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A THESIS

ON

The Reciprocal Polars of Conic Sections

BY

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CHAPTER I.

Introductory.

In reciprocation we start with the auxiliary conic which may be represented by the general equation of the second degree.

The reciprocal polar of a system (A) consisting of a series of points is a system (B) consisting of the polar lines of the points with respect to the auxiliary conic. The reciprocal polar of a system (C) consisting of a series of straight lines is a system (D) consisting of a series of points—the poles of the straight lines with respect to the auxiliary conic.

Again a curve may be regarded either as the locus of a point or as the envelope of a variable straight line moving according to some assigned conditions. In fact, after Plücker, we may regard one and the same curve as described by a point and enveloped by a straight line passing through the point which is advancing along the line at the same time that the line is rotating about the point. The intrinsic equation to the curve furnishes us with the relation between the velocity of the point and the rate of angular rotation of the line. Also we look upon the tangent to a curve as the line passing through two consecutive points on it, and a point on the curve as the point of intersection of two consecutive tangents to the curve. Hence the reciprocal polar curve may be looked upon either as the envelope of the polars of the points on the original curve or the locus of the poles of the tangents to the given curve. Hence the polar reciprocal is the envelope of the corresponding polar line of the point on the original curve rotating about a point in it and the point at the same time is advancing along the straight line and describing the curve. A point in one figure corresponds to a tangent in the other and *vice versâ*. Thus the class of the reciprocal curve must be the same as the degree of the original curve, and *vice versâ* as the reciprocal polar of a series of points lying on a right line is a series of right lines passing through a point and *vice versâ*.

2. Let $ax^2+2hxy+by^2+2gx+2fy+c=0$ be the auxiliary conic.

The reciprocal polar of a point (x', y') is the polar line

$$(ax'+hy'+g)\,x+(hx'+by'+f)\,y+(gx'+fy'+c)=0.$$

Also the reciprocal polar of a right line $\lambda x+\mu y+\nu=0$ is the point (x', y') where

$$\frac{ax'+hy'+g}{\lambda}=\frac{hx'+by'+f}{\mu}=\frac{gx'+fy'+c}{\nu}.$$

Whence

$$x'=-\frac{A\lambda+H\mu+G\nu}{G\lambda+F\mu+C\nu}$$

$$\text{And } y'=-\frac{H\lambda+B\mu+F\nu}{G\lambda+F\mu+C\nu}$$

The reciprocal polar of another conic $a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$ is the locus of a point (x', y') whose polar with respect to the auxiliary conic is a tangent to the second conic. Hence the reciprocal polar is represented by the equation $A'(ax+hy+g)^2+B'(hx+by+f)^2+C'(gx+fy+c)^2+2F'(hx+by+f)(gx+fy+c)+2G'(gx+fy+c)\ (ax+hy+g)+2H'\ (ax+hy+g)(hx+by+f)=0$
or $\{A'a^2+B'h^2+C'g^2+2F'gh+2G'ag+2H'ah\}\ x^2+2\ \{A'ah+B'bh+C'gf+F'(bg+hf)+G'(af+gh)+H'(ab+h^2)\}\ xy+\{A'h^2+B'b^2+C'f^2+2F'bf+2G'hf+2H'bh\}y^2+2\{A'ag+B'hf+C'cg+F'(ch+fg)+G'(ac+g^2)+H'(af+gh)\}\ x+2\{A'gh+B'bf+C'cf+F'(bc+f^2)+G'(gf+ch)+H'(bg+hf)\}\ y+\{A'g^2+B'f^2+C'c^2+2F'cf+2G'cg+2H'fg\}=0$.

The reciprocal conic will be an ellipse, a parabola or an hyperbola according as $\{A'ah+B'bh+C'gf+F'(bg+hf)+G'(af+gh)+H'(ab+h^2)\}^2-\{A'a^2+B'h^2+C'g^2+2F'gh+2G'ag+2H'ah\}\ \{A'h^2+B'b^2+C'f^2+2F'bf+2G'hf+2H'bh\}$ is negative, zero, or positive, *i.e.*, according as $(F'^2-B'C')\,G^2+(G'^2-C'A')\,F^2+(H'^2-A'B')\,C^2-2(G'H'-A'F')CF-2(H'F'-B'G')CG-2(F'G'-C'H')\ FG$ is negative, zero or positive, *i.e.*, according as $-\Delta'\ (a'G^2+b'F^2+c'C^2+2g'CG+2f'CF+2h'FG)$ is negative, zero or positive.

Now it is easy to see that the lines through the origin parallel to the tangents which can be drawn from the centre of the auxiliary conic to the second conic are represented by the equation $x^2(C'F^2-2F'FC+B'C^2)-2xy\ (C'FG-F'GC-G'CF+H'C^2)+y^2(C'G^2-2G'GC+A'C^2)=0$.

These lines are imaginary, coincident or real and distinct according as

$(C'FG - F'GC - G'FC + H'C^2)^2 - (C'F^2 - 2F'FC + B'C^2)(C'G^2 - 2G'GC + A'C^2)$ is negative, zero or positive, *i.e.*, according as $-C^2 \Delta' (a'G^2 + b'F^2 + c'C^2 + 2f'CF + 2g'CG + 2h'FG)$ is negative, zero or positive.

Hence the reciprocal conic is an ellipse, a parabola, or an hyperbola according as the tangents drawn from the centre of the reciprocating conic to the second conic are imaginary, coincident or real and distinct. When the auxiliary conic is a parabola the centre is at infinity in the direction of the axis of the parabola, and in that case the reciprocal polar is an ellipse, a parabola or an hyperbola according as none, only one or two tangents can be drawn to the original conic parallel to the axis of the auxiliary parabola.

3. We can, at our option, imagine the auxiliary conic to be (1) a circle, (2) a parabola, (3) an ellipse, or (4) an hyperbola.

The reciprocal polar of a system will be called its circular, parabolic, elliptic or hyperbolic polar according as the auxiliary conic is a circle, a parabola, an ellipse or an hyperbola.

In this thesis it is intended to discuss these polars in the following order :—

(1) Circular Polars. (2) Parabolic Polars.
(3) Elliptic Polars. (4) Hyperbolic Polars.

CHAPTER II.

CIRCULAR POLARS.

1. Let $x^2 + y^2 = r^2$ be the equation to the auxiliary conic the axes being rectangular.

The circular polar of a point $P(x', y')$ is the line $xx' + yy' = r^2$ which is perpendicular to OP and is at a distance from the origin equal to $\dfrac{r^2}{OP}$.

The circular polar of a line $\lambda x + \mu y = 1$ is the point $(\lambda r^2, \mu r^2)$ such that the line joining the point with the origin is perpendicular to the given line, and the rectangle contained by the distance of the point from the origin, and the perpendicular drawn from the origin on the line is equal to the square of the radius of the auxiliary circle. It is evident that the circular polar is the inverse of the pedal curve, the auxiliary circle being the circle of inversion.

2. To determine the circular polar of the general conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Let $x^2+y^2=r^2$ be the auxiliary circle.

The circular polar is the locus of a point (x', y') whose polar $xx'+yy'-r^2=0$ with respect to the auxiliary circle is a tangent to the conic. Now $\lambda x+\mu y+\nu=0$ touches the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ if $A\lambda^2+2H\lambda\mu+B\mu^2+2G\nu\lambda+2F\nu\mu+C\nu^2=0$. Hence the equation to the circular polar is $Ax^2+2Hxy+By^2-2Gr^2x-2Fr^2y+Cr^4=0$.

3. To determine the foci of the general conic $ax^2+2hxy+by^2+2gx+2fy+c=0$.

The pedal of a central conic with respect to a focus is the auxiliary circle of the conic. Hence the reciprocal polar of a central conic, when a focus is the origin of reciprocation, is the inverse of a circle and is therefore a circle. Again the pedal of a parabola with respect to its focus is the tangent at the vertex; therefore the reciprocal polar of a parabola when the focus is the origin of reciprocation is a circle which is the inverse of the tangent at the vertex. Let (x', y') be a focus of the general conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Transferring the origin to (x', y') the equation to the conic becomes $ax^2+2hxy+by^2+2g'x+2f'y+c'=0$

where
$$g'=ax'+hy'+g,$$
$$f'=hx'+by'+f,$$
$$c'=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c.$$

The circular polar of this conic with respect to $x^2+y^2=r^2$ (referred to the new origin) is

$$A'x^2+2H'xy+B'y^2-2G'r^2x-2F'r^2y+C'r^4=0.$$

In order that this circular polar may be a circle we must have

$$A'=B' \text{ and } H'=0$$

i.e.,
$$bc'-f'^2=ac'-g'^2 \text{ and } f'g'-c'h=0$$

i.e.,
$$\frac{g'^2-f'^2}{a-b}=\frac{g'f'}{h}=c'.$$

i.e.,
$$\frac{(ax'+hy'+g)^2-(hx'+by'+f)^2}{a-b}=\frac{(ax'+hy'+g)(hx'+by'+f)}{h}$$
$$=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c.$$

Hence the co-ordinates x', y' of a focus are determined by the equations

$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}=\frac{(ax+hy+g)\ (hx+by+f)}{h}$$
$$=ax^2+2hxy+by\ \ +2gx+2fy+c.$$

Thus we get the equations determining the foci on the assumption that the circular polar of the general conic is a circle when the origin of reciprocation is a focus.

4. It is easy to see that the general conic has four foci, two of which are real and two imaginary. The foci are given by the equations

$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}=\frac{(ax+hy+g)(hx+by+f)}{h} \quad (1)$$

and $(ax+hy+g)(hx+by+f)=h(ax^2+2hxy+by^2+2gx+2fy+c)$ (2)

The second equation reduces to

$$Cxy-Fx-Gy+H=0$$

which represents an equilateral hyperbola concentric with the general conic. The first equation represents a pair of straight lines through the centre of the general conic and at right angles to each other. These two lines are given by the equation

$$\frac{ax+hy+g}{hx+by+f}=\lambda$$

where
$$\lambda-\frac{1}{\lambda}=\frac{a-b}{h}$$

or
$$h\lambda^2-(a-b)\lambda-h=0.$$

or
$$\lambda=\frac{(a-b)\pm\sqrt{\{(a-b)^2+4h^2\}}}{2h}.$$

Let these values of λ be represented by λ_1 and λ_2. The two lines represented by (1) are therefore

$$(a-\lambda_1 h)x+(h-\lambda_1 b)y+(g-\lambda_1 f)=0$$

and
$$(a-\lambda_2 h)x+(h-\lambda_2 b)y+(g-\lambda_2 f)=0.$$

Now
$$(a-\lambda_1 h)(a-\lambda_2 h)+(h-\lambda_1 b)(h-\lambda_2 b)$$
$$=(a^2+h^2)-h(a+b)(\lambda_1+\lambda_2)+\lambda_1\lambda_2(h^2+b^2)$$
$$=(a^2+h^2)-h(a+b).\frac{a-b}{h}-(h^2+b^2)$$
$$[\because \lambda_1+\lambda_2=\frac{a-b}{h} \text{ and } \lambda_1\lambda_2=-1]$$
$$=(a^2+h^2)-(a^2-b^2)-(h^2+b^2)=0.$$

Hence the two lines represented by (1) are at right angles to each other.

Now of these two rectangular lines drawn through the centre of the equilateral hyperbola represented by (2) one intersects the hyperbola in two real points and the other in two imaginary points. Therefore the general conic has two real and two imaginary foci.

The axes of the general conic are the lines through the centre on which the foci are situated and are therefore determined by the equation

$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}=\frac{(ax+hy+g)(hx+by+f)}{h}$$

When the general conic is a parabola $h^2=ab$ or $C=0$ and then the hyperbola represented by (2) reduces to the right line

$$Fx+Gy=H;$$

also

$$\lambda=\frac{(a-b)\pm\sqrt{(a-b)^2+4h^2}}{2h}.$$

$$=\frac{(a-b)\pm(a+b)}{2h}=\frac{a}{h},\ -\frac{b}{h}.$$

The two lines represented by (1) are in this case

$$\frac{ax+hy+g}{hx+by+f}=\frac{a}{h}\quad\text{or}\quad(gh-af)=0-$$

the line infinity and

$$\frac{ax+hy+g}{hx+by+f}=-\frac{b}{h}$$

or

$$(a+b)hx+(h^2+b^2)y+(gh+bf)=0$$

or

$$hx+by+\frac{gh+bf}{a+b}=0.$$

Hence in the parabola there is only one real focus at a finite distance which is the point of intersection of

$$Fx+Gy=H$$

and

$$hx+by+\frac{gh+bf}{a+b}=0$$

and the other three foci two of which are imaginary are on the line infinity.

5. The circular polar reduces to a pair of straight lines when the discriminant of the equation

$$Ax^2+2Hxy+By^2-2Gr^2x-2Fr^2y+Cr4=0 \text{ vanishes.}$$

Now this discriminant

$$=r^4\,(ABC+2FGH-AF^2-BG^2-CH^2)$$
$$=r^4\,(abc+2fgh-af^2-bg^2-ch^2)^2.$$

The discriminant vanishes when $r=o$, *i.e.*, when the auxiliary circle is a point circle. In this case the circular polar reduces to the pair of lines represented by the equation

$$Ax^2+2Hxy+By^2=0.$$

These lines are perpendicular to the tangents

$$Bx^2-2Hxy+Ay^2=0$$

which can be drawn from the origin (which is also the centre of reciprocation) to the conic.

The discriminant also vanishes when $abc+2fgh-af^2-bg^2-ch^2=0$, *i.e.*, when the original conic breaks up into two straight lines.

In this case $BC-F^2=CA-G^2=AB-H^2=GH-AF=HF-BG=FG-CH=0$ and the circular polar reduces to the pair of coincident lines represented by

$$(x\sqrt{A}+y\sqrt{B}-r^2\sqrt{C})=0.$$

If (x', y') be the pole of this line with respect to the auxiliary circle

$$x'=\sqrt{\frac{A}{C}}=\frac{G}{C}\ \text{ and }\ y'=\sqrt{\frac{B}{C}}=\frac{F}{C},$$

i.e., the pole (x', y') is the point of intersection of the straight lines represented by the given equation.

The polar of the point of intersection is the line joining the poles of the two intersecting lines and the polar of any point lying on this polar line passes through the point of intersection and may therefore be regarded as a tangent to the original conic reduced to a pair of straight lines.

6. The circular polar is an ellipse, a parabola or an hyperbola according as the lines represented by the equation $Ax^2+2Hxy+By^2=0$ are imaginary, coincident or real and distinct. But the tangents from the origin to the given conic are represented by the equation $Bx^2-2Hxy+Ay^2=0$ and are therefore lines at right angles to the pair of lines represented by $Ax^2+2\,Hxy+By^2=0$. Therefore the reciprocal polar is an ellipse, a parabola or an hyperbola according as no tangent, only one or two real and distinct tangents can be drawn from the origin to the given conic.

7. The origin will be the centre of the reciprocal polar if $G=0$ and $F=0$. Hence if the origin be the centre of the given conic the reciprocal polar will be concentric with the given conic.

8 The centre of the circular polar is the point of intersection of the lines

$$Ax+Hy-Gr^2=0$$

and

$$Hx+By-Fr^2=0.$$

If (x, y) be the co-ordinates of the centre

$$x=-\frac{(HF-BG)\,r^2}{AB-H^2}=-\frac{gr^2}{c}$$

and

$$y=-\frac{(HG-AF)\,r^2}{AB-H^2}=-\frac{fr^2}{c}.$$

The polar of this centre with respect to the anxiliary circle is the line $gx+fy+c=0$ which is also the polar of the origin with respect to the given conic.

Transferring the origin to this centre the equation to the circular polar becomes

$$Ax^2+2Hxy+By^2+\frac{r^4}{c^2}(Ag^2+2Hgf+Bf^2+2Ggc+2Ffc+Cc^2)=0$$

or

$$Ax^2+2Hxy+By^2+\frac{r^4}{c^2}\{g(Ag+Hf+Gc)$$

$$+f(Hg+Bf+Fc)+c(Gg+Ff+Cc)\}=0$$

or

$$Ax^2+2Hxy+By^2+\frac{r^4\,\Delta}{c}=0.$$

Let the equation to this circular polar referred to its axis be $A'x^2+B'y^2+\frac{r^4\,\Delta}{c}=0.$

Then

$$\left.\begin{aligned}A'+B'&=A+B\\ A'B'&=AB-H^2.\end{aligned}\right\}$$

and

The area of this reciprocal conic

$$=\pi\sqrt{\frac{r^8\,\Delta^2}{c^2A'B'}}=\frac{\pi\,r^4\,\Delta}{c\sqrt{AB-H^2}}=\frac{\pi\,r^4\,\Delta}{c^{\frac{3}{2}}.\ \Delta^{\frac{1}{2}}}=\pi\,r^4\sqrt{\frac{\Delta}{c^3}}.$$

Also the area of the original conic $=\pi\dfrac{\Delta}{C^{\frac{3}{2}}}$.

