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V

**NUMERICAL ANALYSIS AND
PARTIAL DIFFERENTIAL EQUATIONS**

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NUMERICAL ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS



CONTEMPORARY STATE OF NUMERICAL ANALYSIS

GEORGE E. FORSYTHE

Stanford University

LINEAR PARTIAL DIFFERENTIAL EQUATIONS

PAUL C. ROSENBLOOM

University of Minnesota

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PREFACE

These survey articles, as any of their kind, have the primary objective of accounting in a summary fashion for the state of the fields which they cover, as determined by contributing developments at various times and in many places. Considering the sustained growth of mathematics itself, as well as the intensive use of an ever-increasing number of its branches in diversified applied contexts, the availability of such surveys in selected areas is believed to fill a real need. The present articles are therefore aimed not so much at research specialists, actively contributing to the subjects discussed, as they are aimed at a broader, mathematically literate audience, looking for contemporary information on the important problems and results in these disciplines, whether it be for use in classroom and seminar, or for the sake of possible application to problems in other fields of science and engineering, or simply for reasons of personal interest.

The selection of the areas surveyed, as well as their coverage, was further guided by giving first consideration to developments of whose current state no comprehensive picture could be obtained by going only to the readily accessible literature in familiar languages. It is a unique distinction of the mathematical community to have never been taken in by the myth, now generally shattered, that Russian science merely followed the lead of the West—at a respectful distance. Mathematicians remained aware of the vigorous development during the postwar years of research in their field also in the Communist countries, and they knew of the steady stream of important results which they produced. An early start was made to overcome language and other communication difficulties. The American Mathematical Society was the first of this country's scientific societies to institute, with support from the Office of Naval Research, the systematic selection and translation of

significant articles which appeared in inaccessible journals or unfamiliar languages. A further step was taken two years ago, when the Editorial Office of *Applied Mechanics Reviews*, again with Office of Naval Research support, initiated work on these surveys. It had become apparent by then that there existed major areas of modern mathematics and theoretical mechanics of whose current scope and fabric the Western literature alone—quite apart from missing individual recent results—conveyed only an incomplete and therefore inadequate picture. This is to be remedied in the areas covered by the present Surveys in Applied Mathematics.

Clearly, the success of this venture required that the surveys should be written by authorities in the respective fields, fully conversant with current research and abreast of the international literature. Special thanks must therefore be extended to the distinguished authors of these articles for having given their thought and time to the purpose at hand. The enlistment of their co-operation is due in large measure to the indefatigable leadership of the Editor of *Applied Mechanics Reviews* and his associates, as well as to the effective way in which the Midwest Research Institute and the Southwest Research Institute (after *Applied Mechanics Reviews* had moved there) jointly arranged for the conduct of the editorial work.

F. JOACHIM WEYL, Director
Mathematical Sciences Division
Office of Naval Research

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CONTEMPORARY STATE
OF
NUMERICAL ANALYSIS

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CONTEMPORARY STATE OF NUMERICAL ANALYSIS *

1. What is numerical analysis?

Although it is a much used expression in contemporary mathematics, “numerical analysis” does not have a closely defined meaning, even among its practitioners. Since it clearly has to do with computing, let us first roughly describe the character of scientific computing. A block diagram is useful at this point; see Fig. 1.

Block I represents the scientists and engineers with problems that need a solution in a numerical form—often because an analytic solution is impossible. These are the consumers of numbers. Pure mathematicians occasionally belong to Block I (e.g., number theorists). Although we do not consider business problems in this article, they are not different in principle from scientific and technological problems.

The pressure of the number consumers in Block I has stimulated the development of a wide variety of tools (Block II) with which numbers can be produced by human beings. These include pencil and paper, mathematical tables, desk calculators, punched-card computing equipment, and the new automatically programmed electronic digital computers. (There is a somewhat parallel set of analogue devices which, since they work more by analogy than with numbers, will not be discussed further in this article.) The vast growth in digital computers makes it feasible to compute answers to problems which were unthinkable only five years ago. Hence Block II exerts a large return effect on Block I by bringing many new customers to the number factories.

* The preparation of this work was sponsored at the University of California, Los Angeles, in part by the Office of Naval Research. It was written in part while the author was at the A.E.C. Computing Facility in the Institute of Mathematical Sciences of New York University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

It is often far from obvious how best to use the tools of Block II to find the answers to problems, but over the years a large body of information has been accumulated. The mathematical processes used to solve problems have so far remained about the same during the development of digital computers, and are outlined in Art. 2. What is now called classical numerical analysis includes techniques for carrying out the

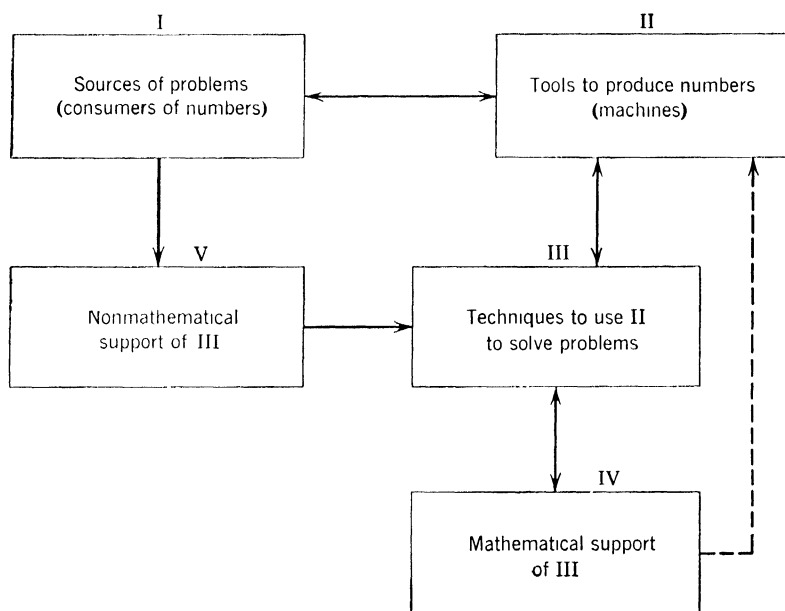


FIG. 1. Numerical analysis is the branch of applied mathematics which uses mathematical ideas to devise and evaluate numerical techniques for employing computers to solve problems, and to study their convergence and errors.

cited processes with pencil and paper or desk calculators. One typical technique is the table of finite differences, an arrangement designed to minimize errors by making them obvious, and to minimize copying time by shortening the numbers actually used. Other developments useful with desk calculators are the compact forms of Gaussian elimination for solving systems of four to ten simultaneous linear algebraic equations. Elimination is ideal for desk work, since it seems to minimize the number of arithmetical operations, and the compact arrangements are designed to minimize the amount of copying of numbers.

