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CELESTIAL MECHANICS

by

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PREFACE

THE book has been designed to provide an introductory course on a subject somewhat neglected in recent years. The first half, approximately, deals in the main with fundamental principles, astronomical and dynamical, and the second half with applications to several of the principal problems of celestial mechanics, selected because of their importance in contemporary astronomy or because of their intrinsic interest. Nearly all the topics discussed have formed the subject-matter of courses of lectures given in recent years at Glasgow, and earlier at Cambridge.

The question of notation is a difficult one to resolve in a subject which has its roots deep down in the great memoirs of the past, each usually with its individualistic set of symbols; however, I have endeavoured to maintain uniformity in the notation consistent as far as possible with modern practice. The book is essentially theoretical in character and makes no pretensions to enter the field of observational astronomy or of the modern methods of computation, each a complete discipline in itself.

The first four chapters deal with the general equations of motion for bodies in an isolated system, the known integrals, the basic formulae for an elliptic orbit and the expansion of various functions in terms of hypergeometric series and Bessel functions. Chapter 5 is devoted to the derivation of Lagrange's planetary equations in considerable detail so as to familiarise the reader with the fundamental processes in passing from an elliptic orbit to planetary perturbations in general; in the following chapter the nature of the various classes of "inequalities"—secular, short-period, long-period—is discussed. Chapter 7 contains an account of the expansion of the disturbing function, first, in lunar theory and, second, in planetary theory. In Chapter 8—on canonic equations—the various theoretical processes are illustrated, step by step, with reference to a simple example; in the following chapter the solution for an elliptic orbit by the canonic method is given in detail. The next two chapters deal with the elements of the theory of contact transformations. In Chapter 12—on Delaunay's lunar theory—the details of a characteristic "operation" are described fully, together with a practical method of obtaining the solutions in a desirable form. The next two chapters are concerned with secular inequalities and include Stockwell's numerical results relating to the orbital eccentricities and inclinations of the planets. Because of the close similarity in analytical methods, discussions of the influence of a resisting medium and of the motion of Mercury's perihelion are taken together in Chapter 15, followed in Chapter 16 by an account of

Adams' and Le Verrier's investigations which led to the discovery of Neptune. De Pontécoulant's lunar theory in Chapter 17 has been chosen to illustrate how the principal perturbations—evection, variation, the parallactic inequality—and the motions of the lunar node and perigee arise directly from the equations. Chapter 18, based on Hill's three classical investigations and on Brown's later researches, deals with the principal features of the Hill-Brown lunar theory. It is followed in Chapter 19 by an account of Adams' famous investigation on the Moon's secular acceleration and its later connection with tidal friction. The final chapter connects dynamical theory with the processes of spherical astronomy which take account of the effects of precession and nutation; it concludes with a section on the latest astronomical concept—ephemeris time.

A present-day writer on celestial mechanics is bound inevitably to acknowledge his debt to Tisserand's *Mécanique Céleste*, which provides in its four majestic volumes a complete account of the classical memoirs of the masters, and to E. W. Brown's *Lunar Theory* and his other researches. I have also been greatly indebted to H. C. Plummer's *Dynamical Astronomy* and to H. Poincaré's *Leçons de Mécanique Céleste* and *Méthodes nouvelles de Mécanique Céleste*. To the inspiring lectures of the late Professor Ludwig Becker, my predecessor at Glasgow, and of Professor H. F. Baker at Cambridge I owe more than I can say for kindling and sustaining an active interest in the subject. Mr. D. H. Sadler has very kindly kept me abreast of recent developments in computational techniques and other matters. To Dr. J. G. Porter I am specially indebted for the calculation of the fundamental constants, for the epoch 1950·0, appearing in the chapter on Precession and Nutation. My warmest thanks are due to Mr. D. G. Ewart, one of my Research Students, who has assisted me most efficiently in reading the proofs. Finally, it is a pleasure as well as a duty to express my thanks to the printers for the excellence of their work.

W. M. S.

UNIVERSITY OBSERVATORY, GLASGOW,
August, 1953

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Chapter 1

INTRODUCTION

1·01. The problems of Celestial Mechanics

For convenience, and also as a matter of tradition, the main problems of celestial mechanics are segregated into two divisions ; in one, *planetary theory* is concerned with the motions of the planets around the Sun, and in the other *satellite theory* deals with the motions of satellites around their primaries, lunar theory being the best known form ; a variant of satellite theory is presented by a close binary star with a distant companion. These two divisions are essentially of the same general character and are based on the same dynamical principles ; they differ only in the practical methods found most convenient for the solution of the various problems within their domains. In the planetary theory our object is to determine the motion, relative to the Sun, of a particular planet under the main gravitational control of the Sun and the less effective attraction of all the other planets and bodies of the solar system. In the lunar theory, for example, our object is to determine the motion, relative to the Earth, of the Moon under the main gravitational control of the Earth and the less effective attraction of the Sun and planets and other bodies of the solar system. In this chapter we shall be concerned, almost exclusively, with planetary theory in its initial stages, although it must be understood that the general principles dealt with are equally applicable to satellite theory.

1·02. Kepler's Laws

The major planets, in order of increasing distances from the Sun, are Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus (1781), Neptune (1846) and Pluto (1930), the year of discovery being shown for the last three. In addition there is a vast number of much smaller bodies, the minor planets, circulating between the orbits of Mars and Jupiter ; about two thousand of these have been discovered so far, and it is possible that the total number may be in the region of a score of thousands.

In the first decades of the seventeenth century, Kepler formulated his famous three laws of planetary motion ; these laws were deduced strictly from observations, the most accurate of which were contributed by Kepler's master, Tycho Brahe.

(i) *First Law.* The first law states that the orbit of a planet around the Sun is an ellipse, the Sun being situated at a focus.

Analytically, if S is the position of the Sun at a focus (Fig. 1) and P

is the position of the planet in its orbit at a given moment, the polar equation of the ellipse is

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\omega)} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where r is the *radius vector* SP, θ is the vectorial angle measured in the direction of the planet's motion from a suitably chosen reference line SN in the plane of the ellipse to SP, ω is the angle between SN and SA where A is the point of the ellipse nearest to S, a is the *semi-major axis* and e the *eccentricity* of the ellipse. In Fig. 1, C is the centre of the ellipse; $CA = CB = a$; also $CS = ae$.

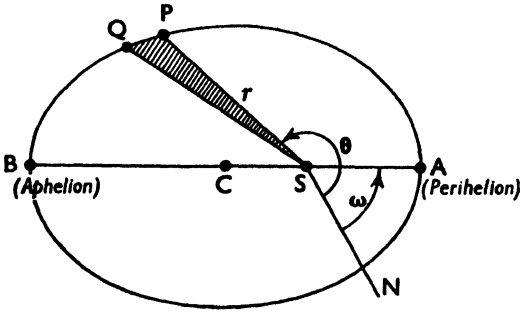


FIG. 1

From (1) the minimum value of r is $a(1-e)$ corresponding to the point A for which the vectorial angle is ω; A is called *perihelion*. The maximum value of r is $a(1+e)$ corresponding to the point B, called *aphelion*.

(ii) *Second Law.* The second law states that the rate of description of area by the moving radius vector is constant for all parts of the orbit. Thus, if in Fig. 1, $P(r, \theta)$ is the position of the planet at time t and $Q(r+dr, \theta+d\theta)$ is the position at time $t+dt$, the rate of description of area is $\frac{dA}{dt}$, where dA is represented by the shaded area SPQ. Now, $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$ and, as the rate of description of area is constant, we can write

$$r^2\dot{\theta} = h \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where h is constant and equal to *twice* the rate of description of area.

We denote by T the time required by the planet to describe the orbit; this is referred to as the *orbital period.* Since the area of the ellipse is πab, where $b = a\sqrt{(1-e^2)}$, b being the semi-minor axis, the constant h is, by the second law, equal to $2\pi ab/T$, so that we write

$$h = \frac{2\pi}{T}a^2\sqrt{(1-e^2)} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In time T the radius vector sweeps out the angle 2π; the *mean angular motion*, denoted by n, is defined by

$$n = 2\pi/T \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Equation (3) can then be written

$$h = na^2\sqrt{(1-e^2)} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

(iii) *Third Law.* The first two laws are concerned with the orbit of a particular planet; the third law states a relation involving the semi-major axes and orbital periods of two or more planets, as follows:

$$\frac{a_1^3}{T_1^2}=\frac{a_2^3}{T_2^2}=\frac{a_3^3}{T_3^2}=\ \ldots\ =\text{constant}\quad , \qquad \ldots\ldots(6)$$

or alternatively, from (4),

$$n_1^2a_1^3=n_2^2a_2^3=\ \ldots\ =\text{constant}\quad , \qquad \ldots\ldots(7)$$

the suffices referring to planets P_1, P_2, etc.

As we shall see in section 2·05, the third law is not quite accurate, although the observations available to Kepler were hardly exact enough to show up its minute deficiencies.

In its amended form the third law is most readily stated in terms of the formula, which will be derived in 2·04,

$$\frac{4\pi^2a^3}{T^2}=G(m_0+m)\quad , \qquad \ldots\ldots(8)$$

or

$$n^2a^3=\mu\quad , \qquad \ldots\ldots(9)$$

where m_0, m are the masses of the Sun and planet respectively, G is the constant of gravitation and

$$\mu=G(m_0+m)\quad . \qquad \ldots\ldots(10)$$

Since the masses of the planets are small in comparison with the solar mass, (8) and (9) become, on neglecting m/m_0,

$$\frac{a^3}{T^2}=\frac{Gm_0}{4\pi^2}\ \text{ and }\ n^2a^3=Gm_0\,;$$

Kepler's third law, in either of the forms (6) and (7), follows immediately.

It is to be remarked that the three laws are three distinct statements with no apparent connection between them. It was left to Newton to provide a single unifying principle—the law of gravitation—from which each of the three laws (the third in its amended form) can be simply deduced.

1·03. The Law of Gravitation

The foundations of celestial mechanics are Newton's law of gravitation and his three laws of motion. The law of gravitation states that a particle P_1 of mass m_1 attracts a particle P_2 of mass m_2 with a force F, acting in the line joining P_1 and P_2, which is proportional to the product of the masses and inversely proportional to the square of the distance, r, between the particles; expressed algebraically, the law is

$$F=\frac{Gm_1m_2}{r^2}\qquad \ldots\ldots(1)$$

where G is called the *constant of gravitation.*

This expression also gives the force of attraction exerted on P_1 by P_2. Further, the acceleration f_1 of m_1 due to the force F is given by $f_1 = Gm_2/r^2$, and the acceleration f_2 of m_2 is given by $f_2 = Gm_1/r^2$.

It is found by processes which we need not describe here that, if $f = 1$ cm./sec.^{2}, $m = 1$ gm. and $r = 1$ cm., then in c.g.s units

$$G = 6{\cdot}66 \,.\, 10^{-8} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In celestial mechanics c.g.s. units are manifestly unsuitable; but whatever units of length, time and mass are adopted, the corresponding value of G in terms of these units can be readily obtained from the numerical value in (2) and the dimensions of G, namely $L^3T^{-2}M^{-1}$. It is to be remarked that after choosing suitable units of length and time we can adjust the unit of mass so that $G = 1$; when this is done the units are sometimes described as *astronomical units.*

In planetary theory the most convenient units are:

for length: the semi-major axis of the Earth's orbit, defining the *astronomical unit* of length;
for time: the mean solar day;
for mass: the Sun's mass.

Consider the Earth's orbit; then, in terms of these units, $a = 1$, $T = 365{\cdot}2564$ and $m = 1/329390$. Accordingly, G is given, from 1·02(8), by

$$G = \frac{329390}{329391} \cdot \frac{4\pi^2}{(365{\cdot}2564)^2}.$$

In earlier literature the constant of gravitation was denoted by k^2, where k is known as the *Gaussian constant*, so that $k = \sqrt{G}$. With the values of m and T available at the time, Gauss calculated the value of k to be given by

$$k = 0{\cdot}017202099.$$

As this value of k has been used over a long period in numerical investigations, the value is retained and the unit of length is adjusted in accordance with the more accurate values of m and T now available; if the unit is denoted by a_1 astronomical units, then it is found that

$$\log a_1 = 0{\cdot}000000013.$$

Again, it is to be remarked that one of the units can be chosen so that $k = 1$, $G = 1$ thereby simplifying analytical formulae; in this system the most convenient units are:

for length: the astronomical unit;
for mass: the Sun's mass;
for time: 58·132441 mean solar days.

In problems concerned with stellar systems the convenient units are

the astronomical unit, the year and the Sun's mass. Hence, from 1·02 (8), on putting $a=1$, $T=1$, $m_0=1$ and ignoring m, we find that $G=4\pi^2$.

In later analytical developments we shall generally retain G as an algebraical symbol, with the units generally unspecified.

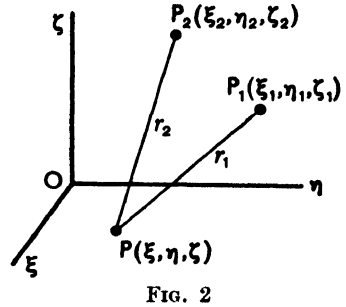


FIG. 2

1·04. Potential

We shall suppose that it is possible to specify unaccelerated axes $O\xi$, $O\eta$, $O\zeta$ to which the positions of the heavenly bodies can be referred (Fig. 2). Let P and P_1 be two of such bodies, which we assume to be point-masses, with coordinates (ξ, η, ζ) and (ξ_1, η_1, ζ_1) and masses m and m_1.

The attraction exerted by P_1 on P is Gmm_1/r_1^2 in the direction $\overrightarrow{PP_1}$, where r_1 is the distance PP_1. Let mX_1, mY_1, mZ_1 be the components of this attractive force in the positive directions of the axes. Then

$$mX_1=\frac{Gmm_1}{r_1^2}\cdot\frac{\xi_1-\xi}{r_1} \quad .$$

But $\quad r_1^2=(\xi-\xi_1)^2+(\eta-\eta_1)^2+(\zeta-\zeta_1)^2$ and $r_1\dfrac{\partial r_1}{\partial \xi}=\xi-\xi_1$.

Hence, $$X_1=-\frac{Gm_1}{r_1^2}\cdot\frac{\partial r_1}{\partial \xi}=\frac{\partial}{\partial \xi}\left(\frac{Gm_1}{r_1}\right) \quad .$$

If there are other bodies P_2, P_3, ... P_n of masses m_2, ... m_n, and if mX is the sum of the components—parallel to $O\xi$—of the attractions of P_1, P_2, ... P_n on P, then

$$X=\frac{\partial}{\partial \xi}\sum_1^n\frac{Gm_i}{r_i} \quad , \quad (i=1, 2, \ldots, n),$$

with similar formulae for Y and Z.

We define the *potential* V at P, arising from the point-masses at P_1, P_2, ... , P_n, by

$$V=\sum_1^n\frac{Gm_i}{r_i} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

It is to be remarked that V is a function of the distances between P and P_1, P_2, ... P_n and is consequently independent of the choice of the co-ordinate-system. We have

$$X=\frac{\partial V}{\partial \xi}, \quad Y=\frac{\partial V}{\partial \eta}, \quad Z=\frac{\partial V}{\partial \zeta} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Thus, the potential at $P(\xi, \eta, \zeta)$ is such that its derivatives with respect to ξ, η and ζ give the components of the attraction on a particle of *unit mass* at P exerted by the n masses m_1, m_2, ... , m_n.

1·05. Potential of a sphere at an external point

We shall suppose that the sphere is of uniform density or, more generally, that the density at any point is a function of its distance from the centre of the sphere. Newton's law of gravitation applies to particles. The Sun and planets and satellites are not particles in the Newtonian sense, with their masses concentrated at points, but immense spherical —or nearly spherical—globes. In many problems it is sufficient to consider these bodies to be truly spherical. Newton proved the remarkable theorem that the attraction of a uniform sphere, or of a sphere in which the density at any point is a function of its distance from the centre of the sphere, is the same as that of a *particle*, whose mass is that of the sphere, situated at the centre. To find the attraction on unit mass at an external point P due to the sphere (Fig. 3) we require to find the potential at P due to the innumerable particles of which the sphere is composed.

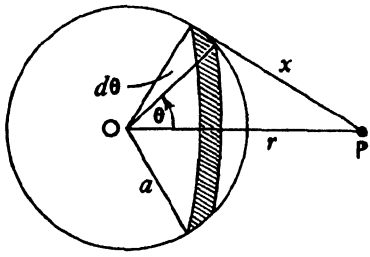


FIG. 3

Consider first a thin uniform spherical shell, centre O and radius a, of mass σ per unit area, and take an annulus bounded by small circles defined by θ and $\theta+d\theta$, the planes of the small circles being perpendicular to OP. The mass of the annulus is $2\pi a^2\sigma \sin\theta\, d\theta$. Since each element of the annulus is at the same distance x from P, the potential at P due to the annulus is

$$2\pi G a^2\sigma \sin\theta\, d\theta/x\ ;$$

hence the potential, V_1, at P due to the uniform spherical shell is given by

$$V_1 = 2\pi G a^2\sigma \int_0^{\pi} \frac{\sin\theta\, d\theta}{x} \quad .$$

Now,
$$x^2 = a^2 + r^2 - 2ar\cos\theta \quad ,$$

from which
$$x\, dx = ar \sin\theta\, d\theta \quad .$$

Hence
$$V_1 = 2\pi G\sigma \frac{a}{r}\int_{x_1}^{x_2} dx = 2\pi G\sigma \frac{a}{r}(x_2 - x_1)$$

where x_1 corresponds to $\theta=0$ and x_2 to $\theta=\pi$. But $x_1 = r-a$ and $x_2 = r+a$. Hence $V_1 = 4\pi G\sigma a^2/r$, or, since the mass of the shell is $4\pi a^2\sigma\ (\equiv M_1)$,

$$V_1 = \frac{GM_1}{r} \quad .$$

Thus the potential of the shell at P is the same as that of a particle M_1 situated at the centre of the shell.

Since we can suppose that the sphere can be divided up into a very great number of uniform thin concentric shells, the potential at P due to the sphere is the same as the potential due to a particle of mass equal to the sum of the masses of all the shells—that is, equal to the mass M of the

sphere—situated at the centre of the sphere. We thus have the formula for the potential V at the external point P, namely

$$V = \frac{GM}{r} \quad .$$

Accordingly, we can replace the Sun for example, for analytical purposes, by a point-mass at its centre. In particular, the attraction of the Sun (mass, m_1) on a planet (mass, m_2) is the same as the attraction of a point-mass m_1, situated at the Sun's centre, on a point-mass m_2, situated at the planet's centre ; the attraction is Gm_1m_2/r^2, where r is the distance between the centres. In the sequel when we refer to the Sun and planets dynamically, we shall be concerned with point-masses the coordinates of which are the coordinates of the centres of the bodies concerned.

1·06. The Equations of Motion

For simplicity we restrict at first the number of bodies to three, namely, P_0, P, P_1 ; we assume them to be point-masses with masses m_0, m_1, m_2. In applying Newton's second law of motion we assume as a fundamental hypothesis that we can choose an unaccelerated and non-rotating axial system $O\xi$, $O\eta$, $O\zeta$ in Euclidean space with respect to which the law is in accord. For brevity we refer to such a system of axes as a Newtonian system. In Fig. 4 the coordinates of P_0, P, P_1 with respect to $O\xi$, $O\eta$, $O\zeta$ are (ξ_0, η_0, ζ_0), (ξ, η, ζ) and (ξ_1, η_1, ζ_1).

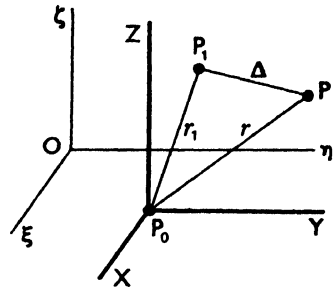


FIG. 4

Consider first the motion of P, distant r and Δ from P_0 and P_1 respectively. By 1·04 (1) the potential V at P arising from a mass m_0 at P_0 and a mass m_1 at P_1 is given by

$$V = \frac{Gm_0}{r} + \frac{Gm_1}{\Delta} \quad .$$

The component of the forces exerted by P_0 and P_1 on *unit mass* at P, parallel to $O\xi$, is $\dfrac{\partial V}{\partial \xi}$, from 1·04 (2). Hence the component of these forces on the mass m at P is $m\dfrac{\partial V}{\partial \xi}$. The rate of change of momentum of P parallel to $O\xi$ is $\dfrac{d}{dt}(m\dot{\xi})$; hence, by the second law of motion,

$$\frac{d}{dt}(m\dot{\xi}) = m\frac{\partial V}{\partial \xi} \quad ,$$

or

$$\ddot{\xi} = \frac{\partial V}{\partial \xi} = Gm_0\frac{\partial}{\partial \xi}\left(\frac{1}{r}\right) + Gm_1\frac{\partial}{\partial \xi}\left(\frac{1}{\Delta}\right) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now

$$r^2=(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2,\quad \Delta^2=(\xi-\xi_1)^2+(\eta-\eta_1)^2+(\zeta-\zeta_1)^2,$$

from which $$\frac{\partial}{\partial\xi}\left(\frac{1}{r}\right)=-\frac{(\xi-\xi_0)}{r^3}\quad,\quad \frac{\partial}{\partial\xi}\left(\frac{1}{\Delta}\right)=-\frac{(\xi-\xi_1)}{\Delta^3}$$

Hence, from (1),

$$\ddot{\xi}=-Gm_0\frac{(\xi-\xi_0)}{r^3}-Gm_1\frac{(\xi-\xi_1)}{\Delta^3} \quad \text{.....(2)}$$

Consider now the motion of P_0. The potential V_0 at P_0 arising from the mass m at P and m_1 at P_1 is given by

$$V_0=\frac{Gm}{r}+\frac{Gm_1}{r_1}\quad .$$

Hence, as before,

$$\ddot{\xi}_0=\frac{\partial V_0}{\partial\xi_0}=Gm\frac{\partial}{\partial\xi_0}\left(\frac{1}{r}\right)+Gm_1\frac{\partial}{\partial\xi_0}\left(\frac{1}{r_1}\right) \quad \text{.....(3)}$$

Now $$r_1^2=(\xi_0-\xi_1)^2+(\eta_0-\eta_1)^2+(\zeta_0-\zeta_1)^2,\ \frac{\partial}{\partial\xi_0}\left(\frac{1}{r_1}\right)=-\frac{(\xi_0-\xi_1)}{r_1^3}\quad .$$

Also, $$\frac{\partial}{\partial\xi_0}\left(\frac{1}{r}\right)=-\frac{(\xi_0-\xi)}{r^3}\quad .$$

Hence (3) becomes

$$\ddot{\xi}_0=-Gm\frac{(\xi_0-\xi)}{r^3}-Gm_1\frac{(\xi_0-\xi_1)}{r_1^3} \quad \text{.....(4)}$$

There are, of course, similar equations in η and ζ symmetrical with (2) and (4).

It is to be emphasised that all these equations of motion are derived with respect to hypothetical axes. Progress can only be made if we combine these equations in such a way as to eliminate the original coordinate axes. This is achieved in the planetary theory by referring the motions of the planets P and P_1 to parallel axes P_0X, P_0Y and P_0Z passing through P_0 which we identify with the Sun, and in the lunar theory by taking the Earth, in a similar way, as the origin of the new coordinate system (these axes are shown in Fig. 4). Sometimes the axes chosen in this way are related simply to the plane of the ecliptic, sometimes to a plane more convenient for the problem in hand such as, in satellite theory, the orbital plane of the parent planet.

1·07. The fundamental equations of planetary motion

In Fig. 4 let P_0 be the origin of the new system of coordinates P_0X, P_0Y and P_0Z; let (x, y, z) be the cordinates of P, and (x_1, y_1, z_1) of P_1. Then

$$x=\xi-\xi_0\quad,\quad x_1=\xi_1-\xi_0\quad,$$

with similar equations for y, y_1, z, z_1. In terms of the new coordinates

we have $r^2 = x^2 + y^2 + z^2$, $r_1^2 = x_1^2 + y_1^2 + z_1^2$,

$$\Delta^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \quad .$$

Now $\ddot{x} = \ddot{\xi} - \ddot{\xi}_0$ and, from 1·06 (2) and (4),

$$\ddot{\xi} = -Gm_0 \frac{x}{r^3} - Gm_1 \frac{(x - x_1)}{\Delta^3} \quad , \quad \ddot{\xi}_0 = Gm \frac{x}{r^3} + Gm_1 \frac{x_1}{r_1^3} \quad ;$$

we thus obtain, by subtraction of these two equations,

$$\ddot{x} + G(m_0 + m)\frac{x}{r^3} = -Gm_1 \frac{(x - x_1)}{\Delta^3} - Gm_1 \frac{x_1}{r_1^3} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

We write $\mu = G(m_0 + m)$. $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Also, $$\frac{x - x_1}{\Delta^3} = -\frac{\partial}{\partial x}\left(\frac{1}{\Delta}\right) \quad ,$$

and, since r_1 and the coordinates x_1, y_1, z_1 are *not* functions of x, y and z, we can write

$$\frac{x_1}{r_1^3} = \frac{\partial}{\partial x}\left(\frac{xx_1 + yy_1 + zz_1}{r_1^3}\right) \quad .$$

The equation (1) then becomes

$$\ddot{x} + \frac{\mu x}{r^3} = \frac{\partial R}{\partial x} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where R is the function defined by

$$R = Gm_1\left(\frac{1}{\Delta} - \frac{xx_1 + yy_1 + zz_1}{r_1^3}\right) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The corresponding equations in y and z are :

$$\ddot{y} + \frac{\mu y}{r^3} = \frac{\partial R}{\partial y} \quad , \quad \ddot{z} + \frac{\mu z}{r^3} = \frac{\partial R}{\partial z} \quad .$$

In these formulae R is called the *disturbing function* associated with the " disturbing " planet P_1. If P_1 were non-existent, the orbit of P around P_0 would be an ellipse according to Kepler's first law, and its equations of motion would be, since now $R = 0$,

$$\ddot{x} + \frac{\mu x}{r^3} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

with two similar equations in y and z. In general terms the effect of P_1 is to impose comparatively small changes in the orbit of P or *perturbations*, as they are called, the magnitude of these changes being proportional to m_1, the mass of the disturbing planet. Now m_1 is small compared with the Sun's mass, m_0, and the factor Gm_1 is small compared with $\mu\,[\equiv G(m_0 + m)]$ so that in equation (3) the term involving R produces comparatively small effects on the Keplerian ellipse ; but it is the determination of these small effects, or perturbations, that constitutes the core of planetary theory.

It is easily seen that if there are n disturbing planets, the equations of motion of P are

$$\ddot{x}+\frac{\mu x}{r^3}=\frac{\partial}{\partial x}\sum_1^n R_i \quad , \qquad \text{.....(6)}$$

where

$$R_i=Gm_i\left(\frac{1}{\Delta_i}-\frac{xx_i+yy_i+zz_i}{r_i^3}\right) \quad , \qquad \text{.....(7)}$$

with the corresponding equations in y and z.

If S_i denotes the angle between P_0P and P_0P_i, then

$$\cos S_i=(xx_i+yy_i+zz_i)/rr_i \quad . \qquad \text{.....(8)}$$

We can then write R_i in the alternative form

$$R_i=Gm_i\left(\frac{1}{\Delta_i}-\frac{r}{r_i^2}\cos S_i\right) \quad . \qquad \text{.....(9)}$$

1·08. The Masses of the Planets

For a solution of the planetary problem the numerical values of the masses of all the bodies contributing to the equations of motion must be known in terms of the Sun's mass as unit; it is only then that the places of the planets in the heavens can be predicted accurately for any given time. As we shall see in section 2·06 the amended form of Kepler's third law enables us to derive accurately from the appropriate observations the masses of those planets which are accompanied by satellites. In the case of Mercury and Venus, which are without satellites, the determination of mass is achieved by comparing the theoretical disturbances which they produce in the orbit of another planet with the observed effects. The mass of Pluto is not known with any approach to accuracy. In 1950 G. P. Kuiper, observing with the 200-inch telescope on Mt. Palomar, found that the angular diameter of the planet's disc was 0″·2, giving a linear diameter of 3600 miles. With a reasonable assumption as to Pluto's average density, the mass is not likely to exceed one-fifth that of the Earth. Its disturbing influence on Neptune's orbit must, consequently, be very small and perhaps undetectable for many years to come.

1·09. The adequacy of the Newtonian Frame

According to the theory of relativity space is not Euclidean as is assumed in the Newtonian scheme. However, in all problems with which we are concerned in celestial mechanics—with one exception—the Newtonian method is adequate. The exception relates to a long-unexplained feature in the motion of the perihelion of the planet Mercury which was only clarified by the application of relativistic principles. We shall consider this particular problem in greater detail in a later Chapter (v. p. 243).

Chapter 2

THE ELLIPTIC ORBIT

2·01. Equations of Motion

In this chapter we deal with the problem of two bodies, regarded as point-masses, which are supposed to be isolated from the gravitational influence of all other bodies.

Let the bodies be P_0 (mass, m_0) and P (mass, m). Our object is to derive the path of P relative to P_0. Let P_0x, P_0y, P_0z be non-rotating axes passing through P_0; the coordinates of P are (x, y, z) with respect to this axial system.

The equations of motion of P relative to P_0 are, by 1·07 (5),

$$\ddot{x}+\frac{\mu x}{r^3}=0 \quad , \quad \ddot{y}+\frac{\mu y}{r^3}=0 \quad , \quad \ddot{z}+\frac{\mu z}{r^3}=0 \quad , \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

where
$$\mu=G(m_0+m) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

These equations give the motion of a planet around the Sun, the motion of the Moon around the Earth, and the motion of one component of a binary star around the other component, in each case on the assumption that the system of two bodies is unaffected gravitationally by any other body.

Considering the equations (1), multiply the second by z and the third by y, and subtract; then

$$y\ddot{z}-\ddot{y}z=0 \quad ,$$

whence
$$y\dot{z}-\dot{y}z=A \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

in which A is the constant of integration.

Similarly,
$$z\dot{x}-\dot{z}x=B \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and
$$x\dot{y}-\dot{x}y=C \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where B and C are constants of integration.

Multiply (3), (4) and (5) by x, y and z respectively, and add; we obtain

$$Ax+By+Cz=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

This last equation shows that the coordinates of P satisfy the equation of a plane through P_0; in other words, the orbit of P lies in a certain plane through P_0.

2·02. Relation of the Orbital Plane to Reference Plane

In Fig. 5 the celestial sphere is drawn with centre at P_0; the axes P_0x, P_0y, P_0z cut the celestial sphere at X, Y, Z; we regard the plane of

the great circle XY as the reference plane, and X the reference point on this great circle.

We suppose that the orbital plane whose equation is given by 2·01 (6) cuts the celestial sphere in the great circle NAM ; the pole of this great circle (in the same hemisphere as Z) is C ; the direction-cosines of P_0C are proportional to A, B and C—the constants of integration introduced in 2·01 (6).

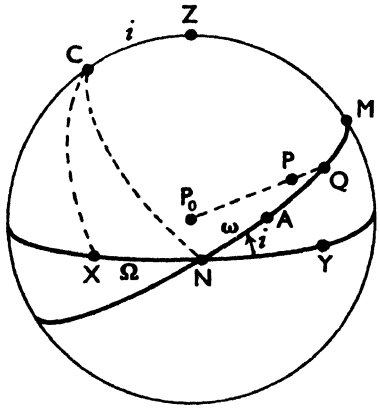


FIG. 5

Let the radius vector P_0P cut the sphere at Q. We suppose that the direction of motion of P around P_0 is in the sense $\overrightarrow{NQ}$ and that, at time τ, the radius vector meets the sphere at A, at an angular distance ω from N.

The position of the orbital plane with respect to the reference plane is specified by (i) the angular distance of N from X, denoted by Ω, and (ii) the inclination of the orbital plane to the reference plane, denoted by i which is the measure of the spherical angle ANY or the angular distance CZ.

In planetary theory, for example, the reference plane is frequently taken to be the plane of the ecliptic, X being identified with ♈, the First Point of Aries ; Z is the pole of the ecliptic on the northward side of that plane. Since the planet's radius vector sweeps from the southern side of the ecliptic through N in the direction NAQ, the point N is called the *ascending node* (usually abbreviated to *node*, simply) and the arc XN (that is, ♈N) is the *longitude of the node*.

We can interpret the constants of integration (A, B and C) of the integrals, given by 2·01 (3), (4) and (5), in terms of Ω and i, and vice versa. The expression $(x\dot{y} - \dot{x}y)$ is twice the rate of description of area swept out by the projection on the XY plane of the orbital radius vector r ; if h denotes twice the rate of description of area swept out in the plane of the orbit by the radius vector r ($h = r^2\dot{\theta}$, where θ is the vectorial angle associated with r), then

$$x\dot{y} - \dot{x}y = h \cos CZ$$

in which CZ denotes the angle between the normals of the orbital plane and the plane XY, that is, $CZ = i$. Hence, by 2·01 (5),

$$C = h \cos i.$$

Similarly, $A = h \cos CX$, $B = h \cos CY$.

From the spherical triangle CXN,

$$\cos CX = \sin \Omega \sin i$$

and from the triangle CYN,

$$\cos CY = -\cos \Omega \sin i \quad .$$

We then have

$$A = h \sin \Omega \sin i, \quad B = -h \cos \Omega \sin i, \quad C = h \cos i \quad , \quad \ldots\ldots\ldots(1)$$

whence $$h^2 = A^2 + B^2 + C^2 \quad .$$

Thus h is constant and Kepler's second law is verified.

From (1), $\tan \Omega = -A/B$ and $\tan i = \sqrt{(A^2 + B^2)}/C$.

2·03. Equations of Motion in the Plane of the Orbit

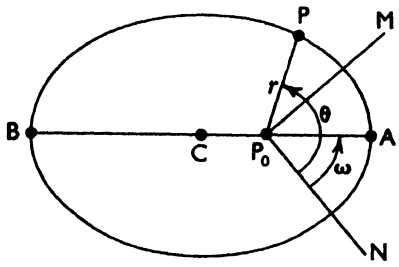


FIG. 6

Take rectangular axes P_0N, P_0M in the plane of the orbit, P_0N being the direction of the node; let the coordinates of P in Fig. 6 be (x_1, y_1). Let θ be the vectorial angle of P_0P with respect to P_0N. Then, the plane of the orbit being regarded for the moment as the reference plane, the equations of motion of P are

$$\ddot{x}_1 + \frac{\mu x_1}{r^3} = 0 \quad , \quad \ddot{y}_1 + \frac{\mu y_1}{r^3} = 0 \quad ,$$

which may be written

$$m\ddot{x}_1 = -\frac{m\mu}{r^2} \cdot \frac{x_1}{r} \quad , \quad m\ddot{y}_1 = -\frac{m\mu}{r^2} \cdot \frac{y_1}{r} \, ;$$

these are the Newtonian equations of motion of a particle of mass m relative to a point P_0 and subjected to an attraction $\frac{m\mu}{r^2}$ in the direction PP_0. Then, if α and β denote the accelerations along P_0P and transverse to P_0P, we have

$$\alpha = -\frac{\mu}{r^2} \quad , \quad \beta = 0 \quad .$$

But, $\alpha = \ddot{r} - r\dot{\theta}^2$ and $\beta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$. Hence the equations of motion of P in polar coordinates are

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \quad \ldots\ldots\ldots(1)$$

and $$r^2\dot{\theta} = h \quad , \quad \ldots\ldots\ldots(2)$$

where h is a constant.

The equation (2) is the analytical expression of Kepler's second law, h being twice the rate of description of area by the radius vector and

equal to $2\pi ab/T$. As in section 1·02 we define the mean angular motion, n, by

$$n = 2\pi/T \quad \ldots\ldots(3)$$

and hence

$$h = na^2\sqrt{(1-e^2)} \quad \ldots\ldots(4)$$

2·04. The polar equation of the orbit

To obtain the polar equation of the orbit we eliminate t between the equations

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \quad \ldots\ldots(1)$$

and

$$r^2\dot{\theta} = h \quad \ldots\ldots(2)$$

Write $u = \frac{1}{r}$; then, by (2), $\dot{\theta} = hu^2$.

Now

$$\dot{r} = \frac{d}{d\theta}\left(\frac{1}{u}\right)\dot{\theta} = -h\frac{du}{d\theta},$$

and

$$\ddot{r} = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Equation (1) then becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \quad \ldots\ldots(3)$$

The general solution of this equation is

$$u = \frac{\mu}{h^2}\{1 + e\cos(\theta - \omega)\}$$

in which e and ω are constants of integration.

Hence

$$r = \frac{h^2/\mu}{1 + e\cos(\theta - \omega)} \quad \ldots\ldots(4)$$

This is the equation of an ellipse of eccentricity e if $0<e<1$, and its semi-latus rectum is h^2/μ; hence

$$h^2 = \mu a(1-e^2) \quad \ldots\ldots(5)$$

But, from 2·03 (4),

$$h = na^2\sqrt{(1-e^2)} \quad \ldots\ldots(6)$$

From (5) and (6) we obtain

$$\mu = n^2a^3 \quad \ldots\ldots(7)$$

or, since $n = 2\pi/T$ and $\mu = G(m_0 + m)$,

$$\frac{a^3}{T^2} = \frac{G(m_0+m)}{4\pi^2} \quad \ldots\ldots(8)$$

which is the amended form of Kepler's third law.

We note that (4) is the general equation of a conic; consequently, the path of a body P, relative to P_0, under the law of gravitation can be

(i) an ellipse, if $0<e<1$ (or a circle, if $e=0$),

(ii) a parabola, if $e=1$,

(iii) a hyperbola, if $e>1$.

We shall be principally concerned with case (i), that is when the orbit is an ellipse with P_0 at a focus ; accordingly, the derivation of (4) constitutes a proof of Kepler's first law.

2·05. Amended Form of Kepler's Third Law

From the previous section we have

$$4\pi^2 \frac{a^3}{T^2} = G(m_0 + m) \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which we associate P_0 (mass, m_0) with the Sun and P (mass, m) with a planet.

If a_1 and T_1 refer to the orbit of another planet (mass, m_1) around the Sun, we have

$$4\pi^2 \frac{a_1^3}{T_1^2} = G(m_0 + m_1) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2),

$$\frac{a^3}{T^2} \Big/ \frac{a_1^3}{T_1^2} = \left(1 + \frac{m}{m_0}\right) \Big/ \left(1 + \frac{m_1}{m_0}\right) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The ratios m/m_0 and m_1/m_0 are small, the maximum value for any such ratio being 1/1047 for the planet Jupiter.

If these small ratios are neglected, (3) becomes

$$\frac{a^3}{T^2} = \frac{a_1^3}{T_1^2} \quad ,$$

which is Kepler's third law relating to two planets. Clearly, the law can be extended to any number of planets. When we refer subsequently to Kepler's third law we shall be concerned with formula (1), the exact embodiment of the law.

2·06. The masses of the planets

We illustrate the method of deriving the mass of a planet, which is accompanied by a satellite, in the case of Mars and its satellite Deimos.

We suppose that the undisturbed orbit of Deimos (mass, m_1) under the central attraction of Mars (mass, m) is an ellipse with semi-major axis a_1 and period T_1. Kepler's third law in its amended form is applicable to the satellite's orbit around Mars and we have

$$4\pi^2 \frac{a_1^3}{T_1^2} = G(m + m_1) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If a and T refer to the orbit of Mars around the Sun (mass, m_0) then

$$4\pi^2 \frac{a^3}{T^2} = G(m_0 + m) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Since m_1 is very small compared with m, we can discard m_1 in (1); then from (1) and (2), we have

$$\frac{m_0+m}{m}=\left(\frac{a}{a_1}\right)^3\left(\frac{T_1}{T}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The ratios a/a_1 and T/T_1 are obtained from a study of the orbits; accordingly, we readily derive the ratio of the mass of Mars to that of the Sun by means of (3).

As we have remarked earlier—section 1·08—the mass of a planet, unaccompanied by a satellite, is obtained by other methods.

2·07. Elements of the elliptic orbit

From 2·04 (4) and (5), the polar equation of the orbit is

$$r=\frac{a(1-e^2)}{1+e\cos(\theta-\omega)} \quad ,$$

in which $a(1-e^2)$ replaces h^2/μ. Now, r is a minimum when $\cos(\theta-\omega)$ is a maximum—that is, when $\theta=\omega$; accordingly, the minimum value of r is $a(1-e)$, corresponding to the perihelion distance P_0A (Fig. 6, p. 13). Similarly, the maximum value of r is obtained when $\theta-\omega=\pi$; it is $a(1+e)$, corresponding to the aphelion distance P_0B. The angle ω thus determines the orientation of the major axis BA with respect to the direction P_0N from which the vectorial angle θ is measured.

a, e and ω are three *elements* of the orbit. A fourth element is τ, the time when the planet passes through perihelion. Two more elements are required to specify the orbit completely with respect to the reference plane and to a reference direction in this plane. When the reference plane is the ecliptic (v. Fig. 5, p. 12) these elements are Ω (the longitude of the node) and i (the inclination of the orbital plane to the ecliptic).

The six elements are, for convenience, arranged in two groups, each with three members. The first group consists of

$$a, \quad e \quad , \quad \tau \quad ,$$

which refer to the ellipse irrespective of its relation to the reference plane; the second group consists of

$$\Omega, \quad \omega, \quad i,$$

the elements Ω and i defining the plane of the ellipse with respect to the ecliptic and the First Point of Aries, and the element ω defining the orientation of the major axis of the orbit with reference to the ascending node N.

These six elements arise naturally from simple geometrical considerations. However, it may be more convenient to regard as constants of the ellipse any six independent functions of a, e, ... i; for example, in some theories the constant $h(\equiv\{\mu a(1-e^2)\}^{1/2})$, which arises naturally in the dynamical solution, is taken as an independent element.

2·08. The true and eccentric anomalies

The *true anomaly* (denoted by f) is defined to be the vectorial angle of the planet measured from P_0A (Fig. 7). With our previous notation we write $f=\theta-\omega$, and the equation of the orbit is then

$$r=\frac{a(1-e^2)}{1+e\cos f} \quad \text{.....(1)}$$

It is evident that the areal equation can be written

$$r^2\dot{f}=h. \quad \text{.....(2)}$$

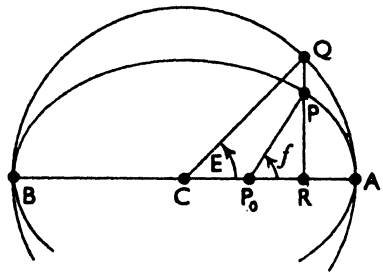


FIG. 7

The polar coordinates of P with respect to P_0A as initial line are (r, f). The corresponding rectangular coordinates are P_0R and RP; denoting these by (ξ, η) we have

$$\xi=r\cos f \quad , \quad \eta=r\sin f \quad \text{.....(3)}$$

With centre C and radius a draw a circle; the ordinate of P cuts this circle at Q. The angle QCA is called the *eccentric anomaly*, denoted by E. Then

$$CR=a\cos E \quad , \quad RQ=a\sin E \quad \text{.....(4)}$$

Since the equation of the ellipse, referred to C as origin and CA as x-axis, is $x^2/a^2+y^2/b^2=1$, where $b^2=a^2(1-e^2)$, the ordinate PR is given in terms of E by

$$PR=\frac{b}{a}(a^2-a^2\cos^2 E)^{1/2}=b\sin E \quad .$$

Hence

$$RP : RQ=b : a \quad \text{.....(5)}$$

Since $CP_0=ae$, we have

$$\xi\equiv r\cos f=a(\cos E-e) \quad \text{.....(6)}$$

and

$$\eta\equiv r\sin f=b\sin E \quad \text{.....(7)}$$

The radius vector, r, is found in terms of E by squaring (6) and (7), and adding; we then obtain

$$r=a(1-e\cos E) \quad \text{.....(8)}$$

From (6),

$$r\left(2\cos^2\frac{f}{2}-1\right)=a(\cos E-e)$$

or, using (8),

$$r\cos^2\frac{f}{2}=a(1-e)\cos^2\frac{E}{2} \quad \text{.....(9)}$$

Similarly, from (7), $r\left(1-2\sin^2\dfrac{f}{2}\right)=a(\cos E-e)$,

from which, using (8),

$$r \sin^2 \frac{f}{2} = a(1+e) \sin^2 \frac{E}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

From (9) and (10),

$$\tan \frac{f}{2} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan \frac{E}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

This last equation expresses the true anomaly in terms of the eccentric anomaly, and vice versa.

2·09. The mean anomaly

At time t we suppose that the planet is at P (Fig. 7); in the interval $(t-\tau)$ the radius vector has swept out the angle PP_0A. The angle swept out in the interval $(t-\tau)$ by a radius vector moving with the mean angular velocity n, the position of the radius vector at time τ being along P_0A, is called the *mean anomaly*. If we denote the mean anomaly by M, then

$$M = n(t-\tau) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

2·10. Kepler's equation

This equation is a relation between the eccentric anomaly E and the mean anomaly M; it is

$$E - e \sin E = M \quad .$$

We derive it as follows:

From $r = a(1 - e \cos E)$

we have $\dot{r} = ae \sin E \,.\, \dot{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

From $\frac{1}{r} = \frac{1 + e \cos f}{a(1-e^2)}$

we have $\frac{\dot{r}}{r^2} = \frac{e \sin f \,.\, \dot{f}}{a(1-e^2)}$

from which, since $r^2\dot{f} = h = na^2\sqrt{(1-e^2)}$,

$$\dot{r} = \frac{nae \sin f}{\sqrt{(1-e^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2), $\dot{E} = \frac{n \sin f}{\sqrt{(1-e^2)} \sin E}$.

But, from 2·08 (3), $\eta \equiv r \sin f = a\sqrt{(1-e^2)} \sin E$.

Hence $r\dot{E} = na$,

or $(1 - e \cos E)\dot{E} = n$.

Integrating, we obtain

$$E - e \sin E = nt + c \quad ,$$

where c is the constant of integration. But $E = 0$ when $t = \tau$; hence $c = -n\tau$. We then have

$$E - e \sin E = M \equiv n(t-\tau) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

which is Kepler's equation, the link between the eccentric anomaly and the mean anomaly.

2·11. Solution of Kepler's equation

We suppose that the elements e and τ of the orbit and the mean motion n are all known. Accordingly, we can calculate the value of M for time t. We have then to determine the value of E, for time t, from the equation

$$E - e \sin E = M. \quad \ldots\ldots(1)$$

(*a*) If e is small, say of order 0·1, as in the case of most of the planetary orbits, we can calculate E by a very useful method due to E. W. Brown.*

We put

$$E = M + x \quad \ldots\ldots(2)$$

where x is to be determined.

Then from (1)

$$x = e \sin (M + x) \quad \ldots\ldots(3)$$

It is evident from (3) that x, expressed in circular measure, is of the order of e.

Now,
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots$$
$$= e \sin (M + x) - \frac{e^3}{6} \sin^3 (M + x) + \frac{e^5}{120} \sin^5 (M + x) - \ \ldots$$

or
$$\sin x (1 - e \cos M) - (e \sin M) \cos x$$
$$= -\frac{e^3}{6} \sin^3 (M + x) + \frac{e^5}{120} \sin^5 (M + x) - \ \ldots\ldots(4)$$

Let C and x_0 be defined by the equations

$$C \sin x_0 = e \sin M \quad \ldots\ldots(5)$$
$$C \cos x_0 = 1 - e \cos M \quad , \quad \ldots\ldots(6)$$

from which
$$\tan x_0 = e \sin M / (1 - e \cos M) \quad , \quad \ldots\ldots(7)$$

$$C \sin (M + x_0) = \sin M \quad , \quad \ldots\ldots(8)$$

and
$$C^2 = 1 - 2e \cos M + e^2 ; \quad \ldots\ldots(9)$$

We calculate x_0 from (7) and C from (8)—unless M is close to 90° or 270° when (9) should be used for deriving C. It is to be noted that x_0 is of order e. Using the formulae (5) and (6) we find that (4) becomes

$$\sin (x - x_0) = -\frac{e^3}{6C} \sin^3 (M + x) + \frac{e^5}{120C} \sin^5 (M + x) \quad \ldots\ldots(10)$$

Formula (9) shows that C is of order unity; accordingly, (10) shows that $x - x_0$ is of order e^3, so that if we write x_0 for x in the first term on the

* M.N.R.A.S., **92**, 104, 1931.

right-hand side of (10) and disregard the second term, we obtain the equation

$$\sin(x - x_0) = -\frac{e^3}{6C}\sin^3(M + x_0) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

which is correct to the order e^4. This equation can be written in an alternative form ; from (5) and (6)

$$\sin x_0 (1 - e\cos M) = e\cos x_0 \sin M \quad ,$$

from which

$$\sin x_0 = e\sin(M + x_0) \, ;$$

we then obtain from (11)

$$\sin(x - x_0) = -\frac{1}{6C}\sin^3 x_0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

From (11) or (12) the value of x can be readily calculated ; let it be denoted by x'. For those planetary orbits with small values of e, this value x' will generally prove sufficiently accurate ; the solution of Kepler's equation then is :

$$E = M + x' \quad .$$

If e has a somewhat larger value, or if greater accuracy is required, we can obtain the appropriate value of x by substituting x' for x in the first term on the right-hand side of (10) and x_0 for x in the second term, and then calculating x from

$$\sin(x - x_0) = -\frac{e^3}{6C}\sin^3(M + x') + \frac{e^5}{120C} \cdot \sin^5(M + x_0) \quad . \quad \ldots\ldots (13)$$

If x'' is the value of x derived from (13), the solution of Kepler's equation is then :

$$E = M + x'' \quad .$$

(*b*) When e is not comparatively small, as in (*a*), the procedure of solution depends on starting with a value of E which gives an approximate solution of Kepler's equation. Such a value may be obtained by one of the many graphical methods devised for this purpose—of which there are about 120—or by means of special tables ; * such tables give the values of M in Kepler's equation calculated for selected values of E and e—for example, E at intervals of 1° and e at intervals of 0·05—and it is then easy to obtain by inspection or interpolation a value E_0 which satisfies Kepler's equation approximately for the given values of e and M.

Let $E_0 + dE$ be the value of E which satisfies the equation accurately. Then

$$E_0 + dE - e\sin(E_0 + dE) = M \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

* For example, J. Bauschinger, *Tafeln zur theoretischen Astronomie* (Leipzig, 1901), or J. J. Astrand, *Hülfstafeln* (Leipzig, 1890).

To the first order in dE, we obtain

$$dE = \frac{M - M_0}{1 - e\cos E_0} \quad , \qquad \text{.....(15)}$$

where $M_0 \equiv E_0 - e\sin E_0$ and is therefore a calculable quantity. Thus the more accurate value, $E_0 + dE$, can be obtained.

To secure a still more accurate value of E satisfying Kepler's equation we can repeat the process described by taking $E_0 + dE$ as the new approximate value.

2·12. Velocity of a planet in its orbit

Let v be the velocity of the planet in its orbit. Then

$$v^2 = \dot{\xi}^2 + \dot{\eta}^2 \quad .$$

Now $$\xi = a(\cos E - e) \quad , \quad \eta = a\sqrt{(1-e^2)}\sin E \; ;$$

hence, $$v^2 = a^2(1 - e^2\cos^2 E)\dot{E}^2 \quad .$$

From Kepler's equation we have

$$(1 - e\cos E)\dot{E} = n \; ; \qquad \text{.....(1)}$$

hence, $$v^2 = n^2a^2\frac{(1+e\cos E)}{(1-e\cos E)} = \frac{\mu}{r}\{2 - (1 - e\cos E)\} \quad ,$$

or $$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \quad . \qquad \text{.....(2)}$$

This is the formula required.

2·13. Modification of the elliptic elements

The orbital elements with which we have been concerned so far are :

$$a,\ e,\ \tau \text{ and } \Omega,\ \omega,\ i \; ;$$

the latter group are shown in Fig. 8 with respect to the reference plane and axes defined by X, Y, and Z.

(i) The element τ occurs in Kepler's equation

$$E - e\sin E = n(t - \tau) \quad ,$$

and it is sometimes convenient to replace it by an element χ, equal to $-n\tau$, so that Kepler's equation becomes

$$E - e\sin E = nt + \chi \quad . \qquad \text{.....(1)}$$

The elements of the orbit are then

$$a,\ e,\ \chi \text{ and } \Omega,\ \omega,\ i.$$

(ii) A more far-reaching modification of the original elements involves the replacement of ω and of τ (or χ). When the reference plane is the ecliptic, the *longitude* of any point on the great circle NAQ is defined in planetary theory as the angle measured from the reference point X (or Υ)

to the ascending node N and then along the orbital great circle to the point concerned. Thus, the *longitude of perihelion*, denoted by ϖ, is the longitude of A and is defined in terms of Ω and ω by

$$\varpi = \Omega + \omega \; ; \quad \ldots\ldots\ldots\ldots (2)$$

we can thus replace ω by $(\varpi - \Omega)$.

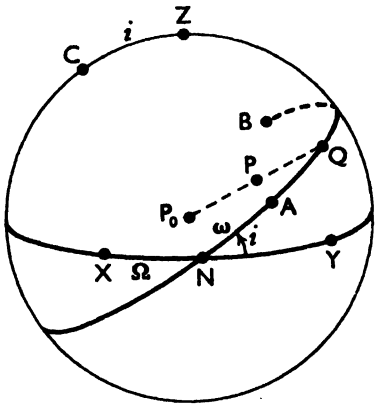


FIG. 8

If Q, in Fig. 8, is the interesection of the radius vector P_0P with the sphere, AQ is the corresponding true anomaly, f; the *true longitude of the planet*, denoted here by L, is the longitude of Q; accordingly,

$$L = \Omega + \omega + f$$

or
$$L = \varpi + f \quad . \quad \ldots\ldots\ldots\ldots (3)$$

In the same way, the *mean longitude* of the planet, denoted here by l, is associated with the mean anomaly M—that is, $n(t-\tau)$ or $nt+\chi$—by the formula

$$l = \varpi + n(t-\tau) \quad . \quad \ldots\ldots\ldots\ldots (4)$$

Instead of τ (or χ) an element ϵ, called the *mean longitude at the epoch*, is introduced; it is the mean longitude at the moment from which t is reckoned; ϵ is then given from (4) by putting $t=0$. Hence

$$\epsilon = \varpi - n\tau \quad \ldots\ldots\ldots\ldots (5)$$

and, eliminating τ between (4) and (5), the mean longitude, l, is given by

$$l = nt + \epsilon \quad . \quad \ldots\ldots\ldots\ldots (6)$$

By (4) the mean anomaly M, is given by

$$M = l - \varpi \; ;$$

hence, using (6)
$$M = nt + \epsilon - \varpi \quad . \quad \ldots\ldots\ldots\ldots (7)$$

In this system, the elements are

$$a,\ e,\ \epsilon,\ \Omega,\ \varpi,\ i.$$

2·14. Summary of the formulae of elliptic motion

The principal formulae derived in the previous sections are collected here for reference:

$$r = \frac{a(1-e^2)}{1+e\cos f} \quad , \quad \ldots\ldots\ldots\ldots (1)$$

$$n^2a^3 = \mu = G(m_0+m) \quad , \quad \ldots\ldots\ldots\ldots (2)$$

$$h^2 = \mu a(1-e^2) \quad , \quad \ldots\ldots\ldots\ldots (3)$$

$$\left.\begin{aligned} E - e\sin E = M &= n(t-\tau) \\ &= nt + \chi \\ &= nt + \epsilon - \varpi \end{aligned}\right\} \quad , \quad \ldots\ldots\ldots\ldots (4)$$

$$r = a(1 - e\cos E), \quad \ldots\ldots\ldots\ldots (5)$$

$$\tan \tfrac{1}{2}f = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan \tfrac{1}{2}E \quad \ldots\ldots\ldots\

(ii) In this stage we derive the heliocentric equatorial coordinates of the planet with respect to the equatorial axes indicated in Fig. 9 by X (or Υ), F and G.

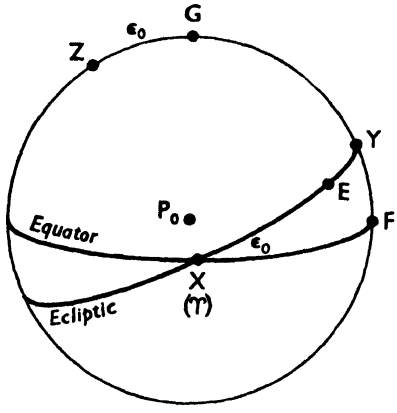


FIG. 9

Let ϵ_0 denote the obliquity of the ecliptic. If (x_1, y_1, z_1) are the coordinates of the planet in the equatorial system, then

$$\left.\begin{aligned} x_1 &= x \\ y_1 &= y\cos\epsilon_0 - z\sin\epsilon_0 \\ z_1 &= y\sin\epsilon_0 + z\cos\epsilon_0 \end{aligned}\right\} \quad \ldots\ldots\ldots(5)$$

From these, ϵ_0 being known, the values of x_1, y_1 and z_1 are derived.

Alternatively, we can combine the principal formulae of (i) and (ii), as follows. From (2) and (5), we have

$$\left.\begin{aligned} x_1 &= aP_x\cos E + bQ_x\sin E - aeP_x \\ y_1 &= aP_y\cos E + bQ_y\sin E - aeP_y \\ z_1 &= aP_z\cos E + bQ_z\sin E - aeP_z \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots(6)$$

where

$$\left.\begin{aligned} P_x &= l_1 \quad , & Q_x &= l_2 \\ P_y &= m_1\cos\epsilon_0 - n_1\sin\epsilon_0 \quad , & Q_y &= m_2\cos\epsilon_0 - n_2\sin\epsilon_0 \\ P_z &= m_1\sin\epsilon_0 + n_1\cos\epsilon_0 \quad , & Q_z &= m_2\sin\epsilon_0 + n_2\cos\epsilon_0 \end{aligned}\right\} \quad \ldots\ldots\ldots(7)$$

The quantities $P_x, \ldots Q_z$ are calculated from the values of $l_1, \ldots n_2$ and the value of ϵ_0. Then x_1, y_1 and z_1 are obtained after the appropriate value of E has been found.

(iii) In the third stage we derive the heliocentric equatorial coordinates of the Earth ; let these be (x_1', y_1', z_1'). In Fig. 9 let E be the point on the ecliptic corresponding to the Earth's radius vector at time t. Let r', f' and ϖ' refer to the Earth in its orbit. Then $XE = \varpi' + f'$ and, since

$$x_1' = r'\cos EX \quad , \quad y_1' = r'\cos EF \quad , \quad z_1' = r'\cos EG \quad ,$$

we obtain

$$\begin{aligned} x_1' &= r'\cos(\varpi' + f') \quad , \\ y_1' &= r'\sin(\varpi' + f')\cos\epsilon_0 \quad , \\ z_1' &= r'\sin(\varpi' + f')\sin\epsilon_0 \quad . \end{aligned}$$

With the elements of the Earth's orbit known, the eccentric anomaly E' can be calculated in the usual way ; then r' and f' are calculated by the formulae in section 2·14.

If (X, Y, Z) are the Sun's geocentric equatorial coordinates, then $X = -x_1'$, $Y = -y_1'$, $Z = -z_1'$, the axes being parallel to, and in the same direction as, the equatorial axes in Fig. 9. The coordinates X, Y, Z are tabulated in the *Nautical Almanac* for the beginning of every day through-

out the year; accordingly, they can be found by interpolation for any time t.

(iv) Let (ξ, η, ζ) be the geocentric equatorial coordinates of the planet

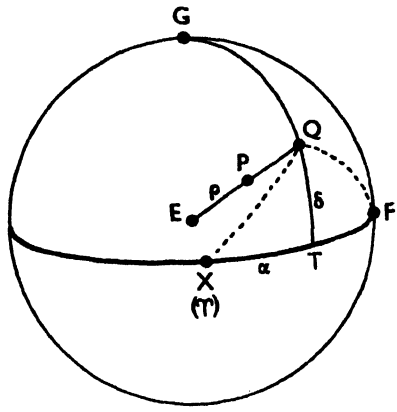


FIG. 10

P at time t (Fig. 10). Then

$$\xi = x_1 - x_1' \quad , \quad \eta = y_1 - y_1' \quad , \quad \zeta = z_1 - z_1' \quad ,$$

or

$$\xi = x_1 + X \quad , \quad \eta = y_1 + Y \quad , \quad \zeta = z_1 + Z \quad ,$$

from which the values of ξ, η and ζ can be obtained.

Let the radius vector EP meet the sphere at Q; GQT is the meridian through Q; XT is the right ascension α, and TQ is the declination δ. Denoting EP by ρ, we have

$$\xi = \rho \cos Q\Upsilon \quad , \quad \eta = \rho \cos QF \quad , \quad \zeta = \rho \cos QG$$

from which we obtain

$$\xi = \rho \cos \alpha \cos \delta \quad , \quad \eta = \rho \sin \alpha \cos \delta \quad , \quad \zeta = \rho \sin \delta \quad .$$

From these,

$$\tan \alpha = \eta / \xi \quad , \quad \tan \delta = \zeta / \sqrt{(\xi^2 + \eta^2)} \quad .$$

Thus, the coordinates α and δ are obtained.

(v) *Summary.* The formulae derived in this section are readily applied to the preparation of an *ephemeris* of a planet—its predicted coordinates (α, δ) at intervals, say, of 10 days. From the known elements of the orbit, the quantities $P_x \ldots Q_z$, which are functions of the elements and the obliquity of the ecliptic, are used in all the calculations for times $t_1, t_2, \ldots$; the values of $E_1, E_2, \ldots$ corresponding to these times have of course to be calculated separately by the procedure outlined in section 2·11. When these computations are completed, we calculate from (6) the heliocentric equatorial coordinates for each of $t_1, t_2, \ldots$. The procedures in stages (iii) and (iv) enable us finally to derive the right ascensions and declinations at the times required.

Chapter 3

EXPANSIONS FOR AN ELLIPTIC ORBIT

3·01. Introduction

In this chapter we deal with the expansions of various functions relating to the elliptic orbit; most of our results form a preparation for the expansion, in Chapter 7, of the disturbing function R—introduced in 1·07 (4)—in a form suitable for the eventual solution of the equations of motion of a planet P disturbed by a planet P_1, or of a satellite around its primary.

From 2·14 we have the following group of formulae associated with the elliptic orbit:

$$r = \frac{a(1-e^2)}{1+e\cos f} \quad , \qquad \text{.....(1)}$$

$$\tan\frac{f}{2} = \sqrt{\left(\frac{1+e}{1-e}\right)}\tan\frac{E}{2} \quad , \qquad \text{.....(2)}$$

$$r = a(1-e\cos E) \qquad \text{.....(3)}$$

$$E - e\sin E = M = n(t-\tau) \quad . \qquad \text{.....(4)}$$

In (1), r is expressed in terms of the true anomaly f and, in (3), in terms of the eccentric anomaly E. Kepler's equation, (4), enables us to express E in terms of the mean anomaly M. To express any function $\psi(r, f)$ directly in terms of the mean anomaly—that is, essentially in terms of t—we make use of the fact that, in planetary and satellite theory generally, the eccentricity e is small and consequently we can develop $\psi(r, f)$ in the form of periodic series in terms of powers of e. This process usually consists of two parts; in the first, $\psi(r, f)$ is expressed as a series of the general form

$$\sum_{n=0}^{\infty}(A_n\cos nE + B_n\sin nE)$$

in which the coefficients A and B are usually power series in e; in the second, $\cos nE$ and $\sin nE$ are converted into periodic series of the forms, respectively,

$$a_0 + a_1\cos M + a_2\cos 2M + \ \ldots$$

and

$$b_1\sin M + b_2\sin 2M + \ \ldots\ ,$$

the coefficients a and b being generally power series in e.

Before proceeding to the development in series we shall devote the following three sections to the consideration of an expansion theorem of Lagrange, Bessel functions, and hypergeometric series which have their place in the subsequent analysis.

3·02. Lagrange's expansion theorem

Consider the relation between x and y given by

$$y = x + \alpha\phi(y) \quad \ldots\ldots(1)$$

in which ϕ denotes a function and α is a small parameter later to be identified with $e/2$. The theorem states that any function $F(y)$ can be developed in a power series in α as follows :

$$F(y) = F_x + \frac{\alpha}{1!}\left(\phi_x \frac{\partial F_x}{\partial x}\right) + \frac{\alpha^2}{2!}\frac{\partial}{\partial x}\left(\phi_x^2 \frac{\partial F_x}{\partial x}\right) + \ldots + \frac{\alpha^n}{n!}\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\phi_x^n \frac{\partial F_x}{\partial x}\right) + \ldots\ldots(2)$$

where $F_x \equiv F(x)$ and $\phi_x \equiv \phi(x)$; it is assumed that ϕ and F and their derivatives are continuous functions.

Regarded as an equation in y, (1) can be *solved* in principle so as to express y in terms of α and x ; for example, if α^2 is neglected then we obtain the approximation $y = x + \alpha\phi(x)$. Suppose then that $y = f(\alpha, x)$ is the result of solving (1). Then $F(y)$ can be expressed as a function of α and x, and the application of Maclaurin's theorem gives

$$F(y) = (F)_0 + \frac{\alpha}{1!}\left(\frac{\partial F}{\partial \alpha}\right)_0 + \frac{\alpha^2}{2!}\left(\frac{\partial^2 F}{\partial \alpha^2}\right)_0 + \ldots + \frac{\alpha^n}{n!}\left(\frac{\partial^n F}{\partial \alpha^n}\right)_0 + \ldots$$

in which the zero suffix indicates that after differentiation α is to be equated to zero.

Let A denote the operator $\dfrac{\partial}{\partial \alpha}$; then the previous equation becomes

$$F(y) = F_x + \frac{\alpha}{1!}(AF)_0 + \frac{\alpha^2}{2!}(A^2F)_0 + \ldots + \frac{\alpha^n}{n!}(A^nF)_0 + \ldots, \quad \ldots\ldots(3)$$

and there also follows from (1)

$$Ay = \phi(y) + \alpha\frac{d\phi}{dy}Ay \quad . \quad \ldots\ldots(4)$$

Let D denote the operator $\dfrac{\partial}{\partial x}$; then, from (1),

$$Dy = 1 + \alpha\frac{d\phi}{dy}Dy \quad . \quad \ldots\ldots(5)$$

Multiply (5) by $\phi(y)$ and subtract from (4) ; then,

$$\left(Ay - \phi Dy\right)\left(1 - \alpha\frac{d\phi}{dy}\right) = 0.$$

Now, $\quad 1 - \alpha\dfrac{d\phi}{dy} \neq 0$; for, if $1 - \alpha\dfrac{d\phi}{dy} = 0$, then

$\alpha\phi = y +$ constant, which is inconsistent with (1) in which ϕ is supposed to be any function of y. Consequently,

$$Ay = \phi\, Dy \quad \ldots\ldots(6)$$

which is the relation between the operators A and D consistent with (1).

Again, $$AF(y) \equiv \frac{\partial F(y)}{\partial \alpha} = \frac{dF}{dy} Ay = \phi \frac{dF}{dy} Dy$$

by means of (6), and

$$DF(y) \equiv \frac{\partial F(y)}{\partial x} = \frac{dF}{dy} Dy ;$$

hence $$AF = \phi\, DF \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

We now prove the formula

$$A^n F = D^{n-1}(\phi^n\, DF) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Suppose that it is true ; operate on (8) with A ; then, since α and x are independent so that A and D are independent operators obeying the commutative law, it follows that

$$A^{n+1} F = D^{n-1}\{A(\phi^n DF)\} \quad .$$

Now,
$$\begin{aligned} A(\phi^n\, DF) &= DF \,.\, A(\phi^n) + \phi^n \,.\, D(AF) \\ &= DF \,.\, \phi D(\phi^n) + \phi^n \,.\, D(AF) \quad , \text{ by (6)}, \\ &= \phi\, DF \,.\, D(\phi^n) + \phi^n\, D(\phi DF) \quad , \text{ by (6)}, \\ &= D(\phi^{n+1}\, DF) \quad . \end{aligned}$$

Hence

$$A^{n+1} F = D^n(\phi^{n+1}\, DF) \quad .$$

Consequently, if (8) is true, it is also true when n is replaced by $n+1$. But (8) is true when $n = 1$, for it then reduces to (7) ; accordingly, (8) is generally true.

Now, in terms of differential coefficients, (8) is

$$\frac{\partial^n F}{\partial \alpha^n} = \frac{\partial^{n-1}}{\partial x^{n-1}} \left(\phi^n \frac{\partial F}{\partial x} \right) ;$$

hence $$\left(\frac{\partial^n F}{\partial \alpha^n} \right)_0 = \frac{\partial^{n-1}}{\partial x^{n-1}} \left(\phi_x^n \frac{\partial F_x}{\partial x} \right) \quad .$$

Inserting this last result in (3), we see that the theorem expressed by (2) is established.

The preceding proof of (2) follows that given by E. W. Brown and C. A. Shook in *Planetary Theory*, p. 37, 1933 ; an extension of the theorem is also given by these authors.

3·03. Bessel functions

Consider the expression

$$U \equiv e^{\frac{x}{2}(z - z^{-1})} = e^{\frac{x}{2} z} \,.\, e^{-\frac{x}{2} z^{-1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^n z^n}{n!} \cdot \sum_{m=0}^{\infty} \frac{\left(-\frac{x}{2}\right)^m z^{-m}}{m!} . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Let $J_p(x)$ denote the coefficient of z^p when the series in (2) are multiplied together. We then have

$$U = J_0(x) + J_1(x)z + J_2(x)z^2 + \dots + J_{-1}(x)z^{-1} + J_{-2}(x)z^{-2} + \quad \dots\dots(3)$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)z^n \quad . \quad \dots\dots(4)$$

It is easily seen from (2) that

$$J_n(x) = \frac{\left(\frac{x}{2}\right)^n}{n!}\left\{1 - \frac{\left(\frac{x}{2}\right)^2}{1\,.\,n+1} + \frac{\left(\frac{x}{2}\right)^4}{1\,.\,2\,.\,n+1\,.\,n+2} - \dots\right\} \quad . \quad \dots(5)$$

The function $J_n(x)$ is called a Bessel Function of the first kind of order n.

We now proceed to obtain several relationships involving Bessel functions.

(i) It is seen from (1) that U remains unaltered if we write $(-z^{-1})$ for z; hence, from (3),

$$U = J_0(x) - J_1(x)z^{-1} + J_2(x)z^{-2} - \dots - J_{-1}(x)z + J_{-2}(x)z^2 - \dots \quad \dots\dots(6)$$

Comparing (3) and (6) we have

$$J_{-1}(x) = -J_1(x)\,; \quad J_{-2}(x) = J_2(x)\,; \quad \dots$$

and generally

$$J_{-n}(x) = (-1)^n J_n(x) \quad . \quad \dots\dots(7)$$

(ii) From (1)

$$\frac{\partial U}{\partial z} = \frac{x}{2}(1 + z^{-2})\,U,$$

and, from (4),

$$\frac{\partial U}{\partial z} = \sum_{n=-\infty}^{\infty} nJ_n(x)z^{n-1} = \sum_{n=-\infty}^{\infty} (n+1)\,J_{n+1}(x)z^n \quad .$$

Hence

$$\frac{x}{2}(1 + z^{-2}) \sum_{n=-\infty}^{\infty} J_n(x)z^n = \sum_{-\infty}^{\infty} (n+1)\,J_{n+1}(x)z^n \quad . \quad \dots\dots(8)$$

Equating coefficients of z^n in (8) we have

$$\frac{x}{2}\{J_n(x) + J_{n+2}(x)\} = (n+1)\,J_{n+1}(x) \quad ,$$

or, on writing $n-1$ for n,

$$nJ_n(x) = \frac{x}{2}\{J_{n-1}(x) + J_{n+1}(x)\} \quad . \quad \dots\dots(9)$$

(iii) Let $J'(x)$ denote $\frac{d}{dx}J(x)$.

From (1) and (4),

$$\frac{\partial U}{\partial x} = \tfrac{1}{2}(z - z^{-1})\,U = \sum_{-\infty}^{\infty} J_n'(x)z^n$$

or

$$\tfrac{1}{2}(z - z^{-1}) \sum_{-\infty}^{\infty} J_n(x)z^n = \sum_{-\infty}^{\infty} J_n'(x)z^n \quad .$$

Equating the coefficients of z^n we have

$$J_n'(x) = \tfrac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

(iv) We now show that $J_n(x)$ satisfies Bessel's differential equation of the second order, namely,

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

From (10),

$$\begin{aligned} J_n''(x) &= \tfrac{1}{4}[\{J_{n-2}(x) - J_n(x)\} - \{J_n(x) - J_{n+2}(x)\}] \\ &= -J_n(x) + \tfrac{1}{4}\{J_{n-2}(x) + J_n(x)\} + \tfrac{1}{4}\{J_n(x) + J_{n+2}(x)\} \end{aligned}$$

so that, by (9) and (10),

$$\begin{aligned} J_n''(x) + J_n(x) &= \frac{1}{2x}\{(n-1)J_{n-1}(x) + (n+1)J_{n+1}(x)\} \\ &= \frac{n^2}{x^2}J_n(x) - \frac{1}{x}J_n'(x) \quad . \end{aligned}$$

It is thus verified that $J_n(x)$ satisfies the differential equation (11).

(v) The Bessel function of order n can be expressed in integral form as follows.

In (1) and (4) put $z = e^{i\theta}$, where $i^2 = -1$.

Then
$$U \equiv e^{ix\sin\theta} = \sum_{-\infty}^{\infty} J_n(x)e^{in\theta}.$$

Multiply both sides by $e^{-in\theta}$ and integrate between 0 and 2π; then, since $\int_0^{2\pi} e^{ik\theta}\,d\theta = 2\pi$ or 0 according as $k = 0$ or as k is a positive or negative integer, we obtain

$$2\pi J_n(x) = \int_0^{2\pi} e^{-i(n\theta - x\sin\theta)}\,d\theta.$$

Write the integral as the sum of two, one with range of integration from 0 to π, and the second with range of integration from π to 2π. In the second write $2\pi - \theta$ for θ; then

$$2\pi J_n(x) = \int_0^{\pi} e^{-i(n\theta - x\sin\theta)}\,d\theta + e^{-2in\pi}\int_0^{\pi} e^{i(n\theta - x\sin\theta)}\,d\theta \quad ,$$

from which,

$$\pi J_n(x) = \int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

3·04. Hypergeometric Series

The hypergeometric series, denoted by $F(a, b, c\,;\, x)$, is defined by

$$F(a, b, c\,;\, x) = 1 + \frac{a\,.\,b}{1\,.\,c}x + \frac{a(a+1)\,.\,b(b+1)}{1\,.\,2\,.\,c(c+1)}x^2 + \ldots\ldots\ldots\ldots(1)$$

which is convergent if $|x| < 1$; in applications x is of order e^2 and so the convergency condition is amply satisfied.

The function F can be changed into a variety of forms amongst which we note for future use

$$F(a, b, c; x) = (1-x)^{-a} F\left(a, c-b, c; -\frac{x}{1-x}\right) \quad \ldots\ldots\ldots(2)$$

It is easily verified that F satisfies the differential equation

$$(x^2 - x)\frac{d^2y}{dx^2} + \{(a+b+1)x - c\}\frac{dy}{dx} + aby = 0. \quad \ldots\ldots\ldots(3)$$

3·05. Method of expressing a function of r and f as a periodic series

Write $$\xi = e^{if}, \qquad \eta = e^{iE} \quad , \ldots\ldots\ldots(1)$$

where $i^2 = -1$ and e is the base of Napierian logarithms. Then

$$\xi^k \equiv e^{ikf} = \cos kf + i \sin kf, \quad \xi^{-k} = \cos kf - i \sin kf \quad \ldots\ldots\ldots(2)$$

There are similar formulae for η^k and η^{-k}.

Write the eccentricity e as:

$$e = \sin\phi \quad \ldots\ldots\ldots(3)$$

and $$\beta = \tan \tfrac{1}{2}\phi; \quad \ldots\ldots\ldots(4)$$

then $$e = \frac{2\beta}{1+\beta^2} \quad \ldots\ldots\ldots(5)$$

and $$\beta = \frac{1}{e}\{1 - \sqrt{(1-e^2)}\} \quad \ldots\ldots\ldots(6)$$

The last formula shows that β is of order $(e/2)$.

Consider the equation

$$\tan \tfrac{1}{2}f = \sqrt{\left(\frac{1+e}{1-e}\right)} \tan \tfrac{1}{2}E = \frac{1+\beta}{1-\beta} \tan \tfrac{1}{2}E \quad \ldots\ldots\ldots(7)$$

Now, $$i \tan \tfrac{1}{2}f = \frac{e^{if}-1}{e^{if}+1} = \frac{\xi - 1}{\xi + 1} \quad ;$$

similarly, $$i \tan \tfrac{1}{2}E = \frac{\eta - 1}{\eta + 1} \quad .$$

Hence (7) becomes

$$\frac{\xi - 1}{\xi + 1} = \frac{1+\beta}{1-\beta} \cdot \frac{\eta - 1}{\eta + 1} \quad ,$$

whence $$\xi = \frac{\eta - \beta}{1 - \beta\eta} = \frac{\eta(1 - \beta\eta^{-1})}{1 - \beta\eta} \quad \ldots\ldots\ldots(8)$$

and $$\eta = \frac{\xi(1 + \beta\xi^{-1})}{1 + \beta\xi} \quad \ldots\ldots\ldots(9)$$

Again, $$r = a(1 - e\cos E) = a\left\{1 - \frac{2\beta}{1+\beta^2}(\eta + \eta^{-1})\right\} \quad ;$$

hence
$$\frac{r}{a}=\frac{(1-\beta\eta)(1-\beta\eta^{-1})}{1+\beta^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

To express r in terms of ξ we have
$$\frac{r}{a}\equiv\frac{1-e^2}{1+e\cos f}=\frac{(1-\beta^2)^2}{1+\beta^2}\cdot\frac{1}{(1+\beta\xi)(1+\beta\xi^{-1})} \quad \ldots\ldots\ldots\ldots(11)$$

3·06. Expansion of β^m in terms of e

We write (5) of the previous section as
$$\beta=\frac{e}{2}+\frac{e}{2}\beta^2 \quad ,$$
or
$$\beta=x+\alpha\phi(\beta) \quad ,$$
where $\alpha=e/2$ and $\phi(\beta)=\beta^2$ so that $\phi(x)=x^2$; in the Lagrange expansion of $F(\beta)\equiv\beta^m$ we finally identify x with $e/2$.

We then have by Lagrange's theorem, section 3·02,
$$\begin{aligned}\beta^m&=x^m+\sum_1^\infty\frac{\alpha^n}{n!}\frac{d^{n-1}}{dx^{n-1}}\left\{x^{2n}\frac{d}{dx}(x^m)\right\}\\&=x^m+m\sum_1^\infty\frac{\alpha^n}{n!}\frac{d^{n-1}}{dx^{n-1}}(x^{2n+m-1})\\&=x^m+m\sum_1^\infty\frac{\alpha^n}{n!}\cdot\frac{(2n+m-1)!}{(n+m)!}x^{n+m}\end{aligned}$$
or, in terms of e,
$$\beta^m=\left(\frac{e}{2}\right)^m\left\{1+m\sum_1^\infty\frac{(2n+m-1)!}{n!\,(n+m)!}\left(\frac{e}{2}\right)^{2n}\right\} \quad \ldots\ldots\ldots\ldots(1)$$

The first few terms of the expansion are
$$\beta^m=\left(\frac{e}{2}\right)^m\left\{1+\frac{m}{4}e^2+\frac{m(m+3)}{4^2\,.\,2!}e^4+\frac{m(m+4)(m+5)}{4^3\,.\,3!}e^6+\ \ldots\right\} \quad \ldots\ldots(2)$$

3·07. Expansion of f in terms of E

We start from 3·05 (8), namely,
$$\xi=\frac{\eta(1-\beta\eta^{-1})}{1-\beta\eta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
Taking logarithms and remembering that $\xi=e^{if}$, $\eta=e^{iE}$ we obtain
$$\begin{aligned}if&=iE+\log(1-\beta\eta^{-1})-\log(1-\beta\eta)\\&=iE+\beta(\eta-\eta^{-1})+\frac{\beta^3}{2}(\eta^2-\eta^{-2})+\ \ldots\end{aligned}$$
Hence
$$f=E+2\beta\sin E+\beta^2\sin 2E+2\frac{\beta^3}{3}\sin 3E+\ \ldots$$
or
$$f=E+2\sum_1^\infty\frac{\beta^n}{n}\sin nE \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Note. The expansion of E in terms of f is obtained at once from (2) for, by 3·05 (9),

$$\eta = \frac{\xi(1+\beta\xi^{-1})}{1+\beta\xi} \equiv \frac{\xi(1-\beta_1\xi^{-1})}{1-\beta_1\xi} \quad \ldots\ldots(3)$$

where $\beta_1 \equiv -\beta$. Since (1) and (3) have the same analytical form, we derive the expansion of E in terms of f by interchanging f and E in (2) and writing $(-\beta)$ for β; hence

$$E = f + 2\sum_1^\infty (-1)^n \frac{\beta^n}{n} \sin nf \quad \ldots\ldots(4)$$

3·08. Expansions of $r^p \cos pf$ and $r^p \sin pf$ in terms of E

From 3·05 (8) and (10) we have

$$r\xi = \frac{a\eta(1-\beta\eta^{-1})^2}{1+\beta^2} \quad .$$

Hence
$$r^p \xi^p = a^p(1+\beta^2)^{-p}\,\eta^p(1-\beta\eta^{-1})^{2p} \quad \ldots\ldots(1)$$

in which p is a positive or negative integer.

Let
$$C_j^k = \frac{k\,.\,k-1\,.\,k-2\,\ldots\,k-j+1}{j!} \quad \ldots\ldots(2)$$

in which j is a positive integer and k is either a positive or negative integer.

If k and j are positive integers, then C_j^k is the coefficient of x^j in the binomial expansion of $(1+x)^k$, it being assumed that $j<k$.

From (1) we then obtain

$$\begin{aligned} r^p\xi^p &= a^p(1+\beta^2)^{-p}\,\eta^p\{1 - C_1^{2p}(\beta\eta^{-1}) + C_2^{2p}(\beta\eta^{-1})^2 - \ldots\} \\ &= a^p(1+\beta^2)^{-p}\{\eta^p - C_1^{2p}\beta\eta^{p-1} + C_2^{2p}\beta^2\eta^{p-2} - \ldots\} \quad . \end{aligned}$$

Hence we obtain, on taking the real and imaginary parts of the previous equation,

$$\begin{aligned} &r^p \cos pf \\ &\quad = a^p(1+\beta^2)^{-p}\{\cos pE - \beta C_1^{2p}\cos(p-1)E + \beta^2 C_2^{2p}\cos(p-2)E - \ldots\} \quad , \\ &r^p \sin pf \\ &\quad = a^p(1+\beta^2)^{-p}\{\sin pE - \beta C_1^{2p}\sin(p-1)E + \beta^2 C_2^{2p}\sin(p-2)E \ldots\} \quad . \end{aligned}$$

Each series terminates if p is a positive integer.

3·09. Expansions of $r^p \cos qf$ and $r^p \sin qf$

From 3·05 (8) and (10)

$$r^p\xi^q = a^p\eta^q(1+\beta^2)^{-p}(1-\beta\eta)^{p-q}(1-\beta\eta^{-1})^{p+q} \quad . \quad \ldots\ldots(1)$$

We here suppose that p is a positive or negative integer and that, without loss of generality, q is a positive integer.

We have

$$\begin{aligned} &(1-\beta\eta)^{p-q}(1-\beta\eta^{-1})^{p+q} \\ &\qquad = \{1 + \sum_{n=1}(-1)^n C_n^{p-q}\beta^n\eta^n\}\{1 + \sum_{m=1}(-1)^m C_m^{p+q}\beta^m\eta^{-m}\} \end{aligned}$$

$$= 1 + \beta^2 C_1^{p-q} C_1^{p+q} + \beta^4 C_2^{p-q} C_2^{p+q} + \ldots$$
$$+ \sum_{n=1} (-1)^n \eta^n \{C_n^{p-q} \beta^n + C_{n+1}^{p-q} C_1^{p+q} \beta^{n+2} \ldots\}$$
$$+ \sum_{m=1} (-1)^m \eta^{-m} \{C_m^{p+q} \beta^m + C_{m+1}^{p+q} C_1^{p-q} \beta^{m+2} + \ldots\}$$
$$= 1 + \sum_{k=1} C_k^{p+q} C_k^{p-q} \beta^{2k} \quad \ldots\ldots(2)$$
$$+ \sum_{n=1} (-\beta\eta)^n \{C_n^{p-q} + \sum_{k=1} C_{n+k}^{p-q} C_k^{p+q} \beta^{2k}\} \quad \ldots\ldots(3)$$
$$+ \sum_{n=1} (-\beta\eta^{-1})^n \{C_n^{p+q} + \sum_{k=1} C_{n+k}^{p+q} C_k^{p-q} \beta^{2k}\} \quad \ldots\ldots(4)$$

Now, from the definition of C_{n+k}^j in which n and k are positive integers,

$$C_{n+k}^j = C_n^j \cdot \frac{j-n \,.\, j-n-1 \ldots j-n-k+1}{n+1 \,.\, n+2 \ldots n+k}$$
$$= C_n^j C_k^{j-n} \frac{k!}{n+1 \,.\, n+2 \ldots n+k} \quad \ldots\ldots(5)$$

The expression within the parenthesis in (3) becomes, by means of (5),

$$C_n^{p-q} \left\{1 + \sum_{k=1} C_k^{p+q} C_k^{p-q-n} \frac{k!}{n+1 \,.\, n+2 \ldots n+k} \beta^{2k}\right\} \quad \ldots\ldots(6)$$

Now, the formula for F, 3·04 (1), becomes (on writing $-a_1$ for a and $-b_1$ for b),

$$F(a, b, c\,;\, x)$$
$$= 1 + \sum_{k=1} (-1)^k \frac{a_1(a_1-1) \ldots (a_1-k+1)}{k!} \cdot (-1)^k \frac{b_1(b_1-1) \ldots (b_1-k+1)}{c(c+1) \ldots (c+k-1)} x^k$$
$$= 1 + \sum_{k=1} C_k^{a_1} C_k^{b_1} \frac{k!}{c(c+1) \ldots (c+k-1)} x^k \quad \ldots\ldots(7)$$

Hence (6) becomes

$$C_n^{p-q} F(-p-q, -p+q+n, n+1\,;\, \beta^2) \,.$$

Write

$$G_n(p, q) \equiv F(-p-q, -p+q+n, n+1\,;\, \beta^2) \quad ; \quad \ldots\ldots(8)$$

then the expression (3) is $\sum_{n=1} (-\beta\eta)^n C_n^{p-q} G_n(p, q)$ $\ldots\ldots(9)$

Similarly, (4) is $\sum_{n=1} (-\beta\eta^{-1})^n C_n^{p+q} G_n(p, -q)$ $\ldots\ldots(10)$

and, from (7), (2) is $G_0(p, q)$ $\ldots\ldots(11)$

We thus obtain

$$(1-\beta\eta)^{p-q} (1-\beta\eta^{-1})^{p+q} = G_0(p, q) + \sum_{n=1} (-\beta\eta)^n C_n^{p-q} G_n(p, q)$$
$$+ \sum_{n=1} (-\beta\eta^{-1})^n C_n^{p+q} G_n(p, -q) \quad \ldots(12)$$

We can write (12) in an alternative form by means of the transformation 3·04 (2), namely

$$G_n(p, q) = (1-\beta^2)^{p+q} F\left(-p-q, p-q+1, n+1\,;\, -\frac{\beta^2}{1-\beta^2}\right)$$

or, writing

$$T_n(p, q) \equiv F\left(-p-q, p-q+1, n+1; -\frac{\beta^2}{1-\beta^2}\right) ,$$

then $$G_n(p, q) = (1-\beta^2)^{p+q} T_n(p, q) ;$$

we now have

$$\begin{aligned}(1-\beta\eta)^{p-q}(1-\beta\eta^{-1})^{p+q} &= (1-\beta^2)^{p+q} T_0(p, q) \\ &+ (1-\beta^2)^{p+q} \Sigma(-\beta\eta)^n C_n^{p-q} T_n(p, q) \\ &+ (1-\beta^2)^{p-q} \Sigma(-\beta\eta^{-1})^n C_n^{p+q} T_n(p, -q) \quad . \quad \ldots\ldots(13)\end{aligned}$$

The advantage of the T function is that it converges more rapidly than the G function when n is comparatively large.

Since $$\frac{1-\beta^2}{1+\beta^2} = \sqrt{(1-e^2)}$$

we find that (1) becomes

$$\begin{aligned}\left(\frac{r}{a}\right)^p \xi^q = (1-e^2)^{p/2}[&(1-\beta^2)^q T_0(p,q)\eta^q + (1-\beta^2)^q \Sigma_{n=1}(-1)^n \beta^n C_n^{p-q} T_n(p,q)\eta^{q+n} \\ &+ (1-\beta^2)^{-q} \Sigma_{n=1}(-1)^n \beta^n C_n^{p+q} T_n(p, -q)\eta^{q-n}] \quad . \quad \ldots\ldots(14)\end{aligned}$$

Equate real and imaginary parts ; then

$$\left(\frac{r}{a}\right)^p \cos qf = A_0 \cos qE + \Sigma_{n=1} A_n \cos(q+n)E + \Sigma_{n=1} B_n \cos(q-n)E \quad , \ldots\ldots(15)$$

$$\left(\frac{r}{a}\right)^p \sin qf = A_0 \sin qE + \Sigma A_n \sin(q+n)E + \Sigma B_n \sin(q-n)E \quad \ldots\ldots(16)$$

where

$$A_0 = (1-e^2)^{p/2}(1-\beta^2)^q T_0(p, q) \quad , \quad \ldots\ldots(17)$$

$$A_n = (-1)^n (1-e^2)^{p/2}(1-\beta^2)^q C_n^{p-q} T_n(p, q)\beta^n \quad , \quad \ldots\ldots(18)$$

$$B_n = (-1)^n (1-e^2)^{p/2}(1-\beta^2)^{-q} C_n^{p+q} T_n(p, -q)\beta^n \quad . \quad \ldots\ldots(19)$$

We consider the following special cases

(i) *Expansion of* $\left(\frac{r}{a}\right)^p$.

Here, $q = 0$; then

$$\left(\frac{r}{a}\right)^p = A_0' + 2\Sigma A_n' \cos nE \quad \ldots\ldots(20)$$

where

$$A_0' = (1-e^2)^{p/2} F\left(-p, p+1, 1; -\frac{\beta^2}{1-\beta^2}\right) \quad , \quad \ldots\ldots(21)$$

$$A_n' = (-1)^n (1-e^2)^{p/2} F\left(-p, p+1, n+1; -\frac{\beta^2}{1-\beta^2}\right) C_n^p \beta^n \quad \ldots(22)$$

(ii) *Expansion of* $\frac{a}{r}$.

Here $p = -1$, $q = 0$; then $C_n^p = (-1)^n$ and $T_n(-1, 0) = 1$.

From (20), (21) and (22)

$$\frac{a}{r}=\frac{1}{\sqrt{(1-e^2)}}\left[1+2\sum_1\beta^n\cos nE\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(23)$$

(iii) *Expansion of* $\frac{a^2}{r^2}$.

In this case $p=-2$, $q=0$; then $C_n^p=(-1)^n(n+1)$ and

$$T_n(p,0)\equiv F\left(2,-1,n+1;\ -\frac{\beta^2}{1-\beta^2}\right)=1+\frac{2\beta^2}{(n+1)(1-\beta^2)}$$

$$=\frac{1}{n+1}\{n+(1-e^2)^{-1/2}\} \quad .$$

From (20),

$$\frac{a^2}{r^2}=(1-e^2)^{-3/2}\left[1+2\sum_1\{1+n\sqrt{(1-e^2)}\}\beta^n\cos nE\right] \quad \ldots\ldots\ldots(24)$$

(iv) *Expansions of cos qf and sin qf*

From (15) and (16) with $p=0$,

$$\cos qf=A_0''\cos qE+\sum_1 A_n''\cos(q+n)E+\sum_1 B_n''\cos(q-n)E \quad , \ldots.(25)$$

$$\sin qf=A_0''\sin qE+\sum_1 A_n''\sin(q+n)E+\sum_1 B_n''\sin(q-n)E \quad \ldots\ldots\ldots(26)$$

where

$$A_0''=(1-\beta^2)^q T_0(0,q) \quad , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

$$A_n''=(-1)^n(1-\beta^2)^q C_n^{-q}T_n(0,q)\beta^n \quad , \ldots\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

$$B_n''=(-1)^n(1-\beta^2)^{-q}C_n^q T_n(0,-q)\beta^n \quad . \ldots\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

(v) *Expansions of cos f and sin f*

From the formula

$$1+e\cos f=\frac{a}{r}(1-e^2)$$

we obtain, by (23),

$$\cos f=-\frac{1-\sqrt{(1-e^2)}}{e}+\frac{2\beta\sqrt{(1-e^2)}}{e}\sum_1\beta^{n-1}\cos nE$$

which can be put in the form

$$\cos f=-\beta+(1-\beta^2)\sum_1\beta^{n-1}\cos nE \quad \ldots\ldots\ldots\ldots\ldots\ldots(30)$$

Since this last equation is obtained by taking the real part of

$$\xi=-\beta+(1-\beta^2)\sum_1\beta^{n-1}\eta^n$$

we obtain, by taking the imaginary part,

$$\sin f=(1-\beta^2)\sum_1\beta^{n-1}\sin nE \quad \ldots\ldots\ldots\ldots\ldots\ldots(31)$$

3·10. Expansions of functions of r, and of M, in terms of f

(i) From 3·05 (9),

$$\eta=\frac{\xi(1+\beta\xi^{-1})}{1+\beta\xi} \quad .$$

Compare this formula with 3·05 (8), namely

$$\xi = \frac{\eta(1-\beta\eta^{-1})}{1-\beta\eta} \quad .$$

It is evident that any function of η when expanded in terms of ξ will be obtained by the same function of ξ when expanded in terms of η by interchanging ξ and η in the second formula—that is, interchanging f and E—and writing $-\beta$ for β.

For example, from (25) and (26) of section 3·09,

$$\cos qE = A_0'' \cos qf + \sum_{n=1} D_n \cos(q+n)f + \sum_{n=1} E_n \cos(q-n)f \quad , \quad \ldots\ldots(1)$$

$$\sin qE = A_0'' \sin qf + \sum_{n=1} D_n \sin(q+n)f + \sum_{n=1} E_n \sin(q-n)f \quad , \quad \ldots\ldots(2)$$

where A_0'' is given by 3·09 (27), and

$$D_n = (-1)^n A_n'' \quad , \quad E_n = (-1)^n B_n'' \quad .$$

In particular, when $q=1$, we derive from 3·09 (30)

$$\cos E = \beta + (1-\beta^2) \sum_{n=1} (-1)^{n-1} \beta^{n-1} \cos nf \quad , \quad \ldots\ldots(3)$$

and, from 3·09 (31),

$$\sin E = (1-\beta^2) \sum_{n=1} (-1)^{n-1} \beta^{n-1} \sin nf \quad . \quad \ldots\ldots(4)$$

Again, to obtain $\left(\frac{r}{a}\right)^{-p}$ in terms of f we proceed as follows.

From 3·09 (11),

$$\left(\frac{r}{a}\right)^{-p} = \frac{(1+\beta^2)^p}{(1-\beta^2)^{2p}} (1+\beta\xi)^p (1+\beta\xi^{-1})^p$$

$$= \left(\frac{1+\beta^2}{1-\beta^2}\right)^{2p} [(1+\beta^2)^{-p} (1+\beta\xi)^p (1+\beta\xi^{-1})^p] \quad . \quad \ldots\ldots(5)$$

But from 3·05 (10) $\left(\frac{r}{a}\right)^p$ is expressed in terms of η by

$$\left(\frac{r}{a}\right)^p \equiv (1+\beta^2)^{-p} (1-\beta\eta)^p (1-\beta\eta^{-1})^p$$

$$= A_0' + 2 \sum A_n' \cos nE, \quad \text{by 3·09 (20)},$$

where A_0' and A_n' are given by 3·09 (21) and (22). It is clear that the expression within the square bracket of (5) is given in terms of f by writing ξ for η—that is, f for E—and $-\beta$ for β; the result will be

$$B_0' + 2 \sum_{n=1} B_n' \cos nf$$

where

$$B_0' = A_0' = (1-e^2)^{p/2} T_0(p, 0) \quad ,$$

$$B_n' = (-1)^n A_n' = (1-e^2)^{p/2} T_n(p, 0) C_n^p \beta^n \quad .$$

Hence, since $\frac{1-\beta^2}{1+\beta^2}=\sqrt{(1-e^2)}$, we derive from (5)

$$\left(\frac{r}{a}\right)^{-p}=\frac{1}{(1-e^2)^{p/2}}\left[T_0(p,0)+2\sum_{n=1}T_n(p,0)\,C_n^p\,\beta^n\cos nf\right] \quad \ldots(6)$$

When $p=-2$, this last equation becomes

$$r^2=a^2\sqrt{(1-e^2)}\left[1+2\sum_{n=1}(-1)^n\{1+n\sqrt{(1-e^2)}\}\,\beta^n\cos nf\right] \quad \ldots\ldots(7)$$

(ii) We can express the mean anomaly M in terms of f. We have

$$\frac{a}{r}=\frac{1+e\cos f}{1-e^2}\;;\;\text{hence}\;\frac{dr}{df}=\frac{er^2\sin f}{a(1-e^2)}\;.$$

But,
$$r\sin f=a\sqrt{(1-e^2)}\sin E\;.$$

Hence
$$\frac{dr}{df}=\frac{er\sin E}{\sqrt{(1-e^2)}} \quad \ldots\ldots(8)$$

Again $r=a(1-e\cos E)$ and $E-e\sin E=M$.

Hence
$$\frac{dr}{dM}=ae\sin E\,\frac{dE}{dM}=\frac{a^2e}{r}\sin E \quad \ldots\ldots(9)$$

From (8) and (9)

$$\frac{dM}{df}=\frac{r^2}{a^2\sqrt{(1-e^2)}}\;.$$

Using (7) and integrating, M vanishing with f, we obtain

$$M=f+2\sum_{n=1}(-1)^n\left\{\frac{1}{n}+\sqrt{(1-e^2)}\right\}\beta^n\sin nf \quad \ldots\ldots(10)$$

The angle $f-M$ is the *equation of the centre* and (10) gives its expansion in terms of the true anomaly.

As far as order e^4, it is found that

$$M=f-2e\sin f+(\tfrac{3}{4}e^2+\tfrac{1}{8}e^4)\sin 2f-\tfrac{1}{3}e^3\sin 3f+\tfrac{5}{32}e^4\sin 4f$$

and reversing the series we can express f in terms of M and e by the formula

$$f=M+(2e-\tfrac{1}{4}e^3)\sin M+(\tfrac{5}{4}e^2-\tfrac{11}{24}e^4)\sin 2M+\tfrac{13}{12}e^3\sin 3M+\tfrac{103}{96}e^4\sin 4M \quad \ldots\ldots(11)$$

3·11. Expansions in terms of the mean anomaly

Let $F(M)$ be a periodic function of M, expressible in the form of the series, supposed convergent,

$$F(M)=\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}A_n\cos nM+\sum_{n=1}^{\infty}\beta_n\sin nM \quad \ldots\ldots(1)$$

Multiply throughout by $\cos nM$ and integrate between 0 and π; then, $(n \neq 0)$,

$$A_n = \frac{2}{\pi}\int_0^\pi F(M)\cos nM\, dM \quad \ldots\ldots(2)$$

Multiply (1) throughout by $\sin nM$ and integrate as before; then

$$B_n = \frac{2}{\pi}\int_0^\pi F(M)\sin nM\, dM \quad \ldots\ldots(3)$$

Also, integrate (1) between 0 and π; then

$$A_0 = \frac{2}{\pi}\int_0^\pi F(M)\, dM \quad \ldots\ldots(4)$$

If $F(M)$ is an *even* function of M, then the coefficients B_r in (1) are zero $(r = 1, 2, \ldots)$.

If $F(M)$ is an *odd* function of M, then the coefficients A_r in (1) are zero $(r = 0, 1, 2, \ldots)$.

We now derive the expansions required.

(i) *Expansion of cos kE.*

E is a function of M through Kepler's equation

$$E - e\sin E = M \quad \ldots\ldots(5)$$

We have, the function being an even function of E and consequently of M,

$$\cos kE = \tfrac{1}{2}A_{k,0} + \sum_{n=1}^{\infty} A_{k,n}\cos nM \quad .$$

By (2),
$$\begin{aligned} A_{k,n} &= \frac{2}{\pi}\int_0^\pi \cos kE\cos nM\, dM \\ &= \frac{2}{n\pi}\Big[\cos kE\sin nM\Big]_0^\pi + \frac{2k}{n\pi}\int_0^\pi \sin kE\sin nM\, dE \\ &= \frac{k}{n\pi}\int_0^\pi \cos(nM - kE)\, dE - \frac{k}{n\pi}\int_0^\pi \cos(nM + kE)\, dE \end{aligned}$$

or, by (5),
$$\begin{aligned} A_{k,n} &= \frac{k}{n\pi}\int_0^\pi \cos\{(n-k)E - ne\sin E\}\, dE \\ &\quad - \frac{k}{n\pi}\int_0^\pi \cos\{(n+k)E - ne\sin E\}\, dE \quad . \end{aligned}$$

But, by 3·03 (12), each integral is a Bessel function and, accordingly,

$$A_{k,n} = \frac{k}{n}[J_{n-k}(ne) - J_{n+k}(ne)] \quad \ldots\ldots(6)$$

Also,
$$\begin{aligned} A_{k,0} &= \frac{2}{\pi}\int_0^\pi \cos kE\,(1 - e\cos E)\, dE \\ &= \frac{2}{\pi}\int_0^\pi \cos kE\, dE - \frac{e}{\pi}\int_0^\pi \cos(k+1)E\, dE - \frac{e}{\pi}\int_0^\pi \cos(k-1)E\, dE \\ &= 0, \quad \text{if } k > 1 \\ &= -e, \text{ if } k = 1. \end{aligned}$$

Hence, if $k>1$,

$$\cos kE = k\sum_{n=1}^{\infty}\frac{1}{n}\{J_{n-k}(ne)-J_{n+k}(ne)\}\cos nM \quad \ldots\ldots\ldots(7)$$

If $k=1$,

$$\cos E = -\tfrac{1}{2}e+\sum_{n=1}^{\infty}\frac{1}{n}\{J_{n-1}(ne)-J_{n+1}(ne)\}\cos nM \quad , \ldots.(8)$$

or, by means of 3·03 (10),

$$\cos E = -\tfrac{1}{2}e+\sum_{n=1}^{\infty}\frac{2}{n^2}\frac{d}{de}\{J_n(ne)\}\cos nM \quad \ldots\ldots\ldots\ldots(9)$$

(ii) *Expansion of sin kE.*

The function is odd and we write

$$\sin kE = \sum_{n=1}^{\infty}B_{k,n}\sin nM \quad \ldots\ldots\ldots\ldots(10)$$

Then, by (3), following the procedure in (i),

$$\begin{aligned}B_{k,n} &= \frac{2}{\pi}\int_0^{\pi}\sin kE\sin nM\,dM\\ &= \frac{k}{n\pi}\int_0^{\pi}\{\cos(kE+nM)+\cos(nM-kE)\}\,dE\\ &= \frac{k}{n}[J_{k+n}(ne)+J_{n-k}(ne)] \quad \ldots\ldots\ldots\ldots(11)\end{aligned}$$

In particular, when $k=1$,

$$\sin E = \sum_{n=1}^{\infty}\frac{1}{n}\{J_{n+1}(ne)+J_{n-1}(ne)\}\sin nM \quad \ldots\ldots\ldots(12)$$

or, by means of 3·03 (9),

$$\sin E = \frac{2}{e}\sum_{n=1}^{\infty}\frac{1}{n}J_n(ne)\sin nM \quad \ldots\ldots\ldots\ldots(13)$$

(iii) *Expansion of r/a.*

Since $r=a(1-e\cos E)$, then by (9)

$$\begin{aligned}\frac{r}{a} &= 1+\tfrac{1}{2}e^2-2e\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{d}{de}\{J_n(ne)\}\cos nM\\ &= 1+\tfrac{1}{2}e^2-(e-\tfrac{3}{8}e^3)\cos M-\tfrac{1}{2}e^2\cos 2M-\tfrac{3}{8}e^3\cos 3M-\ldots \quad \ldots.(14)\end{aligned}$$

(iv) *Expansion of E.*

By Kepler's equation, $E=M+e\sin E$;

hence, by (13),

$$E = M+2\sum_{n=1}^{\infty}\frac{1}{n}J_n(ne)\sin nM \quad \ldots\ldots\ldots\ldots(15)$$

(v) *Expansion of* $\frac{a}{r}$.

From Kepler's equation we have $\frac{dE}{dM}=\frac{a}{r}$.

Hence, from (15) by differentiation,

$$\frac{a}{r} = 1 + 2\sum_{n=1}^{\infty} J_n(ne)\cos nM \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

(vi) *Expansion of* r^2/a^2.

$$\frac{d}{dM}\left(\frac{r^2}{a^2}\right) = \frac{2r}{a^2}\frac{dr}{dE}\cdot\frac{dE}{dM} = 2e\sin E$$

$$= 4\sum_{n=1}^{\infty}\frac{1}{n} J_n(ne)\sin nM \quad .$$

Hence,

$$\frac{r^2}{a^2} = C - 4\sum_{n=1}^{\infty}\frac{1}{n^2} J_n(ne)\cos nM \quad ,$$

where C is a constant.

Now $$\frac{r^2}{a^2} = 1 + \tfrac{1}{2}e^2 - 2e\cos E + \tfrac{1}{2}e^2\cos 2E \quad ,$$

and consequently, by (9), the non-periodic part of $\frac{r^2}{a^2}$ is

$$1 + \tfrac{1}{2}e^2 - 2e(-\tfrac{1}{2}e) \text{ or } 1 + \tfrac{3}{2}e^2 ;$$

hence $$\frac{r^2}{a^2} = 1 + \tfrac{3}{2}e^2 - 4\sum_{n=1}^{\infty}\frac{1}{n^2} J_n(ne)\cos nM \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

(vii) *Expansion of* $\cos f$.

$$e\cos f = -1 + (1-e^2)\frac{a}{r}$$

$$= -1 + (1-e^2)\{1 + 2\sum_{n=1}^{\infty} J_n(ne)\cos nM\} \quad ,$$

by means of (16). Hence

$$\cos f = -e + \frac{2(1-e^2)}{e}\sum_{n=1}^{\infty} J_n(ne)\cos nM \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

(viii) *Expansion of* $\sin f$.

$$r\sin f = a\sqrt{(1-e^2)}\sin E = \frac{1}{e}\sqrt{(1-e^2)}\frac{dr}{dE} \quad .$$

But $$\frac{dr}{dE} = \frac{dr}{dM}\cdot\frac{dM}{dE} = \frac{r}{a}\frac{dr}{dM} \quad .$$

Hence

$$\sin f = \frac{\sqrt{(1-e^2)}}{e}\frac{d}{dM}\left(\frac{r}{a}\right) \quad ,$$

and by (14)

$$\sin f = 2\sqrt{(1-e^2)}\sum_{n=1}^{\infty}\frac{1}{n}\frac{d}{de}\{J_n(ne)\}\sin nM \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

(ix) *Expansions of* $\frac{\cos f}{r^2}$ *and* $\frac{\sin f}{r^2}$.

Let (ξ, η) be the rectangular coordinates of the planet in its orbit ; then

$$\frac{\xi}{a} = \cos E - e \quad , \quad \frac{\eta}{a} = \sqrt{(1-e^2)} \sin E \quad .$$

Then, by (9) and (13),

$$\frac{\xi}{a} = -\frac{3}{2}e + 2\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{d}{de}\{J_n(ne)\} \cos nM \quad , \quad \text{.....(20)}$$

$$\frac{\eta}{a} = \frac{2}{e}\sqrt{(1-e^2)} \sum_{n=1}^{\infty} \frac{1}{n} J_n(ne) \sin nM \quad . \quad \text{.....(21)}$$

Now $\xi = r \cos f \quad , \quad \eta = r \sin f \quad .$

Hence $\frac{\xi}{r^3} = \frac{\cos f}{r^2} \quad , \quad \frac{\eta}{r^3} = \frac{\sin f}{r^2} \quad .$

Also, the equations of motion are

$$\ddot{\xi} + \frac{\mu\xi}{r^3} = 0 \quad , \quad \ddot{\eta} + \frac{\mu\eta}{r^3} = 0 \quad ,$$

or, since $M = n(t-\tau) = (\mu a^{-3})^{1/2}(t-\tau)$,

so that $\frac{d}{dt} = n\frac{d}{dM}$ and $\frac{d^2}{dt^2} = n^2\frac{d^2}{dM^2} = \mu a^{-3}\frac{d^2}{dM^2}$,

then $\frac{d^2\xi}{dM^2} + \frac{a^3\xi}{r^3} = 0 \quad , \quad \frac{d^2\eta}{dM^2} + \frac{a^3\eta}{r^3} = 0 \quad .$

Hence, from (20) and (21),

$$\frac{\xi}{r^3} \equiv \frac{\cos f}{r^2} = \frac{2}{a^2}\sum_{n=1}^{\infty} \frac{d}{de}\{J_n(ne)\} \cos nM \quad , \quad \text{.....(22)}$$

and

$$\frac{\eta}{r^3} \equiv \frac{\sin f}{r^2} = \frac{2\sqrt{(1-e^2)}}{a^2 e} \sum_{n=1}^{\infty} nJ_n(ne) \sin nM \quad \text{.....(23)}$$

The Appendix (p. 378) contains the details of the principal expansions of $J_n(ne)$ and $\frac{d}{de}\{J_n(ne)\}$.

For numerical tables of Bessel Functions the reader is referred to : A. Fletcher, J. C. P. Miller and L. Rosenhead, *An Index of Mathematical Tables*, 244–270 (1946).

Chapter 4

THE GENERAL EQUATIONS OF MOTION AND THEIR KNOWN INTEGRALS

4·01. The force-function

In this chapter we are concerned with the problem of any number of bodies (point-masses) forming an isolated system in space. We suppose that there are n such bodies—$P_1, P_2, \ldots, P_n$—of masses $m_1, m_2, \ldots, m_n$ and with coordinates (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2), $\ldots$, (ξ_n, η_n, ζ_n) with respect to a Newtonian system of axes. If P_i is the body of mass m_i and Δ_{ij} is the distance between P_i and P_j at time t, the potential at P_i is

$$G\left(\frac{m_1}{\Delta_{i1}}+\frac{m_2}{\Delta_{i2}}+\ldots+\frac{m_n}{\Delta_{in}}\right)$$

or

$$G\sum_1^n \frac{m_j}{\Delta_{ij}} \quad . \qquad (j\neq i) \quad .$$

The ξ-equation of motion of P_i is then

$$m_i\ddot{\xi}_i = Gm_i\frac{\partial}{\partial\xi_i}\sum_1^n\frac{m_j}{\Delta_{ij}} \quad , \quad j\neq i, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and, in particular, the ξ-equation of P_1 is

$$m_1\ddot{\xi}_1 = G\frac{\partial}{\partial\xi_1}\left\{\frac{m_1m_2}{\Delta_{12}}+\frac{m_1m_3}{\Delta_{13}}+\ldots+\frac{m_1m_n}{\Delta_{1n}}\right\} \quad .$$

Now, if $i\neq 1$ and $j>i$, Δ_{ij} does not involve the coordinates of P_1; hence, the previous equation is unaltered if we add on the right-hand side the following:

$$G\frac{\partial}{\partial\xi_1}\left[\frac{m_2m_3}{\Delta_{23}}+\frac{m_2m_4}{\Delta_{24}}+\ldots+\frac{m_2m_n}{\Delta_{2n}}\right.$$
$$+\frac{m_3m_4}{\Delta_{34}}+\ldots+\frac{m_3m_n}{\Delta_{3n}}$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$\left.+\frac{m_{n-1}m_n}{\Delta_{n-1,n}}\right] \quad .$$

Let U be defined by

$$U=\sum_{i=1}^n\sum_j\frac{Gm_im_j}{\Delta_{ij}} \quad , \quad j>i \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Here U is a symmetrical function for all the point-masses concerned; it is to be noted that the double summation includes the term involving the product of the masses m_i m_j once only.

The ξ-equation of motion for the mass m_1 is then

$$m_1\ddot{\xi}_1=\frac{\partial U}{\partial \xi_1} \quad .$$

It is evident that the ξ-equation of motion for the mass m_i is

$$m_i\ddot{\xi}_i=\frac{\partial U}{\partial \xi_i} \quad .$$

The equations of motion for m_i are then

$$m_i\ddot{\xi}_i=\frac{\partial U}{\partial \xi_i} \quad , \quad m_i\ddot{\eta}_i=\frac{\partial U}{\partial \eta_i} \quad , \quad m_i\ddot{\zeta}_i=\frac{\partial U}{\partial \zeta_i} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The function U defined in (2) is called the *force-function.*

It is seen from (3) that there are $3n$ equations of motion to be solved and, since each equation is of the second order, the total number of constants of integration which must appear in the complete solution of the equations is $6n$. Only 10 integrals of the equations (3) are known ; these will be found in sections 4·03–4·06. In particular, when $n=3$, the associated problem is known as the *problem of three bodies.*

An important metrical property of the force-function U is that it is independent of the axes selected since it involves only the lengths Δ_{ij}, defined by

$$\Delta_{ij}^2=(\xi_i-\xi_j)^2+(\eta_i-\eta_j)^2+(\zeta_i-\zeta_j)^2 \quad .$$

For example, if the origin is changed or, if a new system of axes is chosen by rotating the original system in any way, then Δ_{ij} is independent of the new axes just as it is of the old.

4·02. Interpretation of U

We shall prove that U is equal to the total work done by the forces of attraction in assembling the system of n point-masses from a state of infinite dispersion to the configuration represented by the points $P_i(\xi_i, \eta_i, \zeta_i)$, where $i=1, 2, \ldots n$, at time t ; further, we shall define $-U$ as the potential energy of the system so constituted.

Suppose that the mass m_1 is fixed at P_1 and that the mass m_2 is brought from a position A to a position B by the force of attraction exerted by m_1, all the other masses being supposed to be at an infinite distance. Let X, Y, Z denote the components of this force acting on m_2 whose coordinates at a point of any path between A and B we assume simply to be (ξ, η, ζ) ; if U_2 is the force-function associated with m_1 and m_2, then

$$X=\frac{\partial U_2}{\partial \xi} \quad , \quad Y=\frac{\partial U_2}{\partial \eta} \quad , \quad Z=\frac{\partial U_2}{\partial \zeta} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where

$$U_2=\frac{Gm_1m_2}{\Delta_{12}} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The work done by the attraction of m_1 in moving m_2 by any path from A to B is

$$\int_A^B (X\,d\xi + Y\,d\eta + Z\,d\zeta)$$

or, by means of (1),

$$\int_A^B \left(\frac{\partial U_2}{\partial \xi}\,d\xi + \frac{\partial U_2}{\partial \eta}\,d\eta + \frac{\partial U_2}{\partial \zeta}\,d\zeta\right)$$

or, since U_2 is a function of (ξ, η, ζ)—the coordinates of m_1 being fixed—the work done is $(U_2)_B - (U_2)_A$.

If A is at an infinite distance from P_1, then $(U_2)_A = 0$, by (2). Hence, on identifying B with the position of m_2 at P_2, the work done by the attraction of m_1 in moving m_2 from an infinite distance to P_2 is U_2, where U_2 is given by (2).

Consider now the work done by the forces of attraction exerted by m_1 and m_2 (both now supposed fixed in position) in bringing m_3 from infinity to its position P_3; the ξ-component of the total attraction (ξ now denoting the ξ-coordinate of P_3) is

$$G\frac{\partial}{\partial \xi}\left(\frac{m_1 m_3}{\Delta_{13}} + \frac{m_2 m_3}{\Delta_{23}}\right)$$

and by the previous argument the work done is evidently

$$G\left(\frac{m_1 m_3}{\Delta_{13}} + \frac{m_2 m_3}{\Delta_{23}}\right) \quad \text{.....(3)}$$

Hence the total work done in assembling m_2 and m_3 into their positions relative to m_1 is

$$U_2 + \text{the expression given by (3)}$$

or, denoting this sum by U_3, we have

$$U_3 = G\left(\frac{m_1 m_2}{\Delta_{12}} + \frac{m_1 m_3}{\Delta_{13}} + \frac{m_2 m_3}{\Delta_{23}}\right) \cdot$$

Thus U_3 is the force-function for the first three bodies.

By continuing this procedure it is evident that the total work done in assembling the n bodies into the positions occupied by them at a given time t is U, where

$$U = G\sum_{i=1}^{n}\sum_{j}\frac{m_i m_j}{\Delta_{ij}} \cdot \qquad (j>i).$$

The *potential energy* of the configuration is defined to be the work done in moving the system, as constituted, to some standard configuration which is taken to be the state of complete dispersion; thus the potential energy of the system is $-U$.

4·03. Integrals associated with the centre of mass of the system

For a particular value of i we have

$$\frac{\partial U}{\partial \xi_i} = Gm_i \frac{\partial}{\partial \xi_i}\left(\frac{m_1}{\Delta_{i1}} + \frac{m_2}{\Delta_{i2}} + \ldots + \frac{m_n}{\Delta_{in}}\right)$$

$$= Gm_i \frac{\partial}{\partial \xi_i} \sum_{j=1}^{n} \frac{m_j}{\Delta_{ij}} \quad , \quad (j \neq i) \quad .$$

Now $\Delta_{ij}^2 = (\xi_i - \xi_j)^2 + (\eta_i - \eta_j)^2 + (\zeta_i - \zeta_j)^2$, $\frac{\partial}{\partial \xi_i}\left(\frac{1}{\Delta_{ij}}\right) = -\frac{\xi_i - \xi_j}{\Delta_{ij}^3}$;

hence
$$\frac{\partial U}{\partial \xi_i} = Gm_i \sum_{j=1}^{n} \frac{m_j(\xi_j - \xi_i)}{\Delta_{ij}^3} \quad , \quad (j \neq i) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly,
$$\frac{\partial U}{\partial \eta_i} = Gm_i \sum_{1}^{n} \frac{m_j(\eta_j - \eta_i)}{\Delta_{ij}^3} \quad , \quad (j \neq i) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) by summing for all values of i from 1 to n we obtain

$$\sum_{1}^{n} \frac{\partial U}{\partial \xi_i} = G \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{m_i m_j (\xi_j - \xi_i)}{\Delta_{ij}^3} \quad ,$$

where $j \neq i$. The double summation on the right is zero for, taking a simple case, the two terms corresponding to $i=2$, $j=3$ and $i=3$, $j=2$ are respectively

$$\frac{Gm_2m_3}{\Delta_{23}^3}(\xi_3 - \xi_2) \text{ and } \frac{Gm_3m_2}{\Delta_{23}^3}(\xi_2 - \xi_3) \quad ;$$

the contribution of these two terms to the sum is zero and since the same is true for all such pairs we have

$$\sum_{1}^{n} \frac{\partial U}{\partial \xi_i} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

By 4·01 (3), this last equation becomes

$$\sum_{1}^{n} m_i \ddot{\xi}_i = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Similarly,
$$\sum_{1}^{n} m_i \ddot{\eta}_i = 0 \quad , \quad \sum_{1}^{n} m_i \ddot{\zeta}_i = 0 \quad .$$

These equations can be integrated at once ; we have

$$\Sigma m_i \dot{\xi}_i = \alpha_1 \quad , \quad \Sigma m_i \dot{\eta}_i = \beta_1 \quad , \quad \Sigma m_i \dot{\zeta}_i = \gamma_1 \quad , \ldots\ldots\ldots\ldots (5)$$

and again

$$\Sigma m_i \xi_i = \alpha_1 t + \alpha_2 \quad , \quad \Sigma m_i \eta_i = \beta_1 t + \beta_2 \quad , \quad \Sigma m_i \zeta_i = \gamma_1 t + \gamma_2 \quad , \ldots . (6)$$

where α_1, β_1, γ_1 and α_2, β_2, γ_2 are constants.

The equations (5) show that the sum of the linear momenta of the bodies in any direction is constant.

Let (X, Y, Z) denote the centre of mass of the n bodies and let M denote $\sum_{1}^{n} m_i$. Then

$$MX=\sum_1^n m_i\xi_i \quad \text{.....(7)}$$

with two similar equations in Y and Z. From (5) we have

$$M\dot{X}=\alpha_1 \quad , \quad M\dot{Y}=\beta_1 \quad , \quad M\dot{Z}=\gamma_1$$

which show that the centre of mass has a constant velocity. Again, from (6) and (7), we have

$$MX=\alpha_1 t+\alpha_2 \quad , \quad MY=\beta_1 t+\beta_2 \quad , \quad MZ=\gamma_1 t+\gamma_2 \quad \text{.....(8)}$$

which show that the centre of mass moves in a straight line.

The constants of integration in (6) and (8) are six in number.

It is evident that a system of parallel axes with its origin at the centre of mass is Newtonian in character ; if (x_i, y_i, z_i) are the coordinates of P_i with respect to this system of axes we shall have

$$m_i\ddot{x}_i=\frac{\partial U}{\partial x_i} \quad ,$$

with similar equations in y and z.

4·04. Integrals of areas

From 4·03 (1) and (2) we obtain

$$\xi_i\frac{\partial U}{\partial \eta_i}-\eta_i\frac{\partial U}{\partial \xi_i}=Gm_i\sum_{j=1}^n\frac{m_j}{\Delta_{ij}^3}(\xi_i\eta_j-\xi_j\eta_i) \quad , \quad (j\neq i) \quad .$$

Hence, by an argument similar to that in deriving 4·03 (3),

$$\sum_{i=1}^n\left(\xi_i\frac{\partial U}{\partial \eta_i}-\eta_i\frac{\partial U}{\partial \xi_i}\right)=0$$

which becomes, by virtue of 4·01 (3),

$$\sum_1^n\{m_i(\xi_i\ddot{\eta}_i-\ddot{\xi}_i\eta_i)\}=0$$

and on integration

$$\sum_1^n m_i(\xi_i\dot{\eta}_i-\dot{\xi}_i\eta_i)=c_3 \quad . \quad \text{.....(1)}$$

Similarly,

$$\sum_1^n m_i(\eta_i\dot{\zeta}_i-\dot{\eta}_i\zeta_i)=c_1 \quad , \quad \text{.....(2)}$$

$$\sum_1^n m_i(\zeta_i\dot{\xi}_i-\dot{\zeta}_i\xi_i)=c_2 \quad . \quad \text{.....(3)}$$

In these equations c_1, c_2 and c_3 are constants. The results (1), (2) and (3) are sometimes known as the *integrals of areas*. A complete interpretation is that the sums of the moments of momentum—or, alternatively, the sums of the angular momenta—of the n masses about each of the axes of reference are constants.

4·05. Alternative proofs of the results in 4·03 and 4·04

(i) Suppose that the origin is changed to (α, β, γ), the new axes being

parallel to the old. Let (x_i, y_i, z_i) be the coordinates of P_i with respect to the new axes. Then

$$\xi_i = x_i + \alpha \quad , \quad \eta_i = y_i + \beta \quad , \quad \zeta_i = z_i + \gamma \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and $U(\xi, \eta, \zeta)$ is transformed into the function $U_1(x, y, z\,;\ \alpha, \beta, \gamma)$.

Hence
$$\frac{\partial U_1}{\partial \alpha} = \sum_i \frac{\partial U}{\partial \xi_i} \cdot \frac{\partial \xi_i}{\partial \alpha} = \sum_i \frac{\partial U}{\partial \xi_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

by means of (1). Now,

$$\Delta_{ij}^2 = \sum_{\xi\eta\zeta} (\xi_i - \xi_j)^2 = \sum_{xyz} (x_i - x_j)^2$$

and, consequently, U_1 does not contain α, β or γ; hence $\dfrac{\partial U_1}{\partial \alpha} = 0$ so that (2) becomes $\sum_i \dfrac{\partial U}{\partial \xi_i} = 0$.

The results (8) in section 4·03 follow.

(ii) Suppose that the ξ and η axes are rotated through a positive angle θ in their plane and that the coordinates of P_i are (x_i, y_i, z_i) with respect to the new axes. Then, dropping the suffices for the moment, we have

$$\xi = x \cos\theta - y \sin\theta \quad , \quad \eta = x \sin\theta + y \cos\theta \quad , \quad \zeta = z \quad . \quad \ldots\ldots\ldots (3)$$

From these,

$$\frac{\partial \xi}{\partial \theta} = -\eta \quad , \quad \frac{\partial \eta}{\partial \theta} = \xi \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$U(\xi, \eta, \zeta)$ is transformed by means of (3) into the function $U_2(x, y, z\,;\ \theta)$; hence

$$\frac{\partial U_2}{\partial \theta} = \sum_i \left\{ \frac{\partial U}{\partial \xi_i} \cdot \frac{\partial \xi_i}{\partial \theta} + \frac{\partial U}{\partial \eta_i} \cdot \frac{\partial \eta_i}{\partial \theta} \right\} = \sum_i \left\{ -\eta_i \frac{\partial U}{\partial \xi_i} + \xi_i \frac{\partial U}{\partial \eta_i} \right\} \quad \ldots\ldots\ldots (5)$$

by means of (4). But, as in (i), $\Delta_{ij}^2 = \sum_{xyz} (x_i - x_j)^2$ so that U_2 does not involve θ; hence $\dfrac{\partial U_2}{\partial \theta} = 0$ and (5) becomes

$$\sum_i \left(\xi_i \frac{\partial U}{\partial \eta_i} - \eta_i \frac{\partial U}{\partial \xi_i} \right) = 0 \quad , \quad \text{or} \quad \sum_i (\xi_i \ddot{\eta}_i - \eta_i \ddot{\xi}_i) = 0$$

whence, by integration,

$$\sum_i (\xi_i \dot{\eta}_i - \dot{\xi}_i \eta_i) = c_3 \quad ,$$

as in 4·04 (1). The integrals of area 4·04 (2) and (3) are established by a similar procedure.

4·06. The energy integral

U is a function of the coordinates (ξ_i, η_i, ζ_i) for $i = 1, 2, \ldots, n$. Also, U does not involve the time explicitly; hence

$$\frac{dU}{dt} = \sum_1^n \left(\frac{\partial U}{\partial \xi_i} \dot{\xi}_i + \frac{\partial U}{\partial \eta_i} \dot{\eta}_i + \frac{\partial U}{\partial \zeta_i} \dot{\zeta}_i \right) = \sum_1^n m_i (\ddot{\xi}_i \dot{\xi}_i + \ddot{\eta}_i \dot{\eta}_i + \ddot{\zeta}_i \dot{\zeta}_i)$$

$$= \frac{1}{2}\frac{d}{dt}\sum_1^n m_i(\dot{\xi}_i^2+\dot{\eta}_i^2+\dot{\zeta}_i^2) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Let T denote the kinetic energy of the n bodies. Then,

$$T = \tfrac{1}{2}\sum_1^n m_i(\dot{\xi}_i^2+\dot{\eta}_i^2+\dot{\zeta}_i^2) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Hence (1) becomes $\dfrac{dU}{dt} = \dfrac{dT}{dt}$, from which

$$T - U = C \ , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where C is a constant.

Since, by 4·02, $-U$ is the potential energy of the system, formula (3) expresses the constancy of the total energy of the system.

Formula (3) gives the last—the tenth—of the known integrals in the general problem of n bodies.

It is to be added that in celestial mechanics the method of solution, which has been partially achieved in deriving the known integrals, does not take us very far. Instead, the solution proceeds by methods of approximation which are rendered practicable either because of geometrical circumstances—as in the lunar theory where the ratio of the geocentric distance of the Moon to that of the Sun is small—or as in the planetary theory which takes account of the smallness of the planetary masses in comparison with the solar mass.

4·07. The formula

$$\frac{d^2}{dt^2}\sum_1^n m_i R_i^2 = 2U + 4C = 2T + 2C \quad .$$

In this formula R_i is the distance of the position P_i, occupied by m_i, from the origin of coordinates—the origin may be the centre of mass of the system of the n bodies.

From 4·01 (3) we have, by multiplying the equations by ξ_i, η_i, and ζ_i, and summing,

$$\sum_1^n m_i(\xi_i\ddot{\xi}_i + \eta_i\ddot{\eta}_i + \zeta_i\ddot{\zeta}_i) = \sum_1^n\left(\xi_i\frac{\partial U}{\partial \xi_i} + \eta_i\frac{\partial U}{\partial \eta_i} + \zeta_i\frac{\partial U}{\partial \zeta_i}\right) \quad .$$

Now U is a homogeneous function of all the coordinates of order -1, and hence, by Euler's theorem,

$$\sum_{\xi\eta\zeta}\sum_1^n \xi_i\frac{\partial U}{\partial \xi_i} = -U \ , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

or

$$\sum_{\xi\eta\zeta}\sum_1^n m_i\xi_i\ddot{\xi}_i + U = 0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now

$$\xi\ddot{\xi} = \frac{d}{dt}(\xi\dot{\xi}) - \dot{\xi}^2 = \frac{1}{2}\frac{d^2}{dt^2}(\xi^2) - \dot{\xi}^2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Hence (2) becomes

$$\frac{1}{2}\frac{d^2}{dt^2}\sum_1^n m_i(\xi_i^2+\eta_i^2+\zeta_i^2)-\sum_1^n m_i(\dot{\xi}_i^2+\dot{\eta}_i^2+\dot{\zeta}_i^2)+U=0 \quad .$$

By 4·06 (2), the second term is $-2T$. Also

$$\xi_i^2+\eta_i^2+\zeta_i^2=R_i^2 \quad .$$

We then obtain

$$\frac{d^2}{dt^2}\Sigma m_iR_i^2-4T+2U=0$$

or, making use of 4·06 (3),

$$\frac{d^2}{dt^2}\Sigma m_iR_i^2=2U+4C=2T+2C \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

In this formula, U and T are positive ; hence if C is positive—or if C is negative so that $|\,C\,|<T$ or $<\frac{1}{2}U$—the right hand side of (4) is positive ; consequently, $\frac{d^2}{dt^2}\Sigma m_iR_i^2$ is positive and therefore $\frac{d}{dt}\Sigma m_iR_i^2$ will increase indefinitely from which it is deduced that at least one of the R_i's increases without limit, resulting in the expulsion of at least one of the masses from the system. If the system is to remain stable in the sense that no dispersion occurs, a necessary condition is that C should be negative and such that $|\,C\,|>T$ or $>\frac{1}{2}U$. This condition, however, is by no means sufficient to ensure the stability of the system.

4·08. The invariable plane

We begin with the " integrals of area " (section 4·04) ; they are

$$\left.\begin{aligned}\Sigma m_i(\eta_i\dot{\zeta}_i-\dot{\eta}_i\zeta_i)&=c_1 \quad ,\\ \Sigma m_i(\zeta_i\dot{\xi}_i-\dot{\zeta}_i\xi_i)&=c_2 \quad ,\\ \Sigma m_i(\xi_i\dot{\eta}_i-\dot{\xi}_i\eta_i)&=c_3 \quad .\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The expressions on the left are directed quantities representing the components of the total angular momentum of the system with respect to the ξ, η and ζ axes respectively. Hence an axis OB can be found with which the *total* angular momentum, A, of the system is associated. If (l_1, m_1, n_1) are the direction-cosines of OB, then

$$l_1A=c_1 \quad , \quad m_1A=c_2 \text{ and } n_1A=c_3 \, ;$$

hence,

$$\frac{l_1}{c_1}=\frac{m_1}{c_2}=\frac{n_1}{c_3} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If the centre of mass of the system is taken as the origin O, the plane through O with direction-cosines given by (2) is called the *invariable plane* of the system. The invariable plane can be determined only if c_1, c_2, c_3 are known and, from (1), this presupposes that at any time t the masses,

the coordinates and the components of velocity of all the n bodies are known. With our present knowledge of the Solar System—gained by the application of the methods to be described in later chapters—the constants c_1, c_2, c_3 can be obtained with considerable accuracy; accordingly, the invariable plane can be specified with reference to the usual reference plane. Any adjustment of the values of c_1, c_2, c_3 in (2) in future will almost certainly arise through the determination of more accurate values of the planetary masses.

If we choose new axes through the centre of mass so that the normal to the invariable plane is the Z axis, the X and Y axes being chosen in that plane, then we obtain equations of the same type as (1). But now c_3 is equal to the total angular momentum of the system and $c_1 = c_2 = 0$. The advantage, of course, in this choice of axes is that two of the constants of integration are zero, thereby leading to a simplifying of the analytical expressions in the general mathematical theory. The use of the invariable plane as the principal plane of reference was advocated by Laplace but the suggestion has not received a great deal of practical application.

The constancy of the angular momentum about the normal to the invariable plane implies certain qualifications. The Sun and the planets are not point-masses but spherical (or nearly spherical) bodies each rotating about an axis, and the rotations must contribute to the angular momentum of the system. If these bodies are rigid spheres in each of which the density is a function of the distance from the centre of the sphere, the total angular momentum of the system remains constant and the invariable plane can be specified and will actually be invariable. This condition does not apply strictly to most of the planets and only approximately to the Sun. Further, even the rotational angular momentum of some of the planets—the Earth, in particular—is subject to progressive changes due to precession and tidal friction; as regards the former, the Earth's axis, for example, changes its orientation with respect to the reference plane and, consequently, the components of its angular momentum with respect to the reference axes are continually changing; as regards the latter, rotation is gradually slowed up, although only at a minute rate.

It is estimated that of the total angular momentum of the Solar System, 98 per cent is attributed to the orbital motions of the four great planets, Jupiter, Saturn, Uranus and Neptune. Accordingly, the variations which we have just mentioned contribute an almost inappreciable amount to the changes in the total angular momentum of the system. The invariable plane is thus, substantially, a fixed plane with reference to the centre of mass of the system.

4·09. The coordinates in the lunar or stellar problem

As an example of the choice of coordinates in a specific instance we derive the relevant equations of motion in the lunar or stellar problem; these will form the basis of more detailed investigations. In the lunar problem we are mainly concerned with three bodies only—the Earth (mass, m_1), the Moon (mass, m_2) and the comparatively remote Sun (mass, m_3), the preponderating mass being the last; in the stellar problem we are concerned with a similar configuration consisting of two close stars with a third star comparatively remote, the masses of all three bodies being of the same order of magnitude.

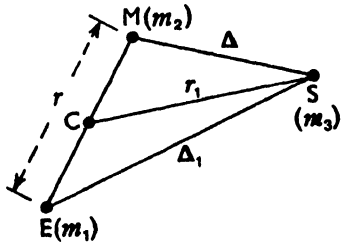


FIG. 11

In Fig. 11 let the three bodies be E, M and S with coordinates (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) and (ξ_3, η_3, ζ_3) with respect to the usual Newtonian frame. The force-function U is given by

$$U = G\left(\frac{m_1 m_2}{r} + \frac{m_2 m_3}{\Delta} + \frac{m_1 m_3}{\Delta_1}\right) \quad , \qquad \ldots\ldots\ldots\ldots (1)$$

where $r = EM$, $\Delta = MS$ and $\Delta_1 = ES$.