Hence the area of the circular polar

$$= \left\{ \frac{r^4}{\sqrt{\Delta}} \cdot \left(\frac{C}{c}\right)^{\frac{3}{2}} \right\} \times \text{area of the original conic.}$$

9. To determine the locus of the centre of reciprocation such that the circular polars of a given conic with respect to an auxiliary circle of constant radius may be of constant area.

Transferring the origin to (x', y') – a point on the locus – the equation to the given conic becomes

$$ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' = 0$$

and the equation to the auxiliary circle becomes

$$x^2 + y^2 = r^2.$$

The area of the circular polar $= \pi r^4 \sqrt{\dfrac{\Delta'}{c'^3}}$.

As r is constant, this area will be constant if

$$\frac{\Delta'}{c'^3} = \frac{1}{k}, \quad \text{where } k \text{ is a constant.}$$

Now $c' = ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c$

and

$$\Delta' = \begin{vmatrix} a, & h, & g' \\ h, & b, & f' \\ g', & f', & c' \end{vmatrix}$$

$$= \begin{vmatrix} a, & h, & ax' + hy' + g \\ h, & b, & hx' + by' + f \\ ax' + hy' + g, & hx' + by' + f, & \begin{matrix} x'(ax' + hy' + g) \\ + y'(hx' + by' + f) \\ + (gx' + fy' + c) \end{matrix} \end{vmatrix}$$

$$= \begin{vmatrix} a, & h, & ax' + hy' + g \\ h, & b, & hx' + by' + f \\ g, & f, & gx' + fy' + c \end{vmatrix} = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = \Delta.$$

Hence the equation to the locus of the centre of reciprocation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = \sqrt[3]{k\Delta}$$

which represents a conic similar, concentric and coaxial with the given conic.

10. The circular polar of the circle

$a(x^2 + y^2) + 2gx + 2fy + c = 0$ is the conic represented by the equation

$$(ca-f^2)x^2+2fgxy+(ca-g^2)y^2+2agr^2x+2afr^2y+a^2r^4=0$$

or
$$(ca-f^2-g^2)(x^2+y^2)+(gx+fy+ar^2)^2=0$$

or
$$x^2+y^2=\frac{(gx+fy+ar^2)^2}{f^2+g^2-ca}.$$

The co-ordinates of the centre of the circle are

$$\left(-\frac{g}{a},-\frac{f}{a}\right)$$

and its polar is the line $gx+fy+ar^2=0$
also for a real circle (f^2+g^2-ca) is positive.

If P (x, y) be any point on the circular polar $OP^2=x^2+y^2$, also if PM be the perpendicular from P on the polar line of the centre $gx+fy+ar^2=0$,

$$PM=\frac{gx+fy+ar^2}{\sqrt{g^2+f^2}}.$$

Hence from the equation to the circular polar we have

$$OP=\sqrt{\frac{f^2+g^2}{f^2+g^2-ca}}\cdot PM$$

$$=e.\ PM \text{ where } e=\sqrt{\frac{f^2+g^2}{f^2+g^2-ca}}.$$

The reciprocal polar conic has therefore the origin for one focus and the polar line of the centre of the given circle is the corresponding directrix. The co-ordinates of the centre of the circular polar are

$\left(-\frac{gr^2}{c},-\frac{fr^2}{c}\right)$ and therefore $\left\{-\frac{2gr^2}{c},-\frac{2fr^2}{c}\right\}$, is the other

focus. Again the eccentricity of the circular polar

$$=\sqrt{\frac{f^2+g^2}{f^2+g^2-ca}}.$$

Assuming "a" to be positive the eccentricity is greater than, equal to or less than unity according as "c" is positive, zero or negative. But "c" is positive, zero or negative according as the origin is outside, on or inside the circle. Hence the circular polar is a hyperbola, a parabola or an ellipse according as the centre is outside, on or within the given circle. This is also evident from the equation

$$e=\sqrt{\frac{f^2+g^2}{f^2+g^2-ca}}=\frac{\sqrt{\dfrac{g^2}{a^2}+\dfrac{f^2}{a^2}}}{\sqrt{\dfrac{g^2}{a^2}+\dfrac{f^2}{a^2}-\dfrac{c}{a}}}$$

$$=\frac{\text{distance of the centre of the given circle from the origin}}{\text{radius of the given circle}}.$$

The eccentricity of the circular polar is independent of the radius of the auxiliary circle which therefore affects the size but not the shape of the circular polar.

11. To determine the eccentricity of the general conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Let (x', y') be a focus; transferring the origin to this focus the equation to the conic becomes

$$ax^2+2hxy+by^2+2g'x+2f'y+c'=0$$

where $g'=ax'+hy'+g$,

$f'=hx'+by'+f$

and $c'=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c$.

The equation to the circular polar with respect to the auxiliary circle $x^2+y^2=r^2$ (referred to the new origin) is the circle

$$A'x^2+2H'xy+B'y^2-2G'r^2x-2F'r^2y+C'r^4=0$$

where $H'=0$ and $A'=B'$, *i.e.*,

$$g'^2-f'^2=c'(a-b)\ldots\ldots(1)$$

$$g'f'=c'h\ldots\ldots\ldots\ldots(2)$$

If "e" be the eccentricity of the original conic we have by the previous article

$$e^2=\frac{G'^2+F'^2}{G'^2+F'^2-C'A'}$$

$$=\frac{(hf'-bg')^2+(hg'-af')^2}{(hf'-bg')^2+(hg'-af')^2-(ab-h^2)(bc'-f'^2)}\ldots\ldots(3)$$

Let $g'=\lambda f'$, then

from (1) $(\lambda^2-1)f'^2=c'(a-b)$

and from (2) $\lambda f'^2=c'h$

and $$\therefore \quad \frac{\lambda^2-1}{\lambda}=\frac{a-b}{h}\ldots\ldots\ldots\ldots(4).$$

From (3) $$1-\frac{1}{e^2}=\frac{(ab-h^2)(bc'-f'^2)}{(hf'-bg')^2+(hg'-af')^2}$$

$$=\frac{(ab-h^2)\left(\frac{b\lambda}{h}-1\right)f'^2}{f'^2\{(h-b\lambda)^2+(h\lambda-a)^2\}}$$

$$=\frac{(ab-h^2)(b\lambda-h)}{h\{(h-b\lambda)^2+(h\lambda-a)^2\}}.$$

From (4) $$h\lambda^2-h=a\lambda-b\lambda$$

$$\therefore \quad h-b\lambda=\lambda(h\lambda-a)$$

and $$h\lambda^2=h+a\lambda-b\lambda.$$

$$\therefore \quad 1-\frac{1}{e^2}=\frac{(ab-h^2)(b\lambda-h)}{h(1+\lambda^2)(h\lambda-a)^2}$$

$$=\frac{-(ab-h^2)\lambda(h\lambda-a)}{h(1+\lambda^2)(h\lambda-a)^2}$$

$$=\frac{-(ab-h^2)\lambda}{(h+h\lambda^2)(h\lambda-a)}$$

$$=\frac{-(ab-h^2)\lambda}{\{2h+(a-b)\lambda\}(h\lambda-a)}$$

$$=\frac{-(ab-h^2)\lambda}{(a-b)h\lambda^2+\lambda(2h^2+ab-a^2)-2ah}$$

$$=\frac{-(ab-h^2)\lambda}{(a-b)(h+a\lambda-b\lambda)+\lambda(2h^2+ab-a^2)-2ah}$$

$$=\frac{-(ab-h^2)\lambda}{-h(a+b)+\lambda(2h^2-ab+b^2)}$$

or $$-(ab-h^2)e^2\lambda=(e^2-1)\{-h(a+b)+\lambda(2h^2-ab+b^2)\}$$

or $$\lambda\{-(ab-h^2)e^2-(e^2-1)(2h^2-ab+b^2)\}=h(a+b)(1-e^2)$$

$$\therefore \quad \lambda=\frac{h(a+b)(e^2-1)}{(ab-h^2)e^2+(e^2-1)(2h^2-ab+b^2)}$$

$$=\frac{h(a+b)(e^2-1)}{(ab-h^2)+(e^2-1)(h^2+b^2)}$$

$\therefore$ from (4)

$$\frac{h(a+b)(e^2-1)}{(ab-h^2)+(e^2-1)(h^2+b^2)}-\frac{(ab-h^2)+(e^2-1)(h^2+b^2)}{h(a+b)(e^2-1)}=\frac{a-b}{h}$$

or
$$h^2(a+b)^2(e^2-1)^2-\{(ab-h^2)+(e^2-1)(h^2+b^2)\}^2$$
$$=(a^2-b^2)(e^2-1)\{(ab-h^2)+(e^2-1)(h^2+b^2)\}$$

or
$$(e^2-1)^2\{h^2(a+b)^2-(h^2+b^2)^2-(a^2-b^2)(h^2+b^2)\}$$
$$-(e^2-1)\{2(ab-h^2)(h^2+b^2)+(a^2-b^2)(ab-h^2)\}-(ab-h^2)^2=0$$

or
$$(e^2-1)^2\{h^2(a+b)^2-(h^2+a^2)(h^2+b^2)\}$$
$$-(e^2-1)(ab-h^2)(2h^2+b^2+a^2)-(ab-h^2)^2=0$$

or
$$\{e^4-2(e^2-1)-1\}\{2abh^2-h^4-a^2b^2\}$$
$$-(e^2-1)(ab-h^2)(2h^2+b^2+a^2)-(ab-h^2)^2=0$$

or $$\{e^4-2(e^2-1)-1\}(ab-h^2)+(e^2-1)(2h^2+b^2+a^2)+(ab-h^2)=0$$

or
$$e^4(ab-h^2)+(e^2-1)\{4h^2+(a-b)^2\}=0$$

or
$$e^4+\frac{(a-b)^2+4h^2}{ab-h^2}(e^2-1)=0$$

which gives the eccentricity of the general conic.

If e_1^2 and e_2^2 be the roots of this equation regarded as a quadratic in e^2 we have

$$e_1^2+e_2^2=e_1^2e_2^2=\frac{(a-b)^2+4h^2}{h^2-ab}.$$

$$\therefore\qquad \frac{1}{e_1^2}+\frac{1}{e_2^2}=1.$$

In the ellipse (h^2-ab) is negative and therefore $e_1^2\,e_2^2$ is negative; hence one of the two quantities e_1 and e_2 is real and the other imaginary.

In the hyperbola (h^2-ab) is positive and therefore $e_1^2\,e_2^2$ is positive; hence e_1 and e_2 are both real in the hyperbola.

12. The equation to the circular polar of $ax^2+2hxy+by^2+2gx+2fy+c=0$ with respect to a circle whose centre is (x', y'), is

$$A'x^2+2H'xy+B'y^2-2G'r^2x-2F'r^2y+C'r^4=0$$

referred to parallel axes, through (x', y').

If ϵ be the eccentricity of this circular polar we have,

$$\epsilon^4 + \frac{(A'-B')^2 + 4H'^2}{A'B' - H'^2}(\epsilon^2 - 1) = 0.$$

Hence the eccentricity will remain unaltered if

$$\frac{(A'-B')^2 + 4H'^2}{A'B' - H'^2} \text{ or } \frac{(A'+B')^2}{A'B' - H'^2} \text{ be constant.}$$

Now $A' + B' = c'(a+b) - (g'^2 + f'^2)$

and $A'B' - H'^2 = (c'b - f'^2)(c'a - g'^2) - (f'g' - c'h)^2$

$= c'(abc' + 2f'g'h - af'^2 - bg'^2 - c'h^2)$

$= c'\Delta.$

Therefore the locus of the centres of reciprocation with respect to which the circular polars of a given conic have the same eccentricity is

$$\lambda\Delta\ (ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$

$$= \{ (a+b)\ (ax^2 + 2hxy + by^2 + 2gx + 2fy + c) - (ax + hy + g)^2$$
$$-(hx + by + f)^2 \}^2 \text{ [where } \lambda \text{ is a constant]}$$

$$= \{ (ab - h^2)(x^2 + y^2) - 2(hf - bg)x - 2(hg - af)y + (ca - g^2) +$$
$$(cb - f^2) \}^2$$

$$= \{ C(x^2 + y^2) - 2Gx - 2Fy + A + B \}^2.$$

The locus is in general a quartic curve. When the original conic is a circle $a = b$ and $h = o$ and then the equation

$$\lambda\Delta(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$
$$= \{ C(x^2 + y^2) - 2Gx - 2Fy + A + B \}^2$$

becomes

$$\lambda\,\Delta\,(ax^2 + ay^2 + 2gx + 2fy + c)$$

$$= a^2\left(ax^2 + ay^2 + 2gx + 2fy + 2c - \frac{g^2 + f^2}{a} \right)^2$$

$$= a^2 \left\{ (ax^2 + ay^2 + 2gx + 2fy + c) - \frac{g^2 + f^2 - ca}{a} \right\}^2$$

which represents a pair of circles concentric with the given circle. This is also evident from the fact that the eccentricity of the circular polar of a circle

$$= \frac{\text{distance of the origin from the centre}}{\text{radius of the circle.}}$$

When the given conic is a parabola $C=0$ and the equation to the locus becomes

$$\lambda\Delta\ (ax^2+2hxy+by^2+2gx+2fy+c)=(2Gx+2Fy-A-B)^2$$

which represents a conic section.

The eccentricity being given by the equation

$$\epsilon^4+(\lambda-4)(\epsilon^2-1)=0$$

if we put $\lambda=0$, the equation becomes

$$\epsilon^4-4\epsilon^2+4=0$$

$$\text{or } \epsilon=\sqrt{2}.$$

The locus is then represented by the equation

$$C(x^2+y^2)-2Gx-2Fy+A+B=0$$

which is the director circle of the given conic and the corresponding reciprocal pola· is an equilateral hyperbola as is otherwise evident from the fact that the tangents drawn from any point on the director circle to the given conic are rectangular.

13. In the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ the foci are given by the equations

$$\frac{\dfrac{x^2}{a^4}-\dfrac{y^2}{b^4}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}=\frac{\dfrac{xy}{a^2b^2}}{o}=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$$

or $\qquad xy=o\ldots\ldots\ldots\ldots(1)$

and $\qquad \dfrac{b^4x^2-a^4y^2}{a^2b^2(b^2-a^2)}=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1\ldots\ldots(2).$

Thus we get the foci

$$\left.\begin{array}{l} y=o \\ x=\pm\sqrt{a^2-b^2}=\pm ae \end{array}\right\} \left.\begin{array}{l} x=o \\ y=\pm\sqrt{b^2-a^2}=\pm aei \end{array}\right\}$$

the second pair of foci being imaginary.

The circular polar of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with respect to the circle

$$(x-ae)^2+y^2=r^2$$

is another circle.

For if (x', y') be a point on the circular polar

$$(x-ae)(x'-ae)+yy'=r^2$$

is a tangent to the ellipse.

Hence we must have

$$a^2(x'-ae)^2+b^2y'^2=\{r^2+ae(x'-ae)\}^2.$$

Therefore the polar reciprocal is the circle

$$y^2+\left\{x-ae\left(1+\frac{r^2}{b^2}\right)\right\}^2=\frac{r^4a^2}{b^4}.$$

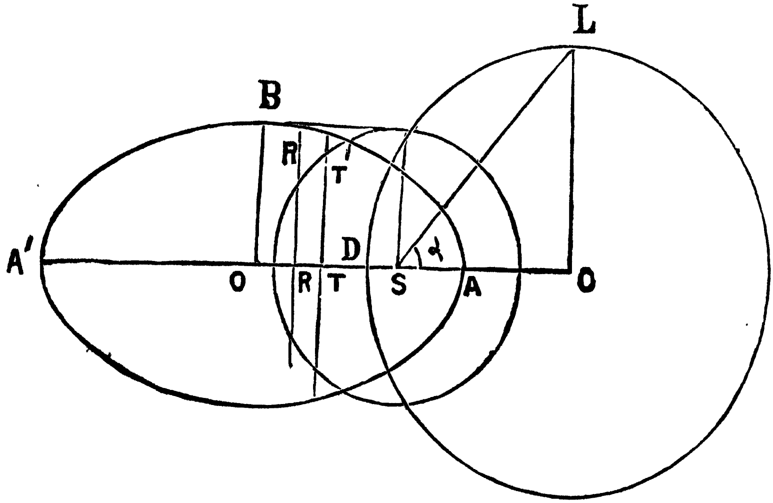


Fig. 1.

If we take $r=b$, the circular polar of the ellipse will be the circle

$$y^2+(x-2ae)^2=a^2.$$

The circular polar of the same ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to the imaginary circle $x^2+(y-aei)^2=r^2$ whose centre is an imaginary focus is another imaginary circle. For if $(x'\ y')$ be a point on the circular polar

$xx' + (y - aei)(y' - aei) = r^2$ is a tangent to the ellipse. Therefore the circular polar is

$$a^2x^2 + b^2(y - aei)^2 = \left\{r^2 + aei(y - aei)\right\}^2$$

$$\text{or } x^2 + \left\{y - aei\left(1 + \frac{r^2}{a^2}\right)\right\}^2 = \frac{r^4b^2}{a^4}$$

The circular polar of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with respect to $(x - ae)^2 + y^2 = b^2$ is the circle

$$(x - 2ae)^2 + y^2 = a^2.$$

The circular polar of the focus S (ae, o) is the line infinity; that of the centre C is the line RR' given by the equation $x = \dfrac{a^2 - 2b^2}{ae}$; and that of the other focus H $(-ae, o)$ is the line TT' given by the equation

$$x = \frac{2a^2 - 3b^2}{2ae}. \quad \text{(Fig. 1).}$$

Hence if O be the centre of the circular polar and R and T the points where RR' and TT' respectively intersect the major axis we have

$$OT = 2ae - \frac{2a^2 - 3b^2}{2ae} = \frac{2a^2 - b^2}{2ae} = \frac{SL^2}{2OS}$$

where OL is the radius drawn perpendicular to the major axis.