The automatic digital computers differ markedly from desk calculators, not only in the absolute times required for comparable operations but even more in the relative times required. For example, with most

machines in current use it takes precisely the same time to multiply 10-decimal numbers as 1-decimal numbers. Thus the principal advantage of finite-difference tables disappears with the new machines, and one does not ordinarily use them. The specific algorithms for carrying out the cited list of mathematical processes change a good deal as new machines are developed. They will probably change much more as we learn how best to use the rapidly developing new computers. And, finally, we expect that the basic mathematical processes themselves will undergo a considerable modification as we become freed from the effect of tradition. Already new processes have been developed to handle the coding needed just to make automatic digital computers operate. For example, the so-called compiler routines can automatically translate a group of nearly conventional algebraic formulas into a complete machine code.

In the United States the new electronic computers already dominate the computing field. As a result, in our country the most significant part of Block III comprises techniques concerned with such computers. In other parts of the world, a greater proportion of computation is being done by slower means; for example, the Soviet Union continues to publish a good many books of multiplication tables. But there is every sign that the dominance of automatic computers will spread widely in the world within a very few years.

The professional support of Block III comes primarily from mathematics proper, and this is represented by Block IV. All areas of mathematics are applied in devising and analyzing computing processes; see Art. 4. The most conspicuous areas are analysis (calculus of variations, differential-equation theory, approximation of functions, etc.), and algebra (mostly the linear algebra of matrices). But geometry (including topology), number theory, logic, probability, and statistics are all much used. In addition to assisting with Block III, mathematics can be quite useful in machine design and in the logic of coding. This connection has not been developed very far, and so is represented by a dotted line in Fig. 1.

The use of practically any computing technique itself raises a number of mathematical problems. There is thus a very considerable impact of computation on mathematics itself, and this may be expected to influence mathematical research to an increasing degree.

Mathematics has no monopoly on ideas for computing. Physicists, for example, ordinarily deal more with numbers than pure mathematicians. Thus physics, electronic engineering, and other disciplines often suggest computing methods; Block V represents this source of ideas. The evaluation and justification of such methods usually fall to mathe-

matics, however. Since the best single source of ideas for computing may be the physical problem itself, a line is shown coupling Block V with Block I.

With this framework it is possible to define numerical analysis. In a strict sense, *numerical analysis is Block III: the branch of applied mathematics which uses mathematical ideas to devise and evaluate numerical techniques for employing computers to solve problems, and to study their convergence and errors.* But, because so many different areas are intimately involved, numerical analysts have to know a good deal about every block in Fig. 1. Therefore, in a broad sense, numerical analysis really covers the whole of Fig. 1.

The practical problems of applied mathematics which numerical analysts are daily called upon to solve are often exceedingly difficult, and lie far beyond the reach of established methods and theorems. In dealing with these problems, the practicing numerical analyst, like others who apply mathematics, operates mostly by intuition and analogy, hoping that his methods and results will be correct enough to be useful. To the degree that he is successful with such difficult problems, he is practicing the *art* of numerical analysis. Fortunately for the computing artist, his customers often have their own independent methods of checking the numbers produced.

For certain standard problems there are guaranteed algorithms for obtaining answers and precise error bounds for confirming them. These constitute the *science* of numerical analysis. When the numerical analyst uses his science, the customer can rely entirely on the accuracy of the computation without independent checks. (But even then the customer may know that the computational model was a gross simplification of a real physical system!)

Now the complexity and difficulty of practical problems have grown colossally in the past 10 or 15 years, just as the machines to solve them have been completely revolutionized. Like any science, the science of numerical analysis is growing only very gradually. As a consequence, there are relatively few practical problems which the practicing numerical analyst really knows how to solve. And it will probably be many years before the chasm between art and science in the computing field can be narrowed to where it was in 1940.

To illustrate this, precise error bounds and stable algorithms are gradually being established for a linear parabolic partial differential equation with one or two space variables, constant coefficients, and simple fixed boundary conditions. But the practical computer is being called on to solve nonlinear heat equations with up to three space variables, complicated coefficients, movable boundaries, and difficult inter-

face conditions. Nevertheless, he is producing answers which seem to make sense, and is learning to guess their validity by tests of internal consistency and variation of parameters; see end of Art. 4.

On the other hand, there are computing areas in which progress is made only through precise theorems. Probably the best illustration of these is number theory, where it is hard to conceive of a gulf between the art and science of computing.

2. Areas of numerical analysis

It is reasonably possible to categorize the majority of the problems from applied mathematics which come to a computing center. The following is the author's outline of such categories. The three main areas are approximation, algebra, and functional equations. Of these, the last two are mainly classical and will be found in such outlines as those of Art. 3. Approximation, however, seems to be newly recognized as a field of numerical analysis, and as such is pretty much a child of automatic digital computers. The outline includes references to relevant articles below.

Approximation

a. Evaluate functions of one or more variables. Integrate, differentiate, and interpolate them. Sum a series. (See Art. 8.)

b. Approximate functions of one or more variables by simpler functions in the sense of least squares, least deviation, or other norms. (See Art. 9.)

c. Test and smooth empirical data; fit them with typical curves. Integrate and differentiate them. Study their spectra. Check for blunders and missing data.

Algebra

d. Solve a number of simultaneous equations.