Let (x, y, z) be the coordinates of M with respect to parallel axes through E, and (x_1, y_1, z_1) the coordinates of S with respect to parallel axes through C, the centre of mass of E and M. Further, let (X, Y, Z) be the coordinates of C with respect to the Newtonian axes; then

$$X = \frac{m_1\xi_1 + m_2\xi_2}{m_1 + m_2} \quad . \qquad \ldots\ldots\ldots\ldots (2)$$

We have

$$x = \xi_2 - \xi_1 \qquad \ldots\ldots\ldots\ldots (3)$$

and

$$x_1 = \xi_3 - X = \xi_3 - \frac{m_1\xi_1 + m_2\xi_2}{m_1 + m_2} \quad . \qquad \ldots\ldots\ldots\ldots (4)$$

From this equation we derive, by means of (3),

$$\xi_3 - \xi_1 = x_1 + \frac{m_2 x}{m_1 + m_2} \quad . \qquad \ldots\ldots\ldots\ldots (5)$$

Also,

$$\xi_3 - \xi_2 \equiv \xi_3 - \xi_1 - (\xi_2 - \xi_1) = x_1 - \frac{m_1 x}{m_1 + m_2} \quad . \qquad \ldots\ldots\ldots\ldots (6)$$

In the sequel we need only consider the equations of motion parallel to the ξ-axis. These equations are

$$\ddot{\xi}_1 = \frac{1}{m_1}\frac{\partial U}{\partial \xi_1} \quad , \quad \ddot{\xi}_2 = \frac{1}{m_2}\frac{\partial U}{\partial \xi_2} \quad , \quad \ddot{\xi}_3 = \frac{1}{m_3}\frac{\partial U}{\partial \xi_3} \quad , \qquad \ldots\ldots\ldots\ldots (7)$$

and from them we derive the corresponding equations in x and x_1.

From (3) and (7), we have at once

$$\ddot{x}=\frac{1}{m_2}\frac{\partial U}{\partial \xi_2}-\frac{1}{m_1}\frac{\partial U}{\partial \xi_1} \quad , \qquad \text{.....(8)}$$

and, from (5) and (7),

$$\ddot{x}_1=\frac{1}{m_3}\frac{\partial U}{\partial \xi_3}-\frac{1}{m_1}\frac{\partial U}{\partial \xi_1}-\frac{m_2\ddot{x}}{m_1+m_2} \quad ,$$

which becomes, by (8),

$$\ddot{x}_1=\frac{1}{m_3}\frac{\partial U}{\partial \xi_3}-\frac{1}{m_1+m_2}\left(\frac{\partial U}{\partial \xi_1}+\frac{\partial U}{\partial \xi_2}\right) \quad .$$

Now, by 4·03 (3) when $n=3$,

$$\frac{\partial U}{\partial \xi_1}+\frac{\partial U}{\partial \xi_2}+\frac{\partial U}{\partial \xi_3}=0 \quad ;$$

hence

$$\ddot{x}_1=\frac{m_1+m_2+m_3}{m_3(m_1+m_2)}\frac{\partial U}{\partial \xi_3} \quad . \qquad \text{.....(9)}$$

Now,

$$r^2=x^2+y^2+z^2 \quad ,$$

$$\Delta^2=\Sigma(\xi_3-\xi_2)^2=\Sigma\left(x_1-\frac{m_1x}{m_1+m_2}\right)^2 \quad ,$$

$$\Delta_1^2=\Sigma(\xi_3-\xi_1)^2=\Sigma\left(x_1+\frac{m_2x}{m_1+m_2}\right)^2$$

by means of (5) and (6).

Accordingly, U becomes a function of x and x_1, y and y_1 and z and z_1. We then obtain, using (3) and (4),

$$\frac{\partial U}{\partial \xi_1}=\frac{\partial U}{\partial x}\cdot\frac{\partial x}{\partial \xi_1}+\frac{\partial U}{\partial x_1}\cdot\frac{\partial x_1}{\partial \xi_1}=-\frac{\partial U}{\partial x}-\frac{m_1}{m_1+m_2}\frac{\partial U}{\partial x_1} \quad .$$

Similarly,

$$\frac{\partial U}{\partial \xi_2}=\frac{\partial U}{\partial x}-\frac{m_2}{m_1+m_2}\frac{\partial U}{\partial x_1}, \quad \frac{\partial U}{\partial \xi_3}=\frac{\partial U}{\partial x_1} \quad .$$

By means of these results, (8) and (9) become

$$\ddot{x}=\frac{m_1+m_2}{m_1m_2}\frac{\partial U}{\partial x} \quad , \qquad \text{.....(10)}$$

$$\ddot{x}_1=\frac{m_1+m_2+m_3}{m_3(m_1+m_2)}\frac{\partial U}{\partial x_1} \quad . \qquad \text{.....(11)}$$

Write

$$\mu=G(m_1+m_2) \quad ; \qquad \text{.....(12)}$$

then (10) becomes, on inserting the expression for U in (1),

$$\ddot{x}+\frac{\mu x}{r^3}=\frac{\partial R}{\partial x} \quad ,$$

where R is the disturbing function for the Moon given by

$$R=\frac{Gm_3(m_1+m_2)}{m_1m_2}\left(\frac{m_2}{\Delta}+\frac{m_1}{\Delta_1}\right) \quad . \qquad \text{.....(13)}$$

Since r is not a function of x_1, y_1, z_1, the equation of motion for the Sun is, from (11),

$$\ddot{x}_1 = \frac{G(m_1+m_2+m_3)}{m_1+m_3}\frac{\partial}{\partial x_1}\left(\frac{m_2}{\Delta}+\frac{m_1}{\Delta_1}\right) \quad .$$

It is convenient to use a function F defined by

$$F = \frac{m_2}{\Delta}+\frac{m_1}{\Delta_1} \quad ,$$

and to replace m_1, m_2 and m_3 by E, M and S respectively. Then F is given by

$$F = \frac{M}{\Delta}+\frac{E}{\Delta_1} \quad , \qquad \text{.....(14)}$$

and the equations of motion for the Moon and the Sun become

$$\ddot{x}+\frac{\mu x}{r^3} = \frac{GS(E+M)}{EM}\frac{\partial F}{\partial x} \quad , \qquad \text{.....(15)}$$

$$\ddot{x}_1 = -\frac{G(E+M+S)}{E+M}\frac{\partial F}{\partial x_1} \quad . \qquad \text{.....(16)}$$

These equations form the foundations of the lunar theory and, as we shall see in due course—section 7·03—they derive their particular power from the circumstance that, to a very high degree of approximation, the motion of the Sun relative to C in Fig. 11 is an ellipse ; accordingly, the coordinates x_1, y_1, z_1 in the function F can be expressed in terms of the time, introduced through the Sun's mean anomaly, by the formulae of Chapter 3.

Chapter 5

LAGRANGE'S PLANETARY EQUATIONS

5·01. Change of variables

The equations of motion of a planet P (mass, m) under the central attraction of the Sun (mass, m_0) and acted upon by planets P_1, P_2, ... (masses, m_1, m_2 ...) have been found in section 1·07 ; they are :

$$\ddot{x}+\frac{\mu x}{r^3}=\frac{\partial R}{\partial x} \quad , \quad \ddot{y}+\frac{\mu y}{r^3}=\frac{\partial R}{\partial y} \quad , \quad \ddot{z}+\frac{\mu z}{r^3}=\frac{\partial R}{\partial z} \quad , \quad \ldots\ldots\ldots\ldots(1)$$

where $\mu=G(m_0+m)=n^2a^3$ and R is the disturbing function ; we assume that the reference plane is the ecliptic with Υ defining the x-axis.

In this chapter we develop one method, due to Lagrange and used extensively in planetary theory, for transforming the equations (1) into forms suitable for their eventual solution.

Consider first the solution of the equations of motion relating to the *undisturbed* orbit of the planet P ; the equations are

$$\ddot{x}+\frac{\mu x}{r^3}=0 \quad , \quad \ddot{y}+\frac{\mu y}{r^3}=0 \quad , \quad \ddot{z}+\frac{\mu z}{r^3}=0 \quad . \quad \ldots\ldots\ldots\ldots(2)$$

The solutions are given by 2·15 (2) in terms of the eccentric anomaly E and the direction-cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) defined in terms of the elements Ω, ω, i by 2·15 (3) and (4). In particular,

$$x=al_1\cos E+bl_2\sin E-ael_1 \quad ; \quad \ldots\ldots\ldots\ldots(3)$$

the corresponding formulae for y and z are in terms of m_1, m_2 and n_1, n_2 respectively.

Reference to 3·11 (9) and (13) shows that $\cos E$ and $\sin E$ are of the forms given by

$$\cos E=-\tfrac{1}{2}e+\sum_{s=1}^{\infty}C_s\cos sM \quad ,$$

$$\sin E=\sum_{s=1}^{\infty}D_s\sin sM \quad ,$$

where M is the mean anomaly and the C's and D's depend on Bessel Functions involving the eccentricity e.

On inserting these expressions in (3) it is readily seen that x is given by a general expression of the form

$$x=A+\Sigma B\cos\,(i_1M+j\Omega+k\omega) \quad ,$$

where A and B are definite functions of a, e, i, and i_1, j, k are positive or negative integers including zero. There are, of course, similar formulae

for y and z. At present we are not so much interested in the details of these expressions as in the fact that x, y and z are given by *definite* functions of the time and the six orbital elements, t being introduced through the mean anomaly M. It is to be remarked that in the early part of the chapter we shall be concerned, for convenience in the analytical developments, with the element χ—rather than with τ or ϵ—as it appears in Kepler's equation

$$E - e \sin E = nt + \chi \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Further, it is convenient to denote the group of elements a, e, χ by α_i $(i = 1, 2, 3)$, and the group of elements Ω, ω, i by $\beta_i (i = 1, 2, 3)$.

The solution of (2) can then be expressed functionally by

$$x = F_1(t, \alpha_i, \beta_i),\ y = F_2(t, \alpha_i, \beta_i),\ z = F_3(t, \alpha_i, \beta_i) \quad \ldots\ldots\ldots\ldots (5)$$

where F_1, F_2, F_3 are definite functions of the time and the elements.

In the same way we can express the components of velocity, $\left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)$, of the planet P in its undisturbed orbit in functional form by

$$\frac{\partial x}{\partial t} = G_1(t, \alpha_i, \beta_i),\ \frac{\partial y}{\partial t} = G_2(t, \alpha_i, \beta_i),\ \frac{\partial z}{\partial t} = G_3(t, \alpha_i, \beta_i) \quad \ldots\ldots\ldots (6)$$

where $\frac{\partial}{\partial t}$ signifies differentiation with respect to the time, the α's and β's being regarded of course as constants ; actually, since

$$(1 - e \cos E)\dot{E} \equiv \frac{r}{a}\dot{E} = n \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

the function G_1, for example, is found from (3) to be

$$G_1 = -\frac{na}{r}(al_1 \sin E - bl_2 \cos E) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

which can be developed into a *definite* function of t and the six elements of a nature similar to that previously found for x.

In considering the solution of (1) in the three variables x, y and z we take the six elements α_i, β_i to be new variables, x, y and z *retaining their functional forms* given by $x = F_1(t, \alpha_i, \beta_i)$ etc. ; from these last three relations we can express any *three* of the α's and β's in terms of x, y, z. To complete the specification of the remaining three α's and β's, we require three additional functional relationships ; we choose these to be

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{\partial x}{\partial t} \equiv G_1(t, \alpha_i, \beta_i) \quad , \\ \frac{dy}{dt} &= \frac{\partial y}{\partial t} \equiv G_2(t, \alpha_i, \beta_i) \quad , \\ \frac{dz}{dt} &= \frac{\partial z}{\partial t} \equiv G_3(t, \alpha_i, \beta_i) \quad . \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

in which $\frac{d}{dt}$ signifies differentiation with respect to the time and the six new variables α_i, β_i which are themselves now supposed to be functions of the time. The method, in general, is known as the method of variation of arbitrary constants.

From the first of (9), we have

$$\frac{d^2x}{dt^2}=\frac{\partial G_1}{\partial t}+\sum_i \frac{\partial G_1}{\partial \alpha_i}\frac{d\alpha_i}{dt}+\sum_i \frac{\partial G_1}{\partial \beta_i}\frac{d\beta_i}{dt} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

where the summations are taken for $i=1$ to 3.

But from (6), writing $\dot{x}$ for $\frac{\partial x}{\partial t}$, we have

$$\frac{\partial^2 x}{\partial t^2}=\frac{\partial G_1}{\partial t} \quad , \quad \frac{\partial \dot{x}}{\partial \alpha_i}=\frac{\partial G_1}{\partial \alpha_i} \text{ and } \frac{\partial \dot{x}}{\partial \beta_i}=\frac{\partial G_1}{\partial \beta_i} \quad ;$$

hence (10) becomes

$$\frac{d^2x}{dt^2}=\frac{\partial^2 x}{\partial t^2}+\sum_i \frac{\partial \dot{x}}{\partial \alpha_i}\cdot\dot{\alpha}_i+\sum_i \frac{\partial \dot{x}}{\partial \beta_i}\cdot\dot{\beta}_i \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Since, in the first equation of (2), the solution is $x=F_1(t, \alpha_i, \beta_i)$ in which the α's and β's are constants, we can rewrite this equation as

$$\frac{\partial^2 x}{\partial t^2}+\frac{\mu x}{r^3}=0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

and (11) becomes

$$\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}=\sum_i \frac{\partial \dot{x}}{\partial \alpha_i}\cdot\dot{\alpha}_i+\sum_i \frac{\partial \dot{x}}{\partial \beta_i}\cdot\dot{\beta}_i \quad ;$$

from (1) we obtain

$$\sum_i \frac{\partial \dot{x}}{\partial \alpha_i}\cdot\dot{\alpha}_i+\sum_i \frac{\partial \dot{x}}{\partial \beta_i}\cdot\dot{\beta}_i=\frac{\partial R}{\partial x} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Again, since $x=F_1(t, \alpha_i, \beta_i)$,

$$\frac{dx}{dt}=\frac{\partial F_1}{\partial t}+\sum_i \frac{\partial F_1}{\partial \alpha_i}\cdot\dot{\alpha}_i+\sum_i \frac{\partial F_1}{\partial \beta_i}\cdot\dot{\beta}_i \quad .$$

But $$\frac{\partial F_1}{\partial t}=\frac{\partial x}{\partial t} \quad , \quad \frac{\partial F_1}{\partial \alpha_i}=\frac{\partial x}{\partial \alpha_i} \quad , \quad \frac{\partial F_1}{\partial \beta_i}=\frac{\partial x}{\partial \beta_i}$$

and, since by (8) $\frac{dx}{dt}=\frac{\partial x}{\partial t}$, we obtain

$$\sum_i \frac{\partial x}{\partial \alpha_i}\cdot\dot{\alpha}_i+\sum_i \frac{\partial x}{\partial \beta_i}\cdot\dot{\beta}_i=0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

The formulae (13) and (14) are important formulae, from which, as we shall see later, we derive equations expressing $\dot{\alpha}_1$, $\dot{\alpha}_2$, ... $\dot{\beta}_3$ each in terms of derivatives of the disturbing function R.

5·02. The osculating ellipse

In Fig. 12 let P be the position of the planet at time t_0. If we suppose that at this moment the disturbing planets cease to exercise any effect on P, the orbit of P will be an ellipse PQ with constant elements $a_0, e_0, \ldots i_0$. The coordinates of P will then be given functionally as $x = F_1(t_0, a_0, \ldots i_0)$ with the two similar functions for y and z. Similarly, the components of velocity in the elliptic orbit at time t_0 are given by $\frac{\partial x}{\partial t} \equiv \dot{x} = G_1(t_0, a_0, \ldots i_0)$ with the corresponding relations in y and z, the notation being that of the previous section. Further, if we suppose that x, y, z and $\dot{x}, \dot{y}, \dot{z}$ are known at time t_0, we have the requisite number of equations for determining the six elements $a_0, \ldots i_0$ of the ellipse.

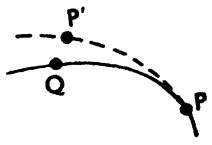


FIG. 12

The actual path of the planet when the action of the disturbing planets is taken into account will be a path somewhat different from the ellipse; it is indicated by the curve PP' in Fig. 12. Since the components of velocity in the actual orbit are the same as the components in the elliptic orbit at time t_0 $\left(\frac{dx}{dt} = \frac{\partial x}{\partial t}, \text{ etc.}\right)$ the tangent to the actual orbit at P will be the same as the tangent to the ellipse at that point. The ellipse PQ is called the *osculating ellipse* at time t_0, and the constants $a_0, e_0, \ldots i_0$ are the *osculating elements* at time t_0. The plane of the osculating ellipse is the *osculating plane* at t_0 and may be used as a reference plane in place of the ecliptic.

At time t_1, just subsequent to t_0, there will be a new osculating ellipse associated with the corresponding coordinates and components of velocity of the planet in its actual path, with new elements $a_1, \ldots i_1$; and so on for times $t_2, t_3, \ldots$. The differences $a_1 - a_0, \ldots i_1 - i_0$ are the *perturbations* of the elements in the interval $t_1 - t_0$, and the rates of change of the elements are precisely the rates $\dot{\alpha}_i, \dot{\beta}_i$ which are the subject of the analysis in the previous section. The *perturbations* of the coordinates (x, y, z) are defined in a similar way.

5·03. Use of the osculating ellipse for short-term predictions

We show that over small intervals of time, the positions of the planet in its actual orbit differ by negligible amounts from the corresponding positions in the osculating ellipse; this fact is taken advantage of in calculating an ephemeris of the planet over the small interval of time envisaged.

Let $P'(x, y, z)$ in Fig. 12 be the true position of the planet at time t and $Q(x', y', z')$ the corresponding position in the osculating ellipse. Let zero suffix denote the evaluation of a quantity at time t_0. Writing $\Delta t \equiv t - t_0$, we have

$$x' = x_0' + \Delta t \left(\frac{\partial x'}{\partial t}\right)_0 + \tfrac{1}{2}(\Delta t)^2 \left(\frac{\partial^2 x'}{\partial t^2}\right)_0 + \ldots$$

and $$x = x_0 + \Delta t \left(\frac{dx}{dt}\right)_0 + \tfrac{1}{2}(\Delta t)^2 \left(\frac{d^2 x}{dt^2}\right)_0 + \ldots$$

Then, approximately, since $x_0 = x_0'$ and $\left(\frac{dx'}{dt}\right)_0 = \left(\frac{\partial x'}{\partial t}\right)_0$ at time t_0,

$$x - x' = \tfrac{1}{2}(\Delta t)^2 \left\{\frac{d^2 x}{dt^2} - \frac{\partial^2 x'}{\partial t^2}\right\}_0 ,$$

or, by 5·01 (1) and (12),

$$x - x' = \tfrac{1}{2}(\Delta t)^2 \left(\frac{\partial R}{\partial x}\right)_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let a be the semi-major axis of P's elliptic (osculating) orbit in astronomical units and T the period in years. Then, if we neglect m/m_0,

$$\frac{4\pi^2 a^3}{T^2} = G m_0 ,$$

where m_0 is the Sun's mass. Then

$$(\Delta t)^2 \equiv \left(\frac{\Delta t}{T}\right)^2 T^2 = \frac{4\pi^2 a^3}{G m_0} \cdot \left(\frac{\Delta t}{T}\right)^2 .$$

For convenience we write $R = G m_1 R_1$, where

$$R_1 = \frac{1}{\Delta} - \frac{x x_1 + y y_1 + z z_1}{r_1^3}$$

from which $$\frac{\partial R_1}{\partial x} = \frac{x_1 - x}{\Delta^3} - \frac{x_1}{r_1^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Then $$x - x' = 2\pi^2 a^3 \left(\frac{m_1}{m_0}\right)\left(\frac{\Delta t}{T}\right)^2 \frac{\partial R_1}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

As an example, consider the orbit of the Earth as disturbed by Jupiter, the most massive planet. Then, $a = 1$, $m_1/m_0 \fallingdotseq 10^{-3}$, $2\pi^2 \fallingdotseq 20$. Suppose that $\Delta t = 0{\cdot}1$ year ; then $\Delta t/T = 0{\cdot}1$. Formula (3) becomes

$$x - x' \fallingdotseq 2 \,.\, 10^{-4} \left(\frac{\partial R_1}{\partial x}\right)_0 .$$

Neglecting the inclination of Jupiter's orbit to the ecliptic we can choose the ecliptic axes so that at t_0 Jupiter's coordinates are $(r_1, 0)$, or—neglecting the eccentricity—$(a_1, 0)$. The expression on the right-hand side of (2) is a maximum at opposition ; then, $x \fallingdotseq a \fallingdotseq 1$ and $\Delta \fallingdotseq a_1 - 1$.

Now, $a_1 \fallingdotseq 5$; hence

$$\left(\frac{\partial R_1}{\partial x}\right)_0 \fallingdotseq \frac{9}{400} .$$

We then have

$$x - x' \fallingdotseq 4{\cdot}5 \,.\, 10^{-6} \text{ astron. units}$$
$$\fallingdotseq 420 \text{ miles.}$$

With the assumptions made, $y=z=0$ and $y_1=z_1=0$ so that the perturbations in the y and z coordinates are zero. The whole perturbation is thus thrown into the x-coordinate; with the value of $(x-x')$ just obtained the perturbation is relatively minute and for many purposes could be neglected. The inference is that for small values of Δt the osculating ellipse will provide nearly accurate values of the coordinates of the Earth on either side of t_0.

The perturbations on the Earth's orbit produced by the other planets of the solar system are very much smaller than that just found for Jupiter, mainly because of the smaller masses concerned.

5·04. The equations for $\dot{\alpha}_i$ and $\dot{\beta}_i$

We start from the equations 5·01 (14) and (13),

namely,
$$\sum_i \frac{\partial x}{\partial \alpha_i}\dot{\alpha}_i + \sum_i \frac{\partial x}{\partial \beta_i}\dot{\beta}_i = 0 \quad \ldots\ldots(1)$$

and
$$\sum_i \frac{\partial \dot{x}}{\partial \alpha_i}\dot{\alpha}_i + \sum_i \frac{\partial \dot{x}}{\partial \beta_i}\dot{\beta}_i = \frac{\partial R}{\partial x} \quad , \quad \ldots\ldots(2)$$

where i takes the values 1, 2 and 3.

Multiply (2) by $\dfrac{\partial x}{\partial \alpha_1}$ and (1) by $\dfrac{\partial \dot{x}}{\partial \alpha_1}$ and subtract; we then have

$$\sum_i \left(\frac{\partial x}{\partial \alpha_1}\cdot\frac{\partial \dot{x}}{\partial \alpha_i} - \frac{\partial x}{\partial \alpha_i}\cdot\frac{\partial \dot{x}}{\partial \alpha_1}\right)\dot{\alpha}_i + \sum_i \left(\frac{\partial x}{\partial \alpha_1}\cdot\frac{\partial \dot{x}}{\partial \beta_i} - \frac{\partial x}{\partial \beta_i}\cdot\frac{\partial \dot{x}}{\partial \alpha_1}\right)\dot{\beta}_i = \frac{\partial R}{\partial x}\cdot\frac{\partial x}{\partial \alpha_1}$$

or, in terms of Jacobians,

$$\sum_i \frac{\partial(x, \dot{x})}{\partial(\alpha_1, \alpha_i)}\dot{\alpha}_i + \sum_i \frac{\partial(x, \dot{x})}{\partial(\alpha_1, \beta_i)}\dot{\beta}_i = \frac{\partial R}{\partial x}\cdot\frac{\partial x}{\partial \alpha_1} \quad . \quad \ldots\ldots(3)$$

There are two similar equations in y and z; adding these to (3) and, writing

$$[\alpha_1, \alpha_i] \equiv \frac{\partial(x, \dot{x})}{\partial(\alpha_1, \alpha_i)} + \frac{\partial(y, \dot{y})}{\partial(\alpha_1, \alpha_i)} + \frac{\partial(z, \dot{z})}{\partial(\alpha_1, \alpha_i)} \quad , \quad \ldots\ldots(4)$$

we have

$$\sum_{i=1}^{3} [\alpha_1, \alpha_i]\dot{\alpha}_i + \sum_{i=1}^{3} [\alpha_1, \beta_i]\dot{\beta}_i = \frac{\partial R}{\partial x}\cdot\frac{\partial x}{\partial \alpha_1} + \frac{\partial R}{\partial y}\cdot\frac{\partial y}{\partial \alpha_1} + \frac{\partial R}{\partial z}\cdot\frac{\partial z}{\partial \alpha_1} . \quad \ldots\ldots(5)$$

The expression for $[\alpha_1, \alpha_i]$, as defined in (4), is known as a *Lagrangian bracket* with reference to α_1, α_i.

Now R is a function of the planet's coordinates (x, y, z) and of the coordinates (x_1, y_1, z_1) of the disturbing planets. When the expressions for x, y, z given by 5·01 (5)—these are: $x=F_1(t, \alpha_i, \beta_i)$, $y=F_2(t, \alpha_i, \beta_i)$, $z=F_3(t, \alpha_i, \beta_i)$—and the similar expressions for x_1, y_1, z_1, etc. are substituted in the disturbing function, R becomes a function of t, α_i, β_i and of the corresponding α's and β's associated with the orbits of the disturbing

planets. Now the element α_1 appears in R through the substitutions for x, y and z just given; hence

$$\frac{\partial R}{\partial x}\cdot\frac{\partial x}{\partial \alpha_1}+\frac{\partial R}{\partial y}\cdot\frac{\partial y}{\partial \alpha_1}+\frac{\partial R}{\partial z}\cdot\frac{\partial z}{\partial \alpha_1}=\frac{\partial R}{\partial \alpha_1} \quad .$$

The equation (5) now becomes

$$\sum_{i=1}^{3}\left\{[\alpha_1, \alpha_i]\,\dot{\alpha}_i+[\alpha_1, \beta_i]\,\dot{\beta}_i\right\}=\frac{\partial R}{\partial \alpha_1} \quad \text{.....(6)}$$

Written out in full this equation is

$$[\alpha_1, \alpha_2]\,\dot{\alpha}_2+[\alpha_1, \alpha_3]\,\dot{\alpha}_3+[\alpha_1, \beta_1]\,\dot{\beta}_1+[\alpha_1, \beta_2]\,\dot{\beta}_2+[\alpha_1, \beta_3]\,\dot{\beta}_3=\frac{\partial R}{\partial \alpha_1} \quad ,$$

the bracket $[\alpha_1, \alpha_1]$, which is the coefficient of $\dot{\alpha}_1$, being zero by definition. However, it is convenient to retain the term involving such a bracket in the general formula (6).

We shall have three equations of the form (6) which we write as

$$\sum_{i=1}^{3}\left\{[\alpha_r, \alpha_i]\,\dot{\alpha}_i+[\alpha_r, \beta_i]\,\dot{\beta}_i\right\}=\frac{\partial R}{\partial \alpha_r} \quad \text{.....(7)}$$

where r takes the values 1, 2 and 3.

Similarly, we shall have three equations with differential coefficients of R with respect to the β's; these are

$$\sum_{i=1}^{3}\left\{[\beta_r, \alpha_i]\,\dot{\alpha}_i+[\beta_r, \beta_i]\,\dot{\beta}_i\right\}=\frac{\partial R}{\partial \beta_r} \quad \text{.....(8)}$$

It will be shown later that any Lagrangian bracket is a function, in general, of the six elements alone. The equations (7) are three in number and the equations (8) are three in number; further, each is linear in $\dot{\alpha}_1, \dot{\alpha}_2, \ldots, \dot{\beta}_3$. Solving these we obtain an equation for $\dot{\alpha}_1$, for example, in terms of the differential coefficients of R and of the various brackets; there are, of course, equations for each of $\dot{\alpha}_2, \ldots, \dot{\beta}_3$ of a similar nature.

The precise algebraical form of R is known and it will be shown later how R can be developed into a series of terms involving α_i and β_i—and involving the time and the α's and β's associated with the disturbing planets. It is sufficient at this stage to note that $\dfrac{\partial R}{\partial \alpha_1}$ etc. can be found explicitly in known forms. To complete the preliminary discussion of the equations which separately give $\dot{\alpha}_1$, etc., we require to evaluate the Lagrangian brackets in terms of the α's and β's.

Note. It is clear from the method of deriving the group of six equations in $\dot{\alpha}_1 \ldots \dot{\beta}_3$ that we would derive equations of precisely the same form if each of $\alpha_1, \alpha_2 \ldots \beta_3$ had been taken, at the outset, to be independent functions of one or more of the six elliptic elements.

5·05. Properties of the Lagrangian brackets

Let p, q be any two of the six elements α_i, β_i or, generally, any two independent functions of the elements. Then, by definition,

$$[p, q] = \sum_{xyz} \frac{\partial(x, \dot{x})}{\partial(p, q)} \quad .$$

(i) $\quad [p, q] = -[q, p] \quad .$

This property follows from that of a Jacobian, namely,

$$\frac{\partial(x, \dot{x})}{\partial(p, q)} = -\frac{\partial(x, \dot{x})}{\partial(q, p)} \quad .$$

(ii) $\quad [p, p] = 0 \quad .$

This property is obvious from the definition of a Jacobian.

(iii) $\quad \frac{\partial}{\partial t}[p, q] = 0 \quad .$

This is a succinct way of expressing the property that, when x, y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$, are expressed in terms of t and the elements by $x = F_1(t, \alpha_i, \beta_i) \ldots$ $\dot{z} = G_3(t, \alpha_i, \beta_i)$,—or in terms of t and the six functions p, q of the elements —and when the differentiations have been performed, then $[p, q]$ is independent of the time.

We start with the three equations of the type given by 5·01 (12), namely,

$$\frac{\partial^2 x}{\partial t^2} + \frac{\mu x}{r^3} = 0 \quad .$$

Let $V = \mu/r$. Then the three equations are

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial V}{\partial x} \quad , \quad \frac{\partial^2 y}{\partial t^2} = \frac{\partial V}{\partial y} \quad , \quad \frac{\partial^2 z}{\partial t^2} = \frac{\partial V}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, $\dfrac{\partial}{\partial t}\left\{\dfrac{\partial(x, \dot{x})}{\partial(p, q)}\right\} = \dfrac{\partial x}{\partial p} \cdot \dfrac{\partial \ddot{x}}{\partial q} - \dfrac{\partial x}{\partial q} \cdot \dfrac{\partial \ddot{x}}{\partial p}$, where $\ddot{x} \equiv \dfrac{\partial^2 x}{\partial t^2}$,

$$= \frac{\partial x}{\partial p} \cdot \frac{\partial}{\partial q}\left(\frac{\partial V}{\partial x}\right) - \frac{\partial x}{\partial q} \cdot \frac{\partial}{\partial p}\left(\frac{\partial V}{\partial x}\right) \quad , \quad \text{from (1)},$$

$$= \frac{\partial}{\partial q}\left(\frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial p}\right) - \frac{\partial}{\partial p}\left(\frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial q}\right) \quad .$$

Hence $\quad \dfrac{\partial}{\partial t}[p, q] = \dfrac{\partial}{\partial q} \sum_{xyz} \dfrac{\partial V}{\partial x} \cdot \dfrac{\partial x}{\partial p} - \dfrac{\partial}{\partial p} \sum_{xyz} \dfrac{\partial V}{\partial x} \cdot \dfrac{\partial x}{\partial q} \quad .$

But $\quad \sum_{xyz} \dfrac{\partial V}{\partial x} \cdot \dfrac{\partial x}{\partial p} = \dfrac{\partial V}{\partial p} \quad , \quad \sum_{xyz} \dfrac{\partial V}{\partial x} \cdot \dfrac{\partial x}{\partial q} = \dfrac{\partial V}{\partial q} \quad .$

Hence $\quad \dfrac{\partial}{\partial t}[p, q] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

which establishes the property.

The significance of this property may be exemplified briefly at this

point—we use the property in detail in later pages. In evaluating a bracket such as $[p, q]$ we have to evaluate $\frac{\partial x}{\partial p}, \frac{\partial \dot{x}}{\partial p}; \frac{\partial y}{\partial p}, \frac{\partial \dot{y}}{\partial p}; \frac{\partial z}{\partial p}, \frac{\partial \dot{z}}{\partial p}$ and the corresponding expressions in q. After performing the differentiations we can put t equal to any value we like, in accordance with the property (iii); in practice it is found most convenient to put $t=\tau$, the time of perihelion passage, for then the eccentric anomaly E, which appears in the expressions for $x \ldots \dot{x} \ldots$ given by 5·01 (3) and (8) is equal to zero.

5·06. Various formulae required for the evaluation of the Lagrangian brackets

(i) *Bracket in terms of* ξ, η

Let (x, y, z) be the coordinates of the planet referred to the axes indicated by X, Y and Z (Fig. 13); O is the Sun. The plane of the osculating ellipse and the direction of perihelion are specified, with reference to these axes, by means of the usual elements Ω, ω and i.

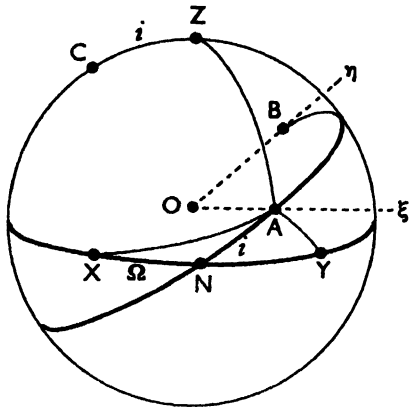


FIG. 13

We take rectangular axes OA, OB in the plane of the ellipse, A being perihelion; C is the pole of the great circle NAB. Let (ξ, η) be the coordinates of the planet with respect to OA, OB.

As in section 2·15, let (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) be the direction-cosines of OA, OB and OC respectively with respect to the axes OX, OY and OZ. Then

$$x=l_1\xi+l_2\eta,\ y=m_1\xi+m_2\eta,\ z=n_1\xi+n_2\eta \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

from which, since $\dot{x} \equiv \frac{\partial x}{\partial t}$, etc.,

$$\dot{x}=l_1\dot{\xi}+l_2\dot{\eta},\ \dot{y}=m_1\dot{\xi}+m_2\dot{\eta},\ \dot{z}=n_1\dot{\xi}+n_2\dot{\eta} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

By means of (1) and (2) a Lagrangian bracket can be expressed in terms of ξ, η, $\dot{\xi}$, $\dot{\eta}$ and $l_1, l_2 \ldots n_2$. We postpone the formal expression of a bracket in this way till later.

(ii) *Formulae for the direction-cosines in terms of* Ω, ω, i

We have

$$\begin{aligned} l_1&=\cos AX\ , & m_1&=\cos AY\ , & n_1&=\cos AZ\ , \\ l_2&=\cos BX\ , & m_2&=\cos BY\ , & n_2&=\cos BZ\ , \\ l_3&=\cos CX\ , & m_3&=\cos CY\ , & n_3&=\cos CZ\ . \end{aligned}$$

We have already found the expressions for $l_1, m_1 \ldots n_2$ in 2·15 (3) and (4); we rewrite these below as formulae (3) and (4). The formulae for l_3,

m_3 and n_3 are found easily from the triangles CXN, CYN, CZN and are given in (5) below. The complete formulae are :

$$\left.\begin{aligned} l_1 &= \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i \\ m_1 &= \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i \\ n_1 &= \sin\omega\sin i \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\left.\begin{aligned} l_2 &= -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i \\ m_2 &= -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i \\ n_2 &= \cos\omega\sin i \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\left.\begin{aligned} l_3 &= \sin\Omega\sin i \\ m_3 &= -\cos\Omega\sin i \\ n_3 &= \cos i \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

(iii) *Differential coefficients of the direction-cosines*

We require the differential coefficients of $l_1, l_2, \ldots, n_2$ with respect to Ω, ω, i.

For example, $\dfrac{\partial l_1}{\partial \Omega} = -\sin\Omega\cos\omega - \cos\Omega\sin\omega\cos i = -m_1$. Proceeding in this way we derive the remaining coefficients ; we collect the results in tabulated form.

Table : Differential Coefficients of the direction-cosines with respect to Ω, ω, i

	l_1	m_1	n_1	l_2	m_2	n_2
Ω	$-m_1$	l_1	0	$-m_2$	l_2	0
ω	l_2	m_2	n_2	$-l_1$	$-m_1$	$-n_1$
i	$l_3\sin\omega$	$m_3\sin\omega$	$n_3\sin\omega$	$l_3\cos\omega$	$m_3\cos\omega$	$n_3\cos\omega$

(iv) *Formulae for $\partial l_2/\partial p$ etc.*

In subsequent work we shall require the differential coefficients of l_2, m_2, n_2 with respect to p, where p is, in general, any function of the six elements but may, in particular cases, be simply an element.

Since l_2 is a function of Ω, ω, i only,

$$\frac{\partial l_2}{\partial p} = \frac{\partial l_2}{\partial \Omega}\cdot\frac{\partial \Omega}{\partial p} + \frac{\partial l_2}{\partial \omega}\cdot\frac{\partial \omega}{\partial p} + \frac{\partial l_2}{\partial i}\cdot\frac{\partial i}{\partial p}$$

which becomes, by means of the appropriate data from the table in (iii),

$$\frac{\partial l_2}{\partial p} = -m_2\frac{\partial \Omega}{\partial p} - l_1\frac{\partial \omega}{\partial p} + l_3\cos\omega\frac{\partial i}{\partial p} \quad \ldots\ldots\ldots\ldots(6)$$

Similarly,

$$\frac{\partial m_2}{\partial p} = l_2\frac{\partial \Omega}{\partial p} - m_1\frac{\partial \omega}{\partial p} + m_3\cos\omega\frac{\partial i}{\partial p} \quad , \ldots\ldots\ldots\ldots(7)$$

$$\frac{\partial n_2}{\partial p} = -n_1\frac{\partial \omega}{\partial p} + n_3\cos\omega\frac{\partial i}{\partial p} \quad \ldots\ldots\ldots\ldots(8)$$

(v) *Formula for* $\underset{l,m,n}{\Sigma} l_1 \dfrac{\partial l_2}{\partial p}$.

From (6), (7) and (8), we have

$$l_1 \frac{\partial l_2}{\partial p} + m_1 \frac{\partial m_2}{\partial p} + n_1 \frac{\partial n_2}{\partial p} = -\cos i \frac{\partial \Omega}{\partial p} - \frac{\partial \omega}{\partial p} \quad , \quad \ldots\ldots\ldots\ldots(9)$$

since $l_1 m_2 - l_2 m_1 \equiv n_3 = \cos i$,

and $l_1^2 + m_1^2 + n_1^2 = 1,\ l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$.

(vi) *Formula for* $\underset{lmn}{\Sigma} \dfrac{\partial (l_1, l_2)}{\partial (p, q)}$.

Differentiate (9) with respect to q, where q is in general, a function of the elements ; then

$$\Sigma \left(\frac{\partial l_1}{\partial q} \cdot \frac{\partial l_2}{\partial p} + l_1 \frac{\partial^2 l_2}{\partial p \, \partial q} \right) = -\cos i \frac{\partial^2 \Omega}{\partial p \, \partial q} + \sin i \frac{\partial \Omega}{\partial p} \cdot \frac{\partial i}{\partial q} - \frac{\partial^2 \omega}{\partial p \, \partial q} \quad \ldots.(10)$$

Writing (9) in terms of q, we have

$$l_1 \frac{\partial l_2}{\partial q} + m_1 \frac{\partial m_2}{\partial q} + n_1 \frac{\partial n_2}{\partial q} = -\cos i \frac{\partial \Omega}{\partial q} - \frac{\partial \omega}{\partial q} \quad ;$$

differentiate this last equation with respect to p ; then

$$\Sigma \left(\frac{\partial l_1}{\partial p} \cdot \frac{\partial l_2}{\partial q} + l_1 \frac{\partial^2 l_2}{\partial p \, \partial q} \right) = -\cos i \frac{\partial^2 \Omega}{\partial p \, \partial q} + \sin i \frac{\partial \Omega}{\partial q} \cdot \frac{\partial i}{\partial p} - \frac{\partial^2 \omega}{\partial p \, \partial q} \quad \ldots(11)$$

Subtracting (10) from (11), we obtain

$$\underset{lmn}{\Sigma} \frac{\partial (l_1, l_2)}{\partial (p, q)} = -\sin i \frac{\partial (\Omega, i)}{\partial (p, q)} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

5·07. Evaluation of $[\beta_r, \beta_s]$.

Let β_r, β_s denote any two of the elements Ω, ω, i ; since ξ, ξ and η, $\dot{\eta}$ do not involve these elements, then—Σ denoting summation for l, m and n—

$$\begin{aligned}
[\beta_r, \beta_s] &= \Sigma \left\{ \left(\xi \frac{\partial l_1}{\partial \beta_r} + \eta \frac{\partial l_2}{\partial \beta_r} \right) \left(\dot{\xi} \frac{\partial l_1}{\partial \beta_s} + \dot{\eta} \frac{\partial l_2}{\partial \beta_s} \right) \right\} \\
&\quad - \Sigma \left\{ \left(\xi \frac{\partial l_1}{\partial \beta_s} + \eta \frac{\partial l_2}{\partial \beta_s} \right) \left(\dot{\xi} \frac{\partial l_1}{\partial \beta_r} + \dot{\eta} \frac{\partial l_2}{\partial \beta_r} \right) \right\} \\
&= (\xi \dot{\eta} - \dot{\xi} \eta) \Sigma \frac{\partial (l_1, l_2)}{\partial (\beta_r, \beta_s)} \quad .
\end{aligned}$$

But $\xi \dot{\eta} - \dot{\xi} \eta = h$, the areal constant defined in 2·04 (5) or (6) ; hence, using 5·06 (12), we obtain

$$[\beta_r, \beta_s] = -h \sin i \frac{\partial (\Omega, i)}{\partial (\beta_r, \beta_s)} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If $\beta_r \equiv \omega$ and $\beta_s \equiv \Omega$ or i, then from (1)

$$[\omega, \Omega] = [\omega, i] = 0 \quad .$$

If $\beta_r \equiv \Omega$ and $\beta_s \equiv i$, then $\dfrac{\partial(\Omega, i)}{\partial(\Omega, i)} = 1$; hence, by (1)

$$[\Omega, i] = -h \sin i \quad .$$

Summary. The values of all brackets of the type $[\beta_r, \beta_s]$ are as follows :

$$[\Omega, i] = -h \sin i \; ; \; [\omega, \Omega] = 0 \; ; \; [\omega, i] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

5·08. Evaluation of $[\alpha, \beta]$.

Here α is any one of the elements a, e, χ and β is any one of the elements Ω, ω, i.

ξ, $\dot{\xi}$ and η, $\dot{\eta}$ are functions of α and l_1, l_2, ... n_2 are functions of β. We then have

$$[\alpha, \beta] = \Sigma \left\{ \left(l_1 \frac{\partial \xi}{\partial \alpha} + l_2 \frac{\partial \eta}{\partial \alpha} \right) \left(\dot{\xi} \frac{\partial l_1}{\partial \beta} + \dot{\eta} \frac{\partial l_2}{\partial \beta} \right) \right\}$$
$$- \Sigma \left\{ \left(l_1 \frac{\partial \dot{\xi}}{\partial \alpha} + l_2 \frac{\partial \dot{\eta}}{\partial \alpha} \right) \left(\xi \frac{\partial l_1}{\partial \beta} + \eta \frac{\partial l_2}{\partial \beta} \right) \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, $l_1^2 + m_1^2 + n_1^2 = 1$; hence

$$\Sigma l_1 \frac{\partial l_1}{\partial \beta} = 0 \quad .$$

Similarly,

$$\Sigma l_2 \frac{\partial l_2}{\partial \beta} = 0 \quad .$$

Also, since $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$, then

$$\Sigma l_1 \frac{\partial l_2}{\partial \beta} = -\Sigma l_2 \frac{\partial l_1}{\partial \beta} \quad .$$

With these results, (1) becomes

$$[\alpha, \beta] = \left(\dot{\eta} \frac{\partial \xi}{\partial \alpha} - \eta \frac{\partial \dot{\xi}}{\partial \alpha} - \dot{\xi} \frac{\partial \eta}{\partial \alpha} + \xi \frac{\partial \dot{\eta}}{\partial \alpha} \right) \Sigma l_1 \frac{\partial l_2}{\partial \beta}$$
$$= \frac{\partial}{\partial \alpha} (\xi \dot{\eta} - \dot{\xi} \eta) \, \Sigma l_1 \frac{\partial l_2}{\partial \beta} \quad .$$

But, from 5·06 (9), $\Sigma l_1 \dfrac{\partial l_2}{\partial \beta} = -\cos i \dfrac{\partial \Omega}{\partial \beta} - \dfrac{\partial \omega}{\partial \beta}$

and $$h = \xi \dot{\eta} - \dot{\xi} \eta = \{ \mu a (1 - e^2) \}^{1/2} \; ;$$

hence $$[\alpha, \beta] = -\left(\cos i \frac{\partial \Omega}{\partial \beta} + \frac{\partial \omega}{\partial \beta} \right) \frac{\partial h}{\partial \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If $\alpha \equiv \chi$, then $\dfrac{\partial h}{\partial \chi} = 0$; hence

$$[\chi, \Omega] = [\chi, \omega] = [\chi, i] = 0 \quad .$$

If $\beta \equiv i$, then $[\alpha, i] = 0$; hence

$$[a, i] = [e, i] = [\chi, i] = 0 \quad .$$

If $\beta \equiv \Omega$, then $[\alpha, \Omega] = -\cos i \dfrac{\partial h}{\partial \alpha}$; hence, since $\mu = n^2 a^3$,

$$[a, \Omega] = -\tfrac{1}{2} na\sqrt{(1-e^2)}\cos i \quad , \quad [e, \Omega] = \frac{na^2 e}{\sqrt{(1-e^2)}}\cos i \quad .$$

If $\beta \equiv \omega$, then $[\alpha, \omega] = -\dfrac{\partial h}{\partial \alpha}$; hence

$$[a, \omega] = -\tfrac{1}{2} na\sqrt{(1-e^2)} \quad , \quad [e, \omega] = \frac{na^2 e}{\sqrt{(1-e^2)}} \quad .$$

This completes the evaluation of all the brackets of the form $[\alpha, \beta]$.

Summary

$$\left.\begin{aligned} &[a, i] = [e, i] = [\chi, \Omega] = [\chi, \omega] = [\chi, i] = 0 \\ &[a, \Omega] = -\tfrac{1}{2} na\sqrt{(1-e^2)}\cos i \; ; \; [a, \omega] = -\tfrac{1}{2} na\sqrt{(1-e^2)} \\ &[e, \Omega] = \frac{na^2 e}{\sqrt{(1-e^2)}}\cos i \; ; \; [e, \omega] = \frac{na^2 e}{\sqrt{(1-e^2)}} \end{aligned}\right\} \quad \ldots\ldots(3)$$

5·09. Evaluation of $[\alpha_r, \alpha_s]$

Since the direction-cosines do not involve the α's, then

$$\begin{aligned} [\alpha_r, \alpha_s] &= \Sigma\left\{\left(l_1 \frac{\partial \xi}{\partial \alpha_r} + l_2 \frac{\partial \eta}{\partial \alpha_r}\right)\left(l_1 \frac{\partial \dot{\xi}}{\partial \alpha_s} + l_2 \frac{\partial \dot{\eta}}{\partial \alpha_s}\right)\right\} \\ &\quad - \Sigma\left\{\left(l_1 \frac{\partial \xi}{\partial \alpha_s} + l_2 \frac{\partial \eta}{\partial \alpha_s}\right)\left(l_1 \frac{\partial \dot{\xi}}{\partial \alpha_r} + l_2 \frac{\partial \dot{\eta}}{\partial \alpha_r}\right)\right\} \\ &= \frac{\partial(\xi, \dot{\xi})}{\partial(\alpha_r, \alpha_s)} + \frac{\partial(\eta, \dot{\eta})}{\partial(\alpha_r, \alpha_s)} \quad . \quad \ldots\ldots(1) \end{aligned}$$

We now use the property that $\dfrac{\partial}{\partial t}[\alpha_r, \alpha_s] = 0$. The bracket is constant for all values of t and, in particular, for a value t such that $n(t-\tau)$ is small; E will also be small. Then we can write Kepler's equation as

$$E - e(E - \tfrac{1}{6}E^3 + \ldots) = n(t-\tau)$$

whence

$$E = \frac{n(t-\tau)}{1-e} + A(t-\tau)^3$$

up to the third order in $n(t-\tau)$, A not requiring further specification.

We then have

$$\cos E = 1 - \frac{n^2}{2(1-e)^2}(t-\tau)^2 \quad , \quad \sin E = \frac{n(t-\tau)}{1-e} + B(t-\tau)^3 \quad ,$$

and hence,

$$\xi = a(1-e) - \frac{an^2}{2(1-e)^2}(t-\tau)^2 + \ldots \quad , \quad \ldots\ldots(2)$$

$$\eta = \frac{na\sqrt{(1-e^2)}}{1-e}(t-\tau) + C(t-\tau)^3 \quad , \quad \ldots\ldots(3)$$

and

$$\dot{\xi} = -\frac{an^2}{(1-e)^2}(t-\tau) + D(t-\tau)^3 \quad , \quad \ldots\ldots(4)$$

$$\dot{\eta} = \frac{na\sqrt{(1-e^2)}}{1-e} + E(t-\tau)^2 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which we need not evaluate B, C, D, and E.

We shall at first take the α's to be a, e and τ. We denote by $\left(\frac{\partial \xi}{\partial a}\right)$ the value of $\frac{\partial \xi}{\partial a}$ when we put $t-\tau=0$ after differentiation with respect to a. From (2) and (4) we have

$$\left(\frac{\partial \xi}{\partial a}\right) = 1-e \quad , \quad \left(\frac{\partial \xi}{\partial e}\right) = -a \quad , \quad \left(\frac{\partial \xi}{\partial \tau}\right) = 0 \quad ,$$

and

$$\left(\frac{\partial \dot{\xi}}{\partial a}\right) = 0 \quad , \quad \left(\frac{\partial \dot{\xi}}{\partial e}\right) = 0 \quad , \quad \left(\frac{\partial \dot{\xi}}{\partial \tau}\right) = \frac{an^2}{(1-e)^2} \quad .$$

Hence, the Jacobians being evaluated for $t=\tau$,

$$\frac{\partial(\xi, \dot{\xi})}{\partial(a, e)} = 0 \quad , \quad \frac{\partial(\xi, \dot{\xi})}{\partial(e, \tau)} = -\frac{a^2n^2}{(1-e)^2} \quad , \quad \frac{\partial(\xi, \dot{\xi})}{\partial(\tau, a)} = -\frac{an^2}{1-e} \quad . \ldots\ldots (6)$$

Again, from (3) and (5),

$$\left(\frac{\partial \eta}{\partial a}\right) = 0 \quad , \quad \left(\frac{\partial \eta}{\partial e}\right) = 0 \quad , \quad \left(\frac{\partial \eta}{\partial \tau}\right) = -\frac{na\sqrt{(1-e^2)}}{1-e} \quad ,$$

and

$$\left(\frac{\partial \dot{\eta}}{\partial a}\right) = \frac{\sqrt{(1-e^2)}}{1-e}\frac{\partial(na)}{\partial a} \quad , \quad \left(\frac{\partial \dot{\eta}}{\partial e}\right) = \frac{na}{(1-e)\sqrt{(1-e^2)}} \quad , \quad \left(\frac{\partial \dot{\eta}}{\partial \tau}\right) = 0 \quad .$$

Hence, the Jacobians being evaluated for $t=\tau$,

$$\frac{\partial(\eta, \dot{\eta})}{\partial(a, e)} = 0 \quad , \quad \frac{\partial(\eta, \dot{\eta})}{\partial(e, \tau)} = \frac{a^2n^2}{(1-e)^2} \quad , \quad \frac{\partial(\eta, \dot{\eta})}{\partial(\tau, a)} = \frac{an^2(1+e)}{2(1-e)} \quad . \ldots (7)$$

From (1), (6) and (7),

$$[a, e] = 0 \quad , \quad [e, \tau] = 0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

and

$$[\tau, a] = -\frac{1}{2}\frac{\mu}{a^2} = -\frac{1}{2}an^2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

It is easily seen that, when ξ, η, $\dot{\xi}$, $\dot{\eta}$ in (1) are expressed in terms of a, e, χ instead of a, e, τ where $\chi = -n\tau$, then

$$[a, e] = 0 \quad , \quad [e, \chi] = 0 \quad , \quad [a, \chi] = -\tfrac{1}{2}na \quad . \quad \ldots\ldots\ldots\ldots (10)$$

5·10. The planetary equations

(i) We have now evaluated all the Lagrangian brackets. Associating α_1, α_2, α_3 with a, e, χ respectively, and β_1, β_2, β_3 with Ω, ω, i respectively, we find that, when the values of the brackets derived in sections 5·07–5·09 are inserted in the equations 5·04 (7) and (8), then these equations become

$$-\tfrac{1}{2}na\dot{\chi} - \tfrac{1}{2}na\sqrt{(1-e^2)}\{\cos i \,.\, \dot{\Omega} + \dot{\omega}\} = \frac{\partial R}{\partial a} \quad ,$$

$$\frac{na^2e}{\sqrt{(1-e^2)}}\{\cos i \,.\, \dot{\Omega} + \dot{\omega}\} = \frac{\partial R}{\partial e} \quad ,$$

$$\tfrac{1}{2}na\dot{a} = \frac{\partial R}{\partial \chi} ,$$

$$\tfrac{1}{2}na\sqrt{(1-e^2)}\cos i\left\{\dot{a} - \frac{2ae}{1-e^2}\cdot\dot{e} - 2a\tan i\,.\,\frac{di}{dt}\right\} = \frac{\partial R}{\partial \Omega} ,$$

$$\tfrac{1}{2}na\sqrt{(1-e^2)}\left\{\dot{a} - \frac{2ae}{1-e^2}\cdot\dot{e}\right\} = \frac{\partial R}{\partial \omega} ,$$

$$na^2\sqrt{(1-e^2)}\sin i\,.\,\dot{\Omega} = \frac{\partial R}{\partial i} .$$

From these we obtain

$$\dot{a} = \frac{2}{na}\frac{\partial R}{\partial \chi} , \qquad \text{.....(1)}$$

$$\dot{e} = \frac{1}{na^2e}\left\{(1-e^2)\frac{\partial R}{\partial \chi} - \sqrt{(1-e^2)}\frac{\partial R}{\partial \omega}\right\} , \qquad \text{.....(2)}$$

$$\dot{\chi} = -\frac{1-e^2}{na^2e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} , \qquad \text{.....(3)}$$

$$\dot{\Omega} = \frac{1}{na^2\sqrt{(1-e^2)}\sin i}\frac{\partial R}{\partial i} , \qquad \text{.....(4)}$$

$$\dot{\omega} = \frac{\sqrt{(1-e^2)}}{na^2e}\frac{\partial R}{\partial e} - \frac{\cot i}{na^2\sqrt{(1-e^2)}}\frac{\partial R}{\partial i} , \qquad \text{.....(5)}$$

$$\frac{di}{dt} = \frac{1}{na^2\sqrt{(1-e^2)}}\left\{\cot i\frac{\partial R}{\partial \omega} - \operatorname{cosec} i\frac{\partial R}{\partial \Omega}\right\} \qquad \text{.....(6)}$$

These are the planetary equations in terms of a, e, χ, Ω, ω, i.

(ii) In planetary theory it is usual to replace the elements ω and χ by ϖ and ϵ. If R' denotes the disturbing function when expressed in terms of a, e, i, Ω, ϖ, ϵ we have the identity

$$R(a, e, i, \Omega, \omega, \chi) \equiv R'(a, e, i, \Omega, \varpi, \epsilon)$$

where, on the right-hand side, $\varpi \equiv \omega + \Omega$ and $\epsilon \equiv \omega + \Omega + \chi$.

Then

$$\frac{\partial R}{\partial \Omega} = \frac{\partial R'}{\partial \Omega} + \frac{\partial R'}{\partial \varpi}\cdot\frac{\partial \varpi}{\partial \Omega} + \frac{\partial R'}{\partial \epsilon}\cdot\frac{\partial \epsilon}{\partial \Omega} = \frac{\partial R'}{\partial \Omega} + \frac{\partial R'}{\partial \varpi} + \frac{\partial R'}{\partial \epsilon} ,$$

$$\frac{\partial R}{\partial \omega} = \frac{\partial R'}{\partial \varpi}\cdot\frac{\partial \varpi}{\partial \omega} + \frac{\partial R'}{\partial \epsilon}\cdot\frac{\partial \epsilon}{\partial \omega} = \frac{\partial R'}{\partial \varpi} + \frac{\partial R'}{\partial \epsilon} ,$$

$$\frac{\partial R}{\partial \chi} = \frac{\partial R'}{\partial \epsilon}\cdot\frac{\partial \epsilon}{\partial \chi} = \frac{\partial R'}{\partial \epsilon} .$$

When these expressions for $\dfrac{\partial R}{\partial \Omega}$, $\dfrac{\partial R}{\partial \omega}$, $\dfrac{\partial R}{\partial \chi}$ are substituted on the right-hand sides of (1) ... (6), and when $\dot{\varpi} - \dot{\Omega}$ and $\dot{\epsilon} - \dot{\varpi}$ replace $\dot{\omega}$ and $\dot{\chi}$ respectively, we obtain the planetary equations as follows (we drop the accent in R'):

$$\dot{a} = \frac{2}{na}\frac{\partial R}{\partial \epsilon} , \qquad \text{.....(7)}$$

$$\dot{\epsilon} = -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{\tan\frac{1}{2}\phi\cos\phi}{na^2}\frac{\partial R}{\partial e} + \frac{\tan\frac{1}{2}i}{na^2\cos\phi}\frac{\partial R}{\partial i} \quad , \qquad \ldots\ldots(8)$$

$$\dot{e} = -\frac{\cot\phi}{na^2}\frac{\partial R}{\partial \varpi} - \frac{\tan\frac{1}{2}\phi\cos\phi}{na^2}\frac{\partial R}{\partial \epsilon} \quad , \qquad \ldots\ldots(9)$$

$$\dot{\Omega} = \frac{1}{na^2\cos\phi\sin i}\frac{\partial R}{\partial i} \quad , \qquad \ldots\ldots(10)$$

$$\dot{\varpi} = \frac{\cot\phi}{na^2}\frac{\partial R}{\partial e} + \frac{\tan\frac{1}{2}i}{na^2\cos\phi}\frac{\partial R}{\partial i} \quad , \qquad \ldots\ldots(11)$$

$$\frac{di}{dt} = -\frac{1}{na^2\cos\phi\sin i}\frac{\partial R}{\partial \Omega} - \frac{\tan\frac{1}{2}i}{na^2\cos\phi}\left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \epsilon}\right) \quad , \qquad \ldots\ldots(12)$$

in which, on the right-hand sides, we have replaced e by $\sin\phi$.

(iii) There are useful modifications of the equations involving e, ϖ; i, Ω.

Let $\quad h = e\sin\varpi \quad , \quad k = e\cos\varpi \quad ;$

$$p = \sin i\sin\Omega \quad , \quad q = \sin i\cos\Omega \quad .$$

Here, h and k are of order e, and p and q are of order i.

Let R_1 denote the disturbing function expressed in terms of h, k and p, q.

Then $\quad \frac{\partial R}{\partial e} = \sin\varpi\frac{\partial R_1}{\partial h} + \cos\varpi\frac{\partial R_1}{\partial k} \quad , \quad \frac{\partial R}{\partial \varpi} = k\frac{\partial R_1}{\partial h} - h\frac{\partial R_1}{\partial k} \quad ;$

and $\quad \frac{\partial R}{\partial i} = \frac{1}{\gamma}\left(p\frac{\partial R_1}{\partial p} + q\frac{\partial R_1}{\partial q}\right) \quad , \quad \frac{\partial R}{\partial \Omega} = q\frac{\partial R_1}{\partial p} - p\frac{\partial R_1}{\partial q} \quad ,$

where $\gamma = \tan i$.

The equations (9) and (11) are replaced by

$$\dot{h} = \frac{\cos\phi}{na^2}\frac{\partial R_1}{\partial k} + \frac{k\tan\frac{1}{2}i}{\gamma na^2\cos\phi}\left(p\frac{\partial R_1}{\partial p} + q\frac{\partial R_1}{\partial q}\right) - \frac{h\cos\phi}{2na^2\cos^2\frac{1}{2}\phi}\frac{\partial R_1}{\partial \epsilon} \quad , \ldots\ldots(13)$$

$$\dot{k} = -\frac{\cos\phi}{na^2}\frac{\partial R_1}{\partial h} - \frac{h\tan\frac{1}{2}i}{\gamma na^2\cos\phi}\left(p\frac{\partial R_1}{\partial p} + q\frac{\partial R_1}{\partial q}\right) - \frac{k\cos\phi}{2na^2\cos^2\frac{1}{2}\phi}\frac{\partial R_1}{\partial \epsilon} \quad , \ldots(14)$$

in which $\cos\phi$, $\cos\frac{1}{2}\phi$, γ and $\tan\frac{1}{2}i$ are to be expressed in terms of h, k and p, q.

In the same way (10) and (12) are replaced by

$$\dot{p} = \frac{\cos i}{na^2\cos\phi}\frac{\partial R_1}{\partial q} - \frac{p\cos i}{2na^2\cos\phi\cos^2\frac{1}{2}i}\left(k\frac{\partial R_1}{\partial h} - h\frac{\partial R_1}{\partial k} + \frac{\partial R_1}{\partial \epsilon}\right) \quad , \ldots\ldots(15)$$

$$\dot{q} = -\frac{\cos i}{na^2\cos\phi}\frac{\partial R_1}{\partial p} - \frac{q\cos i}{2na^2\cos\phi\cos^2\frac{1}{2}i}\left(k\frac{\partial R_1}{\partial h} - h\frac{\partial R_1}{\partial k} + \frac{\partial R_1}{\partial \epsilon}\right) \quad \ldots(16)$$

In the case of planetary perturbations on the Earth's orbit, i is small so that neglecting i^3 we can write

$$p = i\sin\Omega \quad , \quad q = i\cos\Omega \quad .$$

When the secular terms are alone considered it is found that p and q are of the forms given by

$$i \sin \Omega = gt + ht^2 \quad \ldots\ldots(17)$$

$$i \cos \Omega = g_1 t + h_1 t^2 \quad , \quad \ldots\ldots(18)$$

where g, h (not to be confused with $e \sin \varpi$), g_1 and h_1 are small quantities. We shall use (17) and (18) when we deal with precession in Chapter 20.

5·11. Campbell's method of evaluating the Lagrangian brackets*

Write $V = \frac{\mu}{r}$; the equations of motion in the undisturbed ellipse are

$$\ddot{x} = \frac{\partial V}{\partial x} \quad , \quad \ddot{y} = \frac{\partial V}{\partial y} \quad , \quad \ddot{z} = \frac{\partial V}{\partial z} \quad , \quad \ldots\ldots(1)$$

where $\ddot{x} \equiv \frac{\partial^2 x}{\partial t^2}$, etc. If v is the orbital velocity in the ellipse we have, by 2·12 (2),

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) \quad .$$

Write $$T = \tfrac{1}{2} v^2 \quad , \quad V_0 = -\frac{\mu}{2a} \quad ; \quad \ldots\ldots(2)$$

then $$T = V + V_0 \quad .$$

Let p, q be any two of the elements or any two independent functions of them. We have

$$\frac{\partial T}{\partial p} = \frac{\partial V}{\partial p} + \frac{\partial V_0}{\partial p}$$

or, since V is a function of x, y, z,

$$\frac{\partial T}{\partial p} = \sum_{xyz} \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial p} + \frac{\partial V_0}{\partial p} = \Sigma \ddot{x} \frac{\partial x}{\partial p} + \frac{\partial V_0}{\partial p} \quad .$$

Also, by (2), $$\frac{\partial T}{\partial p} = \Sigma \dot{x} \frac{\partial \dot{x}}{\partial p} \quad .$$

Hence, by addition of the last two equations,

$$2 \frac{\partial T}{\partial p} = \frac{\partial V_0}{\partial p} + \Sigma \left(\ddot{x} \frac{\partial x}{\partial p} + \dot{x} \frac{\partial \dot{x}}{\partial p} \right)$$

$$= \frac{\partial V_0}{\partial p} + \frac{\partial}{\partial t} \Sigma \dot{x} \frac{\partial x}{\partial p} \quad ,$$

the summation referring to the sum of symmetrical expressions in x, y and z.

By integrating between the limits τ and t we have

$$2 \int_\tau^t \frac{\partial T}{\partial p} dt = (t - \tau) \frac{\partial V_0}{\partial p} + \Sigma \left(\dot{x} \frac{\partial x}{\partial p} \right)_t - \Sigma \left(\dot{x} \frac{\partial x}{\partial p} \right)_\tau \quad \ldots\ldots(3)$$

Now $$\frac{\partial}{\partial p} \int_\tau^t T \, dt = \int_\tau^t \frac{\partial T}{\partial p} dt - T(\tau) \frac{\partial \tau}{\partial p} \quad ;$$

* A. Y. G. Campbell; M.N.R.A.S., 57, 118 (1897).

hence (3) becomes

$$2\frac{\partial}{\partial p}\int_\tau^t T\,dt - t\frac{\partial V_0}{\partial p} - \Sigma\dot{x}\frac{\partial x}{\partial p} = -2T(\tau)\frac{\partial \tau}{\partial p} - \tau\frac{\partial V_0}{\partial p} - \Sigma\left(\dot{x}\frac{\partial x}{\partial p}\right)_\tau \quad ,\ldots(4)$$

where, in the summation on the left, we have dropped the suffix t.

Now, $2T = \frac{2\mu}{r} - \frac{\mu}{a}$, and, since $r = a(1 - e\cos E)$ and $E = 0$ when $t = \tau$, we have

$$2T(\tau) = \frac{\mu(1+e)}{a(1-e)} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Also, since $V_0 = -\frac{\mu}{2a}$, then $\frac{\partial V_0}{\partial p} = \frac{\mu}{2a^2}\frac{\partial a}{\partial p}$. $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$

From (4), (5) and (6) we have

$$2\frac{\partial}{\partial p}\int_\tau^t T\,dt - t\frac{\partial V_0}{\partial p} - \Sigma\dot{x}\frac{\partial x}{\partial p} = C_p \quad , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where $$C_p = -\frac{\mu(1+e)}{a(1-e)}\cdot\frac{\partial \tau}{\partial p} - \frac{\mu\tau}{2a^2}\cdot\frac{\partial a}{\partial p} - \Sigma\left(\dot{x}\frac{\partial x}{\partial p}\right)_\tau \quad . \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

It is to be noted that C_p is a function of the elements and does not involve the time.

Similarly, q being an element or a function of the elements, we have

$$2\frac{\partial}{\partial q}\int_\tau^t T\,dt - t\frac{\partial V_0}{\partial q} - \Sigma\dot{x}\frac{\partial x}{\partial q} = C_q \quad , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

where C_q is given by an expression analogous to (8).

Differentiate (7) and (9) with respect to q and p respectively, and subtract; then

$$\Sigma\frac{\partial(x, \dot{x})}{\partial(p, q)} = \frac{\partial C_q}{\partial p} - \frac{\partial C_p}{\partial q} \quad ,$$

that is,

$$[p, q] = \frac{\partial C_q}{\partial p} - \frac{\partial C_p}{\partial q} \quad . \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Since C_p and C_q do not involve the time, we have in (10) an independent proof of the characteristic property of a bracket, namely,

$$\frac{\partial}{\partial t}[p,q] = 0 \quad .$$

5·12. Computation of C_p

By (8) of the previous section,

$$C_p = -\frac{\mu(1+e)}{a(1-e)}\cdot\frac{\partial \tau}{\partial p} - \frac{\mu\tau}{2a^2}\cdot\frac{\partial a}{\partial p} - \Sigma\left(\dot{x}\frac{\partial x}{\partial p}\right)_\tau \quad . \ldots\ldots\ldots\ldots\ldots(1)$$

We first evaluate the last term in (1). Express x, y, z in terms of ξ, η by means of

$$x = l_1\xi + l_2\eta \quad , \quad y = m_1\xi + m_2\eta \quad , \quad z = n_1\xi + n_2\eta \quad ,$$

where l_1, l_2, ... n_2 are the direction cosines defined in 5·06 (3) and (4). Then

$$\sum_{xyz} \dot{x} \frac{\partial x}{\partial p} = \sum_{lmn} \left(l_1\dot{\xi} + l_2\dot{\eta}\right)\left(l_1 \frac{\partial \xi}{\partial p} + l_2 \frac{\partial \eta}{\partial p} + \xi \frac{\partial l_1}{\partial p} + \eta \frac{\partial l_2}{\partial p}\right)$$

$$= \dot{\xi} \frac{\partial \xi}{\partial p} + \dot{\eta} \frac{\partial \eta}{\partial p} - (\xi\dot{\eta} - \dot{\xi}\eta) \sum l_1 \frac{\partial l_2}{\partial p} \quad .$$

But $\xi\dot{\eta} - \dot{\xi}\eta = h$; hence, using 5·06 (9), we obtain

$$\sum \left(\dot{x} \frac{\partial x}{\partial p}\right)_\tau = \left(\dot{\xi} \frac{\partial \xi}{\partial p} + \dot{\eta} \frac{\partial \eta}{\partial p}\right)_\tau + h \cos i \frac{\partial \Omega}{\partial p} + h \frac{\partial \omega}{\partial p} \quad . \ldots\ldots\ldots\ldots (2)$$

Now, by 5·09 (4), $(\dot{\xi})_\tau = 0$;

by 5·09 (5), $(\dot{\eta})_\tau = \dfrac{na\sqrt{(1-e^2)}}{1-e}$;

by 5·09 (3), $\left(\dfrac{\partial \eta}{\partial p}\right)_\tau = -\dfrac{na\sqrt{(1-e^2)}}{1-e} \cdot \dfrac{\partial \tau}{\partial p}$.

Hence, since $\mu = n^2a^3$,

$$\left(\dot{\xi} \frac{\partial \xi}{\partial p} + \dot{\eta} \frac{\partial \eta}{\partial p}\right)_\tau = -\frac{\mu(1+e)}{a(1-e)} \cdot \frac{\partial \tau}{\partial p} \quad . \ldots\ldots\ldots\ldots\ldots (3)$$

Using (2) and (3) we find that (1) becomes

$$C_p = -\frac{\mu\tau}{2a^2} \frac{\partial a}{\partial p} - h \cos i \frac{\partial \Omega}{\partial p} - h \frac{\partial \omega}{\partial p} \quad , \ldots\ldots\ldots\ldots\ldots (4)$$

or in terms of χ—by writing χ/n for $-\tau$ and n^2a^3 for μ—

$$C_p = \tfrac{1}{2}na\chi \frac{\partial a}{\partial p} - h \cos i \frac{\partial \Omega}{\partial p} - h \frac{\partial \omega}{\partial p} \quad . \ldots\ldots\ldots\ldots\ldots (5)$$

5·13. General formula for [p, q]

We write (4) of the previous section as

$$C_p = -\tau \frac{\partial}{\partial p}\left(-\frac{\mu}{2a}\right) - h \frac{\partial \omega}{\partial p} - h \cos i \frac{\partial \Omega}{\partial p} \quad . \ldots\ldots\ldots\ldots (1)$$

Similarly,

$$C_q = -\tau \frac{\partial}{\partial q}\left(-\frac{\mu}{2a}\right) - h \frac{\partial \omega}{\partial q} - h \cos i \frac{\partial \Omega}{\partial q} \quad . \ldots\ldots\ldots\ldots (2)$$

Since, by 5·11 (10),

$$[p, q] = \frac{\partial C_q}{\partial p} - \frac{\partial C_p}{\partial q}$$

we obtain from (1) and (2) the general formula for a Lagrangian bracket :

$$[p, q] = \frac{\partial(-\tau, -\mu/2a)}{\partial(p, q)} + \frac{\partial(\omega, h)}{\partial(p, q)} + \frac{\partial(\Omega, h \cos i)}{\partial(p, q)} \quad , \ldots\ldots\ldots (3)$$

in which $h = \sqrt{\{\mu a(1-e^2)\}}$.

Replacing $-\tau$ by χ/n we have the alternative form :

$$[p, q]=\frac{\partial(\chi/n, -\mu/2a)}{\partial(p, q)}+\frac{\partial(\omega, h)}{\partial(p, q)}+\frac{\partial(\Omega, h\cos i)}{\partial(p, q)} \quad \text{.....(4)}$$

If we replace the elements τ (or χ) and ω by ϵ and ϖ, where

$$\tau=-\frac{(\epsilon-\varpi)}{n} \text{ and } \omega=\varpi-\Omega \ ,$$

the formula (1) becomes

$$C_p=\frac{\epsilon-\varpi}{n}\cdot\frac{\mu}{2a^2}\frac{\partial a}{\partial p}-h\frac{\partial \varpi}{\partial p}+h(1-\cos i)\frac{\partial \Omega}{\partial p} \ , \text{.....(5)}$$

and (3) becomes

$$[p, q]=\frac{\partial\left(\frac{\epsilon-\varpi}{n}, -\frac{\mu}{2a}\right)}{\partial(p, q)}+\frac{\partial(\varpi-\Omega, h)}{\partial(p, q)}+\frac{\partial(\Omega, h\cos i)}{\partial(p, q)} \quad \text{.....(6)}$$

5·14. Evaluation of the Lagrangian brackets

Any Lagrangian bracket can be readily evaluated from the general formulae (3) or (4) of the previous section. For example, if we choose $p=a$, $q=\Omega$, then

$$[a, \Omega]=\frac{\partial(\Omega, h\cos i)}{\partial(a, \Omega)} \ ;$$

hence $$[a, \Omega]=-\cos i\frac{\partial h}{\partial a}=-\tfrac{1}{2}na\sqrt{(1-e^2)}\cos i \ ,$$

which agrees with the result in 5·08 (3).

Another method of evaluation of a bracket follows from the definition of C_p in 5·12 (5) or (4). Taking p to be, in turn, a, e, χ, Ω, ω, i we have

$$C_a=\tfrac{1}{2}na\chi \ , \quad C_e=0 \ , \quad C_\chi=0 \ ,$$
$$C_\Omega=-h\cos i \ , \quad C_\omega=-h \ , \quad C_i=0 \ .$$

Then, for example,

$$[a, \Omega]=\frac{\partial C_\Omega}{\partial a}-\frac{\partial C_a}{\partial \Omega}=-\cos i\frac{\partial h}{\partial a}$$
$$=-\tfrac{1}{2}na\sqrt{(1-e^2)}\cos i \ ,$$

as above. We can thus derive the values of the fifteen Lagrangian brackets which have been given in earlier sections.

When the elements are taken to be a, e, i, ϵ, ϖ and Ω the corresponding values of C_p, from 5·13 (6), are

$$C_a=\frac{\epsilon-\varpi}{n}\cdot\frac{\mu}{2a^2} \ , \quad C_e=C_i=C_\epsilon=0 \ ,$$
$$C_\varpi=-h \ , \quad C_\Omega=(1-\cos i)h \ .$$

Nine of the brackets are zero ; the non-zero brackets are as follows :

$$\left.\begin{array}{ll} [a, \epsilon] = -\tfrac{1}{2}na \ , & [a, \varpi] = \tfrac{1}{2}na\{1 - \sqrt{(1-e^2)}\} \\ [a, \Omega] = \tfrac{1}{2}na\sqrt{(1-e^2)}(1-\cos i) \ , & [e, \varpi] = na^2e/\sqrt{(1-e^2)} \\ [e, \Omega] = -na^2e(1-\cos i)/\sqrt{(1-e^2)} \ , & [i, \Omega] = na^2\sqrt{(1-e^2)}\sin i \end{array}\right\} \quad \ldots(1)$$

5·15. Derivation of the canonic equations

Instead of taking p (or q) to be the elements in turn, as in the previous section, we can derive a much simpler set of planetary equations by taking p (or q) to be certain functions of the elements.

Let α_1, α_2, α_3 and β_1, β_2, β_3 now denote independent functions of one or more of the elements defined as follows :

$$\alpha_1 = -\frac{\mu}{2a} \ , \quad \alpha_2 = h \ , \quad \alpha_3 = h\cos i \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\beta_1 = -\tau \ , \quad \beta_2 = \omega \ , \quad \beta_3 = \Omega. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Then, in accordance with the " Note " on p. 61, Lagrange's equations in the general form are, r taking the values 1, 2 and 3,

$$\sum_{i=1}^{3} [\alpha_r, \alpha_i]\dot{\alpha}_i + \sum_{i=1}^{3} [\alpha_r, \beta_i]\dot{\beta}_i = \frac{\partial R}{\partial \alpha_r}$$

and

$$\Sigma\,[\beta_r, \alpha_i]\dot{\alpha}_i + \Sigma\,[\beta_r, \beta_i]\dot{\beta}_i = \frac{\partial R}{\partial \beta_r} \ .$$

In terms of these new constants α_i and β_i, the formula 5·13 (1) for C_p is

$$C_p = \beta_1\frac{\partial \alpha_1}{\partial p} - \alpha_2\frac{\partial \beta_2}{\partial p} - \alpha_3\frac{\partial \beta_3}{\partial p} \ .$$

Similarly,

$$C_q = \beta_1\frac{\partial \alpha_1}{\partial q} - \alpha_2\frac{\partial \beta_2}{\partial q} - \alpha_3\frac{\partial \beta_3}{\partial q} \ .$$

Hence,

$$[p, q] \equiv \frac{\partial C_q}{\partial p} - \frac{\partial C_p}{\partial q} = -\frac{\partial(\alpha_1, \beta_1)}{\partial(p, q)} - \frac{\partial(\alpha_2, \beta_2)}{\partial(p, q)} - \frac{\partial(\alpha_3, \beta_3)}{\partial(p, q)} \quad \ldots\ldots\ldots(3)$$

from which

$$[\alpha_1, \beta_1] = [\alpha_2, \beta_2] = [\alpha_3, \beta_3] = -1 \ ,$$

and all the other brackets are zero.

Lagrange's equations then reduce to :

$$\dot{\beta}_1 = -\frac{\partial R}{\partial \alpha_1} \ , \quad \dot{\beta}_2 = -\frac{\partial R}{\partial \alpha_2} \ , \quad \dot{\beta}_3 = -\frac{\partial R}{\partial \alpha_3} \ ; \quad \ldots\ldots\ldots\ldots(4)$$

$$\dot{\alpha}_1 = \frac{\partial R}{\partial \beta_1} \ , \quad \dot{\alpha}_2 = \frac{\partial R}{\partial \beta_2} \ , \quad \dot{\alpha}_3 = \frac{\partial R}{\partial \beta_3} \ . \quad \ldots\ldots\ldots\ldots(5)$$

The equations (4) and (5) are called *canonic equations* and the variables α_i, β_i are said to be *canonically conjugate*. We deal with canonic equations, their properties and their solutions in later chapters.

We can write (3) in an alternative form in terms of γ_i which we define as $-\beta_i$. Then

$$[p, q] = \sum_{i=1}^{3} \frac{\partial(\alpha_i, \gamma_i)}{\partial(p, q)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

It is to be remembered that p and q are, in general, any independent functions of the elements; they may be taken to be any two of the elements say a_m and a_s. We then have the expression for a Lagrangian bracket which will be used in a subsequent chapter:

$$[a_m, a_s] = \sum_{i=1}^{3} \frac{\partial(\alpha_i, \gamma_i)}{\partial(a_m, a_s)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

5·16. Whittaker's derivation of the general formula for a Lagrangian bracket*

If p and q are any two of the six elements or, more generally, any two functions of the elements, and if A_p is defined by

$$A_p = \dot{x}\frac{\partial x}{\partial p} + \dot{y}\frac{\partial y}{\partial p} + \dot{z}\frac{\partial z}{\partial p} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

then clearly

$$[p, q] = \frac{\partial A_p}{\partial q} - \frac{\partial A_q}{\partial p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Here (x, y, z) are the coordinates of the planet P with respect to the usual reference plane and the axes OX, OY, OZ, the Sun being the centre O (Fig. 14).

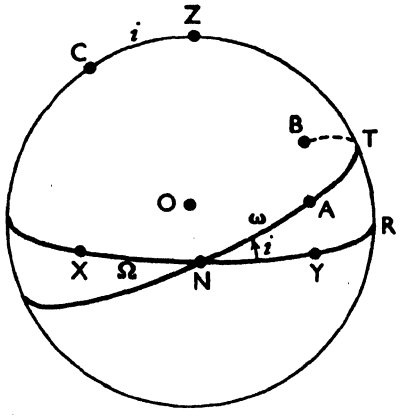


FIG. 14

Let (x_1, y_1, z_1) be the coordinates of P with respect to the axes ON, OR, OZ corresponding to a rotation of OX, OY through the angle Ω. Then

$$x = x_1 \cos\Omega - y_1 \sin\Omega \quad , \quad y = x_1 \sin\Omega + y_1 \cos\Omega \quad , \quad z = z_1 \quad .$$

Let $u = x + iy$, $v = x - iy$; $u_1 = x_1 + iy_1$, $v_1 = x_1 - iy_1$, $i = \sqrt{(-1)}$.

* E. T. Whittaker: *The Messenger of Mathematics*, New Series, No. 309 (January, 1897).

Then $u = u_1 e^{i\Omega}$, $v = v_1 e^{-i\Omega}$, $\dfrac{\partial v}{\partial p} = e^{-i\Omega}\left(\dfrac{\partial v_1}{\partial p} - i v_1 \dfrac{\partial \Omega}{\partial p}\right)$

and $$\dot{u}\frac{\partial v}{\partial p} = \dot{u}_1 \frac{\partial v_1}{\partial p} - i\,\dot{u}_1 v_1 \frac{\partial \Omega}{\partial p} \quad .$$

Taking the real parts we find that

$$\dot{x}\frac{\partial x}{\partial p} + \dot{y}\frac{\partial y}{\partial p} = \dot{x}_1 \frac{\partial x_1}{\partial p} + \dot{y}_1 \frac{\partial y_1}{\partial p} + (x_1\dot{y}_1 - \dot{x}_1 y_1)\frac{\partial \Omega}{\partial p} \quad . \ldots\ldots\ldots\ldots (3)$$

But $(x_1\dot{y}_1 - \dot{x}_1 y_1)$ is equal to $h \cos i$, being the projection on the reference plane of twice the rate of description of area in the orbital plane.

Hence, from (1) and (3),

$$A_p = \dot{x}_1 \frac{\partial x_1}{\partial p} + \dot{y}_1 \frac{\partial y_1}{\partial p} + \dot{z}_1 \frac{\partial z_1}{\partial p} + h \cos i \frac{\partial \Omega}{\partial p} \quad . \ldots\ldots\ldots\ldots (4)$$

Now rotate the y_1 and z_1 axes through the angle i; let (X, Y, Z) be the coordinates of the planet with respect to the new axes ON, OT, OC. Then $x_1 = X$, and by analogy with (3) we have

$$\dot{y}_1 \frac{\partial y_1}{\partial p} + \dot{z}_1 \frac{\partial z_1}{\partial p} = \dot{Y}\frac{\partial Y}{\partial p} + \dot{Z}\frac{\partial Z}{\partial p} + (Y\dot{Z} - \dot{Y}Z)\frac{\partial i}{\partial p} \quad . \ldots\ldots\ldots\ldots (5)$$

But $Z = 0$; hence, from (4) and (5),

$$A_p = \dot{X}\frac{\partial X}{\partial p} + \dot{Y}\frac{\partial Y}{\partial p} + h \cos i \frac{\partial \Omega}{\partial p} \quad . \ldots\ldots\ldots\ldots (6)$$

Now rotate ON, OT through the angle ω to OA and OB, where A is perihelion; let (ξ, η) be the coordinates of the planet with reference to OA and OB. By analogy with (3),

$$\dot{X}\frac{\partial X}{\partial p} + \dot{Y}\frac{\partial Y}{\partial p} = \dot{\xi}\frac{\partial \xi}{\partial p} + \dot{\eta}\frac{\partial \eta}{\partial p} + (\xi\dot{\eta} - \dot{\xi}\eta)\frac{\partial \omega}{\partial p} \quad .$$

But $\xi\dot{\eta} - \dot{\xi}\eta = h$; hence

$$A_p = \dot{\xi}\frac{\partial \xi}{\partial p} + \dot{\eta}\frac{\partial \eta}{\partial p} + h\frac{\partial \omega}{\partial p} + h \cos i \frac{\partial \Omega}{\partial p} \quad . \ldots\ldots\ldots\ldots (7)$$

Now $\xi = a(\cos E - e)$, $\eta = a(1 - e^2)^{1/2} \sin E \equiv b \sin E$

and $\dot{\xi} = -a\dot{E} \sin E$, $\dot{\eta} = b\dot{E} \cos E$.

If F denotes the first two terms on the right of (7), it is easily seen that

$$F/E = a^2 \frac{\partial E}{\partial p}(1 - e^2 \cos^2 E) + a \sin E (1 - e \cos E)\left\{e\frac{\partial a}{\partial p} + a\frac{\partial e}{\partial p}\right\} \quad .$$

But from Kepler's equation

$$\dot{E}(1 - e \cos E) = n = \mu^{1/2} a^{-3/2};$$

hence $$\mu^{-1/2} F = a^{1/2}(1 + e \cos E)\frac{\partial E}{\partial p} + a^{-1/2} e \sin E \frac{\partial a}{\partial p} + a^{1/2} \sin E \frac{\partial e}{\partial p}$$

which can be written as

$$\mu^{-1/2}F = -\tfrac{1}{2}a^{1/2}(1-e\cos E)\frac{\partial E}{\partial p} - \tfrac{3}{4}a^{-1/2}(E-e\sin E)\frac{\partial a}{\partial p} + \tfrac{1}{2}a^{1/2}\sin E\frac{\partial e}{\partial p}$$
$$+\tfrac{1}{2}a^{1/2}(3+e\cos E)\frac{\partial E}{\partial p} + \tfrac{1}{4}a^{-1/2}(3E+e\sin E)\frac{\partial a}{\partial p} + \tfrac{1}{2}a^{1/2}\sin E\frac{\partial e}{\partial p}$$

or
$$F = -\frac{\mu^{1/2}}{2a}\frac{\partial}{\partial p}\{a^{3/2}(E-e\sin E)\} + \frac{\mu^{1/2}}{2}\frac{\partial}{\partial p}\{a^{1/2}(3E+e\sin E)\}$$
$$\equiv -\frac{\mu^{\frac{1}{2}}}{2a}\frac{\partial G}{\partial p} + \frac{\partial H}{\partial p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Now $G \equiv a^{3/2}(E-e\sin E) \equiv na^{3/2}(t+\chi/n) = \mu^{1/2}(t+\chi/n)$

so that
$$\frac{\partial G}{\partial p} = \mu^{1/2}\frac{\partial(\chi/n)}{\partial p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

From (7), (8) and (9) we obtain

$$A_p = -\frac{\mu}{2a}\frac{\partial}{\partial p}(\chi/n) + \frac{\partial H}{\partial p} + h\frac{\partial\omega}{\partial p} + h\cos i\frac{\partial\Omega}{\partial p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

A_q is given by a similar formula. From (2) we then have the general formula for the Lagrangian bracket, namely,

$$[p,q] = \frac{\partial(\chi/n, -\mu/2a)}{\partial(p,q)} + \frac{\partial(\omega,h)}{\partial(p,q)} + \frac{\partial(\Omega, h\cos i)}{\partial(p,q)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

agreeing with 5·13 (4).

Chapter 6

OUTLINE OF THE SOLUTION OF LAGRANGE'S PLANETARY EQUATIONS

6·01. Introduction

We shall use the planetary equations as given in section 5·10, relating to a planet P of mass m, disturbed by a planet P_1, of mass m_1, with reference to the elements a, e, i, Ω, ϖ and ϵ ; we shall be concerned particularly with the equations

$$\dot{a} = \frac{2}{na}\frac{\partial R}{\partial \epsilon} \quad , \qquad \text{.....(1)}$$

$$\dot{\epsilon} = -\frac{2}{na}\frac{\partial R}{\partial a} + P\frac{\partial R}{\partial e} + Q\frac{\partial R}{\partial i} \quad , \qquad \text{.....(2)}$$

$$\dot{\Omega} = A\frac{\partial R}{\partial i} \quad ; \qquad \text{.....(3)}$$

here, n is given by

$$n^2a^3 = \mu = G(m_0 + m), \qquad \text{.....(4)}$$

where m_0 is the Sun's mass ; R is the disturbing function given by

$$R = Gm_1\left(\frac{1}{\Delta} - \frac{xx_1 + yy_1 + zz_1}{r_1^3}\right) \quad , \qquad \text{.....(5)}$$

where (x, y, z) and (x_1, y_1, z_1) are the heliocentric coordinates of P and P_1 respectively ; A, P and Q are functions of a, e, i which need not be specified in detail ; also,

$$\Delta^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \quad . \qquad \text{.....(6)}$$

The solutions of the equations for $\dot{e}$, $\dot{i}$ and $\dot{\varpi}$ have the same general characteristics as for $\dot{\Omega}$—that is the reason why we need consider only (3) in detail in the present discussion.

Consider the disturbing action of P on P_1 ; if a_1, e_1, ..., ϵ_1 are the orbital elements associated with P_1, the equation similar to (1) is

$$\dot{a}_1 = \frac{2}{n_1a_1}\frac{\partial R_1}{\partial \epsilon_1} \quad , \qquad \text{.....(7)}$$

where

$$R_1 = Gm\left(\frac{1}{\Delta} - \frac{xx_1 + yy_1 + zz_1}{r^3}\right) \quad . \qquad \text{.....(8)}$$

We have also equations similar to (2) and (3).

It will be evident that the procedure can be extended to take account of any number of planets.

The practical method of solving the planetary equations is based on the

fact that e, e_1, i and i_1 are small so that the disturbing functions R and R_1 can be expanded in series involving powers of e, e_1, γ and γ_1 where

$$\gamma = \tan i \quad , \quad \gamma_1 = \tan i_1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

We may add the remark that in the lunar theory the ratio of the geocentric distances of the Moon and Sun is small and that the expansion of the disturbing function in this case—the Sun being the principal disturbing body—takes account of this small ratio as well.

6·02. General form of the expansion of the disturbing function

In the following chapter we shall deal in considerable detail with the expansion of the disturbing function R, but meanwhile it may be advantageous to the reader to obtain a preliminary idea, without the introduction of too much refinement, of how R is converted into a form suitable for practical purposes, and of the general characteristics of this form ; in this section we shall ignore third and higher powers of the eccentricities and inclinations.

In the previous chapter we have written the solutions for the elliptic orbit of P as

$$x = F_1(t, \alpha_i, \beta_i), \quad y = F_2(t, \alpha_i, \beta_i), \quad z = F_3(t, \alpha_i, \beta_i) \quad ,$$

where α_i and β_i $(i = 1, 2, 3)$ represent the six elements of the elliptic orbit and F_1, F_2 and F_3 are *definite* functions of the time (or of the mean anomaly M) and the six elements. The planetary equations are then derived, according to the method of variation of arbitrary constants, as in section 5·01 when the above functions of x, y and z are inserted in R, the α's and β's now being regarded as new variables.

The general formula for F_1, for example, is given by 5·01 (3) as

$$x = al_1 \cos E + bl_2 \sin E - ael_1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and if we expand $\cos E$ and $\sin E$ in terms of the mean anomaly M and the eccentricity e we obtain,* to $O(e^2)$,

$$\cos E = -\tfrac{1}{2}e + (1 - \tfrac{3}{8}e^2)\cos M + \tfrac{1}{2}e \cos 2M + \tfrac{3}{8}e^2 \cos 3M \quad , \quad \ldots\ldots\ldots (2)$$

$$\sin E = (1 - \tfrac{1}{8}e^2)\sin M + \tfrac{1}{2}e \sin 2M + \tfrac{3}{8}e^2 \sin 3M \quad . \quad \ldots\ldots\ldots (3)$$

Now, by 5·06 (3), l_1 can be expressed as

$$l_1 = \cos(\Omega + \omega) + 2\sin^2\frac{i}{2}\sin\Omega \sin\omega$$

or, to $O(\gamma^2)$,

$$l_1 = \cos\varpi + \tfrac{1}{4}\gamma^2\{\cos(2\Omega - \varpi) - \cos\varpi\} \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

Similarly,

$$l_2 = -\sin\varpi + \tfrac{1}{4}\gamma^2\{\sin(2\Omega - \varpi) + \sin\varpi\} \quad . \quad \ldots\ldots\ldots\ldots\ldots (5)$$

It is readily seen that, by means of (2)–(5), the formula for x in (1) takes the form

$$x = -\tfrac{3}{2}ae\cos\varpi + \Sigma A \cos(iM + j\Omega + k\varpi) \quad , \quad \ldots\ldots\ldots\ldots\ldots (6)$$

* We use the notation $O(p)$ to signify that the order of magnitude concerned is that of p.

where i, j and k are positive or negative integers, including zero for j, and A is a function of a, e and γ.

In the same way the formula for the coordinate x_1 of the disturbing planet P_1 can be written

$$x_1 = -\tfrac{3}{2}a_1e_1\cos\varpi_1 + \Sigma A_1\cos(i_1M_1 + j_1\Omega_1 + k_1\varpi_1) \quad , \quad \ldots\ldots(7)$$

where i_1, j_1 and k_1 are positive or negative integers, including zero for j_1, and A_1 is a function of a_1, e_1 and γ_1.

Consider now the factor $\frac{x_1}{r_1^3}$ of x occurring in the second term of R in 6·01 (5). The x_1-equation of motion for P_1 in its elliptic orbit is

$$\ddot{x}_1 = -\mu_1\frac{x_1}{r_1^3} = -n_1^2a_1^3\frac{x_1}{r_1^3} \quad .$$

Also, since $M_1 = n_1t + \chi_1$, then $\ddot{x}_1 = n_1^2\dfrac{d^2x_1}{dM_1^2}$;

hence $\dfrac{x_1}{r_1^3} = -\dfrac{1}{a_1^3}\dfrac{d^2x_1}{dM_1^2}$ and, by means of (7),

$$\frac{x_1}{r_1^3} = \frac{1}{a_1^3}\Sigma i_1^2A_1\cos(i_1M_1 + j_1\Omega_1 + k_1\varpi_1) \quad ,$$

where $i_1 \neq 0$. It is then seen from this last equation and (6) that $\dfrac{xx_1}{r_1^3}$ takes the form

$$\Sigma C\cos(iM + i_1M_1 + j\Omega + j_1\Omega_1 + k\varpi + k_1\varpi_1) \quad \ldots\ldots(8)$$

where C is a function of a, a_1, e, e_1, γ and γ_1, and i, but not i_1, can have a zero value. Also, $\dfrac{yy_1}{r_1^3}$ and $\dfrac{zz_1}{r_1^3}$ will contribute series of the same general form as (8).

The expansion of $1/\Delta$ depends on a much more complicated process with which we shall not attempt to deal here even in the most cursory way. We merely state that $1/\Delta$ can be expanded in the form given by (8) with the qualification that i and i_1 may each take a zero value separately or simultaneously. The complete expansion of R can then be written in the form

$$R = m_1\Sigma c\cos\theta_0 + m_1\Sigma C\cos\theta \quad \ldots\ldots(9)$$

where θ_0 and θ are of the forms given by

$$\theta_0 = j\Omega + j_1\Omega_1 + k\varpi + k_1\varpi_1 \quad \ldots\ldots(10)$$

in which j, ... k_1 may take zero values, singly or otherwise,

and $$\theta = iM + i_1M_1 + j\Omega + j_1\Omega_1 + k\varpi + k_1\varpi_1 \quad \ldots\ldots(11)$$

in which i *or* i_1 in particular may be zero. In (9), c and C are functions of a, a_1, e, e_1, γ and γ_1 ; it is assumed for simplicity that the gravitational constant is absorbed in c and C.

The problem of the convergency of the series obtained in (9) in general —that is when all powers of the eccentricities and inclinations are considered—is a difficult one which we shall not attempt to discuss; accordingly, it will be assumed that the general series in (9)—and other series obtained by comparable methods—represent the disturbing function for practical purposes.

It will be noticed that the element ϵ occurs in θ—see (11)—only in association with nt, since $M = nt + \epsilon - \varpi$.

It is convenient to deal with the mean longitude l instead of M, l being given by

$$l = nt + \epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Then θ has the general form given by

$$\theta = il + i_1 l_1 + j\Omega + j_1\Omega_1 + k\varpi + k_1\varpi_1 \quad \ldots\ldots\ldots\ldots\ldots 13)$$

We note from (12) and (13) that

$$\frac{\partial R}{\partial \epsilon} = \frac{\partial R}{\partial l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

6·03. An important modification of the planetary equations

Consider the first of the planetary equations

$$\dot{a} = \frac{2}{na}\frac{\partial R}{\partial \epsilon} = \frac{2}{na}\frac{\partial R}{\partial l} \quad .$$

From 6·02 (9) and (13),

$$\dot{a} = -\frac{2m_1}{na}\Sigma iC \sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since each amplitude on the right-hand side of (1) contains the small factor m_1—small, that is, in comparison with the Sun's mass—we can then assume that, to obtain a first approximation for a, the elements $a, a_1, \ldots \varpi_1$ are constants on the right-hand side of (1) and that

$$\theta = (in + i_1 n_1)t + g \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where g is a constant. Then, on integrating, we obtain

$$a = a_0 + \frac{2m_1}{na}\Sigma\frac{iC}{in + i_1 n_1}\cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The formula (3) shows that a fluctuates about a mean value a_0 and we write concisely

$$a = a_0 + P.T.$$

where $P.T.$ denotes a series of periodic terms.

Consider now the planetary equation 6·01 (3), namely,

$$\dot{\Omega} = A\frac{\partial R}{\partial i} \quad .$$

Now $i(\equiv \tan^{-1}\gamma)$ occurs in the coefficients c and C of 6·02 (9); hence

$$\dot{\Omega} = Am_1 \Sigma \frac{\partial c}{\partial i} \cos\theta_0 + Am_1 \Sigma \frac{\partial C}{\partial i} \cos\theta \quad .$$

For a first approximation, we then obtain

$$\Omega = \Omega_0 + Am_1 t \Sigma \frac{\partial c}{\partial i} \cos\theta_0 + Am_1 \Sigma \frac{1}{in + i_1 n_1} \frac{\partial C}{\partial i} \sin\theta$$

where θ is given by (2). Write λ for the constant coefficient of t in the second term; then

$$\Omega = \Omega_0 + \lambda t + P.T. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

It is easily seen from the appropriate planetary equations that e, i and ϖ are given, to a first approximation, by formulae analogous to (4).

The circumstances are different as regards the remaining planetary equation—that for ϵ—which we rewrite:

$$\dot{\epsilon} = -\frac{2}{na}\frac{\partial R}{\partial a} + P\frac{\partial R}{\partial e} + Q\frac{\partial R}{\partial i} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

On integration the last two terms will yield the part-solution of the form $\lambda t + P.T.$ which, we have seen, is the normal result for the other elements —except in the case of the element a for which $\lambda = 0$. Accordingly, we can ignore the last two terms in the present discussion.

Consider now the first term on the right-hand side of (5); a occurs in the coefficients c and C of the expansion for R and *also* in n appearing in the argument θ, for n is simply, from the analytical point of view, a convenient way of writing $\mu^{1/2}a^{-3/2}$. Forming $\dfrac{\partial R}{\partial a}$ we obtain

$$\dot{\epsilon} = -\frac{2m_1}{na}\Sigma\frac{\partial c}{\partial a}\cos\theta_0 - \frac{2m_1}{na}\Sigma\frac{\partial C}{\partial a}\cos\theta + \frac{2m_1}{na}\cdot\frac{dn}{da}\,t\,\Sigma iC\sin\theta \quad . \ldots.(6)$$

On integration, the first two terms on the right-hand side of this equation produce the solution of the form $\lambda t + P.T.$ which can be incorporated with the earlier part-solution arising from the terms in P and Q of (5).

The new feature to which attention is to be drawn is the form of the last term in (6) which consists of a series of periodic terms with amplitudes increasing with the time. This undesirable complication can be overcome as follows. If we denote by $\left(\dfrac{\partial R}{\partial a}\right)$ the derivative of R with respect to a *only so far as the coefficients c and C are concerned*, then

$$\frac{\partial R}{\partial a} = \left(\frac{\partial R}{\partial a}\right) + \frac{\partial R}{\partial l}\cdot\frac{\partial l}{\partial n}\cdot\frac{dn}{da}$$

or

$$\frac{\partial R}{\partial a} = \left(\frac{\partial R}{\partial a}\right) + t\frac{dn}{da}\frac{\partial R}{\partial \epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

by means of 6·02 (12) and (14). Now

$$\frac{dn}{dt}=\frac{dn}{da}\cdot\frac{da}{dt}=\frac{2}{na}\frac{dn}{da}\frac{\partial R}{\partial \epsilon} \quad \text{.....(8)}$$

by 6·01 (1); hence (5) becomes, on omitting the P and Q terms and using (7) and (8),

$$\dot{\epsilon}=-\frac{2}{na}\left(\frac{\partial R}{\partial a}\right)-t\frac{dn}{dt}$$

or

$$\dot{\epsilon}+t\frac{dn}{dt}=-\frac{2}{na}\left(\frac{\partial R}{\partial a}\right) \quad .$$

Let $\dot{\epsilon}'$ be defined by

$$\dot{\epsilon}'=\dot{\epsilon}+t\frac{dn}{dt} \quad . \quad \text{.....(9)}$$

Then

$$\dot{\epsilon}'=-\frac{2}{na}\left(\frac{\partial R}{\partial a}\right)=-\frac{2m_1}{na}\Sigma\frac{\partial c}{\partial a}\cos\theta_0-\frac{2m_1}{na}\Sigma\frac{\partial C}{\partial a}\cos\theta \quad \text{.....(10)}$$

On integration, we obtain ϵ' given, to a first approximation, by an equation of the form

$$\epsilon'=\epsilon_0+\lambda t+P.T. \quad \text{.....(11)}$$

From (9), $\dot{\epsilon}'=\dot{\epsilon}+\frac{d}{dt}(nt)-n$, so that

$$l\equiv nt+\epsilon=\int n\,dt+\epsilon' \quad . \quad \text{.....(12)}$$

As already remarked, ϵ occurs in R only in its association with l; accordingly, if we substitute $\int n\,dt+\epsilon'$ for $nt+\epsilon$ in R we have

$$\frac{\partial R}{\partial \epsilon'}=\frac{\partial R}{\partial \epsilon} \quad \text{so that} \quad \dot{a}=\frac{2}{na}\frac{\partial R}{\partial \epsilon'} \quad .$$

All the other planetary equations remain unaffected by this transformation. Regarding ϵ' as a new element in place of ϵ we see that the corresponding planetary equation, on referring to (5) and (10), is

$$\dot{\epsilon}'=-\frac{2}{na}\left(\frac{\partial R}{\partial a}\right)+P\frac{\partial R}{\partial e}+Q\frac{\partial R}{\partial i} \quad \text{.....(13)}$$

so that, to a first approximation, ϵ' is of the form given by (11).

To summarise: if p denotes any element (including ϵ') the solution of the planetary equations to a first approximation yields the general result

$$p=p_0+\lambda t+P.T. \quad \text{.....(14)}$$

with one important qualification namely, that if the element is a then the corresponding value of λ in (14) is zero.

Proceeding in the same way with the planetary equations referring

to the planet P_1 disturbed by P, we obtain the solution, to the first approximation, for any element p_1 in the form

$$p_1 = (p_1)_0 + \lambda_1 t + P.T.$$

with, again, the qualification that, if p_1 is a_1, then the corresponding value of λ_1 is zero.

It is to be noted—on referring to the planetary equations 5·10 (7) to (12) and the form of R in 6·02 (9)—that if p is a or e or i then the periodic terms are cosines of θ and that if p is Ω or ϖ or ϵ' then the periodic terms are sines of θ. Thus we can write

$$\left.\begin{matrix} a \\ e \\ i \end{matrix}\right\} = \begin{matrix} a_0 \\ e_0 + \lambda t + m_1 \Sigma J \cos\theta \\ i_0 \end{matrix} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

$$\left.\begin{matrix} \Omega \\ \varpi \\ \epsilon' \end{matrix}\right\} = \begin{matrix} \Omega_0 \\ \varpi_0 + \lambda t + m_1 \Sigma K \sin\theta \\ \epsilon_0 \end{matrix} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

In (15), $\lambda = 0$ for a.

6·04. General method of procedure for higher approximations

For convenience we define ρ by means of

$$\rho = \int n\, dt \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

so that, by 6·03 (12),

$$l \equiv nt + \epsilon = \rho + \epsilon' \quad ; \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

accordingly, in the expansion of R we replace l by $\rho + \epsilon'$ or, dropping the accent in the interests of simplicity, by $\rho + \epsilon$; we must, of course, bear in mind the meaning to be attached to ϵ in this connection. With this notation θ is now given by

$$\theta = i(\rho + \epsilon) + i_1(\rho_1 + \epsilon_1) + j\Omega + j_1\Omega_1 + k\varpi + k_1\varpi_1 \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

the subscripts referring to the quantities associated with the planet P_1.

Remembering that n is a function of a, we see that the definition of ρ in (1) is strictly

$$\rho = \mu^{\frac{1}{2}} \int a^{-3/2}\, dt \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

from which

$$\ddot{\rho} = -\frac{3}{a^2}\frac{\partial R}{\partial \epsilon} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The integral $\int n\, dt$, or ρ, is called the *mean motion in the disturbed orbit.*

As we have seen, the formal solution of the planetary equation for $\dot{a}$, for example, to a first approximation is

$$a = a_0 + \frac{2m_1}{na} \Sigma \frac{iC \cos\theta}{in + i_1 n_1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

on the right of which all the elements are regarded as constants with values a_0, e_0, etc.; to save complications in the notation we omit the zero suffix from each element except, of course, in the first term of (6). The summation terms in (6) are the perturbations in a and their order of magnitude is that of the small quantity m_1. The solution of the corresponding equation for the perturbations of a_1 will be of the same form as (6), the small factor being m in this case.

For simplicity we shall suppose that the order of magnitude of m is the same as that of m_1. We write (6) in the form

$$a = a_0 + \Delta' a \quad \ldots\ldots\ldots\

it follows that

$$\left.\begin{aligned} \rho_0 &= \int n_0\, dt = n_0 t \\ \Delta'\rho &= \int \Delta' n\, dt \\ \Delta''\rho &= \int \Delta'' n\, dt \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

where $\Delta' n$, $\Delta'' n$ are given by (11) and the formula following.

6·05. Characteristics of the perturbations of the first order

We shall consider the planetary equation for $\dot{\Omega}$ as typical. The solution is, by 6·03 (4),

$$\Omega = \Omega_0 + \lambda t + P.T. \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which λ is a constant given by

$$\lambda = m_1 A \, \Sigma \frac{\partial c}{\partial i} \cos\theta_0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which n, a, ... are supposed to have the constant values n_0, a_0, ... ; a typical periodic term is

$$\frac{m_1 A}{in + i_1 n_1} \frac{\partial C}{\partial i} \sin\theta \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The terms in (1) are of three types which we now consider in greater detail.

(i) *Secular terms*

The term λt is called a secular term. If, for example, the coefficient λ is positive then, so far as this term is concerned, the longitude of the node increases at a uniform angular rate λ, where λ is a small quantity of $O(m_1)$; if λ is negative, the longitude diminishes at the constant rate λ. Similar remarks apply to each of the other elements with the exception of a, for which there is no secular term. The consequence of this last result is important. Since the first-order solution gives $a = a_0 + P.T.$, the semi-major axis fluctuates by small amounts about the mean value a_0. Had there been a secular term, say Xt, and *if we suppose for the moment that the solution is exact*, then the semi-major axis would progressively increase or decrease according as X is positive or negative ; in the first case, a would eventually become infinite, the consequence being the escape of the planet from the Sun's control ; in the second case, a would shrink to such an extent that the planet would be finally engulfed by the Sun, assuming that it was not previously vaporised. The absence of such a secular term ensures the general stability of the planetary orbit—so far as our supposition is concerned—since the perturbations in the semi-major axis produce only fluctuations of small amplitude about the mean

value a_0. It follows also that, to the first approximation, there is no secular term associated with the angular motion n.

(ii) *Short-period inequalities*

Consider a particular member of the sum of periodic terms in (1); we can write it, from (3), as

$$\frac{m_1 D}{in+i_1 n_1}\sin\{(in+i_1 n_1)t+q\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where

$$q=i\epsilon+i_1\epsilon_1+j\Omega+j_1\Omega_1+k\varpi+k_1\varpi_1 \quad ,$$

and D has a particular order of magnitude with respect to e, e_1, γ and γ_1; for simplicity, assume that these elements are of the same order of magnitude so that D may be regarded as of order e^p, where p is a positive integer. Then, *provided that* $(in+i_1 n_1)$ *is not a small quantity*, the amplitude of the term (4) is of $O(m_1 e^p)$. Assuming that this condition holds for any pair of values of i and i_1, we define such a term as a *short-period inequality*,* this description arising from the fact that the period of the term is small—at most, of the order of the orbital period of P, or of P_1. Let T' denote the period of (4). Then

$$T'=2\pi/(in+i_1 n_1);$$

also, $n=2\pi/T$ and $n_1=2\pi/T_1$; hence

$$\frac{1}{T'}=\frac{i}{T}+\frac{i_1}{T_1} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

For example, if P refers to Jupiter and P_1 to Saturn then, approximately, $T=11{\cdot}86$ and $T_1=29{\cdot}46$, the unit of time being the year; from (5) we have the following:

$$\begin{array}{lll} i=1 \ , \ i_1=1 \ , & \text{then } T'=8{\cdot}46 \ ; \\ i=1 \ , \ i_1=-1 \ , & \text{then } T'=19{\cdot}85 \ ; \\ i=2 \ , \ i_1=1 \ , & \text{then } T'=4{\cdot}94 \ . \end{array}$$

The amplitude of such terms is $m_1 DT'/2\pi$ and the corresponding perturbations are essentially normal in character provided the condition, mentioned earlier, holds.

(iii) *Long-period inequalities*

These perturbations—of a periodic character—arise through the approximate commensurability, in certain instances, of the mean angular motions n and n_1; if n/n_1 is approximately equal to i_1/i where i and i_1 are particular integers then $in-i_1 n_1$ is a small quantity and the amplitudes of the corresponding periodic terms become very much greater—other things being equal—than in the case of terms discussed in (ii). With the

* " Inequality " is an archaic term signifying *deviation from elliptic motion* and equivalent, in the present connection, to the perturbation of the element concerned.

more accurate values of the orbital periods of Jupiter and Saturn it is found that

$$n = 299''\cdot13 \text{ and } n_1 = 120''\cdot45$$

so that, if $i = -2$ and $i_1 = 5$, then

$$(in + i_1 n_1) = 3''\cdot99 \quad .$$

If T'' denotes the period of the corresponding term, we find that T'' is approximately 75 times the orbital period of Jupiter—that is, about 890 years approximately.

If D_1 is the value of D applicable to this term, the amplitude of the inequality is $m_1 D_1 T''/2\pi$.

Such a term arising from the near-commensurability of n and n_1 is called a *long-period inequality.*

From the discussion of the properties of the disturbing function in the next chapter it is found that D (when $i = 2$, $i_1 = 1$) and D_1 (when $i = -2$ and $i_1 = 5$) are of the same order, e^2. If, for simplicity, we suppose D and D_1 to be equal then the amplitude of the long-period inequality is T''/T' times the amplitude of the corresponding short-period inequality; here $T''/T' \doteqdot 890/4\cdot94 \doteqdot 180$. It is thus evident that the long-period inequalities are responsible for greatly enhanced perturbations.

The diversity in magnitude of the two classes of periodic terms is further exaggerated when we consider the perturbations in mean longitude, l. Since $l = \rho + \epsilon$, the first-order perturbation $\Delta' l$ is given by $\Delta' l = \Delta'\rho + \Delta'\epsilon$. Now, from the discussion in (ii), a typical periodic term in $\Delta'\epsilon$ will have the form

$$\frac{m_1 L}{in + i_1 n_1} \sin\{(in + i_1 n_1)t + q\} \quad , \qquad \text{.....(6)}$$

and we shall obtain short-period and long-period inequalities of the kind already investigated.

Now consider $\Delta'\rho$. From 6·04 (11) and (12),

$$\Delta'\rho = -\frac{3n}{2a}\int \Delta' a \, dt \quad .$$

Also,
$$\Delta' a = m_1 \Sigma \frac{M}{in + i_1 n_1} \cos\{(in + i_1 n_1)t + q\} \quad ;$$

hence a typical term in $\Delta'\rho$ will have the form

$$\frac{m_1 M}{(in + i_1 n_1)^2} \sin\{(in + i_1 n_1)t + q\} \quad . \qquad \text{.....(7)}$$

Consider Jupiter and Saturn once more; let T' denote the period of the short-period term for which $i = 2$, $i_1 = 1$ and let T'' denote the period of the long-period term for which $i = -2$, $i_1 = 5$. As before M is assumed to be of order e^2 for each of these terms; suppose, for simplicity that these

values of M are actually the same. Then by (7) the amplitude of the long-period inequality is $\{T''/T'\}^2$ times the amplitude of the corresponding short-period inequality ; this ratio is thus $(180)^2$: 1, from the numerical results already found. It is evident that the long-period inequalities are of the greatest importance.

It is found that the mean longitude of Saturn, as disturbed by Jupiter, can be affected by as much as 50′ through the influence of long-period inequalities of the type we have been considering.

6·06. The Trojan planets

Another instance—and a more striking one—of the preponderating influence of the long-period perturbations concerns the Trojan planets fifteen of which are known (1952) ; the first, Achilles, was discovered in 1906. Their names have been taken from the heroes of the Graeco-Trojan war described by Homer in the *Iliad*. The Trojan planets are minor planets at very nearly the same mean distance from the Sun as Jupiter, so that their mean motions differ little from the Jovian mean motion. Several of the Trojans are about 60° ahead of Jupiter in longitude, while the remainder are about 60° behind Jupiter in longitude. Most of their orbital planes are included at angles of a few degrees to the orbital plane of Jupiter, although in five instances the angle of inclination exceeds 20°.

If, in Fig. 15, A and B are points on Jupiter's orbit such that the angles ASJ and BSJ are 60°, then A and B are called the *equilateral points* or the *libration points*. To a rough degree of approximation, including the neglect of the orbital inclination referred to, the path of a Trojan *relative to Jupiter* is indicated by the dotted curve at A, or at B, known as the *libration ellipse*, the ratio of the axes of which is the same for all the Trojans. It is perhaps unfortunate that the names of the planets have not been assigned in a systematic manner, for friend and foe find themselves in close proximity at each of the libration points ; in the interests of celestial peace and harmony a more orderly arrangement would have been to keep the Greeks hovering around one libration point and the heroes of Troy around the other. However, with a fine disregard of national prejudices, the entire group of minor planets with which we are now concerned are known simply as Trojan planets.

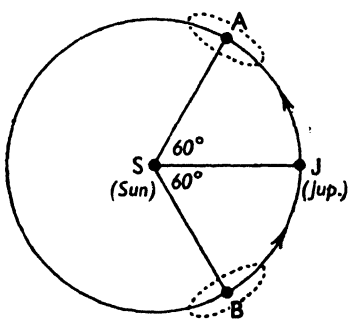


FIG. 15

More than a century and a half before the discovery of the first Trojan, Lagrange had discussed the problem of the motions of three-point masses

with their equilibrium positions at the equilateral vertices; the discovery of the Trojans has now made what was originally a purely theoretical and academic problem one of great practical interest.

The perturbations of Jupiter on a Trojan are very considerable and may amount, in longitude, to as much as 20°. If we consider Nestor, for example, it is known that for this Trojan $n = 301''{\cdot}00$; as n_1 (for Jupiter) is $299''{\cdot}13$ the terms in the disturbing function with the argument $i(n-n_1)t+q$, where $i=1, 2, \ldots$, evidently produce long-period inequalities, for $n-n_1 = 1''{\cdot}87$ so that T'', when $i=1$, is approximately 1900 years. Since the order of magnitude of the corresponding value of D in (4) is e^0, the perturbations are of very much greater magnitude than those discussed in connection with Jupiter and Saturn. For this reason the use of Lagrange's planetary equations is unsuitable for the detailed investigation of the motion of a Trojan and, consequently, more powerful methods have to be employed.

6·07. Equation for second approximation of Ω

The planetary equation for $\dot{\Omega}$ is, with our previous notation,

$$\dot{\Omega} = A\frac{\partial R}{\partial i} = Am_1\Sigma\frac{\partial c}{\partial i}\cos\theta_0 + Am_1\Sigma\frac{\partial C}{\partial i}\cos\theta \quad , \quad \ldots\ldots\ldots\ldots(1)$$

where $$\theta = i(\rho+\epsilon) + i_1(\rho_1+\epsilon_1) + \ldots + k_1\varpi_1 \quad \ldots\ldots\ldots\ldots(2)$$

We write (1) as

$$\dot{\Omega} = m_1 F(\rho+\epsilon,\ \rho_1+\epsilon_1\ ;\ a, a_1\ ;\ \ldots\ ;\ \varpi, \varpi_1) \quad , \quad \ldots\ldots\ldots\ldots(3)$$

where F denotes the function given by

$$F = A\,\Sigma\frac{\partial c}{\partial i}\cos\theta_0 + A\,\Sigma\frac{\partial C}{\partial i}\cos\theta \quad \ldots\ldots\ldots\ldots(4)$$

Write

$$\rho = \rho_0 + \Delta'\rho + \Delta''\rho \quad , \quad \epsilon = \epsilon_0 + \Delta'\epsilon + \Delta''\epsilon \quad ,$$

and so on, where $\Delta'\rho$ etc. are of $O(m)$ and $\Delta''\rho$ etc. are of $O(m^2)$; we assume for simplicity that m and m_1 are of the same order.

Now expand F by Taylor's theorem; then, if α denotes any one of the six orbital elements for P or of the six orbital elements for P_1, then for a second approximation we have

$$\frac{d}{dt}[\Omega_0 + \Delta'\Omega + \Delta''\Omega]$$

$$= m_1 F_0(\rho,\ \rho_1,\ \ldots\ \varpi_1) + m_1\left\{\Delta'\rho\left(\frac{\partial F}{\partial\rho}\right)_0 + \Delta'\rho_1\left(\frac{\partial F}{\partial\rho_1}\right)_0 + \Sigma\Delta'\alpha\left(\frac{\partial F}{\partial\alpha}\right)_0\right\} \ldots(5)$$

in which the suffix zero attached to $F_0, \left(\frac{\partial F}{\partial\rho}\right)_0 \ldots \left(\frac{\partial F}{\partial\alpha}\right)_0$ signifies that $F, \frac{\partial F}{\partial\rho}$, ... are evaluated for $\rho_0, \ldots \alpha_0$. *We note the assumption implicitly made* in the Taylorian expansion, namely, that $\Delta'\rho$, $\Delta'\rho_1$ and $\Delta'\alpha$ are so small that their squares and higher powers can be neglected.

The equation which gives the second-order terms is then

$$\frac{d}{dt}(\Delta''\Omega)=m_1\left\{\Delta'\rho\left(\frac{\partial F}{\partial\rho}\right)_0+\ldots+\Sigma\Delta'\alpha\left(\frac{\partial F}{\partial\alpha}\right)_0\right\} \quad \ldots\ldots\ldots\ldots(6)$$

With our assumption as to m and m_1 the right-hand side is of $O(m^2)$.

6·08. The nature of the second-order perturbations for Ω

We first derive the *forms* of the various products which appear in 6·07 (6).

From 6·07 (4), F is of the form given by

$$F=X+\Sigma Y\cos\theta \quad \ldots\ldots\ldots\ldots(1)$$

where $X\equiv X(a,\ e,\ \ldots,\ \varpi,\ i)$, $Y\equiv Y(a,\ e,\ i)$, $\theta\equiv\theta(\rho,\ \Omega,\ \varpi,\ \epsilon)$, ρ_1 and the corresponding elements for P_1 being supposed incorporated in these functional forms.

According to 6·03 (15) and (16) the elements are of two categories; for precision we suppose e to be characteristic of one group and ϖ of the other. In the sequel we deal only with the *forms* of the various functions.

(i) $\Delta' e=\lambda t+m_1\Sigma J\cos\theta$ and $\left(\frac{\partial F}{\partial e}\right)_0=G+\Sigma H\cos\theta$. Hence, since λ is of $O(m)$,

$$m_1\Delta' e\left(\frac{\partial F}{\partial e}\right)_0=m_1{}^2[A+Bt+P.T.+t\Sigma D\cos\theta] \quad \ldots\ldots\ldots\ldots(2)$$

(ii) $\Delta'\varpi=\lambda t+m_1\Sigma K\sin\theta$ and $\left(\frac{\partial F}{\partial\varpi}\right)_0=G'+\Sigma H'\sin\theta$. Hence,

$$m_1\Delta'\varpi\left(\frac{\partial F}{\partial\varpi}\right)_0=m_1{}^2[A'+B't+P.T.+t\Sigma D'\sin\theta] \quad \ldots\ldots\ldots\ldots(3)$$

(iii) From 6·04 (5) or 6·07 (7), $\Delta'\rho=m_1\Sigma L\sin\theta$; $\left(\frac{\partial F}{\partial\rho}\right)_0=\Sigma L'\sin\theta$.

Hence
$$m_1\Delta'\rho\left(\frac{\partial F}{\partial\rho}\right)_0=m_1{}^2(A''+P.T.) \quad \ldots\ldots\ldots\ldots(4)$$

The constants A, A' and A'' arise through the multiplication of pairs of terms having the same argument.

The formulae (2), (3) and (4) give the forms of characteristic terms on the right-hand side of 6·07 (6). On integration we then find that

$$\Delta''\Omega=A_2t+B_2t^2+P.T.+t\Sigma(D_2\sin\theta+E_2\cos\theta) \quad , \quad \ldots\ldots\ldots\ldots(5)$$

where the suffix 2 denotes the order of the various coefficients.

The first term on the right-hand side of (5) is a second-order part of the secular term. The second term—in t^2—is called the secular acceleration; in the case of the planets this term is minute and unobservable; the longitude of the Moon, however, has a term of this character with a coefficient of non-negligible magnitude revealed by observations over a long interval of time—we deal with this term in Chapter 19. The final

terms in (5), with the factor t, are a series of periodic terms whose amplitudes increase with t; such terms are called *secular-periodic*.

The planetary equations for ϖ, ϵ, e and i will lead, as is easily seen, to second-order solutions of the form given by (5).

6·09. The second-order perturbations for a

From 6·03 (1),

$$\dot{a} = -\frac{2m_1}{na} \Sigma iC \sin\theta \equiv m_1 F'(\rho + \epsilon, \ldots, \alpha),$$

where F' has the form given by

$$F' = \Sigma M \sin\theta \quad .$$

The second-order terms of a are given by the equation

$$\frac{d}{dt}(\Delta'' a) = m_1 \left\{ \Delta'\rho \left(\frac{\partial F'}{\partial \rho}\right)_0 + \ldots + \Sigma \Delta'\alpha \left(\frac{\partial F'}{\partial \alpha}\right)_0 \right\} \quad \ldots\ldots\ldots\ldots(1)$$

We proceed as in the previous section. Now

$$\left(\frac{\partial F'}{\partial e}\right)_0 = m_1 \Sigma N \sin\theta \text{ ; hence}$$

$$m_1 \Delta' e \left(\frac{\partial F'}{\partial e}\right)_0 = m_1^2 [P.T. + t \Sigma P \sin\theta].$$

Similarly, since $\left(\frac{\partial F'}{\partial \varpi}\right)_0 = m_1 \Sigma Q \cos\theta$, then

$$m_1 \Delta' \varpi \left(\frac{\partial F'}{\partial \varpi}\right)_0 = m_1^2 [P.T. + t \Sigma P' \cos\theta] \quad .$$

Also, since $\left(\frac{\partial F'}{\partial \rho}\right)_0 = \Sigma Q' \cos\theta$, then

$$m_1 \Delta' \rho \left(\frac{\partial F'}{\partial \rho}\right)_0 = m_1^2 [P.T.] \quad .$$

The integration of (1) gives

$$\Delta'' a = m_1^2 [P.T.] + t \Sigma (S_2 \sin\theta + T_2 \cos\theta) \quad \ldots\ldots\ldots\ldots(2)$$

6·10. Discussion of the perturbations to the second order of Ω and a

From the previous sections it is seen that the complete expressions for Ω and a to $O(m^2)$ are

$$\Omega = \Omega_0 + A_1 t + B_2 t^2 + P.T. + t \Sigma (D_2 \sin\theta + E_2 \cos\theta) \quad , \quad \ldots\ldots\ldots\ldots(1)$$

$$a = a_0 + P.T. + t \Sigma (S_2 \sin\theta + T_2 \cos\theta) \quad \ldots\ldots\ldots\ldots(2)$$

We add also

$$a = a_0 + m_1 \Sigma J \cos\theta \quad \ldots\ldots\ldots\ldots(3)$$

The perturbations in the longitude of the node given by (1) merely affect the location of the osculating ellipse at any time t with respect to the reference plane and have no effect, by themselves, in modifying the helio-

centric distances of the planets except in so far as to contribute to (2) through their first-order terms. Similar remarks apply to the perturbations in ϖ and ϵ.

Now consider (3) which gives the first-order perturbations in a; the coefficients J are, in general, developments in powers of e, e_1, γ and γ_1. If the series is uniformly convergent a lies between $a_0 + \delta a$ and $a_0 - \delta a$ where $\delta a = m_1 \Sigma \mid J \mid$. The range within which a must lie is thus definitely prescribed, and the same is true for each of the other planets of the system. To this order of approximation, and so far as the semi-major axes are concerned, the planetary system would be described as *stable* in the sense that it would continue to exist with these features, on the average, unchanged.

When we proceed to the second approximation, as represented by (2), the previous argument is no longer appropriate since, for any given value of t, the values of a lie between limits which we can write as

$$a_0 + \delta a_1 + t\, \delta a_2 \text{ and } a_0 - \delta a_1 - t\, \delta a_2 ;$$

the range for a is evidently not closed for all values of t, as in the first approximation, and the characteristics of stability in the sense previously used are absent.

Consider now the solutions for e; the perturbations up to the second-order have the form given by (1). If we assume that this equation is accurate, then the terms $A_1 t + B_2 t^2$ would indicate that e can increase without limit if, for simplicity, we assume that A_1 and B_2 are positive, and decrease without limit if A_1 and B_2 are negative; if A_1 and B_2 are of opposite signs, the argument is the same when t exceeds $\mid A_1 \mid / \mid B_2 \mid$. The orbit, then, would eventually become parabolic and afterwards hyperbolic, and the planet would disappear from the solar system.

At first sight it would seem that the stability of the solar system is not assured. Such a conclusion would be hasty, for it would mean that we had forgotten the assumption implicit in the Taylorian expansion of the disturbing function, namely, that $\Delta'\rho$, $\Delta'\rho_1$, $\Delta'\alpha$ are so small that the squares, products and higher powers of these quantities can be neglected. But owing to the appearance of secular terms in $\Delta'\Omega$, ... $\Delta' e$, ... $\Delta'\gamma$ ... this restriction is only binding provided that the value of t is severely circumscribed, to the order, for example, of a century or two. The analysis of the previous pages will then give the second-order perturbations with reasonable accuracy over the restricted interval concerned—and third-order perturbations with still greater accuracy—but this analysis will not shed any light on the problem of the stability, or instability, of the planetary system. In other words, the failure to deal with the long-range problem of stability is a direct result of the method of approximation adopted.

In Chapter 13 we shall see that there is good reason to believe that the variations of e and the other elements are periodic in character, the periods being measured in scores of thousands of years. Over a short period of, say, a century the true curve of variation of e, for example, is substantially linear ; accordingly, this latter variation is interpreted as arising from a secular term. Conversely, a secular term such as λt in (1) may be thought of as an approximation to sin λt, when λt is small, whatever the element. Accordingly, the so-called secular term finds itself included in the general series of periodic terms ; further, the analysis shows that in this event, secular-periodic terms would not appear in the second-order approximations for any of the elements. These arguments by themselves, it must be emphasised, do not constitute a proof of the stability of the planetary system.

Poincaré has elaborated the conditions* for stability as follows :

(i) The heliocentric distance of any planet cannot increase, or decrease, without limit.

(ii) Close encounters of any pair of planets is ruled out.

(iii) The system can repeatedly pass through the configuration—which it had at time t_0, say—at times t_1, t_2 and so on.

The relevance of (ii) calls for some explanation. If two planets should come very close together the mutual perturbations would then be of a magnitude immensely greater than those with which we have been dealing in this chapter. It is possible, then, for the orbital eccentricity of one planet to be transformed, during the close approach, from a normal value to one exceeding unity, in which event the planet concerned would disappear into interstellar space. Such close encounters are not unknown ; for example, Comet Morehouse made a close approach to Jupiter with the result that its elliptic orbit was transformed into a hyperbolic one.

We make a final remark. The definition of stability takes no cognisance of the effects of frictional forces in the economy of the solar system, one example of which is tidal friction in the terrestrial seas.

* *Les Méthodes Nouvelles de la Mécanique Céleste*, III, 141, 1899.

Chapter 7

EXPANSION OF THE DISTURBING FUNCTION

7·01. Introduction

In this chapter we give in some detail the basic methods for the expansion of the disturbing function R in the lunar and planetary theories; in particular, we shall assume that the eccentricities and inclinations are small and of the same order of magnitude. It is to be realised, of course, that the manner in which R is developed is dependent on the choice of variables in the subsequent transformation of the equations of motion. In many theories the time t is the usual independent variable; on the other hand, in more than one lunar theory the Moon's true longitude—ν, say—is taken as the independent variable. Since the time is normally introduced in the disturbing function explicitly by way of the disturbing body's mean anomaly then, in principle, t has to be translated—by a process of successive approximations—into a series in terms of ν.

In the sequel we shall be content to deal with the expansion of R only when the time is the independent variable. We first discuss the expansion of R in the lunar theory, for there are elements of simplicity in this expansion not to be found in the corresponding problem of the planetary theory; one of the contributory reasons relates to the particular circumstances attaching to the Sun's motion relative to the centre of mass of the Earth and the Moon; this will be discussed specially in section 7·03.

7·02. Expansion of the function $\left(\frac{\mathbf{M}}{\Delta}+\frac{\mathbf{E}}{\Delta_1}\right)$ in the lunar theory

The equations of motion for the Moon and Sun are rewritten from 4·09 (15) and (16); they are

$$\ddot{x}+\frac{\mu x}{r^3}=\frac{GS(E+M)}{EM}\frac{\partial F}{\partial x} \quad , \quad \ldots\ldots\ldots(1)$$

$$\ddot{x}_1=\frac{G(E+M+S)}{E+M}\frac{\partial F}{\partial x_1} \quad , \quad \ldots\ldots\ldots\ldots(2)$$

with similar equations in y, z and y_1, z_1; here (x, y, z) are the coordinates of the Moon with respect to axes through E (Fig. 16), and (x_1, y_1, z_1) are

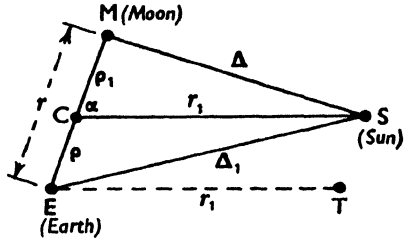


FIG. 16

the coordinates of the Sun with respect to parallel axes through C, which is the centre of mass of E and M; also F is, by 4·09 (14), expressed as

$$F=\frac{M}{\Delta}+\frac{E}{\Delta_1} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

In these formulae E, M and S are the masses of the Earth, Moon and Sun respectively.

In Fig. 16 denote EC by ρ, MC by ρ_1, and the angle MCS by α. We have, by definition,

$$\rho=\frac{Mr}{E+M} \quad , \quad \rho_1=\frac{Er}{E+M} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Now $$\Delta^2=r_1^2-2\rho_1 r_1\cos\alpha+\rho_1^2 \quad ,$$

or $$\frac{r_1}{\Delta}=\left\{1-\frac{2\rho_1}{r_1}\cos\alpha+\left(\frac{\rho_1}{r_1}\right)^2\right\}^{-\frac{1}{2}} \quad ;$$

further, ρ_1/r_1 is small, for r/Δ or r/Δ_1 or r/r_1 is of order 1/400. Hence

$$\frac{r_1}{\Delta}=1+\sum_{i=1}^{\infty}\left(\frac{\rho_1}{r_1}\right)^i P_i(\alpha) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where $P_i(\alpha)$ is the typical Legendre function. In terms of $c(\equiv\cos\alpha)$ we have the well-known formulae :

$$P_1=c\,;\ P_2=\tfrac{3}{2}c^2-\tfrac{1}{2}\,;\ P_3=\tfrac{5}{2}c^3-\tfrac{3}{2}c\,;\ P_4=\tfrac{35}{8}c^4-\tfrac{15}{4}c^2+\tfrac{3}{8} \quad . \ldots\ldots\ldots(6)$$

Again, $$\Delta_1^2=r_1^2+2\rho r_1\cos\alpha+\rho^2$$

from which $$\frac{r_1}{\Delta_1}=1+\sum_{i=1}^{\infty}(-1)^i\left(\frac{\rho}{r_1}\right)^i P_i(\alpha) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

From (5) and (7), we find that (3) is given by

$$r_1F=E+M+\frac{P_1}{r_1}(M\rho_1-E\rho)+\frac{P_2}{r_1^2}(M\rho_1^2+E\rho^2)$$
$$+\frac{P_3}{r_1^3}(M\rho_1^3-E\rho^3)+\frac{P_4}{r_1^4}(M\rho_1^4+E\rho^4)+\ldots$$

or, by means of (4),

$$F=\frac{E+M}{r_1}+\frac{ME}{E+M}\left\{\frac{r^2}{r_1^3}P_2+\frac{E-M}{E+M}\frac{r^3}{r_1^4}P_3+\frac{E^2-EM+M^2}{(E+M)^2}\frac{r^4}{r_1^5}P_4+\ldots\right\}. \quad (8)$$

7·03. The sun's orbit

From 7·02 (2) and (8) we have

$$\ddot{x}_1=G(E+M+S)\frac{\partial}{\partial x_1}\left\{\frac{1}{r_1}+\frac{ME}{(E+M)^2}\frac{r^2}{r_1^3}P_2+\ldots\right\}$$

The ratio of the second term in the bracket to the first is of order

$$\frac{ME}{(E+M)^2}\left(\frac{r}{r_1}\right)^2 \quad ;$$

now, $\frac{M}{E} \doteqdot \frac{1}{81}$ and $\frac{r}{r_1} \doteqdot \frac{1}{400}$; hence the ratio is approximately $8 \cdot 10^{-8}$. Accordingly, the second term has no appreciable effect—and this is also true of the subsequent terms. The equation of motion for the Sun is then

$$\ddot{x}_1 + G(E+M+S)\frac{x_1}{r_1^3} = 0 ;$$

there are similar equations for y_1 and z_1. The motion of the Sun relative to C is thus elliptic ; also, if n_1 and a_1 refer to this elliptic orbit, then

$$G(E+M+S) = n_1^2 a_1^3 \quad \ldots\ldots(1)$$

The normal to the plane of the orbit is in a fixed direction and we choose the z-axis, through E, to be parallel to the normal ; then the x-y plane is the plane of reference—the ecliptic. In Fig. 16 let $ET\,(\equiv r_1)$ be drawn parallel to CS ; then T moves in the ecliptic. The coordinates of T relative to E are the same as the coordinates of the Sun relative to C.

We can then replace the Sun's coordinates (relative to C) appearing in F by elliptic expansions in terms of a_1, e_1, etc. , where a_1, e_1, etc., are *constants*. We notice the difference, in this connection, between the lunar and planetary theories ; in the latter the coordinates of the disturbing body are inserted in the disturbing function in terms of algebraic functions representing the solutions of the undisturbed orbit but a_1, e_1, etc. are *not* constants ; they are, in fact, new variables satisfying Lagrange's planetary equations and coming definitely into prominence in the second-order solutions for the planet under consideration.