The pole of the directrix $x = \dfrac{a}{e}$ is the centre O of the circular polar and that of the other directrix $x = -\dfrac{a}{e}$ is the point D (x', o) where

$$x' = CD = \frac{2ae^3}{1 + e^2}.$$

$$\text{Thus } OD = 2ae\left(1 - \frac{e^2}{1 + e^2}\right) = \frac{2ae}{1 + e^2} = 2ae \cdot \frac{a^2}{a^2 + a^2e^2}$$

$$= 2ae \cdot \frac{OL^2}{SL^2} = 2OS \sin^2 QSL.$$

14. In the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the foci are given by the equations

$$xy = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and
$$\frac{b^4x^2 - a^4y^2}{a^2b^2(a^2+b^2)} = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \ldots\ldots (2).$$

The foci are therefore

$$\left.\begin{array}{l} y=0 \\ x = \pm\sqrt{a^2+b^2} \\ \quad = \pm ae \end{array}\right\} \text{ and } \left.\begin{array}{l} x=0 \\ y = \pm i\sqrt{a^2+b^2} \\ \quad = \pm aei \end{array}\right\}$$

the second pair of foci being imaginary.

The circular polar of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with respect to $(x-ae)^2 + y^2 = r^2$ is a circle.

For if (x', y') be a point on the circular polar $(x-ae)(x'-ae) + yy' = r^2$ is a tangent to the hyperbola and therefore

$$a^2(x'-ae)^2 - b^2y'^2 = \{r^2 + ae(x'-ae)\}^2.$$

Hence the circular polar is

$$y^2 + \left\{x - ae\left(1 - \frac{r^2}{b^2}\right)\right\}^2 = \frac{r^4a^2}{b^4}.$$

If we take $r = b$ the circular polar becomes $x^2 + y^2 = a^2$ which is the auxiliary circle of the hyperbola.

The circular polar of the same hyperbola with respect to $x^2 + (y - aei)^2 = r^2$ whose centre is an imaginary focus is the imaginary circle

$$x^2 + \left\{y - aei\left(1 + \frac{r^2}{a^2}\right)\right\}^2 = -\frac{r^4b^2}{a^4}.$$

Thus when we take $(x-ae)^2 + y^2 = b^2$ for the circle of reciprocation the polar line of the focus S (ae, o) is the line infinity; that of the centre C is the directrix (X) corresponding to the focus S; and the polar line of the other focus H $(-ae, o)$ is the line TT' intersecting the transverse axis at right angles at T where (see Fig. 2)

$$2\,ST.\,SC = b^2 \text{ or } ST = \frac{b^2}{2ae}.$$

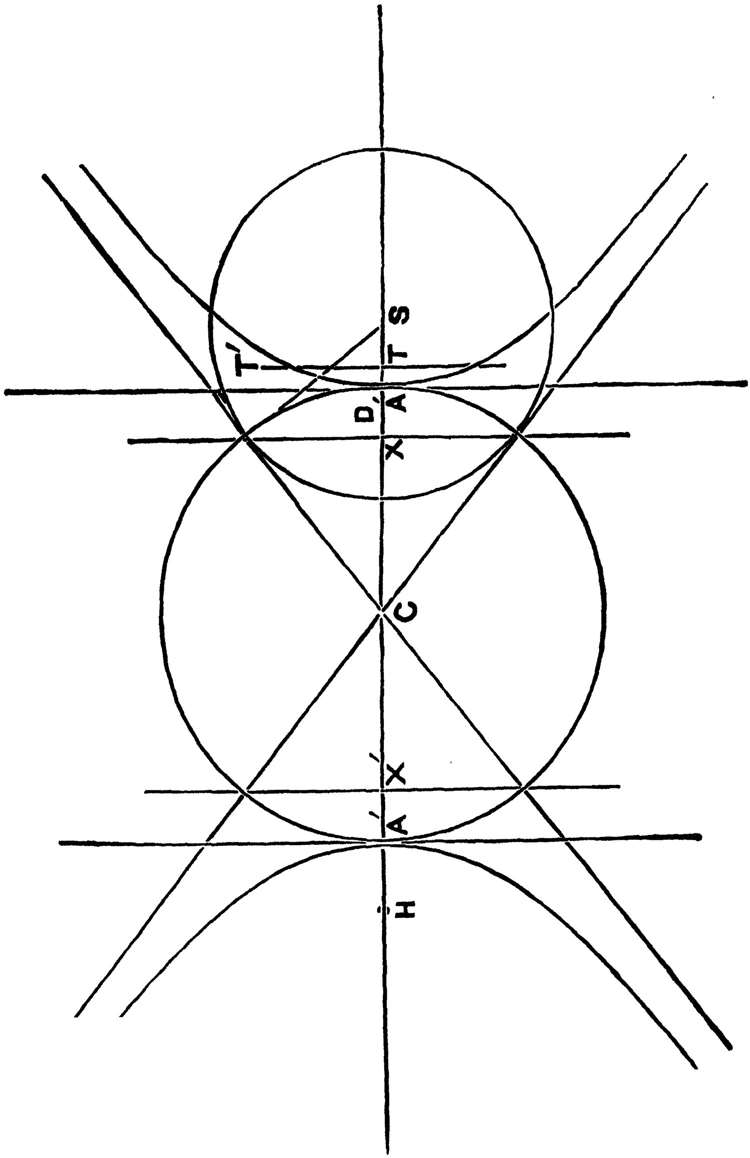


Fig. 2.

$$\text{Thus } CT = ae - \frac{b^2}{2ae} = \frac{2a^2+b^2}{2ae}.$$

Also the circular polar of the other directrix (X') is a point D on the transverse axis where

$$SD = \frac{b^2}{ae + \frac{a}{e}} = \frac{b^2 e}{ae^2 + a} = \frac{b^2 e}{a(1+e^2)}.$$

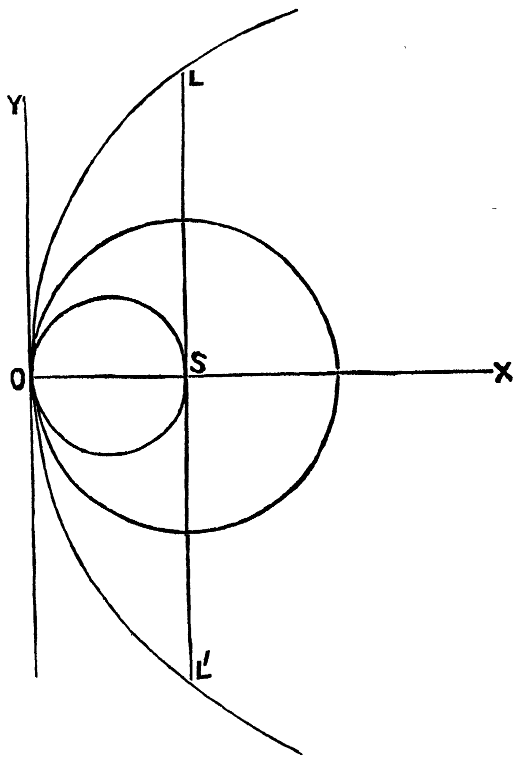


Fig. 3.

15. In the parabola $y^2 = 4ax$ the focus at a finite distance is (a, o) and the circular polar with respect to $(x-a)^2 + y^2 = r^2$ is a circle.

For if (x', y') be a point on the circular polar

$$(x-a)(x'-a) + yy' = r^2$$

is a tangent to the parabola.

Therefore $(x'-a)\{r^2 + a(x'-a)\} + ay'^2 = 0$.

Thus the equation to the circular polar is

$$a(x-a)^2 + ay^2 + r^2 (x-a) = 0$$

$$\text{or} \quad (x-a)^2 + \frac{r^2}{a}(x-a) + y^2 = 0$$

$$\text{or} \quad \left(x - a + \frac{r^2}{2a}\right)^2 + y^2 = \frac{r^4}{4a^2}.$$

If we take $r=a$, the circular polar of $y^2 = 4ax$ is

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

the reciprocating circle being $(x-a)^2 + y^2 = a^2$.

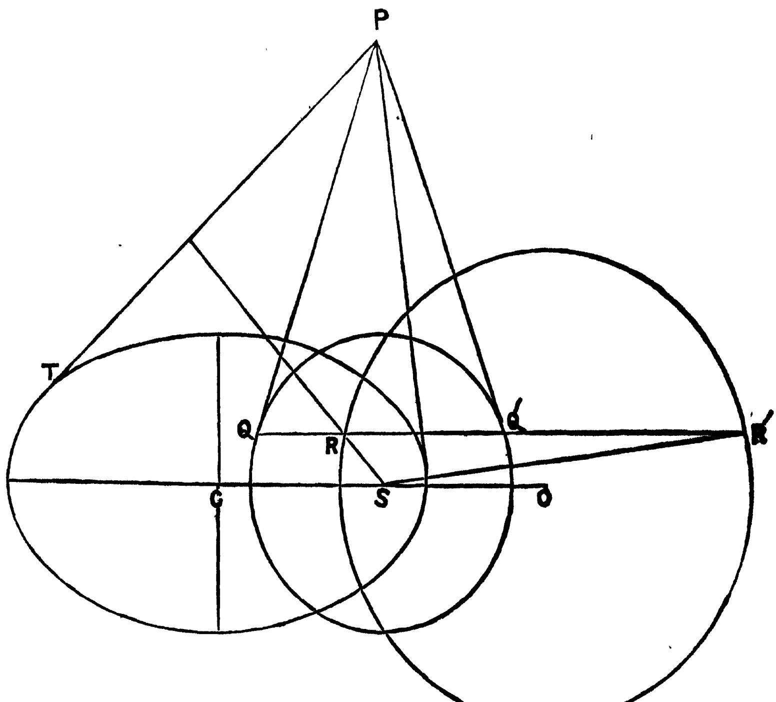


Fig. 4.

16. To draw two tangents to a conic from an external point.

(1) When the given conic is an ellipse.

Produce CS to O, making $SO = CS$.

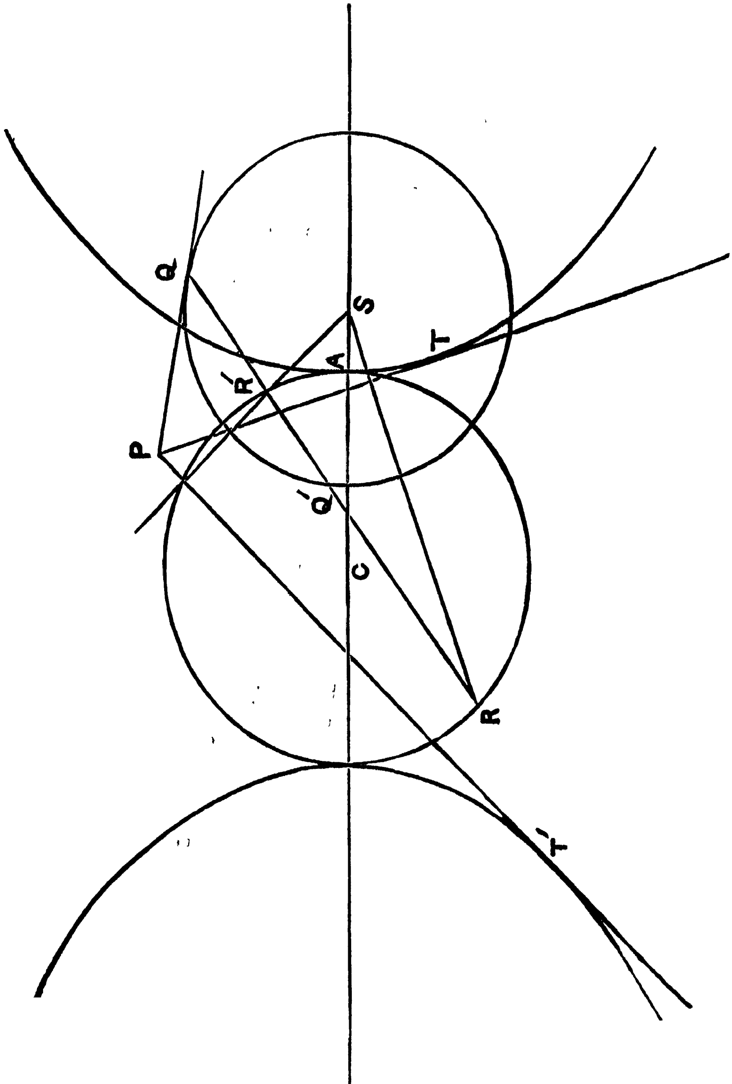


Fig. 5.

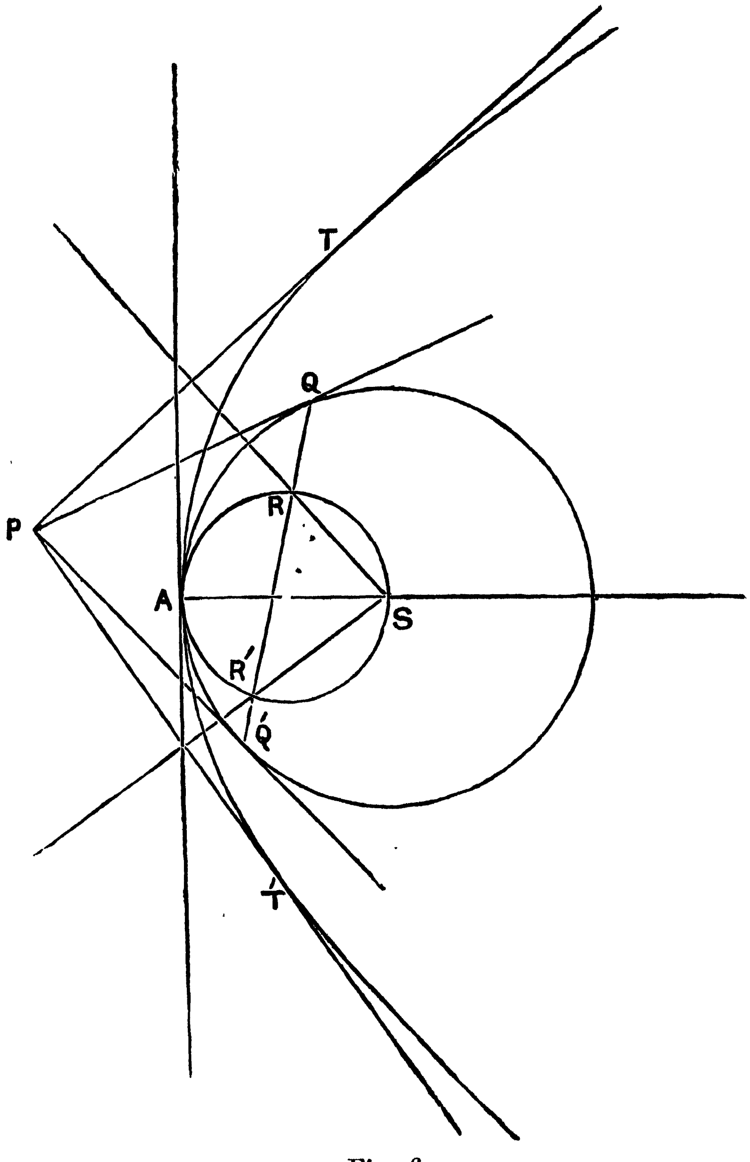


Fig. 6.

About S and O as centres describe circles with radii equal to the semi-minor and semi-major axes of the conic respectively. Then by the previous articles, the latter circle is the circular polar of the conic with respect to the former circle. The same is evident from Geometry. From P draw (if possible) tangents PQ and PQ' and let QQ' the polar of P with respect to the auxiliary circle intersect the circular polar at R and R'. Join SR and SR' and draw PT and PT' perpendiculars to SR and SR' respectively. Then PT and PT' will be tangents to the conic.

As the point R is on the polar of P, P must be on the polar of R which is perpendicular to SR. PT is therefore the polar of R and consequently a tangent to the conic. Similarly, it may be proved that PT' is a tangent.

(2) When the conic is a hyperbola.

With C and S as centres, describe circles having radii equal to the semi-transverse and conjugate axes of the hyperbola. Then in this case (as well as in the former case), it is easy to see by Geometry that the former circle is the circular polar of the conic with respect to the latter circle, for the circular polar is the inverse of the pedal which is the auxiliary circle of the conic. In the hyperbola, the circular polar which is the inverse of the pedal is coincident with the pedal.

