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Vibrations under Variable Couplings Quantitatively Elucidated by Simple Experiments.

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§ 1. *Introduction.*

Let us consider a vibrational system of two degrees of freedom, mechanical or electrical, formed by the coupling of two separate systems. Further, let the restoring forces be proportional to the displacements and let all resistances be negligible. Then it is well known that the simultaneous differential equations of motion of the coupled system lead to a solution expressing the superposition of two simple harmonic motions one or both of whose frequencies differ from those characteristic of the separate systems.

But, obvious as this may be to the trained mathematician, it is extremely difficult to bring it home to the average electrical student, Yet to such the matter is of extreme importance at the present time in connexion with wireless telegraphy. Hence any simple mechanical systems that can be visibly coupled, adjusted, and worked, while valuable in themselves as apparatus for dynamical experiments, become doubly so to students striving to grasp the subtler electrical phenomena which make no immediate appeal to the senses of sight and touch but must be imagined at every stage.

It is in the hope of elucidating the phenomena in question that the present paper is offered. It deals with two kinds of coupled pendulums. These, even when set up very roughly from the materials at hand in any laboratory, serve to yield results suitable for lecture illustration and also afford quantitative exercises for the students' practical work. Of course, with more costly mounting results of higher refinement may be obtained.

As it is difficult to follow the intricacies of the motions while they are being executed, the bobs carry funnels to give sand-traces on a moving board. The paper presents a few photographic reproductions of these traces, some of them showing double traces simultaneously obtained one from each bob of the coupled pendulums whose vibrations are often strikingly different.

Each type of pendulum shows the change in frequency consequent on gradually varying the closeness of the coupling. They also show the changes in relative amplitudes of the superposed vibrations which follow from the different ways of starting the system.

§ 2. *Equations for Electrical Circuits.*

The electrical case which we wish to represent by a mechanical analogy is that of two coupled circuits of negligible resistance each having capacity and inductance. Let us denote the capacities by R and S, the respective inductances by L and N, and their mutual induction by M. Further, let the quantities of electricity on the condensers at time t be y and z respectively. Then the simultaneous equations of motion may be written as follows :—

$$L\frac{d^2y}{dt^2}+\frac{y}{R}=M\frac{d^2z}{dt^2}, \qquad \dots \qquad \dots \quad (1)$$

$$N\frac{d^2z}{dt^2}+\frac{z}{S}=M\frac{d^2y}{dt^2}. \qquad \dots \qquad \dots \quad (2)$$

The coefficient of coupling γ is given by

$$\gamma^2=\frac{M^2}{LN}. \qquad \dots \qquad \dots \quad (3)$$

If the circuits when quite separate give free vibrations proportional to sin mt and sin nt, we have

$$m^2=\frac{1}{LR}, \qquad \dots \qquad \dots \quad (4)$$

and

$$n^2=\frac{1}{NS}. \qquad \dots \qquad \dots \quad (5)$$

On writing in (1)

$$y=e^{xt}, \dots \qquad \dots \qquad \dots \quad (6)$$

we find

$$z=\left(\frac{L}{M}+\frac{1}{MRx^2}\right)e^{xt}. \qquad \dots \qquad \dots \quad (7)$$

Then, substituting (6) and (7) in (2) gives us the auxiliary equation in x, viz.,

$$x^4-(1-\gamma^2)+x^2(m^2+n^2)+m^2n^2=0. \qquad \dots \quad (8)$$

Let this be rewritten in the form

$$x^4+x^2(p^2+q^2)+p^2q^2=0, \qquad \dots \qquad \dots \quad (9)$$

then

$$p^2+q^2=\frac{m^2+n^2}{1-\gamma^2}, \qquad \dots \qquad \dots \quad (10)$$

$$p^2q^2=\frac{m^2n^2}{1-\gamma^2}, \qquad \dots \qquad \dots \quad (11)$$

and

$$x=\pm pi \text{ or } \pm qi, \qquad \dots \qquad \dots \quad (12)$$

where $i=\sqrt{(-1)}$.

Thus the general solution of (1) and (2) may be written

$$y = \mathrm{E}\sin(pt+\epsilon) + \mathrm{F}\sin(qt+\phi), \qquad \dots \quad (13)$$

and

$$z = \frac{\mathrm{E}}{\mathrm{MR}}\left(\frac{1}{m^2}-\frac{1}{p^2}\right)\sin(pt+\epsilon) - \frac{\mathrm{F}}{\mathrm{MR}}\left(\frac{1}{q^2}-\frac{1}{m^2}\right)\sin(qt+\phi), \dots \quad (14)$$

where E, ϵ, F, ϕ are arbitrary constants to be fixed by the initial conditions and p and q are functions of m and n (for the separate systems) and of γ (their coefficient of coupling).

Let us now examine these functions. Dividing (10) by (11) we obtain

$$\frac{1}{p^2}+\frac{1}{q^2}=\frac{1}{m^2}+\frac{1}{n^2}.$$

On changing to the periods μ and ν for the vibrations of the *separate* systems and σ and τ for those of the *coupled* vibrations, this becomes

$$\sigma^2+\tau^2=\mu^2+\nu^2. \qquad \dots \qquad \dots \quad (15)$$

In the same notation, (11) may be written

$$\sigma^2\tau^2=(1-\gamma^2)\mu^2\nu^2 \qquad \dots \qquad \dots \quad (16)$$

Eliminating σ between (15) and (16), we obtain the biquadratic in τ,

$$\tau^4-(\mu^2+\nu^2)\tau^2+(1-\gamma^2)\mu^2\nu^2=0. \qquad \dots \quad (17)$$

Thus solving for τ^2, we have

$$2\tau^2=\mu^2+\nu^2\pm\{(\mu^2-\nu^2)^2+4\gamma^2\mu^2\nu^2\}^{\frac{1}{2}} \qquad \dots \quad (18)$$

The two values of τ^2 here shown may be called τ^2 and σ^2. Accordingly we see that the periods, σ and τ, of the coupled vibrations differ from each other and from those, μ and ν, of the vibrations of the separate systems, and also that the difference between σ and τ exceeds that between μ and ν. Further, even, when μ and ν are alike, σ and τ still differ from each other and from μ (for all finite values of the coupling γ).

As to special cases, we may note the following:—

For $\nu=\mu$, we find

$$\tau=\mu\sqrt{(1+\gamma)} \text{ and } \sigma=\mu\sqrt{(1-\gamma)}. \qquad \dots \qquad \dots \quad (19$$

For $\nu=\mu$ and γ very small this gives

$$\tau=\sigma=\mu. \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

For $\gamma=1$ but μ unlike ν, we find

$$\tau^2=\mu^2+\nu^2 \text{ and } \sigma^2=0. \quad \ldots \quad \ldots \quad (21)$$

Initial Charge.—If one capacity is initially charged, the other uncharged, and both currents zero, which is the usual case, we may write

$$y=0,\ z=b,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=0, \text{ for } t=0. \quad \ldots \quad \ldots \quad (21a)$$

Inserting these conditions in (13) and (14) and in the differentiations of these with respect to the time, we find the following special solutions :—

$$y=\text{MR}\,\frac{p^2q^2}{p^2-q^2}\,b\cos pt-\text{MR}\,\frac{p^2q^2}{p^2-q^2}\,b\cos qt. \quad \ldots \quad \ldots \quad (21b)$$

$$z=\frac{(p^2-m^2)q^2}{(p^2-q^2)m^2}\,b\cos pt+\frac{(m^2-q^2)p^2}{(p^2-q^2)m^2}\,b\cos qt. \quad \ldots \quad (21c)$$

§ 3. *Previous Mechanical Analogies.*

A number of mechanical models have already been devised capable of motions analogous to electrical vibrations or induced currents, and probably all have considerable value, but, perhaps, none has an action that is exactly and completely analogous to the electrical phenomena in the fullest sense of the words. If any proposed model were at once exactly analogous, quite simple, and capable of easy adjustments so as to exhibit all the features of the electrical case, it might put an end to further work in this direction, though other models might still present a purely mechanical interest. But, so far as is known to the present writers, it appears that no such simple, exact and complete analogy has hitherto been put forward.

In order to submit to critical examination a few typical models which have been advanced, let us review the chief features of the electrical case. Each electrical circuit has its own capacity and inductance, the latter (L and N) being usually held to fuction as inertias, while the former (R and S) are likened to the reciprocals of spring factors. Hence each circuit has its definite period. Two such circuits are then brought near enough to involve appreciable electromagnetic induction. They are then said to be *coupled.* In this state the phenomena in each circuit depend partly upon those in the other. The single factor that expresses this dependence or cross-connexion between the variables is M, the coefficient of mutual induction. It is of the same physical nature as the two inductances of the separate circuits, thus it also functions as an inertia. This is all on the assumption that the currents i and j in the circuits are analogous to velocities in the mechanical case, since the kinetic energy T in the coupled circuits is known to be given by

$$T = \tfrac{1}{2}Li^2 + Mij + \tfrac{1}{2}Nj^2. \quad \ldots \quad \ldots \quad (22)$$

Hence, in a model completely analogous to the electrical case, we might naturally look for :

(1) *Masses* in each system to represent the inductances L and N.

(2) *Spring factors* to represent the reciprocals of the capacities R and S, and also

(3) Some *third mass* to represent M, the mutual induction of the circuits as coupled.

Further, might we not legitimately ask that in an exact analogy,

(4) This mass representing M should have to be *greater for closer* coupling and less for looser coupling.

(5) That its presence *should not disturb* the value of the other masses previously used in the separate systems, and

(6) That all relations should be *quantitatively accurate* ?

If we demand from a model these six points we may perhaps look in vain for satisfaction not only in the past but in the near future also. But can any model proposed as analogous be deemed exact and complete if it lack any one of these points ?

Consider now a few typical analogies that have been published. As models for the phenomena of induced currents there is the set of toothed wheels and rack by Sir Oliver Lodge ('Modern Views of Electricity') and the three connected masses sliding on parallel bars by Sir Joseph J. Thomson ('Electricity and Magnetism,' art. 231, Cambridge, 1909). The former appears to be qualitative. The latter is quantitative, indeed its author shows that each of the two connected masses together with a quarter of the connecting mass function as the two self-inductions, and a quarter of the connecting mass as the mutual induction; while the remaining quarter of that mass is left unappropriated.

We may next notice the model due to W. S. Franklin (see fig. 5, p. 558 of "Some Mechanical Analogies in Electricity and Magnetism," *Electrician*, pp. 556-559, July 28, 1916). This may be regarded as a development of that due to Sir J. J. Thomson, by the simple addition of springs to the extreme masses. The model thus imitates electrical circuits having capacities and hence characteristic periods.

A very different model has been proposed and realised by Prof. Thomas R. Lyle (see "On an Exact Mechanical Analogy to the Coupled Circuits used in Wireless Telegraphy, and on &c.," *Phil. Mag.* [6] xxv. pp. 567-592, April 1913). In this device two simple pendulums of lengths l_1 and l_2 with bobs of masses m_1 and m_2 hang from a freely-moving carriage of mass M. The electrical potential differences at the condensers here correspond to the angular displacements θ_1 and θ_2 of the pendulums. It is then shown (p. 575) that the quantities respectively analogous to the self and mutual inductions are :

$$\left.\begin{array}{c}\dfrac{M+m_2}{m_1(M+m_1+m_2)g^2},\quad \dfrac{M+m_1}{m_2(M+m_1+m_2)g^2},\\ \text{and } \dfrac{1}{(M+m_1+m_2)g^2}.\end{array}\right\}\quad \ldots \quad \ldots \quad (23)$$

Thus the inductions are represented by the reciprocals of masses multiplied by the square of gravity. The currents are represented by

$$m_1 l_1 g\dot{\theta}_1 \text{ and } m_2 l_2 g\dot{\theta}_2. \quad \ldots \quad \ldots \quad (24)$$

The electrical and mechanical equations are then shown to be identical in form. Thus the model is indeed an exact analogue in certain very important senses. But when reflecting on the striking properties of this model, it seems difficult to avoid wishing that the inductions were represented throughout by masses simply and the currents by the speeds of those masses. Possibly, however, the designer of a model in very way analogous to the electrical case would have solved by his design the enigma of Nature's electromagnetic mechanism which has hitherto eluded all scrutiny.

But perhaps, since no known model seems able to claim full perfection, there may yet be room for one or two more analogies which are confessedly imperfect.

§ 4. *Theory of Double-Cord Pendulums.*

Description.—We now deal with one of the two very simple devices used by the present writers to imitate in their vibrations the phenomena of coupled electric circuits. No claim for exactness of analogy is made for these. They are remarkable only for their simplicity, facility of adjustment to various desired couplings, and their power of producing simultaneous traces revealing the relative frequencies, amplitudes, and phases of the coupled vibrations executed. They accordingly give vivid illustrations of these mechanical phenomena, and these are in broad outline closely akin to those of the electric circuits.

What may be termed the double-cord pendulum is shown in elevation in figs. 1 and 2, the bobs, board, and traces being shown in perspective in fig. 4 of Plate IV.

In the normal use of the pendulums the oscillations occur only in the plane of fig. 2. In this figure is shown a stiff connector CC′ of fibre-tube which forces the bridles ACA, A′C′A′ to swing together. Each bob consists of a heavy metal ring holding a glass funnel for sand to give the vibration trace on a moving board below. The length of each pendulum may be adjusted by a sliding tightener (T, T′), and the bridles may be set to any desired droop by adjustments at their ends A or A′.

Equations of Motion and Coupling.—Reckoning from C the droop of the bridle, let l be the length of the simple pendulum equivalent to each actual pendulum CB and let the droop DC of the bridle itself be βl. At time t let the

Double-Cord Pendulums.

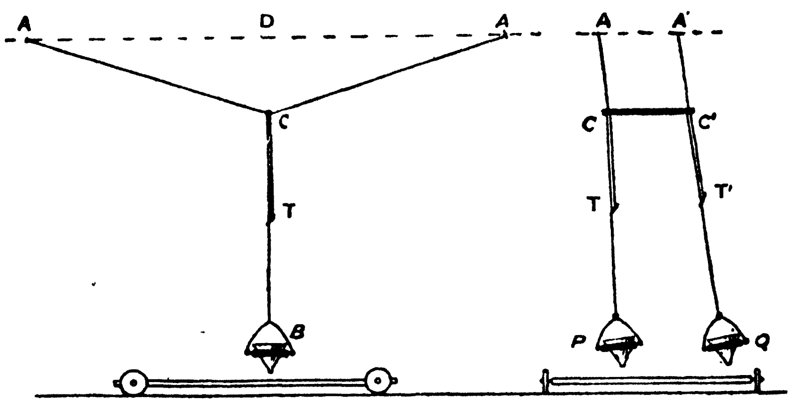

Fig. 1.
Side Elevation.

Fig. 2.
End Elevation.

planes AC and A′C′ of the bridles be inclined ω from the vertical, let the pendulums be inclined θ and ψ, the linear displacements of their bobs of masses P and Q being y and z.

Then their equations of motion for small oscillations may be written

$$P\frac{d^2y}{dt^2}+Pg\theta=0, \quad \ldots \quad \ldots \quad (25)$$

$$Q\frac{d^2z}{dt^2}+Qg\psi=0. \quad \ldots \quad \ldots \quad (26)$$

But θ and ψ are respectively

$$\frac{y-\beta l\omega}{l} \text{ and } \frac{z-\beta l\omega}{l}.$$

Further neglecting the masses of the cords ACA, A′C′A′ and of the connector CC′, we see that ω satisfies the equation :

$$Pg(\theta-\omega)=Qg(\omega-\psi),$$

so that

$$\omega l=\frac{Py+Qz}{(1+\beta)(P+Q)}.$$

Inserting these values in (25) and (26) they may be written :

$$P\frac{d^2y}{dt^2}+\frac{P+Q+\beta Q}{(1+\beta)(P+Q)}P\frac{g}{l}y=\frac{\beta}{(1+\beta)}\frac{PQ}{(P+Q)}\frac{g}{l}z, \quad \ldots \quad \ldots \quad (27)$$

$$Q\frac{d^2z}{dt^2}+\frac{P+\beta P+Q}{(1+\beta)(P+Q)}Q\frac{g}{l}z=\frac{\beta}{(1+\beta)}\frac{PQ}{(P+Q)}\frac{g}{l}y. \quad \ldots \quad \ldots \quad (28)$$

Comparing (27) and (28) with the electrical equations (1) and (2) we see that in our present mechanical case the analogy is not exact. For the mutual induction formed the coefficient of a *second differential* coefficient of quantity of electricity, whereas in our analogue the cross-connecting factor which replaces mutual induction is a coefficient of the *displacement itself* which is taken to represent the quantity of electricity.

This naturally suggests the question as to how the coefficient of coupling is to be estimated in this mechanical case. If this coefficient is to remain a pure number as in the electrical case, the answer is clear. For we must take the product of the coefficients on the right side of the equations and divide by the products of the coefficients of *like terms* on the left sides.

We accordingly obtain

$$\gamma^2 = \frac{\beta^2 PQ}{(P+Q+\beta Q)(P+\beta P+Q)} \cdots \qquad \cdots \quad (29)$$

For the simpler special case where $P = Q$ that will henceforth be dealt with, (27) and (28) may be written

$$(2+2\beta)\frac{d^2y}{dt^2} + (2+\beta)m^2y = \beta m^2 z, \ \ldots \qquad \cdots \quad (30)$$

$$(2+2\beta)\frac{d^2z}{dt^2} + (2+\beta)m^2z = \beta m^2 y, \ \ldots \qquad \cdots \quad (31)$$

where m^2 is written for g/l. For this case the coupling becomes

$$\gamma = \frac{\beta}{2+\beta}. \qquad \cdots \qquad \cdots \quad (32)$$

Solution and Frequencies for Equal Bobs.—On putting in (30)

$$y = \epsilon^{xt}, \ \ldots \qquad \cdots \qquad \cdots \quad (33)$$

we find

$$z = \frac{(2+2\beta)x^2 + (2+\beta)m^2}{\beta m^2} \epsilon^{xt}. \qquad \cdots \qquad \cdots \quad (34)$$

Then (33) and (34) in (31) give

$$\{(2+2\beta)x^2 + (2+\beta)m^2\}^2 = \beta^2 m^4,$$

whence

$$x = \pm mi \text{ or } \pm \frac{mi}{\sqrt{(1+\beta)}}. \qquad \cdots \qquad \cdots \quad (35)$$

Thus, using (35) in (33) and (34) and introducing the usual constants, we may write the general solution in the form

$$y = E \sin (mt + \epsilon) + F \sin \left(\frac{mt}{\sqrt{(1+\beta)}} + \phi\right). \qquad \cdots \quad (36)$$

$$z = -\mathrm{E} \sin (mt + \epsilon) + \mathrm{F} \sin \left(\frac{mt}{\sqrt{(1+\beta)}} + \phi\right), \quad \ldots \quad (37)$$

where E, ϵ. F, ϕ are the arbitrary constants whose values depend upon the initial conditions.

Initial Conditions.—To obtain the special solution for any concrete case we must state the initial displacements and velocities and determine the four constants accordingly. As a preliminary, we write the velocities from (36) and (37) by differentiation with respect to the time. Thus

$$\frac{dy}{dt} = m\mathrm{E} \cos (mt + \epsilon) + \frac{m\mathrm{F}}{\sqrt{(1+\beta)}} \cos \left(\frac{mt}{\sqrt{(1+\beta)}} + \phi\right), \quad \ldots \quad (38)$$

$$\frac{dz}{dt} = -m\mathrm{E} \cos (mt + \epsilon) + \frac{m\mathrm{F}}{\sqrt{(1+\beta)}} \cos \left(\frac{mt}{\sqrt{(1+\beta)}} + \phi\right). \ldots \quad (39)$$

(i) *Single Velocity.*—Take first the case of a single velocity imparted to one bob by a blow when both are at rest in their zero positions. Then we may write

$$y = 0,\ z = 0,\ \frac{dy}{dt} = u,\ \frac{dz}{dt} = 0 \text{ for } t = 0. \quad \ldots \quad (40)$$

These conditions, put in (36)–(39), give equations which are satisfied by

$$\mathrm{E} = \frac{u}{2m},\ \mathrm{F} = \frac{\sqrt{(1+\beta)}u}{2m},\ \epsilon = 0,\ \phi = 0. \quad \ldots \quad (41)$$

Hence for this we have the special solution

$$y = \frac{u}{2m} \sin mt + \frac{\sqrt{(1+\beta)}u}{2m} \sin \frac{mt}{\sqrt{(1+\beta)}}, \quad \ldots \quad (42)$$

$$z = -\frac{u}{2m} \sin mt + \frac{\sqrt{(1+\beta)}u}{2m} \sin \frac{mt}{\sqrt{(1+\beta)}}. \quad \ldots \quad (43)$$

The ratios of the amplitudes of the quick vibrations to those of the slow ones are seen to be

$$1 : \pm\sqrt{(1+\beta)} \quad \ldots \quad \ldots \quad (44)$$

in the y and z vibrations respectively.

(ii) *Single Displacement.*—Now let one bob be pulled horizontally aside while the other is held in the zero position, oth constraints ceasing at the same instant. We may thus rrite

$$y=0,\ z=b,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=0,\ \text{for } t=0. \quad \ldots \quad (45)$$

These conditions inserted in (36–39) yield equations atisfied by

$$E=-\frac{b}{2},\ F=\frac{b}{2},\ \epsilon=\frac{\pi}{2},\ \phi=\frac{\pi}{2}. \quad \ldots \quad \ldots \quad (46)$$

And the values in (36) and (37) give, for this mode of tarting, the special solution

$$y=-\frac{b}{2}\cos mt+\frac{b}{1}\cos\frac{mt}{\sqrt{(1+\beta)}}, \quad \ldots \quad (47)$$

$$z=\ \frac{b}{2}\cos mt+\frac{b}{2}\cos\frac{mt}{\sqrt{(1+\beta)}}. \quad \ldots \quad (48)$$

Thus the ratios of amplitudes of quick and slow vibra-ions in the y and z traces are respectively

$$\mp 1. \quad \ldots \quad \ldots \quad \ldots \quad (49)$$

That is, the amplitudes of the superposed vibrations are umerically equal for any values of the coupling.

The symmetry of the equations shows that the motions will interchange simply if the other pendulums be struck or isplaced.

(iii) *Double Displacement.*—Let one bob be drawn hori-ontally aside, the other hanging motionless in its equilibrium out slightly-displaced position. Thus if

$z=b$ it follows that $y=\dfrac{\beta b}{2+\beta}$,

he other conditions being

$\dfrac{dy}{dt}=0$ and $\dfrac{dz}{dt}=0$. ... (50)

Putting these in (36)—(39) we obtain equations which re satisfied by

$$E=\frac{-b}{2+\beta},\ F=\frac{1+\beta}{2+\beta}b,\ \epsilon=\frac{\pi}{2},\ \phi=\frac{\pi}{2}. \quad \ldots \quad (51)$$

Hence the corresponding special solution may be written

$$y = -\frac{b}{2+\beta}\cos mt + \frac{1+\beta}{2+\beta} b \cos\frac{mt}{\sqrt{(1+\beta)}}, \quad \ldots \quad (52)$$

$$z = \frac{b}{2+\beta}\cos mt + \frac{1+\beta}{2+\beta} b \cos\frac{mt}{\sqrt{(1+\beta)}}. \quad \ldots \quad (53)$$

Accordingly the ratio of amplitudes of quick and slow vibrations in the *y* and *z* traces are respectively

$$\mp 1 : (1+\beta). \quad \ldots \quad \ldots \quad (54)$$

(iv) *Cases of Single Frequency.*—It may be seen immediately that if the phases are *opposite* and amplitudes *equal* the strut CC′ and the bridles ACA and A′C′A′ will all remain at rest. We should accordingly have the quick vibrations alone, which are proportional to cos *mt*. Again, if the phases are *alike* and the amplitudes *equal* we should have both pendulums swinging in unison, each bridle and its lower thread forming one plane. The vibrations are accordingly of the slower type of period $\sqrt{(1+\beta)}$-times the other just noticed.

§ 5. *Theory of Cord and Lath Pendulums.*

Description.—The model here called the cord and lath pendulum consists essentially of two pendulums PR and QS, one suspended by cords from a movable point R on the lath of the other as shown in fig. 3. The bobs are shown by P and Q, the movable and fixed points of suspension by R and S. The cord pendulum is shown by PR. When the apparatus is used for giving double sand-traces a system of four cords and three stretchers is used so as to clear the upper board which takes the trace from the bob Q. Each cord is provided with a tightener for adjusting its length. Of the two upper stretchers the back one is shown in the figure a little shorter and lower so as to make it and its cords visible. In the actual model the cords and stretchers are all symmetrical. The point of suspension R is adjustable on the lath of the pendulum QS by a small sliding metal sleeve with studs to support the cords. This sleeve is set where desired and then fixed in position by a screw-clamp. The suspension S of the lath pendulum consists of two screw points resting in a hole and slot in a metal plate. The two bobs P and Q are the same as those used in the double-cord pendulum, viz., metal rings with glass funnels for sand. The boards shown just beneath the funnels are fixed on the

Cord and Lath Pendulum.

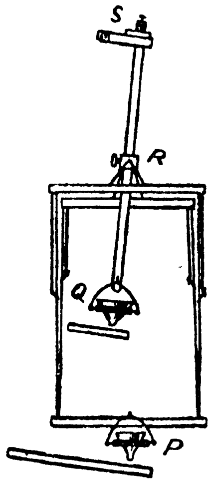

Fig. 3.

same wooden carriage and are capable of simultaneous slow motion on horizontal rails perpendicular to the plane of the diagram. This motion is effected by the rotation of a wheel whose axle winds a cords attached to the carriage. The whole arrangement is shown by the photographic reproduction fig. 5 of Plate IV.

Equations of Motion and Coupling.—For the cord and lath pendulums let the masses of the bobs be P and Q, the lengths of the simple pendulums equivalent to PR and QS be respectively r and s and at time t let their angular displacements be θ and ψ, the linear displacements of their bobs being y and z. Further, let αs denote the distance RS between the movable and fixed points of suspension.

Then for small oscillations we may assimilate sines to angles and regard the total tensions of the cords as always equal to the weights of the bobs simply. The equations of motion may accordingly be written as follows :—

$$P\frac{d^2y}{dt^2}+Pg\theta= \text{ and}$$

$$Q\frac{d^2z}{dt^2}+Qg\psi=Pg(\theta-\psi)\alpha\,;$$

or

$$P\frac{d^2y}{dt^2}+P\frac{g}{r}y=P\frac{g}{r}\alpha z, \quad \ldots \quad \ldots \quad (55)$$

$$Q\frac{d^2z}{dt^2}+\left(\frac{P\alpha+Q}{s}+\frac{P\alpha^2}{r}\right)gz=P\frac{g}{r}\alpha y. \quad \ldots \quad (56)$$

The coefficient of coupling γ is given by

$$\gamma^2=\frac{Ps\alpha^2.}{(P\alpha+Q)r+Ps\alpha^2}\cdot \quad \ldots \quad \ldots \quad (57)$$

In the special case where $P=Q$, and $r=s$, which will henceforth be adhered to, if we write m^2 for g/r, the above become

$$\frac{d^2y}{dt^2}+m^2y=m^2\alpha z, \quad \ldots \quad \ldots \quad \ldots \quad (58)$$

$$\frac{d^2z}{dt^2}+(1+\alpha+\alpha^2)m^2z=m^2\alpha y, \quad \ldots \quad \ldots \quad (59)$$

and

$$\gamma^2=\frac{\alpha^2}{1+\alpha+\alpha^2}\cdot \quad \ldots \quad \ldots \quad \ldots \quad (60)$$

So for very small values of $\gamma=\alpha$ nearly.