7·04. The disturbing function for the moon

From 7·02 (1), the disturbing function R is given by

$$R = \frac{GS(E+M)}{EM} F \quad \ldots\ldots(1)$$

Since r_1, which appears in the first term of F in 7·02 (8), is not a function of the Moon's coordinates (x, y, z), this term can be omitted from F. Also, since $E/S \doteqdot 3 \cdot 10^{-6}$, 7·03 (1) can be written with sufficient accuracy as

$$GS = n_1^2 a_1^3 \quad \ldots\ldots(2)$$

Let n denote the Moon's mean angular motion ; further, let

$$m = \frac{n_1}{n} \quad \ldots\ldots(3)$$

Then m is approximately $1/13$ and is regarded as a small quantity of the first order. Formula (2) then becomes

$$GS = m^2 n^2 a_1^3 \quad \ldots\ldots(4)$$

The formula (1) for R becomes, by means of (4) and 7·02 (8),

$$R=m^2n^2r^2\left(\frac{a_1}{r_1}\right)^3\left\{P_2+\frac{E-M}{E+M}\left(\frac{r}{r_1}\right)P_3+\frac{E^2-EM+M^2}{(E+M)^2}\left(\frac{r}{r_1}\right)^2P_4+\ldots\right\} \quad . (5)$$

If, in the last term of (5), we write $(E-M)^2$ for E^2-EM+M^2 we introduce an error of order $\frac{EM}{(E+M)^2}\left(\frac{r}{r_1}\right)^2$ which, as we have seen in section 7·03, is approximately $8\,.\,10^{-8}$; the effect of this error on the perturbations will be inappreciable.

We then write (5), on introducing a, as

$$R=m^2n^2r^2\left(\frac{a_1}{r_1}\right)^3 \times\left\{P_2+\left(\frac{E-M}{E+M}\cdot\frac{a}{a_1}\right)\frac{r}{a}\cdot\frac{a_1}{r_1}\cdot P_3+\left(\frac{E-M}{E+M}\cdot\frac{a}{a_1}\right)^2\left(\frac{r}{a}\right)^2\left(\frac{a_1}{r_1}\right)^2P_4+\ldots\right\} \quad . (6)$$

It is convenient to write the expression $\left(\frac{E-M}{E+M}\cdot\frac{a}{a_1}\right)$ simply as $\frac{a}{a_1}$ and to bear in mind that henceforth $\frac{a}{a_1}$ is a short-hand expression for the more complicated factor in (6). We thus write

$$R=m^2n^2r^2\left(\frac{a_1}{r_1}\right)^3\left\{P_2+\frac{a}{a_1}\cdot\frac{r}{a}\cdot\frac{a_1}{r_1}\cdot P_3+\left(\frac{a}{a_1}\right)^2\left(\frac{r}{a}\right)^2\left(\frac{a_1}{r_1}\right)^2P_4+\ldots\right\} \quad \ldots.(7)$$

Within the brackets, the order of the coefficient of P_3 is that of $\frac{a}{a_1}$, since $\frac{r}{a}$ and $\frac{a_1}{r_1}$ are approximately unity; further $\frac{a}{a_1}$ is the ratio of the sine of the Sun's parallax to the sine of the Moon's parallax, if we associate a with the semi-axis of the Moon's elliptic orbit or, very approximately, the ratio of the solar and lunar parallaxes. It is convenient to refer to the term in P_3 as the *parallax-term*.

7·05. Order of magnitude of e, e_1, γ, a/a_1 and m

The small quantities occurring in the expansion of R have the approximate values as follows:

$$e=\frac{1}{20} \quad , \quad e_1=\frac{1}{60} \quad , \quad \gamma\equiv\tan i=\frac{1}{11} \quad , \quad m=\frac{1}{13} \text{ and } \frac{a}{a_1}=\frac{1}{400} \quad .$$

We regard the first four as of the same order of smallness and a/a_1 as of the second order. We shall omit the terms in R beyond that in P_3 in the sequel and write, for convenience,

$$R=R_2+R_3 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where

$$R_2=m^2n^2r^2\left(\frac{a_1}{r_1}\right)^3P_2 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and

$$R_3=m^2n^2r^2\frac{a}{a_1}\cdot\frac{r}{a}\cdot\left(\frac{a_1}{r_1}\right)^4P_3 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Accordingly R_2 is of order two owing to the presence of the factor m^2, and R_3 is of order four owing to the presence of the factors m^2 and $\frac{a}{a_1}$.

7·06. Expansion of the lunar disturbing function

(i) *In terms of r, λ and s*

Fig. 17 represents the celestial sphere with the Earth, E, as centre; the great circle ΥNT, with its pole at K, corresponds to the reference plane (the ecliptic) defined in section 7·03; Υ is the fixed point in this plane from which longitudes are measured; M is the Moon's position and T is the point on the reference plane corresponding to the line through E drawn parallel to CS in Fig. 16; also S is the *geocentric* position of the Sun on the sphere.

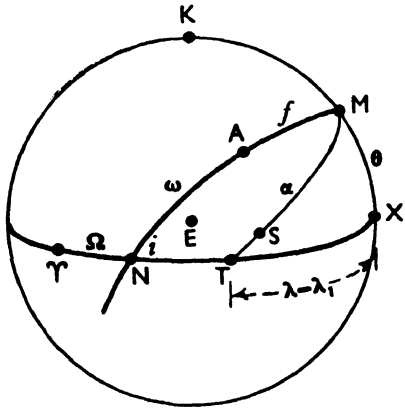


FIG. 17

N is the node of the Moon's instantaneous orbit the inclination of which is i. The Moon's longitude on the reference plane is ΥX, denoted by λ, and its latitude is XM, denoted by θ. The longitude of T is λ_1; hence $TX = \lambda - \lambda_1$. Also, the arc MT is α, as defined in Fig. 16.

From the right-angled triangle MTX, we have

$$\cos\alpha \equiv \cos(\lambda - \lambda_1)\cos\theta \quad .$$

We write

$$s = \tan\theta; \quad \ldots\ldots(1)$$

then, with our notation in 7·02,

$$c \equiv \cos\alpha = \cos(\lambda - \lambda_1)/\sqrt{(1+s^2)} \quad . \quad \ldots\ldots(2)$$

This equation enables us to express the Legendre coefficients P_2, P_3 ... in terms of λ, λ_1 and s.

Also, from the triangle MNX, $\tan\theta = \tan i \sin(\lambda - \Omega)$

or

$$s = \gamma \sin(\lambda - \Omega) \quad \ldots\ldots(3)$$

where

$$\gamma = \tan i \quad \ldots\ldots(4)$$

Thus s is of order γ. We can then expand c in (2) as a series in powers of s and, as a further step, obtain P_2 and P_3 expressed in the same way. For our purposes it will be sufficient to neglect powers of s beyond s^3. We then find that R_2 and R_3, defined by 7·05 (2) and (3), are given by

$$R_2 = \tfrac{1}{2}m^2n^2r^2\left(\frac{a_1}{r_1}\right)^3\{1 - 3s^2 + 3(1-s^2)\cos 2(\lambda - \lambda_1)\} \quad \ldots\ldots(5)$$

and

$$R_3=\frac{m^2n^2r^3}{8a}\left(\frac{a}{a_1}\right)\left(\frac{a_1}{r_1}\right)^4$$
$$\times\{3(1-\tfrac{11}{2}s^2)\cos(\lambda-\lambda_1)+5(1-\tfrac{3}{2}s^2)\cos 3(\lambda-\lambda_1)\} \quad \ldots\ldots(6)$$

In the undisturbed orbit, to $O(e^3)$, r and the true anomaly f are given by 3·11 (14) and 3·10 (11) respectively as follows:

$$r/a=1+\tfrac{1}{2}e^2-(e-\tfrac{3}{8}e^3)\cos M-\tfrac{1}{2}e^2\cos 2M-\tfrac{3}{8}e^3\cos 3M \quad ,\ldots\ldots(7)$$
$$f=M+(2e-\tfrac{1}{4}e^3)\sin M+\tfrac{5}{4}e^2\sin 2M+\tfrac{13}{12}e^3\sin 3M\ ;\ \ldots\ldots(8)$$

in these equations, M is the mean anomaly given by

$$M=nt+\epsilon-\varpi \quad \ldots\ldots(9)$$

(ii) *In terms of the elements*

From $\triangle MNX$ (Fig. 17), $\tan(\lambda-\Omega)=\cos i\tan(f+\omega)$. Set, for simplicity, $V=\lambda-\Omega$, $F=f+\omega$, $x=\tan\frac{1}{2}i$; then

$$\tan V=(1-x^2)\tan F/(1+x^2) \quad , \quad \text{whence}$$
$$\frac{e^{2iV}-1}{e^{2iV}+1}=\frac{1-x^2}{1+x^2}\cdot\frac{e^{2iF}-1}{e^{2iF}+1}$$

in which $i=\sqrt{(-1)}$. A further step gives

$$e^{2i(V-F)}=(1+x^2e^{-2iF})/(1+x^2e^{2iF}) \quad .$$

Take logarithms of both sides and expand in powers of x; then,

$$V=F-x^2\sin 2F+\tfrac{1}{2}x^4\sin 4F-\ldots \quad \ldots\ldots(10)$$

Now, $\gamma\equiv\tan i=2x/(1-x^2)$ from which, to $O(\gamma^4)$, $x^2=\frac{1}{4}\gamma^2-\frac{1}{8}\gamma^4$. To $O(\gamma^3)$, (10) becomes

$$\lambda-\Omega=f+\omega-\tfrac{1}{4}\gamma^2\sin 2(f+\omega) \quad .$$

Since $\omega=\varpi-\Omega$ we then obtain from (8), to the third order,

$$\lambda=nt+\epsilon+(2e-\tfrac{1}{4}e^3)\sin M+\tfrac{5}{4}e^2\sin 2M+\tfrac{13}{12}e^3\sin 3M$$
$$-\tfrac{1}{4}\gamma^2\sin 2\eta-\tfrac{1}{2}e\gamma^2\sin(M+2\eta)-\tfrac{1}{2}e\gamma^2\sin(M-2\eta) \quad \ldots\ldots(11)$$

where
$$\eta=nt+\epsilon-\Omega \quad . \quad \ldots\ldots(12)$$

Similarly, to $O(e_1^2)$, for the Sun's orbit relative to C in Fig. 16—or for T in Fig. 16—

$$r_1=a_1(1-e_1\cos M_1+\tfrac{1}{2}e_1^2-\tfrac{1}{2}e_1^2\cos 2M_1) \quad , \quad \ldots\ldots(13)$$

and
$$\lambda_1=n_1t+\epsilon_1+2e_1\sin M_1+\tfrac{5}{4}e_1^2\sin 2M_1 \quad . \quad \ldots\ldots(14)$$

From (3) and (11), to the third order,

$$s=(1-e^2-\tfrac{1}{8}\gamma^2)\gamma\sin\eta+e\gamma\sin(M+\eta)+e\gamma\sin(M-\eta)$$
$$+\tfrac{1}{2}e^2\gamma\sin(2M-\eta)+\tfrac{9}{8}e^2\gamma\sin(2M+\eta)-\tfrac{1}{8}\gamma^3\sin 3\eta \quad . \quad \ldots(15)$$

It is to be remarked that r_1 and λ_1 differ by very small quantities from the *geocentric* radius vector, Δ_1, and longitude, λ', of the Sun. It is easily seen from Fig. 16 that $\Delta_1\doteqdot r_1(1+\kappa\cos\alpha)$, where $\kappa=\rho/r_1$; now,

$M/E = 1/81\frac{1}{2}$, $r \doteqdot 240{,}000$ miles ; hence $\rho \doteqdot 2900$ miles and $\kappa \doteqdot 3 \,.\, 10^{-5} \doteqdot 6''$. If $\Delta\alpha$ denotes angle ESC in Fig. 16, then $\Delta\alpha = \kappa \sin \alpha$. In Fig. 17, $MT = \alpha$, $ST = \Delta\alpha$; if ψ denotes angle MTX, then

$$\lambda' - \lambda_1 = \Delta\alpha \cos\psi = \kappa \sin\alpha \cos\psi = -\kappa \sin(\eta - \eta_1) \quad ,$$

if we neglect γ^2 and the eccentricities.

Also, the Sun's latitude θ' is given by

$$\theta' = \Delta\alpha \sin\psi = \kappa \sin\alpha \sin\psi = \kappa\gamma \sin\eta \quad ,$$

in which $\kappa\gamma \doteqdot 3 \,.\, 10^{-6} \doteqdot \frac{3}{5}''$.

(iii) Let ξ be defined by

$$\xi = (n - n_1)t + \epsilon - \epsilon_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Then, from (11) and (14),

$$\begin{aligned}\lambda - \lambda_1 = \xi &+ 2(e \sin M - e_1 \sin M_1) \\ &+ \tfrac{5}{4}(e^2 \sin 2M - {e_1}^2 \sin 2M_1) - \tfrac{1}{4}\gamma^2 \sin 2\eta \quad . \quad \ldots\ldots\ldots\ldots (17)\end{aligned}$$

When (7), (13), (15) and (17) are substituted in (5) and (6) the sections R_2 and R_3 of the disturbing function are given by

$$\begin{aligned}R_2 = m^2n^2a^2 \,[&\tfrac{1}{4} + \tfrac{3}{8}(e^2 + {e_1}^2 - \gamma^2) + (\tfrac{3}{4} - \tfrac{15}{8}e^2 - \tfrac{15}{8}{e_1}^2 - \tfrac{3}{8}\gamma^2)\cos 2\xi \\ &+ e\{\tfrac{3}{4}\cos(2\xi + M) - \tfrac{1}{2}\cos M - \tfrac{9}{4}\cos(2\xi - M)\} \\ &+ e_1\{\tfrac{21}{8}\cos(2\xi - M_1) + \tfrac{3}{4}\cos M_1 - \tfrac{3}{8}\cos(2\xi + M_1)\} \\ &+ e^2\{\tfrac{15}{8}\cos(2\xi - 2M) + \tfrac{3}{4}\cos(2\xi + 2M) - \tfrac{1}{8}\cos 2M\} \\ &+ {e_1}^2\{\tfrac{51}{8}\cos(2\xi - 2M_1) + \tfrac{9}{8}\cos 2M_1\} \\ &+ ee_1\{\tfrac{9}{8}\cos(2\xi - M + M_1) + \tfrac{21}{8}\cos(2\xi + M - M_1) - \tfrac{3}{4}\cos(M + M_1) \\ &\quad - \tfrac{3}{8}\cos(2\xi + M + M_1) - \tfrac{3}{4}\cos(M - M_1) - \tfrac{63}{8}\cos(2\xi - M - M_1)\} \\ &+ \tfrac{3}{8}\gamma^2\{\cos(2\xi - 2\eta) + \cos 2\eta\}] \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)\end{aligned}$$

and, with sufficient accuracy for our purpose,

$$R_3 = \frac{m^2n^2}{8a}\left(\frac{a}{a_1}\right)\{3\cos\xi + 5\cos 3\xi\} \quad . \quad \ldots\ldots\ldots\ldots (19)$$

If the expansion of R to higher orders in e, e_1 and γ is required, it is readily seen that R can be expressed as a series with general terms of the form

$$Ar^p \cos qf \,.\, r_1^{-p_1} \cos q_1 f_1 + Br^p \sin qf \,.\, r_1^{-p_1} \sin q_1 f_1$$

where p, p_1, q and q_1 are positive integers such that $p > 1$ and $p_1 > 2$. Then, by 3·09 (15), $r^p \cos qf$ for example can be expressed as a series of the form $\Sigma A_j \cos jE$; in a second step $\cos jE$ can be transformed into a series of the form $\Sigma B_k \cos kM$ by 3·11 (7). The final result will be a series of the form $\Sigma C \cos\theta$, where C is a function of a, a_1, e, e_1 and γ and

$$\theta = iM + i_1M_1 + j\Omega + j_1\Omega_1 + k\varpi + k_1\varpi_1 \,;$$

in this expression for θ, i, i_1, j, j_1, k, k_1 are positive or negative integers one, two, ... all of which can be zero.

7·07. The planetary disturbing function

In the lunar problem, as we have just seen, the expansion of the disturbing function is effected by first developing in powers of r/r_1 and then continuing the development in terms of e, e_1 and γ, γ_1. In the planetary problem r/r_1—or, effectively, a/a_1—may be a considerable fraction; for example, for Venus and the Earth, this fraction is 0·72 and for Jupiter and Saturn it is approximately 0·5. It is evident that, to obtain a degree of accuracy comparable with that of observations, a very large number of terms in series depending on powers of a/a_1—or a_1/a if $a>a_1$—must be taken. The procedure, then, is, *first* to develop R in terms of the eccentricities and inclinations which we shall suppose to be quantities of the first order of smallness—usually of order 0·1 at most—and *then* to deal with the expansions in terms of a/a_1 or a_1/a.

In terms of the heliocentric coordinates, Δ is given by

$$\Delta^2=(x-x_1)^2+(y-y_1)^2+(z-z_1)^2$$

in which z/a and z/a_1 are of $O(\gamma)$ and $O(\gamma_1)$ respectively.

In Fig. 17 let M now refer to the planet P with ecliptic longitude $\lambda(\equiv NX)$ and latitude $\theta(\equiv XM)$; λ_1 and θ_1 refer in the same way to the planet P_1.

Let ρ and ρ_1 be the projections of r and r_1 on the ecliptic; then

$$x^2+y^2=\rho^2 \quad , \quad x_1^2+y_1^2=\rho_1^2 \quad , \quad xx_1+yy_1=\rho\rho_1\cos(\lambda-\lambda_1) \quad .$$

Let $$\lambda-\lambda_1=\sigma \quad , \quad \Delta_1^2=\rho^2-2\rho\rho_1\cos\sigma+\rho_1^2 \quad . \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Then, $$\Delta^2=\Delta_1^2+(z-z_1)^2 \; ;$$

hence $$\frac{1}{\Delta}=\frac{1}{\Delta_1}-\frac{1}{2}\frac{(z-z_1)^2}{\Delta_1^3}+\frac{3}{8}\frac{(z-z_1)^4}{\Delta_1^5}-\ldots \quad . \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Again, the second part of R is

$$-Gm_1\frac{(xx_1+yy_1+zz_1)}{r_1^3}=-Gm_1\frac{(\rho\rho_1\cos\sigma+zz_1)}{(\rho_1^2+z_1^2)^{3/2}}$$

$$=-Gm_1\left[\frac{\rho\cos\sigma}{\rho_1^2}+\frac{zz_1}{\rho_1^3}-\frac{3}{2}\frac{\rho z_1^2\cos\sigma}{\rho_1^4}-\frac{3}{2}\frac{zz_1^3}{\rho_1^5}+\frac{15}{8}\frac{\rho z_1^4\cos\sigma}{\rho_1^6}-\ldots\right] \quad . \ldots(3)$$

The expansions (2) and (3) can be taken to any order in the small quantities z/a and z_1/a_1. In the sequel we shall develop R to the second order of small quantities and, accordingly, we have from (2) and (3)

$$\frac{R}{Gm_1}=\frac{1}{\Delta_1}-\frac{\rho\cos\sigma}{\rho_1^2}-\frac{1}{2}\frac{(z-z_1)^2}{\Delta_1^3}-\frac{zz_1}{\rho_1^3}+\frac{3}{2}\frac{\rho z_1^2\cos\sigma}{\rho_1^4} \quad . \quad \ldots\ldots\ldots(4)$$

7·08. Expansions of ρ, σ and z

(i) From Fig. 17 (p. 100), $\sin\theta=\sin i\sin(L-\Omega)$, where L is the true longitude of the planet, $\Upsilon N+NM$. Now, to the second order,

$$\rho=r\cos\theta=r-\tfrac{1}{2}r\sin^2 i\sin^2(L-\Omega) \quad .$$

In the second term we can replace r by a, L by $nt+\epsilon$ and $\sin i\,(\equiv \tan i \cos i)$ by γ without affecting the order of magnitude of the term; thus, to $O(\gamma^2)$,

$$\rho = r - \tfrac{1}{2}a\gamma^2 \sin^2(nt+\epsilon-\Omega) \equiv r - \tfrac{1}{2}a\gamma^2 \sin^2\eta.$$

Then, using 7·06 (7) we have, to $O(e^2)$,

$$\rho/a = 1 + \tfrac{1}{2}e^2 - \tfrac{1}{4}\gamma^2 - e\cos M - \tfrac{1}{2}e^2\cos 2M + \tfrac{1}{4}\gamma^2\cos 2\eta \quad . \quad \ldots\ldots(1)$$

(ii) The formula for λ is given by 7·06 (11) to the second order; the formula for λ_1 is given by a similar equation. Then $\sigma\,(\equiv \lambda - \lambda_1)$ is given by

$$\sigma = nt+\epsilon-(n_1t+\epsilon_1) + 2e\sin M - 2e_1\sin M_1 + \tfrac{5}{4}e^2\sin 2M - \tfrac{5}{4}e_1^2\sin 2M_1 - \tfrac{1}{4}\gamma^2\sin 2\eta + \tfrac{1}{4}\gamma_1^2\sin 2\eta_1. \quad \ldots\ldots(2)$$

(iii) Since $z = r\sin\theta$, then $z = r\sin i\sin(L-\Omega)$.
Further, as R in 7·07 (4) contains z^2, zz_1 and z_1^2, we need to find the expression for z only to the first order. Hence

$$z = a\gamma\sin\eta \quad . \quad \ldots\ldots(3)$$

Similarly,

$$z_1 = a_1\gamma_1\sin\eta_1 \quad . \quad \ldots\ldots(4)$$

7·09. Method of expanding R

We write

$$\rho = a(1+u) \quad , \quad \rho_1 = a_1(1+u_1) \quad \ldots\ldots(1)$$

$$\sigma = \phi + v \quad \ldots\ldots(2)$$

where*

$$\phi = nt+\epsilon-(n_1t+\epsilon_1) \quad . \quad \ldots\ldots(3)$$

From (1) and (2) of the previous section it is evident that u, u_1 and v are all of the first order of smallness; the actual expressions for u, u_1, v in terms of e, e_1, γ and γ_1 can be picked out from the two formulae just referred to.

Now, in 7·07 (4), R is a function of ρ, ρ_1, σ, z and z_1, and if we write R_0 for the disturbing function when u, u_1, v, z and z_1 are all zero, then from (1), for example,

$$\left(\frac{\partial R}{\partial \rho}\right) = \frac{\partial R_0}{\partial a} \quad ,$$

where $\left(\dfrac{\partial R}{\partial \rho}\right)$ denotes the value of $\dfrac{\partial R}{\partial \rho}$ when u, u_1, v, z and z_1 are all put equal to zero after the differentiation has been completed. Similarly, for example, $\left(\dfrac{\partial^2 R}{\partial\rho\,\partial\sigma}\right) = \dfrac{\partial^2 R_0}{\partial a\,\partial\phi}$.

On expanding R by Taylor's theorem we then have, to the second order of small quantities,

$$R = R_0 + au\frac{\partial R_0}{\partial a} + a_1u_1\frac{\partial R_0}{\partial a_1} + v\frac{\partial R_0}{\partial \phi}$$

* In lunar theory, as we have seen in 7·06 (16), ξ is written for ϕ.

$$+\tfrac{1}{2}(au)^2\frac{\partial^2 R_0}{\partial a^2}+\tfrac{1}{2}(a_1u_1)^2\frac{\partial^2 R_0}{\partial a_1^2}+\tfrac{1}{2}v^2\frac{\partial^2 R_0}{\partial \phi^2}$$

$$+(au)(a_1u_1)\frac{\partial^2 R_0}{\partial a\,\partial a_1}+(au)v\frac{\partial^2 R_0}{\partial a\,\partial \phi}+(a_1u_1)v\frac{\partial^2 R_0}{\partial a_1\,\partial \phi} \quad \ldots\ldots(4)$$

together with the terms in z and z_1.

By definition,
$$\frac{R_0}{Gm_1}=\frac{1}{\Delta_0}-\frac{a\cos\phi}{a_1^2} \quad \ldots\ldots(5)$$

where
$$\Delta_0^2=a^2-2aa_1\cos\phi+a_1^2 \quad \ldots\ldots(6)$$

The terms of $\dfrac{R}{Gm_1}$ depending on z and z_1 are of the second order and can now be written, from 7·07 (4), as

$$-\frac{1}{2}\frac{(z-z_1)^2}{\Delta_0^3}-\frac{zz_1}{a_1^3}+\frac{3}{2}\frac{az_1^2\cos\phi}{a_1^4} \quad \ldots\ldots(7)$$

We shall expand $1/\Delta_0$—the first term in (5)—as a Fourier series of the form

$$\tfrac{1}{2}B_0+\sum_{n=1}^{\infty}B_n\cos n\phi \quad \ldots\ldots(8)$$

Similarly, the factor $1/\Delta_0^3$ occurring in the first term of (7) will be represented by a series of the form

$$\tfrac{1}{2}C_0+\sum_{n=1}^{\infty}C_n\cos n\phi \quad \ldots\ldots(9)$$

Further, when we form $\dfrac{\partial R_0}{\partial a}$ and $\dfrac{\partial^2 R_0}{\partial a^2}$, for example, we shall obtain expressions involving $1/\Delta_0^3$ and $1/\Delta_0^5$. Again, by referring to (4) and (6), it is evident that if we require the expansion of R to higher orders of approximation than the second, we shall have factors $1/\Delta_0^7$, $1/\Delta_0^9$, etc., which can be expressed as series of the form shown in (8) or (9).

We shall suppose that $a>a_1$ and we write

$$\frac{a_1}{a}=\alpha \quad \ldots\ldots(10)$$

and
$$D=1-2\alpha\cos\phi+\alpha^2 \quad \ldots\ldots(11)$$

Then
$$\frac{1}{\Delta_0}=\frac{1}{aD^{1/2}}\ ,\quad \frac{1}{\Delta_0^3}=\frac{1}{a^3D^{3/2}}\ ,\text{ etc.} \quad \ldots\ldots(12)$$

It is convenient to deal with the expansion of the function

$$(1-2\alpha\cos\phi+\alpha^2)^{-s}\ ,$$

s being of the form $n+\frac{1}{2}$ where n is a positive integer including zero.

Before dealing with this particular problem in the next section we collect here for reference the expansions, to the second order, of the

quantities u, u_1, v, u^2, ... u_1v appearing as factors in the expansion of R given by (4).

$$u = -e\cos M + \tfrac{1}{2}e^2 - \tfrac{1}{4}\gamma^2 - \tfrac{1}{2}e^2\cos 2M + \tfrac{1}{4}\gamma^2\cos 2\eta\ ; \quad \text{.....(13)}$$

$$u_1 = -e_1\cos M_1 + \tfrac{1}{2}e_1{}^2 - \tfrac{1}{4}\gamma_1{}^2 - \tfrac{1}{2}e_1{}^2\cos 2M_1 + \tfrac{1}{4}\gamma_1{}^2\cos 2\eta_1\ ; \quad \text{...(14)}$$

$$u^2 = \tfrac{1}{2}e^2 + \tfrac{1}{2}e^2\cos 2M\ ; \quad \text{.....(15)}$$

$$u_1{}^2 = \tfrac{1}{2}e_1{}^2 + \tfrac{1}{2}e_1{}^2\cos 2M_1\ ; \quad \text{.....(16)}$$

$$uu_1 = \tfrac{1}{2}ee_1\cos(\phi - \varpi + \varpi_1) + \cos(M + M_1)\ ; \quad \text{.....(17)}$$

$$v = 2e\sin M - 2e_1\sin M_1 + \tfrac{5}{4}e^2\sin 2M - \tfrac{5}{4}e_1{}^2\sin 2M_1 - \tfrac{1}{4}\gamma^2\sin 2\eta + \tfrac{1}{4}\gamma_1{}^2\sin 2\eta_1\ ; \quad \text{.....(18)}$$

$$uv = -e^2\sin 2M - ee_1\sin(\phi - \varpi + \varpi_1) + ee_1\sin(M + M_1)\ ; \quad \text{.....(19)}$$

$$u_1v = e_1{}^2\sin 2M_1 - ee_1\sin(\phi - \varpi + \varpi_1) - ee_1\sin(M + M_1)\ ; \quad \text{.....(20)}$$

and, for reference later,

$$z_1{}^2 = \tfrac{1}{2}a_1{}^2\gamma_1{}^2 - \tfrac{1}{2}a_1{}^2\gamma_1{}^2\cos 2\eta_1\ ; \quad \text{.....(21)}$$

$$zz_1 = \tfrac{1}{2}aa_1\gamma\gamma_1\cos(\phi - \Omega + \Omega_1) - \tfrac{1}{2}aa_1\gamma\gamma_1\cos(\eta + \eta_1)\ ; \quad \text{.....(22)}$$

$$(z - z_1)^2 = \tfrac{1}{2}a^2\gamma^2 + \tfrac{1}{2}a_1{}^2\gamma_1{}^2 - \tfrac{1}{2}a^2\gamma^2\cos 2\eta - \tfrac{1}{2}a_1{}^2\gamma_1{}^2\cos 2\eta_1 \quad \text{.....(23)}$$

Inserting these in (4) and assuming that we have expressed

$$\frac{\partial R_0}{\partial a}\ , \ \ldots\ \frac{\partial^2 R_0}{\partial a_1\,\partial\phi}$$

as periodic series in ϕ, we then derive, by direct multiplication of such quantities as u and $\dfrac{\partial R_0}{\partial a}$, the expansion of R as a periodic series as far as second order terms in the eccentricities and inclinations.

7·10. Expansion of $(1 - 2\alpha\cos\phi + a^2)^{-s}$

Let $z = e^{i\phi}$; then $2\cos\phi = z + z^{-1}$,

and
$$D = (1 - \alpha z)(1 - \alpha z^{-1})\ ; \quad \text{.....(1)}$$

hence
$$D^{-s} \equiv (1 - 2\alpha\cos\phi + \alpha^2)^{-s} = (1 - \alpha z)^{-s}(1 - \alpha z^{-1})^{-s} \quad \text{.....(2)}$$

$$= \left(1 + s\,.\,\alpha z + \frac{s\,.\,s+1}{2!}\alpha^2 z^2 + \frac{s\,.\,s+1\,.\,s+2}{3!}\alpha^3 z^3 + \ldots\right) \times \left(1 + s\,.\,\alpha z^{-1} + \frac{s\,.\,s+1}{2!}\alpha^2 z^{-2} + \frac{s\,.\,s+1\,.\,s+2}{3!}\alpha^3 z^{-3} + \ldots\right)$$

or
$$D^{-s} = \tfrac{1}{2}B_0^s + \tfrac{1}{2}\sum_{n=1}^{\infty} B_n^s(z^n + z^{-n}) \quad , \quad \text{.....(3)}$$

where
$$\tfrac{1}{2}B_0^s = 1 + s^2\alpha^2 + \frac{(s\,.\,s+1)^2}{(2!)^2}\alpha^4 + \quad \text{.....(4)}$$

and
$$\tfrac{1}{2}B_n^s = \frac{s\,.\,s+1\,.\,\ldots\,.\,s+n-1}{n!}\alpha^n \times \left\{1 + \frac{s}{1}\cdot\frac{s+n}{n+1}\alpha^2 + \frac{s\,.\,s+1}{1\,.\,2}\cdot\frac{s+n\,.\,s+n+1}{n+1\,.\,n+2}\alpha^4 + \ldots\right\} \quad \text{..(5)}$$

Since $\alpha < 1$, the series in (4) and (5) are clearly convergent.

It is evident that (3) can be written in the form

$$D^{-s}=\tfrac{1}{2}\sum_{n=-\infty}^{\infty} B_n^s z^n \quad \ldots\ldots(6)$$

in which n takes all integral values positive or negative, including zero; it is also evident that $B_{-n}^s=B_n^s$.

From (3) we have

$$D^{-s}=\tfrac{1}{2}B_0^s+\sum_{n=1}^{\infty} B_n^s \cos n\phi \quad \ldots\ldots(7)$$

which is the required expansion in terms of ϕ.

The coefficients B_n^s are called Laplace's coefficients, and from (5) it is seen that they are all positive.

It is seen from (5) that

$$\tfrac{1}{2}B_n^s=\frac{s\,.\,s+1\,\ldots\,s+n-1}{n!}\alpha^n F(s,\ s+n,\ n+1\ ;\ \alpha^2) \quad \ldots\ldots(8)$$

or, by means of 3·04 (2),

$$\tfrac{1}{2}B_n^s=\frac{s\,.\,s+1\,\ldots\,s+n-1}{n!}\cdot\frac{\alpha^n}{(1-\alpha^2)^s}F\left(s,\ 1-s,\ n+1\ ;\ -\frac{\alpha^2}{1-\alpha^2}\right) \ldots(9)$$

The expansion of (2) can be written down immediately from 3·09 (12) by writing $q=0$, $-s$ for p and α for β.

7·11. Expression of Laplace's coefficients as integrals

From (7) of the previous section we obtain, by multiplying throughout by $\cos n\phi$ and integrating between the limits 0 and π,

$$B_n^s=\frac{2}{\pi}\int_0^{\pi}\frac{\cos n\phi\, d\phi}{(1-2\alpha\cos\phi+\alpha^2)^s} \quad \ldots\ldots(1)$$

In particular, when $s=\frac{1}{2}$,

$$B_n^{1/2}=\frac{2}{\pi}\int_0^{\pi}\frac{\cos n\phi\, d\phi}{(1-2\alpha\cos\phi+\alpha^2)^{1/2}} \quad \ldots\ldots(2)$$

Now from 7·10 (5) we obtain, on writing $s=\frac{1}{2}$,

$$\tfrac{1}{2}B_n^{1/2}=\left\{\frac{1\,.\,3\,\ldots\,2n-1}{2\,.\,4\,\ldots\,2n}+\frac{1}{2}\cdot\frac{1\,.\,3\,\ldots\,2n+1}{2\,.\,4\,\ldots\,2n+2}\alpha^2+\ldots\right\}\alpha^n$$

and, noting the well-known formula

$$\frac{1}{\pi}\int_0^{\pi}\sin^{2n}\phi\, d\phi=\frac{1\,.\,3\,\ldots\,2n-1}{2\,.\,4\,\ldots\,2n} \quad ,$$

we derive

$$\frac{\pi}{2}B_n^{1/2}=\alpha^n\int_0^{\pi}\sin^{2n}\phi\,\{1+\tfrac{1}{2}\sin^2\phi+\tfrac{3}{8}\sin^4\phi+\ldots\}\,d\phi$$

so that

$$B_n^{1/2}=\frac{2}{\pi}\alpha^n\int_0^{\pi}\frac{\sin^{2n}\phi\, d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} \quad \ldots\ldots(3)$$

From (2) and (3), as a matter of interest, we have

$$\int_0^\pi \frac{\cos n\phi\, d\phi}{\sqrt{(1-2\alpha\cos\phi+\alpha^2)}} = \alpha^n \int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} \quad \ldots\ldots\ldots\ldots(4)$$

From (3), when $n=0$,

$$B_0^{1/2} = \frac{2}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} = \frac{4}{\pi}F(\alpha) \quad , \quad \ldots\ldots\ldots\ldots(5)$$

where

$$F(\alpha) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} \quad , \quad \ldots\ldots\ldots\ldots(6)$$

which is the elliptic integral of the first kind. Again, when $n=1$,

$$\begin{aligned} B_1^{1/2} &= \frac{4\alpha}{\pi}\int_0^{\pi/2} \frac{\sin^2\phi\, d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} \\ &= \frac{4}{\pi\alpha}\left\{\int_0^{\pi/2} \frac{d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} - \int_0^{\pi/2} \sqrt{(1-\alpha^2\sin^2\phi)}\, d\phi\right\} \\ &= \frac{4}{\pi\alpha}\{F(\alpha)-E(\alpha)\} \quad , \quad \ldots\ldots\ldots\ldots(7) \end{aligned}$$

where $E(\alpha)$ is the elliptic integral of the second kind.

The numerical values of $E(\alpha)$, $F(\alpha)$ can be found from tables of these functions, a bibliography of which is given in *An Index of Mathematical Tables* by Fletcher, Miller and Rosenhead (London, 1946).

7·12. Difference formulae for Laplace's coefficients

(i) We have, from 7·10 (6), on writing $z+z^{-1}$ for $2\cos\phi$ in the expression for D,

$$\{1+\alpha^2-\alpha(z+z^{-1})\}^{-s} = \tfrac{1}{2}\sum_{-\infty}^{\infty} B_n^s z^n \quad \ldots\ldots\ldots\ldots(1)$$

Differentiate with respect to z; then

$$\alpha s(1-z^{-2})\{1+\alpha^2-\alpha(z+z^{-1})\}^{-s-1} = \tfrac{1}{2}\sum_{-\infty}^{\infty} nB_n^s z^{n-1} \quad , \quad \ldots\ldots\ldots\ldots(2)$$

whence

$$\alpha s(z-z^{-1})\sum_{-\infty}^{\infty} B_n^s z^n = \{1+\alpha^2-\alpha(z+z^{-1})\}\sum_{-\infty}^{\infty} nB_n^s z^n \quad \ldots\ldots\ldots\ldots(3)$$

Equate coefficients of z^{n-1}; then

$$\alpha s\{B_{n-2}^s - B_n^s\} = (n-1)(1+\alpha^2)B_{n-1}^s - \alpha\{(n-2)B_{n-2}^s + nB_n^s\} \quad ,$$

from which

$$B_n^s = \frac{n-1}{n-s}(\alpha+\alpha^{-1})B_{n-1}^s - \frac{n+s-2}{n-s}B_{n-2}^s \quad \ldots\ldots\ldots\ldots(4)$$

Consequently, if the values of B_0^s, B_1^s are known, the values of B_2^s, B_3^s, ... can be calculated in succession; in particular, when $s=\frac{1}{2}$, the values of $B_2^{1/2}$, $B_3^{1/2}$, ... can be readily found in succession from the values of $B_0^{1/2}$, $B_1^{1/2}$ obtained from tables of the elliptic functions, as mentioned in the previous section.

(ii) We can write (2) as

$$\alpha s(1-z^{-2})\sum_{-\infty}^{\infty} B_n^{s+1} z^n = \sum_{-\infty}^{\infty} nB_n^s z^{n-1} \quad \text{.....(5)}$$

and, by equating coefficients of z^{n-1}, we find that

$$nB_n^s = \alpha s\{B_{n-1}^{s+1} - B_{n+1}^{s+1}\} \quad \text{.....(6)}$$

By writing $n+1$ for n and $s+1$ for s in (4) we have

$$\alpha B_{n+1}^{s+1} = \frac{1}{n-s}\{n(1+\alpha^2)B_n^{s+1} - \alpha(n+s)B_{n-1}^{s+1}\} \quad ,$$

and (6) becomes

$$B_n^s = \frac{s}{n-s}\{2\alpha B_{n-1}^{s+1} - (1+\alpha^2)B_n^{s+1}\} \quad \text{.....(7)}$$

Change n into $n+1$ in this formula and it becomes

$$B_{n+1}^s = \frac{s}{n-s+1}\{2\alpha B_n^{s+1} - (1+\alpha^2)B_{n+1}^{s+1}\} \quad \text{.....(8)}$$

Eliminate B_{n-1}^{s+1} and B_{n+1}^{s+1} from (6), (7) and (8) and we obtain

$$B_n^{s+1} = \frac{(n+s)(1+\alpha^2)B_n^s - 2(n-s+1)\alpha B_{n+1}^s}{s(1-\alpha^2)^2} \quad \text{.....(9)}$$

This formula enables us to calculate the value of B_n^{s+1} when the values of B_n^s and B_{n+1}^s are known.

The combination of the formulae (4) and (9) account for the complete evaluation of Laplace's coefficients when the numerical values of $B_0^{1/2}$ and $B_1^{1/2}$ have been obtained from tables of elliptic functions or by other means.

Using (4)—with $n+1$ replacing n—to eliminate B_{n+1}^s from (9) we find that the latter formula becomes

$$B_n^{s+1} = \frac{2\alpha(n+s-1)B_{n-1}^s - (n-s)(1+\alpha^2)B_n^s}{s(1-\alpha^2)^2} \quad \text{.....(10)}$$

This formula can be used instead of (9) for calculating the values of B_n^{s+1}.

Also, from (9), for values of n and $n+1$, we easily find that

$$s(1+\alpha^2)B_n^{s+1} - 2\alpha s B_{n+1}^{s+1} = (n+s)B_n^s \quad \text{.....(11)}$$

7·13. The differential coefficients of B_n^s with respect to α

(i) Differentiate 7·12 (1) with respect to α; then

$$-s(2\alpha - z - z^{-1})\sum_{-\infty}^{\infty} B_n^{s+1} z^n = \sum_{-\infty}^{\infty} \frac{dB_n^s}{d\alpha} z^n \quad ,$$

from which, on equating coefficients of z^n, we obtain

$$\frac{dB_n^s}{d\alpha} = s\{B_{n-1}^{s+1} + B_{n+1}^{s+1} - 2\alpha B_n^{s+1}\} \quad \text{.....(1)}$$

Use 7·12 (9) to change B_{n-1}^{s+1} in terms of B_{n-1}^s and B_n^s; use 7·12 (10) to change B_{n+1}^{s+1} in terms of B_n^s and B_{n+1}^s and to change B_n^{s+1} in terms of B_{n-1}^s and B_n^s; then use 7·12 (4) to change B_{n+1}^s in terms of B_{n-1}^s and B_n^s.

The formula (1) then becomes

$$\frac{dB_n^s}{d\alpha}=\frac{s}{1-\alpha^2}\left[2(n+s-1)B_{n-1}^s+\{2\alpha s-n(\alpha+\alpha^{-1})\}B_n^s\right] \quad . \quad \ldots\ldots(2)$$

Either (1) or (2) may be used to evaluate $\frac{dB_n^s}{d\alpha}$.

(ii) To find the second differential coefficient of a Laplace coefficient we start from 7·10 (7), namely,

$$(1-2\alpha\cos\phi+\alpha^2)^{-s}=\tfrac{1}{2}B_0^s+\sum_1^\infty B_n^s\cos n\phi \quad .$$

Differentiate with respect to α ; then

$$\frac{2s(\cos\phi-\alpha)}{(1-2\alpha\cos\phi+\alpha^2)^{s+1}}=\tfrac{1}{2}\frac{dB_0^s}{d\alpha}+\sum_1^\infty\frac{dB_n^s}{d\alpha}\cos n\phi \quad .$$

Differentiate again and we have

$$\frac{4s(s+1)(\cos\phi-\alpha)^2-2s(1-2\alpha\cos\phi+\alpha^2)}{(1-2\alpha\cos\phi+\alpha^2)^{s+2}}=\tfrac{1}{2}\frac{d^2B_0^s}{d\alpha^2}+\sum_1^\infty\frac{d^2B_n^s}{d\alpha^2}\cos n\phi. \quad \ldots\ldots(3)$$

The left-hand side can be written as

$$\frac{2s(2s+1)}{D^{s+1}}-\frac{4s(s+1)\sin^2\phi}{D^{s+2}} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The first term of (4) is

$$2s(2s+1)\{\tfrac{1}{2}B_0^{s+1}+\sum_1^\infty B_n^{s+1}\cos n\phi\} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Differentiate

$$D^{-s-1}\equiv\tfrac{1}{2}B_0^{s+1}+\sum_1^\infty B_n^{s+1}\cos n\phi$$

with respect to ϕ ; then

$$\frac{2\alpha(s+1)\sin\phi}{D^{s+2}}=\sum_1^\infty nB_n^{s+1}\sin n\phi \quad .$$

Multiply throughout by $\sin\phi$ and express $\sin n\phi\sin\phi$ as the difference of two cosines ; the second term of (4) becomes

$$-\frac{s}{\alpha}\sum_1^\infty nB_n^{s+1}\{\cos(n-1)\phi-\cos(n+1)\phi\}$$

or

$$-\frac{s}{\alpha}\sum_1^\infty\{(n+1)B_{n+1}^{s+1}-(n-1)B_{n-1}^{s+1}\}\cos n\phi \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Hence, on equating the coefficients of $\cos n\phi$ in (3) and using (5) and (6), we obtain—readily seen to hold also for $n=0$—

$$\frac{d^2B_n^s}{d\alpha^2}=2s(2s+1)B_n^{s+1}-\frac{s}{\alpha}\{(n+1)B_{n+1}^{s+1}-(n-1)B_{n-1}^{s+1}\} \quad . \quad \ldots\ldots(7)$$

7·14. Expansion of R

The terms of zero-order in the expansion of R, 7·09 (4), are given by

$$\frac{R_0}{Gm_1}=\frac{1}{\Delta_0}-\frac{a\cos\phi}{a_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and, expanding $1/\Delta_0 \equiv (aD)^{-1/2}$ and writing $\alpha = a_1/a$, we obtain for these terms

$$\frac{1}{2a}B_0^{1/2}+\frac{1}{a}\left(B_1^{1/2}-\frac{1}{\alpha^2}\right)\cos\phi+\frac{1}{a}\sum_{n=2}B_n^{1/2}\cos n\phi \quad \ldots\ldots\ldots\ldots(2)$$

For terms of first and second order we require $\dfrac{\partial R_0}{\partial a}$ etc.

From (1), omitting temporarily the factor $1/(G/m_1)$,

$$\frac{\partial R_0}{\partial a}=\frac{\partial}{\partial a}\left(\frac{1}{\Delta_0}\right)-\frac{1}{a_1^2}\cos\phi\,;\quad \frac{\partial R_0}{\partial a_1}=\frac{\partial}{\partial a_1}\left(\frac{1}{\Delta_0}\right)+\frac{2a}{a_1^3}\cos\phi$$

and

$$\frac{\partial R_0}{\partial \phi}=\frac{\partial}{\partial \phi}\left(\frac{1}{\Delta_0}\right)+\frac{a}{a_1^2}\sin\phi \quad .$$

Consider first $\dfrac{\partial}{\partial a}\left(\dfrac{1}{\Delta_0}\right)\equiv\dfrac{\partial}{\partial a}\left(\dfrac{D^{-1/2}}{a}\right)=\dfrac{\partial}{\partial a}\left\{\dfrac{1}{a}(\tfrac{1}{2}B_0^{1/2}+\Sigma B_n^{1/2}\cos n\phi)\right\}$.

Now the B's are functions of $\alpha(\equiv a_1/a)$ and hence

$$\frac{\partial}{\partial a}\left(\frac{1}{\Delta_0}\right)=-\frac{1}{2a^2}\left(B_0^{1/2}+\alpha\frac{dB_0^{1/2}}{d\alpha}\right)-\frac{1}{a^2}\sum_{n=1}\left\{B_n^{1/2}+\alpha\frac{dB_n^{1/2}}{d\alpha}\right\}\cos n\phi\,;$$

consequently,

$$\frac{\partial R_0}{\partial a}=-\frac{1}{2a^2}\left(B_0^{1/2}+\alpha\frac{dB_0^{1/2}}{d\alpha}\right)-\frac{1}{a^2}\left(\frac{1}{\alpha^2}+B_1^{1/2}+\alpha\frac{dB_1^{1/2}}{d\alpha}\right)\cos\phi$$
$$-\frac{1}{a^2}\sum_{n=2}\left\{B_n^{1/2}+\alpha\frac{dB_n^{1/2}}{d\alpha}\right\}\cos n\phi \quad .$$

Similarly,

$$\frac{\partial R_0}{\partial a_1}=\frac{1}{2a^2}\frac{dB_0^{1/2}}{d\alpha}+\frac{1}{a^2}\left(\frac{2}{\alpha^3}+\frac{dB_1^{1/2}}{d\alpha}\right)\cos\phi+\frac{1}{a^2}\sum_{n=2}\frac{dB_n^{1/2}}{d\alpha}\cos n\phi\,;$$

$$\frac{\partial R_0}{\partial \phi}=\frac{1}{a}\left(\frac{1}{\alpha^2}-B_1^{1/2}\right)\sin\phi-\frac{1}{a}\sum_{n=2}nB_n^{1/2}\sin n\phi\,;$$

$$\frac{\partial^2 R_0}{\partial a^2}=\frac{1}{2a^3}\left\{2B_0^{1/2}+4\alpha\frac{dB_0^{1/2}}{d\alpha}+\alpha^2\frac{d^2B_0^{1/2}}{d\alpha^2}\right\}$$
$$+\frac{1}{a^3}\sum_{n=1}\left\{2B_n^{1/2}+4\alpha\frac{dB_n^{1/2}}{d\alpha}+\alpha^2\frac{d^2B_n^{1/2}}{d\alpha^2}\right\}\cos n\phi\,;$$

$$\frac{\partial^2 R_0}{\partial a_1^2}=\frac{1}{2a^3}\frac{d^2B_0^{1/2}}{d\alpha^2}-\frac{1}{a^3}\left(\frac{6}{\alpha^4}-\frac{d^2B_1^{1/2}}{d\alpha^2}\right)\cos\phi+\frac{1}{a^3}\sum_{n=2}\frac{d^2B_n^{1/2}}{d\alpha^2}\cos n\phi\,;$$

$$\frac{\partial^2 R_0}{\partial \phi^2}=\frac{1}{a}\left(\frac{1}{\alpha^2}-B_1^{1/2}\right)\cos\phi-\frac{1}{a}\sum_{n=2}n^2B_n^{1/2}\cos n\phi\,;$$

$$\frac{\partial^2 R_0}{\partial a\,\partial a_1}=-\frac{1}{2a^3}\left(2\frac{dB_0^{1/2}}{d\alpha}+\alpha\frac{d^2B_0^{1/2}}{d\alpha^2}\right)+\frac{1}{a^3}\left(\frac{2}{\alpha^3}-2\frac{dB_1^{1/2}}{d\alpha}-\alpha\frac{d^2B_1^{1/2}}{d\alpha^2}\right)\cos\phi$$
$$-\frac{1}{a^3}\sum_{n=2}\left(2\frac{dB_n^{1/2}}{d\alpha}+\alpha\frac{d^2B_n^{1/2}}{d\alpha^2}\right)\cos n\phi\,;$$

$$\frac{\partial^2 R_0}{\partial a\,\partial\phi} = \frac{1}{a^2}\left(\frac{1}{\alpha^2} + B_1^{1/2} + \alpha\frac{dB_1^{1/2}}{d\alpha}\right)\sin\phi + \frac{1}{a^2}\sum_{n=2}\left\{nB_n^{1/2} + n\alpha\frac{dB_n^{1/2}}{d\alpha}\right\}\sin n\phi\,;$$

$$\frac{\partial^2 R_0}{\partial a_1\,\partial\phi} = -\frac{1}{a^2}\left(\frac{2}{\alpha^3} + \frac{dB_1^{1/2}}{d\alpha}\right)\sin\phi - \frac{1}{a^2}\sum_{n=2} n\frac{dB_n^{1/2}}{d\alpha}\sin n\phi.$$

With the values of u, u_1, ... $u_1 v$ given in section 7·09 we can now express $au\cdot\dfrac{\partial R_0}{\partial a}$, ... $a_1u_1\cdot v\dfrac{\partial^2 R_0}{\partial a_1\,\partial\phi}$ as periodic series up to the second order in e, e_1, γ and γ_1 and up to a sufficient order in α depending on the numerical value of this quantity.

Consider now the terms of the disturbing function involving z, z_1; these terms are, from 7·09 (7),

$$-\frac{1}{2}\frac{(z-z_1)^2}{\Delta_0^3} - \frac{zz_1}{a_1^3} + \frac{3}{2}\frac{az_1^2}{a_1^3}\cos\phi \quad \text{.....(3)}$$

The first term is

$$-\frac{1}{2a^3}(z-z_1)^2\left[\tfrac{1}{2}B_0^{3/2} + B_1^{3/2}\cos\phi + \sum_{n=2} B_n^{3/2}\cos n\phi\right],$$

and, when $(z-z_1)^2$ is substituted from 7·09 (23) and products of cosines converted into sums of cosines, the result is

$$-\frac{1}{8a}(\gamma^2+\alpha^2\gamma_1^2)B_0^{3/2} + \frac{\alpha}{4a}\gamma\gamma_1 B_1^{3/2}\cos(\Omega-\Omega_1) \quad \text{.....(4)}$$

+ periodic terms of considerable variety.

The second term in (3) contributes the terms

$$-\frac{1}{2a}\alpha\gamma\gamma_1[\cos(\phi-\Omega+\Omega_1) - \cos(\eta+\eta_1)] \quad \text{.....(5)}$$

and the third term in (3) contributes

$$\frac{3\gamma_1^2}{8a\alpha^2}[2\cos\phi - \cos(\phi-2\eta_1) - \cos(\phi+2\eta_1)] \quad \text{.....(6)}$$

7·15. The non-periodic terms, N, of the disturbing function

An important part of the expansion of R consists of the non-periodic terms; let N denote these terms.

From the terms of zero order—7·14 (2)—and the terms involving the coordinates z, z_1—7·14 (4), (5) and (6)—the contribution to N/Gm_1 is

$$\frac{1}{2a}B_0^{1/2} - \frac{1}{8a}(\gamma^2+\alpha^2\gamma_1^2)B_0^{3/2} + \frac{\alpha}{4a}\gamma\gamma_1 B_1^{3/2}\cos(\Omega-\Omega_1) \quad \text{.....(1)}$$

The contributions of $au\dfrac{\partial R_0}{\partial a}$, ... are as follows:

$$au\frac{\partial R_0}{\partial a}: -\frac{1}{8a}(2e^2-\gamma^2)\left(B_0^{1/2} + \alpha\frac{dB_0^{1/2}}{d\alpha}\right);$$

$$a_1u_1\frac{\partial R_0}{\partial a_1}: \quad \frac{\alpha}{8a}(2e_1^2-\gamma_1^2)\frac{dB_0^{1/2}}{d\alpha};$$

$$\tfrac{1}{2}(au)^2\frac{\partial^2 R_0}{\partial a^2}: \quad \frac{e^2}{8a}\left(2B_0^{1/2}+4\alpha\frac{dB_0^{1/2}}{d\alpha}+\alpha^2\frac{d^2B_0^{1/2}}{d\alpha^2}\right);$$

$$\tfrac{1}{2}(a_1u_1)^2\frac{\partial^2 R_0}{\partial a_1^2}: \quad \frac{e_1^2}{8a}\alpha^2\frac{d^2B_0^{1/2}}{d\alpha^2};$$

$$\tfrac{1}{2}v^2\frac{\partial^2 R_0}{\partial \phi^2}: \quad -\frac{ee_1}{a}\left(\frac{1}{\alpha^2}-B_1^{1/2}\right)\cos(\varpi-\varpi_1);$$

$$au\,.\,a_1u_1\frac{\partial^2 R_0}{\partial a\,\partial a_1}: \quad \frac{ee_1}{4a}\left(\frac{2}{\alpha^2}-2\alpha\frac{dB_1^{1/2}}{d\alpha}-\alpha^2\frac{d^2B_1^{1/2}}{d\alpha^2}\right)\cos(\varpi-\varpi_1);$$

$$au\,.\,v\frac{\partial^2 R}{\partial a\,\partial \phi}: \quad -\frac{ee_1}{2a}\left(\frac{1}{\alpha^2}+B_1^{1/2}+\alpha\frac{dB_1^{1/2}}{d\alpha}\right)\cos(\varpi-\varpi_1);$$

$$a_1u_1\,.\,v\frac{\partial^2 R}{\partial a_1\,\partial \phi}: \quad \frac{\alpha ee_1}{2a}\left(\frac{2}{\alpha^3}+\frac{dB_1^{1/2}}{d\alpha}\right)\cos(\varpi-\varpi_1) \quad .$$

Collecting these results we have

$$\begin{aligned}\frac{N}{Gm_1}=&\frac{1}{2a}B_0^{1/2}+\frac{1}{8a}\left(e^2+e_1^2\right)\left(2\alpha\frac{dB_0^{1/2}}{d\alpha}+\alpha^2\frac{d^2B_0^{1/2}}{d\alpha^2}\right)\\&+\frac{ee_1}{4a}\left(2B_1^{1/2}-2\alpha\frac{dB_1^{1/2}}{d\alpha}-\alpha^2\frac{d^2B_1^{1/2}}{d\alpha^2}\right)\cos(\varpi-\varpi_1)\\&-\frac{\gamma^2}{8a}\left(B_0^{3/2}-B_0^{1/2}-\alpha\frac{dB_0^{1/2}}{d\alpha}\right)-\frac{\gamma_1^2}{8a}\left(\alpha\frac{dB_0^{1/2}}{d\alpha}+\alpha^2B_0^{3/2}\right)\\&+\frac{\gamma\gamma_1}{4a}B_1^{3/2}\cos(\Omega-\Omega_1) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)\end{aligned}$$

We now proceed to simplify this collection of terms. Putting $n=0$, $s=\frac{1}{2}$ in 7·13 (1) and in 7·13 (7), we have

$$\frac{dB_0^{1/2}}{d\alpha}=B_1^{3/2}-\alpha B_0^{3/2} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\frac{d^2B_0^{1/2}}{d\alpha^2}=2B_0^{3/2}-\frac{1}{\alpha}B_1^{3/2} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Similarly, putting $n=1$, $s=\frac{1}{2}$ in the same pair of equations, we have

$$\frac{dB_1^{1/2}}{d\alpha}=\tfrac{1}{2}\{B_0^{3/2}+B_2^{3/2}-2\alpha B_1^{3/2}\} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\frac{d^2B_1^{1/2}}{d\alpha^2}=2B_1^{3/2}-\frac{1}{\alpha}B_2^{3/2} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From (3) and (4), the coefficient of $\frac{e^2}{8a}$ (and of $\frac{e_1^2}{8a}$) in (2) is $\alpha B_1^{3/2}$.

From (5) and (6), the coefficient of $\frac{ee_1}{4a}$ is $-\alpha B_2^{3/2}$.

From (3), the coefficient of $-\gamma^2/8a$ is

$$(1+\alpha^2)B_0^{3/2}-\alpha B_1^{3/2}-B_0^{1/2};$$

but, from 7·12 (9) and (10),

$$B_0^{3/2}=\frac{(1+\alpha^2)B_0^{1/2}-2\alpha B_1^{1/2}}{(1-\alpha^2)^2}\quad,\quad B_1^{3/2}=\frac{2\alpha B_0^{1/2}-(1+\alpha^2)B_1^{1/2}}{(1-\alpha^2)^2}\quad;$$

from these $$(1+\alpha^2)B_0^{3/2}-2\alpha B_1^{3/2}=B_0^{1/2};$$

hence the coefficient of $-\gamma^2/8a$ in (2) is $\alpha B_1^{3/2}$; this is also the coefficient of $-\gamma_1^2/8a$, from (3).

Inserting these values in (2) we obtain the non-periodic terms, N, of the disturbing function, given by

$$N=\frac{Gm_1}{8a}[4B_0^{1/2}+\alpha B_1^{3/2}\{e^2+e_1^2-\gamma^2-\gamma_1^2+2\gamma\gamma_1\cos(\Omega-\Omega_1)\}$$
$$-2\alpha ee_1B_2^{3/2}\cos(\varpi-\varpi_1)]\quad.\quad\ldots\ldots\ldots(7)$$

Now, since $\alpha=a_1/a$, the formula 7·11 (1) can be written

$$B_n^s=\frac{2}{\pi}a^{2s}\int_0^\pi\frac{\cos n\phi\,d\phi}{(a^2-2aa_1\cos\phi+a_1^2)^s}\quad.$$

The integral is a symmetrical function of a and a_1, and we write it as $\frac{\pi}{2}P_n^s$. Then

$$B_n^s=a^{2s}P_n^s\quad.$$

We then have

$$\left.\begin{aligned}\frac{1}{8a}B_0^{1/2}&=\frac{1}{8}P_0^{1/2}\equiv C\quad;\\ \frac{\alpha}{8a}B_1^{3/2}&=\frac{1}{8}aa_1P_1^{3/2}\equiv D\,;\\ \frac{\alpha}{8a}B_2^{3/2}&=\frac{1}{8}aa_1P_2^{3/2}\equiv E\,;\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

in these C, D and E are symmetrical functions of a and a_1. We now write N as follows:

$$N=Gm_1[4C+\{e^2+e_1^2-\gamma^2-\gamma_1^2+2\gamma\gamma_1\cos(\Omega-\Omega_1)\}D$$
$$-2ee_1E\cos(\varpi-\varpi_1)]\quad.\quad\ldots\ldots\ldots(9)$$

This formula is correct to the third order in e, e_1, γ and γ_1.

We shall require this expression for N in Chapter 13. We shall also require the following inequality, namely, $D>E$, which can be easily proved as follows. From 7·12 (9),

$$(1-\alpha^2)^2[B_1^{3/2}-B_2^{3/2}]=3(1+\alpha^2)B_1^{1/2}-\{6\alpha+5(1+\alpha^2)\}B_2^{1/2}-10\alpha B_3^{1/2}\quad.$$

Change the last term by means of 7·12 (4), putting $n=3$ and $s=\frac{1}{2}$; then the right-hand side reduces tc $3(1-\alpha)^2\{B_1^{1/2}+B_2^{1/2}\}$ which is positive since the B's are positive. Hence $B_1^{3/2}>B_2^{3/2}$, that is, by (8),

$$D>E\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

7·16. Proof that the part $Gm_1 r \cos S/r_1^2$ of the disturbing function contains periodic terms only

In all the developments of the disturbing function which we have undertaken up to the present so far as the second-order terms in the eccentricities and inclinations are concerned, it has been evident that the second part of the disturbing function has made no contribution to the non-periodic terms of R; in the planetary theory these non-periodic terms are represented by the expression for N in 7·15 (7). We now show that this particular property is general—that is to say, it is true whatever the order of development of R may be. For simplicity write, in the notation of 1·07 (9),

$$R' = \frac{r \cos S}{r_1^2} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where S is the angle between the radii vectores of the two planets.

The expansions of the previous sections show that R' can be expressed by a series of the form

$$R' = \sum_p \sum_q \{C_{p,q} \cos (pM + qM_1) + S_{p,q} \sin (pM + qM_1)\} \quad \ldots\ldots\ldots (2)$$

in which M and M_1 are the mean anomalies, the C and S coefficients are independent of M, M_1 and are functions of a, e, i, Ω, ϖ and the corresponding elements of the disturbing planet, the q's are positive or negative integers or zero, and the p's—without loss of generality—are positive integers or zero. The non-periodic part of (2) is $C_{0,0}$.

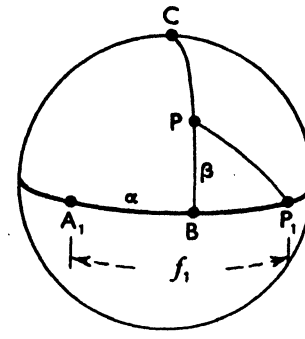


FIG. 18

Regarding M and M_1 as independent variables we have, immediately from (2),

$$\int_0^{2\pi} R' \, dM_1 = 2\pi \sum_p \{C_{p,0} \cos pM + S_{p,0} \sin pM\} \quad \ldots\ldots\ldots\ldots (3)$$

Now, for the disturbing planet,

$$r_1^2 \dot{f}_1 = h_1 = n_1 a_1^2 \sqrt{(1 - e_1^2)} \equiv n_1 c$$

or, since $\dot{M}_1 = n_1$, then $\qquad df_1 = c \, dM_1/r_1^2$;

hence, from (1),

$$\int_0^{2\pi} R' \, dM_1 = \frac{r}{c} \int_0^{2\pi} \cos S \, df_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

In Fig. 18 let A_1BP_1 be the intersection of the plane of P_1's orbit on the celestial sphere with the Sun as centre; let A_1 denote perihelion so that $A_1P_1 = f_1$. Let α, β be the coordinates of P with reference to A_1BP_1. Then $PP_1 = S$, $BP_1 = f_1 - \alpha$ and $BP = \beta$; consequently,

$$\cos S = \cos \beta \cos (f_1 - \alpha)$$

and, since α and β are independent of the position of P_1,

$$\int_0^{2\pi} \cos S \, df_1 = \cos\beta \int_0^{2\pi} \cos(f_1 - \alpha)\, df_1 = 0 \quad .$$

Hence, from (3) and (4),

$$\sum_p \{C_{p,0} \cos pM + S_{p,0} \sin pM\} = 0 \quad .$$

This last equation is true whatever M may be ; hence

$$C_{p,0} = S_{p,0} = 0 \quad , \qquad \ldots\ldots(5)$$

for all values of p.

In particular,

$$C_{0,0} = S_{0,0} = 0$$

which is tantamount to the statement that R' does not contain any non-periodic terms.

By reference to (3) it is clear that if we had formed the integral $\int_0^{2\pi} R' \, dM$ we should have arrived at the result, similar to (5), namely

$$C_{0,q} = S_{0,q} = 0 \quad . \qquad \ldots\ldots(6)$$

The results (5) and (6) show that the general expansion of R' is given by (2) where q is a positive or negative integer, p is a positive integer, *zero values of p and q being excluded.*

7·17. General remarks on the expansion of the disturbing function

When the disturbing function is expanded as a periodic series with the time as independent variable, the discussion in the previous pages shows that the general term is of the form

$$C \cos(iM + i_1 M_1 + j\Omega + j\Omega_1 + k\varpi + k_1\varpi_1) \qquad \ldots\ldots(1)$$

where $M = nt + \epsilon - \varpi$, $M_1 = n_1 t + \epsilon_1 - \varpi_1$, C is a function of a, a_1, e, e_1, γ, γ_1 and i, i_1, j, j_1, k, k_1 are positive or negative integers (including zero). It is readily seen that in general C is a power series involving e, e_1, γ and γ_1, and if we assume that each of these elements is of the same order of magnitude then the order of magnitude of a coefficient C will be that of $e^{\alpha} e_1^{\alpha_1} \gamma^{\beta} \gamma^{\beta_1}$, where $\alpha + \alpha_1 + \beta + \beta_1$ is the smallest sum of the indices of the terms of the series alluded to.

It is pertinent to enquire what is the order of magnitude of a particular term such as (1). For convenience in the exposition we write (1) as

$$C \cos\{(in + i_1 n_1)t + h\} \qquad \ldots\ldots(2)$$

where the value of h—defined in terms of ϵ, ϵ_1, Ω, Ω, ϖ, ϖ_1—is of no particular consequence to us in the present connection.

It is evident from the analysis of the previous sections that, so far as the terms of the disturbing function not involving z and z_1 are concerned, these terms arise through the multiplication of

$$u^p u_1^q v^s \quad \text{and} \quad \frac{\partial^{p+q+s} R_0}{\partial a^p \, \partial a_1^q \, \partial \phi^s}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Consider first the latter expression ; if s is even it is readily seen that this expression is given by a series of which $A_k \cos k\phi$ is the general term, and if s is odd it is given by a series of which $B_k \sin k\phi$ is the general term, the A's and B's being functions of a, a_1 and

$$\phi = (n - n_1)\, t + \epsilon - \epsilon_1. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Consider now the first expression in (3). The general expansion of the radius vector r in terms of the mean anomaly shows that the order of the coefficient of $\cos \alpha M$, say, is that of e^α ; further, when we form $\rho\,(\equiv r \cos\theta)$ and so obtain u, the order of the coefficient of any term $\cos(\alpha nt + q)$ is again α with respect to e or γ. Consider now u^2 which clearly can be developed into a series of which we regard $C \cos(pnt + q)$ to be the general term. This term can arise in one of three ways :

(i) by multiplication of the non-periodic part of u with the term $C_1 \cos(pnt + q)$—if this term occurs ; the order of C is then, at least, p ;

(ii) by multiplication of the terms of the forms

$$C_1 \cos(\alpha nt + \beta) \quad \text{and} \quad C_2 \cos\{(p - \alpha) nt + \delta\}$$

in which $p > \alpha$; the order of C is then the sum of the orders of C_1 and C_2, that is, the sum of α and $(p - \alpha)$, that is, p ;

(iii) by multiplication of the terms of the forms

$$C_1 \cos(\alpha nt + \beta) \quad \text{and} \quad C_2 \cos\{(\alpha - p) nt + \delta\}$$

in which $\alpha > p$; the order of C is then the sum of the orders of α and $(\alpha - p)$, that is $(2\alpha - p)$ which is greater than p.

We conclude, then, that the order of the term $C \cos(pnt + q)$ in u^2 is *at least* equal to p. A similar argument applies to any power of u, to any power of u_1, to any power of v.

Further, if we expand the first term in (3) we shall obtain, if s is even, terms of the forms

$$C_1 \cos\{(pn + qn_1)\, t + \delta\} \quad \text{or} \quad C_2 \cos\{(pn - qn_1)\, t + \delta_1\}$$

in both of which we can suppose that both p and q are positive integers. It is easily seen by the previous arguments that the order of C_1 will be $p + q$ at least and that the order of C_2 will be $|\, p - q \,|$ at least.

If s is odd, we replace cosines by sines and the same conclusions as to orders of magnitude follow.

Consider now the multiplication of the two terms in (3). If s is even we shall have to multiply two cosine series and the product will be a cosine series ; if s is odd we shall have to multiply two sine series and the product will again be a cosine series. We shall discuss only the case of

forming the product of two cosine series since the argument for the product of two sine series is similar.

The multiplication of the two expressions in (3) will produce terms of the forms

$$C \cos \{(pn + qn_1)t + \zeta\} \quad \ldots\ldots(5)$$

$$C' \cos \{(pn - qn_1)t + \zeta'\} \quad \ldots\ldots(6)$$

in which we suppose p and q to be positive integers. A term such as (5) will arise through the multiplication of

$$C_0 \cos k\phi \equiv C_0 \cos \{(kn - kn_1)t + \zeta_0\} \quad \ldots\ldots(7)$$

with either

$$C_1 \cos [\{(p-k)n + (q+k)n_1\}t + \zeta_1] \quad \ldots\ldots(8)$$

or

$$C_2 \cos [\{(p+k)n + (q-k)n_1\}t + \zeta_2] \quad \ldots\ldots(9)$$

Now the order of C and C' in (5) and (6) is the order of C_1 and C_2 in (8) and (9) since the order of C_0 in (7) is zero—C_0 is simply a function of a, a_1.

We consider the following cases :

(i) p and $q > k$; the order of C_1 is at least $(p-k)+(q+k)$ or $p+q$, and a similar result holds for C_2 ;

(ii) $p < k < q$; the order of C_1 is at least $(q+k)-(k-p)$ or $p+q$, and the order of C_2 is at least $(p+k)+(q-k)$ or $p+q$;

(iii) p and $q < k$; the order of C_1 is at least $(q+k)-(k-p)$ or $p+q$ and the order of C_2 is at least $(p+k)-(k-q)$ or $p+q$.

Hence in all cases the order of C in (5) is at least $p+q$.

In the same way the order of C' in (6) is at least $|\, p-q \,|$.

Similar arguments apply to the terms of the disturbing function derived from the coordinates z and z_1.

We conclude finally that the periodic part of the disturbing function can be expressed as a series of cosines of the type

$$C \cos \{(pn + qn_1)t + \zeta\}$$

in which the order of C—p and q being regarded now as positive or negative integers—is at least $|\, p+q \,|$.

Chapter 8

CANONIC EQUATIONS

8·01. Introduction

In some important sections of this chapter we shall be concerned with small variations in coordinates, in components of velocity and in functions of these. As an illustration, consider the simple example of a particle moving in a straight line according to the equation of motion $\ddot{x}=\phi(t, x)$, of which the solution can be represented functionally by

$$x=F(t, \alpha_1, \alpha_2) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where α_1 and α_2 are the constants of integration.

From (1), $$\dot{x}=\frac{\partial F}{\partial t}\equiv G(t, \alpha_1, \alpha_2) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

We may, if we wish—as is usual in many practical problems—relate the constants of integration to prescribed initial conditions, $x=x_0$ and $\dot{x}=(\dot{x})_0$ at $t=0$.

The equations (1) and (2) represent two families of curves with parameters α_1 and α_2, any one curve in each family being associated with particular values of α_1 and α_2. In Fig. 19 we represent the graph of $F(t, \alpha_1, \alpha_2)$ for particular values of α_1 and α_2 by AB, and the graph of $G(t, \alpha_1, \alpha_2)$, similarly, by EF. Neighbouring curves, corresponding to the values $\alpha_1+\delta\alpha_1$ and $\alpha_2+\delta\alpha_2$ of the parameters are represented by CD and GH.

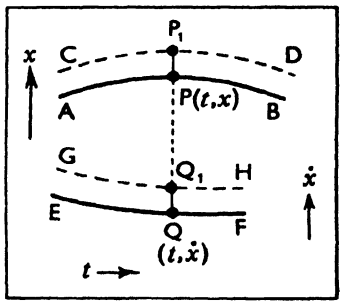


FIG. 19

For any given value of t, a point P on AB will be associated with P_1 on CD. Thus, corresponding to the variations $\delta\alpha_1$ and $\delta\alpha_2$ we have the displacement PP_1. The ordinate of P is x and, if the ordinate of P_1 is denoted by $x+\delta x$, the displacement PP_1 is δx which is a function of $\delta\alpha_1$ and $\delta\alpha_2$ given by

$$\delta x=\frac{\partial F}{\partial \alpha_1}\delta\alpha_1+\frac{\partial F}{\partial \alpha_2}\delta\alpha_2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Similarly, the displacement of a point Q on EF, resulting from the variations $\delta\alpha_1$ and $\delta\alpha_2$, is QQ_1. The ordinate of Q is $\dot{x}$ and, if we denote the ordinate of Q_1 by $\dot{x}+\delta\dot{x}$, the displacement QQ_1 is $\delta\dot{x}$; from (2),

$$\delta\dot{x}=\frac{\partial G}{\partial \alpha_1}\delta\alpha_1+\frac{\partial G}{\partial \alpha_2}\delta\alpha_2 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

or, in terms of F,

$$\delta\dot{x} = \frac{\partial^2 F}{\partial\alpha_1\,\partial t}\,\delta\alpha_1 + \frac{\partial^2 F}{\partial\alpha_2\,\partial t}\,\delta\alpha_2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

From (3), $\dfrac{d}{dt}(\delta x) = \dfrac{\partial^2 F}{\partial t\,\partial\alpha_1}\,\delta\alpha_1 + \dfrac{\partial^2 F}{\partial t\,\partial\alpha_2}\,\delta\alpha_2$; hence, by (5),

$$\frac{d}{dt}(\delta x) = \delta\dot{x} = \delta\left(\frac{dx}{dt}\right) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

If δ and d are regarded as operators, the last formula can be represented by

$$d\,.\,\delta x = \delta\,.\,dx \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

which shows that these independent operators obey the commutative law.

8·02. Variation of a function

First consider the three-dimensional motion of a particle, the equations of motion being

$$\ddot{x} = f_1(t, x, y, z), \quad \ddot{y} = f_2(t, x, y, z), \quad \ddot{z} = f_3(t, x, y, z) \quad .$$

The solutions will involve six constants of integration α_i $(i = 1, 2, \ldots, 6)$ and they can be represented functionally as

$$x = F_1(t, \alpha), \quad y = F_2(t, \alpha), \quad z = F_3(t, \alpha) \quad . \quad \ldots\ldots\ldots\ldots\ldots(1)$$

From these we derive

$$\dot{x} = G_1(t, \alpha), \quad \dot{y} = G_2(t, \alpha), \quad \dot{z} = G_3(t, \alpha) \quad . \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Corresponding to variations $\delta\alpha_i$ $(i = 1, \ldots, 6)$ we obtain

$$\delta x = \sum_1^6 \frac{\partial F_1}{\partial\alpha_i}\,\delta\alpha_i \quad , \quad \delta\dot{x} = \sum_1^6 \frac{\partial G_1}{\partial\alpha_i}\,\delta\alpha_i \quad , \quad \ldots\ldots\ldots\ldots\ldots(3)$$

with similar equations in y and z; thus, associated with independent variations in the six α's we have independent variations in the three coordinates and in the three components of velocity. The equations (3) being six in number we have the requisite number of equations for expressing any $\delta\alpha$ uniquely in terms of $\delta x, \ldots, \delta\dot{z}$.

Consider now a function of t, the coordinates and the components of velocity; denote it by $\phi(t, x, \ldots, \dot{z})$. By means of (1) and (2) the function becomes $\phi(t, \alpha)$ and, corresponding to variations $\delta\alpha_i$, the variation in ϕ is given by

$$\delta\phi = \Sigma\frac{\partial\phi}{\partial\alpha_i}\,\delta\alpha_i \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Also, corresponding to variations $\delta x, \ldots \delta\dot{z}$, we have

$$\delta\phi = \sum_{xyz}\left(\frac{\partial\phi}{\partial x}\,\delta x + \frac{\partial\phi}{\partial\dot{x}}\,\delta\dot{x}\right) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

If δx, $\delta \dot{x}$, ... are given by (3), then the variations of ϕ in (4) and (5) are the same.

Thus we can express by means of (5) the variation of a function in terms of independent variations δx, $\delta \dot{x}$, ... related to the *implied* variations of the six parameters α_i.

The process can evidently be extended to any number of point-masses ; if this number is n the total number of coordinates is $3n$, which we denote by k, and the total number of constants of integration is $2k$. In general the variation of a function $\phi(t, x_i, \dot{x}_i, \ldots)$ where $i = 1, 2, \ldots n$, is given by

$$\delta\phi = \sum_{xyz} \sum_{1}^{n} \left(\frac{\partial \phi}{\partial x_i} \delta x_i + \frac{\partial \phi}{\partial \dot{x}_i} \delta \dot{x}_i \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which δx, $\delta \dot{x}$ ... are independent variations which are supposed related to the implied variations in the $2k$ constants of integration, or parameters, α_i.

8·03. Generalised coordinates

We consider n point-masses with a total of $k (\equiv 3n)$ rectangular coordinates which we suppose are individually functions of the time and k independent variables denoted by $q_1, q_2, \ldots, q_k$; thus we write

$$x_i = F_i(t, q_1, \ldots q_k),\ y_i = G_i(t, q_1, \ldots q_k),\ z_i = H_i(t, q_1, \ldots q_k) \quad \ldots\ldots (1)$$

or, briefly, as

$$x_i = F_i(t, q),\ y_i = G_i(t, q),\ z_i = H_i(t, q) \quad \ldots\ldots\ldots\ldots\ldots (2)$$

The q's are referred to as *generalised coordinates.*

A simple example is the transformation of rectangular coordinates into polar coordinates by means of

$$x = r \cos\phi \sin\theta,\ y = r \sin\phi \sin\theta,\ z = r \cos\theta.$$

This transformation does not involve the time ; here r, ϕ and θ are to be identified with three q's.

A simple example of a transformation in two dimensions involving the time is afforded by

$$x = \xi \cos nt - \eta \sin nt,\ y = \xi \sin nt + \eta \cos nt$$

where ξ and η, coordinates referred to rotating axes, are identified with two q's ; we deal with this particular transformation in later sections.

From (1), $$\dot{x}_i = \frac{\partial F_i}{\partial t} + \sum_{r=1}^{k} \frac{\partial F_i}{\partial q_r} \dot{q}_r \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

there are two similar equations in y and z.

Let T be the kinetic energy of the system, so that

$$2T = \sum_i m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \quad .$$

Then it is clear from (3) that T will be transformed into a function $T(q, \dot{q}, t)$ of the form given by

$$T = \sum_r A_r \dot{q}_r^2 + 2 \sum_r \sum_s B_{r,s} \dot{q}_r \dot{q}_s + \Sigma C_r \dot{q}_r + D \quad , \quad \ldots\ldots\ldots\ldots (4)$$

where in the double summation $s > r$ and the coefficients A, B, C and D are in general functions of q and t, being known in any given problem through the transformations (3).

It is to be noted that the first two summations in (4) constitute a homogeneous quadratic function of the $\dot{q}$'s which we denote by T_2, that the third summation, $\Sigma C_r \dot{q}_r$, is a homogeneous linear function of the $\dot{q}$'s which we denote by T_1, and that the final term D, which we now denote by T_0, is independent of the $\dot{q}$'s. It is convenient to write T as

$$T = T_2 + T_1 + T_0 \quad ; \quad \ldots\ldots\ldots\ldots (5)$$

T_2 and T_1 are functions of q, $\dot{q}$, t and T_0 is a function of q, t.

When the transformations (1) are applied to the force-function U, which in the first instance is a function of the rectangular coordinates, it becomes a function of q, t denoted by $U(q, t)$.

8·04. Lagrange's equations

The equations of motion of n point-masses are

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}, \; m_i \ddot{y}_i = \frac{\partial U}{\partial y_i}, \; m_i \ddot{z}_i = \frac{\partial U}{\partial z_i} \quad , \quad \ldots\ldots\ldots\ldots (1)$$

where U is the force-function, and $i = 1, 2, \ldots n$. These equations are to be transformed into equations involving the k generalised coordinates $q_r (r = 1, 2, \ldots k)$.

It is convenient to write the formulae of transformation given by (2) in the previous section as follows :

$$x_i = x_i(q, t), \; y_i = y_i(q, t), \; z_i = z_i(q, t) \quad . \quad \ldots\ldots\ldots\ldots (2)$$

We then write 8·03 (3) as $\dot{x}_i = \dfrac{\partial x_i}{\partial t} + \sum_r \dfrac{\partial x_i}{\partial q_r} \dot{q}_r$. $\ldots\ldots\ldots\ldots (3)$

From (3) we have $\dfrac{\partial \dot{x}_i}{\partial \dot{q}_r} = \dfrac{\partial x_i}{\partial q_r}$ $\ldots\ldots\ldots\ldots (4)$

and $$\frac{\partial \dot{x}_i}{\partial q_r} = \frac{\partial^2 x_i}{\partial t \, \partial q_r} + \sum_s \frac{\partial^2 x_i}{\partial q_r \, \partial q_s} \dot{q}_s \quad ;$$

also, $$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_r}\right) = \frac{\partial^2 x_i}{\partial t \, \partial q_r} + \sum_s \frac{\partial^2 x_i}{\partial q_r \, \partial q_s} \dot{q}_s \quad .$$

Hence $$\frac{\partial \dot{x}_i}{\partial q_r} = \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_r}\right) \quad \ldots\ldots\ldots\ldots (5)$$

Now T is given by $2T = \Sigma m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$; $\ldots\ldots\ldots\ldots (6)$

consequently,

$$\frac{\partial T}{\partial \dot{q}_r} = \sum_{xyz} \sum_i m_i \left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_r}\right) = \sum_{xyz} \sum_i m_i \left(\dot{x}_i \frac{\partial x_i}{\partial q_r}\right)$$

by (4). Again, from this last equation,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) = \sum_{xyz}\sum_{i} m_i\left[\ddot{x}_i\frac{\partial x_i}{\partial q_r} + \dot{x}_i\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_r}\right)\right]$$

$$= \sum_{xyz}\sum_{i}\left[\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_r} + m_i\dot{x}_i\frac{\partial \dot{x}}{\partial q_r}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

by (1) and (5). But, by (2), $U(x, y, z) \equiv U(q, t)$;

hence $$\sum_{xyz}\sum_{i}\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_r} = \frac{\partial U}{\partial q_r}$$

and (7) becomes, on referring to (6),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) = \frac{\partial U}{\partial q_r} + \frac{\partial T}{\partial q_r} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Now U does not involve $\dot{q}$, so that $\dfrac{\partial U}{\partial \dot{q}} = 0$; hence (8) can be written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) = \frac{\partial L}{\partial q_r} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

where $L = T + U$ and is a function of q, $\dot{q}$, t. The function L is sometimes called the *kinetic potential.*

The equations (8)—or (9)—are Lagrange's equations of motion in generalised coordinates.

From 8·03 (4) it is seen that $\dfrac{\partial L}{\partial \dot{q}_r}$ is of the form given by

$$\frac{\partial L}{\partial \dot{q}_r} = \Sigma E_r\dot{q}_r + C_r \quad .$$

It is then evident that the equation (9) involves (for all values of r and s) $\ddot{q}_r$, $\dot{q}_r\dot{q}_s$, $\dot{q}_r$ and terms independent of the $\dot{q}$'s.

8·05. Example

To illustrate the principles involved in deriving Lagrange's equations and also in preparation for subsequent developments we consider a simply-expressed example of the three-body problem in two dimensions.

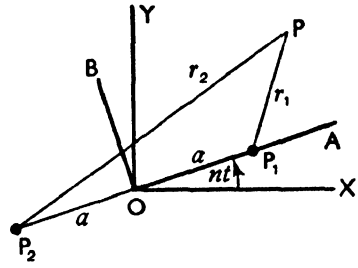


FIG. 20

In Fig. 20 consider two bodies P_1 and P_2, of the same mass M, revolving about their centre of mass O in circular orbits of radius a with constant angular velocity n. Let P be a third body of infinitesimal mass m which is insufficient to affect the motions of P_1 and P_2. The force-function U is given by

$$U = \frac{GM^2}{2a} + GmM\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where r_1 and r_2 are the distances of P from P_1 and P_2. If (x, y) are the coordinates of P with respect to fixed axes OX, OY then the kinetic energy of the system is given by

$$T = Ma^2n^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

We choose as new axes OP_1 (or OA) and OB rotating with angular velocity n; let $\angle AOX = nt$. If (ξ, η) are the coordinates of P with respect to OA, OB then

$$x = \xi \cos nt - \eta \sin nt, \quad y = \xi \sin nt + \eta \cos nt \quad , \ldots\ldots\ldots\ldots (3)$$

from which and (2)

$$T = Ma^2n^2 + \tfrac{1}{2}m\{\dot{\xi}^2 + \dot{\eta}^2 + 2n(\xi\dot{\eta} - \dot{\xi}\eta) + n^2(\xi^2 + \eta^2)\} \quad . \ldots\ldots\ldots (4)$$

Also, $$r_1^2 = (\xi - a)^2 + \eta^2 \quad , \quad r_2^2 = (\xi + a)^2 + \eta^2 \quad .$$

We now regard ξ and η as generalised coordinates q_1 and q_2. It may be noted that the transformation formulae (3) are of the form $x = x(q, t)$ and $y = y(q, t)$.

From (4) we have

$$\frac{\partial T}{\partial \dot{\xi}} = m(\dot{\xi} - n\eta) \quad ; \quad \frac{\partial T}{\partial \dot{\eta}} = m(\dot{\eta} + n\xi) ;$$

$$\frac{\partial T}{\partial \xi} = mn(\dot{\eta} + n\xi) \quad ; \quad \frac{\partial T}{\partial \eta} = mn(-\dot{\xi} + n\eta) \quad .$$

Lagrange's equations are then

$$m\frac{d}{dt}(\dot{\xi} - n\eta) = mn(\dot{\eta} + n\xi) + \frac{\partial U}{\partial \xi} \quad ,$$

$$m\frac{d}{dt}(\dot{\eta} + n\xi) = mn(-\dot{\xi} + n\eta) + \frac{\partial U}{\partial \eta} \quad ,$$

or

$$\ddot{\xi} - 2n\dot{\eta} - n^2\xi = GM\frac{\partial}{\partial \xi}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad , \ldots\ldots\ldots\ldots (5)$$

$$\ddot{\eta} + 2n\dot{\xi} - n^2\eta = GM\frac{\partial}{\partial \eta}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad . \ldots\ldots\ldots\ldots (6)$$

It is to be noticed that in arriving at the equations (5) and (6) the mass m has been divided out; this step is possible only because the *variable* parts of U in (1) and T in (4) have m as a factor. Accordingly, (5) and (6) may be regarded as emanating analytically from U and T as given by

$$U = GM\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

and $$T = \tfrac{1}{2}\{\dot{\xi}^2 + \dot{\eta}^2 + 2n(\xi\dot{\eta} - \dot{\xi}\eta) + n^2(\xi^2 + \eta^2)\} \quad , \ldots\ldots\ldots\ldots (8)$$

these equations differing from (1) and (4), first, by the omission of the constant terms in the latter formulae—these terms are ineffective when U and T are differentiated—and, second, by the disappearance of m.

It may be remarked that an integral of (5) and (6) can be obtained easily; multiply the former by $\dot{\xi}$ and the latter by $\dot{\eta}$, and add. The result is integrable and we obtain

$$\tfrac{1}{2}(\dot{\xi}^2+\dot{\eta}^2)-\tfrac{1}{2}n^2(\xi^2+\eta^2)=GM\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+\text{constant} \quad . \quad \ldots\ldots\ldots(9)$$

This last formula is the energy-integral in terms of the generalised coordinates ξ and η.

8·06. Hamilton's canonic equations

We now effect a change of variables as follows. Let

$$p_i=\frac{\partial T(q,\dot{q},t)}{\partial \dot{q}_i} \quad , \quad \ldots\ldots\ldots(1)$$

for $i=1, 2, \ldots k$; the p's are called generalised momenta or simply momenta. Then, by 8·03 (4), p_i is a linear expression in the $\dot{q}$'s which we can write in the form

$$p_i=a_{i1}\dot{q}_1+a_{i2}\dot{q}_2+\ldots+a_{ik}\dot{q}_k+c_i \; ; \quad \ldots\ldots\ldots(2)$$

here the a's and c_i are, in general, functions of the q's and t, these functions being known in any given problem. Since there are k equations of the form (2), these can be solved in principle to give each of the $\dot{q}$'s in terms of the k p's and the coefficients a and c. We can thus write the solution for any $\dot{q}$ symbolically as

$$\dot{q}_i=\dot{q}_i(q,p,t) \quad , \quad \ldots\ldots\ldots(3)$$

the right-hand side being easily seen to be a linear function of the p's. By means of (3) the function $T(q,\dot{q},t)$ can now be converted into a function of q, p, t.

To discriminate between the two forms in which the kinetic energy function can now be expressed we shall use T to denote that it is a function of $q, \dot{q}, t$ and $\bar{T}$ to denote that it is a function of q, p, t.

Let the q's and $\dot{q}$'s receive small arbitrary and independent variations δq_i and $\delta \dot{q}_i$ $(i=1, 2, \ldots k)$. Then, by 8·02 (5),

$$\delta T=\sum_1^k\left(\frac{\partial T}{\partial q_i}\,\delta q_i+\frac{\partial T}{\partial \dot{q}_i}\,\delta \dot{q}_i\right)$$

or, by (1),

$$\delta T=\sum_i \frac{\partial T}{\partial q_i}\,\delta q_i+\sum_i p_i\,\delta \dot{q}_i \quad . \quad \ldots\ldots\ldots(4)$$

Now, we have seen from 8·03 (5) that T can be expressed as

$$T=T_2+T_1+T_0 \quad , \quad \ldots\ldots\ldots(5)$$

where T_2 is a homogeneous quadratic function of the $\dot{q}$'s, T_1 is a homogeneous linear function of the $\dot{q}$'s, and T_0 is a function of q, t. By Euler's theorem,

$$2T_2=\sum \frac{\partial T_2}{\partial \dot{q}_i}\,\dot{q}_i \quad , \quad T_1=\sum \frac{\partial T_1}{\partial \dot{q}_i}\,\dot{q}_i \quad .$$

Adding these, we have

$$2T_2+T_1=\Sigma\frac{\partial(T_2+T_1)}{\partial\dot{q}_i}\dot{q}_i=\Sigma\frac{\partial T}{\partial\dot{q}_i}\dot{q}_i \quad ,$$

since T_0 is independent of $\dot{q}$. Then, by (1),

$$2T_2+T_1=\sum_i^k p_i\dot{q}_i \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Returning to (2), we note that the right-hand side of this equation is a function of q, $\dot{q}$, t; hence, corresponding to the independent variations δq and $\delta\dot{q}$, p_i will receive the variation δp_i which will evidently be of the form

$$\delta p_i=\sum_j(A_{ij}\,\delta q_j+B_{ij}\,\delta\dot{q}_j) \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where, in general, the A's and the B's are functions of q, $\dot{q}$, t or, by (3), functions of q, p, t.

The equations (7) are k in number; hence they can be solved to give each of the $\delta\dot{q}$'s in the form

$$\delta\dot{q}_i=\sum_j(C_{ij}\,\delta q_j+D_{ij}\,\delta p_j) \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where the C's and D's are functions of q, $\dot{q}$, t or of q, p, t.

Initially, we regarded δq and $\delta\dot{q}$ as the independent variations; it is clear that we can regard δq and δp as independent variations, the variations $\delta\dot{q}$ being given by (8). Further, the variation of a function is the same whatever pair of the variables q, $\dot{q}$ and p are used to describe the function.

We then have from (6)

$$2\,\delta T_2+\delta T_1=\Sigma\dot{q}_i\,\delta p_i+\Sigma p_i\,\delta\dot{q}_i \quad ,$$

and from (5), by means of (1),

$$\delta T\equiv\delta T_2+\delta T_1+\delta T_0=\Sigma\frac{\partial T}{\partial q_i}\delta q_i+\Sigma p_i\,\delta\dot{q}_i \quad .$$

Subtracting these and writing $T^*=T_2-T_0$ $\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$

we obtain $$\delta T^*=\Sigma\dot{q}_i\,\delta p_i-\Sigma\frac{\partial T}{\partial q_i}\delta q_i \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Now, T^* is, in the first instance, a function of q, $\dot{q}$, t which can be transformed by (3) into a function of q, p, t; hence

$$\delta T^*=\Sigma\frac{\partial T^*(q,p,t)}{\partial p_i}\delta p_i+\Sigma\frac{\partial T^*(q,p,t)}{\partial q_i}\delta q_i \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

We now have two expressions for δT^*—(10) and (11); these must be the same and, consequently, equating the coefficients of the δp's and the δq's which are independent, we have

$$\dot{q}_i = \frac{\partial T^*(q, p, t)}{\partial p_i} \quad , \qquad \text{.....(12)}$$

and

$$\frac{\partial T}{\partial q_i} = -\frac{\partial T^*(q, p, t)}{\partial q_i} \quad . \qquad \text{.....(13)}$$

The second of these results—(13)—is remarkable ; written out in full it is :

$$\frac{\partial T(q, \dot{q}, t)}{\partial q_i} = -\frac{\partial (\bar{T}_2 - T_0)(q, p, t)}{\partial q_i} \quad .$$

It is to be noticed that, although T_2 is a homogeneous quadratic function of the $\dot{q}$'s, $\bar{T}(q, p, t)$ is *not* in general a homogeneous quadratic function of the p's.

Let

$$H = T^* - U = \bar{T}_2 - T_0 - U \quad ; \qquad \text{.....(14)}$$

then, H is a function of q, p, t called the *Hamiltonian function.*

Since U does not contain $\dot{q}$ and consequently does not contain p, the formula (12) becomes

$$\dot{q}_i = \frac{\partial H(q, p, t)}{\partial p_i} \quad . \qquad \text{.....(15)}$$

Again, by (1),

$$\dot{p}_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$$

which becomes, by means of the Lagrangian formula 8·04 (8),

$$\dot{p}_i = \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i}$$

or, by (13),

$$\dot{p}_i = -\frac{\partial T^*}{\partial q_i} + \frac{\partial U}{\partial q_i} \quad .$$

Hence, by (14),

$$\dot{p}_i = -\frac{\partial H(q, p, t)}{\partial q_i} \quad . \qquad \text{.....(16)}$$

It will be recalled from 8·04 that Lagrange's equations are of the second order, as indeed are the equations 8·04 (1) in rectangular coordinates. The procedure in this section, fundamentally, has been to replace $\dot{q}$ by a new variable p with the result that the k Lagrangian equations each of the second order have been replaced by the $2k$ linear equations (15) and (16) which we rewrite, for convenient reference,

$$\dot{q}_i = \frac{\partial H(q, p, t)}{\partial p_i} \quad ; \quad \dot{p}_i = -\frac{\partial H(q, p, t)}{\partial q_i} \quad . \qquad \text{.....(17)}$$

These are Hamilton's canonic equations.

In any given problem in which the form of U is known in terms of rectangular coordinates, the form of the Hamiltonian function $H(q, p, t)$

is also known, as is easily seen from the successive transformations of T defined in 8·04 (6).

The theoretical procedure of this section will be illustrated in section 8·08 in detail.

8·07. Energy integral

If the transformations from the rectangular coordinates to the generalised coordinates do not involve the time—that is to say, if $x = x(q)$ —then T does not contain the time explicitly so that $T = T_2$ and, further, U is a function of the q's only. Since $p = \dfrac{\partial T}{\partial \dot{q}}$ it follows that the expression for p in terms of q and $\dot{q}$ does not contain the time explicitly and finally that $H = H(q, p)$. Hence

$$\dot{H} = \Sigma \frac{\partial H}{\partial q_i} \dot{q}_i + \Sigma \frac{\partial H}{\partial p_i} \dot{p}_i = 0$$

by 8·06 (17), and accordingly

$$H \equiv T - U = \text{constant}.$$

This is the energy integral in the circumstances stated, T being the kinetic energy and $-U$ the potential energy.

8·08. Example, continued

In section 8·05 we obtained the Lagrangian equations for the motion of P; in this section we derive the canonic equations. We start with the equations (8) and (7) of section 8·05, writing however—in accordance with the general theory—

$$\xi \equiv q_1 \quad , \quad \eta \equiv q_2 \quad .$$

These equations are now

$$T = \tfrac{1}{2}\{\dot{q}_1^2 + \dot{q}_2^2 + 2n(q_1\dot{q}_2 - \dot{q}_1 q_2) + n^2(q_1^2 + q_2^2)\} \quad , \quad \ldots\ldots\ldots\ldots (1)$$

$$U = GM\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

r_1 and r_2 being functions of q_1 and q_2.

Now $\quad p_1 \equiv \dfrac{\partial T}{\partial \dot{q}_1} = \dot{q}_1 - nq_2 \quad , \quad p_2 \equiv \dfrac{\partial T}{\partial \dot{q}_2} = \dot{q}_2 + nq_1 \quad .$

From these $\quad \dot{q}_1 = p_1 + nq_2 \quad , \quad \dot{q}_2 = p_2 - nq_1 \quad . \ldots\ldots\ldots\ldots (3)$

From (1) $\quad T_2 = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) \quad , \quad \ldots\ldots\ldots\ldots (4)$

and $\quad T_0 = \tfrac{1}{2}n^2(q_1^2 + q_2^2) \quad . \ldots\ldots\ldots\ldots (5)$

Using (3) in (4) we then obtain

$$T^* \equiv \bar{T}_2 - T_0 = \tfrac{1}{2}(p_1^2 + p_2^2 + 2nq_2p_1 - 2nq_1p_2) \quad . \quad \ldots\ldots\ldots\ldots (6)$$

The Hamiltonian function H is then given by

$$H(q, p, t) = \tfrac{1}{2}(p_1^2 + p_2^2 + 2nq_2p_1 - 2nq_1p_2) - U \quad . \quad \ldots\ldots\ldots\ldots (7)$$

Apply now the formulae 8·06 (17) ; the canonic equations are :

$$\dot{q}_1 = p_1 + nq_2 \quad , \quad \dot{q}_2 = p_2 - nq_1 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$\dot{p}_1 = np_2 + \frac{\partial U}{\partial q_1} \quad , \quad \dot{p}_2 = -np_1 + \frac{\partial U}{\partial q_2} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

The two Lagrangian equations, 8·05 (5) and (6), each of the second order are now replaced by four canonic equations each of the first order.

Note

From (1), $$\frac{\partial T}{\partial q_1} = n\dot{q}_2 + n^2 q_1 = np_2, \quad \text{by (3).}$$

From (6), $$\frac{\partial T^*}{\partial q_1} = -np_2.$$

We thus obtain a verification of the general formula 8·06 (13) for the coordinate q_1—and, by a similar procedure, for the coordinate q_2.

8·09. The formal solutions of the canonic equations

The canonic equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $i = 1, 2, \ldots, k$. We now show that a complete solution of these equations follows from the knowledge of a certain function S satisfying a particular differential equation usually known as the Hamilton-Jacobi equation.

The canonic equations in (1) are $2k$ linear equations. Suppose that they are solved completely ; then any q and any p will be expressed as a function of the time and $2k$ constants of integration which we denote by $c_1, c_2, \ldots c_{2k}$. We can thus write

$$q_i = q_i(t, c_1, c_2, \ldots c_{2k}) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Similarly, $$p_i = p_i(t, c_1, c_2, \ldots c_{2k}) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In (1) the Hamiltonian function H is expressed in terms of q, p, t ; when we substitute, by means of (2) and (3), the solution-values for the q's and the p's in terms of the time and the c's, H becomes a function of t and the c's. We then have

$$H(c, t) \equiv H(q, p, t)$$

from which, differentiating with respect to c_j,

$$\frac{\partial H}{\partial c_j} = \sum_{i=1}^{k} \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial c_j} + \sum_{i=1}^{k} \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial c_j} = -\sum_i \dot{p}_i \frac{\partial q_i}{\partial c_j} + \sum_i \dot{q}_i \frac{\partial p_i}{\partial c_j}$$

which becomes, by a simple transformation,

$$\frac{\partial H}{\partial c_j} = -\frac{d}{dt}\sum_{i=1}^{k} p_i \frac{\partial q_i}{\partial c_j} + \frac{\partial}{\partial c_j}\sum_{i=1}^{k} p_i \dot{q}_i \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

But, from 8·06 (6), $\sum_i p_i \dot{q}_i = 2T_2 + T_1$

and, from 8·06 (14), $H = T_2 - T_0 - U$;

hence (4) becomes

$$\frac{d}{dt}\sum_{i=1}^{k} p_i \frac{\partial q_i}{\partial c_j} = \frac{\partial}{\partial c_j}(T_2 + T_1 + T_0 + U) = \frac{\partial}{\partial c_j}(T + U) \quad \ldots\ldots\ldots\ldots (5)$$

Let a function $S(c, t)$ be defined by

$$\frac{dS}{dt} = T + U \quad . \quad \ldots\ldots\ldots\ldots (6)$$

We then write (5) as

$$\frac{\partial}{\partial c_j}\left(\frac{dS}{dt}\right) = \frac{d}{dt}\sum_{i=1}^{k} p_i \frac{\partial q_i}{\partial c_j} \quad .$$

Multiply this equation by the variation δc_j and take the sum of all such equations for $j = 1, 2, \ldots 2k$; then

$$\sum_{1}^{2k} \frac{\partial}{\partial c_j}\left(\frac{dS}{dt}\right) \delta c_j = \frac{d}{dt}\sum_{i=1}^{k} p_i \left\{\sum_{j=1}^{2k} \frac{\partial q_i}{\partial c_j}\delta c_j\right\} \quad \ldots\ldots\ldots\ldots (7)$$

From (2), $$\delta q_i = \sum_{j=1}^{2k} \frac{\partial q_i}{\partial c_j}\delta c_j \quad , \quad \ldots\ldots\ldots\ldots (8)$$

and similarly, the left-hand side of (7) is $\delta\left(\frac{dS}{dt}\right)$ or, by 8·01 (6), $\frac{d}{dt}(\delta S)$. Hence (7) becomes

$$\frac{d}{dt}(\delta S) = \frac{d}{dt}\sum_{i=1}^{k} p_i \, \delta q_i \quad ;$$

on integration we have

$$\delta S = \sum_{i=1}^{k} p_i \, \delta q_i + C \quad , \quad \ldots\ldots\ldots\ldots (9)$$

where C is a constant.

The form of this equation suggests a transformation of the function $S(c, t)$.

We divide the $2k$ c's into two groups (i) c_j and (ii) c_{k+j} ; in each group $j = 1, 2, \ldots k$. For convenience denote the members of the first group by α_j and the members of the second group by γ_j, so that $\alpha_j \equiv c_j$ and $\gamma_j \equiv c_{k+j}$.

Then (2) is $$q_i = q_i(t, \alpha, \gamma) \quad . \quad \ldots\ldots\ldots\ldots (10)$$

There are k equations of this form ; these being supposed solved for the γ's, we can express any γ in the form

$$\gamma_j = \gamma_j(q, \alpha, t) \quad . \quad \ldots\ldots\ldots\ldots (11)$$

Corresponding to the k variations $\delta\alpha$ and the k variations $\delta\gamma$, we have from (10)

$$\delta q_i = \sum_j \frac{\partial q_i}{\partial \alpha_j}\delta\alpha_j + \sum_j \frac{\partial q_i}{\partial \gamma_j}\delta\gamma_j \quad \ldots\ldots\ldots\ldots (12)$$

There are k equations of the type (12) from which each $\delta\gamma$ can be expressed in terms of the k variations $\delta\alpha$ and the k variations δq; these latter can be regarded as independent.

In our new notation the function $S(c, t)$ is expressed as $S(\alpha, \gamma, t)$ which becomes, by means of (11), $S(q, \alpha, t)$. Hence

$$\delta S = \sum_1^k \frac{\partial S}{\partial q_i}\delta q_i + \sum_1^k \frac{\partial S}{\partial \alpha_i}\delta\alpha_i \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

The expressions for δS given in (9) and (13) must be identical; in particular, C is of the form

$$C = \sum_1^k \beta_i\,\delta\alpha_i \quad ,$$

where the β's are constants.

Since δq and $\delta\alpha$ are arbitary variations, comparison of (9) and (13) shows that

$$p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

and

$$\beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

for $i = 1, 2, \ldots k$.

Suppose that by some means we succeed in finding the function $S(q, \alpha, t)$ which we refer to as the *auxiliary function*; then the k equations (15) can be solved to give the k q's each of which is then expressed functionally in terms of the k α's and k β's as $q(t, \alpha, \beta)$; also, substituting $q = q(t, \alpha, \beta)$ in (14) we express p as $p(t, \alpha, \beta)$. This constitutes the formal solution of the canonic equations.

The constants α and β are called *canonic constants*, β_i being referred to as the *canonic conjugate* of α_i, or vice versa.

Also, we can refer to q_i and p_i as a pair of *conjugate variables*.

8·10. The Hamilton-Jacobi equation for S

We now derive a partial differential equation—frequently referred to as the Hamilton-Jacobi equation—of which a solution is the auxiliary function $S(q, \alpha, t)$. By the definition in 8·09 (6), $\frac{dS}{dt} = T + U$.

But $$\frac{dS}{dt} = \frac{\partial S(q, \alpha, t)}{\partial t} + \sum_1^k \frac{\partial S(q, \alpha, t)}{\partial q_i}\dot{q}_i \quad .$$

Hence, by 8·09 (14), $$T + U = \frac{\partial S}{\partial t} + \sum_1^k p_i\dot{q}_i \quad .$$

But, by 8·06 (6), $$2T_2 + T_1 = \sum_1^k p_i\dot{q}_i \quad .$$

Subtracting these two equations we obtain, since $T = T_2 + T_1 + T_0$,

$$\frac{\partial S}{\partial t} + T_2 - T_0 - U = 0 \quad .$$

But, by 8·06 (14), H is defined by $H(q, p, t) = T_2 - T_0 - U$, in which T_2 is expressed as $T_2(q, p, t)$, or $\overline{T}_2$;

hence $$\frac{\partial S(q, \alpha, t)}{\partial t} + H(q, p, t) = 0 \quad .$$

Also, by 8·09 (14), $$p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i} \quad .$$

Consequently, S is a solution of the partial differential equation

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

containing the full number, k, of constants of integration, α—we ignore any constant, say α_0, additive to S.

In any given problem the precise form of $H(q, p, t)$ is known; for, T—given by 8·03 (4)—is a known function of the $\dot{q}$'s, as remarked, and in particular $T_2(q, \dot{q}, t)$ is a known function of the $\dot{q}$'s and T_0 is a known function of q, t; again, the definition of the p's in 8·06 (1) and (2) shows that any p_i is a known linear function of the $\dot{q}$'s and the solution of these k equations allows any $\dot{q}_i$ to be expressed as a linear function of the p's; when these latter expressions are substituted in $T_2(q, \dot{q}, t)$ we obtain a known function of q, p, t. Finally, U being a known function, $H (\equiv T_2 - T_0 - U)$ becomes a known function of q, p, t.

8·11. General remarks on the canonic constants

In section 8·09 the original specification of the $2k$ constants of integration, $c_1, c_2, \ldots c_{2k}$, was entirely general, without any reference to the particular method employed in reaching the solutions.

Consider first the simple problem of a particle moving in a straight line according to the equation of motion $\ddot{x} = -x$. The solution can be expressed as

$$\text{(i)}\ x = \alpha_1 \sin t + \alpha_2 \cos t \quad , \text{ or (ii)}\ x = b \sin (t + \tau).$$

In (i) α_1 and α_2 are the constants of integration which would be identified with c_1 and c_2, and in (ii) b and τ are the constants of integration to be identified with c_1 and c_2. Thus, the general constants c_1 and c_2 have two different interpretations according to the two forms, or methods, of the solution.

Consider now a more complicated example, that of the undisturbed elliptic motion of a planet. If we transform the planet's rectangular coordinates in terms of generalised coordinates q—for example, polar coordinates—the resulting canonic equations can be readily solved, as will be shown in Chapter 9. With the later notation for the constants of integration ($c_i \equiv \alpha_i$ and $c_{k+i} \equiv \gamma_i$; $i = 1, 2, \ldots k$) we can write the solutions as

$$q_i = q_i(t, \alpha, \gamma) \quad , p_i = p_i(t, \alpha, \gamma) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $i=1, 2, 3$. At epoch t_0, $(q_i)_0=q_i(t_0, \alpha, \gamma)$ and $(p_i)_0=p_i(t_0, \alpha, \gamma)$; these are six equations and it is easily seen, by eliminating the three γ's, that the surviving three equations can be solved to give

$$\alpha_i=\alpha_i(t_0, q_0, p_0) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Similarly,

$$\gamma_i=\gamma_i(t_0, q_0, p_0) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Here the α's and γ's are functions of the generalised coordinates and momenta at the epoch t_0. It is then seen that the solutions (1) can be expressed as

$$q_i=q_i(t, q_0, p_0) \quad , p_i=p_i(t, q_0, p_0) \quad .$$

Evidently, the $2k$ constants of integration can be interpreted as (i) the k values of $(q_i)_0$ and the k values of $(p_i)_0$ or (ii) the functions α and γ of the q_0's and p_0's given by (2) and (3) or (iii) still more generally, *any* $2k$ independent functions of the q_0's and p_0's.

The theorem to be proved in the next section deals specifically with the solutions of the canonic equations when the constants of integration appearing in $S(q, \alpha, t)$ take any of the possible forms previously indicated.

8·12. Jacobi's theorem

The theorem is as follows :

If $S(q, \alpha, t)$ is *any* complete solution, expressed as a function of t, the k q's and k independent constants of integration $\alpha_1, \alpha_2, \ldots \alpha_k$ and obtained in any way whatsoever, of the equation

$$\frac{\partial S}{\partial t}+H\left(q, \frac{\partial S}{\partial q}, t\right)=0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

then the equations (for $i=1, 2, \ldots k$)

$$\frac{\partial S(q, \alpha, t)}{\partial \alpha_i}=\beta_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and

$$\frac{\partial S(q, \alpha, t)}{\partial q_i}=p_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

contain the solutions of the $2k$ canonic equations

$$\dot{q}_i=\frac{\partial H(q, p, t)}{\partial p_i} \quad , \quad \dot{p}_i=-\frac{\partial H(q, p, t)}{\partial q_i} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

In (2) the β's are independent constants and it is assumed that, in particular, the p_i's are independent.

The function S being known, the group of equations (2) enable us to find any q in the form

$$q_i=q_i(t, \alpha, \beta) \; ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

from (3), we can then obtain p_i in the form

$$p_i=p_i(t, \alpha, \beta) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Differentiate (2) with respect to t; then, q being a function of t, we obtain

$$\frac{\partial^2 S}{\partial \alpha_i \partial t} + \sum_{j=1}^{k} \frac{\partial^2 S}{\partial \alpha_i \partial q_j} \dot{q}_j = 0 \quad , \quad \text{or, by means of (3),}$$

$$\frac{\partial^2 S}{\partial \alpha_i \partial t} + \sum_j \frac{\partial p_j}{\partial \alpha_i} \dot{q}_j = 0 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Substitute the p's given by (6) in the equation

$$\frac{\partial S(q, \alpha, t)}{\partial t} + H(q, p, t) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

and differentiate with respect to α_i; then

$$\frac{\partial^2 S}{\partial \alpha_i \partial t} + \sum_j \frac{\partial H}{\partial p_j} \cdot \frac{\partial p_j}{\partial \alpha_i} = 0 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Subtract (7) and (9); then

$$\sum_{j=1}^{k} \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right) \frac{\partial p_j}{\partial \alpha_i} = 0 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Denoting $\left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right)$ by x_j, and putting $i = 1, 2, \ldots k$ in succession, we have the k equations, linear in the x's,

$$\begin{aligned}
&\frac{\partial p_1}{\partial \alpha_1} x_1 + \frac{\partial p_2}{\partial \alpha_1} x_2 + \ldots + \frac{\partial p_k}{\partial \alpha_1} x_k = 0 \quad , \\
&\frac{\partial p_1}{\partial \alpha_2} x_1 + \ldots\ldots\ldots + \frac{\partial p_k}{\partial \alpha_2} x_k = 0 \quad , \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad , \\
&\frac{\partial p_1}{\partial \alpha_k} x_1 + \ldots\ldots\ldots + \frac{\partial p_k}{\partial \alpha_k} x_k = 0 \quad .
\end{aligned}$$

Since $p_1, p_2, \ldots p_k$ are independent functions, then

$$\frac{\partial (p_1, p_2, \ldots p_k)}{\partial (\alpha_1, \alpha_2, \ldots \alpha_k)} \neq 0 \quad ,$$

or, expressed otherwise, the k-order determinant consisting of the coefficients of $x_1, x_2, \ldots x_k$ does not vanish. Accordingly, we must have

$$x_1 = x_2 = \ldots = x_k = 0 \quad ,$$

that is,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad . \quad (i = 1, 2, \ldots k). \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Again, differentiate (3) with respect to t; then

$$\dot{p}_i = \frac{\partial^2 S}{\partial t \partial q_i} + \sum_{j=1}^{k} \frac{\partial^2 S}{\partial q_j \partial q_i} \dot{q}_j \qquad \text{or, by means of (11),}$$

$$\dot{p}_i = \frac{\partial^2 S}{\partial t \partial q_i} + \sum_j \frac{\partial H}{\partial p_j} \cdot \frac{\partial^2 S}{\partial q_j \partial q_i} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Also, from (3) p_i can be expressed as $p_i(q, \alpha, t)$; hence, differentiating (8) with respect to q_i and noting that H contains q_i explicitly and implicitly through the p's, we obtain

$$\frac{\partial^2 S}{\partial t\, \partial q_i} + \frac{\partial H}{\partial q_i} + \sum_j \frac{\partial H}{\partial p_j} \cdot \frac{\partial p_j}{\partial q_i} = 0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Since, by (3), $\dfrac{\partial p_j}{\partial q_i} = \dfrac{\partial^2 S}{\partial q_i\, \partial q_j}$, we have at once, from (12) and (13),

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Since (11) and (14) are the canonic equations, the theorem is established.

8·13. Particular cases relating to the Hamilton-Jacobi equation

(i) *H a function of q and p only*

In this case t does not occur explicitly in H, so that $H \equiv H(q, p)$. Then, since $\dot{q} = \dfrac{\partial H}{\partial p}$, $\dot{p} = -\dfrac{\partial H}{\partial q}$, we obtain, as in Section 8·07, the equation of energy, namely,

$$H = \text{constant} = \alpha_1, \text{ say.}$$

We then have from the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} = -\alpha_1 \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

hence $S = -\alpha_1 t + S_1$, where S_1 is a function not involving t explicitly.

(ii) *One coordinate q_i absent from H*

If q_i does not occur explicitly in H, then

$$\frac{\partial H}{\partial q_i} = 0 \quad .$$

Hence, $\dot{p}_i = 0$, and, accordingly,

$$p_i = \text{constant} = \alpha_j, \text{ say.} \quad (j \neq 1).$$

We then have

$$\frac{\partial S}{\partial q_i} = \alpha_j \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

hence $S = \alpha_j q_i + S_2$, where S_2 is a function not involving q_i.

(iii) *One coordinate q_i and t absent from H*

S will evidently be of the form

$$S = -\alpha_1 t + \alpha_j q_i + S' \quad , \quad (j \neq 1), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where S' contains neither t nor q_i.

8·14. Example, continued

In this section we continue to illustrate the general method with

reference to the example considered in Sections 8·05 and 8·08. We rewrite the formula 8·08 (8) for H, namely,

$$H(q, p, t) = \tfrac{1}{2}(p_1^2 + p_2^2 + 2nq_2p_1 - 2nq_1p_2) - U$$
$$= \tfrac{1}{2}(p_1 + nq_2)^2 + \tfrac{1}{2}(p_2 - nq_1)^2 - \tfrac{1}{2}n^2(q_1^2 + q_2^2) - U \quad \ldots\ldots(1)$$

The Hamilton-Jacobi equation is then

$$\frac{\partial S}{\partial t} + \tfrac{1}{2}\left(\frac{\partial S}{\partial q_1} + nq_2\right)^2 + \tfrac{1}{2}\left(\frac{\partial S}{\partial q_2} - nq_1\right)^2 - \tfrac{1}{2}n^2(q_1^2 + q_2^2) - U = 0 \quad \ldots.(2)$$

Now t does not occur explicitly in (1). Hence, by 8·13 (1),

$$\frac{\partial S}{\partial t} = -\alpha_1$$

and thus when $-\alpha_1$ is written for $\dfrac{\partial S}{\partial t}$ in (2), the Hamilton-Jacobi equation becomes

$$\left(\frac{\partial S}{\partial q_1} + nq_2\right)^2 + \left(\frac{\partial S}{\partial q_2} - nq_1\right)^2 = n^2(q_1^2 + q_2^2) + 2U + 2\alpha_1 \quad \ldots\ldots\ldots(3)$$

Since the example is two-dimensional there will be four constants of integration in all, namely, α_1 and α_2, β_1 and β_2.

The mathematical problem consists in determining $\dfrac{\partial S}{\partial q_1}$ and $\dfrac{\partial S}{\partial q_2}$ —in terms of t, q_1, q_2 and the constants of integration α_1 and α_2—which satisfy (3). As it happens in this example, owing principally to the form in which U is expressed in terms of q_1 and q_2, little progress along the lines of the general theory can be made. But formally we can derive the function S for, assuming that $\dfrac{\partial S}{\partial q_1}$ and $\dfrac{\partial S}{\partial q_2}$ are obtainable from (3) and in terms of constants α_1 and α_2, then

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_1}\dot{q}_1 + \frac{\partial S}{\partial q_2}\dot{q}_2 \quad ,$$

so that S is given by

$$S = -\alpha_1 t + \int \frac{\partial S}{\partial q_1}\,dq_1 + \int \frac{\partial S}{\partial q_2}\,dq_2 \quad ,$$

the constant additive to S being ignored. The formal solutions of the problem then follow from the equations

$$\frac{\partial S}{\partial \alpha_i} = \beta_i \quad , \quad \frac{\partial S}{\partial q_i} = p_i \quad ,$$

where $i = 1, 2$.

8·15. General application of the Hamilton-Jacobi method

The general application to planetary and lunar theories is based on the assumption that the canonic equations can be solved by the method already

described when a part only of the Hamiltonian function H appears in the equations ; thus, if we write

$$H = H_0 + (H - H_0) \quad \ldots\ldots(1)$$

and solve the equations

$$\dot{q}_i = \frac{\partial H_0}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H_0}{\partial q_i} \quad \ldots\ldots(2)$$

by the Hamiltonian method in terms of canonic constants α and β, we shall be able to derive, as will shortly be seen, new canonic equations in which the new Hamiltonian function is related to the part $H - H_0$ in (1). It is convenient to write (1) as

$$H = H_0 - H_1 \quad \ldots\ldots(3)$$

where $H_1 = -(H - H_0)$.

The solution of (2) requires us to find a function $S(q, \alpha, t)$ satisfying the equation

$$\frac{\partial S}{\partial t} + H_0\left(q, \frac{\partial S}{\partial q}, t\right) = 0 \quad ; \quad \ldots\ldots(4)$$

we suppose, in accordance with the initial assumption, that the solution of (4) has been achieved. The solutions of the canonic equations (2) are then given by

$$\beta_r = \frac{\partial S(q, \alpha, t)}{\partial \alpha_r} \quad , \quad p_r = \frac{\partial S(q, \alpha, t)}{\partial q_r} \quad \ldots\ldots(5)$$

from which any q_i and p_i can be found in definite analytic forms represented by

$$q_i = q_i(t, \alpha, \beta) \quad , \quad p_i = p_i(t, \alpha, \beta) \quad \ldots\ldots(6)$$

In (6), α and β are conjugate canonic constants.

In accordance with (6) we can rewrite (2) as

$$\frac{\partial q_i}{\partial t} = \frac{\partial H_0}{\partial p_i} \quad , \quad \frac{\partial p_i}{\partial t} = -\frac{\partial H_0}{\partial q_i} \quad . \quad \ldots\ldots(7)$$

The method now to be followed is similar in principle to that employed in deriving Lagrange's planetary equations in Chapter 5.

Let α and β be regarded now as new variables replacing the original variables q and p through substitutions made in accordance with (6), q and p having the same analytical forms in terms of the new variables α and β as they have in the solution of (2)—or (7)—as represented by (6).

The original canonic equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad ;$$

these become by means of (6), where α and β are now variables, and by means of (3),

$$\frac{\partial q_i}{\partial t}+\sum_r\left(\frac{\partial q_i}{\partial \alpha_r}\dot{\alpha}_r+\frac{\partial q_i}{\partial \beta_r}\dot{\beta}_r\right)=\frac{\partial (H_0-H_1)}{\partial p_i} \quad ,$$

$$\frac{\partial p_i}{\partial t}+\sum_r\left(\frac{\partial p_i}{\partial \alpha_r}\dot{\alpha}_r+\frac{\partial p_i}{\partial \beta_r}\dot{\beta}_r\right)=-\frac{\partial (H_0-H_1)}{\partial q_i}$$

Hence, by (7),

$$\sum_r\left(\frac{\partial q_i}{\partial \alpha_r}\dot{\alpha}_r+\frac{\partial q_i}{\partial \beta_r}\dot{\beta}_r\right)=-\frac{\partial H_1}{\partial p_i} \quad , \qquad \text{.....(8)}$$

$$\sum_r\left(\frac{\partial p_i}{\partial \alpha_r}\dot{\alpha}_r+\frac{\partial p_i}{\partial \beta_r}\dot{\beta}_r\right)=\frac{\partial H_1}{\partial q_i} \quad . \qquad \text{.....(9)}$$

Originally, H_1 is a function of q, p ; by means of (6) it can be expressed as a function of the new variables α, β.

Accordingly,

$$\frac{\partial H_1}{\partial \alpha_j}=\sum_i\left(\frac{\partial H_1}{\partial q_i}\cdot\frac{\partial q_i}{\partial \alpha_j}+\frac{\partial H_1}{\partial p_i}\cdot\frac{\partial p_i}{\partial \alpha_j}\right)$$

and, by means of (8) and (9),

$$\frac{\partial H_1}{\partial \alpha_j}=\sum_r\left\{\dot{\alpha}_r\sum_i\frac{\partial (q_i, p_i)}{\partial (\alpha_j, \alpha_r)}+\dot{\beta}_r\sum_i\frac{\partial (q_i, p_i)}{\partial (\alpha_j, \beta_r)}\right\} \quad . \qquad \text{.....(10)}$$

Denote $$\sum_i\frac{\partial (q_i, p_i)}{\partial (\alpha_j, \alpha_r)} \text{ by } [\alpha_j, \alpha_r] \quad , \qquad \text{.....(11)}$$

with a similar definition for the coefficient of $\dot{\beta}_r$. Then these expressions, so defined, are generalised Lagrangian brackets. We now have (10) in the form

$$\frac{\partial H_1}{\partial \alpha_j}=\sum_r\dot{\alpha}_r[\alpha_j, \alpha_r]+\sum_r\dot{\beta}_r[\alpha_j, \beta_r] \quad . \qquad \text{.....(12)}$$

Similarly,

$$\frac{\partial H_1}{\partial \beta_j}=\sum_r\dot{\alpha}_r[\beta_j, \alpha_r]+\sum_r\dot{\beta}_r[\beta_j, \beta_r] \quad . \qquad \text{.....(13)}$$

We now make use of (5). Let $S(q, \alpha, t)$ be transformed, by means of the first group in (6), into a function of t, α, β ; we denote it by $S'(t, \alpha, \beta)$ Since $S(q, \alpha, t)$ contains α_j explicitly we have

$$\frac{\partial S'}{\partial \alpha_j}=\frac{\partial S(q, \alpha, t)}{\partial \alpha_j}+\sum_i\frac{\partial S}{\partial q_i}\cdot\frac{\partial q_i}{\partial \alpha_j}$$

or, by means of (5),

$$\frac{\partial S'}{\partial \alpha_j}=\beta_j+\sum_i p_i\frac{\partial q_i}{\partial \alpha_j} \quad . \qquad \text{.....(14)}$$

Similarly,

$$\frac{\partial S'}{\partial \beta_j}=\sum_i p_i\frac{\partial q_i}{\partial \beta_j} \quad . \qquad \text{.....(15)}$$

$$\text{Now } [\alpha_j, \alpha_r] = \sum_i \left(\frac{\partial q_i}{\partial \alpha_j} \cdot \frac{\partial p_i}{\partial \alpha_r} - \frac{\partial q_i}{\partial \alpha_r} \cdot \frac{\partial p_i}{\partial \alpha_j} \right) = \frac{\partial}{\partial \alpha_r} \sum_i p_i \frac{\partial q_i}{\partial \alpha_j} - \frac{\partial}{\partial \alpha_j} \sum_i p_i \frac{\partial q_i}{\partial \alpha_r}$$

$$= \frac{\partial}{\partial \alpha_r} \left\{ \frac{\partial S'}{\partial \alpha_j} - \beta_j \right\} - \frac{\partial}{\partial \alpha_j} \left\{ \frac{\partial S'}{\partial \alpha_r} - \beta_r \right\}$$

by (14). Since α and β are independent, $[\alpha_j, \alpha_r] = 0$; hence all brackets of this form vanish.

Again, using (15) we find that

$$[\beta_j, \beta_r] = \frac{\partial}{\partial \beta_r} \sum_i p_i \frac{\partial q_i}{\partial \beta_j} - \frac{\partial}{\partial \beta_j} \sum_i p_i \frac{\partial q_i}{\partial \beta_r} = \frac{\partial}{\partial \beta_r} \left(\frac{\partial S'}{\partial \beta_j} \right) - \frac{\partial}{\partial \beta_j} \left(\frac{\partial S'}{\partial \beta_r} \right) = 0 \ ;$$

thus, all brackets of this type vanish.

Finally,
$$[\alpha_j, \beta_r] = \frac{\partial}{\partial \beta_r} \sum_i p_i \frac{\partial q_i}{\partial \alpha_j} - \frac{\partial}{\partial \alpha_j} \sum p_i \frac{\partial q_i}{\partial \beta_r}$$

$$= \frac{\partial}{\partial \beta_r} \left\{ \frac{\partial S'}{\partial \alpha_j} - \beta_j \right\} - \frac{\partial}{\partial \alpha_j} \left(\frac{\partial S'}{\partial \beta_r} \right) \quad .$$

Hence $[\alpha_j, \beta_r] = 0$, if $r \neq j$,

and $[\alpha_j, \beta_j] = -1$,

or, by definition, $[\beta_j, \alpha_j] = 1$.

From (13) and (12) we then have

$$\dot{\alpha}_j = \frac{\partial H_1}{\partial \beta_j} \quad , \quad \dot{\beta}_j = -\frac{\partial H_1}{\partial \alpha_j} \quad . \qquad \text{.....(16)}$$

These are canonic equations. The process can evidently be repeated by solving (16) with a new Hamiltonian function taken to be a convenient part of H_1.

8·16. Canonic equations in terms of the disturbing function

The fundamental equations for the disturbed motion of a planet P are in rectangular coordinates ; that in x is

$$\ddot{x} + \frac{\mu x}{r^3} = \frac{\partial R}{\partial x} \quad , \qquad \text{.....(1)}$$

where R is the disturbing function. R is a function of m_1 and (x_1, y_1, z_1)—the mass and coordinates of the disturbing planet P_1—and the coordinates (x, y, z) of P. We suppose that x_1, y_1 and z_1 are expressed in terms of the time and the elliptic elements of the orbit of P_1. Then $R = R(x, y, z, t)$.

The force-function U and the kinetic function T occurring in the general theory are, in the present instance, given by

$$U = \frac{m\mu}{r} + mR \quad , \quad T = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad .$$

In (1) the factor m has disappeared and so it is sufficient to define U and T—as in the example in 8·05 where the factor m is cancelled—by

$$U=\frac{\mu}{r}+R=U_0+R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $U_0=\frac{\mu}{r}$, and $T=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$.

Expressing x, y, z in terms of three generalised coordinates q_1, q_2, q_3 by equations of the type $x=x(q, t)$, we convert the three equations represented by (1) into the three pairs of canonic equations

$$\dot{q}_i=\frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i=-\frac{\partial H}{\partial q_i} \quad , \quad (i=1, 2, 3), \quad \ldots\ldots\ldots\ldots\ldots(3)$$

where $$H=\bar{T}_2-T_0-U=H_0-R \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

in which $$H_0=\bar{T}_2-T_0-U_0 \quad .$$

If R is neglected, the canonic equations are given by (3) in which H is replaced by H_0. The solution for q_i is then given by $q_i=q_i(t, \alpha, \beta)$, where α and β are the canonic constants; p_i is given in similar form. The canonic constants thus arise in solving the problem of elliptic motion and are clearly, in general, functions of the six orbital elements.

Comparison of (4) and 8·15 (3) show that when α and β are now regarded as variables the derived canonic equations—compare 8·15 (16)—are

$$\dot{\alpha}_i=\frac{\partial R}{\partial \beta_i} \quad , \quad \dot{\beta}_i=-\frac{\partial R}{\partial \alpha_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The process can be further extended; for, if we write $R=R_0-R_1$ and solve the equations (5) with R_0 as the Hamiltonian function obtaining new canonic constants a_i, b_i then the latter, regarded as new variables, will be associated with the canonic equations

$$\dot{a}_i=\frac{\partial R_1}{\partial b_i} \quad , \quad \dot{b}_i=-\frac{\partial R_1}{\partial a_i} \quad .$$

This is the method followed by Delaunay in his lunar theory, the series of operations being carried on in the way indicated until the whole of the disturbing function R is exhausted (see Chapter 12).

8·17. Jacobi's reciprocal relationships

We shall suppose without loss of generality that, as in 8·15, α_i and β_i are the k pairs of canonic constants appearing in the solution of the equations

$$\dot{q}_i=\frac{\partial H_0}{\partial p_i} \quad , \quad \dot{p}_i=-\frac{\partial H_0}{\partial q_i} \quad , \quad (i=1, 2, \ldots k).$$

As we have seen, the solutions are given by

$$q_i = q_i(t, \alpha, \beta) \quad , \quad p_i = p_i(t, \alpha, \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

These equations can be solved, in principle, to give any α or any β in terms of q, p, t; the results are expressed functionally as

$$\alpha_r = \alpha_r(q, p, t) \quad , \quad \beta_r = \beta_r(q, p, t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Jacobi obtained four remarkable relationships between the differential coefficients of q and of p (expressed in terms of α and β) with respect to the α's and the β's and the differential coefficients of α and β (expressed in terms of q and p) with respect to the q's and p's.

We assume, as in 8·15, that the original Hamiltonian function H is $H_0 - H_1$; here H_0 and H_1 are functions of q, p, t.

When α and β are regarded as new variables we obtain the equations (8) and (9) of 8·15; these are

$$\sum_r \left(\frac{\partial q_i}{\partial \alpha_r} \dot{\alpha}_r + \frac{\partial q_i}{\partial \beta_r} \dot{\beta}_r \right) = -\frac{\partial H_1}{\partial p_i} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\sum_r \left(\frac{\partial p_i}{\partial \alpha_r} \dot{\alpha}_r + \frac{\partial p_i}{\partial \beta_r} \dot{\beta}_r \right) = \frac{\partial H_1}{\partial q_i} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Further, from 8·15 (16), the α's and β's are bound together by the canonic equations

$$\dot{\alpha}_r = \frac{\partial H_1}{\partial \beta_r} \quad , \quad \dot{\beta}_r = -\frac{\partial H_1}{\partial \alpha_r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which $H_1 = H_1(t, \alpha, \beta)$

Let δq_i, δp_i be independent variations in q_i, p_i $(i = 1, 2, \ldots k)$. Then, the variation δH_1, when $H_1 = H_1(q, p, t)$, is given by

$$\delta H_1 = \sum_i \left(\frac{\partial H_1}{\partial q_i} \delta q_i + \frac{\partial H_1}{\partial p_i} \delta p_i \right)$$

or, by means of (3) and (4), by

$$\delta H_1 = \sum_i \sum_r \delta q_i \left[\frac{\partial p_i}{\partial \alpha_r} \dot{\alpha}_r + \frac{\partial p_i}{\partial \beta_r} \dot{\beta}_r \right] - \sum_i \sum_r \delta p_i \left[\frac{\partial q_i}{\partial \alpha_r} \dot{\alpha}_r + \frac{\partial q_i}{\partial \beta_r} \dot{\beta}_r \right] \quad . \quad \ldots\ldots (6)$$

By (2), the variations $\delta \alpha_r$, $\delta \beta_r$ are given by

$$\delta \alpha_r = \sum_i \left(\frac{\partial \alpha_r}{\partial q_i} \delta q_i + \frac{\partial \alpha_r}{\partial p_i} \delta p_i \right) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

and

$$\delta \beta_r = \sum_i \left(\frac{\partial \beta_r}{\partial q_i} \delta q_i + \frac{\partial \beta_r}{\partial p_i} \delta p_i \right) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Also, when H_1 is expressed as $H_1(t, \alpha, \beta)$,

$$\delta H_1 = \sum_r \left(\frac{\partial H_1}{\partial \alpha_r} \delta \alpha_r + \frac{\partial H_1}{\partial \beta_r} \delta \beta_r \right) = \sum_r (\dot{\alpha}_r \, \delta \beta_r - \dot{\beta}_r \, \delta \alpha_r) \quad ,$$

by (5); hence, by (7) and (8),

$$\delta H_1 = \sum_i \sum_r \delta q_i \left[\frac{\partial \beta_r}{\partial q_i} \dot{\alpha}_r - \frac{\partial \alpha_r}{\partial q_i} \dot{\beta}_r \right] + \sum_i \sum_r \delta p_i \left[\frac{\partial \beta_r}{\partial p_i} \dot{\alpha}_r - \frac{\partial \alpha_r}{\partial p_i} \dot{\beta}_r \right] \quad \ldots(9)$$

The right-hand sides of (6) and (9) must be identically equal and, since δq_i and δp_i are independent variations, the coefficients of δq_i in (6) and (9) must be equal; similarly, the coefficients of δp_i must be equal. We then have the following results:

$$\sum_r \left(\frac{\partial p_i}{\partial \alpha_r} - \frac{\partial \beta_r}{\partial q_i} \right) \dot{\alpha}_r + \sum_r \left(\frac{\partial p_i}{\partial \beta_r} + \frac{\partial \alpha_r}{\partial q_i} \right) \dot{\beta}_r = 0 \quad , \quad \ldots(10)$$

$$\sum_r \left(\frac{\partial q_i}{\partial \alpha_r} + \frac{\partial \beta_r}{\partial p_i} \right) \dot{\alpha}_r + \sum_r \left(\frac{\partial q_i}{\partial \beta_r} - \frac{\partial \alpha_r}{\partial p_i} \right) \dot{\beta}_r = 0 \quad . \quad \ldots(11)$$

Now, (10) and (11) are k pairs of equations in the k $\dot{\alpha}$'s and the k $\dot{\beta}$'s; since these are independent, the coefficients of $\dot{\alpha}_r$ and $\dot{\beta}_r$ in each of (10) and (11) must vanish. Accordingly,

$$\frac{\partial p_i}{\partial \alpha_r} = \frac{\partial \beta_r}{\partial q_i} \quad ; \quad \frac{\partial p_i}{\partial \beta_r} = -\frac{\partial \alpha_r}{\partial q_i} \quad ; \quad \ldots(12)$$

$$\frac{\partial q_i}{\partial \alpha_r} = -\frac{\partial \beta_r}{\partial p_i} \quad ; \quad \frac{\partial q_i}{\partial \beta_r} = \frac{\partial \alpha_r}{\partial p_i} \quad . \quad \ldots(13)$$

These are Jacobi's reciprocal relationships between pairs of conjugate variables q, p and pairs of conjugate constants α, β.