(i) Linear. (See Art. 10.)

(ii) Nonlinear. Algebraic eigenvalue problems, or other problems. (See Art. 11.)

e. Solve systems of linear or nonlinear inequalities.

(i) Programming problems.

(ii) Problems from game theory.

(iii) Minimax fits of polynomials to functions. (See also (b) above.)

f. Combinatorial problems. (Dealing with functions of permutations.)

Functional equations

- g. Systems of one or more ordinary differential equations.
- h. Systems of one or more partial differential equations. (See Art. 12.)
- i. Other functional equations, like conformal mapping.
- j. Simulate physical systems too complex to be described by known equations (e.g., traffic flow).

Articles 8 to 12 touch on five of the entries in this outline—(a), (b), (i) and (ii) of (d), and (h)—chosen by the author's knowledge and interests. Within the five final articles, the topics mentioned and the treatment are extremely selective, for the space of one article permits no thorough outline of numerical analysis. Soviet material has been given special attention, because it is presumed that readers are less aware of it than of Western work. Also, the author has unabashedly cited his own work far more often than is warranted by its value to numerical analysis!

3. An older Soviet view of numerical analysis

In 1948 there appeared the large and comprehensive "Thirty Years" survey volume [38]. This volume was initiated by the Moscow Mathematical Society as part of a series of works commemorating the progress of Soviet science. The book contains apparently complete bibliographies of Soviet mathematicians for 1917–1947 in both Soviet and foreign journals, though seemingly excluding all mention of mathematicians (like Besicovitch, Tamarkin, and Shohat) who later emigrated. The book is divided into twelve parts: foundations of mathematics and mathematical logic; theory of numbers; algebra; topology; theory of sets and theory of functions; differential equations; functional analysis; theory of probability and mathematical statistics; numerical and graphical methods; geometry; history of mathematics; works of Latvian and Estonian mathematicians. Each section consists of one to four survey articles and a bibliography. There is a smaller companion volume on mechanics, including elasticity, plasticity, and aeromechanics [77].

The material on numerical analysis in [38] is mostly in the section on numerical and graphical methods. This is of average length, and contains three survey articles: *

- a. L. V. Kantorovič and V. I. Krylov: *Approximate methods*.
- b. K. A. Semendyaev: *Auxiliary aids to computation*.
- c. S. V. Bahvalov: *Nomography*.

*The system of Russian transliteration used is that of *Math. Reviews*, **15**, 1141 (1954–1955).

In addition, the modern numerical analyst is interested in one survey in the chapter on set and function theory:

d. S. M. Nikol'skiĭ: Polynomial approximation of functions of a real variable.

The article by Kantorovič and Krylov begins with a tribute to the older mathematicians who were interested in numerical analysis, starting with Newton, Euler, Lobačevskiĭ, and Gauss. P. L. Čebyšev (1821-1894) contributed a number of integration and interpolation techniques, and initiated the theory of the "best" approximation of a function $f(x)$ by a polynomial $P_n(x)$ in the sense that $\max_x |f(x) - P_n(x)|$ is to be minimized. This research was continued by S. N. Bernšteĭn (1880-), A. A. Markov (1856-1922), V. A. Steklov (1864-1926), and A. N. Krylov (1863-1945). [An interest in the questions of the "constructive theory of real functions" has thus characterized Russian mathematics for almost a century. This tradition gives Soviet mathematicians an excellent background in questions of approximation.] *

A. N. Krylov dealt with important numerical problems (improving convergence of trigonometric series, numerical integration of differential equations, etc.), and worked in all areas of applied science: ballistics, stability of ships, etc. His own book [36] is excellent.

The work of the Soviet mathematicians (i.e., after 1918) was first a continuation of the old tradition of classical numerical analysis. But, under the influence of problems of physics, mechanics, geophysics, and various fields of engineering (mechanics, structural mechanics, hydraulics, heat, electricity, artillery), the emphasis is shifting to ordinary and partial differential equations, integral equations, and conformal mapping [just as in the United States]. Another characteristic of more recent research is that significant general results are being discovered more often in connection with the solution of particular applied problems (see work of B. G. Galerkin and S. A. Čaplygin). Moreover, there is an increasing orientation towards the mechanization of computing. There is a tendency of current workers [particularly Kantorovič and his school] to use modern tools of analysis: constructive theory of functions, new results in differential and integral equations, calculus of variations, and functional analysis. For illustrations see Kantorovič and Krylov [34], which is without peer in the world literature. Others will be found in works of Kryloff (Krylov) and Bogoliuboff (Bogolyubov) [37], Mikeladze [53], Leĭbenzon [43], Melent'ev [49], and others. [For an almost complete list of mathematical books, see [18].]

* Material in brackets is added by the author, and is not in [38].

The introduction in [38] concludes with a tribute to the different research centers interested in numerical analysis.

Besides the introduction the Kantorovič-Krylov article in [38] has eight sections: approximate methods in algebra; numerical integration and interpolation; variational and related methods of solving differential equations; the method of finite differences; other methods of solving differential equations; approximate solution of integral equations; conformal mapping; approximate solution of functional equations. Space does not allow us to survey their contents any further.

The article in [38] by Semendyaev has four sections: tables; application of computational devices; mathematical devices; analogue devices for approximate solution of differential equations and certain other mathematical problems.

4. A more recent Soviet view of numerical analysis

Recently an interesting outline appeared of an address by Sobolev [71] in a symposium at Moscow State University.* Being a detailed outline of how pure mathematics supports numerical techniques, it essentially constitutes a definition of numerical analysis. Here is a translation of the outline, published with Professor Sobolev's permission:

1. *The subject of numerical mathematics from a modern viewpoint:* Sets of functions and function spaces. Tables, graphs, approximate formulas, the separation of numerical concepts like finite-dimensional approximations in a function space. How to study sets not confined to the finite-dimensional. The finite ϵ -net in finite-dimensional spaces. Compactness as the most important property of all objects of numerical mathematics.