Solution and Frequencies for Equal Bobs and Lengths.— To solve these equations try in (58)

$$y=\epsilon^{xt}. \quad \ldots \quad \ldots \quad (61)$$

This gives

$$z=\frac{x^2+m^2}{m^2\alpha}\epsilon^{xt}, \quad \ldots \quad \ldots \quad (62)$$

Then (61) and (62) in (59) give the auxiliary equation in x,

$$x^4+x^2(2+\alpha+\alpha^2)m^2+(1+\alpha)m^4=0. \quad \dots \quad (63)$$

If we write this in the form

$$x^4+x^2(p^2+q^2)+p^2q^2=0, \quad \dots \quad \dots \quad (64)$$

we see that

$$x=\pm pi \text{ or } \pm qi. \quad \dots \quad \dots \quad (65)$$

Hence, on inserting the four arbitrary constants, we may write the general solution and its first derivatives as follows :—

$$y=\mathrm{E}\sin(pt+\epsilon)+\mathrm{F}\sin(qt+\phi), \quad \dots \quad \dots \quad \dots \quad (66)$$

$$z=-\frac{p^2-m^2}{m^2\alpha}\mathrm{E}\sin(pt+\epsilon)+\frac{m^2-q^2}{m^2\alpha}\mathrm{F}\sin(qt+\phi), \quad \dots \quad (67)$$

$$\frac{dy}{dt}=p\mathrm{E}\cos(pt+\epsilon)+q\mathrm{F}\cos(qt+\phi), \quad \dots \quad \dots \quad \dots \quad (68)$$

$$\frac{dz}{dt}=-\frac{p^2-m^2}{m^2\alpha}p\mathrm{E}\cos(pt+\epsilon)+\frac{m^2-q^2}{m^2\alpha}q\mathrm{F}\cos(qt+\phi). \quad \dots \quad (69)$$

Returning to the comparison of (63) and (64) we see

and

$$\left.\begin{aligned} p^2+q^2&=(2+\alpha+\alpha^2)m^2=m^2\delta \text{ say,} \\ p^2q^2&=(1+\alpha)m^4=m^4\eta^2 \text{ say,} \end{aligned}\right\} \quad \dots \quad (70)$$

whence

$$\left.\begin{aligned} (p+q)^2&=m^2(\delta+2\eta), \\ (p-q)^2&=m^2(\delta-2\eta), \end{aligned}\right\} \quad \dots \quad (71)$$

$$\left.\begin{aligned} p&=\frac{m}{2}\{\sqrt{(\delta+2\eta)}+\sqrt{(\delta-2\eta)}\}, \\ q&=\frac{m}{2}\{\sqrt{(\delta+2\eta)}-\sqrt{(\delta-2\eta)}\}, \end{aligned}\right\} \quad \dots \quad (72)$$

and

$$\frac{p}{q}=\frac{\sqrt{(\delta+2\eta)}+\sqrt{(\delta-2\eta)}}{\sqrt{(\delta+2\eta)}-\sqrt{(\delta-2\eta)}}. \quad \dots \quad \dots \quad (73)$$

An alternative method is to eliminate q^2 between the equations (70), thus obtaining the quadratic in p^2,

$$p^4-(2+\alpha+\alpha^2)m^2p^2+(1+\alpha)m^4=0. \quad \dots \quad (74)$$

Thus, calling the larger root of this p^2 and the smaller q^2, we have

$$\left.\begin{aligned} p^2&=m^2\left(1+\frac{\alpha}{2}+\frac{\alpha^2}{2}\right)+\frac{m^2\alpha}{2}\sqrt{(\alpha^2+2\alpha+5)}, \\ q^2&=m^2\left(1+\frac{\alpha}{2}+\frac{\alpha^2}{2}\right)-\frac{m^2\alpha}{2}\sqrt{(\alpha^2+2a+5)}, \end{aligned}\right\} \quad \ldots \quad (75)$$

whence

$$\frac{p}{q}=\left\{\frac{2+\alpha+\alpha^2+\alpha(\alpha^2+2\alpha+5)^{\frac{1}{2}}}{2+\alpha+\alpha^2-\alpha(\alpha^2+2\alpha+5)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}, \ldots \quad \ldots \quad (76)$$

which agrees with (73).

Thus by (72) or (75) we see that the superposed vibrations of the coupled system have frequencies which differ from each other and from those of the separate systems even when they are alike.

Hence, in these broad features this system of the cord and lath pendulum is in agreement with the electrical system of coupled circuits. But in the closer details differences show themselves, as may be noted by comparison of (75) with (18).

Initial Conditions.—Since for this cord and lath pendulum the eqations are not symmetrical, the phenomena may depend upon which bob receives a blow or displacement when starting. We accordingly treat each in turn.

(i.) *Upper Bob Struck.*—We may here write as follows:—

$$y=0,\ z=0,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=v, \text{ for } t=0. \quad \ldots \quad (77)$$

These conditions in (66)–(69) give equations satisfied by

$$\epsilon=0,\ \phi=0,\ \mathrm{E}=\frac{-m^2\alpha v}{p(p^2-q^2)},\ \mathrm{F}=\frac{m^2\alpha v}{q(p^2-q^2)}. \quad \ldots \quad (78)$$

So, inserting these values in (66) and (67), we have for the special solution

$$\left.\begin{aligned} y&=-\frac{m^2\alpha v}{(p^2-q^2)p}\sin pt+\frac{m^2\alpha v}{(p^2-q^2)q}\sin qt, \\ z&=\frac{(p^2-m^2)v}{(p^2-q^2)p}\sin pt+\frac{(m^2-q^2)v}{(p^2-q^2)q}\sin qt. \end{aligned}\right\} \quad \ldots \quad (79)$$

If the amplitudes of the quick and slow z-vibrations are G and H, we have

$$\frac{E}{F}=-\frac{q}{p} \text{ and } \frac{G}{H}=\frac{(p^2-m^2)q}{(m^2-q^2)p}. \quad \ldots \quad \ldots \quad (80)$$

(ii.) *Lower Bob Struck.*—Here we may write

$$y=0,\ z=0,\ \frac{dy}{dt}=u,\ \frac{dz}{dt}=0, \text{ for } t=0. \quad \ldots \quad (81)$$

These, inserted in (66) to (69), give equations satisfied by

$$\epsilon=0,\ \phi=0,\ E=\frac{(m^2-q^2)u}{(p^2-q^2)p},\ F=\frac{(p^2-m^2)u}{(p^2-q^2)q}. \quad \ldots \quad (82)$$

And these values put in (66) and (67) give the special solution

$$\left.\begin{aligned} y&=\frac{(m^2-q^2)u}{(p^2-q^2)p}\sin pt+\frac{(p^2-m^2)u}{(p^2-q^2)q}\sin qt, \\ z&=-\frac{(p^2-m^2)(m^2-q^2)u}{m^2\alpha(p^2-q^2)p}\sin pt+\frac{(m^2-q^2)(p^2-m^2)u}{m^2\alpha(p^2-q^2)q}\sin qt. \end{aligned}\right\} \quad \ldots \quad (83)$$

So

$$\frac{E}{F}=\frac{(m^2-q^2)q}{(p^2-m^2)p} \text{ and } \frac{G}{H}=-\frac{q}{p}. \quad \ldots \quad (84)$$

Note the contrast of (84) with (80).

(iii.) *Upper Bob Displaced: Lower Free.*—This case may be represented by

$$y=\alpha b,\ z=b,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=0, \text{ for } t=0. \quad \ldots \quad (85)$$

These put in (66) to (69) give equations satisfied by

$$\epsilon=\frac{\pi}{2},\ \phi=\frac{\pi}{2},\ E=-\frac{aq^2b}{p^2-q^2},\ F=\frac{ap^2b}{p^2-q^2}. \quad \ldots \quad (86)$$

These values in (66) and (67) give the special solution

$$\left.\begin{aligned} y&=-\frac{aq^2b}{p^2-q^2}\cos pt+\frac{ap^2b}{p^2-q^2}\cos qt, \\ z&=\frac{(p^2-m^2)q^2b}{(p^2-q^2)m^2}\cos pt+\frac{(m^2-q^2)p^2b}{(p^2-q^2)m^2}\cos qt. \end{aligned}\right\} \quad \ldots \quad (87)$$

So

$$\frac{E}{F}=-\frac{q^2}{p^2} \text{ and } \frac{G}{H}=\frac{(p^2-m^2)q^2}{(m^2-q^2)p^2}. \quad \ldots \quad \ldots \quad (88)$$

(iv.) *Lower Bob Displaced: Upper Free.*—Let the displacement (a) of the lower bob P be produced by a horizontal force. Then the corresponding value of the displacement (z) of Q when at rest can be found statically. We thus obtain

$$y=a,\ z=\frac{\alpha a}{1+\alpha+\alpha^2},\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=0,\ \text{for } t=0. \qquad \ldots \quad (89)$$

These conditions inserted in (66) to (69) give equations satisfied by

$$\left.\begin{aligned}\epsilon=\frac{\pi}{2},\quad E&=\frac{(1+\alpha)m^2-(1+\alpha+\alpha^2)q^2}{(1+\alpha+\alpha^2)(p^2-q^2)}a,\\ \phi=\frac{\pi}{2},\quad F&=\frac{(1+\alpha+\alpha^2)p^2-(1+\alpha)m^2}{(1+\alpha+\alpha^2)(p^2-q^2)}a.\end{aligned}\right\} \qquad \ldots \quad (90)$$

These values put in (66) and (67) give the special solution

$$\left.\begin{aligned}y=&\frac{(1+\alpha)m^2-(1+\alpha+\alpha^2)q^2}{(1+\alpha+\alpha^2)(p^2-q^2)}a\cos pt\\ &+\frac{(1+\alpha+\alpha^2)-(1+\alpha)m^2}{(1+\alpha+\alpha^2)(p^2-q^2)}a\cos qt,\end{aligned}\right\} \qquad \ldots \quad (91)$$

$$\left.\begin{aligned}z=&-\frac{(p^2-m^2)\{(1+\alpha)m^2-(1+\alpha+\alpha^2)q^2\}}{m^2\alpha(1+\alpha+\alpha^2)(p^2-q^2)}a\cos pt\\ &+\frac{(m^2-q^2)\{(1+\alpha+\alpha^2)p^2-(1+\alpha)m^2\}}{m^2\alpha(1+\alpha+\alpha^2)(p^2-q^2)}a\cos qt.\end{aligned}\right\} \qquad \ldots \quad (92)$$

So the ratios of the amplitudes of the quick and slow vibrations in the y and z motions are given respectively by

$$\frac{E}{F}=\frac{(1+\alpha)m^2-(1+\alpha+\alpha^2)q^2}{(1+\alpha+\alpha^2)p^2-(1+\alpha)m^2}, \qquad \ldots \qquad \ldots \quad (93)$$

and

$$\frac{G}{H}=\frac{-(p^2-m^2)\{(1+\alpha)m^2-(1+\alpha+\alpha^2)q^2\}}{(m^2-q^2)\{(1+\alpha+\alpha^2)p^2-(1+\alpha)m^2\}}. \qquad \ldots \quad (94)$$

Note the contrast of (93) and (94) with (88).

§ 6. *Comparison of the Two Types of Coupled-Pendulums.*

Referring to equations (27) and (28), and (55) and (56), we see that the two types of model under examination have equations of motion of the form

$$P\frac{d^2y}{dt^2}+Ay=Bz, \qquad \ldots \qquad \ldots \quad (95)$$

$$Q\frac{d^2z}{dt^2}+Cz=By. \qquad \ldots \qquad \ldots \quad (96)$$

And in each case we have for the [illegible] γ the relation

$$\gamma^2 = \frac{B^2}{AC}. \qquad \ldots \qquad \ldots \qquad (97)$$

These may be compared with equations (1)-(3) for the electrical case. It is there seen that the coupling involves the inductances L and N but is independent of the capacities. Whereas in (95) and (96) the A and C involve g/l, g/r, and g/s (which are comparable to the reciprocals of the capacities) as well as the masses P and Q (comparable to L and N).

Though both the types of pendulum are broadly alike and fall equally under the equotions (95) and (96), their individual details, as dependent on the values of A, B, and C, are somewhat different as already shown in the separate examinations. It is specially noticeable that in the double-cord pendulum, with equal pendulum lengths and masses, everything is interchangeable and, of the two superposed vibrations when coupled, one has the unaltered period of the pendulums if separated. In the cord and lath pendulum there is no such interchangeability, and both the vibrations superposed when coupled differ in period from those which would occur if the pendulums were separated. This is more like the electrical case.

§ 7. *Forced Vibrations : Special Case of Coupled Vibrations.*

It is interesting to note how the case of coupled vibrations reduces to that of forced vibration when the coupling is small and the driving mass is much greater than the driven mass whose vibrations are forced. Thus in equations (95) and (96) let B be small but so as to be appreciable with respect to the very small mass P but inappreciable with respect to the much larger mass Q.

Then (96) reduces to

$$Q\frac{d^2z}{dt^2} + Cz = 0, \qquad \ldots \qquad \ldots \qquad (98)$$

giving as a solution

$$z = K \sin nt, \text{ say.} \quad \ldots \quad \ldots \quad (99)$$

Then, this used in the right side of (95) gives

$$P\frac{d^2y}{dt^2} + Ay = BK \sin nt, \quad \ldots \quad \ldots \quad (100)$$

which is one form of the equation of motion for forced vibrations.

By hanging a simple pendulum with bob of very small mass near A of the double-cord pendulum, we could imitate the experiment in which Dr. Fleming's Cymometer detects by resonance the two superposed vibrations in a pair of closely-coupled inductive circuits. The pendulum imitating the cymometer would show maximum responses for two lengths corresponding to the quick and slow vibrations.

For cases where P is much less than Q and B small but not quite negligible in comparison with either, the equations (27)-(29) for the double-cord pendulum and (55)-(57) for the cord and lath pendulum assume simpler forms. It is particularly noticeable that the couplings in the two cases are then given by

$$\gamma^2 = \beta^2 P/Q \quad \ldots \quad \ldots \quad (101)$$

and

$$\gamma^2 = \alpha^2 P/Q. \quad \ldots \quad \ldots \quad (102)$$

thus reducing to the same form for each type of pendulum.

§ 8. *Experimental Methods and Results.*

Double-Cord Pendulum.—This pendulum was arranged with its bobs near the floor along which a black board on wheels was slowly pulled by cords so as to receive sand-traces. The board with the sand-traces was then placed on the floor at the foot of a rising stand carrying a camera by which the photographs were taken.

The relations were calculated from the theory already given so as to obtain any desired values of the couplings. The total height BD (see fig. 1) was 229 cm. The results are shown in Table I.

TABLE I.—Double-Cord Pendulum.

Coupling γ.	Ratio CD : BC, or $\beta=\frac{2\gamma}{1-\gamma}$.	Ratio CD BD*, or $\frac{\beta}{1+\beta}=\frac{2\gamma}{1+\gamma}$	Actual Droop CD* for BD=229 cm.	Frequency Ratio $\frac{p}{q}=\sqrt{(1+\beta)}$.	Amplitude Ratio when bob is struck. $\frac{E}{F}=(1+\beta)^{-\frac{1}{2}}$
Per cent.			cm.		
5	2/19	0 095	21·81	1·051	0·952
10	2/9	0·182	41·63	1·105	0·905
15	6/17	0·261	59·77	1·163	0·860
20	1/2	0·333	76·33	1·225	0·816
25	2/3	0·400	91·60	1·291	0·775
33⅓	1	0·500	114·5	1·414	0·707
40	4/3	0·571	130·86	1·527	0·654
50	2	0·667	152·7	1·732	0·577
60	3	0·750	171·5	2	0·500
			Length	Found to give	
>60			173	2	

* For Laboratory work it would be convenient to have a lath of length BD with the positions of C marked on it for each desired coupling.

Plate V. shows sixteen reproductions of the double traces obtained from this pendulum. The couplings used are indicated on each figure as a percentage. The letters F or W indicate that one pendulum-bob was started by a blow from the *finger* or from a *wood-block*. A small circle and stroke show that one pendulum-bob was drawn aside and let go from rest. A small circle against the other trace indicates that it was held in the zero position and both bobs let go together. Arrows along the traces show the sense in which they were described. In some experiments the board was drawn along before the bob was let go. These traces accordingly show their own initial conditions.

Looking at figs. 1-9 of Plate V. we may trace the gradual change from a 5 per cent. coupling to one of 60 per cent The contrast between the first and last is very striking, and at first glance it would seem impossible to bridge the gulf. The first figure with loose (or moderate) coupling exhibits the phenomena of beats and the slow surging of the energy to and fro between the bobs. The vibrations appear to be practically simple harmonic throughout but of slowly-changing ampltude. This is the natural consequence of the superposition of vibrations of only slightly different periods (See Table I.) Moreover, while these beats are recognizable the numbers of vibrations from node to node are in accordance with the theory. Thus, for fig. 1 the first line of Table I. gives the ratio of frequencies as 1·05 or 21 : 20, and the traces show the correct number of vibrations between the nodes. So with the others. It is perhaps worth mentioning that some early traces for the 5 per cent. coupling were found to give about 27 waves from node to node. This was disconcerting, as the following couplings were right. It was thought that the connector CC′ between the two cords might allow a little *lost motion* or *back lash*. It was examined and found not to fit quite tightly, though possibly friction might prevent any shake. Of course, if there were lost motion, a theoretical 5 per cent. might be reduced to an actual 4 per cent. (or less), and the number of waves increased to 25 (or more). The connector was accordingly made quite tight and traces again taken. But it was soon discovered that an electric lead crossing the room for another research had sagged and touched the bridle cords, and thus very slightly vitiated their motion. This disturbance was removed and the trace taken which appears as fig. 1, showing the correct 20 waves.

Turning now to fig. 9 we see that it shows nearly but not quite, the compound harmonic curve characteristic of a tone and its octave. If the 2 : 1 relation were exact the *kink* would not wander on the main curve, but would reappear each time in the same position. Now this case corresponds with the ninth line of Table I, and was calculated to give a frequency ratio of 2 : 1, but on the supposition that the masses of the bridle Cords ACA, A′C′A′, and connector CC′ were negligible. Now for the higher values of the couplings where the droops are very great this can hardly be the case. Hence it is not surprising to find that the droop had to be made (as shown in the Table) 173 cm. instead of the calculated 171·5 cm. to give the exact 2 : 1 ratio. This arrangement furnished fig. 10, in which the kink scarcely wanders perceptibly. The board was also turned through a right angle and left stationary while the pendulums were used as Blackburn's pendulums. They thus gave simultaneously the patterns in the right-hand top corner of the figure and retraced them almost perfectly about ten times.

The gradual change from the case of *beats* in fig. 1 to the exact *tone* and *octave* of fig. 10 is very instructive. It also shows the advantage of taking traces, for in watching the motions of the intermediate cases the eye fails to realize exactly what is happening, though the extremes are easy to recognize.

Several of the remaining traces for this pendulum are for the same coupling, but are varied only by different initial conditions.

Thus fig. 11 has the bob held at zero while the other is pulled aside, then both let go together. (See equations (51)-(54).) In fig. 1 one is allowed to hang freely at rest (but somewhat displaced), while the other is held aside and then let go. (See equatations (47)-(49).) Figs. 13 and 14 each shows simple vibrations instead of two superposed. Fig. 13 was obtained by starting with unlike equal displacements due to a connecting thread which was burnt when all was still. Fig. 14 was obtained by swinging with the hands till the amplitudes were equal and phases alike instead of opposite as in the previous case. Pains were taken to draw the board at the same uniform rate by scale and stop-watch in figs. 13 and 14. Thus the relation of periods 1 : 2 is correctly exhibited along with the phases.

Figs. 15 and 16 show for two important couplings the initial conditions of double displacement, one bob pulled aside the other hanging freely at rest as in fig. 12, but here the board was moved before the bob was let go, as seen by the traces.

In very case in which we tested them the traces were found to give a satisfactory agreement with the theory both qualitatively and quantitatively. These tests were carried out by plotting on squared paper the theoretical curves to be expected and then comparing them with the sand traces obtained.

Cord aud Lath Pendulum.—In the present work this arrangement was used with the lengths, r and s, of the pendulums eqal and the masses, P and Q, of the bobs equal. Accordingly the equations (58)-(60) apply and also those based upon them. Calculating from these we derive the values given in Table II. The length of the simple pen. dulum equivalent to one of the lath pendulums used alone was 112 cm. The lengths SR ($=112a$) were accordingly calculated and are shown in the Table. The positions of R for the various desired couplings were then marked on the lath to facilitate setting. For completeness' sake several values of coupling are included in the table, though they have not been used with the present model. For example, it is diffieultt to set the coupling for values less than 15 per cent. or greater than 55 per cent.

TABLE II—Cord and Lath Pendulums.

Coupling γ.	Ratio Rs: QS $=a$.	Length SR $=a \times 112$ cm.
Per cent.		cm.
5	0 051331	5·75
10	0·105680	11·84
15	0·163660	18·33
20	0·226020	25·31
25	0·293675	32·89
30	0·367800	41·19
35	0·449612	50·36
40	0·541945	60·70
45	0·646610	72·42
50	0·767591	85·97
55	0·910181	101·94
57·7	1	112

Plate VI. presents fourteen sets of traces obtained with various couplings or under different initial conditions. Of these the first ten, figs. 17 to 20, show single traces from the lower bob when the upper bob was struck; the couplings vary from 15 to 55 per cent. In fig. 24 the coupling was about 48 per cent., the adjustment was made by trial to give 2 : 1 as the ratio of frequencies. To show that the exact ratio was practically attained two lines of traces were taken, 12 waves occurring between them. It is seen that the kink shifts its position but slightly on the main wave after the lapse of 23 of the long periods. To trace the slow shift of the kink because the 2 : 1 relation is not fulfilled the figure for the 50 and 55 per cent. are taken in duplicate or triplicate, the board being again passed under while the pendulm was still swinging unchecked.

Figs. 27-30 on the bottom row of Plate VI. show double traces obtained simultaneously from the upper and lower bobs, the two boards being placed together in right relation, for photography. The coupling is the same in each, *viz.* of the order 48 per cent. as to give approximately 2 : 1 as the ratio frequencies. Figs. 27 and 28 show very plainly that the bob which is *not* struck executes the compound harmonic motion whether it is lower or upper, but that the bob which *is* struck has a very different motion according as it is upper or lower. See equations (77) to (84).

Figs. 29 and 30 show the results of pulling one bob aside the other being at rest in its equilibrium position as dealt with in the theory. See equations (85)-(94). The pulling aside was effected by a horizontal thread which was burnt when all was steady.

§ 9. *Summary.*

1. The paper describes two types of coupled pendulums which are considered useful both for lecture demonstration and for quantitative work in the laboratory, especially as they illustrate many important points in the phenomena of nductively-coupled electrical circuits.

2. One of these, called the double-cord pendulum is like a pair of Blackburn's pendulums in parallel planes con-

nected by a ꜱtiff tube at the droop of the bridles. Its vibrations occur perpendicularly to these parallel planes. It may be used with couplings gradually varied from very loose to very tight, say from 1 to 60 per cent. It exhibits, by double sandtraces simultaneously formed on a moving board, the gradual change of the vibrations from the phenomena of slow beats to those of compound harmonic motion. Most of the curves, however, show the superposition of two simple vibrations of incommensurable frequencies.

The pendulum shows also the effects of different initial conditions.

This form is specially suitable for laboratory work, as it may be set up from the simplest materials and yet give results of distinct and quantitative value.

If the lengths are equal and the masses also, then the vibrations of this pendulum are quite interchangeabe.

Photographic reproductions are given of sixteen double traces obtained with this pendulum.

3. The other form of apparatus, or cord and lath pendulum, is yet easier to use for simple lecture illustration, but requires more careful installation for the best work in the laboratory. It consists of a simple pendulum, with a light lath for its rod, from a movable stud on which hangs the cord pendulum.

Even if the lengths are made equal, and also the masses, the vibrations of this pendulum are not interchangeable The effects of various initial conditions are accordingly more striking.

It may be used for couplings varying from about 10 to 55 per cent.

Photographic reproductions are given of ten single and four double traces obtained with this pendulum.

4. The mathematical theories of both pendulums are developed and compared with each other and with the theory of the electrical case it is sought to represent. The experiments are in satisfactory agreement with theory, though this is not always immediately obvious.

5. In the experiments and most of the theory the lengths of the pendulums have been equal and also the masses of the bobs. A large field lies ready for exploration in the cases where the quantities of both classes are unequal and varied at will. These cases are reserved for later papers.

University College, Nottingham,
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Variably-Coupled Vibrations: II. Unequal Masses or Periods.

By

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&

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§ 1. *Introduction.*

In a recent paper* two types pf coupled pendulums were experimented with, their lengths and the masses of their bobs being in each case equal. The present paper, the second of the series, deals with the double-cord pendulum only, but in cases where either the masses of the bobs are unequal or else the lengths of their suspensions are unequal.

These mechanical cases may be regarded as somewhat analogous to the electrical cases of inductively-coupled circuits with unequal inductances or unequal periods respectively.

With unequal masses and equal lengths it is noticeable that with small couplings a great increase in the amplitude of vibration of the small bob entailed very little loss in that of the large bob. Indeed, for masses as 20 : 1 we almost realised the case of forced vibrations.

The funnel of the light bob was here of cardboard and so had an appreciable damping. This rendered it necessary to make corresponding modifications in the theory.

* Phil. Mag. (6) vol. xxxiv. no. 202, pp 246-270 (Oct. 1917).

With unequal lengths and equal masses the response showed a great diminution for small couplings, whereas for larger couplings the mistuning seemed without appreciable effect.

The paper includes twenty-seven photographic reproductions of double sand traces obtained simultaneously one from each bob of the coupled pendulum.

§ 2. *Theory for Unequal Masses.*

Equations of Motion and Coupling.—Throughout the work described in the present paper the double-cord pendulum was used. This was shown in figs. 1, 2, and 4 of the first paper. The equations of motion and coupling were given as (27)-(29) and may now be rewritten here as follows :—

$$\left.\begin{aligned} P\frac{d^2y}{dt^2}+\frac{P+Q+\beta Q}{(1+\beta)\,(P+Q)}P\frac{g}{l}y=\frac{\beta}{(1+\beta)}\cdot\frac{PQ}{(P+Q)}\frac{g}{l}z,\\ Q\frac{dz^2}{dt}+\frac{P+\beta P+Q}{(1+\beta)\,(P+Q)}Q\frac{g}{l}z=\frac{\beta}{(1+\beta)}\cdot\frac{PQ}{(P+Q)}\frac{g}{l}y \end{aligned}\right\} \quad \dots \quad (1)$$

and

$$\gamma^2=\frac{\beta^2PQ}{(P+Q+\beta Q)(P+\beta P+Q)}\cdot \dots \qquad \dots \quad (2)$$

Let us now write in the above

$$\frac{Q}{P}=\rho \qquad \frac{g}{l}=m^2. \qquad \dots \qquad \dots \quad (3)$$

Also divide the two equations (1) by P and Q respectively, and insert the frictional term $2k\ dy/dt$ in the first of them. We then obtain

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+\frac{1+\rho+\beta\rho}{(1+\beta)(1+\rho)}m^2y=\frac{\beta}{(1+\beta)}\cdot\frac{\rho}{(1+\rho)}m^2z, \quad \dots \quad (4)$$

$$\frac{d^2z}{dt^2}+\frac{1+\rho+\beta}{(1+\beta)(1+\rho)}m^2z=\frac{\beta}{(1+\beta)}\cdot\frac{1}{(1+\rho)}m^2y \qquad \dots \quad (5)$$

These may be written

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+ay=\rho bz, \quad \ldots \quad \ldots \quad (6)$$

and

$$\frac{d^2z}{dt^2}+cz=by, \quad \ldots \quad \ldots \quad (7)$$

where

$$\left.\begin{aligned} a&=\frac{1+\rho+\beta\rho}{(1+\beta)(1+\rho)}m^2,\ b=\frac{\beta m^2}{(1+\beta)\,(1+\rho)}, \\ \text{and} \\ c&=\frac{1+\rho+\beta}{(1+\beta)(1+\rho)}m^2 \end{aligned}\right\} \quad \ldots \quad (8)$$

Solution and Frequencies.—To solve (6) and (7) let us write

$$\left.\begin{aligned} z&=\epsilon^{xt}, \\ \text{and, on inserting in (7), we have} \\ y&=\left(\frac{x^2+c}{b}\right)e^{xt}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (9)$$

Then (9) substituted in (6) gives

$$\left(\frac{x^2+c}{b}\right)(x^2+2kx+a)=\rho b,$$

or

$$x^4+2kx^3+(c+a)x^2+2kcx+ca-\rho b^2=0, \quad \ldots \quad (20)$$

which is the auxiliary biquadratic in x. Though this equa--ion has the form of the general biquadritic, an approximate solution, presenting all the accuracy needed for our purpose, may be easily obtained by noting that k is small compared with the other constants. For, as appears from the experiments, k is of the order one-thousandth of the coefficient of x^2 and of the constant term.