From (12) and (13) we have at once

$$\frac{\partial (q_i, p_i)}{\partial (\alpha_r, \beta_r)} = \frac{\partial (\alpha_r, \beta_r)}{\partial (q_i, p_i)} \quad . \quad \ldots(14)$$

Chapter 9

THE CANONIC CONSTANTS FOR THE ELLIPTIC ORBIT

9·01. Derivation of the function S

In this chapter we solve the equations of motion for an undisturbed planet by the Hamilton-Jacobi method with two objects, first, to illustrate the method in detail and, second, to derive the important relations between the orbital elements and the canonic constants which appear in the course of the solution. The first step is to derive the function S satisfying the Hamilton-Jacobi partial differential equation.

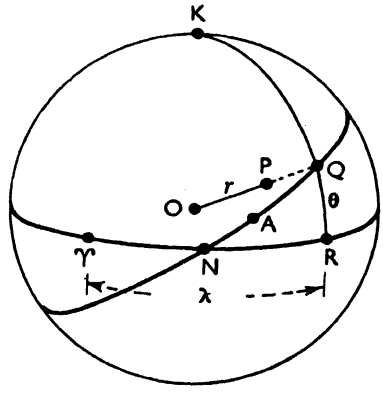


FIG. 21

Let (x, y, z) be the rectangular coordinates of the planet P (mass, m) with respect to ecliptic axes with the origin O at the Sun, the Sun's mass being m_0. In Fig. 21 let r be the radius vector OP, λ the ecliptic longitude ΥR and θ the corresponding latitude RQ. Then

$$x = r\cos\lambda\cos\theta, \quad y = r\sin\lambda\cos\theta, \quad z = r\sin\theta \quad . \quad \ldots\ldots\ldots\ldots(1)$$

The generalised coordinates q_1, q_2 and q_3 are to be identified with r, λ and θ respectively.

The equations of motion in rectangular coordinates are

$$\ddot{x} = \frac{\partial}{\partial x}\left(\frac{\mu}{r}\right) = \frac{\partial U_0}{\partial x} \quad .$$

with two similar equations in y and z; here, $\mu = G(m + m_0)$ and, in the notation of 8·16, $U_0 = \dfrac{\mu}{r}$. Also, in the notation of 8·16, $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ or, by means of (1),

$$T = \tfrac{1}{2}(\dot{r}^2 + r^2\cos^2\theta\,.\,\dot{\lambda}^2 + r^2\dot{\theta}^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This expression for T is a homogeneous quadratic function of the $\dot{q}$'s; hence $T_2 = T$ and $T_0 = 0$.

As in the general theory p_1, p_2 and p_3 are defined by the following:

$$p_1 = \frac{\partial T}{\partial \dot{r}} \quad , \quad p_2 = \frac{\partial T}{\partial \dot{\lambda}} \quad , \quad p_3 = \frac{\partial T}{\partial \dot{\theta}} \quad ;$$

hence $p_1=\dot{r}$, $p_2=r^2\cos^2\theta\,.\,\dot{\lambda}$, $p_3=r^2\dot{\theta}$.

The formula (2) then becomes, when expressed in terms of q and p,

$$T=\tfrac{1}{2}\left(p_1^2+\frac{p_2^2}{r^2\cos^2\theta}+\frac{p_3^2}{r^2}\right) .$$

The Hamiltonian function $H\equiv T_2-T_0-U_0$ is given by

$$H(q,p)=\tfrac{1}{2}\left(p_1^2+\frac{p_2^2}{r^2\cos^2\theta}+\frac{p_3^2}{r^2}\right)-\frac{\mu}{r} \quad , \quad \ldots\ldots\ldots\ldots(3)$$

and the Hamiltonian-Jacobi equation is then

$$\frac{\partial S}{\partial t}+\tfrac{1}{2}\left\{\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2\cos^2\theta}\left(\frac{\partial S}{\partial\lambda}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2\right\}-\frac{\mu}{r}=0 \quad . \quad \ldots\ldots\ldots(4)$$

Now t and λ do not appear explicitly in (3); consequently, by 8·13 (3),

$$S=-\alpha_1 t+\alpha_3\lambda+S'(r,\theta) \quad , \quad \ldots\ldots\ldots\ldots(5)$$

where S' is a function of r and θ only.

By means of (5) the Hamilton-Jacobi equation (4) becomes

$$2\alpha_1 r^2+2\mu r-r^2\left(\frac{\partial S'}{\partial r}\right)^2=\left(\frac{\partial S'}{\partial\theta}\right)^2+\alpha_3^2\sec^2\theta \quad , \quad \ldots\ldots\ldots\ldots(6)$$

in which the variables r, θ have been separated, suggesting that S' may be written in the form

$$S'(r,\theta)=S_1(r)+S_2(\theta) \quad . \quad \ldots\ldots\ldots\ldots(7)$$

It follows that each side of (6) is equal to a constant which we take to be α_2^2. Then

$$r^2\left(\frac{\partial S_1}{\partial r}\right)^2-2\alpha_1 r^2-2\mu r+\alpha_2^2=0 \quad , \quad \ldots\ldots\ldots\ldots(8)$$

and

$$\left(\frac{\partial S_2}{\partial\theta}\right)^2+\alpha_3^2\sec^2\theta=\alpha_2^2 \quad . \quad \ldots\ldots\ldots\ldots(9)$$

We now have

$$S=-\alpha_1 t+\alpha_3\lambda+S_1(r)+S_2(\theta) \quad \ldots\ldots\ldots\ldots(10)$$

where, by (8),

$$S_1=\int_{r_1}^{r}\sqrt{(2\alpha_1 r^2+2\mu r-\alpha_2^2)}\,\frac{dr}{r} \quad \ldots\ldots\ldots\ldots(11)$$

in which r_1 will be specified immediately, and by (9),

$$S_2=\int_0^{\theta}\sqrt{(\alpha_2^2-\alpha_3^2\sec^2\theta)}\,d\theta \quad . \quad \ldots\ldots\ldots\ldots(12)$$

We take r_1 to be the smaller of the two roots r_1 and r_2 of the equation

$$y\equiv 2\alpha_1 r^2+2\mu r-\alpha_2^2=0\,; \quad \ldots\ldots\ldots\ldots(13)$$

we assume that r_1 and r_2 are real and positive.

Then $$r_1+r_2=-\frac{\mu}{\alpha_1} \quad , \quad r_1r_2=-\frac{\alpha_2^2}{2\alpha_1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

so that, by elimination of r_2, $r_1=r_1(\alpha_1, \alpha_2)$.

Thus, no new constant is introduced into $S_1(r)$ in (10). A similar remark applies to $S_2(\theta)$. The choice of the lower limits in each of the integrals (11) and (12) is within our competence since, according to the theorem of 8·12, all we require is *any* function S satisfying the equation (4) and containing three independent constants; these latter are α_1, α_2 and α_3.

Now y, defined by (13), must be positive since the integrand of (11) contains the factor $y^{\frac{1}{2}}$. Now r_1 and r_2 are positive; hence, by (14), α_1 is negative. Then we can write

$$y^{\frac{1}{2}}=(-2\alpha_1)^{\frac{1}{2}}\sqrt{\{(r-r_1)(r_2-r)\}} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

and it is evident that y is positive or zero for all values of r in the range $r_1 \leqslant r \leqslant r_2$.

This restriction in the values of the radius vector, r, suggests an elliptic orbit, r_1 and r_2 corresponding to perihelion and aphelion respectively. The actual proof that the orbit is elliptic comes later.

9·02. Formal expressions of the integrals

The coordinates (r, λ, θ) of the planet at any time t are given by

$$\frac{\partial S}{\partial \alpha_1}=\beta_1 \quad , \quad \frac{\partial S}{\partial \alpha_3}=\beta_3 \quad , \quad \frac{\partial S}{\partial \alpha_2}=\beta_2 \quad .$$

We deal with these in the order indicated.

From 9·01 (10), these are

$$-t+\frac{\partial S_1(r)}{\partial \alpha_1}=\beta_1 \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

since S_2 does not contain α_1;

$$\lambda+\frac{\partial S_2(\theta)}{\partial \alpha_3}=\beta_3 \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

since S_1 does not contain α_3;

and $$\frac{\partial S_1(r)}{\partial \alpha_2}+\frac{\partial S_2(\theta)}{\partial \alpha_2}=\beta_2 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

9·03. The integral $\frac{\partial S}{\partial \alpha_1}=\beta_1$

From (1) of the previous section, $t+\beta_1=\frac{\partial S_1}{\partial \alpha_1}$, where S_1 is given by 9·01 (11). We then have, remembering that the lower limit r_1 of the integral is a function of α_1,

$$t+\beta_1=\int_{r_1}^{r}\frac{r\,dr}{y^{\frac{1}{2}}}-\frac{\partial r_1}{\partial \alpha_1}\left[\frac{1}{r}\sqrt{(2\alpha_1 r^2+2\mu r-\alpha_2^2)}\right]_{r=r_1} .$$

The last term on the right-hand side is zero, since r_1 is a root of the equation $y=0$. Then, making use of 9·10 (15), we obtain

$$t+\beta_1=\frac{1}{\sqrt{(-2\alpha_1)}}\int_{r_1}^{r}\frac{r\,dr}{\sqrt{\{(r-r_1)(r_2-r)\}}} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If our object is simply to identify the canonic constants in terms of the elliptic elements, we see from (1) that $t+\beta_1=0$ when $r=r_1$; accordingly, $\beta_1=-\tau$, where τ is the time of perihelion passage.

We write r_1 and r_2 in terms of two new constants a and e given by

$$r_1=a(1-e) \quad , \quad r_2=a(1+e)$$

in which a and e are identified as the semi-major axes and the eccentricity respectively. From 9·01 (14), we then have

$$\alpha_1=-\frac{\mu}{2a} \quad , \quad \alpha_2=\sqrt{\{\mu a(1-e^2)\}} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

We now proceed to evaluate the integral in (1), being guided by our previous knowledge of the elliptic formulae. Let r be defined in terms of a new variable E by

$$r=a(1-e\cos E), \quad \text{such that} \quad E=0 \quad \text{when} \quad r=r_1. \quad \text{Then}$$
$$r-r_1=ae(1-\cos E) \quad , \quad r_2-r=ae(1+\cos E) \quad , \quad dr=ae\sin E\,dE,$$

and (1) becomes

$$\sqrt{(-2\alpha_1)}\,(t+\beta_1)=a\int_0^E(1-e\cos E)\,dE$$

or

$$E-e\sin E=\frac{1}{a}\sqrt{(-2\alpha_1)}\,.\,(t+\beta_1)\equiv n(t-\tau) \quad ,$$

which is evidently Kepler's equation.

Thus, $n=\frac{1}{a}\sqrt{(-2\alpha_1)}$ so that, from (2),

$$\mu=n^2a^3 \quad .$$

The identifications obtained in this section are as follows :

$$\beta_1=-\tau\equiv\chi/n \quad ; \; \alpha_1=-\frac{\mu}{2a} \quad ; \; \alpha_2=\sqrt{\{\mu a(1-e^2)\}} \quad .$$

9·04. The integral $\frac{\partial S}{\partial\alpha_3}=\beta_3$

From 9·01 (12) and 9·02 (2) we have

$$\beta_3=\frac{\partial S}{\partial\alpha_3}=\lambda-\alpha_3\int_0^{\theta}\frac{\sec^2\theta\,d\theta}{\sqrt{(\alpha_2^2-\alpha_3^2\sec^2\theta)}}$$
$$=\lambda-\int_0^{\theta}\frac{\sec^2\theta\,d\theta}{\sqrt{\left(\frac{\alpha_2^2-\alpha_3^2}{\alpha_3^2}-\tan^2\theta\right)}} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We must have $|\alpha_2| > |\alpha_3|$. Let ϕ be defined by

$$\alpha_2^2 - \alpha_3^2 = \alpha_3^2 \tan^2 \phi \quad \ldots\ldots(2)$$

and we suppose that ϕ is a positive acute angle. Then (1) becomes, on integration,

$$\beta_3 = \lambda - \sin^{-1}\left(\frac{\tan\theta}{\tan\phi}\right) ,$$

where $\phi \geqslant |\theta|$, whence

$$\sin(\lambda - \beta_3) = \tan\theta \cot\phi \quad \ldots\ldots(3)$$

In terms of rectangular coordinates given by 9·01 (1), this equation is

$$y\cos\beta_3 - x\sin\beta_3 = z\cot\phi$$

which shows that the planet moves in a plane containing the Sun, the plane cutting the sphere in Fig, 21 in a great circle.

The formula (3) shows that, when $\theta = 0$, then $\lambda - \beta_3 = 0°$ or $180°$; we define β_3 in terms of the former value so that β_3 is the longitude of the *ascending* node. Thus

$$\beta_3 = \Omega \quad . \quad \ldots\ldots(4)$$

Further, (3) shows that the maximum value of θ is ϕ. But the maximum of θ for points on a great circle is the inclination of the great circle to the reference plane, the ecliptic; hence ϕ is the inclination i. Accordingly, from (2),

$$\alpha_3 = \alpha_2 \cos i \quad , \quad \ldots\ldots(5)$$

or, from 9·03 (3) ,

$$\alpha_3 = \sqrt{\{\mu a(1-e^2)\}}\cos i \quad . \quad \ldots\ldots(6)$$

The identifications of β_3 and α_3 are given by (4) and (6).

9·05. The integral $\dfrac{\partial S}{\partial \alpha_2} = \beta_2$

From 9·02 (3), 9·01 (11) and (12),

$$\beta_2 = \frac{\partial S}{\partial \alpha_2} = \alpha_2 \int_0^\theta \frac{d\theta}{\sqrt{(\alpha_2^2 - \alpha_3^2 \sec^2\theta)}} - \alpha_2 \int_{r_1}^{r} \frac{dr}{r y^{1/2}} - \frac{\partial r_1}{\partial \alpha_2}\left(\frac{y^{1/2}}{r}\right)_{r=r_1} \quad \ldots(1)$$

The final term on the right is zero.

Consider the first term which we denote by I_1; then

$$I_1 = \alpha_2 \int_0^\theta \frac{\cos\theta\, d\theta}{\sqrt{(\alpha_2^2 - \alpha_3^2 - \alpha_2^2 \sin^2\theta)}} = \int_0^\theta \frac{\cos\theta\, d\theta}{\sqrt{(\sin^2 i - \sin^2\theta)}}$$

by means of 9·04 (2), ϕ being replaced by i.

Hence $$I_1 = \sin^{-1}\left(\frac{\sin\theta}{\sin i}\right) \quad .$$

From the triangle QNR in Fig. 21, we have

$$\sin NQ = \frac{\sin\theta}{\sin QNR} = \frac{\sin\theta}{\sin i} \quad ;$$

consequently, $I_1 = NQ$ .....(2)

Let I_2 denote the second term in (1); then

$$I_2 = \frac{\alpha_2}{\sqrt{(-2\alpha_1)}} \int_{r_1}^{r} \frac{dr}{r\sqrt{\{(r-r_1)(r_2-r)\}}} \quad .$$

In terms of E, the integral is

$$\frac{1}{a}\int_0^E \frac{dE}{1-e\cos E} \quad \text{or} \quad \frac{1}{a(1-e)} \int_0^E \frac{\sec^2 \frac{E}{2}\, dE}{1+\frac{1+e}{1-e}\tan^2 \frac{E}{2}} \quad .$$

Let f be defined by

$$\tan \frac{f}{2} = \sqrt{\left(\frac{1+e}{1-e}\right)} \tan \frac{E}{2} \quad .$$

Then, it is easily seen that $I_2 = f$.

It is evident, from our knowledge of the elliptic formulae, that f is the true anomaly AQ. Hence, returning to (1), we see that

$$\beta_2 = NQ - AQ = NA \; ;$$

that is $\beta_2 = \omega$.

9·06. Summary of identifications

The collected results of the previous sections are as follows :

$$\left.\begin{array}{lll} \alpha_1 = -\dfrac{\mu}{2a} & , & \beta_1 = -\tau \equiv \chi/n \\ \alpha_2 = \sqrt{\{\mu a(1-e^2)\}} & , & \beta_2 = \omega \\ \alpha_3 = \sqrt{\{\mu a(1-e^2)\}}\cos i & , & \beta_3 = \Omega \end{array}\right\} \quad .....(1)$$

From these, the elliptic elements in terms of the canonic constants are as follows :

$$\left.\begin{array}{lll} a = -\dfrac{\mu}{2\alpha_1} & , & \tau \equiv -\chi/n = -\beta_1 \\ e = \sqrt{\left\{1 + \dfrac{2\alpha_1\alpha_2^2}{\mu^2}\right\}} & , & \omega = \beta_2 \\ i = \cos^{-1}\left(\dfrac{\alpha_3}{\alpha_2}\right) & , & \Omega = \beta_3 \end{array}\right\} \quad .....(2)$$

In terms of ϵ, ϖ and Ω we have

$$\beta_1 = \frac{1}{n}(\epsilon - \varpi) \quad , \quad \beta_2 = \varpi - \Omega \quad , \quad \beta_3 = \Omega \quad .....(3)$$

and

$$\epsilon = n\beta_1 + \beta_2 + \beta_3 \quad , \quad \varpi = \beta_2 + \beta_3 \quad , \quad \Omega = \beta_3 \quad .....(4)$$

9·07. Poisson's brackets

When the solutions for an elliptic orbit have been obtained, as explained in the previous sections, in terms of canonic constants α_i and β_i

($i = 1, 2, 3$), the equations for the disturbed motion of the planet are

$$\dot{\alpha}_i = \frac{\partial R}{\partial \beta_i} \quad , \quad \beta_i = -\frac{\partial R}{\partial \alpha_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

We shall denote by $a_m (m = 1, 2, \ldots 6)$ any element of the elliptic orbit. Each element is expressed in terms of the canonic constants by the formulae (2) and (4) of the preceding section (we take ϵ, ϖ and Ω as the " angular " elements) ; we rewrite these for convenience :

$$a_1 \equiv a = -\frac{\mu}{2\alpha_1} ; \; a_2 \equiv e = \sqrt{\left(1 + \frac{2\alpha_1 \alpha_2^2}{\mu^2}\right)} ; \; a_3 \equiv i = \cos^{-1}\left(\frac{\alpha_3}{\alpha_2}\right) ; \quad \ldots (2)$$

$$a_4 \equiv \epsilon = n\beta_1 + \beta_2 + \beta_3 ; \; a_5 \equiv \varpi = \beta_2 + \beta_3 ; \; a_6 \equiv \Omega = \beta_3 \quad \ldots\ldots\ldots\ldots (3)$$

Poisson's brackets arise in the derivation of Lagrange's planetary equations by means of (1), (2) and (3). Since, in general, $a_m = a_m(\alpha, \beta)$, then

$$\dot{a}_m = \sum_{r=1}^{3} \left(\frac{\partial a_m}{\partial \alpha_r} \dot{\alpha}_r + \frac{\partial a_m}{\partial \beta_r} \dot{\beta}_r \right) = \sum_r \left(\frac{\partial a_m}{\partial \alpha_r} \cdot \frac{\partial R}{\partial \beta_r} - \frac{\partial a_m}{\partial \beta_r} \cdot \frac{\partial R}{\partial \alpha_r} \right)$$

by (1). Originally, R is a function of a_m and for use in (1) it is converted into $R(\alpha, \beta)$ by means of (2) and (3). We have

$$\frac{\partial R}{\partial \alpha_r} = \sum_{s=1}^{6} \frac{\partial R}{\partial a_s} \cdot \frac{\partial a_s}{\partial \alpha_r} \quad \text{and} \quad \frac{\partial R}{\partial \beta_r} = \sum_{s=1}^{6} \frac{\partial R}{\partial a_s} \cdot \frac{\partial a_s}{\partial \beta_r} \quad .$$

Hence

$$\dot{a}_m = \sum_{s=1}^{6} \frac{\partial R}{\partial a_s} \sum_{r=1}^{3} \left(\frac{\partial a_m}{\partial \alpha_r} \cdot \frac{\partial a_s}{\partial \beta_r} - \frac{\partial a_m}{\partial \beta_r} \cdot \frac{\partial a_s}{\partial \alpha_r} \right) \quad .$$

In Chapter 5 we expressed the Lagrangian bracket $[a_m, a_s]$ in terms of α_i and γ_i, where $\gamma_i = -\beta_i$, by the equation 5·10 (7) :

$$[a_m, a_s] = \sum_{i=1}^{3} \frac{\partial(\alpha_i, \gamma_i)}{\partial(a_m, a_s)} \quad .$$

We define the Poisson bracket in terms of α and γ by

$$\{a_m, a_s\} = \sum_{r=1}^{3} \frac{\partial(a_m, a_s)}{\partial(\alpha_r, \gamma_r)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Then

$$\dot{a}_m = -\sum_{s=1}^{6} \{a_m, a_s\} \frac{\partial R}{\partial a_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

This last equation is simply the typical Lagrangian planetary equation expressed in terms of Poisson's brackets.

The brackets are easily evaluated.

(i) $\{a, a_s\} = -\sum_r \frac{\partial a}{\partial \alpha_r} \cdot \frac{\partial a_s}{\partial \beta_r}$, since a is not a function of β_r. Further, $a \equiv a(\alpha_1)$; therefore the bracket reduces to $-\frac{\partial a}{\partial \alpha_1} \cdot \frac{\partial a_s}{\partial \beta_1}$. From (3) it is seen that β_1 occurs only with ϵ. The only bracket of type (1) is then

$$\{a, \epsilon\} = -\frac{n\mu}{2\alpha_1^2} = -\frac{2}{na} \quad .$$

(ii) $\{e, a_s\} = -\Sigma \frac{\partial e}{\partial \alpha_r} \cdot \frac{\partial a_s}{\partial \beta_r}$, since e is not a function of β_r. It is easily seen that all the brackets are zero except those for which $a_s = \epsilon$ or ϖ. Then

$$\{e, \epsilon\} = -\left(n \frac{\partial e}{\partial \alpha_1} + \frac{\partial e}{\partial \alpha_2}\right) = \frac{\cos\phi(1-\cos\phi)}{na^2 e} ,$$

and

$$\{e, \varpi\} = -\frac{\partial e}{\partial \alpha_2} = \frac{\cos\phi}{na^2 e} ,$$

where $\phi \equiv \sin^{-1}(e)$.

(iii) $\{i, a_s\} = -\frac{\partial i}{\partial \alpha_2} \cdot \frac{\partial a_s}{\partial \beta_2} - \frac{\partial i}{\partial \alpha_3} \cdot \frac{\partial a_s}{\partial \beta_3}$, since i is a function of α_2, α_3 only. It is evident that a_s can be ϵ, ϖ or Ω. Then

$$\{i, \epsilon\} = -\frac{\partial i}{\partial \alpha_2} \cdot \frac{\partial \epsilon}{\partial \beta_2} - \frac{\partial i}{\partial \alpha_3} \cdot \frac{\partial \epsilon}{\partial \beta_3} = \frac{\sec\phi(1-\cos i)}{na^2 \sin i} ;$$

$$\{i, \varpi\} = -\frac{\partial i}{\partial \alpha_2} - \frac{\partial i}{\partial \alpha_3} = \{i, \epsilon\} ;$$

$$\{i, \Omega\} = -\frac{\partial i}{\partial \alpha_3} = \frac{\sec\phi}{na^2 \sin i} .$$

The results in (i), (ii) and (iii) are collected for reference :

$$\left.\begin{aligned} \{a, \epsilon\} &= -\frac{2}{na} \quad ; \quad \{e, \epsilon\} = \frac{\cos\phi(1-\cos\phi)}{na^2 e} \\ \{e, \varpi\} &= \frac{\cos\phi}{na^2 e} \quad ; \quad \{i, \epsilon\} = \frac{\sec\phi(1-\cos i)}{na^2 \sin i} \\ \{i, \varpi\} &= \{i, \epsilon\} \quad ; \quad \{i, \Omega\} = \frac{\sec\phi}{na^2 \sin i} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(6)$$

Lagrange's planetary equations can then be written down by inserting these expressions for the Poisson brackets in (5). We shall later—in Section 9·08—show how these expressions can be easily deduced from the known formulae for the Lagrangian brackets.

9·08. Reciprocal relationships between generalised Lagrangian and Poisson brackets

Let q and p be canonic conjugates. We define the generalised Lagrangian bracket $[a_m, a_r]$ by

$$[a_m, a_r] = \sum_{i=1}^{n} \left(\frac{\partial q_i}{\partial a_m} \cdot \frac{\partial p_i}{\partial a_r} - \frac{\partial q_i}{\partial a_r} \cdot \frac{\partial p_i}{\partial a_m}\right) \quad \ldots\ldots\ldots\ldots(1)$$

and the generalised Poisson bracket $\{a_m, a_s\}$ by

$$\{a_m, a_s\} = \sum_{j=1}^{n} \left(\frac{\partial a_m}{\partial q_j} \cdot \frac{\partial a_s}{\partial p_j} - \frac{\partial a_m}{\partial p_j} \cdot \frac{\partial a_s}{\partial q_j}\right) \quad \ldots\ldots\ldots\ldots(2)$$

In these m, r and s can have any of the values 1, 2, ... $2n$.

For brevity let L_{mr} and P_{ms} denote the Lagrangian and Poisson brackets and X the sum $\sum_{m=1}^{2n} L_{mr}P_{ms}$.

Then

$$X=\sum_i\sum_j\left[\frac{\partial a_s}{\partial p_j}\cdot\frac{\partial p_i}{\partial a_r}\sum_{m=1}^{2n}\frac{\partial q_i}{\partial a_m}\cdot\frac{\partial a_m}{\partial q_j}+\frac{\partial a_s}{\partial q_j}\cdot\frac{\partial q_i}{\partial a_r}\sum_{m=1}^{2n}\frac{\partial p_i}{\partial a_m}\cdot\frac{\partial a_m}{\partial p_j}\right]$$
$$-\sum_i\sum_j\left[\frac{\partial p_i}{\partial a_r}\cdot\frac{\partial a_s}{\partial q_j}\sum_{m=1}^{2n}\frac{\partial q_i}{\partial a_m}\cdot\frac{\partial a_m}{\partial p_j}+\frac{\partial q_i}{\partial a_r}\cdot\frac{\partial a_s}{\partial p_j}\sum_{m=1}^{2n}\frac{\partial p_i}{\partial a_m}\cdot\frac{\partial a_m}{\partial q_j}\right] .$$

We assume that q and p are independent functions of the $2n$ a's and, conversely, that any a is a function of the q's and p's.

Now
$$\sum_{m=1}^{2n}\frac{\partial q_i}{\partial a_m}\cdot\frac{\partial a_m}{\partial q_j}=\frac{\partial q_i}{\partial q_j}=1, \quad \text{if} \quad j=i,$$
$$=0, \quad \text{if} \quad j\neq i.$$

Similarly,
$$\sum_{m=1}^{2n}\frac{\partial p_i}{\partial a_m}\cdot\frac{\partial a_m}{\partial p_j}=1, \quad \text{if} \quad j=i,$$
$$=0, \quad \text{if} \quad j\neq i.$$

Also,
$$\sum_m\frac{\partial q_i}{\partial a_m}\cdot\frac{\partial a_m}{\partial p_j}=\sum_m\frac{\partial p_i}{\partial a_m}\cdot\frac{\partial a_m}{\partial q_j}=0 ,$$

since q and p are independent. Hence

$$X=\sum_{i=1}^{n}\left[\frac{\partial a_s}{\partial p_i}\cdot\frac{\partial p_i}{\partial a_r}+\frac{\partial a_s}{\partial q_i}\cdot\frac{\partial q_i}{\partial a_r}\right]=1, \quad \text{if} \quad s=r,$$
$$=0, \quad \text{if} \quad s\neq r.$$

We then have

$$\sum_{m=1}^{2n}L_{mr}P_{mr}=1 \quad , \qquad \text{.....(3)}$$

and
$$\sum_{m=1}^{2n}L_{mr}P_{ms}=0 \quad , \quad \text{if} \quad s\neq r \ . \qquad \text{.....(4)}$$

It is to be remembered that in these formulae r and s can take any integral value between 1 and $2n$; accordingly, there are $2n$ equations of type (3), and $2n(2n-1)$ of type (4)—for the time being we do not invoke the property : $L_{mr}=-L_{rm}$ etc., with a similar property for the P's ; in all there are $4n^2$ equations.

Suppose that all the L's—which are $4n^2$ in number—are known ; then (3) and (4) are $4n^2$ equations sufficient to enable us to determine the $4n^2$ brackets P.

Write down (3), when $r=1$, and the $(2n-1)$ equations (4) when $s=1$ and $r=2, 3, \ldots 2n$ in succession ; these equations are

$$\left.\begin{aligned} &L_{11}P_{11}+L_{21}P_{21}+\ldots\ldots\ldots+L_{2n,1}P_{2n,1}=1\\ &L_{12}P_{11}+L_{22}P_{21}+\ldots\ldots\ldots+L_{2n,2}P_{2n,1}=0\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &L_{1,2n}P_{11}+L_{2,2n}P_{21}+\ldots+L_{2n,2n}P_{2n,1}=0 \end{aligned}\right\} \quad , \qquad \text{.....(5)}$$

Let Δ be the determinant given by

$$\Delta = \begin{vmatrix} L_{11}, & L_{21}, & \dots & L_{2n,1} \\ L_{12}, & L_{22}, & \dots & L_{2n,2} \\ \vdots & & & \\ L_{1,2n}, & L_{2,2n}, & \dots & L_{2n,2n} \end{vmatrix} \quad \dots\dots\dots\dots(6)$$

The solutions of (5) are given by

$$P_{11} = L'_{11}/\Delta \quad , \quad P_{12} = L'_{12}/\Delta, \ \dots$$

where L'_{11}, L'_{12}, ... are the co-factors of L_{11}, L_{12}, ... respectively in the determinant. The general solution is evidently

$$P_{rs} = \frac{1}{\Delta} L'_{rs} \quad \dots\dots\dots\dots(7)$$

A similar procedure enables us to determine the Lagrangian brackets when the Poisson brackets are known.

The formula (7) can be written in a slightly different form, if we assume that in general the Lagrangian brackets have specific values different from zero ; it is evident that $L'_{rs} = \dfrac{\partial \Delta}{\partial L_{rs}}$ and consequently

$$P_{rs} = \frac{1}{\Delta} \frac{\partial \Delta}{\partial L_{rs}} \quad \dots\dots\dots\dots(8)$$

If we rearrange the columns, for example, of Δ in any way we choose, the resulting determinant Δ_1 is related to Δ in one of the following ways :

$$\Delta_1 = +\Delta \quad \text{or} \quad \Delta_1 = -\Delta \ ;$$

it follows that in all cases of rearrangement :

$$P_{rs} = \frac{1}{\Delta_1} \cdot \frac{\partial \Delta_1}{\partial L_{rs}} \quad \dots\dots\dots\dots(9)$$

or

$$P_{rs} = \frac{1}{\Delta_1} L'_{rs} \quad , \quad \dots\dots\dots\dots(8)$$

where now L'_{rs} is the co-factor of L_{rs} in the determinant Δ_1. We can thus use (7) when Δ is rearranged in any way we choose.

It is to be noticed in deriving the results of this section that we have not invoked any property of the q's and p's except that the q's and p's are independent functions of the $2n$ a's, the converse also holding, namely, that the a's are independent functions of the q's and p's.

9·09. Evaluation of the Poisson brackets from the Lagrangian brackets for the elliptic orbit

We shall identify a_m in terms of the six elliptic elements as follows :

$$a_1 = a \quad , \quad a_2 = e \quad , \quad a_3 = i \ ;$$
$$a_4 = \Omega \quad , \quad a_5 = \varpi \quad , \quad a_6 = \epsilon \ .$$

Reference to 5·14 (1) shows that the non-zero Lagrangian brackets are, on writing $e = \cos\phi$,

$$[a, \epsilon] \equiv L_{16} = -\tfrac{1}{2}na \quad ; \quad [a, \varpi] \equiv L_{15} = \tfrac{1}{2}na(1-\cos\phi) \quad ;$$
$$[a, \Omega] \equiv L_{14} = \tfrac{1}{2}na\cos\phi(1-\cos i) \quad ; \quad [e, \varpi] \equiv L_{25} = na^2\tan\phi \quad ;$$
$$[e, \Omega] \equiv L_{24} = -na^2\tan\phi(1-\cos i) \quad ; \quad [i, \Omega] \equiv L_{34} = na^2\cos\phi\sin i \quad .$$

Further, by definition, $L_{rs} = -L_{sr}$.

The Poisson bracket P_{rs} is then equal to L'_{rs}/Δ_1, where L'_{rs} is the co-factor in the rearranged determinant, derived from 9·08 (6) for $n = 3$,

$$\Delta_1 = \begin{vmatrix} L_{61}, & L_{51}, & L_{41}, & 0, & 0, & 0 \\ 0, & L_{52}, & L_{42}, & 0, & 0, & 0 \\ 0, & 0, & L_{43}, & 0, & 0, & 0 \\ 0, & 0, & 0, & L_{34}, & L_{24}, & L_{14} \\ 0, & 0, & 0, & 0, & L_{25}, & L_{15} \\ 0, & 0, & 0, & 0, & 0, & L_{16} \end{vmatrix}$$

in which the columns are indicated by L_{6r} etc. for $r = 1, 2, \ldots 6$ successively. It is easily seen that Δ_1 is given by its principal diagonal ; hence

$$\Delta_1 = L_{61}L_{52}L_{43}L_{34}L_{25}L_{16} \quad .$$

Also, the co-factor of L_{61} is $L_{52}L_{43}L_{34}L_{25}L_{16}$; hence, by 8·08 (8),

$$P_{61} = 1/L_{61} \quad \text{or} \quad P_{16} = 1/L_{16} = -\frac{2}{na} \quad .$$

Similarly,
$$P_{25} = 1/L_{25} = \frac{\cot\phi}{na^2} \quad ,$$
$$P_{34} = 1/L_{34} = \frac{\sec\phi}{na^2\sin i} \quad .$$

The co-factors of L_{s1} ($s = 5, 4, 3, 2, 1$) from the first row are all zero, since each is equal to a fifth-order determinant in the first row of which the elements are all zero ; hence

$$P_{51} = P_{41} = P_{31} = P_{21} = P_{11} = 0 \quad ,$$

P_{11} being equal to zero by definition.

It is readily seen that the non-zero co-factors are those relating to L_{62}, L_{63} and L_{53}. For example, the co-factor of L_{62} is the determinant given by

$$L'_{62} = \begin{vmatrix} L_{51} & L_{41} & 0 & 0 & 0 \\ 0 & L_{43} & 0 & 0 & 0 \\ 0 & 0 & L_{34} & L_{24} & L_{14} \\ 0 & 0 & 0 & L_{25} & L_{15} \\ 0 & 0 & 0 & 0 & L_{16} \end{vmatrix}$$

Hence
$$P_{62} = L_{51}/(L_{52}L_{61})$$

and, consequently, $$P_{26}=\frac{\cot\phi(1-\cos\phi)}{na^2}\quad.$$

Similarly, $$L'_{63}=-L_{16}L_{25}L_{34}(L_{41}L_{52}-L_{42}L_{51})$$

from which

$$P_{36}=(L_{14}L_{25}-L_{24}L_{15})/(L_{61}L_{52}L_{43})=\frac{\sec\phi(1-\cos i)}{na^2\sin i}\quad.$$

Similarly, we find that $P_{36}=P_{35}$.

The results obtained in this section are in agreement with those in 9·07 (6).

Chapter 10

CONTACT TRANSFORMATIONS

10·01. A canonic criterion

We suppose as in Section 8·09 that the solutions of the canonic equations

$$\dot{q}_i = \frac{\partial H(q, p, t)}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H(q, p, t)}{\partial q_i} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

are given, in terms of t and $2k$ constants $c_1, c_2, \ldots c_{2k}$, by

$$q_i = q_i(c, t) \quad , \quad p_i = p_i(c, t) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Let X be defined by

$$X \equiv \frac{d}{dt} \sum_{i=1}^{k} p_i \frac{\partial q_i}{\partial c_j} - \frac{\partial}{\partial c_j} \sum_{i=1}^{k} p_i \dot{q}_i \quad ; \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

then, on reference to 8·09 (4),

$$X = -\frac{\partial H}{\partial c_j} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

We establish an important theorem, stated as follows.

If, for $i = 1, 2, \ldots k$, q_i and p_i are independent functions of t and $2k$ constants $c_1, c_2, \ldots c_{2k}$ such that for any c_j

$$X \equiv \frac{d}{dt} \sum_{i=1}^{k} p_i \frac{\partial q_i}{\partial c_j} - \frac{\partial}{\partial c_j} \sum_{i=1}^{k} p_i \dot{q}_i = -\frac{\partial H}{\partial c_j} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where H is a function of q, p, t in the first instance and transformed into a function of t and the $2k$ c's by means of (2), then q and p are canonically conjugate variables satisfying the canonic equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad .$$

We have, since $H(c, t) \equiv H(q, p, t)$,

$$\frac{\partial H}{\partial c_j} = \sum_{i=1}^{k} \left(\frac{\partial H}{\partial q_i} \cdot \frac{\partial q_i}{\partial c_j} + \frac{\partial H}{\partial p_i} \cdot \frac{\partial p_i}{\partial c_j} \right) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Now, the left-hand side of (5) can be written as

$$\sum_{i=1}^{k} \left(\dot{p}_i \frac{\partial q_i}{\partial c_j} - \dot{q}_i \frac{\partial p_i}{\partial c_j} \right) \quad ;$$

hence, by means of (6), we can write (5) as

$$\sum_{i=1}^{k} \left\{ x_i \frac{\partial p_i}{\partial c_j} + y_i \frac{\partial q_i}{\partial c_j} \right\} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

where $$x_i \equiv \frac{\partial H}{\partial p_i} - \dot{q}_i \text{ and } y_i \equiv \frac{\partial H}{\partial q_i} + \dot{p}_i \quad .$$

There are $2k$ equations of this type, corresponding to the $2k$ values of c_j, $(j = 1, 2, \ldots 2k)$, in which the k x's and the k y's are regarded as unknowns. Since the q's and the p's are independent functions of the c's, then

$$\frac{\partial(p_1, p_2, \ldots p_k, q_1, q_2, \ldots q_k)}{\partial(c_1, c_2, \ldots, c_{2k})} \neq 0 ;$$

consequently, $$x_1 = x_2 = \ldots = x_k = y_1 = y_2 \ldots = y_k = 0 \quad .$$

Hence $$\dot{q}_i = \frac{\partial H(q, p, t)}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H(q, p, t)}{\partial q_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

The theorem is thus established ; it will be used as a criterion in the powerful method associated with contact transformations with which we shall deal in the following sections.

10·02. Contact transformations

We begin with the canonic equations (8) of the previous section. Suppose that we change the variables q and p to new variables Q and P by means of the general functional equations

$$q_i = q_i(Q, P, t) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$p_i = p_i(Q, P, t) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which there are k variables Q and k variables P.

If the change of variables is such that Q_i and P_i are canonically conjugate with reference to a Hamiltonian function $H'(Q, P, t)$ the transformation of variables is called a *contact transformation.*

From the two groups of equations in (1) and (2) we can solve, in principle, for each Q and P in terms of q and p, obtaining in functional form

$$Q_i = Q_i(q, p, t) \quad , \quad P_i = P_i(q, p, t) \quad \ldots\ldots\ldots\ldots (3)$$

The solutions of 10·01 (8) are

$$q_i = q_i(c, t) \quad , \quad p_i = p_i(c, t) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

the constants of integration, the c's, being $2k$ in number ; accordingly, by (3), the Q's and P's can be expressed in terms of the $2k$ constants c by

$$Q_i = Q_i(c, t) \quad , \quad P_i = P_i(c, t) \quad . \quad \ldots\ldots\ldots\ldots\ldots (5)$$

Consider now the group of equations (1) ; these can be solved in principle for each P_i which will then be expressed as $P_i = P_i(q, Q, t)$.

Suppose now that a function $F(q, Q, t)$ exists so that

$$\sum_{i=1}^{k} p_i \, dq_i - \sum_{i=1}^{k} P_i \, dQ_i \equiv \sum_{i=1}^{k} \left(\frac{\partial F}{\partial q_i} dq_i + \frac{\partial F}{\partial Q_i} dQ_i \right) \equiv dF \quad , \quad \ldots\ldots\ldots (6)$$

dF being a total differential with respect to the q's and Q's.

Then $$p_i = \frac{\partial F(q, Q, t)}{\partial q_i} \quad , \quad P_i = -\frac{\partial F(q, Q, t)}{\partial Q_i} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

We now apply the canonic criterion of the previous section. Let

$$Y \equiv \frac{d}{dt} \sum_{i=1}^{k} P_i \frac{\partial Q_i}{\partial c_j} - \frac{\partial}{\partial c_j} \sum_{i=1}^{k} P_i \dot{Q}_i \quad .$$

Then, by means of 10·01 (3), we have

$$X - Y = \frac{d}{dt} \sum_i \left(p_i \frac{\partial q_i}{\partial c_j} - P_i \frac{\partial Q_i}{\partial c_j} \right) - \frac{\partial}{\partial c_j} \sum_i (p_i \dot{q}_i - P_i \dot{Q}_i)$$

whence, by (7),

$$X - Y = \frac{d}{dt} \sum_i \left(\frac{\partial F}{\partial q_i} \cdot \frac{\partial q_i}{\partial c_j} + \frac{\partial F}{\partial Q_i} \cdot \frac{\partial Q_i}{\partial c_j} \right) - \frac{\partial}{\partial c_j} \sum_i \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial Q_i} \dot{Q}_i \right) \quad . \ldots . (8)$$

Now, the function $F(q, Q, t)$ can be transformed into the function $F(c, t)$ by means of the first group in (4) and the first group in (5). We then have

$$\frac{\partial F(c, t)}{\partial c_j} = \sum_i \left(\frac{\partial F}{\partial q_i} \cdot \frac{\partial q_i}{\partial c_j} + \frac{\partial F}{\partial Q_i} \cdot \frac{\partial Q_i}{\partial c_j} \right)$$

and $$\frac{dF(c, t)}{dt} = \frac{\partial F(q, Q, t)}{\partial t} + \sum_i \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial Q_i} \dot{Q}_i \right) \quad .$$

By means of these (8) becomes

$$X - Y = \frac{d}{dt} \left\{ \frac{\partial F(c, t)}{\partial c_j} \right\} - \frac{\partial}{\partial c_j} \left\{ \frac{dF(c, t)}{dt} - \frac{\partial F(q, Q, t)}{\partial t} \right\} = \frac{\partial}{\partial c_j} \left\{ \frac{\partial F(q, Q, t)}{\partial t} \right\} \quad .$$

Now, by 10·01 (3) and (4),

$$X = -\frac{\partial H}{\partial c_j} \quad ; \quad \text{hence, } Y = -\frac{\partial H'}{\partial c_j} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

where $$H' = H(q, p, t) + \frac{\partial F(q, Q, t)}{\partial t} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

By means of (1) and (2), H' can be expressed as $H'(Q, P, t)$; accordingly (9) is an expression of the canonic criterion and, consequently,

$$\dot{Q}_i = \frac{\partial H'(Q, P, t)}{\partial P_i} \quad , \quad \dot{P}_i = -\frac{\partial H'(Q, P, t)}{\partial Q_i} \quad ; \quad \ldots\ldots\ldots\ldots (11)$$

in other words, the new variables Q and P satisfy canonic equations with reference to the Hamiltonian function H' given by (10), the function $F(q, Q, t)$ being defined in accordance with (6).

10·03. Derivation of the Hamilton-Jacobi equation

Suppose that the Hamiltonian function H' in 10·02 (10) is zero, so that

$$H(q, p, t) + \frac{\partial F(q, Q, t)}{\partial t} = 0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Then, by 10·02 (11), $\dot{Q}_i = \dot{P}_i = 0$; hence Q_i and P_i are constants, and we write

$$Q_i = \alpha_i \quad , \quad P_i = -\beta_i \quad , \quad (i = 1, 2, \ldots k).$$

The formulae 10·02 (7) then become

$$p_i = \frac{\partial F(q, \alpha, t)}{\partial q_i} \quad , \quad \beta_i = \frac{\partial F(q, \alpha, t)}{\partial \alpha_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

It is evident that in this case the function F is identical with the function S introduced in Section 8·09. In (1) replace F by S and Q by α ; then this equation becomes

$$\frac{\partial S(q, \alpha, t)}{\partial t} + H(q, p, t) = 0$$

and thus, on referring to the first group of equations in (2), S is a solution of the equation

$$\frac{\partial S(q, \alpha, t)}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0 \quad ,$$

which is the Hamiltonian-Jacobi equation derived in 8·10.

10·04. A further application

In this section we derive by the method of contact transformations the results obtained in 8·15.

The canonic equations from which we start are

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

As before, we write $H = H_0 - H_1$ ……(2)

and we solve the canonic equations with H_0 as Hamiltonian function. The solutions are

$$q = q(t, \alpha, \beta) \quad , \quad p = p(t, \alpha, \beta) \quad , \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

derivable from

$$p = \frac{\partial S(q, \alpha, t)}{\partial q} \quad , \quad \beta = \frac{\partial S(q, \alpha, t)}{\partial \alpha} \quad ,$$

where S is a solution of

$$\frac{\partial S}{\partial t} + H_0\left(q, \frac{\partial S}{\partial q}, t\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Now regard α and β as variables so that the analytical expressions for q and p in (3) satisfy the equations (1). Identify Q_i with α_i and P_i with $-\beta_i$; then

$$\sum_{i=1}^{k} p_i \, dq_i - \sum_i P_i \, dQ_i \equiv \sum_i (p_i \, dq_i + \beta_i \, d\alpha_i)$$

$$= \sum_i \left(\frac{\partial S}{\partial q_i} dq_i + \frac{\partial S}{\partial \alpha_i} d\alpha_i\right) \quad .$$

The right-hand side is the total differential dS with respect to q and α; hence, by the principles of 10·02,

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} \quad , \quad \dot{P}_i = -\frac{\partial H'}{\partial Q_i} \quad ,$$

where $$H' = H + \frac{\partial S(q, \alpha, t)}{\partial t} = H - H_0 \quad , \quad \text{by (4)},$$

$$= -H_1 \quad , \quad \text{by (2)}.$$

Hence $$\dot{Q}_i = -\frac{\partial H_1}{\partial P_i} \quad , \quad \dot{P}_i = \frac{\partial H_1}{\partial Q_i}$$

which become, on replacing Q by α and P by $-\beta$,

$$\dot{\alpha}_i = \frac{\partial H_1}{\partial \beta_i} \quad , \quad \dot{\beta}_i = -\frac{\partial H_1}{\partial \alpha_i} \quad .$$

These are the formulae derived in 8·15.

If H_0 is the Hamiltonian function associated with the elliptic orbit so that α and β are the corresponding canonic constants, the equations for the disturbed orbit are then found as in 8·15 to be

$$\dot{\alpha}_i = \frac{\partial R}{\partial \beta_i} \quad , \quad \dot{\beta}_i = -\frac{\partial R}{\partial \alpha_i} \quad ,$$

where α and β are now taken as new canonic variables.

10·05. Conditions for a contact transformation in terms of Lagrangian and Poisson brackets

For a contact transformation from variables q, p to variables Q, P we have from 10·02 (6)

$$\sum_{i=1}^{k} p_i \, dq_i - \sum_{i=1}^{k} P_i \, dQ_i \equiv dF(q, Q, t) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where dF is a perfect differential with respect to q and Q.

The functional transformations are

$$q \equiv q(Q, P, t) \text{ and } p \equiv p(Q, P, t)$$

From the first of these F becomes a function of Q, P, t so that

$$dF = \sum_{i=1}^{k} \left(\frac{\partial F}{\partial Q_i} dQ_i + \frac{\partial F}{\partial P_i} dP_i \right) \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

also, $$dq_r = \sum_{i=1}^{k} \left(\frac{\partial q_r}{\partial Q_i} dQ_i + \frac{\partial q_r}{\partial P_i} dP_i \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In (1) write $\sum_{i=1}^{k} p_i \, dq_i$ as $\sum_{r=1}^{k} p_r \, dq_r$; then we have, by means of (2) and (3),

$$\sum_{r=1}^{k} \sum_{i=1}^{k} p_r \left(\frac{\partial q_r}{\partial Q_i} dQ_i + \frac{\partial q_r}{\partial P_i} dP_i \right) - \sum_{i=1}^{k} P_i \, dQ_i \equiv \sum_{i=1}^{k} \left(\frac{\partial F}{\partial Q_i} dQ_i + \frac{\partial F}{\partial P_i} dP_i \right)$$

from which, on equating coefficients of dQ_i and dP_i,

$$\sum_r p_r \frac{\partial q_r}{\partial Q_i} - P_i = \frac{\partial F}{\partial Q_i} \quad , \qquad \ldots\ldots(4)$$

$$\sum_r p_r \frac{\partial q_r}{\partial P_i} = \frac{\partial F}{\partial P_i} \qquad \ldots\ldots(5)$$

We define the Lagrangian bracket $[u, v]$ by

$$[u, v] = \sum_{r=1} \frac{\partial (q_r, p_r)}{\partial (u, v)} \quad . \qquad \ldots\ldots(6)$$

From (5), $$\frac{\partial}{\partial P_j} \sum_r p_r \frac{\partial q_r}{\partial P_i} = \frac{\partial}{\partial P_i} \sum_r p_r \frac{\partial q_r}{\partial P_j} \quad ,$$

that is, $$\sum_r \frac{\partial (q_r, p_r)}{\partial (P_i, P_j)} \equiv [P_i, P_j] = 0 \quad . \qquad \ldots\ldots(7)$$

Again, from (4), since the P's and Q's are independent functions,

$$\frac{\partial}{\partial Q_j} \sum_r p_r \frac{\partial q_r}{\partial Q_i} = \frac{\partial}{\partial Q_i} \sum_r p_r \frac{\partial q_r}{\partial Q_j}$$

from which $$[Q_i, Q_j] = 0 \quad . \qquad \ldots\ldots(8)$$

Again, differentiate (4) and (5) with respect to P_i and Q_i respectively and subtract; then

$$\frac{\partial}{\partial P_i} \sum_r p_r \frac{\partial q_r}{\partial Q_i} - 1 = \frac{\partial}{\partial Q_i} \sum_r p_r \frac{\partial q_r}{\partial P_i}$$

from which $$[Q_i, P_i] = 1 \quad . \qquad \ldots\ldots(9)$$

The relationships (7), (8) and (9) constitute the set of conditions for the contract transformations.

If we refer to the determinant Δ in 9·08 which controls the reciprocal relationships between Lagrangian and Poisson brackets, we see that Δ reduces to the expression given by

$$\Delta = \prod_{i=1}^{2n} L_{rr} = \prod_i [Q_i, P_i].$$

From 9·08 (7), $$P_{rr} = \frac{1}{\Delta} L'_{rr} = 1/L_{rr} \; ;$$

hence, by (9), $$\{Q_i, P_i\} = 1 \quad . \qquad \ldots\ldots(10)$$

Also, since $L_{rs} = 0$ $(r \neq s)$, the co-factor of L_{rs} is zero; hence

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0 \quad . \qquad \ldots\ldots(11)$$

The formulae (10) and (11) specify the conditions for contact transformations in terms of the Poisson brackets.

In particular, if Q_i and P_i are identified with α_i and $-\beta_i$ respectively, the conditions in terms of Lagrangian brackets become

$$[\alpha_i, \beta_i] = -1 \qquad \ldots\ldots(12)$$

and $$[\alpha_i, \alpha_j] = [\beta_i, \beta_j] = 0 \quad . \qquad \ldots\ldots(13)$$

The conditions in terms of Poisson brackets are

$$\{\alpha_i, \beta_i\} = -1 \quad \ldots\ldots(14)$$

and

$$\{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = 0 \quad \ldots\ldots(15)$$

10·06. A particular case of contact transformation

Because of its importance later we consider the special case when one pair only—say (q_1, p_1)—of canonic variables are transformed into the pair (Q_1, P_1).

Formally,

$$q_1 = q_1(Q_1, P_1, t) \quad , \quad p_1 = p_1(Q_1, P_1, t) \quad \ldots\ldots(1)$$

and

$$q_r = Q_r \quad , \quad p_r = P_r \quad , \quad (r = 2, 3, \ldots k).$$

From the previous section the conditions are

$$\frac{\partial(q_1, p_1)}{\partial(Q_1, P_1)} + \sum_{r=2}^{k} \frac{\partial(q_r, p_r)}{\partial(Q_1, P_1)} = 1 \quad \ldots\ldots(2)$$

and

$$[Q_i, Q_j] = [P_i, P_j] = 0 \quad \ldots\ldots(3)$$

Since q_r and p_r $(r > 1)$ are independent of Q_1 and P_1, the conditions (3) are automatically satisfied and, further, each term of the summation in (2) is zero ; the condition required is then, from (2),

$$\frac{\partial(q_1, p_1)}{\partial(Q_1, P_1)} = 1 \quad \ldots\ldots(4)$$

10·07. Alternative proof

We give an alternative proof of the result in 10·06, based on simple principles, and further we derive the general form of the new Hamiltonian function to be associated with Q_1 and P_1.

Since q_1 and p_1 are conjugate variables, we have

$$\dot{q}_1 = \frac{\partial H(q, p, t)}{\partial p_1} \quad \text{and} \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1} \quad .$$

Solving 10·06 (1), we obtain in functional form

$$Q_1 = Q_1(q_1, p_1, t) \quad , \quad P_1 = P_1(q_1, p_1, t) \quad \ldots\ldots(1)$$

Hence, omitting suffices for simplicity,

$$\dot{Q} = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} + \frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial q} \cdot \frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p} \cdot \frac{\partial H}{\partial q} + \frac{\partial Q}{\partial t} \quad .$$

By 10·06 (1) H can be expressed as a function of Q, P, t ; hence

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial Q} \cdot \frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P} \cdot \frac{\partial P}{\partial p} \quad , \quad \frac{\partial H}{\partial q} = \frac{\partial H}{\partial Q} \cdot \frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P} \cdot \frac{\partial P}{\partial q} \quad .$$

Hence

$$\dot{Q} = \frac{\partial H}{\partial P} \cdot \frac{\partial(Q, P)}{\partial(q, p)} + \frac{\partial Q}{\partial t} \quad .$$

Similarly,

$$\dot{P} = -\frac{\partial H}{\partial Q} \cdot \frac{\partial(Q, P)}{\partial(q, p)} + \frac{\partial P}{\partial t} \quad .$$

From these, restoring the suffices, we can write

$$\dot{Q}_1 = \frac{\partial H'}{\partial P_1} \quad , \quad \dot{P}_1 = -\frac{\partial H'}{\partial Q_1}$$

if
$$\frac{\partial (Q_1, P_1)}{\partial (q_1, p_1)} = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and H' is a function such that

$$H' = H + \phi \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where
$$\frac{\partial \phi}{\partial P_1} = \frac{\partial Q_1}{\partial t} \quad \text{and} \quad \frac{\partial \phi}{\partial Q_1} = -\frac{\partial P_1}{\partial t} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The condition (2) states that the Poisson bracket $\{Q_1, P_1\}$ is equal to unity, all the other terms in the series of the form $\frac{\partial (Q_1, P_1)}{\partial (q_r, p_r)}$ being zero $(r>1)$. By the reciprocal theorem, $\{Q_1, P_1\} = 1/[Q_1, P_1]$; hence (2) is equivalent to

$$\frac{\partial (q_1, p_1)}{\partial (Q_1, P_1)} = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

as found in 10·06 (4).

From (4) the function $\phi(Q, P, t)$ is given by

$$\phi = -\int \frac{\partial P_1}{\partial t}\, dQ_1 + \int \frac{\partial Q_1}{\partial t}\, dP_1 \quad .$$

Now,
$$dQ_1 = \frac{\partial Q_1}{\partial q_1}\, dq_1 + \frac{\partial Q_1}{\partial p_1}\, dp_1 \quad ,$$

with a similar expression for dP_1. Hence, when ϕ is expressed in terms of q_1, p_1, t we have

$$\phi = -\int \frac{\partial (Q_1, P_1)}{\partial (q_1, t)}\, dq_1 - \int \frac{\partial (Q_1, P_1)}{\partial (p_1, t)}\, dp_1 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Thus the new Hamiltonian function H' is found by means of (3) and (6). The new canonic equations are then

$$\dot{Q}_1 = \frac{\partial H'}{\partial P_1} \quad , \quad \dot{P}_1 = -\frac{\partial H'}{\partial Q_1} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

where H' is supposed to be expressed in terms of Q_1 and P_1.

Since ϕ in (6) is *not* a function of Q_r, P_r $(r=2, 3, \ldots)$ the remaining canonic equations can be written

$$\dot{Q}_r = \frac{\partial H'}{\partial P_r} \quad , \quad \dot{P}_r = -\frac{\partial H'}{\partial Q_r} \quad .$$

Corollary. If the transformation from q and p to Q and P does not involve the time explicitly, $\frac{\partial Q_1}{\partial t}$ and $\frac{\partial P_1}{\partial t}$ are both zero; hence $\phi = 0$ and the new canonic equations are

$$\dot{Q}_r = \frac{\partial H}{\partial P_r} \quad , \quad \dot{P}_r = -\frac{\partial H}{\partial Q_r} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

the Hamiltonian function remaining unchanged.

10·08. Extended linear point transformations

We begin with the canonic equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad .$$

If the transformation to new canonic variables Q and P is such that Q_r for example is a function of the q's alone and P_r is a function of the p's alone, then the transformation is called an *extended point transformation.* Formally we have

$$Q_r = Q_r(q) \quad , \quad P_r = P_r(p) \quad .$$

Further, if $Q_r(q)$ is linear in the q's and $P_r(p)$ is linear in the p's the transformation is called an *extended linear point transformation* ; it is this latter transformation we now consider.

We write for the linear transformations when r takes the values 1, 2, ... n,

$$Q_r = A_{r1}q_1 + A_{r2}q_2 + \ldots + A_{rn}q_n \equiv \sum_{s=1}^{n} A_{rs}q_s \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$P_r = a_{r1}p_1 + a_{r2}p_2 + \ldots + a_{rn}p_n \equiv \sum_{s} a_{rs}p_s \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where the A's and a's are constants.

For a contact transformation we have

$$\sum_r p_r\, dq_r - \sum_r P_r\, dQ_r = dF \quad .$$

This is true if $F = 0$; hence

$$\sum p_r\, dq_r - \sum P_r\, dQ_r = 0 \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

which becomes by means of (1) and (2)—on writing t for s in (1)—

$$\sum_r p_r\, dq_r - \sum_r \sum_s a_{rs}p_s \sum_t A_{rt}\, dq_t = 0 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The triple summation on the left can be written

$$\sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}A_{rs}p_s\, dq_s + \sum_r \sum_s \sum_t a_{rs}A_{rt}p_s\, dq_t$$

where $t \neq s$; (4) then becomes, on writing $\sum_r p_r\, dq_r$ as $\sum_s p_s\, dq_s$,

$$\sum_s p_s\, dq_s - \sum_s p_s\, dq_s \sum_r a_{rs}A_{rs} - \sum_s \sum_t p_s\, dq_t \sum_r a_{rs}A_{rt} = 0$$

in which r and s can take all values from 1 to n and $t \neq s$.

The last equation is an identity if

$$\sum_{r=1}^{n} a_{rs}A_{rs} = 1 \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and $$\sum_{r=1}^{n} a_{rs}A_{rt}=0 \quad , \quad (t\neq s). \qquad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

These are the conditions for contact transformation and, since $F=0$, the new canonic equations are

$$\dot{Q}_r=\frac{\partial H}{\partial P_r} \quad , \quad \dot{P}_r=-\frac{\partial H}{\partial Q_r} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

If definite values are assigned, for example, to the a's, then (5) and (6) provide the requisite number of equations for determining the corresponding values of the A's; for, there are n equations (5)—corresponding to the n separate values of s—and $n(n-1)$ equations (6), making in all a total of n^2 equations from which the n^2 values of the A's can be found. It is easily seen, by writing out the equation (5) for $s=1$ and the $(n-1)$ equations (6) for $t=1$ and $s=2, 3, \ldots n$, that A_{11}, for example, is the co-factor of a_{11}, divided by Δ, in the determinant

$$\Delta=\begin{vmatrix} a_{11}, & a_{21}, & \ldots & a_{n1} \\ a_{12}, & a_{22}, & \ldots & a_{n2} \\ \ldots\ldots & \ldots\ldots & & \\ \ldots\ldots & \ldots\ldots & & \\ a_{1n}, & a_{2n}, & \ldots & a_{nn} \end{vmatrix}$$

and, generally, that

$$A_{rs}=\frac{1}{\Delta}a'_{rs} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where a'_{rs} is the co-factor of a_{rs}.

The conditions (5) and (6) can be summarised in a single and important formula. Since Q_r is a linear expression in the q's it is evident that the conditions whereby

$$\sum_r p_r q_r - \sum_r P_r Q_r \equiv 0$$

are precisely the same conditions whereby

$$\sum_r p_r\, dq_r - \sum_r P_r\, dQ_r \equiv 0\,;$$

these latter conditions are given by (5) and (6). Hence, for an extended linear point transformation the sufficient condition is

$$\sum_r Q_r P_r = \sum_r q_r p_r \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

10·09. Orthogonal transformations

If, in 10·08 (2), $a_{rs}=A_{rs}$ the transformation is said to be *orthogonal*. The conditions (5) and (6) of the previous section are then

$$\sum_{r=1}^{n} A^2_{rs}=1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and $$\sum_{r=1}^{n} A_{rs}A_{rt}=0 \quad , \quad t\neq s. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If $n=3$, then the A's have the properties of direction-cosines. Thus, if we write

$$l_s=A_{1s} \quad , \quad m_s=A_{2s} \quad , \quad n_s=A_{3s} \quad ,$$

the conditions (1) and (2) become

$$l_s^2+m_s^2+n_s^2=1 \quad , \quad (s=1, 2, 3)$$

and

$$l_sl_t+m_sm_t+n_sn_t=0 \quad , \quad (t\neq s),$$

which are the well-known formulae in three-dimensional geometry relating to direction-cosines.

10·10. Infinitesimal contact transformations

Let

$$Q_i=q_i+\epsilon x_i \quad , \quad P_i=p_i+\epsilon y_i \quad , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where ϵ is an arbitrary infinitesimal quantity independent of the q's and p's whose square can be neglected. For a contact transformation we must have

$$\sum_i p_i\,dq_i-\sum_i P_i\,dQ_i=dF \quad ,$$

where dF is a total differential ; by (1) we obtain

$$\sum_i p_i\,dq_i-\sum_i (p_i+\epsilon y_i)(dq_i+\epsilon\,dx_i)=dF$$

or

$$-\epsilon\sum_i (y_i\,dq_i+p_i\,dx_i)=dF \quad .$$

The function F can be assumed to contain ϵ as a factor ; write $F=\epsilon W$; then

$$\Sigma(y_i\,dq_i+p_i\,dx_i)=-dW$$

which can be written as

$$\sum_i (y_i\,dq_i-x_i\,dp_i)=-d\{W+\sum_i p_ix_i\}=-dK \quad ,$$

where $K(q, p, t)$ is the function within the brackets. Hence

$$x_i=\frac{\partial K(q, p, t)}{\partial p_i} \quad , \quad y_i=-\frac{\partial K(q, p, t)}{\partial q_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Replace ϵ by Δt which is independent of the q's and p's. Then, by (1) and (2), the most general infinitesimal contact transformation is given, in terms of an arbitrary function K, by

$$Q=q_i+\frac{\partial K(q, p, t)}{\partial p_i}\cdot\Delta t \quad , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$P_i=p_i-\frac{\partial K(q, p, t)}{\partial q_i}\cdot\Delta t \quad . \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Now the coordinates and associated components of momentum are q and p at time t ; the equations (3) and (4) show that, at time $t+\Delta t$, the state of the system has been reached by an infinitesimal contact

transformation. The whole motion of the system can thus be described in terms of a succession of infinitesimal transformations, resembling in general character the description of a planet's disturbed path in terms of a succession of infinitesimal arcs of osculating orbits.

Chapter 11

THE DELAUNAY AND POINCARÉ VARIABLES

11·01. Need for transformation of the canonic variables

We consider the motion of a planet P, disturbed by a planet P_1. The equations of motion are

$$\dot{\alpha}_i = \frac{\partial R}{\partial \beta_i} \quad , \quad \dot{\beta}_i = -\frac{\partial R}{\partial \alpha_i} \quad , \quad (i = 1, 2, 3) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where, in the first instance α and β are the canonic constants arising in the solution for elliptic motion and are now, in (1), canonic variables.

The disturbing function R—the Hamiltonian function in (1)—consists of non-periodic and periodic terms, and of these that need only be considered in the present connection are the terms in which the mean anomaly M of the planet P appears in the argument. Such terms are of the form $C \cos\theta$ where $C = C(\alpha_1, \alpha_2, \alpha_3)$ and $\theta = jM + u$; the latter can be written

$$\theta = jn(t + \beta_1) + u \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

j being a positive integer and u a linear function of β_2, β_3, M_1, ϖ_1 and Ω_1—the last three referring to P_1.

Now, in (2), n is a function of α_1 since

$$n^2 a^3 = \mu \quad \text{and} \quad a = -\mu/2\alpha_1 \; ; \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

hence, so far as the aggregate of such terms as $C \cos\theta$ is concerned, the second equation of (1) becomes, when $i = 1$,

$$\dot{\beta}_1 \equiv -\frac{\partial R}{\partial \alpha_1} = -\Sigma \frac{\partial C}{\partial \alpha_1} \cos\theta + (t + \beta_1) \Sigma j \frac{dn}{d\alpha_1} \sin\theta \quad . \quad \ldots\ldots\ldots (4)$$

The presence of the factor $(t + \beta_1)$ evidently causes embarrassment of the kind encountered in 6·03 (6) in connection with the element ϵ. It is to be noted that (4) is the only equation of the six equations in (1) where this particular feature arises. The difficulty is resolved by regarding the mean anomaly $n(t + \beta_1)$ as a new canonic variable—denoted * by l in Delaunay's notation—and discovering a conjugate variable, denoted by L ; then L and l replace α_1 and β_1, all the other variables α_2, β_2, α_3, β_3 remaining unaltered.

11·02. The conjugate of the mean anomaly

The mean anomaly l is given by

$$l \equiv l(\alpha_1, \beta_1, t) = n(t + \beta_1) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

n being expressed in terms of α_1, by the formulae 11·01 (3).

* Not to be confused with the mean longitude as used, for example, in 6·02 (12).

Let $L \equiv L(\alpha_1, \beta_1, t)$ be the conjugate of l. It is required to find the precise form of L such that

$$\dot{L} = \frac{\partial R'}{\partial l} \quad , \quad l = -\frac{\partial R'}{\partial L} \quad , \qquad \ldots\ldots(2)$$

where R' is a new Hamiltonian function.

The condition that L and l shall be conjugate has been derived in sections 10·06 and 10·07 if we associate α_1, β_1 with q_1, p_1 and L, l with Q_1, P_1. The condition given by 10·07 (2) is then

$$\frac{\partial(L, l)}{\partial(\alpha_1, \beta_1)} = 1 \quad ; \qquad \ldots\ldots(3)$$

the function R' is $R + \phi$, where ϕ is defined by 10·07 (6) and is given in the new notation by

$$\phi = -\int \frac{\partial(L, l)}{\partial(\alpha_1, t)} d\alpha_1 - \int \frac{\partial(L, l)}{\partial(\beta_1, t)} d\beta_1 \quad . \qquad \ldots\ldots(4)$$

With l given by (1), the condition (3) becomes

$$n \frac{\partial L}{\partial \alpha_1} - (t + \beta_1) \frac{dn}{d\alpha_1} \cdot \frac{\partial L}{\partial \beta_1} = 1 \quad . \qquad \ldots\ldots(5)$$

This equation is satisfied if $L \equiv L(\alpha_1)$ so that $\frac{\partial L}{\partial \beta_1} = \frac{\partial L}{\partial t} = 0$; accordingly, we can write (5) as $n\, dL = d\alpha_1$ or, by means of 11·01 (3), $dL = d\{\sqrt{(\mu a)}\}$,

whence $$L = \sqrt{(\mu a)} = \mu \sqrt{\left(-\frac{1}{2\alpha_1}\right)} \quad . \qquad \ldots\ldots(6)$$

Then (4) becomes

$$\phi = -\int n \frac{dL}{d\alpha_1} \cdot d\alpha_1 = -\alpha_1 \quad . \qquad \ldots\ldots(7)$$

Consequently, $$R' = R - \alpha_1 = R + \frac{\mu^2}{2L^2} \quad , \qquad \ldots\ldots(8)$$

by means of (6). The canonic equations involving L and l are then given by (2), R' being now supposed to be transformed into $R'(L, l)$.

Since $\mu^2/(2L^2)$ is not a function of α_2, β_2 the corresponding canonic equations can be written as $\dot{\alpha}_2 = \frac{\partial R'}{\partial \beta_2}$, $\dot{\beta}_2 = -\frac{\partial R'}{\partial \alpha_2}$.

The remaining pair (in α_3 and β_3) can be written in a similar form.

11·03. Alternative derivation of L

The reader may consider it instructive to have the results of the previous section derived from first principles without having recourse to the general formulae of the previous chapter.

We begin with

$$L \equiv L(\alpha_1, \beta_1, t) \quad , \quad l = n(t + \beta_1) \quad , \qquad \ldots\ldots(1)$$

where $n \equiv n(\alpha_1)$. We have at once

$$\dot{L} = \frac{\partial L}{\partial \alpha_1}\dot{\alpha}_1 + \frac{\partial L}{\partial \beta_1}\dot{\beta}_1 + \frac{\partial L}{\partial t} = \frac{\partial L}{\partial \alpha_1}\cdot\frac{\partial R}{\partial \beta_1} - \frac{\partial L}{\partial \beta_1}\cdot\frac{\partial R}{\partial \alpha_1} + \frac{\partial L}{\partial t} \quad \ldots\ldots\ldots(2)$$

Similarly
$$l = \frac{\partial l}{\partial \alpha_1}\cdot\frac{\partial R}{\partial \beta_1} - \frac{\partial l}{\partial \beta_1}\cdot\frac{\partial R}{\partial \alpha_1} + \frac{\partial l}{\partial t} \quad \ldots\ldots\ldots(3)$$

Now, $R(\alpha_1, \beta_1, t) \equiv R(L, l, t)$ when (1) are substituted ; hence

$$\frac{\partial R}{\partial \alpha_1} = \frac{\partial R}{\partial L}\cdot\frac{\partial L}{\partial \alpha_1} + \frac{\partial R}{\partial l}\cdot\frac{\partial l}{\partial \alpha_1} \quad , \quad \frac{\partial R}{\partial \beta_1} = \frac{\partial R}{\partial L}\cdot\frac{\partial L}{\partial \beta_1} + \frac{\partial R}{\partial l}\cdot\frac{\partial l}{\partial \beta_1} \quad .$$

Substitute these in (2) ; then

$$\dot{L} = \frac{\partial (L, l)}{\partial (\alpha_1, \beta_1)}\cdot\frac{\partial R}{\partial l} + \frac{\partial L}{\partial t} \quad .$$

Similarly,
$$l = -\frac{\partial (L, l)}{\partial (\alpha_1, \beta_1)}\cdot\frac{\partial R}{\partial L} + \frac{\partial l}{\partial t} \quad .$$

Then
$$\dot{L} = \frac{\partial R'}{\partial l} \quad , \quad l = -\frac{\partial R'}{\partial L}$$

if
$$\frac{\partial (L, l)}{\partial (\alpha_1, \beta_1)} = 1 \quad , \quad \ldots\ldots\ldots(4)$$

and
$$R' = R + \phi \quad , \quad \text{where} \quad \frac{\partial \phi}{\partial l} = \frac{\partial L}{\partial t} \quad , \quad \frac{\partial \phi}{\partial L} = -\frac{\partial l}{\partial t} \quad . \quad \ldots\ldots\ldots(5)$$

If $L \equiv L(\alpha_1)$, (4) is equivalent to $n\,dL = d\alpha_1$, from which we obtain $L = \sqrt{(\mu a)}$, as in 11·02 (6).

From (5), $\frac{\partial \phi}{\partial l} = 0$ and $\frac{\partial \phi}{\partial L} \equiv \frac{d\phi}{dL} = -n$; hence $\phi = -\alpha_1$. R' is then given by 11·02 (8).

11·04. Delaunay variables

Delaunay introduced the following notations, now regarded as standard :

$$\left.\begin{array}{lll} L = \sqrt{(\mu a)} & ; & l = n(t + \beta_1) \\ G \equiv \alpha_2 = \sqrt{\{\mu a (1 - e^2)\}} & ; & g \equiv \beta_2 = \omega \\ H \equiv \alpha_3 = \sqrt{\{\mu a (1 - e^2)\}}\cos i & ; & h \equiv \beta_3 = \Omega \end{array}\right\} \quad \ldots\ldots\ldots(1)$$

The canonic equations are then

$$\left.\begin{array}{lll} \dot{L} = \dfrac{\partial R'}{\partial l} & ; & l = -\dfrac{\partial R'}{\partial L} \\ \dot{G} = \dfrac{\partial R'}{\partial g} & ; & \dot{g} = -\dfrac{\partial R'}{\partial G} \\ \dot{H} = \dfrac{\partial R'}{\partial h} & ; & \dot{h} = -\dfrac{\partial R'}{\partial H} \end{array}\right\} \quad , \quad \ldots\ldots\ldots(2)$$

where
$$R' = R + \frac{\mu^2}{2L^2} \quad . \quad \ldots\ldots\ldots(3)$$

These equations are the starting-point of Delaunay's lunar theory.

l, g and h are *angular variables*; we refer to L, G and H as the non-angular variables.

11·05. Modified Delaunay variables

These variables are connected with extended linear point transformations of the Delaunay variables L, G, H; l, g, h. The general theory has been dealt with in 10·08, to which we now refer.

Identify L, G, H with q_r $(r=1, 2, 3)$ and l, g, h with p_r; L', G', H' with Q_r and l', g', h' with P_r.

We have from 10·08 (2) and 10·08 (1) the general linear transformations

$$\left.\begin{aligned} l' &= a_{11}l + a_{12}g + a_{13}h \\ g' &= a_{21}l + a_{22}g + a_{23}h \\ h' &= a_{31}l + a_{32}g + a_{33}h \end{aligned}\right\} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and a similar group

$$L' = A_{11}L + A_{12}G + A_{13}H \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

with two similar equations for G' and H'.

L', l'; G', g'; H', h' are pairs of canonic variables, the Hamiltonian function remaining unchanged, if—by 10·08 (5) and (6)—

$$\sum_{r=1}^{3} a_{rs}A_{rs} = 1 \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\sum_{r=1}^{3} a_{rs}A_{rt} = 0 \quad , \quad (t \neq s), \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

s and t taking the values 1, 2 and 3.

If we are given the values of the a's in (1), the values of the A's in (2) are given by 10·08 (8), namely,

$$A_{rs} = \frac{1}{\Delta} a'_{rs} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where a'_{rs} is the co-factor of a_{rs} in the determinant

$$\Delta = \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} \quad .$$

Further, by 10·08 (9), the conditions (3) and (4) are equivalent to the condition

$$L'l' + G'g' + H'h' = Ll + Gg + Hh \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The object of such transformations is to simplify the Hamilton-Jacobi equation to the greatest possible extent; we illustrate briefly as follows.

Let R_1 denote the aggregate of terms $C_0 + \Sigma C_s \cos s\theta$ $(s=1, 2, \ldots)$ in the disturbing function, where the C's are functions of L, G, H and

$$\theta = il + jg + kh + un_1 t + v \quad ,$$

i, j, k, u being positive or negative integers (including zero) and v is a linear function of the elements ϖ_1, Ω_1 and ϵ_1 relating to the disturbing body.

To solve the equations with R_1 as Hamiltonian function, let

$$l' = il + jg + kh$$

in which i, j and k are now identified with a_{11}, a_{12}, a_{13} in (1).

Suppose further that the a-coefficients in the formulae for g' and h' in (1) are chosen in any way—although in practice we pay attention in our selection to considerations of simplicity or to the taking advantage of some desirable property. The A-coefficients can then be found by means of (5), and then L, G, H can each be expressed in terms of L', G', H' by means of the three equations (2). The equations to be solved are

$$\left.\begin{aligned} \dot{L}' &= \frac{\partial R_1}{\partial l'} \quad , \quad l' = -\frac{\partial R_1}{\partial L'} \\ \dot{G}' &= \frac{\partial R_1}{\partial g'} \quad , \quad g' = -\frac{\partial R_1}{\partial G'} \end{aligned}\right\} \quad , \qquad \text{.....(7)}$$

with a similar pair for H', h' ; here

$$R_1 = C_0 + \Sigma C_s(L', G' H') \cos s\ (l' + un_1 t + v) \quad .$$

Hence, $\dot{G}' = \dot{H}' = 0$ so that G' and H' are constants—say, α_2, α_3. The Hamilton-Jacobi equation reduces to

$$\frac{\partial S}{\partial t} + C_0 + \Sigma C_s(L', \alpha_2, \alpha_3) \cos\left\{s\left(\frac{\partial S}{\partial L'} + un_1 t + v\right)\right\} = 0 \quad ,$$

an equation which involves only one variable L'. The solution can be effected by methods which we shall describe more fully in Chapter 12 ; in the process a third constant, α_1, will be introduced.

The solution of the equations (7) is thus formally achieved, the conjugate constants being α_i and β_i $(i = 1, 2, 3)$.

When α_i and β_i are taken as new canonic variables, the equations can be at once written down in terms of a Hamiltonian function, given by the original function from which the terms $\Sigma C_s \cos s\theta$ have been removed and to which is added a part similar to $\mu^2/(2L^2)$ added to R in 11·04 (3).

It is evident, without going into further details at present, that the operation described can be repeated so as to eliminate successive single terms or groups of terms from the original disturbing function.

The practical advantages of the method just sketched reside, of course, in the fact that two modified Delaunay variables, G' and H', are constants with the consequence that the Hamiltonian-Jacobi is much simplified.

11·06. An important set of modified Delaunay variables

It is evident from the expressions, in 11·04 (1), of the Delaunay variables in terms of the elliptic elements that $L - G$, or $G - L$, is of order e^2 ; hence if G' is chosen to be $L - G$, or $G - L$, then G' is $O(e^2)$.

Similarly, since

$$G - H \equiv G(1 - \cos i) = 2G \sin^2 \frac{i}{2} \quad ,$$

then $G - H$, or $H - G$, is $O(\gamma^2)$, where $\gamma = \tan i$.

We make the following convenient choice of modified variables :

$$L' = L \quad , \quad G' = G - L \quad , \quad H' = H - G \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

We then use the condition 11·05 (6) to derive suitable forms of the conjugates l', g' and h'.

Since $L' = L$, then

$$\begin{aligned} L'(l+g+h) &= L(l+g+h) \\ &\equiv Ll + Gg + Hh - (g+h)(G-L) - h(H-G), \quad \ldots\ldots\ldots (2) \end{aligned}$$

or, by means of (1),

$$L'(l+g+h) + G'(g+h) + H'h = Ll + Gg + Hh \quad .$$

Hence the condition

$$L'l' + G'g' + H'h' = Ll + Gg + Hh$$

is satisfied if

$$l' = l + g + h \quad , \quad g' = g + h \text{ and } h' = h \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In terms of the elliptic elements

$$l' = l + \varpi \quad , \quad g' = \varpi \quad \text{and} \quad h' = \Omega \; ; \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

also l' is the mean longitude. The arguments of the periodic terms in the disturbing function are thus readily expressed in terms of l', g' and h'. It is easily seen from (2) that if we had chosen G' and H' to be defined by

$$G' = L - G \quad , \quad H' = G - H$$

then the corresponding conjugates would be given by

$$g' = -(g+h) = -\varpi \quad , \quad h' = -h = -\Omega \quad ,$$

thus necessitating changes of sign in the arguments of the periodic terms of the disturbing function so far as ϖ and Ω are concerned. The convenience of the choice in (1) is thus apparent.

Defining G' as in (1), we have

$$G' \equiv G - L = L\{\sqrt{(1-e^2)} - 1\} = L(-\tfrac{1}{2}e^2 - \tfrac{1}{8}e^4 \ldots) \; ; \quad \ldots\ldots\ldots\ldots (5)$$

G' is negative. To a first approximation, $e^2 = -2G'/L$.

To a second approximation, $$e^2 = -\frac{2G'}{L} - \left(\frac{G'}{L}\right)^2 \; . \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Higher approximations, if required, can be obtained by similar simple methods. In the same way, any power of e can be expressed as a series in $\sqrt{\left(-\frac{2G'}{L}\right)}$. For example,

$$e=\sqrt{\left(-\frac{2G'}{L}\right)}\left\{1+\frac{G'}{4L}\ldots\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Similarly, $$H' \equiv -G(1-\cos i) = -2(G'+L)\sin^2\frac{i}{2}$$

from which $$\sin^2\frac{i}{2} = -\frac{H'}{2L}\left(1+\frac{G'}{L}\right)^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Hence, $\sin^2\frac{i}{2}$—or, alternatively, γ—can be expressed to any degree of approximation as a series in $(-H'/2L)$ and G'/L. The disturbing function is then of the form $C_0+\Sigma C'\cos\theta'$, where

$$\theta' = il'+jg'+kh'+un_1t+v \quad .$$

We can then proceed with the operation described in the previous section by choosing a modified variable l'' given by

$$l'' = il'+jg'+kh' \quad .$$

For reference we collect the expressions for the modified Delaunay variables in terms of the elliptic elements.

$$\left.\begin{aligned} L' &\equiv L = \sqrt{(\mu a)} \quad , \\ G' &\equiv G-L = \sqrt{(\mu a)}\,\{\sqrt{(1-e^2)}-1\} \quad , \\ H' &\equiv H-G = \sqrt{\{(\mu a)(1-e^2)\}}\,(\cos i-1) \\ l' &= l+g+h = \text{mean longitude}, \\ g' &= g+h = \varpi \quad , \\ h' &= h = \Omega \quad . \end{aligned}\right\} \quad , \ldots\ldots\ldots\ldots(9)$$

11·07. Poincaré variables

We denote these variables, in the first instance, by L_1, G_1, H_1; l_1, g_1, h_1. The first pair are selected, in the notation of the previous section, as follows:

$$L_1 = L' = L\,;\ l_1 = l' = l+g+h \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We assume that the second pair are defined by

$$G_1 = A\sin g' \quad , \quad g_1 = A\cos g' \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $g' = g+h$, and A is in general a function of G' and g'. By analogy with 11·03 (4), G_1 and g_1 are conjugate if

$$\frac{\partial(G_1, g_1)}{\partial(G', g')} = 1 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

from which it is easily found that $A\dfrac{\partial A}{\partial G'} = -1$. Hence, we can take A to be given by $A = \sqrt{(-2G')}$.

The third pair are defined by

$$H_1 = B\sin h' \quad , \quad h_1 = B\cos h' \quad .$$

By a similar process to that just described for evaluating A, B is given by $B = \sqrt{(-2H')}$.

The complete set of variables is then

$$\left.\begin{array}{lll} L_1 = L & ; & l_1 = l+g+h\,; \\ G_1 = \sqrt{(-2G')}\sin g' & ; & g_1 = \sqrt{(-2G')}\cos g' \\ H_1 = \sqrt{(-2H')}\sin h' & ; & h_1 = \sqrt{(-2H')}\cos h' \end{array}\right\} . \quad \ldots\ldots\ldots\ldots (4)$$

Since the transformations do not involve the time explicitly the Hamiltonian function for the new variables is unaltered.

For convenience we change the notation as follows :

For the first pair, write λ for l_1 ; then $\lambda = l+\varpi$, so that λ is the mean longitude.

For the second pair, write ξ for G_1 and η for g_1, ρ for $\sqrt{(-2G')}$ and replace g' by ϖ ; then $\xi = \rho\sin\varpi$, $\eta = \rho\cos\varpi$.

For the third pair, write u for H_1, and v for h_1, σ for $\sqrt{(-2H')}$ and replace h' by Ω ; then $u = \sigma\sin\Omega$, $v = \sigma\cos\Omega$.

Collected, the Poincaré variables are

$$\left.\begin{array}{lll} L = \sqrt{(\mu a)} & ; & \lambda = l+\varpi \\ \xi = \rho\sin\varpi & ; & \eta = \rho\cos\varpi \\ u = \sigma\sin\Omega & ; & v = \sigma\cos\Omega \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

We have seen in the previous section that G' is $O(e^2)$ and H' is $O(\gamma^2)$; thus ρ is $O(e)$ and σ is $O(\gamma)$; consequently, ξ and η are $O(e)$, and u and v are $O(\gamma)$.

ξ and η are called by Poincaré the *eccentric* variables, and u and v the *oblique* variables.

It is to be remarked that of the new variables *only one is an angular variable* ; this is λ denoting the mean longitude.

In 5·10 (ii) we introduced functions of the elements e and ϖ given by

$$h = e\sin\varpi \quad , \quad k = e\cos\varpi\,;$$

these are seen to be closely related to ξ and η in (5), since ρ is of order e.

We also introduced functions of i and Ω given by

$$p = \sin i\sin\Omega \quad , \quad q = \sin i\cos\Omega \quad ,$$

or, neglecting γ^3,

$$p = \gamma\sin\Omega \quad , \quad q = \gamma\cos\Omega\,;$$

these are seen to be closely related to u and v in (5), since σ is of order γ.

When the disturbing function is expanded in terms of the Poincaré variables, it takes the form* given by

$$R = \Sigma A\xi^a\eta^b\xi_1^{\,c}\eta_1^{\,d}u^e v^f u_1^{\,g}v_1^{\,h}\begin{matrix}\cos\\ \sin\end{matrix}(j\lambda + j_1\lambda_1) \quad \ldots\ldots\ldots\ldots (6)$$

where A is a function of L, L_1 (a suffix relates to the disturbing planet) and

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ j \text{ and } j_1$$

are positive integers (with the exception of j_1 which may be negative) one or more of which can be zero.

* Poincaré, *Méthodes Nouvelles de la Mécanique Céleste*, I, 32 (1892).

In (6) the cosine is to be taken if

$$a+c+e+g=2s$$

where s is a positive integer ; the sine is to be taken if $a+c+e+g$ is of the form $2s+1$.

If we suppose for the sake of simplicity that e, e_1, γ and γ_1 are all of the same order of magnitude, the order of magnitude O of a term in (6) is $a+b+c+d+e+f+g+h$, and is related to the numbers j and j_1 appearing in the argument by

$$O=|j+j_1|+2r \quad ,$$

where r is a positive integer in general and may be zero.

Chapter 12

DELAUNAY'S LUNAR THEORY

12·01. Introduction

The most exhaustive application of the canonic method in celestial mechanics was made by Delaunay in connection with the Moon's orbit; the result—the fruit of twenty year's work—is the most complete literal solution ever achieved in this complicated problem. Because of their nature the final formulae can be applied to satellite problems in general.

We start from the canonic equations with Delaunay's variables L, G, H; l, g, h and we write them succinctly in the forms

$$\frac{d}{dt}(L, G, H) = \frac{\partial R'}{\partial (l, g, h)} \quad ; \quad \frac{d}{dt}(l, g, h) = -\frac{\partial R'}{\partial (L, G, H)} \quad \ldots\ldots\ldots\ldots (1)$$

where

$$R' \equiv R + \frac{\mu^2}{2L^2} \quad , \quad \ldots\ldots\ldots\ldots (2)$$

R being the original disturbing junction.

We add, for reference, the expressions for the Delaunay variables in terms of the elliptic elements:

$$L = \sqrt{(\mu a)} \quad ; \quad G = \sqrt{\{\mu a (1 - e^2)\}} \quad ; \quad H = \sqrt{\{\mu a (1 - e^2)\}} \cos i \quad ; \ldots (3)$$

$$l = nt + \epsilon - \varpi \quad ; \quad g = \varpi - \Omega \quad ; \qquad h = \Omega \quad . \quad \ldots\ldots\ldots\ldots (4)$$

In 7·06 (18) and (19) we have given the expansion of R up to the fourth order of the small quantities m, e, γ, e_1 and a/a_1, the last being taken to be of second order; in these formulae ξ, for example, is defined by

$$\xi = (n - n_1)t + \epsilon - \epsilon_1 = l + \varpi - n_1 t - \epsilon_1$$

and M, the mean anomaly, is the same as l.

We take as a typical periodic term from 7·06 (18) that with argument $(2\xi + M)$, or $(3l + 2g + 2h - 2n_1 t - 2\epsilon_1)$. We can transform this argument in terms of the modified Delaunay variables l', g', h' given by 11·06 (3); these are:

$$l' = l + g + h \quad , \quad g' = g + h \quad , \quad h' = h \quad ,$$

or

$$l' = \text{mean longitude} \quad , \quad g' = \varpi \quad , \quad h' = \Omega \quad .$$

The argument is then $(3l' - g' - 2n_1 t - 2\epsilon_1)$.

It is convenient to drop the dashes and to write the argument as $(3l - g - 2n_1 t - 2\epsilon_1)$ so that now

$$l = \text{mean longitude} \quad , \quad g = \varpi \quad , \quad h = \Omega \quad . \quad \ldots\ldots\ldots\ldots (5)$$

From (3) and 11·06 (1), the corresponding non-angular variables are, the dashes being dropped,

$$L=\sqrt{(\mu a)}\ ;\ G=\sqrt{(\mu a)}\{\sqrt{(1-e^2)}-1\}\ ;\ H=\sqrt{\{\mu a(1-e^2)\}}(\cos i-1). \quad \ldots(6)$$

Here G is $O(e^2)$ and H is $O(\gamma^2)$.

The canonic equations are then given by (1), where L, G, H ; l, g, h have now the meanings given by (6) and (5).

The amplitude of the term referred to is $\frac{3}{4}m^2n^2a^2e$, from 7·06 (18), or $\dfrac{3\mu^2m^2e}{4L^2}$ on writing $n^2=\mu a^{-3}$; by 11·06 (7),

$$e=\sqrt{\left(-\frac{2G}{L}\right)}\left\{1+\frac{G}{4L}\ldots\right\} \quad .$$

The amplitude in this case is thus expressed in terms of L and G.

In the same way, from 7·06 (18), the non-trigonometrical part of R', the term $\mu^2/(2L^2)$ being necessarily included, is

$$\frac{\mu^2}{2L^2}+\frac{\mu^2m^2}{4L^2}\left[1+\tfrac{3}{2}(e^2-\gamma^2+e_1^2)\right] \quad \ldots\ldots(7)$$

which can be expressed in terms of L, G, H by (6) ; here e_1 is a constant.

Delaunay's procedure is equivalent to : (i) writing R' in the form

$$R'=-B-\Sigma A\cos(il+jg+kh+qn_1t+q') \quad \ldots\ldots(8)$$

in which i, j, k and q are positive or negative integers (including zero), q' is a linear expression in terms of ϖ_1, Ω_1 and ϵ_1 with numerical coefficients, and B and A are functions of L, G and H defined by (6), (ii) choosing one trigonometrical term—such as that referred to—from the summation in (8), and (iii) solving (1) by the canonic method when R' is replaced by R_0 which we write, with the selected term in general form, as

$$R_0=-B-A\cos(il+jg+kh+qn_1t+q') \quad . \quad \ldots\ldots(9)$$

Delaunay's expansion for R', in (8), consisted of 320 periodic terms all at least to the seventh order of small quantities and several to the eighth order.

12·02. The auxiliary function S

We choose new canonic variables L', G', H' ; l', g', h' which are connected linearly with the old so that, by 11·05 (6),

$$L'l'+G'g'+H'h'=Ll+Gg+Hh \quad . \quad \ldots\ldots(1)$$

We first choose l' and L' as follows :

$$l'=il+jg+kh \quad , \quad \ldots\ldots(2)$$

$$L'=\frac{1}{i}L \quad . \quad \ldots\ldots(3)$$

In (3) it is supposed that $i\neq 0$.

Then (1) becomes

$$G'g' + H'h' = g\left(G - \frac{j}{i}L\right) + h\left(H - \frac{k}{i}L\right) \quad .$$

The condition (1) is satisfied if

$$g' = g \quad , \quad h' = h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and

$$G' = G - \frac{j}{i}L \quad , \quad H' = H - \frac{k}{i}L \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Accordingly, we obtain the new canonic equations

$$\frac{d(L', G', H')}{dt} = \frac{\partial R_0}{\partial(l', g', h')} \quad , \quad \frac{d}{dt}(l', g', h') = -\frac{\partial R_0}{\partial(L', G', H')} \quad , \ldots (6)$$

in which

$$R_0 = -B - A\cos(l' + qn_1t + q') \; ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

it is now supposed that A and B are expressed, by means of (3) and (5), in terms of L', G', H'.

Since (7) does not contain g' and h', we have from (6), $\dot{G}' = \dot{H}' = 0$; thus G' and H' are constants which we shall later identify with two of the canonic constants arising in the solution of (6).

From the method of choice of the new variables we regard L' and l' as the principal canonic variables.*

In terms of the general theory we identify L', G', H' with the generalised coordinates q_1, q_2, q_3 and l', g', h' with the corresponding conjugates p_1, p_2, p_3 and R_0 with the Hamiltonian function $H(q, p, t)$.

The solution of (6) is then achieved if we can find *any* function

$$S(q, \alpha, t) \equiv S(L', \alpha, t)$$

containing three independent constants α_1, α_2, α_3 and satisfying the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} - B - A\cos\left(\frac{\partial S}{\partial L'} + qn_1t + q'\right) = 0 \quad . \quad \ldots\ldots\ldots\ldots\ldots (8)$$

Write

$$S = S' - (qn_1t + q')L' \; ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

then (8) becomes

$$\frac{\partial S'}{\partial t} - qn_1L' - B - A\cos\left(\frac{\partial S'}{\partial L'}\right) = 0 \quad . \quad \ldots\ldots\ldots\ldots\ldots (10)$$

Since t does not appear explicitly in this equation, then $\frac{\partial S'}{\partial t} = C$, where C is a constant which we identify with α_1 of the general theory. Let

$$B_1 = B + qn_1L' \; ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

then, from (10),

$$\frac{\partial S'}{\partial L'} = \cos^{-1}\left(\frac{C - B_1}{A}\right) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

* If $i = 0$, and $j \neq 0$, we would use—instead of (2) and (3)—$g' = jg + kh$, $G' = G/j$ and then regard G' and g' as the principal canonic variables.

Hence, $$S' = Ct + \int_X^{L'} \cos^{-1}\left(\frac{C - B_1}{A}\right) dL' + D \quad , \quad \ldots\ldots\ldots\ldots(13)$$

where D is a constant and the lower limit, X, of the integral is a function of C, G', H' later to be defined specifically. The integrand is a function of L', G', H' and C of which G', H' and C are constants; the integral can thus be evaluated, in principle at least.

In solving (8) we have, up to the present, introduced only one of the three α's, namely C; consequently, D must be a function of α_2 and α_3 and, of course, must be constant. The simplest form for D is

$$D = \alpha_2 G' + \alpha_3 H' \quad \ldots\ldots\ldots\ldots(14)$$

which complies with the requirements that S is *any* function, with three α's, which satisfies (8).

Adopting Delaunay's notation we write: $\alpha_2 = (g)$, $\alpha_3 = (h)$. Then, from (9), (13) and (14), we have

$$S = Ct - (qn_1 t + q')L' + S_1 + (g)G' + (h)H' \quad , \quad \ldots\ldots\ldots\ldots(15)$$

where S_1 denotes the integral

$$S_1 = \int_X^{L'} \cos^{-1}\left(\frac{C - B_1}{A}\right) dL' \quad . \quad \ldots\ldots\ldots\ldots(16)$$

12·03. The formal solutions

The solutions of 12·02 (6) are, from 12·02 (15),

$$\beta_1 \equiv \frac{\partial S}{\partial C} = t + \frac{\partial S_1}{\partial C} \quad , \quad \ldots\ldots\ldots\ldots(1)$$

$$\beta_2 \equiv \frac{\partial S}{\partial (g)} = G' \quad , \quad \ldots\ldots\ldots\ldots(2)$$

$$\beta_3 \equiv \frac{\partial S}{\partial (h)} = H' \quad , \quad \ldots\ldots\ldots\ldots(3)$$

together with

$$l' \equiv \frac{\partial S}{\partial L'} = \frac{\partial S_1}{\partial L'} - (qn_1 t + q') \quad , \quad \ldots\ldots\ldots\ldots(4)$$

$$g' \equiv \frac{\partial S}{\partial G'} = (g) + \frac{\partial S_1}{\partial G'} \quad , \quad \ldots\ldots\ldots\ldots(5)$$

$$h' \equiv \frac{\partial S}{\partial H'} = (h) + \frac{\partial S_1}{\partial H'} \quad . \quad \ldots\ldots\ldots\ldots(6)$$

We have seen earlier that G' and H' are constants, and (2) and (3) show that G' and H' are canonic constants which are conjugate to α_2 and α_3, that is, to (g) and (h).

It is convenient to set out here the identification of the canonic constants introduced (except for β_1 which we identify later):

$$\alpha_1 = C \; ; \; \alpha_2 = (g) \; ; \; \alpha_3 = (h) \; ;$$
$$\beta_2 = G' \; ; \; \beta_3 = H' \quad .$$

Since S_1 is a function of the variable L' and the constants G', H' and C, the equation (1) can be solved in principle so as to express L' as a function of $(t-\beta_1)$, G', H' and C; thus we write

$$L' \equiv L'(t-\beta_1\,;\ G',\ H',\ C) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Let θ be defined by $\theta = l' + qn_1 t + q'$. $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$

Then, θ is the argument in 12·02 (7) and (8); we then have, from (4),

$$\theta = \frac{\partial S_1}{\partial L'} = \cos^{-1}\left(\frac{C-B_1}{A}\right) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

or $$C - B_1 = A\cos\theta \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Thus θ can be expressed as a function of L', G', H' and C. We can then write 12·02 (16) in the forms

$$S_1 = \int_X^{L'} \theta\, dL' = \int_X^{L'} \cos^{-1}\left(\frac{C-B_1}{A}\right) dL' \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

We now specify X as a value of L' which corresponds to the zero value of θ; accordingly, from (10), X is a root of the equation

$$C - B_1 = A$$

and is a function of C, G' and H', as previously stated.

12·04. Form of the solutions for L' and l'

We consider the two principal canonic equations

$$\dot{L}' = \frac{\partial R_0}{\partial l'} \quad , \quad l' = -\frac{\partial R_0}{\partial L'} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $$R_0 = -B - A\cos(l' + qn_1 t + q') = -B - A\cos\theta$$

and θ is given by 12·03 (8).

Then, $$\dot{L}' = A\sin\theta \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and $$l' = \dot{\theta} - qn_1 = \frac{\partial B}{\partial L'} + \frac{\partial A}{\partial L'}\cos\theta \quad ;$$

or, by 12·02 (11), $$\dot{\theta} = \frac{\partial B_1}{\partial L'} + \frac{\partial A}{\partial L'}\cos\theta \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

which, for convenience, we write as

$$\dot{\theta} = B' + A'\cos\theta,$$

where A' and B' are functions of L', G', H'.

Now, the expression for the disturbing function shows that A—and, consequently, A'—is of order m^2 at least and that B—and B_1—is of zero order, since in R' we have the additive term $\mu^2/(2L^2)$. Suppose, for the sake of explanation, that A is $O(m^2)$. Hence, if we omit terms of $O(m^2)$, we can write (2) as $\dot{L}' = 0$, from which the first approximation to L' is given by $L' = L_0$, where L_0 is a constant and is of order zero.

The first approximation to θ can then be found from (3) by omitting A' and by putting L_0 for L' in B', the latter now being written as B_0', a constant. Then,

$$\dot{\theta} = B'_0 \equiv \theta_0 \quad , \quad \text{say,}$$

whence

$$\theta = \theta_0(t+c) \equiv \lambda \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which c is the constant of integration. It is evident that θ_0 is of order zero, since B_0' is of order zero.

Consider now the second approximations ; we write

$$L' = L_0 + L_1' \quad , \quad \theta = \lambda + \theta_1' \quad ,$$

where L_1' and θ_1' are $O(m^2)$. Then, to the order concerned, (2) becomes $\dot{L}_1' = A_0 \sin\lambda$, A_0 being the value of A when L_0 is substituted for L' ; hence,

$$L' = L_0 + L_1 \cos\lambda \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where $L_1 = -A_0/\theta_0$ and is of $O(m^2)$.

Again, (3) becomes

$$\dot{\theta} \equiv \dot{\lambda} + \dot{\theta}_1' = B' + A_0' \cos\lambda \, ;$$

now, to $O(m^2)$, $B' = B_0' + \left(\frac{\partial B'}{\partial L'}\right)_0 L_1'$ by Taylor's theorem. But, $L_1' = L_1 \cos\lambda$; hence

$$\dot{\lambda} + \dot{\theta}_1' = \theta_0 + \left\{A_0' + \left(\frac{\partial B'}{\partial L'}\right)_0 L_1\right\} \cos\lambda$$

from which, since $\dot{\lambda} = \theta_0$ from the first approximation, $\theta_1' = \theta_1 \sin\lambda$,

where

$$\theta_1 = \frac{1}{\theta_0}\left\{A_0' + \left(\frac{\partial B'}{\partial L'}\right)_0 L_1\right\} \quad .$$

The solutions, so far, are—λ being given by $\lambda = \theta_0(t+c)$—

$$L' = L_0 + L_1 \cos\lambda \quad , \quad \theta = \lambda + \theta_1 \sin\lambda \quad .$$

It is evident that by proceeding to higher approximations we can express the solutions of (2) and (3) as the two series

$$L' = L_0 + \sum_{p=1} L_p \cos p\theta_0(t+c) \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$\theta = \theta_0(t+c) + \sum_{p=1} \theta_p \sin p\theta_0(t+c) \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

in which L_0, θ_0, L_p, θ_p are functions of C, G' and H'.

From 12·03 (8), the series for l' is given by

$$l' = \theta_0(t+c) - qn_1 t - q' + \Sigma\theta_p \sin p\theta_0(t+c) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

It is to be understood that an important part of the operation is the precise establishment of the series in (6) and (7) in terms of m and according to powers of e, e_1, γ, γ_1 and a/a_1 ; we deal more fully with this aspect of the problem in 12·17.

As already stated, θ_0 is of order zero, and L_1 and θ_1 are $O(m^2)$; in (6)

and (7), L_p and θ_p $(p>1)$ are successively of higher orders than $O(m^2)$. In making these assessments of orders of magnitude we are taking A to be of the lowest possible order, that is, $O(m^2)$.

12·05. Identification of β_1

Consider 12·03 (1); it is

$$\beta_1 = t + \frac{\partial S_1}{\partial C} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $S_1 = \int_X^{L'} \theta \, dL'$. We then have

$$\beta_1 - t = \int_X^{L'} \frac{\partial \theta}{\partial C} dL' - (\theta) \frac{\partial X}{\partial C} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where (θ) denotes the value of θ when $L' = X$; by definition, $(\theta) = 0$.

Now, $\cos\theta = \dfrac{C - B_1}{A}$; hence, $\dfrac{\partial\theta}{\partial C} = -\dfrac{1}{A\sin\theta} = -\dfrac{1}{\dot{L}'}$, by 12·04 (2). We then have, from (2),

$$\beta_1 - t = -\int_{t_0}^{t} dt = t_0 - t \quad ,$$

where t_0 $(\equiv \beta_1)$ is the value of t when $L' = X$—that is, when $\theta = 0$. But, from 12·04 (7), $\theta = 0$ when $t = -c$; we then have $\beta_1 = t_0 = -c$.

The formula (1) can now be written

$$t + c = -\frac{\partial S_1}{\partial C} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now, S_1 is a function of L', G', H' and C, the only variable being L'. Thus, (3) is an equation which enables us, in principle, to express L' in terms of $(t+c)$, G', H' and C; L' is, in fact, given by the series in 12·04 (6).

12·06. Solutions for g' and h'

Consider 12·03 (5), namely,

$$g' = (g) + \frac{\partial S_1}{\partial G'} = (g) + \int_X^{L'} \frac{\partial \theta}{\partial G'} dL' - (\theta) \frac{\partial X}{\partial G'} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

As in the preceding section, $(\theta) = 0$. Now, $\cos\theta = (C - B_1)/A$ in which B_1 and A are functions of G'; hence

$$A\sin\theta \frac{\partial\theta}{\partial G'} = \frac{\partial B_1}{\partial G'} + \frac{C - B_1}{A} \cdot \frac{\partial A}{\partial G'} \equiv F(C, L', G', H') \quad . \quad \ldots\ldots\ldots\ldots (2)$$

Then, since $\dot{L}' = A\sin\theta$ and $t = -c$ when $L' = X$, (1) becomes

$$g' = (g) + \int_{-c}^{t} F(C, L', G', H') \, dt \quad .$$

Since L' is given by 12·04 (6) as a cosine-series, it is evident that F can be developed as a cosine series of the form

$$F = g_0 + \Sigma D_p \cos p\theta_0(t+c) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Hence, on integration,

$$g' = (g) + g_0(t+c) + \Sigma g_p \sin p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots\ldots(4)$$

where $g_p = D_p/(p\theta_0)$. It is to be noticed that the constant of integration in (4) is $g_0 c$ which is such that $g' = (g)$ in accordance with (1) when $L' = X$, that is, when $t = -c$.

Similarly,

$$h' = (h) + h_0(t+c) + \Sigma h_p \sin p\theta_0(t+c).$$

It is important to know the order of magnitude of g_0 and h_0; we consider the former.

Since $A \cos\theta = C - B_1$, the function F can be expressed as

$$F = \frac{\partial B_1}{\partial G'} + \frac{\partial A}{\partial G'} \cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

With sufficient accuracy for our purpose we write

$$L' = L_0 + L_1 \cos\lambda \quad ,$$

where $\lambda = \theta_0(t+c)$; then, approximately,

$$\frac{\partial B_1}{\partial G'} = \left(\frac{\partial B_1}{\partial G'}\right)_0 + \left(\frac{\partial^2 B_1}{\partial L' \partial G'}\right)_0 L_1 \cos\lambda \quad , \quad \ldots\ldots\ldots\ldots\ldots(6)$$

the zero suffix denoting that L' is put equal to L_0 after differentiation.

Similarly,

$$\frac{\partial A}{\partial G'} = \left(\frac{\partial A}{\partial G'}\right)_0 + \left(\frac{\partial^2 A}{\partial L' \partial G'}\right)_0 L_1 \cos\lambda \quad . \quad \ldots\ldots\ldots\ldots\ldots(7)$$

It is sufficient to write, in (5), $\cos\theta = \cos\lambda$. Now, g_0 is the non-periodic part of F; hence, from (6) and (7)—to a sufficient degree of accuracy—g_0 is given by

$$g_0 = \left(\frac{\partial B_1}{\partial G'}\right)_0 + \tfrac{1}{2} L_1 \left(\frac{\partial^2 A}{\partial L' \partial G'}\right)_0 \quad . \quad \ldots\ldots\ldots\ldots\ldots(8)$$

First consider B_1; according to 12·01 (7) and 12·02 (11), it is given by

$$B_1 = qn_1 L' - \frac{\mu^2}{2L^2} - \frac{\mu^2 m^2}{4L^2}\left[1 + \tfrac{3}{2}(e^2 - \gamma^2 + e_1^2)\right] \quad .$$

We need only consider the term in e^2 in the present connection. Since $G = L\{\sqrt{(1-e^2)} - 1\} \doteqdot -\frac{1}{2}e^2 L$ and $G' = G - jL'$ it is evident that $\left(\frac{\partial B_1}{\partial G'}\right)_0$ is $O(m^2)$.

We take A to be the coefficient in the disturbing function of lowest order containing e, and we can write $A = \alpha m^2 e$, where α is independent of m and e. Then

$$\frac{\partial A}{\partial G'} = \frac{\partial A}{\partial e} \cdot \frac{\partial e}{\partial G'} = -\frac{\alpha m^2}{eL} \equiv O(m) \quad ,$$

if we suppose that e and m are of the same order. Further, $\dfrac{\partial^2 A}{\partial L' \partial G'}$ is $O(m)$, since differentiation with respect to L' leaves the order of magnitude unchanged. Since L_1 is at least of $O(m^2)$, the order of g_0 is that of the first term on the right-hand side of (8). We conclude then that g_0 is $O(m^2)$; similarly, h_0 is $O(m^2)$.

It is easily seen that g_1 and h_1 are of still higher order and, further, that the order of g_p, h_p increases with p.

12·07. Summary of results

It is convenient to collect the chief results of the operation just concluded; they are as follows (p taking the values 1, 2, ...):

$$L' = L_0 + \Sigma L_p \cos p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$G' = \text{constant} \quad , \quad H' = \text{constant} \quad ,$$

$$l' = \theta_0(t+c) - qn_1 t - q' + \Sigma \theta_p \sin p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$g \equiv g' = (g) + g_0(t+c) + \Sigma g_p \sin p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots\ldots (3)$$

$$h \equiv h' = (h) + h_0(t+c) + \Sigma h_p \sin p\theta_0(t+c) \quad . \quad \ldots\ldots\ldots\ldots\ldots (4)$$

The original modified Delaunay variables are given by 12·01 (6) and (5); they can be easily expressed in the form of series. Thus, $L \equiv iL', G \equiv jL' + G'$, $H = kL' + H'$ are given at once by means of (1); also g and h are given by (3) and (4).

Since $l' = il + jg + kh$, we can write

$$l = (l) + l_0(t+c) - \frac{1}{i}(qn_1 t + q') + \Sigma l_p \sin p\theta_0(t+c) \quad \ldots\ldots\ldots\ldots\ldots (5)$$

where

$$(l) = -\frac{1}{i}\{j(g) + k(h)\} \quad , \quad l_0 = \frac{1}{i}\{\theta_0 - jg_0 - kh_0\} \quad ,$$

$$l_p = \frac{1}{i}\{\theta_p - jg_p - kh_p\} \quad . \quad \ldots\ldots\ldots\ldots\ldots (6)$$

For reference the canonic constants introduced into the solutions are identified with the canonic constants of the general theory as follows:

$$\left.\begin{array}{lll} \alpha_1 = C & ; \quad \alpha_2 = (g) & ; \quad \alpha_3 = (h) \\ \beta_1 = -c & ; \quad \beta_2 = G' & ; \quad \beta_3 = H' \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (7)$$

Having solved the canonic equations for the part, R_0, of the complete Hamiltonian function R', which we write as

$$R' = R_0 - (-R_1) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

we regard the equivalents of the α's and β's as new canonic variables; the new canonic equations with $(-R_1)$ as the Hamiltonian function are

$$\dot{C} = \frac{\partial R_1}{\partial c} \quad , \quad \dot{G}' = \frac{\partial R_1}{\partial (g)} \quad , \quad \dot{H}' = \frac{\partial R_1}{\partial (h)} \quad ; \quad \ldots\ldots\ldots\ldots\ldots (9)$$

$$\dot{c} = -\frac{\partial R_1}{\partial C} \quad , \quad (\dot{g}) = -\frac{\partial R_1}{\partial G'} \quad , \quad (\dot{h}) = -\frac{\partial R_1}{\partial H'} \quad . \quad \ldots\ldots\ldots\ldots (10)$$

in which R_1 is now supposed to be expressed in terms of the new variables C, G', H' ; c, (g), (h).

12·08. The second operation

Consider a typical term of R_1 before the substitutions, given by the solutions summarised in the previous section, are made ; such a term is

$$D \cos (i_1 l + j_1 g + k_1 h + q_1 n_1 t + q_1') \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where D is expressed as $D(L, G, H)$. It is evident from the previous section that D can be developed as $D_0 + \Sigma D_p \cos p\theta_0 (t+c)$, in which the D's are functions of C, G' and H', and $p = 1, 2, \ldots$.

The argument in (1) is, by 12·07 (3), (4) and (5),

$$[i_1(l) + j_1(g) + k_1(h) + (q_1 - i_1 q) n_1 t + q_1' - i_1 q' + (t+c)\{i_1 l_0 + j_1 g_0 + k_1 h_0\} + \Sigma E_p \sin p\theta_0 (t+c)] \quad ,$$

where $E_p = i_1 l_p + j_1 g_p + k_1 h_p$. Accordingly, since E_p is of $O(m^2)$ at least, the term (1) can be expanded as a series of the form

$$\Sigma A^* \cos \theta^* \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where $\theta^* = i_1(l) + j_1(g) + k_1(h) + (q_1 - i_1 q) n_1 t + q_1' - i_1 q' + (t+c)\{i_1 l_0 + j_1 g_0 + k_1 h_0 \pm p\theta_0\}$ $\ldots\ldots\ldots\ldots (3)$

and $A^* = A^*(C, G', H')$.

In the same way all the other terms in R_1 will be expressible as series of the form (2) ; consequently, we can regard (2) as the general form assumed by R_1 when all the substitutions are made in terms of the new canonic variables C, G', H' ; c, (g), (h).

Consider the canonic equation, in 12·07 (10), $\dot{c} = -\dfrac{\partial R_1}{\partial C}$; then, denoting the coefficient of $(t+c)$ in (3) by Q, we obtain, from (2),

$$\dot{c} = -\Sigma \frac{\partial A^*}{\partial C} \cos \theta^* + (t+c) \Sigma A^* \frac{\partial Q}{\partial C} \sin \theta^* \quad , \quad \ldots\ldots\ldots\ldots (4)$$

Q being a function of C, G', H'.

It is evident that $(\dot{g})$, $(\dot{h})$ will be given by equations similar in form to (4).

The presence of the factor $(t+c)$ outside the second summation in the three equations of type (4) causes the same analytical embarrassment as we encountered in Section 11·01 ; there we introduced a new pair of variables, L and l, to overcome the difficulty. In the present instance, which corresponds to the most general case, we require to make a much more comprehensive change of the variables.

12·09. The new variable Λ

We return to the formula 12·05 (3)

$$t+c=-\frac{\partial S_1}{\partial C} \quad \ldots\ldots(1)$$

where
$$S_1 \equiv S_1(L', G', H', C)=\int_X^{L'} \theta\, dL' \quad ; \quad \ldots\ldots(2)$$

the function S_1 is supposed to have been derived explicitly.

Now express S_1 in terms of the time by means of

$$L'=L_0+\Sigma L_p \cos p\lambda \quad \ldots\ldots(3)$$

where $\lambda=\theta_0(t+c)$; let $K(t+c\,;\ C, G', H')$ denote the function so transformed. Then we have the identity

$$K(t+c\,;\ C, G', H') \equiv S_1(L', G', H', C) \quad \ldots\ldots(4)$$

when (3) is substituted in S_1 ; hence

$$\frac{\partial K}{\partial C}=\frac{\partial S_1}{\partial L'}\cdot\frac{\partial L'}{\partial C}+\frac{\partial S_1}{\partial C} \quad .$$

Now, by 12·03 (9),
$$\frac{\partial S_1}{\partial L'}=\theta \quad \ldots\ldots(5)$$

and, using (1), we obtain

$$t+c+\frac{\partial K}{\partial C}=\theta\frac{\partial L'}{\partial C} \quad \ldots\ldots(6)$$

We then have the identity

$$\begin{aligned} t+c+\frac{\partial K}{\partial C} &= \{\theta_0(t+c)+\Sigma\theta_p \sin p\lambda\} \\ &\quad \times\left\{\frac{\partial L_0}{\partial C}+\Sigma\frac{\partial L_p}{\partial C}\cos p\lambda-(t+c)\frac{\partial\theta_0}{\partial C}\Sigma pL_p \sin p\lambda\right\} \\ &=(t+c)\left\{\theta_0\frac{\partial L_0}{\partial C}-\tfrac{1}{2}\frac{\partial\theta_0}{\partial C}\Sigma pL_p\theta_p\right\}+X \quad , \quad \ldots\ldots(7) \end{aligned}$$

where X denotes terms of the form given by

$$X=P.T.+(t+c)\,[P.T.]+(t+c)^2\,[P.T.] \quad . \quad \ldots\ldots(8)$$

Again, from (2),
$$S_1=\int_X^{L'}\theta\, dL'=\int_{-c}^{l}\theta\frac{dL'}{dt}\,dt \quad ;$$

hence
$$K=\int_0^{\lambda}\theta\frac{dL'}{d\lambda}\,d\lambda \quad . \quad \ldots\ldots(9)$$

Now,
$$\begin{aligned} \theta\frac{dL'}{d\lambda} &= -\{\lambda+\Sigma\theta_p \sin p\lambda\}\{\Sigma pL_p \sin p\lambda\} \\ &= -\tfrac{1}{2}(L_1\theta_1+2L_2\theta_2+3L_3\theta_3+\ldots\}+P.T.+(t+c)\,[P.T.] \quad . \ldots(10) \end{aligned}$$

From (9) and (10) we obtain by integration

$$K=-\tfrac{1}{2}\theta_0(t+c)\,\Sigma pL_p\theta_p+P.T.+(t+c)[P.T.] \quad , \ldots\ldots(11)$$

the constant of integration being zero since, by (9), $K=0$ when $t=-c$.

From (11), $$\frac{\partial K}{\partial C} = -\tfrac{1}{2}(t+c)\frac{\partial}{\partial C}\{\theta_0 \Sigma p L_p \theta_p\} + Y \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

where Y has the form given by X in (8).

Insert (12) in (7) and equate the non-periodic coefficients of $(t+c)$; then

$$1 - \tfrac{1}{2}\frac{\partial}{\partial C}\{\theta_0 \Sigma p L_p \theta_p\} = \theta_0 \frac{\partial L_0}{\partial C} - \tfrac{1}{2}\frac{\partial \theta_0}{\partial C}\Sigma p L_p \theta_p \quad \ldots\ldots\ldots\ldots (13)$$

from which $$1 = \theta_0 \frac{\partial}{\partial C}\{L_0 + \tfrac{1}{2}\Sigma p L_p \theta_p\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Let Λ be defined by

$$\Lambda = L_0 + \tfrac{1}{2}\Sigma p L_p \theta_p \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Λ is thus a function of C, G', H'. From (14),

$$\frac{\partial \Lambda}{\partial C} = \frac{1}{\theta_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Λ will later be taken to be a new canonic variable replacing C.

12·10. Properties of Λ

Consider now the formula 12·03 (5); it is

$$g' = (g) + \frac{\partial S_1}{\partial G'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, by 12·09 (4), since $L' \equiv L'(C, G', H')$,

$$\frac{\partial K}{\partial G'} = \frac{\partial S_1}{\partial L'} \cdot \frac{\partial L'}{\partial G'} + \frac{\partial S_1}{\partial G'}$$

whence, by (1) and 12·09 (5),

$$g' = (g) + \frac{\partial K}{\partial G'} - \theta \frac{\partial L'}{\partial G'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

This becomes an identity when we replace g' by

$$(g) + g_0(t+c) + \Sigma g_p \sin p\lambda, \quad \text{and} \quad L' \quad \text{by 12·09 (3).}$$

We then have, from (2) ,

$$-g_0(t+c) + \frac{\partial K}{\partial G'} = \theta \frac{\partial L'}{\partial G'} + P.T. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Compare this equation with 12·09 (6) which is

$$(t+c) + \frac{\partial K}{\partial C} = \theta \frac{\partial L'}{\partial C} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The subsequent analysis in transforming (3) will be identical with that in deriving 12·09 (13) if we replace C by G' and the coefficient, unity, of t in (4) by $-g_0$ in (3). We then obtain

$$-g_0 = \theta_0 \frac{\partial \Lambda}{\partial G'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

There is a similar formula for h_0.

Thus Λ, a function of C, G', H', has the following properties—we rewrite 12·09 (16) and also (5) above :

$$\frac{\partial \Lambda}{\partial C} = \frac{1}{\theta_0} \quad , \quad \frac{\partial \Lambda}{\partial G'} = -\frac{g_0}{\theta_0} \quad , \quad \frac{\partial \Lambda}{\partial H'} = -\frac{h_0}{\theta_0} \quad \text{.....(6)}$$

12·11. The new canonic variables

Since $$\Lambda \equiv \Lambda(C, G', H') \quad , \quad \text{.....(1)}$$

then $$d\Lambda = \frac{\partial \Lambda}{\partial C} dC + \frac{\partial \Lambda}{\partial G'} dG' + \frac{\partial \Lambda}{\partial H'} dH',$$

or, by means of 12·10 (6),

$$dC = \theta_0 d\Lambda + g_0 dG' + h_0 dH' \quad . \quad \text{.....(2)}$$

But, from (1), we can express C as $C(\Lambda, G', H')$; hence

$$dC = \frac{\partial C}{\partial \Lambda} d\Lambda + \frac{\partial C}{\partial G'} dG' + \frac{\partial C}{\partial H'} dH' \quad .$$

Comparing this last equation with (2), we have

$$\frac{\partial C}{\partial \Lambda} = \theta_0 \quad , \quad \frac{\partial C}{\partial G'} = g_0 \quad , \quad \frac{\partial C}{\partial H'} = h_0 \quad . \quad \text{.....(3)}$$

We keep G' and H' unchanged as new canonic variables ; the non-angular variables are then Λ, G', H'.

Let λ, χ and η denote * the non-periodic terms in the solutions for l', g' and h' ; these are, by 12·07 (2), (3) and (4) :

$$\lambda = \theta_0(t+c) - qn_1 t - q', \quad \text{.....(4)}$$
$$\chi = (g) + g_0(t+c), \quad \text{.....(5)}$$
$$\eta = (h) + h_0(t+c). \quad \text{.....(6)}$$

We have now to show that Λ, λ ; G', χ ; H', η are conjugate pairs.

12·12. The equations for $\dot{\Lambda}$ $\dot{G}'$, $\dot{H}'$

The equations to be transformed are given by 12·07 (9) ; they are :

$$\dot{C} = -\frac{\partial R_1}{\partial c} \quad , \quad \dot{G}' = \frac{\partial R_1}{\partial (g)} \quad , \quad \dot{H}' = \frac{\partial R_1}{\partial (h)} \quad . \quad \text{.....(1)}$$

We suppose that the Hamiltonian function R_1 is transformed into a function of the new variables which we denote by R'. Then we have, identically,

$$R_1(C, G', H' ; c, (g), (h)) \equiv R'(\Lambda, G', H' ; \lambda, \chi, \eta). \quad \text{.....(2)}$$

Consider first the second equation in (1). Now, from 12·11 (5), (g) only occurs in association with χ ; hence, from (2),

* λ in (4) is not to be confused with the λ used in previous sections merely as a short-hand means of writing $\theta_0(t+c)$.

$$\frac{\partial R_1}{\partial (g)} = \frac{\partial R'}{\partial \chi} \cdot \frac{\partial \chi}{\partial (g)} = \frac{\partial R'}{\partial \chi} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Similarly,
$$\frac{\partial R_1}{\partial (h)} = \frac{\partial R'}{\partial \eta} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Hence, from (1), (3) and (4),

$$\dot{G}' = \frac{\partial R'}{\partial \chi} \quad , \quad \dot{H}' = \frac{\partial R'}{\partial \eta} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Consider now the first equation in (1). From 12·11 (4), (5) and (6), c occurs in association with λ, χ and η; hence

$$\frac{\partial R_1}{\partial c} = \frac{\partial R'}{\partial \lambda} \cdot \frac{\partial \lambda}{\partial c} + \frac{\partial R'}{\partial \chi} \cdot \frac{\partial \chi}{\partial c} + \frac{\partial R'}{\partial \eta} \cdot \frac{\partial \eta}{\partial c} \quad ,$$

that is,
$$\dot{C} = \theta_0 \frac{\partial R'}{\partial \lambda} + g_0 \frac{\partial R'}{\partial \chi} + h_0 \frac{\partial R'}{\partial \eta} \quad .$$

Now, from 12·11 (2), $\dot{C} = \theta_0 \dot{\Lambda} + g_0 \dot{G}' + h_0 \dot{H}'$.
These two expressions for $\dot{C}$ must be identical and, since by (5) the coefficients of g_0 and h_0 are the same, we then obtain $\dot{\Lambda} = \frac{\partial R'}{\partial \lambda}$.

Collecting the principal results, we have

$$\dot{\Lambda} = \frac{\partial R'}{\partial \lambda} \quad , \quad \dot{G}' = \frac{\partial R'}{\partial \chi} \quad , \quad \dot{H}' = \frac{\partial R'}{\partial \eta} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

12·13. The equations for $\dot{\chi}$ and $\dot{\eta}$

The equations to be transformed are the second and third of 12·07 (10); they are:

$$(\dot{g}) = -\frac{\partial R_1}{\partial G'} \quad , \quad (\dot{h}) = -\frac{\partial R_1}{\partial H'} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

From 12·11 (5) we have

$$\dot{\chi} = (\dot{g}) + g_0 + g_0 \dot{c} + (t+c)\dot{g}_0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and 12·07 (10), this equation becomes

$$\dot{\chi} = -\frac{\partial R_1}{\partial G'} + g_0 - g_0 \frac{\partial R_1}{\partial C} + (t+c)\dot{g}_0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Since g_0 is originally a function of C, G', H', it becomes, when C is expressed as $C(\Lambda, G', H')$, a function of Λ, G', H'. Hence

$$\dot{g}_0 = \frac{\partial g_0}{\partial \Lambda} \dot{\Lambda} + \frac{\partial g_0}{\partial G'} \dot{G}' + \frac{\partial g_0}{\partial H'} \dot{H}' \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

But, by 12·11 (3), $\frac{\partial g_0}{\partial \Lambda} = \frac{\partial^2 C}{\partial \Lambda \, \partial G'} = \frac{\partial \theta_0}{\partial G'}$, $\frac{\partial g_0}{\partial H'} = \frac{\partial^2 C}{\partial H' \, \partial G'} = \frac{\partial h_0}{\partial G'}$.

Hence (4) becomes, when we use 12·12 (6),

$$\dot{g}_0 = \frac{\partial \theta_0}{\partial G'} \cdot \frac{\partial R'}{\partial \lambda} + \frac{\partial g_0}{\partial G'} \cdot \frac{\partial R'}{\partial \chi} + \frac{\partial h_0}{\partial G'} \cdot \frac{\partial R'}{\partial \eta} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now, from the identity

$$R_1(C, G', H' : c, (g), (h)) \equiv R'(\Lambda, G', H' ; \lambda, \chi, \eta) \quad \ldots\ldots(6)$$

when $C \equiv C(\Lambda, G', H')$ is substituted in the left-hand side and the expressions for λ, χ and η, given by 12·11 (4), (5) and (6), are substituted in the right-hand side, we obtain, by differentiation with respect to G',

$$\frac{\partial R_1}{\partial C} \cdot \frac{\partial C}{\partial G'} + \frac{\partial R_1}{\partial G'} = \frac{\partial R'}{\partial G'} + (t+c)\left\{\frac{\partial R'}{\partial \lambda} \cdot \frac{\partial \theta_0}{\partial G'} + \frac{\partial R'}{\partial \chi} \cdot \frac{\partial g_0}{\partial G'} + \frac{\partial R'}{\partial \eta} \cdot \frac{\partial h_0}{\partial G'}\right\}$$
$$= \frac{\partial R'}{\partial G'} + (t+c)\dot{g}_0 \quad , \quad \text{by (5).}$$

Subtracting this equation from (3), we obtain

$$\dot{\chi} - \frac{\partial R_1}{\partial C} \cdot \frac{\partial C}{\partial G'} = g_0 - g_0 \frac{\partial R_1}{\partial C} - \frac{\partial R'}{\partial G'} \quad .$$

But, by 12·11, $g_0 = \dfrac{\partial C}{\partial G'}$; hence

$$\dot{\chi} = -\frac{\partial}{\partial G'}(R' - C) \quad . \quad \ldots\ldots(7)$$

Similarly,

$$\dot{\eta} = -\frac{\partial}{\partial H'}(R' - C) \quad . \quad \ldots\ldots(8)$$

12·14. The equation for $\dot{\lambda}$

The equation to be transformed is

$$\dot{c} = -\frac{\partial R_1}{\partial C} \quad . \quad \ldots\ldots(1)$$

Since $\lambda = \theta_0(t+c) - qn_1 t - q'$,

then $$\dot{\lambda} = \theta_0 - qn_1 + \theta_0 \dot{c} + (t+c)\dot{\theta}_0 \quad . \quad \ldots\ldots(2)$$

By the same process followed in establishing 12·13 (5), it is found that

$$\dot{\theta}_0 = \frac{\partial \theta_0}{\partial \Lambda} \cdot \frac{\partial R'}{\partial \lambda} + \frac{\partial g_0}{\partial \Lambda} \cdot \frac{\partial R'}{\partial \chi} + \frac{\partial h_0}{\partial \Lambda} \cdot \frac{\partial R'}{\partial \eta} \quad . \quad \ldots\ldots(3)$$

From the identity 12·13 (6), we obtain, on differentiating with respect to Λ,

$$\frac{\partial R_1}{\partial C} \cdot \frac{\partial C}{\partial \Lambda} = \frac{\partial R'}{\partial \Lambda} + (t+c)\left\{\frac{\partial R'}{\partial \lambda} \cdot \frac{\partial \theta_0}{\partial \Lambda} + \frac{\partial R'}{\partial \chi} \cdot \frac{\partial g_0}{\partial \Lambda} + \frac{\partial R'}{\partial \eta} \cdot \frac{\partial h_0}{\partial \Lambda}\right\}$$
$$= \frac{\partial R'}{\partial \Lambda} + (t+c)\dot{\theta}_0, \quad \text{by (3).}$$

Subtracting this equation from (2), we obtain

$$\dot{\lambda} - \frac{\partial R_1}{\partial C} \cdot \frac{\partial C}{\partial \Lambda} = \theta_0 - qn_1 + \theta_0 \dot{c} - \frac{\partial R'}{\partial \Lambda} \quad .$$

But, from 12·11 (3), $\theta_0 = \dfrac{\partial C}{\partial \Lambda}$; hence, using (1), we obtain

$$\dot{\lambda} = -\frac{\partial R'}{\partial \Lambda} + \frac{\partial C}{\partial \Lambda} - qn_1$$

$$= -\frac{\partial}{\partial \Lambda}(R' - C + qn_1\Lambda) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

12·15. The new canonic equations

In this section we collect the results of the preceding three sections; these results are the equations given by 12·12 (6), 12·13 (7) and (8), and 12·14 (4).

Let R'' be defined by

$$R'' = R' - C + qn_1\Lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

in which C is supposed to be expressed as $C(\Lambda, G', H')$.

Consider the first group in 12·12 (6). Since C and Λ are independent of λ, χ and η, then R' in these equations can be replaced by R''. In the same way we can replace $R' - C$ on the right of 12·13 (7) and (8) by R''. The canonic equations are then:

$$\left.\begin{aligned} \dot{\Lambda} &= \frac{\partial R''}{\partial \lambda} \quad , \quad \dot{\lambda} = -\frac{\partial R''}{\partial \Lambda} \\ \dot{G}' &= \frac{\partial R''}{\partial \chi} \quad , \quad \dot{\chi} = -\frac{\partial R''}{\partial G'} \\ \dot{H}' &= \frac{\partial R''}{\partial \eta} \quad , \quad \dot{\eta} = -\frac{\partial R''}{\partial H'} \end{aligned}\right\} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

These equations form the starting-point for the second Delaunay operation for which the complete Hamiltonian function R'' can be expressed in the form

$$R'' = -B' - \Sigma A' \cos\theta' \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where A' and B' are functions of Λ, G', H' and θ' is of the form given by

$$\theta' = i'\lambda + j'\chi + k'\eta + q'n_1 t + q'' \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

in (4) i', j', k', q' are not necessarily integers.

12·16. Special cases

(i) Suppose that $i = 0$ and $j \neq 0$ in the argument of the term selected in 12·01 (9). Write $g' = jg + kh$; then the argument is $\theta \equiv g' + qn_1 t + q'$. Applying the condition

$$L'l' + G'g' + H'h' = Ll + Gg + Hh$$

for the linear transformation of canonic variables,* we see that the following satisfy the condition:

$$l' = l \quad , \quad g' = jg + kh \quad , \quad h' = h \quad ;$$

* In *any* operation we use the notation L, G, H; l, g, h and L', G', H'; l', g', h' to denote the variables *before* and *after* the transformation and R_0, B, A, q, q' to denote the corresponding functions, etc.

$$L' = L \quad , \quad G' = \frac{1}{j} G \quad , \qquad H' = H - \frac{k}{j} G \quad .$$

The solution of the Hamilton-Jacobi equation is

$$S = Ct - (qn_1 t + q')G' + S_1 + (l)L' + (h)H'$$

where L', H' are constants, and $S_1 = \int_{\gamma}^{G'} \theta \, dG'$, γ being the value of G' when $\theta = 0$.

The subsequent work follows the procedure in the previous sections.

(ii) Suppose $i = j = k = 0$ and $q \neq 0$.

We take all the terms in the disturbing function subject to these conditions and write

$$R_0 = -\Sigma A_r \cos(rn_1 t + q_r) \quad , \quad (r = 1, 2, \ldots) \quad ,$$

in which the A's are functions of L, G, H. Since R_0 is not a function of l, g and h, then L, G and H are constants. The Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} - \Sigma A_r \cos(rn_1 t + q_r) = 0,$$

a solution of which, in terms of three α's denoted by (l), (g) and (h), is

$$S = \Sigma \frac{1}{rn_1} A_r \sin(rn_1 t + q_r) + (l)L + (g)G + (h)H \quad .$$

The solutions of the canonic equations are

$$l \equiv \frac{\partial S}{\partial L} = (l) + \Sigma \frac{1}{rn_1} \frac{\partial A_r}{\partial L} \sin(rn_1 t + q_r) \quad ,$$

with two similar equations in g and h, and—in terms of the β's—$L = \beta_1$, $G = \beta_2$, $H = \beta_3$.

(iii) When all the periodic terms have been eliminated, the final canonic equations are

$$\dot{L} = \frac{\partial R_0}{\partial l} \quad , \quad l = -\frac{\partial R_0}{\partial L} \quad ,$$

with similar equations in G, g ; H, h. Here $R_0 = -B$; hence L, G and H are constants. The Hamilton-Jacobi equation is then $\frac{\partial S}{\partial t} - B(L, G. H) = 0$ of which a solution is

$$S = Bt + (l)L + (g)G + (h)H \quad .$$

The integrals of the canonic equations are then

$$l = (l) + l_0 t \quad , \quad g = (g) + g_0 t \quad , \quad h = (h) + h_0 t$$

where $l_0 = \frac{\partial B}{\partial L}$, $g_0 = \frac{\partial B}{\partial G}$ and $h_0 = \frac{\partial B}{\partial H}$, and

$$L = \text{Constant} = L_0 \quad , \quad G = G_0 \quad , \quad H = H_0 \quad , \quad \text{say.}$$

12·17. The practical method of obtaining the solutions of the first operation

We shall deal here with the first Delaunay operation although the method is applicable to any operation.

Since the disturbing function is developed in terms of e and γ ($\equiv \tan i$), the method of solving the principal canonic equations as described in 12·04 involves the transformations of a, e and γ, in terms of L', G', H', in the functions B and A of the Hamiltonian function R_0 given by

$$R_0 = -B - A\cos\theta \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

In practice, it is advantageous to retain e and γ in R_0 unchanged and to effect the final solution by the method now to be described which consists essentially in replacing L' (or L/i) by either e or γ ; we consider the former.

(i) We rewrite the modified non-angular Delaunay variables given in 12·01 (6) :

$$L = iL' = \sqrt{(\mu a)} \quad ; \quad G = \sqrt{(\mu a)}\{\sqrt{(1-e^2)} - 1\} \quad ; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$H = \sqrt{\{\mu a(1-e^2)\}}(\cos i - 1) = -2\sin^2\frac{i}{2}\sqrt{\{\mu a(1-e^2)\}} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Two of the integrals of the canonic equations are, from section 12·07,

$$G = \frac{j}{i}L + G' = jL' + G' \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$H = \frac{k}{i}L + H' = kL' + H' \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

in these, G' and H' are constants.