Numerical mathematics as one of the branches of functional analysis. New methods, immediately transferable from functional analysis to practical computation.

2. *Numerical mathematics and discrete functions of a discrete argument:* The binary representation of numbers. Two-valued functions of several variables, assuming two values 0, 1.

Connections between numerical mathematics and mathematical logic.

Messages and information. Problematics of the theory of information, as connected with long messages. Estimates of algorithms by their complexity (by the numbers of operations).

3. *Mathematical machines:* Universal high-speed electronic computing machines. Widening of the classes of computable problems. Programming; its theory and practice. The reverse influence of machine engineering on the problematics of the mathematical sciences as a whole.

Mathematical logic and its applications.

The appearance of a requirement for solving complex mathematical problems simultaneously with the widening possibility of solving them.

Spatial and nonlinear problems.

* The author is a professor of mathematical physics.

4. *Theory of approximations*: New problems in the theory of approximation of functions, connected with introducing them into computations. Problems of constructing algorithms for minimax approximations.

Interpolation of functions of many variables.

5. *Special questions on the approximation of operators*: Quadrature formulas and the expression of derivatives by means of differences for functions of many variables.

Inverse operators for approximations; approximations for inverses.

Explicit forms of certain inverse operators.

6. *Cauchy problems for differential and net equations*: Marching problems, their stability, the stability of computation via difference schemes. Purely computational effects connected with rounding off.

7. *Systems of a large number of algebraic equations*: Border problems between algebra and analysis. Systems of a large number of equations, corresponding to a given integral.

Equations of elliptic type and the net systems corresponding to them.

Methods of analysis in algebraic equations. Algorithmization of classical analysis as a result of the development of the possibility of computing.

8. *Conclusion*.

Note Sobolev's thesis that numerical analysis is a branch of functional analysis. It indicates a strong concern for precise problems like bounding the error in approximating integral equations by systems of linear algebraic equations, and so on. One characteristic of American numerical analysis, in contrast to that of Europe and the Soviet Union, is a property of most American applied mathematics and perhaps of our entire technology, i.e., the tendency to use elementary methods to solve problems rather than advanced analysis. Thus, in numerical analysis, many young American mathematicians search for workable machine algorithms but comparatively few make penetrating analyses of their properties. The emphasis is on the energy to produce correct codes, and on having enough machine time to learn how they work empirically. In contrast, the European numerical analyst is likely to think for a long time, and devise an ingenious algorithm the behavior of which is well understood within a certain framework. The algorithm may or may not solve a practical problem, but the real significance of the contribution has been an *idea* or two to join the pool of world information.

For other remarks on the contrast between American and Soviet applied mathematics, see J. R. Kline's article in [86], *Soviet mathematics*.

5. Automatic digital computers in the world

Since contemporary numerical analysis is becoming almost exclusively concerned with automatic electronic computers, it is essential to have some idea of their world-wide development. There are two main classes

of existing modern scientific computers. The first (the *drum computer*) uses a magnetic drum to store some 2000 to 16,000 *words* (i.e., numbers or instructions), each of which can be retrieved in a few milliseconds; there is an arithmetic unit capable of multiplying two numbers in something like 0.01 second. Such a computer retailed for over \$150,000 in the United States in the middle of 1956, plus additional charges for desirable extra input-output equipment. There are many manufacturers of such computers.

The second (the *microsecond computer*) has a store (= memory) from which a word can be retrieved in a few millionths of a second. These are made from acoustic delay lines, cathode-ray (Williams) tubes, or magnetic cores (which are dominating the field), and store from 200 up to as many as 32,000 words. The arithmetic unit can multiply two numbers out of this high-speed storage unit in something like 0.0001 second, or add them in about 0.00006 second. To be balanced, these computers must carry some reasonably fast input-output equipment and, including this and a moderate amount of high-speed storage, they are likely to retail for over \$1,500,000. The leading commercial computers of this class in late 1957 were International Business Machines' Model 704 and Sperry Rand's Model 1103A (Univac Scientific).

A third class of computer is under development in the United States. This may have more than 100,000 words of storage on magnetic cores, any of which it could retrieve in about 1 microsecond. It may multiply in perhaps 0.000005 second (5 microseconds), and probably do this with floating-point arithmetic in the same time. This machine would surely cost several million dollars. The specifications mentioned here are mostly guesses, and the rapid development of the computer art may make them obsolete before this work is published!

Between the drum and microsecond machines, there is another class of computers, of which the Sperry Rand Univac I is a good example. The high-speed memory consists of acoustic delay lines. The arithmetic operations are approximately ten times as slow as those of the microsecond computers. A very large intermediate memory on magnetic tapes permits substantial problems to be performed quite efficiently.

In the United States at the end of 1956 there existed several hundred drum computers, with about 2000 more on order. It appeared that most sizeable organizations with computing problems felt they could justify a drum computer. The United States also had well over 100 microsecond computers, and representatives of one manufacturer informally referred to over 200 orders on hand in May 1956. One large aircraft company wants over a half-dozen computers of this class, probably partly for business work. These computers are quite expensive

to operate, and the demand for them for scientific computing is backed by very large research projects, mostly government-sponsored.

It is a reasonable estimate that an average high-speed computer requires about ten persons with some mathematical skill to prepare and run problems. This amounts to about 30,000 full-time persons to man the computers listed above as being on order in late 1956. By contrast, the combined memberships of the nation's main mathematical, statistical, and computer societies are about 17,000!

For a recent survey of United States computers, see [81].

In Europe one sees much less computer activity. In 1954, few countries seemed willing to commit themselves to a microsecond computer, but most were developing or buying one or more drum computers. In England, where the Williams tube was invented, people were going ahead with several computers of both classes. Sweden had one of the microsecond computers. At Zurich, where Professor Stiefel's Institute is a leading world center of numerical analysis, it was felt that the slower speed of a drum computer permitted and encouraged a greater development of mathematical ingenuity. (This probably meant that people were not attacking problems so large as to compel the use of a microsecond computer.)

