lip-int. Since this generalized surface is then distant $< \epsilon$ from $L(f)$, this completes the proof.

PROOF OF (2.2). (i) First suppose (a) that $x(u,v)$ is linear—so that it describes on the perimeter of R the oriented perimeter C of a parallelogram whose oriented area j_0 is the constant Jacobian of $x(u,v)$ —and that $M_{u,v}(f)$ is a polyhedral micro-disc obtained by concentrating, at the point $x(u,v) \in X$, a flat-rimmed polyhedral disc (possibly depending on (u,v)) whose rim is C —and whose oriented area is therefore j_0. And suppose further (b_1) that the shape of $M_{u,v}(f)$ is constant. This requires that the flat-rimmed polyhedral disc of rim C from which $M_{u,v}(f)$ is obtained by concentrating it at the point $x(u,v) \in X$ is independent of (u,v). We denote it by $L^*(f)$, and we denote by $x^*(u,v)$ a piecewise linear representation of $L^*(f)$ whose restriction to the perimeter of R is *linear*; the existence of such a representation follows easily from the fact that C is the perimeter of a parallelogram. We now divide R into n^2 equal squares R_ν which we map linearly onto R by a function $w_\nu = w_\nu(u,v)$, $(u,v) \in R_\nu$, such that similarly situated sides correspond. We denote by $x_\nu^*(u,v)$, $(u,v) \in R_\nu$, the composite function $x^*(w_\nu)$, and we write $x_n(u,v)$, $(u,v) \in R_\nu$, for the function of the form $ax_\nu^* + b_\nu$ where the constants $a = 1/n$ and b_ν are such that $x_n(u,v) = x(u,v)$ on the perimeter of R_ν; this choice of (a and) b_ν is possible because $x(u,v)$ describes linearly on this perimeter a translation of a suitable magnification of C. By fitting together the $x_n(u,v)$ for $(u,v) \in R_\nu$ which correspond to the various values of ν and which evidently agree (with $x(u,v)$ and so with one another) on a common side or corner of two R_ν, we obtain a function $x_n(u,v)$ defined for $(u,v) \in R$. This function constitutes a piecewise linear parametric representation of a certain flat-rimmed polyhedral disc $L_n(f)$ satisfying our requirements: for, on the one hand, $x_n(u,v)$ coincides with $x(u,v)$ on the perimeter of R; and on the other hand, writing $f_\nu(x,j)$ for $f(ax+b_\nu, j)$ and $f_{u,v}(j) = f_{u,v}(x,j)$ for $f(x(u,v),j)$, we see that for each $f \in F$, the expression

$$L_n(f) - L(f) = \sum_\nu \iint_{R_\nu} L^*(f - f_{uv})\,du\,dv$$

tends to zero as $n \longrightarrow \infty$.

(ii) Next suppose (a) as before, and suppose further (b_2) that the shape of $M_{u,v}(f)$ is *piecewise constant*, in fact that R is the sum of a finite number of squares R_ν in the interior of each of which $M_{u,v}(f)$ has a constant shape. Writing $L^\nu(f)$ for the integral of $M_{(u,v)}(f)$ over R_ν, we apply stage (i) to $L^\nu(f)$ after mapping R_ν suitably on R so that similarly situated sides correspond. Then $L^\nu(f)$ is the limit of a flat-rimmed polyhedral disc $L_n^\nu(f)$ whose parametric representation on R becomes, after mapping R back onto R_ν, a function $x_n(u,v)$ $(u,v) \in R_\nu$ whose restriction to the perimeter of R_ν agrees with that of $x(u,v)$. As before we can fit together the functions $x_n(u,v)$, $(u,v) \in R_\nu$, to a single function $x_n(u,v) \in R$, and the latter agrees with $x(u,v)$ on the perimeter of R and constitutes a piecewise linear parametric representation of the flat-rimmed polyhedral disc $L_n(f) = \Sigma_\nu L_n^\nu(f)$ which tends to $L(f)$.

(iii) Now suppose merely that $x(u, v)$ is linear. In this case it will suffice to show (A) that $L(f)$ is a repeated limit of a generalized surface of the form (2.1) for which $x(u, v)$ is the same and for which the assumptions (a) and (b_2) are satisfied. For by case (ii) $L(f)$ is then a repeated limit of a flat-rimmed polyhedral disc with a Lipschitzian representation which agrees with $x(u, v)$ on the perimeter of R; we can clearly arrange further that this flat-rimmed polyhedral disc lies in some fixed neighborhood of the parallelogram with the perimeter C, and therefore in some fixed sphere, and this ensures that convergence be expressible in terms of McShane metric [Young 18, p. 62], so that our repeated limit becomes, by the well-known diagonal process, a single limit as required. We observe that $L(f)$ is distant less than ϵ from an $L^*(f)$ of the same form for which the micro-disc at (u, v) has bounded area, and therefore that (A) need only be established in the case in which the area of $M_{u,v}(f)$ is a bounded function of (u, v). This being so, we denote by $S_{u,v}(f)$ the shape of $M_{u,v}(f)$, by E the set of the $S_{u,v}(f)$ for varying $(u, v) \in R$, and by $\{f_k\}$ a dense subsequence of those integrands $f(x, j)$ whose values and difference ratios for $x = 0$ and $j \in J_1$ do not exceed unity in absolute value. Given $\epsilon > 0$ and $S \in E$, we note the measurability of the set $W(\epsilon, S)$ consisting of the points $(u, v) \in R$ for which ϵ exceeds the McShane distance [Young 18, p. 62] of S and $S_{u,v}$ (considered as generalized surfaces with carriers at the origin); in fact the complement of $W(\epsilon, S)$ is the sum of a null set (in which S_{uv} is not defined) and of the sets W_k consisting of the points $(u, v) \in R$ for which $| S_{u,v}(f_k) - S(f_k) | \geq \epsilon$, and each of these W_k is clearly measurable. Furthermore, since the members of E have uniformly bounded areas, and carriers at the origin, every infinite subset contains a convergent sequence, and therefore there is a finite subset E_ϵ of E such that every $S \in E$ has McShane distance $< \epsilon$ from some member S^* of E_ϵ. We can therefore partition all but measure zero of R into a finite number of disjoint measurable sets Q, consisting of differences of the $W(\epsilon, S^*)$ where $S^* \in E_\epsilon$, such that in each Q $S_{u,v}$ has McShane distance $< \epsilon$ from some constant $S^* \in E_\epsilon$; denoting by $M^*_{u,v}(f)$ the micro-disc at $x(u, v)$ with this shape S^*, and writing now $L^*(f)$ for its integral over R, we see that the McShane distance of $L^*(f)$ from $L(f)$ cannot exceed ϵ. Thus (A) need only be established in the case in which $M_{u,v}(f)$ has a constant shape in each of a finite number of disjoint measurable sets Q which cover almost all of R. We may clearly suppose, by a suitable approximation, that each of these constant shapes is that of a flat-rimmed polyhedral disc with the rim C, and to derive (A) it is now sufficient to approximate further in well-known fashion each of the measurable sets Q by a suitable sum of squares.

(iv) Now suppose merely that $x(u, v)$ is Lipschitzian. We apply a result proved elsewhere [Young 12 (12.1) p. 98]. (Incidently, although we need not use the fact here, its proof remains valid for vector-functions if the appeal to a lemma of McShane [Young 12, (9.2) p. 93] is replaced by an appeal to one due to Mickle [5].) Accordingly, there exists a constant K exceeding the Lipschitz constant of $x(u, v)$ (by a factor which is the number of components of our vectors if we use the result quoted in its scalar form) and for each $\epsilon > 0$ a Lipschitzian vector-function $x_\epsilon(u, v)$ with Lipschitz constant less than K and with partial derivatives differing from those of $x(u, v)$ by less than ϵ^3 outside a subset W_1 of R of measure less than ϵ, such that $x_\epsilon(u, v)$ coincides with $x(u, v)$ on the perimeter of R, that $x_\epsilon(u, v)$ never differs from $x(u, v)$ by more than ϵ, and that $x_\epsilon(u, v)$ is linear in each of a finite number of disjoint squares R_ν whose sum covers all but a subset W_2 of R of measure less than ϵ. The Jacobians $j(u, v)$ and $j_\epsilon(u, v)$ of $x(u, v)$ and $x_\epsilon(u, v)$ then differ by less than ϵ^2 if ϵ is small. We denote by W a measurable subset of R of measure less than 3ϵ which contains $W_1 + W_2$ such that $M_{u,v}(f)$ has bounded area outside W. We observe further that the *oriented* area of $M_{u,v}(f)$ is $j(u, v)$ and so is exceeded by K^2 in absolute value. By an argument similar to the one used in case (iii), there is a finite set $E_{\epsilon,\nu}$ of shapes S^* of polyhedral discs L^* with square rims of sides smaller than K, such that every $M_{u,v}(f)$ for $(u, v) \in R_\nu - W$ has its shape at a McShane distance $< \epsilon^2$ from some corresponding member S^* of $E_{\epsilon,\nu}$; in particular its oriented area differs by less than ϵ^2 from that of the square with the same rim as the corresponding L^*. Again by an argument similar to that of case (iii), we can express $R_\nu - W$ as a finite sum of disjoint measurable sets Q in which an S^* with the property just stated is independent of (u, v). In particular the oriented area of this S^*, and so, that of the square with the same rim as the corresponding L^*, differs by less than $2\epsilon^2$ from the constant value (in R_ν and so in Q) of the Jacobian $j_\epsilon(u, v)$ of $x_\epsilon(u, v)$. A simple calculation shows that we can construct a square with the oriented area $j_\epsilon(u, v)$ and therefore with sides smaller than K, which has one corner at a corner of the rim of L^* and the other three corners distant less than 2ϵ from the corresponding corners of this rim. Hence by fitting on to L^* six triangular faces of total area $< 6\epsilon K$ we can construct a flat-rimmed polyhedral disc L' with the oriented area $j_\epsilon(u, v)$ whose shape has McShane distance less than $6\epsilon K$ from that of $M_{u,v}(f)$. Writing $M_{u,v,\epsilon}(f)$ for the micro-disc obtained by concentrating L' at $x_\epsilon(u, v)$ where $(u, v) \in Q$, we define $M_{u,v,\epsilon}(f)$ in each Q and we complete the definition by writing $M_{u,v,\epsilon}(f) = f(x_\epsilon(u, v), j_\epsilon(u, v))$ almost everywhere in W. We find that the McShane distance of $M_{u,v,\epsilon}(f)$ and $M_{u,v}(f)$ is at most $\epsilon K^2 + 6\epsilon K$ for $(u, v) \in R - W$. Writing $L_\epsilon(f)$ for the integral of $M_{u,v,\epsilon}(f)$ extended over R, we see that the McShane distance of $L_\epsilon(f)$ from $L(f)$ exceeds by at most $(\epsilon K^2 + 6\epsilon K) + 3\epsilon K^2$ the integral over W of the area of $M_{u,v}(f)$ and therefore tends to zero as $\epsilon \to 0$. The same is true if $x_\epsilon(u, v)$ is modified in the interior of

$R - \Sigma R_\nu$ but remains Lipschitzian with constant $<K$, since the McShane distance in question is then increased by at most another $2\epsilon K$. Remembering that $x_\epsilon(u,v)$ is linear in each R_ν, we can arrange this modification so that $x_\epsilon(u,v) \in$ Pol-int: to this effect it is sufficient to triangulate suitably, with a countable number of triangles, the set $R - \Sigma R_\nu$ and to replace in each triangle the original $x_\epsilon(u,v)$ by the linear interpolation of its values at the vertices. Similarly in the case in which $x(u,v)$ is piecewise linear on the perimeter of R we arrange that $x_\epsilon(u,v) \in$ Pol. Finally, by applying case (iii) to the part of $L_\epsilon(f)$ arising from each square R_ν in which $x_\epsilon(u,v)$ is linear, we see that $L(f)$ is a repeated limit of an $L_{\epsilon,n}(f)$, and therefore the single limit of an $L_n(f)$ which satisfies our requirements.

(v) We pass on to the general case. If $x(u,v)$ is Lipschitzian on the perimeter of R, (2.3) shows that we may suppose $x(u,v) \in$ Lip without loss of generality, so that our assertion is then true on account of (iv). If $x(u,v)$ is not Lipschitzian on the perimeter of R, we may suppose $x(u,v) \in$ Lip-int, and moreover, reference to the proof of (2.3) shows that we may suppose that, at all points interior to R which are sufficiently near to the perimeter, the partial derivatives x_u, x_v are continuous and that $M_{u,v}(f) = f[x(u,v), j(u,v)]$. We can now triangulate, with a countable number of triangles, a small annulus whose outer rim is that of R, so that the set of limiting points of the vertices consists of the two rims and so that in each triangle the oscillation of $x_u^2 + x_v^2$ is < 1. The function derived from $x(u,v)$ by linear interpolation, in each triangle, of the values at the vertices then clearly belongs to D and agrees with $x(u,v)$ on the two rims. We may replace $x(u,v)$ by it and modify $M_{u,v}(f)$ accordingly in the annulus, since the generalized surface thus obtained is at an arbitrarily small distance from $L(f)$. Denoting by R_0 a square concentric to R whose perimeter is interior to our annulus, we find that $x(u,v)$ is Lipschitzian in R_0 and piecewise linear on its perimeter, and our final assertion follows by applying case (iv), with R_0 in place of R.

3. The change of variable, sewing and patching theorems. In this section we make some use of classical methods and definitions to establish results that will be much used. By a Fréchet representation $x(u,v)$ we mean a continuous function defined in R and whose values lie in Euclidean m-space. Fréchet equivalences, Fréchet curves, Fréchet surfaces will be understood to be *oriented* [7a, pp. 158–159], but otherwise defined as in Rado's book [7a]. The term *monotone*, applied to a mapping, has the meaning there assigned to it [7a, Chap. II]; a mapping is continuous but not necessarily one to one, and such a mapping of one plane domain onto another will be termed *positively oriented* if its degree is $+1$ [Alexandroff-Hopf, Topologie].

A set of points will be termed *k-dimensionally thin*, if it is of zero k-dimensional measure, in the sense of Carathéodory (Hausdorff). A Fréchet curve or surface is k-dimensionally thin if the set of its points is so.

(3.1) *Let T be a monotone, positively oriented, mapping of R onto itself and suppose that the functions $x^*(u, v)$ and $x^*(T[u, v]) = x(u, v)$ are continuous in R and belong to D. Then they are representations of a same parametric surface $L(f)$.*

(3.2) *Let $L_1(f)$, $L_2(f)$ be parametric surfaces with representations*

$$x_1 \in \text{quasi-Lip}, \ x_2 \in \text{quasi-Lip},$$

and suppose that the restrictions $x_1(1, v), x_2(0, v)$ describe, for $0 \leq v \leq 1$, a same 2-dimensionally thin oriented Fréchet curve γ. Denote further by C_1 and C_2 the oriented Fréchet curves described, on the perimeter of R, by $x_1(u, v)$ for $u \neq 1$ and by $x_2(u, v)$ for $u \neq 0$ respectively. Then $L_1(f) + L_2(f)$ has a continuous representation $x \in D$ which describes C_1 and C_2 on two complementary arcs of the perimeter of R.

The first statement (3.1), which has been formulated so as to be most useful to us and is entirely adequate for this purpose, appears almost trivial by comparison with what is known in 3-space in the case of continuous representations related to one another by the notion of Fréchet equivalence, which is far more general and far more complex than our change of variable [7a, II 1–66 (4) p. 75 and Youngs 23]. However the proofs of the known results in 3-space use an elementary property of bi-vectors [7(b), §5 p. 74] which is no longer valid in m-space for $m > 3$, and they are obtained via semi-continuity theorems which our theory is designed to supersede, so that, if such an approach were possible, as doubtless it should be, we would still be reluctant to use it.

It is convenient to relegate the proof of (3.1) to the end of this section, so that our first, and main, effort will be directed towards proving the second of our statements (3.2). The latter is deep in the present state of our knowledge, but it will undoubtedly be superseded by a much stronger sewing theorem in which the term quasi-Lipschitzian is replaced by continuous and the assumption concerning γ is slightly weakened: for we could do this, and shorten our proof, if we knew that the Lebesgue area of a surface does not exceed the sum of the Lebesgue areas of its projections on the coordinate planes. This result, which is inevitable as area theory progresses, is at present only available for a surface situated in 3-space and its proof then depends on a deep topological "four-line" property of 3-space [1c, footnote 26, XI p. 38]; instead, we employ a temporarily adequate, but less elegant, substitute–and incidentally the notion of Banach area [14] (a similar notion, based however on a different measure, was defined by Federer [3a]). This substitute is the following lemma:

(3.3) *Let $x(u, v)$ be a Fréchet representation which satisfies the conditions:*

(a) *There is a finite constant k which exceeds the Lebesgue area of the orthogonal projection of $x(u, v)$ on every 3-dimensional linear subspace.*

(b) *$x(u, v)$ is regular in the sense of Cesari, i.e., there is a dense set of parallels to the axes in R, on each of which the projection of $x(u, v)$ on each coordinate plane assumes only a 2-dimensionally thin set of values.*

(c) *The Banach area of $x(u, v)$ on the interior of R is finite.*
Then the Lebesgue area of $x(u, v)$ is less than a bound depending only on (k, m).

From the point of view of our applications, we are not interested in reducing the hypotheses except by the total removal of (c). In proving the lemma as stated, we may actually strengthen this last hypothesis without loss of generality, by supposing that the Banach area of $x(u, v)$ on the closed unit square R is finite: to see this we need only approximate to $x(u, v)$ by the function $x(au, av)$ where $0 < a < 1$ and use semi-continuity of the Lebesgue area as $a \to 1$.

If we then weaken this assumption by supposing merely that the set of values of $x(u, v)$ on R is 3-dimensionally thin and at the same time weaken (a) by supposing merely that the Lebesgue area of the orthogonal projection of $x(u, v)$ on every 2-dimensional subspace is less than a fixed finite constant, and finally delete (b) altogether, our statement becomes a corollary of a result announced by Federer [3(b), (14) p. 93]. Federer's statement contains a misprint reported in Mathematical Reviews [vol. 13 p. 831], and the wording of his reference to Cesari [1d, 1362–1381] does not account for the omission of assumption (b); however, we have formulated lemma (3.3) in such a weak manner that it cannot fail to be included in the final versions promised by both Federer and Cesari.

PROOF of (3.2). We denote by $\phi(u, v)$ the complex-valued function defined in R by the expression $(u + iv)\exp(-u^{-1})$ for $u > 0$ and by continuity for $u = 0$ so that $\phi(0, v) = 0$. We have for $u > 0$ $|\phi_v| = \exp(-u^{-1}) < 1$ and moreover $|\phi_u| \leq |1 + u^{-1} + iu^{-2}| \exp(-u^{-1}) < 1$ because, writing $t = u^{-1} \geq 1$, we have $(\phi_u)^2 = (1 + 2t + t^2 + t^4)\exp(-2t) < 1$ since the function $\exp(2t) - (1 + 2t + t^2 + t^4)$ and its two first derivatives are > 0 at $t = 1$ while its third derivative is > 0 for $t \geq 1$. Hence ϕ is Lipschitzian.