Then we may write for the roots of x in the biquadratic (10) the values

$$-r\pm ip \quad \text{and} \quad -s\pm iq, \quad \ldots \quad \ldots \quad (11)$$

where i denotes $\sqrt{(-1)}$, and r and s (being comparable to k) are to be treated as small quantities whose squares or products are negligible in comparison with p and q which depend upon the larger constants of the equation.

Thus, with the roots from (11) we may write instead of (10) the equivalent equation

$$(x+r-ip)(x+r+ip)(x+s-iq)(x+s+iq)=$$

or

$$x^4+2(r+s)x^3+(p^2+q^2+r^2+s^2+4rs)x^2$$
$$+2(p^2s+q^2r+r^2s+rs^2)x+(p^2+r^2)(q^2+s^2)=0. \quad \ldots \quad (12)$$

This, on omitting the negligible quantities, becomes the approximate equation sufficiently accurate for our purpose,

$$x^2+2(r+s)x^3+(p^2+q^2)x^2+2(p^2s+q^2r)x+p^2q^2=0. \quad \ldots \quad (13)$$

The comparison of coefficients in (10) and (13) yields

$$r+s=k, \quad \ldots \quad \ldots \quad (14)$$
$$p^2+q^2=c+a, \quad \ldots \quad \ldots \quad (15)$$
$$p^2s+q^2r=ck, \quad \ldots \quad \ldots \quad (16)$$
$$q^2q^2=ca-\rho b^2. \quad \ldots \quad \ldots \quad (17)$$

From (15) to (17) we may eliminate q^2 and obtain a quadratic in p^2 whose roots may be called p^2 and q^2. We thus find

$$\left.\begin{aligned} 2p^2&=c+a+\sqrt{\{(a-c)^2+4\rho b^2\}}, \\ \text{and} \quad 2q^2&=c+a-\sqrt{\{(a-c)^2+4\rho b^2\}}. \end{aligned}\right\} \quad \ldots \quad (18)$$

Again, from (14) and (16) we obtain

$$r=\frac{p^2-c}{p^2-q^2}k. \textit{ and } s=\frac{c-q^2}{p^2-q^2}k.$$

And by use of (18) these become

$$r=\frac{a-c+\sqrt{\{(a-c)^2+4\rho b^2\}}}{2\sqrt{\{(a-c)^2+4\rho b^2\}}}k. \quad \ldots \quad (19)$$

and

$$s=\frac{c-a+\sqrt{\{(a-c)^2+4\rho b^2\}}}{2\sqrt{\{(a-c)^2+4\rho b^2}}k. \quad \ldots \quad (20)$$

Then, inserting the values of a, b, and c from (8) in (18), (19), and (20) we obtain

$$p=m, \quad q=\frac{m}{\sqrt{(1+\beta)}}. \quad \ldots \quad \ldots \quad (21)$$

$$\frac{p}{q}=\sqrt{(1+\beta)}, \quad \ldots \quad \ldots \quad (22)$$

$$r=\frac{pk}{1+\rho}, \quad \ldots \quad \ldots \quad (23)$$

$$s=\frac{k}{1+\rho}. \quad \ldots \quad \ldots \quad (24)$$

Thus, using (11) in (9) and introducing the usual constants, the general solution may be written in the form

$$z=e^{-rt}(Ae^{pit}+Be^{-pit})+e^{-st}(Ce^{qit}+De^{-qit}), \quad \ldots \quad (25)$$

and, omitting r^2 and s^2,

$$y+\frac{(-p^2+c)}{b}e^{-rt}(Ae^{pit}+Be^{-pit})+\frac{(-q^2+c)}{b}e^{-st}(Ce^{qit}+De^{-qit})$$

$$+\frac{2pri}{b}e^{-rt}(-Ae^{pit}+Be^{-pit})+\frac{2qsi}{b}e^{-st}(-Ce^{qit}+De^{-pit}). \quad \ldots \quad (26)$$

Or, by transformation of (25) and (26) and use of (21)-(24), we may write the general solution in the form

$$z=Ee^{-\rho st}\sin(mt+\epsilon)+Fe^{-st}\sin\left(\frac{mt}{\sqrt{(1+\beta)}}+\phi\right), \quad \ldots \quad (27)$$

and

$$y=-\rho E'e^{-\rho st}\sin(mt+\epsilon')+F'e^{-st}\sin\left(\frac{mt}{\sqrt{(1+\beta)}}+\phi'\right), \quad \ldots \quad (28)$$

where

$$(E')^2=E^2\frac{\beta^2m^2+4(1+\beta)^2k^2}{\beta^2m^2},$$

and

$$\tan(\epsilon'-\epsilon)=\frac{2(1+\beta)k}{\beta m}; \quad \ldots \quad \ldots \quad (29)$$

also

$$(F')^2=F^2\frac{\beta^2m^2+4(1+\beta)k^2}{\beta^2m^2},$$

and

$$\tan(\phi'-\phi)=\frac{-2\sqrt{(1+\beta)}k}{\beta m}; \quad \ldots \quad \ldots \quad (30)$$

the exponential coefficient s being given by (24), and E, ϵ, F, and ϕ being the arbitrary constants dependent on the initial conditions. In many of the experimental cases E′ may be assimilated to E and F′ to F without appreciable error. The changes $(\epsilon'-\epsilon)$ and $(\phi'-\phi)$ of the phase angles may be distinctly appreciable for very small values of β. But in these cases the vibrations show a slow waxing and waning of amplitude and the phase is of very little importance. On the other hand, for β equal to unity, we have

$$\tan(\epsilon'-\epsilon)=4k/m \quad \text{and} \quad \tan(\phi'-\phi)=-2\sqrt{2k/m}.$$

And the numerical values of these are of the order 0·020 and 0·014, hence $\epsilon'-\epsilon=1°\ 10'$ and $\phi'-\phi=0°\ 48'$ nearly. Hence for all our present experimental cases, we may drop the four accents in equation (28).

Initial Conditions. Case I.—Suppose the heavy bob of mass Q (which $=\rho$P) is pulled aside and that the light one of mas P is allowed to hang at rest in its more or less displaced position according to the coupling in use. Then we may write :

$$\left.\begin{array}{l}\text{For } t=0 \text{ let } z=f,\\ \text{then it follows statically that } y=\dfrac{\beta\rho f}{1+\rho+\beta\rho};\\ \text{also put}\\ \qquad \dfrac{dz}{dt}=0 \text{ and } \dfrac{dy}{dt}=0.\end{array}\right\} \quad \ldots \quad \ldots \quad (31)$$

Differentiating with respect to time (27) and (28) without its accents, and writing in the latter n for $m/\sqrt{(1+\beta)}$, we find

$$\frac{dz}{dt}=\mathrm{E}e^{-\rho st}[m\cos(mt+\epsilon)-\rho s\sin(mt+\epsilon)]$$
$$+\mathrm{F}e^{-st}[n\cos(nt+\phi)-s\sin(nt+\phi)], \quad \ldots \quad (32)$$
$$\frac{dy}{dt}=-\rho\mathrm{E}e^{-\rho st}[m\cos(mt+\epsilon)-\rho s\sin(mt+\epsilon)]$$
$$+\mathrm{F}e^{-st}[n\cos(nt+\phi)-s\sin(nt+\phi)]. \quad \ldots \quad (33$$

The conditions (31) introduced in eqations (27), (28), (32), and (33) give

$$f=\mathrm{E}\sin\epsilon+\mathrm{F}\sin\phi, \quad \ldots \quad \ldots \quad (34)$$

$$\frac{\beta\rho f}{1+\rho+\beta\rho}=-\rho\mathrm{E}\sin\epsilon+\mathrm{F}\sin\phi, \quad \ldots \quad \ldots \quad (35)$$

$$0 = E\,(m\cos\epsilon - \rho s\sin\epsilon) + F\,(n\cos\phi - s\sin\phi), \qquad \ldots \quad (36)$$

$$0 = -\rho E\,(m\cos\epsilon - \rho s\sin\epsilon) + F\,(n\cos\phi - s\sin\phi). \qquad \ldots \quad (37)$$

But, by reason of the smallnes of ρs in comparison with m (of the order 0·01) and of s in comparison with n (still less), we may write instead of (36) and (37) the following :

$$0 = Em\cos\epsilon + Fn\cos\phi, \qquad \ldots \qquad \ldots \quad (38)$$

and

$$0 = -\rho Em\cos\epsilon + Fn\cos\phi. \quad \ldots \qquad \ldots \quad (39)$$

These are satisfied by

$$\epsilon = \frac{\pi}{2} \quad \text{and} \quad \phi = \frac{\pi}{2}. \qquad .. \qquad \ldots \quad (40)$$

These values inserted in (34) and (35) give

$$f = E + F,$$

and

$$\frac{\beta\rho f}{1+\rho+\beta\rho} = -\rho E + F;$$

whence

$$E = \frac{f}{1+\rho+\beta\rho} \quad \text{and} \quad F = \frac{(1+\rho)\beta f}{1+\rho+\beta\rho}. \qquad \ldots \quad (41)$$

Hence, for the special solution with these initial conditions, we have

$$z = \frac{f}{1+\rho+\beta\rho}e^{-\rho st}\cos mt + \frac{(1+\rho)\beta f}{1+\rho+\beta\rho}e^{-st}\cos\frac{mt}{\sqrt{(1+\beta)}}, \ \ldots \qquad \ldots \quad (42)$$

$$y = -\frac{\rho f}{1+\rho+\beta\rho}e^{-\rho st}\cos mt + \frac{(1+\rho)\beta f}{1+\rho+\beta\rho}e^{-st}\cos\frac{mt}{\sqrt{(1+\beta)}}, \qquad \ldots \quad (43)$$

where $s = \frac{k}{1+\rho}$.

Thus the ratios of the amplitudes of the quick and slow components in the y and z vibrations are respectively given by

$$\frac{-\rho}{(1+\rho)\beta}e^{-(\rho-1)st} \quad \text{and} \quad \frac{1}{(1+\rho)\beta}e^{-(\rho-1)st}. \qquad \ldots \quad (43\,a)$$

Case II.—Suppose now that the heavy bob (of mass $Q = \rho P$) is pulled aside while the light one (of mass P) is held undisplaced. Then we have :

For

$$\left.\begin{array}{l} t=0, \quad z=f, \quad y=0, \\ \dfrac{dy}{dt}=0 \text{ and } \dfrac{dz}{dt}=0. \end{array}\right\} \quad \ldots \quad \ldots \quad (44)$$

Putting (44) in (27), (28) without accents, (32) and (33), and omitting small quantities as before, we find

$$\left.\begin{array}{l} f=\mathrm{E}\sin\epsilon+\mathrm{F}\sin\phi, \\ 0=-\rho\mathrm{E}\sin\epsilon+\mathrm{F}\sin\phi\,; \end{array}\right\} \quad \ldots \quad \ldots \quad (45)$$

$$\left.\begin{array}{l} 0=\mathrm{E}m\cos\epsilon+\mathrm{F}n\cos\phi, \\ 0=-\rho\mathrm{E}m\cos\epsilon+\mathrm{F}n\cos\phi. \end{array}\right\} \quad \ldots \quad \ldots \quad (46)$$

Then (46) is satisfied by

$$\epsilon=\frac{\pi}{2} \quad \text{and} \quad \phi=\frac{\pi}{2},$$

and putting these values in (45), we obtain

$$\mathrm{E}=\frac{f}{1+\rho} \quad \text{and} \quad \mathrm{F}=\frac{\rho f}{1+\rho}.$$

Hence, for the special solution with these initial conditions, we have

$$y=-\frac{\rho f}{1+\rho}e^{-\rho st}\cos mt+\frac{\rho f}{1+\rho}e^{-st}\cos\frac{mt}{\sqrt{(1+\beta)}}, \quad \ldots \quad (47)$$

$$z=\frac{f}{1+\rho}e^{-\rho st}\cos mt+\frac{\rho f}{1+\rho}e^{-st}\cos\frac{mt}{\sqrt{(1+\beta)}}. \quad \ldots \quad (48)$$

Accordingly the ratios of the amplitudes of the quick and slow vibrations in the y and z traces are respectively

$$-\epsilon^{-(\rho-1)st} \text{ and } \frac{\epsilon^{(\rho-1)st}}{\rho}. \quad \ldots \quad \ldots \quad (49)$$

Relation of Dampings in the Vibrations separate and coupled.—The vibrations of a separate damped pendulum of length l are derived from tne equation of motion

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+m^2y=0, \quad \ldots \quad \ldots \quad (50)$$

where $m^2=g/l$.

The solution of this involves simple harmonic vibrations of approximate period

$$\tau = 2\pi/m,$$

and of damping factor

$$\epsilon^{kt}.$$

Thus the ratio of successive amplitudes is

$$\epsilon^{k\tau/2} = \epsilon^{k\pi/m}$$

nearly.

But the logarithmic decrement λ (per half wave) is defined as the logarithm to base e of this ratio.

Hence

$$\lambda = \frac{k\pi}{m} \text{ or } k = \frac{m\lambda}{\pi}, \qquad \ldots \qquad \ldots \quad (51)$$

which gives the relation between damping coefficient and logarithmic decrement for a separate pendulum.

We have now to express in terms of λ the two damping coefficients r and s which apply to the superposed vibrations when the pendulums are coupled. Thus, combining (23) and (24) with (51), we find

$$r = \frac{\rho}{1+\rho} \frac{m\lambda}{\pi}, \qquad \ldots \qquad \ldots \quad (52)$$

and

$$s = \frac{1}{1+\rho} \cdot \frac{m\lambda}{\pi}. \qquad \ldots \qquad \ldots \quad (53)$$

§ 3. *Theory for Uueqıal Periods.*

Equations of Motion and Coupling.—Still using the double-cord pendulum, as shown in figs. 1, 2, and 4 of the first paper, we now make the masses of the bobs equal, but the lengths of the suspensions unequal. (The droops of the two bridles always remain equal.) In other words Q = P or $\rho = 1$, while the lengths of the suspensions for the y and z vibrations are now denoted by ηl and l respectively, the droop of each bridle being βl as before.

Then the aquations of motion of the pendulums may be written at first in the form :

$$P\frac{d^2y}{dt^2}+Pg\theta=0, \qquad \ldots \qquad \ldots \quad (54)$$

$$Q\frac{d^2z}{dt^2}+Qg\psi=0, \qquad \ldots \qquad \ldots \quad (55)$$

where θ and ψ are the inclinations of the suspensions to the vertical.

But we have also

$$\theta=\frac{y-\beta l\omega}{\eta l} \text{ and } \psi=\frac{z-\beta l\omega}{l} \quad \ldots \qquad \ldots \quad (56)$$

where ω is the inclination to the vertical of the planes of the bridles.

Neglecting masses of bridles, connector, and suspensions, ω must satisfy

$$Qg(\psi-\omega)=Pg(\omega-\theta)=Pg(\psi-\omega). \qquad \ldots \quad (57)$$

Then (56) in (57) gives

$$l\omega=\frac{y+\eta z}{\beta+\beta\eta+2\eta}. \qquad \ldots \qquad \ldots \quad (58)$$

And (58) in (56) yields

$$\theta=\frac{(2+\beta)y-\beta z}{l(\beta+\beta\eta+2\eta)} \text{ and } \psi=\frac{(\beta+2\eta)z-\beta y}{l(\beta+\beta\eta+2\eta)}. \qquad \ldots \quad (59)$$

Then by (59), equations (54) and (55) become

$$\frac{d^2y}{dt^2}+\frac{2+\beta}{\beta+\beta\eta+2\eta}m^2y=\frac{\beta m^2}{\beta+\beta\eta+2\eta}z, \qquad \ldots \quad (60)$$

$$\frac{d^2z}{dt^2}+\frac{\beta+2\eta}{\beta+\beta\eta+2\eta}m^2z=\frac{\beta m^2}{\beta+\beta\eta+2\eta}y, \qquad \ldots \quad (61)$$

where m^2 is written for g/l.

So, for the coupling γ, we have

$$\gamma^2=\frac{\beta^2}{(2+\beta)(\beta+2\eta)}. \qquad \ldots \qquad \ldots \quad (62)$$

Hence, for $\eta=1$, we recover the original relation

$$\gamma=\frac{\beta}{2+\beta}, \qquad \ldots \qquad \ldots \quad \ldots(63)$$

which agrees with (32) of the first paper.

Solution and Frequencies.—In equation (61) try

then we have

$$\left.\begin{array}{l} z=e^{xt} \\ y=\dfrac{x^2(\beta+\beta\eta+2\eta)+(\beta+2\eta)m^2}{\beta m^2}e^{xt} \end{array}\right\} \quad \ldots \quad (64)$$

And, by (64) in (60), we obtain

$$\{x^2(\beta+\beta\eta+2\eta)+(\beta+2\eta)m^2\}\{x^2(\beta+\beta\eta+2\eta)\}+(2+\beta)m^2\}$$
$$=\beta^2m^4.$$

This reduces to the auxiliary biquadratic in x,

$$x^4(\beta+\beta\eta+2\eta)+2(1+\beta+\eta)m^2x^2+2m^4=0. \quad \ldots \quad (65)$$

Solving this as a quadratic in x^2, we have

$$x^2=-m^2\frac{1+\beta+\eta\pm\sqrt{\{(1-\eta)^2+\beta^2\}}}{\beta+\beta\eta+2\eta}. \quad \ldots \quad (66)$$

Or, let us write

$$x=\pm pi \text{ or } \pm qi. \quad \ldots \quad \ldots \quad (67)$$

Then, for the sake of brevity putting $\triangle^2$ for $(1-\eta)^2+\beta^2$ we have

$$\left.\begin{array}{l} p^2=\dfrac{1+\beta+\eta+\triangle}{\beta+\beta\eta+2\eta}m^2, \\ q^2=\dfrac{1+\beta+\eta-\triangle}{\beta+\beta\eta+2\eta}m^2, \\ \dfrac{p}{q}=\left\{\dfrac{1+\beta+\eta+\triangle}{1+\beta+\eta-\triangle}\right\}^{1/2} \end{array}\right\} \quad \ldots \quad \ldots \quad (68)$$

and

Thus, using (67) in (64) and introducing the usual constants, we obtain

$$z=\text{E}\sin(pt+\epsilon)+\text{F}\sin(qt+\phi), \ldots \quad \ldots \quad (69)$$

and

$$y=-\frac{1-\eta+\triangle}{\beta}\text{E}\sin(pt+\epsilon)+\frac{\triangle-(1-\eta)}{\beta}\text{F}\sin(qt+\phi), \ldots \quad (70)$$

p and q being defined by (68).

Initial Conditions.—Consider the case of pulling aside the bob Q of the pendulum of length l whose vibrations are denoted by z, the other bob hanging at rest in a more or less displaced position according to the magnitude of the coupling.

Thus, we may write :

$$\left.\begin{array}{l} \text{For} \quad t=0, \quad z=f, \\ \text{when it follows statically that, } y=\dfrac{\beta f}{2+\beta}, \\ \text{and we have also} \quad \dfrac{dy}{dt}=0, \quad \dfrac{dz}{dt}=0. \end{array}\right\} \quad \ldots \quad \ldots \quad (71)$$

Differentiating (69) and (70) with respect to the time, and introducing (71) gives equations which are satisfied by

$$\epsilon=\frac{\pi}{2} \text{ and } \phi=\frac{\pi}{2}. \quad \ldots \quad \ldots \quad (72)$$

Then, introducing (71) and (72) in (69) and (70) we find

$$\left.\begin{array}{l} E=\dfrac{(2+\beta)(-1+\eta+\triangle)-\beta^2}{2(2+\beta)\triangle}f, \\ F=\dfrac{(2+\beta)(1-\eta+\triangle)+\beta^2}{2(2+\beta)\triangle}f. \end{array}\right\} \quad \ldots \quad \ldots \quad (73)$$

Finally, (72) and (73) in (69) and (70) give as the required special solution

$$z=\frac{(2+\beta)(-1+\eta+\triangle)-\beta^2}{2(2+\beta)\triangle}f\cos pt$$
$$+\frac{(2+\beta)(1-\eta+\triangle)+\beta^2}{2(2+\beta)\triangle}f\cos qt, \quad \ldots \quad \ldots \quad (74)$$

and

$$y=-\frac{1+\beta+\eta-\triangle}{2(2+\beta)\triangle}\beta f\cos pt$$
$$+\frac{1+\beta+\eta+\triangle}{2(2+\beta)\triangle}\beta f\cos qt. \quad \ldots \quad \ldots \quad (75)$$

§ 4. *Relations among Variables.*

It is instructive to plot graphs with the values of the coupling γ as ordinates, the abscissæ being the corresponding values of β (ratio of droop of bridle to pendulum length). A different graph is needed for each value of η and ρ (which are respectively the ratios of pendulum lengths and masses of bobs).

The data for these graphs are derived from the equations and are given in Tables I., II.

TABLE I.—Masses 20 : 1 and lengths equal.

Coupling γ.	$\frac{\text{Bridle Droop}}{\text{Pendulum length.}}=\beta$	Actual Droop for total length 229 cm.	Frequency Ratio $p:q=\sqrt{(1+\beta)}$.
Per cent.		cm.	
0	0·0	0·0	1 00
1	0·05	10·9	1·025
2	0·10	20 8	1·05
3	0·151	29·9	1·07
4	0 207	39·7	1·10
5	0 265	48·1	1·12
10	0·60	85·9	1·27
20	1·53	138·6	1·59
30·4	3·00	171·1	2·00

TABLE II.—Masses 5 : 1 and lengths equal.

Coupling γ	$\frac{\text{Bridle Droop}}{\text{Pendulum length.}}=\beta$	Actual Droop for total length 229 cm.	Frequency Ratio $p:q=\sqrt{(1+\beta)}$.
Per cent.		cm.	
0	0	0	1·00
1	0·027	6·0	1·014
2	0·058	13 3	1·029
5·2	0·141	28·2	1·07
9·4	0·259	47·1	1·12
25·5	1	114·5	1·414
39·5	2	152·7	1·732
48·8	3	171·8	2

TABLE III.—Masses equal and lengths 3 : 4.

Coupling γ	$\frac{\text{Bridle Droop}}{\text{Pendulum length.}} = \beta$	Actual Droop for total length 229 cm.	Frequency Ratio $p:q$.
Per cent.		cm.	
0	0	0	1·154
5·5	0·1	20·8	1·16
10·3	0·2	38·2	1·175
22·4	0·5	76·3	1·29
36·5	1	114·5	1·48
59·9	2·6	165·4	2·00
63·2	3	171·8	2·11

The graphs referred to are given in fig. 1.

We may now, from the data in the same tables, plot graphs with the values of the frequency ratios $p : q$ as ordinates, the abscissæ being the corresponding values of the coupling γ.

These are shown in fig. 2, separate graphs being plotted for mass ratios 1, 5, and 20 and lengths equal, and also for lengths 3 : 4 and masses equal.

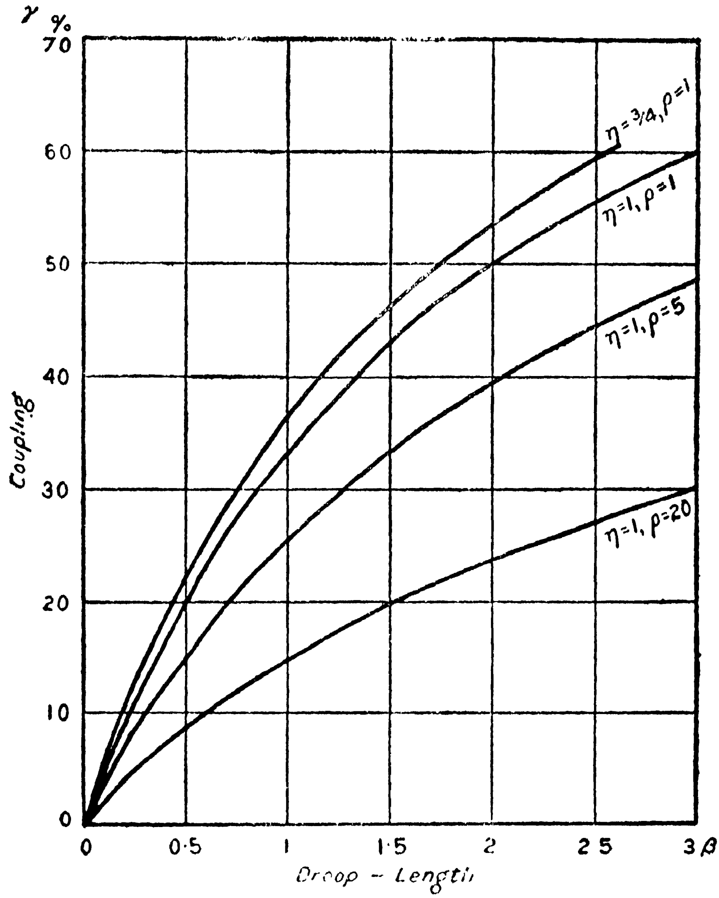

Fig. 1.—Coupling and Droop.

With the separate frequencies equal and a given coupling, it may be noted that the greater the inequality of the masses the greater is the inequality of the frequencies of the resulting superposed vibrations of the coupled system.

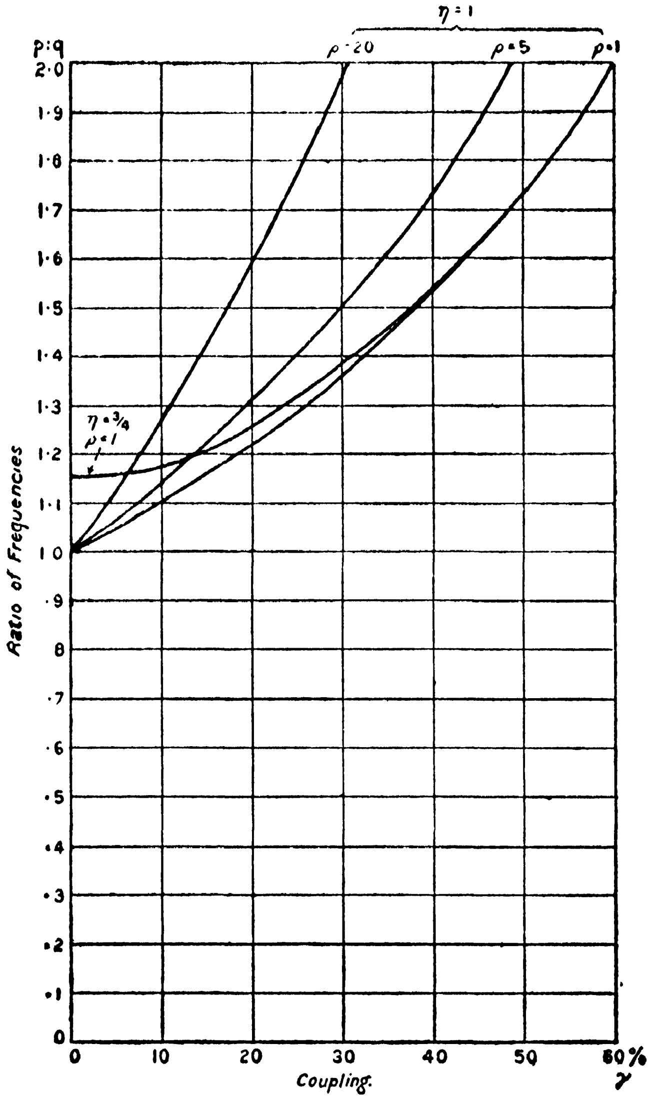

Fig. 2.—Frequency Ratios and Coupling.

When the coupling vanishes the frequencies of the separate vibrations are of course undisturbed. Thus for equal lengths, but any ratio of masses, we have for $\gamma=0$, $p:q$ equals unity. But for different separate frequencies (*i.e.*, η not equal to unity) we have for $\gamma=0$, $p:q$ greater than unity. But with large couplings the effect of unequal separate frequencies gradually disappears.

§ 5. *Experimental Results.*

Masses 20 : 1.—The bobs used in these experiments were of the order 1000 gms. and 50 gms. respectively. Figs. 1-11 of Plate I. give photographic reproductions of the double sand traces simultaneously obtained when the masses of the bobs Q and P were as 20 : 1, *i.e.*, $\rho = 20$.