By 1956 almost every major country on either side of the Iron Curtain had drum computers. And several more countries were building computers of the microsecond class, including Norway, Denmark, Israel, and perhaps India. In late 1955, the Soviet Union finally released information on its developments in the computer field.

During the decade 1945-1955 of furious development of high-speed automatic digital computers in England, the United States, and more recently elsewhere, the Soviet Union seems to have made no public reference to its work in this area. There were many articles on Soviet analogue machines, and a few translations of Western works on digital machines. By about 1953 or 1954, published tables of the Institute of Precision Mechanics and Computational Mathematics (Moscow) began to show signs of having been computed by very fast computers. But it was not until 1955 that one noticed public reference to Soviet fast computers. By the spring of 1956, such references had become very numerous. It appears also (*Literaturnaya Gazeta*, March 24, 1956) that the 20th Session of the Communist Party's Supreme Soviet directed an increased development and production of electronic computers—not only for scientific computation but also for the automatic direction of the processes of production.

V. Pekelis [61] lists as computers BESM, Strela, URAL, Pogoda, and Kristall. BESM and URAL are described in Art. 6. Strela is described

as having 2048 decimal numbers stored on 43 cathode-ray tubes, with up to 200,000 words stored on magnetic tape. The speed of operation is about 2000 operations per second. There is also discussion of mechanical translation of English sentences into Russian. A week-long conference on computers took place in Moscow in early 1956, attended by some 700 persons, including delegates from China, East Germany, and Yugoslavia.

Another two machines, *M-1* and *M-2*, are described in *Literaturnaya Gazeta*, March 24, 1956.* *M-2* seems to be mainly made of transistors.

From these articles and from Art. 6 below, it is clear that in 1956 the Soviet Union was concentrating on digital computers for scientific computation and for data processing. We may expect an effect on scientific thinking comparable to that in this country. A big pressure in the United States in 1956 is on business applications, including filing, continuous inventory, production control, etc., but there is no evidence of a parallel interest in the Soviet Union at the present time. It does appear that in both countries there is a good deal of thought about using machines for bibliographic reference, filing, retrieval of general information, etc. One Soviet article even mentions the use of computers to correct the grammar of Russian writing. But it would appear that these aspects of data processing remain speculative everywhere in the world.

In the other countries within the Soviet zone of influence there seem to be some developments of drum computers, but no microsecond computer is known to the author. In 1956 drum computers were to be found in Prague, Warsaw, and Dresden.

6. Two Soviet computers

At a conference on digital computing held at Professor Walther's Institute in Darmstadt in the fall of 1955, Soviet delegates presented papers on two Soviet digital computers: BESM and URAL. Both papers have now appeared in English [2, 41]. These papers are of especial interest, since they were the first scientific releases about Soviet digital computer progress. In a word, they indicate a level of electronic computing development comparable with that in the United States.

The general performance and speed of the microsecond computer BESM is comparable with that of SWAC in Los Angeles.† BESM is a three-address parallel binary machine with built-in floating-point operations. Binary-decimal conversions are performed by subroutines. In addition to a cathode-ray-tube memory of 1023 words, there is a static

* The author is indebted to Professor K. E. Harper of the University of California, Los Angeles, for references to *Literaturnaya Gazeta*.

† SWAC, built by the National Bureau of Standards, is located at the University of California, Los Angeles. For details, see [81].

memory of 276 commands, consisting of germanium-diode switches which are selected by means of holes in punched cards. This appears to be a form of card-programming.

The magnetic drum holds 5120 words and revolves at 750 revolutions per minute. The recording density is about 3 bits per millimeter. The air gap of 35 microns between the read-write heads and the drum is comparable with the gap of 18 microns on SWAC's new drum built in 1956. There is a magnetic tape memory with 4 tape recorders, a tape width of 6.5 millimeters, and tape length of 200 meters. Eight bits are stored per millimeter.

The initial loading of the machine is done with perforated tapes in a serial code at 20 words per second. Output on magnetic tape can be reloaded by magnetic tape. The results of the computations are output on magnetic tape, and are subsequently printed on cinema film or a special photo-printing device. The last two forms of output do not appear to have been widely developed in our country. In other respects, BESM seems sufficiently like United States machines to preclude further comment.

Only one BESM appears to have been built, but the other machine described at Darmstadt is intended to be duplicated. This is URAL [2], a drum computer. It has 1024 words of memory on a magnetic drum and 40,000 words on magnetic tape. It has 800 tubes and 3000 germanium diodes, consuming 8 kilowatts of power. Its speed is about 10 milliseconds per one-address command. The plate voltage is 140 volts. In general, URAL appears representative of the large number of drum machines being constructed all over the world. The floating-decimal subroutines are said to be very handy.

As examples of the performance of URAL, Bazilevskii mentions the following: multiplying two matrices of order 20 and printing out the answers requires 20 minutes; the inversion of a matrix of order 60 with floating-decimal representation requires 16 hours.

7. Literature on numerical analysis

One way to define numerical analysis is to point to the material published in certain books and articles. Bibliographies of such books and papers have recently appeared by Householder [31 (Bibliography), 32], Hildebrand [30], and by the author in [58]. In other articles [15, 16] the author has attempted to suggest reading to orient mathematicians towards machine computation.*

* The following periodicals contain a good many articles on the subject: *Computers and Automation*, *J. Assoc. Comput. Mach.*, *J. Math. Phys.*, *J. Res. Nat. Bur. Standards*, *J. Soc. Indust. Appl. Math.*, *Math. Reviews* (numerical analysis section), *Math. Tables*

Very few books on numerical analysis have yet appeared which are primarily oriented towards machine computation, but one can think of the following: Householder [31], Meyer (editor) [50], Engineering Research Associates [11], and Curtiss (editor) [8].

There are now also many books on numerical analysis which give some consideration to high-speed machines. We shall not list them here. The author knows of no Soviet numerical analysis books in which automatic computers are explicitly considered.