Further, for $0 < u \leq u'$, the relation $\phi(u, v) = \phi(u', v')$ implies $1 \geq u/u' = \exp\{u^{-1} - (u')^{-1}\} \geq 1$, which necessitates $u = u'$ and consequently $v = v'$. Thus 0 is the only value taken at more than one point by $\phi(u, v)$. We observe further that the relation $\phi \neq 0$ implies (real part ϕ) > 0; in particular 0 is the only value taken by both ϕ and $-\phi$.

This being so, let $y_1(u, v) = [x_1(u, v), \phi(1-u, v)]$ a vector-function whose values lie in a space Y of dimension $m+2$ where Y is the topological product of X with the complex plane Φ. Clearly $y_1 \in D$ since ϕ is Lipschitzian and $x_1 \in D$; in particular the projection of y_1 on any 3-space of Y belongs to D and so has finite Lebesgue area. Clearly also $y_1(u, v)$ is quasi-Lipschitzian, and so of finite Banach area equal to its Lebesgue area.

Writing similarly $y_2(u, v) = [x_2(u, v), -\phi(u, v)]$, we find that the vector-function $y_2(u, v)$ has the corresponding properties and we note that the restrictions $y_1(1, v)$ and $y_2(0, v)$, which coincide with $x_1(1, v)$ and $x_2(0, v)$ respectively, describe, for $0 \le v \le 1$ the same oriented Fréchet curve y.

We now appeal to known results [7a, II–5–6 p. 160] according to which there is an ordered correspondence $\alpha \leftrightarrow \beta$ between the maximal continua of constancy α of $y_1(1, v)$ and those β of $y_2(0, v)$, ordered for increasing v on $0 \le v \le 1$, such that $y_1(1, v)$ takes on α the same value as $y_2(0, v)$ on β. Denoting by α^*, β^* the sets of (u, v) for which $[u=0,\ v \in \alpha]$ and $[u=1,\ v \in \beta]$ respectively, we define in R a function $y_3(u, v)$, equal to $y_1(1, v)$ on $u=0$ and to $y_2(0, v)$ on $u=1$, by making it constant on the convex closure (segment, triangle or trapezium) of each set $\alpha^* + \beta^*$ where $\alpha \leftrightarrow \beta$.

We define further in R a function $y^*(u, v)$ by the expressions: $y_1(3u, v)$ for $0 \le u \le 1/3$, $y_3(3u-1, v)$ for $1/3 \le u \le 2/3$, $y_2(3u-2, v)$ for $2/3 \le u \le 1$. We write x^* for the projection of y^* on X.

By its definition, $y^*(u, v)$ is a basic [17(a), p. 321] representation. Moreover $y^*(u, v)$ has *finite Lebesgue area* on R. This last follows from (3.4) if we verify (a) that the projection of $y^*(u, v)$ on any coordinate 3-space is of finite Lebesgue area and (c) that the Banach area of $y^*(u, v)$ on the interior of R is finite. (The hypothesis (b) of (3.4) is clearly satisfied.) Here (a) is derived from the corresponding verification for $y_1(u, v)$ and $y_2(u, v)$ by known additivity theorems valid in 3-space [7a, V–1–14, V–1–17, V–2–67, pp. 468–506 where the assumption $A(S) < \infty$ is superflous by V–2–65], and therefore applicable to our 3-dimensional projections. While (c) follows at once by addition from the corresponding properties of $y_1(u, v)$ and $y_2(u, v)$.

This being so, the (oriented) Fréchet surface defined by $y^*(u, v)$ has, by Morrey's theorem [17(a), p. 324], a quasi-conformal representation $y \in D$ on R. We write x and ξ for the projections of y on X and on Φ. Clearly, $x \in D$ and $x(u, v)$ describes, like $x^*(u, v)$, the given oriented Fréchet curves C_1 and C_2 as (u, v) describes two suitable complementary arcs of the perimeter of R. It remains to show that $x(u, v)$ is a representation of the parametric surface $L_1(f) + L_2(f)$. Denoting by $j(u, v)$ the Jacobian of $x(u, v)$ and by Γ, O', O'' the sets of points interior to R in which the real part of $\xi(u, v)$ is respectively equal, greater, less than zero, it is sufficient to show that:

(3.4) *The integrals of the function* $f[x(u, v), j(u, v)]$ *over the sets* Γ, O', O'' *have the respective values* 0, $L_1(f)$, $L_2(f)$.

From the properties established for the function ϕ, it follows that, in Γ, $\xi(u, v) = 0$ and the values of $x(u, v)$ lie on y. We now express Γ as the sum of a set of plane measure zero and a sequence of closed sets Γ_n in each of which $x(u, v)$ is Lipschitzian. On Γ_n the Banach area of the projection of $x(u, v)$ on any coordinate plane is zero; this Banach area is, however, equal to the classical area integral on Γ_n [14, Th. 4.3 p. 281], so that the integrand has to be zero almost everywhere on Γ_n and therefore on Γ. This shows that each component j_{rs} of the bi-vector $j(u, v)$ is zero almost everywhere on Γ and therefore that $j(u, v)$ itself is zero almost everywhere on Γ. This clearly implies that the integral over Γ of the function $f[x(u, v), j(u, v)]$ is zero. To establish (3.4), and so (3.2), it is now sufficient by symmetry to establish the part concerning the integral over O'.

We denote by O the open set derived from O' by removing all continua of constancy of $y(u, v)$ which meet the perimeter of R. By an argument established elsewhere [14, pp. 299–301, in particular (9.7) p. 300], $O' - O$ is the sum of a set of plane measure zero, a set in which $y(u, v)$ has Jacobian zero, and a sequence of disjoint closed sets possessing no point of density on a continuum of constancy of $y(u, v)$ which does not reduce to a single point. It follows at once that at almost all points of $O' - O$ the Jacobian of $y(u, v)$ is zero, and therefore that $j(u, v)$ and consequently $f[x(u, v), j(u, v)]$ vanish. Writing $j^*(u, v)$ for the Jacobian of $x^*(u, v)$ and O^* for the interior of the first third $0 \leq u \leq 1/3$ of R, it is therefore sufficient to prove that

$$(3.5) \qquad \iint_O f[x(u, v), j(u, v)]\, dudv = \iint_{O^*} f[x^*(u, v), j^*(u, v)]\, dudv;$$

for the left-hand side coincides with the same integral over O', while the right-hand side clearly equals $L_1(f)$.

To this effect we first observe that there is a positively oriented, monotone, continuous mapping T of O onto O^*, such that

$$(3.6) \qquad y(u, v) = y^*(T[u, v]) \text{ for } (u, v) \in O.$$

In fact y and y^* have monotone-light factorizations [7a, pp. 59–60] with the same middle space, the same light factor, and with Fréchet equivalent monotone factors M and M^*. In our case, the nature of the projection $\phi(1-u, v)$ ensures that M^* is one to one on O^* and that $M^{-1}M^*O^* = O$. The relation (3.6) now follows with $T = M^{*-1}M$ for $(u, v) \in O$, and we verify that T is positively oriented as stated.

This being so, we express the open rectangle O^* as the limit of an expanding closed rectangle, whose interior and perimeter we denote by O_n^* and Π_n^* respectively. The disjoint sets $T^{-1}(O_n^*)$ and $T^{-1}(\Pi_n^*)$ are respectively open and closed and their sum is closed; this follows from the continuity of T. Since they are both connected [7a, p. 46], O_n is necessarily a simply-connected domain whose boundary is mapped by T into the rectifiable curve Π_n^*. This ensures in particular that $T(u, v)$ is constant on each prime end of O_n [15, (11.3) p. 15].

Denoting by U a positively oriented conformal transformation of the interior of R onto O_n, we may now conclude from the theory of prime ends that TU is uniformly continuous: in fact, if w_n is an interior point of R which tends to a boundary point, the point Uw_n tends to a prime end P of O_n and in particular the distance of Uw_n from the nearest point of P tends to zero; TUw_n therefore tends to a limit by uniform continuity of T in the closure of O_n (which is interior to O). This ensures that TU is uniformly continuous.

Let now V be a positively oriented linear mapping of the closure of O_n^* onto R. Then VTU is a positively oriented uniformly continuous mapping of the interior of R onto itself and so possesses an extension T^* which maps the closed unit square R onto itself. We define, for $(u, v) \in R$,

$$z^*(u, v) = x^*(V^{-1}[u, v]), \quad z(u, v) = z^*(T^*[u, v]),$$

and we observe that $z(u, v) = x(U[u, v])$ when (u, v) is interior to R.

From this observation it follows, according to McShane [4b, p. 597], that the function $z_a(u, v) = z\,(au, av)$, where $0 < a < 1$, satisfies for an N independent of a the condition $z_a \in D(N)$; hence, making $a \to 1$, we find that $z \in D$. And it follows further that the parametric surface represented by $z(u, v)$ is the functional

$$\iint_{O_n} f[x(u, v), j(u, v)]\, du dv$$

which tends to the left-hand side of (3.5) as $n \to \infty$, since $O_n = T^{-1}(O_n^*)$ then tends to $T^{-1}(O^*) = O$. On the other hand it is clear that $z^* \in D$ and that the parametric surface which it represents is the functional

$$\iint_{O_n^*} f[x^*(u, v), j^*(u, v)]\, du dv$$

which tends to the right-hand side of (3.5) as $n \to \infty$. To complete the proof of

(3.5) and so of (3.2), it is now sufficient to verify that the same parametric surface is represented by $z \in D$ and by $z^* \in D$, where z and z^* are connected by the relation $z(u, v) = z^*(T^*[u, v])$. This is a special case of (3.1) and the proof of (3.2) thus reduces to that of (3.1).

PROOF of (3.1). We have to establish the equality of the integrals over R of $f[x(u, v), j(u, v)]$ and $f[x^*(u, v), j^*(u, v)]$ where j, j^* are the Jacobians of x, x^*, and it is enough to prove this for an f with continuous partial derivatives, since such functions are dense in F. We may suppose f defined in (X, J^*) and homogeneous of degree 1.

Given a positive $\epsilon < 1$, we determine $\delta > 0$ so that ϵ exceeds the Dirichlet integrals of $x(u, v)$ and $x^*(u, v)$ over any subset of R of plane measure $< \delta$; in particular ϵ then exceeds the integrals, over such a subset, of $|j(u, v)|$ and $|j^*(u, v)|$. We determine further, by Egoroff's theorem, an open subset W_0 of R of plane measure $< \epsilon\delta$ such that the restrictions of $j(u, v)$ and $j^*(u, v)$ to $R - W_0$ are continuous, and we denote by $\eta > 0$ a number such that ϵ exceeds the oscillations of $x(u, v)$, $x^*(u, v)$ in any subset of R of diameter $< \eta$ and also those of $j(u, v)$, $j^*(u, v)$ in any such a subset of $R - W_0$. Finally we denote by K and by $\omega(\epsilon)$ the maximum modulus, and the oscillation in distance ϵ, of the vector-derivative $f_j(x, j)$ for $j \in J^*$ and $x \in E$, where E is the set of values taken by $x(u, v)$ and $x^*(u, v)$ in R. Since $f(x, j)$ is positively homogeneous in j of order 1, its derivative f_j is positively homogeneous in j of order 0, and hence K is finite and $\omega(\epsilon) \to 0$ as $\epsilon \to 0$.

Denoting by O^* the interior of R and writing $O = T^{-1}O^*$, we see exactly as in the preceding proof that it suffices to establish the formula (3.5) for our present functions and sets. We denote by G a grating consisting of a finite number of parallels to the axes in R, on each of which $x(u, v)$ and $x^*(u, v)$ are absolutely continuous functions of one variable, and we choose it so as to divide R into rectangles of diameter $< \eta$. We write $W^* = G + T(G)$ and $W = T^{-1}W^*$. According to [14, (4.3) p. 281 and (9.3) p. 299], $j(u, v)$ and $j^*(u, v)$ vanish almost everywhere in the sets W and W^* respectively, since the corresponding Banach areas are zero. Denoting by O_n^* the constituent domains of the open set $O^* - W^*$ and writing $O_n = T^{-1}O_n^*$, we observe that the O_n are the constituent domains of $O = T^{-1}O^*$ [7a, p. 46]. Thus if a_n and a_n^* are the integrals

$$\iint_{O_n} f[x(u, v), j(u, v)]\, du dv \quad \text{and} \quad \iint_{O_n^*} f[x^*(u, v), j^*(u, v)]\, du dv,$$

it remains to show that $\Sigma a_n = \Sigma a_n^*$. It will clearly be sufficient to show that $\Sigma(a_n - a_n^*)$ tends to zero with ϵ. We write j_n and j_n^* for the integrals over O_n and O_n^* of the respective Jacobians $j(u, v)$ and $j^*(u, v)$, and we denote by x_n

the value taken by $x(u, v)$ at some fixed point (u_n, v_n) of O_n^*, so that x_n is then also the value taken by $x^*(u, v)$ at the point $(u_n^*, v_n^*) = T(u_n, v_n)$ of O_n.

Let m_n be the measure of O_n and let $j'(u, v)$ denote in each O_n the constant j_n/m_n. Further let I_n and I be the Dirichlet integrals of $x(u, v)$ over O_n and over R. We observe that, for $(u, v) \in O_n - W_0$,

$$|j_n - m_n j(u, v)| = \Big|\iint_{O_n} \{j(u', v') - j(u, v)\}\, du'dv'\Big|$$

$$\leq \iint_{O_n - W_0} |j(u', v') - j(u, v)|\, du'dv' + \iint_{O_n W_0} \{|j(u', v')| + |j(u, v)|\}\, du'dv'$$

$$\leq \epsilon m_n + \iint_{O_n W_0} |j(u', v')|\, du'dv' + |j(u, v)| \iint_{O_n W_0} du'dv'$$

and that, for $(u, v) \in O_n W_0$,

$$|j_n - m_n j(u, v)| \leq |j_n| + m_n |j(u, v)| \leq I_n + m_n |j(u, v)|.$$

Hence, integrating over $O_n - W_0$ and $O_n W_0$ and adding, we obtain

$$m_n \iint_{O_n} |j'(u, v) - j(u, v)|\, dudv \leq \epsilon m_n^2 + 2I_n \iint_{O_n W_0} dudv + 2m_n \iint_{O_n W_0} |j(u, v)|\, dudv.$$

Thus

$$\iint_{O_n} |j'(u, v) - j(u, v)|\, dudv \leq \epsilon m_n + 2\epsilon I_n + 2 \iint_{O_n W_0} |j(u, v)|\, dudv$$

whenever the measure of $O_n W_0$ is $\leq \epsilon m_n$; and in any case

$$\iint_{O_n} |j'(u, v) - j(u, v)|\, dudv \leq \epsilon m_n + 2I_n + 2 \iint_{O_n W_0} |j(u, v)|\, dudv.$$

It follows that

$$\sum_n \iint_{O_n} |j'(u, v) - j(u, v)|\, dudv \leq \epsilon + 2\epsilon I + 2 \iint_{W_0} |j(u, v)|\, dudv + 2\lambda$$

where λ is the sum $\sum I_n$ extended over the values of n for which $O_n W_0$ has measure $> \epsilon m_n$. Since however the sum of the sets O_n for such n has measure at most ϵ^{-1} times the measure of W_0, and so measure $< \delta$, it follows that $\lambda < \epsilon$.

Thus we find that

$$\sum_n \iint_{O_n} |j'(u,v) - j(u,v)|\, dudv \leq 5\epsilon + 2\epsilon I.$$

If we observe further that, for $(u,v) \in O_n$,

$$|f[x(u,v), j_n] - f(x_n, j_n)| \leq \omega(\epsilon)|j_n| \leq \omega(\epsilon) I_n$$

and that

$$|f[x(u,v), m_n j(u,v)] - f[x(u,v), j_n]| \leq K m_n |j(u,v) - j'(u,v)|,$$

it follows that

$$|a_n - f(x_n, j_n)| \leq \omega(\epsilon) I_n + K \iint_{O_n} |j(u,v) - j'(u,v)|\, dudv$$

and therefore that

$$\sum |a_n - f(x_n, j_n)| \leq 5K\epsilon + \{2\epsilon K + \omega(\epsilon)\} I.$$

A similar calculation shows that

$$\sum |a_n^* - f(x_n, j_n^*)| \leq 5K\epsilon + \{2\epsilon K + \omega(\epsilon)\} I.$$

Hence to show that $\Sigma(a_n - a_n^*)$ tends to zero with ϵ, and so to complete the proof of (3.1), it is enough to show that $j_n = j_n^*$ for all n.

Keeping n fixed, we approximate to O_n^* from the interior by an open polygonal domain Q^* whose rim is a polygon formed by segments parallel to the axes on which $x^*(u,v)$ is absolutely continuous. Then $T^{-1}Q^*$ approximates to O_n and therefore it is enough to prove that

$$\iint_{Q^*} j^*(u,v)\, dudv = \iint_{T^{-1}Q^*} j(u,v)\, dudv. \tag{3.7}$$

Since the rim of Q^* is a polygon, and so of finite length, the transformation T is constant on each prime end of the domain $T^{-1}Q^*$, so that if U is a positively oriented conformal transformation of the interior of R onto $T^{-1}Q^*$, TU is uniformly continuous in the interior of R. We denote by V a positively oriented transformation of the closure of Q^* onto R which is conformal in Q^*. Then VTU is a uniformly continuous map of the interior of R onto itself which can be extended to a map T^* of R into itself. We write $z^*(u,v) = x^*(V^{-1}[u,v])$ and $z(u,v) = z^*(T^*[u,v])$. Then $z(u,v) = x(U[u,v])$ for u, v interior to R. The formula (3.7) to be proved thus reduces to

$$\iint_R \zeta^*(u,v)\,dudv = \iint_R \zeta(u,v)\,dudv$$

where ζ^*, ζ are the Jacobians of $z^*(u,v)$ and $z(u,v) = z^*(T^*[u,v])$, and where in addition z^* describes a rectifiable curve on the perimeter of R. To complete the proof it is now sufficient to appeal to Stokes' theorem.[7]

Note. We may establish similarly to (3.2) a corresponding sewing theorem for a finite number of parametric surfaces with quasi-Lipschitzian representations. Moreover in (3.2) we may slightly strengthen the statement that $x(u,v)$ describes C_1 and C_2 on two complementary arcs of the perimeter of R. In fact, if we denote these arcs by β_1 and β_2, and we write a_1 and a_2 for the corresponding arcs $u \neq 1$ and $u \neq 0$ of the perimeter on which C_1 and C_2 are described by $x_1(u,v)$ and $x_2(u,v)$ respectively, the proof of (3.2) shows that the auxiliary functions $y(u,v)$, $y_1(u,v)$, $y_2(u,v)$, there introduced, are so related that $y(u,v)$ describes on β_1 and on β_2 the same oriented Fréchet curves as $y_1(u,v)$ on a_1 and $y_2(u,v)$ on a_2, respectively. Since $y_1(u,v)$ and $y_2(u,v)$ are nowhere constant on a_1 and a_2 respectively, this implies [7a, II. 1. 66(1), p. 75] the existence of sense-preserving monotone mappings T_n of β_n onto a_n such that $y(u,v) = y_n(T_n[u,v])$ on β_n $(n = 1, 2)$ and therefore such that

$$x(u,v) = x_n(T_n[u,v]) \text{ on } \beta_n \quad (n = 1, 2). \tag{3.8}$$

4. Topological types. We recall briefly the familiar process described in Seifert and Threlfall [9, §40, p. 142] for identifying certain sets of vertices and certain pairs of edges of a finite system of disjoint plane polygons. To suit our applications to parametric and to generalized surfaces, we shall indicate a few modifications of the usual definitions; they are necessary to us, but do not affect the elementary theory presented in Seifert and Threlfall to any material extent.