The couplings vary from 1 per cent. in the first to a little over 30 per cent. in the last, and are shown as percentages on every figure.

Figs. 1-8 were obtained by drawing the heavy bob aside horizontally, the light bob being allowed to hang at rest in its more or less displaced position according as the coupling was tight or loose. In figs. 9-11, while the heavy bob was pulled aside, the light one was held in its undisplaced position. Figs. 1-6 show a very marked effect due to the inequality of the masses. For, as the resultant vibrations of the light bob wax and wane in amplitude, those of the heavy bob scarcely change. Thus showing that with masses 20 : 1 we have in this respect almost reached the limiting case of forced vibrations in which the reaction of the driven on the driver is negligible. The frequencies, however, are still appreciably affected. The contrast with the case of equal masses may be seen by referring to figs. 1-5 in Plate V. where the waxings and wanings occur equally and alternately in both traces. Figs. 1-8 show that as the coupling increases the inequality of the frequencies of the superposed vibrations increases also. Hence there are fewer vibrations in the beat cycle and this fulfils the theory.

In fig. 9 the initial displacement of the heavy bob was so great that a collision occurred between the two as indicated. But its effect passed away after a few vibrations. This may be seen by fig. 10, in which with a slightly smaller displacement the collision was avoided.

Fig. 11 shows appreciable damping of the vibration of the light bob which was held undisplaced whlle the heavy one was drawn aside, whereas that of the heavy bob is not appreciably damped. This is exactly what might be expected from general considerations. But it seems at first sight in direct contradiction to the theory which shows that the y and z vibrations for the light and heavy bobs respectively involve the selfsame damping factors. But by eqations (23) and (24) we see that one damping coefficient is ρ-times the other. Again, by equation (48) the amplitude of the slow vibrations of the heavy bob is ρ-times that of its quick ones. In the present experimental case ρ equals 20, hence almost all the vibration visible is the slow one with the negligibly small damping coefficient. On the other hand, by equation (47) we see that the amplitudes of the slow and quick vibrations of the light bob are numerically equal. Consequently the large damping coefficient, which is 20 times the small one, affects at least half of the amplitude visible.

Logarithmic decrements.—The lower trace on fig. 11 just dealt with, led to the theoretical introduction of the damping of the light bob as expressed by the constant k in eqation (4). It also became necessary to estimate the experimental value of k. To do this one pendulum with a light bob was allowed to oscillate alone, the other being meanwhile disconnected. The traces for the lighter bobs P were taken when their masses were respectively as used in the experiments, so as to be one-twentieth and one-fifth of those of the corresponding heavy ones. The results are given in fig. 12. From the upper trace with the very light bob consisting simply of a cardboard funnel, a few weights and sand (total mass about 50 gms.), we find that the logarithmic decrement is of the order $\lambda = 0{\cdot}017$.

Then by (57) we have

$$k = \frac{m\lambda}{\pi} = (0{\cdot}005)m. \qquad \ldots \qquad \ldots \quad (76)$$

The lower trace with bob about 120 gms. shows considerably less damping and the decrement need not be evaluated.

Masses 5 : 1.—The masses of the bobs used in these experiments were of the order 600 gms. and 120 gms. respectively.

Figs. 13-19 in Plate II. show double traces obtained with this arrangement. In figs. 13-16 we see very plainly the beat effects on the lower trace which is left by the lighter bob. The traces of the heavier bob also show distinct but much slighter fluctuations of amplitude. In this respect they are seen to present an intermediate state between the cases of equal masses and masses as 20 : 1. And this is just what we should naturally expect. Further, the beat cycles contain fewer and fewer vibrations as the coupling increases. This again is in accord with theory, for the frequencies of the superposed vibrations are then more unequal and therefore gain more quickly on each other.

Lengths 3 : 4.—Figs. 20-28 show double traces simultaneously obtained with the masses of the bobs equal, but the lengths of the suspensions as 3 : 4. The lower trace on each figure is that made by the shorter pendulum. In the case of fig. 20, the short pendulum was pulled aside, the long one hanging still in its slightly displaced position. In the cases of figs. 21-25, the long one was drawn aside while the short one hung at rest in its more or less displaced position. In figs. 26-28 the long pendulum was pulled aside while the short one was held in its zero position, as this favoured the exhibition of the compound harmonic trace which it was then sought to obtain.

The couplings in this set vary from about 5 per cent. to over 60 per cent. In the 5 and 10 per cent. couplings the response of the second pendulum is feeble and the beat cycles contain very few vibrations. These are the effects of the inequality of lengths. But as the coupling is further increased these effects of the inequality of the separate frequencies are seen to be overpowered. This is exactly in accord with the theory as exhibited in the graphs on fig. 2.

Fig. 26 shows an accidental collision of the lighter bob with the releasing apparatus. But the effect of the blow is seen to pass away after a few vibrations, as shown by comparison with fig. 27, which is a repetition of the conditions first intended. Figs. 26 and 27 are seen to present almost the appearance of the compound harmonic motion of a tone and its octave, the latter being too sharp. Fig. 28 shows the coupling reduced to 60 per cent., and this gives the relation of frequencies almost exactly 2 : 1.

The pair of simultaneous traces in fig. 28 is almost identical in type with those in fig. 11 of Plate V. in which latter case the lengths were equal. It may well seem surprising that the effect of the present mistuning (in which the frequency ratio exceeds 8 : 7) should be so completely obliterated by this coupling. But experiment and theory agree that it should be so.

§ 6. *Summary.*

1. This second paper describes further experiments with the double-cord pendulum, but with the masses unequal as 20 : 1 and as 5 : 1, or the lenths unequal as 3 : 4. These

are somewhat analogous to coupled electrical circuits with different inductances or different periods.

2. The case of masses 20 : 1 is seen to be very nearly that of forced vibrations in which the light bob is driven by receiving energy from the heavy bob or driver, while the latter's loss, though equal in energy, entails only a very small decrease of amplitude. The case of masses as 5 : 1 is about midway in character between that of 20 : 1 and equal masses. Eighteen photographic reproductions of double traces are given for unequal masses.

3. It was noticeable on one of the traces that the light bob showed diminution of amplitude as the traces proceeded. This led to taking resistance into account in the equation of motion. It was also necessary to determine experimentally the actual damping of the light bob when vibrating separately. The theory thus developed and numerically applied fitted the observed facts.

4. In the case of unequal lengths but equal masses, a feebler response and a shorter beat cycle may naturally be expected if mistuning were absent. Both these effects are quite striking with loose couplings. But with the tighter couplings the effect of mistuning is practically unnoticeable. The theory agrees with this experimental result. Nine sets of double traces are given for the unequal periods.

5. It is hoped that these methods may be shortly applied to the illustrations of important phenomena in other branches of Physics.

Notingham,
Nov. 19, 1917.

Variably-Coupled Vibrations: III Both Masses and Periods Unequal.

By

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&

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§ 1. *Indroduction.*

In the work described in previous papers, the double-cord pendulum was experimented with: (1) when the masses of the bobs and the periods of vibration of the separate pendulums were equal: (2) when either the masses of the bobs or the separate periods were unequal. The present paper deals with the cases where the masses of the bobs and the periods of the sparate vibrations are both unequal.

The mechanical case may be regarded as somewhat analogus to the electrical case of coupled circuits in which the inductions and periods are both unequal.

A series of photographs was taken from sand traces obtained when the masses were 20 : 1 and the length of the pendulum with the heavier bob as 4 : 3 of that with the lighter bob. The ratio of the frequencies then slightly exceeds 8 : 7, *i.e.*, to put the matter in acoustical terms, the pendulums are out of tune by 248 logarithmic cents or approximately a tone and a quarter.

Other photographs were taken with masses 20 : 1 and pendulum lengths as 9 : 8, the lighter bob still being on the shorter pendulum. The ratio of the frequencies slightly exceeds 21 : 20, *i.e.*, the pendulums are out of tune by 102 logarithmic cents or approximately an equal-tempered semitone.

In both cases it was noticeable for small couplings that very little of the energy of the heavy bob was required to build up in the lesser bob an amplitude nearly equal to that of the heavier bob. Further, that for couplings about 30 per cent. the curves obtained were almost identical in the two cases and almost indistinguishable from that of 30 per cent. coupling shown in Paper II. for masses 20 : 1 and lengths equal.

The pendulum with the heavy bob was altered in length until it was 3 : 4 times as long as that with the light bob, the masses remaining as 20 : 1. The results of theory and experiment were rather striking. The ratio of the frequencies of the separate pendulums slightly exceeds 8 : 7. As coupling was increased from one to about six per cent. the ratio of the frequencies diminished to about 13 : 12, and the two pendulums had greater action and reaction on one another. When the coupling was further increased, the ratio increased to 2 : 1 at coupling about 30 per cent. as in the other cases.

Quenched Spark.—Prof. J. A. Fleming has pointed out that by means of a rapidly damped spark discharge in a primary circuit a slowly damped electrical vibration may be produced in the secondary or antenna. In this paper a photograph of a mechanical analogue of such a discharge is reproduced from a sand trace.

§ 2. *Theory of General Case.*

Equations of Motion and Coupling.—The double-cord pendulum was shown in figs. 1 and 2, of the first paper. The equations of motion and coupling were there given as equations (25), (26), and (29). They may now be re-written as follows :—

$$P\frac{d^2y}{dt^2}+Pg\theta=0, \quad \dots \quad \dots \quad (1)$$

$$Q\frac{d^2z}{dt^2}+Qg\psi=0, \quad \dots \quad \dots \quad (2)$$

$$\gamma^2=\frac{\beta PQ}{(P+Q+\beta Q)(P+\beta P+Q)}. \quad \dots \quad \dots \quad (3)$$

The ratio of the masses of the bobs may be expressed by $\rho=Q/P$ and the lengths of the suspesions for the y and z vibrations by ηl and l respectively, the droop of each bridle being βl.

Then

$$\theta=\frac{y-\beta l\omega}{\eta l}\ ;\ \psi=\frac{z-\beta l\omega}{l}. \quad \dots \quad \dots \quad (4)$$

Further, neglecting masses of suspensions, connector, and bridles, ω must satisfy

$$\left.\begin{array}{l} Qg(\psi-\omega)=Pg(\omega-\theta) \\ \text{or} \\ \rho g(\psi-\omega)=g(\omega-\theta) \end{array}\right\} \quad \dots \quad \dots \quad (5)$$

Then (4) in (5) gives

$$\omega l=\frac{y+\eta\rho z}{\beta+\eta+\eta\rho+\beta\eta\rho}. \quad \dots \quad \dots \quad (6)$$

And (6) in (4) gives

$$\theta=\frac{(1+\rho+\rho\beta y)-\beta\rho z}{l(\beta+\eta+\eta\rho+\beta\rho\eta)}, \quad \dots \quad \dots \quad (7)$$

$$\psi=\frac{(\beta+\eta+\eta\rho)z-\beta y}{l(\beta+\eta+\eta\rho+\beta\rho\eta)}. \quad \dots \quad \dots \quad (8)$$

Inserting frictional term $2kPdy/dt$ in (1), putting $g/l=m^2$ and dividing (1) by P and (2) by Q, then (7) and (8) in (1) and (2) give

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+\frac{1+\rho+\beta\rho}{\beta+\eta+\eta\rho+\beta\eta\rho}m^2y=\frac{\beta\rho m^2}{\beta+\eta+\eta\rho+\beta\eta\rho}z, \quad \dots \quad (9)$$

$$\frac{d^2z}{dt^2}+\frac{\beta+\eta+\eta\rho}{\beta+\eta+\eta\rho+\beta\eta\rho}m^2z=\frac{\beta m^2}{\beta+\eta+\eta\rho+\beta\eta\rho}y. \quad \ldots \quad (10)$$

Further, the coupling may be written

$$\gamma^2=\frac{\beta^2\rho}{(1+\rho+\beta\rho)(\beta+\eta+\eta\rho)}. \quad \ldots \quad \ldots \quad (11)$$

To simplify, (9) and (10) may be abbreviated thus:

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+ay=\rho bz, \quad \ldots \quad \ldots \quad (12)$$

$$\frac{d^2z}{dt^2}+\quad cz \quad =by. \quad \ldots \quad \ldots \quad (13)$$

Where

$$a=\frac{1+\rho+\beta\rho}{\beta+\eta+\eta\rho+\beta\eta\rho}m^2; \quad b=\frac{\beta m^2}{\beta+\eta+\eta\rho+\beta\eta\rho};$$

$$\text{and } c=\frac{\beta+\eta+\eta\rho}{\beta+\eta+\eta\rho+\beta\eta\rho}m^2. \quad \ldots \quad \ldots \quad (14)$$

Equations (12) and (13) are the same as (6) and (7) of Paper II., but the values of a, b, and c are different.

Solution and Frequencies.—To solve (12) and (13) we write

Then
$$\left.\begin{aligned} z&=e^{xt} \\ y&=\left(\frac{x^2+c}{b}\right)e^{xt} \end{aligned}\right\}. \quad \ldots \quad \ldots \quad (15)$$

From equation (11) of Paper II., we see that the values of x may be written

$$-r\pm ip \quad \text{and} \quad -s\pm iq. \quad \ldots \quad \ldots \quad (16)$$

Hence, omitting small quantities, we have

$$\left.\begin{aligned} 2p^2&=[c+a+\sqrt{\{(a-c)^2+4\rho b^2\}}] \\ 2q^2&=[c+a-\sqrt{\{(a-c)^2+4\rho b^2\}}] \end{aligned}\right\}. \quad \ldots \quad (17)$$

$$\left.\begin{aligned} r&=\frac{a-c+\sqrt{\{(a-c)^2+4\rho b^2\}}}{2\sqrt{\{(a-c)^2+4\rho b^2\}}}k \\ s&=\frac{c-a+\sqrt{\{(a-c)^2+4\rho b^2\}}}{2\sqrt{\{(a-c)^2+4\rho b^2\}}}k \end{aligned}\right\}. \quad \ldots \quad (18)$$

Thus using (15) and (16) and introducing the usual constants, the general solution may be written in the form

$$z=e^{-rt}(Ae^{pit}+Be^{-pit})+e^{-st}(Ce^{qit}+De^{-qit}), \quad \ldots \quad (19)$$

and

$$y=\frac{(-p^2+c)}{b}e^{-rt}(Ae^{pit}+Be^{-pit})+\frac{(-q^2+c)}{b}e^{-st}(Ce^{qit}+De^{-qit})$$
$$+\frac{2pri}{b}e^{-rt}(-Ae^{pit}+Be^{-pit})+\frac{2qsi}{b}e^{-st}(Ce^{qit}+De^{-qit}). \quad \ldots \quad (20)$$

When small quantities are further neglected, these will simplify to

$$z=Ee^{-rt}\sin(pt+\epsilon)+Fe^{-st}\sin(qt+\phi), \quad \ldots \quad (21)$$

and

$$y=Ge^{-rt}\sin(pt+\epsilon)+He^{-st}\sin(qt+\phi). \quad \ldots \quad (22)$$

(21) and (22) are each equtions of two superposed vibrations, of which the frequency ratio is

$$\frac{p}{q}=\left[\frac{c+a+\sqrt{\{(a-c)^2+4\rho b^2\}}}{c+a-\sqrt{\{(a-c)^2+4\rho b^2\}}}\right]^{1/2}, \quad \ldots \quad (23)$$

where a, b, and c are given by (14).

Initial Conditions.—Let the heavy bob of mass Q on the pendulum of length l be pulled aside a distance f by a horizontal force, the light bob on the other pendulum hanging freely at rest. The displacement of the light bob can then be written at once from equation (31) of the previous paper, since the length of this pendulum makes no difference to the quantity in question. Hence we have

for $t=0$,

$$\left.\begin{array}{l} z=f, \quad y=\dfrac{\beta\rho f}{1+\rho+\beta\rho}=\dfrac{\beta}{1+\beta}f \text{ neary for } \rho=20 \\ \dfrac{dz}{dt}=0, \quad \text{and} \quad \dfrac{dy}{dt}=0. \end{array}\right\} \quad \ldots \quad (24)$$

Then, introducing these conditions in (21) and (22) and into the differentiations of these with respect to the time and, as before, omitting small quantities, we find

$$\left.\begin{aligned} f&=\mathrm{E}\sin\epsilon+\mathrm{F}\sin\phi,\\ \frac{\beta f}{1+\beta}&=\mathrm{G}\sin\epsilon+\mathrm{H}\sin\phi.\end{aligned}\right\} \quad \ldots \quad \ldots \quad (25)$$

$$\left.\begin{aligned} 0&=\mathrm{E}p\cos\epsilon+\mathrm{F}q\cos\phi,\\ 0&=\mathrm{G}p\cos\epsilon+\mathrm{H}q\cos\phi.\end{aligned}\right\} \quad \ldots \quad \ldots \quad (26)$$

Equations (26) are satisfied by

$$\epsilon=\frac{\pi}{2}, \qquad \phi=\frac{\pi}{2}. \quad \ldots \quad \ldots \quad (27)$$

From (17), (20) and (22) we have

$$\left.\begin{aligned} \mathrm{G}&=\frac{-p^2+c}{b}\mathrm{E}=\frac{c-a-\sqrt{\{(a-c)^2+4\rho b^2\}}}{2b}\mathrm{E},\\ \mathrm{H}&=\frac{-q^2+c}{b}\mathrm{F}=\frac{c-a+\sqrt{\{(a-c)^2+4\rho b^2\}}}{2b}\mathrm{F}.\end{aligned}\right\} \quad \ldots \quad (28)$$

Equations (27) and (28) in (25) give

$$f=\mathrm{E}+\mathrm{F}, \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

and

$$\frac{\beta f}{1+\beta}=\frac{c-a-\sqrt{\{(a-c)^2+4\rho b^2\}}}{2b}\mathrm{E}$$

$$+\frac{c-a+\sqrt{\{(a-c)^2+4\rho b^2\}}}{2b}\mathrm{F}; \quad (30)$$

whence

$$\frac{\mathrm{E}}{\mathrm{F}}=\frac{(c-a+\delta)(1+\beta)-2b\beta}{(-c+a+\delta)(1+\beta)+2b\beta}, \quad \ldots \quad \ldots \quad (31)$$

and

$$\frac{\mathrm{G}}{\mathrm{H}}=-\frac{4\rho b^2(1+\beta)+2b\beta(c-a-\delta)}{4\rho b^2(1+\beta)+2b\beta(c-a+\delta)}, \quad \ldots \quad (32)$$

where

$$\delta^2=(c-a)^2+4\rho b^2. \quad \ldots \quad \ldots \quad (33)$$

These give the values of the *ratios* of the constants determined by the initial conditions in question, and this is all that we need check the records experimentally obtained.

§ 3. *Experimental Results.*

Masses 20 : 1, ***Lengths*** 4 : 3 ($\eta=3:4$).—The relations were calculated from the theory given so as to obtain any desired values of the coupling and frequencies, the results are shown in Table I. For the longer pendulum the sum of pendulum length and droop of bridle was 229 cm., and it had the heavier bob.

TABLE I.—Masses 20 : 1, Lengths 4 : 3 ($\eta=3:4$).

Coupling $=\gamma$.	$\frac{\text{Bridle Droop}}{\text{Long Pendulum Length.}}=\beta$	Frequency Ratio $p:q$.
Per cent.		
0	0	1·154
4·245	0·2	1·255
9·96	0·5	1·403
17·07	1	1·62
28	2·12	2
34·43	3	2·29

Figures 1-5 (Pl. II.) shows photographic reproductions of the double sand-traces simultaneously obtained, with masses 20 : 1, *i. e.*, $\rho=20$ and the length of the pendulum carrying the lighter bob 3 : 4 of that with the heavier bob. The first four photographs (with couplings 4 per cent. to 28 per cent.) were obtained by drawing aside the heavy bob and allowing the lighter one to settle in its more or less displaced position, according as the coupling was tight or loose. The fifth (with coupling 28 per cent.) was obtained by holding the light bob in its undisplaced position and pulling the heavy one aside.

In all the curves it is noticeable that there is very little fluctuation of the amplitude of the heavier bob, although the amplitude of the lighter one waxes and wanes considerably. Comparing fig. 1 of this paper with figs. 6 and 21 of the previous paper, it is seen that the amplitude of the lighter bob in fig. 6, Paper II., is much greater than that attained when the lengths are unequal as well as the masses. But the shorter pendulum in fig. 21, Paper II., has an amplitude much less than that of the shorter pendulum in the pesent case. Fig. 4 in this paper is almost identical with fig. 8 of Paper II., the amplitudes in the two cases are nearly the same and the couplings are almost alike. Fig. 5 of this paper is also similar to fig. 9 of Paper II.

Masses 20 : 1, *Lengths* 9 : 8 ($\eta = 8 : 9$).—Table II. shows the frequencies for certain couplings with the masses of the bobs 20 : 1 and the lengths of the pendulums 9 : 8, the longer one having the heavier bob and being 229 cm. long if the droop of the bridle were zero.

TABLE II.—Masses 20 : 1, Lengths 9 : 8.

Coupling $=\gamma$.	$\frac{\text{Bridle Droop}}{\text{Long Pendulum Length.}} = \beta$	Frequency Ratio $p : q$.
Per cent.		
0	0	1·06
4	0·19	1·162
10	0·55	1·32
15.75	1	1·494
29.5	2·6	2

Figs. 7-12 (Pl. III.) show photographs taken with masses still 20 : 1, but the length of the pendulum with the lighter bob 8/9 that of the one with the heavier bob. Again we see very little fluctuation of the amplitude of the heavy bob throughout. Figs. 7-9 were taken with the heavy bob held aside and the light one free to hang in its more or less displaced position. Fig. 10 was taken with the light bob held aside and the heavy one allowed to hang freely. Figs. 11

and 12 were obtained with the heavy bob drawn aside and the light one held in its zero position. It may be seen that fig. 6 in Paper II. is more like fig. 7 than like fig. 1, both of the present paper. On the other hand, fig. 21 of Paper II. is less like fig. 7 than like fig. 1, both of this paper. This is because the separate frequencies of the component pendulums are more nearly in tune with each other. Fig. 10 shows the effect of drawing aside the lighter bob, little energy is given to the heavy bob and there is but little action or reaction of the one bob on the other. Fig. 12 is almost identical with fig. 5 of the present paper and with fig. 9 of Paper II., and the couplings in all cases are very nearly the same.

Masses 20 : 1 *Lengths* 3 : 4 ($\eta = 4 : 3$).—Table III. shows the frequencies, couplings, and bridle droops with bob masses 20 : 1 and pendulum length 3 : 4, the longer one having the lighter bob, and its length being 137 cm. if the bridle droop were zero.

TABLE III.—Masses 20 : 1, Lengths 3 : 4.

Coupling $=\gamma$.	$\frac{\text{Bridle Droop}}{\text{Short Pendulum Length.}}=\beta$	Frequency Ratio. $p:q$	Ratio of Amplitudes. E : F.	Ratio of Amplitudes. G : H.
Per cent.				
0	0	1·154	∞	Indeterminate.
1·76	0·1	1·106	104·6	−0·809
3·37	0·2	1·07	12·9	
4·85	0·3	1·054	2·283	−0·893
6·3	0·4	1·065		
7·5	0·5	1·09	0·2875	−0·836
9·9	0·7	1·154		
12·97	1·0	1·243	0·0696	−0·640
31·47	4·0	1·952		

The figures illustrating the cases in Table III. were obtained with the new portable apparatus shown in fig. 13 (Pl. V.) and which is described in detail later. Figs. 14-22 (Pls. IV. & V.) show traces taken with masses 20 : 1 and the length of the pendulum with the heavy bob 3/4 of that with the light one. Figs. 14-21 were obtained by drawing aside the heavy bob and showing the light one to rest in its more or less displaced position. In figs. 14-17 it is seen that the light bob is almost undisplaced. In fig. 22 the light bob was held undisplaced while the heavy one was drawn aside.

It is noticeable that with couplings between 2 and 13 per cent. the fluctuations of amplitude of the heavy bob are distinctly marked especially about 6 per cent. In this case the heavy bob gives up nearly all its energy to the light bob, which then attains an amplitude more than three times that with which the heavy bob started. For very small or very large couplings there is very little fluctuation of amplitude in the vibration of the heavy bob. This is seen in figs. 14, 15, and 20 22. This is in accord with the theory. For the ratio between E and F, the amplitudes of the driver's superposed vibrations have been calculated for the initial conditions in use. The results are given in Table III., which shows that E/F has values near unity for couplings about six per cent. Whereas for very small couplings much exceeding six per cent. E/F is very small. And either a lage or small value of E/F means inappreciable fluctuation of the driver's resultant amplitude.

Let us now consider the question of the ratio (p/q) of frequencies of the superposed vibrations and the variation of this ratio with coupling. When the coupling is zero this ratio naturally has that value which applies to the pendulums when separate. When the bobs were equal and lengths unequal, the value of this ratio increased with the coupling until p/q almost merged into the value for equal pendulum lengths. When the bobs were unequal as well as the lengths but the heavy bob was on the long pendulum, the same behaviour was noticeable in the ratio p/q and its dependence on coupling (see Tables I, and II.).

On the other hand, when bobs are unequal as well as lengths but the heavy bob is on the short pendulum, a quite new feature is theoretically predicted (see Table III.). Thus when the coupling is gradually increased from zero, the value of p/q at first diminishes, reaches a minimum and then increases. These striking features are to a first approximation upheld by the experiments. For, as seen in passing along figs. 14-20, the number of vibrations in the beat cycle at first increases and then decreases. The maximum number of vibrations in the cycle is about 13 and occurs in fig. 17 for a coupling of 6·3 per cent. Accordingly this coupling should correspond to a minimum value of about 1·08 of the ratio p/q. From Table III., however, it is seen that the minimum value of p/q is about 1·054 and occurs for a coupling of about 5 per cent. These slight discrepancies are easily accounted for by the presence of the sand in the funnels and a possible error in estimating the lengths of the simple pendulums equivalent to those in use. Thus, if with the average amounts of sand in the funnels the masses were in the ratio 19 : 1 and if the lengths were really 11 : 16 (instead of 20 : 1 and 12 : 16 respectively), the minimum value of p/q as calculated would be in sensible agreement with that experimentally observed and would occour for practically the the same coupling as that in actual use. Table IV. is calculated from the above data and is found to agree fairly well with the observations.

TABLE IV.—Masses 19 : Lengths 11 : 16.

Coupling $=\gamma$.	$\dfrac{\text{Bridle Droop}}{\text{Short Pendulum Length.}}=\beta$	Frequency Ratio $p:q$.
Per cent.		
0	0	1·21
1·724	0·1	1·155
3·301	0·2	1·115
4·758	0·3	1·086
6·112	0·4	1·072
7·379	0·5	1·076

Figs. 21 and 22 show traces with 32 per cent. coupling, which gives a ratio of p/q almost equal to 2 : 1 or a tone and its octave. In fig. 21 the conditions of starting masked the compound character of the vibrations, but this is clearly revealed in fig. 22.

Quenched Spark.—Fig. 52, p. 714 of Professor J. A. Fleming's 'Principles of Electric Wave Telegraphy and Telephony," 2nd ed., shows "the electrical beats produced in the primary and secondary circuits when a sustained primary spark is used and the single periodic oscillations in the secondary circuit when the Quenched Spark is employed." The mechanical analogue of beats was obtained on the double-cord pendulum. The damping was not so marked as in Prof. Fleming's case, because our damping factor was almost negligible.

To produce the effect of the quenched spark the masses of the bobs were equal and also their separate frequencies; further their coupling was 10 per cent. One of the bobs was drawn aside and the other allowed to hang in its slightly displaced position. The bob was then freed and its oscillations were quickly diminished by the transference of its energy to the other pendulum, which in about six vibrations had attained an amplitude equal to that with which the other pendulum started. The first pendulum had at this instant lost all amplitude, and it was then suddenly raised by hand and held in this position while the other bob oscillated with the single period. Fig. 6 (Pl II.) is a photographic reproduction of the sand traces thus obtained. The lower trace represents the quenched spark and the upper one shows the vibrations set up in the secendary circuit or antenna.

§ 4. *Portable Apparatus*

The work with the double-cord pendulum up to fig. 12 inclusive was done with rough apparatus suspended from beams of the roof. At this point it seemed desirable to have an apparatus that was portable and so arranged as to facilitate the various adjustments required. This was accomplished by the new apparatus shown in fig. 13 (Pl. V.).