The principal equations are, by 12·04 (2) and (3),

$$\dot{L}' = A\sin\theta \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$\dot{\theta} = \frac{\partial B_1}{\partial L'} + \frac{\partial A}{\partial L'}\cos\theta \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

(ii) From (2) and (4), we obtain

$$L' = G'/\{\sqrt{(1-e^2)} - 1 - j/i\} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Thus L' can be expressed as a power series in e and in terms of G'. Now, $L^2 = \mu a = i^2L'^2$; hence a can be expressed as $a(e, G')$, a power series in e.

The form of (3) suggests the convenience of dealing with $\sin\frac{i}{2}$, which we denote by γ_1, instead of $\tan i$ or γ ; accordingly, we shall suppose that the disturbing function is expressed in terms of γ_1. From (3) and (5), we have

$$\gamma_1^2 = -\frac{k + H'/L'}{2i\sqrt{(1-e^2)}} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Insert the expression for L' given by (8) ; then γ_1^2 can be found as a power series in e and in terms of G' and H'.

(iii) By means of (8), we can transform the equation (6); it becomes

$$\frac{d(e^2)}{dt}=2\sqrt{(1-e^2)}\{\sqrt{(1-e^2)}-1-j/i\}^2\frac{A}{G'}\sin\theta \quad \ldots\ldots\ldots\ldots(10)$$

A is in general a function of a, e and γ_1; hence, since $a=a(e, G')$ and $\gamma_1=\gamma_1(e, G', H')$, the right-hand side of the equation (10) can be expressed as $A_1(e, G', H')\sin\theta$. Then

$$\frac{d(e^2)}{dt}=A_1(e, G', H')\sin\theta \quad \ldots\ldots\ldots\ldots(11)$$

Consider now the equation (7). Since B_1 and A are functions of a, e and γ_1 we have, for example,

$$\frac{\partial A}{\partial L'}=\frac{\partial A}{\partial a}\cdot\frac{\partial a}{\partial L'}+\frac{\partial A}{\partial e}\cdot\frac{\partial e}{\partial L'}+\frac{\partial A}{\partial \gamma_1}\cdot\frac{\partial \gamma_1}{\partial L'} \quad \ldots\ldots\ldots\ldots(12)$$

Now, $\mu a=L^2=i^2L'^2$; hence, $\frac{\partial a}{\partial L'}=\frac{2i^2L'}{\mu}$. Accordingly, $\frac{\partial a}{\partial L'}$ can be expressed as a function of e and G'. Again, from (8), $\frac{e}{\sqrt{(1-e^2)}}\frac{\partial e}{\partial L'}=\frac{G'}{L'^2}$; thus $\frac{\partial e}{\partial L'}$ can be expressed in terms of e and G'. Further, from (9) we can obtain $\frac{\partial \gamma_1}{\partial L'}$ in a rather more complicated form which again can be expressed in terms of e, G' and H'. In the same way, $\frac{\partial A}{\partial a}$, $\frac{\partial A}{\partial e}$ and $\frac{\partial A}{\partial \gamma_1}$ can be expressed in terms of e, G' and H'. Evidently, we can write (7) as

$$\dot\theta=f(e, G', H')+F(e, G', H')\cos\theta, \quad \ldots\ldots\ldots\ldots(13)$$

in which f and F are power series in e.

By a process of successive approximations the solutions of (11) and (13) are found to be of the forms

$$e^2=e_0^2+\Sigma e_p\cos p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots(14)$$
$$\theta=\theta_0(t+c)+\Sigma A_p\sin p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots(15)$$

where e_0 and c are constants of integration and e_p, θ_0 and θ_p are functions of e_0, G' and H'.

It is evident that, by (8) and (14), we can express a in the form of a series as

$$a=a_0+\Sigma a_p\cos p\theta_0(t+c). \quad \ldots\ldots\ldots\ldots(16)$$

Similarly, by (9), we can write

$$\gamma_1^2=\gamma_0^2+\Sigma\gamma_p'\cos p\theta_0(t+c). \quad \ldots\ldots\ldots\ldots(17)$$

In (16) and (17) the a's and γ''s are functions of e_0, G' and H'; accordingly, we write

$$a_0=a_0(e_0, G', H') \quad , \quad \gamma_0=\gamma_0(e_0, G', H') \quad ,$$

and, solving these functional equations, we derive

$$G' = G'(a_0, e_0, \gamma_0) \quad , \quad H' = H'(a_0, e_0, \gamma_0) \quad . \quad \ldots\ldots\ldots\ldots\ldots(18)$$

Replacing G' and H' by means of (18) in the coefficients of (16), (14) and (17) and in θ_0, we have the group of solutions

$$a = a_0 + \Sigma A_p \cos p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots\ldots(19)$$

$$e^2 = e_0^2 + \Sigma E_p \cos p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots\ldots(20)$$

$$\gamma_1^2 = \gamma_0^2 + \Sigma \Gamma_p \cos p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots\ldots(21)$$

where A, E, Γ and θ_0 are functions of a_0, e_0 and γ_0.

It is evident that we can obtain series for e and γ_1 from (20) and (21); these we write as

$$e = e_0 + \Sigma E_p' \cos p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots\ldots(22)$$

$$\gamma_1 = \gamma_0 + \Sigma \Gamma_p' \cos p\theta_0(t+c) \quad . \quad \ldots\ldots\ldots\ldots\ldots(23)$$

(iv) We now consider the angular variables l', g', h'.

Since $\theta = l' + qn_1 t + q'$, we have at once from (15)

$$l' = \theta_0(t+c) - qn_1 t - q' + \Sigma \theta_p \sin p\theta_0(t+c) \quad . \quad \ldots\ldots\ldots\ldots\ldots(24)$$

The equation to determine g' is

$$\dot{g}' \equiv \dot{g} = -\frac{\partial R_0}{\partial G'} = \frac{\partial B_1}{\partial G'} + \frac{\partial A}{\partial G'} \cos\theta.$$

By the procedure adopted in (12) we can write this equation as

$$\dot{g} = f(e, G', H') + F(e, G', H') \cos\theta \quad . \quad \ldots\ldots\ldots\ldots\ldots(25)$$

By means of (15), $\cos\theta$ can be developed as a cosine series in $\theta_0(t+c)$ and, by (18) and (20), f and F can be developed in the same way. Equation (25) takes the form

$$\begin{aligned} \dot{g} &= f_1(a_0, e_0, \gamma_0) + \Sigma F_p(a_0, e_0, \gamma_0) \cos p\theta_0(t+c) \\ &= \quad g_0 \quad\quad + \Sigma p g_p \theta_0 \cos p\theta_0(t+c) \end{aligned}$$

where $pg_p\theta_0 \equiv F_p$. On integration we obtain

$$g = (g) + g_0(t+c) + \Sigma g_p \sin p\theta_0(t+c) \quad , \quad \ldots\ldots\ldots\ldots\ldots(26)$$

in which the constant of integration is $(g) + g_0 c$.

Similarly,

$$h = (h) + h_0(t+c) + \Sigma h_p \sin p\theta_0(t+c) \quad \ldots\ldots\ldots\ldots\ldots(27)$$

In (24), (26) and (27), θ_r and g_r and h_r ($r = 0, 1, 2, \ldots$) are all functions of a_0, e_0, γ_0.

As in 12·07, we obtain

$$l = (l) + l_0(t+c) - \frac{1}{i}(qn_1 t + q') + \Sigma l_p \sin p\theta_0(t+c) \quad . \quad \ldots\ldots\ldots\ldots\ldots(28)$$

In (28), (26) and (27), l is the mean longitude, $g = \varpi$ and $h = \Omega$.

To complete the list of principal formulae, we rewrite (15):

$$\theta = \theta_0(t+c) + \Sigma\, \theta_p \sin p\theta_0(t+c) \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

in which θ_0, θ_p are functions of a_0, e_0, γ_0.

(v) The new variables for the next operation are

$$\Lambda, G', H' \; ; \quad \lambda, \chi, \eta \quad .$$

The Hamiltonian function is R'', given by

$$R'' = R' - C + qn_1\Lambda \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

We shall first show how Λ and C are transformed into functions of a_0, e_0, γ_0.

Referring to 12·07 (1), we have

$$L' = L_0 + \Sigma\, L_p \cos p\theta_0(t+c) \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

and, by 12·09 (15), Λ is given in terms of L_0, L_p, θ_p by

$$\Lambda = L_0 + \tfrac{1}{2}\Sigma\, pL_p\theta_p \quad . \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

Now, $L = \sqrt{(\mu a)} = iL'$ and, by (19), $a = a_0 + \Sigma\, A_p \cos p\theta_0(t+c)$; hence we can express L' as

$$L' = L_0' + \Sigma\, L_p' \cos p\theta_0(t+c) \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

where

$$i^2[L_0' + \Sigma\, L_p' \cos p\theta_0(t+c)]^2 = \mu[a_0 + \Sigma\, A_p \cos p\theta_0(t+c)].$$

The expressions for L_0' and L_p' in terms of a_0 and A_p are readily found by elementary methods; accordingly, L_0' and L_p' are now known functions of a_0, e_0, γ_0.

Now (31) and (33) are identical if we suppose L_0, L_p to be expressed as functions of a_0, e_0, γ_0; hence, (31) becomes

$$\Lambda = L_0' + \tfrac{1}{2}\Sigma\, pL'_p\theta_p \quad ,$$

a function of a_0, e_0, γ_0.

Again, C occurs in the equation, 12·03 (10),

$$C = B_1 + A\cos\theta \quad ;$$

originally, A and B_1 are functions of a, e and γ (or γ_1) and, consequently, $C = B_1 + A$ when, for $t+c=0$, a, e and γ_1 are replaced by

$$a_0 + \Sigma\, A_p \quad , \quad e_0 + \Sigma\, E'_p \quad \text{and} \quad \gamma_0 + \Sigma\, \Gamma'_p$$

respectively, the last two expressions coming from (22) and (23). Thus C is expressed in terms of a_0, e_0, γ_0.

Consider now R' in (30); it is the complete summation of trigonometrical terms in the original disturbing function *less* the particular cosine term selected for the first operation. R' is of the form

$$\Sigma\, A \cos(i_1 l + j_1 g + k_1 h + q_1 n_1 t + q'_1) \quad ,$$

where A is, in the first instance, a function of a, e and γ_1 and now, by (19),

(22) and (23), expressible in the form $D_0 + \Sigma D_p \cos p\theta_0(t+c)$, where θ_0, D_0 and D_p are functions of a_0, e_0, γ_0.

In terms of the new angular variables R' is then developed as

$$R' = -\Sigma A_1 \cos (i'\lambda + j'\chi + k'\eta + q'_1 n_1 t + q'') \quad \ldots\ldots\ldots\ldots (34)$$

where i', ... , q_1' are not necessarily integers, and A_1 is a function of a_0, e_0, γ_0.

The second operation consists of solving the canonic equations in Λ, G', H' ; λ, χ, η with the Hamiltonian function R_0' consisting of one term from (34) together with $(-C + qn_1\Lambda)$ from (30) ; thus

$$R_0' = -B_1' - A' \cos \theta'$$

where $B_1' = C - qn_1\Lambda$ and θ' is the argument in (34).

The solutions are then obtained by the procedures previously outlined.

Chapter 13

SECULAR INEQUALITIES

13·01. Introduction

In the earlier sections of Chapter 6 we derived the solutions of the planetary equations to a first approximation, regarding the elements $a, e, \ldots$ as constants in the disturbing function. For example, the solution for the eccentricity is of the form

$$e = e_0 + \alpha t + P.T. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

When we proceed to the second approximation which involves the Taylorian expansion of the disturbing function by means of (1)—and similar equations for the other elements (except that $\alpha = 0$ in the equation for the semi-major axis)—we have to stipulate that (αt) is so small that $(\alpha t)^2$ can be neglected. In absolute measure α is a small quantity of, perhaps, order 10^{-5} and it therefore follows that the validity of the procedure is jeopardised if t is taken so large that $(\alpha t)^2$ can no longer be neglected in comparison with (αt); in other words, the solution obtained is applicable for what is in fact a severely limited interval before and after a selected epoch.

If we assume that (αt) is small then, analytically, the secular term could be replaced, without prejudice to the accuracy of the solution, by $\sin \alpha t$, the period of which is $2\pi/\alpha$. It may then be conjectured that the appearance of the secular term in (1) is possibly the result of the particular way in which the solutions have been developed and that the perturbations in e—and in a and γ—are purely periodic in character. The conjecture is strengthened by the results of the previous chapter where, in the three-body problem involving the Sun, Earth and Moon, the complete expressions for a, e and γ (or γ_1) are entirely periodic.

The term (αt) in (1) arises from the non-periodic part of the disturbing function. Accordingly, we investigate rigorously the solutions of the equations when the non-periodic part of the disturbing function is alone considered, with the reservation that we exclude terms of order higher than the third in the eccentricities and inclinations.

We first consider the solutions in the case of two planets and later the solutions for any number of planets.

13·02. The equations for two planets

For the planet P (mass, m) we write in the notation of 5·10 (ii)

$$h = e \sin \varpi \ , \qquad k = e \cos \varpi \ ; \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$p = \sin i \sin \Omega \ , \qquad q = \sin i \cos \Omega \ ; \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and for the planet P_1 (mass, m_1),

$$h_1 = e_1 \sin \varpi_1 \quad , \qquad k_1 = e_1 \cos \varpi_1 \quad ; \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$p_1 = \sin i_1 \sin \Omega_1 \quad , \quad q_1 = \sin i_1 \cos \Omega_1 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

We denote by N the non-periodic part of the disturbing function; then, by 7·11 (11) we have

$$N = Gm_1[4C + (e^2 + e_1^2)D - 2ee_1 E \cos(\varpi - \varpi_1) - (\gamma^2 + \gamma_1^2)D + 2\gamma\gamma_1 D \cos(\Omega - \Omega_1)] \qquad \ldots\ldots\ldots\ldots (5)$$

in which C, D and E are symmetrical functions of a and a_1, and $\gamma = \tan i$, $\gamma_1 = \tan i_1$.

Now, by 5·10 (7), $\dot{a} = \dfrac{2}{na}\dfrac{\partial R}{\partial \epsilon} = 0$; hence a and n are constants so far as the present investigation is concerned; similarly, a_1 and n_1 are constants. Further, C, D and E are constants.

Replacing R_1 by N in 5·10 (13) and (14), we have

$$\dot{h} = \frac{\cos\phi}{na^2}\frac{\partial N}{\partial k} + \frac{k \tan \frac{1}{2} i}{\gamma na^2 \cos\phi}\left(p\frac{\partial N}{\partial p} + q\frac{\partial N}{\partial q}\right) \quad ,$$

$$\dot{k} = -\frac{\cos\phi}{na^2}\frac{\partial N}{\partial h} - \frac{h \tan \frac{1}{2} i}{\gamma na^2 \cos\phi}\left(p\frac{\partial N}{\partial p} + q\frac{\partial N}{\partial q}\right) \quad ,$$

in which $\cos\phi = \sqrt{(1 - e^2)}$.

In these equations the terms involving $\dfrac{\partial N}{\partial p}$ and $\dfrac{\partial N}{\partial q}$ are—with reference to e and γ and, consequently, to h, k, p and q—of order two higher than the terms involving $\dfrac{\partial N}{\partial h}$ and $\dfrac{\partial N}{\partial k}$; since we suppose that e and γ are small and of the same order of magnitude, the terms referred to can be neglected. Further, to the order aimed at, we can write: $\cos\phi = 1$. The equations are then

$$\dot{h} = \frac{1}{na^2}\frac{\partial N}{\partial k} \quad , \quad \dot{k} = -\frac{1}{na^2}\frac{\partial N}{\partial h} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

In the same way, we derive from 5·10 (15) and (16)

$$\dot{p} = \frac{1}{na^2}\frac{\partial N}{\partial q} \quad , \quad \dot{q} = -\frac{1}{na^2}\frac{\partial N}{\partial p} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Also, to the order concerned, it is sufficient to write

$$p = \gamma \sin \Omega \quad , \qquad q = \gamma \cos \Omega \quad ; \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$p_1 = \gamma_1 \sin \Omega_1 \quad , \quad q_1 = \gamma_1 \cos \Omega_1 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

By means of (1), (3), (8) and (9), the function N, given by (5), becomes—on omitting C, which is a constant—

$$N = Gm_1 D[h^2 + k^2 + h_1^2 + k_1^2 - p^2 - q^2 - p_1^2 - q_1^2 + 2pp_1 + 2qq_1] - 2Gm_1 E(hh_1 + kk_1) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Since D and E are symmetrical functions of a and a_1, N/m_1 is a symmetrical function of a, a_1 and h, h_1 ; k, k_1 ; p, p_1 ; q, q_1.

If N_1 denotes the non-periodic part of the disturbing function for the planet P_1, as affected by the planet P, then

$$\frac{N_1}{m} = \frac{N}{m_1} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

From (6) and (10) we obtain

$$\dot{h} = \frac{2Gm_1}{na^2}(kD - k_1E) \quad , \quad \dot{k} = -\frac{2Gm_1}{na^2}(hD - h_1E) \quad .$$

Similarly, with the help of (11), we obtain

$$\dot{h}_1 = \frac{2Gm}{n_1 a_1^2}(k_1D - kE) \quad , \quad \dot{k}_1 = -\frac{2Gm}{n_1 a_1^2}(h_1D - hE) \quad .$$

Write

$$\alpha = \frac{2Gm_1D}{na^2} \quad , \quad \beta = \frac{2Gm_1E}{na^2} \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

$$\alpha_1 = \frac{2GmD}{n_1 a_1^2} \quad , \quad \beta_1 = \frac{2GmE}{n_1 a_1^2} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Then the equations become

$$\dot{h} = \alpha k - \beta k_1 \quad ; \quad \dot{k} = \beta h_1 - \alpha h \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

$$\dot{h}_1 = \alpha_1 k_1 - \beta_1 k \quad ; \quad \dot{k}_1 = \beta_1 h - \alpha_1 h_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

In the same way, the equations involving the inclinations and the longitudes of the nodes become

$$\dot{p} = \alpha(q_1 - q) \quad ; \quad \dot{q} = \alpha(p - p_1) \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

$$\dot{p}_1 = \alpha_1(q - q_1) \quad ; \quad \dot{q}_1 = \alpha_1(p_1 - p) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

It is to be remarked that, in (12) and (13), D, E and n, n_1 are positive ; hence α, α_1 and β, β_1 are all positive.

13·03. Solutions of the equations in h and k

The equations to be solved are (14) and (15) of the previous section.

Write $$X = k + ih \quad , \quad X_1 = k_1 + ih_1 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $i^2 = -1$. Then we obtain

$$\dot{X} - i\alpha X = -i\beta X_1 \quad , \quad \dot{X}_1 - i\alpha_1 X_1 = -i\beta_1 X \quad ,$$

or, in terms of the operator $D \left(\equiv \frac{d}{dt} \right)$,

$$(D - i\alpha)X = -i\beta X_1 \quad , \quad (D - i\alpha_1)X_1 = -i\beta_1 X \quad .$$

We operate on the first of these equations with $(D - i\alpha_1)$ and use the second to eliminate X_1 ; the result is

$$\ddot{X} - i(\alpha + \alpha_1)\dot{X} - (\alpha\alpha_1 - \beta\beta_1)X = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Put $X = Me^{i(gt+c)}$ in (2); then

$$g^2 - (\alpha + \alpha_1)g + \alpha\alpha_1 - \beta\beta_1 = 0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The roots of this equation are real since

$$(\alpha + \alpha_1)^2 - 4(\alpha\alpha_1 - \beta\beta_1) \equiv (\alpha - \alpha_1)^2 + 4\beta\beta_1 \quad ,$$

a positive quantity. It is evident also that the roots are positive for, by 7·15 (10), $D^2 > E^2$ or, by 13·02 (12) and (13), $\alpha\alpha_1 - \beta\beta_1 > 0$.

If g_1 and g_2 are the roots of (3), the solution of (2) is

$$X = M_1 e^{i(g_1 t + c_1)} + M_2 e^{i(g_2 t + c_2)} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where M_1, M_2 and c_1, c_2 are constants of integration. That there are four constants of integration—and not two, as may appear at first sight—is due to the fact that (2) is equivalent to two independent equations, each of the second order, in real variables h and k.

From (1) and (4) we obtain, by equating imaginary and real parts, the pair of solutions

$$h = M_1 \sin(g_1 t + c_1) + M_2 \sin(g_2 t + c_2) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$
$$k = M_1 \cos(g_1 t + c_1) + M_2 \cos(g_2 t + c_2) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

In the same way we derive the solutions for h_1 and k_1; they are

$$h_1 = M_1' \sin(g_1 t + c_1) + M_2' \sin(g_2 t + c_2) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$
$$k_1 = M_1' \cos(g_1 t + c_1) + M_2' \cos(g_2 t + c_2) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

in which M_1' and M_2' are not independent of M_1 and M_2 for, on substituting (5), (6) and (8) in the first equation of 13·02 (14), we deduce at once that

$$\beta M_1' = (\alpha - g_1)M_1 \quad , \quad \beta M_2' = (\alpha - g_2)M_2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

13·04. Evaluation of the constants of integration

The numerical values of M_1, M_2 and c_1, c_2 can be readily found when the masses and the orbital elements of the two planets are known at a given epoch. The formulae of section 7·15 enable us to calculate D and E; with these the values of α, β and α_1, β_1 in 13·02 (12) and (13) are derived; the values of g_1 and g_2 are then found from the quadratic equation 13·03 (3). From the known values of e and ϖ at the selected epoch, which we take to be given by $t = 0$, the values of $h\,(\equiv e \sin \varpi)$ and k $(\equiv e \cos \varpi)$ at the epoch are derived; denote these, and the corresponding values of h_1 and k_1, by h_0, k_0 and $(h_1)_0$, $(k_1)_0$. Further, write

$$\left.\begin{aligned} P_1 &= M_1 \sin c_1 \quad , \quad P_2 = M_2 \sin c_2 \\ Q_1 &= M_1 \cos c_1 \quad , \quad Q_2 = M_2 \cos c_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Then, 13·03 (5) and (7)—the latter with the help of 13·03 (9)—give the following:

$$h_0 = P_1 + P_2 \quad , \quad \beta(h_1)_0 = (\alpha - g_1)P_1 + (\alpha - g_2)P_2 \quad ,$$

from which P_1 and P_2 are easily calculated.

In the same way, we form two equations in k and Q from which Q_1 and Q_2 can be found.

The group of equations (1) then furnishes the means to derive the values of M_1, M_2 and c_1, c_2.

13·05. The eccentricities

From 13·03 (5) and (6) we have at once

$$e^2 \equiv h^2 + k^2 = M_1^2 + M_2^2 + 2M_1M_2 \cos\{(g_1 - g_2)t + c_1 - c_2\} \quad \ldots\ldots(1)$$

Also,

$$\tan \varpi \equiv \frac{h}{k} = \frac{M_1 \sin(g_1t + c_1) + M_2 \sin(g_2t + c_2)}{M_1 \cos(g_1t + c_1) + M_2 \cos(g_2t + c_2)} \quad \ldots\ldots\ldots(2)$$

There are two similar equations for e_1^2 and ϖ_1.

The equation (1) shows that e is periodic ; the maximum value of e is $|M_1| + |M_2|$, and the minimum value is $|M_1| \sim |M_2|$. The period of oscillation is $2\pi/(g_1 \sim g_2)$.

It has been found for Jupiter and Saturn that the period of oscillation is about 70,400 years and that the maximum and minimum values* of e for Jupiter are 0·060 and 0·026 ; the corresponding values for Saturn are 0·084 and 0·013.

The fact that the values of e and e_1 lie within restricted limits can be demonstrated directly from the equations 13·02 (14) and (15). It is easily seen that

$$\beta_1(h\dot{h} + k\dot{k}) + \beta(h_1\dot{h}_1 + k_1\dot{k}_1) = 0$$

which becomes—on inserting the expressions for β and β_1 given in 13·02 (12) and (13), and integrating—

$$mna^2e^2 + m_1n_1a_1^2e_1^2 = C \quad , \quad \ldots\ldots\ldots(1)$$

where C is a constant.

This equation shows (i) that, since n and n_1 have the same sign, the values of e and e_1 are restricted in magnitude, and (ii) that e is a maximum when e_1 is a minimum, and vice versa.

13·06. Perihelion longitudes

From 13·05 (2) it is easily found that

$$e^2\dot{\varpi} = g_1M_1^2 + g_2M_2^2 + (g_1 + g_2)M_1M_2 \cos(\theta_1 - \theta_2) \quad \ldots\ldots\ldots(1)$$

in which $\theta_1 = g_1t + c_1 \quad , \quad \theta_2 = g_2t + c_2 \quad .$

The equation (1) can be written in the alternative form

$$\dot{\varpi} = \frac{(g_1 - g_2)(M_1^2 - M_2^2)}{2e^2} + \tfrac{1}{2}(g_1 + g_2) \quad \ldots\ldots\ldots(2)$$

Without loss of generality we can suppose that $g_1 > g_2$. Also, it is to be

* The values obtained from the analysis embracing all the planets are to be found on p. 215.

recognised that M_1 and M_2 may be positive or negative. There are two cases to be considered.

Case (i). $(g_1+g_2) \mid M_1M_2 \mid \geqslant (g_1M_1^2+g_2M_2^2)$.

Then (1) can be expressed as

$$e^2\dot{\varpi} = (g_1+g_2)\dot{M}_1M_2[\cos(\theta_1-\theta_2)-\cos\alpha] \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where α is the least positive angle $(0<\alpha<\pi)$ such that

$$\cos\alpha = -\frac{g_1M_1^2+g_2M_2^2}{(g_1+g_2)M_1M_2} \quad .$$

The equation (3) shows that ϖ has real turning-values, corresponding to

$$\theta_1-\theta_2 = \alpha+2r\pi \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

or to

$$\theta_1-\theta_2 = 2r\pi-\alpha \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where r takes integral values.

From (3),

$$e^2\ddot{\varpi}+\dot{\varpi}\frac{d}{dt}(e^2) = -(g_1^2-g_2^2)M_1M_2\sin(\theta_1-\theta_2) \quad ;$$

hence, if M_1M_2 is positive, maximum turning-values correspond to (4) and minimum turning-values correspond to (5); if M_1M_2 is negative the turning-values are interchanged for the values of $(\theta_1-\theta_2)$ given above.

The extreme values of ϖ can then be calculated by means of the formulae

$$h = e\sin\varpi \quad , \quad k = e\cos\varpi$$

when h and e have been evaluated for the values of t given by (4) and (5).

Case (ii). $(g_1+g_2) \mid M_1M_2 \mid < (g_1M_1^2+g_2M_2^2)$.

The right-hand side of (1) is positive for all values of t. It follows that $\dot{\varpi}$ is positive and that ϖ increases with the time.

It is found that case (ii) applies to Jupiter and Saturn.

13·07. Solutions for the inclinations

From 13·02 (16) and (17), the equations are

$$\dot{p} = \alpha(q_1-q) \quad ; \quad \dot{q} = \alpha(p-p_1) \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\dot{p}_1 = \alpha_1(q-q_1) \quad ; \quad \dot{q}_1 = \alpha_1(p_1-p) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The solution of these equations can be found from the results of section 13·03 by writing α for β and α_1 for β_1, and then applying a negative sign to α and α_1. If $X = q+ip$, then by 13·03 (2),

$$\ddot{X}+i(\alpha+\alpha_1)\dot{X} = 0.$$

Put $X = Ne^{i(ft+c)}$; then, $f^2+f(\alpha+\alpha_1) = 0$ of which the roots are $-(\alpha+\alpha_1)$ $(\equiv f)$ and 0; thus f is negative. We then have

$$p = N_1\sin(ft+c_1)+N_2\sin c_2 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$q = N_1\cos(ft+c_1)+N_2\cos c_2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Similarly, $$p_1 = N_1' \sin(ft + c_1) + N_2' \sin c_2 \quad ,$$
$$q_1 = N_1' \cos(ft + c_1) + N_2' \cos c_2 \quad .$$

Since, by (1) and (2), $\alpha_1 \dot{p} + \alpha \dot{p}_1 = 0$, then $N_1' = -\alpha_1 N_1/\alpha$. Also, from the first equation of (1) $N_2' = N_2$. The solutions for p_1 and q_1 can then be written as

$$p_1 = -\frac{\alpha_1}{\alpha} N_1 \sin(ft + c_1) + N_2 \sin c_2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$q_1 = -\frac{\alpha_1}{\alpha} N_1 \cos(ft + c_1) + N_2 \cos c_2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From (3) and (4) we derive

$$\gamma^2 = N_1^2 + N_2^2 + 2N_1 N_2 \cos(ft + c_1 - c_2) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

There is a similar equation for γ_1.

From (7) it is seen that γ oscillates between

$$|N_1| + |N_2| \quad \text{and} \quad |N_1| \sim |N_2| \quad .$$

Similarly, γ_1^2 oscillates between

$$\frac{\alpha_1}{\alpha}|N_1| + |N_2| \quad \text{and} \quad \frac{\alpha_1}{\alpha}|N_1| \sim |N_2| \quad .$$

The period of oscillation is $2\pi/(\alpha + \alpha_1)$.

From (1) and (2), it is easily seen that

$$\alpha_1(p\dot{p} + q\dot{q}) + \alpha(p_1\dot{p}_1 + q_1\dot{q}_1) = 0 \quad ;$$

hence $$mna^2\gamma^2 + m_1 n_1 a_1^2 \gamma_1^2 = C' \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where C' is a constant. This equation is analogous to 13·05 (1).

The numerical values of N_1, N_2 and c_1, c_2 can be derived by a procedure similar to that described in 13·04.

For Jupiter and Saturn the period of oscillation is about 50,700 years ; for Jupiter, the maximum and minimum values of the inclination i ($\equiv \tan^{-1}\gamma$) are about 2° 2′ and 1° 17′ ; the corresponding values for Saturn are about 2° 33′ and 0° 47′.

13·08. The longitudes of the nodes

Since $\tan \Omega = p/q$, we obtain from the solutions of the previous section

$$\tan \Omega = \frac{N_1 \sin(ft + c_1) + N_2 \sin c_2}{N_1 \cos(ft + c_1) + N_2 \cos c_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

from which, θ being written for $ft + c_1 - c_2$,

$$\gamma^2 \dot{\Omega} = fN_1^2 + fN_1 N_2 \cos\theta \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Case (i). $|N_2| > |N_1|$.

Let β denote the least positive angle ($0 < \beta < \pi$) such that

$$\cos\beta = -N_1/N_2 \quad .$$

Then, from (2), $\gamma^2\dot{\Omega} = fN_1N_2(\cos\theta - \cos\beta)$. If N_1 and N_2 have the same sign, Ω has a maximum value when $\theta = \beta + 2r\pi$, and a minimum value when $\theta = 2r\pi - \beta$, r being an integer.

If N_1 and N_2 are of opposite signs the turning-values are interchanged for the values of θ stated.

The extreme values of Ω can be calculated by the method described in the case of ϖ.

It is found that for Jupiter and Saturn Ω oscillates, the range for Jupiter being about 13°·2 and about 31°·9 for Saturn.

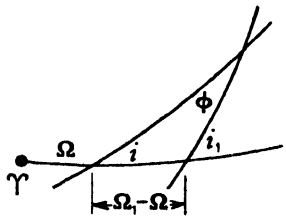


FIG. 22

Case (ii). $|N_1| > |N_2|$.

In this case, since f is negative being equal to $-(\alpha + \alpha_1)$, then Ω is negative for all values of t; accordingly, Ω decreases with the time.

13·09. The mutual inclination of two orbits

In Fig. 22 let ϕ be the inclination of one orbit on the other. Then, by the polar analogue of the cosine-formula,

$$\cos\phi = \cos i \cos i_1 + \sin i \sin i_1 \cos(\Omega_1 - \Omega) \quad ,$$

from which

$$1 - 2\sin^2\frac{\phi}{2} = \left(1 - 2\sin^2\frac{i}{2}\right)\left(1 - 2\sin^2\frac{i_1}{2}\right) + pp_1 + qq_1 \quad .$$

Neglect terms of order γ^4. Then

$$4\sin^2\frac{\phi}{2} = p^2 + q^2 + p_1^2 + q_1^2 - 2pp_1 - 2qq_1$$
$$= (p - p_1)^2 + (q - q_1)^2 \quad ,$$

which becomes, by formulae (4)–(7) of 13·07,

$$4\sin^2\frac{\phi}{2} = \left(1 + \frac{\alpha_1}{\alpha}\right)^2 N_1^2.$$

Thus, to the order of approximation indicated, the mutual inclination of the two orbits is constant.

13·10. The equations for n planets

We consider planets $P_1, P_2, \ldots, P_n$ of masses $m_1, m_2, \ldots, m_n$. Let N_1 denote the non-periodic part of the disturbing function associated with P_1. Then, by 13·02 (10), N_1 is given by

$$N_1 = G\sum_{j=2}^{n} m_j D_{1j}(h_1^2 + k_1^2 - p_1^2 - q_1^2 + h_j^2 + k_j^2 - p_j^2 - q_j^2 + 2p_1p_j + 2q_1q_j)$$
$$- 2G\sum_{j=2}^{n} m_j E_{1j}(h_1h_j + k_1k_j) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which D_{1j} and E_{1j} are symmetrical functions of a_1 and a_j, the latter

being regarded—as in 13·02—as constants so far as the present investigation is concerned.

The equations for h_1 and k_1 are

$$\dot{h}_1 = \frac{1}{n_1 a_1^2}\frac{\partial N_1}{\partial k_1} \quad , \quad \dot{k}_1 = -\frac{1}{n_1 a_1^2}\frac{\partial N_1}{\partial h_1} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Write

$$\frac{2Gm_j D_{ij}}{n_i a_i^2} = (i,j) \quad , \quad \frac{2Gm_j E_{ij}}{n_i a_i^2} = [i,j] \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

We shall suppose that m, n and a are all known for the individual planets; accordingly, (i,j) and $[i,j]$ are all known.

The first of the equations (2) becomes

$$\dot{h}_1 - k_1 \sum_{j=2}^{n} (1,j) + \sum_{j=2}^{n} [1,j] k_j = 0$$

or

$$\dot{h}_1 - C_1 k_1 + \sum_{2}^{n} [1,j] k_j = 0$$

where, in general,

$$C_i = \sum_{j=1}^{n} (i,j) \quad , \quad j \neq i \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Similarly, the second of the equations (2) becomes

$$\dot{k}_1 + C_1 h_1 - \sum_{j=2}^{n} [1,j] h_j = 0 \quad .$$

In general, we have the equations, $(j \neq i)$,

$$\dot{h}_i - C_i k_i + \sum_{j} [i,j] k_j = 0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$\dot{k}_i + C_i h_i - \sum_{j} [i,j] h_j = 0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

From (3),

$$\sqrt{(m_i n_i)} a_i [i,j] \equiv \frac{2G\sqrt{(m_i m_j)}\, E_{ij}}{\sqrt{(n_i n_j)}\, a_i a_j} \cdot \sqrt{(m_j n_j)}\, a_j = B_{ij}\sqrt{(m_j n_j)}\, a_j \quad \ldots\ldots (7)$$

where B_{ij} is a symmetrical function of m_i and m_j, n_i and n_j, a_i and a_j; accordingly, we have the important property:

$$B_{ij} = B_{ji} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Also, since m, n and a are supposed known, the value of any B can be found by means of (7).

Write

$$H_i = \sqrt{(m_i n_i)}\, a_i h_i \quad , \quad K_i = \sqrt{(m_i n_i)}\, a_i k_i \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Multiply (5) and (6) by $\sqrt{(m_i n_i)}\, a_i$; then we obtain, using (7)

$$\dot{H}_i - C_i K_i + \sum_{j} B_{ij} K_j = 0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$\dot{K}_i + C_i H_i - \sum_{j} B_{ij} H_j = 0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

in which $j \neq i$ and the values of the B's and C's are supposed known.

Let

$$2F=\sum_{i=1}^{n} C_i(H_i^2+K_i^2)-\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}(H_iH_j+K_iK_j) \quad , \quad \ldots\ldots\ldots\ldots(12)$$

where $j\neq i$. Then, utilising the property in (8), we find that for all values of i the equations (10) and (11) become

$$\dot{H}_i=\frac{\partial F}{\partial K_i} \quad , \quad \dot{K}_i=-\frac{\partial F}{\partial H_i} \quad . \quad \ldots\ldots\ldots\ldots(13)$$

These equations are canonic in form, F being the Hamiltonian function. From (10) and (11),

$$\sum_{i=1}^{n}(H_i\dot{H}_i+K_i\dot{K}_i)+\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}(H_iK_j-K_iH_j)=0 \quad .$$

The double summation in which $j\neq i$, vanishes as a result of the property (8); hence

$$\sum_i (H_i^2+K_i^2)=\kappa^2 \quad , \quad \ldots\ldots\ldots\ldots(14)$$

where κ is a constant, or

$$\sum_i m_in_ia_i^2e_i^2=\kappa^2 \quad . \quad \ldots\ldots\ldots\ldots(15)$$

This last equation is the generalised form of 13·05 (1). Since the values of n_i have the same sign in the planetary system, (15) shows that the values of the eccentricities are restricted or, in other words, that any e has no associated secular term.

It is to be remarked that, since F does not involve the time explicitly, an integral of the equations (13) is

$$F=\text{constant} \quad . \quad \ldots\ldots\ldots\ldots(16)$$

13·11. Solutions of the equations in H and K

Let $$U_s=K_s+iH_s \quad \ldots\ldots\ldots\ldots(1)$$

where now i denotes $\sqrt{(-1)}$. Then, from (10) and (11) of the previous section we have

$$\dot{U}_s-iC_sU_s+i\sum_j B_{sj}U_j=0 \quad , \quad j\neq s \quad . \quad \ldots\ldots\ldots\ldots(2)$$

Let $$U_s=P_s\,e^{i(gt+c)} \quad \ldots\ldots\ldots\ldots(3)$$

Then $$(g-C_s)P_s+\sum_j B_{sj}P_j=0 \quad , \quad j\neq s \quad , \quad \ldots\ldots\ldots\ldots(4)$$

and, putting $s=1, 2, \ldots, n$ in succession, we have the group of n equations

$$\left.\begin{array}{llllll} (g-C_1)P_1 & +B_{12}P_2 & +B_{13}P_3+\ldots+ & B_{1n}P_n & =0 \\ B_{21}P_1 & +(g-C_2)P_2+B_{23}P_3+\ldots+ & B_{2n}P_n & =0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ B_{n1}P_1 & +B_{n2}P_2+\ldots\ldots+ & (g-C_n)P_n & =0 \end{array}\right\} \quad \ldots\ldots(5)$$

Hence, g is a root of the equation

$$\Delta = 0 \quad \ldots\ldots\ldots\

from which, on equating the coefficients of E_t, we have

$$(g_t - C_s)M_{ts} + \sum_j B_{sj}M_{tj} = 0 \quad .$$

Similarly

$$(g_r - C_s)M_{rs} + \sum_j B_{sj}M_{rj} = 0 \quad .$$

Eliminate C_s between these two equations ; then

$$(g_t - g_r)M_{ts}M_{rs} = -\sum_j B_{sj}(M_{tj}M_{rs} - M_{ts}M_{rj}) \quad .$$

Sum for $s = 1, 2, \ldots, n$; then

$$(g_t - g_r)\sum_{s=1}^{n} M_{ts}M_{rs} = -\sum_j \sum_{s=1}^{n} B_{sj}(M_{tj}M_{rs} - M_{ts}M_{rj})$$

in which $j \neq s$. Since $B_{sj} = B_{js}$, the double summation on the right is zero. Hence, if the roots g_t and g_r are distinct,

$$\sum_{s=1}^{n} M_{ts}M_{rs} = 0 \quad , \quad t \neq r \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

or, on interchanging r and s,

$$\sum_{r=1}^{n} M_{sr}M_{tr} = 0 \quad , \quad t \neq s \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

13·13. Evaluation of the constants of integration

From 13·12 (1), $U_s = \sum_{t=1}^{n} M_{ts}E_t$. Multiply by M_{rs} and sum for $s = 1, 2, \ldots, n$; then

$$\sum_{s=1}^{n} M_{rs}U_s = \sum_{s=1}^{n}\sum_{t=1}^{n} M_{rs}M_{ts}E_t$$

$$= \sum_{s=1}^{n} M^2{}_{rs}E_r + \sum_t E_t \sum_{s=1}^{n} M_{rs}M_{ts}$$

where, in the double summation on the right, $t \neq r$. By 13·12 (2), the double summation vanishes ; hence

$$\sum_{s=1}^{n} M_{rs}U_s = \sum_{s=1}^{n} M^2{}_{rs}E_r.$$

Equate imaginary and real parts ; then, for a given r,

$$M_{r1}H_1 + M_{r2}H_2 + \ldots + M_{rn}H_n = (M^2{}_{r1} + \ldots + M^2{}_{rn}) \sin(g_r t + c_r) \quad , \quad \ldots (1)$$

$$M_{r1}K_1 + M_{r2}K_2 + \ldots + M_{rn}K_n = (M^2{}_{r1} + \ldots + M^2{}_{rn}) \cos(g_r t + c_r) \quad . \quad \ldots (2)$$

Now, at epoch $t = 0$, the values of h_i ($\equiv e_i \sin \varpi_i$) and k_i ($\equiv e_i \cos \varpi_i$) can be found ; thereafter the values of H_i and K_i can be found. Substitute these values in (1) and (2)—we do not indicate them in a special way in the interests of simplification of nomenclature. Then

$$M_{r1} \sin c_r = \frac{H_1 + \frac{M_{r2}}{M_{r1}}H_2 + \ldots + \frac{M_{rn}}{M_{r1}}H_n}{1 + \left(\frac{M_{r2}}{M_{r1}}\right)^2 + \ldots + \left(\frac{M_{rn}}{M_{r1}}\right)^2} \quad , \quad \ldots\ldots\ldots\ldots\ldots (3)$$

with a similar equation for $M_{r1} \cos c_r$ in K.

Let X_r be the value of the right-hand side of (3), and Y_r the corresponding value in the equation for K. Then

$$M_{r1} \sin c_r = X_r \quad , \quad M_{r1} \cos c_r = Y_r \quad ;$$

in these equations the values of X_r and Y_r are supposed to be known by the method described in 13·11. The evaluation of M_{r1} and c_r then follows at once.

The solutions in H and K are now formally complete ; from 13·10 (9) we deduce the solutions in h and k. The former are, from 13·11 (1) and (8)—on equating imaginary and real parts—

$$H_s = \sum_{r=1}^{n} M_{rs} \sin (g_r t + c_r) \quad , \quad K_s = \sum_{r=1}^{n} M_{rs} \cos (g_r t + c_r) \quad , \ldots\ldots\ldots (4)$$

or, from 13·10 (9), on writing $D_{rs} = M_{rs}/\{\sqrt{(m_s n_s)} a_s\}$,

$$h_s = \sum_{r=1}^{n} D_{rs} \sin (g_r t + c_r) \quad , \quad k_s = \sum_{r=1}^{n} D_{rs} \cos (g_r t + c_r) \quad .$$

13·14. The roots of the equation in g

We have assumed that the roots $g_1, g_2, \ldots, g_n$ of the equation $\Delta = 0$—13·11 (7)—are all real. Suppose that two of them are complex, say g_1 and g_2 ; then they must be of the forms

$$g_1 = u - iv \quad , \quad g_2 = u + iv$$

where we assume v to be positive, u and v being real. The formula for U_s in 13·11 (8) is now

$$U_s = M_{1s} e^{vt} \cdot e^{i(ut+c_1)} + M_{2s} e^{-vt} \cdot e^{i(ut+c_2)} + P.T.$$

Write $\theta_1 = ut + c_1 \quad , \quad \theta_2 = ut + c_2 \quad , \quad \theta_r = g_r t + c_r \quad ;$

then, equating imaginary and real parts we obtain

$$H_s = M_{1s} e^{vt} \sin \theta_1 + M_{2s} e^{-vt} \sin \theta_2 + \sum_{r=3}^{n} M_{rs} \sin \theta_r \quad ,$$

$$K_s = M_{1s} e^{vt} \cos \theta_1 + M_{2s} e^{-vt} \cos \theta_2 + \sum_{r=3}^{n} M_{rs} \cos \theta_r \quad .$$

Hence,

$$\sum_{s=1}^{n} (H_s^2 + K_s^2) = e^{2vt} \sum_s M_{1s}^2 + e^{-2vt} \sum_s M_{2s}^2 + e^{vt}[P.T.] + e^{-vt}[P.T.] + P.T. + R$$

where R is a constant. But, by 13·10 (14), the left-hand side is equal to κ^2, κ being a constant. Hence we have identically, in an obvious notation,

$$\kappa^2 \equiv P^2 e^{2vt} + Q^2 e^{-2vt} + e^{vt}[P.T.] + e^{-vt}[P.T.] + P.T. + R.$$

But e^{2vt} and e^{vt} increase without limit ; accordingly, the right-hand side cannot be equated to a constant. We conclude that the roots g_1 and g_2 must be real since we must have $v = 0$, and that, generally, all the roots $g_1, \ldots, g_n$ must be real.

13·15. Canonic equations in H and K

With a slight change in notation, the canonic equations are, from 13·10 (13),

$$\dot{H}_r = \frac{\partial F}{\partial K_r} \quad , \quad \dot{K}_r = -\frac{\partial F}{\partial H_r} \quad ,$$

in which F, by 13·10 (12), is given by

$$2F = \sum_{r=1}^{n} C_r(H_r^2 + K_r^2) - \sum_{r=1}^{n}\sum_{s=1}^{n} B_{rs}(H_rH_s + K_rK_s) \quad , \quad s \neq r \quad \ldots\ldots(1)$$

Here, F is a homogeneous quadratic function of the H's and an *identical* homogeneous quadratic function of the K's. We can, consequently, make an orthogonal linear transformation of the variables H, K to new variables L, l. Let

$$H_r = A_{r1}L_1 + A_{r2}L_2 + \ldots + A_{rn}L_n \quad , \quad \ldots\ldots(2)$$

$$K_r = A_{r1}l_1 + \ldots + A_{rn}l_n \quad . \quad \ldots\ldots(3)$$

By means of these we can express F as $F(L, l)$. The new variables will be canonically conjugate if, by 10·09 (1) and (2),

$$\sum_{r=1}^{n} A^2{}_{rs} = 1 \quad , \quad \ldots\ldots(4)$$

$$\sum_{r=1}^{n} A_{rs}A_{rt} = 0 \quad , \quad t \neq s \quad , \quad \ldots\ldots(5)$$

where s and t can take the values 1, 2, ... , n. We then have

$$\dot{L}_r = \frac{\partial F}{\partial l_r} \quad , \quad \dot{l}_r = -\frac{\partial F}{\partial L_r} \quad . \quad \ldots\ldots(6)$$

It is evident that F when transformed is given by

$$2F = \sum_{s=1}^{n} a_r(L_r^2 + l_r^2) + \sum_{r=1}^{n}\sum_{s=1}^{n} f_{rs}(L_rL_s + l_rl_s) \quad , \quad \ldots\ldots(7)$$

where the a's and f's are functions of the A's and B's and C's, and further that, since $B_{rs} = B_{sr}$, then $f_{rs} = f_{sr}$.

The equations (4) are n in number and the equations (5) are $\frac{1}{2}n(n-1)$ in number (since, for example, $\sum_r A_{r1}A_{r2} \equiv \sum_r A_{r2}A_{r1}$). Hence, since the A's are n^2 in number, then $\frac{1}{2}n(n-1)$ of the A's can be regarded as arbitrary.

Now, in (7), the f's are $\frac{1}{2}n(n-1)$ in number ; hence we can choose the arbitrary A's so that all the f's are zero ; in other words, we have $\frac{1}{2}n(n-1)$ new relations which, with (4) and (5) enable us to determine each A explicitly. We can then write

$$2F = \sum_{1}^{n} a_r(L_r^2 + l_r^2) \quad . \quad \ldots\ldots(8)$$

The canonic equations are : $\dot{L}_r = a_r l_r$, $\dot{l}_r = -a_r L_r$, from which $\ddot{L}_r + a_r^2 L_r = 0$, the solution of which is

$$L_r = P_r \sin(a_r t + c_r) \quad . \quad \ldots\ldots(9)$$

Referring to (8) and to the energy-integral given by 13·10 (16), we see that we must have

$$l_r = P_r \cos(a_r t + c_r) \quad .$$

We then have, from (2) and (3),

$$H_r = \sum_{s=1}^{n} A_{rs} P_r \sin(a_r t + c_r) \quad , \quad K_r = \sum_{s=1}^{n} A_{rs} P_r \cos(a_r t + c_r) \quad . \quad \ldots(10)$$

Compare these last equations with 13·13 (4). It is then seen that the a's are to be identified with the g's—the roots of $\Delta = 0$ given by 13·11 (7). Also, $M_{rs} = A_{rs} P_r$.

Suppose that the transformations (2) and (3) are carried out; then, by (8)—on replacing a_r by g_r—it is evident that the equation $\Delta = 0$ in 13·11 (6) is transformed into $\Delta' \equiv \prod_{r=1}^{n} (g - g_r) = 0$.

13·16. Case of two equal roots of $\Delta = 0$

Consider the determinant Δ defined by 13·11 (7); with its variegated values of the quantities C and B it would seem next to impossible that the equation $\Delta = 0$ could have two roots equal. At one time it had been thought that if two roots were actually equal, the purely periodic character of the solutions of H and K would be imperilled. To examine the nature of the solutions when two roots—say g_1 and g_2—are equal we confine the analysis to the case of three planets, that is, when $n = 3$. In this case, the A-coefficients in 13·15 (2) and (3) have the properties of, and can be replaced by, direction-cosines $(\lambda_r, \mu_r, \nu_r)$, $r = 1, 2, 3$.

The reduction of F in 13·15 (1) to the form given by 13·15 (8) is the analytical equivalent of reducing the equation of the ellipsoid

$$2F \equiv C_1 x^2 + C_2 y^2 + C_3 z^2 - B_1 yz - B_2 zx - B_3 xy = 1$$

to the equation

$$2F \equiv a_1 x^2 + a_2 y^2 + a_3 z^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

referred to its principal axes. If $g_1 = g_2$—that is, if $a_1 = a_2$—the equation (1) becomes, on replacing a by g,

$$2F \equiv g_1 x^2 + g_1 y^2 + g_3 z^2 = 1 \quad ,$$

the equation of an ellipsoid of revolution; the new x and y axes lying in the central circular section are entirely arbitrary. As in the previous section, the equations in L are

$$\ddot{L}_1 + g_1^2 L_1 = 0 \quad , \quad \ddot{L}_2 + g_1^2 L_2 = 0 \quad , \quad \ddot{L}_3 + g_3^2 L_3 = 0 \quad ,$$

of which the solutions are

$$L_1 = P_1 \sin(g_1 t + c_1) \quad , \quad L_2 = P_2 \sin(g_1 t + c_2) \quad , \quad L_3 = P_3 \sin(g_3 t + c_3) \quad .$$

There are similar equations in l. It follows that the solutions in H and K are periodic.

13·17. **The equations for the inclinations**

For a planet P_1 the equations now under consideration are

$$\dot{p}_1 = \frac{1}{n_1 a_1^2}\frac{\partial N_1}{\partial q_1} \quad , \quad \dot{q}_1 = -\frac{1}{n_1 a_1^2}\frac{\partial N_1}{\partial p_1} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

the relevant part of N_1 being

$$-\sum_{j=2}^{n} Gm_j D_{1j}(p_1^2 + q_1^2 + p_j^2 + q_j^2 - 2p_1 p_j - 2q_1 q_j) \quad .$$

With the notation of 13·10 (3) and (4) we obtain from (1)

$$\dot{p}_1 + C_1 q_1 - \sum_{2}^{n}(1,j)q_j = 0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\dot{q}_1 - C_1 p_1 + \sum_{2}^{n}(1,j)p_j = 0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Set $\alpha_i = \sqrt{(m_i n_i)}\, a_i$. Then

$$\alpha_i \,.\, (i,j) = \frac{2Gm_i m_j D_{ij}}{\alpha_i \alpha_j} \,.\, \alpha_j = -b_{ij}\alpha_j \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where b_{ij} is a symmetrical function with the property

$$b_{ij} = b_{ji} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Multiply (2) by α_1 and apply (4); then, setting

$$\alpha_i p_i = P_i \quad , \quad \alpha_i q_i = Q_i \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

we obtain

$$\dot{P}_1 + C_1 Q_1 + \sum_{2}^{n} b_{1j} Q_j = 0 \quad .$$

In general, we have

$$\dot{P}_i + C_i Q_i + \sum_{j} b_{ij} Q_j = 0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

and, similarly,

$$\dot{Q}_i - C_i P_i - \sum_{j} b_{ij} P_j = 0 \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

in these, $j \neq i$.

From (7) and (8) we have

$$\sum_{i=1}^{n}(P_i\dot{P}_i + Q_i\dot{Q}_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(P_i Q_j - P_j Q_i) = 0 \quad ;$$

the double summation vanishes by reason of (5); hence $\sum_{1}^{n}(P_i^2 + Q_i^2) = k^2$, where k is a constant. This last equation is equivalent to

$$\sum_{1}^{n} m_i n_i a_i^2 \gamma_i^2 = k^2 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

which is the generalised form of 13·07 (8). Thus, since n_i has the same sign for all values of i, the values of γ are restricted; accordingly, the inclinations have no associated secular terms.

13·18. Solutions for the inclinations

The solutions of 13·17 (7) and (8) are evidently similar to those derived for H and K in 13·11. Let

$$P_i = N_i \sin(gt+c) \quad , \quad Q_i = N_i \cos(gt+c) \quad ; \quad \ldots\ldots\ldots\ldots (1)$$

then, on substituting these expressions for P_i and Q_i in 13·17 (7) we obtain, for successive values of i,

$$\begin{aligned} (g+C_1)N_1 + \; b_{12}N_2 \; + \ldots + \; b_{1n}N_n \; &= 0 \quad , \\ b_{21}N_1 \; + (g+C_2)N_2 + \ldots + \; b_{2n}N_n \; &= 0 \quad , \\ \ldots\ldots\ldots\

We can regard $N_{11}, N_{21}, \ldots, N_{n1}$ and the c's as the constants of integration.

The procedure in evaluating the constants in the general solution is similar to that described in 13·13 in the case of the eccentricities. A similar remark applies to the arguments based on the solution of the canonic equations to which 13·17 (7) and (8) can be reduced.

13·19. Numerical results

It is evident that the amount of labour in effecting the numerical solutions is prodigious when we are concerned with the planetary system as we know it at present. If we restrict the investigation to the major planets, we have to determine, in the case of the eccentricities, (i) the nine values of the roots, g, of a determinant of order 9—13·11 (7)—(ii) the values of the 18 constants of integration, M_{1s}, c_s, and (iii) the values of the remaining coefficients, M_{rs} $(r \neq 1)$. The equations for the inclinations lead to equally laborious calculations. In earlier sections the results for two planets (Jupiter and Saturn) have been given. The complete solutions for the eight major planets known at the time—Pluto had not yet been discovered—was effected by J. N. Stockwell.* The principal results are shown in the accompanying table, the inclinations being referred to the invariable plane of the planetary system. As regards the eccentricities, the values of g—the roots of the equation $\Delta = 0$, 13·11 (6)—lie between 0″·616 and 22″·46 when expressed in seconds of arc. The corresponding periods are about 2,100,000 years and 58,000 years. The periods for the inclinations are comparable.

Table: Maximum and minimum values of the eccentricities and inclinations

Planet	*Eccentricity*		*Inclination*	
	Max.	*Min.*	*Max.*	*Min.*
Mercury	0·2317	0·1215	9° 10′·7	4° 44′·4
Venus	0·0706	0·0000	3° 16′·3	*
Earth	0·0677	0·0000	3° 6′·0	*
Mars	0·1397	0·0185	5° 56′·0	*
Jupiter	0·0608	0·0255	0° 28′·9	0° 14′·4
Saturn	0·0843	0·0124	1° 0′·6	0° 47′·3
Uranus	0·0780	0·0118	1° 7′·2	0° 54′·4
Neptune	0·0145	0·0056	0° 47′·3	0° 33′·7

* Undecided for reason of the smallness of certain coefficients in the analysis.

13·20. Longitude of the epoch

From 5·10 (8), R being replaced by the non-periodic part N, we have

* *Smithsonian Contributions to Knowledge*, Vol. 18, 1873.

$$\dot{\epsilon} = -\frac{2}{na}\frac{\partial N}{\partial a} + \frac{\tan\frac{\phi}{2}\cos\phi}{na^2}\frac{\partial N}{\partial e} + \frac{\tan\frac{i}{2}}{na^2\cos\phi}\frac{\partial N}{\partial i}$$

where $\cos\phi = \sqrt{(1-e^2)}$ or, with sufficient accuracy,

$$\dot{\epsilon} = -\frac{2}{na}\frac{\partial N}{\partial a} + \frac{e}{2na^2}\frac{\partial N}{\partial e} + \frac{\gamma}{2na^2}\frac{\partial N}{\partial i} \quad .$$

When h, k and p, q are introduced this equation becomes

$$\dot{\epsilon} = -\frac{2}{na}\frac{\partial N}{\partial a} + \frac{1}{2na^2}\left(h\frac{\partial N}{\partial h} + k\frac{\partial N}{\partial k} + p\frac{\partial N}{\partial p} + q\frac{\partial N}{\partial q}\right) \quad .$$

Insert now the solutions for h, k and p, q; it is seen that the last equation has the form

$$\dot{\epsilon} = A + \Sigma B\cos(\alpha t + \beta)$$

where the α's are related to the g's from the periodic terms in the eccentricities and in the inclinations and are small. A and the B's are small constants. Thus, for a particular planet,

$$\epsilon = \epsilon_0 + At + P.T.$$

from which the mean longitude l is given by

$$l = (n+A)t + \epsilon_0 + P.T.$$

The observed mean motion is the coefficient of t in this last formula.

13·21. General remarks

It is to be remembered that in the investigations of this chapter we have omitted (i) all periodic terms from the disturbing function, and (ii) all terms of order four or higher in the eccentricities and inclinations from the non-periodic part of the disturbing function. Within the limits set we have seen that h and k can be reduced to a number of periodic terms which, for a particular planet, we can write simply as

$$h \equiv e\sin\varpi = \Sigma M_r \sin(g_r t + c_r) \quad ,$$
$$k \equiv e\cos\varpi = \Sigma M_r \cos(g_r t + c_r) \quad .$$

Let e_0 and ϖ_0 be the values of these elements at epoch $t=0$. At a subsequent time t, let

$$e = e_0 + \Delta e \quad , \quad \varpi = \varpi_0 + \Delta\varpi \quad .$$

If t is small compared with the shortest period associated with the g's so that $(g_r t)^2$ can be neglected, then

$$\Delta e\sin\varpi_0 + e_0\Delta\varpi\cos\varpi_0 = t\,\Sigma g_r M_r\cos c_r \quad ,$$
$$\Delta e\cos\varpi_0 - e\Delta\varpi\sin\varpi_0 = -t\,\Sigma g_r M_r\sin c_r \quad ;$$

from these

$$\Delta e = t\,\Sigma g_r M_r\sin(\varpi_0 - c_r) \quad .$$

This last result shows that the true change in e is practically indistinguishable from the change associated with a secular term within, of course, the limitations stated.

Similar results follow from the solutions for the inclinations.

The suggestion, then, is that the appearance of a secular term in the eccentricity, for example, found in the solution of Lagrange's planetary equations by the methods of Chapter 6 is the result of the analytic procedure adopted. A similar remark applies to the inclination.

If the restrictions, mentioned at the beginning of this section, are no longer imposed the problem of the stability of the planetary system—in the sense that the elements a, e and γ are expressible in terms of convergent periodic series—is clothed with immense difficulties and has, hitherto, evaded exact solution. On the other hand, Delaunay's literal solution of the lunar problem points to gravitational stability in the sense stated—although the question of convergency of the various series obtained is complicated beyond measure—and indicates, in particular, that secular and secular-periodic terms are disguised forms of genuine periodic terms.

Chapter 14

GAUSS' METHOD FOR THE CALCULATION OF SECULAR INEQUALITIES

14·01. Introduction

Consider the planetary equation, 5·10 (10),

$$\dot{\Omega} = \frac{1}{na^2\sqrt{(1-e^2)}\sin i}\frac{\partial R}{\partial i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If the disturbing function R has been expanded in the usual way, the integration of (1) enables us to write—as in 6·03 (4)—

$$\Omega = \Omega_0 + \lambda t + P.T. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

when the elements on the right of (1) are regarded as constants ; this is equivalent to saying that the solution (2) is correct to the first power of the masses of the two planets concerned, a degree of accuracy which is sufficient for the present purpose. The term λt in (2) is the secular inequality. R is, of course, a complicated function and the calculation of the numerical value of λ is long and tedious. The same remark applies to the secular inequalities of the other elements. The method originally due to Gauss* enables the numerical value of λ to be found without recourse to the expansion of the disturbing function and depends on a knowledge of the numerical values of the elliptic coordinates of the planet P—or any other body—and of the disturbing planet P_1 at various selected times. The method has been applied to a variety of problems.

14·02. The orthogonal components S, T and W of the attraction

In Fig. 23, let O be the Sun, P the planet (or object) concerned and A the corresponding point of P on the celestial sphere with its centre at O. With respect to the usual reference plane the coordinates of P are (x, y, z) and the coordinates of the disturbing planet P_1, of mass m_1, are (x_1, y_1, z_1). We write for convenience : $\kappa = Gm_1$.

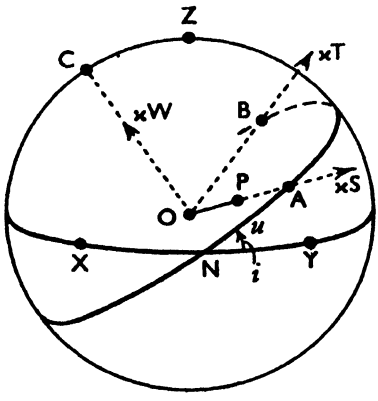


FIG. 23

Let κS, κP and κW be the components of the attraction of P_1 on unit mass at P defined as follows ; κS is along the radius vector OP (or OA) ; κT is transverse to the radius vector and in the orbital plane, and is parallel

* Tisserand, *Mécanique Céleste*, I, 431 (1889).

to OB; κW is perpendicular to the orbital plane and parallel to OC.

Let the direction cosines of OA, OB and OC be (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3). Then, if u denotes the arc NA,

$$l_1 = \cos\Omega\cos u - \sin\Omega\sin u\cos i \quad .$$

The remaining equations are those of 5·06 (3), (4) and (5) when ω is replaced by u. Further, the differential coefficients of l_1 etc. with respect to Ω, u and i are given in the table in 5·06 (c), ω being replaced by u.

If f is the true anomaly,

$$u = \omega + f = \varpi - \Omega + f \quad \text{.....(1)}$$

Now $\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}, \frac{\partial R}{\partial z}$ are the components of the attraction of P_1 on unit mass at P along OX, OY, OZ; hence

$$\left.\begin{aligned} \frac{1}{\kappa}\frac{\partial R}{\partial x} &= l_1 S + l_2 T + l_3 W \\ \frac{1}{\kappa}\frac{\partial R}{\partial y} &= m_1 S + m_2 T + m_3 W \\ \frac{1}{\kappa}\frac{\partial R}{\partial z} &= n_1 S + n_2 T + n_3 W \end{aligned}\right\} \quad \text{, ...(2)}$$

from which

$$S = \frac{1}{\kappa}\left(l_1\frac{\partial R}{\partial x} + m_1\frac{\partial R}{\partial y} + n_1\frac{\partial R}{\partial z}\right) \quad \text{, ...(3)}$$

with two similar equations for T and W.

Now $$R = \kappa\left(\frac{1}{\Delta} - \frac{xx_1 + yy_1 + zz_1}{r_1^3}\right) \quad ,$$

where $$\Delta^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \quad ;$$

hence, for example, $$\frac{\partial R}{\partial x} = -\frac{\kappa(x - x_1)}{\Delta^3} - \kappa\frac{x_1}{r_1^3} \quad .$$

The value of S at any time can then be calculated by means of (3) when the elements of P's orbit and the coordinates of P and P_1 are known.

In Lagrange's equations, differential coefficients of R with respect to the elements occur; if σ is any element we require to express $\frac{\partial R}{\partial \sigma}$ in terms of S, T and W.

14·03. Expression of $\frac{\partial R}{\partial \sigma}$ in terms of S, T and W

We have: $x = l_1 r$, $y = m_1 r$, $z = n_1 r$, where r is the radius vector of the planet P; hence, σ being any element,

$$\frac{\partial x}{\partial \sigma} = r\frac{\partial l_1}{\partial \sigma} + l_1\frac{\partial r}{\partial \sigma} \quad ,$$

with two similar equations for $\frac{\partial y}{\partial \sigma}$ and $\frac{\partial z}{\partial \sigma}$.

Now $$\frac{\partial R}{\partial \sigma} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial \sigma} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial \sigma} \quad ;$$

hence, by (2) of the previous section,

$$\begin{aligned}\frac{1}{\kappa}\frac{\partial R}{\partial \sigma} &= \sum_{l,m,n}\left(r\frac{\partial l_1}{\partial \sigma} + l_1\frac{\partial r}{\partial \sigma}\right)\left(l_1 S + l_2 T + l_3 W\right) \\ &= S\frac{\partial r}{\partial \sigma} + rT\,\Sigma\, l_2\frac{\partial l_1}{\partial \sigma} + rW\,\Sigma\, l_3\frac{\partial l_1}{\partial \sigma} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)\end{aligned}$$

Now $$\begin{aligned}\frac{\partial l_1}{\partial \sigma} &= \frac{\partial l_1}{\partial \Omega}\cdot\frac{\partial \Omega}{\partial \sigma} + \frac{\partial l_1}{\partial u}\cdot\frac{\partial u}{\partial \sigma} + \frac{\partial l_1}{\partial i}\cdot\frac{\partial i}{\partial \sigma} \\ &= -m_1\frac{\partial \Omega}{\partial \sigma} + l_2\frac{\partial u}{\partial \sigma} + l_3\sin u\frac{\partial i}{\partial \sigma}\end{aligned}$$

by the table in section 5·06 (*c*) in which ω is replaced by u. The differential coefficients of m_1 and n_1 with respect to σ are obtained in a similar way. The coefficients of rT and rW in (1) are then easily found to be

$$n_3\frac{\partial \Omega}{\partial \sigma} + \frac{\partial u}{\partial \sigma} \quad \text{and} \quad -n_2\frac{\partial \Omega}{\partial \sigma} + \sin u\frac{\partial i}{\partial \sigma} \quad \text{respectively,}$$

where $n_3 = \cos i$ and $n_2 = \cos u \sin i$. Hence

$$\frac{1}{\kappa}\frac{\partial R}{\partial \sigma} = S\frac{\partial r}{\partial \sigma} + rT\left(n_3\frac{\partial \Omega}{\partial \sigma} + \frac{\partial u}{\partial \sigma}\right) + rW\left(-n_2\frac{\partial \Omega}{\partial \sigma} + \sin u\frac{\partial i}{\partial \sigma}\right) \ldots\ldots(2)$$

We require to find $\frac{\partial r}{\partial \sigma}$ and $\frac{\partial u}{\partial \sigma}$.

(i) $\frac{\partial r}{\partial \sigma}$. We have : $r = a(1 - e\cos E)$, from which

$$\frac{\partial r}{\partial \sigma} = \frac{r}{a}\frac{\partial a}{\partial \sigma} - a\cos E\frac{\partial e}{\partial \sigma} + ae\sin E\frac{\partial E}{\partial \sigma} \quad .$$

Now $$E - e\sin E = \int n\,dt + \epsilon - \varpi$$

in which the first term on the right is *not* to be differentiated with regard to a. Then, in general,

$$\frac{r}{a}\frac{\partial E}{\partial \sigma} - \sin E\frac{\partial e}{\partial \sigma} = \frac{\partial(\epsilon - \varpi)}{\partial \sigma} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and, accordingly, after a little reduction,

$$\frac{\partial r}{\partial \sigma} = \frac{r}{a}\frac{\partial a}{\partial \sigma} - a\cos f\frac{\partial e}{\partial \sigma} + \frac{a^2 e\sin E}{r}\frac{\partial(\epsilon - \varpi)}{\partial \sigma} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

(ii) $\frac{\partial u}{\partial \sigma}$. From 14·02 (1),

$$\frac{\partial u}{\partial \sigma} = \frac{\partial(\varpi - \Omega)}{\partial \sigma} + \frac{\partial f}{\partial \sigma} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Now, $\tan \frac{1}{2}f = \sqrt{\left(\frac{1+e}{1-e}\right)} \tan \frac{1}{2}E$, from which we derive :

$$\frac{1}{\sin f}\frac{\partial f}{\partial \sigma} = \frac{1}{1-e^2}\frac{\partial e}{\partial \sigma} + \frac{1}{\sin E}\frac{\partial E}{\partial \sigma} \quad .$$

But $r \sin f = b \sin E$, where $b = a(1-e^2)^{1/2}$; hence, using (3), we find

$$\frac{\partial f}{\partial \sigma} = \sin f \left(\frac{1}{1-e^2} + \frac{a}{r}\right)\frac{\partial e}{\partial \sigma} + \frac{ab}{r^2}\frac{\partial(\epsilon - \varpi)}{\partial \sigma} \quad \text{.....(6)}$$

The expression for $\frac{\partial u}{\partial \sigma}$ is then obtained from (5) and (6).

Inserting the expressions for $\frac{\partial r}{\partial \sigma}$, $\frac{\partial u}{\partial \sigma}$, n_3 and n_2 in (2) we find that

$$\begin{aligned}\frac{1}{\kappa}\frac{\partial R}{\partial \sigma} = {} & S\left[\frac{r}{a}\frac{\partial a}{\partial \sigma} - a\cos f \frac{\partial e}{\partial \sigma} + \frac{a^2 e \sin E}{r}\frac{\partial(\epsilon - \varpi)}{\partial \sigma}\right] \\ & + rT\left[\sin f\left(\frac{1}{1-e^2} + \frac{a}{r}\right)\frac{\partial e}{\partial \sigma} + \frac{ab}{r^2}\frac{\partial \epsilon}{\partial \sigma} + \left(1 - \frac{ab}{r^2}\right)\frac{\partial \varpi}{\partial \sigma} - (1-\cos i)\frac{\partial \Omega}{\partial \sigma}\right] \\ & + rW\left[-\cos u \sin i \frac{\partial \Omega}{\partial \sigma} + \sin u \frac{\partial i}{\partial \sigma}\right] \quad \text{.....(7)}\end{aligned}$$

From this last equation we immediately deduce the following :

$$\frac{1}{\kappa}\frac{\partial R}{\partial a} = \frac{r}{a} . S \quad , \quad \frac{1}{\kappa}\frac{\partial R}{\partial e} = -a\cos f . S + r \sin f \left(\frac{1}{1-e^2} + \frac{a}{r}\right) T \quad ,$$

$$\frac{1}{\kappa}\frac{\partial R}{\partial i} = r \sin u . W \quad , \quad \frac{1}{\kappa}\frac{\partial R}{\partial \Omega} = -2r \sin^2 \frac{i}{2} . T - r \cos u \sin i . W \quad ,$$

$$\frac{1}{\kappa}\frac{\partial R}{\partial \epsilon} = \frac{ae \sin f}{(1-e^2)^{1/2}} . S + \frac{a^2(1-e^2)^{1/2}}{r} . T \quad , \quad \frac{1}{\kappa}\frac{\partial R}{\partial \varpi} = -\frac{1}{\kappa}\frac{\partial R}{\partial \epsilon} + rT \quad .$$

14·04. The equations for $\dot{a}$, $\dot{e}$, . . . $\dot{\epsilon}$

We now insert the expressions for $\frac{\partial R}{\partial a}$ etc., derived in the previous section, in the planetary equations 5·10 (7)–(12). The results are :

$$\dot{a} = \frac{2Gm_1}{n(1-e^2)^{1/2}}\left[Se \sin f + \frac{b^2}{ar}T\right] \quad ,$$

$$\dot{e} = \frac{Gm_1(1-e^2)^{1/2}}{na}[S \sin f + T(\cos E + \cos f)] \quad ,$$

$$\dot{i} = \frac{Gm_1}{na^2(1-e^2)^{1/2}} . Wr \cos u \quad ,$$

$$\dot{\Omega} = \frac{Gm_1}{na^2(1-e^2)^{1/2}\sin i} . Wr \sin u \quad ,$$

$$\dot{\varpi} = 2 \sin^2 \frac{i}{2} . \dot{\Omega} + \frac{Gm_1(1-e^2)^{1/2}}{nae}\left[-S \cos f + T\left(1 + \frac{ar}{b^2}\right)\sin f\right] \quad ,$$

$$\dot{\epsilon} = -\frac{2Gm_1}{na^2} . rS + \frac{e^2}{1+(1-e^2)^{1/2}}\dot{\varpi} + 2(1-e^2)^{1/2} \sin^2 \frac{i}{2} . \dot{\Omega} \quad .$$

Since the mean longitude—which we denote here by ζ—is given by $\zeta = \int n\,dt + \epsilon$, we have $\dot{\zeta} = n + \dot{\epsilon}$.

These formulae are useful in mechanical quadrature for, as explained in 14·02, the numerical values of S, T and W can be calculated for any instant ; with the elements known, the right-hand sides of these equations can be calculated for the instant concerned. We can then obtain a series of values of $\dot{e}$, for example, for times t_0, $t_0 + \tau$, $t_0 + 2\tau$, ... from which by numerical integration we can express e as a function of t.

14·05. Application to secular inequalities

We have seen in 7·16 that the second term in the disturbing function

$$R \equiv Gm_1 \left(\frac{1}{\Delta} - \frac{xx_1 + yy_1 + zz_1}{r_1^3} \right)$$

is wholly periodic ; consequently, in dealing with secular perturbations, the disturbing function is effectively given by

$$R = Gm_1 \left(\frac{1}{\Delta} \right) \equiv \kappa \left(\frac{1}{\Delta} \right) \quad .$$

Hence we can regard S, T and W as the projections of $\frac{1}{\Delta^2}$ along the axes OA, OB and OC in Fig. 23.

Now, in general, when the whole of the disturbing function is involved, the planetary equation for an element σ can be written in the form

$$\dot{\sigma} = A_0 + \sum_i \sum_j B \cos(i\zeta + j\zeta_1 + q) \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where ζ, ζ_1 are the mean longitudes of the planets P and P_1, and i, j are integers, positive or negative, including zero, with the proviso that i and j are not simultaneously zero.

Multiply (1) by $d\zeta\, d\zeta_1$ and integrate between 0 and 2π for ζ and ζ_1 ; then, writing $[\dot{\sigma}]$ for A_0, we have

$$[\dot{\sigma}] = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \dot{\sigma}\, d\zeta\, d\zeta_1 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Gauss introduced the functions S_0, T_0, W_0, the first being defined by

$$S_0 = \frac{m_1}{2\pi} \int_0^{2\pi} S\, d\zeta_1 \quad ; \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

T_0 and W_0 are given by analogous expressions. Since S, T and W are functions of ζ and ζ_1, the new functions as represented by (3) are functions of ζ.

To the first order of the masses, the element a is periodic ; hence

$[\dot{a}]=0$. The secular inequalities for e, i, etc. are then given, from the formulae of section 14·04 and (3) above, by

$$[\dot{e}]=\frac{G(1-e^2)^{1/2}}{na}\cdot\frac{1}{2\pi}\int_0^{2\pi}[S_0\sin f+T_0(\cos E+\cos f)]\,d\zeta \quad ,$$

$$[\dot{i}]=\frac{G}{na^2(1-e^2)^{1/2}}\cdot\frac{1}{2\pi}\int_0^{2\pi}W_0\,r\cos u\,d\zeta \quad ,$$

$$[\dot{\Omega}]=\frac{G}{na^2(1-e^2)^{1/2}\sin i}\cdot\frac{1}{2\pi}\int_0^{2\pi}W_0\,r\sin u\,d\zeta \quad , \qquad \ldots\ldots(4)$$

$$[\dot{\varpi}]=2\sin^2\frac{i}{2}\,[\dot{\Omega}]+\frac{G(1-e^2)^{1/2}}{nae}\cdot\frac{1}{2\pi}\int_0^{2\pi}\left[-S_0\cos f+T_0\left(1+\frac{ar}{b^2}\right)\sin f\right]d\zeta \quad ,$$

$$[\dot{\epsilon}]=-\frac{2G}{na^2}\cdot\frac{1}{2\pi}\int_0^{2\pi}S_0\,r\,d\zeta+\frac{e^2}{1+(1-e^2)^{1/2}}\,[\dot{\varpi}]+2(1-e^2)^{1/2}\sin^2\frac{i}{2}[\dot{\Omega}] \quad .$$

14·06. The functions S_0, T_0 and W_0

It will be sufficient to deal with S_0, defined by

$$S_0=\frac{m_1}{2\pi}\int_0^{2\pi}S\,d\zeta_1 \quad . \qquad \ldots\ldots(1)$$

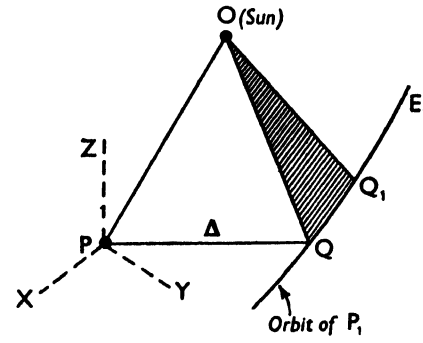


FIG. 24

In Fig. 24, let QQ_1 be an element of the orbit of the disturbing planet P_1. Suppose that the mass m_1 is distributed over the orbit in such a way that the mass $d\mu$ on the element QQ_1 is proportional to the time dt taken by P_1 to move from Q to Q_1. If the orbital period is T_1 and the area OQQ_1 is dA, then we have

$$\frac{d\mu}{m_1}=\frac{dt}{T_1}=\frac{d\zeta_1}{2\pi}=\frac{dA}{\pi a_1 b_1} \quad , \qquad \ldots\ldots(2)$$

where a_1 and b_1 are the semi-axes of the orbit of P_1. Then (1) and (2) give

$$S_0=\int S\,d\mu \quad , \qquad \ldots\ldots(3)$$

in which the integration is taken over the elliptic ring loaded in the way described.

Let PX, PY, PZ be any axes with P as origin; let (x, y, z) be the coordinates of Q. The attraction at P of $d\mu$ at Q has components

$$G\,d\mu\,\frac{x}{\Delta^3} \quad , \quad G\,d\mu\,\frac{y}{\Delta^3} \quad , \quad G\,d\mu\,\frac{z}{\Delta^3}$$

with respect to the new axes. Denote $d\mu\,\frac{x}{\Delta^3}$ by $d\phi_x$; then

$$\left.\begin{aligned}\phi_x&=\int\frac{x}{\Delta^3}\,d\mu \quad .\\ \text{Similarly,}\qquad \phi_y&=\int\frac{y}{\Delta^3}\,d\mu \quad , \quad \phi_z=\int\frac{z}{\Delta^3}\,d\mu\end{aligned}\right\} \quad . \qquad \ldots\ldots(4)$$

Now, $GS\,d\mu$ is the radial component (that is, along OP) of the attraction of $d\mu$ at Q on P. If (l_1, m_1, n_1) are the direction-cosines of OP with respect to PX, PY, and PZ then

$$GS\,d\mu = G[l_1 d\phi_x + m_1 d\phi_y + n_1 d\phi_z] \quad ;$$

hence, by (3),

$$S_0 = l_1\phi_x + m_1\phi_y + n_1\phi_z \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Later we shall choose the axes PX, PY and PZ to be the principal axes of the cone with P as vertex and P_1's orbital ellipse as a plane section; also it will be shown how ϕ_x etc. can be calculated for a given position of P. Thus, S_0 can be found by means of (5). Also, the functions T_0 and W_0 are given by equations similar to (5).

It is to be noted that, in (4), Δ is given by

$$\Delta^2 = x^2 + y^2 + z^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

14·07. The functions ϕ_x, ϕ_y, ϕ_z

By means of (2) and (4) of the previous section we have

$$\phi_x = \frac{m_1}{\pi a_1 b_1}\int \frac{x}{\Delta^3}\,dA \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let p be the perpendicular from P to the orbital plane of P_1; then p is also the perpendicular from P to the area OQQ_1 and to all such areas. If dV denotes the volume of the tetrahedron with vertex P and base OQQ_1, then

$$dV = \tfrac{1}{3}p\,dA \quad .$$

Let the coordinates of Q_1 and the Sun, O, be $(x+dx, y+dy, z+dz)$ and (x_0, y_0, z_0) respectively; then

$$dV = \tfrac{1}{6}[x_0(y\,dz - z\,dy) + y_0(z\,dx - x\,dz) + z_0(x\,dy - y\,dx)] \quad .$$

Equating the two expressions of dV, we obtain dA expressed in terms of x, x_0, dx etc. We write, for convenience,

$$g = \frac{m_1}{\pi a_1 b_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

We then obtain from (1)

$$\phi_x = \frac{g}{2p}(x_0 P_x + y_0 Q_x + z_0 R_x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where

$$\left.\begin{aligned} P_x &= \int \frac{x(y\,dz - z\,dy)}{\Delta^3} \quad , \\ Q_x &= \int \frac{x(z\,dx - x\,dz)}{\Delta^3} \quad , \quad R_x = \int \frac{x(x\,dy - y\,dx)}{\Delta^3} \end{aligned}\right\} \quad ; \ldots\ldots (4)$$

in these the integrals are taken round the orbital ellipse, E_1, of P_1. Let

$$x = uz \quad , \quad y = vz \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Then, by 14·06 (6), $\Delta^2 = z^2 U^2$,

where $$U^2 = 1 + u^2 + v^2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

From (5) we obtain at once

$$y\,dz - z\,dy = -z^2\,dv \quad , \quad z\,dx - x\,dz = z^2\,du \quad , \quad x\,dy - y\,dx = z^2(u\,dv - v\,du) \quad .$$

Hence, from (4),

$$P_x = -\int \frac{u\,dv}{U^3} \quad , \quad Q_x = \int \frac{u\,du}{U^3} \quad , \quad R_x = \int \frac{u(u\,dv - v\,du)}{U^3} \quad . \quad \ldots(7)$$

Similarly, we write

$$P_y = \int \frac{y(y\,dz - z\,dy)}{\Delta^3} \quad , \quad P_z = \int \frac{z(y\,dz - z\,dy)}{\Delta^3} \quad ,$$

with similar formulae for Q_y, Q_z, R_y, R_z. We then have the group of formulae—rewriting (7)—

$$\left.\begin{aligned} P_x &= -\int \frac{u\,dv}{U^3} \quad , \quad Q_x = \int \frac{u\,du}{U^3} \quad , \quad R_x = \int \frac{u(u\,dv - v\,du)}{U^3} \\ P_y &= -\int \frac{v\,dv}{U^3} \quad , \quad Q_y = \int \frac{v\,du}{U^3} \quad , \quad R_y = \int \frac{v(u\,dv - v\,du)}{U^3} \\ P_z &= -\int \frac{dv}{U^3} \quad , \quad Q_z = \int \frac{du}{U^3} \quad , \quad R_z = \int \frac{u\,dv - v\,du}{U^3} \end{aligned}\right\} \quad ; \quad \ldots(8)$$

From these we have at once

$$P_x + Q_y + R_z = 0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Again, on applying (6),

$$P_y - Q_x = -\int \frac{u\,du + v\,dv}{U^3} = -\int \frac{dU}{U^2} = \left[\frac{1}{U}\right] \quad ,$$

where $\left[\frac{1}{U}\right]$ signifies the result of integrating round the ellipse E_1. But E_1 is closed; hence $\left[\frac{1}{U}\right] = 0$. Consequently,

$$P_y = Q_x \quad .$$

Similarly, $$P_z = R_x \quad \text{and} \quad Q_z = R_y \quad .$$

Let

$$F = \frac{g}{4p}\left[x_0^2 P_x + y_0^2 Q_y + z_0^2 R_z + 2y_0 z_0 R_y + 2z_0 x_0 P_z + 2x_0 y_0 Q_x\right] \quad \ldots(10)$$

Then, from (3) and its analogues, we have

$$\phi_x = \frac{\partial F}{\partial x_0} \quad , \quad \phi_y = \frac{\partial F}{\partial y_0} \quad , \quad \phi_z = \frac{\partial F}{\partial z_0} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

The cone with vertex P and with the section E_1 has the equation $f(x, y, z) = 0$ where f is a homogeneous quadratic function of x, y and z. Consider, for example, the integral in (8) for P_x; the integrand is of order zero in x, y and z, and since $x = uz$, $y = vz$ the form of P_x in terms of u and v

shows that the integration can be carried round *any* plane section of the cone.

14·08. The cone with vertex P and section E_1

The equation of P_1's ellipse referred to SA_1 as ξ-axis, the η-axis as shown in Fig. 25 and the ζ-axis perpendicular to the orbital plane, is

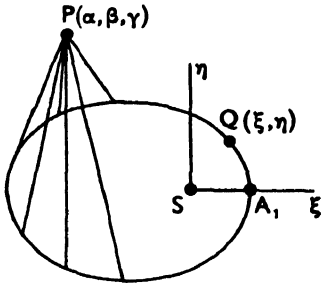


FIG. 25

$$\frac{(\xi+a_1e_1)^2}{a_1^2}+\frac{\eta^2}{b_1^2}=1 \quad , \quad \zeta=0 \quad .$$

Let (α, β, γ) be the coordinates of P referred to these axes. Any straight line through P is of the form

$$\frac{\xi-\alpha}{l}=\frac{\eta-\beta}{m}=z-\gamma \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

if this line intersects E_1 then

$$\frac{(\alpha-l\gamma+a_1e_1)^2}{a_1^2}+\frac{(\beta-m\gamma)^2}{b_1^2}=1 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The equation of the cone with P as vertex and E_1 as section is obtained by eliminating l and m between (1) and (2); it is

$$\frac{1}{a_1^2}\left[\alpha+a_1e_1-\frac{\gamma(\xi-\alpha)}{\zeta-\gamma}\right]^2+\frac{1}{b_1^2}\left[\beta-\frac{\gamma(\eta-\beta)}{\zeta-\gamma}\right]^2=1 \quad . \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Change the origin to P; then, if (x, y, z) are the coordinates of any point on the cone,

$$x=\xi-\alpha \quad , \quad y=\eta-\beta \quad , \quad z=\zeta-\gamma \quad .$$

The equation (3) becomes

$$\frac{1}{a_1^2}[z(\alpha+a_1e_1)-\gamma x]^2+\frac{1}{b_1^2}[\beta z-\gamma y]^2=z^2$$

which can be written in the form

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx=0$$

where, for example, $A=\gamma^2/a_1^2$.

If (l, m, n) denote the direction-cosines of the normal to a principal plane, then

$$\frac{Al+Gn}{l}=\frac{Bm+Fn}{m}=\frac{Gl+Fm+Cn}{n}=\lambda$$

so that λ is a root of the equation

$$\begin{vmatrix} A-\lambda, & 0, & G \\ 0, & B-\lambda, & F \\ G, & F, & C-\lambda \end{vmatrix}=0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

It is easily seen that (4) reduces to

$$\lambda^3 + P\lambda^2 + Q\lambda + a_1^4 b_1^4 \gamma^4 = 0 \quad ;$$

hence, of the three roots λ_1, λ_2, λ_3, two are positive and one is negative. Let λ_3 be the negative root so that $\lambda_3 = -\lambda$, where λ is positive.

The equation of the cone referred to its principal axes is then

$$\lambda_1 x^2 + \lambda_2 y^2 - \lambda z^2 = 0 \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

the z-axis is the internal principal axis.

We now identify the axes PX, PY, PZ in Fig. 24 with the principal axes of the cone. The direction-cosines (l, m, n) of one of these axes with reference to the ξ, η, ζ axes of Fig. 25 are given by

$$\frac{l}{G(\lambda - B)} = \frac{m}{F(\lambda - A)} = \frac{n}{(\lambda - A)(\lambda - B)} \quad ,$$

where λ now denotes the appropriate root of (4).

14·09. Evaluation of P_x etc.

From 14·07 (7), $$P_x = -\int \frac{u\, dv}{U^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $u = x/z$ and $v = y/z$ and the integration is taken round any section of the cone

$$\lambda_1 x^2 + \lambda_2 y^2 - \lambda z^2 = 0 \quad .$$

Take the section to be $z = 1$; this is the ellipse

$$\lambda_1 x^2 + \lambda_2 y^2 - \lambda = 0 \quad .$$

Write $$\mu_1^2 = \lambda/\lambda_1 \quad , \quad \mu_2^2 = \lambda/\lambda_2 \quad .$$

Since λ_1, λ_2 and λ are supposed to be known from the solution of the cubic equation 14·08 (4), then μ_1 and μ_2 are known ; the ellipse now becomes

$$\frac{x^2}{\mu_1^2} + \frac{y^2}{\mu_2^2} = 1 \quad .$$

We shall assume that $\mu_1 > \mu_2$; the procedure is similar when $\mu_1 < \mu_2$. Let θ be the eccentric angle ; then $x = \mu_1 \cos\theta$, $y = \mu_2 \sin\theta$. Hence, since $z = 1$,

$$u = \mu_1 \cos\theta \quad , \quad v = \mu_2 \sin\theta \quad .$$

Also, from 14·07 (6),

$$U^2 \equiv 1 + u^2 + v^2 = 1 + \mu_1^2 - (\mu_1^2 - \mu_2^2) \sin^2\theta = \mu^2 (1 - k^2 \sin^2\theta) \quad ,$$

where $$\mu^2 = 1 + \mu_1^2 \quad , \quad k^2 = \frac{\mu_1^2 - \mu_2^2}{\mu^2} \quad .$$

Then, (1) becomes

$$P_x = -\frac{\mu_1 \mu_2}{\mu^3} \int_0^{2\pi} \frac{\cos^2\theta\, d\theta}{(1 - k^2 \sin^2\theta)^{3/2}} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Similarly,

$$Q_x \equiv \int \frac{u\,du}{U^3} = -\frac{{\mu_1}^2}{2\mu^3}\int_0^{2\pi} \frac{\sin 2\theta\,d\theta}{(1-k^2\sin^2\theta)^{3/2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

We consider Q_x first.

(i) *Evaluation of* Q_x.

By dividing the range of integration in (3) into the two ranges 0 to π and π to 2π, it is easily seen that the integral vanishes. Hence $Q_x=0$.

Similarly, $$P_y=Q_z=R_y=P_z=R_x=0 \quad .$$

The function in 14·07 (10) now becomes

$$F=\frac{g}{4p}[x_0{}^2P_x+y_0{}^2Q_y+z_0{}^2R_z] \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Since, by 14·07 (9), $P_x+Q_y+R_z=0$, we need to calculate only two of these, say, P_x and Q_y.

(ii) *Evaluation of* P_x.

Let X be given by

$$X=\frac{d}{d\theta}\left[\frac{\sin\theta\cos\theta}{\sqrt{(1-k^2\sin^2\theta)}}\right] \quad .$$

It is easily deduced that

$$X=\frac{1-2\sin^2\theta+k^2\sin^4\theta}{(1-k^2\sin^2\theta)^{3/2}} \quad ,$$

the numerator of which can be written as

$$\frac{1}{k^2}[(1-k^2\sin^2\theta)^2-(1-k^2)] \quad .$$

Hence, since $\int_0^{\pi/2} X\,d\theta=0$, we have

$$\int_0^{\pi/2}\frac{d\theta}{(1-k^2\sin^2\theta)^{3/2}}=\frac{1}{1-k^2}\int_0^{\pi/2}\sqrt{(1-k^2\sin^2\theta)}\,d\theta=\frac{1}{1-k^2}E(k) \quad ,\ldots(5)$$

where $E(k)$ is the second elliptic integral.

Now, it is easily seen from (2) that

$$P_x=-\frac{4\mu_1\mu_2}{\mu^3}\int_0^{\pi/2}\frac{\cos^2\theta\,d\theta}{(1-k^2\sin^2\theta)^{3/2}}$$

and, since $$\cos^2\theta\equiv\frac{1}{k^2}[1-k^2\sin^2\theta-(1-k^2)] \quad ,$$

the integral is, by means of (5), equal to $\frac{1}{k^2}[F(k)-E(k)]$, where $F(k)$ is the first elliptic integral defined by

$$F(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}} \quad .$$

Hence $$P_x=\frac{4\mu_1\mu_2}{k^2\mu^3}[E(k)-F(k)] \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

(iii) *Evaluation of Q_y and R_z.*

$$Q_y=\int\frac{v\,du}{U^3}=-\frac{4\mu_1\mu_2}{\mu^3}\int_0^{\pi/2}\frac{\sin^2\theta\,d\theta}{(1-k^2\sin^2\theta)^{3/2}} \quad .$$

But, $$P_x+Q_y=-\frac{4\mu_1\mu_2}{\mu^3}\int_0^{\pi/2}\frac{d\theta}{(1-k^2\sin^2\theta)^{3/2}} \quad ;$$

hence, by (5), $$R_z=\frac{4\mu_1\mu_2}{(1-k^2)\mu^3}E(k) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

We then obtain

$$Q_y=\frac{4\mu_1\mu_2}{(1-k^2)k^2\mu^3}[(1-k^2)F(k)-E(k)] \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Since k is supposed known, the numerical values of $E(k)$ and $F(k)$ can be found from tables of elliptic functions. Hence, P_x, Q_y and R_z can be evaluated.

14·10. Calculation of the secular inequalities

We shall consider the secular inequality $[\dot{\Omega}]$ given by 14·05 (4), namely,

$$[\dot{\Omega}]=\frac{G}{na^2\sqrt{(1-e^2)}\sin i}\cdot\frac{1}{2\pi}\int_0^{2\pi}W_0\,r\sin u\,dM \quad , \quad \ldots\ldots\ldots\ldots(1)$$

in which we have replaced the mean longitude ζ by the mean anomaly M.

(i) By analogy with 14·06 (5), W_0 is given by

$$W_0=l_3\phi_x+m_3\phi_y+n_3\phi_z \quad ,$$

where (l_3, m_3, n_3) are the direction-cosines of the normal to the orbital plane of P_1 with reference to the principal axes of the cone ; the values of the direction-cosines can be readily calculated. From 14·07 (11) and 14·09 (4) we then have

$$W_0=\frac{g}{2p}[l_3x_0P_x+m_3y_0Q_y+n_3z_0R_z] \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Here g is a constant given by 14·07 (2). Also, if (X, Y, Z) are the co-ordinates of P with respect to the usual axes of reference,

$$p=X\sin\Omega_1\sin i_1-Y\cos\Omega_1\sin i_1+Z\cos i_1$$

where Ω_1, i_1 refer to the orbit of P_1 ; p can then be readily calculated for any time t. The terms within the square brackets of (2) can also be calculated. We assume then that the numerical value of W_0 can be found for any time t.

(ii) Consider the integrand, $W_0\,r\sin u$, in (1). Analytically, W_0 is evidently a complicated function of M and, since $u=\omega+f=\varpi-\Omega+f$, where f is the true anomaly, the whole integrand can be expressed as a

function of M which we denote by $\psi(M)$. We can then express $\psi(M)$ as follows :

$$\psi(M) = a_0 + \sum_{s=1}^{\infty} (a_s \cos sM + b_s \sin sM) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where s is a positive integer. It follows that

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \psi(M)\, dM$$

and consequently, by (1), that

$$[\dot{\Omega}] = \frac{Ga_0}{na^2 \sqrt{(1-e^2)} \sin i} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

We now suppose that W_0 and $r \sin u$—and hence the function $\psi(M)$—are calculated for each of the j values of M given by

$$0, \frac{2\pi}{j}, \frac{4\pi}{j}, \ldots, \frac{2r\pi}{j}, \ldots, \frac{2\pi(j-1)}{j} \quad .$$

Let ψ_r be the value of $\psi(M)$ when $M = \frac{2r\pi}{j} \equiv M_r$. Also let $s = k + qj$ where $k = 1, 2, \ldots, j$ and $q = 0, 1, 2, \ldots$. Then

$$\sum_{s=1}^{\infty} a_s \cos sM_r = \sum_{k=1}^{j} \sum_{q=0}^{\infty} a_{k+qj} \cos (k+qj) M_r \quad .$$

But, $\cos (k+qj) M_r = \cos kM_r$ and $\cos (j+qj) M_r = 1$. Hence

$$\begin{aligned} \sum_s a_s \cos sM_r &= a_j + a_{2j} + a_{3j} + \ldots + \sum_{k=1}^{j-1} [\sum_{q=0}^{\infty} a_{k+qj}] \cos kM_r \\ &= a_j + a_{2j} + a_{3j} + \ldots + \sum_{k=1}^{j-1} A_k \cos kM_r \end{aligned}$$

where $$A_k = \sum_{q=0}^{\infty} a_{k+qj} \quad .$$

We obtain in a similar way

$$\sum_s b_s \sin sM_r = \sum_{k=1}^{j-1} B_k \sin kM_r \quad .$$

Hence

$$\psi_r = a_0 + a_j + a_{2j} + \ldots + \sum_{k=1}^{j-1} (A_k \cos kM_r + B_k \sin kM_r) \quad ,$$

from which

$$\sum_{r=0}^{j-1} \psi_r = j[a_0 + a_j + a_{2j} + \ldots] + \sum_{k=1}^{j-1} [A_k \sum_{r=0}^{j-1} \cos kM_r + B_k \sum_{r=0}^{j-1} \sin kM_r] \quad .$$

Set $\alpha = \frac{2\pi k}{j}$; then $kM_r = r\alpha$. Let $C = \sum_0^{j-1} \cos r\alpha$, $S = \sum_0^{j-1} \sin r\alpha$; then, if $i = \sqrt{(-1)}$,

$$C + iS = \sum_0^{j-1} e^{ir\alpha} = \frac{1 - e^{ij\alpha}}{1 - e^{i\alpha}} = 0 \quad .$$

Hence $C=S=0$, and

$$\sum_{r=0}^{j-1}\psi_r = j[a_0+a_j+a_{2j}+\ldots] \quad .$$

If j is taken to be 12, say, the coefficients a_j, a_{2j}, ... are of high order with respect to the small quantities e and i (the inclination) entering into the expansion of $W_0 r \sin u$, and can be neglected. Hence

$$a_0 = \frac{1}{j}[\psi_0+\psi_1+\ldots+\psi_{j-1}]$$

and finally

$$[\dot{\Omega}] = \frac{G}{na^2\sqrt{(1-e^2)}\sin i}\cdot\frac{\psi_0+\psi_1+\ldots+\psi_{j-1}}{j} \quad .$$

This completes our discussion of the various steps in the investigation.

(iii) Amongst the many applications of the method may be mentioned Adams's calculation of the secular inequality of the longitude of the node of the Leonids* which was found to be $+28'$ during the adopted period of $33\frac{1}{4}$ years ; to this amount Jupiter, Saturn and Uranus contributed respectively $+20'$, $+7'$ and $+1'$. The value of $[\dot{\Omega}]$ deduced from observations was $+29'$.

Another important investigation† is that of G. W. Hill : " On Gauss's method of computing secular perturbations with an application to the action of Venus on Mercury."

For further developments of the method, the reader is referred to H. C. Plummer's *Dynamical Astronomy*, 206 (1918).

* M. N. 27, 247 (1867) ; *Collected Works*, I, 269 (1896).

† *Collected Works*, II, 1 (1906).

Chapter 15

THE INFLUENCE OF A RESISTING MEDIUM AND THE MOTION OF MERCURY'S PERIHELION

15·01. Introduction

The problem associated with a resisting medium first came into prominence in connection with Encke's Comet, the mean angular motion of which appeared to be increasing, the perturbations of the planets having been allowed for in the usual way. Encke suggested that this effect could quite well be explained in terms of a diffuse envelope of matter surrounding the Sun. That such matter exists is well known ; there is, first, the irregular solar corona extending to a distance of several radii beyond the Sun's surface and, second, the much more extensive envelope giving rise to the " zodiacal light ".

For simplicity, the term " planet " will be used to denote any member of the solar system whose motion is to be investigated in the present connection. The resistance to a planet's motion will depend on several factors : the projection of the planet's surface area perpendicular to the direction of motion (we take this to be constant for the planet concerned), the velocity v, and the density ρ of the medium. It will be assumed that, in general, ρ is an isotropic function of distance from the Sun's centre, so that we can write $\rho = \phi(r)$. Further, it can be taken for granted that there is no correlation between v and ρ ; accordingly, we can express R, the resistance per unit mass of the planet, in the form

$$R = cF(v)\phi(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which it will be assumed that c is an extremely small quantity, constant for the planet concerned.

Since Encke's time the discovery in the Galaxy of the vast diffuse nebulae, dark and luminous, through which the stars plough their way and, more recently, of the more tenuous interstellar matter has brought the problem again into prominence, more especially in the realms of cosmogony. In most of the current theories as to the formation and evolution of the solar system it is recognised that in the initial stages the Sun must have been surrounded by a chaotic cloud of matter through which the primeval planets moved. Now, one feature of the solar system is the extremely small values, in general, of the eccentricities of the planetary orbits ; one effect of the resisting medium, as we shall see, is the progressive reduction of the orbital eccentricities.

We shall first consider the easily integrable case when, in (1), $F(v) = v$

and $\phi(r) = 1/r^2$; later, we investigate the more general problem. Since the resistance has no component perpendicular to the plane of the orbit, it is sufficient to take this plane as the reference plane.

Because of the similarity of the methods employed we include at the end of the chapter a discussion on the motion of Mercury's perihelion.

15·02. Equations of motion ($R = cv/r^2$)

Let OX be a reference line in the orbital plane, the Sun being at O. Let (r, θ) be the polar coordinates of the planet, of mass m, and ϖ the "longitude" of perihelion so that $\theta = f + \varpi$, where f is the true anomaly. If ds is an element of the length of the orbit, the components of the resistance, radially outwards and transverse to the radius vector, are $-mR\frac{dr}{ds}$ and $-mRr\frac{d\theta}{ds}$ respectively, or $-mR\dot{r}/v$ and $-mRr\dot{\theta}/v$. The equations of motion are then, when $R = cv/r^2$,

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} - c\frac{\dot{r}}{r^2} \quad \ldots\ldots(1)$$

and

$$\frac{d}{dt}(r^2\dot{\theta}) = -c\dot{\theta} \quad \ldots\ldots(2)$$

Here $\mu = G(m_0 + m)$, where m_0 is the Sun's mass.

Let $$u = \frac{1}{r} \quad \text{and} \quad H = r^2\dot{\theta} \quad , \quad \ldots\ldots(3)$$

so that $\dot{\theta} = Hu^2$. Equation (2) becomes

$$H = h - c\theta \quad \ldots\ldots(4)$$

where h is the constant of integration.

Again, $$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -H\frac{du}{d\theta} \quad \ldots\ldots(5)$$

and $$\ddot{r} = -\frac{d}{d\theta}\left(H\frac{du}{d\theta}\right)\dot{\theta} = -Hu^2\frac{d}{d\theta}\left(H\frac{du}{d\theta}\right) \quad .$$

Hence, to $O(c)$, (1) becomes by means of (3),

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{H^2} = \frac{\mu}{h^2}\left(1 + \frac{2c\theta}{h}\right) \quad .$$

The solution of this equation is

$$u = \frac{1}{p}\left[1 + e\cos(\theta - \varpi) + \frac{2c\theta}{h}\right] \quad \ldots\ldots(6)$$

where e and ϖ are constants of integration and

$$p = \frac{h^2}{\mu} \quad ; \quad \ldots\ldots(7)$$

since h is a constant, it follows that p is a constant.

15·03. Variation of the osculating elements e and ϖ

If the resistance suddenly ceases at time t_0, the planet P will describe an elliptic orbit—the osculating ellipse—whose equation can be written as

$$\frac{1}{r_0} \equiv u_0 = \frac{1}{p_0}[1 + e_0 \cos(\theta - \varpi_0)] \quad \ldots\ldots(1)$$

where we use r_0 and u_0 to refer to the osculating ellipse. In (1),

$$p_0 = a_0(1 - e_0^2) \quad \ldots\ldots(2)$$

in which a_0 and e_0, together with ϖ_0 in (1), are the osculating elements. If H_0 denotes the areal constant for the osculating ellipse, then

$$H_0 = r_0^2\dot{\theta} \quad \ldots\ldots(3)$$

and

$$p_0 = \frac{H_0^2}{\mu} \quad \ldots\ldots(4)$$

The definition of the osculating ellipse ordains that (i) the coordinates of P and (ii) the components of velocity are the same in the osculating ellipse and in the true orbit.

Consider (i). Let θ_0 be the common vectorial angle at t_0; then $r_0(\theta_0) = r(\theta_0)$ or $u_0(\theta_0) = u(\theta_0)$. From (1) and 15·02 (6), we have

$$\frac{1}{p_0}[1 + e_0 \cos(\theta_0 - \varpi_0)] = \frac{1}{p}\left[1 + e\cos(\theta_0 - \varpi) + \frac{2c\theta_0}{h}\right] \quad \ldots\ldots(5)$$

Consider (ii). By (3), $H_0 = r_0^2\dot{\theta}$ and, by 15·02 (3), $H = r^2\dot{\theta}$; since $\dot{\theta}$ is the same for both orbits and $r = r_0$ at t_0, we have $H_0 = H(\theta_0)$ or, by 15·02 (4),

$$H_0 = h - c\theta_0 \quad ;$$

hence, to $O(c)$, by (4) and 15·02 (7),

$$p_0 = p\left(1 - \frac{2c\theta_0}{h}\right) \quad \ldots\ldots(6)$$

Again, by 15·02 (5), $\dfrac{du}{d\theta} = -\dfrac{\dot{r}}{H}$ and, similarly, for the osculating ellipse $\dfrac{du_0}{d\theta} = -\dfrac{\dot{r}_0}{H_0}$; hence, since $\dot{r}_0 = \dot{r}$ and $H = H_0$ at t_0, then

$$\frac{du_0}{d\theta} = \frac{du}{d\theta} \quad \text{when} \quad \theta = \theta_0 \quad \ldots\ldots(7)$$

From (1) and 15·02 (6) we then obtain

$$\frac{1}{p_0}[e_0 \sin(\theta_0 - \varpi_0)] = \frac{1}{p}\left[e\sin(\theta_0 - \varpi) - \frac{2c}{h}\right] \quad \ldots\ldots(8)$$

By means of (6), the formulae (5) and (8) become, to $O(c)$,

$$e_0 \cos(\theta_0 - \varpi_0) = e\cos(\theta_0 - \varpi)\left[1 - \frac{2c\theta_0}{h}\right] \quad , \quad \ldots\ldots(9)$$

$$e_0 \sin(\theta_0 - \varpi_0) = e\sin(\theta_0 - \varpi)\left[1 - \frac{2c\theta_0}{h}\right] - \frac{2c}{h} \quad \ldots\ldots(10)$$

Multiply (9) by $\cos(\theta_0 - \varpi)$ and (10) by $\sin(\theta_0 - \varpi)$, and add ; then

$$e_0 \cos(\varpi_0 - \varpi) = e\left[1 - \frac{2c\theta_0}{h}\right] - \frac{2c}{h}\sin(\theta_0 - \varpi) \quad \ldots\ldots\ldots\ldots(11)$$

Multiply (9) by $\sin(\theta_0 - \varpi)$ and (10) by $\cos(\theta_0 - \varpi)$, and subtract ; then

$$e_0 \sin(\varpi_0 - \varpi) = \frac{2c}{h}\cos(\theta_0 - \varpi) \quad \ldots\ldots\ldots\ldots(12)$$

Let $e_0 = e + \Delta_0 e$, $\varpi_0 = \varpi + \Delta_0 \varpi$; then, to $O(c)$, (11) and (12) become

$$\Delta_0 e = -\frac{2ce\theta_0}{h} - \frac{2c}{h}\sin(\theta_0 - \varpi) \quad \ldots\ldots\ldots\ldots(13)$$

and

$$e \,.\, \Delta_0 \varpi = \frac{2c}{h}\cos(\theta_0 - \varpi) \quad \ldots\ldots\ldots\ldots(14)$$

At some subsequent time when θ has increased from θ_0 to θ_1, let the new osculating elements be e_1 and ϖ_1, or $e + \Delta_1 e$ and $\varpi + \Delta_1 \varpi$. Then, by (13) and (14),

$$\Delta_1 e = -\frac{2ce\theta_1}{h} - \frac{2c}{h}\sin(\theta_1 - \varpi)$$

and

$$e \,.\, \Delta_1 \varpi = \frac{2c}{h}\cos(\theta_1 - \varpi) \quad .$$

If δe and $\delta \varpi$ denote the increase in the osculating elements while θ increases from θ_0 to θ_1 then, since $\delta e = \Delta_1 e - \Delta_0 e$,

$$\delta e = -\frac{2ce}{h}(\theta_1 - \theta_0) - \frac{2c}{h}\{\sin(\theta_1 - \varpi) - \sin(\theta_0 - \varpi)\} \quad \ldots\ldots\ldots(15)$$

and, similarly,

$$e\delta\varpi = \frac{2c}{h}\{\cos(\theta_1 - \varpi) - \cos(\theta_0 - \varpi)\} \quad \ldots\ldots\ldots\ldots(16)$$

Formula (15) shows that the variation of the eccentricity, corresponding to successive osculating orbits, consists of a secular term and periodic terms, and that during a revolution of the planet in its orbit—that is, when $\theta_1 - \theta_0 = 2\pi$—the eccentricity *diminishes* by $4\pi ce/h$. The effect of the resisting medium is thus to ensure a continuous reduction of the eccentricity of the osculating ellipses during successive orbital periods.

From (16), the variation in ϖ is purely periodic and is zero over an orbital period.

15·04. Variation of a and n

To investigate the nature of the variation in the semi-major axis of the osculating ellipse, we have, from (2) and (6) of the previous section,

$$p_0 \equiv a_0(1 - e_0^2) = p\left(1 - \frac{2c\theta_0}{h}\right)$$

in which p ($\equiv h^2/\mu$) is a constant. Write $p=a(1-e^2)$, where a is a constant. Then, if $a_0=a+\Delta_0 a$, we find that, to $O(c)$,

$$(1-e^2)\Delta_0 a=2ae\Delta_0 e-\frac{2ac\theta_0}{h}(1-e^2)$$

from which

$$\frac{\Delta_0 a}{a}=-\frac{2c\theta_0}{h}\left(\frac{1+e^2}{1-e^2}\right)-\frac{4ce}{1-e^2}\sin(\theta_0-\varpi) \quad .$$

We have a similar formula for $\Delta_1 a$. Hence in one period, the variation of the semi-major axis is given by

$$\delta a\equiv\Delta_1 a-\Delta_0 a=-\frac{4\pi ca}{h}\left(\frac{1+e^2}{1-e^2}\right) \quad .$$

Accordingly, the semi-major axis of the osculating ellipse decreases through successive orbital periods.

Since $\mu=n_0^2a_0^3$ the variation δn of the mean angular motion is given by $\delta n=-\frac{3}{2}\frac{n}{a}\delta a$ in which a and n are constants; consequently, the mean angular motion increases by $\frac{6\pi nc}{h}\left(\frac{1+e^2}{1-e^2}\right)$ during each orbital period.

15·05. General equations for $\dot{a}$ etc.

In this section we shall suppose that the resistance is given by the general expression in 15·01 (1). We denote the outward radial component and the transverse component by S' and T' respectively; then

$$S'=-R\frac{\dot{r}}{v}, \quad T'=-R\frac{r\dot{\theta}}{v} \quad . \qquad \ldots\ldots\ldots\ldots(1)$$

In 14·04 we have derived the formulae for $\dot{a}, \dot{e}, \ldots$ in terms of the components S, T and W of the attraction of the disturbing planet of mass m_1. Associated with S, T and W is the factor Gm_1 and it is evident that we can use the general equations in 14·04 by replacing Gm_1S and Gm_1T by S' and T' and putting $W=i=0$.

The relevant equations are as follows (we replace b^2 by ap and $\sqrt{(1-e^2)}$ by $\cos\phi$, where $\sin\phi=e$):

$$\dot{a}=\frac{2}{n\cos\phi}\left[S'e\sin f+\frac{p}{r}T'\right] \quad , \qquad \ldots\ldots\ldots\ldots(2)$$

$$\dot{e}=\frac{\cos\phi}{na}[S'\sin f+T'(\cos E+\cos f)] \quad , \qquad \ldots\ldots\ldots\ldots(3)$$

$$e\dot{\varpi}=\frac{\cos\phi}{na}\left[-S'\cos f+T'\left(1+\frac{r}{p}\right)\sin f\right] \quad , \qquad \ldots\ldots\ldots\ldots(4)$$

$$\dot{\epsilon}=\frac{e}{1+\cos\phi}(e\dot{\varpi})-\frac{2}{na^2}rS' \quad . \qquad \ldots\ldots\ldots\ldots(5)$$

The equation of the ellipse is

$$\frac{p}{r} = 1 + e\cos f \quad \ldots\ldots(6)$$

where $p = b^2/a = a(1-e^2) = a\cos^2\phi$.

Also, $r^2\dot{\theta} \equiv r^2\dot{f} = h$, where $h^2 = \mu p = n^2a^4\cos^2\phi$.

From (6), $p\dot{r} = r^2 e \sin f \,.\, \dot{f}$, from which $\dot{r} = \dfrac{eh}{p}\sin f$.

Then $$v^2 \equiv \dot{r}^2 + r^2\dot{\theta}^2 = \frac{h^2}{p^2}(1 + 2e\cos f + e^2) \quad ,$$

or $$v = \frac{h}{p}U \quad , \quad \ldots\ldots(7)$$

where $$U^2 = 1 + 2e\cos f + e^2 \quad . \quad \ldots\ldots(8)$$

The equations (2)–(5) then become, on using (1) and the formulae for $\dot{r}$ etc. just found,

$$\frac{\dot{a}}{a} = -\frac{2}{1-e^2}\cdot\frac{R}{v}U^2 \quad , \quad \ldots\ldots(9)$$

$$\dot{e} = -2\frac{R}{v}(e + \cos f) \quad , \quad \ldots\ldots(10)$$

$$e\dot{\varpi} = -2\frac{R}{v}\sin f \quad , \quad \ldots\ldots(11)$$

$$\dot{\epsilon} = -2\frac{R}{v}e\sin f\left(\frac{1}{1+\cos\phi} - \frac{r}{p}\cos\phi\right) \quad . \quad \ldots\ldots(12)$$

15·06. Variations of a etc. when $R = cv^\alpha r^{-\beta}$

To make further progress it is now assumed that the form of R is given by

$$R = cv^\alpha r^{-\beta} \quad . \quad \ldots\ldots(1)$$

Then, for example, $$\frac{\dot{a}}{a} = -\frac{2c}{1-e^2}v^{\alpha-1}r^{-\beta}U^2 \quad .$$

Now, $$\frac{da}{df} = \frac{\dot{a}}{\dot{f}} = \frac{r^2}{h}\dot{a} \quad ;$$

hence $$\frac{da}{df} = -\frac{2acU^2}{1-e^2}\left(\frac{v^{\alpha-1}r^{2-\beta}}{h}\right) \quad .$$

The expression within the bracket is easily seen to be $KU^{\alpha-1}V^{\beta-2}$ where, as before,

$$U^2 = 1 + 2e\cos f + e^2 \quad , \quad \ldots\ldots(2)$$

and $$K = h^{\alpha-2}p^{3-\alpha-\beta} \quad , \quad \ldots\ldots(3)$$

$$V = 1 + e\cos f \quad . \quad \ldots\ldots(4)$$

The equations (9)–(12) of the previous section become

$$\frac{da}{df} = -\frac{2caK}{1-e^2}U^{\alpha-1}V^{\beta-2}\,.\,U^2 \quad , \quad \ldots\ldots(5)$$

$$\frac{de}{df} = -2cKU^{\alpha-1}V^{\beta-2}(e+\cos f) \quad , \qquad \text{.....(6)}$$

$$e\frac{d\varpi}{df} = -2cKU^{\alpha-1}V^{\beta-2}\sin f \quad , \qquad \text{.....(7)}$$

$$\frac{d\epsilon}{df} = -2cKU^{\alpha-1}V^{\beta-2}\left(\frac{1}{1+\cos\phi} - \frac{\cos\phi}{U^2}\right)e\sin f \quad . \qquad \text{.....(8)}$$

Now, $U^{\alpha-1}V^{\beta-2}$ is an even function which can be expressed by the Fourier expansion

$$A_0 + A_1\cos f + A_2\cos 2f + \ldots$$

where

$$A_0 = \frac{1}{\pi}\int_0^\pi U^{\alpha-1}V^{\beta-2}\,df \quad , \qquad \text{.....(9)}$$

$$A_1 = \frac{2}{\pi}\int_0^\pi U^{\alpha-1}V^{\beta-2}\cos f\,df \quad , \qquad \text{.....(10)}$$

etc., Hence, from (5),

$$\frac{da}{df} = -\frac{2caK}{1-e^2}[(A_0 + A_1\cos f + A_2\cos 2f + \ldots)(1 + 2e\cos f + e^2)]$$

$$\equiv -\frac{2caK}{1-e^2}[C + C_1\cos f + C_2\cos 2f + \ldots]$$

where $C = A_0(1+e^2) + eA_1$.

It is evident that we can write C as

$$C = \frac{1}{\pi}\int_0^\pi U^{\alpha+1}V^{\beta-2}\,df \quad ; \qquad \text{.....(11)}$$

since U and V are positive for all values of e and f, it follows that C is positive whatever the values of α and β may be. The same is true for A_0.

Now, the true anomaly f is expressed in terms of the mean anomaly M by the equation of the centre

$$f = M + (2e - \tfrac{1}{4}e^3)\sin M + \ldots \quad ;$$

hence

$$df = n\,dt[1 + P.T.] \quad .$$

Accordingly, if Δa denotes the variation in a during time t,

$$\Delta a = -\frac{2canK}{1-e^2} \,.\, Ct + P.T. \qquad \text{.....(12)}$$

Since C is positive for all values of α and β, the semi-major axis decreases when R is defined by (1).

Similarly, from (6),

$$\frac{de}{df} = -2cK[A_0 + A_1\cos f + \ldots][e + \cos f]$$

$$= -2cK[D + D_1\cos f + \ldots]$$

where

$$D = eA_0 + \tfrac{1}{2}A_1 \quad ; \qquad \text{.....(13)}$$

then

$$\Delta e = -2cnK \,.\, Dt + P.T. \qquad \text{.....(14)}$$

We now examine the sign of D. Integrate (10) by parts ; then

$$\frac{\pi}{2} A_1 = \Big[U^{\alpha-1} V^{\beta-2} \sin f \Big]_0^\pi + e \int_0^\pi U^{\alpha-3} V^{\beta-3} [(\alpha-1)V + (\beta-2)U^2] \sin^2 f \, df \ .$$

The integrated part vanishes and the integral is positive whatever the values of e and f may be if $\alpha \geqslant 1$ and $\beta \geqslant 2$. Since A_0 is positive, so also is D, given by (13) ; accordingly, the eccentricity decreases if α and β have the restrictions on their values just mentioned.

It is evident from (7) and (8) that $\Delta\varpi$ and $\Delta\epsilon$ are purely periodic.

15·07. Discussion when the eccentricity is small

We shall suppose that, as in the case of many of the planetary orbits, e is so small that e^2 can be neglected. Then

$$U^2 = 1 + 2e\cos f \ , \quad U^{\alpha-1} = 1 + (\alpha-1)e\cos f \ , \quad V^{\beta-2} = 1 + (\beta-2)\, e\cos f \ .$$

Hence,

$$\pi A_0 = \int_0^\pi [1 + (\alpha+\beta-3)e\cos f] \, df$$

and

$$\pi A_1 = 2\int_0^\pi [1 + (\alpha+\beta-3)e\cos f] \cos f \, df \ ;$$

from these, $A_0 = 1$ and $A_1 = (\alpha+\beta-3)e$. Hence, to $O(e)$, $C = 1$ and, by 15·06 (13), $D = \frac{1}{2}(\alpha+\beta-1)e$. Accordingly, Δe is negative if $\alpha+\beta > 1$, which is a much less severe restriction than in the general case discussed in the previous section.

15·08. Encke's hypothesis

Encke assumed that $R = cv^2/r^2$ or, in our notation, that $\alpha = 2$ and $\beta = 2$; these values are in conformity with the restrictions on the values of α and β found in 15·06.

From 15·06 (11), $\pi C = \int_0^\pi U^3 \, df$, or, by 15·05 (7), $\pi C = \dfrac{p^3}{h^3} \int_0^\pi v^3 \, df$.

We express the integral in terms of the eccentric anomaly E as independent variable. Then

$$df = \frac{\dot{f}}{\dot{E}} dE = \frac{h}{r^2} \cdot \frac{r}{a} \cdot \frac{dE}{n} = \frac{h}{na^2} \left(\frac{1 + e\cos E}{1 - e^2\cos^2 E} \right) dE \ .$$

Also,

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) = \frac{n^2 a^3}{r} (1 + \cos E) = \frac{n^2 a^2 (1 + \cos E)^2}{1 - e^2 \cos^2 E} \ .$$

Hence,

$$\pi C = (1 - e^2)^2 \int_0^\pi \frac{(1 + e\cos E)^4}{(1 - e^2\cos^2 E)^{5/2}} \, dE \ .$$

If the numerator of the integrand is expanded it is evident that odd powers of $\cos E$ make no contribution to the integral. Put $E = \pi/2 - \theta$ and

$$X^2 = 1 - e^2\cos^2 E = 1 - e^2 \sin^2 \theta \ ;$$

then

$$\frac{\pi C}{2(1-e^2)^2} = \int_0^{\pi/2} \frac{1+6e^2\sin^2\theta+e^4\sin^4\theta}{X^5}\,d\theta \quad . \quad \ldots\ldots\ldots\ldots(1)$$

The numerator of the integrand is $[1+6(1-X^2)+(1-X^2)^2]$; hence, if I denotes the integral, then

$$I = \int_0^{\pi/2} \left(\frac{8}{X^5} - \frac{8}{X^3} + \frac{1}{X}\right) d\theta \quad . \quad \ldots\ldots\ldots\ldots(2)$$

Let $$Z = \frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{X^{r-1}}\right) \quad \text{and} \quad \alpha = e^2 \quad ,$$

so that $X^2 = 1-\alpha\sin^2\theta$ and $X\dfrac{\partial X}{\partial\theta} = -\alpha\sin\theta\cos\theta$. Then

$$Z = \frac{1-2\sin^2\theta}{X^{r-1}} + \frac{\alpha(r-1)}{X^{r+1}}\sin^2\theta\cos^2\theta$$

and, consequently,

$$\alpha Z = \frac{\alpha-2(1-X^2)}{X^{r-1}} + \frac{r-1}{X^{r+1}}[\alpha(1-X^2)-(1-X^2)^2]$$
$$= \frac{(\alpha-1)(r-1)}{X^{r+1}} - \frac{(\alpha-2)(r-2)}{X^{r-1}} - \frac{(r-3)}{X^{r-3}} \quad .$$

Integrate from 0 to $\pi/2$; then, denoting $\int_0^{\pi/2} \dfrac{d\theta}{X^r}$ by I_r we obtain, since $\int_0^{\pi/2} Z\,d\theta = 0$,

$$(r-1)(1-e^2)I_{r+1} - (r-2)(2-e^2)I_{r-1} + (r-3)I_{r-3} = 0 \quad .$$

When $r=2$, then

$$(1-e^2)I_3 = I_{-1} = E(e) \quad , \quad \ldots\ldots\ldots\ldots(3)$$

where $$E(e) = \int_0^{\pi/2} (1-e^2\sin^2\theta)^{1/2}\,d\theta \quad .$$

When $r=4$, then

$$3(1-e^2)I_5 - 2(2-e^2)I_3 + I_1 = 0$$

and, from the previous result,

$$I_5 = \frac{2(2-e^2)}{3(1-e^2)^2}E(e) - \frac{1}{3(1-e^2)}F(e) \quad , \quad \ldots\ldots\ldots\ldots(4)$$

where $$F(e) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} \quad .$$

E and F are the Lagrangian elliptic functions.

From (1), (2), (3) and (4), C is given by

$$3\pi C = 16(1+e^2)E(e) - 2(1-e^2)(5+3e^2)F(e) \quad . \quad \ldots\ldots\ldots\ldots(5)$$

Similarly,

$$\pi D = \int_0^{\pi} U(e+\cos f)\,df = (1-e^2)^2 \int_0^{\pi} \frac{(1+e\cos E)^3\cos E}{X^5}\,dE$$

$$= 2(1-e^2)^2 \int_0^{\pi/2} \frac{(3e\cos^2 E + e^3 \cos^4 E)}{X^5}\, dE = \frac{2}{e}(1-e^2)^2[4I_5 - 5I_3 + I_1]$$

$$= \frac{2}{3e}[(1+7e^2)E(e) - (1-e^2)(1+3e^2)F(e)] \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

15·09. Application to Encke's comet

With the value of 0·85 for e—the approximate value for Encke's Comet—Tisserand* found that, in our notation, $C/2D = 0{\cdot}97$ and, further, that if α and β were given integral values greater than 1 and 2 respectively, then the ratio $C : 2D$ remained substantially unaltered.

When the secular terms in 15·06 (12) and (14) are alone considered,

$$\frac{1}{a}\frac{\Delta a}{\Delta e} = \frac{1}{1-e^2} \cdot \frac{C}{D} \quad .$$

Also,
$$2\frac{\Delta n}{n} = -3\frac{\Delta a}{a}$$

and, since $\sin\phi = e$, then $\Delta e = \cos\phi\, \Delta\phi$. We then obtain

$$\frac{\Delta n}{n} = -\frac{3}{1-e^2} \cdot \frac{C}{2D} \cos\phi\, \Delta\phi \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

According to V. Asten† the observations of Encke's Comet between 1819 and 1865 showed that

$$\Delta n = 0''{\cdot}1044, \quad \Delta\phi = -3''{\cdot}68$$

per revolution. Since $n = 1070''$, $\phi = 57°\ 49'$, the formula (1) gave

$$\frac{C}{2D} = 0{\cdot}97 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Also,
$$\frac{\Delta n}{n} = 9{\cdot}7 \,.\, 10^{-5} \quad , \quad \Delta e = -9{\cdot}4 \,.\, 10^{-6} \quad . \quad \ldots\ldots\ldots\ldots(3)$$

Although the result in (2) agrees with the theoretical value when $R = cv^2/r^2$, the similarity of the results when α and β are both greater than 2 does not make any conclusion possible as to the law governing the assumed resistance R.

Later observations of the comet show that the average value of $\Delta n/n$ for the interval 1865–1901 is practically identical with the average value for the interval 1901–1934; this value is $4{\cdot}2 \,.\, 10^{-5}$ which is rather less than half that given in (3). The value of $\Delta\phi$ for each of the two later intervals is $-1''{\cdot}6$ and the corresponding value of Δe is $-4 \,.\, 10^{-6}$.

The only known envelopes surrounding the Sun are, as mentioned in 15·01, the solar corona and the matter associated with the Zodiacal Light. Some years ago, Jeffreys‡ made an estimate of the densities of these

* *Mécanique Céleste*, IV, 223 (1896).
† Tisserand, *Mécanique Céleste*, IV, 226 (1896).
‡ M.N.R.A.S., **80**, 138 (1919).

envelopes in connection with a possible explanation of the part of the motion of Mercury's perihelion unaccounted for by gravitational theory but explicable by the theory of relativity, as we shall show in section 15·11. For the first envelope the density was shown to be about

$$3 \cdot 10^{-12} r^{-5} \text{ gm./cm.}^3,$$

where r is the heliocentric distance in terms of the Sun's radius as unit ; for the second envelope the average density was stated to be about $1{\cdot}5 \cdot 10^{-18}$ gm./cm.^{3}

The heliocentric distance of Encke's Comet at perihelion is easily found from the known period of about $3\frac{1}{4}$ years and the value of the eccentricity already mentioned ; it is one-third of an astronomical unit or, say, $30 \cdot 10^6$ miles—a distance far exceeding the dimensions of the corona. The exiguous density of the second and more extensive envelope suggests that, even if the comet passed through the medium, the resisting effect would be far too small to be detected.

Although the changes in n and e are undoubtedly real—the change in n may be interpreted in the sense that the expression for the mean longitude contains an acceleration term of the form σt^2—it would seem that the hypothesis of a resisting medium, introduced to account for the observed phenomena associated with Encke's Comet, must be abandoned. This conclusion has been enormously strengthened in recent years by investigations on other comets in some of which $\Delta n/n$ is found to be *negative* and Δe *positive*, that is, of opposite signs to those found for Encke's Comet and one or two others.

In the next section we describe briefly a recent investigation which suggests an explanation for the diverse characteristics of the cometary mean motions.

15·10. Whipple's Model of a comet

In the preceding sections we have tacitly assumed that a comet is a point-mass and that the mass is constant. This is very far from being the case. The existence of an extensive tail is almost conclusive evidence that cometary matter is escaping into space ; accordingly, the dynamical problem must take account of the continuous diminution of mass especially when the comet comes within effective distance of the solar radiation. In a recent investigation,* F. L. Whipple begins with the conception of a cometary nucleus consisting of a loose aggregation of meteoric matter and such compounds in the solid state as water, carbon dioxide, ammonia, etc. which the spectroscope reveals as constituents ; these he refers to, generically, as " ices ". As the comet approaches perihelion the solar radiation will vaporise the " ices ", the molecules of which will be dissipated into space with kinetic velocities appropriate to the

* *Ap. J.*, 111, 375 (1950).

temperatures prevailing, the velocity of escape from the feebly gravitational nucleus being practically negligible. The general effect is a non-uniform recoil action over the nucleus. Two necessary features of the theory are, first, the slow rotation of the nucleus and, second, a time lag in the transfer of heat through the outer layers to the " ices ". The general direction of the recoil-action will then be dependent on the *sense* of rotation of the nucleus ; a retrograde rotation will be associated with an acceleration of the mean motion as for Encke's Comet, and a forward rotation with a retardation of the mean motion as observed for other comets.

15·11. The secular motion of Mercury's perihelion

A noteworthy discrepancy between observation and the results of planetary theory based on the Newtonian law of gravitation relates to the motion of the perihelion of Mercury. The application of the planetary equations gives the secular term for the perturbation in ϖ, the coefficient of this term being about 43″ per century less than that deduced from observations. Many attempts, invoking all sorts of hypotheses, including that of a resisting medium, had failed to elucidate the discrepancy when confronted with the observational information relating to the particular assumptions introduced. The discrepancy has been cleared up, within the limits of observational error, by the theory of relativity. The method of procedure is similar to that adopted in 15·03.

With the notation of 15·02, the differential equation for the orbit of a planet revolving around the Sun is, on relativity theory, given* by

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}+\alpha u^2 \quad \text{.....(1)}$$

where

$$\alpha=\frac{3\mu}{c^2} \quad , \quad \text{.....(2)}$$

in which c is the velocity of light ; α is a small quantity, the squares and higher powers of which can be neglected.

For a first approximation, α being omitted in (1), the solution is

$$u=\frac{1}{p}[1+e\cos(\theta-\varpi)] \quad \text{.....(3)}$$

in which e and ϖ are constants and $p=h^2/\mu$.

Inserting this expression for u in the term αu^2 in (1), we have

$$\frac{d^2u}{d\theta^2}+u=\frac{1}{p}+\frac{\alpha_1}{p}+\frac{2\alpha e}{p^2}\cos(\theta-\varpi)+\frac{\alpha_2}{p}\cos 2(\theta-\varpi) \quad \text{.....(4)}$$

where

$$\alpha_1=\frac{\alpha}{p}(1+\tfrac{1}{2}e^2) \quad \text{and} \quad \alpha_2=\tfrac{1}{2}\frac{\alpha e^2}{p} \quad .$$

* Eddington, *The Mathematical Theory of Relativity*, p. 88 (1923).

If the solution of (4) is denoted by $u+\Delta u$, where u is given by (3), then

$$\frac{d^2}{d\theta^2}(\Delta u)+\Delta u=\frac{\alpha_1}{p}+\frac{2\alpha e}{p^2}\cos(\theta-\varpi)+\frac{\alpha_2}{p}\cos 2(\theta-\varpi) \quad .$$

All that we require is the particular integral of this equation which is easily found by elementary methods to be given by

$$\Delta u=\frac{\alpha_1}{p}+\frac{\alpha e}{p^2}\theta\sin(\theta-\varpi)-\tfrac{1}{3}\frac{\alpha_2}{p}\cos 2(\theta-\varpi) \quad .$$

The complete solution of (1), to $O(\alpha)$, is then

$$u=\frac{1}{p}[1+e\cos(\theta-\varpi)]+\frac{\alpha_1}{p}+\frac{\alpha e}{p^2}\theta\sin(\theta-\varpi)-\tfrac{1}{3}\frac{\alpha_2}{p}\cos 2(\theta-\varpi) \quad \ldots.(5)$$

Let
$$u_0=\frac{1}{p_0}[1+e_0\cos(\theta-\varpi_0)] \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

be the equation of the osculating ellipse for the epoch given by $\theta=\theta_0$. Then, by the principles of 15·03,

$$u_0(\theta_0)=u(\theta_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

and
$$\frac{du_0}{d\theta}=\frac{du}{d\theta} \quad , \quad \text{when} \quad \theta=\theta_0 \quad . \ldots\ldots\ldots\ldots(8)$$

Now, p_0 will differ from p by a quantity of order α; we can then write $\frac{p_0}{p}=1+k\alpha$, where k is a constant. The application of (7) to (6) and (5) then gives, to $O(\alpha)$,

$$e_0\cos(\theta_0-\varpi_0)$$
$$=k_1\alpha+e(1+k\alpha)\cos(\theta_0-\varpi)+\frac{\alpha e}{p}\theta_0\sin(\theta_0-\varpi)-\tfrac{1}{3}\alpha_2\cos 2(\theta_0-\varpi)\ldots(9)$$

where k_1 is a constant.

The application of (8) to (6) and (5) gives, in a similar way,

$$e_0\sin(\theta_0-\varpi_0)$$
$$=e(1+k_2\alpha)\sin(\theta_0-\varpi)-\frac{\alpha e}{p}\theta_0\cos(\theta_0-\varpi)-\tfrac{2}{3}\alpha_2\sin 2(\theta_0-\varpi) \quad \ldots\ldots(10)$$

where k_2 is another constant.

Multiply (9) by $\sin(\theta_0-\varpi)$ and (10) by $\cos(\theta_0-\varpi)$, and subtract; then

$$e_0\sin(\varpi_0-\varpi)=\frac{\alpha e}{p}\theta_0+\alpha\,\Sigma\,C_i\sin i(\theta_0-\varpi) \quad \ldots\ldots\ldots\ldots(11)$$

where $i=1$, 2 and 3 and the C's are constants.

Multiply (9) by $\cos(\theta_0-\varpi)$ and (10) by $\sin(\theta_0-\varpi)$ and add; then

$$e_0\cos(\varpi_0-\varpi)=e+\kappa\alpha+\alpha\,\Sigma\,D_i\cos i(\varpi_0-\varpi) \quad \ldots\ldots\ldots\ldots(12)$$

Write
$$\varpi_0-\varpi=\Delta_0\varpi \quad , \quad e_0-e=\Delta_0 e \quad .$$

From (11) and (12) it is seen at once that $\Delta_0\varpi$ and $\Delta_0 e$ are of order α and further that $\Delta_0 e$ is purely periodic.