With the same construction as before, it can be easily proved that PT and PT' are tangents to the conic.

(3) When the conic is a parabola.

In the parabola, the pedal with respect to the focus is the tangent at the vertex.

If the auxiliary circle be taken to be one of which the centre is S and radius SA, the circular polar will be the circle on SA as diameter.

With the same construction, it may be easily proved that PT and PT' are tangents to the parabola.

When the point is one on the curve, the line QQ' will be a tangent and the line corresponding to the point of contact will be the tangent at the point. When the point is inside the given conic, the polar line will not intersect the circular polar in real points and the corresponding tangents will be imaginary.

17. We have seen that the circular polars of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to the auxiliary circles

$$(x-ae)^2+y^2=r^2$$

and

$$x^2+(y-aei)^2=r^2$$

are respectively

$$y^2 + \left\{ x - ae\left(1 + \frac{r^2}{b^2}\right) \right\}^2 = \frac{r^4a^2}{b^4} \quad (1)$$

and

$$x^2 + \left\{ y - aei\left(1 + \frac{r^2}{a^2}\right) \right\}^2 = \frac{r^4b^2}{a^4} \quad (2).$$

In (1), if we make $r^2 = -b^2$, the circular polar is $x^2 + y^2 = a^2$. In this case the radius of the circle of reference is imaginary but the polar is a real circle, the polar lines being drawn on the side of the origin remote from the points.

In (2), if we make $r^2 = -a^2$, the circular polar becomes $x^2 + y^2 = b^2$ a real circle. Here the centre as well as the radius of reciprocation are imaginary but the polar thence formed is real.

The circular polar of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with respect to $x^2 + (y - aei)^2 = r^2$, is

$$x^2 + \left\{ y - aei\left(1 + \frac{r^2}{a^2}\right) \right\}^2 = -\frac{r^4b^2}{a^4}.$$

If we take $r^2 = -a^2$, the circular polar will become $x^2 + y^2 = -b^2$, an imaginary circle.

18. To determine the circular polars of a series of confocal conics.

Let $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$ be a conic of the system. The common foci are given by

$$\left.\begin{aligned} y &= 0 \\ x &= \pm\sqrt{a^2 - b^2} \\ &= \pm c \text{ say} \end{aligned}\right\}$$

Let $(x - c)^2 + y^2 = k^2$ be the reciprocating circle.

If $x'\ y'$ be a point on the circular polar, $(x - c)(x' - c) + yy' = k^2$ will be a tangent to the conic.

The condition of tangency gives us,

$$\{k^2 + c(x' - c)\}^2 = (a^2 + \lambda)(x' - c)^2 + (b^2 + \lambda)y'^2.$$

Therefore the circular polar is,

$$k^4 + 2ck^2(x - c) = (b^2 + \lambda)\{(x - c)^2 + y^2\}$$

or

$$(x - c)^2 + y^2 - \frac{2ck^2}{b^2 + \lambda}(x - c) - \frac{k^4}{b^2 + \lambda} = 0$$

or

$$(x - c)^2 + y^2 - \frac{2ck^2}{b^2 + \lambda}\left(x - c + \frac{k^2}{2c}\right) = 0.$$

The circular polars are therefore a system of co-axial circles having

$$x=c-\frac{k^2}{2c},$$

for the radical axis which, it is easy to see, corresponds to the other focus.

19. The circular polar of $x^2+y^2=e^2(a-x)^2$, a conic having the origin for one focus and $x=a$, for the corresponding directrix, with respect to $x^2+y^2=r^2$, is a circle.

For let $x'\,y'$ be a point on the circular polar; then $xx'+yy'=r^2$ is a tangent to the conic.

From the condition of tangency, we have,

$$(a\,e^2\,y'^2-r^2\,x')^2=(x'^2+y'^2-e^2\,y'^2)\,(r^4-e^2\,a^2\,y'^2).$$

The reciprocal polar is therefore

$$y^2+\left(x-\frac{r^2}{a}\right)^2=\left(\frac{r^2}{ea}\right)^2.$$

Therefore, the circular polars of a system of circles having the same centre is a system of conics having the same focus and directrix.

It is easy to see, that $(2e\,\alpha)$ is the parameter of the original conic. Hence, the circular polars of all equal circles with respect to the same circle have the same parameter.

20. To determine the centre of reciprocation in order that the polar triangle of a given triangle may be similar to another given triangle.

Let $(x'\,y')$, $(x''\,y'')$, $(x'''\,y''')$ be the vertices of the first triangle and λ, μ, ν the angles of the second triangle.

Let $\alpha\,\beta$ be the required co-ordinates of the centre of reciprocation, the reciprocating circle being $(x-\alpha)^2+(y-\beta)^2=r^2$.

The sides of the polar triangle are respectively

$$\left.\begin{aligned}(x-\alpha)\,(x'-\alpha)+(y-\beta)\,(y'-\beta)=r^2\ldots.(1)\\(x-\alpha)\,(x''-\alpha)+(y-\beta)\,(y''-\beta)=r^2\ldots.(2)\\(x-\alpha)\,(x'''-\alpha)+(y-\beta)\,(y'''-\beta)=r^2\ldots.(3)\end{aligned}\right\}$$

We must have λ, μ, and ν for the angles between these three lines.

$$\therefore\ \cos\lambda=\frac{(x''-\alpha)\,(x'''-\alpha)+(y''-\beta)\,(y'''-\beta)}{\pm\sqrt{\{(x''-\alpha)^2+(y''-\beta)^2\}}\sqrt{\{(x'''-\alpha)^2+(y'''-\beta)^2\}}}$$

$$\cos\mu = \frac{(x''' - \alpha)(x' - \alpha) + (y''' - \beta)(y' - \beta)}{\pm\sqrt{\{(x''' - \alpha)^2 + (y''' - \beta)^2\}}\sqrt{\{(x' - \alpha)^2 + (y' - \beta)^2\}}}$$

$$\cos\nu = \frac{(x' - \alpha)(x'' - \alpha) + (y' - \beta)(y'' - \beta)}{\pm\sqrt{\{(x' - \alpha)^2 + (y' - \beta)^2\}}\sqrt{\{(x'' - \alpha)^2 + (y'' - \beta)^2\}}}.$$

The lines joining $(x'\ y')$, $(x''\ y'')$, $(x'''\ y''')$ with $(\alpha\ \beta)$ are

$$\frac{x-\alpha}{x'-\alpha} = \frac{y-\beta}{y'-\beta} \quad \ldots\ldots \quad (4)$$

$$\frac{x-\alpha}{x'-\alpha} = \frac{y-\beta}{y''-\beta} \quad \ldots\ldots \quad (5)$$

$$\frac{x-\alpha}{x'''-\alpha} = \frac{y-\beta}{y'''-\beta} \quad \ldots\ldots \quad (6).$$

If $\lambda'\ \mu'\ \nu'$ be the angles formed by the lines (4), (5) and (6), we have

$$\cos\lambda' = \frac{(x'' - \alpha)(x''' - \alpha) + (y'' - \beta)(y''' - \beta)}{\pm\sqrt{\{(x'' - \alpha)^2 + (y'' - \beta)^2\}}\sqrt{\{(x''' - \alpha)^2 + (y''' - \beta)^2\}}}$$

$$\cos\mu' = \frac{(x''' - \alpha)(x' - \alpha) + (y''' - \beta)(y' - \beta)}{\pm\sqrt{\{(x''' - \alpha)^2 + (y''' - \beta)^2\}}\sqrt{\{(x' - \alpha)^2 + (y' - \beta)^2}}$$

$$\cos\nu' = \frac{(x' - \alpha)(x'' - \alpha) + (y' - \beta)(y'' - \beta)}{\pm\sqrt{\{(x' - \alpha)^2 + (y' - \beta)^2\}}\sqrt{\{(x'' - \alpha)^2 + (y'' - \beta)^2}}$$

$$\therefore \left.\begin{aligned} \cos\lambda' &= \pm\cos\lambda \\ \cos\mu' &= \pm\cos\mu \\ \cos\nu' &= \pm\cos\nu \end{aligned}\right\}$$

$$i.e., \left.\begin{aligned} \lambda' &= \lambda \text{ or } 180^\circ - \lambda \\ \mu' &= \mu \text{ or } 180^\circ - \mu \\ \nu' &= \nu \text{ or } 180^\circ - \nu \end{aligned}\right\}$$

Now from the geometrical point of view, the three angles formed by three lines meeting at a point are either together equal to four right angles or are such that one of them is equal to the sum of the other two.

Clearly there are four possible centres of reciprocation corresponding to the four sets of values of λ', μ', ν', *viz.*—

$$\left.\begin{aligned} \lambda' &= 180^\circ - \lambda \\ \mu' &= 180^\circ - \mu \\ \nu' &= 180^\circ - \nu \end{aligned}\right\}(1) \qquad \left.\begin{aligned} \lambda' &= \lambda \\ \mu' &= 180^\circ - \mu \\ \nu' &= \nu \end{aligned}\right\}(3)$$

$$\left.\begin{aligned} \lambda' &= 180^\circ - \lambda \\ \mu' &= \mu \\ \nu' &= \nu \end{aligned}\right\}(2) \qquad \left.\begin{aligned} \lambda' &= \lambda \\ \mu' &= \mu \\ \nu' &= 180^\circ - \nu \end{aligned}\right\}(4).$$

Let ABC be the given triangle; let arcs BOC and $BO'C$, COA and $CO''A$, AOB and $AO'''B$ be described on the sides BC, CA, AB containing angles equal to $180°-\lambda$, $180°-\mu$, and $180°-\nu$ respectively, λ, μ, ν being the angles of the other given

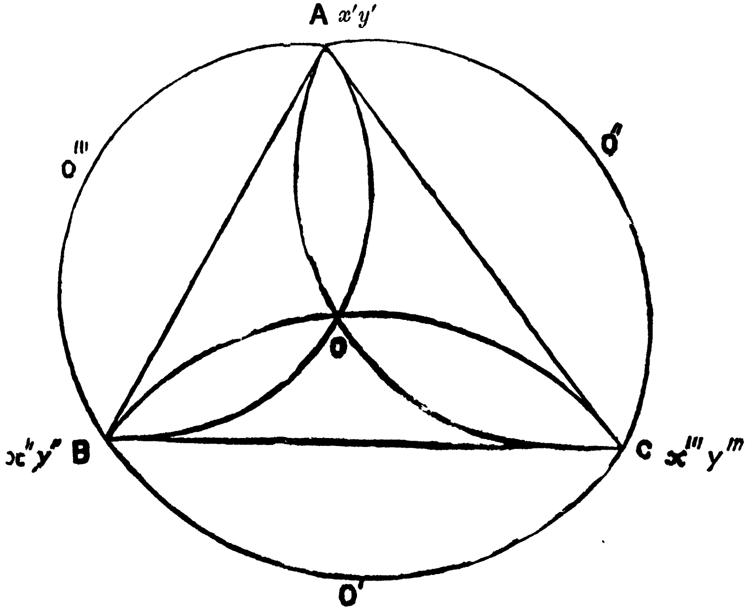


Fig. 7.

triangle. Evidently, the arcs on BC, CA, AB will intersect each other at O, which is one possible centre of reciprocation. The centres of reciprocation corresponding to the cases (2), (3)

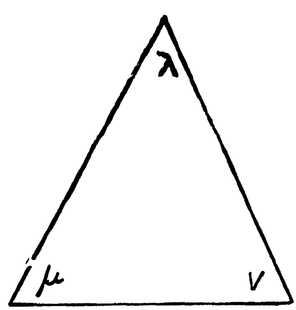


Fig. 8.

and (4) will be situated on the three arcs as represented in the diagram. We shall discuss these four cases in order.

Case I. In this case the point must be inside the given triangle, for the sum of two angles can not be equal to the third. Hence, it is evident that this centre of reciprocation will be possible if

$$180° - \lambda > A,\ 180° - \mu > B \text{ and } 180° - \nu > C \ldots\ldots\ldots (\alpha)$$

If these inequalities be not satisfied, there can be no required centre of reciprocation inside the triangle.

Case II. The centre of reciprocation in this case must be on the arc $BO'C$, the $\angle CO'A$ being equal to μ and the $\angle AO'B$ to ν.

I.

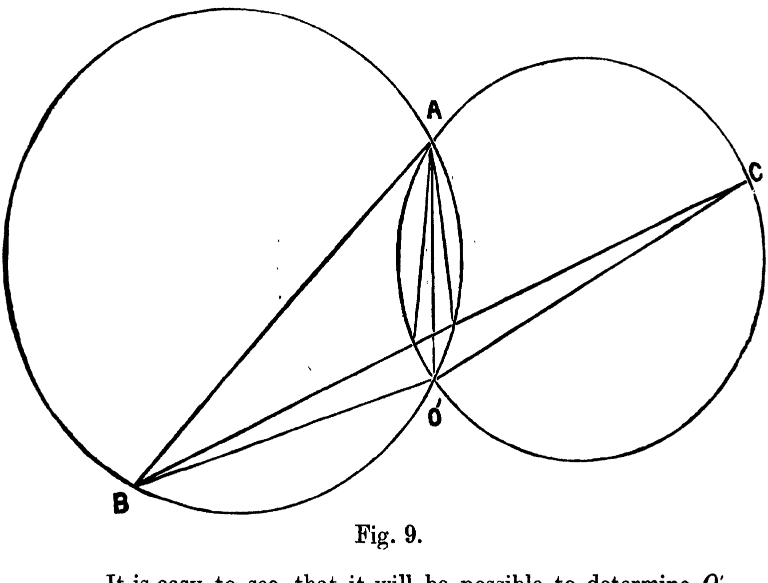


Fig. 9.

It is easy to see, that it will be possible to determine O' if

Fig. I.		Fig. II.	
either $\mu > B$ and $\nu > C$	or	$B > \mu$ and $C > \nu$	.... (β).

Case III. In this case, the centre of reciprocation will be some point O'' on the arc $CO''A$, the angles $CO''B$ and $AO''B$

being equal to λ and ν respectively, and it is easy to see in the same way that O'' will be a possible centre if

Fig. I		Fig. II	
either $\nu > C$ and $\lambda > A$	or	$C > \nu$ and $A > \lambda$	.... (γ).

Case IV. In this case, the centre of reciprocation will be some point O''' on the arc $AO'''B$, the angles $CO'''A$ and $BO'''C$ being respectively equal to μ and λ.

It is easy to see that O''' will be a possible centre of reciprocation, if

Fig. I		Fig. II	
either $\lambda > A$ and $\mu > B$	or	$A > \lambda$ and $B > \mu$	.... (δ).

II.

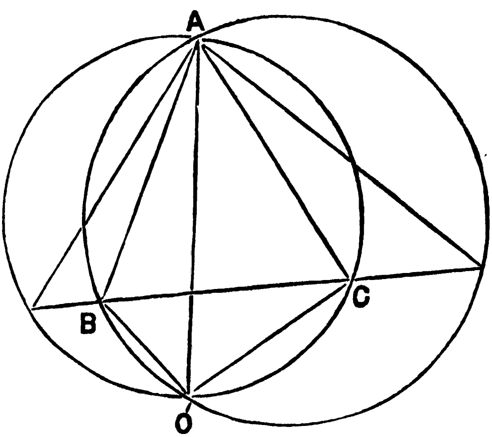


Fig. 10.

It is easy to see that the conditions (β), (γ) (δ) can not hold good simultaneously, as $\lambda + \mu + \nu = A + B + C =$ two right angles. Any one of these conditions (six in number) holding good, the others cannot be true. Thus, only one of the three centres O', O'', O''' is a possible centre of reciprocation, the other two being imaginary. The same proposition may be established in another way. The points O', O'', O''', B, C are concyclic, as also the two systems of points O', O'', O''', C, A and O', O'', O''', A, B. Therefore the points O', O'', O''', A, B, C are all concyclic, which is impossible since λ, μ, ν are not equal to A, B, C respectively. The fallacy lies in the fact that two of the points O', O'', O''' are imaginary, *viz.*, the circular points at infinity, and therefore the proposition converse to Prop. 21. Book III [Euclid] is not applicable.

21. To determine the centre of reciprocation in order that the circular polar of a given quadrilateral may be a parallelogram.