It consists essentially of a braced framework of deal, one and a half inches square, the main rods being each six feet long. The bridles are of whipcord and fastened off on cleats fixed on the end frame. The pendulum suspensions in actual use are wires of various lengths with hooks at each end, the fine adjustment being attained by a thin cord and tightener as used for tent ropes. In the photograph these working bridles and suspensions would have been scarcely visible and so were replaced by course white cords. The two longitudinal rods at the base of the frame are provided with rails made of hoop-iron set edgewise and let into saw-gates along their length. These rails carry four ball-bearing sheaves, which are fixed on the under side of the board 31 by 23 inches arranged to carry the detachable cards which receive the sand traces. To draw this board along, a cord passes from the centre of one end through two tension-eyes to a bobbin on one side of the end frame. This bobbin is turned by a handle slowly or quickly as may be desired for the purpose in view.

§ 5. *Coupling Graph for Cord and Lath Pendulum.*

Both in the electrical case and for the double-cord pendulum the coupling may approach but cannot reach the value unity. But in the case of the cord and lath pendulum the conditions are somewhat different. Thus we have

$$\gamma^2 = \frac{a^2}{1+a+a^2},$$

where γ is the coupling and a is the ratio in which the second suspension divides the lath.

So that here also with increasing positive values of a, γ only approaches but never reaches unity except for $a=\infty$. But for negative values of a we see that γ may reach unity for $a=-1$.

This suggested plotting a graph giving γ as ordinates, a being the abscissæ. This is shown in fig. 23 (Pl. V.). The graph has a maximum and a minimum at $a=-2$ and points of inflexion at $a=-0\cdot344$ and $-2\cdot906$ nearly, and it also asymptotes to $\gamma=\pm 1$.

§ 6. *Possible Further Work.*

The vibrations of two coupled pendulums have hitherto been developed for their own sake and as an analogue to electrical vibrations in coupled circuits. It appears, however, that by modification of the pendulums the analogue may be usefully extended so as to illustrate phenomena in various other branches of physics.

The following have occurred to us as worthy of investigation and plans of attack of several have already been matured :—

1. Kater Pendulum for "g" and the possible disturbance of period due to vibration of bracket. Theory and experiment will enable us to evaluate the possible error and eliminate it.

2. Large vibrations with restoring forces not simply proportional to the displacement but involving its squares or cubes.

3. Such a system under double forcing.

4. Optical Dispersion.

5. Dynamical Analogies to Colour Vision and Hearing.

6. Any of the above but further specialized by damping ing where necessary.

Nottingham,
March 16, 1918.

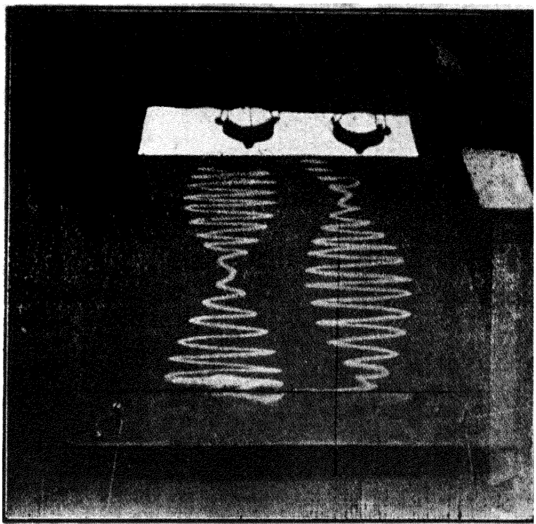


Fig. 4.

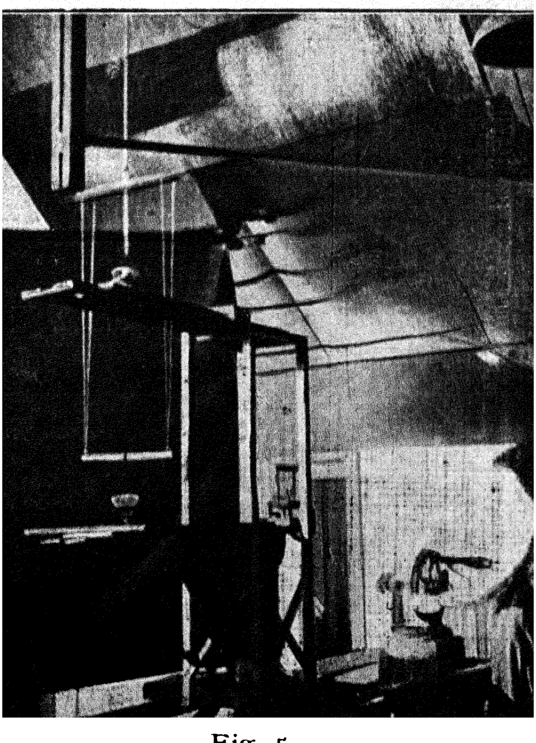
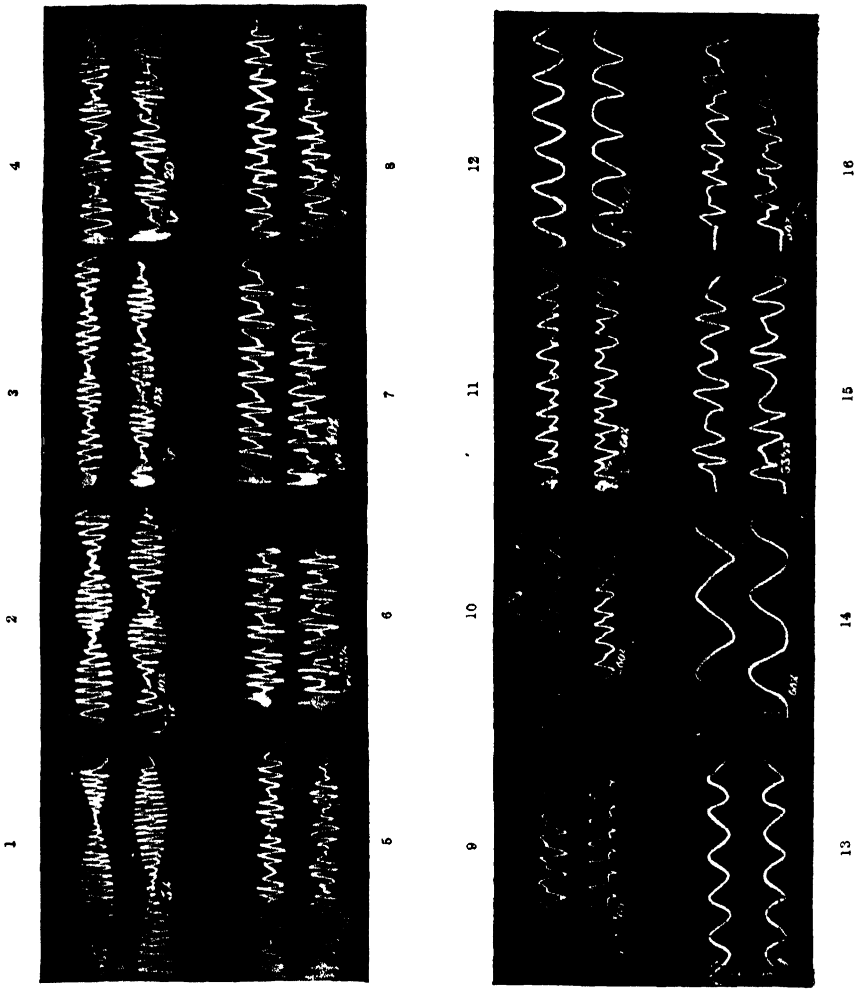
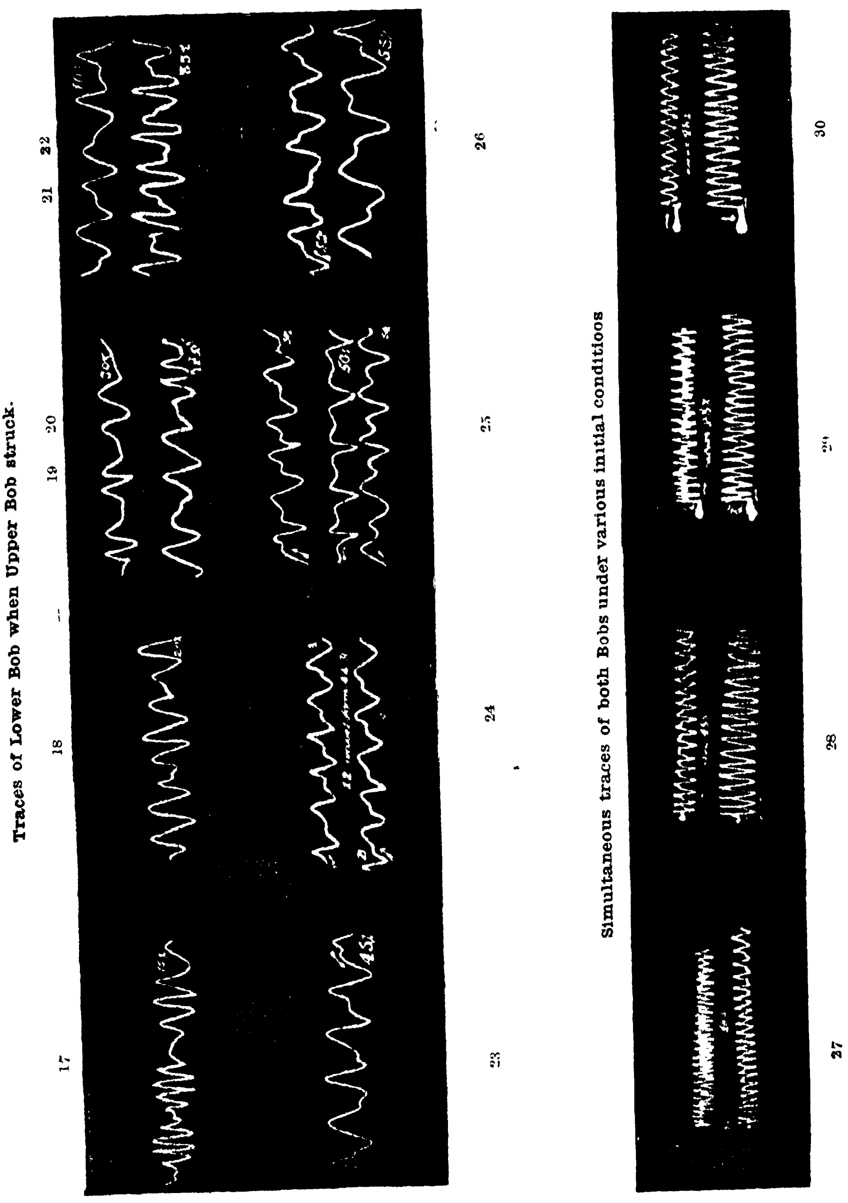


Fig. 5



DOUBLE CORD PENDULUM.
(COUPLINGS GIVEN AS PERCENTAGES.)



LATH AND CORD PENDULUM.
(COUPLINGS GIVEN AS PERCENTAGES.)

PROCEEDINGS

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Vol. V. PART II.

Some Observations of the Ionisation of the Air in India.

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In the years 1917 and 1918, I examined the ionisation of the air in five different places in India. My object was to collect some data about the electric charge of the air and to ascertain to what extent the nature of the rock and of the soil influences the ionisation of the air. The ionisation of the air near the ground is due largely, if not completely, to the various radiations from the radio-active substances present in the air and in the upper layers of the soil. This question has been carefully investigated by Eve[1]. The radio-active substances in the air ultimately come from the soil or rock and from the ocean. Hence we may expect to find a variation of the electric charge of the air with a variation of the composition of the ground in contact with the air.

In my investigations I used an Ebert's ion-counter, an instrument which is well known and largely used for this kind of work. The principle of the instrument is this: A known quantity of air is sucked through a cylindrical condenser charged to a known potential. The air passing

(1) *Phil. Mag.*, vol. 22, p. 26, 1911.

through the condenser partly discharges the latter. From the fall of potential and the capacity of the instrument the quantity of electricity given to the instrument by the known quantity of air is easily calculated. From this again we calculate the quantity of electricity in 1 m^3. of air. The charge so found is, however, not the whole charge of the air.

The atmospheric ions have been classified according to their mobilities into the following three types[2]:

Type of ions.	Mean Value of their Mobility.
Small	1·8 to 1·02
Intermediate	0·1 to 0·006
Large	0·0008 to 0·0003

Ebert's ion-counter records all the small ions and a portion of the intermediate ions. In my measurements all the ions with a mobility equal to or greater than 0·027 cms. per second per volt centimeter were caught. This quantity is easily determined from the equation:

$$v = \frac{\log_e \frac{R_0}{R_i}}{L \times 2\pi} \times \frac{\frac{C.C.}{Sec}}{V}$$

where v = specific velocity;

R_0 = radius of the outer cylinder;

R_i = radius of inner cylinder;

L = length of inner cylinder;

$\frac{C.C.}{Sec.}$ = quantity of air in C.C. which passes through the instrument in one second;

V = potential of the charge in volts.

Every measurement for ions of one sign lasted from 15 to 20 minutes when only one measurement for ions of one sign

(2) *Phys. Rev.*, Ser. II, vol. VII. p. 50.

was made. But in most cases I made two measurements for ions of the same sign one immediately after the other in order to find out the rapid changes of the charge. In that case each measurement lasted only 10 minutes. Of the values from two consecutive measurements the mean value was taken. Corrections were made for the natural leak of the instrument. This was found to be not the same for a + charge as for a − charge.

Although the instrument fails to record the total charge of the air it may still serve for comparing the electrification of the air in different places, if proper care is taken to avoid smoke and other impurities of the air as much as possible. As Ebert's ion-counter has been used in many parts of the world the results given in this paper may be of some interest.

By E, I denote the electric charge of the air per m^3. in e. s. u. The number of ions per C. C. is easily calculated from E; it depends on the value which we attribute to the elementary charge *e*. In the following tables this number is calculated for three different values of e.

n_1 = number of ions per c.c. for $e = 3{\cdot}4 \times 10^{-10}$

n_2 = " " " " " " $e = 4{\cdot}65 \times 10^{-10}$

n_3 = " " " " " " $e = 4{\cdot}774 \times 10^{-10}$

The extreme values of E in e. s. u. are also given in the tables. They are the mean values of consecutive measurements when more than one observations were made. Finally the column *d* gives the number of days on which observations were made.

I. Mount Abu.

24°36′ N.; 72°42′ E.;

Elevation 4,000 ft. above sea-level, 3000 ft. above the plain; Rock—Archæan gneiss.

The high plateau of Mt. Abu (12 miles in length and from 2 to 3 miles in breadth) is completely detached from the Aravalli range by a valley 7 miles across. To the E. of the plateau are the Aravalli mountains, to the N., W., and S. the sandy plain of Rajputana. The rock is a highly felspathic, massive and crystalline gneiss. Sandstorms are of frequent occurrence and the plains at the foot of the mountains are covered with a dense veil of dust. There is, as a rule, a great amount of very fine dust in the air, particularly so after a sandstorm. So it was at least when I was there.

1st Series.

(2-6 to 8-6-17)

The observations were made in a small room 3·12 by 3·56 ms.; the height of the room was 2·88 and 4·17 ms. (slanting ceiling). The room was a corner-room with windows in two walls and doors in the two other walls. The doors and the windows were kept open so that the wind could blow in directly through the two windows upon the instrument, placed in the middle of the room. The walls and the floor were of bricks made locally. The whole building was some 50 years old. Of the long series of observations the mean values of E for the last 7 days only are given. The values of E on the other days was of the same orders of magnitude. Two sets of measurements for $_{+}E$ and $_{-}E$ were made in the morning, one between 8 and 9, the other between 11 and 12 noon. In the afternoon observations were made between 3 and 4 o'clock.

TABLE I.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min. e. s. u.	d.
$_{+}E$ a. m.	2·165	6368	4656	4535	2·897	1·279	7
$_{+}E$ p. m.	2·539	7468	5460	5318	3·242	1·971	7
$_{-}E$ a. m.	2·116	6224	4550	4432	2·983	1·307	7
$_{-}E$ p. m.	2·463	7244	5297	5159	2·678	1·634	7

$$E_{a.m.} < E_{p.m.} ;\quad {}_{+}E > {}_{-}E.$$

2nd Series.

(10-6 to 15-6-17.)

The observations were made in the open air, away from the house, in the shade of a tree. As a rule 3 sets of observations were made for $_{+}E$ and $_{-}E$ in the morning, and as many in the afternoon.

TABLE II.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min. e. s. u.	d.
$_{+}E$ a. m.	1·257	3697	2703	2633	1·378	1·071	6
$_{+}E$ p. m.	1·402	4124	3015	2937	1·576	1·232	6
-E a. m.	1·241	3650	2669	2598	1·391	0·944	6
$_{-}E$ p. m.	1·412	4153	3037	2958	1·553	1·185	6

$$E_{a.m.} < E_{p.m.}; \quad {}_{+}E = {}_{-}E \text{ nearly.}$$

The great difference between the values of E found inside the room and those found outside, in the open air, seems to be due to a powerful radiation from the old brick walls. It was certainly not due to a change in the electrification of the air outside, because a series of observations made on the same day in which readings were taken in the room and immediately afterwards outside the room in the open air showed the same difference. The mean velocity of the ions measured by Mache's method was practically the same in the room and in the open air. Also the conductivity of the air measured by the rate of discharge of a charged insulated conductor was found to be considerably greater in the room with all the windows shut than in the open air.

The very high values of E found at Mt. Abu will be considered later.

In all the other places observations were made, as a rule, once only in the morning between 9 and 10 and once in the afternoon between 4 and 5 o'clock.

II. Bombay.

18°53′49″ N.; 72°48′56″ E.;
Elevation sea-level.
Rock—Deccan Trap (basalt).

Bombay is not a very favourable place for this kind of work on account of the great amount of smoke and dust always present in the air.

1st Series.

(14-10-17 to 10-1-18).

The observations were made on a flat roof 25 ms. above the ground. In the morning the observations were all taken in the open air, in the afternoon under shelter, the air which passed through the instrument being sucked from outside through a glass-tube 1·5 ms. long and 3·1 cms. in inner diameter, so that no air from the shelter passed through the instrument. The glass-tube had been washed with hydrochloric acid.

TABLE III.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min e. s. u.	d.
$_+E$ a. m.	0·415	1221	892	869	0 610	0·306	18
$_+E$ p. m.	0·363	1068	781	760	0·506	0·264	31
$_-E$ a. m.	0·279	821	600	584	0·442	0·159	18
$_-E$ p. m.	0 289	850	621	605	0·448	0·160	31

$_+E_{a.m.} > {}_+E_{p.m.}$; $_-E_{a.m.} < {}_-E_{p.m.}$; $_+E > {}_-E$.

2nd Series.

(11-1 to 25-2-18).

The observations were made on the same flat roof The glass-tube was discarded and the instrument also in the afternoon placed in the open air in the shade of a vertical canvas-screen.

TABLE IV.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min. e. s. u.	d.
$_+E$ a. m.	0·495	1456	1065	1087	0·697	0·350	8
$_+E$ p. m.	0·435	1279	935	911	0·592	0·327	34
$_-E$ a. m.	0·348	1024	748	729	0·552	0·229	8
$_-E$ p. m.	0·309	909	665	647	0·531	0·108	34

$E_{a.m.} > E_{p.m.}$; $_+E > {}_-E$.

3rd Series.

(7-4 to 21-4-18).

The observations were made under the same conditions as those of the preceding series.

TABLE V.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min. e. s, u.	d.
$_+E$ a. m.	0·430	1265	925	901	0·538	0·325	6
$_+E$ p. m.	0·367	1079	789	769	0·430	0·297	7
$_-E$ a. m.	0·333	979	716	698	0·454	0·269	6
$_-E$ p. m.	0·347	1021	746	727	0·508	0·229	7

$E_{a.m.} > E_{p.m.}$; $_+E_{a.m.} > {}_+E_{p.m.}$; $_-E_{a.m.} < {}_-E_{p.m.}$.

The values of $E_{p.m.}$ at the end of the 1st series (air sucked through a glass-tube) and these at the beginning of the 2nd series (without glass-tube) show no difference, although there is a difference in the mean values from the two series. But there is a similar difference between the mean values of $E_{a.m.}$ from the two series although all the observations in the morning were made in the open air. From this it seems to follow that the method used in the 1st series may be followed without fear. A similar method has been used by Langevin and Moulin[1] in Paris. They used metal tubes. In a foot-note they state that the results are not appreciably affected by this arrangement, provided that the time of passage of the air through the tube is sufficiently short.

The influence of the humidity of the air on E is here in Bombay obliterated by other uncontrolable factors.

When plotting the individual values of +E and -E against time, I found that the changes of +E are less sudden than those of -E. This indicates that +E is less affected by the condifions of the air (humidity, smoke, dust,) than -E.

When comparing $E_{a.m.}$ and $E_{p.m.}$ we find:

$$+E_{a.m.} > +E_{p.m.}$$

or

$$-E_{a.m.} > -E_{p.m.} \qquad -E_{a.m.} = -E_{p.m.} \text{ nearly.}$$

$$(+E_{a.m.})_{max.} > (+E_{p.m.})_{max.}$$

(1) *C.R.*, vol. 140, pp. 305, 1905.

or

$$\left({}_{-}E_{a.\,m.}\right)_{max.} > \left({}_{-}E_{p.\,m.}\right)_{max.}$$

$$\left({}_{-}E_{a.\,m.}\right)_{max.} = \left({}_{-}E_{p.\,m.}\right)_{max.} \text{ nearly.}$$

$$\left({}_{+}E_{a.\,m.}\right)_{min.} > \left({}_{+}E_{p.\,m.}\right)_{min.}$$

$$\left({}_{-}E_{a.\,m.}\right)_{min.} > \left({}_{-}E_{p.\,m.}\right)_{min.}$$

or

$$\left({}_{-}E_{a.\,m.}\right)_{min.} = \left({}_{-}E_{p.\,m.}\right)_{min.}$$

i.e. $E_{a.\,m.} > E_{p.\,m.}$ with the one exception

$\left({}_{-}E_{a.\,m.}\right)_{max.} < \left({}_{-}E_{p.\,m.}\right)_{max.}$ in Table *V*.

These relations exist inspite of the greater humidity and the much greater amount of smoke in the air in the morning. They show that the electrification of the air in Bombay is greater in the morning than in the afternoon. The difference seems to me to be due to the fact that in the morning the wind blows from the land-side, in the afternoon it comes from the sea.

III. Khandala.

18°45′ N.; 73°25′E

Elevation 1790 ft.;

Rock—Deccan Trap (basalt).

Distance from the sea over 30 miles.

Khandala is not a very favourable place for this kind of work. The numerous trains that come up the Ghats produce much smoke. Moreover the villagers, as everywhere in India, often light fires. I have tried, and I think with a fair amount of success, to choose time and place of observation so as to escape the smoke. Whenever there was danger of smoke, I stopped and moved the instrument to a safer place.

St. Xavier's Sanatarium is situated N. W. of the village of Khandala, about half a mile distant, on the edge of the plateau. To the N. W. there is a deep ravine and the Reversing Station of the G. I. P. Ry. From St. Xavier's Sanatarium a good view can be had of the thick atmosphere in the plain below.

St. Mary's Sanatarium is not far from the village and to the N. of it. To the N. of the Sanatarium there is a deep ravine and to the N W. the G. I. P. Ry.

1st Series.

(28-10 to 14-11-17)

The observations were made in the compound of St. Xavier's Sanatarium,

TABLE VI.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min. e. s. u.	d.
$_+E$ a. m.	0·435	1279	935	911	0·561	0·297	15
$_+E$ p. m.	0·452	1329	972	947	0·611	0·312	15
$_-E$ a. m.	0·457	1344	983	957	0·570	0·374	15
$_-E$ p. m.	0·463	1362	996	970	0·601	0·307	15

$E_{a.\,m.} < E_{p.\,m.}$; $_+E < {_-E}$ slightly.

It is puzzling to find in this series that -E is so often greater than +E. I have no explanation to offer, except perhaps the fact that the observations were made soon after the monsoon and that the rapid changes of E were very great at times.

2nd Series.

(28-2 to 11-3-18)

The observations were made in the compound of St. Mary's Sanatarium.

TABLE VII.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min. e. s. u.	d.
$_+E$ a. m.	0·467	1374	1004	978	0·608	0·344	12
$_+E$ p. m.	0·408	1200	877	855	0·605	0·296	12
$_-E$ a. m.	0·358	1053	775	750	0·440	0·291	12
$_-E$ p. m.	0·355	1044	763	744	0·583	0·217	12

$$E_{a.m.} > E_{p.m.}; \quad {}_+E > {}_-E;$$

3rd Series.

(27-4 to 4-5-18)

The observations were made in the compound of St. Xavier's Sanatarium.

TABLE VIII.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min. e. s. u.	d.
$_+E_{a.m.}$	0·415	1211	892	869	0·515	0·289	7
$_+E_{p.m.}$	0·277	815	596	580	0·376	0·231	8
$_-E_{a.m.}$	0·342	1006	735	716	0·404	0·239	7
$_-E_{p.m.}$	0·232	682	499	486	0·273	0·175	8

$$E_{a.m.} > E_{p.m.}\ ; \quad {}_+E > {}_-E\ .$$

Here we find $E_{a.m.}$ much greater than $E_{p.m.}$. This may be explained by the fact that the peasants were already burning their fields to prepare them for sowing. Many fires could be seen on the horizon. The atmosphere in the afternoon contained more dust and looked much thicker than in the morning.

4th Series.

(22-5 to 25-5-18)

The observations were made in the compound of St. Xavier's Sanatarium. There had been heavy rain at Khandala about the middle of May. Observations could not be taken in the afternoon.

TABLE IX.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min. e. s. u.	d.
$_+E$ a. m.	0·541	1591	1163	1133	0·625	0·440	3
$_-E$ p. m.	0·461	1356	991	966	0·571	0·292	3

$$_+E > {}_-E\,.$$

Comparison of the values of E *found in Bombay and at Khandala.*

We omit the values of E found at Khandala about the end of May 1918 and given in Table IX. The series is too short. The observations made at Khandala and in Bombay extend over the same time of the year, so that we may be justified when forming the mean values of E for each set separately. We find in this way :

	Bombay.	Khandala.
$_+E$ a. m.	0·447	0·439
$_+E$ p. m.	0·388	0·369 (0·430)
$_-E$ a. m.	0·320	0·386
$_-E$ p. m.	0·315	0·350 (0·409)

$$_+E_B = {}_+E_K \text{ nearly}$$

$$_-E_B < {}_-E_K$$

When we omit the obviously abnormal values of $E_{p.m}$ found in the 3rd series at Khandala and given in Table VIII we obtain for $E_{p.m.}$ at Khandala the values added between brackets. Then we get the following relations :

$$(1) \quad \left({}_{+}E_{a.\,m.} \right)_B = \left({}_{+}E_{a.\,m.} \right)_K \text{ nearly}$$

$$(2) \quad \left({}_{+}E_{p.\,m.} \right)_B < \left({}_{+}E_{p.\,m.} \right)_K$$

$$(3) \quad {}_{-}E_B < {}_{-}E_K .$$

These inequalities suggest the following conclusions :

The influence of smoke and dust is greater on — ions than on + ions. This seems to follow from (1) and (3) as there is more smoke and dust in Bombay than at Khandala.

The ionisation of the air coming from the sea is less than that of the air which passes over the the land. This follows from (1) and (2). At Khandala the air always passes over the land, in Bombay it comes from the sea in the afternoon.

The difference between the elevations of Bombay and Khandala (over 1700 ft.) has no influence on the ionisation. This suggests itself from (1). The rock is the same in the two places.

IV. *Tumrikop.*

15°26′ N. ; 75°7′ E. ;

Elevation over 2000 ft.

Geological. Formation : Dharwars (old sediments) surrounded by Archaean gneiss.

Distance from the sea over 60 miles.

Tumrikop is a small village in an undulating country about 17 miles S.S.W. of Dharwar. The geographical co-ordinates given here are those of Dharwar. My place of observation was about a quarter of a mile from the village. The air was very clear for Indian conditions. The early break of the monsoon put an end to my work. The observations began on the 8th May and ended on the 17th May 1918.