An identification process $\mathfrak{J}$ on the perimeter of R, or more generally on the perimeters of a finite number of disjoint plane polygonal domains G_n, consists of the following:

7. The theorem appealed to is, more precisely, the formula for an oriented area of W. H. Young [22], but we need it under slightly weaker assumptions. It is clearly sufficient to establish the formula for the corresponding oriented area of the projection on a coordinate plane. In this form it is established in Rado's book [7a, III 3.88 p. 253, IV 4.10 p. 368, IV 4.21 iii_3) p. 375] under assumptions satisfied in our case.

(a) a *subdivision* of the perimeters (oriented positively, i.e., counter-clockwise)[8] by a finite set W of points of division termed *vertices*, into a finite set of oriented arcs termed *positive edges*; by reversing the orientation we define the negative edges, and we term set of edges the combined set Λ of the positive and the negative edges; every edge has an initial, and a final, vertex;

(b) a *partition* of the sets W and Λ of the vertices and the edges into disjoint subsets by the relations $W = \Sigma W_i$, $\Lambda = \Sigma\Lambda_k$; members of a same subset W_i or Λ_k are termed *identified* (by $\mathfrak{I}$); members of different such subsets W_i are termed *distinct* vertices, while edges are distinct if neither member of a pair of them is identified with the other or with the other's negative; finally we term *free* an edge which is the sole member of a subset Λ_k.

The identification process $\mathfrak{I}$ is said to be *admissible* if (i) identified edges have identified negatives and identified initial vertices, (ii) no edge is identified with its own negative, nor with more than one other edge, and (iii) the following condition is satisfied: given any vertex A, each pair (α, β) of edges, with a same initial vertex identified with A, can be so ordered, and the set E of these ordered pairs can be so ordered, or so circularly ordered, that in any two consecutive pairs (α, β), (α', β') the edges β and α' are identified by $\mathfrak{I}$; if A is an initial or final vertex of a free edge, E is ordered and in that case we stipulate further that the first member of the first pair and the second member of the last pair are respectively positive and negative edges (necessarily free); if no free edge has an initial or final vertex identified with A, E is circularly ordered. Moreover $\mathfrak{I}$ is said to be *oriented* if in addition (iv) each positive edge, other than a free edge, is identified with a negative edge.

It follows from (iii) that an initial vertex of a free positive edge is identified with the final vertex of exactly one free positive edge, and therefore not identified with the initial vertex of any other free positive edge. Hence the free positive edges can be grouped into a finite number of circularly ordered sets, termed *rims*, in each of which the final vertex of an edge is identified by $\mathfrak{I}$ with the initial vertex of the next. We denote by s the *number of rims* of an admissible identification process $\mathfrak{I}$ and by κ the *characteristic* $-n_d + n_e - n_v$, where n_d is the number of domains G_n on whose perimeters $\mathfrak{I}$ is defined, n_e is the number of distinct edges and n_v is the number of distinct vertices.

Let $\mathfrak{I}$ and $\mathfrak{I}^*$ be admissible identification processes on the same, or on different, plane perimeters. We say that $\mathfrak{I}$ is *similar* to $\mathfrak{I}^*$ if there is a one to one correspondence between the two sets W, W^* of their vertices and also between the two sets of their positive edges and between those of their negative edges,

8. or, if a G_n has several perimeters, the outer one only counter-clockwise. We exclude this from now on, so as to avoid similar additional stipulations.

such that corresponding edges have corresponding negatives and corresponding initial vertices, and that vertices or edges, identified by $\mathfrak{I}$, correspond to vertices or edges, identified by $\mathfrak{I}^*$, and vice-versa. We say that $\mathfrak{I}^*$ is derived from $\mathfrak{I}$ by a simple *subdivision* if: the edges of one set Λ_k identified by $\mathfrak{I}$, and also their negatives, are subdivided into, and replaced by, pairs of edges, and the pairs of their negatives, from which the first members [9] of a pair, and the second members of a pair, are each identified by $\mathfrak{I}^*$ as are also the new vertices of the subdivision; otherwise edges or vertices identified by $\mathfrak{I}^*$ are those identified by $\mathfrak{I}$. We say that $\mathfrak{I}^*$ is derived from $\mathfrak{I}$ by a *simple dissection* if: one of the polygonal figures G_n, on whose perimeters $\mathfrak{I}$ is defined, is divided into two pieces by joining, by a simple polygonal arc interior to it, two of its vertices, and the two pieces are then moved apart, so as to be disjoint from one another and from the remaining G_n; the two new positive edges, arising from the dividing polygonal arc, are identified by $\mathfrak{I}^*$ with one another's negatives, while the remaining edges and vertices identified by $\mathfrak{I}^*$ are those identified by $\mathfrak{I}$ or arising from them by moving the two pieces apart. We say that $\mathfrak{I}^*$ is derived from $\mathfrak{I}$ by a *simple reflection* if: one of the polygonal figures, on whose perimeters $\mathfrak{I}$ is defined, is reflected in the u-axis and then moved, if necessary, to a position disjoint from the others; edges and vertices identified by $\mathfrak{I}^*$ are those identified by $\mathfrak{I}$ or arising from them by the reflection and the moving. (We note that the negative orientation of the perimeter becomes by reflection the positive orientation; and hence that the figure to be reflected must contain on its perimeter *all* or *none* of the free edges of a same rim, since we suppose both $\mathfrak{I}$ and $\mathfrak{I}^*$ admissible.) Finally $\mathfrak{I}$ and $\mathfrak{I}^*$ are termed *combinatorially equivalent* if there exist a finite number of admissible identification processes $\mathfrak{I} = \mathfrak{I}_0, \mathfrak{I}_1, \mathfrak{I}_2, \cdots, \mathfrak{I}_p = \mathfrak{I}^*$, such that, for each $r = 1, \cdots, p$, one of the identification processes $\mathfrak{I}_{r-1}, \mathfrak{I}_r$ is either similar to the other, or else derived from it by a simple subdivision, dissection, or reflection; if we exclude reflection, we say that $\mathfrak{I}$ and $\mathfrak{I}^*$ are *combinatorially equivalent with orientation*.

Given an admissible identification process $\mathfrak{I}$, defined on the perimeters of a finite number of polygonal domains G_n, we shall say of a vector-function $x(u, v)$, which takes values in X, that it is *subject to* $\mathfrak{I}$, if $x(u, v)$ is defined in the sum of the closures of the G_n and satisfies the conditions $x(T_n[u, v]) \in D$ where T_n is a conformal orientation-preserving transformation of R onto G_n, and if further $x(u, v)$ takes the same value at identified vertices and describes, on any two identified edges, a same 2-dimensionally thin, oriented Fréchet curve. Extending slightly the notation of the preceding sections, we say in this case that $x(u, v)$

9. The order in a pair is determined in the natural manner by that of the vertices of the relevant original positive or negative edge, prior to the subdivision.

belongs to the class $D_{\mathfrak{J}}$ and *represents the parametric surface* $L(f) = \Sigma L_n(f)$, where $L_n(f)$ is the parametric surface represented in previous sense by the function $x(T_n[u, v])$ defined on R. We observe that as (u, v) describes in their circular order the free positive edges of a rim of $\mathfrak{J}$, the function $x(u, v)$, subject to $\mathfrak{J}$, describes oriented arcs which fit together to form a closed Fréchet curve C: we say that $x(u, v)$ describes C on the rim in question. Given s such closed oriented Fréchet curves $C_1, \cdots, C_s$, where s is the number of rims of $\mathfrak{J}$, we denote by $D_{\mathfrak{J}}(C_1, \cdots, C_s)$ the class of functions $x \in D_{\mathfrak{J}}$ which describe $C_1, \cdots, C_s$ on these rims, and by $D_{\mathfrak{J}}(\pm C_1, \cdots, \pm C_s)$ the class of those which describe, on the rims, Fréchet curves which differ at most in orientation from $C_1, \cdots, C_s$.

(4.1) *Let* $\mathfrak{J}$ *and* $\mathfrak{J}^*$ *be oriented identification processes, combinatorially equivalent with orientation and let* $x \in D_{\mathfrak{J}}(C_1, \cdots, C_s)$ *represent the parametric surface* $L(f)$. *Then given* $\epsilon > 0$ *there is an* $x^* \in D_{\mathfrak{J}^*}(C_1, \cdots, C_s)$ *which represents a parametric surface* $L^*(f)$ *distant* $< \epsilon$ *from* $L(f)$.

(4.2) *Let* $\mathfrak{J}$ *and* $\mathfrak{J}^*$ *be admissible identification processes which are combinatorially equivalent, and let* $x \in D_{\mathfrak{J}}(\pm C_1, \cdots, \pm C_s)$ *represent the non-oriented parametric surface* $L(f)$. *Then given* $\epsilon > 0$ *there is an* $x^* \in D_{\mathfrak{J}^*}(\pm C_1, \cdots, \pm C_s)$ *which represents a non-oriented parametric surface* $L^*(f)$ *distant* $< \epsilon$ *from* $L(f)$.

PROOF of (4.1) and (4.2). We verify easily that the conclusion holds with $L^*(f) = L(f)$ in both assertions, if one of $\mathfrak{J}$, $\mathfrak{J}^*$ is similar to the other, or else derived from it by a simple subdivision, and that it holds in (4.2) if one of $\mathfrak{J}$, $\mathfrak{J}^*$ is derived from the other by a simple reflection. Hence we need only treat the case in which one of $\mathfrak{J}$, $\mathfrak{J}^*$ is derived from the other by a simple dissection. We may suppose by conformal transformation that the single polygonal figure affected by the dissection is the unit square R and that the dividing polygonal arc λ_0 is the segment $u = 1/2$ of R. We shall ignore the polygonal figures unaffected by the dissection without loss of generality, and we shall suppose in (4.2) that

$$x \in D_{\mathfrak{J}}(C_1, \cdots, C_s).$$

We shall distinguish two cases: (a) that in which $\mathfrak{J}^*$ is derived from $\mathfrak{J}$ by the dissection, (b) that in which $\mathfrak{J}$ is so derived from $\mathfrak{J}^*$.

Case (a). We cover λ_0 by a strip of area $< \delta$ on whose sides u has constant values for which $x(u, v)$ is absolutely continuous. For small δ, a parametric surface $L^*(f)$ distant $< \epsilon$ from $L(f)$ is then represented by the modification of $x(u, v)$ which differs from $x(u, v)$ only in the strip and is there defined by harmonic interpolation. Since this modification of $x(u, v)$ describes a 2-dimensionally thin curve on λ_0, we easily construct from it the required $x^*(u, v)$.

Case (b). Let R_n, Π_n be, for $n = 1$ or 2, the first or second half of R and its perimeter determined by the dissection. Let U_n be the transformation which moves R_n to the position it takes up for $\mathfrak{J}$; and let V_n be the positively oriented conformal transformation of R onto $U_n R_n$ which maps for $n = 1$ the side $u = 1$ onto $U_1 \lambda_0$, and for $n = 2$ the side $u = 0$ onto $U_2 \lambda_0$. Writing $x_n(u, v) = x(U_n[u, v])$, we conclude from our assumptions that $L(f)$ is the sum of the parametric surfaces $L_n(f)$ represented by $x_n \in D$ for $n = 1, 2$, and that $x_1(u, v)$, $x_2(u, v)$ describe for $0 \leq v \leq 1$ the same oriented, 2-dimensionally thin, Fréchet curve γ on the sides $u = 1$, $u = 0$ respectively.

By (3.3) there exists, for $n = 1$, 2, a quasi-Lipschitzian $x_n^*(u, v)$ which coincides with $x_n(u, v)$ on the perimeter of R and which represents a parametric surface $L_n^*(f)$ distant $< \epsilon/2$ from $L_n(f)$.

By (3.2) the parametric surface $L^*(f) = L_1^*(f) + L_2^*(f)$, distant $< \epsilon$ from $L(f)$, has a representation $x^{*\prime} \in D$, which, according to the note at the end of section 3, has on the perimeter of R the form $x_n^*(T_n^*[u, v])$ for $(u, v) \in \beta_n$ $(n = 1, 2)$, where β_1, β_2 are complementary arcs of the perimeter of R mapped monotonely by T_1^*, T_2^* respectively, onto the parts $u \neq 1$ and $u \neq 0$ of this perimeter. Thus on β_n we have

$$x^{*\prime}(u, v) = x(T_n[u, v]) \qquad (n = 1, 2)$$

where $T_n = V_n T_n^*$ is a monotone mapping of β_n onto $U_n \Pi_n$.

We now associate with the extremities of Π_n the corresponding extremities of β_n, and with each remaining vertex w^* of $\mathfrak{J}^*$, which therefore lies in Π_n for only one value of $n = 1, 2$, a vertex $w^{*\prime}$ arbitrarily selected from the appropriate set $(T_n)^{-1} U_n w^*$. We thus obtain an admissible identification process $\mathfrak{J}^{*\prime}$ similar to $\mathfrak{J}^*$ by associating edges correspondingly: an edge $\lambda^{*\prime}$ of $\mathfrak{J}^{*\prime}$ is then contained in β_n for $n = 1$ or 2, and is associated with the edge $\lambda^* = (U_n)^{-1} \lambda$ of $\mathfrak{J}^*$, where $\lambda = T_n \lambda^{*\prime}$ is an edge of $\mathfrak{J}$. Since $x^{*\prime}(u, v)$ describes on $\lambda^{*\prime}$ the same oriented Fréchet curve as $x(u, v)$ on λ, it follows that $x^{*\prime}(u, v)$ is subject to $\mathfrak{J}^{*\prime}$ and describes on its rims the curves $C_1, \cdots, C_s$. Thus $x^{*\prime} \in D_{\mathfrak{J}^{*\prime}}(C_1, \cdots, C_s)$ represents the parametric surface $L^*(f)$, and we conclude from the verifications at the beginning of this proof that there is an $x^* \in D_{\mathfrak{J}^*}(C_1, \cdots, C_s)$ which also represents $L^*(f)$, since $\mathfrak{J}^{*\prime}$ is similar to $\mathfrak{J}^*$. This completes the proof.

From now on, we shall only be concerned with admissible identification processes on the perimeter of R and we term these simply *identifications*. Such an identification is termed *orientable* if it is combinatorially equivalent to some oriented identification; otherwise it is *non-orientable*. Two identifications are combinatorially equivalent, if and only if (i) both are orientable or else both are non-orientable, and (ii) they have the same characteristic k and the same number s of rims; two oriented identifications are combinatorially equivalent with orientation, if and only if they have the same characteristic κ and the same number s of rims. The characteristic has the form $\kappa = s + 2r - 2$ in the oriented, or orientable, case where r is a non-negative integer (number of handles), and the form $\kappa = s + q - 2$ in the non-oriented case, where q is a positive integer (number of cross-caps).

A few comments on these statements are necessitated by the slight differences between our definitions and those adopted by Seifert and Threlfall. It is enough to show that an identification can be reduced to a certain normal form. To this effect we first reduce it to one in which each rim is a single edge, without the use of reflection. This is substantially as set out in Seifert and Threlfall. In the subsequent reductions described there, reflection is needed only in the non-oriented case, to transform certain positive edges into negative ones.

A parametric representation $x \in D$ will be said to be *of the type with* r *handles and* s *rims*, if there is an identification $\mathfrak{J}$ with r handles and s rims and a disjoint system of s oriented simple closed Fréchet curves $C_1, \cdots, C_s$, such that $x \in D_{\mathfrak{J}}(C_1, \cdots, C_s)$. If this $\mathfrak{J}$ can be chosen oriented, we say that $x \in D$ is of the oriented type. Instead of specifying the number of handles and that of rims, we may, of course, specify one of them and the characteristic. We say further, when we specify the disjoint system $C_1, \cdots, C_s$, that $x \in D$ is of the type with r handles and the rims $C_1, \cdots, C_s$. Similarly a parametric representation $x \in D_{\mathfrak{J}}(C_1, \cdots, C_s)$ where $\mathfrak{J}$ has q cross-caps and s rims and where the $C_1, \cdots, C_s$ are disjoint and simple, is said to be *of the type with* q *cross-caps and* s *rims*, or with the rims $C_1, \cdots, C_s$.

A parametric, or generalized, oriented surface $L(f)$ will be said to be *of the type with* r *handles and* s *rims* (or with the rims $C_1, \cdots, C_s$) if $L(f) = \lim L_n(f)$ where each $L_n(f)$ is a parametric surface with at least one representation $x_n \in D$ of the oriented type with r handles and the rims $C_1, \cdots, C_s$ (where the latter do not depend on n) and if further $L(f)$ is not expressible as such a limit when each $L_n(f)$ satisfies the corresponding condition with $r - 1$ in place of r. Similarly we define for *non-oriented* $L(f)$ the types with r handles and s rims, or with q cross-caps and s rims. An $L(f)$ of one of these types is termed of *finite topological type*. We observe that:

(4.3) *A parametric, or generalized, oriented surface* $L(f)$ *of the type with no handle and the one rim* C *is of the type of the disc with rim* C*; while one of the type with no handle and no rim is of the type of the sphere.*

PROOF. In the case in which there is no handle and one rim, we have merely to apply (4.1) after observing that an oriented identification with no handle and one rim is combinatorially equivalent with orientation to the identification in which the whole perimeter, divided at one point, is a single free positive edge, since the two identifications have the same characteristic (-1) and the same number of rims $(+1)$.

In the case in which there is no handle and no rim, we may suppose that the parametric representation $x_n(u, v)$, which occurs in the definition, is constant on some segment λ_0 of the perimeter. This can be arranged by a substitution affecting only the variable u, by which a small interval is transformed into a value u_0 such that the function of v, $x_n(u_0, v)$, is absolutely continuous and has a derivative almost everywhere whose square is integrable. We may suppose λ_0 interior to an edge λ of the identification with no handle and no rim to which x_n is subject by hypothesis. We subdivide this edge into three parts one of which is λ_0 and we subdivide suitably in two parts the edge identified with $-\lambda$, so as to construct an identification to which x_n is subject and which has characteristic -1 and (only) one free edge λ_0. Since this last identification is now one with no handle and one rim, and since x_n is constant on this rim, the conclusion now follows as before.

For brevity, we shall speak of a *parametric, or generalized, sphere* and of a *parametric, or generalized, disc*, instead of a parametric, or generalized, surface of the type of a sphere and a parametric, or generalized, surface of the type of a disc.

A generalized sphere is said to be *connected*, if it is not expressible as the sum of two non-null generalized spheres, otherwise it is *disconnected*; moreover it is termed *totally disconnected* if it is not expressible as the sum of a generalized sphere and a connected non-null generalized sphere. (We have changed here the terminology of [18], where a connected generalized sphere was termed a cycle, or cyclic element, by analogy with the terminology used in [11a] for generalized curves and with that of the cyclic decomposition theory for Fréchet surfaces. Moreover a totally disconnected generalized sphere was there termed a singular surface. Except for the change in terminology, the following statement merely repeats results established in [18].)