Let x_1y_1, x_2y_2, x_3y_3, x_4y_4 be the *vertices* of the given quadrilateral. The sides of the reciprocal polar, the centre of reciprocation being $(\alpha\beta)$ are :—

$$(x-\alpha)(x_1-\alpha)+(y-\beta)(y_1-\beta)=r^2\ldots\ldots\ldots(1)$$

$$(x-\alpha)(x_2-\alpha)+(y-\beta)(y_2-\beta)=r^2\ldots\ldots\ldots(2)$$

$$(x-\alpha)(x_3-\alpha)+(y-\beta)(y_3-\beta)=r^2\ldots\ldots\ldots(3)$$

$$(x-\alpha)(x_4-\alpha)+(y-\beta)(y_4-\beta)=r^2\ldots\ldots\ldots(4)$$

These lines must be parallel in pairs, and as we can choose two pairs of equations from the four in three ways, there are three possible centres of reciprocation given by

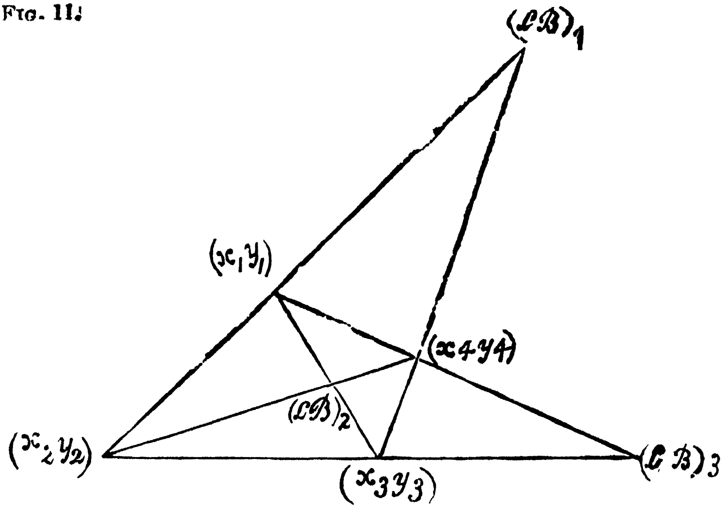
$$\left.\begin{array}{l}\dfrac{x_1-\alpha}{x_2-\alpha}=\dfrac{y_1-\beta}{y_2-\beta}\\[2ex] \dfrac{x_3-\alpha}{x_4-\alpha}=\dfrac{y_3-\beta}{y_4-\beta}\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\left.\begin{array}{l}\dfrac{x_1-\alpha}{x_3-\alpha}=\dfrac{y_1-\beta}{y_3-\beta}\\[2ex] \dfrac{x_2-\alpha}{x_4-\alpha}=\dfrac{y_2-\beta}{y_4-\beta}\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\left.\begin{array}{l}\dfrac{x_1-\alpha}{x_4-\alpha}=\dfrac{y_1-\beta}{y_4-\beta}\\[2ex] \dfrac{x_2-\alpha}{x_3-\alpha}=\dfrac{y_2-\beta}{y_3-\beta}\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The three centres of reciprocation are the points of intersections of the diagonals and opposite sides.

FIG. 11.



Hence, we can reciprocate a system of conics passing through four given points into a system inscribed in the same parallelogram.

22. Let $ax^2+2hxy+by^2+2gx+2fy+c=0=\phi(x, y)$........(1)
and $a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0=\phi'(x, y)$..........(2)
be two conics.

$$\text{Let } (x-x')^2+(y-y')^2=r^2$$

be the reciprocating circle. The circular polars are respectively

$$A'x^2+2H'xy+B'y^2-2G'r^2x-2F'r^2y+C'r^4=0 \text{........(3)}$$
$$A''x^2+2H''xy+B''y^2-2G''r^2x-2F''r^2y+C''r^4=0 \text{.....(4)}$$

The circular polar of (1), *i.e.* (3), will be similar and similarly situated with respect to (2)

if

$$\frac{A'}{a'}=\frac{H'}{h'}=\frac{B'}{b'},$$

i.e. if the centre of reciprocation be given by

$$\frac{b\phi(x, y)-(hx+by+f)^2}{a'}=\frac{(ax+hy+g)(hx+by+f)-h\phi(x, y)}{h'}$$

$$=\frac{a\phi(x, y)-(ax+hy+g)^2}{b'}.$$

i.e. if the centre of reciprocation be given by the equations :

$$\frac{Cx^2-2Gx+A}{a'}=\frac{Cxy-Fx-Gy+H}{h'}=\frac{Cy^2-2Fy+B}{b'},$$

or,
$$\frac{\left(x-\frac{G}{C}\right)^2+\frac{AC-G^2}{C^2}}{a'}=\frac{\left(x-\frac{G}{C}\right)\left(y-\frac{F}{C}\right)+\frac{CH-FG}{C^2}}{h'}$$

$$=\frac{\left(y-\frac{F}{C}\right)^2+\frac{BC-F^2}{C^2}}{b'}=\lambda \quad \text{(say)}$$

Then,
$$\left(x-\frac{G}{C}\right)^2=\lambda a'-\frac{AC-G^2}{C^2}$$

$$=\lambda a'-\frac{b\Delta}{C^2}\dots\dots\dots\dots(5)$$

$$\left(x-\frac{G}{C}\right)\left(y-\frac{F}{C}\right)=\lambda h'+\frac{FG-CH}{C^2}$$

$$=\lambda h'+\frac{h\Delta}{C^2}\dots\dots\dots\dots(6)$$

$$\left(y-\frac{F}{C}\right)^2=\lambda b'-\frac{BC-F^2}{C^2}$$

$$=\lambda b'-\frac{a\Delta}{C^2}\dots\dots\dots\dots(7)$$

Whence,
$$\left(\lambda a'-\frac{b\Delta}{C^2}\right)\left(\lambda b'-\frac{a\Delta}{C^2}\right)=\left(\lambda h'+\frac{h\Delta}{C^2}\right)^2$$

or,
$$\lambda^2(a'b'-h'^2)-\lambda\frac{\Delta}{C^2}(aa'+bb'+2hh')$$

$$+\frac{\Delta^2}{C^4}(ab-h^2)=0\dots\dots\dots\dots(8)$$

Equation (8) determines λ, and from (5) and (6) we get the co-ordinates of the four possible centres of reciprocation.

The two values of λ as given by equation (8) will be real and distinct if

$$(aa' + bb' + 2hh')^2 > 4(ab - h^2)(a'b' - h'^2),$$

i.e. if $\quad (aa' - bb')^2 + 4(a'h + bh')(ah' + b'h)$

be positive.

The circular polars of (3) and (4) will be similar and similarly situated if

$$\frac{A'}{A''} = \frac{H'}{H''} = \frac{B'}{B''}$$

i.e. if the centre of reciprocation be determined by the equations:—

$$\frac{b\phi(x, y) - (hx + by + f)^2}{b'\phi'(x, y) - (h'x + b'y + f')^2}$$

$$= \frac{(ax + hy + g)(hx + by + f) - h\phi(x, y)}{(a'x + h'y + g')(h'x + b'y + f') - h'\phi'(x, y)},$$

$$= \frac{a\phi(x, y) - (ax + hy + g)^2}{a'\phi'(x, y) - (a'x + h'y + g')^2},$$

or,

$$\frac{Cx^2 - 2Gx + A}{C'x^2 - 2G'x + A'} = \frac{Cxy - Fx - Gy + H}{C'xy - F'x - G'y + H'}$$

$$= \frac{Cy^2 - 2Fy + B}{C'y^2 - 2F'y + B'} = \mu \text{ (say)}$$

or,

$$\left.\begin{aligned} &(C - \mu C')x^2 - 2(G - \mu G')x + (A - \mu A') = 0 \\ &(C - \mu C')xy - (F - \mu F')x - (G - \mu G')y + (H - \mu H') = 0 \\ &(C - \mu C')y^2 - 2(F - \mu F')y + (B - \mu B') = 0 \end{aligned}\right\}$$

Whence,

$$\left(x - \frac{G - \mu G'}{C - \mu C'}\right)^2 = \left(\frac{G - \mu G'}{C - \mu C'}\right)^2 - \frac{A - \mu A'}{C - \mu C'}$$

$$= \frac{1}{(C - \mu C')^2} \cdot \left\{(G - \mu G')^2 - (A - \mu A')(C - \mu C')\right\}$$

$$= \frac{1}{(C - \mu C')^2} \cdot$$

$$\left\{(G^2 - AC) + \mu(AC' + A'C - 2GG') + \mu^2(G'^2 - A'C')\right\} \ldots\ldots (9)$$

And

$$\left(x - \frac{G - \mu G'}{C - \mu C'}\right)\left(y - \frac{F - \mu F'}{C - \mu C'}\right)$$

$$=\frac{1}{(C-\mu C')^2}\cdot\left\{(G-\mu G')(F-\mu F')-(C-\mu C')(H-\mu H')\right\}$$

$$=\frac{1}{(C-\mu C')^2}\cdot\left\{(GF-CH)-\mu(GF'+G'F-CH'-C'H)\right.$$

$$\left.+\mu^2(G'F'-C'H')\right\}\ldots\ldots(10)$$

also $\left(y-\dfrac{F-\mu F'}{C-\mu C'}\right)^2$

$$=\frac{1}{(C-\mu C')^2}\cdot\left\{(F-\mu F')^2-(B-\mu B')(C-\mu C')\right\}.$$

$$=\frac{1}{(C-\mu C')^2}\cdot\left\{(F^2-BC)+\mu(BC'+B'C-2FF')\right.$$

$$\left.+\mu^2(F'^2-B'C')\right\}\ldots\ldots\ldots(11)$$

From (9), (10) and (11) we get for the determination of μ, the biquadratic equation

$$\left\{(AC-G^2)-\mu(AC'+A'C-2GG')+\mu^2(A'C'-G'^2)\right\}$$

$$\times\left\{(BC-F^2)-\mu(BC'-B'C-2FF')+\mu^2(B'C'-F'^2)\right\}$$

$$=\left\{(GF-CH)-\mu(GF'+G'F-CH'-C'H)+\mu^2(G'F'-C'H')\right\}^2$$

or, $\left\{b\Delta-\mu(AC'+A'C-2GG')+\mu^2.\,b'\Delta'\right\}$

$$\times\left\{a\Delta-\mu(BC'+B'C-2FF')+\mu^2.\,a'\Delta'\right\}$$

$$=\left\{h\Delta-\mu(GF'+G'F-CH'-C'H)+\mu^2h.'\Delta'\right\}^2.$$

or, $\mu^4(a'b'-h'^2)\Delta'^2-\mu^3\Delta'\{a'(AC'+A'C-2GG')+b'(BC'+B'C$
$-2FF')-2h'(GF'+G'F-CH'-C'H)\}$

$+\mu^2\{(ab'+a'b-2hh')\Delta.\Delta'+(AC'+A'C-2GG')(BC'+B'C-$

$2FF')-(GF'+G'F-CH'-C'H)^2\}-\mu\Delta\{b(BC'+B'C-2FF')$

$+a(AC'+A'C-2GG')\ -2h(GF'+G'F-CH'-C'H)\}$

$+(ab-h^2)\Delta^2=0\ldots.(12)$

Equation (12) gives the values of μ and from (9) and (10) we get the co-ordinates of the eight possible centres of reciprocation.

The circular polars (3) and (4) will be concentric

$$\left.\begin{aligned} &\text{if} \quad \frac{H'F'-B'G'}{A'B'-H'^2}=\frac{H''F''-B''G''}{A''B''-H''^2}\,. \\ &\text{and} \quad \frac{H'G'-A'F'}{A'B'-H'^2}=\frac{H''G''-A''F''}{A''B''-H''^2}\,. \end{aligned}\right\}$$

$$i.e. \quad \text{if } \frac{g'}{c'}=\frac{g''}{c''} \text{ and } \frac{f'}{c'}=\frac{f''}{c''}\,,$$

$$i.e. \quad \text{if } \frac{g'}{g''}=\frac{f'}{f''}=\frac{c'}{c''}\,.$$

i.e., if the co-ordinates of the centre of reciprocation be determined by the equations

$$\frac{ax+hy+g}{a'x+h'y+g'}=\frac{hx+by+f}{h'x+b'y+f'}=\frac{gx+fy+c}{g'x+f'y+c'}=\nu \text{ (say)}$$

Whence,

$$\left.\begin{aligned} (a-a'\nu)x+(h-h'\nu)y+(g-g'\nu)=0 \\ (h-h'\nu)x+(b-b'\nu)y+(f-f'\nu)=0 \\ (g-g'\nu)x+(f-f'\nu)y+(c-c'\nu)=0 \end{aligned}\right\}$$

Eliminating x, y we get a cubic equation for the determination of ν

$$\begin{vmatrix} a-a'\nu, & h-h'\nu, & g-g'\nu \\ h-h'\nu, & b-b'\nu, & f-f'\nu \\ g-g'\nu, & f-f'\nu, & c-c'\nu \end{vmatrix}=0 \quad \ldots\ldots\ldots\ldots (13)$$

Thus there are three possible centres of reciprocation in order that the circular polars of two given conics may be concentric.

It is easy to see that the three values of ν given by (13) are the values of ν for which $\phi(x, y)-\nu\phi'(x, y)=0$ breaks up into two right lines.

The three centres of reciprocation are the points of intersection of the diagonals and opposite sides of the common inscribed quadrilateral of the two given conics. The polars of each of these three points with respect to the two given conics are coincident right lines and therefore the corresponding circular polars are concentric. This is also evident from Art 21.

23. In this article, we shall establish by circular polars, properties of conics, and in so doing we shall simply state the corresponding reciprocal property in the circle.

1. Any focal chord of a conic is divided harmonically by the curve, the focus and the directrix.

The portion of a straight line intercepted between two parallel tangents to a circle at right angles to it is bisected by the line through the centre parallel to the tangents.

2. If any chord QQ' of a conic intersects the directrix in D, SD bisects the exterior angle between SQ and SQ'.

The line joining the centre to the point of intersection of two tangents makes equal angles with the two tangents.

3 The angle at the focus subtended by a tangent intercepted by the directrix is a right angle.

In the circle, the radius is perpendicular to the tangent.

4. The subtangent at any point of a parabola is bisected at the vertex.

Any straight line is divided harmonically by the point, the polar and the circumference of the circle.

5. The tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.

The angle in a semicircle is a right angle.

6. Two tangents subtend equal angles at the focus.

The tangents make equal angles with the chord of contact in a circle.

7. The exterior angle between two tangents is equal to the angle subtended at the focus of the parabola.

The angle between the tangent and the chord is equal to the angle in the alternate segment.

8. The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

The parabola has for a tangent the line infinity.

Or, the feet of the perpendiculars dropped from any point on the circumscribed circle of a triangle on the three sides are collinear.

9. The lines joining the extremities of any two focal chords of a conic intersect on the directrix and the focal distances of their intersections are at right angles.

The point of intersection of the diagonals of a parallelogram circumscribed about a circle is the centre of the circle and the diagonals intersect each other at right angles.

10. If two conics have a common focus their common chord or chords will pass through the point of concourse of their directrices.

The point or points of intersection of common tangents to two circles will lie on the straight line joining the centres of the two circles.

11. The vertex of a circumscribed triangle whose base subtends a constant angle at the focus lies on a conic having the same focus and directrix.

The envelope of the base of a triangle inscribed in a circle having a given vertical angle is a circle.

12. Given the focus and the directrix of a conic, shew that the polar of a given point with respect to it passes through a fixed point.

The poles of a fixed straight line with respect to a series of concentric circles lie on another fixed right line.

13. Conics having the same focus and directrix do not meet nor have common tangents.

Concentric circles do not intersect and have no common tangent.

14. Given four right lines passing through a point and a fixed point O. The envelope of a line intersecting the given lines at A, B, C, D, in such a manner that $\angle AOB = \angle COD$ is a conic having the origin O for a focus.

A, B, C, D are four given points in a straight line and P is a point such that $\angle APB = \angle CPD$. The locus of P is a circle.

24. The method of reciprocal polars may with advantage be used in solving some problems. We shall consider here the following problem :—

To find the envelope of a line moving in such a manner that the product of its distances from two fixed points may be constant.

Let us take one of the fixed points for origin and let the other be $(\alpha\beta)$.

Let $x^2 + y^2 = r^2$ be the auxiliary circle.

If p_1 and p_2 be the two perpendicular distances and $x'y'$ be the pole of the variable line in any position, we have

$$p_1 \sqrt{x'^2 + y'^2} = r^2 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$p_2 = \frac{\alpha x' + \beta y' - r^2}{\sqrt{x'^2 + y'^2}} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

But $p_1\, p_2 = b^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

$\therefore$ the circular polar of the envelope is the circle

$$x^2+y^2=\frac{r^2}{b^2}(\alpha x+\beta y-r^2)\ldots\ldots\ldots\ldots(4)$$

The envelope which is the circular polar of (4) must be a central conic having the fixed points for foci.

25. In this article we shall establish some metrical properties by the method of circular polars.

Let $x'y'$, $x''y''$ be two points; their circular polars with respect to $x^2+y^2=r^2$ are $xx'+yy'=r^2$, $xx''+yy''=r^2$ respectively.