TABLE X.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min. e. s. u.	d.
$_+E$ a. m.	0·631	1856	1357	1322	0·782	0·428	6
$_+E$ p. m.	0·532	1565	1144	1114	0·667	0·357	9
$_-E$ a. m.	0·521	1532	1120	1091	0·665	0·348	6
$_-E$ p. m.	0·565	1662	1215	1184	0·682	0·449	10

$$_+E_{a.m.} > {}_+E_{p.m.}\,; \quad {}_-E_{a.m.} < {}_-E_{p.m.}\,;$$

$$_+E_{a.m.} > {}_-E_{a.m.}\,; \quad {}_+E_{p.m.} < {}_-E_{p.m.}\,.$$

The sky was often heavily clouded and then as a rule the difference between $_+E$ and $_-E$ great. Sometimes $_+E > {}_-E$ at other times $_+E < {}_-E$. The amplitudes of the variations of E can hardly be called very abnormal, so that we may assume that inspite of the approaching monsoon the mean values of E give us a fair idea of the ionisation of the air at Tumrikop.

Comparison of the values of E *found at Khandala and at Tumrikop.*

The observations of series 3 and 4 at Khandala were made for the sake of comparison with the observations at

Tumrikop. We must however leave out of consideration the obviously abnormal values of $E_{p.m.}$ in the 3rd series. We obtain the following results :

	Khandala. 27·4—1·5.	Tumrikop. 8·5—17·5.	Khandala. 22·5—25·5.
$_{+}E_{a.m.}$	0·415	0·631	0·541
$_{-}E_{a.m.}$	0·342	0·521	0·461

A glance at this table shows that the values of E are distinctly higher at Tumrikop than at Khandala. The conditions of the atmosphere were not more disturbed at Tumrikop than at Khandala at the end of May.

V. *Anand.*

22°34′ N., 73°1′ E. ;

Elevation—135 ft. ;

Soil—Alluvium (Pleistocene).

Distance from the sea—20 miles.

There was very little wind all the time and its direction often unsteady. The air was never quite free from smoke and dust. It seemed to me to be purer in the afternoons than in the mornings. Observations were made from the 11th to the 23rd October, 1918.

	Mean e. s. u.	n_1	n_2	n_3	Max. e. s. u.	Min. e. s. u.	d.
$_{+}E_{a.m.}$	0·465	1368	1000	974	0·597	0·346	12
$_{+}E_{p.m.}$	0·636	1871	1368	1332	0·771	0·522	12
$_{-}E_{a.m.}$	0·443	1303	953	928	0·548	0·312	12
$_{-}E_{p.m.}$	0·536	1575	1153	1123	0·637	0·412	12

$$E_{a.m.} < E_{p.m.}\,; \quad {}_{+}E > {}_{-}E\,.$$

Comparison of the values of E *found at Anand and at Khandala.*

	$_{+}E_{a.m.}$	$_{-}E_{a.m.}$	$_{+}E_{p.m.}$	$_{-}E_{p.m.}$
Anand. 11-10—23-10-18.	0·465	0·443	0·636	0·536
Khandala.				
28-10—14-11-17	0·435	0·457	0·452	0·463
28-2—11-3-18	0·467	0·358	0·408	0·355
27-4—4-5-18	0·415	0·342	0·277	0·232

It appears from this table that $E_{a.m.}$ is much the same in the two places, and that $E_{p.m.}$ is distinctly greater at Anand than at Khandala. This is so, also when we omit the abnormal values of $E_{p.m.}$ in series 3 at Khandala.

Connection between E *and the nature of the rock.*

We consider only +E which is less affected by impurities of the air than -E and form the means of the mean values in the different places. We omit the observations made in Bombay and the obviously abnormal values of $E_{p.m}$ in Table VIII. We thus get the following result:

Locality.	Rock or Soil.	+E
Mt. Abu ...	Archaean gneiss	1·32
Khandala ...	Volcanic rock—Deccan Trap (basalt) ...	0·44
Tumrikop ...	Old Sediments—Dharwars	0·58
Anand ...	Alluvium (Pleistocene)	0·55

The great difference between the value of +E found at Mt. Abu and those found in the other localities may be attributed to the nature of the rock, which seems to be rich in radio-active substances. Samples of water from various springs and wells which I examined showed an unusually high radio-activity.

The difference between the value of +E found at Tumrikop and that found at Khandala is sufficiently well marked and it is quite possible that part of the difference is due to the different nature of the rock in the two places. The narrow belt of Dharwars in which Tumrikop is situated is surrounded by Archaean gneiss. But considering that the air at Tumrikop was purer than at Khandala it is also possible that the rock has little or nothing to do with the higher values of +E found at Tumrikop.

The mean value of $+E$ found at Anand is practically the same as that found at Tumrikop. The air is more polluted at Anand than at Tumrikop, but less so than at Khandala. It is hard to say whether the difference between the values of $+E$ found at Anand and at Khandala is due to the difference in the soil, considering that the value of $+E_{a.m.}$ is practically the same in the two places. The difference is marked only in the afternoon when the air at Anand is purer.

It has been pointed out by many observers that the ionisation of the air is often greater on mountains than in the plains[1]. It would be intersting to examine whether this may be due to the different nature of the rock on high mountains.

Rapid Changes of E.

When making two measurements for ions of the same sign one immediately after the other, the two values obtained very often are very different from one another. I have found these changes wherever I looked for them. As an instance I give the extreme values of these changes observed at Anand in October 1918.

	E_1	E_2	E_1-E_2	
$+E$ a. m.	0·580	0·420	0·160	**Max.**
$+E$ a. m.	0·511	0·509	0·002	**Min.**
$+E$ p. m.	0·950	0·591	0·359	**Max.**
$+E$ p. m.	0·524	0·520	0·004	**Min.**
$-E$ a. m.	0·526	0·400	0·126	**Max.**
$-E$ a. m.	0·481	0·476	0·005	**Min.**
$-E$ p. m.	0·744	0·496	0·248	**Max.**
$-E$ p. m.	0·505	0·495	0·010	**Min.**

[1] Gockel, Luftelektrizität, pp. 33, 34.

Rapid changes of E have also been found by other observers[2]. They do not necessarily imply real changes of E, although such changes may exist. The air is in constant motion. Therefore the air which is examined in two consecutive measurements is not the same air. There is nothing which forces us to assume an even distribution of the ions in the air. Also smoke, dust, &c., are not distributed quite evenly in the air. We may compare the distribution of ions with a fog which is also of varying density. This explanation will also account for the fact that simultaneous observations with similar instruments placed apart a distance of only 10 metres occasionally yield different results[3].

These rapid changes are, no doubt, also partly due to the uneven distribution of impurities (smoke, dust, &c.) in the air, which change small ions into large ones which are not caught by the instrument and so give no sign of their presence.

Results.

1. The ionisation of the air has been measured with an Ebert's ion-counter in 5 different places.

2. In general $+E > -E$ or $+E = -E$ nearly; only exceptionally $+E < -E$.

3. In certain places and at certain times in the same place $E_{a.m.} > -E_{p.m.}$; in other places $E_{a.m.} < E_{p.m.}$ The inequality depends much on the state of the air as to smoke, dust, humidity and on the direction of the wind in the morning and in the afternoon.

4. $-E$ is more affected by impurities of the air than $+E$.

5. Great and rapid changes of E were observed.

6. No influence of the difference in elevation on E has been observed in Bombay and at Khandala when the

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[2] C. R. vol. 140, p. 305, 1905.

[3] Gockel, l. c.

wind in Bombay comes from the land-side. The rock is the same in the two places.

7. In Bombay the air coming from the land-side seems to carry a higher electric charge than that coming from the sea-side.

8. In one place at least (Mt. Abu) the nature of the rock seems to have a great influence on the ionisation of the air. It is doubtful whether the differences of E observed in the other places are due to the difference of the rocks or to other causes.

9. The ionisation of the air in a small well-ventilated room has been found to be far greater than that of the air outside the house.

Forced Vibrations Experimentally Illustrated.

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&

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§ 1. *Introduction.*

It is well known that forced vibrations play an important part in most branches of physics. We may mention in this connexion : resonance tubes, fluorescence, Lodge's syntonic jars, Hertz's oscillator and resonator, wireless telegraphy. Possibly we might be justified in adding to this list the sensitive parts of the ear and eye.

It accordingly seems desirable to have some quite simple vivid mechanical illustration that exhibits qualitatively and quantitatively the chief phenomena concerned. Types of such experiments are here described.

The apparatus consists of a single heavy driving pendulum and a number of light driven ones, of graduated lengths all suspended from the same tightly stretched cord. Thus the various effects of tuning and mistuning may be observed simultaneously. By a steady view the variation of amplitude with tuning is seen in spite of the phase differences involved. By a stroboscopic view (or illumination) the variation of phase with tuning is exhibited to a small class (or a larger audience).

The above remarks refer to the experiment in various forms. For confirmation of exact quantitative relations the experiment needs arranging with special attention to certain details. It is then found to confirm the theory in every respect.

By changing to responding bobs of greater density the increased sharpness of resonance with smaller damping is shown.

Photographic reproductions are given showing four time-exposures and eight instantaneous views of the responding pendulums. These exhibit all the features enumerated above.

§ 2. *General Theory.*

The equation of a single particle of mass m with restoring force s times its displacement y and r times its velocity under the action of a sustained harmonic impressed force, may be written

$$m\frac{d^2y}{dt^2}+r\frac{dy}{dt}+sy=F\sin nt, \quad \ldots \quad \ldots \quad (1)$$

or

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+p^2y=f\sin nt, \quad \ldots \quad \ldots \quad (2)$$

where

$$2k=\frac{r}{m},\ p^2=\frac{s}{m},\ \text{and } f=\frac{F}{m}. \quad \ldots \quad \ldots \quad (3)$$

The solution of this may be written

$$y=\frac{f\sin(nt-\delta)}{\sqrt{\{(p^2-n^2)^2+(2kn)^2\}}}+Ee^{-kt}\sin(qt+\epsilon), \ldots \quad \ldots \quad (4)$$

or,

$$\left\{\begin{matrix}\text{Resultant}\\\text{Displacement}\end{matrix}\right\}=(\text{Forced Vibration})+(\text{Free Vibration}).$$

In equation (4)

$$\tan\delta=\frac{2kn}{p^2-n^2},\ q^2=p^2-k^2, \quad \ldots \quad \ldots \quad (5)$$

and E and ϵ are arbitrary constants to be chosen to fit the initial conditions.

As indicated, the first term on the right side of (4) represents the forced vibration with which we are here chiefly concerned. The second term denotes the free vibration of the system, and this must be present to complete the solution. If the responding particles were at rest in the zero position when the impressed force was started, then the values of the E and ϵ would have to be such as to express a free vibration which would annul both displacement and velocity as given by the forced vibration, whose amplitude and phase have nothing arbitrary.

If the forced and free vibrations co-exist of differing periods and comparable amplitudes, beats will occur between them. These are easily obtained but are usually best avoided.

When, in virtue of the damping factor involving k, the free vibration has practically disappeared, the forced vibration is left in possession of the field. No beats are then possible. While the free vibration is dying away, the resultant motion which is under observation grows from nothing to the fixed amplitude and phase of the forced vibration.

Considering now the forced vibration itself, we may note, from the first term on the right side of equation (4), the following points.

1. The period of the forced vibration is identical with that of the impressed forces whatever the period natural to the responding system.

2. The best response occurs for the best tuning. This is a brief statement which may convey the right idea with sufficient accuracy for our present purpose. To make the statement precise we must define best as applied both to response and

to tuning. This has already been done by one of the present writers in "Range and Sharpness of Resonance, &c." (Phil. Mag, July, 1913).

3. The phase of the forced vibration varies continuously between o and π with the tuning. Thus the phase angle δ is almost zero for p^2 much greater than n^2, *i. e.*, for a responding system whose natural frequency is much greater than that of the impressed force. On the other hand, δ is almost π for p^2 much less than n^2, *i. e.*, for a responding system of natural frequency much less than that of the impressed force. Finally, for $p^2=n^2$, $\delta=\pi/2$, and this corresponds with the case of maximum amplitude of response.

4. The smaller the damping of the responding system the sharper is its resonance, the greater the damping the greater is its range of resonance. That is to say, the smaller the value of k the greater is the falling off of the response for a given mistuning, and *vise versa*. For it is seen from the first term on the right side of equation (4) that when $p^2=n^2$ the amplitude is a maximum, for n constant while p varies. Further, when p^2-n^2 is finite and of a given value it has a less effect on the amplitude, if the other term in the denominator ($2kn^2$) is large.

By reference to the second term on the right side of equation (4) we see that the ratio of successive amplitudes of the free vibrations is $e^{k\pi/q}=e^{k\pi/p}$ nearly. But the logarithmic increment λ (per half wave) for this system is the logarithm to the base e of this ratio. Hence we have

$$\lambda=\frac{k\pi}{p} \text{ or } k=\frac{p\lambda}{\pi}=\frac{n\lambda_0}{\pi}, \quad \ldots \quad \ldots \quad (6)$$

where $\lambda_0=$ the log. dec. for the responding pendulum of the same period as the forces. Thus by observations on the free vtbrations of a responding pendulum the value of k may be found.

It might be urged that in the experimental arrangement specified we have strictly speaking an instance of coupled vibrations and have not reached the ideal of forced vibrations. That this is not the case may be ascertained as follows.

On reference to "Coupled vibrations, II." (Phil. Mag. Jan. 1918) we see that in coupled systems two superposed vibrations occur, the ratio of their frequencies being $p/q=\surd(1+\beta)$. Also by equation (24) p. 65 and (43*a*) p. 68 of the same paper, we see that the ratio of the amplitudes of these quick and slow vibrations for our responding systems is given by

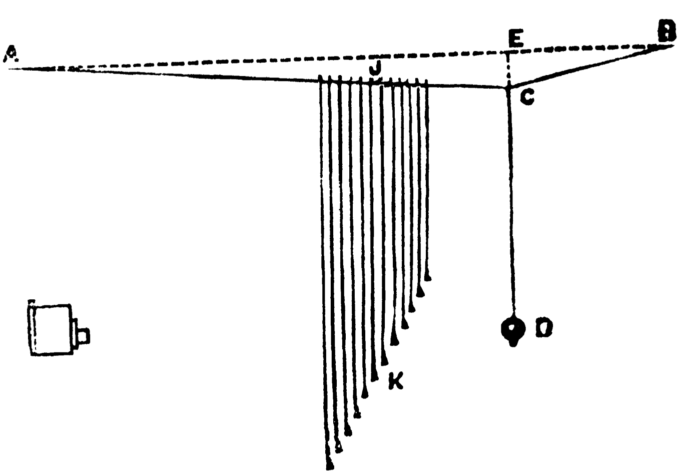
$$\frac{-\rho}{(1+\rho)\beta}e^{-(\rho-1)kt/(1+\rho)}=\frac{e^{-kt}}{\beta} \text{ nearly for } \rho \text{ large.} \quad \ldots \quad (7)$$

In our experimental case ρ exceeds 2000 (being 700 gm. /0.3 gm. k is of the order one fifth, and β about one third. Thus after 20 and 40 seconds, the ratio in question has fallen to 1/20 and 1/1000 respectively.

§ 3. *Illustrative Experiments.*

In fig. 1 is shown an experimental arrangement that was found convenient when exact quantitative work was not the aim but rather a lecture demonstration of the general features involved was required.

Fig. 1.



Forced Vibration Apparatus with Differing Forces.

The tightly stretched cord A B is drawn down to a peak at C by the weight of the driving pendulum CD about 60 cm. long with bob of iron about 6 cm. diameter. The responding systems are pendulums of graduated lengths and with very light bobs so that their free vibrations are quickly damped. These pendulums should be placed fairly near to the driver and the point A kept far from them, so that their points of suspension all have approximately the same motion from the vibrations of the heavy bob D. The bobs of these responding pendulums may be—

(*a*) of solid cork about 2·3 cm. long, 1·2 cm. diameter, and 0·4 gm. mass.

(*b*) of hollow paper cones, semi-vertical angle 20^0, of mass 0·2 gm.

(*c*) of paper cones, semi-vertical angle 45^0, of mass 0·3 gm.

(*d*) of blown-glass spheres in imitation of pearls, diameter 6 mm.

The attachments to the cord AC may be made by passing the cotton suspension through it with a needle and leaving the end free. They are then sufficiently held by friction and may be adjusted, at will, by simple pulling. The paper cones may each have a little soft wax inside and the cotton suspension passed through by a needle and the end left free. The cone may then be slid up and down the cotton at pleasure to adjust in line with the others, and will stay where left.

The white paper cones are much easier seen (or photographed) stroboscopically than the corks, and are on the whole more satisfactory than the corks or the blown-glass spheres.

Since the phase of the forced vibration varies with the natural period and therefore with the length of the responding pendulum, the full displacements are not attained simultaneously. But on viewing the apparatus steadily from one end, A say, the full displacements are seen to be reached successively. Hence one may see a resonance curve in which the squares of the various periods or lengths of the pendulums are disposed vertically while the corresponding amplitudes exhibit themselves horizontally. Thus the limits to which the light bobs swing on each side form there a resonance curve in which the squares of the periods are the vertical abscissæ and the amplitudes are the horizontal ordinates. Thus a time exposure will give a photograph exhibiting this resonance curve in duplicate to right and left of the central line. The effect is shown for various types of responding pendulums in figs. 1, 2, and 3 of Plate I. It is seen that the blunt cones (fig. 1) give curves showing the sharpest resonance, the small blown-glass spheres (fig. 3) give the greatest range of reasonance, and the sharp cones (fig. 2) show an intermediate type of resonance. This is in accordance with theory, since the values of k for these three kinds of bob (as found from their logarithmic decrements when vibrating alone) are 0·16, 0·265, and 0·2 respectively.

In order to appreciate the various phases of the vibrating systems of differing periods an instantaneous view of the bobs is needed. The motion is so slow that it seemed quite unnecessary to make any elaborate electric timing arrangement. At first the camera was instantaneously exposed 40 times at the desired instant as judged by sight, and this gave the result reproduced in fig. 4, Plate I. Better results shown in figs. 5 and 6 were obtained by the ordinary flash-light process. One of these (fig. 5) corresponds to the central position of the driver, and exhibits what may be called an exaggerated resonance curve. This is because when the driver is at the centre, the driven bob of about the same length and having maximum response is then at one end of its swing and therefore shows its full amplitude. But as we pass to pendulums shorter or longer than this one, we gradually change to like phase with the driver or opposite phase respectively. Hence the horizontal ordinates of the curve rapidly diminish from their maximum both because the amplitude is less and the phase is not right to exhibit it fully. The Comparison of fig. 5 with fig. 2 makes this point clearer.

Special interest attaches to the instantaeous view shown in fig. 6, and taken when the driving bob was at one end of its swing. The responding bob of about the same length as the driver is then at the centre, the much shorter responders are nearly in phase with the driver, the much longer ones in the opposite phase nearly. The resulting curve may be approximately represented by

$$y = \frac{\pm x}{a^2 + x^2}.$$

Other powers of x would be needed to represent more precisely the exact curve for any given arrangement of the experiment. This will be dealt with later.

To exhibit these instantaneous effects to a single observer stroboscopic vision is desirable. This was easily arranged by using a card with a vertical slit in its centre, each end of the card being carried by a pendulum. The period of this pendulum should bear a simple relation to that of the driver. In the actual experiments it was made of four times the length of the driver, as that suited the position of a purlin in the roof. The moving slit at the middle of its swing passes a slit of the same size in a fixed card. The period of coincidence of these slits can be shortened at will by increasing the amplitude of the pendulums carrying the moving card. For about six observers we may use a camera and focussing screen instead of a fixed slit. For a larger audience the same arrangement of fixed and moving cards may be used as for a single observer, but the light from an arc-lamp should be passed through the slits on to the bobs while the room is otherwise in darkness.

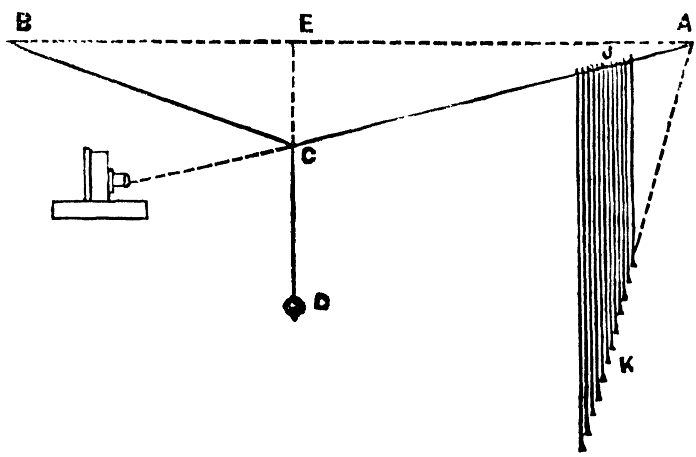
§ 4. *Detailed Theory.*

Let us now pass from general ideas as illustrated by the apparatus in fig. 1 to an experimental arrangement more suitable for a strict quantitative examination of the phenomena involved. Referring to equations (1) to (3) we see that in the set of responding pendulums we naturally keep m and r constant throughout the series but vary the natural periods. For quantitative work it is also desirable to keep f constant throughout.

Now the period depends upon $s = mp^2 = mg/x$ where x is the length JK of the pendulum in question. Further, $F = mg \times$ (inclination of JK) due to full amplitude of heavy driving bob D.

Now this inclination of JK (to the vertical) is the displacement of J divided by the length $JK = x$. Hence to keep F and f of same value for all the responding pendulums we must have their inclinations equal for a given displacement of D. And this is obviously obtained by arranging the bobs so that the straight line AK passes through them all as shown

Fig. 2.



Forced vibration Apparatus with Equal Forces.

in fig. 2. For when the pendulum length is halved the displacement of the point of suspension is halved also, and thus the inclination retains the same value.

Again, to have on the photographic plate coincidence of all the points J we must have the camera-lens in the line AJC when all is at rest. This coincidence is desirable so that the length x for each pendulum shall reckon from a definite invariable origin. Further, to have the displacements of the bobs K measured from the same vertical line on the plate, we must have the centre of the camera in the vertical plane through ABCD when at rest. But this has already been secured.

Finally, to avoid unequal treatment of the displacements of the various bobs K, their distances from the camera must be nearly equal. Hence they should be set well away from he camera but as close together as will avoid entanglement.

As regards the length JK for the best tuning with DE, it should be noticed that no equality will be apparent on the photographs. First, because E is not shown at all, and second, because the length DC which is shown is greatly magnified relatively to the lengths JK. To confirm the theory in this respect actual measurements of these lengths should be made on the apparatus itself.

Consider the time after the dying away of the free vibrations. Then equation (4) has reduced to

$$y=\frac{l \sin (nt-\delta)}{\sqrt{\{(p^2-n^2)^2+(2kn)^2\}}}, \quad \ldots \quad \ldots \quad (8)$$

which expresses the forced vibration only.

***Case* I.** Take first the variation of amplitude y, of the forced vibration with frequency natural to the responding system. We have already from (6), $k=n\lambda_0/\pi$, let us now write

$$p^2=g/x, \quad n^2=g/l, \quad \text{then } kn=g\lambda_0/\pi l. \quad \ldots \quad (9)$$

And (9) in (8) leads to

$$y_1^2 = \frac{\pi^2 f^2 l^2 x^2}{g^2\{\pi^2(l-x)^2+4\lambda_0^2 x^2\}} \quad \ldots \quad \ldots \quad (10).$$

Case II. For the second case take the instant when the heavy bob D is undisplaced but is moving in the positive direction. Then we may write sin $nt=0$ and cos $nt=1$. Inserting these in (8) we have

$$y_2 = \frac{-f\sin\delta}{\sqrt{\{(p^2-n^2)^2+(2kn)^2}} = \frac{-f(2kn)}{(p^2-n^2)^2+(2kn)^2}. \quad \ldots \quad (11)$$

Then using (9), (11) becomes

$$y^2 = \frac{-2\pi f\lambda_0 l x^2}{g\{\pi^2(l-x)^2+4\lambda_0^2 x^2\}}, \quad \ldots \quad \ldots \quad (12)$$

Case III. Consider next the instant when the heavy bob D has its maximum displacement in the positive direction. Then we may write cos $nt=0$, and sin $nt=1$. Substituting these in (8) we have

$$y_3 = \frac{f\cos\delta}{\sqrt{\{(p^2-n^2)^2+(2kn)^2}} = \frac{f(p^2-n^2)}{(p^2-n^2)^2+(2kn)^2}. \quad \ldots \quad (13)$$

Inserting the values of p^2, n^2, and kn given in (9) equation (13) becomes

$$y_3 = \frac{\pi^2 f l(l-x)x}{g\{\pi^2(l-x)^2+4\lambda_0^2 x^2\}}. \quad \ldots \quad \ldots \quad (14)$$

§ 5. *Experimental Results.*

The photographs shown in figs. 7-12, Pl. II., were taken with the apparatus arranged as in fig. 2 so as to keep the value of f due to the big bob the same for each r pendulum. One kind of light bob only was used, viz., the sharp-angled paper cones. The curves obtained on the plates differed so little from those used in the first arrangement that it seemed unnecessary to repeat experiments with bobs having different dampings. Fig. 8 represents the resonance curve in duplicate to the right and left of the central line, and was obtained by a time exposure. The maximum swing of the lower cones is seen to be greater than that of the upper cones. This is because the vertical abscissæ are lengths as x in (10) and not the squares of the frequencies as p^2 in (8). These curves agree with equation (10).

Figs. 7, 9, 10, 11, 12, show instantaneous views taken by flash-powder. Fig. 7 shows the state when the heavy bob was passing the centre towards the right. The figure shows the bob slightly beyond the centre, but this is a small fraction of the amplitude and involves a still smaller fraction of the quarter period. Then it is well seen from the curve (*a*) that the upper responding bobs are in phase with the driving bob and therefore at the middle of their swing towards the right, (*b*) that the lower ones are also at the middle of their swing though they are in opposite phase and moving to the left, and (*c*) the middle bobs are at the end of their swing with a lag of about 90° phase angle behind the driver.

Fig. 9 shows the curve obtained with the large bob at the end of its swing to the right. It will be noticed that the upper bobs are less displaced from the centre than the lower ones. This asymmetry was to be expected from the form of equation (14), with which it is in entire accord.

Figs. 10-12 are intermediate stages with the large bob partly displaced. They show the gradual melting of the curve from the case of exaggerated resonance with the bobs all on one side, fig. 7, to the state of fig. 9 with half the bobs on each side. The set of figures 7, 10, 11, 12, 9 correspond to intervals of about the tenth of a second in the motions of the actual pendulums.

Nottingham,

May 28, 1918.

Mechanical "Resonators" under Double Forcing.

By

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&

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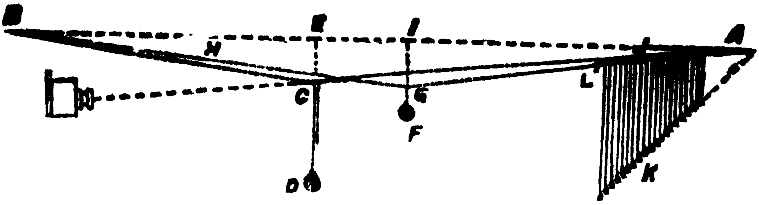
Following up the idea of a recent paper on Forced Vibrations, it seemed desirable to use a similar set of pendulum "resonators" to exhibit experimentally the effects upon them of two simultaneous harmonic forcings of different periods. Four sets of photographs of six each were taken. The first set had resonators highly damped. The second set had denser bobs and so less damping. In each case the two drivers differed greatly in their periods. The third set of six photographs had the less damped resonators and drivers whose periods differed slightly and were made to approach and finally coalesce. The final set of six had but one driver and was taken to demonstrate that the resonators could discriminate several periods of driver between those of adjacent resonators.

Experimental Arrangements.—Fig. 1 shows the experimental arrangement used throughout. Twenty "resonators" were made as pendulums with small paper cones as bobs and black threads as suspensions. These cones were

* Communicated by the Authors.

† Phil. Mag. [6] vol. xxxvi, pp. 169-178 (Aug. 1918).

used alone for the first set of six photographs, and afterwards wfth rings of copper wire on them to lessen their damping in the subsequent photographs. Their logarithmic



decrements in the two states were respectively of the order 0·1 and 0·025 per half wave, but varied slightly from end to end of the syetem.