(4.4) (i) *Every generalized sphere is the sum of a totally disconnected sphere and a countable number of connected generalized spheres with bounded carriers such that the sum of their areas converges.* (ii) *Conversely, every countable sum of generalized spheres with bounded carriers such that the sum of their areas converges is a generalized sphere.* (iii) *A micro-sphere is totally disconnected. More generally, in order that a generalized surface be a totally disconnected sphere it* *is necessary and sufficient that it have the form*

$$\int M_x(f)\, d\mu$$

where μ *is a bounded measure with bounded carrier in the space* X *and* $M_x(f)$ *is a micro-sphere whose carrier is the variable point* $x \in X$.

For generalized surfaces other than spheres the term connected is defined in a less natural way. We first observe that:

(4.5) *Let* $s = s' + s''$ *and let* $(C_1', \cdots, C_{s'}')$, $(C_1'', \cdots, C_{s''}'')$ *be a partition of the system* $(C_1, \cdots, C_s)$ *of disjoint simple closed oriented Fréchet curves. Suppose further that* $L'(f)$ *and* $L''(f)$ *are of the oriented types with* r' *handles and the rims* $(C_1', \cdots, C_{s'}')$ *and with* r'' *handles and the rims* $(C_1'', \cdots, C_{s''}'')$. *Then* $L'(f) + L''(f)$ *is of a type with* r *handles and the rims* $(C_1, \cdots, C_s)$ *where* r *is a non-negative integer not exceeding* $r' + r''$.

PROOF. We express $L'(f)$ and $L''(f)$ as limits of parametric surfaces $L_n'(f)$ and $L_n''(f)$ with representations $x_n' \in D$ and $x_n'' \in D$ which are respectively of the type with r' handles and the rims $(C_1', \cdots, C_{s'}')$ and of the type with r'' handles and the rims $(C_1'', \cdots, C_{s''}'')$ and each of which is constant on some interval of the perimeter of R. We may suppose these intervals of constancy interior to two corresponding edges of the appropriate identifications and by conformal representation we may suppose that they are two opposite sides of the square R. The representation $x_n \in D$ derived from x_n' and x_n'' by a linear mapping on the two outer thirds $u \leq 1/3$ and $u \geq 2/3$ of R and by linear interpolation in u in the middle third then defines $L_n'(f) + L_n''(f)$ and is of the type with $r' + r''$ handles and the rims $(C_1, \cdots, C_s)$. This completes the proof.

There is a similar statement concerning the addition of non-oriented generalized surfaces (in terms of either handles or cross-caps, it being remembered that a handle counts as two cross-caps in a non-orientable identification). We observe however that in a variety of simple examples the sum $L'(f) + L''(f)$ has fewer handles or cross-caps than would be obtained by adding those of $L'(f)$ to those of $L''(f)$.

A generalized oriented surface $L(f)$ (other than a sphere) of the type T with r handles and the rims $C_1, \cdots, C_s$, will be said to be of the *disconnected* type T if it is expressible as a sum $L'(f) + L''(f)$ of two generalized surfaces, neither of which is a sphere, which satisfy the hypotheses of (4.5) and for which $r = r' + r''$; it is said to be of the *connected* type T if it is not of the disconnected type T; in the latter case it is termed *connected* if it is not the sum of a generalized sphere $\neq 0$ and a generalized surface of type T. The definitions in the non-oriented case are similar. From these definitions there follows:

(4.6) *A generalized surface $L(f)$ of finite topological type is the sum of a generalized sphere and a finite number of connected generalized surfaces with disjoint systems of rims whose union is the system of rims of $L(f)$, and with numbers of handles or cross-caps whose sum is the number of handles or cross-caps of $L(f)$ (a handle counting where necessary as two cross-caps).*

5. A decomposition theorem for certain limits of parametric surfaces. The main result of this section greatly strengthens a theorem previously stated [18, p. 75].

(5.1) *Let the generalized surface $L(f)$ be the limit of a parametric surface $L_n(f)$ with a representation $x_n \in D(N)$, whose restriction to the perimeter of R has uniformly bounded length and converges uniformly. Suppose further that $x_n(u, v)$ converges in the BL sense* [17 (b)] *to an $x(u, v) \in D$. Finally let*

$$L_n(f, \Lambda) = \iint_{\Lambda} f[x_n(u, v),\ j_n(u, v)]\,dudv$$

where Λ is an arbitrary subinterval of R and where $j_n(u, v)$ is the Jacobian of $x_n(u, v)$. Then (i) *there exists a subdivision of R into a finite number of subintervals Δ of arbitrarily small diameters, and a subsequence $\{n_k\}$, $k = 1, 2, \cdots$, of the positive integers n, such that for each Δ of this subdivision the limit of $L_n(f, \Lambda)$ as n describes this subsequence has the form*

$$L^*(f, \Delta) + \iint_{\Delta} M_{u,v}(f)\,dudv,$$

where $L^*(f, \Delta)$ is a generalized sphere and where $M_{u,v}(f)$ is a micro-representation whose carrier is $x(u, v)$. In particular (ii) *$L(f)$ is the sum of a generalized sphere $L^*(f)$ and a generalized surface possessing the micro-representation $M_{u,v}(f) \epsilon D^*$ with the carrier $x(u, v)$.*

We shall comment briefly on this statement before passing on to its proof. In the first place, the very stringent hypothesis concerning uniformly bounded lengths will not affect our ultimate applications where we deal with rims which need not even be rectifiable. In the second place, our principal conclusion is (ii), which provides an important decomposition for $L(f)$ on which nearly all our applications will be based. However (ii) reduces to triviality in the case in which $L(f)$ is itself a generalized sphere, and this is where we shall require (i).

In stating that (ii) is a particular case of (i), we mean that (ii) follows at once from (i) by adding together the contributions of the relevant Λ. Conversely however, we shall find that (i) follows relatively easily from (ii) and we shall ignore (i) from then on: this will enable us to reduce the general case of our theorem to one in which the generalized sphere $L^*(f)$ is absent, a reduction which is essential to the success of our methods.

In the earlier statement mentioned [18, p. 75] $x_n(u, v)$ was assumed to converge uniformly to a continuous $x(u, v)$ and $L^*(f)$ was then stated to be totally disconnected: this is an easy consequence of (ii). The main difference between this earlier statement and the present one consists in the assertion that $M_{u,v}(f)$ is a micro-representation and consequently a micro-disc for the relevant (u, v). The main force of our proof is directed towards establishing this fact, which constitutes, in view of, and together with, (2.2), the principal technical advance of this paper. (The technical difficulties to be overcome arise from the absence of a simple characterization of a micro-disc among the micro-surfaces with a given bivector as oriented area: such a characterization is known at present only when the surfaces are in 3-space.)

PROOF of (5.1). We divide the proof into several parts.

(a) Notation. $|W|$ denotes the plane measure of W where $W \subset R$, and $I(g, W)$ is the Dirichlet integral over W of a function $g \in D$. We write further $\lambda(W)$ for the linear measure of W and, in the case in which W consists of a subset of a finite number of segments parallel to the axes, we write $I_1(g, W)$, and term simple Dirichlet integral of $g(u, v)$ over W, the integral $\int g_u^2 du + g_v^2 dv$ on W. Finally we denote by $G(u_0, v_0; a)$ the intersection of the interior of R with the systems of lines $u = u_0 + ka$, $v = v_0 + ka$ $(k = 0, 1, 2, \cdots)$ where we suppose (u_0, v_0) interior to aR (i.e., $0 < u_0 < a$, $0 < v_0 < a$) and the distance a sufficiently small. We note that the double integrals with respect to (u_0, v_0) in aR of the quantities $\lambda(GW)$ and $I_1(g, G)$, where $G = G(u_0, v_0; a)$, have the values $2a\ |W|$ and $2aI(g, R)$.

(b) It will be sufficient to prove (ii). To see this, let $E(r, g)$ be the set of $(u_0, v_0) \in aR$ for which the function $g \in D$ is absolutely continuous on the grating $G = G(u_0, v_0; a)$ and satisfies the inequality $I_1(g, G) > 2rN/a$, and let $W_0 = \Pi_r \Sigma_s \Pi_n E(r, x_{s+n})$ extended over the positive integers r, s, n. Since $I(x_n, R) < N$ we have $|E(r, x_n)| < a^2/r$ and therefore $|W_0| = 0$. Choosing (u_0, v_0) outside W_0, we obtain, for an arbitrarily small a, a grating $G = G(u_0, v_0; a)$ such that $I_1(x_n, G) \leq 2rN/a$ for an r and a subsequence of n which depend on G. Furthermore $x_n(u, v)$ is absolutely continuous, and so continuous, on the segments of G, and tends uniformly [17(b), (5.2)] to $x(u, v)$ on G, as n describes the subsequence. Since the lengths of the curves described by these $x_n(u, v)$ on the perimeter of each interval Λ determined by the grating G are uniformly bounded (by Schwarz's inequality), we can derive the statement (i) from the corresponding statement (ii) by mapping each Δ onto R in the natural manner. This establishes (b).

(c) Let $D(N, l)$ be the class of functions defined in R which are members of $D(N)$ and which describe on the perimeter of R corresponding curves of lengths $\leq l$. We may suppose that $x_n(u, v)$ and $x(u, v)$ belong to $D(N, l)$ and that $L(f)$ is not decomposable in the form $L^*(f) + L_0(f)$ where $L^*(f)$ is a generalized sphere $\neq 0$ and where $L_0(f)$ is the limit of a parametric surface $L_{0n}(f)$ with a representation $x_{0n} \in D(N, l)$ which tends to $x(u, v)$ in the BL sense and tends uniformly to $x(u, v)$ on the perimeter of R.

For if $L(f)$ be so decomposable, we can write it $L_p^*(f) + L_{0p}(f)$ where as $p \longrightarrow \infty$ the areas of the generalized spheres $L_p^*(f)$ tend to the relevant least upper bound; the limit of a subsequence of $L_p^*(f)$ is a generalized sphere $L^*(f)$ of maximal area such that $L(f) = L^*(f) + L_0(f)$ where $L_0(f)$ is of the form stated. This clearly implies that $L_0(f)$ is not further decomposable, and it is sufficient, on account of (b), to prove our theorem for $L_0(f)$.

(d) We may suppose that the limit $L(f, \Delta)$ of $L_n(f, \Delta)$ exists for all f and Δ. To this effect we first select a subsequence such that this limit exists for countable dense sets of f and Δ, where the set of f includes in particular the function $f(x, j) = |j|$, and we write $\gamma(\Delta)$ for $L(f, \Delta)$ when f is this particular function. Since $\gamma(\Delta)$ is defined for a dense set of Δ and is additive, it has an extension γ^* as a measure of subsets of R, uniquely defined by the condition $\gamma^*(\Delta') \leq \gamma(\Delta) \leq \gamma^*(\Delta'')$ where Δ', Δ'' are the sets consisting of the interior and the closure of Δ. We denote by $\mathcal{E}^*$ the set of parallels to the axes interior to R on which γ^* is $\neq 0$. By standard arguments the limit $L(f, \Delta)$ for our subsequence exists whenever Δ does not have a side on $\mathcal{E}^*$. By selecting fresh subsequences the limit exists similarly whenever the sides of Δ on $\mathcal{E}^*$ are situated on the first k lines of $\mathcal{E}^*$. By the diagonal process we then obtain a subsequence such that $L_n(f, \Delta)$ tends to a limit for all f and Δ. By restricting ourselves to this subsequence, we justify (d).

(e) Let G be the grating $G(u_0, v_0; a)$ where a is sufficiently small and where (u_0, v_0) is selected as in the proof of (b), then for each of the intervals into which R is subdivided by G the expression $L(f, \Delta) + \iint_\Delta f[x(u, v), -j(u, v)]\, dudv$ is a generalized sphere. To see this let $\{n_k\}$, $k = 1, 2, \cdots$, and r be the subsequence of n and the positive integer defined in the proof of (b). Since $l_1(x_n, G) \leq 2rN/a$, the length of the curve described by $x_n(u, v)$ on the perimeter of Δ cannot exceed $l + (4a \cdot 2rNa^{-1})^{1/2}$ $(n \in \{n_k\})$. Since $x_n(u, v)$ tends uniformly to $x(u, v)$ on this perimeter as n describes $\{n_k\}$, this same bound is not less than the length of the corresponding curve described by $x(u, v)$. On a slightly larger interval concentric and similar to Δ we can therefore define an extension of the function $x_n(u, v)$ in

Δ so that the curve described on the outer perimeter coincides with that described by $x(u, v)$ on the perimeter of Δ and so that the area integral on the annulus thus added to Δ tends to zero as $n \to \infty$ in the subsequence $\{n_k\}$. This implies that there is a parametric surface whose distance from $L_n(f, \Delta)$ tends to zero as n describes $\{n_k\}$, and which possesses a representation $\in D$ (according to our sewing theorem) whose restriction to the perimeter describes the same rectifiable curve as $x(u, v)$ on the perimeter of Δ. Since the opposite curve is described by a suitable representation $\in D$ of the parametric surface $\iint_\Delta f[x(u, v), -j(u, v)]\,dudv$ on the perimeter, it follows (from our sewing theorem) that, for $n \in \{n_k\}$,

$$L_n(f, \Delta) + \iint_\Delta f[x(u, v), -j(u, v)]\,dudv$$

has from a corresponding parametric sphere a distance which tends to zero as n describes $\{n_k\}$, and therefore that the limit

$$L(f, \Delta) + \iint_\Delta f[x(u, v), -j(u, v)]\,dudv$$

is a generalized sphere. This establishes (e).

(f) Let G be as before and let G' be the subset of G consisting of the sum of the perimeters of those intervals of the subdivision of R by G which do not intersect the perimeter of R; then for large $n \in \{n_k\}$ there exist $x_n' \in D(N, l)$ such that $x_n'(u, v) = x(u, v)$ on G' and that for each interval Δ of the subdivision of R by G we have

$$L(f, \Delta) = \lim \iint_\Delta f(x_n', j_n')\,dudv$$

as n describes $\{n_k\}$, where j_n' is the Jacobian of x_n'; moreover $x_n' = x_n$ except in a set G'' depending on n such that $|G''| \to 0$ as $n \to \infty$.

This is very similar to (e) except that this time the actual values of $x_n'(u,v)$ on G' are prescribed (so that our sewing theorems are inapplicable) and we have to appeal to an elementary procedure typified by the following statement: Let $g(u)$ and $h(u)$ be absolutely continuous for $0 \le u \le 1$ and let the integrals of $g_u^2/2$, $h_u^2/2$, $(h-g)^2/2$ be not greater than A, B, $\epsilon^2 C$ respectively; then the function $\phi(u, v)$ equal for each u to $g(u) + \{h(u) - g(u)\}\,v/\epsilon$ for $0 \le v \le \epsilon$ has on the rectangle $0 \le u \le 1$, $0 \le v \le \epsilon$ a Dirichlet integral $\le \epsilon(A+B+C)$; similarly if we map the rectangle on the trapezium $0 \le u \le 1 - v$, $0 < v < \epsilon$ and write

$$\psi(u, v) = \phi\left(\frac{1+v}{1-v}\,u, v\right)$$

we find that the Dirichlet integral of ψ on this trapezium $\to 0$ as $\epsilon \to 0$. To establish (f) we form a set G'' (depending on n) by slightly "thickening" the lines of G', the "thickness" being of the order of magnitude of the maximum of $|x_n(u, v) - x(u, v)|$ on G' where we suppose $n \in \{n_k\}$ and sufficiently large. We denote by T the most natural linear mapping of the closure Q'' of any complementary domain (square or annulus) of G'' in R onto the closure Q' of the corresponding complementary domain of G' in R and we write $x'_n(u, v) = x(u, v)$ on G' and $x'_n(u, v) = x_n(T[u, v])$ for $(u, v) \in Q''$ for each Q''. By the elementary procedure described above, we now complete the definition of x'_n in the interior of $G'' - G'$ which we may suppose expressed as a sum of thin rectangles, so as to ensure that $I(x'_n, G'') \to 0$ as n describes $\{n_k\}$, and we then verify (f).

(g) γ^* is absolutely continuous. We shall suppose the contrary, i.e., that for each $p = 1, 2, \cdots$ there is a finite sum W_p of subintervals of R such that $|W_p| < 1/p$ while $\gamma^*(W_p)$ exceeds a fixed positive constant. We enclose W_p in the sum of a system S_p of intervals Δ determined by the grating G, defined as before, where the fineness a is chosen sufficiently small, in such a manner that $\Sigma |\Delta| < 1/p$ for the $\Delta \in S_p$ and that further S_p includes all those Δ, determined by the grating, which intersect the perimeter $\dot{R}$. Let $x_n^p(u, v) = x(u, v)$ in each $\Delta \in S_p$ and $x_n^p(u, v) = x'_n(u, v)$ elsewhere in R. Let $L^p(f)$ be the limit, as n describes $\{n_k\}$, of the parametric surface defined by $x_n^p(u, v)$. According to (e) and (f), $x_n^p(u, v)$ belongs to $D(N, l)$ for large $n \in \{n_k\}$ and tends to $x(u, v)$ as n describes $\{n_k\}$, and moreover, by summing $L(f, \Delta)$ for the Δ determined by G, we find that the sum of $L^p(f)$ and the generalized sphere

$$L^{*p}(f) = \sum_{\Delta \in S_p} L(f, \Delta) + \iint_{\Delta} f[x(u, v), -j(u, v)]\, du dv$$

differs from $L(f)$ by the expression

$$\sum_{\Delta \in S_p} \iint_{\Delta} \{ f[x(u, v), j(u, v)] + f[x(u, v), -j(u, v)]\}\, du dv$$

whose norm

$$\sum_{\Delta \in S_p} \iint_{\Delta} |j(u, v)|\, du dv$$

tends to zero as $p \to \infty$, since $\Sigma |\Delta|$ for $\Delta \in S_p$ tends to zero. According to (c), the generalized sphere $L^{*p}(f) \to 0$ as $p \to \infty$ and therefore

$$\sum_{\Lambda \in S_p} L(f, \Delta) \to 0$$

and in particular $\Sigma \gamma(\Lambda) \to 0$ for $\Lambda \in S_p$, which is impossible since $\Sigma \gamma(\Lambda)$ for $\Lambda \in S_p$ is not less than $\gamma^*(W_p)$ which exceeds a fixed positive constant. This contradiction establishes (g).

(h) The limit $M_{u,v}(f)$ of $L(f, \Lambda)/|\Delta|$, for regular sequences of intervals Δ tending to (u, v), exists, simultaneously in f, for almost every $(u, v) \in R$ and we have

$$L(f) = \iint_R M_{u,v}(f)\, du dv.$$

Moreover $M_{u,v}(f)$ has the value $j(u, v)$ when $f(x, j) = j$.