If δ be the distance between the two points, θ the angle between the two lines, and p_1 and p_2 the perpendiculars on them from the origin,

$$\delta^2=(x'-x'')^2+(y'-y'')^2\ldots\ldots\ldots\ldots(1)$$

$$\cos\theta=\frac{x'x''+y'y''}{\sqrt{(x'^2+y'^2)}\sqrt{(x''^2+y''^2)}}\ldots\ldots(2)$$

$$\left.\begin{aligned}\text{and } p_1&=\frac{r^2}{\sqrt{x'^2+y'^2}}\\ p_2&=\frac{r^2}{\sqrt{x''^2+y''^2}}\end{aligned}\right\}\ldots\ldots\ldots(3)$$

Whence, $$\delta^2=\frac{r^4}{p_1^2p_2^2}\cdot(p_1^2+p_2^2-2p_1p_2\cos\theta)$$

or $$\delta=\frac{r^2}{p_1p_2}\sqrt{(p_1^2+p_2^2-2p_1p_2\cos\theta)}.$$

It is easy to see that the points A, B, Q, P are concyclic. The triangles OAB and OQP are similar and we have

$$\frac{PQ}{AB}=\frac{OP}{OB}=\frac{OP\cdot OA}{OA\cdot OB}=\frac{r^2}{OA\cdot OB}\qquad\text{or, } PQ=\frac{r^2}{p_1p_2}\cdot AB.$$

Also if PM and QN be dropped perpendiculars to the polars of Q and P respectively,

$$PM=\frac{x'x''+y'y''-r^2}{\sqrt{x''^2+y''^2}}\text{ and }QN=\frac{x'x''+y'y''-r^2}{\sqrt{x'^2+y'^2}}.$$

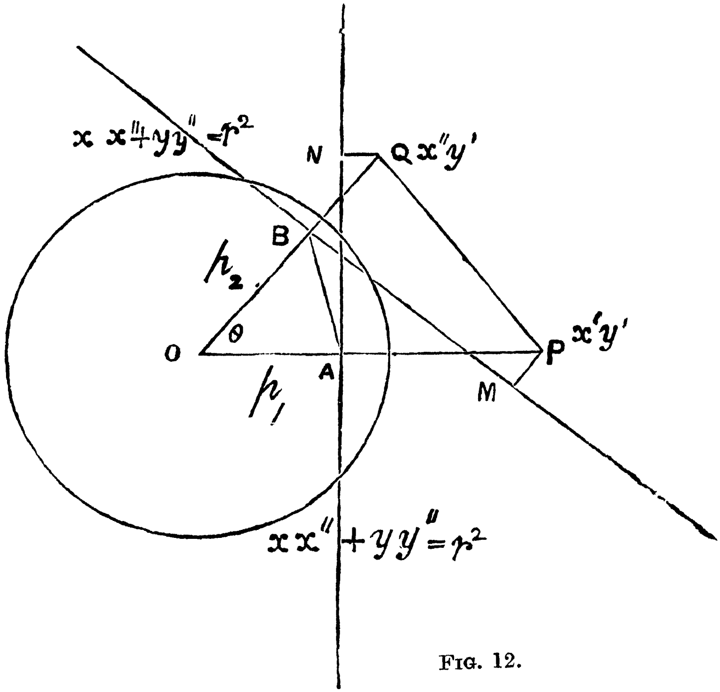


FIG. 12.

We have therefore

$$PM \cdot OQ = QN \cdot OP$$

$$\text{or,} \quad \frac{PM}{OP} = \frac{QN}{OQ} \cdot$$

Again if δ be the distance between the two parallel lines

$$\left.\begin{array}{r} xx' + yy' = r^2 \\ \lambda xx' + \lambda yy' = r^2 \end{array}\right\}$$

and δ' the distance between their poles $(x'y')(\lambda x', \lambda y')$,

$$\left.\begin{array}{l} \delta = \dfrac{r^2}{\sqrt{(x'^2 + y'^2)}} \cdot \left(1 - \dfrac{1}{\lambda}\right) \\ \delta' = (\lambda - 1)\sqrt{x'^2 + y'^2} \end{array}\right\}$$

and therefore $\lambda\delta\delta' = r^2 \cdot (\lambda - 1)^2$.

As instances of theorems relating to metrical properties which can be inferred one from the other by the method of circular polars, we state the following:—

1. The locus of a point moving in such a manner that the product of its distances from two fixed lines is constant, is a hyperbola to which the fixed lines are asymptotes.

The envelope of a variable line such that the product of its distances from two fixed points is proportional to the square of its distance from a third fixed point, is a conic section.

2 The distance between two parallel tangents to two concentric circles is constant.

If two conics have the same focus S and directrix and a line SPQ be drawn cutting the conics at P and Q,

$$PQ \text{ is proportional to } SP.\ SQ.$$

3. If any number of chords of a circle be drawn through a given point within or without a circle, the rectangle contained by the segments of the chords are equal.

The rectangle contained by the perpendiculars dropped from a focus of a central conic on parallel tangents to the conic is constant.

4. In the ellipse, $SP + S'P$ is constant.

S is a point inside a circle centre O and a radius OL is drawn at right angles to OS. A point T is taken in OS produced such that $2\,OT \cdot OS = SL^2$. T' is the foot of the perpendicular dropped from S on any tangent to the circle, then

$$ST' \propto TT' + ST. \quad \text{(Vide figure 13.)}$$

5. In the hyperbola $S'P - SP$ is constant. (Vide figure 14.)

S is a point outside a circle, centre C. In CS a point T is taken, such that $2ST \cdot CS$ = the square on the tangent from S to the circle. T' is the foot of the perpendicular from S to any tangent to the circle, then

$$ST' \propto TT' - ST.$$

6. In the central conic $S'P \pm SP$ is constant.

Given two fixed lines and a fixed point O; the perpendiculars OA and OB are drawn to the fixed lines and OP is drawn perpendicular to a third variable line so that

$\frac{1}{OP}\left(\frac{AP}{OA} \pm \frac{BP}{OB}\right)$ is constant. The envelope of the variable line is a central conic.

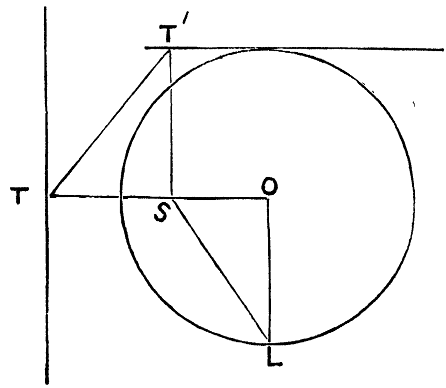


FIG. 13.

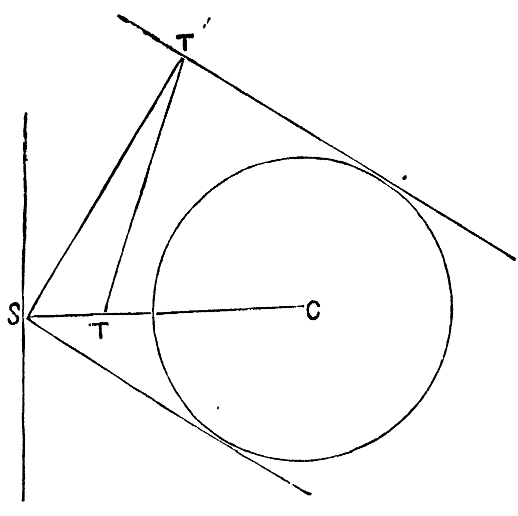


FIG. 14.

CHAPTER III.

Circular Polars—Oblique Axes.

1. The reciprocating circle is

$$x^2 + 2xy \cos \omega + y^2 = r^2.$$

If $x'y'$ be a point on the circular polar of the general conic,

$$x(x' + y' \cos \omega) + y(y' + x' \cos \omega) = r^2$$

must be a tangent to

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

The circular polar of the general conic is therefore

$$A(x + y \cos \omega)^2 + 2H(x + y \cos \omega)(y + x \cos \omega)$$
$$+ B(y + x \cos \omega)^2 - 2Gr^2(x + y \cos \omega) - 2Fr^2(y + x \cos \omega)$$
$$+ Cr^4 = 0,$$

or,

$$(A + 2H \cos \omega + B \cos^2\omega)x^2$$
$$+ 2(A \cos \omega + H + H \cos^2\omega + B \cos \omega)xy$$
$$+ (B + 2H \cos \omega + A \cos^2\omega)y^2 - 2(G + F \cos \omega)r^2x$$
$$- 2(F + G \cos \omega)r^2y + Cr^4 = 0.$$

2. Transferring the origin to $x'y'$, the equation to the conic becomes

$$ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' - 0.$$

The circular polar of the conic with respect to

$x^2 + y^2 + 2xy \cos \omega = r^2$ referred to the new origin is

$$(A' + 2H' \cos \omega + B' \cos^2\omega)x^2$$
$$+ 2(A' \cos \omega + H' + H' \cos^2\omega + B' \cos \omega)xy$$
$$+ (B' + 2H' \cos \omega + A' \cos^2\omega)y^2$$
$$- 2(G' + F' \cos \omega)r^2x - 2(F' + G' \cos \omega)r^2y + C'r^4 = 0.$$

This will represent a circle, if

$$A' + 2H' \cos \omega + B' \cos^2\omega = B' + 2H' \cos \omega + A' \cos^2\omega \ldots\ldots (1)$$

and $A' \cos \omega + H' + H' \cos^2\omega + B' \cos \omega$

$$= \cos \omega(A' + 2H' \cos \omega + B' \cos^2\omega) \ldots\ldots (2)$$

From (1) $A' = B'$

and (2) $H' + B' \cos\omega = 0$

$$\therefore \quad A' = B' = -\frac{H'}{\cos\omega}.$$

i.e.

$$bc' - f'^2 = ac' - g'^2 = \frac{f'g' - c'h}{\cos\omega}.$$

or

$$\cos\omega(bc' - f'^2) = \cos\omega\,(ac' - g'^2) = c'h - f'g'.$$

$$\therefore \quad c'(h - b\cos\omega) = f'(g' - f'\cos\omega).$$

$$\text{and} \quad c'(h - a\cos\omega) = g'(f' - g'\cos\omega).$$

The circular polar is a circle only when the centre of reciprocation is a focus. Hence, the equations determining the foci when the axes are oblique and inclined at an angle ω are :—

$$\frac{(ax + hy + g)(\overline{hx + by + f} - \overline{ax + by + g}\cos\omega)}{h - a\cos\omega}$$

$$= \frac{(hx + by + f)(\overline{ax + hy + g} - \overline{hx + by + f}\cos\omega)}{h - b\cos\omega}$$

$$= ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

The first equation represents a pair of lines through the centre and must therefore represent the axes of the general conic on which the foci are situated. The axes are therefore given by

$$\frac{\dfrac{\delta\phi}{\delta x}\left(\dfrac{\delta\phi}{\delta y} - \dfrac{\delta\phi}{\delta x}\cos\omega\right)}{h - a\cos\omega} = \frac{\dfrac{\delta\phi}{\delta y}\left(\dfrac{\delta\phi}{\delta x} - \dfrac{\delta\phi}{\delta y}\cos\omega\right)}{h - b\cos\omega}.$$

3. The circular polar of the circle

$$a(x^2 + y^2 + 2xy\cos\omega) + 2gx + 2fy + c = 0$$

is

$$\{(ac - f^2) + 2(fg - ca\cos\omega)\cos\omega + (ac - g^2)\cos^2\omega\}x^2$$
$$+ 2\{(ac - f^2)\cos\omega + (fg - ca\cos\omega) + (fg - ca\cos\omega)\cos^2\omega$$
$$+ (ac - g^2)\cos\omega\}xy$$
$$+ \{(ac - g^2) + 2(fg - ca\cos\omega)\cos\omega + (ac - f^2)\cos^2\omega\}y^2$$

$$-2\{(af\cos\omega-ga)+(ga\cos\omega-af)\cos\omega\}r^2x$$
$$-2\{(ga\cos\omega-af)+(fa\cos\omega-ag)\cos\omega\}r^2y$$
$$+a^2\sin^2\omega r^4=0$$

or,
$$(f^2+g^2-2fg\cos\omega-ac\sin^2\omega)(x^2+y^2+2xy\cos\omega)$$
$$=\sin^2\omega(gx+fy+ar^2)^2.$$

If PM be drawn pependicular to $gx+fy+ar^2=0$ from P, a point $(x\ y)$ on the locus,

$$PM^2=\frac{\sin^2\omega\ (gx+fy+ar^2)^2}{f^2+g^2-2fg\cos\omega}$$

and therefore
$$OP^2=e^2.\ PM^2$$

where
$$e^2=\frac{f^2+g^2-2fg\cos\omega}{f^2+g^2-2fg\cos\omega-ac\sin^2\omega}.$$

Hence, the circular polar is a conic having the origin for focus and polar of the centre for directrix and the eccentricity

$$e=\frac{\text{the distance of the centre from the origin}}{\text{the radius of the circle}}.$$

4. The eccentricity of the general conic referred to oblique axes can be determined as in Art. 11, Chapter II.

Transferring the origin to a focus $(x'y')$ the equation becomes

$$ax^2+2hxy+by^2+2g'x+2f'y+c'=0.$$

The circular polar of it, which is a circle, is

$$(A'+2H'\cos\omega+B'\cos^2\omega)\,x^2+(B'+2H'\cos\omega+A'\cos^2\omega)y^2$$
$$+2(A'\cos\omega+H'+H'\cos^2\omega+B'\cos\omega)xy-2(G'+F'\cos\omega)r^2x$$
$$-2(F'+G'\cos\omega)r^2y+C'r^4=0.$$

Therefore, if e be the eccentricity of the general conic,

$$e^2=\frac{F'^2+G'^2+2F'G'\cos\omega}{F'^2+G'^2+2F'G'\cos\omega-C'(A'+2H'\cos\omega+B'\cos^2\omega)}$$

or

$$\frac{e^2-1}{e^2}=\frac{(ab-h^2)\{(bc'-f'^2)+2(f'g'-c'h)\cos\omega+(ac'-g'^2)\cos^2\omega\}}{(g'h-af')^2+(hf'-bg')^2+2(hg'-af')(hf'-bg')\cos\omega.}\quad(1)$$

where $\dfrac{g'(f'-g'\cos\omega)}{h-a\cos\omega}=\dfrac{f'(g'-f'\cos\omega)}{h-b\cos\omega}=c'$ .... (2)

$$\text{Let } g'=\lambda f', \ldots\ldots\ldots\ldots$$

then from (2)

$$\left.\begin{aligned} f'^2(\lambda-\lambda^2\cos\omega)&=c'(h-a\cos\omega)\\ \text{and}\quad f'^2(\lambda-\cos\omega)&=c'(h-b\cos\omega)\end{aligned}\right\} \ldots\ldots\ldots\ldots (3)$$

Whence

$$\frac{\lambda-\lambda^2\cos\omega}{\lambda-\cos\omega}=\frac{h-a\cos\omega}{h-b\cos\omega} \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Making these substitutions in (2) we get

$$\frac{e^2-1}{e^2}=\frac{(ab-h^2)(b\lambda-h)\sin^2\omega}{(h-b\cos\omega)\{(h\lambda-a)^2+(h-b\lambda)^2+2(h\lambda-a)(h-b\lambda)\cos\omega\}}$$

where λ is given by (4).

It is also easy to see that the equation to the director-circle, the axes being oblique, is

$$4(a+b-2h\cos\omega)\,\phi(x,y)$$

$$=\left(\frac{\delta\phi}{\delta x}\right)^2+\left(\frac{\delta\phi}{\delta y}\right)^2-2\frac{\delta\phi}{\delta x}\cdot\frac{\delta\phi}{\delta y}\cdot\cos\omega,$$

the circular polar being an equilateral hyperbola in this case.

CHAPTER IV.

Parabolic Polars.

Let $y^2=4ax$ be the auxiliary conic.

If $x'\ y'$ and $x''\ y''$ be two points, their polars with respect to the parabola are:—

$$\left.\begin{aligned} yy'-2a\,(x+x')&=0\\ \text{and } yy''-2a\,(x+x'')&=0\end{aligned}\right\}$$

The points where these polars intersect the axis of x are

$$(-x', 0) \text{ and } (-x'', 0).$$

The points where they intersect the axis of y are

$$\left(0,\ \frac{2ax'}{y'}\right) \text{ and } \left(0,\ \frac{2ax''}{y''}\right).$$

Therefore the intercept on the axis of x made by the polars is equal to the difference of the abscissæ of the two points.

Again when the points are on the parabola, the intercept on the axis of y made by the tangents at the points is equal to one-half of the difference of the ordinates of the points.

The vertex of the reciprocating parabola will be referred to as the origin of parabolic reciprocation.

1. Let the parabola $y^2 = 4ax$ be the auxiliary conic.

We shall determine the parabolic polar of the general conic,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If $x'y'$ be a point on the parabolic polar,

$$yy' - 2a(x + x') = 0$$

must be a tangent to the general conic and therefore

$$4Aa^2 + By^2 + 4Ca^2x^2 - 4Faxy + 8Ga^2x$$
$$- 4Hay = 0 \text{ is the parabolic polar.}$$

2. Transferring the origin to $(x'y')$ the equation to the conic becomes

$$ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' = 0.$$

The parabolic polar of this conic with respect to $y^2 - 4ax = 0$, referred to the new origin, is

$$4C'a^2x^2 - 4F'axy + B'y^2 + 8G'a^2x$$
$$- 4H'ay + 4A'a^2 = 0 \quad \text{.................(1)}$$

The discriminant of this equation

$$= 16a^4(A'B'C' + 2F'G'H' - A'F'^2 - B'G'^2 - C'H'^2)$$
$$= 16a^4 \quad \Delta^2.$$

Similar inferences can be drawn as in Art. 5, Chap. II.