The pendulums JK were hung at equal distances and their lengths arranged so that a line drawn through the bobs pointed to A, one of the fixed ends of the cord. The extreme lengths of these responding pendulums were about as three to one. The direction of this cord AC pointed to the lens of the camera. At the point C on this cord ACB was hung a pendulum with heavy bob D. On an adjacent cord AGH was hung at G a similar pendulum, with bob F about equal in mass to D. This pendulum was made to affect the resonators by means of a wooden connector at L. This connector is far from both the large driving pendulums, and so the effects of the coupling between them were always small and never obtruded themselves.

Thus the two forcing pendulums or drivers could be of various lengths at will, and so subject the resonators to the corresponding double harmonic impressed forces. For each one was sufficiently connected to the resonators while the two were almost free from action and reaction on each other. The experiments consisted in adjusting the lengths of the drivers, starting their oscillations, and then taking either flash photographs or time exposures of the resonators.

It should be noted that the effective lengths of the driving pendulums are DE and FI respectively, those of the resonators being typified by JK.

Results.—Plate III. figs. 1-6 shows the effects obtained on the responding pendulums with plane paper cones by two drivers of distinctly different periods and kept of the same lengths throughout. The responders were highly damped, and soon settled to their steady state corresponding to their forced vibrations only. It is highly instructive to watch the resonators in these cases. They show two distinct humps or places of maximum resonance as exhibited in fig. 1, which is a time exposure. But also, when watching them, the quick vibrations of those pendulums in tune with the short driver are in striking contrast to the slower vibrations of those in tune with the longer driver. Thus the vibrators exhibiting the two humps are almost always out of phase. But near each hump or maximum there is the usual state of ordered phase relation corresponding to the resonance in question. These effects are partly indicated by the flash photographs of figs. 2-6. These five reproductions illustrate one of the effects shown in the paper already cited. Namely, that we have sometimes a hump all on one side of the central line, and sometimes the quasi-cubic with smaller humps one on each side. It should be noticed that in figs. 1-6 of this Plate the resonance is not sharp but spreads over a considerable distance up and down owing to the very strong damping of the resonators in use.

Plate III. figs. 7-12 shows the effects obtained when the paper cones each carried a ring of copper wire and so were much less damped. Consequently the resonances are much sharper instead of being widely spread as before. This is best seen in the two time exposures, figs. 9 & 10. The flash photographs again show the random phases of the resonators responding to the two different periods of forcing in use. The cases of a hump at one side and the cubic are again illustrated. The increased sharpness of resonance suggested pushing the periods of the two drivers nearer to equality with the view of noting how closely two maxima of resonance could occur, and yet be detected without coalescence of behaviour on the part of the responders. This detection of separate maxima in spite of some spreading of the resonance is somewhat analogous to the case of resolution of points by lenses and lines by prisms in spite of diffraction.

The idea of such resolution was accordingly followed up in the experiments whose results are given in Plate IV. figs. 13-18, consisting of four time exposures and two flash photographs. Of these four time exposures, the driving pendulums had the greatest diffrence of lengths for fig. 16, they are closer for fig. 13, closer still for fig. 17, and are in exact agreement of lengths and periods in fig. 18. Indeed the result shown in fig. 18 is indistinguishable in character from that of fig. 19 (also on Plate IV.), in which only a single driver was employed. The two flash photographs, figs. 14 & 15 are for the same drivers as used in the time exposure fig. 13, and show the change of phase introduced by the slightly different periods of forcing.

In Plate IV. figs. 19-24 are given the results obtained by using a single driver whose length was very slightly changed from each figure to the next. Thus the discrimination of frequency of a driving vibrator which can be effected by a set of graduated responding vibrators is seen to be carried to *a fineness much closer than that of the difference of frequencies of the successive responders themselves.* For in passingr through this series of six figures (19-24) we do not cover the interval between the frequency of one vibrator and its neighbour.

Conclusion.— Some find difficulties in accepting a resonance theory of audition on the ground that there may not be in the internal ear sufficient responding vibrators provided with separate nerves to give one nerve for each perceptible pitch throughout the range of hearing. To such the above simple experiments with pendulums may afford some help by analogy. These experiments, at any rate, show that the resonance theory of audition, whether it stands or falls, does not fall on this account.

It is hoped to deal more particularly with this and other points as to the resonance theory of hearing in another paper for which the experiments are already completed, and to which this paper forms a natural introduction.

Nottingham,
February 14th, 1919.

The Resonance Theory of Audition subjected to Experiments.

By

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&

H. M. Browning, M. Sc.

Theories of audition have been recently under considerable discussion*: indeed this controversial subject appears to be of perennial interest. Helmholtz long ago advanced his hypothesis of sympathetic resonance, and supposed at first that the rôle of resonator was played by each of the arches of Corti. This latter detail was afterwards modified, the basilar membrane being then considered to act somewhat like a set of resonators or harp strings. It was shown that this was possible owing to its fibrous nature with high lateral tension and relative slackness longitudinally. There are of course difficulties in the hypothesis in matters of detail. But this is only to be expected in the case of so small a mechanism working at such high frequencies and with such minute displacements. Indeed, there are difficulties in any hypothesis, and that of sympathetic resonance seems to deserve careful examination from a new standpoint. Some anatomists have felt considerable difficulty in accepting it, others accept it unreservedly. Some of its critics have obviously not quite grasped the meaning of the hypothesis, and so unfortunately base upon their mistaken view a criticism which falls wide of its mark.

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* 'Analytical Mechanism of the Internal Ear' : Sir T. Wrightson and Dr. A. Keith. London, 1918. "The Internal Ear," Nature, Aug. 8 1918. Letters in 'Nature,' Oct. 17 & 31, Nov. 7 & 21, Dec. 5 & 19, 1918, Jan. 9, 1919. "On Sir T. Wrightson's Theory of Hearing." W. B. Morton, Phys. Soc. Proc. xxxi. Part III. April 1919.

The term "resonance" used in the present connexion is open to misunderstanding. In the minds of some, it recalls simply the familiar case of the actual *resounding* for several seconds by a lamp-shade of a musical sound originally due to the voice or piano. Taken in this crude sense of the actual reproduction of a sound probably no one, competent to judge, has ever believed in a resonance theory of hearing. But the essential facts of the hypothesis are present in the case just referred to. The lamp-shade has a certain period of vibration natural to it. But when practically the same note is sung, the very feeble vibrations of the air reaching it, being of the right period and repeated hundreds of times, elicit a powerful response.

If the periods had been utterly different instead of nearly alike, the response would have been unnoticeable instead of arresting.

Hence the sufficiently powerful vibrational response of an elastic system to very weak forces, owing to the almost exact tuning between the period of the forces and that of the responder, is of the essence of the theory of sympathetic resonance, whether that responder makes any sound or not. Perhaps *sympathetic response* would have described more precisely what is intended by the commoner phrase sympathetic resonance.

Further, it must be borne in mind that the degree of falling off in response among resonators, owing to their mistuning with the forces impressed upon them, depends upon the damping of their own natural vibrations. Thus, theory shows that the more highly the vibrations natural to a responder are damped, the less is the falling off of their response owing to a mistuning of the impressed forces. In other words, the response is more widely spread among a graduated set of responders when they are highly damped, but is more concentrated when the responders are but slightly damped. This may be clearly seen by reference to the plates of previous papers*.

Hypotheses of audition may be approached in a variety of ways. Perhaps the most natural basis is that of dissection and microscopic examination of the anatomical structure of the ear. These investigations have been carried out by a number of workers, among whom it may suffice to mention Bowman, Corti, Deiters, Hasse, Henle, Hensen, Kölliker Kuile, Reissner, Retzius, Schultze.

But it is not sufficient to regard the internal ear as a structure merely. It must be recognized that it is a working mechanism. Further, it must not be looked upon as a mechanism capable only of *slow* displacements. For it is of the very essence of this mechanism that it is movable at acoustic frequencies, highly susceptible to very feeble forces provided they alternate at any such frequency.

The question may be asked here,—Is it easy to imagine these mechanisms responding to such feeble forces as are usually present except on the principle of forced vibrations? Further, the presence of a graduation in these mechanisms suggests that they are elastic systems with natural periods of vibration which form a series according to their dimensions and other conditions.

Without at all prejudging the case for or against the resonance hypothesis, it is allowable to consider what are the chief facts of audition, and whether they are explicable on the resonance theory. If they appear to be so, we may

* See figs. 1. 2, & 3, Plate VIII. Forced Vibrations, &c, Phil. Mag., August 1918. Plate V, Mechanical "Resonators," &c., Phil. Mag. April, 1919.

further ask what number and disposition of responders are needed.

Since the mathematical theory of forced vibrations remains essentially unchanged for a great variety in the forms of the vibrators, we may reduce the problem to its simplest terms by arranging a set of ***simple pendulums*** of graduated periods to represent these vibrators. Then their behaviour may be compared with the facts of audition and the agreement or conflict noted. Any conflict, if observed, would seriously discredit the resonance hypothesis. On the other hand, any agreement that may be observed will essentially support the hypothesis in general terms, and might conceivably give some clue as to which parts of the ear could act as responders. For the facts of audition might be reproducible only by a certain number of responders with given frequencies and dampings, and the properties requisite might be possible to certain anatomical structures only.

The anatomical method of studying the subject would probably be best of all could it be carried out in its entirety on a living subject. But as this is impossible and the alternative post mortems are in some respects inconclusive, we seem justified in taking any indirect method of approach that is available. Hence the method of using a set of pendulums, though they are confessedly unlike any structure in the ear, may throw a valuable side-light on the subject by revealing and displaying what number and arrangement of responders would prove adequate to account for the known facts of the case.

It would appear from the experiments thus made and their accompanying photographic records, that about twelve reponders to the octave or a total of about one hundred in all, if of suitable damping, would probably suffice to account for some of the chief facts of audition.

And this number is only about one thirtieth of the number of Corti arches present in the human ear. Accordingly on this view the resonance theory would not make the demand for so large a number of structures with separate nerves as its adherents have usually supposed. On the contrary, it leaves a liberal allowance for the possibility of the number of nerve-fibres being much smaller than the number of corti arches. So that if a single nerve-fibre is distributed to a number of arch segments (as found by Held), this would not necessarily invalidate the resonance theory.

§ 2. *Fundamental Facts of Audition.*

We may now review some of the basal facts of audition, so as to set up a standard such that the success or failure of the resonance theory to account for these facts would afford a confirmation or disproof of the hypothesis.

For those whose hearing is normal the following may be taken as fairly representative of the fundamental fact with which we are now concerned.

1. When two different notes at a considerable interval are sounded together, we can hear both notes and estimate their interval, but do not mistake them for a single note of intermediate pitch. Thus C and G sounded togather are recognized as forming the interval of the fifth and are not mistaken for a single note of pitch E or E♭. (This is the direct contratry of the case with colour vision in some parts of the spectrum. For, when beams of red and green light are converged on to the same white srceen, the impression received is that of yellow, and the unassisted eyes furnishes no hint of the dual nature of this composite light which might be a monochromatic yellow for aught we are able to perceive.)

2. When two near notes are sounded successively the small irterval between them can be perceived by a specially keen ear down to something of the order of two vibrations in a thousand or one-twentieth of an equal-tempered semitone.

3. When two very near notes of almost equal intensities are sounded simultaneously, the difference of their frequencies can be recognized by anyone as the number of beats, per second. And this may serve to discriminate an interval of say one vibration in a thousand, or about the fortieth of an equal-tempered semitione.

4. The range of audition is limited at both ends, each limit varying with the individual, but about eleven octaves are usually audible.

5. Before either limit of audition is absolutely reached the note recognized to be very high or very low, but the power of distinct location of pitch is lost, only about seven octaves being musically audible.

6. A musical shake of about ten notes per second on a tone of frequency a hundred and ten per second is heard quite distinctly.

§ 3. *Variables of Responding System.*

If a set of vibrating responders is provisionally postulated as existing in the ear and if in order to test the validity of the postulate we are to make a working model on this principle, it is evident that many variables are at our disposal, and probably the success of the model in accounting for the actual facts of audition, should that prove possible, well depend somewhat upon the right choice of these variables.

The chief variables in question may be stated as follows :—

(*b*) The musical intervals between adjacent responders.

(*c*) The damping natural to these responders.

(*d*) The constancy or otherwise of the intervals and of the damping throughout the range.

We must also suppose that

(*e*) a certain order of discrimination of relative amplitude of vibrations of different responpers is possible by means of the nevers attched to them.

Obviously these variables must be chosen so as to accord as far as possible with the facts of audition previously enumerated. Thus the facts under headings 1, 2, and 3 give some clue to the smallness of the interval between adjacent responders and also show that the damping must not be *too large*. Otherwise the sharpness of resonance would not be great enough to facilitate location of pitch.

The facts of limited range of audition and loss of exact sense of pitch near ends (headings 4 and 5) show that the range of responders should extend to about seven octaves.

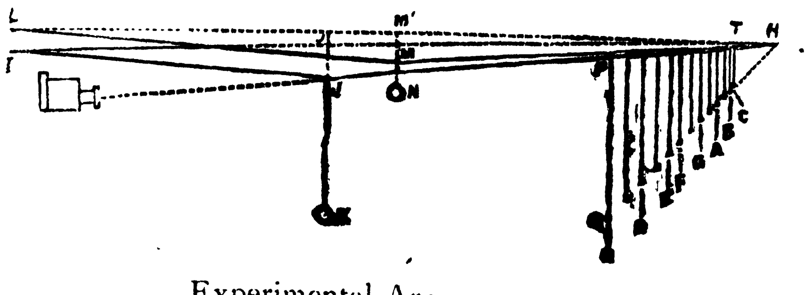
From the sixth fact as to the clearness of a shake, Helmholtz concluded that the expiring tone is reduced to one-tenth of its original amount in the fifth of a second. If therefore the ear has vibrational responders, their natural damping at this pitch must correspond to a logarithmic decrement of the order $\lambda = 0{\cdot}06$. This shows that the damping must not be *too small*.

We have thus obtained some light on (*a*), (*b*), and (*c*), but not upon (*d*), which would require more refined examination.

Further, it should be pointed out that these facts of audition do not lead us to any exact determination of the variables at our disposal in the set of responders. On the contarary, they suggest values of a certain order of magnitude, or furnish us with approximate upper and lower limits. Thus it would be sometimes possible to account for the facts of the case with certain values of the variables or to account for them equally well by changing one variable, some other being adjusted in compensation. For example, the less the damping of the responders, the sharper is the resonance and the easier would it be to locate the pitch of maximum responce. But a greater damping of the responders, leading to less sharpness of resonance, could be compensated by an enhanced discrimination of relative amplitudes of responding vibrations near the maximum.

§ 4. *Experimental Arrangements.*

No attempt has been made to set up the whole seven octaves of responders postulated, but only a single representative octave, which suffices for experimental tests. In order to have a definite and constant interval between adjacent responders throughout the octave, they were set at distances from one end, and adjusted to lengths, which formed a geometrical progression. And it seemed desirable to make the intervals corespond musically to those of the tempered chromatic scale. Thus the ratio of adjacent pendulam lengths was $\sqrt[6]{2}$, the ratio of periods being accord-



Experimental Arrangement.

ingly $\sqrt[12]{2}$. Thus the thirteen responders for the one octave may be referred to by the letters used for the notes of the scale with sharps and flats where required. Eight of these are indicated in the figure by C, D, E, F, G, A, B, C, the five corresponding to the sharps and flats being left unlettered.

All these responding pendulums have bobs in the form of paper cones and weighted with a ring of copper wire (like those used in Mechanical "Resonators," Phil. Mag., April 1919). These hang by suspensions of black thread from the stout cord HJI, to which is attached the driving pendulum JK whose true length must be reckoned from J′. It is adjustable to various required lengths by a tightener as shown. A second cord is shown by H M L, from which is suspended a second driving pendulum M N (of virtual lenth M N) and whose bob N is equal in mass to the bob K. Thse two cords are connected by the wooden bridge T, thus the two driving pendulums are only loosely coupled to each other but each act quite distinctly upon the set of responders.

The camera lens is along the line HJ produced so that in the photographs all the responding pendulums will seem to hang from the same point. Further, the responding bobs all lie along a straight line QH in order that each responder shall experience the same driving influence (see "Forced Vibrations," p. 176, Phil. Mag., Aug. 1918).

§ 5. *Results and their Significance.*

Plate V, gives six reproductions of time exposures of the responders actuated by two drivers of widely differing periods, the slower of the two being gradually increased in length from figure to figure throughout the series. In fig. 1 it is obvious that the length of the driver is about midway between those of the responders whose bobs are third and fourth from the bottom. Or, in musical terms, we might say that the pitch of the driver was about midway between E♭and D, the E♭ being slightly favoured. In the second figure we may, in like manner, refer to the pitch of the driver as being slightly nearer D than E♭, since the bob third from the bottom responds better than the fourth. In the third figure, the third bob from the bottom responds much better than any other, but the fourth shows a distinctly better response than the second. Hence the pitch of the driver is recognized as distinctly sharper than D. In the fourth figure the driver is still slightly sharper than D, whereas in the fifth it is slightly flatter than D. In the sixth figure the driver is distinctly flatter than D. Hence, in five steps we have passed over about a quarter cf a tone, giving an average interval of a twentieth of a tone, or ten logarithmic cents.

It is evident by inspection of these figures that pitches midway between the adjacent ones given would be discernible as differing one from the other. Thus between figures 1 and 2 with fourth bob favoured and third bob favoured we might have had another case with neither favoured. Hence, with nervous discrimination of relative amplitudes equivalent to our perception of these figures, it would be possible to discriminate between successive notes differing in pitch by only the twentieth of an equal-tempered semitone or five logarithmic cents. And this is just about what a good ear can accomplish.

In the case of the six figures of Plate V. it should be noted that the shorter driver is allowed to swing the whole time, and in no way interferes with the discrimination of pitch of the longer one as seen throughout. And this is known to be the case with hearing when the other note is neither too near in pitch nor too loud.

Hence the six figures of this Plate corroborate the facts of audition 1 and 2 in our list. Indeed, fact 1 was supported also by photographs 1-18 on plates V. and VI. of the paper on Mechanical "Resonators, &c." (Phil. Mag., April 1919). Further, photographs 14 and 15 of plate VI. in that paper showed by flash exposures the presence of beats, which obviously allow of a finer discrimination of pitch between simultaneous notes. And this constitutes our *third* fact of audition.

As for fact 4 of audition, any finite set of responders would obviously accord with that experience.

Photographs 7-12 (Pl. VI.) are devoted to the test of fact 5, which is the failure to recognize with precision pitches lying near either limit of audition. Thus in figures 7-9 we test the upper limit on the supposition that our single octave represents the top of the whole set of responders in the ear. In fig. 7 the responder second from the top has maximum amplitude, in fig. 8 the driving pendulum has been shortened so that the shortest pendulum responds best. In fig. 9 the driving pendulum was shorter than any responder, but that fact can scarcely be inferred from inspection of the figure so that the exact location of pitch is lost. Indeed, as soon as the pitch of the driver passes beyond either limit of the pitches of the set of responders, so that no one bob exhibits a maximum vibrations with a falling off above and below, the exact location of pitch must be lost.

For the lower limit of this single representative octave of responders the gradual failure to locate the pitch is illustrated by photographs 10-12. In fig. 10 the pitch is evidently between the lowest and the second, that is between what we have called C and C♯. In fig. 11 it is about at the lowest C, and in fig. 12 it is still lower, but by an amount which cannot be precisely inferred from the figure.

It may be noted that although by a set of responders, the exact pitch is not detectable for notes near either limit, it is yet clearly recognizable whether the note in question is near the upper or lower limit of the range. And this corresponds with one of the facts (5) of audition as already pointed out.

Referring to fact (6) of audition, the test corresponding to a musical shake was carried out as follows. The two driving pendulums were set to what we many call the notes D and E (that is the bobs third and fifth from the bottom of the series). One driver was started while the other remained at rest, and in a short time the maximum vibration of the corresponding bob was elicited, the other bobs exhibiting the ordered amplitudes and phases characteristic of forced vibrations. Then the first driver was stopped and the second started. After five or ten vibrations the new pitch was clearly established, as shown by the ordered state of things with the new maximum. So there is here quite sufficient damping to make clear shakes possible. And we have previously seen that the damping is small enough to give fairly sharp resonance and so render possible a fine discrimination of pitch.

§ 6. *Sammary and Conclusion.*

1. The present position of the resonance theory of audition is reviewed. The subject is acknowledged to be controversial. But the endeavour is made to throw a side-light upon it by the trial of a graduated set of pendulums used as responders to other pendulums as drivers. In actual form these pendulums make no pretensions to respresent any structures to be found in the ear; but in their essential behaviour they do typify such mechanisms as are postulated for the ear by the resonance theory. This typical representation may be possible and useful, because we can apply to it the theory of forced vibrations in its most essential aspects.

2. Six facts of audition are then recognized as fundamental. These include a much finer discrimination of pitch

than one for each responder and the failure to locate pitches quite exactly when they lie near either limit of audition.

3. Twelve photographs of the responding pendulums in action are taken and here reproduced.

4. These experimental results nowhere conflict with the above six facts of audition.

Indeed, for their explanation it suffices to have a set of suitably damped responders and their associated nerves of about twelve to the octave over a range of seven octaves, or say about a hundred in all. The supposed necessity for a much larger number of nerves and consequent conflict with some anatomical results are thus removed.

5. Any hypothesis like the resonance theory in question must be very difficult to prove to the hilt by any number of confirmatory experiments. But, if it is essentially at variance with facts, its disproof should be comparatively easy.

Nottingham,
March 20, 1919.

A Syntonic Hypothesis of Colour Vision with Mechanical Illustrations.

By

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&

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The conception of tri-colour vision due to Young, and further developed by Helmholtz and Maxwell, has proved so successful as to be almost universally accepted. But in one respect it appears incomplete, since red, green, and violet *sensations* are always referred to without any indication being given of the *type of mechanism* to which the initiation of those sensations is ascribed.

Theories of audition often involve some kind of resonance, but are occasionally attacked as demanding a larger number of separate vibrators and nerves than are actually present. An attempt was recently made* to reduce very considerably the number of separate aural mechanisms and associated nerves needed on such an hypothesis.

This led us to imagine that a similar view for the explanation of colour vision might be successful provided the number of vibrating responders for each element of the retina could be reduced to *three*; being one each for red, green and violet.

Probably the formulation of any such syntonic hypothesis has been deferred owing to the impossibility of imagining vibrators of such high frequencies in any molar system and the very considerable difficulty in the way of such high frequencies for molecular or atomic vibrators. These diffi-

* "The Resonance Theory of Audition," Phil. Mag. July 1919.

culties are distinctly lessened by the advent of the electron theory and the "resonarors" Planck.

But in adopting for the eye the so-called resonance theory started for the ear, this hypothesis cannot by itself suffice for the former as it may do for the latter. It my be, however, that syntonic responses initiate changes of a physiological or chemical character which in turn affect the nerves.

Experiments are here described with a set of three vibrating responders in the shape of pendulums whose relative vibrational frequencies correspond to red, green, and violet. This crude experimental representation is allowable and may be highly useful because the mathematical theory of forced vibrations is almost independent of the exact form of the vibrators themselves.

These three pendelum responders are found to be sufficient to imitate most of the fundamental facts of colour vision. Others, such as persistence of vision and irradiation, are then supposed to be due to the comparative slowness of the physiological changes and their spreading round the spot on the retina where they originated.

§ 2. *Some Fundamental Facts of Vision.*

For the present purpose we may summarize the most important facts of colour vision as follows :—

(*a*) The range of visual sensations corresponds to rather less than an octave, or the wave-length varies from about $0{\cdot}76\,\mu$ to $0{\cdot}4\,\mu$, the frequencies being nearly $(4 \text{ to } 7.6) \times 10^{14}$ per second.

(*b*) The possibility of making white and matching any colour by the true addition of red, green, and violet.

(*c*) The true addition of read and green give the appearance of yellow.

(*d*) The true addition of blue and yellow do not give green but white (or pink).

(*e*) The true addition of red and violet give purple which is not found in the spectrum.

(*f*) About thirty colours can be discriminated in the spectrum, each one appearing to be a monochromatic patch.

(*g*) A period of about one tenth of a second is needed for fully acquiring or losing visual sensations.

(*h*) The occurrence of irradiation or the spreading of very bright images on the retina.

All the above are for persons with both sight and colour vision normal. For colour-blind persons one (or more) colour sensation seems to be absent.

It is well here to contrast the eye and the ear.

The visual analysis of colours into their components is practically lacking, but the special properties of the eye and associated judgment have great power in perceiving direction and estimating range.

The ear, on the other hand, has an astonishing development of analytical power, but its perception of direction does not approach that of the eye.

§ 3. *Three Vibratory Responders postulated.*

As shown in a previous paper on "The Resonance Theory of Audition"*, we saw that the resonators must have a range comparable to that of distinct recognition of pitch. Similarly, if a "resonance theory" is to play any part in accounting for colour vision the "resonators" must be supposed to have a range approximately equivalent to that of the visible spectrum. The exact relations between the range of perception and that of the resonators may be

* Phil. Mag. July 1919.

slightly different in the case of the ear and the eye because of the different dampings which are probable in each case.

Obviously if the facts of colour vision depend upon syntony the number of responders (for each element of the retina) will probably be three only, since the tri-colour theory of Young-Helmholtz has been so successful. Accordingly if only three responders are provided for about one octave, the resonance of each music be very much spread instead of very sharp, as the latter would involve gaps in the spectrum seen by the eye. From this it follows that the damping of the responders (as measured by logarithmic decrement) must be greater for the eye, than in the case of those that have any measure of success in explaining audition.

These considerations point to the postulation of a set of three vibratory responders tuned respectively to red, green, and violet, their precise frequencies and dampings being chosen to accord with the facts of the case.

Having previously used, for the imitation of the ear's mechanism, a set of pendulum responders tuned to each semitone of an octave, it now seemed well to adopt, as a trial for the eye, three of the same responders, namely, those then referred to as C♯, F♯, and B.

For the purpose of imitating vision these responders may be taken as corresponding to wave-lengths of 0·76μ. 0·55 μ and 0·4 μ, the colours being red, green, and violet respectively. Possibly a nearer imitation would be afforded if the range were still less.

§ 4. *Mathematical Theory.*

It is obviously desirable to associate some specified values of the dampings with the "resonators" postulated, and then derive mathematically the consequent resonance curves.

The equation of motion of forced vibrations may be written

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+p^2y=f\sin nt, \quad \ldots \quad \ldots \quad (1)$$

where y is the displacement of an element of the responder, $2k$ is the frictional resisting force per unit mass per unit velocity, p^2 is the elastic force per unit mass per unit displacement, f is the maximum value per unit mass of the harmonic impressed force of frequency $n/2\pi$; the frequency natural to the responder, if devoid of friction, being $p/2\pi$.

The complete solution of this equation may be written

$$y=\frac{f\sin(nt-\phi)}{\sqrt{\{(p^2-n^2)^2+(2kn)^2\}}}+Ae^{-kt}\sin(qt+\alpha), \quad \ldots \quad (2)$$

(*Displacement*)) (*Forced Vibration*) (*Free Vibration*)

where

$$\tan\phi=\frac{2kn}{p^2-n^2}, \; q^2=p^2-k^2, \quad \ldots \quad \ldots \quad (3)$$

and A and α are arbitrary constants depending upon initial conditions.

We are here concerned with incident radiations of variable frequency $n/2\pi$ throughout the visible spectrum, but with only ***three*** assumed values for the frequenceis of the responders. These may be denoted by $p_1/2\pi$, $p_2/2\pi$, $p_3/2\pi$ respectively and their wave-lengths by λ_1, λ_2, λ_3.

Then if v be written for the speed of light we shall have $p_1\lambda_1/v=2\pi$ or

$$p_1=\frac{2\pi v}{\lambda_1} \text{ and } n=\frac{2\pi v}{x}, \quad \ldots \quad \ldots \quad (4)$$

where x is written for the variable wave-length of the incident light.

From equation (2) we may be obtain the value of δ, the logarithmic decrement per half period natural to the responders.