To $O(\alpha)$, (11) can be written as

$$\Delta_0 \varpi = \frac{\alpha}{p}\theta_0 + P.T. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

If e_1 and ϖ_1 are the osculating elements at a subsequent epoch given by θ_1, then

$$\Delta_1 \varpi = \frac{\alpha}{p}\theta_1 + P.T. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Further, if $\theta_1 = \theta_0 + 2\pi$, the formulae (13) and (14) show that the osculating element ϖ_0 has increased by $\frac{2\pi\alpha}{p}$ in one orbital period. If λ denotes the coefficient of the secular term resulting from this effect, measured in seconds of arc per century, then

$$\lambda = \frac{2\pi\alpha}{p} \cdot \frac{100}{T} \text{ cosec } 1''$$

in which T is the orbital period in years; or, using (2) and putting $\mu = n^2a^3 = \frac{4\pi^2}{T^2}a^3$ and $p \equiv h^2/\mu = a(1-e^2)$, then

$$\lambda = \frac{2400\pi^3 a^2 \text{ cosec } 1''}{c^2 T^3 (1-e^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Now, $c = 186{,}271$ miles/sec. $= 5{\cdot}88 \cdot 10^{12}$ miles/year

$$= \frac{5{\cdot}88 \cdot 10^6}{93} \text{ A.U./year.}$$

If a is expressed in A.U. and T in years, then $a^3 = T^2$ when planetary masses are neglected. Hence (15) becomes, on inserting the numerical values of c, π and cosec $1''$,

$$\lambda = \frac{3{\cdot}841}{(1-e^2)aT} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

For Mercury, $a = 0{\cdot}3871$, $T = 0{\cdot}2408$ and $e = 0{\cdot}2056$. Hence $\lambda = 43''{\cdot}0$ per century, which agrees very closely with the defect between observation and Newtonian theory.

Actually, the theoretical quantity to be compared with observations is not λ but $e\lambda$. The following table gives the calculated values of $e\lambda$ and λ for the four planets nearest the Sun.

	Mercury	*Venus*	*Earth*	*Mars*
$e\lambda$ - -	8″·82	0″·06	0″·06	0″·12
λ - - -	43·03	8·63	3·84	1·34

As has been mentioned, the relativity effect has been confirmed in the case of Mercury; recently, a similar confirmation has been obtained in the

case of the Earth despite the fact that the observed quantity, $e\lambda$ per century, is no more than $0''{\cdot}06$. Since the observed effect increases with the time, it may be anticipated that Mars at least will furnish confirmation in due course. Formula (16) shows that for Jupiter and the still more distant planets the calculated values of λ are at most only a few hundredths of a second of arc ; the values of $e\lambda$ are practically negligible.

Chapter 16

THE DISCOVERY OF NEPTUNE

16·01. Introduction

The planet Uranus was discovered on March 13, 1781, by W. Herschel during one of his systematic sweeps of the heavens. The other planets nearer to the Sun than Uranus had been known from remote antiquity and Herschel's achievement ushered in a fruitful era of astronomical discovery to which he himself made the most notable contributions. As the new planet was little fainter than stars just visible to the naked eye it occurred to the German astronomer Bode that possibly Uranus had been mistaken for a star in the past, in which event its positions would be recorded in the contemporary catalogues. When the elements of the new planet's orbit had been derived with sufficient exactitude for the purpose of calculating its positions in the years preceding its discovery, a search of the older catalogues proved more successful than had been anticipated, for Uranus had been observed on no fewer than 19 different occasions, the observation most remote in time being that made in 1690 by Flamsteed, the first Astronomer Royal.

Towards the end of the second decade of the nineteenth century the French astronomer Bouvard undertook the task of deriving the elements of the new planet's orbit ; he had, of course, at his disposal a large number of accurate observations of the planet's positions in the post-discovery period—these are referred to as the " modern " observations—and also the 19 observations alluded to, these being designated " ancient " observations. Most of these latter observations were undoubtedly less accurate than their " modern " counterparts ; but they had the advantage of covering rather more than one complete revolution of the planet in its orbit, while the " modern " observations covered rather less than half a revolution. Bouvard immediately encountered difficulties of an unexpected nature ; when he used only the " modern " observations for determining the elements he found that the differences between the calculated and observed positions of Uranus in the pre-discovery years were so excessively large that no reasonable observational error could account for them ; further, when he used only the positions given by the " ancient " observations to calculate the elements, there were serious discrepancies between the calculated and observed positions of the " modern " series which could not possibly be attributed to observational error. Distrusting the " ancient " observations, Bouvard based his tables of Uranus, published in 1821, on the orbital elements derived from the

" modern " observations alone. This procedure, of course, did not solve the mystery surrounding the " ancient " observations and when, shortly after 1821, the planet began to deviate more and more from its predicted positions—in 1846 the discrepancy in longitude amounted to 2′—astronomers were confronted with a challenge which did not long remain unanswered.

While still an undergraduate at St. John's College, Cambridge, John Couch Adams penned his celebrated memorandum of July 3, 1841, in which he recorded his determination, as soon as he had taken his degree, to investigate the irregularities in the motion of Uranus on the hypothesis that these arose from the perturbations of an unknown planet. By 1845 Adams had obtained a solution—between 1843 and 1845 he had made six solutions of ever-increasing accuracy—which would undoubtedly have led to the visual detection of the unknown planet had telescopic observations been undertaken at once. In 1845, the rising young French astronomer, U. J. J. Le Verrier, embarked on a similar investigation, independent of and unknown to Adams. By different analytical methods, both succeeded in solving the difficult problem of deducing the position, at a specified date, of the hypothetical planet. Although there is not the slightest doubt that Adams reached his conclusions considerably earlier than Le Verrier—after needless delay Challis had begun a laborious search for the planet at Cambridge Observatory in the July preceding the planet's discovery—Le Verrier was the first to publish his results, and it was on his initiative that the search for the planet was undertaken by Galle at Berlin Observatory with immediate success. The planet, later named Neptune, was discovered on September 23, 1846, close to the positions assigned by Adams and Le Verrier. The history* of the discovery of Neptune and the subsequent controversies form one of the most remarkable episodes in the history of astronomy ; but when the wordy battle had subsided the claims on behalf of Adams as co-discoverer with Le Verrier were universally admitted.

In this chapter we give an account of the procedures adopted by Adams and Le Verrier in their investigations. It will be convenient here to make one or two remarks. First, the discrepancies in the *mean* longitudes of Uranus formed the basis of Adams' work, while the discrepancies in *true* longitude were utilised by Le Verrier. Second, since the inclinations of the orbits of the planets then known were all very small—that for Uranus, for example, is $\frac{3}{4}^\circ$ approximately—both astronomers assumed that the same applied to the hypothetical planet ; accordingly, neglecting the inclinations of Uranus and Neptune, they supposed that both planets

* For a detailed account see the author's " John Couch Adams and the Discovery of Neptune ", *Occasional Notes of the Royal Astronomical Society*, No. 11, August 1947.

moved in the ecliptic ; the elements concerned, then, are a, e, ϵ and ϖ for Uranus and similar elements a_1, e_1, ϵ_1 and ϖ_1 for the unknown planet together with the mass m_1 of the latter ; also, since the mean motions n and n_1 are given by

$$n^2a^3 = G(m_0 + m) \quad , \quad n_1{}^2a_1{}^3 = G(m_0 + m_1)$$

in which m and m_1 can be neglected in comparison with the Sun's mass m_0, it is convenient to regard n and n_1, replacing a and a_1, as elements. Third, both astronomers began, for reasons to be stated later, by assuming that $a/a_1 = 1/2$, or alternatively, $n^2 = 8n_1{}^2$.

16·02. Effect of errors in the elements of Uranus on the true longitude

Since the observations of Uranus from which Bouvard derived the orbital elements were, unknown to him, affected by the perturbations of Neptune, the values of these elements on which his tables were based carried errors which we denote by Δn, Δe, $\Delta\epsilon$ and $\Delta\varpi$; the true elements are then $n + \Delta n$, $e + \Delta e$, $\epsilon + \Delta\epsilon$ and $\varpi + \Delta\varpi$.

Let λ denote the true longitude of Uranus, f the true anomaly and M the mean anomaly. We have

$$M = nt + \epsilon - \varpi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and the equation of the centre is, by 3·10 (11),

$$f - M = 2e \sin M + \tfrac{5}{4}e^2 \sin 2M$$

as far as e^2. Now, $\lambda = f + \varpi$; hence

$$\lambda = nt + \epsilon + 2e \sin M + \tfrac{5}{4}e^2 \sin 2M \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If $\Delta\lambda$ is the error in Bouvard's longitude λ arising from Δn etc., then, from (2),

$$\Delta\lambda = t\,\Delta n + \Delta\epsilon + (2e \cos M + \tfrac{5}{2}e^2 \cos 2M)\,\Delta M + (2 \sin M + \tfrac{5}{2}e \sin 2M)\,\Delta e \quad .$$

But, by (1), $\Delta M = t\,\Delta n + \Delta\epsilon - \Delta\varpi$; hence

$$\Delta\lambda = (t\,\Delta n + \Delta\epsilon)(1 + 2e \cos M + \tfrac{5}{2}e^2 \cos 2M) + \Delta e(2 \sin M + \tfrac{5}{2}e \sin 2M) - e\,\Delta\varpi(2 \cos M + \tfrac{5}{2}e \cos 2M) \quad . \quad \ldots (3)$$

We can write this last equation as

$$\Delta\lambda = \alpha_1\Delta n + \alpha_2\Delta\epsilon + \alpha_3\Delta e + \alpha_4\Delta\varpi \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which the coefficients α_1 , ... , α_4 can be easily calculated for each observation.

Let P_1 denote the perturbation in true longitude arising from the action of Neptune. If λ_0 and λ_c are, respectively, the observed true longitude of Uranus and the true longitude calculated from Bouvard's elements, then the equation of condition is $\lambda_0 - \lambda_c = \Delta\lambda + P_1$, or, on writing

$$c_1 = \lambda_0 - \lambda_c \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$c_1 = \alpha_1 \Delta n + \alpha_2 \Delta\epsilon + \alpha_3 \Delta e + \alpha_4 \Delta\varpi + P_1 \quad . \quad \ldots\ldots\ldots\ldots(6)$$

We shall consider the expression for P_1 later.

Equation (6) is the equation of condition used by Le Verrier; it is to be understood that for each observation the appropriate value of c_1 is known.

16·03. Effect of errors in the elements of Uranus on the mean longitude

We denote the mean longitude by l. Let δl denote the increment in mean longitude corresponding to an increment $\delta\lambda$ in true longitude. By Kepler's second law,

$$r^2 \dot{f} = h \quad \text{or} \quad nr^2 \frac{d\lambda}{dl} = h \quad ;$$

hence

$$\delta l = \frac{r^2}{a^2\sqrt{(1-e^2)}}\,\delta\lambda \quad . \quad \ldots\ldots\ldots\ldots(1)$$

Now, the discrepancies in the true longitude, which we have denoted by c_1 in 16·02 (5), are known; identifying $\delta\lambda$ with c_1 Adams used the corresponding discrepancies in the mean longitude as calculated by (1); we denote these latter by c; then

$$c = \frac{r^2}{a^2\sqrt{(1-e^2)}}\,c_1 \quad , \quad \ldots\ldots\ldots\ldots(2)$$

r being calculated for each observation from Bouvard's elements. The values of c, given by (2) are now supposed to be known.

Further, if Δl denotes the effect of the errors of the elements of Uranus corresponding to $\Delta\lambda$ then, by (1),

$$\Delta l = \frac{r^2}{a^2\sqrt{(1-e^2)}}\,\Delta\lambda \quad . \quad \ldots\ldots\ldots\ldots(3)$$

Neglecting e^3 and higher powers of e we find, since $r = a(1 - e\cos E)$ and $\cos E = \cos M - e\sin^2 M$ [or directly, from 3.11 (17)], that the coefficient of $\Delta\lambda$ in (3) is

$$1 + 2e^2 - 2e\cos M - \tfrac{1}{2}e^2\cos 2M \quad . \quad \ldots\ldots\ldots\ldots(4)$$

From (3), (4) and 16·02 (3) we obtain, after a little reduction and to the order aimed at,

$$\begin{aligned}\Delta l = {} & t\,\Delta n + \Delta\epsilon + 2e^2\,\Delta\varpi + \Delta e(2\sin M + \tfrac{1}{2}e\sin 2M) \\ & - e\,\Delta\varpi(2\cos M + \tfrac{1}{2}e\cos 2M) \quad \ldots\ldots\ldots\ldots(5)\end{aligned}$$

At this stage Adams incorporated the small correction $2e^2\,\Delta\varpi$ with $\Delta\epsilon$. Write $\epsilon - \varpi = \beta$; then β is a known angle; also, $M = nt + \beta$.

The formula (5) now becomes

$$\begin{aligned}\Delta l = {} & t\,\Delta n + \Delta\epsilon + \cos nt\,\{2\Delta e\sin\beta - 2e\,\Delta\varpi\cos\beta\} \\ & + \sin nt\,\{2\Delta e\cos\beta + 2e\,\Delta\varpi\sin\beta\} \\ & + \cos 2nt\,\{\tfrac{1}{2}e\,\Delta e\sin 2\beta - \tfrac{1}{2}e^2\,\Delta\varpi\cos 2\beta\} \\ & + \sin 2nt\,\{\tfrac{1}{2}e\,\Delta e\cos 2\beta + \tfrac{1}{2}e^2\,\Delta\varpi\sin 2\beta\} \quad . \quad \ldots\ldots\ldots\ldots(6)\end{aligned}$$

Let

$$\Delta x_1 = 2\Delta e \sin\beta - 2e\,\Delta\varpi\cos\beta \quad , \qquad \ldots\ldots(7)$$
$$\Delta y_1 = 2\Delta e \cos\beta + 2e\,\Delta\varpi\sin\beta \quad ; \qquad \ldots\ldots(8)$$

thus we can regard Δx_1 and Δy_1 as replacing Δe and $\Delta\varpi$. From (7) and (8) we have immediately

$$\Delta x_1\cos\beta + \Delta y_1\sin\beta = 2\Delta e\sin 2\beta - 2e\,\Delta\varpi\cos 2\beta \quad , \qquad \ldots\ldots(9)$$
$$\Delta x_1\sin\beta - \Delta y_1\cos\beta = -2\Delta e\cos 2\beta - 2e\,\Delta\varpi\sin 2\beta \quad . \qquad \ldots\ldots(10)$$

Let the coefficients of $\cos 2nt$ and $\sin 2nt$ in (6) be denoted by Δx_2 and Δy_2 respectively ; then, by (9) and (10),

$$\Delta x_2 = \tfrac{1}{4}e(\Delta x_1\cos\beta + \Delta y_1\sin\beta) \quad , \qquad \ldots\ldots(11)$$
$$\Delta y_2 = -\tfrac{1}{4}e(\Delta x_1\sin\beta - \Delta y_1\cos\beta) \quad , \qquad \ldots\ldots(12)$$

or

$$\Delta x_2 = \gamma_1\,\Delta x_1 + \gamma_2\,\Delta y_1 \quad , \qquad \ldots\ldots(13)$$
$$\Delta y_2 = -\gamma_2\,\Delta x_1 + \gamma_1\,\Delta y_1 \quad , \qquad \ldots\ldots(14)$$

in which the numerical values of γ_1 and γ_2 are supposed known. The equation (6) can now be written

$$\Delta l = t\,\Delta n + \Delta\epsilon + \Delta x_1\cos nt + \Delta y_1\sin nt + \Delta x_2\cos 2nt + \Delta y_2\sin 2nt \quad . \quad \ldots(15)$$

If P denotes the perturbation in mean longitude arising from the action of Neptune, the equation of condition as used by Adams is

$$c = t\,\Delta n + \Delta\epsilon + \Delta x_1\cos nt + \Delta y_1\sin nt + \Delta x_2\cos 2nt + \Delta y_2\sin 2nt + P \quad \ldots(16)$$

in which the value of c is a known angle for each observation of Uranus ; also, for each observation the coefficients of Δn, Δx_1, Δx_2, Δy_1, Δy_2 in (16) can easily be calculated.

16·04. Perturbation in mean longitude

The disturbing function R from which the perturbations in the orbital elements of Uranus are to be found is given by

$$R = m_1 B + m_1\,\Sigma\, C\cos\theta \quad , \qquad \ldots\ldots(1)$$

where

$$\theta = il + i_1 l_1 + k\varpi + k_1\varpi_1 \quad ; \qquad \ldots\ldots(2)$$

B and C are functions of a, a_1, e, e_1, and i, i_1, k, k_1 are positive or negative integers, including zero ; also, $l = \rho + \epsilon$, $l_1 = \rho_1 + \epsilon_1$, where

$$\rho = \int n\,dt \quad , \quad \rho_1 = \int n_1\,dt \quad .$$

As the nature of the problem requires the consideration of first-order perturbations only, we can write $nt + \epsilon$ and $n_1 t + \epsilon_1$ for l and l_1 respectively in (2) so that

$$\theta = i(nt + \epsilon) + i_1(n_1 t + \epsilon_1) + k\varpi + k_1\varpi_1 \quad . \qquad \ldots\ldots(3)$$

If $\Delta_1 l$ denotes the first-order perturbation in mean longitude, then

$$\Delta_1 l = \Delta_1\rho + \Delta_1\epsilon \quad . \qquad \ldots\ldots(4)$$

From 6·04 (5), $\ddot{\rho} = -\frac{3}{a^2}\frac{\partial R}{\partial \epsilon} = \frac{3m_1}{a^2}\Sigma iC \sin\theta$, from which and (3),

$$\Delta_1\rho = -\frac{3m_1}{a^2}\Sigma\frac{iC\sin\theta}{(in+i_1n_1)^2} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Also, by 6·03 (16), $\Delta_1\epsilon$ is of the form given by

$$\Delta_1\epsilon = \lambda t + \Sigma K \sin\theta \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The perturbation P in mean longitude is then the sum of the terms on the right-hand sides of (5) and (6); further, the secular term, λt, in (6) can be incorporated with the term $t\,\Delta n$ in 16·03 (16). We can then regard P as being expressed by periodic terms of the form given by

$$\rho = m_1 \Sigma E \sin\theta \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

We shall later give the details of the terms in (7) which were used by Adams in his equations of condition.

It is to be remarked that E is a function of a, a_1, e and e_1; both Adams and Le Verrier confined themselves to terms up to the first power of e and e_1 only. Further, a and a_1 enter into the coefficients E in such a complicated way that little progress could be achieved in solving the equations of condition if a_1 remained an unknown. As stated in 16·01, both astronomers invoked the aid of Bode's rule which appeared to represent with fair accuracy the relative mean heliocentric distances of the planets according to the formula $4+3\,.\,2^n$, where $n=0, 1, 2, \ldots, 6$ for Venus, Earth, Mars, the average for the minor planets, Jupiter, Saturn and Uranus respectively. When, on its discovery, Uranus was seen to fit into the rule satisfactorily it seemed reasonable —at least as a working hypothesis—to expect the unknown planet to conform also, with 7 as the value of n. In this event, $a/a_1 = \frac{1}{2}$ approximately. In their first solutions Adams and Le Verrier put simply $a_1 = 2a$, from which, with sufficient accuracy,

$$n_1 = n\,.\,2^{-3/2} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Accordingly, the factors of E in (7) which depend on a, a_1—and e—can be calculated explicitly; similarly, the coefficient of t in the expression for θ in (3) can be calculated by means of (8). Thus, the unknowns in the expression for P in (7) are reduced to four in number, namely, m_1, ϵ_1, e_1 and ϖ_1.

16·05. Perturbation in true longitude

It is convenient to give here the general form of the perturbation, P_1, in true longitude as used by Le Verrier in his solution with which we deal in 16·10. From 16·02 (2), the true longitude λ is given by

$$\lambda = l + 2e\sin(l-\varpi) + \tfrac{5}{4}e^2\sin 2(l-\varpi) \quad ,$$

since $M = l - \varpi$. With the notation of the previous section we obtain, since $P_1 = \Delta_1 \lambda$,

$$\begin{aligned} P_1 = {} & \Delta_1 l\{1 + 2e\cos(l-\varpi) + \tfrac{5}{2}e^2\cos 2(l-\varpi)\} \\ & + \Delta_1 e\{2\sin(l-\varpi) + \tfrac{5}{2}e\sin 2(l-\varpi)\} \\ & - e\,\Delta_1\varpi\{2\cos(l-\varpi) + \tfrac{5}{2}e\cos 2(l-\varpi)\} \quad . \end{aligned}$$

Now, by (7) of the previous section, since $\Delta_1 l \equiv P$,

$$\Delta_1 l = m_1 \Sigma E \sin\theta \quad .$$

By 6·03 (16) and (17), the periodic part of $\Delta_1 e$ and $\Delta_1 \varpi$ are given by expressions of the forms

$$m_1 \Sigma E' \cos\theta \quad \text{and} \quad m_1 \Sigma E'' \sin\theta$$

respectively ; the secular terms are incorporated with $t\,\Delta n$ as was done in the previous section with the secular term of $\Delta_1 \epsilon$.

Hence, P_1 is of the form

$$P_1 = m_1 \Sigma E_1 \sin\theta_1 \quad , \qquad \ldots\ldots(1)$$

where $\theta_1 = \theta \pm j(l-\varpi)$, in which $j = 0, 1, 2$. This expression for P_1 is now to be inserted in 16·02 (6) which is the equation of condition used by Le Verrier ; the details of (1) will be given later in 16·10.

16·06. Adams' method

We rewrite Adams' equation of condition in 16·03 (16) :

$$c = t\,\Delta n + \Delta\epsilon + \Delta x_1 \cos nt + \Delta y_1 \sin nt + \Delta x_2 \cos 2nt + \Delta y_2 \sin 2nt + P \quad ,$$

where P is given, in general form, by 16·04 (7) ; the terms of P used by Adams are :

$$\begin{aligned} P = {} & m_1 u_1 \sin(nt - n_1 t + \epsilon - \epsilon_1) + m_1 u_2 \sin\{2(nt - n_1 t + \epsilon - \epsilon_1)\} \\ & + m_1 u_3 \sin\{3(nt - n_1 t + \epsilon - \epsilon_1)\} + m_1 u_4 \sin(n_1 t + \epsilon_1 - \varpi) \\ & + m_1 e_1 u_5 \sin(n_1 t + \epsilon_1 - \varpi_1) + m_1 u_6 \sin(nt - 2n_1 t + \epsilon - 2\epsilon_1 + \varpi) \\ & + m_1 e_1 u_7 \sin(nt - 2n_1 t + \epsilon - 2\epsilon_1 + \varpi_1) \\ & + m_1 u_8 \sin(2nt - 3n_1 t + 2\epsilon - 3\epsilon_1 + \varpi) \\ & + m_1 e_1 u_9 \sin(2nt - 3n_1 t + 2\epsilon - 3\epsilon_1 + \varpi_1) \\ & + m_1 e_1 u_{10} \sin(3nt - 3n_1 t + 3\epsilon - 3\epsilon_1 - \varpi + \varpi_1) \quad \ldots\ldots(1) \end{aligned}$$

in which $u_1, u_2, \ldots$ are numerical coefficients (in seconds of arc) obtained by inserting Bouvard's values of a and e and the adopted value of a/a_1 in the corresponding terms of the above expression for P.

In his final solution Adams used a modified value of a/a_1—0·515—suggested by his earlier solutions. Also, for numerical convenience he defined m_1 and e_1 (with, of course, the requisite numerical adjustments) as 5000 times the mass of the unknown planet and as twenty times the eccentricity respectively. It is noteworthy that Le Verrier adopted a similar expedient as regards m_1, the factor in this case being 10,000.

The four unknowns relating to Neptune are m_1, e_1, ϵ_1 and ϖ_1 which appear explicitly in the expression (1) for P. We can write P as follows :

$$\begin{aligned} P = &\sum_{i=1}^{3} (h_i \cos i\theta_1 + k_{1i} \sin i\theta_1) \\ &+ p_1 \cos \theta_2 + p_2 \cos \theta_3 + p_3 \cos \theta_4 \\ &+ q_1 \sin \theta_2 + q_2 \sin \theta_3 + q_3 \sin \theta_4 \quad \ldots\ldots(2) \end{aligned}$$

where $\theta_1 = (n - n_1)t$, $\theta_2 = n_1 t$, $\theta_3 = (n - 2n_1)t$, $\theta_4 = (2n - 3n_1)t$; ...(3)

in (3) the coefficients of t are all known.

Let $\theta = \epsilon - \epsilon_1$, $\beta = \epsilon - \varpi$, $\beta_1 = \epsilon - \varpi_1$ . .....(4)

Then, from (1) and (2),

$$\begin{aligned} h_1 &= m_1 u_1 \sin \theta \quad , \quad k_1 = m_1 u_1 \cos \theta \quad , \\ h_2 &= m_1 u_2 \sin 2\theta \quad , \quad k_2 = m_1 u_2 \cos 2\theta \quad . \quad \ldots\ldots(5) \end{aligned}$$

Also $h_3 = m_1 u_3 \sin 3\theta + m_1 e_1 u_{10} \sin (3\theta + \beta - \beta_1)$

$= m_1 u_3 \sin 3\theta + m_1 e_1 u_{10} \{\sin (3\theta - \beta_1) \cos \beta + \cos (3\theta - \beta_1) \sin \beta\}$,

$p_3 = m_1 u_8 \sin (3\theta - \beta) + m_1 e_1 u_9 \sin (3\theta - \beta_1)$ .....(6)

and $q_3 = m_1 u_8 \cos (3\theta - \beta) + m_1 e_1 u_9 \cos (3\theta - \beta_1)$ . .....(7)

Hence, eliminating β_1 from the expression for h_3 by means of (6) and (7) we obtain, after a little reduction,

$$h_3 = m_1 \left(u_3 - \frac{u_8 \cdot u_{10}}{u_9} \right) \sin 3\theta + \frac{u_{10}}{u_9} (p_3 \cos \beta + q_3 \sin \beta)$$

which is of the form

$$h_3 = m_1 v_1 \sin 3\theta + v_2 p_3 + v_3 q_3 \quad \ldots\ldots(8)$$

where the coefficients v_1, v_2, v_3 have certain numerical values, β being known from Bouvard's elements. In the same way we find that

$$\left.\begin{aligned} k_3 &= m_1 v_1 \cos 3\theta - (v_3 p_3 - v_2 q_3) \\ p_1 &= m_1 w_1 \sin (\theta - \beta) + w_2 (p_3 \cos 2\theta - q_3 \sin 2\theta) \\ q_1 &= -m_1 w_1 \cos (\theta - \beta) - w_2 (p_3 \sin 2\theta + q_3 \cos 2\theta) \\ p_2 &= m_1 z_1 \sin (2\theta - \beta) + z_2 (p_3 \cos \theta - q_3 \sin \theta) \\ q_2 &= m_1 z_1 \cos (2\theta - \beta) + z_2 (p_3 \sin \theta + q_3 \cos \theta) \end{aligned}\right\} , \quad \ldots\ldots(9)$$

in which the values of the v's, w's and z's are known.

The original unknowns m_1, e_1, ϵ_1, ϖ_1 are now replaced by m_1, θ, p_3, q_3 since h_1, h_2, k_1, k_2 are simple functions of m_1 and θ by (5), and h_3, k_3, p_1, q_1, p_2, q_2 are functions of m_1, θ, p_3, q_3 in (8) and (9).

16·07. The equations of condition

The equation of condition is, on writing θ' for nt,

$$\begin{aligned} c = &\Delta\epsilon + \Delta x_1 \cos \theta' + \Delta x_2 \cos 2\theta' + t\,\Delta n + \Delta y_1 \sin \theta' + \Delta y_2 \sin 2\theta' \\ &+ h_1 \cos \theta_1 + h_2 \cos 2\theta_1 + h_3 \cos 3\theta_1 + k_1 \sin \theta_1 + k_2 \sin 2\theta_1 + k_3 \sin 3\theta_1 \\ &+ p_1 \cos \theta_2 + p_2 \cos \theta_3 + p_3 \cos \theta_4 + q_1 \sin \theta_2 + q_2 \sin \theta_3 + q_3 \sin \theta_4 \quad . \quad \ldots(1) \end{aligned}$$

Adams chose the mean opposition of Uranus in 1810 as the origin of time ; this is 1810·328. He also chose the unit of time to be three synodic

periods of Uranus, namely, 3·0362 years; this procedure, of course, required the adjustment of the constant of gravitation, G, in the various numerical coefficients $u_1, u_2, \ldots$ occurring in the disturbing function. With $a/a_1 = 0{\cdot}515$ and n, n_1 expressed in terms of the new unit of time, it was found that

$$\theta' = 13^\circ\ 0'{\cdot}6t\ ,\ \theta_1 = 8^\circ\ 12'{\cdot}1t\ ,\ \theta_2 = 4^\circ\ 48'{\cdot}5t\ ,\ \theta_3 = 3^\circ\ 23'{\cdot}6t\ ,\ \theta_4 = 11^\circ\ 35'{\cdot}7t\ .$$

The " modern " observations were grouped to give the appropriate values for $1810{\cdot}328 \pm t$, where $t = 0, 1, \ldots, 10$; the corresponding values of c in (1) were then deduced from the values of l derived from Bouvard's tables. Adams had thus 21 equations of condition based on the " modern " observations* from 1780 to 1840.

As regards the " ancient " observations these were combined to give 9 equations of condition.

16·08. Treatment of the equations of condition

We first remark that in the equation 16·07 (1) Δx_2 and Δy_2 are not independent unknowns; they are given in terms of Δx_1, Δy_1 by 16·03 (13) and (14), namely,

$$\Delta x_2 = \gamma_1 \Delta x_1 + \gamma_2 \Delta y_1\ ,\quad \Delta y_2 = -\gamma_2 \Delta x_1 + \gamma_1 \Delta y_1$$

in which γ_1 and γ_2 are known numerical quantities.

Adams' treatment of the equations 16·07 (1) for the " modern " observations was as follows.

Each of the 21 equations was multiplied by the coefficient of Δx_1 and the sum formed as in deriving normal equations. This procedure was repeated for $\Delta\epsilon, \Delta n, \Delta y_1$; $h_1, k_1, h_2, k_2, h_3, k_3$; p_2, q_2, p_3, q_3. Use was then made of the following easily-established formulae in which the summations are for $t = -10, -9, \ldots, -1, 0, 1, 2, \ldots, 10$;

$$2\,\Sigma \cos \xi t = \frac{\sin \frac{21}{2}\xi}{\sin \frac{1}{2}\xi}\ ,$$

$$4\,\Sigma t \sin \xi t = \frac{11 \sin 10\,\xi - 10 \sin 11\,\xi}{\sin^2 \frac{1}{2}\xi}\ ,$$

$$4\,\Sigma \cos \xi t \cos \eta t = \frac{\sin \frac{21}{2}(\xi-\eta)}{\sin \frac{1}{2}(\xi-\eta)} + \frac{\sin \frac{21}{2}(\xi+\eta)}{\sin \frac{1}{2}(\xi+\eta)}\ ,$$

$$4\,\Sigma \sin \xi t \sin \eta t = \frac{\sin \frac{21}{2}(\xi-\eta)}{\sin \frac{1}{2}(\xi-\eta)} - \frac{\sin \frac{21}{2}(\xi+\eta)}{\sin \frac{1}{2}(\xi+\eta)}\ ,$$

$$4\,\Sigma \cos^2 \xi t = 21 + \frac{\sin 21\,\xi}{\sin \xi}\ ,$$

$$4\,\Sigma \sin^2 \xi t = 21 - \frac{\sin 21\,\xi}{\sin \xi}\ .$$

* The mean observation for the opposition of 1780 was derived from observations lying on either side of the discovery-date of Uranus and, strictly, was only partially a 'modern' observation.

Also, summations of " odd " combinations such as $\Sigma \cos \xi t \sin \eta t$ and $\Sigma t \cos \xi t$ are all zero.

By this procedure, the 21 equations of condition were reduced to 14 pseudo-normal equations forming two groups of 7 equations, the first group being of the general form

$$C = a_1 \Delta\epsilon + a_2 \Delta x_1 + a_3 \Delta x_2 + \sum_{r=1}^{3} (b_r h_r + c_r p_r) \quad , \quad \ldots\ldots\ldots\ldots (1)$$

and the second group of the general form

$$C' = a_1' \Delta n + a_2' \Delta y_1 + a_3' \Delta y_2 + \sum_{r=1}^{3} (b_r' k_r + c_r' q_r) \quad . \quad \ldots\ldots\ldots\ldots (2)$$

Taking advantage of certain numerical features of (1) and (2) and eliminating $\Delta\epsilon$, Δn, Δx_1, Δy_1, Δx_2, Δy_2 Adams was left with two equations of the form

$$D = \sum_{1}^{3} (\alpha_r h_r + \alpha_r' k_r + \beta_r p_r + \beta_r' q_r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Consider now the " ancient " observations which, as already stated, Adams combined to form 9 equations of condition ; these had not, of course, the systematic features associated with the equations of condition for the " modern " observations. By a process in general similar to that in deriving (3), Adams obtained two equations of the form

$$E = \sum_{1}^{3} (A_r h_r + A_r' k_r + B_r p_r + B_r' q_r) \quad . \quad \ldots\ldots\ldots\ldots\ldots (4)$$

The pair of equations (3) and the pair of equations (4) are thus sufficient to determine the 4 unknowns m_1, θ, p_3 and q_3.

Adams then eliminated the left-hand members of the second equation (3) and the two equations (4) by means of the first equation (3), thus obtaining a group of three equations of the form

$$0 = \sum_{1}^{3} (C_r h_r + C_r' k_r + D_r p_r + D_r' q_r) \quad . \quad \ldots\ldots\ldots\ldots\ldots (5)$$

Now, by 16·06 (5), (8) and (9), h_1, h_2, h_3, k_1, k_2, k_3, p_1, p_2, q_1, q_2 are given in terms of m_1, θ, p_3 and q_3—the known numerical value of β being substituted in these expressions. If we write p and q for p_3/m_1 and q_3/m_1 respectively, it is seen that each of the equations (5) is reduced to the form

$$\begin{aligned} 0 = \sum_{1}^{3} (a_r \sin r\theta + b_r \cos r\theta) \\ + cp + dq + e(p \cos\theta - q \sin\theta) + f(p \sin\theta + q \cos\theta) \\ + g(p \cos 2\theta - q \sin 2\theta) + l(p \sin 2\theta + q \cos 2\theta) \quad \ldots\ldots\ldots\ldots (6) \end{aligned}$$

in which the numerical values of the coefficients a, b, c, d, e, f, g and l are all known.

16·09. Adams' solutions

By a process of successive approximations Adams solved the three equations (6) of the previous section, obtaining the following results :

$$\theta \equiv \epsilon - \epsilon_1 = -46^\circ\ 55' \quad , \quad p \equiv p_3/m_1 = 138''\cdot92 \quad , \quad q \equiv q_3/m_1 = -109''\cdot83.$$

At the epoch 1810·328, the value of ϵ is 217° 55′ ; hence $\epsilon_1 = 264^\circ\ 50'$, from the value of θ just stated.

The mean longitude l_1 ($\equiv n_1 t + \epsilon_1$) can now be found at any opposition, since n_1 is known from the adopted value 0·515 of the ratio $a : a_1$. Now, in our notation $\theta_2 = n_1 t$ and from 16·07 we have : $n_1 = 4^\circ\ 48'\cdot5$. With $t = 12$—that is, after 36 synodic periods, corresponding to the opposition at 1846·762 or October, 6, 1846—$n_1 t = 57^\circ\ 42'$ and, accordingly, the mean longitude l_1 of the unknown planet is given, a correction of 30′ for precession being applied, by

$$l_1 = 323^\circ\ 2' \quad , \quad \text{October 6, 1846} \quad .$$

From 16·06 (6) and (7), on inserting the numerical values of u_8 and u_9, we have

$$p \equiv \frac{p_3}{m_1} = 33''\cdot93 \sin(3\theta - \beta) - 63''\cdot41 e_1 \sin(3\theta - \beta_1) = 138''\cdot92 \quad ,$$

$$q \equiv \frac{q_3}{m_1} = 33''\cdot93 \cos(3\theta - \beta) - 63''\cdot41 e_1 \cos(3\theta - \beta_1) = -108''\cdot83$$

in which $\beta_1 = \epsilon - \varpi_1$. With θ and β known, the solution of these equations gives

$$e_1 = 2\cdot4123 \quad , \quad \beta_1 = 279^\circ\ 14' \quad .$$

The second result gives $\varpi_1 = 298^\circ\ 41'$ at the selected epoch and, after applying precession, the longitude of perihelion for the unknown planet on October 6, 1846, is given by

$$\varpi_1 = 299^\circ\ 11'.$$

Remembering that a factor of 20 has been applied to the eccentricity, we have

$$e_1 = 0\cdot1206 \quad .$$

The mass of the unknown planet is given by

$$m_1 = \frac{1}{6666} \quad ,$$

the Sun's mass being the unit.

The condensed equations of condition in terms of Δn, $\Delta\epsilon$, Δe, $\Delta\varpi$; m_1, ϵ_1, e_1, ϖ_1 now enable the values of the first four quantities to be calculated.

When the values of all the unknowns are substituted in the individual equations of condition the deviations, in each case, between Adams' theory ($a/a_1 = 0\cdot515$) and observations are derived ; in almost every instance the outstanding deviations are very small, comparable in magni-

tude with the probable errors of observation. The last observations used by Adams were those of 1842; after his solution had been completed, it was found that the deviations for 1843–1845 increased significantly; Adams, without repeating the calculations involving the new observations inferred that with $a/a_1 = 0{\cdot}574$ all discrepancies would substantially disappear.

16·10. Le Verrier's equations of condition

The terms used by Le Verrier to express the perturbation, P_1, in true longitude are given by

$$\begin{aligned} P_1 = {} & m_1 \Sigma A_i \sin \{i(nt - n_1 t + \epsilon - \epsilon_1)\} \\ & + m_1 e \Sigma B_i \sin \{(i-1)(nt+\epsilon) - in_1 t - i\epsilon_1 + \varpi\} \\ & + m_1 e_1 \Sigma C_i \sin \{(i-1)(nt+\epsilon) - in_1 t - i\epsilon_1 + \varpi_1\} \quad \ldots\ldots\ldots(1) \end{aligned}$$

in which $i = 1, 2, 3$ and the coefficients A, B and C are functions of a and a_1. As stated previously Le Verrier, guided by Bode's rule, assumed at first that $a_1 = 2a$; accordingly, the numerical values of n_1 and of the coefficients of A, B and C could be calculated. The terms in (1) are of the same form as those used by Adams, except that Adams' term in u_{10} is not included in (1) by Le Verrier.

When Bouvard's values of ϵ, ϖ and e are inserted in (1)—and also the values of a and a_1—the first two summations of (1) can be written as $m_1 L$, where L is of the form

$$L = \Sigma D_i \sin i\epsilon_1 + \Sigma E_i \cos i\epsilon_1 \quad \ldots\ldots\ldots(2)$$

in which $i = 1, 2, 3$ and the numerical values of the D's and E's are known.

Let $$h = e_1 \sin \varpi_1 \quad , \quad k = e_1 \cos \varpi_1 \quad ; \ldots\ldots\ldots(3)$$

then the third summation of (1) can be written as

$$m_1 hH + m_1 kK \quad ,$$

where H and K have each the general form of (2) with, of course, different numerical values of the coefficients D, E. We can then write P_1 in the form

$$P_1 = m_1 hH + m_1 kK + m_1 L \quad \ldots\ldots\ldots(4)$$

in which H, K, L are trigonometric functions of ϵ_1.

Let σ denote $(\lambda_0 - \lambda_c)$, the difference between the observed and Bouvard's true longitudes. Le Verrier's equation of condition is

$$\sigma = \alpha_1 \Delta n + \alpha_2 \Delta\epsilon + \alpha_3 \Delta e + \alpha_4 \Delta\varpi + m_1 hH + m_1 kK + m_1 L \quad , \quad \ldots\ldots(5)$$

the first four terms on the right arising from the errors Δn, $\Delta\epsilon$, Δe, $\Delta\varpi$ in Bouvard's elements as given by 16·02 (6).

By combining the "ancient" and "modern" observations into suitable groups—the latter at intervals of 7 years—Le Verrier had 18 equations of the form (5) in which there are 8 unknowns Δn, $\Delta\epsilon$, Δe, $\Delta\varpi$;

m_1, m_1h, m_1k, ϵ_1, the last being concealed in the expressions for H, K, L.

Le Verrier's first attempted solution was based on the following procedure. If each of the equations (5) is assumed to be exact in the sense that there are no observational errors in λ_0, and, consequently, in σ then the elimination of the first seven unknowns from any eight equations of the form (5) will yield a transcendental—and highly complicated—equation in ϵ_1 alone ; this equation can, theoretically, be solved to give a value of ϵ_1, which, when substituted in seven of the eight equations, will enable the values of Δn, $\Delta\epsilon$, ... , m_1k to be obtained. However, he was unable to obtain a satisfactory solution of the transcendental equation—owing, presumably, to the presence of non-negligible accidental errors of observation—which would lead to a *positive* value of m_1.

16·11. Le Verrier's solution

(i) Le Verrier now proceeded to deal with the problem in a totally different way, recognising that errors of observation in the " ancient " true longitudes must be considered.

He first chose 4 equations (5) at the widely-separated epochs 1715, 1775, 1810 and 1845 ; he assumed (i) that the errors in σ for 1715 and 1775 were p and q respectively, and (ii) that the errors for 1810 and 1845 were zero, the observations concerned being very numerous and made with instruments of high precision. Writing F for the right-hand side of (5) in the previous section he had then 4 equations,

$$F_1=\sigma_1+p \quad , \quad F_2=\sigma_2+q \quad , \quad F_3=\sigma_3 \quad , \quad F_4=\sigma_4 \quad , \ldots\ldots\ldots(1)$$

where σ_1, ... , σ_4 are the values of σ at the epochs concerned. Solving these 4 equations for Δn, $\Delta\epsilon$, Δe and $\Delta\varpi$ he obtained

$$\Delta n=f(m_1,\ m_1h,\ m_1k,\ p,\ q,\ \epsilon_1) \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

with three similar equations for $\Delta\epsilon$, Δe and $\Delta\varpi$; in these equations the functions f are linear in m_1, m_1h, m_1k and they contain ϵ_1 transcendentally.

Le Verrier then substituted the expressions for Δn, $\Delta\epsilon$, Δe and $\Delta\varpi$—given by (2)—in the remaining 14 equations of condition, thus obtaining 14 equations linear in m_1, m_1h, m_1k, p and q, and containing ϵ_1 transcendentally. Of these equations he grouped the four for 1782, 1789, 1796 and 1801 together, and the four for 1817, 1824, 1831 and 1838 together, thereby condensing 8 equations to 2, each linear in m_1, m_1h, m_1k, p and q ; solving these two equations for m_1h and m_1k, he obtained results of the forms

$$m_1h=G_1m_1+G_2p+G_3q+\kappa_1 \quad , \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$m_1k=g_1m_1+g_2p+g_3q+\kappa_2 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

in which the G's and g's are trandscendental functions of ϵ_1 and κ_1, κ_2 are numerical constants associated with the σ's.

The expressions for m_1h, m_1k in (3) and (4) were then substituted in the 4 equations (2), thus obtaining expressions for Δn, $\Delta\epsilon$, Δe, $\Delta\varpi$ each of

the form given by the right-hand side of (3) or (4). When this has been done, we now have 6 equations giving Δn, $\Delta\epsilon$, Δe, $\Delta\varpi$ and $m_1 h$, $m_1 k$ by equations of the form of (3) or (4).

Up to this point a total of 12 equations of condition have been specifically used—8 in the procedure just described and 4 represented by the group of equations (1); accordingly, 6 equations of condition remain. We write the right-hand side of 16·10 (5) as F, so that an equation of condition can be expressed as

$$\sigma = F \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

In F substitute the expressions for Δn, $\Delta\epsilon$, Δe, $\Delta\varpi$ and $m_1 h$, $m_1 k$; then, if ρ denotes the residual $F - \sigma$, each equation reduces to

$$\rho = a + bm_1 + cp + dq \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which b, c and d are transcendental functions of ϵ_1.

Of the observations associated with the six equations (6) Le Verrier regarded the two "ancient" observations of 1690 and 1747 as crucial, and for these the values of a, b, c and d were found for the successive values 0°, 9°, 18°, ..., 351° of ϵ_1. The residuals ρ in (6) for $\epsilon_1 = 0°$ and $\epsilon_1 = 252°$ are shown below to illustrate the results obtained.

ϵ_1	1690	1747
0°	$324'' + 87''m_1 + 0''\cdot4p - 2''\cdot0q$	$-261'' - 16''m_1 - 1''\cdot3p - 1''\cdot6q$
252°	$-8'' - 45''m_1 - 0''\cdot5p - 2''\cdot3q$	$-24'' \ -3''m_1 \ -0''\cdot7p - 1''\cdot3q$

As stated previously, Le Verrier used, as unit for m_1, 10^{-4} times the Sun's mass, the necessary numerical adjustments having been made in the equations of condition. It might then be anticipated that the extreme range of the possible values of m_1 would be, say, 0·1 to 10; m_1 is hardly likely to exceed 10—the approximate mass of Jupiter on this scale—for the disturbing planet would then be a comparatively bright object which could hardly have escaped detection in the past; nor could m_1 be less than 0·1 for, if so, the effect of its perturbation on Uranus would be relatively insignificant.

A discussion of contemporary observations round about 1715 and 1775 suggested that p was unlikely to exceed 15″ numerically and q to exceed 10″ numerically.

Consider now the residual, ρ_1, for 1690 when $\epsilon_1 = 0$; ρ_1 is smallest if the errors p and q are $-15''$ and $+10''$ respectively and then

$$\rho_1 = 298'' + 87''m_1 \quad .$$

Similarly, if ρ_2 is the residual for 1747 when $\epsilon_1 = 0$, the smallest value of ρ_2 occurs if $p = -15''$, $q = -10''$, and then

$$\rho_2 = -225'' - 16''m_1 \quad .$$

If ϵ_1 is, in fact, equal to zero, these residuals ρ_1 and ρ_2 should be of the

same order of magnitude as the errors p and q ; but, since m_1 is *positive*, the least values of the residuals for all values of m_1 are 298″ and − 225″ which are very many times larger than any possible real observational error with which any of the observations could be charged. With this evidence Le Verrier concluded that $\epsilon_1 = 0$ was *not* a permissible solution.

Proceeding in this way he calculated ρ_1 and ρ_2 for values of ϵ_1 at 9° intervals and it became evident from the complete results that the actual value of ϵ_1 must be in the neighbourhood of 252°. For example—on referring to the data already set out—when $\epsilon_1 = 252°$ and p and q are taken to be − 15″ and − 10″ respectively, then

$$\rho_1 = 22'' - 45''m_1$$

which, for any reasonable value of the residual not exceeding 22″—a generous estimate—will give a positive value of m_1 of the right order of magnitude.

In the same way, by increasing very slightly the maximum numerical values of p and q, the residual ρ_2 can, without straining the probabilities, provide a positive value of m_1.

(ii) In the final stage Le Verrier introduced corrections to a_1 and ϵ_1 in the forms

$$\frac{a}{a_1} = \frac{1}{2} + \frac{1}{5}\gamma \quad , \quad \epsilon_1 = 252° + \frac{\pi}{10}\beta \quad .$$

By means of the former the coefficients of the terms in the expression for the perturbation in true longitude were expanded in terms of the small quantity $\frac{1}{5}\gamma$; by means of the latter, sines and cosines of $i\epsilon_1$ $(i = 1, 2, 3)$ could be expanded in terms of the small angle $\frac{\pi}{10}\beta$. Suppose that we ignore squares and products of $\frac{1}{5}\gamma$ and $\frac{\pi}{10}\beta$ and the products of these with $\Delta n, \Delta\epsilon, \ldots, m_1 k, m_1$; we then find that the equation of condition is linear in the unknowns

$$\Delta n, \Delta\epsilon, \Delta e, \Delta\varpi \; ; \; m_1 h, m_1 k, m_1, \beta \text{ and } \gamma \quad .$$

The procedure of the method of least squares can then be followed to determine the values of the unknowns from the 18 equations of condition. Actually, Le Verrier's procedure was more elaborate, for he developed the equation of condition to take account of squares and products of β and γ.

Le Verrier's principal results were :

$$a_1 = 36{\cdot}16 \text{ astronomical units}, \quad e_1 = 0{\cdot}1076 \quad ,$$

$$m_1 = \frac{1}{9300} \quad , \quad \lambda_1 = 326° \; 32' \; (1847{\cdot}0).$$

The true longitude given by Le Verrier's solution for the date on which Neptune was discovered at Berlin was about 1° from the value deduced from observation.

As we have seen in 16·09, Adams' value of the mean longitude on October 6, 1846, was 323° 2′ ; after applying the mean motion between this date and 1947·0 and applying further the equation of the centre, the true longitude for 1947·0 is given by

$$\lambda_1 = 329° 57' \quad (1847{\cdot}0) \quad .$$

The true longitude of Neptune derived from the elements of the orbit when a sufficient number of observations became available is given by

$$\lambda_1 = 327° 34' \quad (1847{\cdot}0) \quad .$$

The positions predicted by Adams and Le Verrier were thus very close to the observed position.*

* For further discussion on the solutions of Adams and Le Verrier and their relation to the actual elements of Neptune's orbit the reader is referred to the author's " John Couch Adams and the Discovery of Neptune ", *Occasional Notes of the Royal Astronomical Society*, No. 11, August 1947 and E. W. Brown : "On a criterion for the prediction of an unknown planet ", M.N.R.A.S., **92**, 80 (1931).

Chapter 17

DE PONTÉCOULANT'S LUNAR THEORY

17·01. Equations of motion

In this section, we derive the equations of motion as used by de Pontécoulant* and in later sections we describe the principal steps leading to the final solutions, with some discussion of such famous perturbations as the evection, variation, and so on.

In rectangular coordinates the equations of motion for the Moon are

$$\ddot{x}+\frac{\mu x}{r^3}=\frac{\partial R}{\partial x} \quad , \quad \ddot{y}+\frac{\mu y}{r^3}=\frac{\partial R}{\partial y} \quad , \quad \ddot{z}+\frac{\mu z}{r^3}=\frac{\partial R}{\partial z} \quad ,$$

where R is the disturbing function arising from the Sun and $\mu \equiv G(E+M)$ is regarded as known, E and M being the masses of the Earth and Moon ; multiply these equations by $\dot{x}$, $\dot{y}$, $\dot{z}$ and add ; then

$$\sum_{xyz}\left\{\dot{x}\ddot{x}+\mu\frac{x\dot{x}}{r^3}\right\}=\sum_{xyz}\frac{\partial R}{\partial x}\dot{x} \quad .$$

On integration we obtain

$$2T \equiv \dot{x}^2+\dot{y}^2+\dot{z}^2=2\frac{\mu}{r}-\frac{\mu}{a}+2\int\frac{d'R}{dt}\,dt+K \quad , \quad \ldots\ldots\ldots\ldots(1)$$

where $(K-\mu/a)$ is the constant of integration and $\dfrac{d'R}{dt}$ denotes differentiation of R with respect to the time only in so far as the Moon's coordinates are concerned.

Let v and θ denote the ecliptic longitude and latitude ; then

$$x=r\cos v\cos\theta \quad , \quad y=r\sin v\cos\theta \quad , \quad z=r\sin\theta \quad ,$$

from which

$$2T=\dot{r}^2+r^2\dot{v}^2\cos^2\theta+r^2\dot{\theta}^2 \quad . \quad \ldots\ldots\ldots\ldots(2)$$

In terms of the coordinates r, v and θ and $U \equiv \mu/r+R$, Lagrange's equations are

$$\ddot{r}+\frac{\mu}{r^2}=r\dot{v}^2\cos^2\theta+r\dot{\theta}^2+\frac{\partial R}{\partial r}\equiv\frac{1}{r}\{2T-\dot{r}^2\}+\frac{\partial R}{\partial r} \quad , \quad \ldots\ldots\ldots\ldots(3)$$

$$\frac{d}{dt}(r^2\dot{v}\cos^2\theta)=\frac{\partial R}{\partial v} \quad , \quad \ldots\ldots\ldots\ldots(4)$$

$$\frac{d}{dt}(r^2\dot{\theta})=\frac{\partial R}{\partial\theta}-r^2\dot{v}^2\sin\theta\cos\theta \quad . \quad \ldots\ldots\ldots\ldots(5)$$

* *Théorie Analytique du Système du Monde*, IV (1846).

By means of (1) and (2) we write (3) as

$$\frac{1}{2}\frac{d^2}{dt^2}(r^2)=\frac{\mu}{r}-\frac{\mu}{a}+r\frac{\partial R}{\partial r}+2\int\frac{d'R}{dt}\,dt+K \quad \ldots\ldots(6)$$

Let $$s=\tan\theta \quad \ldots\ldots(7)$$

Then (4) becomes on integration

$$\dot{v}=\frac{h}{r^2}(1+s^2)+\frac{1+s^2}{r^2}\int\frac{\partial R}{\partial v}\,dt \quad , \ldots\ldots(8)$$

where h is a constant of integration. The right-hand side of (8) can be developed as a series of periodic terms and a non-periodic part which we denote by n ; hence

$$v=nt+\epsilon+P.T. \quad \ldots\ldots(9)$$

where ϵ is the constant of integration.

We now modify (5). Since $z=r\sin\theta$, we have

$$\dot{z}=\dot{r}\sin\theta+r^2\dot{\theta}\left(\frac{\cos\theta}{r}\right) \quad , \quad \ddot{z}=(\ddot{r}-r\dot{\theta}^2)\sin\theta+\frac{\cos\theta}{r}\frac{d}{dt}(r^2\dot{\theta}) \quad ;$$

hence, from (3) and (5),

$$\ddot{z}=-\frac{\mu}{r^2}\sin\theta+\sin\theta\frac{\partial R}{\partial r}+\frac{\cos\theta}{r}\frac{\partial R}{\partial\theta} \quad .$$

But $\dfrac{\partial R}{\partial\theta}=\dfrac{\partial R}{\partial s}\sec^2\theta$; consequently we obtain, using (7),

$$\ddot{z}+\frac{\mu z}{r^3}=\frac{s}{\sqrt{(1+s^2)}}\frac{\partial R}{\partial r}+\frac{\sqrt{(1+s^2)}}{r}\frac{\partial R}{\partial s} \quad . \quad \ldots\ldots(10)$$

This last equation can be written alternatively as

$$\frac{d^2}{dt^2}\left\{\frac{rs}{\sqrt{(1+s^2)}}\right\}+\frac{\mu}{r^3}\cdot\frac{rs}{\sqrt{(1+s^2)}}=\frac{s}{\sqrt{(1+s^2)}}\frac{\partial R}{\partial r}+\frac{\sqrt{(1+s^2)}}{r}\frac{\partial R}{\partial s} \quad \ldots(11)$$

17·02. The constants of integration

Our derived equations of motion are three in number, each of the second order—they are (6), (4) and (10) or (11) of the previous section—and there must be six constants of integration. In deriving 17·01 (8) and (9) from 17·01 (4) we have introduced two constants of integration, h and ϵ. The remaining two equations of motion will introduce four more ; two of these we take to be e and γ ($\equiv\tan i$) and the others ϖ and Ω. But K and a have been introduced in addition—in 17·01 (1). Now, returning to 17·01 (8) and (9), the constant n in the latter equation is clearly a function of h. We now define a by

$$\mu=n^2a^3 \quad ; \quad \ldots\ldots(1)$$

consequently a is a function of h. We now have as independent consonants : h, e, i, K ; ϵ, Ω, ϖ ; these are seven in number ; accordingly, there must be a relation amongst these seven constants. This relation

can be found by means of 17·01 (3) which we write in terms of s, noting that $\dot{s}=\dot{\theta}(1+s^2)$, as

$$\frac{\ddot{r}}{r}+\frac{\mu}{r^3}-\frac{\dot{v}^2}{(1+s^2)}-\frac{\dot{s}^2}{(1+s^2)^2}=\frac{1}{r}\frac{\partial R}{\partial r} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The solutions for r, v and s—expressed in terms of the seven constants —will, when substituted in (2), lead to the desired relation.

It is evident that h can be replaced as one of the six independent constants, by a—or, if need be, by n.

The relation between n and h can be easily deduced in the case of the elliptic orbit. We have, from 17·01 (8), $r^2\dot{v}\cos^2\theta=h_0$, where we write h_0 for the constant of integration for the elliptic case. Let r_1 be the projection of r on the ecliptic; then $r_1=r\cos\theta$ and $r_1^2\dot{v}=h_0$. But $r_1^2\dot{v}$ is the projection of $\sqrt{\{\mu a(1-e^2)\}}$ on the ecliptic; hence

$$h_0=\sqrt{\{\mu a(1-e^2)\}}\cos i=\frac{na^2\sqrt{(1-e^2)}}{\sqrt{(1+\gamma^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

It is convenient to write the constant of integration, h, appearing in 17·01 (8), as

$$h=h_0(1+\delta h) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

We can then regard δh as the independent constant of integration in place of h, h_0 being given by (3). Further, we write $a^2n^2\delta b$ for K in 17·01 (6).

17·03. Summary of the equations

For convenience of reference, we gather together the principal equations. They are, μ being replaced by n^2a^3, and K by $a^2n^2\delta b$:

$$\frac{1}{2}\frac{d^2}{dt^2}(r^2)-\frac{n^2a^3}{r}+n^2a^2=r\frac{\partial R}{\partial r}+2\int\frac{d'R}{dt}\,dt+a^2n^2\delta b \quad , \quad \ldots\ldots\ldots(1)$$

$$\dot{v}=\frac{h(1+s^2)}{r^2}+\frac{1+s^2}{r^2}\int\frac{\partial R}{\partial v}\,dt \quad , \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{d^2}{dt^2}\left\{\frac{rs}{\sqrt{(1+s^2)}}\right\}+\frac{n^2a^3}{r^3}\cdot\frac{rs}{\sqrt{(1+s^2)}}=\frac{s}{\sqrt{(1+s^2)}}\frac{\partial R}{\partial r}+\frac{\sqrt{(1+s^2)}}{r}\frac{\partial R}{\partial s} \quad , \quad \ldots(3)$$

or

$$\ddot{z}+n^2a^3\cdot\frac{z}{r^3}=\frac{s}{\sqrt{(1+s^2)}}\frac{\partial R}{\partial r}+\frac{\sqrt{(1+s^2)}}{r}\frac{\partial R}{\partial s} \quad , \quad \ldots\ldots\ldots\ldots(4)$$

and, in addition,

$$\frac{\ddot{r}}{r}+\frac{n^2a^3}{r^3}-\frac{\dot{v}^2}{1+s^2}-\frac{\dot{s}^2}{(1+s^2)^2}=\frac{1}{r}\frac{\partial R}{\partial r} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

In (2), the constant of integration h is written as

$$h=h_0(1+\delta h) \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where h_0 is the ecliptic value given by

$$h_0=\frac{na^2\sqrt{(1-e^2)}}{\sqrt{(1+\gamma^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

It is convenient to refer to (1) as the equation in radius vector, to (2) as the longitude equation, to (3) or (4) as the latitude equation.

17·04. Motions of Perigee and the Node

If we consider the undisturbed orbit of the Moon, the longitude of perigee, ϖ, and the longitude of the node, Ω, are of course constants. But observations show that the effect of the solar perturbations is to produce a rapid increase in ϖ and a rapid diminution in Ω with respect to the time ; thus, regarding $\dot{\varpi}$ and $\dot{\Omega}$ to be constants we can write ϖ and Ω at time t as

$$\varpi = \varpi_0 + (1-c)\,nt \quad , \quad \Omega = \Omega_0 + (1-g)\,nt \quad \ldots\ldots\ldots\ldots\ldots(1)$$

where c and g are constants to be determined, and ϖ_0 and Ω_0 are constants, these being the values of ϖ and Ω when $t=0$.

Consider first the function M given by $M = nt + \epsilon - \varpi$; then

$$M = nt + \epsilon - \varpi_0 - (1-c)\,nt = cnt + \epsilon - \varpi_0 \quad .$$

Thus M is the mean anomaly measured from the moving perigee.

Similarly,

$$\eta \equiv nt + \epsilon - \Omega = nt + \epsilon - \Omega_0 - (1-g)\,nt = gnt + \epsilon - \Omega_0 \quad .$$

From what has been said, $(1-c)$ is positive and $(1-g)$ is negative. Perigee and the node will describe 2π in times $2\pi/\{(1-c)n\}$ and $2\pi/\{(g-1)n\}$; these periods are approximately 9 years and 18 years respectively.

Dropping the zero suffices in ϖ_0 and Ω_0, we write

$$\phi = cnt + \epsilon - \varpi \quad , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\eta = gnt + \epsilon - \Omega \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

in these ϖ and Ω are now regarded as constants ; as we shall see they are constants of integration arising in the solutions of the equations of motion.

In the next two sections we determine approximate values of c and g.

17·05. The mean motion of perigee

In this section we derive approximate solutions of the equations in 17·03, neglecting s, γ, e_1 and a/a_1 ; also, we neglect e^2 in 17·03 (7) so that $h_0 = na^2$, and $h = na^2(1+\delta h)$. Further, we neglect the longitude terms in R given by 7·06 (5) and keep only the principal term so that

$$R = \tfrac{1}{4}m^2n^2r^2 \quad . \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Here $m = n_1/n$ and is a quantity of the first order of smallness.

We write $r = \sigma + x$, where σ is of order a and x is periodic ; we assume further that the amplitude of x is small and, accordingly, we neglect $x^2, x^3, \ldots$.

The equation in radius vector—17·03 (1)—becomes

$$\sigma\ddot{x} - \frac{n^2a^3}{\sigma}\left(1 - \frac{x}{\sigma}\right) + n^2a^2 = m^2n^2(\sigma^2 + 2\sigma x) + n^2a^2\delta b \quad ,$$

since $r\frac{\partial R}{\partial r}+2\int\frac{d'R}{dt}\,dt=m^2n^2r^2$, by (1).

Since x is periodic we have

$$\ddot{x}+n^2\left\{\left(\frac{a}{\sigma}\right)^3-2m^2\right\}x=0 \quad \text{.....(2)}$$

and

$$a^2-a^3/\sigma=m^2\sigma^2+a^2\delta b \quad \text{.....(3)}$$

We now write $\sigma=a(1-\delta a)$, where δa is a small quantity; we neglect $(\delta a)^2$. We then find that (2) can be written

$$\ddot{x}+c^2n^2x=0 \quad \text{.....(4)}$$

where

$$c^2=1+3\delta\alpha-2m^2 \quad \text{.....(5)}$$

Formula (3) becomes :

$$\delta\alpha+m^2+\delta b=0 \quad \text{.....(6)}$$

The solution of (4) is then

$$x=Aa\cos(cnt+\epsilon-\varpi) \quad \text{.....(7)}$$

where A and ϖ are new constants of integration.

Now consider the longitude equation—17·03 (2)—in which R is given by (1). Then

$$\dot{v}=\frac{h_0(1+\delta h)}{\sigma^2}\left(1-\frac{2x}{\sigma}\right)=n\left(1+2\delta\alpha+\delta h-\frac{2x}{a}\right) \quad .$$

By 17·01 (9), the solution is : $v=nt+\epsilon+P.T.$; accordingly, we must have

$$2\delta\alpha+\delta h=0 \quad \text{.....(8)}$$

and

$$v=nt+\epsilon-2\frac{A}{c}\sin(cnt+\epsilon-\varpi) \quad \text{.....(9)}$$

We now use 17·03 (5) ; it is sufficient to substitute the non-periodic parts of r and v ; we then have

$$\frac{n^2a^3}{\sigma^3}-n^2\equiv 3n^2\delta\alpha=\tfrac{1}{2}m^2n^2 \quad .$$

Hence $\delta\alpha=\frac{1}{6}m^2$. From (8), $\delta h=-\frac{1}{3}m^2$. From (6), $\delta b=-\frac{7}{6}m^2$.

Also, from (5), $c^2=1-\frac{3}{2}m^2$ from which, to $O(m^2)$,

$$c=1-\tfrac{3}{4}m^2 \quad \text{.....(10)}$$

Writing the constant of integration A as $-e$, in analogy with the formula for the radius vector in the undisturbed orbit, the approximate solutions are

$$r=a[1-\tfrac{1}{6}m^2-e\cos(cnt+\epsilon-\varpi)] \quad ,$$
$$v=nt+\epsilon+2e\sin(cnt+\epsilon-\varpi) \quad ,$$

em^2 being neglected in the amplitude of (9) ; in terms of ϕ, defined by 17·04 (2), these are

$$r=a(1-\tfrac{1}{6}m^2-e\cos\phi) \quad , \quad v=nt+\epsilon+2e\sin\phi \quad \text{.....(11)}$$

17·06. The mean motion of the node

We now consider the latitude equation—17·03 (3). We neglect s^3, e, e_1 and a/a_1. From 7·06 (5) we have :

$$\frac{\partial R}{\partial r} = \tfrac{1}{2}m^2n^2r \quad \text{and} \quad \frac{\partial R}{\partial s} = -\tfrac{3}{2}m^2n^2r^2s \quad .$$

Hence
$$\frac{d^2}{dt^2}(rs) + \frac{n^2a^3}{r^3}(rs) = -m^2n^2(rs) \quad .$$

Since $e=0$, then $r=a$; hence

$$\ddot{s} + n^2(1 + 3\delta a + m^2)s = 0$$

and
$$s = \gamma \sin(gnt + \epsilon - \Omega) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where γ and Ω are constants of integration and

$$g^2 = 1 + 3\delta a + m^2 = 1 + \tfrac{3}{2}m^2 \quad ,$$

or, to $O(m^2)$,
$$g = 1 + \tfrac{3}{4}m^2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

To $O(\gamma)$, we have from (1)

$$z = a\gamma \sin(gnt + \epsilon - \Omega)$$

or
$$z = a\gamma \sin \eta \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

17·07. The modified variables

In the undisturbed orbit, the radius vector r and the longitude v are given by

$$r/a = 1 + \tfrac{1}{2}e^2 - e\cos M - \tfrac{1}{2}e^2\cos 2M \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$v = nt + \epsilon + 2e\sin M + \tfrac{5}{4}e^2\sin 2M - \tfrac{1}{4}\gamma^2\sin 2\eta' \quad , \quad \ldots\ldots\ldots(2)$$

where
$$M = nt + \epsilon - \varpi \quad , \quad \eta' = nt + \epsilon - \Omega \quad ;$$

in these equations we include, for purposes of illustration, only terms of the second order* in e and γ.

We now denote by ρ/a the series on the right-hand side of (1) in which M is replaced by ϕ ; thus, to the second order

$$\rho/a = 1 + \tfrac{1}{2}e^2 - e\cos\phi - \tfrac{1}{2}e^2\cos 2\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Similarly, we denote by w the series on the right-hand side of (2) in which M is replaced by ϕ and η' by $\eta\,(\equiv gnt + \epsilon - \Omega)$; thus, to the second order,

$$w = nt + \epsilon + 2e\sin\phi + \tfrac{5}{4}e^2\sin 2\phi - \tfrac{1}{4}\gamma^2\sin 2\eta \quad \ldots\ldots\ldots\ldots(4)$$

The third-order terms in (3) and (4) are readily obtained from 7·06 (7) and 7·07 (11).

Also, in the undisturbed orbit, to the second order

$$s = \gamma\sin\eta' + e\gamma\{\sin(M + \eta') + \sin(M - \eta')\} \quad ; \ldots\ldots\ldots\ldots(5)$$

* The formulae to the third order are given by 7·06 (7) for r and 7·06 (11) for v which is denoted by λ in section 7·06 ; it is also to be noted that η in the formulae of section 7·06 is the same as η' of the present section, that is, $nt + \epsilon - \Omega$.

this is formula 7·06 (15) in terms of η'. We define σ by

$$\sigma = \gamma \sin \eta + e\gamma \{\sin(\phi + \eta) + \sin(\phi - \eta)\} \quad \ldots\ldots(6)$$

in which $\eta = gnt + \epsilon - \Omega$ and ϕ replaces M.

We refer to ρ, w and σ as *modified variables*.

For the sake of symmetry we write ϕ_1 for M_1 in the equations for r_1 and v_1; then

$$r_1/a_1 = 1 + \tfrac{1}{2}e_1^2 - e_1 \cos\phi_1 - \tfrac{1}{2}e_1^2 \cos 2\phi_1 \ldots \quad , \quad \ldots\ldots(5)$$

$$v_1 = n_1 t + \epsilon_1 + 2e_1 \sin\phi_1 + \tfrac{5}{4}e_1^2 \sin 2\phi_1 \ldots \quad ; \quad \ldots\ldots(6)$$

here $\phi_1 = n_1 t + \epsilon_1 - \varpi_1$ and, of course, $\gamma_1 = 0$.

The arguments of the terms involving ϕ and η take account of the constant motion of perigee and of the node; consequently ρ, w and σ may be regarded as approximations to r, v and s for the disturbed motion. We write

$$r = \rho + \delta r \quad , \quad v = w + \delta v \quad , \quad s = \sigma + \delta s \quad \ldots\ldots(7)$$

where δr, δv and δs are of second order at least; we then modify the equations of motion in terms of δr, δv and δs.

As we shall see, there will be a function R' associated with the modified variables and with the equations of motion; this may be designated the modified disturbing function. R' is, in fact, what R becomes when M is replaced by ϕ and $nt + \epsilon - \Omega$ by $\eta (\equiv gnt + \epsilon - \Omega)$. In general we write

$$R' \equiv R'(\rho, w, \sigma) \equiv R'(\phi, \eta) \quad .$$

It is to be noted that, by 7·06 (18) and (19), R' is a function of ξ also: accordingly, $R' \equiv R'(\xi, \phi, \eta)$.

17·08. Method of solving the equations of motion

In the earlier operations, the inclination of the Moon's orbit to the ecliptic is neglected—that is to say, $\gamma = s = z = 0$. The relevant equations of 17·03 are, on writing D for d/dt,

$$\tfrac{1}{2}D^2(r^2) - \frac{n^2a^3}{r} + n^2a^2 = Q(r, v) + n^2a^2\delta b \quad , \quad \ldots\ldots(1)$$

$$Dv - \frac{h}{r^2} = S(r, v) \quad , \quad \ldots\ldots(2)$$

$$\frac{1}{r}D^2r + \frac{n^2a^3}{r^3} - \dot{v}^2 = T(r, v) \quad , \quad \ldots\ldots(3)$$

where $$Q = r\frac{\partial R}{\partial r} + 2\int \frac{d'R}{dt}\,dt \quad , \quad S = \frac{1}{r^2}\int \frac{\partial R}{\partial v}\,dt \quad , \quad T = \frac{1}{r}\frac{\partial R}{\partial r} \quad ; \quad \ldots\ldots(4)$$

here, $R(r, v) = R_2 + R_3$. By 7·06 (5) and 7·06 (6) we have, when $\gamma = 0$,

$$R_2 = m^2n^2r^2\left(\frac{a_1}{r_1}\right)^3 \{\tfrac{1}{4} + \tfrac{3}{4}\cos 2(v - v_1)\} \quad , \quad \ldots\ldots(5)$$

$$R_3 = \frac{m^2n^2}{a}\left(\frac{a}{a_1}\right)r^3\left(\frac{a_1}{r_1}\right)^4 \{\tfrac{3}{8}\cos(v - v_1) + \tfrac{5}{8}\cos 3(v - v_1)\} \quad . \quad \ldots\ldots(6)$$

R_2 is $O(m^2)$, and R_3 is $O(m^4)$ since we assume that a/a_1 is $O(m^2)$.

By Euler's theorem, we have the general formulae

$$r\frac{\partial R_2}{\partial r}=2R_2\,,\quad r\frac{\partial R_3}{\partial r}=3R_3\,;\quad r_1\frac{\partial R_2}{\partial r_1}=-3R_2\,,\quad r_1\frac{\partial R_3}{\partial r_1}=-4R_3\,;\ \ldots(7)$$

hence,
$$r\frac{\partial R}{\partial r}=2R_2+3R_3\quad;\quad \frac{\partial R}{\partial r_1}=-\frac{1}{r_1}(3R_2+4R_3)\ \ldots\ldots\ldots\ldots(8)$$

The formulae 7·06 (5) and (6) show that in general R is a function of $v-v_1$; this is true, in particular, of (5) and (6) above; accordingly, $\frac{\partial R}{\partial v}=-\frac{\partial R}{\partial v_1}$. Now, R is developed by writing, as in 7·06 (17),

$$v-v_1=\xi+2e\sin M-2e_1\sin M_1\ldots\quad;$$

hence
$$\frac{\partial R}{\partial v}=-\frac{\partial R}{\partial v_1}=\frac{\partial R}{\partial \xi}\quad,\quad \frac{\partial^2 R}{\partial r\,\partial v}=\frac{\partial^2 R}{\partial r\,\partial \xi}\quad\text{etc.}\ \ldots\ldots\ldots\ldots(9)$$

We consider now the integral which appears in the expression for Q in (4). Now,

$$\frac{dR}{dt}=\frac{d'R}{dt}+\frac{\partial R}{\partial r_1}\dot r_1+\frac{\partial R}{\partial v_1}\dot v_1\quad;$$

hence, by (8) and (9),

$$\frac{d'R}{dt}=\frac{d}{dt}(R_2+R_3)+(3R_2+4R_3)\frac{\dot r_1}{r_1}+\frac{\partial R}{\partial \xi}\dot v_1$$

and
$$Q=4R_2+5R_3+2\int(3R_2+4R_3)\frac{\dot r_1}{r_1}dt+2\int\frac{\partial R}{\partial \xi}\dot v_1\,dt\ \ldots\ldots(10)$$

Similarly,
$$S=\frac{1}{r^2}\int\frac{\partial R}{\partial \xi}\,dt\quad,\quad T=\frac{1}{r^2}(2R_2+3R_3)\ \ldots\ldots\ldots\ldots(11)$$

For convenience, we write (10) as

$$Q=4R_2+5R_3+Q_0\ \ldots\ldots\ldots\ldots(12)$$

where Q_0 is the two final terms in (10).

The method of solution can now be briefly indicated at this point—it proceeds by successive approximations. We illustrate the method with reference to (1). We write $r=\rho+\delta r$, $v=w+\delta v$, where ρ is the modified radius vector and, as we shall see, δr is $O(m^2)$.

We first develop Q. We write it as follows:

$$Q=Q(\rho,w)+\delta_1 Q+\delta_2 Q+\ldots\ \ldots\ldots\ldots\ldots(13)$$

where
$$\delta_1 Q=\left(\frac{\partial Q}{\partial r}\right)\delta r+\left(\frac{\partial Q}{\partial v}\right)\delta v\quad,\ \ldots\ldots\ldots\ldots(14)$$

$$\delta_2 Q=\frac{1}{2}\left[\left(\frac{\partial^2 Q}{\partial r^2}\right)(\delta r)^2+2\left(\frac{\partial^2 Q}{\partial r\,\partial v}\right)\delta r\,\delta v+\left(\frac{\partial^2 Q}{\partial v^2}\right)(\delta v)^2\right]\quad,\ \ldots\ldots(15)$$

in which $\left(\frac{\partial Q}{\partial r}\right)$ etc. denotes the replacement of r and v by ρ and w after differentiation.

From (10)

$$Q(\rho, w) = 4R_2' + 5R_3' + 2\int (3R_2' + 4R_3')\frac{\dot{r}_1}{r_1}\,dt + 2\int \frac{\partial R'}{\partial \xi}\,\dot{v}_1\,dt \quad , \quad \ldots(16)$$

where R_2', R_3' are the expansions in 7·06 (18) and (19)—with $\gamma = 0$—where M is replaced by ϕ and $\eta = gnt + \epsilon - \Omega$. Thus $Q' \equiv Q(\rho, w)$ can be found to any desired degree of accuracy.

Consider now $\left(\frac{\partial Q}{\partial r}\right)$ appearing in (14). From (10), by means of (7), we have

$$\frac{\partial Q}{\partial r} = \frac{1}{r}(8R_2 + 15R_3) + 2\int \frac{1}{r}(6R_2 + 12R_3)\frac{\dot{r}_1}{r_1}\,dt + 2\int \frac{1}{r}(2R_2 + 3R_3)\,\dot{v}_1\,dt \quad \ldots(17)$$

Hence $\left(\frac{\partial Q}{\partial r}\right)$ is obtained by writing R_2', R_3' for R_2, R_3 in this equation. Similarly,

$$\left(\frac{\partial Q}{\partial v}\right) \equiv \left(\frac{\partial Q}{\partial \xi}\right) = 4\frac{\partial R_2'}{\partial \xi} + 5\frac{\partial R_3'}{\partial \xi} + 2\int\left(3\frac{\partial R_2'}{\partial \xi} + 4\frac{\partial R_3'}{\partial \xi}\right)\frac{\dot{r}_1}{r_1}\,dt + 2\int \frac{\partial^2 R}{\partial \xi^2}\,dt \quad . \quad (18)$$

Thus, the coefficients of δr and δv in (14) can be found to any desired degree of approximation. In the same way we develop $\delta_2 Q$ in (15). For example, from (17),

$$\left(\frac{\partial^2 Q}{\partial r\,\partial v}\right) = \frac{1}{\rho}\frac{\partial}{\partial \xi}(8R_2' + 15R_3') + 2\int \frac{1}{\rho}\frac{\partial}{\partial \xi}(6R_2' + 12R_3')\frac{\dot{r}_1}{r_1}\,dt + 2\int \frac{1}{\rho}\frac{\partial}{\partial \xi}(2R_2' + 3R_3')\,\dot{v}_1\,dt \quad \ldots(19)$$

Equation (1) is then a differential equation in δr given by

$$\tfrac{1}{2}D^2\{(\rho + \delta r)^2\} - \frac{n^2a^3}{\rho + \delta r} + n^2a^2 = Q' + \delta_1 Q + \delta_2 Q + \ldots + n^2a^2\,\delta b \quad \ldots(20)$$

The first approximation to δr is obtained by neglecting $(\delta r)^2$; higher approximations follow by the usual methods. Equations (2) and (3) are treated in a similar way.

17·09. Equations satisfied by ρ and w ($\gamma = 0$)

It is convenient to transform the equations 17·08 (20) and the others derived from 17·08 (2) and (3) by utilising equations satisfied by ρ and w, when $\gamma = 0$. Then

$$\rho/a = 1 - e\cos\phi + \tfrac{1}{2}e^2(1 - \cos 2\phi)\ldots$$
$$w = nt + \epsilon + 2e\sin\phi + \tfrac{5}{4}e^2\sin 2\phi + \ldots$$

where $\phi = cnt + \epsilon - \varpi$. Let

$$w_0 = cnt + \epsilon + 2e\sin\phi\ldots$$

so that

$$w = w_0 + n(1 - c)t \quad .$$

Further, let $\tau = ct$. Then

$$\rho/a = 1 - e\cos(n\tau + \epsilon - \varpi) + \dots$$
$$w_0 = n\tau + \epsilon + 2e\sin(n\tau + \epsilon - \varpi) + \dots \quad .$$

Evidently, ρ and w_0 refer to an undisturbed orbit in the ecliptic when the time-variable is τ. Put $R = 0$ in 17·08 (1), (2) and (3); then ρ and w_0 satisfy the equations

$$\frac{1}{2}\frac{d^2}{d\tau^2}(\rho^2) - \frac{n^2a^3}{\rho} + n^2a^2 = 0 \quad , \quad \frac{dw_0}{d\tau} = \frac{h_0}{\rho^2} \quad , \quad \frac{1}{\rho}\frac{d^2\rho}{d\tau^2} + \frac{n^2a^3}{\rho^3} - \left(\frac{dw_0}{d\tau}\right)^2 = 0 \quad ,$$

from which

$$\frac{1}{2}\frac{d^2}{dt^2}(\rho^2) = c^2n^2a^2\left(\frac{a}{\rho} - 1\right) \quad , \quad \frac{dw_0}{dt} = \frac{ch_0}{\rho^2} \quad , \quad \frac{1}{\rho}\frac{d^2\rho}{dt^2} + c^2n^2\frac{a^3}{\rho^3} - \frac{c^2h_0^2}{\rho^4} = 0 \quad ;$$

in these equations $h_0 = na^2\sqrt{(1-e^2)}$.

Since $(c^2 - 1)$ and $(c - 1)$ are each of $O(m^2)$, we write the previous equations as follows:

$$\tfrac{1}{2}D^2\left(\frac{\rho}{a}\right)^2 - n^2\frac{a}{\rho} + n^2 = n^2(c^2-1)\left(\frac{a}{\rho} - 1\right) \quad , \quad \dots\dots\dots\dots(1)$$

$$Dw - \frac{h_0}{\rho^2} = (c-1)\left(\frac{h_0}{\rho^2} - n\right) \quad , \quad \dots\dots\dots\dots(2)$$

$$D^2\rho + n^2\frac{a^3}{\rho^2} - \frac{h_0^2}{\rho^3} = (c^2-1)\left(\frac{h_0^2}{\rho^3} - n^2\frac{a^3}{\rho^2}\right) \quad . \quad \dots\dots\dots\dots(3)$$

Since $(a/\rho) - 1$ is $O(e)$, the right-hand side of each of these equations is of order m^2e.

17·10. Equation for the radius vector

We start with 17·08 (1). Instead of developing the equation in powers of δr, where δr is defined by $r = \rho + \delta r$, it is more convenient for later interpretation to write

$$\frac{a}{r} = \frac{a}{\rho} + \delta u \quad \dots\dots\dots\dots(1)$$

from which it is easily found that $\delta r = -\frac{\rho^2}{a}\delta u + \frac{\rho^3}{a^2}(\delta u)^2 - \frac{\rho^4}{a^3}(\delta u)^3 \dots$.

Then

$$\frac{1}{2}\frac{r^2}{a^2} = \frac{1}{2}\left(\frac{\rho}{a}\right)^2 - \left(\frac{\rho}{a}\right)^3\delta u + \frac{3}{2}\left(\frac{\rho}{a}\right)^4(\delta u)^2 \dots \quad .$$

Let P be defined by

$$P = -\left\{\left(\frac{\rho}{a}\right)^3 - 1\right\}\delta u + \frac{3}{2}\left(\frac{\rho}{a}\right)^4(\delta u)^2 \dots \quad . \quad \dots\dots\dots\dots(2)$$

Then

$$\frac{1}{2}\frac{r^2}{a^2} = \frac{1}{2}\frac{\rho^2}{a^2} - \delta u + P$$

and 17·08 (1) becomes

$$\tfrac{1}{2}D^2\left(\frac{\rho}{a}\right)^2 - n^2\frac{a}{\rho} + n^2 - D^2(\delta u) - n^2\,\delta u + P'' = \frac{Q}{a^2} + n^2\,\delta b$$

where $P'' = \dfrac{d^2}{dt^2}P$. We now use 17·09 (1) and so obtain

$$D^2(\delta u) + n^2\,\delta u = n^2(c^2-1)\left(\frac{a}{\rho} - 1\right) - \frac{Q}{a^2} + P'' - n^2\,\delta b \quad . \quad \ldots\ldots\ldots(3)$$

In this equation Q is $O(m^2)$ and δb is $O(m^2)$; the first term on the right is of order m^2e; from (2) the coefficient of δu is of order e. It is evident that δu is $O(m^2)$. For a first approximation, to $O(m^3)$—the order of e being taken to be that of m—we need only consider the first term of P in (2).

De Pontécoulant's method of solving (3) is to assume that δu is given by a series of terms with arguments involving ξ, ϕ and ϕ_1, and with undetermined coefficients as amplitudes. This general expression for δu is substituted in (3) which then becomes an identity; a series of equations involving the undetermined coefficients *and* c and δb is then obtained. A similar procedure is applied to 17·08 (2) and (3). There is then a sufficient number of equations for deriving the values of the coefficients *and* c and δb in terms of m, e and e_1. Before we illustrate the method in some detail, we first make some remarks in the next section on certain terms appearing on the right-hand side of (3) arising from the disturbing function in the first instance.

17·11. On certain terms in the disturbing function

A typical term appearing on the right-hand side of 17·10 (3) may be written

$$n^2k_i\cos(\alpha\xi + \beta\phi + \gamma\phi_1) \equiv n^2k_i\cos(pnt + q) \quad \ldots\ldots\ldots(1)$$

where α, β and γ are positive or negative integers, and k_i is $O(m^i)$ with respect to e, e_1, m, a/a_1. Now

$$\begin{aligned} \xi &= (n-n_1)t + \ldots = n(1-m)t + \ldots \quad , \\ \phi &= cnt + \ldots \qquad\quad = n(1-\tfrac{3}{4}m^2\ldots)t + \ldots \quad , \\ \phi_1 &= n_1t + \ldots \qquad\quad = mnt + \ldots \quad . \end{aligned}$$

Hence, p in (1) is given by

$$p = \alpha + \beta - m(\alpha - \gamma) - \tfrac{3}{4}\beta m^2\ldots \quad \ldots\ldots\ldots(2)$$

The equation in δu—17·10 (3)—can then be written

$$D^2(\delta u) + n^2\,\delta u = n^2A + n^2\,\Sigma\,k_i\cos(pnt + q) \quad ; \ldots\ldots\ldots(3)$$

in arriving at this form we shall suppose for the sake of simplicity that P is ignored. Now de Pontécoulant's procedure is tantamount to obtaining the particular integral of (3); accordingly,

$$\delta u = A + \Sigma\frac{k_i}{1-p^2}\cos(pnt + q) \quad .$$

Case (i). If p is of the form

$$p = 1 + \alpha_1 m + \beta_1 m^2 + \dots \quad , \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4)$$

where $\alpha_1 \neq 0$, then $k_i/(1-p^2) \fallingdotseq -k_i/(2\alpha_1 m)$. Hence, the term (1) in this case provides the corresponding part of the particular integral to order $(i-1)$; consequently, to obtain δu to order i we must include terms of order $(i+1)$ in the disturbing function and elsewhere.

From (2) the conditions that p shall have the form in (4) are : $\alpha + \beta = 1$, $\alpha - \gamma \neq 0$. Arguments of such terms are ξ, $2\xi - \phi$, $3\xi - 2\phi$ in particular and, in general

$$\xi + (1+\gamma_1)\phi_1, \qquad 2\xi - \phi + (2+\gamma_1)\phi_1 \quad , \quad 3\xi - 2\phi + (3+\gamma_1)\phi_1 \quad , \dots$$

in which $\gamma_1 \neq 0$.

There will be similar terms in the latitude equation with $\eta\,(\equiv gnt + \epsilon - \Omega)$ replacing ϕ.

Case (ii). If p is of the form

$$p = 1 + \beta_1 m^2 + \dots \quad ,$$

then $k_i/(1-p^2) \fallingdotseq -k_i/(2\beta_1 m^2)$. Hence the term (1) in this case provides the corresponding part of the particular integral to order $(i-2)$. Consequently, if we require δu to order i, we must include certain terms to order $(i+2)$. In this case, we must have from (2),

$$\alpha + \beta = 1 \quad , \quad \alpha - \gamma = 0 \quad , \quad \beta \neq 0 \quad .$$

The arguments are then of the form : $(1-\beta)\xi + \beta\phi + (1-\beta)\phi_1$. Typical arguments are : ϕ, $2\xi - \phi + 2\phi_1$, $3\xi - 2\phi + 3\phi_1$, ... In the latitude equation there will be similar arguments, η replacing ϕ.

17·12. Calculation of $Q(\rho, w)$

In this section we omit the parallax term in R—that is, R_3—and we have, simply, $R = R_2$. From 17·08 (16),

$$Q' \equiv Q(\rho, w) = 4R_2' + 6\int R_2' \frac{\dot{r}_1}{r_1}\,dt + 2\int \frac{\partial R_2'}{\partial \xi}\,\dot{v}_1\,dt \quad . \qquad \dots\dots\dots (1)$$

(i) Since $r_1/a_1 = 1 - e_1 \cos\phi_1 + \frac{1}{2}e_1^2\,(1 - \cos 2\phi_1)$,

$$a_1/r_1 = 1 + e_1 \cos\phi_1 + e_1^2 \cos 2\phi_1 \quad \text{and} \quad \dot{\phi}_1 = n_1 = mn$$

then, to $O(e_1^2)$,

$$\dot{r}_1/r_1 = mne_1(\sin\phi_1 + \tfrac{3}{2}e_1 \sin 2\phi_1) \quad .$$

Now,
$$R_2' = \tfrac{1}{4}m^2n^2a^2(1 + 3\cos 2\xi + 3e_1 \cos\phi_1) \qquad \dots\dots\dots (2)$$

with sufficient approximation for our purpose here. Hence

$$R_2'\,\dot{r}_1/r_1 = \tfrac{1}{4}m^3n^3a^2e_1[\sin\phi_1 + \tfrac{3}{2}\sin(2\xi + \phi_1) - \tfrac{3}{2}\sin(2\xi - \phi_1) + 3e_1 \sin 2\phi_1].$$

Since $\xi = n(1-m)t + \dots$, $\phi_1 = mnt + \dots$, we find that the second term of (1) is, to $O(m^2e_1^2) \equiv O(m^4)$,

$$-\tfrac{3}{2}m^2n^2a^2[e_1\cos\phi_1+\tfrac{3}{4}me_1\{\cos(2\xi+\phi_1)-\cos(2\xi-\phi_1)\}$$
$$+\tfrac{5}{2}e_1^2\cos 2\phi_1] \quad . \quad .....(3)$$

It is seen that the integration of the term in $\sin\phi_1$ reduces the order by one ; hence the inclusion of the final term in (2).

(ii) Consider now the second integral in (1). Since

$$v_1=n_1t+\epsilon_1+2e_1\sin\phi_1+\tfrac{5}{4}e_1^2\sin 2\phi_1 \quad ,$$

then
$$\dot{v}_1=mn(1+2e_1\cos\phi_1+\tfrac{5}{2}e_1^2\cos 2\phi_1) \quad . \quad .....(4)$$

Since $\dot{r}_1/r_1$ is one order higher than $\dot{v}_1$, we consider terms of R_2' one order higher than in (2).

The relevant terms in R_2' which depend on ξ are given by

$$R_2'=m^2n^2a^2[\tfrac{3}{4}\cos 2\xi-\tfrac{9}{4}e\cos(2\xi-\phi)+\tfrac{3}{4}e\cos(2\xi+\phi)$$
$$+\tfrac{21}{8}e_1\cos(2\xi-\phi_1)-\tfrac{3}{8}e_1\cos(2\xi+\phi_1)+\tfrac{15}{2}e^2\cos(2\xi-2\phi)] \ ...(5)$$

from which $\dfrac{\partial R_2'}{\partial\xi}$ is obtained at once ; this series is multiplied by the series in (3). On integration we then find that

$$2\int\frac{\partial R_2'}{\partial\xi}\dot{v}_1\,dt=2m^3n^2a^2\left[\frac{3\cos 2\xi}{4(1-m)}-\frac{9}{2}e\frac{\cos(2\xi-\phi)}{2-2m-c}+\frac{3}{2}e\frac{\cos(2\xi+\phi)}{2-2m+c}\right.$$
$$\left.+\frac{27}{4}e_1\frac{\cos(2\xi-\phi_1)}{2-2m-m}+\frac{3}{4}e_1\frac{\cos(2\xi+\phi_1)}{2-2m+m}+\frac{15}{8}e^2\frac{\cos(2\xi-2\phi)}{1-m-c}\right] \quad . \ ...(6)$$

Since $c\doteqdot 1-\tfrac{3}{4}m^2$, the denominator of the final term is $O(m)$; hence the inclusion of this term in e^2. The right-hand side of the previous formula can be written down to $O(m^4)$; the coefficient of the first term, for example, within the bracket is $\tfrac{3}{4}(1+m)$; the coefficient of the second term is $-\tfrac{9}{2}e$, and so on.

Writing as before $Q'=4R_2'+Q_0$, where Q_0 is the sum of the last two terms in (1), we find that

$$Q_0=m^2n^2a^2[\tfrac{3}{2}m(1+m)\cos 2\xi-9me\cos(2\xi-\phi)+me\cos(2\xi+\phi)$$
$$-\tfrac{15}{4}e^2\cos(2\xi-2\phi)-\tfrac{3}{2}e_1\cos\phi_1+\tfrac{63}{8}me_1\cos(2\xi-\phi_1)$$
$$-\tfrac{3}{8}me_1\cos(2\xi+\phi_1)] \quad . \quad .....(7)$$

Q' is then obtained by adding $4R_2'$ to this last expression.

17·13. The solution of the equation for the radius vector

To illustrate de Pontécoulant's method we shall keep only the terms in Q and P'' which have the following arguments : ϕ, 2ξ, $2\xi-\phi$ and $2\xi-2\phi$. We then assume that δu is given, in de Pontécoulant's notation, by

$$\delta u=a_0+a_1e\cos\phi+a_2\cos 2\xi+a_3e\cos(2\xi-\phi)+a_4e^2\cos(2\xi-2\phi) \ ...(1)$$

in which the a's are to be determined.*

* De Pontécoulant used an equation (1) with 76 a's with factors depending on one or more of e, e_1 and a/a_1.

(i) *Calculation of Q'.*

Keeping only the terms in R_2'—given by 7·06 (18)—and in Q_0—given by 17·12 (7)—which have the selected arguments, we find that

$$\frac{Q'}{a^2} = m^2n^2[1 - 2e\cos\phi + 3(1+\tfrac{1}{2}m)\cos 2\xi - 9e(1+m)\cos(2\xi-\phi) + \tfrac{15}{4}e^2\cos(2\xi-2\phi)] \quad .$$

(ii) *Calculation of P''.*

By 17·10 (2), $P = -\left\{\left(\frac{\rho}{a}\right)^3 - 1\right\}\delta u = (3e\cos\phi - 3e^2)\,\delta u$, to the order concerned. Substitute the expression for δu given by (1). On simplification we keep only the terms with the selected arguments. It is then found that

$$\begin{aligned} P'' = &-n^2[3a_0c^2e\cos\phi + 4(1-m)^2(\tfrac{3}{2}a_3 - 3a_2)e^2\cos 2\xi \\ &+ (\tfrac{3}{2}a_2e - 3a_3e^3)(2-2m-c)^2\cos(2\xi-\phi) \\ &+ 6(1-m-c)^2a_3e^2\cos(2\xi-2\phi)] \quad . \end{aligned}$$

(iii) *Calculation of* $n^2(c^2-1)\left(\frac{a}{\rho}-1\right)$.

We keep only the term in ϕ; the expression is then

$$n^2(c^2-1)(e\cos\phi) \quad .$$

(iv) $D^2(\delta u)$.

From (1),

$$\begin{aligned} D^2(\delta u) = &-n^2[a_1c^2e\cos\phi + 4(1-m)^2a_2\cos 2\xi \\ &+ (2-2m-c)^2a_3e\cos(2\xi-\phi) + 4(1-m-c)^2a_4e^2\cos(2\xi-2\phi)] \quad . \end{aligned}$$

These expressions are now substituted in 17·10 (3), the equation for the radius vector; equating the coefficients of the various terms (including the non-periodic) we obtain 5 equations connecting c, δb and the five a's $(a_0, \ldots, a_4)$; they are

$$\left.\begin{aligned} &a_0 + m^2 + \delta b = 0 \\ &a_1(1-c^2) - 2m^2 + 3a_0c^2 - (c^2-1) = 0 \\ &a_2\{1 - 4(1-m)^2\} + 3m^2\left(1+\frac{m}{2}\right) + 4(1-m)^2(\tfrac{3}{2}a_3 - 3a_2)e^2 = 0 \\ &a_3\{1 - (2-2m-c)^2\} - 9m^2(1+m) + \tfrac{3}{2}(2-2m-c)^2(a_2 - 2a_3)e^2 = 0 \\ &a_4\{1 - 4(1-m-c)^2\} + \tfrac{15}{4}m^2 + 6(1-m-c)^2a_3 = 0 \end{aligned}\right\} \quad \ldots(2)$$

To evaluate the seven unknowns—c, δb and the five a's—we require two more equations. These are found, as we shall show in the next section, from the longitude equation which, as we have seen earlier, introduces another unknown, δh. The necessary additional equation—making eight in all—is provided by 17·08 (3); accordingly, the eight unknowns δc, δb, the five a's and δh, can be evaluated.

17·13. The longitude equation

This equation is, from 17·08 (2),

$$Dv - \frac{h_0(1+\delta h)}{r^2} = S(r, v) \quad \ldots\ldots(1)$$

in which we have written $h_0(1+\delta h)$ for h and S is, by 17·08 (4), given by

$$S = \frac{1}{r^2}\int \frac{\partial R}{\partial v}\,dt = \frac{1}{r^2}\int \frac{\partial R}{\partial \xi}\,dt \quad \ldots\ldots(2)$$

Now, $v = w + \delta v$ and, by 17·10 (1), $a/r = a/\rho + \delta u$; also, w satisfies the equation 17·09 (2), namely,

$$Dw - \frac{h_0}{\rho^2} = (c-1)\left(\frac{h_0}{\rho^2} - n\right) \quad \ldots\ldots(3)$$

Subtracting (3) from (1), we obtain the equation for δv:

$$D(\delta v) = \frac{h_0(1+\delta h)}{a^2}\left\{\left(\frac{a}{\rho}\right)^2 + 2\frac{a}{\rho}\,\delta u + (\delta u)^2\right\} + \frac{1}{r^2}\int \frac{\partial R}{\partial \xi}\,dt - \frac{h_0}{\rho^2} - n(c-1)\left(\frac{a^2}{\rho^2} - 1\right) \quad \ldots\ldots(4)$$

This equation is correct to all orders. As before, we neglect the parallax terms in R so that $R = R_2$; further, for a first approximation we neglect $(\delta u)^2$ in (4). Now, δu is $O(m^2)$, and δh is $O(m^2)$ as found in 17·05. Further $h_0 = na^2\sqrt{(1-e^2)}$ and, neglecting $(\delta u)^2$,

$$\frac{1}{r^2} = \frac{1}{a^2}\left\{\left(\frac{a}{\rho}\right)^2 + 2\frac{a}{\rho}\,\delta u\right\} \quad \ldots\ldots(5)$$

To $O(m^3)$, e and m being of the same order, equation (4) becomes

$$\frac{1}{n}D(\delta v) = \left(\frac{a}{\rho}\right)^2 \delta h + 2\frac{a}{\rho}\,\delta u + \frac{1}{na^2}\left\{\left(\frac{a}{\rho}\right)^2 + \frac{2a}{\rho}\,\delta u\right\}\int \frac{\partial R_2'}{\partial \xi}\,dt - 2e(c-1)\cos\phi \quad \ldots\ldots(6)$$

As in the case of the equation for the radius vector, we keep only those terms with arguments 2ξ, $2\xi - \phi$ and $2\xi - 2\phi$—the others do not involve ξ. We then find that, to $O(m^3)$,

$$\int \frac{\partial R_2'}{\partial \xi}\,dt = m^2na^2\left[\tfrac{3}{4}(1+m)\cos 2\xi - \tfrac{9}{2}e\cos(2\xi - \phi) + \tfrac{15}{8}e^2\frac{\cos(2\xi - 2\phi)}{1-m-c}\right] \quad .(7)$$

The coefficient in the last term within the square brackets is simply $-\frac{15e^2}{m}$ to the order concerned. Since the expression on the right-hand side of (7) is $O(m^2)$, it will be sufficient in (5) to discard δu and to write simply $a^2/\rho^2 = 1 + 2e\cos\phi$. We then find that

$$\frac{1}{nr^2}\int \frac{\partial R_2'}{\partial \xi}\,dt = m^2\left\{\tfrac{3}{4}(1+m)\cos 2\xi - \tfrac{15}{4}e\cos(2\xi-\phi) - 15\frac{e^2}{m}\cos(2\xi-2\phi)\right\} \quad \ldots\ldots\ldots(8)$$

We have also, in (6),

$$\left(\frac{a}{\rho}\right)^2 \delta h = \delta h + 2e\,\delta h\cos\phi \quad .$$

We now substitute the expression for δu, given by 17·13 (1), in (6). We then have

$$\frac{1}{n}D(\delta v) = A_0 + A_1 e\cos\phi + A_2\cos 2\xi + A_3 e\cos(2\xi-\phi) + A_4 e^2\cos(2\xi-2\phi),$$

where

$$\left.\begin{aligned} A_0 &= \delta h + 2a_0 + a_1e^2 \\ A_1 &= 2\delta h + 2a_1 + a_1e^2 + 2a_0 - 2(c-1) \\ A_2 &= 2a_2 + a_3e^2 + \tfrac{3}{4}m^2(1+m) \\ A_3 &= 2a_3 + a_2 - \tfrac{15}{4}m^2 \\ A_4 &= a_2 + a_3 + 2a_4 - \tfrac{15}{8}m \end{aligned}\right\} \quad \ldots\ldots\ldots(9)$$

Hence,

$$\delta v = A_0 nt + B_0 + \frac{A_1}{c}e\sin\phi + \frac{A_2\sin 2\xi}{2(1-m)} + A_3\frac{e\sin(2\xi-\phi)}{2-2m-c} + \tfrac{1}{4}A_4\frac{e^2\sin(2\xi-2\phi)}{1-m-c} \quad \ldots\ldots\ldots(10)$$

and, accordingly, since $w = nt + \epsilon + 2e\left(1-\frac{e^2}{8}\right)\sin\phi$,

$$v \equiv w + \delta v = n(1+A_0)t + \epsilon + B_0 + 2e\left(1 - \frac{e^2}{8} + \frac{A_1}{2c}\right)\sin\phi + \delta v' \quad , \ldots(11)$$

where $\delta v'$ denotes the terms in (10) with arguments 2ξ, $2\xi-\phi$, $2\xi-2\phi$.

B_0 is an unnecessary constant of integration since we have ϵ already in that capacity.

We assume that the non-periodic part of v in (11) is $nt+\epsilon$ and that the coefficient of $\sin\phi$ is the same as that of $\sin M$ in the undisturbed orbit, namely, $2e(1-\frac{1}{8}e^2)$. Hence $A_0 = A_1 = 0$, that is,

$$\delta h + 2a_0 + a_1e^2 = 0 \quad , \ldots\ldots\ldots(12)$$

$$2\delta h + 2a_1 + a_1e^2 + 2a_0 - 2(c-1) = 0 \quad . \ldots\ldots\ldots(13)$$

This procedure as regards the coefficient of $\sin\phi$ differs from that followed by de Pontécoulant who keeps A_1 as a general coefficient later to be derived in terms of m, e as in the case of A_2, A_3, ... The procedure adopted here is advantageous when observations are invoked for the evaluation of the constants of integration.

17·14. Equation in T

This equation, from 17·08 (3), is

$$\ddot{r}+\frac{n^2a^3}{r^2}-r\dot{v}^2=rT \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From 17·08 (4), $rT=\dfrac{\partial R}{\partial r}=\dfrac{2}{r}R_2$; as before, we neglect R_3. It is to be remarked that (1) is not an independent equation; it becomes an identity when the solutions for r and v, previously obtained, are substituted. Accordingly, the non-periodic terms arising in (1) will provide a relation involving at least two of the eight unknowns δb, δh, c and the five a's $(a_0, \ldots, a_4)$. We discard $\ddot{r}$ in (1), since $\ddot{r}$ contains no constant term, and write (1) as

$$r\dot{v}^2=\frac{n^2a^3}{r^2}-\frac{2}{r}R_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

We first derive the non-periodic terms on the right-hand side of (2) to the third order.

(i) $$\frac{n^2a^3}{r^2}=\frac{n^2a^3}{\rho^2}+2n^2\frac{a^2}{\rho}\delta u$$
$$=n^2a\{1+\tfrac{1}{2}e^2+\ldots+2(1+e\cos\phi)(a_0+a_1e\cos\phi+\ldots)\}$$
$$=n^2a(1+\tfrac{1}{2}e^2+2a_0+a_1e^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

(ii) $$\frac{2}{r}R_2=\frac{2}{a}(1+e\cos\phi)\,.\,m^2n^2a^2(\tfrac{1}{4}-\tfrac{1}{2}e\cos\phi)$$
$$=\tfrac{1}{2}m^2n^2a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

(iii) $\dot{v}=\dot{w}+D(\delta v)$. From 17·09 (2),

$$\dot{w}=\frac{h_0}{\rho^2}+(c-1)\left(\frac{h_0}{\rho^2}-n\right)$$

where $h_0=na^2\sqrt{(1-e^2)}$. To the third order,

$$\dot{v}^2=\frac{c^2h_0^2}{\rho^4}+\frac{2nch_0(1-c)}{\rho^2}+\frac{2ch_0}{\rho^2}D(\delta v) \quad .$$

Hence, since $r=\rho-\rho\,\delta u/a^2$ and $c=1+O(m^2)$,

$$r\dot{v}^2=\frac{n^2c^2a^4(1-e^2)}{\rho^3}+\frac{2n^2a^2(1-c)}{\rho}-\frac{n^2a^3}{\rho^2}\delta u+2n\frac{a^2}{\rho}D(\delta v) \quad \ldots\ldots(5)$$

The first term in (5) contributes

$$n^2c^2a(1-e^2)(1+\tfrac{3}{2}e^2) \quad \text{or} \quad n^2c^2a(1+\tfrac{1}{2}e^2) \quad .$$

The second term contributes $2n^2a(1-c)$.

The third term is $-n^2a[1+\frac{1}{2}e^2+2e\cos\phi+\ldots][a_0+a_1e\cos\phi+\ldots]$ and it contributes $-n^2a\{a_0(1+\frac{1}{2}e^2)+a_1e^2\}$.

The fourth term makes no contribution, since $A_0=A_1=0$, by the results of 17·13.

The total contribution of $r\dot{v}^2$ is then

$$n^2a[c^2(1+\tfrac{1}{2}e^2)+2(1-c)-a_0(1+\tfrac{1}{2}e^2)-a_1e^2]$$

or
$$n^2a[(c-1)^2+\tfrac{1}{2}e^2+1-a_0(1+\tfrac{1}{2}e^2)-a_1e^2] \quad \ldots\ldots(6)$$

from which we can discard $(c-1)^2$ which is $O(m^4)$.

From (3), (4) and (5) we then obtain

$$3a_0(1+\tfrac{1}{6}e^2)+2a_1e^2-\tfrac{1}{2}m^2=0 \quad . \quad \ldots\ldots(7)$$

This is the final—the eighth—relation required.

17·15. Evaluation of the eight constants

We have the five equations given by 17·12 (2), the two equations 17·13 (12) and (13) and the equation 17·14 (7). In the first group of five equations we simplify some of the coefficients paying attention to the fact that $c=1+O(m^2)$ and neglecting unnecessary powers of m. The eight equations are then :

$$a_0+m^2+\delta b=0 \quad ; \quad \ldots\ldots(1)$$

$$3a_0c^2+a_1(1-c^2)-2m^2-(c^2-1)=0 \quad ; \quad \ldots\ldots(2)$$

$$-3a_2(1-\tfrac{8}{3}m+\tfrac{4}{3}m^2)+3m^2\left(1+\frac{m}{2}\right)+4(1-2m)(\tfrac{3}{2}a_3-3a_2)e^2=0 \quad ; \quad \ldots\ldots(3)$$

$$4m(1-\tfrac{11}{8}m)a_3-9m^2(1+m)+\tfrac{3}{2}(1-4m)(a_2-2a_3e^2)=0 \quad ; \quad \ldots\ldots(4)$$

$$a_4(1-4m^2)+\tfrac{15}{4}m^2+6m^2a_3=0 \quad ; \quad \ldots\ldots(5)$$

$$\delta h+2a_0+a_1e^2=0 \quad ; \quad \ldots\ldots(6)$$

$$2\delta h+2a_1+a_1e^2+2a_0-2(c-1)=0 \quad ; \quad \ldots\ldots(7)$$

$$3a_0(1+\tfrac{1}{6}e^2)+2a_1e^2-\tfrac{1}{2}m^2=0 \quad . \quad \ldots\ldots(8)$$

Eliminate δh in (6) and (7) ; then

$$2a_0-a_1(2-e^2)+2(c-1)=0 \quad . \quad \ldots\ldots(9)$$

These equations are solved by successive approximations. Assuming that $c=1+O(m^2)$, we see from (9) that a_0 and a_1 are at least $O(m^2)$. Then, from (8),

$$a_0=\tfrac{1}{6}m^2 \quad ,$$

which is correct to third order.

From (9), neglecting a_1e^2, we have

$$a_1=a_0+(c-1) \quad . \quad \ldots\ldots(10)$$

Neglecting $a_1(1-c^2)$ in (2), we find that this equation can be written

$$\tfrac{1}{2}m^2(c^2-1)-\tfrac{3}{2}m^2-(c^2-1)=0$$

from which, to $O(m^2)$, $c^2=1-\tfrac{3}{2}m^2$, or

$$c=1-\tfrac{3}{4}m^2 \quad , \quad \ldots\ldots(11)$$

a result we obtained earlier. From (10), $a_1=-\tfrac{7}{12}m^2$, which is correct to $O(m^3)$. From (6), $\delta h=-\tfrac{1}{3}m^2$, which is correct to $O(m^3)$.

The first approximations to the coefficients a_2, a_3, a_4 are easily found.

The complete results, so far as the first approximation is concerned, are

$$a_0 = \tfrac{1}{6}m^2 \quad , \quad a_1 = -\tfrac{7}{12}m^2 \quad , \quad a_2 = m^2 \quad , \quad a_3 = \tfrac{15}{8}m \quad , \quad a_4 = -\tfrac{15}{4}m^2 \quad ,$$
$$c = 1 - \tfrac{3}{4}m^2 \quad , \quad \delta h = -\tfrac{1}{3}m^2 \quad , \quad \delta b = -\tfrac{7}{6}m^2 \quad .$$

The second approximation for a_2 is easily derived from (3) by the usual procedure for such occasions ; the result is :

$$a_2 = m^2 + \tfrac{19}{6}m^3 + \tfrac{15}{4}me^2.$$

Similarly, $$a_3 = \tfrac{15}{8}m + \tfrac{329}{64}m^2 \quad ;$$

all the other quantities have been determined with the accuracy required.

The values of A_2, A_3, A_4 in 17·13 (9) can then be calculated ; the coefficients of the terms with arguments 2ξ, $2\xi - \phi$ and $2\xi - 2\phi$ in the expression 17·09 (10) for the perturbation δv in longitude are then readily computed.

17·16. The solutions for radius vector and longitude

We have $$\frac{a}{r} = \frac{a}{\rho} + \delta u \quad , \quad v = w + \delta v$$

where a/ρ and w are the expressions for the elliptic orbit with ϕ replacing the mean anomaly M.

The expansions are given to the third order except for terms with arguments $2\xi - 2\phi$, $k\phi$ ($k = 1, 2, 3, 4$) which are given to the fourth order, for reasons already stated. We add also the terms with argument $2\xi + \phi$ which we omitted from the analysis from considerations of simplicity. The results are as follows.

$$\begin{aligned}\frac{a}{r} = 1 &+ \tfrac{1}{6}m^2 + e(1 - \tfrac{1}{8}e^2 - \tfrac{7}{12}m^2)\cos\phi \\ &+ (m^2 + \tfrac{19}{6}m^3 + \tfrac{15}{4}me^2)\cos 2\xi + em(\tfrac{15}{8} + \tfrac{329}{64}m)\cos(2\xi - \phi) \\ &+ \tfrac{33}{16}em^2\cos(2\xi + \phi) - \tfrac{15}{4}e^2m^2\cos(2\xi - 2\phi) \\ &+ e^2(1 - \tfrac{1}{3}e^2)\cos 2\phi + \tfrac{9}{8}e^3\cos 3\phi + \tfrac{4}{3}e^4\cos 4\phi \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)\end{aligned}$$

$$\begin{aligned}v = nt + \epsilon &+ 2e(1 - \tfrac{1}{8}e^2)\sin\phi + (\tfrac{11}{8}m^2 + \tfrac{59}{12}m^3 + \tfrac{75}{16}e^2m)\sin 2\xi \\ &+ em(\tfrac{15}{4} + \tfrac{2637}{160}m)\sin(2\xi - \phi) + \tfrac{17}{8}em^2\sin(2\xi + \phi) \\ &+ \tfrac{33}{32}e^2m\sin(2\xi - 2\phi) + \tfrac{5}{4}(1 - \tfrac{11}{30}e^2)e^2\sin 2\phi \\ &+ \tfrac{13}{12}e^3\sin 3\phi + \tfrac{1033}{960}e^4\sin 4\phi \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)\end{aligned}$$

17·17. Terms depending on e_1

(i) The equation for δu is, from 17·10 (3),

$$D^2(\delta u) + n^2\delta u + \frac{Q}{a^2} - P'' + n^2\,\delta b - n^2(c^2 - 1)\left(\frac{a}{\rho} - 1\right) = 0 \quad , \quad \ldots\ldots (1)$$

where, to the third order, $Q = 4R_2' + Q_0$. We consider only the terms depending on the first power of e_1. From 17·12 (7),

$$Q_0 = -\tfrac{3}{2}m^2n^2a^2e_1\cos\phi_1 \quad ;$$

also, R_2' is given from 7·06 (18), ϕ replacing M and ϕ_1 replacing M_1. Let δu_1 be the part of δu depending on e_1 only. Then

$$D^2(\delta u_1)+n^2(\delta u_1)=-m^2n^2e_1\{\tfrac{3}{2}\cos\phi_1+\tfrac{21}{2}\cos(2\xi-\phi_1)-\tfrac{3}{2}\cos(2\xi+\phi_1)\}\,.$$

Now, $\phi_1\equiv n_1t+\epsilon_1-\varpi_1=mnt+\ldots$. Hence the particular integral is

$$\delta u_1=-\tfrac{3}{2}\frac{m^2e_1}{1-m^2}\cos\phi_1-\tfrac{21}{2}\frac{m^2e_1\cos(2\xi-\phi_1)}{1-(2-3m)^2}+\tfrac{3}{2}\frac{m^2e_1\cos(2\xi+\phi_1)}{1-(2-m)^2}$$

or, to the third order,

$$\delta u_1=-\tfrac{3}{2}m^2e_1\cos\phi_1+\tfrac{7}{2}m^2e_1\cos(2\xi-\phi_1)-\tfrac{1}{2}m^2e_1\cos(2\xi+\phi_1)\quad\ldots(2)$$

(ii) If δv_1 is the perturbation in longitude depending on e_1, the equation for δv_1 is deduced from 17·13 (6) to be

$$\frac{1}{n}D(\delta v_1)=2\frac{a}{\rho}\delta u_1+\frac{1}{nr^2}\int\frac{\partial R_2'}{\partial\xi}\,dt\quad\ldots\ldots(3)$$

It is sufficient to write $\rho=a$ in the first term and $r=a$ in the second, since R_2' is $O(m^2e_1)$. Now

$$\frac{\partial R_2'}{\partial\xi}=-m^2n^2a^2\{\tfrac{21}{4}e_1\sin(2\xi-\phi_1)-\tfrac{3}{4}e_1\sin(2\xi+\phi_1)\}\,.$$

The second term on the right of (3) is then, to $O(m^2e_1)$,

$$m^2e_1\{\tfrac{21}{8}\cos(2\xi-\phi_1)-\tfrac{3}{8}\cos(2\xi+\phi_1)\,.$$

Add these terms to $2\delta u_1$ given by (2) and integrate; then

$$\delta v_1=-3me_1\sin\phi_1+\tfrac{77}{16}m^2e_1\sin(2\xi-\phi_1)-\tfrac{11}{16}m^2e_1\sin(2\xi+\phi_1)\quad\ldots(4)$$

17·18. Terms depending on a/a_1

In the previous sections we have found the solutions to the third order of small quantities—a first approximation—and have not proceeded further; in particular, in the equation of the radius vector—17·10 (3)—we have omitted P'' and omitted δ_1Q, ... in deriving the solution for δu. In the present connection we illustrate much more fully the general method followed by de Pontécoulant.

Here we are concerned with the part R_3 of the disturbing function which we write, for convenience, as

$$R_3=m^2n^2a^2(A\cos\xi+B\cos3\xi)\quad\ldots\ldots(1)$$

where

$$A=\frac{3}{8}\frac{a}{a_1}\,,\quad B=\frac{5}{8}\frac{a}{a_1}\,;\quad\ldots\ldots(2)$$

since we reckon a/a_1 as $O(m^2)$, A and B are of the second order.

It is to be remembered that $\dfrac{a}{a_1}$ is a short-hand form of the accurate expression $\dfrac{E-M}{E+M}\cdot\dfrac{a}{a_1}$.

(i) *Equation in radius vector.*

For a first approximation we neglect P'' and $\delta_1 Q$. If δu_2 denotes the part of δu which involves the parallax terms—that is, the terms in a/a_1—the equation for δu_2 reduces to

$$D^2(\delta u_2) + n^2\,\delta u_2 = -Q'/a^2 \equiv -Q_2/a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Now, Q_2 is given by

$$Q_2 = 5R_3 + 3\int R_3 \cdot \frac{\dot r_1}{r_1}\,dt + 2\int \frac{\partial R_3}{\partial \xi}\,\dot v_1\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The first integral is of the sixth order, since $\dot r_1/r_1$ is $O(me_1)$, and we neglect it. In the second integral, it is sufficient to write $\dot v_1 = n_1 = mn$. Since $\xi = n(1-m)t + \ldots$, the final term in (4) is $2mR_3$, to $O(m^5)$. Hence,

$$Q_2 = (5+2m)R_3 \quad .$$

The equation for δu_2 is then

$$D^2(\delta u_2) + n^2\,\delta u_2 = -m^2n^2[(5+2m)A\cos\xi + (5+2m)B\cos 3\xi] \quad \ldots(5)$$

By the principles of section 17·11, case (i), the particular integral of this equation will be such that the coefficient of $\cos\xi$ will be reduced in order by one. We have

$$\begin{aligned}\delta u_2 &= -m^2\left[\frac{5+2m}{1-(1-m)^2}A\cos\xi + \frac{5+2m}{1-9(1-m)^2}B\cos 3\xi\right]\\ &= -\tfrac{15}{16}m(1+\tfrac{9}{10}m)\frac{a}{a_1}\cos\xi - \tfrac{25}{64}m^2\frac{a}{a_1}\cos 3\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)\end{aligned}$$

to $O(m^4)$. We shall see in the next section that the fourth-order part of the coefficient of $\cos\xi$ is incomplete.

(ii) *Longitude equation.*

If δv_2 denotes the perturbation in longitude arising from R_3, we have, by 17·17 (3),

$$\frac{1}{n}D(\delta v_2) = 2\left(\frac{a}{\rho}\right)\delta u_2 + \frac{1}{nr^2}\int \frac{\partial R_3}{\partial \xi}\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

or, for a first approximation,

$$(1-m)\frac{d}{d\xi}(\delta v_2) = 2\delta u_2 + \frac{1}{n^2a^2(1-m)}R_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

from which, to $O(m^4)$, it is readily deduced that

$$\delta v_2 = -(\tfrac{15}{8}m + \tfrac{51}{16}m^2)\frac{a}{a_1}\sin\xi + \tfrac{15}{32}m^2\frac{a}{a_1}\sin 3\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

As in the case of δu_2, the fourth-order part of the coefficient of $\sin\xi$ is incomplete.

17·19. The second approximation to δu_2 and δv_2

The general equation for δu is, by 17·10 (3),

$$D^2(\delta u) + n^2\,\delta u + \frac{Q}{a^2} - P'' + n^2\,\delta b - n^2(c^2-1)\left(\frac{a}{\rho}-1\right) = 0 \quad . \quad \ldots\ldots(1)$$

Our object being to find δu_2 and δv_2 correct to $O(m^4)$, we examine P'' and the full expression for Q to see if they contribute terms in $\cos\xi$ to $O(m^5)$; such terms will produce coefficients of $O(m^4)$ in the particular integral.

(i) By 17·10 (2),

$$P = -\left\{\left(\frac{\rho}{a}\right)^3 - 1\right\}\delta u + \frac{3}{2}\left(\frac{\rho}{a}\right)^4(\delta u)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

in which δu refers to the solution when the disturbing function *as a whole* is involved; δu thus includes the totality of solutions which we have obtained up to date. It is evident that new terms in $\cos\xi$ can only arise through $(\delta u)^2$. The only terms in the part of δu given by 17·16 (1) to be considered are

$$\tfrac{1}{6}m^2 + m^2\cos 2\xi \quad .$$

The terms in e_1 clearly make no contribution—at least, to the order concerned.

In the last section, we have a part of δu, which we designated δu_2, given by $\delta u_2 = -\frac{15}{16}m\dfrac{a}{a_1}\cos\xi$, to $O(m^3)$. Thus, we find that the relevant part of $(\delta u)^2$ is obtained by squaring

$$\tfrac{1}{6}m^2 + m^2\cos 2\xi - \tfrac{15}{16}m\frac{a}{a_1}\cos\xi \quad .$$

The term in $\cos\xi$, to $O(m^5)$, is easily seen to be $-\frac{5}{4}m^3\dfrac{a}{a_1}\cos\xi$. The contribution of $-P''$ in (1) is thus $-\frac{15}{8}m^3n^2\dfrac{a}{a_1}\cos\xi$.

(ii) We consider now $\delta_1 Q$ given by

$$\delta_1 Q = \left(\frac{\partial Q}{\partial r}\right)\delta r + \left(\frac{\partial Q}{\partial v}\right)\delta v \quad ;$$

here $Q = 4R_2 + 5R_3$ to the fourth order; we can clearly neglect R_3. For our purpose it is sufficient to take

$$Q = 4R_2 = m^2n^2r^2(1 + 3\cos 2\xi) \quad .$$

Since $\delta r = -\rho^2\,\delta u/a$, we then have

$$\delta_1 Q = -2m^2n^2a^2(1 + 3\cos 2\xi)\,\delta u - 6m^2n^2a^2\sin 2\xi\,\delta v \quad .$$

The only parts of the complete expressions for δu and δv which can contribute terms in $\cos\xi$ to $O(m^3)$ are

$$\delta u_2\left(\equiv -\tfrac{15}{16}m\frac{a}{a_1}\cos\xi\right) \quad \text{and} \quad \delta v_2 = -\tfrac{15}{8}m\frac{a}{a_1}\sin\xi \quad .$$

We then find easily, to $O(m^5)$, that

$$\delta Q_1 = \tfrac{165}{16} m^3 n^2 a^2 \left(\frac{a}{a_1}\right) \cos \xi \quad .$$

Accordingly, $(\delta_1 Q/a^2 - P'')$ contributes $\tfrac{135}{16} m^3 n^2 \dfrac{a}{a_1} \cos \xi$. This is the fifth-order term to be added to 17·18 (5). The equation for δu_2 is then

$$D^2(\delta u_2) + n^2 \delta u_2 = -m^2 n^2 \frac{a}{a_1} [\{\tfrac{3}{8}(5 + 2m) + \tfrac{135}{16} m\} \cos \xi + \tfrac{25}{8} \cos 3\xi]$$

of which the particular integral, to $O(m^4)$, is now found by the usual procedure to be

$$\delta u_2 = -(\tfrac{15}{16} m + \tfrac{81}{16} m^2) \frac{a}{a_1} \cos \xi + \tfrac{25}{64} m^2 \frac{a}{a_1} \cos 3\xi \quad . \quad \ldots\ldots\ldots\ldots (3)$$

(iii) Since we are finding δv_2 to $O(m^4)$, the equation 17·18 (8) is adequate provided we replace δu_2 by (3) above. We then obtain

$$\delta v_2 = -(\tfrac{15}{8} m + \tfrac{93}{8} m^2) \frac{a}{a_1} \sin \xi + \tfrac{15}{32} m^2 \frac{a}{a_1} \sin 3\xi \quad . \quad \ldots\ldots\ldots\ldots (4)$$

17·20. The latitude equation

(i) The latitude equation is given by 17·03 (3). We have, neglecting s^3, e_1 and a/a_1,

$$D^2(rs) + n^2 \left(\frac{a}{r}\right)^3 (rs) = \frac{\partial R}{\partial r} + \frac{1}{r} \frac{\partial R}{\partial s} \quad \ldots\ldots\ldots\ldots (1)$$

where $$R = m^2 n^2 r^2 \{\tfrac{1}{4} - \tfrac{3}{4} s^2 + \tfrac{3}{4}(1 - s^2) \cos 2(v - v_1)\} \quad .$$

The right-hand side of (1) is easily seen to be $-m^2 n^2 (rs)$. Since

$$z = \frac{rs}{\sqrt{(1 + s^2)}} \doteqdot rs \quad ,$$

we write (1) as

$$D^2 z + n^2 \left(\frac{a^3}{r^3} + m^2\right) z = 0 \quad \ldots\ldots\ldots\ldots (2)$$

Now, $a/r = a/\rho + \delta u$ and, neglecting e so that $\rho = a$, we find that (2) becomes, to the fourth order,

$$D^2 z + n^2 \{1 + 3\,\delta u + 3(\delta u)^2\} z = 0 \quad . \quad \ldots\ldots\ldots\ldots (3)$$

When $e = 0$, the expression for δu, to $O(m^3)$, is easily recognised. We shall, in this section, assume the result to $O(m^4)$; it is

$$\delta u = \tfrac{1}{6} m^2 - \tfrac{179}{288} m^4 + (m^2 + \tfrac{19}{6} m^3 + \tfrac{131}{18} m^4) \cos 2\xi + \tfrac{7}{8} m^4 \cos 4\xi$$

from which $(\delta u)^2 = m^4 (\tfrac{19}{36} + \tfrac{1}{3} \cos 2\xi + \tfrac{1}{2} \cos 4\xi)$.

With these results (3) becomes

$$D^2 z + n^2 (1 + Am^2 + Bm^2 \cos 2\xi + Cm^4 \cos 4\xi) z = 0 \quad , \quad \ldots\ldots\ldots\ldots (4)$$

where $$A = \tfrac{3}{2} - \tfrac{9}{32} m^2 \quad , \quad B = 3 + \tfrac{19}{2} m + \tfrac{137}{6} m^2 \quad , \quad C = \tfrac{33}{8} \quad . \quad \ldots\ldots\ldots\ldots (5)$$

We have seen in section 17·06 that, to a first approximation,

$$z/a = \gamma \sin \eta \quad ,$$

where $\eta = gnt + \epsilon - \Omega$ and $g = 1 + \frac{3}{4}m^2$. Assume that z is given by

$$\frac{z}{a} = b[\sin \eta + P \sin (2\xi - \eta) + Q \sin (2\xi + \eta) + R \sin (4\xi - \eta)] \quad \ldots\ldots(6)$$

where, to a first approximation, $b = \gamma$.

We shall find P, Q and R to $O(m^3)$. If an additional term $bS \sin (4\xi + \eta)$ is included, it would be found that S is $O(m^4)$.

It is to be remarked that (4) is a particular case of the general Mathieu equation

$$\frac{d^2z}{dt^2} + n^2[1 + \sum_1^\infty A_j \cos 2jt]\, z = 0 \quad \ldots\ldots(7)$$

of which the solution is

$$z = \sum_{-\infty}^{\infty} B_r \cos (gnt + 2rt + \alpha) \quad ; \quad \ldots\ldots(8)$$

B_0 and α are constants of integration and g is a quantity to be derived from the A's ; the latter remark also applies to B_r $(r \neq 0)$. We deal with a similar equation in section 18·30 below.

Substitute (6) in (4) and equate coefficients of $\sin \eta$, $\sin 2\xi - \eta$, ... ; then

$$1 - g^2 + Am^2 - \tfrac{1}{2}BPm^2 + \tfrac{1}{2}BQm^2 = 0 \quad , \quad \ldots\ldots(9)$$

$$P[1 + Am^2 - (2 - 2m - g)^2] - \tfrac{1}{2}Bm^2 + \tfrac{1}{2}BRm^2 = 0 \quad , \quad \ldots\ldots(10)$$

$$Q[1 + Am^2 - (2 - 2m - g)^2] + \tfrac{1}{2}Bm^2 = 0 \quad , \quad \ldots\ldots(11)$$

$$R[1 + Am^2 - (4 - 4m - g)^2] + \tfrac{1}{2}BPm^2 - \tfrac{1}{2}Cm^4 = 0 \quad . \quad \ldots\ldots(12)$$

These are four equations from which we derive, by successive approximations, the values of g, P, Q and R. It is to be remembered that A, B and C—defined by (5)—are of zero order. It is evident that P, Q and R are at least of $O(m)$.

First approximation. From (9), $g^2 = 1 + Am^2 = 1 + \frac{3}{2}m^2$; $g = 1 + \frac{3}{4}m^2$. We then obtain : $P = \frac{3}{8}m$; $Q = \frac{3}{16}m^2$; $R = \frac{9}{128}m^3$.

Second approximation. We simply state the results :

$$P = \tfrac{3}{8}m + \tfrac{41}{32}m^2 + \tfrac{5389}{1536}m^3 \quad ;$$

$$Q = \tfrac{3}{16}m^2 + \tfrac{7}{8}m^3 + \tfrac{989}{384}m^4 \quad ;$$

$$R = \tfrac{9}{128}m^3 \quad .$$

We give g to $O(m^4)$, which we would have obtained if we had added the term $bS \sin (4\xi + \eta)$ to (6) :

$$g = 1 + \tfrac{3}{4}m^2 - \tfrac{9}{32}m^3 - \tfrac{273}{128}m^4 \quad . \quad \ldots\ldots(13)$$

(ii) We now find the corresponding expression for s. Now $z = rs$, neglecting s^3. To $O(m^2 b)$,

$$s = \frac{z}{a} \cdot \frac{a}{r} = \frac{z}{a}[1 + \tfrac{1}{6}m^2 + m^2 \cos 2\xi] \quad .$$

Hence, by (6),

$$s=(1+\tfrac{1}{6}m^2)b\sin\eta+(\tfrac{3}{8}m+\tfrac{25}{32}m^2)b\sin(2\xi-\eta)+\tfrac{11}{16}m^2b\sin(2\xi+\eta) \quad ....(14)$$

(iii) Since $s=\tan\theta$, where θ is the Moon's latitude, we have, by a well-known series,

$$\begin{aligned}\theta&=s-\tfrac{1}{3}s^3+\tfrac{1}{5}s^5\ldots\\&=s-\tfrac{1}{3}b^3\sin^3\eta\quad,\qquad\text{to } O(b^3)\quad.\end{aligned}$$

Using (14), we then obtain

$$\theta=(1+\tfrac{1}{6}m^2-\tfrac{1}{4}b^2)b\sin\eta+(\tfrac{3}{8}m+\tfrac{25}{32}m^2)b\sin(2\xi-\eta)+\tfrac{11}{16}m^2b\sin(2\xi+\eta)+\tfrac{1}{12}b^3\sin3\eta \quad .....(15)$$

As in the case of the coefficient of ϕ in the longitude, it is convenient and desirable to have the coefficients of $\sin\eta$ in the undisturbed and disturbed orbits the same; hence we write

$$\gamma=b(1+\tfrac{1}{6}m^2-\tfrac{1}{4}b^2)$$

from which, to the third order,

$$b=\gamma(1-\tfrac{1}{6}m^2+\tfrac{1}{4}\gamma^2) \quad .....(16)$$

The coefficients in (6) and (14) can then be expressed in terms of γ by means of (16).

The formula (15) for θ becomes

$$\theta=\gamma\sin\eta+(\tfrac{3}{8}m+\tfrac{25}{32}m^2)\gamma\sin(2\xi-\eta)+\tfrac{11}{16}m^2\gamma\sin(2\xi+\eta)+\tfrac{1}{12}\gamma^3\sin3\eta \quad .....(17)$$

This procedure, as in longitude, differs from that followed by de Pontécoulant.

17·21. General remarks

We regard e, e_1, γ as of the same order as m, and a/a_1 as of $O(m^2)$. The final literal expression for the radius vector—or rather for the more convenient form a/r—derived by de Pontécoulant is given by a multitude of terms to $O(m^5)$, the longitude to $O(m^6)$ generally with several of the principal terms to $O(m^7)$ or even $O(m^8)$, and the latitude to $O(m^6)$.

To prevent confusion we shall suppose that de Pontécoulant's constants e and γ are denoted by e_0 and γ_0. His coefficient of $\sin\phi$ in the final solution is a series in m, e_0, γ_0, e_1 and a/a_1 which we denote by

$$2C(m, e_0, \gamma_0, e_1, a/a_1) \quad ;$$

our coefficient is $2e(1-\frac{1}{8}e^2)$ which is the coefficient of $\sin M$ in the undisturbed orbit. Hence

$$e-\tfrac{1}{8}e^3=C(m, e_0, \gamma_0, e_1, a/a_1) \quad .....(1)$$

Again, de Pontécoulant's coefficient of $\sin\eta$ in his formula for the latitude is a similar series which we denote by $D(m, e_0, \gamma_0, e_1, a/a_1)$; our coefficient is γ. Hence

$$\gamma=D(m, e_0, \gamma_0, e_1, a/a_1) \quad .....(2)$$

From (1) and (2), e_0 and γ_0 can be separately found, by a process of successive approximations, in terms of m, e, γ, e_1, a/a_1. These latter quantities are derivable from observations; consequently, de Pontécoulant's e_0 and γ_0 can be found.

17·22. The constants

(i) *The observed coordinates (longitude and latitude).*

The longitude, v, and the latitude, θ, of the Moon are derived from observations at any time t. The theoretical formulae with which the observations are to be compared are of the forms

$$v = nt + \epsilon + \Sigma A \sin(pt+q) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\theta = \quad \Sigma B \sin(pt+q) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which we suppose that the amplitudes A and B, which are in absolute measure in the theoretical formulae, are converted into seconds of arc by the application of the factor cosec $1''$ ($\equiv 206265{\cdot}8$).

From an analysis of a very large number of observations of v, for example, the numerical values of n, ϵ, the coefficients A and the periods $2\pi/p$ can be derived. In particular, the coefficient of $\sin\phi$ is found; since its theoretical expression is $2e(1-\frac{1}{8}e^2)$, the value of e is then easily calculated. Similarly, γ, ϖ, Ω are derived.

It is found that the Moon's sidereal period T ($\equiv 2\pi/n$) is 27·321661 days (mean solar) and T_1 ($\equiv 2\pi/n_1$) for the Sun is 365·2564 days; from these we evaluate the important constant m ($\equiv n_1/n$); de Pontécoulant's value is

$$m = 0{\cdot}07480130 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

His values of e_0, γ_0 and e_1 are as follows:

$$e_0 = 0{\cdot}05473074 \quad ; \quad \gamma_0 = 0{\cdot}08967336 \quad ; \quad e_1 = 0{\cdot}01679182 \quad \ldots\ldots\ldots (4)$$

(ii) *The observed parallax.*

A third observable quantity is the Moon's equatorial horizontal parallax Π. If a_0 is the Earth's equatorial radius, $\sin\Pi = \frac{a_0}{r} = \frac{a_0}{a} \cdot \frac{a}{r}$.

If we write $a/r = \alpha + P.T.$, the mean value of $\sin\Pi$ is then $\alpha\frac{a_0}{a}$, which can now be compared with its observed value, namely, $3422{\cdot}7 \sin 1''$; since α can be calculated—it is a function of m, e and e_1—the numerical value of a_0/a can be found at once. De Pontécoulant's value of a_0/a is 0·0165617. Since a_0 is 3963·35 miles, it is then found that

$$a = 239300 \text{ miles.}$$

It is to be remarked that this value of a is *not* the Moon's mean geocentric distance—the latter, denoted by a', is found by expressing r/a as $a'/a + P.T.$ The result is: $a' = 238200$ miles.

If the solar parallax is assumed to be known, then $a/a_1 = (a_0/a_1) \cdot (a/a_0)$,

the first factor on the right being the solar parallax. With the above value of a_0/a and a rather low value of the solar parallax, de Pontécoulant obtained :

$$\frac{a}{a_1} = 0{\cdot}00252551 \fallingdotseq \frac{1}{396} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

This process can be reversed. Consider the principal term in (1) involving a/a_1 or rather $\frac{E-M}{E+M} \cdot \frac{a}{a_1}$; it is $D \sin \xi$ where

$$D = A(m, e, e_1, \gamma) \frac{E-M}{E+M} \cdot \frac{a}{a_1} \quad .$$

We suppose that D is obtained from the observations of longitude—by harmonic analysis or otherwise ; according to de Pontécoulant,

$$D = 123''{\cdot}6 \quad .$$

Now A can be evaluated and if we assume that the ratio $M : E$ is known,* then a/a_1 can be calculated ; combining this with the value of a_0/a previously found, we derive the value of the solar parallax. This method of deducing the solar parallax is now discarded in favour of the direct method based on observations of the minor planet Eros.