Compared with the United States, the Soviet Union publishes a proportionately larger number of expository mathematics books at the graduate school to professional level. These are written in great detail by good mathematicians, and often run from 400 to 800 pages long. The editions are extremely large. The detail in such books makes them well suited for self-teaching. It is discouragingly expensive to publish correspondingly lengthy books in this country, unless they will sell as textbooks. In the author's opinion, a graduate student who invests in a reading knowledge of mathematical Russian (an easier investment than one might imagine) will get an immediate dividend in consulting some of these expositions in his courses. And, in the author's opinion, a government which wants to improve technical education could very well subsidize the translation of a number of these expositions. The Russian editions are low in price.

8. Numerical integration and interpolation

Numerical integration consists in the approximate evaluation of integrals by sums:

$$(1) \quad \int_a^b p(x) f(x) dx = \sum_{k=1}^n A_k^{(n)} f(x_k^{(n)}) + R_n.$$

The main problems, according to the authors of the "Thirty Years" survey [38], are: to establish conditions for convergence of R_n to 0 (as $n \rightarrow \infty$); to bound $|R_n|$ for finite n ; and to find new formulas of smaller error or better applicability. This material is closely related to that on the approximation of functions by polynomials, as discussed in Art. 9.

Aids Comput., Naval Res. Logist. Quart., Proc. Cambridge Philos. Soc., Quart. Appl. Math., Quart. J. Mech. Appl. Math., Referativnyi Zhurnal (Matematika), Vychisl. Mat., Vychisl. Mat. Vychisl. Tehn., Z. angew. Math. Mech., Z. angew. Math. Phys. One can profitably also consult the quarterly progress reports of such computer organizations as the National Bureau of Standards Mathematics Division, and the Electronic Computer Project (now inactive) of the Institute for Advanced Study.

This has been a field of very active interest in the Soviet Union, and a good deal of activity is recorded in the "Thirty Years" book. Most of it is presented in Mikeladze's excellent book [52], and we begin with a discussion of that book. It is both narrower and deeper than an all-around survey of numerical analysis, as it completely omits linear algebra and differential equations. But this enables Mikeladze to give a thorough account of the topics he treats: (a) finite differences and sums; (b) interpolation and numerical differentiation; (c) numerical integration; (d) Čebyšev and least-squares approximations; (e) functions of several variables. It is a numerical analyst's reference book, written with theorems and good notation, rather than a text or a handbook for the numerical technician. The following is a synopsis of the less familiar material treated: use of Bernštejn polynomials for approximation, Čebyšev approximation in general; convergence theorems for polynomial interpolation (Jackson, Bernštejn); new interpolation formulas (Mikeladze); interpolation with continued fractions; convergence of trigonometric interpolation (Bernštejn); multiple Fourier series; new integration formulas (Mikeladze); investigations by Steklov, Markov, Bernštejn, and others on integration formulas of the Čebyšev type (limited number of distinct weights); multiple integration.

The original Čebyšev integration formulas are of type

$$\int_{-1}^1 f(x) dx = w \sum_{k=1}^n f(x_k^{(n)}) + R_n$$

and have the advantage of equal weights. The abscissas $x_k^{(n)}$ are chosen so $R_n = 0$ for polynomials of as high degree as possible. As Bernštejn showed in the 1930's, the $x_k^{(n)}$ lie in the interval $(-1, 1)$ only for $n = 1, 2, 3, 4, 5, 6, 7, 9$, and so the Čebyšev formulas seem useless for other n . Less well known are some researches of Bernštejn, Ahiezer, and Ahiezer and Krejn [51] on similar formulas with only two distinct weights. For example, one can choose x_i so that the formula

$$\int_{-1}^1 f(x) dx = \frac{1}{4} \sum_{i=1}^6 f(x_i) + \frac{1}{8} \sum_{i=7}^{10} f(x_i)$$

is exact for polynomials of degree 11.

The postwar Soviet researches mentioned by Mikeladze are not numerous, but several deal with numerical evaluation of multidimensional or iterated integrals [9, 46, 51]. The recent expository books include those by Vetčinkin and Kogan [76], and Serebrennikov [70].

For balance let us mention some recent Western approaches to formulas of numerical integration. Blanc [3] asks that the coefficients in (1) be

selected so that the expected error $E(R_n)$ be zero and the variance $E(R_n^2)$ be minimized over a certain ensemble of functions f . In a series of papers Sard [65] considers formulas of type (1) which are to be exact for polynomials of some given degree $p < 2n - 1$. Then he notes that the error can be written:

$$R_n = \int_a^b f^{(p+1)}(x) k(x) dx.$$

The "best" formula relative to a given norm is defined as one for which $\|k(x)\|$ is minimized. Greenwood [25] finds coefficients so that (1) is exact when $f(x)$ is a certain sum of exponential terms.

9. Approximation of functions

In probably no other area are electronic computers causing a bigger revision of classical numerical analysis than in the field of approximation and representation of functions. Indeed, the approximation of functions was hardly a part of numerical analysis before computers appeared. In desk computing, a function can be represented by a table of values, and this table can be as long as necessary to give the function to the accuracy desired, with or without interpolation. The desk-computing person has no trouble in locating the proper page of a 600,000-entry table and copying a needed item in a few multiplication times. But an electronic computer would require many thousands of multiplication times to scan a magnetic tape for a needed entry and, indeed, some computers (like SWAC) could not store more than a few thousand entries anyway.

On the other hand, even the simplest interpolation can slow up a desk-computing person a good deal in extracting the value of a function. But an electronic computer, with its accelerated arithmetic, can do the most intricate interpolation in a tiny fraction of its table-look-up time.