To see this, we observe that, for each f, $L(f, \Lambda)$ does not exceed the product $\gamma(\Lambda)\ \mathrm{Max}\,|f|$, where the maximum is for $j \in J_1$ and for x in a suitable bounded set, and therefore that $L(f, \Lambda)$ is absolutely continuous. Hence $L(f)$ is the integral of its derivative $M_{u,v}(f)$, and the latter exists, simultaneously for a dense countable set of f, and so simultaneously for all f, almost everywhere in R. Finally, for each square Δ determined by G and whose sides lie on G', we have $L(f, \Delta) = \mathrm{Lim} \iint_\Delta f(x_n', j_n')\, du dv$ as n describes $\{n_k\}$ and so

$$L(f, \Lambda) = \iint_\Delta j_n'(u, v)\, du dv = \iint_\Delta j(u, v)\, du dv$$

when $f(x, j) = j$, on account of Stokes's theorem,[10] since $x_n'(u, v) = x(u, v)$ on the perimeter of Δ. Since almost every $(u, v) \in R$ is limit of a square Δ, determined by G and with sides on G', for which the width a of G tends to zero, it follows that, when $f(x, j) = j$, $M_{u,v}(f)$ is almost everywhere the derivative of

$$\iint_\Lambda j(u, v)\, du dv,$$

and so has the value $j(u, v)$. This establishes (h).

(i) Given a sufficiently small $\epsilon > 0$, we determine [17 (a), (15.3), p. 329] an open set O of plane measure $< \epsilon^{10}$, which contains the perimeter of R and is such that $x(u, v)$ is Lipschitzian in $R - O$. Then the grating G defined in (b) can be chosen for each small a to satisfy the further conditions: $\lambda(OG) \leq 4\,\epsilon^{10}/a$, $l_1(x, G) \leq 4N/a$.

10. The form of Stokes's theorem actually used is essentially the formula for an area of W. H. Young [22]. On account of [17(b), (7.2)] this formula extends to discontinuous Dirichlet representations. See also [7a] and §3 above.

In fact, with the notation used in (b) the points (u_0, v_0) for which $G(u_0, v_0; a)$ violates, respectively, these two conditions, occupy two subsets of aR whose plane measures are both $< a^2/2$ according to (a).

(j) We denote by G^* the system of the squares of side a interior to R which are determined by the grating G defined in (i), where a is sufficiently small. We associate with this grating an integer $n^* \in \{n_k\}$ exceeding ϵ^{-1} so large that each $L(f, \Lambda)$, for which Δ is an interval determined by G, has a McShane distance $< \epsilon|\Delta|$ from $\iint_\Lambda f(x'_n, j'_n)\, du dv$ for $n = n^*$ and that further $|G''| < \epsilon$. This is possible according to (f).

(k) Let A_1 be the sum of those squares $\Delta \in G^*$ for which $I(x'_{n^*}, \Delta) < a^2 \epsilon^{-3}$ and for which there exists a linear function $Z_\Delta(u, v)$ such that, on the perimeter of Δ, the functions $x(u, v)$ and $Z_\Delta(u, v)$ differ by $< 3a\epsilon^3$ and describe curves of length $< 5a\epsilon^{-1}$. Then $|R - A_1| < \epsilon$. To see this, we first determine [12, (9.2)] a Lipschitzian function $z(u, v)$ which coincides with $x(u, v)$ in $R - O$, and we approximate [12, (12.2) p. 98] to the gradient $\phi(u, v) = (z_u, z_v)$ by a piecewise constant vector-function $\Phi(u, v)$ with $2m$ components, such that $|\phi - \Phi| < \epsilon^{10}$ except possibly in plane measure $< \epsilon^{10}$ of R. For small enough a, we may suppose, by altering Φ in small measure if necessary, that Φ is constant in each $\Delta \in G^*$. Hence [12, p. 99] there is a finite sum A of squares $\Delta \in G^*$ such that $|R-A| < \epsilon^2$ and that in each square $\Delta \in G^*$ situated in A the function $z(u, v)$ differs by at most $\epsilon^3 a$ from a corresponding function $Z_\Delta(u, v)$ linear in this square. We now determine in A a set A_0 consisting of the sum of the squares $\Delta \in G^*$ situated in A such that $I(x'_{n^*}, \Delta) < a^2\, \epsilon^{-3}$, $I_1(x, P) < 2a\epsilon^{-2}$, $\lambda(PO) < a\epsilon^8$, where P is the perimeter of Δ. A simple calculation shows that $|A - A_0| < (N + 8N + 8)\epsilon^2$ and hence that $|R - A_0| < 9(N+1)\epsilon^2 < \epsilon$ if ϵ is sufficiently small. Hence we need only show that A_1 contains A_0. To this effect, we observe that, by Schwarz's inequality, $x(u, v)$ describes, on P a curve of length $\leq \{4aI_1(x, P)\}^{1/2} < 4a\epsilon^{-1}$, and on PO arcs of total length $\leq \{\lambda(PO) I_1(x, P)\}^{1/2} < 2a\epsilon^3$; from this last inequality it follows that $x(u, v)$ differs on P from $Z_\Delta(u, v)$ by $< 3a\epsilon^3$. Consequently $Z_\Delta(u, v)$ describes on P a parallelogram of length $< 4a\epsilon^{-1} + 12a\epsilon^3 < 5a\epsilon^{-1}$. This establishes (k).

(l) Keeping ϵ fixed, we select our grating G to have sufficiently small width a, and we construct a certain $y(u, v) \in D(N, l)$ with the following properties: 1) outside measure 3ϵ of R, $y(u, v)$ coincides with one of the given functions $x_n(u, v)$ where $n > \epsilon^{-1}$; 2) on G', $y(u, v)$ coincides with $x(u, v)$; 3) on each $\Delta \in G^*$ situated in A_1, the oscillation of $y(u, v)$ is $< \epsilon$; 4) for each interval Δ determined by G, there exists a parametric sphere $L^*(f, \Delta)$ such that $L(f, \Delta)$ is at a McShane distance $< 5\epsilon|\Delta|$ from the parametric surface

$$L^*(f, \Delta) + \iint_\Delta f[y(u, v), i(u, v)]\, dudv$$

where $\iota(u, v)$ denotes the Jacobian of $y(u, v)$.

For this purpose, we define $L^*(f, \Delta) = 0$ and $y(u, v) = x'_{n^*}(u, v)(u, v) \in \Delta$, when Δ is an interval determined by G not situated in A_1. While for $\Delta \in G^*$ situated in A_1 we make use of a lemma in the theory of surfaces [14, (8.2) p. 297] with $\epsilon^4 K^{-1}$ in place of the parameter there denoted by ϵ, to obtain $y(u, v)$ for $(u, v) \in \Delta$, by a certain process of harmonic interpolation in small sets, from $x'_{n^*}(u, v)$. We apply this lemma, not to $x'_{n^*}(u, v)$ itself, but to the vector-function derived from it by adding the two further components u and v, and we denote by $y(u, v)$ the result of removing these two further components after the modification has been carried out. If W is the set of $(u, v) \in \Delta$ at which $y(u, v) \neq x'_{n^*}(u, v)$ then both $|W|$ and $\iint_W |i(u, v)|\, dudv$ are $< \epsilon^4|\Delta| + I(x'_{n^*}, \Delta) < 2a^2\epsilon$; consequently $y(u, v)$ coincides with $x_n(u, v)$, for $n = n^* > \epsilon^{-1}$, outside measure 3ϵ (since $x'_{n^*} = x_{n^*}$ outside G'' and $|G''| < \epsilon$). Further $y(u, v)$ coincides with $x_{n^*}(u, v)$ on the perimeter of R and its Dirichlet integral does not exceed that of $x'_{n^*}(u, v)$, so that $y(u, v) \in D(N, l)$. Also on $G'_1\, y(u, v)$ coincides with $x'_{n^*}(u, v)$ and so with $x(u, v)$. Also on each $\Delta \in G^*$ situated in A_1 the oscillation of $y(u, v)$, according to the lemma quoted, is less than $2(4a\epsilon^{-1} + 4a) \exp(K\epsilon^{-4})$ where K is an absolute constant; therefore, for sufficiently small a, this oscillation is $< \epsilon$. Finally by adding (for each $\Delta \in G^*$ situated in A_1) to the parametric surface defined by $x'_{n^*}(u, v)$ on the portions of the corresponding surface defined by $y(u, v)$ for which $x'_{n^*}(u, v) \neq y(u, v)$ and also their opposite sides, we make it split up into the sum of a parametric sphere $L^*(f, \Delta)$ and the surface $\iint_\Delta f[y(u, v), \iota(u, v)]\, dudv$ defined by $y(u, v)$ on Δ; consequently this sum is distant $< 4a^2\epsilon$ from $\iint_\Delta f(x'_{n^*}, j'_{n^*})\, dudv$ and is therefore at a McShane distance $< 5a^2\epsilon$ from $L(f, \Delta)$. Since the same inequality clearly holds in the intervals Δ determined by G not situated in A_1, this establishes (l).

(m) Let t^2 be the sum of the areas of the parametric spheres $L^*(f, \Delta)$ corresponding to the $\Delta \in G^*$ in A_1 according to (l), and let B be the sum of the squares $\Delta \in G^*$ in A_1 for which the area of $L^*(f, \Delta)$ is $\leq ta^2$. Then $|A_1 - B| < t$ and so $|R - B| < t + \epsilon$.

(n) Let $\{\epsilon_\nu\}$ $(\nu = 1, 2, \cdots)$ be a sequence of positive numbers tending to zero. The sum $\Sigma L^*(f, \Delta)$, summed for the Δ determined by G, will be denoted by $L^*_\nu(f)$ when ϵ_ν is substituted for ϵ, and similarly we shall write $y_\nu(u, v)$, $i_\nu(u, v)$, a_ν, G_ν, B_ν for $y(u, v)$, $i(u, v)$, a, G, B. Further we write $(t_\nu)^2$ for the area of $L^*_\nu(f)$ and b_ν for $t_\nu + \epsilon_\nu$. According to [17 (b)] there is a subsequence of the y_ν which converges in the BL sense, and also almost everywhere, to a function $\in D$ which, by (l) 1), can be identified with $x(u, v)$, since y_ν converges almost everywhere to the same limit as x_n. According to (c), this implies that $L_\nu^*(f) \to 0$ as ν describes the subsequence. We can therefore select this subsequence, and then identify it without loss of generality with the sequence of positive integers ν, so that moreover Σb_ν converges and is less than an arbitrarily given $b > 0$.

(o) In the set $E = \Pi B_\nu$ of measure $> 1 - b$, each point (u, v) is limit of a square Δ_ν determined by G_ν, such that $L(f, \Delta_\nu)$ has, from $\iint f(y_\nu, i_\nu)\, dudv$ over Δ_ν, a McShane distance $< 5\epsilon_\nu|\Delta_\nu| + t_\nu|\Delta_\nu| < 5b_\nu|\Delta_\nu|$ and moreover $y_\nu(u,v)$ has in Δ_ν an oscillation $< \epsilon_\nu < b_\nu$ and coincides on the perimeter of Δ_ν with $x(u,v)$. Finally, on this perimeter, there exists a linear function $Z_\nu(u, v)$ which differs from $x(u, v)$ by $< 3a_\nu(\epsilon_\nu)^3$ and the lengths of the curves described on the perimeter of Δ_ν by $x(u, v)$ and by $Z_\nu(u, v)$ are $< 5a_\nu(\epsilon_\nu)^{-1}$.

(p) Arguing similarly to (e), we obtain a parametric surface $\Lambda_\nu(f)$, with a representation $\in D$ whose restriction to the perimeter describes the same parallelogram as $Z_\nu(u, v)$ describes on the perimeter of Δ_ν, and whose diameter is $< 6a_\nu(\epsilon_\nu)^3 + \epsilon_\nu < 2\epsilon_\nu < 2b_\nu$, such that $\Lambda_\nu(f)$ is distant less than a constant multiple of $3a_\nu(\epsilon_\nu)^3\ 10a_\nu\epsilon_\nu^{-1} = 30(a_\nu)^2(\epsilon_\nu)^2 < \epsilon_\nu|\Lambda_\nu|$ from $\iint f(y_\nu, i_\nu)\,dudv$ over Δ_ν, and therefore such that $\Lambda_\nu(f)$ has a McShane distance $< 6b_\nu|\Delta_\nu|$ from $L(f, \Delta_\nu)$.

(q) According to (2.2) and the Note following (2.3) there is a flat-rimmed polyhedral disc $\Omega_\nu(f)$ whose diameter is $< 2b_\nu$ and whose McShane distance from $\Lambda_\nu(f)$ is $< b_\nu|\Delta_\nu|$. Consequently $\Omega_\nu(f)$ has a McShane distance $< 7b_\nu|\Lambda_\nu|$ from $L(f, \Delta_\nu)$, and $|\Delta_\nu|^{-1}\Omega_\nu(f)$ has a McShane distance $< 7b_\nu$ from $|\Delta_\nu|^{-1}L(f, \Delta_\nu)$.

(s) Let now (u', v') be any point of the set E, defined in (o), such that $x(u, v)$ is absolutely continuous on a parallel to the axes through the point and that $L(f, \Delta)$ has the derivative $M_{u',v'}(f)$. According to (q), $M_{u',v'}(f)$ is the limit of $|\Delta_\nu|^{-1}\,\Omega_\nu(f)$. Moreover the diameter of $\Omega_\nu(f)$ is $< 2b_\nu$ and its rim is distant $< 3a_\nu(\epsilon_\nu)^3 < b_\nu$ from the curve described by $x(u, v)$ on the perimeter of Δ_ν and therefore the carrier of $\Omega_\nu(f)$ has from the point $x(u', v')$ a distance which tends to zero as $\nu \to \infty$.

(t) According to (s) the limit of $|\Delta_\nu|^{-1}\,\Omega_\nu(f)$ is also that of its concentration at the point $x(u', v')$. Since this concentration is also that of the flat-rimmed polyhedral disc derived from $\Omega_\nu(f)$ by magnification in the scale $|\Delta_\nu|^{-1}$, the limit of $|\Delta_\nu|^{-1}\,\Omega_\nu(f)$ is a micro-disc, and therefore $M_{u',v'}(f)$ is a micro-disc with the carrier $x(u', v')$.

(u) Finally since according to (s), (u', v') is almost any point of E, and since $|R - E| < b$ which is arbitrarily small it follows that $M_{u,v}(f)$ is a micro-disc with carrier $x(u, v)$ almost everywhere in R.

(v) This completes the proof of (5.1).

6. The representation of a connected rimless generalized surface of finite

type. The main result of this section is:

(6.1) *Let $L(f)$ be a connected rimless generalized surface of finite topological type, oriented or non-oriented. Then $L(f)$ possesses a micro-representation whose carrier $x(u, v) \in D$ is of the same topological type. Furthermore, as (u, v) describes the perimeter of R, $x(u, v)$ describes a rectifiable curve which reduces to a point if $L(f)$ is a generalized sphere.*

A more general result will be established in a later section, partly by means of the same machinery, but we shall then need the following corollary of (6.1), which is the main object of the present section:

(6.2) *A rimless generalized surface of finite topological type, whose carrier is a 2-dimensionally thin set of points, is a totally disconnected sphere.*

PROOF that (6.1) implies (6.2). Let $x(u, v)$ be the carrier of the micro-representation $M_{u,v}(f)$ which our generalized surface possesses by hypothesis according to (6.1). The set of values taken by $x(u, v)$ is then contained in the sum of a 2-dimensionally thin set A and a set B such that the integral of $M_{u,v}(f)$, and therefore that of $|j(u, v)|$ the absolute value of the Jacobian of $x(u, v)$, is zero when extended over the set of (u, v) for which $x(u, v)$ takes values in B. According to [14, (4.3) p. 281], the integral of $|j(u, v)|$ vanishes also on the set of (u, v) for which $x(u, v)$ takes values in A. Thus $j(u, v)$ is zero for almost all (u, v); and $M_{u,v}(f)$ is therefore almost everywhere a micro-sphere and its integral is a totally disconnected generalized sphere. This completes the proof.

For reasons indicated in connection with (5.1), we divide the proof of (6.1) into two parts which deal, respectively, with the case in which $L(f)$ is a generalized sphere and with that in which $L(f)$ is of higher topological type. The first part will depend on the following lemma:

(6.3) *Given a parametric representation $x(u, v) \in D$, there is a parametric representation $x^*(u, v)$ derived from $x(u, v)$ by conformal transformation of R into itself, such that the two diagonals of R divide R into four triangles on which the area integrals $\iint |j^*(u, v)|\, du dv$ of $x^*(u, v)$ are equal.*

The proof of (6.3), which is elementary, will be given in the Appendix.

PROOF of (6.1) for the type of a sphere. Let $L(f)$ be a generalized sphere and let N be a constant exceeding its area. According to (6.3) and [17 (a), (10.2) p. 325], $L(f)$ is the limit of a sequence of parametric spheres $L_n(f)$, with continuous representations $x_n \in D(N)$ whose restrictions to the perimeter of R are constant, such that the four triangles into which R is divided by its diagonals contribute equal amounts to the area integral $\iint |j_n(u,v)|\,dudv$ of $x_n(u,v)$, where j_n is the Jacobian of x_n.

Let (Γ) denote, for any circular disc Γ of centre the origin, the conformal transformation of Γ onto R such that the centres, and the directions through the centres parallel to the positive axis of u, correspond; under (Γ), the segments of the two domains, through the centres, parallel to the axes or to their bisectors, correspond also, for reasons of symmetry. Define $y_n(r,\theta)$ to be constant in the complement of the interior of a particular $\Gamma = \Gamma_0$ and to be given (in polar coordinates) in Γ_0 by the value $x_n(u,v) \in X$ for which r, θ are the polar coordinates of $T_0(u,v)$ where $T_0 = (\Gamma_0)$ and suppose Γ_0 chosen with centre at $r = 0$ and radius so large that the concentric unit circular disc contributes just half the area integral on Γ_0 of $y_n(r,\theta)$ (in polar coordinates).

We denote by Γ_1 a concentric circular disc of radius r_1 such that $1/2 < r_1 < 1$ and that, for a subsequence of n, a fixed multiple of N exceeds the expressions

$$\int_0^{2\pi} \left[\frac{\partial y_n(r_1,\theta)}{\partial \theta}\right]^2 d\theta;$$

the existence of Γ_1 follows as in part (b) of the proof of (5.1) and we find, as there, that, for a further subsequence of n which we identify for notational purposes with that of all positive integers, the function y_n tends uniformly to a limit function on $r = r_1$ and describes on this circle a curve whose length does not exceed a constant l independent of n.

Let now $x_n'(u,v)$ and $x_n''(u,v)$ be, for $(u,v) \in R$, the values of $y_n(r,\theta)$ and $y_n(r_1^2 r^{-1},\theta)$ when (r,θ) are the polar coordinates of $T_1(u,v)$, where $T_1 = (\Gamma_1)$, and let γ be the transform under T_1 of the circle $r = 1/2$. We observe that $x_n' \in D(N)$ and $x_n'' \in D(N)$, and that further:

(a) on the perimeter of R the functions $x_n'(u,v)$ and $x_n''(u,v)$ coincide for each n and converge uniformly to a limit function as $n \to \infty$; moreover they describe a curve of length $< l$;

(b) the sum of the area integrals of $x_n'(u,v)$ and $x_n''(u,v)$ is divided into four equal parts when R is divided into four triangles by its diagonals;

(c) the area integral of $x_n'(u,v)$ over R coincides with that of $x_n''(u,v)$ on a domain contained in R which has γ in its interior, where γ is, from its definition given above, a simple closed curve independent of n which surrounds the centre of R;

(d) we have $L_n(f) = L_n'(f) + L_n''(f)$ where

$$L_n'(f) = \iint_R f(x_n', j_n')\,dudv, \quad L_n''(f) = \iint_R f(x_n'', -j_n'')\,dudv,$$

and where j_n', j_n'' are the Jacobians of x_n', x_n''.