Thus, for the indices of e in (2), we have

$$k_1 t = k_1(\lambda_1/2v) = \delta,$$

whence

$$k_1 = \frac{2v\delta}{\lambda_1} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Referring to equation (2) and using (4) and (5), we may write for the amplitude of the forced vibration

$$Y_1 = \frac{f}{\sqrt{\{(p_1^2 - n^2)^2 + (2k_1 n)^2\}}}$$

$$= \frac{f\lambda_1 x}{8\pi v^2 \delta \sqrt{\left\{1 + \frac{\pi^2}{4\delta^2}\left(\frac{x}{\lambda_1} - \frac{\lambda_1}{x}\right)^2\right\}}}, \quad \ldots \quad (6)$$

and similar equations for Y_2 and Y_3, the value of δ being assumed the same throughout.

For the shape of the "resonance" curves we are concerned only with relative responses, so may fitly take the ratio of Y_1 to its maximum value Y_0 attained for $x = \lambda_1$.

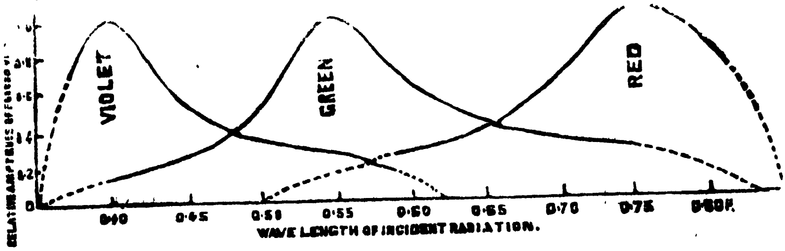
Then, from (6) we have

$$Y_0 = \frac{f\lambda_1^2}{8\pi v^2 \delta}. \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Thus for the ordinate u_1 of the relative curves we have

$$u_1 = \frac{Y_1}{Y_0} = \frac{x/\lambda_1}{\sqrt{\left\{1 + \frac{\pi^2}{4\delta^2}\left(\frac{x}{\lambda_1} - \frac{\lambda_1}{x}\right)\right\}}}. \quad \ldots \quad \ldots \quad (8)$$

Fig. 1.



Resonance Curves.

The three "resonance" curves shown by the full lines of fig. 1 are plotted from equation (8) by giving u the three values corresponding to wave-lengths $0{\cdot}4\mu$, $0{\cdot}55\mu$, and $0{\cdot}76\mu$ respectively, thus representing violet, green. and red; the value of the log. dec. per half period being retained throughout $\delta = 0{\cdot}2$.

It will be seen that these resonance curves in their general features are very similar to those current as representing the various degrees of excitation of the colour sensations. The following contrasts are, however, to be noted, (*a*) these resonance curves do not fall completely to zero, (*b*) there is no special convexity of the curve for the red "responder" for very short wave-lengths.

From the curves shown it is thus seen that the damping has to be considerable even for the half period. Accordingly the absolute time to reduce the natural vibrations of one of these responders to a negligible quantity would be extremely short. But vision persists for about a tenth of a second and also the full normal perception is not reached under about the tenth of a second. Hence in addition to the syntonic vibration as a primary response to the stimulus of incident light we must also postulate a secondary effect executed by another mechanism whose operation requires time of the order of a tenth of a second for its completion.

We may therefore suppose that some atomic or electronic vibatory mechanism first responds in syntonic fashion to the stimulus of the light received, and that this response initiates some physiological or chemical change, the completion of which occupies an appreciable fraction of a second. The nerves are then supposed to be excited by the results of this change. Further, it may be supposed that, after the light is cut off, the vision persists until the ordinary physiological operations reinstate the normal condition of things. And this process may be imagined as occupying about the tenth of a second.

Referring again to the "resonance," curves plotted in fig. 1, their drooping ends might be continued to zero, on the supposition that when the vibratory response fell below a certain limit the physiological change initiated by it fell still more abruptly and soon became imperceptible.

Such hypothetical continuations are shown on fig. 1 by broken lines. Thus the full line curves may be taken to represent both the syntonic responses and the physiogical changes supposed to be directly proportional to them. The broken lines, on the other hand, represent only the values of the physlological changes supposed to be much smaller than would result from simple proportionality to the vibrations that inintiate them. The whole of the curved lines, full and broken together, are then closely like those current as representing colour sensations.

It may be well to note here the flexibility of the present hypothesis of syntonic vibrators of three frequencies. The frequency of each responder and its damping are entirely at our disposal; thus giving six variables which may be chosen to accord as closely as possible with experimental facts.

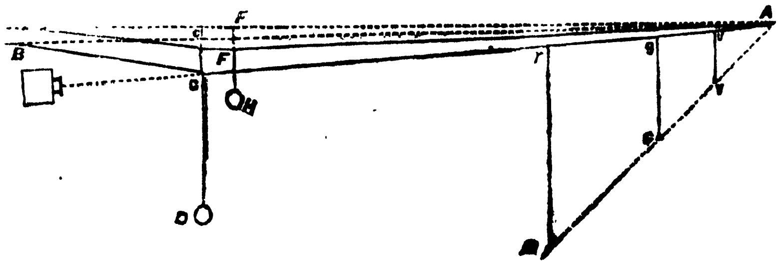
It should be remarked that the full line curves gives *anplitudes* of the responding virbations postulated. Their intensities would be proportional to the square of the amplitude mulitipled by the square of the frequency. Consequently, curves of intensity would be much more sharply peaked and exhibit some asymmetry.

Any further details as to the possible mechanism of such syntonic responders or that of the subsequent physiological changes we do not presume to enter upon.

§ 5. *Experimental Arrangements.*

Among the facts of vision already enumerated, the arly ones show that the range of the responders should be slightly within an octave, and that their relative frequencies should correspond to those of red, green, and violet. Further, to leave on gaps in the visual spectrum necessitates considerable damping in the responders in order that their "resonance" should be sufficiently spread.

Fig. 2.



Experimental Arrrngement.

The mathematical theory of forced vibrations holds for almost any type of syntonic responder. Hence we here adopt as an experimental test and illustration of the hypothesis the simplest mechanical vibrator. The arrangment consists essentially of pendulums hanging from an overhead cord, as shown in fig, 2.

A stout horizontal cord ACB has a pendulum with heavy bob D hanging from C. A second cord AFE has a second pendulum hanging from F with heavy bob H equal to D. Each pendulum is adjustable in length so as to represent any incident radiation desired. It may be observed that the virtual lengths of these heavy pendulums are C′D and F′H respectively. The set of three responders supposed presnt at every part of the retina, are represented by the pendulums R*r*, G*g*, and V*v*, with very light bobs consisting of paper cones. These are made of such lengths as to correspond, in relative frequencies, with the three colours red, green, and violet.

These light-bobbed responders have a logarithmic decerement of the order used in plotting the resonance curves of fig. 1. The heavy driving pendulums have negligible dampings and so can represent the sustained amplitudes of incident radiations.

In order that each responder may be equally influenced by the heavy pendulum these light bobs R, G, and V are arranged along a straight line through A. (See "Forced Vibrations," Phil. Mag. Aug., 1918.)

If only one incident radiation is to be imitated, the cord ACB and pendulum CD are sufficient. In order to imitate a second radiation simultaneosly incident, the second order AFE and pendulum FH are provided. This second pendulum influences the responders by means of a wooden bridge near *v*, conneeting the two cords ACB and AFE

In order to record the response of these light pendulums under different conditions of stimulations, it is necessary to photograph them when in action. To insure that all these pendulums shall appear to hang from the same point in the photograph, the lens of the camera must be on the line AC produced, as shown in fig. 2.

§ 6. *Results and their Significance.*

Eighteen photographs are here resproduced (Pl. VII.) showing time exposures of the responders vibrating under various conditions of excitation. Photographs 1 to 6 show tests as to colour mixtures. Photographs 7 to 18 show the responders under the action of a single driver, changed in frequency by a number of small steps.

In dealing with these effects it will be convenient to designate the pendulum or pendulums by the names of colours : thus a driver, of such length as to equal that of the red responder will be called a "red" driver, and will imitate the emission of red radiation. Pendulums of other lengths will in like manner be referred to under the names of other colours.

Colour Mixtures.

Red and Green make Yellow.—Photo 1 shows the response to a yellow driver. Photo 2 shows a response to two drivers, red and green, acting simultaneously. It is seen that the responses in 1 and 2 are precisely alike in character as might be expected from the resonance curves of figure 1. Thus, on our hypothesis, it is shown that red and green ***make*** yellow. That is to say, this set of only three responders, red, green, and violet, cannot differentiate between (i.) the simultaneous stimuli of red and green drivers, and (ii.) the single stimulus of a yellow driver, whose frequency is intermediate between those of red and green. Photo 3 shows the response to drivers representing red and bluish green.

Blue and Yellow fail to make Green.—It might be supposed that on our hypothesis any two colours of the spectrum being used as simultaneous stimuli, the system would give the same response as if under a single stimulus of intermediate frequency. If so, blue and yellow would make green, whereas, with the eye it is not so. On the contrary, the true, addition of blue and yellow give the sensation of white or a faint pink. It is therefore a crucial test of the present hypothesis to ascertain the response of the three vibrators when under the action of blue and yellow drivers. This is shown in photo 4. It is seen that all the responders move, the middle or green responder least of all, and the red one most. Thus, so far as the colour departs from white, it is certainly not green but painkish. It therefore accords with the known facts of colour vision as exhibited by colour tops and converging beams. It is easily seen why with this set of three responders, blue and yellow drivers do not give the effect of a single intermediate frequency as the red and green did. The latter have each a special responder, neither of the former has a special responder in tune with it.

Blue and Red give a Colour not found in the Spectrum.—Photo 5 shows the effect of blue and red drivers acting simultaneously. As might be expected, we have here a response that could not be due to any single spectral colour, *i. e.* to the vibrations of any single driving pendulum of length anywhere in the range of the responders. This result accords with the known fact that the true addition of red and blue gives a purple which does not occur in the spectrum.

Hering's Theory tested.—On Hering's Theory, in addition to red, green, and violet, there are also yellow, black, and white sensations. To test this, as regards the yellow, a fourth responder was used of relative frequency to correspond with yellow. Then with drivers representing blue and yellow lght, the effects shown in Photo 6 were obtained. Here the amplitudes of responses are unequal but the yellow predominates. It is difficult to believe that this would correspond to the impression of white light or pink as was the case in Photo 4, where only the red, green, and violet responders were used.

Fineness of Discrimination of Colour.—Dr. Edridge-Green states that in the spectrum the eye can distinguish eighteen to twenty-seven different regions each of which appears monochromatic but different from its neighbours. It was therefore necessary to test the present hypothesis as to the adequacy of three responders to provide for a fineness of discrimination of this order. This was done by starting with the driving pendulum of period equal to that of the middle responder representing the green. The effect of this is shown in photo 7. The suspension of the driving pendulum was then repeatedly lengthened by small steps till in photo 18 it equalled that of the red responder. The intermediate effects are shown in photos 8 to 17. The differences of relative amplitudes of the responders are usually clearly marked as we pass from plate to plate. Hence, there are twelve states shown in about half the range of the spectrum, which would give a possible twenty-four in the whole spectrum, and this approaches the order of discrimination stated by Dr. Edridge-Green as being actually accomplished by the eye in the most favourable cases. A glance at some of the photos (*e. g.* 7, 8, and 9) will show that still finer discrimination than that actually exhibited would be possible. So that, on the present hypothesis of vibrating responders of only three frequencies, provision is made for a fineness of discrimination quite equal to that known to exist. This is on the supposition that the relative amplitudes of the vibratory responses can be appreciated through the changes occurring in the eye just as well as one can estimate the relative dimensions exhibited on these photographic prints.

In the light of these results the present hypothesis of colour vision may be compared with that put forward by Dr. R. A. Houstoun ("A Theory of Colour Vision," Proc. Roy. Soc. ser. A, vol. xcii, no. A 642, pp 424-32)..

§ 7. *Summary and Conclusion.*

The hypothesis is here advanced that each element of the retina may possess syntonic vibrators of frequencies corresponding to those of red, green, and violet light respectively. These responders are supposed to be of molecular, atomic, or electronic nature somewhat like the resonators of Planck. The adequacy of three such responders to yield the great variety of effects corresponding to those of visual experience is tested by an experimental arrangement. This consists of a horizontal cord from which hang three light pendulums. These are sympathetically stimulated by one or more heavy pendulums hung from the same or an associated cord. The light pendulums represent the vibrators supposed present in the eye. The slight displacements of the cord from which they hang represent the light incident upon the eye. The heavy pendulums which produce these displacements are like luminous sources.

In order that a vibration of any frequency throughout the range corresponding to the visible spectrum should appreciably stimulate these responders, their "resonance" must not be too sharp. The responders must accordingly have considerable damping. To account for the persistence of vision and some sluggishness in its inception it is further supposed that the responders in the eye do not act directly on the nerves. On the contrary it is imagined that they only start some physiological change which lasts about a tenth of a second. And this change in its turn is supposed to stimulate the nerves.

The behaviours of these three responders under various conditions of stimulus are photographically recorded and compared with the facts of colour vision. It is thus found that on these suppositions red and greed would make yellow, blue and yellow would make a pinkish tint, and so on for other facts of normal colour vision.

The cases of colour-blindness would be met by the supposition of the absence of "certain frequencies."

No claim is made that the resonators postulated have been proved to exist. The considerations here put forward simply show that if the mechanisms postulated were present colour vision would be in the main as we now find it.

Nottingham,
May 30, 1919.

Plate I.

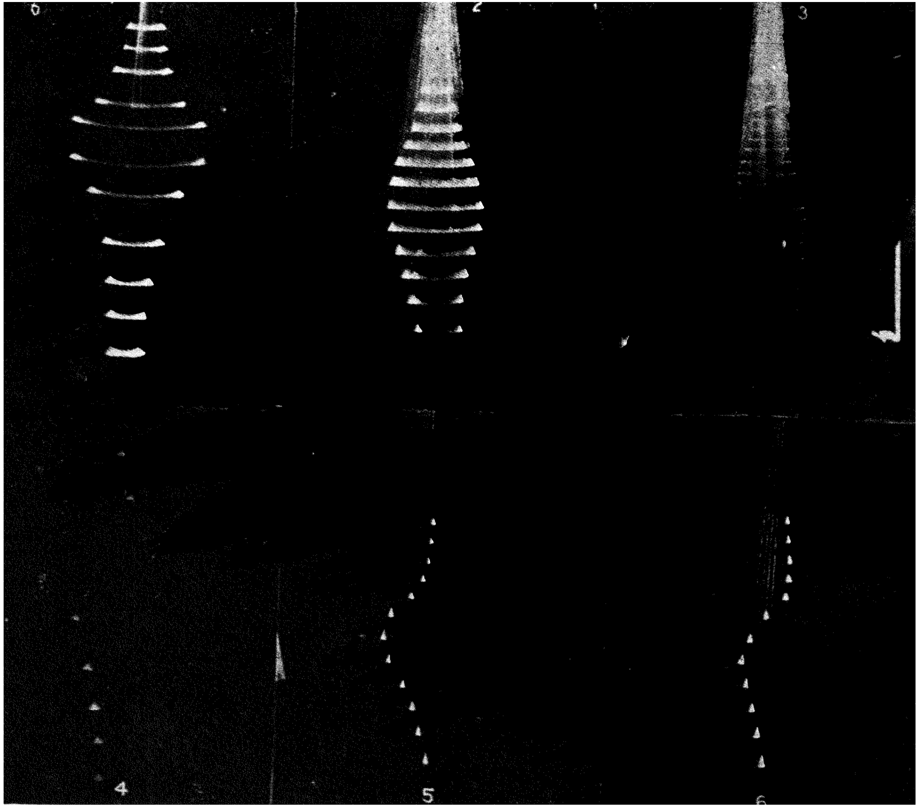
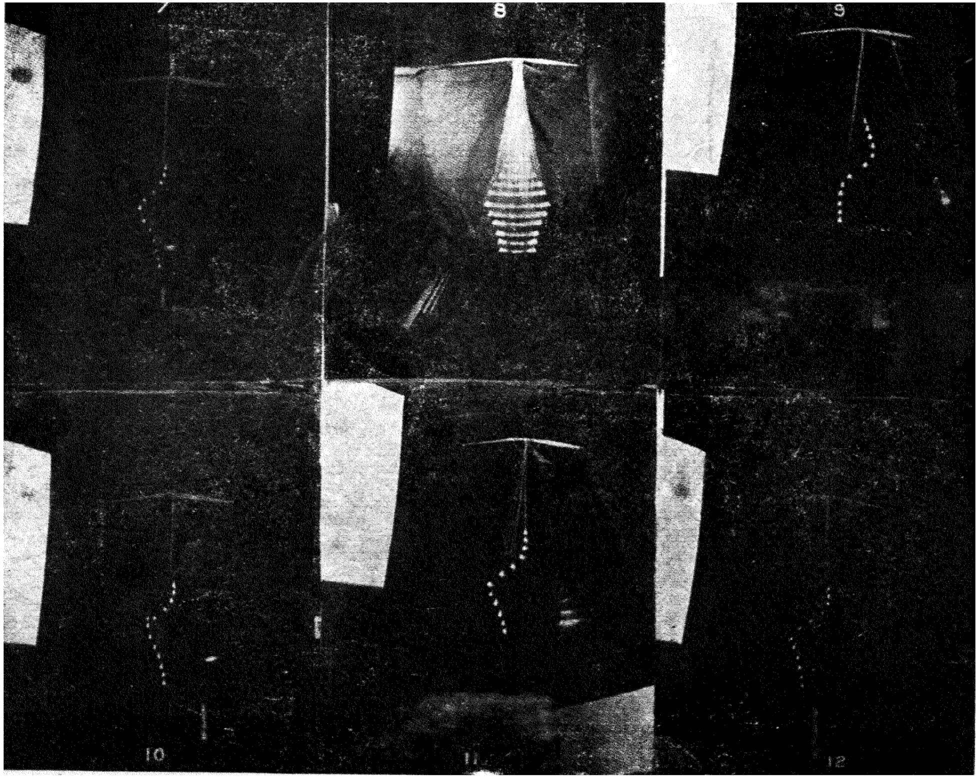
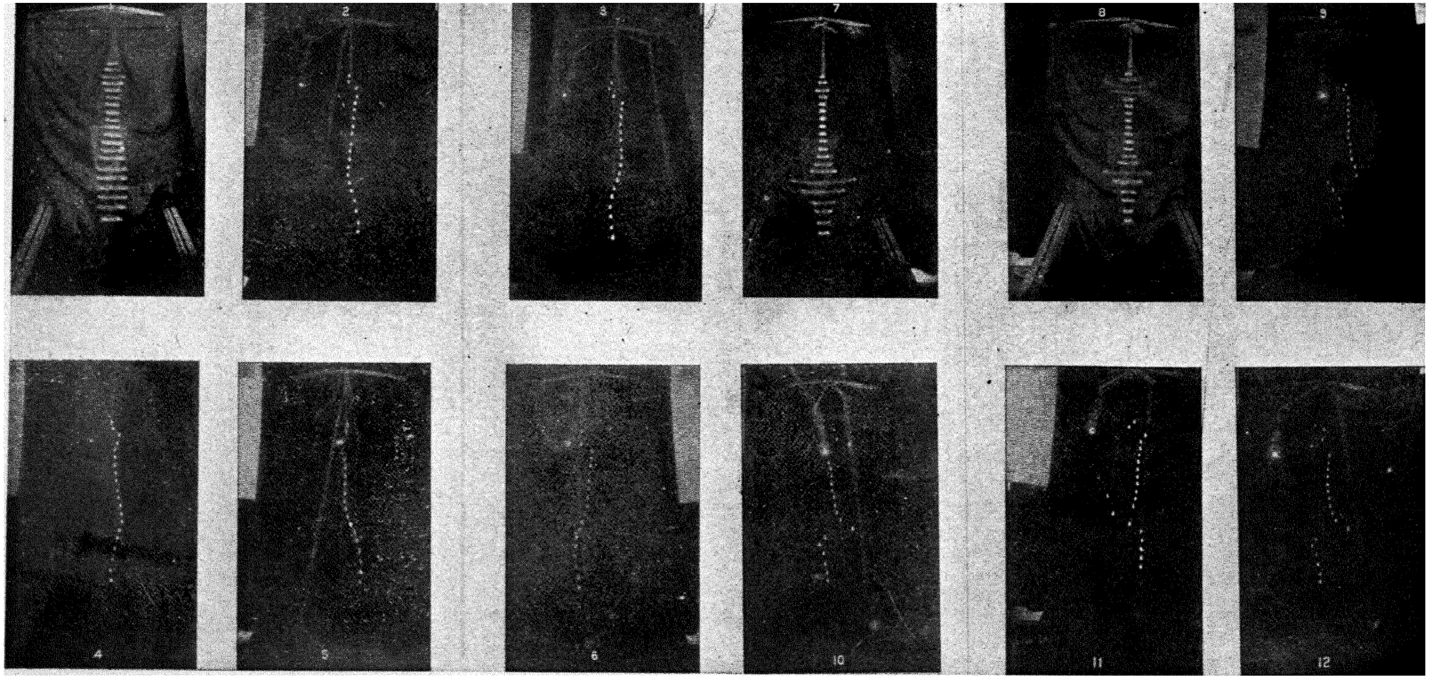


Plate II.



Barton & Browning

Plate III.



Barton & Browning

Plate IV.

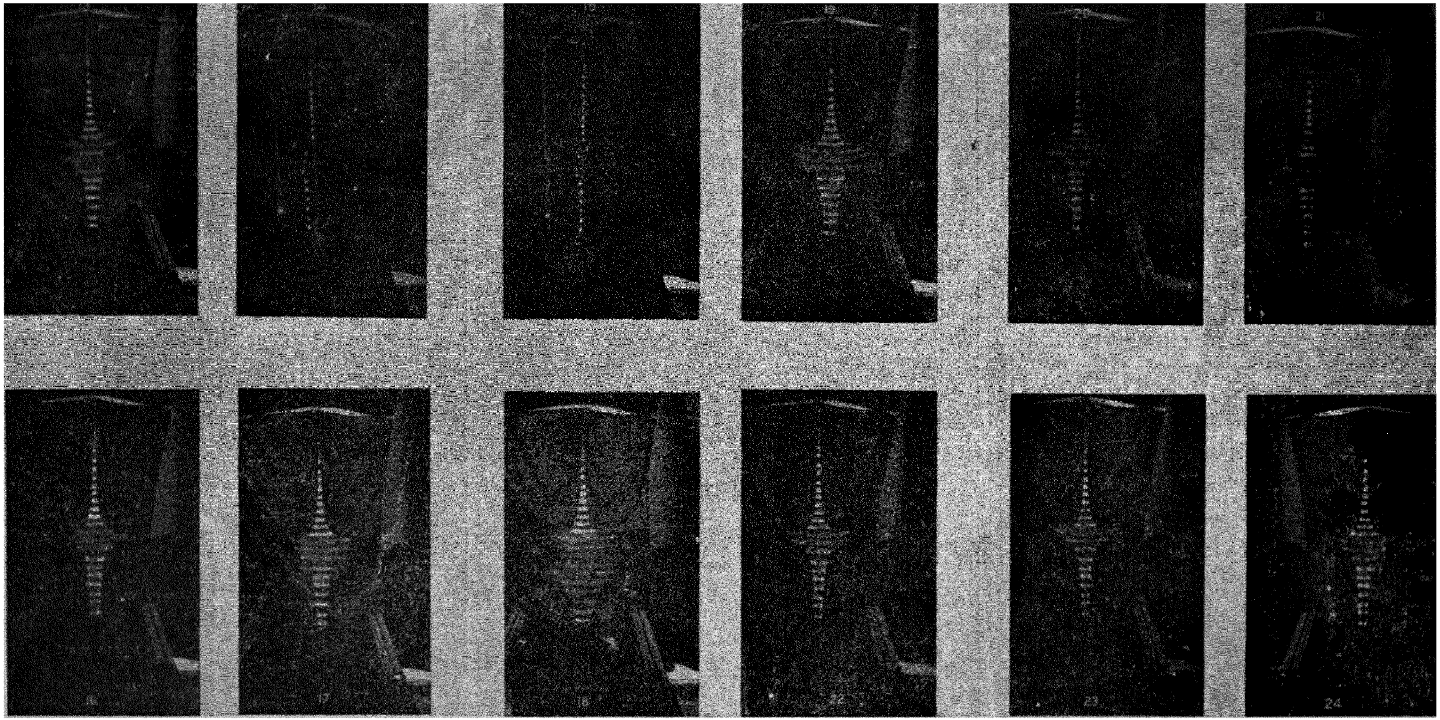
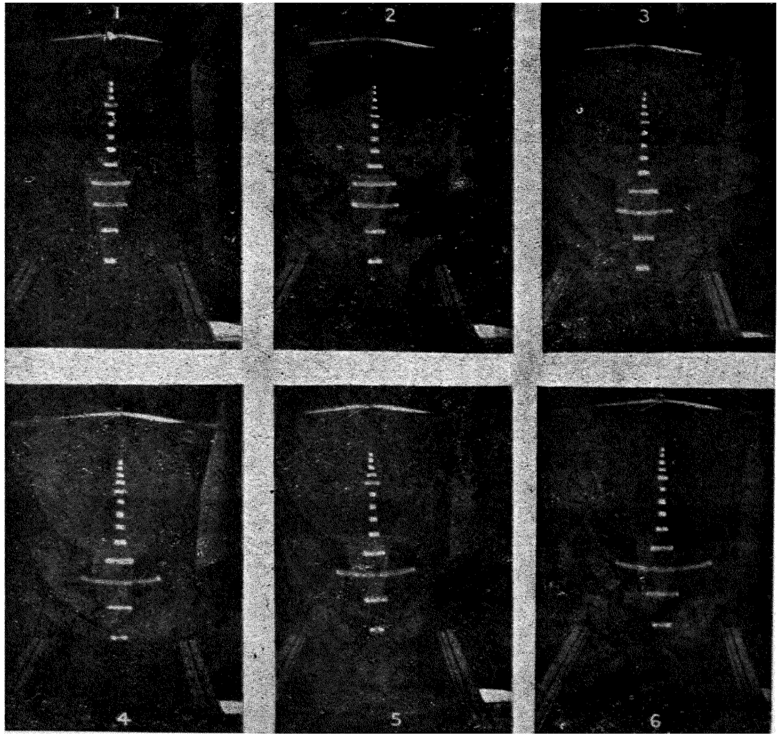
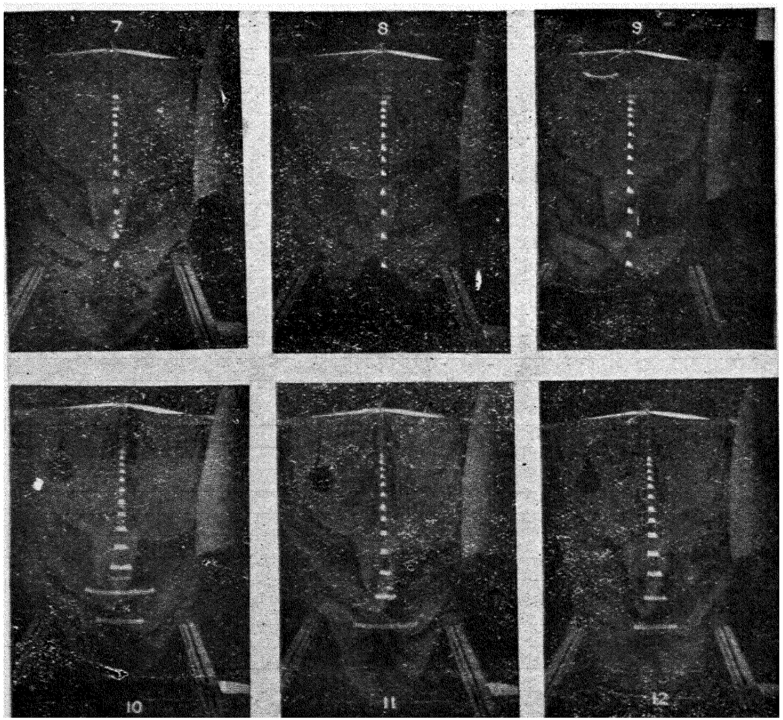


Plate V.

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BARTON & BROWNING

Plate VII.