A clever feature of classical numerical analysis is the *optimum-interval table*, in which functional values $f(x_i)$ are stored for unequally spaced abscissas x_i , so picked that (say) linear interpolation between the entries just suffices to yield (say) 4-decimal accuracy in $f(x)$. One of the classical problems of the constructive theory of functions is, given $f(x)$ for $0 \leq x \leq 1$, given $\epsilon > 0$, and given n sufficiently large, to find a polynomial $P_n(x)$ of degree n such that $|P_n(x) - f(x)| < \epsilon$. Now the purpose of both $P_n(x)$ and the optimum-interval table is to *condense the information* in $f(x)$. To an automatic digital computer, there is no difference between these two methods of condensation, and the choice between them is made in terms of which condenses the data more effectively, with respect to storage space and retrieval time. Indeed, one can regard the above $P_n(x)$ as essentially representing $n + 1$ values of $f(x)$,

from which $f(x)$ is to be obtained for an arbitrary x by n th order interpolation. At this point there is not even a difference in principle between the two methods of condensation.

When a digital computer is to generate $\sin x$, e^x , $J_0(x)$, etc., only the elementary operations of addition, subtraction, multiplication, and division are available. It is usual to approximate such a function $f(x)$ over a given interval I by a rational function of x or, in particular, by a polynomial. To be definite, let us fix our attention on approximation of a continuous $f(x)$ by a polynomial $P_n(x)$ of degree n . The customary measures of approximation are

$$\max_{x \in I} |f(x) - P_n(x)| = \|f - P_n\|_\infty,$$

and

$$\left\{ \int_I |f(x) - P_n(x)|^2 dx \right\}^{1/2} = \|f - P_n\|_2.$$

The maximum norm $\|\cdot\|_\infty$ is ordinarily considered preferable, because one customarily wants to assert that $|f(x) - P_n(x)|$ nowhere exceeds a certain ϵ . This brings up the computing problem: given n , how can one generate the polynomial $P_n(x)$ which best approximates $f(x)$ over I (called the *minimax* approximation)? This problem, first posed by Čebyšev (= Tschebyscheff = Tchebichef = Chebyshev), has now left the realm of pure mathematics and joined numerical analysis.

There is a classical theory giving criteria for $P_n(x)$ to be a polynomial of best approximation [75]. A recent paper of Motzkin [54] has extended the classical theory to nonlinear families of functions $P_n(x)$. But, until recently, the theory did not emphasize methods of constructing $P_n(x)$ in practice. Hastings and his collaborators [27] have produced a number of polynomial and rational approximations to common functions $f(x)$, but the methods used to find these are a bit obscure and certainly were not automatic. The formulas are widely used for practical approximations in computers, and few installations yet have their own codes to generate approximations. Hastings' remarkable book also contains some theory of minimax approximation, expressed mainly in pictures.

For functions $f(x, y, \dots)$ of several variables, the basic theory of minimax approximation is hardly developed, and seems likely to be fairly difficult. For the minimization of $\|f(x) - P_n(x)\|_\infty$, the best P_n is always unique and can be obtained by interpolating $f(x)$ at $n + 2$ abscissas, where $P_n(x) - f(x)$ attains equal moduli with opposite signs. But, for functions of several variables, it is possible that P_n is not unique, as the number of points where $|f - P_n|$ attains its extrema may be too few to define P_n uniquely. The property of alternation, so simple with one

variable, goes over into an intractable separation property with two or more variables. There is much need, however, for investigation of these matters.

Soviet mathematicians have given some attention to the construction of minimax approximations to functions of one variable. One excellent paper is by Novodvorskiĭ and Pinsker [56]. Let E_n be a set of $n + 2$ points where $f(x) - P_n(x)$ has equal absolute values with alternating signs. They concentrate on finding a set E^k of $n + 2$ abscissas in which $f(x)$ is interpolated by a polynomial $P^k(x)$ of degree n . As $k \rightarrow \infty$, the set E^k is forced to tend to $n + 2$ points of E_n . Then the corresponding polynomials $P^k(x) \rightarrow P_n(x)$, as $k \rightarrow \infty$. The particular feature of the algorithm is that the set E^{k+1} is obtained by dropping just one point of E^k and adding a single point where $P^k(x) - f(x)$ attains its maximum absolute value. P. C. Curtis (unpublished) has a similar algorithm which seems to converge more rapidly because all the points of E^k are changed at each step. Curtis has coded his method for a microsecond computer.

A paper by Wenzl [83] considers minimax approximation of rational functions $f(x)$ by rational functions $R^k(x)$ of lower degree. He shows how to simplify the algebraic equations for computing $R^k(x)$ by use of certain symmetries. Carlson and Goldstein [5] report another algorithm for rational approximation of continuous $f(x)$. They have had good experience with a digital computer, but give no proof of convergence.

In spite of the importance and mathematical interest of the subject, there is relatively little in the literature on the constructive calculation of minimax approximations. It is to be hoped that more will appear. There is an especial need for both basic theory and practical algorithms for approximation of functions of two and more variables. The work of the Russian and Ukrainian school should provide an excellent background. An attractive book by Remez [63] was the source of the Novodvorskiĭ-Pinsker algorithm, and appears to have more material available for exploitation. Since Ukrainian is even less familiar than Russian, a translation would be most welcome.

The least-squares polynomial approximation to a function $f(x)$ over an interval I or finite set of points $x_1, \dots, x_N$ would appear to be well understood. Yet in practice, computing personnel are continually running into the difficulty that the normal equations defining the coefficients are exceedingly hard to solve when the order of the polynomial exceeds 6 or 8. The reason for this is easy to see, and the cure is well known to professionals in the field. How can one pass this information along to the large number of persons from divers fields who are casual users of least-squares methods?

Suppose we want to fit $f(x)$ by a polynomial $P_n(x)$ of degree n , so that

$$\int_0^1 |f(x) - P_n(x)|^2 dx = \text{minimum}.$$

Suppose we decide to represent P_n in the form $P_n(x) = \sum_{k=0}^n a_k x^k$. It is easily shown that the a_k are found by solving the system of *normal equations*

$$\sum_{j=0}^n g_{ij} a_j = b_i \quad (i = 0, 1, \dots, n),$$

where

$$g_{ij} = \int_0^1 x^i x^j dx = (i + j + 1)^{-1} \quad (\text{all } i, j),$$

and where

$$b_i = \int_0^1 x^i f(x) dx \quad (i = 0, 1, \dots, n).$$

Now the matrix of coefficients $(i + j + 1)^{-1}$ ($i, j = 0, 1, \dots, n$) is known as the *Hilbert matrix* (of order $n + 1$). It is a most unruly matrix [73], and the system of normal equations involving it will be solved by elimination only if an extraordinarily large number of significant digits is carried in the computation. Other methods of solution will be correspondingly difficult. One must expect almost comparable difficulty if the distribution dx over the interval $[0, 1]$ of the fit is replaced by another distribution.