We select, in accordance with [17(b), (5.4)], a subsequence of n, which we again identify with that of all positive integers, such that x_n', x_n' converge in the BL sense to limit functions x', x'' belonging to D, and that moreover $L_n'(f)$ tends to a limit $L'(f)$ for each f.

Then by (5.1), $L'(f)$ is the sum of a generalized sphere $L'^*(f)$ and a generalized surface possessing a micro-representation $M'_{u,v}(f)$ with carrier $x'(u, v)$. Moreover, by (c) above, the area of $L'(f)$ and so that of $L'^*(f)$ is at most half that of $L(f)$.

Similarly, after interchanging the variables u and v so as to change the signs of the Jacobians, we see that $L''(f)$ is the sum of a generalized sphere $L''^*(f)$ and a generalized surface, possessing a micro-representation $M''_{u,v}(f)$ with carrier $x''(u, v)$. We observe further that if $L''^*(f)$ is connected then it follows from (5.1) (i) that there exist an arbitrarily small interval Δ and a subsequence of n such that the area of $L''^*(f)$ does not exceed that of the expression

$$\lim_n \iint_\Delta f(x_n'', -j_n'')\, dudv$$

and therefore that this area does not exceed half that of $L(f)$ on account of (b) and (c) above. Thus $L''^*(f)$ is either disconnected or of area at most half that of $L(f)$.

Consequently $L(f) = L^*(f) + \iint_R \{M'_{u,v}(f) + M''_{u,v}(f)\}\, dudv$ where $L^*(f) = L'^*(f) + L''^*(f)$ is a generalized sphere which is either disconnected or else of at most half the area of $L(f)$, and therefore in any case different from $L(f)$; and where $M'_{u,v}(f)$ and $M''_{u,v}(f)$ are micro-discs with the carriers $x'(u, v)$ and $x''(u, v)$. Now according to (2.2) and (4.3), the expression $\iint_R \{M'_{u,v}(f) + M''_{u,v}(f)\}\, dudv$ is a generalized sphere. Since $L(f)$ is connected this implies that $L^*(f) = 0$.

This being so, we determine a parallel to the axis of u on which the simple Dirichlet integrals of x' and x'' are finite and on which x' and x'' are absolutely continuous and we denote by H a strip of R with this parallel as its central line such that a preassigned $\epsilon > 0$ exceeds the sum $I(x', H) + I(x'', H)$ of the Dirichlet integrals of x' and x'' in the strip. We denote further by T a mapping of R onto itself by a piecewise linear function $\phi(v)$ of v only, such that T maps the central half of H onto its central line and H onto itself and that T is the identity outside H.

We write $x^{(1)}(u, v) = x'(T[u, v])$, $x^{(2)}(u, v) = x''(T[u, v])$, $M^{(1)}_{u,v}(f) = M'_{T[u,v]}(f) \cdot \phi'(v)$, $M^{(2)}_{v,u}(f) = M''_{ST[u,v]}(f) \cdot \phi'(v)$, where $\phi'(v)$ denotes the derivative of $\phi(v)$ and S is the operation which permutes the two cartesian coordinates of a point of the plane. Then $M^{(1)}_{u,v}(f)$ and $M^{(2)}_{u,v}(f)$ are micro-discs with the carriers $x^{(1)}(u, v)$ and $x^{(2)}(u, v)$. Moreover the functions $x^{(1)}(u, v)$ and $x^{(2)}(u, v)$ belong to D, coincide on the perimeter and are constant on an interval of this perimeter. Finally $L(f)$ has in terms of $M^{(1)}_{u,v}(f)$ and $M^{(2)}_{u,v}(f)$ an expression wholly analogous to its expression in terms of $M'_{u,v}(f)$ and $M''_{u,v}(f)$. This state of affairs persists when R is transformed conformally onto itself so as to map such an interval of constancy onto the arc formed by three sides of R, and we retain for simplicity the same notations after the conformal transformation and take the fourth side of R to be the line $v = 0$.

Since $x^{(1)}(u, v)$ and $x^{(2)}(u, v)$ then coincide for $v = 0$, the functions $x(u, v) = x^{(1)}(u, 2v-1)(v \geq 1/2)$, $x(u, v) = x^{(2)}(u, 1-2v)(v \leq 1/2)$ belongs to D and is the carrier of the micro-disc

$$M_{u,v}(f) = 2M^{(1)}_{u,2v-1}(f)(v \geq 1/2),\ M_{u,v}(f) = 2M^{(2)}_{1-2v,u}(f)(v \leq 1/2).$$

Moreover we have $L(f) = \iint_R M_{u,v}(f)dudv$ and this completes the proof of (6.1) for the type of a sphere.

For the higher topological types, we use a lemma that we intend to prove in a sharper form in [21]. It concerns a "pinching constant" for a polyhedron, substantially the minimal length of its one-cycles not homotopic to 0.

(6.4) *Let $\mathfrak{T}$ be the class of rimless polyhedra of the connected type T. Then there exist $k = k(T) > 0$ and, for each $L(f) \in \mathfrak{T}$ of area $A > 0$, a "pinching" constant $c = c(L) > 0$ with the following properties:*

(i) *there is a totally disconnected polyhedral sphere of area $\leq c^2/2$ such that $L(f) + L^*(f)$ is either of the disconnected type T or of lower type;*

(ii) *there is a representation $x(u, v) \in$ Pol of $L(f)$, subject to an identification of the type T such that the perimeter of R is subdivided into $\leq k$ parts, which describes a length $\leq kA/c$ on the perimeter of R and, for positive each $\epsilon < c/k$, a length $\leq k\,\epsilon$ on every arc of the perimeter of R whose extremities can be joined in R by an arc on which $x(u, v)$ describes a length ϵ.*

In conjunction with (6.4) we use the following result

(6.5) *Let $L(f)$ be a parametric surface of finite topological type with no rim (or with polygonal rims). Then there is a sequence of polyhedra $L_n(f)$ of the same type with no rim (or with the same rims) such that $L(f) = \lim L_n(f)$.*

The proof of (6.5) need only be sketched, since the sewing argument of (3.2), on which it depends, is now sufficiently familiar. By a small alteration of $L(f)$ if necessary, we may suppose that $L(f)$ has a quasi-Lipschitzian representation $x(u, v)$, subject to an identification $\mathfrak{J}$ of the topological type possessed by $L(f)$ with no rim (or with polygonal rims). We may suppose further that $x(u, v)$ has a rectifiable restriction to the perimeter of R.

This last can be arranged, without altering $L(f)$, as follows: we first alter $x(u, v)$ near the vertices, by cutting off from R, for each set of identified vertices of $\mathfrak{J}$, small polygonal neighborhoods of these identified vertices, and by fitting these polygonal neighborhoods together as a separate polygonal domain in which the identified vertices become a single (vertex or) interior point and finally by substituting, for this point, another more satisfactory point in its neighborhood and then reversing the previous procedure; in this way we arrange that $x(u, v)$ describes rectifiable curves on sufficiently small arcs of the perimeter which contain the relevant vertices of $\mathfrak{J}$, next we cut off portions of R, abbutting on slightly shortened arcs of the perimeter between two consecutive vertices, and sew them onto corresponding portions of identified arcs. The final perimeter after each alteration may be supposed to be again that of R, by making a conformal transformation. Ultimately we thus obtain a new representation $x(u, v)$ which describes a rectifiable curve on the perimeter of R.

Again by a small alteration of $L(f)$, we can arrange that the neighborhood of the perimeter of R is the sum of a finite number of simply-connected domains on whose rims $x(u, v)$ is rectifiable and in whose interiors $x(u, v)$ is harmonic. Finally, keeping $x(u, v)$ fixed except in the immediate neighborhood of the perimeter, we approximate to $x(u, v)$ by a piecewise linear function on the perimeter and by the corresponding harmonic interpolation near the perimeter; according to the theorem of Carlson and Young [16 (b)], this alters only a small area of the surface represented, provided the length of the curve described on the perimeter is not increased; hence by a small alteration of $L(f)$ such as just described, we may suppose $x(u, v)$ piecewise linear on the perimeter of R. This final reduction now renders the statement to be established a corollary of (2.2) and so completes its proof.

We need one further lemma, which we take from [17 (a)] but which is not stated explicitly there. We denote (in this lemma only) by $\lambda[f(W)]$ and $d[f(W)]$ respectively the length and the diameter of the curve described by $f(u, v)$ when (u, v) describes an arc $W \subset R$. We say further that a sequence of functions $f_n(u, v)$ satisfy a 3-point condition on the perimeter of R if there exist three points a_n, b_n, c_n of this perimeter whose mutual distances exceed some fixed $\delta > 0$, such that the values of f_n at these three points tend to three distinct limits as $n \to \infty$.

(6.6) *Given an infinite sequence S of functions $\in D(N)$ whose restrictions to the perimeter of R are not equi-continuous and which satisfy a 3-point condition, there is a subsequence $\{f_n\}$ of S, a corresponding sequence $\{W_n\}$ of cross-*

cuts of R, and a positive number a, such that, if W_n', W_n'' are the two arcs of the perimeter of R with the same extremities as W_n, we have $d[f_n(W_n')] \geq a$, $d[f_n(W_n'')] \geq a$ and $\lambda[f_n(W_n)] \to 0$ as $n \to \infty$.

This statement is established by the same proof as a weaker one given in [17 (a), (5.1) (i) p. 320].

PROOF of (6.1) for a type other than that of a sphere. According to (6.5), $L(f)$ is the limit of a sequence of polyhedra $L_n(f)$ of the same topological type. Clearly by (6.4) (i) the pinching constants of the polyhedra $L_n(f)$ have a positive lower limit; we may therefore suppose that they have a positive lower bound δ_0. Moreover the areas of the $L_n(f)$ remain less than a constant N. Consequently the polyhedra $L_n(f)$ possess representations $x_n(u, v)$ with the properties stated in (6.4) (ii).

According to [17 (a), (10.2) p. 325], $L_n(f)$ has a representation $x_n' \in D(N)$ Fréchet equivalent to x_n, and we can arrange, by conformal transformation and by the selection of a subsequence of n, that the $x_n'(u, v)$ satisfy a 3-point condition on the perimeter of R, since $L(f)$ is not a generalized sphere.

In view of (6.6), the functions $x_n'(u, v)$ are equi-continuous on the perimeter of R, while by (6.4) (ii) they describe on this perimeter curves of length uniformly bounded by a constant l and are subject to corresponding identifications $\mathcal{J}_n$ for each of which the perimeter of R is subdivided into a uniformly bounded number of arcs.

According to [17 (b) (5.4)] there is a subsequence of the x_n' converging in the BL sense in R, and converging uniformly on the perimeter of R, to an $x(u, v) \in D(N, l)$, of which we see at once that it is subject to an identification $\mathcal{J}$ of the same type as all the $\mathcal{J}_n$, i.e., of the type of $L(f)$. On account of (5.1), this completes the proof.

7. An application. We can now derive an important variant of (5.1).

(7.1) *Let $x_n \in D(N)$ describe for $n = 1, 2, \cdots$ a sequence of representations of parametric surfaces $L_n(f)$ which tend to a generalized surface $L(f)$. Suppose further that the x_n are subject to identifications $\mathcal{J}_n$, which arise from subdivisions of the perimeter into a bounded number of parts and which belong to a same topological type T, and that they describe on the perimeter:*

on the free arcs, a same system $C_1, \cdots, C_s$ of 2-dimensionally thin, disjoint rims;

on the rest of the perimeter, rectifiable curves whose total length does not exceed a fixed constant, independent of n.

Then (a), *if the x_n converge uniformly to a continuous $x \in D$ there exists a* *totally disconnected sphere* $L^*(f)$ *and a micro-disc* $M_{u,v}(f)$ *with carrier* $x(u,v)$, *such that*

$$L(f) = L^*(f) + \iint_R M_{u,v}(f)\,dudv.$$

Moreover (b) *this conclusion holds with* $L^*(f) = 0$ *if we merely suppose that the* x_n *converge in the* BL *sense to an* $x \in D$ *and uniformly on the perimeter of* R, *provided that* $L(f)$ *is connected of the type* T.

PROOF. (a) As in (5.1) we select a subsequence of n which may be supposed to consist of all positive integers, along which the parametric surface

$$L_n(f, \Lambda) = \iint_\Lambda f(x_n, j_n)\,dudv,$$

where j_n is the Jacobian of x_n, tends for each subinterval Δ of R to a limit $L(f, \Delta)$, and we denote by $M_{u,v}(f)$ the derivative of $L(f, \Lambda)$ for almost every (u, v). We shall extend the notation $L_n(f, \Lambda)$, $L(f, \Delta)$ by addition to the case in which Δ is a figure expressible as the sum of a finite number of subintervals of R.

For each positive $\epsilon < 1$, let R_ϵ denote an interval interior to R which contains the square concentric to R of sides $1 - \epsilon$, such that the sides of R_ϵ are situated on four lines in R, on each of which, with the notation of the proof of (5.1), the simple Dirichlet integrals of the x_n are bounded functions of n. The existence of R_ϵ follows from [17 (b)] as in the definition of BL convergence.

According to (5.1), applied to the representations derived from the x_n when R_ϵ is mapped onto R, there is a totally disconnected sphere $L^*_\epsilon(f)$ such that

$$L(f, R_\epsilon) = L^*_\epsilon(f) + \iint_{R_\epsilon} M_{u,v}(f)\,dudv.$$

Let $L^*_0(f)$ be the limit of $L^*_\epsilon(f)$ when ϵ describes a suitable sequence tending to zero. By selecting R_ϵ anew for the other values of ϵ if necessary, we may suppose that $L^*_0(f)$ is the limit of $L^*_\epsilon(f)$ as $\epsilon \to 0$ in any manner. Then $L^*_0(f)$ is a totally disconnected sphere. The functional

$$L^*(f) = L(f) - \iint_R M_{u,v}(f)\,dudv$$

may now be written

$$L^*(f) = \lim_\epsilon L(f) - L(f, R_\epsilon) + L^*_0(f) = L^{**}(f) + L^*_0(f),$$

and we have only to show that the generalized surface

$$L^{**}(f) = \lim_{\epsilon} L(f, R-R_\epsilon) = \lim_{\epsilon} \lim_{n} L_n(f, R-R_\epsilon)$$

is a totally disconnected sphere. According to (6.2) we need only verify two things:

(i) The carrier of $L^{**}(f)$ is two-dimensionally thin;

(ii) L^{**} is of finite topological type with no rim.

The verification of (i) is immediate: since the x_n converge uniformly to a continuous limit $x(u, v)$, the carrier of $L^{**}(f)$ is clearly a subset of the set of values taken by $x(u, v)$ on the perimeter of R and the latter is situated on the sum of the rims $C_1, \cdots, C_s$ and a bounded number of rectifiable curves; this set is therefore 2-dimensionally thin.

In order to verify (ii), we first modify $L_n(f, R-R_\epsilon)$ in accordance with the argument of part (f) of the proof of (5.1). We do this by an elementary mapping T of $R-R_\epsilon$ onto the annulus derived from $R-R_\epsilon$ by slightly thickening the inner perimeter and by defining a representation $y_{n,\epsilon}(u, v)$ equal in $T(R-R_\epsilon)$ to $x_n(T^{-1}[u, v])$ and completed in the rest of $R-R_\epsilon$ by interpolation, as in the argument referred to, so as to coincide with $x(u, v)$ on the inner perimeter. If the thickening is a sufficiently small function of n and we denote by $i_{n,\epsilon}$ the Jacobian of $y_{n,\epsilon}$, we find that the distance of $L_n(f, R-R_\epsilon)$ from

$$\int\int_{R-R_\epsilon} f(y_{n,\epsilon}, i_{n,\epsilon})\, du dv$$

tends to zero as $n \to \infty$. Since $f[x(u, v), -j(u, v)]$, where $j(u, v)$ is the Jacobian of $x(u, v)$, is integrable in R, it follows that

$$L^{**}(f) = \lim_{\epsilon} \lim_{n} L_{n,\epsilon}(f)$$

where

$$L_{n,\epsilon}(f) = \int\int_{R-R_\epsilon} f(y_{n,\epsilon}, i_{n,\epsilon})\, du dv + \int\int_{R-R_\epsilon} f[x(u, v), -j(u, v)]\, du dv.$$

Since $L_{n,\epsilon}$ is easily seen to be of a bounded topological type with no rim, the same is true of $L^{**}(f)$ and we verify (ii). This completes the proof of (a).

(b) It is clear that $x(u, v)$ is subject to an identification of the type T and describes the rims $C_1, \cdots, C_s$ on the free arcs. We have to prove that $L(f)$ has a micro-representation $M_{u,v}(f)$ with the carrier $x(u, v)$. We shall derive this result from (7.1). By applying the procedure of [17 (a), (20.2) p. 333] as modified slightly in [17 (b), (7.1)], we derive from the x_n, for each $\epsilon > 0$, a corresponding sequence of equi-continuous representations $x_{n,\epsilon} \in D(N)$ with the following properties:

(i) we have $x_n(u, v) = x_{n,\epsilon}(u, v)$ except at most in an open subset O_n of the interior of R such that the area integral of $x_{n,\epsilon}(u, v)$ on O_n is $< \epsilon$;

(ii) the passage from x_n to $x_{n,\epsilon}$ occurs in a countable number of steps at most, and each step consists in replacing the representation which results from previous steps, by its harmonic interpolation in a simply-connected subdomain of the interior of R; moreover if ϵ_ν denotes, at the ν-th step, the area integral of the harmonic interpolation over the subdomain, we have $\Sigma \epsilon_\nu < \epsilon$.

Now we may regard each such harmonic interpolation as effecting a corresponding modification of the parametric surface $L_n(f)$ defined by $x_n(u, v)$ for $(u, v) \in R$, the modifications being made successively. At the ν-th stage, the modification consists in adding the surface integral of the new (the $(\nu + 1)$-st) representation for the integrand $f(x, j)$ over the subdomain in which interpolation occurs, and in removing the corresponding part of the old (the ν-th) surface integral. If we add also the corresponding surface integral of the new representation for the integrand $f(x, -j)$ and then remove it again, the modification may be described as the addition of a totally disconnected sphere of area $2\epsilon_\nu$ followed by the removal of a parametric sphere. Hence $L_{n,\epsilon}(f)$, the parametric surface represented by $x_{n,\epsilon}$, arises from $L_n(f)$ by adding a totally disconnected sphere of area $2\Sigma\epsilon_\nu < 2\epsilon$ and then by removing a generalized or parametric sphere. Keeping ϵ fixed, and making n describe a suitable subsequence, we deduce that

$$L(f) + L'_\epsilon(f) = L''_\epsilon(f) + \lim_n L_{n,\epsilon}(f), \tag{7.2}$$

where $L'_\epsilon(f)$ is a totally disconnected sphere of area $\leq 2\epsilon$ and $L''_\epsilon(f)$ is a generalized sphere.