Again, the important constants c and g are found by de Pontécoulant to $O(m^7)$ in terms of m, e_0, γ_0, e_1, a/a_1 ; their values are given by the following :

$$1 - c = 0{\cdot}008\ 450\ 821 \quad ; \quad g - 1 = 0{\cdot}004\ 021\ 678 \quad .$$

These results are, of course, based on the contemporary values of the various constants derived from observation.

17·23. The principal inequalities

(i) *Evection.* This inequality is the term $C \sin (2\xi - \phi)$ in the longitude. It was discovered by Ptolemy and is discussed in his *Almagest* (Book IV). The coefficient C is given by de Pontécoulant to $O(m^7)$ in m, e, e_1, γ and a/a_1 and its value is found to be 1° 16′ 27″·0. The period of the evection is $2\pi/\{n(2 - 2m - c)\}$ or about $31\frac{4}{5}$ days.

(ii) *Variation.* This is the inequality in the longitude with argument 2ξ. The amplitude is found to $O(m^7)$ and its value is 39′ 30″·8. The period of the term is one-half of a synodic period or about $14\frac{3}{4}$ days. The inequality was discovered by Tycho Brahe although it is possible he was forestalled by Abul Wafa, a Baghdad astronomer of the tenth century A.D.

(iii) *The parallactic inequality.* This is the inequality in the longitude with argument ξ and amplitude containing $\frac{E-M}{E+M} \cdot \frac{a}{a_1}$ as a factor. The period is one synodic month or $29^d\ 12^h\ 44^m$. The amplitude, according to de Pontécoulant, is $-122''{\cdot}38$.

* In Chapter 20, section 22, M/E is found to be 1/81·6.

(iv) *The annual equation.* Discovered by Tycho Brahe, it is the inequality with argument ϕ_1 ($\equiv n_1 t + \epsilon_1 - \varpi_1$). Its period is thus one year and its coefficient, as given by de Pontécoulant, is $-11'\ 8''{\cdot}93$.

(v) *The motion of perigee.* The period of a complete revolution is $2\pi/\{n(1-c)\}$ or $\frac{m \,.\, 2\pi/n_1}{1-c}$ or $\frac{m}{1-c}$ years; this period is 3232·4 days or about 8·85 years.

(vi) *The motion of the node.* The period of a complete revolution (retrograde) is $2\pi/\{n(g-1)\}$ or $\frac{m}{g-1}$ years; the period is 6794·4 days or about 18·60 years.

Chapter 18

THE HILL-BROWN LUNAR THEORY

18·01. Introduction

In the majority of lunar theories devised since the time of Newton, the equations of motion are formulated fundamentally in terms of polar coordinates—spherical or cylindrical—or in terms of the orbital elements associated with these. A notable exception is Euler's theory of 1772 based on the use of rectangular coordinates, the x and y axes rotating in the ecliptic with the Moon's mean angular motion. Euler's theory appears to have received little attention until a century later when G. W. Hill demonstrated the potentiality of the method based on rectangular co-ordinates, with this difference from Euler's theory, however, that the axes rotate with the *Sun's* mean angular motion n_1, the x-axis pointing to the Sun's mean position. Hill's three classic investigations* formed the basis of E. W. Brown's exhaustive researches† which completed the theory of the Moon's motion and found their expression in the tables‡ which have been used since 1923 in the almanacs.

In these tables, the coefficients of all periodic terms in longitude, latitude and parallax are derived to the order 0″·01. This involved keeping in the disturbing function terms of $O(m^6)$, $O(e^6)$, $O(e_1^4)$, $O(\gamma^6)$ and $O\left\{\left(\frac{a}{a_1}\right)^3\right\}$; e_1^2 is assumed to be of the same order as e^3 or m^3, and a/a_1 of the same order as m^2; here, $m = n_1/n$. In addition, the direct and indirect effects of the action of the planets and the deviations of the Earth and Moon from the spherical form are taken into account.

In the following sections, we shall describe, in some detail, the general principles involved in dealing with the chief types of perturbations arising through the Sun's action.

18·02. Equations of motion referred to rotating axes

We have seen in section 7·03 that, so far as the motion of the Moon is concerned, the Sun can be assumed to describe an elliptic orbit around the centre of mass, C, of the Earth and the Moon. The equations of motion of the Moon with respect to axes $O\xi$, $O\eta$, $O\zeta$ through the Earth's centre,

* " Researches in the Lunar Theory ", *Amer. Jour. of Mathematics*, I, 5 and I, 129 (1878); " On the part of the motion of the Lunar Perigee which is a function of the mean motions of the Sun and Moon ", *Acta. Math.*, VIII, 1 (1886).

† *Memoirs, R. A. S.*, LIII, 39 (1897); LIII, 163 (1899); LIV, 1 (1900); LVII, 51 (1905); LIX, 1 (1908).

‡ *Tables of the Motion of the Moon*, Yale U. P. (1919).

the $\xi\eta$ plane (the ecliptic) being parallel to the plane of the Sun's orbit around C, are

$$\ddot{\xi}=\frac{\partial U}{\partial \xi} \quad , \quad \ddot{\eta}=\frac{\partial U}{\partial \eta} \quad , \quad \ddot{\zeta}=\frac{\partial U}{\partial \zeta} \qquad \text{.....(1)}$$

where $$U=\frac{\mu}{r}+R \quad ,$$

R being the disturbing function due to the Sun.

As in section 7·06 (ii), we identify the Sun with T in Fig. 16 (p. 96), thus assuming that the Sun moves in an elliptic orbit in the ecliptic.

Consider now axes OX, OY in the ecliptic rotating with mean angular velocity n_1, the axis OX being directed towards the Sun's mean position ; the angle ξOX is the Sun's mean longitude $(n_1 t+\epsilon_1)$ which, for convenience, we have denoted by l. Let (x, y, z) be the Moon's coordinates with respect to OX, OY and to OZ, the latter being the same as $O\zeta$; then

$$\xi=x\cos l-y\sin l \quad , \quad \eta=x\sin l+y\cos l \quad , \quad \zeta=z$$

from which

$$\dot{\xi}=\dot{x}\cos l-\dot{y}\sin l-n_1\eta \quad , \quad \dot{\eta}=\dot{x}\sin l+\dot{y}\cos l+n_1\xi \quad , \quad \dot{\zeta}=\dot{z}$$

The kinetic function $T\equiv\frac{1}{2}(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)$ is then given by

$$2T=\dot{x}^2+\dot{y}^2+\dot{z}^2+n_1^2(x^2+y^2)-2n_1(\dot{x}y-x\dot{y})$$

The application of Lagrange's dynamical equations gives, as the equations of motion following from (1),

$$\ddot{x}-2n_1\dot{y}-n_1^2x=-\frac{\mu x}{r^3}+\frac{\partial R}{\partial x} \quad , \qquad \text{.....(2)}$$

$$\ddot{y}+2n_1\dot{x}-n_1^2y=-\frac{\mu y}{r^3}+\frac{\partial R}{\partial y} \quad , \qquad \text{.....(3)}$$

$$\ddot{z}=-\frac{\mu z}{r^3}+\frac{\partial R}{\partial z} \quad . \qquad \text{.....(4)}$$

18·03. The disturbing function

From 7·04 (6) we have, replacing mn by n_1^2,

$$R=n_1^2\left[r^2\left(\frac{a_1}{r_1}\right)^3 P_2(\alpha)+\frac{r^3}{a_1}\frac{E-M}{E+M}\left(\frac{a_1}{r_1}\right)^4 P_3(\alpha)+\ldots\right] \qquad \text{.....(1)}$$

where α is the angle between EM and EP in Fig. 16 (p. 96). If $(x_1, y_1, 0)$ are the Sun's coordinates, the Sun being identified with T in Fig. 16, then

$$rr_1\cos\alpha=xx_1+yy_1 \quad . \qquad \text{.....(2)}$$

Now, from (2) and 7·02 (6),

$$r^2P_2(\alpha)=\frac{3}{2}\cdot\frac{(xx_1+yy_1)^2}{r_1^2}-\tfrac{1}{2}r^2 \quad , \qquad \text{.....(3)}$$

so that, since $r^2 = x^2 + y^2 + z^2$, we can write

$$2r^2 P_2(\alpha) = 2x^2 - y^2 - z^2 + 3x^2 \left\{ \left(\frac{x_1}{r_1}\right)^2 - 1 \right\} + 3y^2 \left(\frac{y_1}{r_1}\right)^2 + \frac{6xyx_1y_1}{r_1^2} . \quad \ldots(4)$$

The last three terms on the right-hand side of (4) involve the eccentricity e_1 of the solar orbit ; they vanish when e_1 is neglected, for then $x_1 = r_1 = a_1$ and $y_1 = 0$.

The first term in (1) can be written as

$$n_1^2 \left[r^2 P_2(\alpha) + r^2 P_2(\alpha) \left\{ \left(\frac{a_1}{r_1}\right)^3 - 1 \right\} \right]$$

Hence (1) can be written, by means of (3) and (4),

$$R = \tfrac{1}{2} n_1^2 (2x^2 - y^2 - z^2) + \Omega \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where

$$\Omega = n_1^2 \left[r^2 P_2(\alpha) \left\{ \left(\frac{a_1}{r_1}\right)^3 - 1 \right\} + \frac{3x^2}{2} \left\{ \left(\frac{x_1}{r_1}\right)^2 - 1 \right\} + \frac{3}{2} y^2 \left(\frac{y_1}{r_1}\right)^2 + \frac{3xyx_1y_1}{r_1^2} + \frac{r^3}{a_1} \frac{E - M}{E + M} \left(\frac{a_1}{r_1}\right)^4 P_3(\alpha) + \ldots \right] . \quad \ldots(6)$$

Ω is a function of the coordinates when we suppose that $r^2 P_2(\alpha)$ in the first term is replaced by the expression on the right of (3). The order of Ω is evidently that of $m^2 e_1$. We write

$$\Omega = \Omega_2 + \Omega_3 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where Ω_p denotes a homogenous function of order p in the Moon's coordinates. Inserting (5) in the equations (2), (3) and (4) of the previous section, we obtain

$$\ddot{x} - 2n_1\dot{y} - 3n_1^2 x = -\frac{\mu x}{r^3} + \frac{\partial \Omega}{\partial x} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$\ddot{y} + 2n_1\dot{x} \qquad = -\frac{\mu y}{r^3} + \frac{\partial \Omega}{\partial y} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

$$\ddot{z} \qquad + n_1^2 z = -\frac{\mu z}{r^3} + \frac{\partial \Omega}{\partial z} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Multiply these, in order, by $\dot{x}$, $\dot{y}$, $\dot{z}$ and add ; then, since

$$\frac{d\Omega}{dt} = \dot{x}\frac{\partial \Omega}{\partial x} + \dot{y}\frac{\partial \Omega}{\partial y} + \dot{z}\frac{\partial \Omega}{\partial z} + \frac{\partial \Omega'}{\partial t} \quad ,$$

where $\dfrac{\partial \Omega'}{\partial t}$ denotes differentiation with respect to t, only as introduced by the Sun's coordinates, we obtain on integration

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 3n_1^2 x^2 + n_1^2 z^2 = \frac{2\mu}{r} + 2\Omega - 2\int \frac{\partial \Omega'}{\partial t}\, dt + C \quad . \quad \ldots\ldots(11)$$

This is Jacobi's integral in the general case.

The first term in (5) contains the principal part of the disturbing function ; it is, in fact, the complete expression of R if we neglect e_1 and

a/a_1. The solution of the equations (8), (9) and (10), in which Ω is omitted, will then give the principal perturbations in the Moon's motion arising from the Sun's action.

18·04. Introduction of complex variables

We write, i denoting $\sqrt{(-1)}$,

$$u = x + iy \quad , \quad s = x - iy \quad \ldots\ldots(1)$$

Then
$$r^2 = us + z^2 \quad \ldots\ldots(2)$$

Also,
$$\frac{\partial \Omega}{\partial x} = \frac{\partial \Omega}{\partial u} + \frac{\partial \Omega}{\partial s} \quad , \quad \frac{\partial \Omega}{\partial y} = i\left(\frac{\partial \Omega}{\partial u} - \frac{\partial \Omega}{\partial s}\right) \quad \ldots\ldots(3)$$

Multiply 18·03 (9) by i ; then, adding to, and afterwards subtracting from, the equation 18·03 (8) we obtain, by means of (3),

$$\ddot{u} + 2in_1\dot{u} - \tfrac{3}{2}n_1^2(u+s) = -\mu\frac{u}{r^3} + 2\frac{\partial \Omega}{\partial s} \quad , \quad \ldots\ldots(4)$$

$$\ddot{s} - 2in_1\dot{s} - \tfrac{3}{2}n_1^2(u+s) = -\mu\frac{s}{r^3} + 2\frac{\partial \Omega}{\partial u} \quad . \quad \ldots\ldots(5)$$

Now, let
$$\xi = (n - n_1)t + \epsilon - \epsilon_1 = (n - n_1)(t - t_0) \quad \ldots\ldots(6)$$

and
$$\zeta = e^{i\xi} \quad . \quad \ldots\ldots(7)$$

Further, let D denote the operator $\zeta \dfrac{d}{d\zeta}$. Then

$$\dot{u} = (n - n_1)\frac{du}{d\xi} = i(n - n_1)\zeta\frac{du}{d\zeta} = i(n - n_1)Du \quad .$$

Similarly,
$$\ddot{u} = -(n - n_1)^2 D^2 u \quad .$$

Let
$$\mathrm{m} = \frac{n_1}{n - n_1} \quad , \quad \ldots\ldots(8)$$

$$\kappa = \frac{\mu}{n - n_1} \quad , \quad \ldots\ldots(9)$$

and
$$W = \frac{2\Omega}{(n - n_1)^2} \quad . \quad \ldots\ldots(10)$$

Then, (4), (5) and 18·03 (10) become

$$D^2u + 2\mathrm{m}\,Du + \tfrac{3}{2}\mathrm{m}^2(u+s) - \kappa\frac{u}{r^3} = -\frac{\partial W}{\partial s} \quad , \quad \ldots\ldots(11)$$

$$D^2s - 2\mathrm{m}\,Ds + \tfrac{3}{2}\mathrm{m}^2(u+s) - \kappa\frac{s}{r^3} = -\frac{\partial W}{\partial u} \quad , \quad \ldots\ldots(12)$$

$$D^2z - \mathrm{m}^2z - \kappa\frac{z}{r^3} = -\frac{1}{2}\frac{\partial W}{\partial z} \quad . \quad \ldots\ldots(13)$$

These are Hill's equations in terms of the new complex variables u, s and ζ. They are further transformed as follows.

(i) Multiply (11) by s and (12) by u, and subtract ; then,

$$u\,D^2s - s\,D^2u - 2\mathrm{m}(u\,Ds + s\,Du) + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) = s\frac{\partial W}{\partial s} - u\frac{\partial W}{\partial u}$$

But, $D(u\,Ds) = u\,D^2s + Du\,Ds$, $D(s\,Du) = s\,D^2u + Du\,Ds$;
hence, we obtain

$$D(u\,Ds - s\,Du - 2\mathrm{m}us) + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) = s\frac{\partial W}{\partial s} - u\frac{\partial W}{\partial u} \quad . \quad \ldots\ldots(14)$$

(ii) Multiply (11) by s, (12) by u and (13) by $2z$, and add ; on the right we have $-\left(u\frac{\partial W}{\partial u} + s\frac{\partial W}{\partial s} + z\frac{\partial W}{\partial z}\right)$ which becomes $-\Sigma pW_p$ by Euler's theorem applied to the functions W_2, W_3, ... homogeneous in u, s and z. We now have

$$s\,D^2u + u\,D^2s + 2z\,D^2z + 2\mathrm{m}(s\,Du - u\,Ds) + \tfrac{3}{2}\mathrm{m}^2(u+s)^2 - 2\mathrm{m}^2z^2 - \frac{2\kappa}{r} = -\Sigma pW_p \quad . \quad \ldots\ldots(15)$$

(iii) Multiply (11) by Ds, (12) by Du and (13) by $2Dz$ and add. Then

$$Ds\,D^2u + Du\,D^2s + \tfrac{3}{2}\mathrm{m}^2(u+s)(Du+Ds) + 2Dz\,D^2z - 2\mathrm{m}^2z\,Dz - \frac{\kappa}{r^2}(u\,Ds + s\,Du + 2z\,Dz) = -\left(\frac{\partial W}{\partial u}Du + \frac{\partial W}{\partial s}Ds + \frac{\partial W}{\partial z}dz\right) \quad . \ldots(16)$$

The term in κ is $-\frac{\kappa}{r^3}D(us + z^2) = -\frac{\kappa}{r^3}D(r^2) = 2\kappa D\left(\frac{1}{r}\right)$.

Also, since W is a function of u, s, z and ζ (the last introduced by the time t occurring in the Sun's coordinates),

$$DW = \frac{\partial W}{\partial u}Du + \frac{\partial W}{\partial s}Ds + \frac{\partial W}{\partial z}Dz + \frac{\partial W}{\partial \zeta}D\zeta \quad .$$

But $$\frac{\partial W}{\partial \zeta}D\zeta = \zeta\frac{\partial W}{\partial \zeta} = D_tW \equiv D(D^{-1}D_tW) \quad ,$$

where D_tW denotes the operation on W only so far as t (or ζ) is introduced into W by the Sun's coordinates, and D^{-1} is the operator inverse to D. Making use of these results in (16) and integrating, we obtain

$$Du\,Ds + (Dz)^2 + \tfrac{3}{4}\mathrm{m}^2(u+s)^2 - \mathrm{m}^2z^2 + \frac{2\kappa}{r} = C - W + D^{-1}(D_tW) \quad .$$

Add this equation to (15) ; then there results

$$D^2(us + z^2) - Du\,Ds - (Dz)^2 + 2\mathrm{m}(s\,Du - u\,Ds) + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 - 3\mathrm{m}^2z^2 = C - \Sigma(p+1)W_p + D^{-1}(D_tW) \quad . \quad \ldots\ldots(17)$$

The equations (14), (17) and (13) are Hill's equations in new form. It is to be noticed that in (14) and (17) κ has disappeared and that a new constant C has been introduced. We refer to these points later.

18·05. The intermediate orbit

In planetary theory it is usual to regard the elliptic orbit, the disturbing forces being ignored, as the first approximation. In de Pontécoulant's

lunar theory the " modified elliptic orbit " represents the basic approximation and this takes into account the uniform motion of the node and perigee. Hill's theory is based on a particular solution of the equations of motion when the solar eccentricity, parallax and z are all neglected, that is, when $\Omega = W = z = 0$. The curve representing this particular solution is called the *intermediate orbit* ; as we shall see, this particular solution contains only two constants of integration. The intermediate orbit is, of course, only an approximation to the lunar orbit ; its importance is derived from two considerations : (i) it takes into account, at the outset, the principal part of the solar perturbation, and (ii) the resulting coordinates can be expressed readily as convergent periodic series in which the coefficients obey comparatively simple rules ; the coefficients are functions of m, the numerical value of which is known to a very high degree of accuracy, and accordingly they can be evaluated with all necessary exactitude.

Reverting to the equations 18·03 (8) and (9), we rewrite them in terms of $\xi[\equiv (n-n_1)(t-t_0)]$ putting $\Omega = z = 0$; they become, since

$$\mathrm{m} = n_1/(n-n_1) \quad \text{and} \quad \kappa = \mu/(n-n_1)^2 \quad ,$$

$$\frac{d^2x}{d\xi^2} - 2\mathrm{m}\frac{dy}{d\xi} - 3\mathrm{m}^2 x = -\frac{\kappa x}{r^3} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{d^2y}{d\xi^2} + 2\mathrm{m}\frac{dx}{d\xi} = -\frac{\kappa y}{r^3} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In these equations we regard m as a small quantity—its value is 0·081 approximately. If we neglect m^2, it is easily seen that particular solutions of (1) and (2) are

$$x = \mathrm{a}\cos\xi \quad , \quad y = \mathrm{a}\sin\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

provided that

$$\frac{\kappa}{\mathrm{a}^3} = 1 + 2\mathrm{m} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The complete solutions of (1) and (2) involve, of course, four arbitrary constants ; the solutions (3) involve into two which are introduced in the definition of ξ—these are n and t_0. The constant a is given in terms of m, n and μ, the latter being defined by $\mu = G(E+M)$ which we regard as known.

A more accurate solution can be written as

$$x = \mathrm{a}(1+p)\cos\xi \quad , \quad y = \mathrm{a}(1+q)\sin\xi$$

where p and q are small quantities of $O(\mathrm{m})$ at least. Then, squares and products of p and q being neglected,

$$r^2 = \mathrm{a}^2\{1 + p + q + (p-q)\cos 2\xi\}$$

and

$$\frac{\kappa x}{r^3} = \frac{\kappa}{\mathrm{a}^2}(1+p)\cos\xi\{1 - \tfrac{3}{2}(p+q) - \tfrac{3}{2}(p-q)\cos 2\xi\} \quad ;$$

the latter, when simplified, contains a term in $\cos\xi\cos 2\xi$ which gives rise to terms in $\cos\xi$ and $\cos 3\xi$. This feature suggests that x could be expressed more accurately by

$$x = \mathrm{a}(\cos\xi + p\cos\xi + r\cos 3\xi)$$

and, similarly,

$$y = \mathrm{a}(\sin\xi + q\sin\xi + s\sin 3\xi) \quad .$$

It is evident that if p and q are $O(\mathrm{m})$ at least, then r and s are $O(\mathrm{m}^2)$ at least. The process can clearly be extended and we assume with Hill that x and y can be expressed in series as follows :

$$x = A_0\cos\xi + A_1\cos 3\xi + A_2\cos 5\xi + \ldots \quad ,$$
$$y = B_0\sin\xi + B_1\sin 3\xi + B_2\sin 5\xi + \ldots \quad ,$$

in which the order of A_j and B_j diminishes as j increases. Hill writes :

$$A_j = \mathrm{a}(a_j + a_{-j-1}) \quad , \quad B_j = \mathrm{a}(a_j - a_{-j-1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

in which, without loss of generality, $a_0 = 1$; however, for the sake of symmetry, a_0 will be retained and its value substituted when required. The series for x and y can now be written as

$$x = \mathrm{a}\sum_{-\infty}^{\infty} a_j\cos(2j+1)\xi \quad , \quad y = \mathrm{a}\sum_{-\infty}^{\infty} a_j\sin(2j+1)\xi \quad .$$

We then have the series for u and s :

$$u = \mathrm{a}\sum_{-\infty}^{\infty} a_j\zeta^{2j+1} \quad , \quad s = \mathrm{a}\sum_{-\infty}^{\infty} a_j\zeta^{-2j-1} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

These expressions for u and s are the *particular* solutions of the equations 18·04 (11) and (12) when $W = z = 0$. These equations are :

$$D^2u + 2\mathrm{m}\,Du + \tfrac{3}{2}\mathrm{m}^2(u+s) - \kappa\frac{u}{r^3} = 0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$D^2s - 2\mathrm{m}\,Ds + \tfrac{3}{2}\mathrm{m}^2(u+s) - \kappa\frac{s}{r^3} = 0 \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

a and κ are related, as in (4), in terms of a function of m.

Similarly, the transformed equations 18·04 (14) and (17) are :

$$D(u\,Ds - s\,Du - 2\mathrm{m}us) + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) = 0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$
$$D^2(us) - Du\,Ds + 2\mathrm{m}(s\,Du - u\,Ds) + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 = C \quad . \quad \ldots\ldots(10)$$

The equations (9) and (10) are used to derive the expressions for a_j in terms of m. When the series for u and s have been obtained the expression for κ/a^3 can be obtained by substituting the two series, (6), in either (7) or (8). The value of C, if it is required, can be obtained in a similar way from (10).

It is to be remarked that (7) and (8) are not independent for each, being complex, gives rise to the same pair of equations in real coordinates.

18·06. Multiplication and division of series

The substitution of the series,* which we write as

$$u = \mathrm{a} \Sigma a_i \zeta^{2i+1} \quad , \quad s = \mathrm{a} \Sigma a_i \zeta^{-2i-1} \quad \ldots\ldots(1)$$

in the transformed equations (9) and (10) of the previous section, involves the multiplication of certain series. Later, we shall have to deal with the quotient of two series. We consider here general principles together with some simple applications with reference to the substitutions to be made in 18·05 (9) and (10).

We note, first, that if we write $2r+1$ for $-2i-1$, the series for s can be expressed alternatively by

$$s = \mathrm{a} \Sigma a_{-r-1} \zeta^{2r+1} \quad , \quad \ldots\ldots(2)$$

and, second, that since s is the conjugate of u, we can write

$$s = \mathrm{a} \Sigma a_i \zeta_1^{2i+1} \quad , \quad \ldots\ldots(3)$$

where ζ_1 is the conjugate of ζ, that is, $e^{-\sqrt{(-1)}\xi}$.

We consider the two series :

$$U = \Sigma A_i \zeta^{2i+1} \quad , \quad V = \Sigma B_r \zeta^{2r+1} \quad .$$

(i) *Product of U and V.*

$$UV = \sum_i \sum_r A_i B_r \zeta^{2i+2r+2} \quad .$$

Write $2j$ for $(2i+2r+2)$; then

$$UV = \sum_i \sum_j A_i B_{j-i-1} \zeta^{2j} \equiv \Sigma C_j \zeta^{2j} \quad .$$

Hence

$$C_j = \sum_i A_i B_{j-i-1} \quad , \quad \ldots\ldots(4)$$

from which any C can be calculated from the known values of the A's and B's.

(ii) u^2. Here $U = V = u$; $A_i = a_i$, $B_r = a_r$; hence, by (4)

$$u^2 = \mathrm{a}^2 \sum_i \sum_j a_i a_{j-i-1} \zeta^{2j} \quad . \quad \ldots\ldots(5)$$

(iii) us. Here $U = u$, $V = s$; $A_i = a_i$ and, from (2), $B_r = a_{-r-1}$; hence,

$$us = \mathrm{a}^2 \sum_i \sum_j a_i a_{i-j} \zeta^{2j} \quad . \quad \ldots\ldots(6)$$

(iv) s^2. Since s is the conjugate of u, then, by (5),

$$s^2 = \mathrm{a}^2 \sum_i \sum_r a_i a_{r-i-1} \zeta_1^{2r}$$

or, on writing $-j$ for r,

$$s^2 = \mathrm{a}^2 \sum_i \sum_j a_i a_{-i-j-1} \zeta^{2j} \quad \ldots\ldots(7)$$

* Unless otherwise stated in the remainder of this chapter the sign Σ denotes summation from $-\infty$ to $+\infty$ for integral values, including zero, of i or of any other symbol used ; when it is necessary to discriminate between two or more symbols, the appropriate one is attached to the symbol of summation.

(v) $s\,Du$. Since $D=\zeta\frac{d}{d\zeta}$, then $Du=\mathrm{a}\,\Sigma\,(2i+1)a_i\zeta^{2i+1}$.

Hence, by analogy with (iii),

$$s\,Du=\mathrm{a}^2\sum_i\sum_j(2i+1)a_ia_{i-j}\zeta^{2j} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

We also have the following, which are easily derived :

(vi) $$u\,Ds=\mathrm{a}^2\sum_i\sum_j(2j-2i-1)a_ia_{i-j}\zeta^{2j} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

(vii) $$Du\,Ds=\mathrm{a}^2\sum_i\sum_j(2i+1)(2j-2i-1)a_ia_{i-j}\zeta^{2j} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

(viii) *Quotient of U and V.*

It is evident that U/V is of the form $\Sigma\,D_j\zeta^{2j}$. Then

$$\Sigma\,A_i\zeta^{2i}=\sum_r\sum_j B_rD_j\zeta^{2r+2j}=\sum_i\sum_j B_{i-j}D_j\zeta^{2i} \quad .$$

Equate coefficients of ζ^{2i} ; then

$$\sum_j B_{i-j}D_j=A_i \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

We have separate equations when i takes the values 0, +1, −1, +2, −2, ... in succession. In practice the values of the D's are found by successive approximations.

18·07. A general expression for a_j

We now substitute the expressions for u^2, us, s^2, $s\,Du$, etc., derived in the previous section, in the equations 18·05 (9) and (10), and performing the operation D as required we obtain

$$\sum_i\sum_j[a_ia_{i-j}\{4j(2i-j+1)+4j\mathrm{m}\}-\tfrac{3}{2}\mathrm{m}^2a_i(a_{j-i-1}-a_{-j-i-1})]\zeta^{2j}=0 \quad , \ldots(1)$$

$$\sum_i\sum_j[a_ia_{i-j}\{4j^2-(2i+1)(2j-2i-1)+4\mathrm{m}(2i-j+1)+\tfrac{9}{2}m^2\}$$
$$+\tfrac{9}{4}\mathrm{m}^2a_i(a_{j-i-1}+a_{-j-i-1})]\zeta^{2j}=C \quad . \quad \ldots\ldots\ldots\ldots(2)$$

These are identities and the coefficients of ζ^{2j} in (1) and (2) are zero, except in the latter when $j=0$. Accordingly, we have from (1) and (2), multiplying the former by 3 and the latter by 2,

$$\sum_i[a_ia_{i-j}\{12j(2i-j+1)+12j\mathrm{m}\}-\tfrac{9}{2}\mathrm{m}^2a_i(a_{j-i-1}-a_{-j-i-1})]=0 \quad , \quad \ldots\ldots(3)$$

$$\sum_i[a_ia_{i-j}\{8j^2-(4i+2)(2j-2i-1)+8\mathrm{m}(2i-j+1)+9\mathrm{m}^2\}$$
$$+\tfrac{9}{2}\mathrm{m}^2a_i(a_{j-i-1}+a_{-j-i-1})]=0 \quad . \quad \ldots\ldots\ldots\ldots(4)$$

Subtract (3) from (4), and then add (3) and (4) ; we obtain the pair of equations

$$\sum_i a_ia_{i-j}A_{ij}+9\mathrm{m}^2\sum_i a_ia_{j-i-1}=0 \quad , \quad \ldots\ldots\ldots\ldots(5)$$

$$\sum_i a_ia_{i-j}B_{ij}+9\mathrm{m}^2\sum_i a_ia_{-j-i-1}=0 \quad , \quad \ldots\ldots\ldots\ldots(6)$$

where

$$A_{ij} = 20j^2 - 16j(2i+1) + 2(2i+1)^2 + \mathrm{m}(8+16i-20j) + 9\mathrm{m}^2 \quad , \quad \ldots(7)$$

$$B_{ij} = -4j^2 + 8j(2i+1) + 2(2i+1)^2 + \mathrm{m}(8+16i+4j) + 9\mathrm{m}^2 \quad \ldots\ldots(8)$$

It is to be remembered that in these formulae $j \neq 0$. Also, i takes all positive and negative integral values including zero.

Let Σ' denote a summation *excluding* zero. We write (5) and (6) as

$$a_0 a_{-j} A_{0j} + \Sigma' a_i a_{i-j} A_{ij} + 9\mathrm{m}^2 \Sigma a_i a_{j-i-1} = 0 \quad , \quad \ldots\ldots\ldots(9)$$

$$a_0 a_{-j} B_{0j} + \Sigma' a_i a_{i-j} B_{ij} + 9\mathrm{m}^2 \Sigma a_i a_{-j-i-1} = 0 \quad . \quad \ldots\ldots\ldots(10)$$

Multiply (9) by B_{0j} and (10) by A_{0j}, and subtract; then

$$\Sigma' C_{ij} a_i a_{i-j} + 9\mathrm{m}^2 \Sigma a_i D_{ij} = 0 \quad \ldots\ldots\ldots(11)$$

where

$$C_{ij} = B_{0j} A_{ij} - A_{0j} B_{ij}$$

$$= -48ij\{4j^2 + 4j - 2 + 4i(j-1) + 4\mathrm{m}(j-i-1) + \mathrm{m}^2\} \quad \ldots\ldots(12)$$

and

$$D_{ij} = B_{0j} a_{j-i-1} - A_{0j} a_{-j-i-1} \quad \ldots\ldots\ldots(13)$$

Now, let Σ'' denote a summation when $i \neq 0$ and $i \neq j$. Then (11) becomes, when we put $i = j$ in one term of Σ',

$$a_0 a_j C_{jj} + \Sigma'' a_i a_{i-j} C_{ij} + 9\mathrm{m}^2 \Sigma a_i D_{ij} = 0 \quad . \quad \ldots\ldots\ldots(14)$$

Set

$$[j, i] = -\frac{C_{ij}}{C_{jj}} \quad , \quad [j] = -\frac{9B_{0j}}{C_{jj}} \quad , \quad (j) = \frac{9A_{0j}}{C_{jj}} \quad . \quad \ldots\ldots\ldots(15)$$

Then, since $a_0 = 1$, (14) becomes, with the help of (13),

$$a_j = \Sigma''[j, i] a_i a_{i-j} + \mathrm{m}^2[j] \Sigma a_i a_{j-i-1} + \mathrm{m}^2(j) \Sigma a_i a_{-j-i-1} \quad \ldots\ldots(16)$$

This is the required formula for a_j with specified value of j. The expressions defined in (15) are as follows:

$$[j, i] = -\frac{i\{4j^2 + 4j - 2 + 4i(j-1) + 4\mathrm{m}(j-i-1) + \mathrm{m}^2\}}{j(8j^2 - 2 - 4\mathrm{m} + \mathrm{m}^2)} \quad , \quad \ldots\ldots(17)$$

$$[j] = \frac{3(-4j^2 + 8j + 4\mathrm{m}j + 2 + 8\mathrm{m} + 9\mathrm{m}^2)}{16j^2(8j^2 - 2 - 4\mathrm{m} + \mathrm{m}^2)} \quad , \quad \ldots\ldots\ldots(18)$$

$$(j) = -\frac{3(20j^2 - 16j - 20\mathrm{m}j + 2 + 8\mathrm{m} + 9\mathrm{m}^2)}{16j^2(8j^2 - 2 - 4\mathrm{m} + \mathrm{m}^2)} \quad \ldots\ldots\ldots(19)$$

All three functions as defined* are of zero order with respect to m. Since $j \neq 0$, the denominators in these expressions never vanish and so each function can be expanded, if need be, in convergent series of m, or directly evaluated when the numerical value of m is substituted.

The value of C is then obtained, if required, by putting $j = 0$ in (2).

18·08. Evaluation of the a's

(i) Put $j = 1$ in the formula (16) of the previous section. Then,

$$a_1 = \Sigma''[1, i] a_i a_{i-1} + \mathrm{m}^2[1] \Sigma a_i a_{-i} + \mathrm{m}^2(1) \Sigma a_i a_{-i-2}$$

* Hill defined $[j]$ and (j) to be the coefficients of the second and third summations in (16); Hill's functions are thus m^2 times the functions defined in (15). It seems more convenient, when assessing the orders of the various terms that arise, to define $[j]$ and (j) as in (15).

In the first summation, $i \neq 1$. We have

$$\begin{aligned} a_1 = \quad & [1, 2]a_1a_2 \quad + \quad [1, 3]a_2a_3 \quad + \ldots \\ & + [1, -1]a_{-1}a_{-2} + [1, -2]a_{-2}a_{-3} + \ldots \\ & + \mathrm{m}^2[1]\{a_0^2 + 2a_1a_{-1} + 2a_2a_{-2} + \ldots\} \\ & + \mathrm{m}^2(1)\{a_0a_{-2} + a_1a_{-3} + \ldots\} \quad . \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

It is evident from 18·05 (5) that $a_1, a_{-1}, \ldots$ are of higher order than a_0 ($\equiv 1$); consequently, the order of a_1 is given by the first term in the third line of (1), namely, $\mathrm{m}^2[1]$. Thus a_1 is $O(\mathrm{m}^2)$.

(ii) Put $j = -1$ in 18·07 (16). Then

$$\begin{aligned} a_{-1} = & [-1, 1]a_1a_2 + \ldots + \mathrm{m}^2[-1]\{a_{-1}a_{-1} + 2a_0a_{-2} + \ldots\} \\ & + \mathrm{m}^2(-1)\{a_0^2 + 2a_1a_{-1} + \ldots\} \quad . \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Hence, a_{-1} is $O(\mathrm{m}^2)$.

(iii) Put $j = 2$. Then

$$\begin{aligned} a_2 = & [2, 1]a_1a_{-1} + [2, 3]a_3a_1 + \ldots + [2, -1]a_{-1}a_{-3} + \ldots \\ & + \mathrm{m}^2[2]\{2a_0a_1 + a_2a_{-1} + a_{-1}a_2 + \ldots\} \\ & + \mathrm{m}^2(2)\{a_0a_{-3} + a_1a_{-4} + \ldots\} \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

If we assume that the magnitude of a_j diminishes as $|j|$ increases, the principal part of a_2 is given by

$$a_2 = [2, 1]a_1a_{-1} + 2\mathrm{m}^2[2]a_1 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Thus, a_2 is $O(\mathrm{m}^4)$.

(iv) By continuing steps of this kind, it is seen that

$$a_r \text{ is } O(\mathrm{m}^{|2r|})$$

(v) Correct to $O(\mathrm{m}^5)$, $\quad a_1 = \mathrm{m}^2[1]$, from (1),

and $\quad a_{-1} = \mathrm{m}^2(-1)$, from (2).

From 18·07 (18),

$$[1] = \frac{3}{16} \cdot \frac{6 + 12\mathrm{m} + 9\mathrm{m}^2}{6 - 4\mathrm{m} + \mathrm{m}^2}$$

Expressing this as a power-series in m, it is then found that

$$a_1 = \frac{3}{16}\mathrm{m}^2 + \frac{1}{2}\mathrm{m}^3 + \frac{7}{12}\mathrm{m}^4 + \frac{11}{36}\mathrm{m}^5 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Similarly,

$$a_{-1} = -\frac{19}{16}\mathrm{m}^2 - \frac{5}{3}\mathrm{m}^3 - \frac{43}{36}\mathrm{m}^4 - \frac{14}{27}\mathrm{m}^5 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The coefficient a_2 given by (4) can now be expressed in terms of m; a_{-2} can be obtained in a similar manner.

(vi) Correct to $O(\mathrm{m}^7)$, a_1 is now seen from (1) to be given by

$$a_1 = [1, 2]a_1a_2 + [1, -1]a_{-1}a_{-2} + \mathrm{m}^2[1]\{1 + 2a_1a_{-1}\} + \mathrm{m}^2(1)a_{-2}$$

The first approximations to a_1, a_2, a_{-1}, a_{-2} are inserted on the right-hand side and then a_1 is derived as a power series to $O(\mathrm{m}^7)$.

(vii) By successive approximations of the kind just described, the expressions of the coefficients to any desired order can be rapidly obtained. Hill has given such expressions to m^9 and, taking the value of m to be given by

$$\mathrm{m} = 0{\cdot}080\ 848\ 933\ 808\ 312 \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

he has calculated the values of the corresponding coefficients to 15 places of decimals.

We add the series, correct to m^6, for the relevant coefficients.

$$a_1 = (5) \text{ above} - \frac{30749}{2^{12}\,3^3}\,\mathrm{m}^6 \quad ; \quad a_{-1} = (6) - \frac{7381}{2^{10}\,3^4}\,\mathrm{m}^6 \quad ;$$

$$a_2 = \frac{25}{256}\,\mathrm{m}^4 + \frac{803}{1920}\,\mathrm{m}^5 + \frac{6109}{7200}\,\mathrm{m}^6 \quad ; \quad a_3 = \frac{833}{3\,.\,2^{12}}\,\mathrm{m}^6 \quad ;$$

$$a_{-2} = \frac{23}{640}\,\mathrm{m}^5 + \frac{299}{2400}\,\mathrm{m}^6 \quad ; \quad a_{-3} = \frac{1}{192}\,\mathrm{m}^6 \quad .$$

18·09. The constant a

We begin with the equation 18·05 (7), in which we write $r^2 = us$, since we are considering the solutions when $z = 0$:

$$(D^2 + 2\mathrm{m}D + \tfrac{3}{2}\mathrm{m}^2)\,u + \tfrac{3}{2}\mathrm{m}^2 s = \frac{\kappa u}{r^3} = \kappa u^{-1/2} s^{-3/2} \quad .$$

Substitute the series for u and s ; then

$$\Sigma\,\{(2i+1)^2 + 2\mathrm{m}(2i+1) + \tfrac{3}{2}\mathrm{m}^2\}\,a_i\zeta^{2i+1} + \tfrac{3}{2}\mathrm{m}^2\,\Sigma\,a_i\zeta^{-2i-1}$$
$$= \frac{\kappa}{\mathrm{a}^3}\,[\{\Sigma\,a_i\zeta^{2i+1}\}^{-1/2}\,\{\Sigma\,a_i\zeta^{-2i-1}\}^{-3/2}] \quad .$$

This is an identity, and true for all values of ζ. Put $\zeta = 1$; then

$$\frac{\kappa}{\mathrm{a}^3} = \{\Sigma\,a_i\}^2\,[\Sigma\,\{(2i+1)^2 + 2\mathrm{m}(2i+1) + 3\mathrm{m}^2\}\,a_i] \quad \ldots\ldots\ldots\ldots (1)$$

Define a by

$$\mu = n^2 a^3 \quad ; \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

then, since $\kappa = \mu/(n - n_1)^2$, we have

$$\kappa = (1+\mathrm{m})^2 a^3 \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Formula (1) becomes

$$\left(\frac{a}{\mathrm{a}}\right)^3 = (1+\mathrm{m})^{-2}\,\{\Sigma\,a_i\}^2\,[\Sigma\{(2i+1)^2 + 2\mathrm{m}(2i+1) + 3\mathrm{m}^2\}a_i] \quad \ldots\ldots (4)$$

from which a/a can be evaluated when the numerical value of m is substituted or expressed as a series in powers of m. It is found that, to $O(\mathrm{m}^3)$,

$$\mathrm{a} = a(1 - \tfrac{1}{6}\mathrm{m}^2 + \tfrac{1}{3}\mathrm{m}^3) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

This equation—or, rather, its extension with the necessary degree of accuracy—enables us to compare the present theory, in which a occurs explicitly, with any other theory expressed in terms of a.

18·10. Variational terms

In this section, we shall be concerned with the polar coordinates r and v, the latter denoting the Moon's true longitude ; the Sun's mean longitude is $n_1 t + \epsilon_1$ and consequently, since $z = 0$, the angle between the x-axis and the direction of the Moon is $v - n_1 t - \epsilon_1$. Hence

$$x = r\cos(v - n_1 t - \epsilon_1) \quad , \quad y = r\sin(v - n_1 t - \epsilon_1) \quad .$$

Now,

$$\begin{aligned} r\cos(v - nt - \epsilon) &\equiv r\cos(v - n_1 t - \epsilon_1 - \xi) \\ &= x\cos\xi + y\sin\xi = \tfrac{1}{2}(u\zeta^{-1} + s\zeta) \quad . \quad \ldots\ldots\ldots\ldots(1) \end{aligned}$$

Similarly,

$$r\sin(v - nt - \epsilon) = -\tfrac{1}{2}i(u\zeta^{-1} - s\zeta) \quad . \quad \ldots\ldots\ldots\ldots(2)$$

Inserting the expansions for u and s given in 18·05 (6), we have

$$r\cos(v - nt - \epsilon) = \mathrm{a}\{1 + (a_1 + a_{-1})\cos 2\xi + (a_2 + a_{-2})\cos 4\xi + \ldots\} \quad , \quad \ldots(3)$$

$$r\sin(v - nt - \epsilon) = \mathrm{a}\{\quad (a_1 - a_{-1})\sin 2\xi + (a_2 - a_{-2})\sin 4\xi + \ldots\} \quad . \quad \ldots(4)$$

(i) Since $r^2 = us$ $(z = 0)$, r^2 can be rapidly obtained by means of 18·06 (6).

The equation for κ/r can be found readily from 18·04 (15) in which we put $W = z = 0$; the equation is then

$$\frac{2\kappa}{r} = s\,D^2u + u\,D^2s + 2\mathrm{m}(s\,Du - u\,Ds) + \tfrac{3}{2}\mathrm{m}^2(u + s)^2 \quad . \quad \ldots\ldots\ldots(5)$$

(ii) To find $(v - nt - \epsilon)$—the equation of the centre which we denote temporarily by θ—we have, from (1) and (2),

$$i\tan\theta = (u\zeta^{-1} - s\zeta)/(u\zeta^{-1} + s\zeta) \quad ;$$

consequently,

$$\frac{1 + i\tan\theta}{1 - i\tan\theta} \equiv e^{2i\theta} = \frac{u\zeta^{-1}}{s\zeta} = \frac{\Sigma\, a_j\zeta^{2j}}{\Sigma\, a_j\zeta^{-2j}} = \frac{\Sigma\, a_j\zeta^{2j}}{\Sigma\, a_j\zeta_1^{2j}} \quad ,$$

from which

$$2i\theta = \log\{\Sigma\, a_j\zeta^{2j}\} - \log\{\Sigma\, a_j\zeta_1^{2j}\} \quad . \quad \ldots\ldots\ldots\ldots(6)$$

Now, $\quad \Sigma\, a_j\zeta^{2j} = 1 + (a_1\zeta^2 + a_{-1}\zeta^{-2}) + (a_2\zeta^4 + a_{-2}\zeta^{-4}) + \ldots$

and

$$\log\{\Sigma\, a_j\zeta^{2j}\} = -a_1a_{-1} + a_1\zeta^2 + a_{-1}\zeta^{-2} + (a_2 - \tfrac{1}{2}a_1^2)\zeta^4 + (a_{-2} - \tfrac{1}{2}a_{-1}^2)\zeta^{-4} + \ldots .$$

We then have, from (6),

$$2i\theta = a_1(\zeta^2 - \zeta_1^2) + a_{-1}(\zeta^{-2} - \zeta_1^{-2}) + \ldots$$

and

$$\theta = (a_1 - a_{-1})\sin 2\xi + (a_2 - a_{-2} - \tfrac{1}{2}a_1^2 + \tfrac{1}{2}a_{-1}^2)\sin 4\xi + \ldots \quad .$$

The true longitude is then given by an equation of the form

$$v = nt + \epsilon + A_2\sin 2\xi + A_4\sin 4\xi + \ldots \quad , \quad \ldots\ldots\ldots\ldots(7)$$

where A_{2j} is of order $\mathrm{m}^{|2j|}$.

(iii) It is found, to $O(\mathrm{m}^3)$, that

$$r = \mathrm{a}\{1 - (\mathrm{m}^2 + \tfrac{7}{6}\mathrm{m}^3)\cos 2\xi\} \quad , \quad \ldots\ldots\ldots\ldots(8)$$

$$v = nt + \epsilon + (\tfrac{11}{8}\mathrm{m}^2 + \tfrac{13}{6}\mathrm{m}^3)\sin 2\xi \quad . \quad \ldots\ldots\ldots\ldots(9)$$

Perturbations in r and v depending only on terms with arguments $2j\xi$ are the *variational terms.*

The principal term in the longitude in (9)—that depending on sin 2ξ—is the *variation* as usually defined (v. section 17·23).

18·11. The lunar perigee

We begin with the equation 18·05 (7), namely,

$$(D^2 + 2\mathrm{m}D + \tfrac{3}{2}\mathrm{m}^2)u + \tfrac{3}{2}\mathrm{m}^2 s = \kappa \frac{u}{r^3} \quad . \quad \ldots\ldots\ldots\ldots(1)$$

The particular solutions of (1), corresponding to the intermediate orbit, are

$$u = \mathrm{a}\,\Sigma\, a_i \zeta^{2i+1} \quad , \quad s = \mathrm{a}\,\Sigma\, a_i \zeta^{-2i-1} \quad ,$$

in which n and t_0 are the only constants of integration introduced so far. The complete solution of (1) which is equivalent to two equations in real coordinates, each of the second order, must contain four constants of integration, one of which is evidently closely related to the eccentricity in the instantaneous lunar orbit.

Let $u + \Delta u$, $s + \Delta s$ be the required solutions of (1) where u and s are given by (2). For convenience, we write $\rho^2 = us$; also, $r^2 = (u + \Delta u)(s + \Delta s)$. We omit squares and products of Δu and Δs. Then

$$r^2 = \rho^2 + u\,\Delta s + s\,\Delta u \quad . \quad \ldots\ldots\ldots\ldots(2)$$

Since (1) is satisfied by u and s with ρ replacing r, the equation in Δu and Δs is

$$(D^2 + 2\mathrm{m}D + \tfrac{3}{2}\mathrm{m}^2)\Delta u + \tfrac{3}{2}\mathrm{m}^2 \Delta s = \frac{\kappa(u + \Delta u)}{r^3} - \frac{\kappa u}{\rho^3} \quad .$$

By means of (2), the right-hand side is easily seen to be

$$-\tfrac{1}{2}\frac{\kappa}{\rho^3}\left(\Delta u + 3\frac{u}{s}\Delta s\right) \quad .$$

We then have

$$\left(D^2 + 2\mathrm{m}D + \tfrac{3}{2}\mathrm{m}^2 + \frac{\kappa}{2\rho^3}\right)\Delta u + \left(\tfrac{3}{2}\mathrm{m}^2 + \tfrac{3}{2}\frac{\kappa}{\rho^3}\cdot\frac{u}{s}\right)\Delta s = 0 \quad \ldots\ldots\ldots(3)$$

Now $DX \equiv D\{\zeta(\zeta^{-1}X)\} = \zeta D(\zeta^{-1}X) + \zeta(\zeta^{-1}X) = \zeta(D+1)(\zeta^{-1}X)$;

$D^2X = \zeta(D^2 + D)(\zeta^{-1}X) + \zeta(D+1)(\zeta^{-1}X) = \zeta(D+1)^2(\zeta^{-1}X)$.

By means of these, we transform (3) ; it becomes

$$\left[(D + 1 + \mathrm{m})^2 + \tfrac{1}{2}\left(\mathrm{m}^2 + \frac{\kappa}{\rho^3}\right)\right](\zeta^{-1}\Delta u) + N(\zeta\,\Delta s) = 0 \ldots\ldots\ldots(4)$$

where $$N=\frac{3}{2}\mathrm{m}^2\zeta^{-2}+\frac{3}{2}\frac{\kappa}{\rho^3}\cdot\frac{u\zeta^{-1}}{s\zeta} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

We shall show later that the solution of (4) involves the motion of the lunar perigee. Before preceeding further, however, we require the expansions of $\mathrm{m}^2+\frac{\kappa}{\rho^3}$ and N.

18·12. Expansions of $(\kappa/\rho^3+\mathrm{m}^2)$ and N

(i) It is evident that if we square 18·10 (3) and (4) and add then r^2 is a series in cos $2j\xi$; here, r has the meaning attached to ρ. It is also clear that ρ^{-3} is a similar cosine series.

From 18·11 (1), r being replaced by ρ,

$$\frac{\kappa}{\rho^3}=\tfrac{3}{2}\mathrm{m}^2+\frac{(D^2+2\mathrm{m}D)u+\frac{3}{2}\mathrm{m}^2 s}{u} \quad .$$

Now, $$(D^2+2\mathrm{m}D)u=\mathrm{a}\zeta\ \Sigma[(2j+1)^2+2\mathrm{m}(2j+1)]a_j\zeta^{2j}$$
$$=(1+2\mathrm{m})u+\mathrm{a}\zeta\ \Sigma[4j(j+1+\mathrm{m})a_j\zeta^{2j}] \quad .$$

Hence,

$$\frac{\kappa}{\rho^3}+\mathrm{m}^2=1+2\mathrm{m}+\tfrac{5}{2}\mathrm{m}^2+\frac{\Sigma[4j(j+1+\mathrm{m})a_j+\frac{3}{2}\mathrm{m}^2a_{-j-1}]\zeta^{2j}}{\Sigma a_r\zeta^{2r}} \quad . \ldots\ldots(1)$$

Let the quotient on the right be denoted by $\Sigma A_i\zeta^{2i}$.

Then $$[\underset{r}{\Sigma} a_r\zeta^{2r}][\Sigma A_i\zeta^{2i}]\equiv\underset{i}{\Sigma}\ \underset{j}{\Sigma} a_{j-i}A_i\zeta^{2j} \quad .$$

By equating coefficients of ζ^{2j}, we obtain

$$\underset{i}{\Sigma} a_{j-i}A_i=4j(j+1+\mathrm{m})a_j+\tfrac{3}{2}\mathrm{m}^2a_{-j-1} \quad . \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This is true for all positive or negative integral values of j, including zer.o For example, for $j=0,\ 1,\ -1$, we have in succession

$$a_0A_0+a_{-1}A_1+a_1A_{-1}+\ldots\ =\tfrac{3}{2}\mathrm{m}^2a_{-1}\ , \ldots\ldots\ldots\ldots\ldots\ldots(3)$$
$$a_0A_1+a_1A_0+a_2A_{-1}+\ldots\ =4(2+\mathrm{m})a_1+\tfrac{3}{2}\mathrm{m}^2a_{-2}\ , \ldots\ldots\ldots(4)$$
$$a_0A_{-1}+a_{-1}A_0+a_{-2}A_1+\ldots=-4\mathrm{m}a_{-1}+\tfrac{3}{2}\mathrm{m}^2a_0 \quad . \ldots\ldots\ldots\ldots(5)$$

From (4) and (5) it is evident that A_1 and A_{-1} are $O(\mathrm{m}^2)$ and from (3) that A_0 is $O(\mathrm{m}^4)$.

Putting $j=2$ in (2) and then $j=-2$, we find that A_2 and A_{-2} are $O(\mathrm{m}^4)$.

Now, since (1) represents a cosine series, it follows that $A_i=A_{-i}$.

We add some results.

$$A_1=A_{-1}=\tfrac{3}{2}\mathrm{m}^2-4\mathrm{m}a_{-1}\ ,\ \text{to } O(\mathrm{m}^5)\ ;$$
$$A_0=-\tfrac{3}{2}\mathrm{m}^2a_1+4\mathrm{m}a_{-1}(a_1+a_{-1})\ ,\ \text{to } O(\mathrm{m}^7)\ ;$$
$$A_2=A_{-2}=8(1-\mathrm{m})a_{-2}+\tfrac{3}{2}\mathrm{m}^2(a_1-a_{-1})+4\mathrm{m}a_{-1}^2\ ,\ \text{to } O(\mathrm{m}^5)\ .$$

We now write (1) as

$$\frac{\kappa}{\rho^3}+\mathrm{m}^2=1+2\mathrm{m}+\tfrac{5}{2}\mathrm{m}^2+A_0+\Sigma' A_i\zeta^{2i} \quad . \ldots\ldots\ldots\ldots(6)$$

It is convenient to write (6) in the alternative form

$$\frac{\kappa}{\rho^3}+\mathrm{m}^2=2\,\Sigma\, M_i\zeta^{2i}=2M_0+2\,\Sigma'\, M_i\zeta^{2i} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where $$2M_0=1+2\mathrm{m}+\tfrac{5}{2}\mathrm{m}^2+A_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

and $$2M_i\equiv 2M_{-i}=A_i \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

We also write

$$\frac{\kappa}{\rho^3}=2\,\Sigma\, M_i'\zeta^{2i} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

where $2M_0'=2M_0-\mathrm{m}^2$ and $M_i'=M_i$ if $i\neq 0$.

(ii) We now consider N in 18·11 (5); we have

$$N=\tfrac{3}{2}\mathrm{m}^2\zeta^{-2}+\frac{3}{2}\frac{\kappa}{\rho^3}\cdot\frac{u\zeta^{-1}}{s\zeta} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Let E denote the second term. Then, by (10),

$$\tfrac{2}{3}E=\frac{2[\Sigma\, M_i'\zeta^{2i}][\Sigma\, a_r\zeta^{2r}]}{\Sigma\, a_{-r}\zeta^{2r}}\equiv\Sigma\,\nu_i\zeta^{2i} \quad .$$

By a procedure similar to that in dealing with the quotient in (1) we obtain the formula for the ν's—analogous to (2)—

$$\sum_i \nu_i a_{i-j}=2\sum_i M_i' a_{j-i} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

We write $$N=\Sigma\, N_i\zeta^{2i} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

where

$$N_{-1}=\tfrac{3}{2}\mathrm{m}^2+\tfrac{3}{2}\nu_{-1} \quad , \quad N_i=\tfrac{3}{2}\nu_i \quad , \quad (i\neq -1) \quad . \quad \ldots\ldots\ldots\ldots(14)$$

Also, N_i is of order $\mathrm{m}^{|2i|}$. To order m^4 we easily find from (12) that

$$\begin{aligned}\nu_0&=2M_0'\{1-(a_1-a_{-1})^2\} \quad ,\\ \nu_1&=2M_1+2M_0'(a_1-a_{-1}) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(15)\\ \nu_{-1}&=2M_1-2M_0'(a_1-a_{-1}) \quad .\end{aligned}$$

18·13. Forms of Δu and Δs

Our principal equation is 18·11 (4) which becomes, by means of 18·12 (7) and (13),

$$[(D+1+\mathrm{m})^2+\Sigma\, M_i\zeta^{2i}](\zeta^{-1}\Delta u)+[\Sigma\, N_i\zeta^{2i}](\zeta\,\Delta s)=0 \quad , \quad \ldots\ldots\ldots(1)$$

in which the M's and the N's are supposed to be known functions of m. We assume, as solutions,

$$\zeta^{-1}\Delta u=\mathrm{a}[e^{-\omega\sqrt{(-1)}}\,\Sigma\, e_r\zeta^{2r+\mathrm{c}}+e^{\omega\sqrt{(-1)}}\,\Sigma\, f_r\zeta^{2r-\mathrm{c}}] \quad , \quad \ldots\ldots\ldots\ldots(2)$$

$$\zeta\,\Delta s=\mathrm{a}[e^{\omega\sqrt{(-1)}}\,\Sigma\, e_r\zeta^{-2r-\mathrm{c}}+e^{-\omega\sqrt{(-1)}}\Sigma\, f_r\zeta^{-2r+\mathrm{c}}] \quad . \quad \ldots\ldots\ldots\ldots(3)$$

We take ω and e_0 as the two additional and necessary arbitrary constants; c is at present an undetermined constant. Substitute (2) and (3) in (1), and write $i+r=j$ in the products associated with M and N, and $r=j$ elsewhere. Then we have

$$\sum_j e^{-\omega(\sqrt{-1})}\,\zeta^{2j+c}[e_j(2j+c+1+m)^2+\sum_i (M_i e_{j-i}+N_i f_{i-j})]$$
$$+\sum_j e^{\omega\sqrt{(-1)}}\,\zeta^{2j-c}[f_j(2j-c+1+m)^2+\sum_i M_i f_{j-i}+N_i e_{i-j}]=0 \quad .$$

This is an identity ; hence, for any value of j,

$$e_j(2j+c+1+m)^2+\sum_i (M_i e_{j-i}+N_i f_{i-j})=0 \quad , \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$f_j(2j-c+1+m)^2+\sum_i (M_i f_{j-i}+N_i e_{i-j})=0 \quad . \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

We write (4) as

$$e_j\{(2j+c+1+m)^2+M_0\}+N_0 f_{-j}+\sum_i{}' (M_i e_{j-i}+N_i f_{i-j})=0 \quad , \ldots.(6)$$

and (5), j being replaced by $-j$, as

$$f_{-j}\{(2j+c-1-m)^2+M_0\}+N_0 e_j+\sum_i{}' (M_i f_{-j-i}+N_i e_{j+i})=0 \quad . \ldots\ldots(7)$$

By eliminating f_{-j} from (6) and (7) we obtain an equation giving e_j ; similarly, by eliminating e_j we obtain an equation giving f_{-j}. These equations can be solved by successive approximations—the method is similar to that in deriving the values of a_j in 18·07.

When $j=0$ it is clear from (6) and (7) that there is a linear relationship between e_0 and the remaining e's and all the f's ; we can evidently determine the values of e_j/e_0 and f_j/e_0 for all values of j. We assume, then, that e_0 is an arbitrary constant, the fourth being ω.

18·14. Approximate value of c

It is seen from (6) and (7) of the previous section after a little consideration that e_1/e_0 and f_{-1}/e_0 are $O(m)$ at least ; we assume that the other ratios are of progressively higher orders. Put $j=0$ in 18·13 (6) and (7) ; then, to $O(m^2)$,

$$e_0[(c+1+m)^2+M_0]+N_0 f_0=0 \quad , \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$e_0 N_0+f_0[(c-1-m)^2+M_0]=0 \quad , \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

from which

$$[(c+1+m)^2+M_0][(c-1-m)^2+M_0]-N_0^2=0 \quad . \ldots\ldots\ldots\ldots(3)$$

This equation enables us to determine c to $O(m^2)$. Now, by 18·12 (8) and (14), to $O(m^2)$,

$$2M_0=1+2m+\tfrac{5}{2}m^2 \quad , \quad N_0^2=9M_0^2-\tfrac{9}{2}m^2 \quad ;$$

hence (3) becomes

$$[c^2-(1+m)^2]^2+2M_0[c^2+(1+m)^2]=8M_0^2-\tfrac{9}{2}m^2$$

or

$$[c^2-(1+m)^2]^2+2M_0[c^2-(1+m)^2]=4M_0\{2M_0-(1+m)^2\}-\tfrac{9}{2}m^2$$

The right-hand side is $O(m^2)$; hence, to $O(m)$,

$$=1+m \quad .$$

A second approximation is easily found to be

$$c = 1 + m - \tfrac{3}{4}m^2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

We can now calculate the values of the more important e's and f's.

(i) f_0. From (2) and (4), $f_0 = -e_0 \cdot \dfrac{N_0}{M_0} = -3(1-m^2)e_0$.

(ii) e_{-1} and f_1. In 18·13 (6) and (7), put $j = -1$. Then, to $O(m^2)$,

$$e_{-1}[(c-1+m)^2 + M_0] + N_0 f_1 = -M_{-1}e_0 - N_{-1}f_0$$
$$e_{-1}N_0 + f_1[(3+m-c)^2 + M_0] = -M_1 f_0 - N_1 e_0$$

When the expression for c in (4) is inserted in these equations and f_1 eliminated from them, it is found with little difficulty that e_{-1} is $O(m)$ and equal to $-\frac{45}{8}me_0$. It is then deduced that $f_1 = \frac{15}{8}me_0$.

(iii) In 18·13 (6) and (7), put $j = 1$. It is then easily seen that e_1 and f_{-1} are $O(m^2)$. In general, if r is a *positive* integer, e_{-r} and f_r are $O(m^{2r-1})$ and e_r and f_{-r} are $O(m^{2r})$.

18·15. The evection

In this section, we consider the expressions for Δu and Δs to $O(m)$; of the e's and f's we then neglect all except e_{-1} and f_1. By 18·13 (2) and (3), writing i for $\sqrt{(-1)}$,

$$\zeta^{-1}\frac{\Delta u}{a} = e^{-i\omega}[e_0\zeta^c + e_{-1}\zeta^{c-2}] + e^{i\omega}[f_0\zeta^{-c} + f_1\zeta^{2-c}] \quad , \quad \ldots\ldots\ldots\ldots(1)$$

$$\zeta\frac{\Delta s}{a} = e^{i\omega}[e_0\zeta^{-c} + e_{-1}\zeta^{2-c}] + e^{-i\omega}[f_0\zeta^c + f_1\zeta^{c-2}] \quad . \quad \ldots\ldots\ldots\ldots(2)$$

Also, to $O(m)$, $u = a\zeta$, $s = a\zeta^{-1}$, so that $\rho = a$ and, from 18·11 (2),

$$\left(\frac{r}{a}\right)^2 = 1 + \frac{s\,\Delta u + u\,\Delta s}{a^2} = 1 + \frac{(\zeta^{-1}\Delta u + \zeta\,\Delta s)}{a}$$

Hence, by means of (1) and (2),

$$\left(\frac{r}{a}\right)^2 = 1 + 2(e_0 + f_0)\cos(c\xi - \omega) + 2(e_{-1} + f_1)\cos(\overline{c-2}\xi - \omega)$$

Insert the values of f_0, e_{-1} and f_1 in terms of e_0 derived in the previous section; write $e = 2e_0$ and use 18·09 (5). Then

$$\left(\frac{r}{a}\right)^2 = 1 - 2e\cos(c\xi - \omega) - \tfrac{15}{4}me\cos(\overline{c-2}\xi - \omega)$$

from which

$$\frac{r}{a} = 1 + \tfrac{1}{2}e^2 - e\cos(c\xi - \omega) + \tfrac{1}{2}e^2\cos 2(c\xi - \omega)$$
$$- \tfrac{15}{8}me\cos(\overline{c-2}\xi - \omega) \quad \ldots\ldots\ldots(3)$$

In this formula, e corresponds to the eccentricity in the elliptic orbit; the

final term—that in me—is the principal perturbation in radius vector arising from m.

We now consider the perturbation in longitude. In 18·10 (ii) we denoted the equation of the centre by θ ($\equiv v - nt - \epsilon$) and we had

$$e^{2i\theta} = \frac{u\zeta^{-1}}{s\zeta}$$

If θ_0 is the part of θ that depends on m alone and $\Delta\theta$ the part depending on e, then

$$e^{2i(\theta_0+\Delta\theta)} = \frac{\zeta^{-1}(u+\Delta u)}{\zeta(s+\Delta s)}$$

from which

$$2i\,\Delta\theta = \frac{\Delta u}{u} - \frac{\Delta s}{s}$$

or, since $u\zeta^{-1} = s\zeta = \mathrm{a}$ to $O(\mathrm{m})$,

$$2i\,\Delta\theta = \frac{1}{\mathrm{a}}(\zeta^{-1}\,\Delta u + \zeta\,\Delta s)$$

From (1) and (2) we then obtain

$$\Delta\theta = (e_0 - f_0)\sin(\mathrm{c}\xi - \omega) + (e_{-1} - f_1)\sin(\overline{\mathrm{c}-2}\xi - \omega)$$

or, in terms of e,

$$\Delta\theta = 2\mathrm{e}\sin(\mathrm{c}\xi - \omega) - \tfrac{15}{4}\mathrm{me}\sin(\overline{\mathrm{c}-2}\xi - \omega) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The final term in (4) is the *evection*.

18·16. Motion of perigee

With the notation of the previous section the equation of the centre, θ, is $\theta_0 + \Delta\theta$. The part θ_0 is given by 18·10 (9) and, to $O(\mathrm{m})$, $\theta_0 = 0$, that is, $v = nt + \epsilon$. Adding $\Delta\theta$ from (4) of the previous section, we have, to $O(\mathrm{m})$,

$$v = nt + 2\mathrm{e}\sin(\mathrm{c}\xi - \omega) + \text{evection}.$$

Also, from (3) of 18·15,

$$r/a = 1 + \tfrac{1}{2}\mathrm{e}^2 - \mathrm{e}\cos(\mathrm{c}\xi - \omega) + \tfrac{1}{2}\mathrm{e}^2\cos 2(\mathrm{c}\xi - \omega) + \text{evection}$$

In the instantaneous elliptic orbit we have, identifying e with the eccentricity e,

$$v = nt + 2\mathrm{e}\sin M \quad ,$$
$$r = a(1 + \tfrac{1}{2}\mathrm{e}^2 - \mathrm{e}\cos M + \tfrac{1}{2}\mathrm{e}^2\cos 2M) \quad ,$$

where M ($\equiv nt + \epsilon - \varpi$) is the mean anomaly.

In the sets of formulae for v and r the arguments and $(\mathrm{c}\xi - \omega)$ are comparable. Denote the latter by ψ; then, we can write

$$\psi = nt + \left(\frac{\mathrm{c}}{1+\mathrm{m}} - 1\right)nt + \epsilon - \varpi_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $\varpi_0 \equiv \mathrm{c}\epsilon_1 + (1-\mathrm{c})\epsilon - \omega \equiv \text{constant}$.

Compare (1) with $nt+\epsilon-\varpi$ (that is, M); then

$$\varpi-\varpi_0+\left(1-\frac{c}{1+m}\right)nt$$

or, from 18·14 (4) and to $O(m^2)$,

$$\varpi=\varpi_0+\tfrac{3}{4}m^2nt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Thus the longitude of perigee is increasing at the rate $\left(1-\frac{c}{1+m}\right)n$ or $\frac{3}{4}m^2n$. From (1) it is evident that the relationship between c of the present theory and de Pontécoulant's c is

$$c=(1+m)c \quad .$$

18·17. Remarks on the previous solutions

In 18·14, we indicated the method of determining the coefficients e_j, f_j in terms of m and as we have seen the method involves the evaluation of c *pari passu*. The work is evidently tedious and laborious even when it is a matter of say the first and second approximations. If we could derive c accurately, either numerically or as a power series in m, say to m^8, the determination of e_j and f_j would be achieved with a very great saving in labour. Our next procedure is to describe the evaluation of c by an independent method. This involves reference to new axes and the derivation of an equation for c introducing an infinite determinant.

18·18. Preliminary formulae

Referred to the system of axes rotating with angular velocity n_1, the equations of motion for a point P (x, y) on the intermediate orbit CP in Fig. 26 are, from 18·03 (8) and (9) in which $\Omega=z=0$,

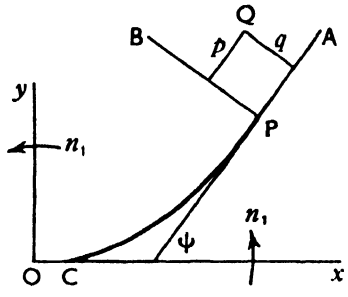


Fig. 26

$$\ddot{x}-2n_1\dot{y}-3n_1^2x=-\frac{\mu x}{r^3} \quad , \quad \ldots\ldots\ldots\ldots(1)$$

$$\ddot{y}+2n_1\dot{x} \qquad =-\frac{\mu y}{r^3} \quad , \quad \ldots\ldots\ldots\ldots(2)$$

or

$$\ddot{x}-2n_1\dot{y}=\frac{\partial F}{\partial x} \quad , \quad \ldots\ldots\ldots\ldots(3)$$

$$\ddot{y}+2n_1\dot{x}=\frac{\partial F}{\partial y} \quad , \quad \ldots\ldots\ldots\ldots(4)$$

where

$$F=\frac{\mu}{r}+\tfrac{3}{2}n_1^2x^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Multiply (3) and (4) by $\dot{x}$ and $\dot{y}$ respectively; add and integrate; then we obtain Jacobi's integral

$$\dot{x}^2+\dot{y}^2\equiv v^2=2F+k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where k is a constant and v is the velocity of P relative to the rotating axes. Since CP is a part of the intermediate orbit we assume that the solutions for x and y—they are given in terms of the complex coordinates u and s—are known.

Let the tangent at P make the angle ψ with the x-axis; then

$$\dot{x} = v\cos\psi \quad , \quad \dot{y} = v\sin\psi \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

For simplicity we write: $C \equiv \cos\psi$, $S = \sin\psi$. From (6),

$$v\dot{v} = \frac{dF}{dt} = \dot{x}\frac{\partial F}{\partial x} + \dot{y}\frac{\partial F}{\partial y} \quad ;$$

hence

$$\dot{v} = C\frac{\partial F}{\partial x} + S\frac{\partial F}{\partial y} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Multiply (3) by $\dot{y}$ and (4) by $\dot{x}$, and subtract; then

$$\dot{x}\ddot{y} - \ddot{x}\dot{y} + 2n_1 v^2 = v\left(-S\frac{\partial F}{\partial x} + C\frac{\partial F}{\partial y}\right) \quad .$$

But from (7) it is found at once that

$$\dot{x}\ddot{y} - \ddot{x}\dot{y} = v^2\dot{\psi} \quad ; \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

hence

$$v(\dot{\psi} + 2n_1) = -S\frac{\partial F}{\partial x} + C\frac{\partial F}{\partial y} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

from which, using (8),

$$v\ddot{\psi} + \dot{v}(\dot{\psi} + 2n_1) = -\dot{v}\dot{\psi} - S(\dot{x}F_{xx} + \dot{y}F_{xy}) + C(\dot{x}F_{xy} + \dot{y}F_{yy})$$

or

$$v\ddot{\psi} + 2\dot{v}(\dot{\psi} + n_1) = vP \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

where

$$P = -SC(F_{xx} - F_{yy}) + (C^2 - S^2)F_{xy} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

18·19. Jacobi's integral for the general solution

In Fig. 26, let $Q(X, Y)$, corresponding to $P(x, y)$ on the intermediate orbit, be associated with the actual orbit which is expressible in terms of the proper number (four) of arbitrary constants. We write: $X = x + \Delta x$, $Y = y + \Delta y$. Then (X, Y) satisfy 18·18 (3) and (4), so that

$$\ddot{x} + \frac{d^2}{dt^2}(\Delta x) - 2n_1\left(\dot{y} + \frac{d}{dt}(\Delta y)\right) = F_x + \Delta x\, F_{xx} + \Delta y\, F_{xy} \quad ,$$

$$\ddot{y} + \frac{d^2}{dt^2}(\Delta y) + 2n_1\left(\dot{x} + \frac{d}{dt}(\Delta x)\right) = F_y + \Delta x\, F_{xy} + \Delta y\, F_{yy} \quad ;$$

hence the equations satisfied by Δx, Δy are

$$\frac{d^2}{dt^2}(\Delta x) - 2n_1\frac{d}{dt}(\Delta y) = \Delta x\, F_{xx} + \Delta y\, F_{xy} \equiv \Delta F_x \quad , \qquad \ldots\ldots\ldots\ldots (1)$$

$$\frac{d^2}{dt^2}(\Delta y) + 2n_1\frac{d}{dt}(\Delta x) = \Delta x\, F_{xy} + \Delta y\, F_{yy} \equiv \Delta F_y \quad . \qquad \ldots\ldots\ldots\ldots (2)$$

Multiply 18·18 (3) and (4) by $\frac{d}{dt}(\Delta x)$ and $\frac{d}{dt}(\Delta y)$, and (1) and (2) by $\dot{x}$ and $\dot{y}$, and add ; then

$$\sum_{xy}\left\{\ddot{x}\frac{d}{dt}(\Delta x)+\dot{x}\frac{d^2}{dt^2}(\Delta x)\right\}=F_x\frac{d}{dt}(\Delta x)+F_y\frac{d}{dt}(\Delta y)$$
$$+\Delta x(\dot{x}F_{xx}+\dot{y}F_{xy})+\Delta y(\dot{x}F_{xy}+\dot{y}F_{yy}).$$

Since the last two terms are $\Delta x\frac{d}{dt}(F_x)+\Delta y\frac{d}{dt}(F_y)$, we obtain

$$\sum_{xy}\frac{d}{dt}\left\{\dot{x}\frac{d}{dt}(\Delta x)\right\}=\frac{d}{dt}(\Delta x F_x+\Delta y F_y)$$

and, on integrating,

$$\dot{x}\frac{d}{dt}(\Delta x)+\dot{y}\frac{d}{dt}(\Delta y)=\Delta x F_x+\Delta y F_y \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The constant of integration is zero, since both sides must vanish when $\Delta x=\Delta y=0$.

It is to be remarked that Δx and Δy are related to the complex quantities Δu and Δs of the previous sections by

$$\Delta u=\Delta x+i\,\Delta y \quad , \quad \Delta s=\Delta x-i\,\Delta y \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

18·20. Change of axes

In Fig. 26, PA is the tangent at P to the intermediate orbit and PB is the normal 90° ahead of PA in the sense in which ψ is increasing. Let the coordinates of Q be (p, q) referred to PA and PB—for simplicity we do not write these coordinates as differentials. Then,

$$\Delta x=pC-qS \quad ; \quad \Delta y=pS+qC \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and
$$p=C\,\Delta x+S\,\Delta y \quad ; \quad q=-S\,\Delta x+C\,\Delta y \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1),
$$\frac{d}{dt}(\Delta x)=\dot{p}C-\dot{q}S-\dot{\psi}\,\Delta y \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\frac{d}{dt}(\Delta y)=\dot{p}S+\dot{q}C+\dot{\psi}\,\Delta x \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Multiply (3) by $\dot{x}$ and (4) by $\dot{y}$, and add. Then, since $\dot{x}=vC$, $\dot{y}=vS$, Jacobi's integral, 18·19 (3), becomes

$$v(\dot{p}-q\dot{\psi})=p(CF_x+SF_y)+q(-SF_x+CF_y)$$

or, by means of 18·18 (8) and (10),

$$v(\dot{p}-q\dot{\psi})=p\dot{v}+qv(\dot{\psi}+2n_1)$$

or
$$v\dot{p}-p\dot{v}=2vq(\dot{\psi}+n_1) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

From (3) and (4) we have

$$C\frac{d}{dt}(\Delta x)+S\frac{d}{dt}(\Delta y)=\dot{p}-q\dot{\psi} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$-S\frac{d}{dt}(\Delta x)+C\frac{d}{dt}(\Delta y)=\dot{q}+p\psi \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Differentiate (7) and use (6); then

$$E\equiv -S\frac{d^2}{dt^2}(\Delta x)+C\frac{d^2}{dt^2}(\Delta y)=\psi(\dot{p}-q\psi)+\ddot{q}+\dot{p}\psi+p\dot{\psi} \quad . \quad \ldots\ldots\ldots(8)$$

But, from 18·19 (1) and (2) and also (6) above

$$E=-2n_1(\dot{p}-q\psi)-S\Delta F_x+C\Delta F_y \quad .$$

Hence, (8) becomes

$$\ddot{q}+p\dot{\psi}+2\dot{p}(\psi+n_1)-q\psi(\psi+2n_1)=-S\Delta F_x+C\Delta F_y \quad . \quad \ldots\ldots\ldots(9)$$

It is easily seen that the right-hand side is $pP+qQ$, where P is given by 18·18 (12) and Q by

$$Q=S^2F_{xx}-2CSF_{xy}+C^2F_{yy} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

We then find that (9) becomes

$$\ddot{q}+p(\dot{\psi}-P)+2\dot{p}(\psi+n_1)=q[\psi(\psi+2n_1)+Q]$$

or, using 18·18 (11),

$$\ddot{q}-2\left(p\frac{\dot{v}}{v}-\dot{p}\right)(\psi+n_1)=q[\psi(\psi+2n_1)+Q]$$

or, using 18·20 (5),

$$\ddot{q}+4q(\psi+n_1)^2=q[\psi(\psi+2n_1)+Q]$$

which we write as

$$\ddot{q}+(n-n_1)^2\Theta\,.\,q=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

where

$$(n-n_1)^2\Theta=4(\psi+n_1)^2-\psi(\psi+2n_1)-Q \quad . \quad \ldots\ldots\ldots\ldots(12)$$

We find an equivalent expression for Θ as follows. We have, from 18·18 (8),

$$\dot{v}=CF_x+SF_y \quad ;$$

hence

$$\ddot{v}=\psi[-SF_x+CF_y]+C[\dot{x}F_{xx}+\dot{y}F_{xy}]+S[\dot{x}F_{xy}+\dot{y}F_{yy}]$$

or

$$\frac{\ddot{v}}{v}=\psi(\psi+2n_1)+(C^2F_{xx}+2CSF_{xy}+S^2F_{yy})$$

by means of 18·18 (10). This last equation is equivalent to

$$\frac{\ddot{v}}{v}=\psi(\psi+2n_1)+F_{xx}+F_{yy}-Q \quad .$$

We then have from (12) on eliminating Q

$$(n-n_1)^2\Theta=\frac{\ddot{v}}{v}+2(\psi+n_1)^2+2n_1{}^2-(F_{xx}+F_{yy}) \quad . \quad \ldots\ldots(13)$$

In the next section we transform Θ into a function of u and s and, eventually, of ζ. As we shall see, the determination of c is linked up with the solution of (11).

Suppose that (11) is solved so that $q=q(t)$ in terms of four arbitrary constants. Then the corresponding solution for p is given by means of 18·20 (5) which can be written as

$$\frac{d}{dt}\left(\frac{p}{v}\right)=\frac{2q}{v}(\dot{\psi}+n_1) \quad ,$$

from which

$$p=2v\int\frac{q}{v}(\dot{\psi}+n_1)\,dt \quad .$$

As we shall see later, v and ψ are expressible in terms of t or ζ; hence p is obtained. The solutions for Δx and Δy follow from 18·20 (1); from these Δu and Δs are obtained.

18·21. The function Θ

The function Θ is defined by equation 18·20 (13). We derive the following transformations.

(i) $\ddot{v}/v$. We have

$$\frac{\ddot{v}}{v}=\frac{d}{dt}\left(\frac{\dot{v}}{v}\right)+\left(\frac{\dot{v}}{v}\right)^2 \quad .$$

Now, $v^2=\dot{u}\dot{s}$; hence

$$2\frac{\dot{v}}{v}=\frac{\ddot{u}}{\dot{u}}+\frac{\ddot{s}}{\dot{s}} \quad .$$

Also, $\frac{d}{dt}=i(n-n_1)D$; we then obtain

$$\frac{\ddot{v}}{v}=-\tfrac{1}{2}(n-n_1)^2D\left\{\frac{D^2u}{Du}+\frac{D^2s}{Ds}\right\}-\tfrac{1}{4}(n-n_1)^2\left\{\frac{D^2u}{Du}+\frac{D^2s}{Ds}\right\}^2 \quad .$$

(ii) $(\dot{\psi}+n_1)$. From 18·18 (9),

$$v^2\dot{\psi}=\dot{x}\ddot{y}-\ddot{x}\dot{y} \quad ,$$

which becomes—since $2x=u+s$, $2iy=u-s$ and $v^2=\dot{u}\dot{s}$—

$$\dot{\psi}=\frac{1}{2iv^2}(\ddot{u}\dot{s}-\dot{u}\ddot{s})=\frac{1}{2i}\left(\frac{\ddot{u}}{\dot{u}}+\frac{\ddot{s}}{\dot{s}}\right) \quad ;$$

hence

$$\dot{\psi}+n_1=(n-n_1)\left\{\frac{1}{2}\left(\frac{D^2u}{Du}-\frac{D^2s}{Ds}\right)+\mathrm{m}\right\} \quad .$$

(iii) $F_{xx}+F_{yy}$. From 18·18 (5) it is found at once that

$$F_{xx}+F_{yy}=\frac{\mu}{r^3}+3n_1^2=(n-n_1)^2\left\{\frac{\kappa}{r^3}+3\mathrm{m}^2\right\} \quad ,$$

since, by 18·04 (8) and (9), $n_1=\mathrm{m}(n-n_1)$ and $\mu=\kappa(n-n_1)$.

Inserting these results in 18·20 (13), we obtain

$$\Theta=-\left(\frac{\kappa}{r^3}+\mathrm{m}^2\right)+2\left\{\frac{1}{2}\left(\frac{D^2u}{Du}-\frac{D^2s}{Ds}\right)+\mathrm{m}\right\}^2$$
$$-\tfrac{1}{2}D\left\{\frac{D^2u}{Du}+\frac{D^2s}{Ds}\right\}-\frac{1}{4}\left\{\frac{D^2u}{Du}+\frac{D^2s}{Ds}\right\}^2 \quad . \qquad \text{.....(1)}$$

18·22. The equation for q

From 18·20 (11),

$$\ddot{q} + (n - n_1)^2 \Theta q = 0$$

which becomes

$$D^2 q = \Theta q \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where Θ is given by (1) of the previous section. In this latter equation, we substitute

$$u = \Sigma\, a_j \zeta^{2j+1} \quad , \quad s = \Sigma\, a_j \zeta^{-2j-1}$$

and thus determine Θ as a function of ζ and the a's.

We note from 18·12 (10) that $\left(\frac{\kappa}{r^3} + \mathrm{m}^2\right)$, where $r^2 \equiv \rho^2 = us$, is of the form $\Sigma\, P_i \cos 2i\xi$, of which the coefficients P_i are known functions of m. As we shall see, the other parts of the function Θ in 18·21 (1) are also of this form. Then Θ can be written as

$$\Theta = \Theta_0 + 2\Theta_1 \cos 2\xi + 2\Theta_2 \cos 4\xi + \ldots = \Sigma\, \Theta_i \zeta^{2i} \qquad \ldots\ldots\ldots\ldots (2)$$

in which $\Theta_{-i} = \Theta_i$.

18·23. The coefficients Θ_i

We make use of the principles in 18·06 for the products and quotients of series.

(i) We have

$$\frac{D^2 u}{Du} = \frac{\Sigma\,(2j+1)^2 a_j \zeta^{2j}}{\Sigma\,(2j+1) a_j \zeta^{2j}} = \Sigma\, Q_i \zeta^{2i}$$

from which the coefficients Q_i can be obtained as in 18·06 (viii).

Also,

$$\frac{D^2 s}{Ds} = -\frac{\Sigma\,(2j+1)^2 a_j \zeta^{-2j}}{\Sigma\,(2j+1) a_j \zeta^{-2j}} = -\frac{\Sigma\,(2j+1)^2 a_j \zeta_1^{2j}}{\Sigma\,(2j+1) a_j \zeta_1^{2j}}$$
$$= -\Sigma\, Q_i \zeta_1^{2i} = -\Sigma\, Q_{-i} \zeta^{2i} \quad .$$

Hence

$$E \equiv \frac{D^2 u}{Du} - \frac{D^2 s}{Ds} = \Sigma\, Q_i (\zeta^{2i} + \zeta^{-2i})$$

which is of the form $\Sigma\, P_i \cos 2i\xi$; here, it is easily seen that $Q_0 = 1$. Writing E in the form

$$E = \Sigma\,(Q_i + Q_{-i})\zeta^{2i} = 2\,\Sigma\, V_i \zeta^{2i} \quad ,$$

where $V_0 = 1$ and $2V_i = Q_i + Q_{-i}$, we then have

$$E_1 \equiv \frac{1}{2}\left(\frac{D^2 u}{Du} - \frac{D^2 s}{Ds}\right) + \mathrm{m} = \Sigma\, V_i \zeta^{2i} \quad ,$$

where now V_0 denotes $1 + \mathrm{m}$. Consequently,

$$E_1^2 = \Sigma\, V_j \zeta^{2j} \,.\, \Sigma\, V_r \zeta^{2r} = \Sigma\, W_i \zeta^{2i} \quad .$$

The coefficients W can be determined by the method of 18·06 (i), first, in

terms of the V's and, second, in terms of the Q's. Since E_1 is a cosine-series, so also is E_1^2 so that $W_i = W_{-i}$.

(ii) $$F \equiv \frac{D^2u}{Du} + \frac{D^2s}{Ds} = \Sigma\, Q_i(\zeta^{2i} - \zeta^{-2i})$$

Hence $$DF = \Sigma\, 2iQ_i(\zeta^{2i} + \zeta^{-2i})$$

which is of the form $\Sigma\, P_i \cos 2i\xi$.

(iii) The final part of Θ is F^2, which is readily seen to be a cosine-series.

Hill's calculations give the following expressions for Θ_0, Θ_1 and Θ_2 to $O(\mathrm{m}^5)$ in terms of the a's :

$$\Theta_0 = 1 + 2\mathrm{m} - \tfrac{1}{2}\mathrm{m}^2 + \tfrac{3}{2}\mathrm{m}^2 a_1 + 54a_1^2 + (12 - 4\mathrm{m})a_1 a_{-1} + (6 - 4\mathrm{m})a_{-1}^2 \quad ;$$
$$\Theta_1 = (6 + 12\mathrm{m})a_1 + (6 + 8\mathrm{m})a_{-1} - \tfrac{3}{2}\mathrm{m}^2 \quad ;$$
$$\Theta_2 = 20\mathrm{m}a_2 + (16 + 20\mathrm{m})a_{-2} - (9 + 40\mathrm{m})a_1^2 + 6a_1 a_{-1} + (7 + 4\mathrm{m})a_{-1}^2 - \tfrac{3}{2}\mathrm{m}^2(a_1 - a_{-1}) \quad .$$

The coefficients Θ can then be evaluated when the value of m is substituted. It is to be remarked that Θ_i is $O(\mathrm{m}^{|2i|})$.

18·24. The form of the solution for q

It is to be recollected that q is a differential of the length of the normal at P (Fig. 26) related to Δx and Δy by 18·20 (2) which we write in the form—we write $C = \dot{x}/v$ and $S = \dot{y}/v$—

$$q = \frac{1}{v}(\dot{x}\,\Delta y - \dot{y}\,\Delta x) \quad .$$

Express x and y in terms of u and s, and write

$$2\Delta x = \Delta u + \Delta s \quad , \quad 2i\,\Delta y = \Delta u - \Delta s \quad ;$$

then

$$q = \frac{1}{2iv}(\dot{s}\,\Delta u - \dot{u}\,\Delta s) = \frac{n - n_1}{2v}[\zeta\, Ds\,.\,(\zeta^{-1}\Delta u) - \zeta^{-1}\, Du\,.\,(\zeta\,\Delta s)] \quad \ldots(1)$$

We thus link q with the solutions $\zeta^{-1}\Delta u$ and $\zeta\,\Delta s$ in 18·13 (2) and (3) which we write for convenience, taking the terms in e only, as

$$\zeta^{-1}\Delta u = \Sigma\, E_r \zeta^{2r+c} \quad , \quad \zeta\,\Delta s = \Sigma\, E_r' \zeta_1^{2r+c}$$

where $E_r = e_r \exp\{-\omega\sqrt{(-1)}\}$, and E' and ζ_1 are the conjugates of E and ζ respectively. Then

$$\zeta\, Ds\,.\,(\zeta^{-1}\Delta u) = -[\Sigma\,(2i+1)a_i\zeta^{-2i}][\Sigma\, E_r\zeta^{2r+c}] = -\sum_i \sum_j (2i+1)a_i E_{i+j}\zeta^{2j+c}$$

Similarly, $$\zeta^{-1}\,\Delta u\,.\,(\zeta\,\Delta s) = \sum_i \sum_j (2i+1)a_i E'_{i+j}\zeta_1^{2j+c}$$

The expression within the squared bracket in (1) is thus of the form $\Sigma\, P_j \cos\{(2j+c)\xi - \omega\}$ where $P_{-j} = P_j$.

Again, $v^2 = -(n-n_1)^2 Du \,.\, Ds$; it is readily seen that v^2 is expressible in the form $\Sigma Q_k \cos 2k\xi$ and, further, that $1/v$ is expressible in the same form. It follows that, in (1), q is of the form given by

$$q = \Sigma R_j \cos\{(2j+c)\xi - \omega\}$$

or, in terms of ζ, by

$$q = \Sigma b_j \zeta^{2j+c} + \Sigma d_j \zeta^{2j-c} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The inclusion of the terms in f, in 18·13 (2) and (3), leads to a similar result. Thus, (2) represents the general form of q, the two additional constants of integration being concealed within the coefficients b and d.

18·25. The equation for c (in terms of m)

From 18·22 (1) and (2), the equations for q is, in terms of integers k,

$$D^2 q = q \Sigma \Theta_k \zeta^{2k} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

We substitute in this equation the general expression for q given by 18·22 (2). We then have

$$\begin{aligned}
&\Sigma (2j+c)^2 b_j \zeta^{2j+c} + \Sigma (2j-c)^2 d_j \zeta^{2j-c} \\
&\quad = \sum_k \sum_i \Theta_k b_i \zeta^{2k+2i+c} + \sum_k \sum_i \Theta_k d_i \zeta^{2k+2i-c} \\
&\quad = \sum_j \sum_i \Theta_{j-i} b_i \zeta^{2j+c} + \sum_j \sum_i \Theta_{j-i} d_i \zeta^{2j-c}
\end{aligned}$$

Equating coefficients of ζ^{2j+c}, we obtain

$$(2j+c)^2 b_j = \sum_i \Theta_{j-i} b_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Similarly, by equating coefficients of ζ^{2j-c}, we obtain

$$(2j-c)^2 d_j = \sum_i \Theta_{j-i} d_i$$

or, on writing $-j$ for j and $-i$ for i and remembering that $\Theta_{-k} = \Theta_k$,

$$(2j+c)^2 d_{-j} = \sum_i \Theta_{j-i} d_{-i} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The equation for c, so far as it depends on m alone, is obtained by eliminating the b's from the series of linear equations given by (2). The same result follows from the elimination of the d's from the series of linear equations given by (3). We then need consider only (2) which we write as

$$\ldots - \Theta_2 b_{j-2} - \Theta_1 b_{j-1} + \{(c+2j)^2 - \Theta_0\} b_j - \Theta_1 b_{j+1} - \Theta_2 b_{j+2} - \ldots = 0 \quad , \ldots (4)$$

and which we suppose to be divided throughout by $(4j^2 - \Theta_0)$. Put $j = \ldots, -2, -1, 0, +1, +2, \ldots$ in succession ; we thus obtain a set of linear equations in the b's with the appropriate denominators. If we suppose that the ordinary rules apply to an infinite number of equations of the form (4)—with the appropriate denominators—we obtain, by eliminating the b's, the equation

$$\Delta(c) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where $\Delta(c)$ is the symmetrical infinite determinant given by

$$\Delta(c) =$$

$$\begin{vmatrix}
\cdots & \dfrac{(c-4)^2-\Theta_0}{4^2-\Theta_0} & -\dfrac{\Theta_1}{4^2-\Theta_0} & -\dfrac{\Theta_2}{4^2-\Theta_0} & -\dfrac{\Theta_3}{4^2-\Theta_0} & -\dfrac{\Theta_4}{4^2-\Theta_0} & \cdots \\
\cdots & -\dfrac{\Theta_1}{2^2-\Theta_0} & \dfrac{(c-2)^2-\Theta_0}{2^2-\Theta_0} & -\dfrac{\Theta_1}{2^2-\Theta_0} & -\dfrac{\Theta_2}{2^2-\Theta_0} & -\dfrac{\Theta_3}{2^2-\Theta_0} & \cdots \\
\cdots & -\dfrac{\Theta_2}{0^2-\Theta_0} & -\dfrac{\Theta_1}{0^2-\Theta_0} & \dfrac{c^2-\Theta_0}{0^2-\Theta_0} & -\dfrac{\Theta_1}{0^2-\Theta_0} & -\dfrac{\Theta_2}{0^2-\Theta_0} & \cdots \\
\cdots & -\dfrac{\Theta_3}{2^2-\Theta_0} & -\dfrac{\Theta_2}{2^2-\Theta_0} & -\dfrac{\Theta_1}{2^2-\Theta_0} & \dfrac{(c+2)^2-\Theta_0}{2^2-\Theta_0} & -\dfrac{\Theta_1}{2^2-\Theta_0} & \cdots \\
\cdots & -\dfrac{\Theta_4}{4^2-\Theta_0} & -\dfrac{\Theta_3}{4^2-\Theta_0} & -\dfrac{\Theta_2}{4^2-\Theta_0} & -\dfrac{\Theta_1}{4^2-\Theta_0} & \dfrac{(c+4)^2-\Theta_0}{4^2-\Theta_0} & \cdots
\end{vmatrix} \quad \ldots\ldots\ldots(6)$$

It is evident that (5) has an infinite number of roots; however, the root we seek is that in the neighbourhood of unity, the approximate value of which we found in 18·14 (4) to be $1+m-\frac{3}{4}m^2$.

Infinite determinants were first introduced into analysis by Hill in the present connection. We shall assume that the determinant is convergent; the procedure of dividing (4) by the appropriate expression ensures that elements in the leading diagonal tend to unity as j tends to infinity, c having an assigned value.

18·26. The roots of $\Delta(c)=0$

(i) If c_0 is a root of $\Delta(c)=0$, so also is $-c_0$. For, writing $-j$ for j in 18·25 (4), we obtain

$$\ldots - \Theta_2 b_{-j-2} - \Theta_1 b_{-j-1} + \{(c-2j)^2 - \Theta_0\} b_{-j} - \Theta_1 b_{-j+1} - \ldots = 0 \quad .$$

The coefficient of b_{-j} can be written as $\{(-c)+2j\}^2 - \Theta_0$, and hence on the elimination of the b's, we have $\Delta(-c)=0$, the factors in the denominators remaining unchanged. The roots of the equation $\Delta(c)=0$ thus occur in pairs and the equation may then be regarded as an equation in c^2.

(ii) If c_0 is a root, so also are $c_0 \pm 2r$, where r is a positive integer. Write $c+2j$ in 18·25 (4) as $\{c-2+2(j+1)\}$; then

$$\ldots - \Theta_1 b_{j-1} + [\{c-2+2(j+1)\}^2 - \Theta_0]\, b_j - \Theta_1 b_{j+1} - \ldots = 0$$

which, on division by $\{4(j+1)^2 - \Theta_0\}$ and the elimination of the b's, leads to $\Delta(c-2)=0$. In the same way, $\Delta(c+2)=0$. By repeated applications of these results, we have

$$\Delta(c \pm 2r) = 0$$

so that, if c_0 is a root, so are $c_0 \pm 2r$.

(iii) The roots of $\Delta(c)=0$ are also the roots of

$$\cos \pi c - \cos \pi c_0 = 0 \quad .$$

The roots of the latter equation are : $c = c_0 \pm 2r$. Hence, we must have

$$\Delta(c) = k(\cos \pi c - \cos \pi c_0) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where k is independent of c.

(iv) The determinant $\Delta_0(c)$.

This determinant is defined to be derived from $\Delta(c)$ when the elements $\Theta_1, \Theta_2, \ldots$ are replaced by zero ; then $\Delta_0(c)$ is simply the product of the elements in the leading diagonal of $\Delta(c)$. Let T_j, T_{-j} denote the two elements of the leading diagonal corresponding to a given value of j ; then

$$\Delta_0(c) = T_0 \prod_1^\infty (T_j T_{-j})$$

where $T_0 = 1 - \dfrac{c^2}{\Theta_0}$. For convenience, we write

$$\Theta_0 = 4\theta^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now,
$$T_j T_{-j} = \frac{(c+2j)^2 - 4\theta^2}{4(j^2-\theta^2)} \cdot \frac{(c-2j)^2 - 4\theta^2}{4(j^2-\theta^2)} \quad ,$$

which, on recombining the linear factors of the numerator and of the denominator, is easily seen to become

$$T_j T_{-j} = \frac{c^2 - (2j+2\theta)^2}{4(j+\theta)^2} \cdot \frac{c^2 - (-2j+2\theta)^2}{4(-j+\theta)^2} \quad ,$$

Hence, since $T_0 = 1 - \dfrac{c^2}{4\theta^2}$,

$$\Delta_0(c) = \prod_{-\infty}^{\infty} \left[1 - \frac{c^2}{4(j+\theta)^2} \right] \quad .$$

By a well-known theorem of trigonometry,

$$\frac{\cos x - \cos y}{1 - \cos y} = \prod_{-\infty}^{\infty} \left[1 - \frac{x^2}{(y+2\pi j)^2} \right] \quad ;$$

write $x = \pi c$, $y = 2\pi\theta = \pi\sqrt{\Theta_0}$; then

$$\Delta_0(c) = \frac{\cos \pi c - \cos (\pi\sqrt{\Theta_0})}{1 - \cos (\pi\sqrt{\Theta_0})} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

(v) *The equation for* c_0.

Let $\Delta^n(c)$ denote the symmetrical determinant with $2n+1$ rows and columns derived from 18·25 (6). If n is large we can write (1), with sufficient accuracy, as

$$\Delta^n(c) = k(\cos \pi c - \cos \pi c_0) \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

This equation will then be true for all values of c ; accordingly, the value of k can be obtained by equating the coefficient of the highest power of c

in $\Delta^n(c)$ to the coefficient of the corresponding power of c in the product-expansion of $\cos \pi c$. Now, the coefficient of the highest power of c in $\Delta^n(c)$ is obtained from the leading diagonal and is the same as the coefficient of the highest power of c in ${\Delta_0}^n(c)$—the symmetrical determinant with $2n+1$ rows and columns and with the elements outside the leading diagonal equal to zero. Hence, from (3) and (4),

$$k = \frac{1}{1 - \cos(\pi\sqrt{\Theta_0})} \quad \ldots\ldots(5)$$

Accordingly,

$$\Delta(c) = \frac{\cos \pi c - \cos \pi c_0}{1 - \cos(\pi\sqrt{\Theta_0})} \quad \ldots\ldots(6)$$

This last equation is true for all values of c so that, when $c = 0$,

$$\Delta(0) = \frac{\sin^2 \frac{1}{2}\pi c_0}{\sin^2(\frac{1}{2}\pi\sqrt{\Theta_0})} \quad \ldots\ldots(7)$$

This equation gives c_0 in terms of the determinant $\Delta(0)$. The calculation of c_0 is thus dependent on the evaluation of $\Delta(0)$ which we write in the form

$$\Delta(0) = \begin{vmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots, & 1, & -B_2\Theta_1, & -B_2\Theta_2, & -B_2\Theta_3, & -B_2\Theta_4, \ldots \\ \ldots, & -B_1\Theta_1, & 1, & -B_1\Theta_1, & -B_1\Theta_2, & -B_1\Theta_3, \ldots \\ \ldots, & -B_0\Theta_2, & -B_0\Theta_1, & 1, & -B_0\Theta_1, & -B_0\Theta_2, \ldots \\ \ldots, & -B_1\Theta_3, & -B_1\Theta_2, & -B_1\Theta_1, & 1, & -B_1\Theta_1, \ldots \\ \ldots, & -B_2\Theta_4, & -B_2\Theta_3, & -B_2\Theta_2, & -B_2\Theta_1, & 1, \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad \ldots(8)$$

in which

$$B_j = \frac{1}{4j^2 - \Theta_0} \equiv \frac{1}{4(j^2 - \theta^2)} \quad \ldots\ldots(9)$$

18·27. Evaluation of $\Delta(0)$

In evaluating $\Delta(0)$ we remember that the coefficients Θ_j are $O(m^{2j})$, j being positive. For example, if we restrict our calculation to $O(m^4)$, it is clear that all elements involving Θ_r, for $r > 2$, can all be put equal to zero; $\Delta(0)$ is then effectively reduced to the leading diagonal and the elements with factor Θ_1 on either side of it. Proceeding in this way, it can be seen that

$$\Delta(0) = 1 + A\Theta_1^2 + \{B\Theta_2^2 + C\Theta_1^2\Theta_2 + D\Theta_1^4\} + \ldots \quad \ldots\ldots(1)$$

in which the coefficient of A is $O(m^4)$, the terms within the brackets are $O(m^8)$, and so on. Thus, $\Delta(0)$ converges very rapidly.

The coefficient A arises through product-terms of the form $-B_jB_{j-1}$; thus

$$A = -\sum_{-\infty}^{\infty} B_jB_{j-1} \quad .$$

Now,

$$X_j \equiv B_jB_{j-1} = \frac{1}{16(\theta^2 - j^2)[\theta^2 - (j-1)^2]}$$

which can be readily expressed in partial fractions, giving

$$16X_j = \frac{1}{2\theta(2\theta-1)}\left\{\frac{1}{\theta-j}+\frac{1}{\theta+j-1}\right\} - \frac{1}{2\theta(2\theta+1)}\left\{\frac{1}{\theta+j}+\frac{1}{\theta-j+1}\right\} \quad .$$

We introduce the well-known formula

$$\pi \cot \pi\phi = \sum_{-\infty}^{\infty} \frac{1}{\phi+j} \quad ; \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

hence

$$\Sigma X_j = \frac{1}{32\theta}\left[\frac{2\pi \cot \pi\theta}{2\theta-1} - \frac{2\pi \cot \pi\theta}{2\theta+1}\right]$$

and

$$A = -\frac{\pi \cot \pi\theta}{8\theta(4\theta^2-1)} \quad .$$

Thus, omitting the terms in B, C, D, etc., we obtain the approximate value of $\Delta(0)$ given by

$$\Delta(0) = 1 + \frac{\pi\Theta_1^2 \cot(\frac{1}{2}\pi\sqrt{\Theta_0})}{4(1-\Theta_0)\sqrt{\Theta_0}} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Since $(1-\Theta_0)$ is $O(m)$, then $(1-\Theta_0)^{-1} \cot \frac{1}{2}\pi\sqrt{\Theta_0}$ is of zero order; consequently (3) gives $\Delta(0)$ to $O(\mathrm{m}^7)$.

The terms in B, C, D in (1) can be treated by a similar method—that is to say, we convert such terms into partial fractions using (2) to obtain closed expressions such as we have found in (3). The process is evidently more complicated than that for deriving the expression for A in (1), and we refer the reader to E. W. Brown's *Lunar Theory*, p. 223, for further details.

18·28. Calculation of c

With m known the numerical values of Θ_0, Θ_1 are found; the numerical value of $\Delta(0)$ given by 18·27 (3) is then obtained to the degree of accuracy implied.

We have seen that the roots of the equation $\Delta(\mathrm{c}) = 0$ are, dropping the zero suffix, of the form $\mathrm{c} \pm 2r$, where r is an integer; we choose as the significant root that value of c nearest unity, for that is the value which is related to the motion of the lunar perigee.

When $\Delta(0)$ has been calculated, the value of c, to $O(\mathrm{m}^7)$, is then derived from

$$\sin^2 \tfrac{1}{2}\pi\mathrm{c} = \Delta(0) \sin^2(\tfrac{1}{2}\pi\sqrt{\Theta_0}) \quad .$$

Hill's value of c, found in this way, is :

$$\mathrm{c} = 1{\cdot}07158\ 38 \quad .$$

Taking terms of higher order in the development of $\Delta(0)$, Hill obtained—to 15 places of decimals—

$$\mathrm{c} = 1{\cdot}07158\ 32774\ 16012 \quad .$$

18·29. General solutions depending on m

The relevant equations are 18·05 (9) and (10); they are

$$D(u\,Ds - s\,Du - 2\mathrm{m}us) + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) = 0 \quad , \qquad \ldots\ldots(1)$$

$$D^2(us) - Du\,.\,Ds + 2\mathrm{m}(s\,Du - u\,Ds) + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 = C. \qquad \ldots\ldots(2)$$

From 18·13 the approximate solutions of the equivalent equations in terms of Δu, Δs are $u+\Delta u$, $s+\Delta s$ where u, s are the solutions for the intermediate orbit and Δu is given by

$$\zeta^{-1}\Delta u = e^{-\omega\sqrt{-1}}\,\Sigma\, e_i\zeta^{2i+\mathrm{c}} + e^{\omega\sqrt{-1}}\,\Sigma\, f_i\zeta^{2i-\mathrm{c}} \quad ; \qquad \ldots\ldots(3)$$

there is a corresponding expression for $\zeta\,\Delta s$. Considering the first summation in (3), we write it as

$$\Sigma\, e_i\zeta^{2i}\zeta_0{}^{\mathrm{c}} \quad ,$$

where
$$\zeta_0{}^{\mathrm{c}} = \exp\{\sqrt{-1}\,(\mathrm{c}\xi - \omega)\} \qquad \ldots\ldots(4)$$

Now,
$$\mathrm{c}\xi - \omega = \mathrm{c}(n-n_1)(t-t_0) - \omega \equiv \mathrm{c}(n-n_1)(t-t_1) \quad , \qquad \ldots\ldots(5)$$

where
$$\mathrm{c}(n-n_1)t_1 = \mathrm{c}(n-n_1)t_0 + \omega \quad ;$$

hence
$$\zeta_0 = \exp\{\sqrt{-1}\,(n-n_1)(t-t_1)\} \qquad \ldots\ldots(6)$$

Thus t_1 can replace ω as one of the arbitrary constants.

Referring to (3) we see that the solutions of (1) and (2), involving first powers of Δu and Δs only, can be written as

$$u = \mathrm{a}\,\Sigma\, A_{2i+p\mathrm{c}}\zeta^{2i+1}\zeta_0{}^{p\mathrm{c}} \quad , \quad s = \mathrm{a}\,\Sigma\, A_{2i+p\mathrm{c}}\zeta^{-(2i+1)}\zeta_0{}^{-p\mathrm{c}} \quad , \qquad \ldots\ldots(7)$$

where p takes the values $-1, 0, +1$; the value 0 corresponds to the intermediate orbit and then $A_{2i} \equiv a_i$.

The *general* solutions of (1) and (2), when we withdraw the restriction that powers of Δu and Δs beyond the first are omitted, have the forms given by (7) where p has now any integral value, positive or negative and including zero.

If D_0 denotes the operator $\zeta_0\dfrac{d}{d\zeta_0}$, we have

$$\frac{d}{dt} = (n-n_1)D\sqrt{-1} = (n-n_1)D_0\sqrt{-1} \quad ;$$

hence
$$D(\zeta^{2i+1}\zeta_0{}^{p\mathrm{c}}) = (2i+1+p\mathrm{c})\zeta^{2i+1}\zeta_0{}^{p\mathrm{c}} \quad ,$$

and
$$D^2(\zeta^{2i+1}\zeta_0{}^{p\mathrm{c}}) = (2i+1+p\mathrm{c})^2\zeta^{2i+1}\zeta_0{}^{p\mathrm{c}} \quad .$$

As an analytical device, we put $\zeta_0 = \zeta$ and write (7) as

$$\zeta^{-1}u = \mathrm{a}\sum_i\sum_p A_{2i+p\mathrm{c}}\zeta^{2i+\mathrm{c}} \quad , \quad \zeta s = \mathrm{a}\sum_i\sum_p A_{2i+p\mathrm{c}}\zeta^{-(2i+p\mathrm{c})} \quad . \qquad \ldots\ldots(8)$$

These expressions are substituted in (1) and (2); equating coefficients of powers of ζ we then obtain linear equations in the A's from which, by successive approximations, the expressions of the A's in terms of m and c and one of them—corresponding to e_0 in 18·13—can be obtained. The procedure is related to that adopted for determining the coefficients a_i of the intermediate orbit; the general lines to be followed are given in Brown's *Lunar Theory*, pp. 207–211.

18·30. The equation in latitude

We neglect the solar eccentricity and parallax; then $W=0$ and the equation in latitude is, from 18·04 (13),

$$D^2z-\left(\frac{\kappa}{r^3}+\mathrm{m}^2\right)z=0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now, $r^2=us+z^2$; if we neglect z^2, then $r^2=us$ and, by 18·12 (7),

$$\frac{\kappa}{r^3}+\mathrm{m}^2=2\,\Sigma\, M_j\zeta^{2j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $M_j=M_{-j}$. Now z is $O(\gamma)$, where $\gamma=\tan i$, and in the undisturbed orbit z is expressible in terms of a sine-series. Let $Z=iz$ where $i=\sqrt{-1}$; then, from (1) and (2),

$$D^2Z=2(\Sigma\, M_j\zeta^{2j})Z \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

This equation is of the same form as 18·22 (1), and the solution, involving two arbitrary constants γ and β, is

$$Z=\mathrm{a}\gamma e^{i\beta}\,\Sigma\, C_j\zeta^{2j+\mathrm{g}} \quad ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

z is then the imaginary part of (4), so that

$$z=\mathrm{a}\gamma\,\Sigma\, C_j\sin\{(2j+\mathrm{g})\xi+\beta\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

in which C_0 can be taken, without loss of generality, to be unity. Substituting (4) in (3) and equating coefficients of $\zeta^{2j+\mathrm{g}}$ in the usual way we obtain

$$(\mathrm{g}+2j)^2C_j=2\sum_i M_{j-i}C_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

This equation is similar to 18·25 (2), C_j replacing b_j, g replacing c and $2M_{j-i}$ replacing Θ_{j-i}. The determination of g is then parallel to that of c, the determinant-equation being

$$\Delta(\mathrm{g})=0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

It is found that to $O(\mathrm{m}^3)$, for example,

$$\mathrm{g}=1+\mathrm{m}+\tfrac{3}{4}\mathrm{m}^2-\tfrac{33}{32}\mathrm{m}^3 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

this being the root of (6) which is close to unity—in accordance with the observed perturbation of the lunar orbit in this particular connection. The numerical value of g was first given by J. C. Adams;* it is:

$$\mathrm{g}=1{\cdot}08517\ 13927\ 46869 \quad .$$

Expressing g in terms of m, as in (8), we can readily derive the expressions for C_j in terms of m by successive approximations. For example, since $C_0=1$, $M_i=M_{-i}$ and $M_i=O(\mathrm{m}^{|2i|})$, we find on putting $j=0$ in (6) that

$$\tfrac{1}{2}\mathrm{g}^2-M_0=M_1(C_1+C_{-1})+M_2(C_2+C_{-2})+\ldots \quad .$$

* M.N.R.A.S., 38, 43 (1877).

Similarly, when $j=1$ and $j=-1$,

$$\{\tfrac{1}{2}(g+2)^2-M_0\}C_1=M_1(1+C_2)+M_2C_{-1}+\dots \quad , \dots\dots\dots(9)$$

$$\{\tfrac{1}{2}(g-2)^2-M_0\}C_{-1}=M_1(1+C_{-2})+M_2C_1+\dots \quad . \dots\dots\dots(10)$$

Now, from 18·12 (8), to $O(m^2)$, $2M_0=1+2m+\frac{5}{2}m^2$. The coefficient of C_1 in (9) is thus of the form $a+bm+dm^2$; hence C_1 is $O(m^2)$. Also, the coefficient of C_{-1} in (10) is of the form $am+bm^2$; hence C_{-1} is $O(m)$ and it is easily seen that, to $O(m)$, $C_{-1}=-\frac{3}{8}m$. Generally, when j is positive, C_j is $O(m^{2j})$ and C_{-j} is $O(m^{2j-1})$.

18·31. First approximation for motion of the node

If l ($\equiv nt+\epsilon$) is the Moon's mean longitude and θ its latitude and if the orbital eccentricity is neglected, then

$$\tan\theta=\tan i\sin(l-\Omega)=\gamma\sin(nt+\epsilon-\Omega) \quad .$$

Also, $z=r\tan\theta$, where $r=\sqrt{(us+z^2)}$. If we consider first order terms only, then $r=a$ and

$$z=a\gamma\sin(nt+\epsilon-\Omega) \quad . \dots\dots\dots(1)$$

From 18·30 (5) the principal term of the solution is, for $j=0$,

$$z=a\gamma\sin(g\xi+\beta) \quad . \dots\dots\dots(2)$$

Now,
$$\xi=(n-n_1)t+\epsilon-\epsilon_1=\frac{nt}{1+m}+\epsilon-\epsilon_1 \quad ;$$

hence
$$g\xi+\beta=nt+\epsilon-\Omega_0+\left(\frac{g}{1+m}-1\right)nt \quad ,$$

where
$$-\Omega_0=(g-1)\epsilon+\beta-g\epsilon_1 \quad .$$

Comparison of (1) and (2) shows that

$$\Omega=\Omega_0-\alpha nt \quad ,$$

where
$$\alpha=\frac{g}{1+m}-1 \dots\dots\dots(3)$$

To $O(m^3)$ it is seen that

$$\alpha=\tfrac{3}{4}m^2-\tfrac{57}{32}m^3 \quad . \dots\dots\dots(4)$$

The secular term in the formula for the retrogression of the node is given by $\dot{\Omega}=-\alpha n$.

18·32. General treatment for the motion of the node

Let l_1 ($\equiv n_1t+\epsilon_1$) be the Sun's mean longitude with reference to fixed axes in the reference plane; if (X, Y) denote the Moon's coordinates with respect to these axes, then

$$X=x\cos l_1-y\sin l_1 \quad , \quad Y=x\sin l_1+y\cos l_1 \quad .$$

Let
$$U=X+iY \quad , \quad S=X-iY \quad ; \dots\dots\dots(1)$$

then
$$U=(x+iy)e^{il_1} \quad , \quad S=(x-iy)e^{-il_1} \quad . \dots\dots\dots(2)$$

Now,
$$l_1=n_1t+\epsilon_1=m(n-n_1)t+\epsilon_1=m(n-n_1)(t-t_0)+\delta$$

where $\delta = \mathrm{m}(n - n_1)t_0 + \epsilon_1$;

thus $l_1 = \mathrm{m}\xi + \delta$.

Set $d = e^{i\delta}$, $d_1 = e^{-i\delta}$; then (2) become

$$U = du\zeta^{\mathrm{m}} \quad , \quad V = d_1 s\zeta^{-\mathrm{m}} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

or $$U = d\,\Sigma\, a_i\zeta^{2i+1+\mathrm{m}} \quad , \quad S = d_1\,\Sigma\, a_i\zeta^{-(2i+1+\mathrm{m})} \quad . \quad \ldots\ldots\ldots\ldots\ldots (4)$$

If (ξ, η, ζ) denote current coordinates associated with the fixed axes OX, OY, OZ then the instantaneous plane passing through O (the Earth), the Moon and the directions of the Moon's motion is

$$\begin{vmatrix} \xi & , & \eta & , & \zeta \\ X & , & Y & , & Z \\ \dot{X} & , & \dot{Y} & , & \dot{Z} \end{vmatrix} = 0 \quad .$$

This plane cuts the reference plane $\zeta = 0$ in the straight line

$$\xi(Y\dot{Z} - \dot{Y}Z) = \eta(X\dot{Z} - \dot{X}Z) \quad .$$

The longitude of the node (measured from OX) is such that $\tan \Omega = \eta/\xi$; hence

$$\tan \Omega = \frac{Y\dot{Z} - \dot{Y}Z}{X\dot{Z} - \dot{X}Z}$$

from which, by means of (1) and on writing z for Z,

$$e^{2i\Omega} = \frac{U\dot{z} - z\dot{U}}{S\dot{z} - z\dot{S}} = \frac{U\,Dz - z\,DU}{S\,Dz - z\,DS} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

From 18·30 (5),

$$\frac{2i}{\mathrm{a}\gamma} z = \Sigma\, C_r e^{i\beta}\zeta^{2r+\mathrm{g}} - \Sigma\, C_r e^{-i\beta}\zeta^{-(2r+\mathrm{g})} \quad .$$

Let $C_r e^{i\beta} = K_r$, $C_r e^{-i\beta} = K_r'$; K_r' is the conjugate of K_r. Then

$$\frac{2i}{\mathrm{a}\gamma} z = \zeta^{\mathrm{g}}\,\Sigma\, K_r\zeta^{2r} - \zeta^{-\mathrm{g}}\,\Sigma\, K'_{-r}\zeta^{2r} \quad .$$

From this equation and (4) we find, after some reduction,

$$\frac{2i}{\mathrm{a}\gamma}(UDz - z\,DU) = d\zeta^{1+\mathrm{m}-\mathrm{g}} \sum_r \sum_j (2j - 4r + 1 + \mathrm{m} + \mathrm{g}) a_{j-r} K'_{-r}\zeta^{2j}$$
$$- d\zeta^{1+\mathrm{m}+\mathrm{g}} \sum_r \sum_j (2j - 4r + 1 + \mathrm{m} - \mathrm{g}) a_{j-r} K_r \zeta^{2j} \quad .$$

Let $e^{i\beta} = b$, $e^{-i\beta} = b_1$; then $K_r = bC_r$, $K_r' = b_1 C_r$. Further, let

$$P_{jr} = (2j - 4r + 1 + \mathrm{m} + \mathrm{g}) a_{j-r} C_{-r} \quad , \quad \ldots\ldots\ldots\ldots\ldots (6)$$
$$Q_{jr} = (2j - 4r + 1 + \mathrm{m} - \mathrm{g}) a_{j-r} C_r \quad . \quad \ldots\ldots\ldots\ldots\ldots (7)$$

The values of the coefficients a and C are supposed known ; we can then calculate the coefficients P and Q. We now have

$$\frac{2i}{\mathrm{a}\gamma}(UDz - z\,DU) = b_1 d\zeta^{1+\mathrm{m}-\mathrm{g}} \left[\sum_r \sum_j \{P_{jr}\zeta^{2j} - b^2 Q_{jr}\zeta^{2j+2\mathrm{g}}\}\right] \quad . \quad \ldots\ldots (8)$$

Since the denominator of (5) is the conjugate of the numerator, we have

$$\frac{2i}{a\gamma}(S\,Dz - z\,DS) = bd_1\zeta_1^{1+m-g}[\sum_r \sum_j \{P_{jr}\zeta_1^{2j} - b_1^2 Q_{jr}\zeta_1^{2j+2g}\}] \quad \ldots\ldots(9)$$

The equation (5) now becomes, on replacing b and d by their exponential expressions,

$$e^{2i(\Omega-\delta+\beta)} = \zeta^{2(1+m-g)} \cdot \frac{F}{F_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

where F and F_1 denote the expressions within the squared brackets on the right-hand sides of (8) and (9) respectively.

Since $a_0 = C_0 = 1$ the value of $P_{0,0}$ from (6) is $1 + m + g$; from 18·30 (8) it is seen that $P_{0,0}$ is $O(m^0)$. Similarly, $Q_{0,0}$ is $O(m^2)$.

Let $$\frac{1}{P_{0,0}} \sum_r P_{jr} = R_j \quad , \quad \frac{1}{P_{0,0}} \sum_r Q_{jr} = S_j \quad ; \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

then $$F/P_{0,0} = 1 + \Sigma' R_j\zeta^{2j} - b^2 \sum S_j\zeta^{2g+2j} \quad . \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

When the coefficients R and S are expanded in powers of m and multiplied by $P_{0,0}$, F can be written as

$$F = 1 + mA(\zeta) + m^2B(\zeta) + \ldots$$

where $A(\zeta)$, $B(\zeta)$, ... are functions of ζ ; hence we can write

$$\log F = ma(\zeta) + m^2b(\zeta) + \ldots$$

where

$$a(\zeta) = A(\zeta) \quad , \quad b(\zeta) = B(\zeta) - \tfrac{1}{2}\{A(\zeta)\}^2 \quad ,$$

and so on. Similarly,

$$\log F_1 = ma(\zeta_1) + m^2b(\zeta_1) + \ldots \quad .$$

We then obtain from (10),

$$\Omega - \delta + \beta = (1 + m - g)\xi + \frac{m}{2i}\{a(\zeta) - a(\zeta_1)\} + \frac{m^2}{2i}\{b(\zeta) - b(\zeta_1)\} + \ldots \quad .(13)$$

It is evident that the right-hand side is a sine-series.

As an illustration we shall find the formula for Ω when m^2 and higher powers are omitted in F and F_1. From (12),

$$F = 1 + R_1\zeta^2 + R_{-1}\zeta^{-2} + \ldots - b^2\zeta^{2g}(S_0 + S_1\zeta^2 + S_{-1}\zeta^{-2})\ldots \quad .$$

It is easily seen that S_{-1} is the only coefficient of order m in this expression; the significant part of S_{-1} is $\frac{1}{P_{0,0}} Q_{-1,-1}$ and, from (6) and (7), we find—actually correct to $O(m^2)$—

$$S_{-1} = \frac{3 + m - g}{1 + m + g} C_{-1} \equiv fC_{-1} \quad ,$$

or, since $C_{-1} = -\frac{3}{8}m$, then $S_{-1} = -\frac{3}{8}mf$. Then

$$F = 1 + \tfrac{3}{8}mfb^2\zeta^{2g-2} \quad . \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

The expression for F_1 is the same as (14) with ζ_1 and b_1 replacing ζ and b. We then derive—as in (13)—

$$\Omega - \delta + \beta - (1 + \mathrm{m} - \mathrm{g})\,\xi = \tfrac{3}{8}\mathrm{m}f \cdot \frac{1}{2i}\{b^2\zeta^{2g-2} - b_1{}^2\zeta_1{}^{2g-2}\}$$
$$= \tfrac{3}{8}\mathrm{m}f \sin\{(2\mathrm{g} - 2)\,\xi + 2\beta\} \quad .$$

The motion of the node is then given by

$$\dot{\Omega} = -\alpha n + \tfrac{3}{4}(\mathrm{g} - 1)\,\frac{\mathrm{m}n}{1 + \mathrm{m}} \cdot \frac{3 + \mathrm{m} - \mathrm{g}}{1 + \mathrm{m} + \mathrm{g}} \cos\{(2\mathrm{g} - 2)\,\xi + 2\beta\} \quad \ldots\ldots(15)$$

where α is defined by 18·31 (3) or (4). Since $\mathrm{g} = 1 + \mathrm{m} + \ldots$, the amplitude of the periodic term is $O(\mathrm{m}^2)$. The periodic terms of higher order can be obtained by the procedure already described.

18·33. Perturbations depending on m and the solar eccentricity e_1

We neglect z, a/a_1 and the lunar eccentricity e. From 18·03 (1),

$$R_2 = n_1{}^2 r^2 \left(\frac{a_1}{r_1}\right)^3 P_2(\alpha) \quad ,$$

and hence the function Ω, defined by 18·03 (5), is given by

$$\Omega = n_1{}^2 r^2 \left(\frac{a_1}{r_1}\right)^3 P_2(\alpha) - \tfrac{1}{2} n_1{}^2 (2x^2 - y^2 - z^2)$$

and the function W [$\equiv 2\Omega/(n - n_1)^2$], defined in 18·04 (10), is given by

$$W = 2\mathrm{m}^2 r^2 \left(\frac{a_1}{r_1}\right)^3 P_2(\alpha) - \mathrm{m}^2 (3x^2 - r^2) \quad .$$

Now, $r^2 P_2(\alpha) = \tfrac{3}{2} r^2 C^2 - \tfrac{1}{2} r^2$, where $C \equiv \cos\alpha$ and, by 18·03 (2),

$$rC = \frac{x x_1 + y y_1}{r_1} \quad .$$

Let v_1 denote the Sun's true longitude; then $x_1 = r_1 \cos\phi$, $y_1 = r_1 \sin\phi$, where

$$\phi = v_1 - n_1 (t - t_0) = f_1 - M_1 \quad \ldots\ldots(1)$$

Then $$rC = \tfrac{1}{2}(u + s)\cos\phi + \frac{1}{2i}(u - s)\sin\phi = \tfrac{1}{2}(ue^{-i\phi} + se^{i\phi}) \quad .$$

Hence, W becomes, on writing $r^2 = us$,

$$W = Au^2 + 2Bus + Cs^2 \quad , \quad \ldots\ldots(2)$$

where

$$A = \tfrac{3}{4}\mathrm{m}^2 \left[\left(\frac{a_1}{r_1}\right)^3 e^{-2i\phi} - 1\right] \quad , \quad \ldots\ldots(3)$$

$$B = \tfrac{1}{4}\mathrm{m}^2 \left[\left(\frac{a_1}{r_1}\right)^3 - 1\right] \quad , \quad \ldots\ldots(4)$$

$$C = \tfrac{3}{4}\mathrm{m}^2 \left[\left(\frac{a_1}{r_1}\right)^3 e^{2i\phi} - 1\right] \quad . \quad \ldots\ldots(5)$$

The equations 18·04 (14) and (17) are, since $z=0$ and W is a homogeneous quadratic function in u and s,

$$D(u\,Ds - s\,Du - 2\mathrm{m}us) + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) = s\frac{\partial W}{\partial s} - u\frac{\partial W}{\partial u} \quad , \quad \ldots\ldots(6)$$

$$D^2(us) - Du\,.\,Ds + 2\mathrm{m}(s\,Du - u\,Ds) + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 = K - 3W + D^{-1}(D_t W) \quad . \quad \ldots\ldots(7)$$

in which D_t denotes operating with D on the terms in t introduced by the Sun's coordinates, and K (the constant of integration) is written for C.

The terms on the right-hand side of (6) are

$$2Cs^2 - 2Au^2 \quad \ldots\ldots(8)$$

and the terms on the right-hand side of (7) are

$$-3(Au^2 + 2Bus + Cs^2) + D^{-1}(u^2 DA + 2usDB + s^2 DC) \quad , \quad \ldots\ldots(9)$$

the constant K being omitted since we associate it with the solution when $W=0$, corresponding to the intermediate orbit.

18·34. The functions A, B and C

The definition of ϕ in 18·33 (1) shows that ϕ is the equation of the centre so that $\phi = f_1 - M_1$, where f_1, M_1 are the Sun's true and mean anomalies respectively; the functions of the Sun's coordinates in A, B and C can thus be expressed in terms of M_1.

(i) B. For an elliptic orbit, we can write, in general,

$$\left(\frac{a}{r}\right)^3 - 1 = \alpha_0 + 2\sum_1^\infty \alpha_p \cos pM = \Sigma\,\alpha_p e^{ipM} \quad ,$$

where $\alpha_{-p} = \alpha_p$. Hence we have, for the solar orbit,

$$B = \tfrac{1}{4}\mathrm{m}^2\,\Sigma\,\alpha_p e^{ipM_1} \quad \ldots\ldots(1)$$

in which the coefficient α_p is, in general, a series in powers of e_1, of $O(e_1^{|p|})$, which can be explicitly determined by the usual procedures.

(ii) A and C. From 18·33 (3) and (4), we have—dropping the suffix temporarily—

$$C + A = \tfrac{3}{2}\mathrm{m}^2\left[\left(\frac{a}{r}\right)^3 \cos 2\phi - 1\right] \quad , \quad \ldots\ldots(2)$$

$$C - A = \tfrac{3}{2}i\mathrm{m}^2\left[\left(\frac{a}{r}\right)^3 \sin 2\phi\right] \quad . \quad \ldots\ldots(3)$$

Now, $\phi = f - M$, which can be expressed as a trigonometrical series with arguments which are multiples of M; it is found that the function within the squared brackets in (2) can be expanded as

$$\left(\frac{a}{r}\right)^3 \cos 2\phi - 1 = \Sigma\,\beta_p \cos pM \quad \ldots\ldots(4)$$

in which the coefficients β_p $(-\infty < p < \infty)$ are given in terms of e_1. Also,

$$\left(\frac{a}{r}\right)^3 \sin 2\phi = \Sigma\, \beta_p \sin pM \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

for, from (2)-(5), since A is the conjugate of C,

$$C = \tfrac{3}{4}\mathrm{m}^2\, \Sigma\, \beta_p e^{ipM_1} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$A = \tfrac{3}{4}\mathrm{m}^2\, \Sigma\, \beta_p e^{-ipM_1} = \tfrac{3}{4}\mathrm{m}^2\, \Sigma\, \beta_{-p} e^{ipM_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

in which the suffix has now been restored.

Now, $M_1 = n_1(t - t_0) - \varpi_1$, where ϖ_1 is the longitude of perihelion for the solar orbit. Also, since $n_1 = \mathrm{m}(n - n_1)$ and $\xi = (n - n_1)(t - t_0)$, then

$$M_1 = \mathrm{m}\xi - \varpi_1 \quad .$$

Denote $e^{-ip\varpi_1}$ by L_p; then $e^{ipM_1} = L_p \zeta^{p\mathrm{m}}$. We then have, from (6), (7) and (1),

$$\left.\begin{aligned} A &= \tfrac{3}{4}\mathrm{m}^2\, \Sigma\, \beta_{-p} L_p \zeta^{p\mathrm{m}} \\ B &= \tfrac{1}{4}\mathrm{m}^2\, \Sigma\, \alpha_p L_p \zeta^{p\mathrm{m}} \\ C &= \tfrac{3}{4}\mathrm{m}^2\, \Sigma\, \beta_p L_p \zeta^{p\mathrm{m}} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Since L_p is associated with $\zeta^{p\mathrm{m}}$ in a simple way we can omit it in the subsequent analysis restoring it when we transform to real variables x and y.

18·35. Solutions in terms of m and e_1

The solutions of 18·33 (6) and (7) are of the form

$$u = \mathrm{a} \sum_i \sum_p A_{i,p} \zeta^{2i+1+p\mathrm{m}} \quad , \quad s = \mathrm{a} \sum_i \sum_p A_{i,p} \zeta^{-(2i+1+p\mathrm{m})} \quad . \qquad \ldots\ldots\ldots\ldots (1)$$

When these are substituted in the equations and the coefficients of $\zeta^{2i+1+p\mathrm{m}}$ considered together, we obtain linear equations in the A's which are solved by the method of successive approximations.

It is to be remarked that, when $p = 0$, we obtain the solution for the intermediate orbit, for then $A_{i,0} = a_i$, with which the constant K in 18·33 (7) is associated.

A further remark is concerned with the right-hand side of 18·33 (7) in which the operator D^{-1} appears; the relevant terms are given in 18·33 (9). Consider first the term $D^{-1}(u^2\, DA)$. By 18·34 (8), DA is of the form $\Sigma'\, \gamma_p \zeta^{p\mathrm{m}}$, the prime indicating that $p \neq 0$. By (1) u^2 can be developed in the form $\sum_i \sum_p B_{i,p} \zeta^{2i+p\mathrm{m}}$. Hence $u^2\, DA$ will be of the form $\Sigma\, \Sigma\, \mathrm{C}_{i,p} \zeta^{2i+p\mathrm{m}}$ in which $p \neq 0$. We then have

$$D^{-1}(u^2\, DA) = \Sigma\, \Sigma \frac{C_{i,p} \zeta^{2i+p\mathrm{m}}}{2i + p\mathrm{m}} \quad , \quad (p \neq 0) \quad .$$

Since m is incommeasurable, $2i + p\mathrm{m} \neq 0$. The smallest numerical value of the divisor occurs when $i = 0$ and $p = \pm 1$; accordingly, the order (say, m^q) of the coefficients $C_{0,1}$ and $C_{0,-1}$ are reduced to m^{q-1} by the divisor.

18·36. Parallactic inequalities

These depend on m and $1/a_1$; accordingly, we put $z=0$ and $e_1=0$ so that $r_1=a_1$; also, $W_2=0$ and $D_tW=0$. We then have $W=W_3+W_4$, where

$$W_3=2\mathrm{m}^2\frac{r^3}{a_1}P_3(\alpha) \quad , \quad W_4=2\mathrm{m}^2\frac{r^4}{a_1^2}P_4(\alpha) \quad ;$$

in these we omit the factor $(E-M)/(E+M)$ associated with $1/a_1$ in W_3 and its square associated with $1/a_1^2$ in W_4. Now,

$$rC\equiv r\cos\alpha=x=\tfrac{1}{2}(u+s) \quad ;$$

hence, since

$$P_3(\alpha)=\tfrac{5}{2}C^3-\tfrac{3}{2}C \quad , \quad P_4(\alpha)=\tfrac{35}{8}C^4-\tfrac{15}{4}C^2+\tfrac{3}{8} \quad ,$$

we obtain

$$W_3=\frac{\mathrm{m}^2}{8a_1}[5(u^3+s^3)+3us(u+s)] \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$W_4=\frac{\mathrm{m}^2}{64a_1^2}[35(u^4+s^4)+20us(u+s)^2+18u^2s^2] \quad . \quad \ldots\ldots\ldots(2)$$

The equations to be solved are 18·04 (14) and (17) which reduce in the present instance to

$$D(u\,Ds-s\,Du-2\mathrm{m}us)+\tfrac{3}{2}\mathrm{m}^2(u^2-s^2)=s\frac{\partial W}{\partial s}-u\frac{\partial W}{\partial u} \quad , \quad \ldots\ldots\ldots(3)$$

$$D^2(us)-Du\,.\,Ds+2\mathrm{m}(s\,Du-u\,Ds)+\tfrac{9}{4}\mathrm{m}^2(u+s)^2 = C-4W_3-5W_4 \quad \ldots\ldots\ldots(4)$$

If, for u and s, we substitute power-series in ζ in (3) and (4), the left-hand sides will consist of terms involving even powers of ζ. Consider W_3 on the right of (3) and (4)—in (3) $W=W_3+W_4$; in each case we have a homogeneous cubic function of u and s; the only way in which such terms can produce even powers of ζ (with factor $1/a_1$) requires that the series associated with W_3—that is, with $1/a_1$—shall consist of terms with even powers of ζ. Similarly, it is easily seen that the series for u and s associated with W_4 will consist of terms with odd and even powers of ζ.

The *general* solutions of (3) and (4) must contain the solutions for the intermediate orbit—corresponding to $1/a_1=0$—and can be written in the forms

$$u=\mathrm{a}\,\Sigma\,a_i\zeta^{2i+1}+\mathrm{a}\,\Sigma\,B_j\zeta^j \quad , \quad s=\mathrm{a}\,\Sigma\,a_i\zeta^{-(2i+1)}+\mathrm{a}\,\Sigma\,B_j\zeta^{-j} \quad , \quad \ldots(5)$$

in which the coefficients B are associated with the factors $1/a_1$ and $1/a_1^2$. The expressions in (5) are to be substituted in (3) and (4) and, equating coefficients of powers of ζ in the usual way, we derive linear equations in the B's involving m—through the a's—$1/a_1$ and $1/a_1^2$. First approximations are obtained by neglecting $1/a_1^2$; in this case the B's—now of the form $\ldots$—are derived.