3. The parabolic polar will be a circle, the origin of reciprocation being $(x'y')$ and axes rectangular,

if $\qquad 4C'a^2 = B' \quad \text{......................(1)}$

and $\qquad F' = O \quad \text{......................(2)}$

From (2) $F' = hg' - af'$

$$= h(ax' + hy' + g) - a(hx' + by' + f)$$
$$= 0.$$

or, $(ab - h^2)y' = hg - af$

or, $Cy' = F.$

From (1) $4(ab - h^2)\alpha^2 = ac' - g'^2$

$$= a(ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c) - (ax' + hy' + g)^2$$
$$= (ab - h^2)y'^2 - 2(hg - af)y' + (ac - g^2).$$

The parabolic polar will be a circle if the origin of reciprocation be determined by the equations.

$$\left.\begin{array}{r} Cy = F \\ \text{and} \quad 4C\alpha^2 = Cy^2 - 2Fy + B \end{array}\right\} \qquad \text{(A)}$$

Eliminating y between these equations we get

$$4C^2\alpha^2 = BC - F^2$$

or,
$$\alpha^2 = \frac{a \,.\, \Delta}{4(ab - h^2)^2} \qquad \text{(B)}$$

or,
$$\alpha = \frac{\sqrt{a \,.\, \Delta}}{2(ab - h^2)}$$

In order that the parabolic polar of the general conic may be a circle, we must have the parameter of the reciprocating parabola equal to $\dfrac{2\sqrt{a.\Delta}}{ab - h^2}$; at the same time, the origin of reciprocation must be some point on the line $CY = F$.

4. The parabolic polar will be an ellipse, a parabola or an hyperbola, according as $B'C' - F'^2$ is positive, zero or negative, i.e. according as $(a.\Delta)$ is positive, zero or negative.

Now $y = \beta$ will be a tangent to the general conic if

$$C\beta^2 - 2F\beta + B = 0.$$

The two values of β as given by this equation will be imaginary. equal or real and distinct, according as $BC - F^2$ is positive, zero or negative, i e. according as $(a.\Delta)$ is positive, zero or negative.

Hence, the parabolic polar will be an ellipse, a parabola or an hyperbola according as the tangents to the general conic parallel to the axis of the parabola are imaginary, coincident or real and distinct.

5. The parabolic polar will be an equilateral hyperbola, if

$$4C'\alpha^2 + B' = 0$$

or,
$$4C\alpha^2 + Cy'^2 - 2Fy' + B = 0.$$

The parabolic polar will be an equilateral hyperbola, if the origin of reciprocation be anywhere on one of the two lines parallel to the axis of x and given by the equation

$$Cy^2 - 2Fy + 4C\alpha^2 + B = 0$$

or,
$$y = \frac{F \pm \sqrt{(F^2 - BC - 4C^2\alpha^2)}}{C}.$$

6. The parabolic polar of the circle

$$ax^2 + ay^2 + 2gx + 2fy + c = 0,$$

with respect to
$$y^2 - 4\alpha x = 0,$$

is
$$4a^2\alpha^2x^2 + 4af\alpha xy + (ac - g^2)y^2$$
$$- 8ag\alpha^2x - 4fg\alpha y + 4(ca - f^2)\alpha^2 = 0$$

This will be an hyperbola, a parabola or an ellipse according as $4a^2\alpha^2 (f^2 + g^2 - ac)$ is positive, zero or negative, i.e. according as the given circle is a circle with a real and finite radius, a point circle or an imaginary circle. The parabolic polar of a real circle is an hyperbola as is otherwise evident from Art. 4.

7. In order that the parabolic polar of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

may be a circle, we must have

$$\alpha^2 = \frac{\frac{1}{a^2}\left(\frac{1}{a^2} \cdot \frac{1}{b^2}\right)}{4\left(-\frac{1}{a^2b^2}\right)^2} = \frac{b^2}{4}$$

or,
$$\alpha = \frac{b}{2}.$$

i.e. the parameter of the reciprocating parabola must be equal to the conjugate axis of the hyperbola. Also the ordinate y of the origin of reciprocation must be $\frac{F}{C}=0$, i.e. the origin of reciprocation must lie on the axis of x, i.e. the transverse axis.

8. If $x'y'$ be a point on the parabolic polar of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

with respect to the parabola $y^2-2xb=0$,

$$yy'-bx-bx'=0$$

is a tangent to the hyperbola and therefore, from the condition of tangency, we get

$$\frac{a^2}{x'^2}-\frac{b^2y'^2}{b^2x'^2}=1$$

or,

$$x'^2+y'^2=a^2,$$

i.e. the parabolic polar is the auxiliary circle of the hyperbola, viz. the circle $x^2+y^2=a^2$.

From what has been established in this article, it is easy to see that the following pairs of propositions may be deduced one from the other by the method of parabolic polars, remembering that in the parabola the difference between the abscissæ of two points is equal to the intercept on the axis of x—the axis of the parabola—by the polars of the points.

Examples.

1. The rectangle contained by the distances of any point on an hyperbola from the asymptotes, is of constant magnitude.

The rectangle contained by the intercepts on two parallel tangents to a circle by a variable tangent to it, is constant.

2. PQ is a chord of the circle $(x-\alpha)^2+y^2=a^2$ passing through either of the two fixed points $(\pm\sqrt{\alpha^2-a^2},0)$.

PM and QN are perpendiculars drawn from P and Q to the axis of y. Then $PM\,.\,QN=\alpha^2-a^2=$ constant.

From any point in either of the two right lines $x=\pm\sqrt{\alpha^2-a^2}$, tangents are drawn to the hyperbola intersecting the axis of x at T and T', the equation to the hyperbola being

$$\frac{(x+\alpha)^2}{a^2}-\frac{y^2}{b^2}=1.$$

Then,

$$OT\,.\,OT'=\alpha^2-a^2=\text{constant}.$$

9. To determine the parabolic polar of $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, with respect to $y^2 - 2bx = 0$.

Let $x'y'$ be a point on the parabolic polar, then

$$yy' - b(x + x') = 0$$

is a tangent to the hyperbola.

Therefore, $$\frac{b^2 y'^2}{b^2 x'^2} - \frac{a^2}{x'^2} = 1.$$

The parabolic polar is therefore the rectangular hyperbola,

$$y^2 - x^2 = a^2.$$

The following propositions may by inferred one from the other by the method of parabolic polars as established in this article.

EXAMPLE.

If $pP\ P'p'$ be a double ordinate of an hyperbola cutting the curve at P and P' and the asymptotes at p and p', then

$$pP = p'P'$$

and $$pP.\ pP' = \text{constant}.$$

If the parallel tangents PT and PT' to a rectangular hyperbola intersect the conjugate axis at T and T' and lines At and $A't'$ be drawn through A and A', the extremities of the transverse axis parallel to the tangents intersecting the conjugate axis at t and t', then

$$tT = t'T'$$

and $$tT.\ tT' = \text{constant}.$$

10. To determine the parabolic polar of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ with respect to } y^2 = 2bx.$$

If $x'y'$ be a point on the parabolic polar,

$$yy' - b(x + x') = 0$$

is a tangent to the ellipse.

Therefore, $$\frac{a^2}{x'^2} + \frac{b^2 y'^2}{b^2 x'^2} = 1,$$

or, the parabolic polar is the rectangular hyperbola,

$$x^2 - y^2 = a^2$$

The following propositions are parabolic polars with respect to one another :—

In an equilateral hyperbola, the product of the distances of any point on the curve from the asymptotes is constant.

In an ellipse, the rectangle contained by the intercepts on two fixed parallel tangents by a variable tangent is constant.

11. To determine the parabolic polar of $y^2 = 4ax$ with respect to $y^2 = 2ax$.

If $x'y'$ be a point on the parabolic polar,

$$yy' - a(x + x') = 0$$

is a tangent to the parabola $y^2 = 4ax$,

i.e. $$y'^2 = ax'.$$

Therefore, the parabolic polar is another parabola having the same axis and vertex.

ILLUSTRATION.

Two fixed tangents to a parabola are cut proportionally by any variable tangent.

If P, Q be two fixed points on a parabola, and R a third variable point on it, the portions of the axis intercepted by the line PQ and the tangents at P and Q, will be cut proportionally by the lines RP and RQ respectively.

12. To determine the parabolic polar of the circle $x^2 + y^2 = b^2$ with respect to $y^2 = 2bx$.

Let $x'y'$ be a point on the parabolic polar, then

$$yy' - b(x + x') = 0$$

is a tangent to the parobola.

Therefore $$y'^2 + b^2 = x'^2.$$

The parabolic polar is therefore the equilateral hyperbola

$$x^2 - y^2 = b^2.$$

EXAMPLE.

In the equilateral hyperbola if $pPP'p'$ be an ordinate intersecting the asymptotes at p and p' and the curve at P and P',

$$Pp = P'p'$$

and $$Pp \cdot Pp' = b^2 = \text{constant}.$$

If two parallel tangents be drawn to a circle intersecting a diameter at T and T', and if also parallel lines be drawn through the extremities of the perpendicular diameter intersecting the former at t and t', then

$$Tt = T't'$$

and $$Tt \,.\, Tt' = (\text{radius})^2.$$

13. To determine the parabolic polar of $x^2 + y^2 = a^2 + \lambda$ with respect to $y^2 = 2bx$.

Let $x'y'$ be a point on the parabolic polar, then

$$yy' - b(x + x') = 0$$

must be a tangent to the circle.

Therefore $$b^2x'^2 = (a^2 + \lambda)\ (y'^2 + b^2).$$

The parabolic polar is therefore

$$\frac{x^2}{a^2 + \lambda} - \frac{y^2}{b^2} = 1.$$

Hence, the parabolic polar of a series of concentric circles is a series of concentric and co-axial hyperbolas having the same conjugate axis.

ILLUSTRATION.

The points of contact of tangents drawn from a point to a series of concentric circles are situated on the circumference of a circle.

If a series of hyperbolas be described having the same conjugate axis, the envelope of the tangents at the points of intersection with a straight line parallel to the conjugate axis is a hyperbola.

14. To determine the parabolic polar of $x^2 - y^2 = a^2 + \lambda$ with respect to $y^2 = 2bx$.

Let $x'y'$ be a point on the parabolic polar, then

$$yy' - b(x + x') = 0$$

is a tangent to the hyperbola.

Therefore, $$\frac{a^2 + \lambda}{x'^2} - \frac{(a^2 + \lambda)y'^2}{b^2} = 1.$$

or, $$\frac{x'^2}{a^2+\lambda}+\frac{y'^2}{b^2}=1.$$

Therefore, the parabolic polar is $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2}=1.$

Hence, the parabolic polar of a series of rectangular hyperbolas having the same asymptotes but different transverse axes is a series of concentric and co-axial ellipses having the same minor axis.

Illustration.

The points of contact of tangents to a series of concentric and co-axial equilateral hyperbolas drawn from a point on the transverse axis lie on another rectangular hyperbola having parallel axes.

The envelope of the tangents at the points of intersection of a series of ellipses having the same minor-axis with a line parallel to the common minor-axis is an ellipse having parallel axes.

15. To determine the parabolic polar of $2xy=a^2$, with respect to $y^2=2ax$.

If $x'y'$ be a point on the parabolic polar $yy'-a(x+x')=0$ is a tangent to $2xy=a^2$.

Eliminating y, we get

$$2x\ (x+x')-ay'=0$$

Therefore, $$4x'^2+4\ .\ 2ay'=0$$

or, $$x'^2+2ay'=0.$$

The parabolic polar is therefore a parabola $x^2+2ay=0$.

Examples of Reciprocation.

1. The lines joining a variable point on an hyperbola to two fixed points on it intercept a constant length on either asymptote.

The oblique projection of the intercept of a variable tangent to a parabola between two fixed tangents on another fixed tangent is equal to the intercept of the fixed tangent, the projecting lines being drawn parallel to the diameter of the parabola.

2. Any circle which touches both branches of a hyperbola makes an intercept equal to the transverse axis on either asymptote.

The intercept on any fixed tangent to a parabola by tangents to a hyperbola having double contact with the parabola (the points of contact being on the two arms of the parabola) and parallel to the axis of the parabola, is constant.

CHAPTER V.

Elliptic Polars.

1. The auxiliary conic is the ellipse

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1.$$

The elliptic polar of the general conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is $$\frac{A}{\alpha^4}x^2+2\frac{H}{\alpha^2\beta^2}xy+\frac{B}{\beta^4}y^2-2\frac{G}{\alpha^2}x-2\frac{F}{\beta^2}y+C=0.$$

Transferring the origin to $x'y'$, the equation to the conic becomes

$$ax^2+2hxy+by^2+2g'x+2f'y+c'=0$$

and the elliptic polar of this conic with respect to

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1,$$

referred to the new origin, is

$$\frac{A'}{\alpha^4}x^2+2\frac{H'}{\alpha^2\beta^2}xy+\frac{B'}{\beta^4}y^2-2\frac{G'}{\alpha^2}x-2\frac{F'}{\beta^2}y+C'=0.$$

The discriminant of this equation $=\dfrac{\Delta^2}{\alpha^4\beta^4}$.

It vanishes when the original conic is a pair of right lines, the elliptic polar being then a pair of coincident right lines.

The elliptic polar will be an ellipse, a parabola or an hyperbola according as the tangents drawn from the centre are imaginary, coincident or real and distinct.

The elliptic polar will be a circle, if the centre of reciprocation be determined by the equations

$$\frac{\alpha^4(ax+hy+g)^2-\beta^4(hx+by+f)^2}{\alpha^4 a-\beta^4 b}$$

$$=(ax+hy+g)(hx+by+f)/h$$

$$=ax^2+2hxy+by^2+2gx+2fy+c.$$

It is evident that the axes of the elliptic polar will be parallel to the co-ordinate axes, if the centre of reciprocation be anywhere on the rectangular hyperbola,

$$Cxy-Fx-Gy+H=0.$$

This condition remains true for circular and hyperbolic polars.

The elliptic polar will be an equilateral hyperbola, if the centre of reciprocation be anywhere on the ellipse,

$$\left(\frac{b}{\alpha^4}+\frac{a}{\beta^4}\right)(ax^2+2hxy+by^2+2gx+2fy+c)$$

$$=\frac{(ax+hy+g)^2}{\beta^4}+\frac{(hx+by+f)^2}{\alpha^4}.$$

2. The auxiliary ellipse referred to the conjugate diameter is $x^2+y^2=r^2$, the axes being oblique and the included angle ω being equal to the angle between the equi-conjugate diameters.

The elliptic polar of the general conic will, in this case, be

$$Ax^2+2Hxy+By^2-2Gr^2x-Fr^2y+Cr^4=0.$$

If $x'y'$ be the centre of reciprocation, the elliptic polar will be

$$A'x^2+2H'xy+B'y^2-2G'r^2x-2F'r^2y+C'r^4=0.$$

This will represent a circle, if

$$\left.\begin{array}{r}A'=B'\\ \text{and}=H'\ \ A'\cos\omega\end{array}\right\}$$

i.e. if the centre of reciprocation be given by

$$\frac{\dfrac{\delta\phi}{\delta y}\left(\dfrac{\delta\phi}{\delta x}+\dfrac{\delta\phi}{\delta y}\cdot\cos\omega\right)}{h-b\cos\omega}$$

$$=\frac{\frac{\delta\phi}{\delta x}\left(\frac{\delta\phi}{\delta y}+\frac{\delta\phi}{\delta x}\cos\omega\right)}{h+a\cos\omega}$$

$$=4\phi(x, y).$$

The elliptic polar will be an equilateral hyperbola, if

$$A'+B'-2H'\cos\omega=0,$$

i.e. if the centre of reciprocation be on the conic,

$$4(a+b+2h\cos\omega)\,\phi\,(x, y)$$

$$=\left(\frac{\delta\phi}{\delta x}\right)^2+\left(\frac{\delta\phi}{\delta y}\right)^2+2\frac{\delta\phi}{\delta x}\cdot\frac{\delta\phi}{\delta y}\cos\omega.$$

3. To determine the elliptic polar of $\frac{x^2}{ma^2}+\frac{y^2}{mb^2}=1$, with respect to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let $x'y'$ be a point on the elliptic polar; then $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$ is a tangent to the ellipse.

Therefore
$$ma^2\left(\frac{x'}{a^2}\right)^2+mb^2\left(\frac{y'}{b^2}\right)^2=1,$$

i.e. the elliptic polar is

$$\frac{x^2}{\frac{a^2}{m}}+\frac{y^2}{\frac{b^2}{m}}=1,$$

a conic similar and similarly situated with the given conics.

CHAPTER VI.

Hyperbolic Polars.