We write, as previously, $L_n(f, \Delta)$ for the double integral over Δ of $f(x_n, j_n)$, where j_n is the Jacobian of x_n and where Δ is a figure expressible as the sum of a finite number of subintervals of R. Similarly we define $L_{n,\epsilon}(f, \Delta)$ in terms of $x_{n,\epsilon}$ and we write $L(f, \Delta)$ for the limit of $L_n(f, \Delta)$ for a suitable subsequence of n, independent of Δ: according to part (d) of the proof of (5.1), this limit exists, and so does, for a fixed ϵ, that of $L_{n,\epsilon}(f, \Delta)$. We denote their derivatives in Δ respectively by $M_{u,v}(f)$ and $M_{u,v,\epsilon}(f)$.

We determine further, as in part (b) of the proof of (5.1), a grating G of side a on which, for a suitable subsequence of n (to which we restrict ourselves), the sum of the simple Dirichlet integrals of $x_n(u, v)$ and $x_{n,\epsilon}(u, v)$ is a bounded function of n. We denote by R_0 the subinterval of R formed by the squares determined by the grating G whose sides are interior to R, and we choose the side a of our grating so small that the double integrals over $R - R_0$ of the areas of $M_{u,v}(f)$ and $M_{u,v\epsilon}(f)$ are both $< \epsilon$; this is possible since these functions are certainly integrable in R by standard theorems.

We now finally restrict further our subsequence of n so as to make the $x_{n,\epsilon}(u, v)$ converge uniformly to a continuous limit function in R. We can apply (5.1) with R_0 in place of R to the two sequences $L_n(f)$ and $L_{n,\epsilon}(f)$ and we can apply (7.1) to the sequence $L_{n,\epsilon}(f)$. We thus obtain the three equations

$$L(f, R_0) = L_0^*(f) + \iint_{R_0} M_{u,v}(f)\,dudv,$$

$$\lim_n L_{n,\epsilon}(f, R_0) = L'''_{\epsilon,0}(f) + \iint_{R_0} M_{u,v,\epsilon}(f)\,dudv, \tag{7.3}$$

$$\lim_n L_{n,\epsilon}(f) = L'''_{\epsilon}(f) + \iint_{R} M_{u,v,\epsilon}(f)\,dudv,$$

where $L_0^*(f)$, $L'''_{\epsilon,0}(f)$, $L'''_{\epsilon}(f)$ are generalized spheres, the last two totally disconnected. Moreover $M_{u,v}(f)$ and $M_{u,v,\epsilon}(f)$ are micro-discs with the respective carriers $x(u, v)$ and $\lim_n x_{n,\epsilon}(u, v)$. Combining the last two equations with (7.2) and with the corresponding relation on R_0

$$L(f, R_0) + L'_{\epsilon,0}(f) = L''_{\epsilon,0}(f) + \lim_n L_{n,\epsilon}(f, R_0)$$

where $L'_{\epsilon,0}(f)$ and $L''_{\epsilon,0}(f)$ are generalized surfaces whose areas do not exceed those of $L'_\epsilon(f)$ and $L''_\epsilon(f)$ respectively, and writing $L''_\epsilon(f) + L'''_\epsilon(f) = L^*_\epsilon(f)$, $L''_{\epsilon,0}(f) + L'''_{\epsilon,0}(f) = L^*_{\epsilon,0}(f)$, we obtain in their place

$$\begin{cases} L(f, R_0) + L'_{\epsilon,0}(f) = L^*_{\epsilon,0}(f) + \displaystyle\iint_{R_0} M_{u,v,\epsilon}(f)\,dudv, \\[2ex] L(f) + L'_\epsilon(f) = L^*_\epsilon(f) + \displaystyle\iint_{R} M_{u,v,\epsilon}(f)\,dudv, \end{cases} \tag{7.4}$$

where $L'_\epsilon(f)$, $L^*_\epsilon(f)$ are generalized spheres, the former totally disconnected and of area $\leq 2\epsilon$, and where $L'_{\epsilon,0}(f)$, $L^*_{\epsilon,0}(f)$ are generalized surfaces, the former of area $\leq 2\epsilon$.

Let $\omega(\epsilon)$ denote the area of $L^*_\epsilon(f)$, and let ϵ describe a sequence tending to zero such that the terms occuring in the second equation (7.4) tend to corresponding generalized surfaces. Since $\lim L'_\epsilon(f) = 0$, $\lim L^*_\epsilon(f)$ is a generalized sphere, and $\iint_R M_{u,v,\epsilon}(f)\,dudv$ according to (2.2) has for each $\epsilon > 0$ the same topological type and the same rims as $L(f)$, we can now conclude, from the fact that $L(f)$ is connected, that the generalized sphere $\lim L^*_\epsilon(f)$ vanishes, and therefore that $\omega(\epsilon) \to 0$.

By substracting the first equation (7.4) from the second, we obtain

$$L(f, R-R_0) = \iint_{R-R_0} M_{u,v,\epsilon}(f)\,dudv + L^*_\epsilon(f) + L'_{\epsilon,0}(f) - L^*_{\epsilon,0}(f) - L'_\epsilon(f)$$

so that the area of the generalized surface $L(f, R-R_0)$ is at most $\epsilon + \omega(\epsilon) + 2\epsilon$, which tends to zero for our sequence of ϵ.

Hence finally by the first equation (7.3), denoting by $L^*(f)$ a suitable limit of $L^*_0(f)$ as $\epsilon \to 0$, we find that

$$L(f) = L^*(f) + \iint_R M_{u,v}(f)\,dudv$$

which in its turn requires $L^*(f) = 0$ since $L^*(f)$ is a generalized sphere and $\iint_R M_{u,v}(f)\,dudv$ is, by (2.2), of the same type as $L(f)$ and possessed of the same rims, while $L(f)$ is connected by hypothesis. This shows that $L(f)$ possesses the micro-representation $M_{u,v}(f)$ with the carrier $x(u, v)$, and so completes the proof.

8. Representation of a connected generalized surface of finite topological type whose rims are 2-dimensionally thin. In this section we follow in part the treatment of the special case of a connected generalized disc with rectifiable rim [18, (12.4) p. 78–82]. Our main theorem is as follows:

(8.1) *Let $L(f)$ be a connected generalized surface of finite topological type T whose rims are disjoint and 2-dimensionally thin. Then there exists a micro-representation*

$$L(f) = \iint_R M_{u,v}(f)\,dudv$$

where $M_{u,v}(f)$ has a carrier $x(u, v) \in D$ subject to an identification of the type T with the same rims.

With regard to this theorem, we note that a connected generalized surface of type T may well be a disconnected generalized surface of a higher type according to our definitions (§4). In proof we shall appeal to an elaboration of (6.4) which runs as follows:

(8.2) *Let* $\mathfrak{T}$ *be the class of generalized surfaces of the connected type* T *whose rims consist of a given set* $C = (C_1, \cdots, C_s)$ *of disjoint 2-dimensionally thin simple closed curves. Then there exist:* $k = k(T) > 0$; $a = a(C) \geq 0$; $b = b(\epsilon, A, T) > 0$ *defined for* $\epsilon > 0$, $A > 0$; *and for each* $L(f) \in \mathfrak{T}$ *of area* $A - a > 0$, *a constant* $c = c(L) > 0$ *with the following properties:*

(i) *the inequality* $c < b$ *implies the existence of a totally disconnected sphere* $L^*(f)$ *of area* $< \epsilon$ *such that* $L(f) + L^*(f)$ *is either of the disconnected type* T *or of lower type with the same rims;*

(ii) $L(f)$ *is the limit of a parametric surface with a continuous representation* $x_n \in D$, *which is subject to an identification of the type* T *such that the perimeter of* R *is subdivided into* $\leq k$ *parts* λ, *and which describes, on the perimeter of* R, *the rims* $C_1, \cdots, C_s$ *on the sum of the free arcs* λ *and rectifiable curves of total length* $\leq kA/c$ *on the rest of the perimeter of* R, *and finally which describes, for each positive* $\epsilon < c/k$, *a rectifiable curve of length* $\leq k\epsilon$ *on every arc of the perimeter of* R, *disjoint from the free arcs, whose extremities can be joined in* R *by an arc on which* $x_n(u, v)$ *describes a rectifiable curve of length* ϵ.

The proof of (8.2), like that of (6.4), will be omitted, since we intend to present it in [21] for parametric surfaces $L(f)$ with representations $\in$ Pol-int subject to an identification of the type T, and the general case easily reduces to this one, on account of (2.2).

PROOF of (8.1). We may suppose by (6.1) that the set of rims is not empty. According to (8.2) (ii), $L(f)$ is the limit of a parametric surface $L_n(f)$ with a representation $x_n \in D$ as stated there.

According to [17 (a), (10.2) p, 325], we may further suppose that there is a fixed constant N such that $x_n \in D(N)$. Moreover we may clearly suppose that the x_n satisfy a 3-point condition on the perimeter of R. According to (6.6), the $x_n(u, v)$ have equi-continuous restrictions to the perimeter of R. By selecting a subsequence, we may now suppose that the x_n converge in the BL sense to an $x(u, v)$ whose restriction to the perimeter is the uniform limit of the corresponding restrictions. Clearly $x(u, v)$ is subject to an identification of the type T and describes the rims $C_1, \cdots, C_s$ on the free arcs. Our assertion is now a direct consequence of (7.1) (b).

9. Existence of continuous variational solutions. In this section, we denote by $\mathfrak{K}$ the class of generalized surfaces of finite topological types with given rims and with at most a given number of handles or cross-caps, and we suppose that the rims consist of a finite (possibly empty) system of disjoint 2-dimensionally thin simple closed curves $C_1, \cdots, C_s$. A micro-representation of a member of $\mathfrak{K}$ will be termed *admissible* if its carrier $x(u, v) \in D$ is of a type with the given rims and at most the given number of handles or cross-caps; it will be termed *continuous* if $x(u, v)$ is continuous. Finally, we term an $L(f) \in \mathfrak{K}$ *at most finitely disconnected*, if its decomposition according to (4.6) and (4.4) (i) consists of a finite number of connected generalized surfaces, described in (4.6), together with a *finite number only* of connected generalized spheres. We write for brevity $\mathfrak{K}_c$ for the subclass of $\mathfrak{K}$ whose members are at most finitely disconnected and possess continuous admissible micro-representations.

To avoid unnecessary distinctions, a parametric integrand $f_0(x, j)$ will as usual be understood to be symmetric when the members of $\mathfrak{K}$ are non-oriented. We term the parametric integrand $f_0(x, j)$ *totally irregular* if there exist a non-vanishing micro-sphere $M(f)$ such that $M(f_0) \le 0$. A generalized surface $L(f) \in \mathfrak{K}$ will be termed *totally non-critical* if there exist a sequence of generalized surfaces $L_n(f) \in \mathfrak{K}$ tending strongly to $L(f)$, such that, given any parametric integrand $f_0(x, j)$ which is *not* totally irregular, the inequality $L_n(f_0) < L(f_0)$ holds for all large n. A class of generalized surfaces will be termed totally non-critical if each of its members is so. The main result of this section is:

(9.1) $\mathfrak{K} - \mathfrak{K}_c$ *is totally non-critical.*

Given a finite non-empty ordered set of parametric integrands $f_0, f_1, \cdots, f_k$, let us denote by $\mathfrak{P}$ the problem of the minimum of $L(f_0)$ for surfaces $L(f) \in \mathfrak{K}$ of uniformly bounded areas situated in a bounded portion of the space X, subject to the k subsidiary inequalities $L(f_\nu) \le a$ $(\nu = 1, \cdots, k)$ where k may be zero. This problem will be termed *acceptable* if there exists a member of K which satisfies the subsidiary inequalities, and if moreover none of the integrands f_0, $f_1, \cdots, f_k$ is totally irregular. Clearly (9.1) implies that for such a problem the minimum cannot be attained in $\mathfrak{K} - \mathfrak{K}_c$. On account of the principle of minimum [18, (5.2) p. 64], it follows that (9.1) has as a corollary:

(9.2) *(Existence of minimizing continuous micro-representation.) If $\mathfrak{P}$ is acceptable, the minimum is attained in $\mathfrak{K}_c$, but never in $\mathfrak{K} - \mathfrak{K}_c$.*

We observe that the condition of acceptability excludes only problems which are to some extent trivial. Except in so far as we suppose the rims 2-dimensionally thin,[11] this condition replaces enormously more stringent conditions imposed

11. The elimination of this hypothesis is one of the aims of the theory of the boundary which originates in [20] and in [19].

by Danskin, Sigalov and Cesari in the existence theorem referred to in the introduction, which of course concerns only the type of the disc. It goes without saying that the minimum is attained for a *parametric* surface in $\mathfrak{R}_c$ if the integrands $f_\nu (\nu = 0, \cdots, k)$ satisfy the weak convexity condition in j: $M(f_\nu) \geq f(x, j)$ for every polyhedral micro-disc of oriented area j whose carrier is the point x.

Before establishing (9.1) we introduce some auxiliary results needed in its proof. We first remark that:

(9.3) *In order that the integrand $f_0(x, j)$ be not totally irregular, it is necessary and sufficient that to each bounded portion E of the space X there correspond positive numbers b and c such that $L(f_0) \geq cA$ for every polyhedral sphere $L(f)$ of area $A \leq b$ situated in E.*

In fact, if the condition is satisfied, we find that, by a simple passage to the limit, every micro-sphere $M(f)$ of area $A \leq b$ whose carrier lies in E satisfies the condition $M(f_0) \geq cA > 0$, and therefore that $f_0(x, j)$ is not totally irregular. On the other hand, if the condition is violated, there exists for

$$b = c = 1/n \, (n = 1, 2, \cdots)$$

a polyhedral sphere $L_n(f)$ of area $A_n \leq 1/n$ such that $L_n(f_0) < n^{-1} A_n$; writing $L_n^*(f) = k_n L_n(f)$ where k_n is the integral part of A_n^{-1}, and denoting by $L^*(f)$ the limit of $L_n^*(f)$ for a suitable subsequence of n, we see that $L^*(f)$ is a totally disconnected sphere of area 1 for which

$$L^*(f_0) \leq \lim n^{-1} k_n A_n \leq \lim n^{-1} = 0,$$

and therefore, on account of (4.4) (iii), that there exists a non-vanishing micro-sphere $M(f)$ such that $M(f_0) \leq 0$; in other words, $f_0(x, j)$ is then totally irregular.

Our next lemma is:

(9.4) *Let $x(u, v)$ be a continuous special Dirichlet representation whose oscillation on the perimeter of R is d, and let $a(W)$ denote the area integral of $x(u, v)$, i.e., the integral of the absolute value of its Jacobian, on an arbitrary measurable subset W of R. Then given $\epsilon > 0$, there exist a finite number of continua $\gamma_1, \cdots, \gamma_k$ interior to R, which bound disjoint simply-connected subdomains of R whose sum is an open set O, such that*

(i) *the square of the total intrinsic length described by $x(u, v)$ on $\gamma_1, \cdots, \gamma_k$ does not exceed $\pi \epsilon a(O)$,*

(ii) *the oscillation of $x(u, v)$ on $R - O$ is $\leq d + 4b$ where $b^2 = \epsilon^{-1} a(R)$.*

This is really a variant, due to Cesari, of the lemma in the theory of surfaces [13; 14, (8.2) p. 297; 17, (16.1) p. 330] on which so much of our machinery is ultimately based, but we have not, so far, found a satisfactory single version to replace both. For a proof of (9.4), we refer to the Appendix, since Cesari's is restricted to the case of a polyhedron, in accordance with his general scheme, so beautifully carried out, of reducing everything to polyhedra.

Since [14, (7.2) p. 294] the boundary length of O described by $x(u, v)$ is at most twice the length described on $\gamma_1 + \cdots + \gamma_k$, it follows from the isoperimetric property of a harmonic surface $L^2 \leq 4\pi A$, and from (9.4) (i) that we have the following corollary:

(9.5) *With the notation and hypotheses of* (9.4), *let* $x^*(u, v)$ *denote the function which coincides with* $x(u, v)$ *outside* O *and with its harmonic interpolation in each of the constituent domains of* O, *and let* $a^*(W)$ *denote the area integral of* $x^*(u, v)$ *on an arbitrary measurable subset* W *of* R. *Then* $a^*(O) \leq \epsilon a(O)$.

From these results, we deduce:

(9.6) *Given a positive* $\epsilon < 1$, *let* $L(f)$ *be a generalized surface which possesses a micro-representation* $M_{u,v}(f)$ *with a continuous quasi-Lipschitzian carrier* $x(u, v) \in D(N)$. *Suppose further that the values of* $x(u, v)$ *lie in a bounded convex set* E, *that the oscillation of* $x(u, v)$ *on the perimeter of* R *is* $< \epsilon$ *and that the area of* $L(f)$ *is* $< \epsilon^3/100$. *Denote by* O *and by* $x^*(u, v)$ *the open set and the function defined in correspondence with* $x(u, v)$ *in lemmas* (9.4) *and* (9.5), *and write* $L^*(f)$ *for the generalized surface with the micro-representation* $M^*_{u,v}(f) = M_{u,v}(f)$ *for* (u, v) *outside* O *and* $M^*_{u,v}(f) = f[x^*(u, v), j^*(u, v)]$ *for* (u, v) *in* O, *where* $j^*(u, v)$ *is the Jacobian of* $x^*(u, v)$. *Finally let* A *denote the integral over* O *of the area of* $M_{u,v}(f)$. *Then*

(i) $x^*(u, v)$ *is the carrier of the micro-representation of* $L^*(f)$;

(ii) $x^*(u, v) \in D(N)$ *and coincides with* $x(u\ v)$ *on the perimeter of* R;

(iii) $x^*(u, v)$ *has oscillation* $< 2\epsilon$ *in* R;

(iv) *the distance of* $L^*(f)$ *from* $L(f)$ *is* $\leq 2A < \epsilon$;

(v) *for every parametric integrand* $f_0(x, j)$ *which satisfies* (9.3) *in* E *with positive constants* $b \geq \epsilon$ *and* $c \geq 3\epsilon Q(f_0, E)$, *where* $Q(f_0, E)$ *denotes the maximum of* $f_0(x, j)$ *for* $x \in E$ *and* $j \in J_1$, *we have*

$$L^*(f_0) \leq L(f_0) - \epsilon^2 A\, Q(f_0, E).$$

We need only prove (v). It follows from the sewing theorem (3.2), coupled with conformal transformation onto R of each constituent domain of O, that the integrals over O of

$$M_{u,v}(f) + f[x^*(u, v), - j^*(u, v)] \text{ and } M^*_{u,v}(f) + f[x^*(u, v), - j^*(u, v)]$$

are each expressible as the sum of k generalized spheres, and so as a single generalized sphere, which we denote, in the two cases under consideration, by $L''(f)$ and $L'(f)$. The area of $L''(f)$ is $\leq (\epsilon^3/100) + \epsilon a(O) \leq \epsilon^3/50 < b$ and that of $L'(f)$ is $\leq 2\epsilon a(O)$; hence

$$L'(f_0) \leq 2\epsilon a(O) Q(f_0, E) \leq ca(O) - \epsilon a(O) Q(f_0, E) \leq L''(f_0) - \epsilon a(O) Q(f_0, E).$$

Therefore, by putting $f = f_0$ in the identity $L^*(f) + L''(f) = L(f) + L'(f)$, we obtain $L^*(f_0) \leq L(f_0) - \epsilon a(O) Q(f_0, E)$. In the case in which $\epsilon A \leq a(O)$ this is our assertion; in the opposite case we define a generalized surface $L^0(f)$ by assigning to it the micro-representation $M^0_{u,v}(f) = M_{u,v}(f)$ for (u, v) outside O and $M^0_{u,v}(f) = f[x(u, v), j(u, v)]$ for $(u, v) \in O$ where $j(u, v)$ is the Jacobian of $x(u, v)$, and we obtain as above $L^*(f_0) \leq L^0(f_0) - \epsilon a(O) Q(f_0, E) \leq L^0(f_0)$; moreover arguing as before with the generalized spheres expressed as integrals over O of

$$M_{u,v}(f) + f[x(u, v), - j(u, v)] \text{ and } M^0_{u,v}(f) + f[x(u, v), - j(u, v)],$$

we find, in the case in which $\epsilon A > a(O)$, that $L^0(f_0) \leq L(f_0) - \epsilon A\, Q(f_0, E)$ from which our assertion again follows since $L^*(f_0) \leq L^0(f_0)$ and $\epsilon < 1$.