18·37. Other perturbations

In the preceding sections of this chapter an account has been given of the derivation of the principal perturbations in the motion of the Moon. The final theory, as completed by E. W. Brown, deals in detail with the following :

(i) Terms of the disturbing function depending on m and combinations of the lunar and solar eccentricities, the inclination ($\tan^{-1}\gamma$) and $1/a_1$;

(ii) the direct disturbing effects of the planets on the lunar orbit ;

(iii) the indirect planetary action appearing as perturbations of the solar orbit ;

(iv) the motion of the ecliptic ;

(v) the ellipsoidal figure of the Earth ;

(vi) the Moon's secular acceleration.

The last will be considered in some detail in Chapter 19. For details concerned with (i)–(v) the reader is referred to Brown's *Lunar Theory*, or to the Memoirs noted in 18·01.

Chapter 19

SECULAR ACCELERATION OF THE MOON

19·01. Halley's discovery

The mean longitude of a planet or of the Moon in an undisturbed orbit is given by $l = nt + \epsilon$, n and ϵ being constants. It is to be understood in what follows that, as calculated by this formula, l will be represented in part by $360°j$, where j denotes the integral number of sidereal periods in the interval t reckoned from a particular epoch. When perturbations are in evidence, l is written

$$l = \int n\, dt + \epsilon \quad \text{.....(1)}$$

where n and ϵ are no longer constants; in effect, we can write $l = l_0 + n_0 t + P$, where l_0 and n_0 are constants, the latter incorporating the coefficient of the secular term arising from the element ϵ, and P represents the non-secular perturbations in longitude.

The values of l deduced from observation at three different epochs are then

$$l_1 = l_0 + n_0 t_1 + P_1 \quad , \quad l_2 = l_0 + n_0 t_2 + P_2 \quad , \quad l_3 = l_0 + n_0 t_3 + P_3 \quad .$$

From the first two and from the last two we have

$$n_0 = (l_2 - l_1 - P_2 + P_1)/(t_2 - t_1) \quad , \quad \text{.....(2)}$$
$$n_0 = (l_3 - l_2 - P_3 + P_2)/(t_3 - t_2) \quad . \quad \text{.....(3)}$$

Ideally, these two expressions should give the same value of n_0, as is found in the case of the planets.

When similar considerations are applied to the Moon's motion, the identity of (2) and (3) no longer holds, a fact demonstrated first by Halley in 1693. The accuracy of the evaluation of n_0 by (2) and (3) clearly depends on the lengths of the intervals $t_2 - t_1$ and $t_3 - t_2$ concerned, and if these are, for example, of the order of several hundreds of years, the perturbation terms P play an insignificant role in the determination of n_0. Halley, in his investigation of the Moon's mean motion, chose three epochs as far apart as possible; to obtain accurate values of l in pre-telescopic times, he had recourse to the records of ancient eclipses for, the circumstances of any eclipse being known, the Moon's longitude could be readily ascertained from the easily derivable coordinates of the Sun. One group of eclipses used by Halley referred to those recorded in Ptolemy's *Almagest*, and a second group to those observed by Arabian astronomers at the end of the ninth century A.D. The individual eclipses of each of these groups

could, by a simple process, be related to a single epoch. The third epoch was a contemporary one.

Halley found, in effect, that the two equations (2) and (3) gave different values of n_0, the second being the greater. The suggestion followed that the expression for the mean longitude of the Moon should include a term in t^2—the so-called acceleration term. For the Moon, then, we can write, on this assumption,

$$l = l_0 + n_0 t + \sigma \left(\frac{t}{100}\right)^2 + P \quad , \qquad \text{.....(4)}$$

where t is measured in Julian years of $365\frac{1}{4}$ days from some selected epoch. In this formula, σ is called the coefficient of the secular acceleration. The term under consideration can also be written as σT^2, where T is the number of Julian centuries from the adopted epoch.

After Halley's time the value of the coefficient σ was found from more elaborate studies of eclipses to be in the neighbourhood of $11''$.

It is to be noted that the discovery of the secular acceleration came, not from gravitational theory, but from observations. Theory was now confronted with the challenge to show how Newtonian gravitation could account for the phenomenon.

19·02. Laplace's theoretical investigation of the acceleration

In investigating the motions of Jupiter's satellites Laplace noticed that the secular term in the expression for the eccentricity of Jupiter's orbit produced a secular acceleration in the mean longitude of a satellite and, realising that the secular term in the Earth's orbital eccentricity arising from planetary perturbations would produce a similar effect in the case of the Moon, he calculated the value of σ in 19·01 (4) and found it to be in good agreement with the observed value. It was at once concluded that the phenomenon of the acceleration had been satisfactorily accounted for.

We now derive very simply the expression for σ as found by Laplace.

From 19·01 (1), $\qquad l = n + \dot{\epsilon} \quad . \qquad \text{.....(1)}$

Now the Lagrangian formula for $\dot{\epsilon}$ can be written as

$$\dot{\epsilon} = -\frac{2}{na}\frac{\partial R}{\partial a} + A\frac{\partial R}{\partial e} + B\frac{\partial R}{\partial i}$$

of which we need consider only the first term; the relevant part of the disturbing function is given by 7·06 (18)—we ignore periodic terms and terms involving e and γ; accordingly,

$$R = \tfrac{1}{4}m^2 n^2 a^2 (1 + \tfrac{3}{2}e_1^2)$$

whence $\qquad \dot{\epsilon} = -m^2 n (1 + \tfrac{3}{2}e_1^2) \qquad \text{.....(2)}$

where $\qquad m = n_1/n \quad , \qquad \text{.....(3)}$

n_1 being the Sun's mean angular motion. As we are disregarding periodic terms arising in (1) we can regard n as constant in (2).

Now, as has been remarked, e_1 is not constant; so far as secular terms are concerned, we can write $e_1 = e_0' - \alpha t$. For the epoch 1850·0 the value of e_0' is 0·016771 and the value of α is $4{\cdot}245 \cdot 10^{-7}$. Thus, α being an extremely small quantity we have, with sufficient accuracy,

$$e_1^2 = e_0'^2 - 2e_0'\alpha t \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Inserting this expression in (2) and integrating, we find that

$$\epsilon = \epsilon_0 + ct + \tfrac{3}{2}m^2 n e_0'\alpha t^2 \quad ,$$

where c is a constant which we need not specify in detail.

Thus, to the Moon's longitude given by 19·01 (1), we add the secular acceleration term $\sigma\left(\frac{t}{100}\right)^2$, where

$$\sigma = \tfrac{3}{2} \cdot 10^4 m^2 n e_0'\alpha \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Here, σ is given in absolute measure; its value in seconds of arc is given by multiplying the right-hand side of (5) by cosec 1″. Now

$$m = \tfrac{3}{400} \quad \text{and} \quad n = \frac{2\pi}{27{\cdot}322} \cdot 365\tfrac{1}{4} \quad ;$$

with the values of e_0' and α previously stated, it is found that $\sigma = 10''{\cdot}4$, a result apparently confirming the observational value of σ.

At first sight it would appear that the effect of σ in increasing the Moon's mean angular motion and consequently in diminishing the semi-major axis a would be to bring the Moon ever closer to the Earth. It is to be remembered, however, that the secular form for e_1—namely, $e_0' - \alpha t$—is not intrinsically secular as we have found in Chapter 13; in fact, the secular term is merely the short-range form of periodic terms with a principal period, in this case, of about 24,000 years. Accordingly, *so far as purely gravitational theory is concerned*, the mean angular motion of the Moon and, consequently, the semi-major axis of its orbit each fluctuate about a mean value in the long period mentioned; the appearance of the secular acceleration term thus arises from the limitations of the mathematical analysis.

Theoretically, similar arguments should apply to the mean motion of a planet, disturbed by other planets but the corresponding value of σ is too small to be detected. It is only in the peculiar circumstances of satellite theory—particularly in the case of the Moon—that the value of σ is large enough to be deduced from observation.

19·03. Adams's disagreement with the Laplacian value of σ

In deriving the value of σ by the method of the previous section we are of course finding the first approximation* only which involves the

* Laplace was evidently of the opinion that a higher approximation, in terms of m, would produce little effect on the result.

factor m^2, where m is regarded as a small quantity comparable with the lunar eccentricity for example. In proceeding to a higher approximation we have to replace the elements in the full expression for R, including the periodic terms, by the expressions for their first approximations ; the new form of R will involve non-periodic terms with higher powers of m, arising in part from the multiplication of terms with the same arguments. The next step is evidently a highly complicated one.

It was Adams who first pointed out that, despite the apparent reconciliation between the observed value of σ and its theoretical value as calculated by Laplace, the latter value was seriously in error. Proceeding to higher approximations, Adams* showed that σ can be expressed in the form

$$\sigma = am^2 + bm^4 + cm^5 + dm^6 + em^7 + \ldots$$

(there is no contribution of the form $a_1 m^3$), that this series converged very slowly, and further that the coefficients b, c, ... were all negative. With the contemporary values of the constants, he found that the values of the successive terms in the above expression for σ were $10''{\cdot}66$, $-2''{\cdot}34$, $-1''{\cdot}58$, $-0''{\cdot}71$, $-0''{\cdot}25$, and, estimating the contribution of the remaining terms of order m^8 and higher to be $-0''{\cdot}06$, he arrived at the theoretical value $5''{\cdot}72$ for σ, a result significantly different from the accepted value derived from observations. This value was at first challenged by de Pontécoulant, Plana and Hansen who obtained substantially the same result as Laplace and, of course, the same as that derived from observations. In the ensuing controversy Adams pointed out that the question at issue was a purely mathematical one to be decided on mathematical grounds alone, independently of the agreement or disagreement of theory with observation ; in particular, since e_1 is not constant, then this fact should be taken into account in integrating the complete equations of motion—a feature not recognised in the investigations of the three astronomers mentioned. If, then, the theoretical value of σ is unmistakeably different from that obtained from observations Adams concluded that the motion of the Moon is affected by some other cause of a non-gravitational nature. " This fact ", he added, " might put us on the traces of an important physical discovery." It was only after the first World War that Adams's faith was justified when G. I. Taylor† and H. Jeffreys‡ showed that the divergence between the observed value of σ and Adams's value (subsequently confirmed by a variety of methods) could be attributed to the slowing-up of the Earth's rotation by tidal friction in shallow seas, thus producing an apparent acceleration of about half that deduced from eclipse observations.

* Phil. Trans. R. S., **143**, 397 (1853) ; M.N.R.A.S., **20**, 225, 279 (1860) ; **40**, 411 (1880).

† M.N.R.A.S., **80**, 308 (1920). ‡ M.N.R.A.S., **80**, 309 (1920).

In the following sections we give the essential steps in Adams's original investigation* the details of which were not published till 1880 ; we limit the analysis to terms of order m^4.

19·04. Equations of motion (Adams)

We neglect the inclination of the Moon's orbit to the ecliptic. Let (r, θ) and (r_1, θ_1) be the polar coordinates of the Moon and the Sun respectively with respect to the usual axis of reference $O\Upsilon$ in the ecliptic, the Earth being at the origin O ; θ and θ_1 are then the true longitudes of the Moon and Sun. Omitting the parallax terms—that is, terms in in $(a_1/r_1)^4$ etc.—we find that the disturbing function is given accurately by

$$R = \frac{Gm_1 r^2}{2r_1^3}[3\cos^2(\theta - \theta_1) - 1] \quad .$$

The equations of motion in polar coordinates are

$$\ddot{r} - r\dot{\theta}^2 = \frac{\partial F}{\partial r} \quad , \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\frac{\partial F}{\partial \theta}$$

where $$F = \frac{\mu}{r} + R \quad .$$

Then $$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} + \frac{Gm_1 r}{2r_1^3}[3\cos 2(\theta - \theta_1) + 1] \quad , \quad \text{.....(1)}$$

$$\frac{d}{dt}(r^2\dot{\theta}) = -\frac{3Gm_1 r^2}{2r_1^3}\sin 2(\theta - \theta_1) \quad . \quad \text{.....(2)}$$

We write $$r = \frac{1}{u} \quad , \quad H = r^2\dot{\theta} \quad .$$

Then $$\dot{r} = -H\frac{du}{d\theta} \quad , \quad \ddot{r} = -H^2u^2\frac{d^2u}{d\theta^2} - Hu^2\frac{dH}{d\theta}\frac{du}{d\theta} \quad , \quad \dot{H} = Hu^2\frac{dH}{d\theta} \quad .$$

Also, $$Gm_1 = n_1^2 a_1^3 = m^2 n^2 a_1^3 \quad .$$

The equation (2) becomes

$$H\frac{dH}{d\theta} = -\frac{3m^2n^2}{2u^4}\left(\frac{a_1}{r_1}\right)^3 \sin 2(\theta - \theta_1) \quad .$$

The equation (1) becomes, with the help of this last result,

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{H^2} - \frac{m^2n^2}{2H^2u^3}\left(\frac{a_1}{r_1}\right)^3[3\cos 2(\theta - \theta_1) + 1] + \frac{3m^2n^2}{2H^2u^4}\frac{du}{d\theta}\left(\frac{a_1}{r_1}\right)^3 \sin 2(\theta - \theta_1) \quad .$$

Now, to order e_1^2,

$$\left(\frac{a_1}{r_1}\right)^3 = 1 + \tfrac{3}{2}e_1^2 + 3e_1\cos\phi_1 + \tfrac{9}{2}e_1^2\cos 2\phi_1$$

where ϕ_1 is the Sun's mean anomaly $(n_1 t + \epsilon_1 - \varpi_1)$.

* M.N.R.A.S., **40**, 472 (1880).

Similarly,

$$\left(\frac{a_1}{r_1}\right)^3 \cos 2(\theta-\theta_1) = (1-\tfrac{5}{2}e_1^2)\cos 2(\theta-l_1) + \tfrac{7}{2}e_1\cos(2\theta-2l_1-\phi_1) - \tfrac{1}{2}e_1\cos(2\theta-2l_1+\phi_1) + \tfrac{17}{2}e_1^2\cos(2\theta-2l_1-2\phi_1)$$

and

$$\left(\frac{a_1}{r_1}\right)^3 \sin 2(\theta-\theta_1) = (1-\tfrac{5}{2}e_1^2)\sin 2(\theta-l_1) + \tfrac{7}{2}e_1\sin(2\theta-2l_1-\phi_1) - \tfrac{1}{2}e_1\sin(2\theta-2l_1+\phi_1) + \tfrac{17}{2}e_1^2\sin(2\theta-2l_1-2\phi_1)$$

where $l_1 (\equiv n_1 t + \epsilon_1)$ is the Sun's mean longitude.

It is found that the terms involving $2\phi_1$ can be omitted without affecting the argument—such terms do not combine later with other terms to give non-periodic terms of the order contemplated.

For the sake of brevity, we write

$$\left.\begin{aligned} \theta_2 &= 2(\theta-l_1) \equiv 2\theta - 2n_1 t - 2\epsilon_1 \\ \theta_3 &= 2\theta - 2l_1 - \phi_1 \equiv 2\theta - 3n_1 t - 3\epsilon_1 + \varpi_1 \\ \theta_1 &= 2\theta - 2l_1 + \phi_1 \equiv 2\theta - n_1 t - \epsilon_1 - \varpi_1 \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots(3)$$

the suffix in θ being related to the numerical coefficient of $n_1 t$. The equations of motion then become

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{H^2} - \frac{m^2n^2}{2H^2u^3}(1+\tfrac{3}{2}e_1^2+3e_1\cos\phi_1) - \frac{3m^2n^2}{2H^2u^3}[(1-\tfrac{5}{2}e_1^2)\cos\theta_2 + \tfrac{7}{2}e_1\cos\theta_3 - \tfrac{1}{2}e_1\cos\theta_1] + \frac{3m^2n^2}{2H^2u^4}\frac{du}{d\theta}[(1-\tfrac{5}{2}e_1^2)\sin\theta_2 + \tfrac{7}{2}e_1\sin\theta_3 - \tfrac{1}{2}e_1\sin\theta_1] \quad ; \ldots(4)$$

$$\frac{d(H^2)}{d\theta} = -\frac{3m^2n^2}{u^4}[(1-\tfrac{5}{2}e_1^2)\sin\theta_2 + \tfrac{7}{2}e_1\sin\theta_3 - \tfrac{1}{2}e_1\sin\theta_1] \quad . \ldots\ldots(5)$$

The subsequent procedure will be, first, to solve these equations and to obtain certain general expressions when e_1 is regarded as a constant and second, to derive subsidiary equations—which will take account of the variability of e_1, a and n.

19·05. Solution when e_1 is constant

In this section we regard e_1, a and n as constants and we derive the non-periodic part of u in terms of e_1 as far as order $m^4 e_1^2$. In the sequel the terms in u and θ which involve the eccentricity of the lunar orbit are omitted, as such terms do not combine with others to give a non-periodic part involving e_1.

By substituting general expressions in the equations 19·04 (4) and (5) Adams showed that θ and u can be expressed, up to order m^2 in m—all that is required in the sequel—by the following formulae :

$$nt + \epsilon = \theta + 3me_1\sin\phi_1 - \tfrac{11}{8}m^2(1-\tfrac{5}{2}e_1^2)\sin\theta_2 - \tfrac{77}{16}m^2e_1\sin\theta_3 + \tfrac{11}{16}m^2e_1\sin\theta_1 \quad , \ldots\ldots\ldots\ldots\ldots(1)$$

$$u=\frac{1}{a}[1-\tfrac{3}{2}m^2e_1\cos\phi_1+m^2(1-\tfrac{5}{2}e_1^2)\cos\theta_2$$
$$+\tfrac{7}{2}m^2e_1\cos\theta_3-\tfrac{1}{2}m^2e_1\cos\theta_1] \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Differentiate (1) with respect to t; then, remembering that

$$\phi_1=n_1t+\epsilon_1-\varpi_1$$

and that, for example, $\theta_2=2\theta-2n_1t-2\epsilon_1$ so that

$$\frac{d}{dt}\sin\theta_2=2(\dot\theta-mn)\cos\theta_2 \quad ,$$

we have

$$n[1-3m^2e_1\cos\phi_1-\tfrac{11}{4}m^3(1-\tfrac{5}{2}e_1^2)\cos\theta_2-\tfrac{231}{16}m^3e_1\cos\theta_3+\tfrac{11}{16}m^3e_1\cos\theta_1]$$
$$=\dot\theta[1-\tfrac{11}{4}m^2(1-\tfrac{5}{2}e_1^2)\cos\theta_2-\tfrac{77}{8}m^2e_1\cos\theta_3+\tfrac{11}{8}m^2e_1\cos\theta_1]$$

from which, keeping only the essential terms,

$$\frac{n}{\dot\theta}=1+\tfrac{9}{2}m^4e_1^2+3m^2e_1\cos\phi_1-\tfrac{11}{4}m^2(1-\tfrac{5}{2}e_1^2)\cos\theta_2$$
$$-\tfrac{77}{8}m^2e_1\cos\theta_3+\tfrac{11}{8}m^2e_1\cos\theta_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Now, $H=r^2\dot\theta$, so that $H^2=u^{-4}\dot\theta^2$; from (2) and (3) we obtain to $O(m^4)$, after some algebraical manipulation, the following expression for H^2:

$$H^2=n^2a^4[1+\tfrac{171}{32}m^4+\tfrac{2421}{64}m^4e_1^2$$
$$+\tfrac{3}{2}m^2(1-\tfrac{5}{2}e_1^2)\cos\theta_2+\tfrac{21}{4}m^2e_1\cos\theta_3-\tfrac{3}{4}m^2e_1\cos\theta_1] \quad \ldots(4)$$

By a similar process, the expression for μ/H^2 is found to be given by

$$\frac{\mu}{H^2}=\frac{\mu}{n^2a^4}[1-\tfrac{135}{32}m^4-\tfrac{1881}{64}m^4e_1^2$$
$$-\tfrac{3}{2}m^2(1-\tfrac{5}{2}e_1^2)\cos\theta_2-\tfrac{21}{4}m^2e_1\cos\theta_3+\tfrac{3}{4}m^2e_1\cos\theta_1] \quad \ldots(5)$$

19·06. Calculation of μ/H^2 from the first equation of motion

We consider now the first equation of motion, 19·04 (4). By means of 19·05 (2) we can calculate $\frac{du}{d\theta}$ and $\frac{d^2u}{d\theta^2}$, and by means of 19·05 (4) we can calculate $\frac{1}{H^2u^3}$ and $\frac{1}{H^2u^4}\frac{du}{d\theta}$ to $O(m^2)$ which is all that is required for these last two terms. We then derive from 19·04 (4) an expression for μ/H^2 which must be identically equal to that given by 19·05 (5). From 19·05 (2),

$$\frac{du}{d\theta}=\frac{1}{\dot\theta}\frac{du}{dt}=\frac{1}{a}\left[\frac{3}{2\dot\theta}m^3e_1\sin\phi_1-2m^2(1-\tfrac{5}{2}e_1^2)\left(1-\frac{mn}{\dot\theta}\right)\sin\theta_2\right.$$
$$\left.-\tfrac{7}{2}m^2e_1\left(2-\frac{3mn}{\dot\theta}\right)\sin\theta_3+\tfrac{1}{2}m^2e_1\left(2-\frac{mn}{\dot\theta}\right)\sin\theta_1\right] \quad .$$

It is seen from 19·05 (3) that it is sufficient to write $\dot\theta=n$ in this equation; hence, to $O(m^2)$,

$$\frac{du}{d\theta}=\frac{1}{a}[-2m^2(1-\tfrac{5}{2}e_1^2)\sin\theta_2-7m^2e_1\sin\theta_3+m^2e_1\sin\theta_1] \quad \ldots.(1)$$

In the same way, it is found that

$$\frac{d^2u}{d\theta^2}=\frac{1}{a}[-4m^2(1-\tfrac{5}{2}e_1^2)\cos\theta_2-14m^2e_1\cos\theta_3+2m^2e_1\cos\theta_1] \quad , (2)$$

$$\frac{m^2n^2}{2H^3u^3}=\frac{m^2}{2a}[1+\tfrac{9}{2}m^2e_1\cos\phi_1-\tfrac{9}{2}m^2(1-\tfrac{5}{2}e_1^2)\cos\theta_2 - \tfrac{63}{4}m^2e_1\cos\theta_3+\tfrac{9}{4}m^2e_1\cos\theta_1] \quad ,.....(3)$$

and

$$\frac{m^2n^2}{H^2u^4}\frac{du}{d\theta}=\frac{m^2}{a}[-2m^2(1-\tfrac{5}{2}e_1^2)\sin\theta_2-7m^2e_1\sin\theta_3+m^2e_1\sin\theta_1] \quad ....(4)$$

Inserting these expressions, and also that for u given by 19·05 (2), in 19·04 (5) we obtain for the non-periodic part of μ/H^2 the expression given by

$$\frac{\mu}{H^2}=\frac{1}{a}[1+\tfrac{1}{2}m^2(1+\tfrac{3}{2}e_1^2)-\tfrac{15}{8}m^4(1-5e_1^2)-\tfrac{321}{16}m^4e_1^2] \quad . .....(5)$$

Equating this to 19·05 (5) we obtain, after some reduction and noting that μ/n^2a^3 is very close to unity,

$$\frac{\mu}{n^2a^3}=1+\tfrac{1}{2}m^2(1+\tfrac{3}{2}e_1^2)+\tfrac{75}{32}m^4+\tfrac{11907}{64}m^4e_1^2 \quad , ...(6)$$

which is correct to order m^4. It is to be remarked that, as a check on the working, the periodic terms—not given in (6)—in the two expressions for μ/H^2 are in agreement.

19·07. New form of first equation of motion (e_1 variable)

We now assume that $e_1=e_0'-\alpha t$ and that we can neglect α^2. We choose n and a (now regarded as variables) so that the non-periodic parts of u and H^2 have the same form as in section 19·05; thus, we shall find that n and a are related by the formula 19·06 (6). It is to be remarked that n is the mean motion at the time corresponding to the longitude θ. To satisfy the equations of motion when the variability of e_1 is taken into account, it is evident that periodic terms must be added to the expressions for u—in 19·05 (2)—and for H^2—in 19·05 (4). We write

$$u=\frac{1}{a}[1+\delta\nu+P_1] \quad .....(1)$$

where P_1 denotes the periodic terms in 19·05 (2), and

$$H^2=n^2a^4[1+2\delta\eta+P_0+P_2] \quad .....(2)$$

where P_0 and P_2 denote, respectively, the non-periodic and the periodic terms in 19·05 (4); in these $\delta\nu$ and $\delta\eta$ are periodic functions.

Then $\dfrac{du}{d\theta}$ will be given by the terms in 19·06 (1) together with the additional terms

$$-\frac{1}{a^2}[1+\delta\nu+P_1]\frac{da}{d\theta}+\frac{1}{a}\frac{d}{d\theta}(\delta\nu)+\frac{1}{a}\frac{dP_1}{de_1}\cdot\frac{de_1}{d\theta} \quad ,$$

or $$\frac{1}{\dot\theta}\left[-\frac{\dot a}{a^2}(1+\delta\nu+P_1)-\frac{\alpha}{a}\frac{dP_1}{de_1}\right]+\frac{1}{a}\frac{d}{d\theta}(\delta\nu) \quad .$$

It will be found later that $\dot a$ and $\delta\nu$ are of order $m^2\alpha$; we can thus omit $\delta\nu$ and P_1 in the coefficient of $\dot a$. As we require $\frac{du}{d\theta}$ to $O(m^2)$ only, it is sufficient to write $\dot\theta = n$. The additional terms of $\frac{du}{d\theta}$ are then, in full,

$$\frac{1}{a}\frac{d}{d\theta}(\delta\nu)+\frac{1}{na}\left[-\frac{\dot a}{a}+\tfrac{3}{2}m^2\alpha\cos\phi_1+m^2\alpha(5e_1\cos\theta_2-\tfrac{7}{2}\cos\theta_3+\tfrac{1}{2}\cos\theta_1)\right] \quad .(3)$$

Owing to its association with the small quantity α as factor and since we are neglecting α^2, e_1 ($\equiv e_0' - \alpha t$) is to be regarded in expression (3), and subsequently, as a constant.

Similarly, $\frac{d^2u}{d\theta^2}$ will contain the additional terms

$$\frac{1}{na}[-m^2\alpha(10e_1\sin\theta_2-7\sin\theta_3+\sin\theta_2)]$$
$$+\frac{1}{na}[-m^2\alpha(10e_1\sin\theta_2-7\sin\theta_3+\sin\theta_2)]+\frac{1}{a}\frac{d^2}{d\theta^2}(\delta\nu) \quad ,$$

the first line coming from 19·06 (1) and the remaining terms from (3) above. Accordingly, $\left(\frac{d^2u}{d\theta^2}+u\right)$ contains the additional terms

$$\frac{1}{a}\left[\frac{d^2}{d\theta^2}(\delta\nu)+\delta\nu\right]+\frac{m^2\alpha}{na}[-20e_1\sin\theta_2+14\sin\theta_3-2\sin\theta_2] \quad .$$

Also, μ/H^2 contains the additional term $\frac{\mu}{n^2a^4}(-2\delta\eta)$ or, with sufficient accuracy, $-\frac{2}{a}\delta\eta$. The other terms in the differential equation produce no additional terms to the order concerned. The first equation of motion is now replaced by

$$\frac{d^2}{d\theta^2}(\delta\nu)+\delta\nu+\frac{m^2\alpha}{n}(-20e_1\sin\theta_2+14\sin\theta_3-2\sin\theta_1)=-2\delta\eta \quad . \quad ...(4)$$

19·08. New form of second equation of motion

We now find $\frac{d}{d\theta}(H^2)$ by a similar process. We keep terms up to order $m^4e_1\alpha$ in the non-periodic part and terms up to order $m^2e_1\alpha$ in the periodic part. $\frac{d(H^2)}{d\theta^2}$ has then the following additional terms:

$$\frac{n^2a^4}{\dot\theta}\left[\frac{2\dot n}{n}+\frac{4\dot a}{a}-\tfrac{3}{2}\tfrac{4}{2}\tfrac{2}{2}\tfrac{1}{1}m^4e_1\alpha+m^2\alpha(\tfrac{15}{2}e_1\cos\theta_2-\tfrac{21}{4}\cos\theta_3+\tfrac{3}{4}\cos\theta_1)\right]$$
$$+2n^2a^4\frac{d}{d\theta}(\delta\eta) \quad .$$

Also, since $u^{-4}=a^4(1-4\delta\nu)$ so far as $\delta\nu$ is concerned, the right-hand side of the second differential equation 19·04 (5) contains the following additional terms :

$$12m^2n^2a^4\delta\nu(\sin\theta_2+\tfrac{7}{2}e_1\sin\theta_3-\tfrac{1}{2}e_1\sin\theta_1) \quad .$$

The new form of the second equation is now :

$$\frac{1}{\dot\theta}\left[\frac{2\dot n}{n}+\frac{4\dot a}{a}-\tfrac{3}{32}\tfrac{21}{2}m^4e_1\alpha+m^2\alpha(\tfrac{15}{2}e_1\cos\theta_2-\tfrac{21}{4}\cos\theta_3+\tfrac{3}{4}\cos\theta_1)\right]+2\frac{d}{d\theta}(\delta\eta)$$
$$=12m^2\delta\nu(\sin\theta_2+\tfrac{7}{2}e_1\sin\theta_3-\tfrac{1}{2}e_1\sin\theta_1) \quad \text{.....(1)}$$

We first consider the periodic terms of order $m^2\alpha$ and $m^2\alpha e_1$ in (1) ; it is sufficient then to write n for $\dot\theta$ and to neglect the right-hand side of (1) since $\delta\nu$ is of order $m^2\alpha$.

Then we have

$$2\frac{d}{d\theta}(\delta\eta)=-\frac{m^2\alpha}{n}(\tfrac{15}{2}e_1\cos\theta_2-\tfrac{21}{4}\cos\theta_3+\tfrac{3}{4}\cos\theta_1)$$

from which, on integrating,

$$2\delta\eta=-\frac{m^2\alpha}{n}(\tfrac{15}{4}e_1\sin\theta_2-\tfrac{21}{8}\sin\theta_3+\tfrac{3}{8}\sin\theta_1) \quad \text{.....(2)}$$

Insert this expression for $2\delta\eta$ in 19·07 (4), putting $\mu=n^2a^3$ in the periodic terms—these are required to $O(m^2)$ only ; then we find that this latter equation becomes

$$\frac{d^2}{d\theta^2}(\delta\nu)+\delta\nu=\frac{m^2\alpha}{n}(\tfrac{95}{4}e_1\sin\theta_2-\tfrac{133}{8}\sin\theta_3+\tfrac{19}{8}\sin\theta_1) \quad \text{.....(3)}$$

of which the solution is

$$\delta\nu=-\frac{m^2\alpha}{n}(\tfrac{95}{12}e_1\sin\theta_2-\tfrac{133}{24}\sin\theta_3+\tfrac{19}{24}\sin\theta_1) \quad \text{.....(4)}$$

Consider now the non-periodic terms in (1) after we substitute the expressions for $\delta\eta$ and $\delta\nu$ given by (2) and (4) and the expression for $\dot\theta$ given by 19·05 (3) ; after a little reduction we find that

$$\frac{\dot n}{n}+\frac{2\dot a}{a}=\tfrac{21}{32}\tfrac{03}{2}m^4e_1\alpha \quad \text{.....(5)}$$

19·09. The secular acceleration to order m⁴

When the expressions for $\delta\nu$ and $\delta\eta$ are substituted in the first differential equation, 19·04 (4), there are no non-periodic terms depending on α ; consequently, μ/n^2a^3 retains its original form given by 19·06 (6), so that

$$\log\left(\frac{\mu}{n^2a^3}\right)=\log[1+\tfrac{1}{2}m^2(1+\tfrac{3}{2}e_1^2)+\tfrac{75}{32}m^4+\tfrac{1197}{64}m^4e_1^2]$$
$$=\tfrac{1}{2}m^2(1+\tfrac{3}{2}e_1^2)+\tfrac{71}{32}m^4+\tfrac{1173}{64}m^4e_1^2 \quad . \quad \text{.....(1)}$$

Now $$mn = n_1 \quad ; \quad \text{hence} \quad \frac{\dot{m}}{m} = -\frac{\dot{n}}{n} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Differentiate (1) with respect to t and use (2); then

$$\frac{2\dot{n}}{n} + \frac{3\dot{a}}{a} = \tfrac{1}{2}m^2 e_1 \alpha (3 + \tfrac{1173}{16}m^2) + \frac{\dot{n}}{n} \cdot m^2 (1 + \tfrac{3}{2}e_1{}^2 + \tfrac{71}{8}m^2 + \tfrac{1173}{16}m^2 e_1)$$

from which

$$\frac{3\dot{a}}{a} + \frac{\dot{n}}{n}[2 - m^2(1 + \tfrac{3}{2}e_1{}^2 + \tfrac{71}{8}m^2 + \tfrac{1173}{16}m^2 e_1)] = \tfrac{3}{2}m^2 e_1 \alpha (1 + \tfrac{391}{16}m^2) \quad \ldots (3)$$

Combining (3) with 19·08 (5) we find

$$\frac{\dot{n}}{n} = 3m^2 e_1 \alpha - \tfrac{3771}{32}m^4 e_1 \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and

$$\frac{\dot{a}}{a} = -\tfrac{3}{2}m^2 e_1 \alpha + \tfrac{2937}{32}m^4 e_1 \alpha \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

We write (4) as

$$\dot{n} = n(3m^2 e_1 \alpha - \tfrac{3771}{32}m^4 e_1 \alpha)$$

where, on the right-hand side, we can regard n as a constant. Then the part of $\int n\,dt$ arising from the variability of e_1 in the expression for the mean longitude is Pt^2, where

$$P = \tfrac{3}{2}m^2 n e_1 \alpha - \tfrac{3771}{64}m^4 n e_1 \alpha \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The secular acceleration, up to order m^4, is then given by $\sigma = 10^4 P$. In (6), the ratio of the second term to the first is $1257m^2$ numerically, or 0·221 when the value of m is inserted. Thus, since the first term is 10″·4, the term of order m^4 contributes the value $-2''{\cdot}30$ to σ, agreeing with Adams's value of $-2''{\cdot}34$ in 19·03, when account is taken of the slightly different values of the constants used by Adams.

As stated in 19·03, the final value of σ, found by Adams, by including terms of higher order in m in his analysis, is 5″·72; the value of σ derived from eclipse observations is about 11″; the balance of about 5″·3 is to be ascribed to the effect of tidal friction in slowing up the Earth's rotation thereby increasing the fundamental unit of time and leading to an apparent acceleration of the Moon's mean motion by about 5″·3 per century.

It may be added that owing to the variability of e_1, the longitude of perihelion, ϖ, and the longitude of the node, Ω, contain secular acceleration terms; the values of these according to Delaunay are $-40''\left(\frac{t}{100}\right)^2$ and $+6''{\cdot}8\left(\frac{t}{100}\right)^2$ respectively.

Chapter 20

PRECESSION AND NUTATION

20·01. Introduction

We have seen in section 1·05 that, if a planet is regarded as a sphere, either of uniform density or so constructed that the density, at any point distant r from the centre O, is a function of r, then the potential at an external point is the same as if the planet were replaced by a point-mass at O; accordingly, the development of planetary theory is based on the mutual attractions of point-masses. The Earth, however, is not quite spherical and one result of the attractions of the Moon and Sun on the Earth—we can ignore the planets in this connection—is to change the direction of the Earth's rotational axis with respect to the background of the stars or, more particularly, to some fixed coordinate system of our choice. The phenomenon thus briefly indicated is that of precession and nutation, the former being associated with the secular rate of change of the rotational axis and the latter with the accompanying periodic characteristics.

Precession was discovered by Hipparchus more than 2,000 years ago as an observational phenomenon disclosed in the progressive increase in the longitudes of the stars at the rate of about 36″ per annum (the modern value is about 50″ per annum), there being no perceptible change in their latitudes. The interpretation followed that the ecliptic was a fixed plane and that the position of the equator was variable, the vernal equinox moving backwards along the ecliptic at the constant rate previously mentioned. Consequently, it appeared that the pole of the equator describes uniformly a small circle about the pole of the ecliptic in a period, as we now know, of about 26,000 years. The explanation of precession in terms of dynamical theory, was first given by Newton in the *Principia* and represents one of his most remarkable achievements. The radius of the small circle alluded to—or, what is the same thing, the angular distance between the poles of the ecliptic and the equator—is the obliquity of the ecliptic which is supposed in the crude theory to remain constant. The more refined theory, following the discovery of nutation by Bradley in 1748, shows that the ecliptic is not quite a fixed plane, that the obliquity is not constant and that the backward motion of the equinox is not uniform.

In the development of the dynamical arguments we assume that the Earth is a non-spherical *rigid* body for which we can specify a system of axes fixed in the Earth and with the origin at the Earth's centre. Our investigations refer then to the precise manner in which these axes alter

with respect to axes fixed in space ; we shall identify the latter with axes associated with the ecliptic defined for a particular epoch which we denote for the present by t_0. The subsequent analysis will be based on Lagrange's equations of motion (section 8·04) for which the force-function U, arising from the action of the Moon and the Sun, and the kinetic function T—both expressed in terms of suitable variables—will be required. In the earlier sections we shall consider, for simplicity, the Moon alone and derive the appropriate equations of motion. The Sun's contribution to the complete equations can be written down at once by analogy with the lunar contribution.

20·02. The force-function U

The Earth is, very approximately, an oblate spheroid in form but for general analytical purposes we regard it as an ellipsoid with two slightly unequal axes OX and OY (Fig. 27) associated with the equator ; the smallest of the three principal axes is the polar axis OZ. It is assumed further that as regards distribution of mass the Earth is symmetrical with respect to each of OX, OY and OZ ; accordingly, O is the centre of mass of the Earth. It is to be emphasised that OX, OY, OZ are axes *fixed* in the Earth ; we refer to them as the axes of figure.

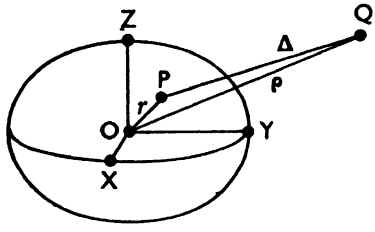


FIG. 27

Since O is the centre of mass then, if $d\mu$ is the element of mass at a point $P(x, y, z)$, we have

$$\int x\,d\mu = \int y\,d\mu = \int z\,d\mu = 0 \quad , \qquad \ldots\ldots(1)$$

where each integration is taken throughout the entire volume of the Earth. Further, with the assumptions made, it is evident that

$$\int xy\,d\mu = \int yz\,d\mu = \int zx\,d\mu = 0 \quad , \qquad \ldots\ldots(2)$$

and

$$\int x^3\,d\mu = \int x^2y\,d\mu = \int xy^2\,d\mu = 0 \quad , \qquad \ldots\ldots(3)$$

with results of a like nature involving y and z in addition.

The moments of inertia of the Earth about OX, OY, OZ are denoted by A, B, C respectively and defined by

$$A = \int (y^2+z^2)\,d\mu \quad , \quad B = \int (z^2+x^2)\,d\mu \quad , \quad C = \int (x^2+y^2)\,d\mu \quad . \quad \ldots(4)$$

It may be added that C is greater than A and B.

In Fig. 27, let Q be the Moon's centre ; it is sufficient to assume that the Moon is an isotropic sphere and replaced by a point-mass M at Q. The

potential at P arising from the Moon is then GM/Δ, where Δ denotes the distance PQ. Let dU denote the element of the force-function arising from $d\mu$ at P and M at Q; then $dU = GM\, d\mu/\Delta$. The force-function U for the Earth and the Moon is then given by

$$U = GM \int \frac{d\mu}{\Delta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Let (ξ, η, ζ) denote the coordinates of P with respect to O as origin and OQ as ξ-axis, the η and ζ axes being chosen in any way so long as the three axes form a rectangular system. Then

$$\Delta^2 = \rho^2 - 2\xi\rho + r^2$$

where $\rho = OQ$ and $r = OP$. Since ξ/ρ and r/ρ are small (at most of order 1/60 for the Moon—and very much smaller for the Sun), we write

$$\frac{1}{\Delta} = \frac{1}{\rho}\left(1 - \frac{2\xi\rho - r^2}{\rho^2}\right)^{-1/2}$$

and expand by the binomial theorem; accordingly, (5) becomes—to the fourth order in ξ/ρ (or r/ρ)—

$$U = \frac{GM}{\rho} \int d\mu \left(1 + \frac{\xi}{\rho} + \frac{3\xi^2 - r^2}{2\rho^2} + \frac{5\xi^3 - 3\xi r^2}{2\rho^3} + \frac{35\xi^4 - 30\xi^2 r^2 + 3r^4}{8\rho^4}\right)$$

which, for convenience, we write in the form

$$U = \frac{GM}{\rho}\left[X_0 + \frac{X_1}{\rho} + \frac{X_2}{2\rho^2} + \frac{X_3}{2\rho^3} + \frac{X_4}{8\rho^4}\right] .$$

Also, let (l, m, n) be the direction-cosines of OQ with respect to OX, OY, OZ; then

$$\xi = lx + my + nz \quad .$$

We note the following:

(i) $X_0 = \int d\mu = E$, where E is the Earth's mass.

(ii) $X_1 = \int \xi\, d\mu = \int (lx + my + nz)\, d\mu = 0$, by (1).

(iii) $X_2 = \int (3\xi^2 - r^2)\, d\mu = \int \{2r^2 - 3(\eta^2 + \zeta^2)\}\, d\mu$.

Now, $\int 2r^2\, d\mu = A + B + C$, by (4) .

Also, $\int (\eta^2 + \zeta^2)\, d\mu$ is the moment of inertia of the Earth about OQ; denote it by I. Then

$$X_2 = A + B + C - 3I \quad .$$

(iv) Since $l^2 + m^2 + n^2 = 1$, then

$$X_3 = \int [5(lx + my + nz)^3 - 3(lx + my + nz)(l^2 + m^2 + n^2)(x^2 + y^2 + z^2)]\, d\mu \quad .$$

Representative parts are of the forms

$$\int x^3\,d\mu \quad , \quad \int x^2y\,d\mu \quad , \quad \int xy^2\,d\mu$$

which, from (3), are all zero. Hence $X_3 = 0$.

(v) The integral X_4 is evidently of order Ea^4, where a denotes any one of the Earth's semi-axes; the corresponding part of U—say, U_4—is thus of order $\frac{GME}{\rho}\left(\frac{a}{\rho}\right)^4$. Similarly, the part U_2 of U corresponding to X_2 is of order $\frac{GME}{\rho}\left(\frac{a}{\rho}\right)^2$. Thus U_4/U_2 is of order $\left(\frac{a}{\rho}\right)^2$ which, in the case of the Moon is $(1/60)^2$ and still smaller in the case of the Sun. The effects produced by U_4 are so small that they can be neglected. Collecting all the previous results we have

$$U = GM\left[\frac{E}{\rho} + \frac{A+B+C-3I}{2\rho^3}\right] \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

which is correct to order $\left(\frac{a}{\rho}\right)^3$.

We now express U in terms of the coordinates of Q with reference to the terrestrial axes of figure. Now

$$I = \int (\eta^2 + \zeta^2)\,d\mu = \int (r^2 - \xi^2)\,d\mu$$
$$= \int [(l^2+m^2+n^2)(x^2+y^2+z^2) - (lx+my+nz)^2]\,d\mu$$

which, by means of (2) and (4), becomes

$$I = Al^2 + Bm^2 + Cn^2 = A - (A-B)m^2 + (C-A)n^2 \quad .$$

We now denote the coordinates of Q with respect to the axes OX, OY, OZ by (X, Y, Z). Then, $m = Y/\rho$ and $n = Z/\rho$; consequently,

$$U = GM\left[\frac{E}{\rho} + \frac{B+C-2A}{2\rho^3} + \frac{3(A-B)Y^2 - 3(C-A)Z^2}{2\rho^5}\right] \qquad \ldots\ldots\ldots (7)$$

In applying Lagrange's equations we shall be concerned with differential coefficients of the form $\frac{\partial U}{\partial q}$, where q denotes one of the variables defining the positions of the Earth's axes of figure in relation to fixed axes; since ρ is independent of these variables the first two terms of (7) do not contribute to $\frac{\partial U}{\partial q}$. Consequently, we can assume that for our purpose U is given by

$$U = \frac{3GM}{2\rho^5}\{(A-B)Y^2 - (C-A)Z^2\} \quad . \qquad \ldots\ldots\ldots\ldots\ldots (8)$$

20·03. Moving axes and the kinetic function

In Fig. 28, let OX, OY, OZ be the Earth's axes of figure. After an interval of time dt the directions of the axes will be slightly different and we assume that the new positions result from simultaneous angular velocities α, β, γ about the instantaneous positions of OX, OY, OZ respectively. Let (x, y, z) be the coordinates of *any* point P with respect to the moving axes at any time t. Also, let (u, v, w) be the components of the linear velocity of P with respect to fixed axes momentarily coincident with OX, OY, OZ at time t. Draw perpendiculars PA, PB, PC to the coordinate planes and complete the rectangular figure with parallel faces $ODCE$ and $FBPA$. Then

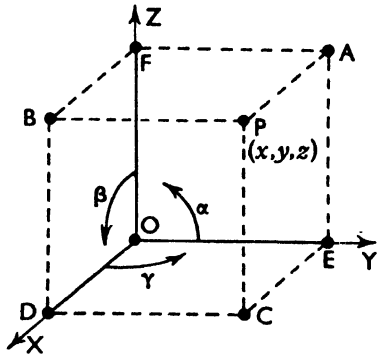


FIG. 28

u = velocity of P relative to O and parallel to the fixed axis instantaneously coincident with OX
= velocity of P relative to A and parallel to OX
+ velocity of A relative to F and parallel to OX
+ velocity of F relative to O and parallel to OX.

The first term is $\dot{x}$. The second term is $-\gamma \,.\, FA$ or $-\gamma y$, for FA is rotating about OZ with angular velocity γ in the sense $\overrightarrow{PA}$. Similarly, the third term is $+\beta z$. Hence we have

$$u = \dot{x} - \gamma y + \beta z \quad .$$

Similarly, $v = \dot{y} - \alpha z + \gamma x$, $w = \dot{z} - \beta x + \alpha y$.

If P is now specified as a fixed point within the rigid Earth, then

$$\dot{x} = \dot{y} = \dot{z} = 0 \quad ;$$

accordingly,

$$u = \beta z - \gamma y \quad , \quad v = \gamma x - \alpha z \quad , \quad w = \alpha y - \beta x \quad . \ldots\ldots\ldots\ldots (1)$$

Now, the kinetic function, T, in connection with a rotating body is defined in terms of *fixed* axes, and these we choose to be momentarily coincident with OX, OY, OZ ; hence, $d\mu$ being the element of mass at P, the kinetic function is given by

$$2T = \int (u^2 + v^2 + w^2)\, d\mu$$

or, by (1),

$$2T = \int [(\beta z - \gamma y)^2 + (\gamma x - \alpha z)^2 + (\alpha y - \beta x)^2]\, d\mu$$

$$= \int [\alpha^2 (y^2 + z^2) + \beta^2 (z^2 + x^2) + \gamma^2 (x^2 + y^2)]\, d\mu \quad ,$$

terms of the form $\int yz\, d\mu$ vanishing by 20·02 (2). Using 20·02 (4) we have

$$2T = A\alpha^2 + B\beta^2 + C\gamma^2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

20·04. The Eulerian angles

In Fig. 29, let OX_0, OY_0, OZ_0—with reference to a sphere centred at O—be fixed axes associated with our fixed plane of reference, the ecliptic at the epoch t_0; Z_0 is the pole of the ecliptic. We shall specify the position of X_0 more particularly later. OX, OY, OZ are the Earth's axes of figure, Z corresponding to the north pole. The great circle XY cuts the fixed great circle X_0Y_0 at N which is thus the pole of the great circle Z_0ZAB. The positions of X, Y and Z are defined with reference to the fixed axes by means of three angles θ, ϕ and ψ, known as the Eulerian angles.

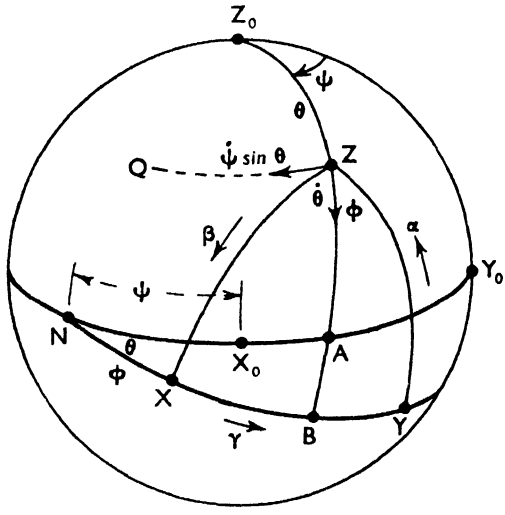


FIG. 29

θ is the angle between the two great circles XY and X_0Y_0 and is equal to Z_0Z, the angle between the poles of these great circles; without entering at present into precise definitions we observe that θ is evidently related to the obliquity of the ecliptic and N to the vernal equinox.

ϕ is the angle NX, measured from N in the direction $\overrightarrow{XY}$ and is equal to the angle BZY which is the angle between the great circles of which N and X are the poles.

ψ is the angle Y_0Z_0A or Y_0A measured from Y_0 in the direction $\overrightarrow{Y_0A}$; it is easily seen that ψ is the angle X_0N measured in the direction $\overrightarrow{X_0N}$.

We shall later require a particular relation involving the Eulerian angles. Let (l_3, m_3, n_3) be the direction cosines of OZ with respect to the fixed axes; then it is easily seen from the appropriate spherical triangles that

$$l_3 \equiv \cos ZX_0 = \sin\theta \sin\psi \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$m_3 \equiv \cos ZY_0 = \sin\theta \cos\psi \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$n_3 \equiv \cos ZZ_0 = \cos\theta \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In the formulae 20·02 (7) and (8) we used (X, Y, Z) as the coordinates of the Moon's centre—Q, in Fig. 27—with respect to the axes of figure;

it is convenient to write these now as (x, y, z). Let (ξ, η, ζ) denote the coordinates of the Moon's centre with respect to the fixed axes OX_0, OY_0, OZ_0. Then

$$z = l_3\xi + m_3\eta + n_3\zeta$$

or, from (1), (2) and (3),

$$z = \xi \sin\theta \sin\psi + \eta \sin\theta \cos\psi + \zeta \cos\theta \quad . \quad \ldots\ldots\ldots\ldots(4)$$

This is the relationship required ; it will be used in 20·12.

In terms of the new notation the force-function in 20·02 (8) is now expressed as

$$U = \frac{3GM}{2\rho^5}\{(A-B)y^2 - (C-A)z^2\} \quad \ldots\ldots\ldots\ldots(5)$$

20·05. The angular velocities α, β, γ in terms of the Eulerian angular velocities

In this section we derive the relations between α, β, γ and $\dot\theta$, $\dot\phi$, $\dot\psi$.

Consider first the motion of Z in Fig. 29. Referred to fixed axes instantaneously coincident with the moving axes OX, OY, OZ the angular motion of Z is (i) α along $\overrightarrow{YZ}$, and (ii) β along $\overrightarrow{ZX}$; hence, resolving along $\overrightarrow{ZA}$ and at right angles to Z_0Z in the direction ZQ, we find that these angular velocities combine to give

$$\beta \sin\phi - \alpha \cos\phi \quad , \quad \text{along } ZA \quad ,$$

and

$$\beta \cos\phi + \alpha \sin\phi \quad , \quad \text{along } ZQ \quad .$$

But, in terms of the Eulerian angles, the angular motions of Z, with respect to the fixed axes, are

$$\dot\theta \quad \text{along } ZA \quad \text{and} \quad \dot\psi \sin\theta \quad \text{along } ZQ \quad .$$

Hence

$$\dot\theta = \beta \sin\phi - \alpha \cos\phi \quad \ldots\ldots\ldots\ldots(1)$$

and

$$\dot\psi \sin\theta = \beta \cos\phi + \alpha \sin\phi \quad . \quad \ldots\ldots\ldots\ldots(2)$$

Consider now the motion of B in a similar way. Referred to fixed axes instantaneously coincident with OX, OY, OZ we see that B has an angular velocity γ along $\overrightarrow{BY}$. In terms of the Eulerian angles, B has an angular velocity $\dot\phi$ along $\overrightarrow{BY}$ and an angular velocity $\dot\psi \sin Z_0B$ or $\dot\psi \cos\theta$ along $\overrightarrow{BX}$. Hence

$$\dot\phi - \dot\psi \cos\theta = \gamma \quad . \quad \ldots\ldots\ldots\ldots(3)$$

It is convenient to collect these, expressing $\dot\phi$ in (3) in terms of α, β, γ by means of (2) ; we have

$$\dot\theta = \beta \sin\phi - \alpha \cos\phi \quad , \quad \ldots\ldots\ldots\ldots(4)$$

$$\dot\psi \sin\theta = \beta \cos\phi + \alpha \sin\phi \quad , \quad \ldots\ldots\ldots\ldots(5)$$

$$\dot\phi = \gamma + \cot\theta(\alpha \sin\phi + \beta \cos\phi) \quad . \quad \ldots\ldots\ldots\ldots(6)$$

We require also the formulae for α, β and γ in terms of θ, ϕ, and ψ; they are, from (4) and (5), and re-writing (3):

$$\alpha = \dot{\psi} \sin\theta \sin\phi - \dot{\theta}\cos\phi \quad , \quad \ldots\ldots(7)$$

$$\beta = \dot{\psi} \sin\theta \cos\phi + \dot{\theta}\sin\phi \quad , \quad \ldots\ldots(8)$$

$$\gamma = \dot{\phi} - \dot{\psi}\cos\theta \quad . \quad \ldots\ldots(9)$$

20·06. Euler's equations for a rotating rigid body

We apply Lagrange's equations (section 8·04), T being given in 20·03 (2) by

$$2T = A\alpha^2 + B\beta^2 + C\gamma^2 \quad . \quad \ldots\ldots(1)$$

(i) Consider first the Eulerian angle ϕ. We have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial U}{\partial \phi}$$

so that, from (1),

$$\Sigma A\dot{\alpha}\frac{\partial \alpha}{\partial \dot{\phi}} + \Sigma A\alpha\left\{\frac{d}{dt}\left(\frac{\partial \alpha}{\partial \dot{\phi}}\right) - \frac{\partial \alpha}{\partial \phi}\right\} = \frac{\partial U}{\partial \phi} \quad \ldots\ldots(2)$$

where the summations signify the inclusion of symmetrical expressions in B, β and C, γ.

From 20·05 (7), (8) and (9) we have

$$\frac{\partial \alpha}{\partial \dot{\phi}} = \frac{\partial \beta}{\partial \dot{\phi}} = 0 \quad , \quad \frac{\partial \gamma}{\partial \dot{\phi}} = 1 \quad ; \quad \frac{\partial \alpha}{\partial \phi} = \beta \quad , \quad \frac{\partial \beta}{\partial \phi} = -\alpha \quad , \quad \frac{\partial \gamma}{\partial \phi} = 0 \quad .$$

Hence

$$C\dot{\gamma} - (A - B)\alpha\beta = \frac{\partial U}{\partial \dot{\phi}} \quad . \quad \ldots\ldots(3)$$

(ii) Consider now the Eulerian angle θ. By the analogue of (2) we have

$$\Sigma A\dot{\alpha}\frac{\partial \alpha}{\partial \dot{\theta}} + \Sigma A\alpha\left\{\frac{d}{dt}\left(\frac{\partial \alpha}{\partial \dot{\theta}}\right) - \frac{\partial \alpha}{\partial \theta}\right\} = \frac{\partial U}{\partial \theta} \quad . \quad \ldots\ldots(4)$$

Now, by 20·05 (9),

$$\frac{\partial \gamma}{\partial \dot{\theta}} = 0 \quad .$$

Also,

$$\frac{d}{dt}\left(\frac{\partial \alpha}{\partial \dot{\theta}}\right) - \frac{\partial \alpha}{\partial \theta} = \sin\phi(\dot{\phi} - \dot{\psi}\cos\theta) = \gamma\sin\phi \quad .$$

Similarly, the coefficient of $B\beta$ in the second summation is $\gamma\cos\phi$.

Finally, $\frac{\partial \gamma}{\partial \theta} = \dot{\psi}\sin\theta = \alpha\sin\phi + \beta\cos\phi$, by 20·05 (5). Equation (4) then becomes

$$-A\dot{\alpha}\cos\phi + B\dot{\beta}\sin\phi + (A - C)\alpha\gamma\sin\phi + (B - C)\beta\gamma\cos\phi = \frac{\partial U}{\partial \theta} \quad . \ldots(5)$$

(iii) Consider the Eulerian angle ψ. By the analogue of (2) we have, since α, β and γ are not functions of ψ,

$$\Sigma A\dot{\alpha}\frac{\partial \alpha}{\partial \dot{\psi}} + \Sigma A\alpha\frac{d}{dt}\left(\frac{\partial \alpha}{\partial \dot{\psi}}\right) = \frac{\partial U}{\partial \psi} \quad . \quad \ldots\ldots(6)$$

In the second summation, the coefficient of $A\alpha$ is

$$\frac{d}{dt}(\sin\theta\sin\phi)=\cos\theta\sin\phi(\beta\sin\phi-\alpha\cos\phi)$$
$$+\cos\phi(\alpha\cos\theta\sin\phi+\beta\cos\theta\cos\phi+\gamma\sin\theta)$$
$$=\beta\cos\theta+\gamma\sin\theta\cos\phi.$$

Similarly, the coefficient of $B\beta$ is $-\alpha\cos\theta-\gamma\sin\theta\sin\phi$, and the coefficient of $C\gamma$ is $-\sin\theta(\beta\sin\phi-\alpha\cos\phi)$.

The first summation is

$$A\dot{\alpha}\sin\theta\sin\phi+B\dot{\beta}\sin\theta\cos\phi-C\dot{\gamma}\cos\theta \quad ,$$

the last term of which is, by (3), $-\cos\theta(A-B)\alpha\beta-\cos\theta\dfrac{\partial U}{\partial\phi}$. Making these substitutions in (6) we have, after dividing through by $\sin\theta$,

$$A\dot{\alpha}\sin\phi+B\dot{\beta}\cos\phi-(B-C)\beta\gamma\sin\phi+(A-C)\alpha\gamma\cos\phi$$
$$=\cot\theta\frac{\partial U}{\partial\phi}+\frac{1}{\sin\theta}\frac{\partial U}{\partial\psi} \quad ....(7)$$

Solving (5) and (7) for $\dot{\alpha}$ and $\dot{\beta}$ and rewriting (3) we obtain the group of equations

$$A\dot{\alpha}-(B-C)\beta\gamma=\frac{\sin\phi}{\sin\theta}\left(\cos\theta\frac{\partial U}{\partial\phi}+\frac{\partial U}{\partial\psi}\right)-\cos\phi\frac{\partial U}{\partial\theta} \quad , ...(8)$$

$$B\dot{\beta}-(C-A)\gamma\alpha=\frac{\cos\phi}{\sin\theta}\left(\cos\theta\frac{\partial U}{\partial\phi}+\frac{\partial U}{\partial\psi}\right)+\sin\phi\frac{\partial U}{\partial\theta} \quad , ...(9)$$

$$C\dot{\gamma}-(A-B)\alpha\beta=\frac{\partial U}{\partial\phi} \quad . .....(10)$$

20·07. The modified equations

As we have remarked previously, the Earth closely approximates to an oblate spheroid and for our final purposes it is sufficient to take the moments of inertia A and B to be equal. The expression for T then becomes

$$2T=A(\alpha^2+\beta^2)+C\gamma^2$$

or, by means of 20·05 (7), (8) and (9),

$$2T=A(\dot{\theta}^2+\dot{\psi}^2\sin^2\theta)+C(\dot{\phi}-\dot{\psi}\cos\theta)^2 \quad . .....(1)$$

Referring to the expression for U in 20·04 (5), U now becomes

$$-\frac{3GM(C-A)}{2\rho^5}z^2 \quad ;$$

further z is given by 20·04 (4) which does not involve ϕ. Consequently $\dfrac{\partial U}{\partial\phi}=0$ and, from 20·06 (10), we have immediately

$$\gamma\equiv\dot{\phi}-\dot{\psi}\cos\theta=\omega \quad .....(2)$$

where ω is a constant. This result also follows by applying the Lagrangian equation for ϕ to (1).

We can derive the remaining equations from 20·06 (8) and (9), or directly from (1). Adopting the latter method we have the Lagrangian equation in ψ :

$$\frac{d}{dt}[A\psi \sin^2\theta - C\cos\theta(\dot{\phi} - \psi\cos\theta)] = \frac{\partial U}{\partial \psi}$$

or, using (2),

$$A\ddot{\psi}\sin^2\theta + 2A\dot{\theta}\dot{\psi}\sin\theta\cos\theta + C\omega\dot{\theta}\sin\theta - C\cos\theta(\ddot{\phi} - \ddot{\psi}\cos\theta + \dot{\theta}\dot{\psi}\sin\theta) = \frac{\partial U}{\partial \psi} \quad .$$

It is found, as will be seen later, that $\ddot{\psi}$, $\dot{\theta}\dot{\psi}$, $\ddot{\phi}$ are all small compared with $\omega\dot{\theta}$. The previous equation then reduces to

$$\dot{\theta} = \frac{1}{C\omega\sin\theta}\frac{\partial U}{\partial \psi} \quad \text{.....(3)}$$

Again, taking the θ equation, we have

$$A\ddot{\theta} - A\dot{\psi}^2\sin\theta\cos\theta - C\omega\dot{\psi}\sin\theta = \frac{\partial U}{\partial \theta}$$

and, assuming that $\ddot{\theta}$ and $\dot{\psi}^2$ are small compared with $\omega\dot{\psi}$, the equation becomes

$$\dot{\psi} = -\frac{1}{C\omega\sin\theta}\frac{\partial U}{\partial \theta} \quad \text{.....(4)}$$

20·08. The complete force-function for Moon and Sun

The force-function for the Moon, when $A = B$, is

$$-\frac{3GM(C-A)z^2}{2\rho^5}$$

by 20·04 (5), where z is the z-coordinate of the Moon referred to the axes of figure OX, OY, OZ. Similarly, the force-function arising from the Sun is

$$-\frac{3GS(C-A){z_1}^2}{2{\rho_1}^5}$$

where S is the Sun's mass, z_1 its z-coordinate, and ρ_1 its geocentric distance. We then write as the complete force-function :

$$U = -\frac{3(C-A)}{2}\left[GM\frac{z^2}{\rho^5} + GS\frac{{z_1}^2}{{\rho_1}^5}\right] \quad \text{.....(1)}$$

If a_1 and n_1 denote the semi-major axis of the Earth's orbit and the corresponding mean angular motion then, with sufficient accuracy,

$$GS = {n_1}^2{a_1}^3 \quad .$$

We then have

$$U = -\frac{3(C-A)}{2}{n_1}^2{a_1}^3\left[\frac{M}{S}\cdot\frac{z^2}{\rho^5} + \frac{{z_1}^2}{{\rho_1}^5}\right] \quad ,$$

or, if a denotes the semi-major axis of the Moon's orbit,

$$U = -\tfrac{3}{2}(C-A)n_1^2\left[\frac{M}{S}\left(\frac{z}{\rho}\right)^2\left(\frac{a}{\rho}\right)^3\left(\frac{a_1}{a}\right)^3+\left(\frac{z_1}{\rho_1}\right)^2\left(\frac{a_1}{\rho_1}\right)^3\right] .$$

Define K and L as follows :

$$K = \frac{3}{2}\frac{C-A}{C}\frac{n_1^2}{\omega} , \quad \text{.....(2)}$$

$$L = \frac{M}{S}\left(\frac{a_1}{a}\right)^3 . \quad \text{.....(3)}$$

Then
$$-\frac{U}{C\omega} = K\left[L\left(\frac{a}{\rho}\right)^3\left(\frac{z}{\rho}\right)^2+\left(\frac{a_1}{\rho_1}\right)^3\left(\frac{z_1}{\rho_1}\right)^2\right] \quad \text{.....(4)}$$

This is the form to be used with equations (3) and (4) of the previous section.

20·09. Order of magnitude of U/T

In the formula 20·07 (1) for T, $\dot{\theta}$ and $\dot{\phi}$ are small compared with ω defined by 20·07 (2) ; thus, approximately,

$$2T = C\omega^2 \quad \text{.....(1)}$$

Neglecting the eccentricities of the lunar and terrestrial orbits, we have, from 20·08 (1),

$$U = -\tfrac{3}{2}(C-A)\left[GM\frac{\sin^2\delta}{a^3}+GS\frac{\sin^2\delta_1}{a_1^3}\right]$$

where δ and δ_1 are the declinations of the Moon and Sun. The maximum values of δ and δ_1 are $28\frac{1}{2}°$ and $23\frac{1}{2}°$ approximately and the maximum values of $\sin\delta$ and $\sin\delta_1$ are approximately $\frac{1}{2}$ and $\frac{2}{5}$.

Writing $G(E+M) = n^2a^3$ and $GS = n_1^2a_1^3$, we have

$$|U| < \tfrac{3}{2}(C-A)\left[\frac{1}{4}\frac{M}{E+M}n^2+n_1^2\right] .$$

Hence,
$$\frac{|U|}{T} < \frac{3(C-A)}{C}\left(\frac{n_1}{\omega}\right)^2\left[1+\frac{1}{4}\frac{M}{E}\left(\frac{n}{n_1}\right)^2\right] .$$

Now, as we shall see later,

$$\frac{C-A}{C} \doteqdot \frac{1}{304} , \quad \frac{M}{E} \doteqdot \frac{1}{81} ;$$

also $n_1 = \dfrac{2\pi}{366\frac{1}{4}}$ per sidereal day and we can take ω to be 2π per sidereal day so that $n_1/\omega = 1/366\frac{1}{4}$; again, $n/n_1 \doteqdot 13$, being the ratio of the year to the sidereal orbital period of the Moon. Inserting these numbers, we find that, in round figures,

$$\frac{|U|}{T} < 2 \cdot 10^{-7} .$$

20·10. The Eulerian free period

The final result for the previous section shows that U is extremely small in comparison with T. If we neglect U and $A - B$, the equations (8) and (9) of section 20·06 are, on writing ω for γ,

$$A\dot{\alpha} - (A - C)\omega\beta = 0 \quad , \quad A\dot{\beta} - (C - A)\omega\alpha = 0 \quad ,$$

or

$$\dot{\alpha} + p\beta = 0 \quad , \quad \dot{\beta} - p\alpha = 0$$

where

$$p = \omega(C - A)/A \quad .$$

The solutions are easily seen to be of the forms

$$\alpha = \sigma\omega \cos(pt + q) \quad , \quad \beta = \sigma\omega \sin(pt + q) \quad , \ldots\ldots\ldots\ldots\ldots(1)$$

where $\sigma\omega$ and q are the constants of integration.

Now

$$p = \frac{C - A}{C} \cdot \frac{C}{A}\omega \doteq \frac{1}{303}\omega$$

and the period T_1 associated with p is $\frac{2\pi}{p}$ and, since $\omega = 2\pi$ per sidereal day, the period T_1 is 303 sidereal days or approximately 302 mean solar days. This is the Eulerian free period of about 10 months.

20·11. The Eulerian free period and the variation of latitude

Referred to the axes of figure, the equation of the instantaneous axis is

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$$

or, from (1) of the previous section, since $\gamma \equiv \omega$,

$$\frac{x}{\sigma \cos(pt + q)} = \frac{y}{\sigma \sin(pt + q)} = \frac{z}{1} \quad . \ldots\ldots\ldots\ldots\ldots(1)$$

As will be shown later, σ is a very small quantity ; accordingly, the line (1) will cut the Earth's surface close to the north pole P at the point I which is the instantaneous pole (Fig. 30).

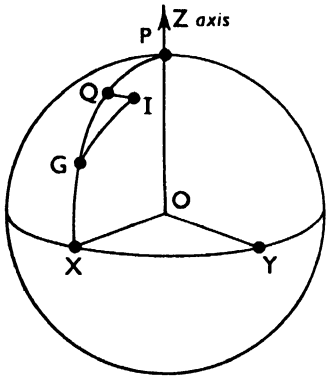


FIG. 30

Consider a place G. As we are now assuming the Earth to be a spheroid we can choose the X axis to be defined by the point of intersection of the equator and the meridian PG. Let IQ be the small circle through I parallel to the equator. Then we have, with sufficient accuracy, $GQ = GI$. Further, the z-coordinate of I differs little from OP ; hence, taking OP to be the unit of distance, we have from (1),

$$x \equiv PQ = \sigma \cos(pt + q)$$

or, expressed in seconds of arc,

$$PQ = a'' \cos(pt + q) \quad \ldots\ldots(2)$$

where $a = \sigma \operatorname{cosec} 1''$.

Since we are dealing with small arcs PQ and QI we can disregard for the present purpose the eccentricity of the meridian of G. Hence the instantaneous colatitude of G is

$$GI = GQ = PG - PQ \quad \ldots\ldots(3)$$

But PG is the colatitude of G with reference to the fixed pole P; hence, if ϕ_0 is the corresponding latitude and ϕ the instantaneous latitude, then by (2) and (3),

$$\phi = \phi_0 + a'' \cos(pt + q) \quad \ldots\ldots(4)$$

Now astronomical observations in which the latitude of the observer is involved refer to the instantaneous pole I. Accordingly, in those observations which are sufficiently accurate there should be evidence of fluctuations in latitude represented by the periodic term in (4), if a is at least of the order of one or two tenths of a second of arc. In 1888 Küstner detected small periodical changes in the latitude of Berlin, and since then the problem of the *variation of latitude*, as the phenomenon is called, has exercised the observational skill of many astronomers in different parts of the globe. The chain of observing stations in the northern hemisphere and the special techniques associated with the Cookson floating telescope at the Royal Greenwich Observatory and the new photographic zenith telescopes installed in several observatories are in themselves evidence of the importance attached to this problem in fundamental astronomy. It is now known that at any place the variation of latitude is given as the sum of two periodic terms so that

$$\phi = \phi_0 + a_1 \cos(p_1 t + q_1) + a_2 \cos(p_2 t + q_2) \quad \ldots\ldots(5)$$

in which the period of the first term is one year and the period of the second term is 432 days or about 14 months; also, the variation of latitude—that is, the departure from ϕ_0—does not at the most exceed $0''{\cdot}3$ numerically. The observations as represented by (5) evidently show that, inasmuch as the observed periods of variation are totally different from the Eulerian free period, the periodic term—if it exists—represented in (4) must have an amplitude so small as to be unrevealed by observations. Thus σ (in absolute measure) must be an extremely small quantity and, for practical purposes, the theoretical variation of latitude in terms of the Eulerian free period can be entirely ignored.

The *observed* variation of latitude is, of course, a different matter. The first term in (5)—that with a period of one year, the amplitude being $0''{\cdot}09$—is attributed to meteorological causes producing periodical changes in the principal moments of inertia of the Earth. Again, the dynamical

theory, it is to be recollected, is based on the assumption that the Earth is a perfectly rigid body, and the actual departure from rigidity, which must evidently be small, is believed to be reflected in the second periodic term of (5) of period 14 months; the amplitude of this term is $0''{\cdot}18$.

If we assume that the specialised observations, to which we have referred, give at any time the position of the instantaneous pole I, we can then suppose that all observations can be corrected for the variation of latitude so that we need only consider the motion of the axes of figure when we deal with the equations of motion involving the force-function U. Our solutions will then be the appropriate particular integrals of the equations of motion.

20·12. Expansion of U in terms of θ and ψ

To express U in terms of the Eulerian angles and the elements of the lunar and solar orbits it is seen from 20·08 (4) that we require to develop $(a/\rho)^3(z/\rho)^2$, first, when this function refers to the Moon and, second, when it refers to the Sun. We deal with the Moon in detail and deduce the corresponding result for the Sun.

Now, by 20·04 (4),

$$z = \xi \sin\theta \sin\psi + \eta \sin\theta \cos\psi + \zeta \cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where (ξ, η, ζ) are the coordinates of the Moon with respect to the fixed axes OX_0, OY_0, OZ_0. In this formula ξ, η and ζ are independent of θ and ψ.

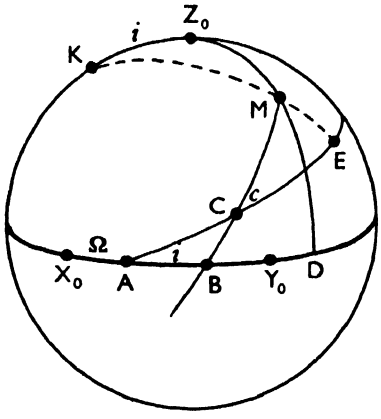


FIG. 31

In Fig. 31, X_0Y_0 is the fixed ecliptic at the epoch t_0; AE is the true ecliptic at time t, its pole being K. The variations in the ecliptic are the results of planetary perturbations on the Earth's orbit. The position of the true ecliptic is defined with reference to the fixed ecliptic by means of i, the inclination of the moving ecliptic to the fixed ecliptic, and Ω the longitude of the ascending node A.

M is the position of the Moon at time t and BCM its orbit; BCM makes an angle c with the ecliptic ACE; C is the node of the lunar orbit on the moving ecliptic. Draw the great circles Z_0MD and KME. Let $X_0D = L$ and $DM = B$; then

$$\xi = \rho \cos L \cos B \quad , \quad \eta = \rho \sin L \cos B \quad , \quad \zeta = \rho \sin B \quad .$$

Hence, from (1),

$$\frac{z}{\rho} = \cos B \sin\theta \sin(L+\psi) + \sin B \cos\theta \quad . \quad \ldots\ldots\ldots\ldots (2)$$

We transform the right-hand side of (2) in the following stages.

(i) Let $X_0A + AE = l$ and $EM = b$. Then, in the triangle KZ_0M,

$$KZ_0 = i\ ;\quad Z_0M = 90^\circ - B\ ;\quad KM = 90^\circ - b\ ;\quad KZ_0M = 90^\circ + L - \Omega\ ;$$
$$Z_0KM = 90^\circ - (l - \Omega)\ .$$

We then have :

$$\sin B = \sin b \cos i + \cos b \sin i \sin (l - \Omega)\ , \quad \ldots\ldots(3)$$
$$\cos B \sin (L - \Omega) = -\sin b \sin i + \cos b \cos i \sin (l - \Omega)\ , \quad \ldots\ldots(4)$$
$$\cos B \cos (L - \Omega) = \cos b \cos (l - \Omega)\ . \quad \ldots\ldots(5)$$

(ii) Let $X_0A + AC = N$, $X_0A + AC + CM = v$. Then, in the triangle MCE,

$$MC = v - N\ ;\quad ME = b\ ;\quad CE = l - N\ ;\quad MCE = c\ ;\quad MEC = 90^\circ\ .$$

We then have

$$\cos (v - N) = \cos b \cos (l - N)\ , \quad \ldots\ldots(6)$$
$$\sin (v - N) \cos c = \cos b \sin (l - N)\ , \quad \ldots\ldots(7)$$
$$\sin (v - N) \sin c = \sin b\ . \quad \ldots\ldots(8)$$

(iii) In (2), write $(L - \Omega) + (\psi + \Omega)$ for $L + \psi$; then

$$\frac{z}{\rho} = \sin\theta \cos (\Omega + \psi)[\cos B \sin (L - \Omega)]$$
$$+ \sin\theta \sin (\Omega + \psi)[\cos B \cos (L - \Omega)] + \cos\theta [\sin B] \quad \ldots\ldots(9)$$

We transform the expressions within squared brackets by means of (4) (5) and (3).

Now i is a very small angle and it is sufficient to write :

$$\sin i = i\ , \quad \cos i = 1\ .$$

(iv) From (4) we now have

$$\cos B \sin (L - \Omega) = -i \sin b + \cos b \sin (l - \Omega)\ .$$

Write $(l - \Omega) = (l - N) + (N - \Omega)$; then, by means of (8), (6) and (7), we obtain

$$\cos B = -i \sin c \sin (v - N)$$
$$+ \cos c \sin (v - N) \cos (N - \Omega) + \cos (v - N) \sin (N - \Omega)$$

The angle c is about $5^\circ{\cdot}2$ so that $\sin c$ is of order $1/11$. We omit terms with the factor $i \sin c$ and, writing $\sin c \equiv s$, we keep terms up to order s^3 ; accordingly, $\cos c = 1 - \frac{1}{2}s^2$. With these approximations we easily find that

$$\cos B \sin (L - \Omega) = (1 - \tfrac{1}{4}s^2) \sin (v - \Omega) - \tfrac{1}{4}s^2 \sin (v - 2N + \Omega)\ . \quad \ldots(10)$$

(v) Proceeding in the same way we find that

$$\cos B \cos (L - \Omega) = (1 - \tfrac{1}{4}s^2) \cos (v - \Omega) + \tfrac{1}{4}s^2 \cos (v - 2N + \Omega)\ . \quad \ldots(11)$$

(vi) Similarly,

$$\sin B = s \sin (v - N) + i \sin (v - \Omega)\ . \quad \ldots\ldots(12)$$

(vii) Inserting the expressions (10), (11) and (12) in (9) we obtain, after some reduction,

$$\frac{z}{\rho}=(1-\tfrac{1}{4}s^2)\sin\theta\sin(v+\psi)+s\cos\theta\sin(v-N)$$
$$+i\cos\theta\sin(v-\Omega)-\tfrac{1}{4}s^2\sin\theta\sin(v-2N+\psi) \quad \text{.....(13)}$$

(viii) We now form $(z/\rho)^2$ from (13). We omit here terms with the arguments $(2v-2N)$, $(2v-3N-\psi)$ and $(2v-N+\psi)$ whose coefficients are of order i or of s^2 at least; such terms of short period produce trifling results in the final integrations although in a rigorous solution they would be retained. It is then found that

$$\left(\frac{z}{\rho}\right)^2=\tfrac{1}{2}s^2+(\tfrac{1}{2}-\tfrac{3}{4}s^2)\sin^2\theta-(\tfrac{1}{2}-\tfrac{1}{4}s^2)\sin^2\theta\cos(2v+2\psi)$$
$$-\tfrac{1}{4}s^2\sin^2\theta\cos(2N+2\psi)+s(1-\tfrac{1}{2}s^2)\sin\theta\cos\theta\cos(N+\psi)$$
$$+i\sin\theta\cos\theta\cos(\Omega+\psi) \quad \text{.....(14)}$$

(ix) We now express $(a/\rho)^3$ in terms of the mean anomaly $M(=nt+\epsilon-\varpi)$. Then, to $O(e^2)$,

$$\left(\frac{a}{\rho}\right)^3=1+\tfrac{3}{2}e^2+3e\cos M+\tfrac{9}{2}e^2\cos 2M \quad \text{.....(15)}$$

Also v, which appears in (14), is given by

$$v=nt+\epsilon+2e\sin M+\tfrac{5}{4}e^2\sin 2M$$

(x) From (14) and (15) we have, keeping the principal terms and omitting the constant terms which do not contribute to $\dfrac{\partial U}{\partial\theta}$ and $\dfrac{\partial U}{\partial\phi}$,

$$\left(\frac{a}{\rho}\right)^3\left(\frac{z}{\rho}\right)^2=(\tfrac{1}{2}+\tfrac{3}{4}e^2-\tfrac{3}{4}s^2)\sin^2\theta+s(1-\tfrac{1}{2}s^2+\tfrac{3}{2}e^2)\sin\theta\cos\theta\cos(N+\psi)$$
$$-\tfrac{1}{4}s^2\sin^2\theta\cos(2N+2\psi)+i\sin\theta\cos\theta\cos(N+\psi)$$
$$+\tfrac{3}{2}e\sin^2\theta\cos M-\tfrac{1}{2}\sin^2\theta\cos(2M+2\varpi+2\psi) \quad .$$

We apply this formula to the Sun, putting $s=\sin c=0$ and writing M_1, e_1, ϖ_1, ϵ_1 in the appropriate terms.

The formula for U, expressed by 20·08 (4), is then finally given by

$$-\frac{U}{K\omega C}=[L(\tfrac{1}{2}+\tfrac{3}{4}e^2-\tfrac{3}{4}s^2)+\tfrac{1}{2}+\tfrac{3}{4}e_1^2]\sin^2\theta$$
$$+[Ls(1-\tfrac{1}{2}s^2+\tfrac{3}{4}e^2)]\sin\theta\cos\theta\cos(N+\psi)$$
$$-\tfrac{1}{4}Ls^2\sin^2\theta\cos(2N+2\psi)$$
$$-\tfrac{1}{2}[L\cos(2M+2\varpi+2\psi)+\cos(2M_1+2\varpi_1+2\psi)]\sin^2\theta$$
$$+i(L+1)\cos(\psi+\Omega)\sin\theta\cos\theta$$
$$+\tfrac{3}{2}(Le\cos M+e_1\cos M_1)\sin^2\theta \quad \text{.....(16)}$$

where K and L are defined by 20·08 (2) and (3).

The only inequality in the lunar elements that need be considered is

that relating to N which retrogrades on the ecliptic with a period of about 18·6 years. In the case of the Sun and the moving ecliptic we take into account the secular inequalities of e_1, i and Ω; we write $e_1 = e_0 + e't$; here e' is small and it is sufficient to write

$$e_1^2 = e_0^2 + 2e_0e't \quad .$$

Also, from planetary theory—5·10 (17) and (18)—we can write, ignoring periodic terms,

$$i \sin \Omega = gt + ht^2 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

$$i \cos \Omega = g_1t + h_1t^2 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

where g, g_1, h, h_1 are small quantities.

The term in (16) involving i has a factor

$$\begin{aligned} i \cos (\psi + \Omega) &\equiv \cos \psi (i \cos \Omega) - \sin \psi (i \sin \Omega) \\ &= (g_1 \cos \psi - g \sin \psi)t \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19) \end{aligned}$$

where we neglect the extremely small quantities h and h_1.

Substituting these expressions for e_1, e_1^2 and $i \cos (\psi + \Omega)$ in (16), we write

$$F = K[L(\tfrac{1}{2} + \tfrac{3}{4}e^2 - \tfrac{3}{4}s^2) + \tfrac{1}{2} + \tfrac{3}{4}e_0^2] \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

$$G = K(L + 1) \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

$$H = \tfrac{3}{2}Ke_0e' \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

$$\begin{aligned} V/K = {} & Ls(1 - \tfrac{1}{2}s^2 + \tfrac{3}{2}e^2) \sin \theta \cos \theta \cos (N + \psi) - \tfrac{1}{4}Ls^2 \sin^2 \theta \cos (2N + 2\psi) \\ & - \tfrac{1}{2}[L \cos (2M + 2\varpi + 2\psi) + \cos (2M_1 + 2\varpi_1 + 2\psi)] \sin^2 \theta \\ & + \tfrac{3}{2}[Le \cos M + e_1 \cos M_1] \sin^2 \theta \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23) \end{aligned}$$

so that U is given by

$$-\frac{U}{C\omega} = F \sin^2 \theta + [G(g_1 \cos \psi - g \sin \psi) \sin \theta \cos \theta + H \sin^2 \theta]t + V \quad \ldots(24)$$

20·13. Solution of the equations

The equations for precession and nutation are, from 20·07 (4) and (3),

$$\dot{\psi} = -\frac{1}{C\omega \sin \theta} \frac{\partial U}{\partial \theta} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\dot{\theta} = \frac{1}{C\omega \sin \theta} \frac{\partial U}{\partial \psi} \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

which, by means of 20·12 (24), become

$$\dot{\psi} = 2F \cos \theta + \left[G(g_1 \cos \psi - g \sin \psi) \frac{\cos 2\theta}{\sin \theta} + 2H \cos \theta\right]t + \frac{1}{\sin \theta} \frac{\partial V}{\partial \theta} \quad ,\ldots(3)$$

$$\dot{\theta} = G \cos \theta (g_1 \sin \psi + g \cos \psi)t - \frac{1}{\sin \theta} \frac{\partial V}{\partial \psi} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Now, the X_0 axis on the fixed ecliptic at the epoch t_0 has not so far been specified and we can choose it in any way we like. At the moment we choose it to be close to the node N so that ψ is a small angle; then,

in (3) and (4) it is sufficient to write $\cos\psi = 1$ and we neglect $g \sin\psi$ and $g_1 \sin\psi$.

We now write

$$\psi = \psi_m + \Psi \quad , \qquad \ldots\ldots(5)$$

$$\theta = \theta_m + \Theta \qquad \ldots\ldots(6)$$

where Ψ and Θ depend on the periodic terms in V.

It is sufficient to replace θ by a constant θ_0 on the right of (3) and (4) and to deal with ψ as just indicated. Omitting V, we get from (5) and (6),

$$\dot{\psi}_m = 2F\cos\theta_0 + \left(Gg_1 \frac{\cos 2\theta_0}{\sin\theta_0} + 2H\cos\theta_0\right)t \quad ,$$

$$\dot{\theta}_m = Gg\cos\theta_0 \,.\, t$$

in which t is measured from the epoch t_0. From these we have

$$\psi_m = \psi_0 + at + bt^2 \quad , \qquad \ldots\ldots(7)$$

$$\theta_m = \theta_0 + ct^2 \quad , \qquad \ldots\ldots(8)$$

where

$$a = 2F\cos\theta_0 \quad , \qquad \ldots\ldots(9)$$

$$b = \tfrac{1}{2}\left(Gg_1 \frac{\cos 2\theta_0}{\sin\theta_0} + 2H\cos\theta_0\right) \quad , \qquad \ldots\ldots(10)$$

$$c = \tfrac{1}{2}Gg\cos\theta_0 \quad . \qquad \ldots\ldots(11)$$

The equations (7) and (8) relate to precession.

We consider now the equations for Ψ and Θ; they are

$$\dot{\Psi} = \frac{1}{\sin\theta}\frac{\partial V}{\partial\theta} \quad , \quad \dot{\Theta} = -\frac{1}{\sin\theta}\frac{\partial V}{\partial\psi} \quad ,$$

where V is given by 20·12 (23). Replacing θ by θ_0 after differentiation, we find that

$$\begin{aligned}\dot{\Psi} = {} & KLs\left(1 - \tfrac{1}{2}s^2 + \tfrac{3}{2}e^2\right)\frac{\cos 2\theta_0}{\sin\theta_0}\cos(N+\psi) \\ & - K\cos\theta_0\left[\tfrac{1}{2}Ls^2\cos(2N+2\psi) + L\cos(2M+2\varpi+2\psi)\right. \\ & \left. + \cos(2M_1+2\varpi_1+2\psi) - 3(Le\cos M + e_1\cos M_1)\right] \qquad \ldots\ldots(12)\end{aligned}$$

and

$$\begin{aligned}\dot{\Theta} = {} & KLs\left(1 - \tfrac{1}{2}s^2 + \tfrac{3}{2}e^2\right)\cos\theta_0\sin(N+\psi) - K\sin\theta_0\left[\tfrac{1}{2}Ls^2\sin(2N+2\psi)\right. \\ & \left. + L\sin(2M+2\varpi+2\psi) + \sin(2M_1+2\varpi_1+2\psi)\right] \quad . \qquad \ldots\ldots(13)\end{aligned}$$

Now the lunar node retrogrades on the ecliptic in a period of 18·6 years approximately; hence we can write N in the form

$$N = N_0 - N't \quad . \qquad \ldots\ldots(14)$$

Also, $\psi = \psi_0 + at$, with sufficient accuracy. Hence

$$N + \psi = \beta - \alpha t \qquad \ldots\ldots(15)$$

where

$$\beta = N_0 + \psi_0 \quad \text{and} \quad \alpha = N' - a \quad . \qquad \ldots\ldots(16)$$

In (12) we write, for convenience,

$$\alpha D \equiv KLs(1 - \tfrac{1}{2}s^2 + \tfrac{3}{2}e^2)\frac{\cos 2\theta_0}{\sin \theta_0} \quad \text{.....(17)}$$

and in (13)

$$\alpha E \equiv KLs(1 - \tfrac{1}{2}s^2 + \tfrac{3}{2}e^2)\cos \theta_0 \quad \text{.....(18)}$$

Then, integrating (12) and (13) and neglecting a when it appears with n and n_1, we obtain—using (15)—

$$\begin{aligned}\Psi = &-D \sin (N + \psi) \\ &+ K \cos \theta_0 \left[\frac{Ls^2}{4\alpha} \sin (2N + 2\psi) - \frac{L}{2n} \sin (2M + 2\varpi + 2\psi)\right. \\ &\left. - \frac{1}{2n_1} \sin (2M_1 + 2\varpi_1 + 2\psi) + 3\left(\frac{Le}{n} \sin M + \frac{e_1}{n_1} \sin M_1\right)\right]\end{aligned} \quad \text{.....(19)}$$

$$\begin{aligned}\Theta = &\quad E \cos (N + \psi) \\ &- K \sin \theta_0 \left[\frac{Ls^2}{4\alpha} \cos (2N + 2\psi) - \frac{L}{2n} \cos (2M + 2\varpi + 2\psi)\right. \\ &\left. - \frac{1}{2n_1} \cos (2M_1 + 2\varpi_1 + 2\psi)\right]\end{aligned} \quad \text{.....(20)}$$

These expressions for Ψ and Θ give the principal terms for the nutation in longitude and in the obliquity respectively.

20·14. The mean equator at time t

In defining the mean equator at any time we ignore the nutational terms Ψ and Θ. Accordingly, the position of the mean equator is specified in relation to the fixed axes by ψ_m and θ_m, the latter being the mean obliquity. We have found in (7) of the previous section that

$$\psi_m = \psi_0 + at + bt^2$$

with reference to the fixed axis X_0 which we now designate specifically as the ascending node of the fixed ecliptic on the mean ecliptic at the epoch t_0; consequently, $\psi_m = 0$ when $t = 0$ and hence $\psi_0 = 0$. We now have

$$\psi_m = at + bt^2 \quad \text{.....(1)}$$

By (8) of section 20·13,

$$\theta_m = \theta_0 + ct^2 \quad ; \quad \text{.....(2)}$$

consequently, θ_0 is the obliquity of the fixed ecliptic with reference to the mean equation at the epoch t_0.

ψ_m is the *luni-solar precession* and θ_m the associated obliquity.

In (1) and (2) b and c are very small compared with a; their values are given in 20·19.

20·15. General precession

In Fig. 32, we show the ecliptic at t_0 and at t, and the mean equator at t. For simplicity, we drop temporarily the suffix m pertaining to ψ and θ

in the following analysis. We have: $X_0N=\psi$, $\widehat{AN}E=\theta$, $NAN_1=i$ and $X_0A=\Omega$.

Also, let $NN_1=\lambda$, $\widehat{AN}_1E=\theta'$ and $X_0D=\psi'$, where D is the foot of the perpendicular great circle from N_1 to NA.

The angle ψ' is the *general precession*.

From 20·12 (17) and (18),

$$i\sin\Omega=gt+ht^2 \quad , \qquad \text{.....(1)}$$

$$i\cos\Omega=g_1t+h_1t^2 \quad , \qquad \text{.....(2)}$$

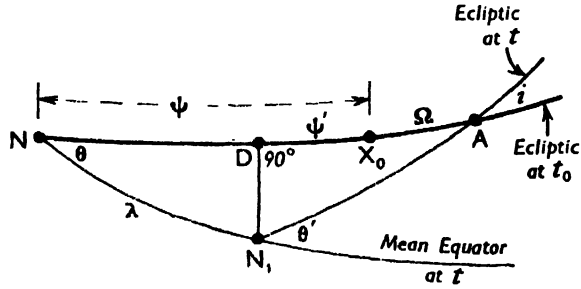


FIG. 32

in which we regard i, g and g_1 as small quantities of the first order and h, h_1 as of the second order.

(i) From the polar analogue of the cosine-formula we have

$$\cos\theta'=\cos\theta\cos i-\sin\theta\sin i\cos(\Omega+\psi) \quad .$$

Let $\theta'=\theta+\Delta\theta$; then, to the second order, we find that

$$\Delta\theta\sin\theta+\tfrac{1}{2}(\Delta\theta)^2\cos\theta=i\sin\theta\cos(\Omega+\psi)+\tfrac{1}{2}i^2\cos\theta \quad .$$

The first approximation is:

$$\Delta\theta=i\cos(\Omega+\psi) \quad . \qquad \text{.....(3)}$$

To the second order we then have

$$\Delta\theta=i\cos(\Omega+\psi)+\tfrac{1}{2}i^2\cot\theta\sin^2(\Omega+\psi) \quad . \qquad \text{.....(4)}$$

It is sufficient to write at for ψ in the first term of this equation, and θ_0 for θ and $\psi=0$ in the second term—we recollect that here $\psi\equiv\psi_m=at+bt^2$, from 20·14 (1). Formula (4) then becomes, by means of (1) and (2),

$$\Delta\theta=g_1t+(h_1-ag+\tfrac{1}{2}g^2\cot\theta_0)t^2 \quad . \qquad \text{.....(5)}$$

Also, $\theta\equiv\theta_m=\theta_0+ct^2$, by 20·14 (2). Hence we can write

$$\theta'=\theta_0+Qt+Q_1t^2 \qquad \text{.....(6)}$$

where $$Q=g_1 \quad , \quad Q_1=c+h_1-ag+\tfrac{1}{2}g^2\cot\theta_0 \quad . \qquad \text{.....(7)}$$

(ii) In the triangle NN_1D, $NN_1=\lambda$, $ND=\psi-\psi'$; hence

$$\tan(\psi-\psi')=\tan\lambda\cos\theta \quad .$$

Now $(\psi-\psi')$ and λ are small angles and, consequently, if we neglect $(\psi-\psi')^3$ and λ^3, we obtain simply

$$\psi-\psi'=\lambda\cos\theta\equiv\sin\lambda\cos\theta_0 \quad , \qquad \text{.....(8)}$$

in which it is sufficient to replace θ by θ_0—we shall see later that the constant c is an extremely small quantity.

From the triangle NAN_1, in which $NA=\Omega+\psi$, we have

$$\sin\lambda=\frac{\sin i\sin(\Omega+\psi)}{\sin\theta'} \quad . \qquad \text{.....(9)}$$

In this equation it is sufficient to expand $\sin\theta'$ to the first order; by (3),

$$\sin\theta'=\sin\theta[1+\cot\theta_0\,.\,i\cos(\Omega+\psi)] \quad . \qquad \text{.....(10)}$$

Hence, from (8), (9) and (10)—replacing θ by θ_0 in (10)—we obtain

$$\psi-\psi'=i\sin(\Omega+\psi)\cot\theta_0-i^2\sin(\Omega+\psi)\cos(\Omega+\psi)\cot^2\theta_0 \quad . \quad \text{...(11)}$$

To the second order the first term becomes, on writing $\psi=at$,

$$(gt+ht^2+ag_1t^2)\cot\theta_0 \quad .$$

The second term is of the second order, and is equal to $-gg_1t^2\cot^2\theta_0$ when we write $\psi=0$. Hence (11) becomes on replacing $\psi\equiv\psi_m$ by $at+bt^2$,

$$\psi'=Pt+P_1t^2 \quad , \qquad \text{.....(12)}$$

where

$$P=a-g\cot\theta_0 \quad , \qquad \text{.....(13)}$$

$$P_1=b-(h+ag_1)\cot\theta_0+gg_1\cot^2\theta_0 \quad . \qquad \text{.....(14)}$$

We now write ψ_m' for ψ' and θ_m' for θ' to indicate that ψ_m' and θ_m' refer to the mean equator in relation to the ecliptic at time t; then, from (12) and (6) we have

$$\psi_m'=Pt+P_1t^2 \quad , \qquad \text{.....(15)}$$

$$\theta_m'=\theta_0+Qt+Q_1t^2 \quad , \qquad \text{.....(16)}$$

where P, P_1 are given by (13) and (14), and Q, Q_1 by (7).

ψ_m' is the *general precession* and θ_m' the associated obliquity.

20·16. Planetary precession

In this section we derive an expression for λ in terms of the time. As before, we keep terms up to the second order of small quantities.

From 20·15 (8),

$$\lambda=(\psi-\psi')\sec\theta=(\psi-\psi')\sec\theta_0 \quad . \qquad \text{.....(1)}$$

The expansion for $(\psi-\psi')$ is given by 20·15 (11), of which the first and second terms have been separately evaluated; (1) then becomes

$$\lambda=(gt+ht^2+ag_1t^2)\operatorname{cosec}\theta_0-gg_1t^2\cot\theta_0\operatorname{cosec}\theta_0 \quad ,$$

or

$$\lambda=\lambda't+\lambda''t^2 \quad , \qquad \text{.....(2)}$$

where

$$\lambda'=g\operatorname{cosec}\theta_0 \quad , \qquad \text{.....(3)}$$

$$\lambda''=(h+ag_1-gg_1\cot\theta_0)\operatorname{cosec}\theta_0 \quad . \qquad \text{.....(4)}$$

λ is the *planetary precession*, on the equator.

The planetary perturbations on the Earth's orbit which are relevant in the present connection are defined by $i \sin \Omega$ and $i \cos \Omega$, that is, in terms of g, h and g_1, h_1 and t. The constants appearing in (3) and (4) are all calculable and, consequently, λ can be evaluated for any value of t.

20·17. Precession in R.A. and declination

In Fig. 33, X' and N_1 are the nodes of the fixed ecliptic and of the true ecliptic N_1AB at time t, respectively, on the mean equator at time t. Let OX', OY', OZ' be axes associated with the mean equator. S is a star and $Z'ST$ is the star's meridian. If we neglect nutation, N_1T is the right ascension, α, at time t, and TS is the declination, δ; (α, δ) are the mean equatorial coordinates at time t. Also, $X'N_1$ is the planetary precession λ; hence $X'T = \alpha + \lambda$.

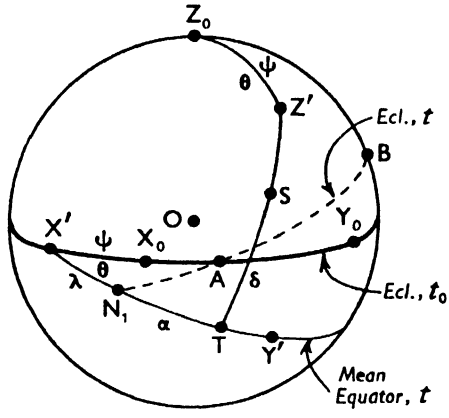


FIG. 33

Let (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) be the direction-cosines of OX', OY', OZ' with respect to OX_0, OY_0, OZ_0. Then, from Fig. 33,

$$l_1 = \cos\psi \ , \quad m_1 = -\sin\psi \ , \quad n_1 = 0 \ ;$$
$$l_2 = \sin\psi \cos\theta \ , \quad m_2 = \cos\psi \cos\theta \ , \quad n_2 = -\sin\theta \ ;$$
$$l_3 = \sin\psi \sin\theta \ , \quad m_3 = \cos\psi \sin\theta \ , \quad n_3 = \cos\theta \ .$$

We require the differential coefficients of l_1, l_2, etc., with respect to ψ and θ; they are given in the following table:

	l_1	m_1	n_1	l_2	m_2	n_2
ψ	m_1	$-l_1$	0	m_2	$-l_2$	0
θ	0	0	0	$-l_3$	$-m_3$	$-n_3$

Let (ξ, η, ζ) denote the coordinates of S with respect to the axes OX, OY', OZ', and (X_0, Y_0, Z_0) its coordinates with respect to the fixed axes OX_0, OY_0, OZ_0. Then, the radius of the sphere being unity,

$$\xi \equiv \cos(\alpha+\lambda)\cos\delta = l_1X_0 + m_1Y_0 + n_1Z_0 \quad \text{.....(1)}$$
$$\eta \equiv \sin(\alpha+\lambda)\cos\delta = l_2X_0 + m_2Y_0 + n_2Z_0 \quad \text{.....(2)}$$
$$\zeta \equiv \sin\delta = l_3X_0 + m_3Y_0 + n_3Z_0 \quad \text{.....(3)}$$

from which

$$X_0 = l_1\xi + l_2\eta + l_3\zeta \ , \quad \text{.....(4)}$$
$$Y_0 = m_1\xi + m_2\eta + m_3\zeta \ ; \quad \text{.....(5)}$$

these will be required later. Since X_0, Y_0, Z_0 are independent of θ, ψ we have from (1) and (2),

$$\dot{\xi} = -(\dot{\alpha}+\dot{\lambda})\eta - \dot{\delta}\xi \tan\delta \quad ,$$
$$\dot{\eta} = (\dot{\alpha}+\dot{\lambda})\xi - \dot{\delta}\eta \tan\delta \quad ,$$

from which

$$\xi\dot{\eta} - \dot{\xi}\eta = (\dot{\alpha}+\dot{\lambda})\cos^2\delta \quad , \qquad \ldots\ldots(6)$$
$$\xi\dot{\xi} + \eta\dot{\eta} = -\dot{\delta}\sin\delta\cos\delta \quad . \qquad \ldots\ldots(7)$$

Now, (X_0, Y_0, Z_0) are constants—we disregard proper motions here. Hence, from (1),

$$\dot{\xi} = \dot{l}_1 X_0 + \dot{m}_1 Y_0 + \dot{n}_1 Z_0 \quad .$$

But,
$$\dot{l}_1 = \frac{\partial l_1}{\partial \psi}\dot{\psi} + \frac{\partial l_1}{\partial \theta}\dot{\theta} \quad , \quad \text{etc.}$$

Then, using the entries in the preceding table and also (4) and (5), we obtain

$$\dot{\xi} = (m_1 X_0 - l_1 Y_0)\dot{\psi} = [(l_2 m_1 - l_1 m_2)\eta + (l_3 m_1 - l_1 m_3)\zeta]\dot{\psi} \quad ,$$

or
$$\dot{\xi} = (-n_3\eta + n_2\zeta)\dot{\psi} \quad .$$

Similarly,

$$\dot{\eta} = (m_2 X_0 - l_2 Y_0)\dot{\psi} - (l_3 X_0 + m_3 Y_0 + n_3 Z_0)\dot{\theta} = n_3\xi\dot{\psi} - \zeta\dot{\theta} \quad .$$

Using these in (6) and (7), we find that

$$\dot{\alpha} = \dot{\psi}\cos\theta - \dot{\lambda} + \dot{\psi}\sin\theta\sin(\alpha+\lambda)\tan\delta - \dot{\theta}\cos(\alpha+\lambda)\tan\delta \quad , \quad \ldots(8)$$
$$\dot{\delta} = \dot{\psi}\sin\theta\cos(\alpha+\lambda) + \dot{\theta}\sin(\alpha+\lambda) \quad . \qquad \ldots\ldots(9)$$

In these, it is sufficient to write : $\cos\lambda = 1$, $\sin\lambda = \lambda$. Then we have

$$\dot{\alpha} = \dot{\psi}\cos\theta - \dot{\lambda} + \sin\alpha\tan\delta[\dot{\psi}\sin\theta + \lambda\dot{\theta}] - \cos\alpha\tan\delta[\dot{\theta} - \lambda\dot{\psi}\sin\theta] \quad ,$$

and

$$\dot{\delta} = \cos\alpha[\dot{\psi}\sin\theta + \lambda\dot{\theta}] + \sin\alpha[\dot{\theta} - \lambda\dot{\psi}\sin\theta] \quad .$$

We write, replacing ψ and θ by ψ_m and θ_0 (since c is extremely small),

$$\left.\begin{aligned} m &= \dot{\psi}_m\cos\theta_0 - \dot{\lambda} \\ n &= \dot{\psi}_m\sin\theta_0 + \lambda\dot{\theta} \\ p &= \dot{\theta} - \lambda\dot{\psi}_m\sin\theta \end{aligned}\right\} \quad ; \qquad \ldots\ldots(10)$$

then

$$\dot{\alpha} = m + n\sin\alpha\tan\delta - p\cos\alpha\tan\delta \quad , \qquad \ldots\ldots(11)$$
$$\dot{\delta} = \qquad n\cos\alpha \quad + \quad p\sin\alpha \quad . \qquad \ldots\ldots(12)$$

It is found that p is of order $3 \cdot 10^{-7}t$ and that $\lambda\dot{\theta}$ is of order $6 \cdot 10^{-12}t^2$; these can be safely neglected in the present connection and (11) and (12) become

$$\dot{\alpha} = m + n\sin\alpha\tan\delta \quad , \qquad \ldots\ldots(13)$$
$$\dot{\delta} = n\cos\alpha \quad , \qquad \ldots\ldots(14)$$

where
$$m = \dot{\psi}_m \cos\theta_0 - \dot{\lambda} \quad , \qquad \ldots\ldots(15)$$
$$n = \dot{\psi}_m \sin\theta_0 \quad . \qquad \ldots\ldots(16)$$

When the unit of time is a year, $\dot{\alpha}$ and $\dot{\delta}$—as given by (13) and (14)—are referred to as the *annual precession* in right ascension and declination.

If μ_α and μ_δ are the components of annual proper motion, $\dot{\alpha}+\mu_\alpha$ and $\dot{\delta}+\mu_\delta$ are called the *annual variations* in right ascension and declination, the values of which are given for individual stars in the *Nautical Almanac* and in the various catalogues.

The *secular variation* in right ascension is defined to be the rate of change of $\dot{\alpha}$ per century ; if it is denoted by s, then

$$\frac{s}{100} = \ddot{\alpha} = \frac{d}{dt}(m + n \sin\alpha \tan\delta) \quad .$$

Hence, if α_0 is the right ascension of a star at time t_0, then t years later the right ascension, α, is given by

$$\alpha = \alpha_0 + t\left[(\dot{\alpha})_0 + \frac{st}{200}\right] \quad .$$

There is a similar formula for the declination. The values of the secular variations in both coordinates are given in many catalogues.

20·18. The nutation ellipse

The expressions for the nutation Ψ in longitude and for the nutation Θ in the obliqrity are given by 20·13 (19) and (20). The terms in $(N+\psi)$ are long-period terms and the remaining terms are of short period ; for example, the period of the term sin $(N+\psi)$ in Ψ is about $18\frac{1}{2}$ years and the remaining terms not involving N have periods of a lunar month or year or fractions of these. The principal terms we write as follows :

$$x = E\cos(N+\psi) \quad , \quad y = -D\sin(N+\psi) \quad , \ldots\ldots(1)$$

the former referring to the nutation in the obliquity.

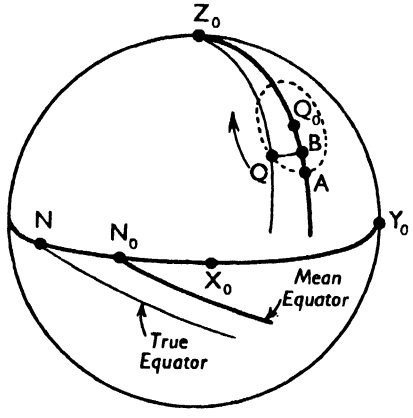


FIG. 34

In Fig. 34, N_0 is the node of the mean equator at time t with reference to the fixed ecliptic at t_0. The node of the *true* equator, which takes account of nutation, is N. The poles of the mean and true equators are Q_0 and Q respectively. Let QB be drawn perpendicular to Z_0Q_0. Then Q_0B is the nutation in the obliquity and the angle QZ_0Q_0 is the nutation N_0N in longitude. Hence $Q_0B = x$ and $QB = y \sin Z_0B$, or, with sufficient accuracy, $QB = y\sin\theta_0$.

Thus, we see from (1) that, relative to Q_0 , Q describes the small ellipse

$$\frac{x^2}{E^2}+\frac{y^2}{D^2\sin^2\theta_0}=1 \quad \text{.....(2)}$$

This is the *nutation ellipse*; since $\cos\theta_0>\cos 2\theta_0$, we see from the expressions for D, E in 20·13 (17) and (18) that E is the major axis; E is called the *constant of nutation*. The arrow shows the direction in which Q moves relatively to Q_0.

The constant of nutation, E, is an observable quantity and we shall assume it to be $9''\cdot210$.

From 20·13 (18),

$$E=\frac{KLs}{\alpha}(1-\tfrac{1}{2}s^2+\tfrac{3}{2}e^2)\cos\theta_0 \quad ; \quad \text{.....(3)}$$

since the values of α, s, e and θ_0 are known in addition to that of E, the equation (3) enables us to determine the product KL, where K and L are defined by 20·08 (2) and (3).

20·19. Numerical values of the constants

We have already remarked that the constant of nutation is an observable quantity; the value $9''\cdot210$ was adopted in 1896 by a conference of Superintendents of Nautical Almanacs and since confirmed, almost exactly, by several long and exhaustive observational programmes amongst which may be mentioned the Greenwich observations with the Cookson floating telescope in 1911–1931 and the observations, 45,000 in number, made at Poulkowa between 1904 and 1941.

The constant of general precession—P in 20·15 (15)—is known from observations with high accuracy. A similar remark applies to the mean obliquity θ_0 (corresponding to epoch t_0).

So far we have not defined specifically the epoch t_0. In earlier investigations the epoch was taken to be

1850 January 0, Greenwich Mean Noon .

Newcomb's *Tables of the Sun* are based on the epoch

1900 January 0, Greenwich Mean Noon .

It is to be remembered that previous to 1925, the astronomical day began at Greenwich Mean Noon.

The values of the constants given below are for the epoch

$t_0=$ 1950 January 0, Greenwich Mean Noon

and I am indebted to Dr. J. G. Porter of H.M. Nautical Almanac Office for their calculation.

Further, the time t used in our dynamical and other equations requires definite specification; we take the unit of time t to be the Julian year of

365·25 mean solar days—later we shall consider the definition of a mean solar day in greater detail.

(*a*) *Principal constants.*

$P = 50''\cdot2675$; $E = 9''\cdot210$; $\theta_0 = 23^\circ\ 26'\ 44''\cdot84$.

(*b*) *Constants associated with the Moon.*

(i) Eccentricity of orbit :

$$e = 0\cdot0549005 \quad .$$

(ii) Angle c in arc :

$$c = 5^\circ\ 8'\ 43''\cdot4 \quad , \quad \text{from which}$$
$$s \equiv \sin c = 0\cdot0896835 \quad .$$

(iii) Mean angular motion, n, in radians per Julian year :

$$n = 83\cdot9970927 \quad .$$

(iv) Relating to motion of node :

$$N' = 0\cdot3375711 \quad , \quad \alpha = 0\cdot3373280 \quad ,$$

both in radians per Julian year.

(*c*) *Constants associated with the Sun.*

(i) Eccentricity $e_1 \equiv e_0 + e't$:

$$e_0 = 0\cdot01673011 \quad , \quad e' = -4\cdot18\ .\ 10^{-7} \quad .$$

(ii) Mean angular motion, n_1, in radians per Julian year :

$$n_1 = 6\cdot2830759 \quad \text{(sidereal)}$$
$$= 6\cdot2833196 \quad \text{(tropical)} \quad .$$

(*d*) *Luni-solar Precession.*

$$\psi_m = at + bt^2 \quad ,$$
$$\theta_m = \theta_0 + ct^2 \quad ;$$
$$a = 50''\cdot3732 \quad , \quad b = -0''\cdot0001072 \quad ;$$
$$\theta_0 = 23^\circ\ 26'\ 44''\cdot84 \quad , \quad c = 5''\cdot608\ .\ 10^{-6} \quad .$$

(*e*) *General Precession.*

$$\psi_m' = Pt + P_1t^2 \quad , \quad \theta_m' = \theta_0 + Qt + Q_1t^2 \quad ;$$
$$P = 50''\cdot2675 \quad , \quad P_1 = 0''\cdot000111 \quad ;$$
$$Q = -0''\cdot4685 \quad , \quad Q_1 = -3''\cdot5\ .\ 10^{-7} \quad .$$

(*f*) *Planetary Precession.*

$$\lambda = \lambda't + \lambda''t^2 \quad ;$$
$$\lambda' = 0''\cdot1153 = 0^s\cdot007687 \quad ,$$
$$\lambda'' = -2''\cdot38\ .\ 10^{-4} = -1^s\cdot59\ .\ 10^{-5} \quad .$$

(*g*) *Annual precessions in R.A. and Dec.*

$$m = \psi_m \cos\theta_0 - \dot{\lambda} \equiv m_1 + 2m_2t \quad , \quad \text{.....(1)}$$
$$n = \psi_m \sin\theta_0 \quad \equiv n_1 + 2n_2t \quad ;$$

$$m_1 = 46''\cdot0990 \quad = 3^s\cdot07327 \ ,$$
$$m_2 = 1''\cdot4 \ . \ 10^{-4} \quad = 9^s\cdot3 \ . \ 10^{-6} \ ;$$
$$n_1 = 20''\cdot0426 \quad = 1^s\cdot33617 \ ,$$
$$n_2 = -4''\cdot2 \ . \ 10^{-5} = -2^s\cdot85 \ . \ 10^{-6} \ .$$

(*h*) *Planetary Perturbations* (*secular*).

$$i \sin \Omega = gt + ht^2 \ ; \ i \cos \Omega = g_1 t + h_1 t^2 \ .$$

(i) $a = P + g \cot \theta_0$.

With the values of a and P in (*d*) and (*e*), the value of g is easily seen to be given by $g = 0''\cdot04584$.

(ii) Similarly, from the relations for Q, Q_1 and P_1 in 20·15 (7) and (14), the values of h, g_1 and h_1 are deduced as follows :

$$h = 1''\cdot955 \ . \ 10^{-5} \ ; \ g_1 = -0''\cdot46849 \ ; \ h_1 = 5''\cdot33 \ . \ 10^{-7} \ .$$

20·20. Calculation of KL and K

From 20·18 (3), since $E = 9''\cdot210$,

$$KL = \frac{9''\cdot210\alpha}{s(1 - \frac{1}{2}s^2 + \frac{3}{2}e^2) \cos \theta_0} = 37''\cdot74 \ . \ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We have, from 20·13 (9), $a = 2F \cos \theta_0$

where $2F = KL(1 + \frac{3}{2}e^2 - \frac{3}{2}s^2) + K(1 + \frac{3}{2}e_0^2)$.

Inserting the numerical values of a, θ_0, etc., we find

$$54\cdot92 = KL(1 - 0\cdot007544) + K(1 + 0\cdot00042)$$

from which, by means of (1),

$$17\cdot46 = K(1 + 0\cdot00042) \ .$$

Hence, $K = 17''\cdot45$ . .....(2)

From (1) and (2), $L = 2\cdot163$ . .....(3)

From 20·13 (17), the value of D is found to be given by

$$D = 17''\cdot23 \ . \ \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The semi-minor axis of the aberration-ellipse ($D \sin \theta_0$) is found to be $6''\cdot861$.

20·21. The nutation terms

The expressions for the nutations Ψ and Θ in 20·13 (19) and (20) can now be calculated. The notation adopted in the *Nautical Almanac* is as follows :

$$N + \psi \equiv \Omega \ ; \ M \equiv g_1 \ ; \ M + \varpi \equiv \mathbb{C} \ ;$$
$$M_1 + \varpi_1 \equiv L \ ; \ \varpi_1 \equiv \pi \ ; \ M_1 = L - \pi \ .$$

We then have :

$$\Psi = -17''\cdot23 \sin \Omega + 0''\cdot21 \sin 2\Omega - 1''\cdot27 \sin 2L$$
$$- 0''\cdot21 \sin 2\mathbb{C} + 0''\cdot07 \sin g_1 + 0''\cdot13 \sin (L - \pi) \ , \ \ldots\ldots\ldots\ldots(1)$$

and $\Theta = 9''\cdot21 \cos \Omega - 0''\cdot09 \cos 2\Omega + 0''\cdot55 \cos 2L + 0''\cdot09 \cos 2\mathbb{C}$ ....(2)

These are the principal terms in the nutations. Terms neglected in our analysis have amplitudes less than $0''\cdot05$; such terms are of course taken account of in the precise calculation of Ψ and Θ.

20·22. The mass of the moon and the mechanical ellipticity of the earth

(i) From 20·08 (3), we have

$$L = \frac{M}{S}\left(\frac{a_1}{a}\right)^3 \equiv \frac{M}{E+M} \cdot \frac{E+M}{S} \cdot \left(\frac{a_1}{a}\right)^3 \quad .$$

But $$G(E+M) = n^2a^3 \quad , \quad G(S+E) = n_1{}^2a_1{}^3 \quad ;$$

hence, with sufficient accuracy, since $E/S \doteqdot 3 \,.\, 10^{-6}$,

$$L = \frac{M}{E+M} \cdot \frac{n^2}{n_1{}^2} \quad .$$

Inserting the values of n and n_1, and the value of L in 20·20 (3), we then obtain

$$\frac{M}{E} = \frac{1}{81\cdot6} \quad .$$

The ratio of the mass of the Moon to that of the Earth can also be found from the observations of Eros in connection with the determination of the solar parallax; the ratio $M : E$ appears in the equations of condition, since it is the centre of mass of the Earth-Moon system which describes the elliptic orbit around the Sun. The results derived by Hinks* from the 1900–1 observations and by Spencer Jones† from the 1930–1 observations are respectively 1/81·5 and 1/81·3.

(ii) From 20·08 (2) we have

$$K = \frac{3}{2}\frac{C-A}{C}\left(\frac{n_1}{\omega}\right) n_1 \quad .$$

Now, to a high degree of approximation, the period of ω is a sidereal day (v. 20·27); also, that of n_1 is the year or $366\frac{1}{4}$ sidereal days. Thus

$$\frac{n_1}{\omega} = \frac{1}{366\frac{1}{4}} \quad .$$

Also, $n_1 = 6\cdot2833$ radians per Julian day
$= 6\cdot2833$ cosec $1''$ seconds of arc per Julian day.

Inserting the value of K ($17''\cdot45$) from 20·20 (2), we find that

$$\frac{C-A}{C} = \frac{1}{304\cdot2} \quad .$$

The function, $(C-A)/C$, of the principal moments of inertia is referred to as the *mechanical ellipticity* of the Earth.

* M.N., **70**, 63, 1909. † M.N., **101**, 356, 1941.

20·23. Sidereal time

Since we are assuming that the Earth is a spheroid, the terrestrial equator is a circle ; accordingly, we can choose the X and Y equatorial axes of figure as we please. In Fig. 35, let the intersection of the meridian, ZGX, of Greenwich with the equator define the X-axis.

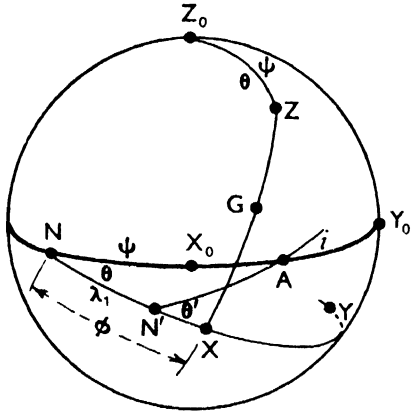


FIG. 35

In the figure, NXY is the *true* equator at time t, and N' is the node of the *true* ecliptic on NXY at time t. The *true* obliquity is θ ; $NX_0=\psi$; $NX=\phi$; $X_0A=\Omega$. Then

$$\theta=\theta_m+\Theta \quad , \quad \psi=\psi_m+\Psi \quad \text{.....(1)}$$

Let $NN'=\lambda_1$ and $\widehat{AN'X}=\theta'$.

The Greenwich sidereal time is XN', which we denote by τ. Then $\tau=\phi-\lambda_1$; hence, $\dot\tau=\dot\phi-\dot\lambda_1$, or, by 20·07 (2),

$$\dot\tau=\omega+\dot\psi\cos\theta-\dot\lambda_1 \quad \text{.....(2)}$$

We consider the expansions of the second and third terms on the right-hand side of (2) to the second order of the small quantities ψ, Ψ, Θ.

(i) From (1),

$$\begin{aligned}\dot\psi\cos\theta&=(\dot\psi_m+\dot\Psi)\{\cos\theta_m-\Theta\sin\theta_m\}\\&=(\dot\psi_m+\dot\Psi)\cos\theta_0-n\Theta-\Theta\dot\Psi\sin\theta_0\end{aligned}$$

in which it is sufficient to write θ_0 for θ_m ; also, by 20·17 (16), we have written n for $\dot\psi_m\sin\theta_0$. The principal terms of Ψ and Θ are given by 20·13 (19) and (20) :

$$\Psi=-D\sin(N+\psi) \quad , \quad \Theta=E\cos(N+\psi)$$

in which $D=17''\cdot23$, $E=9''\cdot21$. We then obtain

$$\dot\psi\cos\theta=(\dot\psi_m+\dot\Psi)\cos\theta_0-c_1\cos(N+\psi)-c_2-c_2\cos2(N+\psi) \quad ,$$

where $c_1=9''\,.\,10^{-4}$ and $c_2=5''\cdot3\,.\,10^{-5}$. We can ignore immediately the small periodic terms in $(N+\psi)$; then

$$\dot\psi\cos\theta=(\dot\psi_m+\dot\Psi)\cos\theta_0-c_2 \quad \text{.....(3)}$$

(ii) From the triangle $NN'A$ we have

$$\lambda_1=\frac{i\sin(\Omega+\psi)}{\sin\theta'} \quad .$$

Write $\theta'=\theta+\Delta\theta$; as in 20·15 (3), $\Delta\theta=i\cos(\Omega+\psi)$; hence

$$\lambda_1=\frac{i\sin(\Omega+\psi)}{\sin\theta'}-gg_1t^2\cot\theta_0\operatorname{cosec}\theta_0 \quad .$$

From 20·15 (8) and (11), it is easily seen that

$$\lambda = \frac{i \sin (\Omega + \psi_m)}{\sin \theta_0} - gg_1 t^2 \cot \theta_0 \operatorname{cosec} \theta_0 \quad ;$$

it is to be remembered that in 20·15 (11), $\psi \equiv \psi_m$.

By means of (1), it is readily found that

$$\lambda_1 - \lambda = -t \operatorname{cosec} \theta_0 \{g_1 D \sin (N + \psi) + gE \cos (N + \psi)\} \quad .$$

Here $g_1 D = 4''\,.\,10^{-5}$ and $gE = 2''\,.\,10^{-6}$. We ignore the terms on the right-hand side of the previous equation, and write $\lambda_1 = \lambda$.

(iii) *Formula for* τ.

From (2) and (3) we now have

$$\begin{aligned} \dot{\tau} &= \omega + \dot{\psi}_m \cos \theta_0 - \dot{\lambda} + \dot{\Psi} \cos \theta_0 - c_2 \\ &= \omega + m + \dot{\Psi} \cos \theta_0 - c_2 \quad . \end{aligned}$$

Write, as in 20·19 (1), $m = m_1 + 2m_2 t$; then

$$\tau = \tau_0 + (\omega + m_1 - c_2)t + m_2 t^2 + \Psi \cos \theta_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The term in c_2 is generally omitted since its effect is to alter the fourth decimal place of m_1 by only half a unit.

The final term in (4) is periodic ; considering only the principal term we write

$$\Psi \cos \theta_0 = -D \sin (N + \Psi) \cos \theta_0 = -1^s{\cdot}054 \sin (N + \psi) \quad .$$

The period of this term is $2\pi/\alpha$ or $18\frac{1}{2}$ years approximately ; thus the term varies very slowly from day to day. The value of the aggregate of the nutational terms in (4) can, of course, be calculated for any time t.

(iv) *Uniform sidereal time.*

We define uniform sidereal time, τ_1, by

$$\tau_1 = \tau_0 + (\omega + m_1)t \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The *true* sidereal time is then given by

$$\tau = \tau_1 + m_2 t^2 + \Psi \cos \theta_0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

A perfect sidereal clock may be assumed to keep uniform sidereal time, an ideal to which the modern quartz-clock conforms very closely. The variations between the time shown by such a clock and the true sidereal time over a period of a few months would be inappreciable so far as the term $m_2 t^2$ is concerned. On the other hand the variations arising from the nutational term are, over a similar interval of time, of the order of a few milli-seconds, and the detection of such variations comes within the capacity of the quartz-clock ; these variations, accordingly, can be allowed for when great precision is required.

20·24. Mean time

Mean time is based on the conception of a fictitious body—the *mean sun*—which is defined to move on the true equator with a constant sidereal angular velocity, which we denote by μ, and to be related to the Sun's mean longitude in a manner to be specified. If A denotes the right ascension of the mean sun, measured with respect to the *true* equinox, then, by 20·17 (13), the declination being zero, and 20·19 (1),

$$\dot{A} = \mu + m_1 + 2m_2 t + \dot{\Psi} \cos \theta_0 \quad .$$

Hence
$$A = A_0 + (\mu + m_1)t + m_2 t^2 + \Psi \cos \theta_0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let H denote the Greenwich hour angle of the mean sun ; then, τ being the Greenwich sidereal time, $H = \tau - A$. Now, τ is given by 20·23 (4) ; hence

$$H = \tau_0 - A_0 + (\omega - \mu)t$$

which shows that H increases uniformly with the time. Further, since t is measured from the Greenwich mean noon associated with the adopted epoch t_0, we have

$$\tau_0 = A_0 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Hence
$$H = (\omega - \mu)t \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

This last formula is the basis of the measurement of Greenwich Mean Time* which, since 1925, is defined to be $H + 12^h$.

Referred to the fixed ecliptic the Sun's mean longitude is of the form

$$l_0 + l_1 t + l_2 t^2 \quad ,$$

the final term arising from planetary perturbations, and referred to the moving mean equinox it is

$$l_0 + l_1 t + l_2 t^2 + Pt + P_1 t^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The mean longitude affected by aberration, denoted by L, is given by the expression (4) in which l_0 is replaced by $l_0 - \kappa$, where κ is the constant of aberration ; the value of κ used by Newcomb in his fundamental researches was 20″·501, a value which is still used in the *Nautical Almanac* for L. The expression for L is of the form given by

$$L = L_0 + L_1 t + L_2 t^2 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

We now complete the specification of the mean sun, remembering that we have introduced an angular motion μ the value of which has still to be determined. The mean sun is defined to be such that its right ascension, given by (1) but without the nutational term, is equal to L given by (5) *so far as this is possible*. Ideally, this definition implies an identity between

$$A_0 + (\mu + m_1)t + m_2 t^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

* Or Universal Time (U.T.).

and L, which we write in the form

$$L_0 + L_1 t + m_2 t^2 + (L_2 - m_2)t^2 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

which is only possible if $L_2 = m_2$ or, alternatively, if $(L_2 - m_2)$ is so small that it can be neglected.

According to Newcomb, for the epoch 1900 January 0, G.M. noon,

$$L_2 = 7^s{\cdot}26 \,.\, 10^{-6} \quad , \quad m_2 = 9^s{\cdot}28 \,.\, 10^{-6}$$

so that $$L_2 - m_2 = -2^s{\cdot}02 \,.\, 10^{-6} \quad .$$

The neglect of the term in t^2 in (7) thus produces an error of only $2^s{\cdot}02$ after 1000 years. Newcomb remarked in this connection that we must leave to the astronomers of the future the further necessary adjustments in the definition of mean time.

From (2) and the identity given by (6) and (7), we have

$$A_0 = L_0 = \tau_0 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$\mu + m_1 = L_1 \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

For our adopted epoch t_0, we have, in time-measure,

$$L_0 = 18^h\ 38^m\ 19^s{\cdot}85 \quad ,$$

$$L_1 = 86401^s{\cdot}84635 \quad , \quad L_2 = 7^s{\cdot}26 \,.\, 10^{-6} \quad ,$$

$$m_1 = 3^s{\cdot}07327 \quad , \quad m_2 = 9^s{\cdot}3 \,.\, 10^{-6} \quad .$$

From (9) we then have

$$\mu = 86398^s{\cdot}77308 \quad , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

which is the mean sun's sidereal angular motion in the true equator, the unit of time being the Julian year.

20·25. Evaluation of ω and $\omega + m_1$

From 20·24 (3), $H = (\omega - \mu)t$ or, if D is the number of mean solar days corresponding to t Julian years so that $t = \dfrac{D}{365\frac{1}{4}}$,

$$H = \frac{\omega - \mu}{N} D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which we write N for $365\frac{1}{4}$; a mean solar day, it should be added, is the interval corresponding to the increase of H by 24^h.

After one mean solar day, the hour angle is $H + 24^h$ so that

$$H + 24^h = \frac{\omega - \mu}{N}(D + 1) \quad .$$

Hence, subtracting (1) from this last result, we have

$$\frac{\omega - \mu}{N} = 24^h \equiv 86400^s \quad ,$$

from which $$\frac{\omega}{N} = 24^h + \frac{\mu}{N} = 24^h + 236^s{\cdot}54695 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Also, $$\frac{\omega + m_1}{N} = 24^h + \frac{L_1}{N} = 24^h + 236^s{\cdot}55536 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

20·26. Relation between Mean Solar Time (M.S.T.) and Sidereal time

Since $\tau_0 = L_0$, the formula for the sidereal time, τ, is

$$\tau = L_0 + (\omega + m_1)t + m_2 t^2 + \Psi \cos\theta_0$$

or, if t is expressed in terms of D mean solar days so that $t = \frac{D}{N}$, then

$$\tau = L_0 + \frac{(\omega + m_1)}{N} D + \frac{m_2}{N^2} D^2 + \Psi \cos\theta_0 \quad .$$

After 1 mean solar day, the sidereal time τ' is now given by

$$\tau' = L_0 + \frac{\omega + m_1}{N}(D+1) + \frac{m_2}{N^2}(D+1)^2 + \Psi \cos\theta_0$$

in which we ignore the change in Ψ in 1 mean solar day. Hence

$$\tau' - \tau = \frac{\omega + m_1}{N} + \frac{m_2}{N^2}(2D+1) \quad ;$$

accordingly

24^h M.S.T. $= (24^h + 236^s{\cdot}55536 + 5{\cdot}1 \,.\, 10^{-7}t + 6{\cdot}9 \,.\, 10^{-11})$ sidereal time.

Considering only uniform sidereal time we have the result:

$$24^h \text{ M.S.T.} = 24^h\ 3^m\ 56^s{\cdot}55536 \text{ sidereal time} \quad .$$

We have also the relation

$$\frac{1 \text{ mean solar day}}{1 \text{ sidereal day}} = \frac{24^h\ 3^m\ 56^s{\cdot}55536}{24^h} = 1{\cdot}002737909 \,.$$

20·27. Relation between the sidereal day and the rotation period of the earth

If S denotes the length of a sidereal day and R the rotation period of the Earth then, in absolute units,

$$S = \frac{2\pi}{\omega + m_1} \quad , \quad R = \frac{2\pi}{\omega} \quad .$$

Hence $$\frac{R}{S} = \frac{\omega + m_1}{\omega}$$

and, using the values of $\omega + m_1$ and ω given in 20·26 (3) and (2), we find

$$\frac{R}{S} = \frac{86636{\cdot}55536}{86636{\cdot}54695} = 1 + 9{\cdot}72 \,.\, 10^{-8} \quad .$$

20·28. The tropical year

The tropical year is defined with reference to the Sun's mean longitude L, as affected by aberration, by

$$L = L_0 + L_1 t + L_2 t^2 \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and is the interval ($\equiv T$ mean solar days) during which L has increased by 360° or, in time measure, 24^h. Hence

$$L + 24^h = L_0 + L_1\left(t + \frac{T}{N}\right) + L_2\left(t + \frac{T}{N}\right)^2$$

and, consequently, by means of (1),

$$24^h \equiv 86400^s = \frac{T}{N}\left\{L_1 + 2L_2 t + L_2 \frac{T}{N}\right\} \quad .$$

The final term is exceedingly small compared with L_1. Then,

$$T = \frac{86400N}{L_1 + 2L_2 t} \quad ;$$

on inserting the values of L_1, L_2 from 20·24, we find that

$$T = 365{\cdot}242\ 1957 - 6{\cdot}14\ .\ 10^{-8} t \quad .$$

This is the number of mean solar days in a tropical year. The secular term will have a negligible effect for many years to come ; accordingly, we can take the tropical year to be 365·2422 mean solar days.

The beginning of the *Besselian year*, whose length is the tropical year, is defined as the moment when $L = 280° = 18^h\ 40^m$. The corresponding date is in the neighbourhood of January 1 ; for example, the beginning of the Besselian year 1954 is

$$1954 \text{ January } 0^d{\cdot}892 \quad ,$$

which is written, for convenience, as 1954·0.

20·29. Ephemeris time

The theoretical discussion of precession and nutation has proceeded on the assumption that the Earth is a rigid body and that, in particular, the moments of inertia—A, B and C—remain constant ; in these circumstances the mean solar day is taken to be the standard unit of time, the Earth's rotational period being regarded as constant, apart from the effect of tidal friction discussed in Chapter 19.

Recent investigations have shown indubitably that there are small outstanding fluctuations between gravitational theory and observations in the longitudes of the Moon, the Sun and the planets Mercury and Venus —all of a like nature—which appear to indicate that the Earth is not a quite perfect clock. These fluctuations are now attributed to minute changes in the moments of inertia of the Earth, arising from meteorological, seismic and other causes. The mean solar day, accordingly, is not a

perfectly satisfactory unit of time. In its place the length of the sidereal year at 1900·0 has recently been adopted as the standard unit of time, and time reckoned in terms of this unit is designated *ephemeris time*.

The predictions* in the almanacs will be based in the immediate future on ephemeris time instead of G.M.T. or Universal Time as at present. The change from mean solar time to ephemeris time is effected by the following correction† Δt, given by

$$\Delta t = +24^{s}\cdot349 + 72^{s}\cdot3165T' + 29^{s}\cdot949T'^{2} + 1\cdot821B$$

where T' is in Julian centuries reckoned from 1900 January 0, Greenwich mean noon, and B is a quantity designated by H. Spencer Jones‡ as the *fluctuation* in the Moon's longitude.

* D. H. Sadler: M.N.R.A.S., **111**, 624 (1951).

† Transactions of the International Astronomical Union, VIII (in press).

‡ M.N.R.A.S., **99**, 542 (1939).

APPENDIX

Expansions of $J_n(ne)$ and $\frac{d}{de}\{J_n(ne)\} \equiv J_n'(ne)$, to $O(e^7)$

$$J_0(e) = 1 - \frac{1}{4}e^2 + \frac{1}{64}e^4 - \frac{1}{2304}e^6 \quad .$$

$$J_1(e) = \frac{1}{2}e\left[1 - \frac{1}{8}e^2 + \frac{1}{192}e^4 - \frac{1}{9216}e^6\right] \quad .$$

$$J_2(2e) = \frac{1}{2}e^2\left[1 - \frac{1}{3}e^2 + \frac{1}{24}e^4\right] \quad .$$

$$J_3(3e) = \frac{9}{16}e^3\left[1 - \frac{9}{16}e^2 + \frac{81}{640}e^4\right] \quad .$$

$$J_4(4e) = \frac{2}{3}e^4\left[1 - \frac{4}{5}e^2\right] \quad .$$

$$J_5(5e) = \frac{625}{768}e^5\left[1 - \frac{25}{24}e^2\right] \quad .$$

$$J_6(6e) = \frac{81}{80}e^6 \quad .$$

$$J_7(7e) = \frac{(343)^2}{92160}e^7 \quad .$$

$$J_0'(e) = -\frac{1}{2}e\left[1 - \frac{1}{8}e^2 + \frac{1}{192}e^4 - \frac{1}{9216}e^6\right] \quad .$$

$$J_1'(e) = \frac{1}{2}\left[1 - \frac{3}{8}e^2 + \frac{5}{192}e^4 - \frac{7}{9216}e^6\right] \quad .$$

$$J_2'(2e) = e\left[1 - \frac{2}{3}e^2 + \frac{1}{8}e^4 - \frac{1}{90}e^6\right] \quad .$$

$$J_3'(3e) = \frac{27}{16}e^2\left[1 - \frac{15}{16}e^2 + \frac{189}{640}e^4\right] \quad .$$

$$J_4'(4e) = \frac{8}{3}e^3\left[1 - \frac{6}{5}e^2 + \frac{8}{15}e^4\right] \quad .$$

$$J_5'(5e) = \frac{3125}{768}e^4\left[1 - \frac{35}{24}e^2\right] \quad .$$

$$J_6'(6e) = \frac{243}{40}e^5\left[1 - \frac{12}{7}e^2\right] \quad .$$

$$J_7'(7e) = \frac{7(343)^2}{92160}e^6 \quad .$$

$$J_8'(8e) = \frac{4096}{315}e^7 \quad .$$

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