1. The auxiliary conic is the hyperbola

$$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1.$$

The hyperbolic polar, the origin being $x'y'$, is

$$\frac{A'}{\alpha^4}x^2 - \frac{2H'}{\alpha^2\beta^2}xy + \frac{B'}{\beta^4}y^2 - 2G'\frac{x}{\alpha^2} + 2F'\frac{y}{\beta^2} + C' = 0$$

The hyperbolic polar will represent an ellipse, a parobola or an hyperbola, according as the tangents from the centre of reciprocation to the original conic are imaginary, coincident or real and distinct.*

The hyperbolic polar will be a circle if the centre of reciprocation be determined by the equations

$$\frac{\alpha^4(ax+hy+g)^2 - \beta^4(hx+by+f)^2}{\alpha^4 a - \beta^4 b}$$

$$= (ax+hy+g)(hx+by+f)/h$$

$$= ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

The hyperbolic polar will be an equilateral hyperbola if the centre of reciprocation be situated anywhere on the conic,

$$\left(\frac{a}{\beta^4} + \frac{b}{\alpha^4}\right)(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$

$$= \frac{(ax+hy+g)^2}{\beta^4} + \frac{(hx+by+f)^2}{\alpha^4}.$$

2. When the auxiliary conic is an equilateral hyperbola $x^2 - y^2 = r^2$, the hyperbolic polar of the general conic is

$$Ax^2 - 2Hxy + By^2 - 2Gr^2x + 2Fr^2y + Cr^4 = 0$$

If $x'y'$ be taken as the origin, the hyperbolic polar, with respect to $x^2 - y^2 = r^2$ referred to the new origin, will be

$$A'x^2 - 2H'xy + B'y^2 - 2G'r^2x + 2F'r^2y + C'r^4 = 0$$

In this case the hyperbolic polar will be a circle, if the centre of reciprocation be given by

$$\frac{(ax+hy+g)^2 - (hx+by+f)^2}{a-b}$$

$$= (ax+hy+g)(hx+by+f)/h$$

$$= ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

i.e. if the centre of reciprocation be one of the foci.

The hyperbolic polar will be an equilateral hyperbola, if the centre of reciprocation be anywhere on the director circle,

$$C(x^2+y^2)-2Gx-Fy+A+B=0.$$

The rectangular hyperbolic polar of the circle

$$a(x^2+y^2)+2gx+2fy+c=0,$$

is the conic $$x^2+y^2=\frac{(gx-fy+ar^2)^2}{g^2+f^2-ac}$$

of which the origin is a focus and eccentricity

$$=\sqrt{\frac{g^2+f^2}{g^2+f^2-ac}}.$$

3. When the auxiliary conic is the hyperbola $2xy=r^2$, referred to the asymptotes, the hyperbolic polar of the general conic is

$$Ay^2+2Hxy+Bx^2-2Gr^2y-2Fr^2x+Cr^4=0.$$

When $x'y'$ is the origin and $2xy=r^2$ is the reciprocating conic referred to that origin, the hyperbolic polar is

$$A'y^2+2H'xy+B'x^2-G'r^2y-2F'r^2x+C'r^4=0.$$

The hyperbolic polar will be a circle, if the centre of reciprocation is given by the equations

$$\frac{\frac{\delta\phi}{\delta x}\left(\frac{\delta\phi}{\delta y}+\frac{\delta\phi}{\delta x}\cos\omega\right)}{h+a\cos\omega}$$

$$=\frac{\frac{\delta\phi}{\delta y}\left(\frac{\delta\phi}{\delta x}+\frac{\delta\phi}{\delta y}\cos\omega\right)}{h+b\cos\omega}$$

$$=4\phi(x,y),$$

where 'ω' is the angle between the asymptotes.

The hyperbolic polar will be an equilateral hyperbola, if

$$\left(\frac{\delta\phi}{\delta x}\right)^2+\left(\frac{\delta\phi}{\delta y}\right)^2+2\,\frac{\delta\phi}{\delta x}\cdot\frac{\delta\phi}{\delta y}\cos\omega$$

$$=4\,(a+b+2h\cos\omega)\,\phi\,(x,y).$$

CHAPTER VIII.

THEOREMS RELATING TO AREAS.

1. The sides of the polar triangle with respect to $x^2+y^2=r^2$ of the triangle of which the vertices are $(x'y')$, $(x''y'')$, $(x'''y''')$ are respectively

$$xx'+yy'-r^2=0$$

$$xx''+yy''-r^2=0$$

$$xx'''+yy'''-r^2=0$$

If Δ and Δ' denote the areas of the original and the polar triangle respectively,

$$2\Delta=\begin{vmatrix} x', & y', & 1 \\ x'', & y'', & 1 \\ x''', & y''', & 1 \end{vmatrix}.$$

and
$$2\Delta'=\frac{r^4\begin{vmatrix} x', & y', & 1 \\ x'', & y'', & 1 \\ x''', & y''', & 1 \end{vmatrix}^2}{\begin{vmatrix} x', & y' \\ x'', & y'' \end{vmatrix}\begin{vmatrix} x'', & y'' \\ x''', & y''' \end{vmatrix}\begin{vmatrix} x''', & y''' \\ x', & y' \end{vmatrix}}$$

$$=\frac{4r^4\Delta^2}{(x'y''-x''y')(x''y'''-x'''y'')(x'''y'-x'y''')}$$

$$\therefore \Delta'=\frac{2r^4\Delta^2}{(OA\,.\,OB\,.\,OC)^2\sin BOC.\sin COA.\sin AOB.}.$$

Observation 1.

If a triangle ABC be inscribed in the reciprocating circle, the polar triangle $A'B'C'$ will be circumscribed about it, and we have

in this case $\Delta'=\dfrac{2r^4\Delta^2}{r^6\sin AOB\sin BOC\sin COA}$

$$=\frac{2\Delta^2}{r^2\sin 2A\sin 2B\sin 2C}.$$

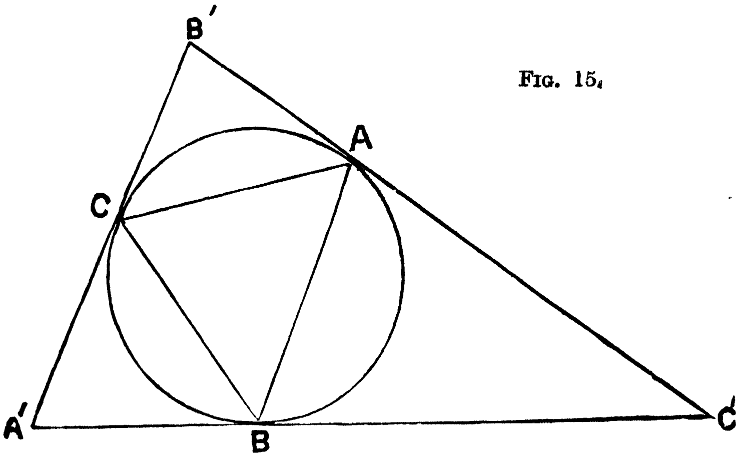


FIG. 15.

Observation 2.

The area of the self-conjugate triangle ABC with respect to $x^2+y^2=r^2$ is

$$\frac{(OA\,.\,OB\,.\,OC)^2 \sin BOC\,.\sin COA\,.\sin AOB}{2r^4}.$$

Observation 3.

If the reciprocating circle be the incircle of the triangle ABC, the polar triangle will be the corresponding inscribed triangle $A'B'C'$.

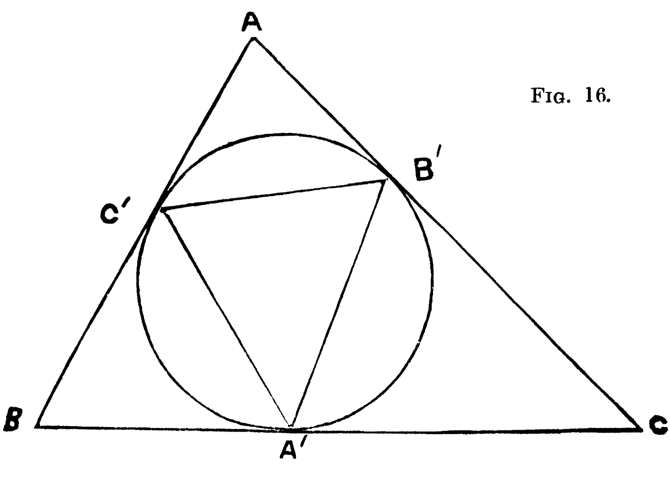


FIG. 16.

In this case,

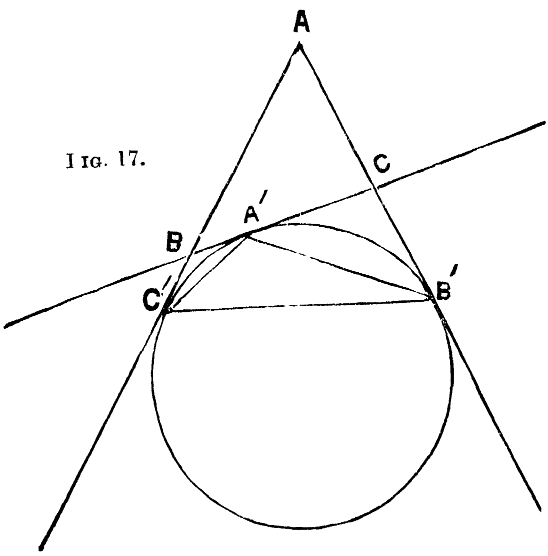
$$\Delta' = \frac{2r^4 \Delta^2}{(OA.OB.OC)^2 . \sin BOC . \sin COA . \sin AOB}$$

$$= \frac{2\Delta^2 . \sin^2\frac{A}{2} . \sin^2\frac{B}{2} . \sin^2\frac{C}{2}}{r^2 \cos\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}} .$$

Observation 4.

If the reciprocating circle be one of the escribed circles, the triangle will be one inscribed in the escribed circle.

I IG. 17.



In this case,

$$\Delta'_a = \frac{2r_a^4 \Delta^2}{(OA . OB . OC)^2 . \sin BOC . \sin COA . \sin AOB}$$

$$= \frac{2r_a^4 \Delta^2}{\left(\frac{r_a}{\sin\frac{A}{2}} . \frac{r_a}{\cos\frac{B}{2}} . \frac{r_a}{\cos\frac{C}{2}}\right)^2 \cos\frac{A}{2} . \sin\frac{B}{2} . \sin\frac{C}{2}}$$

$$=\frac{2\Delta^2 \,.\, \sin^2\frac{A}{2} \cdot \cos^2\frac{B}{2} \cdot \cos^2\frac{C}{2}}{r_a^2 \,.\, \cos\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}}.$$

Similarly,

$$\Delta'_b=\frac{2\Delta^2 \,.\, \sin^2\frac{B}{2} \cdot \cos^2\frac{C}{2} \cdot \cos^2\frac{A}{2}}{r_b^2 \,.\, \cos\frac{B}{2} \cdot \sin\frac{C}{2} \cdot \sin\frac{A}{2}}.$$

and $$\Delta'_c=\frac{2\Delta^2 \,.\, \sin^2\frac{C}{2} \cdot \cos^2\frac{A}{2} \cdot \cos^2\frac{B}{2}}{r_c^2 \,.\, \cos\frac{C}{2} \cdot \sin\frac{A}{2} \cdot \sin\frac{B}{2}}.$$

2. The sides of the polar triangle with respect to

$$\frac{x^2}{\alpha^2}\pm\frac{y^2}{\beta^2}=1,$$

a central conic, are respectively

$$\frac{xx'}{\alpha^2}\pm\frac{yy'}{\beta^2}=1$$

$$\frac{xx''}{\alpha^2}\pm\frac{yy''}{\beta^2}=1$$

$$\frac{xx'''}{\alpha^2}\pm\frac{yy'''}{\beta^2}=1$$

If Δ and Δ' denote the areas of the original and the polar triangle, (ABC and $A'B'C'$)

$$2\Delta'=\frac{\alpha^2\beta^2(2\Delta)^2}{(OA\,.\,OB\,.\,OC)^2.\ \sin BOC\,.\,\sin COA\,.\,\sin AOB}$$

or $$\Delta'=\frac{2\alpha^2\beta^2\Delta^2}{(OA\,.\,OB\,.\,OC)^2.\ \sin BOC\,.\,\sin COA\,.\,\sin AOB}.$$

Obs. The area of the self-conjugate triangle ABC

$$=\frac{(OA\,.\,OB\,.\,OC)^2\sin BOC\,.\,\sin COA\,.\,\sin AOB}{2\alpha^2\beta^2}.$$

3. To determine the area of the polar triangle with respect to the parabola $y^2 = 4ax$ of a given triangle.

Let $(x'y')$, $(x''y'')$, $(x'''y''')$ be the vertices of the given triangle. The sides of the polar triangle are

$$\left.\begin{aligned} 2a(x+x')-yy' &= 0 \\ 2a(x+x'')-yy'' &= 0 \\ 2a(x+x''')-yy''' &= 0 \end{aligned}\right\}$$

If Δ and Δ' denote the areas of the original and the polar triangle,

$$2\Delta' = \frac{\begin{vmatrix} 2a, & -y', & 2ax' \\ 2a, & -y'', & 2ax'' \\ 2a, & -y''', & 2ax''' \end{vmatrix}^2}{\begin{vmatrix} 2a, & -y' \\ 2a, & -y'' \end{vmatrix} \begin{vmatrix} 2a, & -y'' \\ 2a, & -y''' \end{vmatrix} \begin{vmatrix} 2a, & -y''' \\ 2a, & -y' \end{vmatrix}}$$

$$= \frac{16a^4 \begin{vmatrix} 1, & y', & x' \\ 1, & y'', & x'' \\ 1, & y''', & x''' \end{vmatrix}^2}{8a^3 \begin{vmatrix} 1, & y' \\ 1, & y'' \end{vmatrix} \begin{vmatrix} 1, & y'' \\ 1, & y''' \end{vmatrix} \begin{vmatrix} 1, & y''' \\ 1, & y' \end{vmatrix}}.$$

$$= \frac{2a\,.\,(2\Delta)^2}{(y'-y'')(y''-y''')(y'''-y')}$$

$$\therefore\ \Delta' = \frac{4a\Delta^2}{(y'-y'')(y''-y''')(y'''-y')}.$$

Obs. The area of the self-conjugate triangle is

$$\frac{(y'-y'')(y'-y''')(y'''-y')}{4a},$$

a function of the differences of the ordinates of the vertices of the triangle.

4. To determine the area of the polar triangle with respect to $x^2 - y^2 = r^2$, of a given triangle.

If $(x'y')$, $(x''y'')$, $(x'''y''')$ be the vertices of the original triangle, the sides of the polar triangle are

$$\left.\begin{aligned} xx'-yy'-r^2 &= 0 \\ xx''-yy''-r^2 &= 0 \\ xx'''-yy'''-r^2 &= 0 \end{aligned}\right\}$$

and

$$\Delta' = \frac{2r^4 \Delta^2}{(OA \cdot OB \cdot OC)^2 \cdot \sin BOC \cdot \sin COA \cdot \sin AOB}$$

Cor. The area of the self-conjugate triangle with respect to an equilateral hyperbola

$$= \frac{(OA \cdot OB \cdot OC)^2 \cdot \sin BOC \cdot \sin COA \cdot \sin AOB}{2r^4}.$$

5. To determine the area of the reciprocal polar conic of the circle $a(x^2+y^2)+2gx+2fy+c=0$, with respect to

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1.$$

The reciprocal is,

$$\frac{ac-f^2}{\alpha^4} x^2 + 2 \cdot \frac{fg}{\alpha^2\beta^2} xy + \frac{ac-g^2}{\beta^4} y^2 + 2 \cdot \frac{agr^2}{\alpha^2} x + 2 \cdot \frac{afr^2}{\beta^2} y + a^2 = 0$$

The area of this conic

$$= \pi \cdot \frac{\Delta'}{(a'b'-h'^2)^{\frac{3}{2}}}$$

where
$$\Delta' = \frac{a^2}{\alpha^4\beta^4}(g^2+f^2-ac)^2$$

and
$$a'b'-h'^2 = \frac{ca(ca-g^2-f^2)}{\alpha^4\beta^4}.$$

The area of the elliptic polar

$$\pi\alpha^2\beta^2 \left\{\frac{a^3}{c^3} \cdot \frac{ca-g^2-f^2}{a^2}\right\}^{\frac{1}{2}}$$

$$= \pi R \alpha^2\beta^2 \left(-\frac{a}{c}\right)^{\frac{3}{2}}.$$

where R is the radius of the circle.

The area of the hyperbolic polar with respect to

$$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1,$$

$$= \pi R \alpha^2\beta^2 \left(-\frac{a}{c}\right)^{\frac{3}{2}}.$$

The area of the circular or rectangular hyperbolic polar with respect to $x^2 \pm y^2 = r^2$

$$= \pi R r^4 \left(-\frac{a}{c} \right)^{\frac{3}{2}}.$$

Assuming 'a' to be positive 'c' must be negative, i.e. the origin of reciprocation must lie inside the circle in order that the expressions for the areas may be real. This is also evident from the geometrical point of view. For when the origin is outside the circle the reciprocal polar is an hyperbola. It is easy to see that the corresponding geometrical interpretation of the expression for the area is that the modulus of the imaginary expression is the expression for the area included by the reciprocal hyperbola and its conjugate. For it can easily be established by the principles of the Integral Calculus that the area included by the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and its conjugate $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ is πab.

Again, the locus of the centres of reciprocation with respect to which the areas of the polar conics of the given circle is constant, is in each case a concentric circle.

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