Finally from (9.6) we deduce yet another lemma:

(9.7) *Given a positive $\epsilon < 1$, let $L(f)$ be a generalized surface which possesses a micro-representation $M_{u,v}(f)$ with an arbitrary carrier $x(u, v) \in D(N)$. Suppose further that the values of $x(u, v)$ lie in a bounded convex set E, that the oscillation of $x(u, v)$ on the perimeter of R is $<\epsilon$, and that the area of $L(f)$ is $< \epsilon^3/100$. Then either $M_{u,v}(f)$ has a second carrier $x^*(u, v) \in D(N)$ which coincides with $x(u, v)$ on the perimeter of R and has in R an oscillation $\leq 2\epsilon$; or else there exists, at a distance $\leq \epsilon$ from $L(f)$, a generalized surface $L^*(f)$ with the following properties:*

(i) *$L^*(f)$ is the limit of a sequence of generalized surfaces $L^*_n(f)$ which possess micro-representations with corresponding carriers $x^*_n(u, v)$, where the latter belong to $D(N)$, coincide with $x(u, v)$ on the perimeter of R, and have in R oscillations $\leq 2\epsilon$;*

(ii) *we have*

$$L^*(f_0) < L(f_0)$$

whenever $f_0(x, j)$ *is a parametric integrand which satisfies* (9.1) *in* $'E$ *with positive constants* $b \geq \epsilon$ *and* $c \geq 3\epsilon Q(f_0, E)$, *where* $Q(f_0, E)$ *denotes the maximum of* $f_0(x, j)$ *for* $x \in E$ *and* $j \in J_1$.

PROOF. According to [17 (b), (7.2)] and the remarks preceding (2.2) above, there exists a sequence of continuous quasi-Lipschitzian representations $x_n(u, v) \in D(N)$, such that $x_n(u, v)$ coincides with $x(u, v)$ outside a subset W_n interior to R on which $\iint \{1 + |j_n(u, v)|\} dudv \to 0$ as $n \to \infty$, where $j_n(u, v)$ denotes the Jacobian of $x_n(u, v)$. We denote by $L_n(f)$ the generalized surface with the micro-representation $M_{u,v}(f)$ for (u, v) outside W_n and $f[x_n(u, v), j_n(u, v)]$ for $(u, v) \in W_n$; we observe that the distance of $L_n(f)$ from $L(f)$ tends to zero, since the integral over W_n of $|j_n(u, v)|$ tends to zero, and since that of the area of $M_{u,v}(f)$ tends to zero with the measure of W_n.

Applying (9.6), where we place the additional suffix n on all the expressions concerned, we obtain a generalized surface $L_n^*(f)$ with a micro-representation whose carrier is a function $x_n^*(u, v) \in D(N)$, where $x_n^*(u, v)$ coincides with $x_n(u,v)$ and with $x(u, v)$ on the perimeter of R and has oscillation $< 2\epsilon$ in R; moreover $L_n^*(f)$ is at a distance $\leq 2A_n < \epsilon$ from $L_n(f)$, where A_n is such that for every parametric integrand $f_0(x, j)$ which satisfies the stated conditions we have

$$L_n^*(f_0) \leq L_n(f_0) - \epsilon^2 A_n Q(f_0, E).$$

In the case in which A_n does not tend to zero, our conclusions now follow by taking for $L^*(f)$ the limit of a suitable subsequence of the $L_n^*(f)$.

On the other hand if $A_n \to 0$ then the distance of $L(f)$ from $L_n^*(f)$ tends to zero and moreover a suitable BL limit $x^*(u, v)$ of $x_n^*(u, v)$ has oscillation $\leq 2\epsilon$ in R. Outside $W_n + O_n$ we have $x^*(u, v) = x_n^*(u, v) = x(u, v)$, so that $x^*(u, v)$ is the carrier of $M_{u,v}(f)$ outside $\Pi(W_n + O_n)$ and therefore almost everywhere outside the set $\Pi(O_n - W_n)$ on which, for each n, the integral of the area of $M_{u,v}(f)$ is $\leq 2A_n$ by the definition of A_n. Thus the area of $M_{u,v}(f)$ is zero almost everywhere in $\Pi(O_n - W_n)$, so that its carrier is still $x^*(u, v)$ in this set wherever the Jacobian $j^*(u, v)$ of $x^*(u, v)$ vanishes. To complete the proof it is therefore sufficient to show that the assumption $A_n \to 0$ implies $j^*(u, v) = 0$ almost everywhere in $\Pi(O_n - W_n)$, i.e., that the integral over $\Pi(O_n - W_n)$ of $|j^*(u, v)|$ vanishes. Now this integral is certainly not greater than

$$\iint_{O_n + W_n} |j^*(u, v)|\, dudv = \iint_{O_n + W_n} |j_n^*(u, v)|\, dudv + B_n$$

where

$$B_n = \iint_R |j^*(u, v)|\, dudv - \iint_R |j_n^*(u, v)|\, dudv,$$

and since

$$\int\int_{O_n+W_n} |j_n^*(u, v)|\, dudv = \int\int_{O_n} |j_n^*(u, v)|\, dudv + \int\int_{W_n-O_n} |j_n^*(u, v)|\, dudv$$

$$\leq \epsilon A_n + \int\int_{W_n} |j_n(u, v)|\, dudv$$

tends to zero, it is sufficient to prove that $\limsup B_n \leq 0$. This is a consequence of the following semi-continuity lemma for area of parametric surfaces, which we state separately:

(9.8) *Let* $x_n(u, v) \in D$ *converge to* $x(u, v)$ *in the* BL *sense in* R *and uniformly on its perimeter, and let* a_n *and* a *denote the areas of the parametric surfaces defined by* $x_n(u, v)$ *and by* $x(u, v)$. *Then* $a \leq \liminf a_n$.

This result is actually a corollary of [17 (b), (7.1)] and of the classical semi-continuity theorem for Lebesgue area, so that the proof of (9.7) is now completed.

Our final lemma is a proposition worth stating for its own sake:

(9.9) *Suppose that* $L(f)$ *possesses a micro-representation* $M_{u,v}(f)$, *with a carrier* $x(u, v) \in D(N)$ *but no continuous carrier* $\in D(N)$ *which coincides with* $x(u, v)$ *on the perimeter of* R. *Then* $L(f)$ *is a strong limit of generalized surfaces of the form* $L_n(f) = \lim_p L_{n,p}(f)$, *where each* $L_{n,p}(f)$ *possesses a micro-representation with a quasi-Lipschitzian carrier* $\in D(N)$ *which coincides with* $x(u, v)$ *on the perimeter of* R, *and where, for every parametric integrand* f_0 *not totally irregular, we have* $L_n(f_0) < L(f_0)$ *for all large* n.

PROOF. We write $L(f, \Delta)$ for the integral of $M_{u,v}(f)$ over any figure $\Delta \subset R$.

In the case in which $x(u, v)$ is continuous on the perimeter of a subinterval Δ of R we write $\omega(\Delta)$ for the least oscillation in Δ of a carrier $x^*(u, v)$ of $M_{u,v}(f)$ such that $x^*(u, v)$ coincides with $x(u, v)$ on the perimeter of Δ. We write further $\omega(\epsilon)$ for the supremum of $\omega(\Delta)$ when Δ is a subinterval of R, on whose perimeter $x(u, v)$ is continuous and has oscillation $\leq \epsilon$, such that the area of $L(f, \Delta)$ is $\leq \epsilon^3/100$.

Suppose first that $\omega(\epsilon) > 2\epsilon$ for some arbitrarily small $\epsilon = \epsilon_n$ tending to 0. There would then be, for $\epsilon = \epsilon_n$ an interval Δ_ϵ such that the area of $L(f, \Delta_\epsilon)$ is $\leq \epsilon^3/100$, that $x(u, v)$ is continuous on the perimeter of Δ_ϵ and has oscillation $\leq \epsilon$ on this perimeter but no carrier of $M_{u,v}(f)$ which coincides with $x(u, v)$ on this perimeter has oscillation $\leq 2\epsilon$ in Δ_ϵ. By applying (9.7) to the generalized surface $L(f, \Lambda_\epsilon)$ after a simple mapping of Δ_ϵ onto R, we deduce that there exists, at a distance $<\epsilon$ from $L(f, \Delta_\epsilon)$, a generalized surface $L^*(f, \Lambda_\epsilon)$ such that $L(f, R-\Delta_\epsilon) + L^*(f, \Delta_\epsilon)$ constitutes for $\epsilon=\epsilon_n$ a generalized surface $L_n(f)$ with the properties stated.

It remains to treat the case in which $\omega(\epsilon) \leq 2\epsilon$ is true for $\epsilon \leq \epsilon_0$ where ϵ_0 is sufficiently small. We shall show that, contrary to our hypotheses, this in its turn implies that $M_{u,v}(f)$ has a continuous carrier $x^*(u, v)$ which coincides with $x(u,v)$ on the perimeter of R. To this effect we determine for $\epsilon=\epsilon_0$ a corresponding $\delta'>0$ such that $\epsilon^3/100$ exceeds the integral of the area of $M_{u,v}(f)$ on every measurable subset of R of measure $<\delta'$; we determine further $\delta''>0$, where $4(\delta'')^2<\delta'$, such that the oscillation of $x(u, v)$ on any segment of length δ'' of the perimeter of R is $<\epsilon/8$; and finally in accordance with [17(a), (3.1)] and [17 (b)], $\delta>0$ and a grating G consisting of a finite number of parallels to the axes on each of which $x(u, v)$ is absolutely continuous, such that G subdivides R into intervals Δ whose sides lie between δ and 2δ and that, when (u, v) describes any segment of length $<\delta$ on G the length of the curve described by $x(u, v)$ is $<\epsilon/8$. All this clearly implies that $\omega(\Delta) \leq \omega(\epsilon) \leq 2\epsilon$ for each Δ determined by the grating. Consequently there exists a carrier $x_1(u, v) \in D(N)$ of $M_{u,v}(f)$, which coincides with $x(u,v)$ on the grating G and on the perimeter of R, such that $x_1(u, v)$ has oscillation $\leq 2\epsilon$ on each Δ determined by G.

Since $x_1(u, v)$ is a carrier of $M_{u,v}(f)$ it satisfies in its turn an inequality of the form $\omega_1(\epsilon) \leq 2\epsilon$ for $\epsilon \leq \epsilon_1$ where ϵ_1 is sufficiently small. We choose $\epsilon_1<\epsilon_0/2$ and define, in relation to $x_1(u, v)$ and $\epsilon=\epsilon_1$, corresponding numbers δ', δ'', δ and a grating G, and finally a carrier $x_2(u, v) \in D(N)$ of $M_{u,v}(f)$.

The process is continued indefinitely.

We obtain a sequence $\epsilon_0, \epsilon_1, \cdots, \epsilon_n, \cdots$ of positive numbers where $\epsilon_{n+1} \leq 2^{-1}\epsilon_n$, and corresponding sequences of positive numbers $\delta_0, \delta_1, \cdots, \delta_n, \cdots$, of gratings $G_0, G_1, \cdots, G_n, \cdots$ and of carriers $x_0(u, v)$, $x_1(u, v)$, $\cdots$, $x_n(u, v)$, $\cdots \in D(N)$ of $M_{u,v}(f)$, such that each interval Δ determined by G_n has sides between δ_n and $2\delta_n$ and that $x_n(u, v)$ coincides with $x_{n-1}(u, v)$ on the perimeter of such a Δ and has oscillation $\leq 2\epsilon_n$ in it. Clearly the oscillation of x_{n+p} on a Δ determined by G_n is then $\leq 2(\epsilon_n + \epsilon_{n+1} + \cdots + \epsilon_{n+p}) \leq 4\epsilon_n$ and hence the difference of the values of x_{n+p}, at two points (u', v') and (u'', v'') whose distance does not exceed δ_n, cannot exceed $12\epsilon_n$.

It follows at once that if $x^*(u,v)$ is the BL limit of a suitable subsequence of the $x_n(u, v)$, then $x^*(u, v)$ is equal almost everywhere to a continuous function, and we may replace it by the latter (which is also the BL limit). At almost every point at which $M_{u,v}(f)$ has area $\neq 0$ we have $x_n(u, v) = x(u, v) = x^*(u, v)$ for all n

and therefore $x^*(u, v)$ is the carrier of $M_{u,v}(f)$ at such a point. At the points at which $M_{u,v}(f)$ has area 0 the carriers $x_n(u,v)$ have Jacobians 0; and arguing as at the end of the proof of (9.7) and using (9.8), we find that at almost every such (u, v), $x^*(u, v)$ has Jacobian 0, and therefore is a carrier of $M_{u,v}(f)$. Since $x^*(u,v)$ is continuous and coincides with $x(u, v)$ on the perimeter of R, we have the desired contradiction.

PROOF of (9.1). Let $L(f) \in \mathfrak{R} - \mathfrak{R}_c$. We have to show that $L(f)$ is totally non-critical. By (9.3) coupled with (4.4) (i) and (4.6), and by (4.4) (iii), this is certainly the case when $L(f)$ is not at most finitely disconnected. By simple constructions, we may therefore suppose that $L(f)$ is a connected member of $\mathfrak{R} - \mathfrak{R}_c$ and consequently that $L(f)$ possesses, according to (8.1), a micro-representation $M_{u,v}(f)$ with a carrier $x(u, v) \in D(N)$ for some N, and the conclusion now follows from (9.9).

10. **A combinatorial problem.** One factor which has contributed to the depth and difficulty of our proofs is our ignorance of the precise nature of a micro-sphere and a micro-disc in the case in which the dimension of our space X exceeds 3, and more particularly our ignorance of the precise nature of a polyhedral micro-sphere and a polyhedral micro-disc. This is made clear by comparing our present results and their proofs with those of its predecessor [18]. In terms of the classical calculus of variations, the problem is that of defining the notion of regularity of a parametric integrand, or, what comes to the same, that of defining a satisfactory E-function.

Our formulation of the problem enables its statement in quite elementary terms. Essentially it is a variant of the problem of characterising a sum of bi-vectors consisting of the oriented areas of a polyhedral sphere or of a flat-rimmed polyhedral disc, a problem of whose solution we are ignorant even when the space has dimension 3. Our variant is presumably an "easier" variant, since it is solved for 3-space quite simply, and it consists of the following problem: Given a sum of bi-vectors $j_1, \cdots, j_n$ no two of which are parallel, what are the necessary and sufficient algebraic relations which ensure that the following situation arises: Given $\epsilon > 0$, there exists a polyhedral sphere, or again a polyhedral flat-rimmed disc, such that the oriented areas of its faces parallel to j_k have, for each $k = 1, \cdots, n$, a sum distant $< \epsilon$ from j_k and the areas of its faces parallel to no

$$j_k (k = 1, \cdots, n)$$

have the sum $< \epsilon$. In this statement of our problem we mean by a bi-vector parallel to a given bi-vector j, one which has the form cj where $c \geq 0$.

The available evidence seems to suggest that the condition $\Sigma j_k = 0$ is necessary and sufficient as regards the case of the polyhedral sphere, but that the condition "Σj_k is to be a bi-vector" is not sufficient as regards the case of the polyhedral flat-rimmed disc.

Appendix. (i) PROOF of (6.3). R has a conformal map on the circular disc Δ for which its diagonals become orthogonal diameters; we term x-area of a subset of Δ the area integral of $x(u, v)$ on the corresponding subset of R. Let $C(z)$, $C'(z)$ be circles, orthogonal to one another and to the rim of Δ, each of which bisects the x-area of Δ, such that $C(z)$ passes through a given point z of this rim, and let z' be the first intersection with $C'(z)$ of a counterclockwise arc of the rim of Δ originating at z. Further let $f(z)$ be the proportion of the x-area of Δ arising from the part of Δ bounded by this arc from z to z' and by arcs of $C(z)$, $C'(z)$. Since $C'(z') = C(z)$, we find that $f(z) + f(z') = 1/2$ whence $f(z)$ must assume the value $1/4$ for some z, and the corresponding pair of circles $C(z)$, $C'(z)$ then divides the x-area of Δ into four equal parts. Since this pair of circles become, by a suitable conformal map of Δ into itself, lines at right angles through the centre, this completes the proof.

(ii) PROOF of (9.4). By [1(e)], $\bar{a}(t) \geq \int_t^\infty L(u)\,du$, where $\bar{a}(t)$ is the area of the part of the surface distant at least t from the boundary curve, and where $L(u)$ is the intrinsic length of the part at distance u. We can choose values $t < b$ and $t' < t+b$ so that $2L^2(t) \leq \epsilon\pi a(R)$ and $L^2(t') \leq \epsilon\pi\bar{a}(t')$, since otherwise the definition of b contradicts the implied inequalities $a(R) \geq \int_0^b L(t)\,dt \geq b\{\epsilon\pi a(R)\}^{1/2}$, or else $(\epsilon\pi)^{-1} L^2(t+u) \geq \int_u^b L(t+v)\,dv$ $(0 \leq u < b)$, which, by [1(a), Lemma 15, p. 272], implies in its turn $b \leq 2L(t)/(\epsilon\pi) \leq \{2a(R)/(\epsilon\pi)\}^{1/2}$. Now let H be the component, containing the perimeter of R, of the set of (u, v) at which $x(u, v)$ is distant $\leq t'$ from the boundary curve, and let W be the sum of H and those of its complementary domains on which the oscillation of $x(u, v)$ is $< 2b - t'$. The sum of the boundary of W, relative to the interior of R, and the continua of constancy of $x(u, v)$ which meet it, consists of a finite number of disjoint continua $\gamma_1, \cdots, \gamma_k$, and these bound simply-connected subdomains whose sum is an open set O, containing $R - H$, for which therefore $a(O) \geq \bar{a}(t') \geq (\epsilon\pi)^{-1} L^2(t')$, while the total intrinsic length of the mapping by $x(u, v)$ of the sets $\gamma_1, \cdots, \gamma_k$ is $\leq L(t')$; moreover, since $R - O \subset W$, the oscillation of $x(u, v)$ on $R - O$ exceeds by at most $4b - 2t'$ that on H, and therefore by at most $4b$ that on the perimeter of R; this oscillation is thus $\leq d + 4b$ as asserted. This completes the proof.

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