One cure is to represent $P_n(x)$ not as a sum of powers x^k but, instead, as a sum of polynomials $q_k(x)$ which are orthogonal with respect to the distribution of the problem. A recursive technique for generating such orthogonal polynomials is described by Stiefel [72]. Alternatively, one may reduce the difficulty by using polynomials which are approximately orthogonal or, at any rate, not nearly so “parallel” as the powers x^k .

10. Solving linear algebraic equations

There is no computing problem more universally posed than the solution of simultaneous algebraic equations. Because linear functions are simplest, systems of linear equations are the commonest of the algebraic systems. The two most frequent sources of linear systems seem to be: (1) the normal equations of a least squares problem; and (2) the difference-equation approximation to a system of differential equations. The types of linear systems differ markedly in the two sources. Those arising from differential equations are likely to be *sparse* (with many zero elements) but of high order (as high as can be treated). Those arising

from statistical problems usually have matrices which are *dense* but of relatively low order (say 3 to 30). In addition to the order and sparseness, it is important to know whether the elements of the matrix must be *filed* as data, or whether they are so simple that they can be *generated* as needed. An example of the latter is the matrix of Laplace's difference operator over a rectangular grid, where the elements are -4 , 1 , or 0 , according to a simple rule.

The choice of a method of solving a linear system depends crucially on the type of matrix, and mainly on whether it is sparse or dense, filed or generated. For a dense filed matrix it is clear that all n^2 elements are recorded in some part of the memory. There is, therefore, room to store in the same place all the transformed elements that will arise in a Gaussian elimination of the linear system. Gaussian elimination requires but $n^3/3 + O(n^2)$ multiplications to solve a general system of order n , and apparently no method can solve a general system in fewer multiplications. (Can some one prove this?) Hence Gaussian elimination remains the most popular method of solving linear systems with a dense filed matrix. It has the advantage of well over a century of numerical experience with substantial problems, in the course of which various computational efficiencies have been discovered [10], mainly to reduce the transcription of data. This experience is very helpful, though it is not ordinarily of decisive benefit to one using an automatic computer, because data transcription occurs at a different point in the calculation and in a different way.

From a theoretical point of view, the understanding of Gaussian elimination was enhanced by the discovery by Benoit, Banachiewicz, and others of its basis [74] in the "*LDU* theorem," which is not widely known to pure mathematicians. Suppose A is a square matrix of order n , whose upper-left-hand principal minors of orders $1, 2, \dots, n-1$ do not vanish. Then A has a unique representation in the form $A = LDU$; here L has zeros above the main diagonal and ones on the main diagonal; D is a diagonal matrix; and U has zeros below the main diagonal and ones on the main diagonal. Gaussian elimination can be described roughly as a constructive process for finding the elements of L , D , and U recursively.

If an upper-left-hand principal minor A_m of order $m < n$ vanishes, but A is of rank $n-1$ or n , then the *LDU* construction can be carried out, but only after certain rows or columns of A have been permuted. Moreover, if the determinant $\det A_m$, while nonzero, is close to zero compared to other minors of order m , the *LDU* process will apparently cause a great loss of significant digits unless a permutation is performed so that another minor becomes A_m . This permutation, called "choosing a pivot" in

numerical work, seems to exert a considerable influence on the precision of Gaussian elimination. Most practical elimination processes are distinguished by how pivots are chosen and how numbers are scaled. While those choices are undoubtedly important, the author knows of no thorough theoretical or practical study of these questions.

There are two general methods of scaling numbers for matrix operations with a digital computer. One is to use a *fixed radical* (i.e., decimal or binary) *point*, and then scale down all elements of a row or column or matrix when one of them gets too large. The other is to use a *floating radical point*. In a typical form of this (used on SWAC), 8 binary digits of the 37-digit "word" are devoted to $128 + p$, and 29 binary digits to q and its sign. The number represented by such a word is interpreted as $q \cdot 2^p$, where $2^{-1} \leq |q| < 1$ and $-128 \leq p < 128$. The elementary arithmetic operations of a computer are used to synthesize subroutines which work directly with these *floating numbers*. (However, in many new machines floating operations are built into the hardware.) The enormous range of the floating numbers normally prevents "overflow." Of the two systems, the floating numbers are the safer in a problem where the number range is unknown *a priori*. One price of using them is the loss of the significant digits in the numbers which are devoted to the exponent p . (On SWAC, for example, in the floating mode one effectively works with 28 significant digits instead of 36.) This loss of precision is tolerable in many fields of computation. In matrix work, however, it can be disastrous for a number of applications. To combat this loss of precision there is an increasing tendency to use *double-precision floating numbers*. On SWAC, for example, there might be 8 digits devoted to the exponent, with $28 + 36 = 64$ significant digits for the number, stored in two machine words. Although such numbers increase the precision, operating with them on microsecond machines is about 50 to 100 times slower than single-precision, non-floating operations. Such a cost in time may become critical for certain problems.

For matrix problems there is a third scaling alternative which is sometimes exploited: the use of *floating vectors*. Here a vector of n components is represented in $n + 1$ words. The first of these is a positive or negative integer p such that 2^p is a multiplicative factor common to the other n components. The last n words are the n components of the vector, so normalized that the largest is between $\frac{1}{2}$ and 1 in absolute value. Each of the n rows of a matrix can be independently represented as a floating vector. The few scalars that enter into a matrix calculation can be represented in two cells as floating vectors of one component. This use of floating vectors permits a full word length of significant digits without the loss of time of double-precision floating operations.

For dense filed matrices, a promising new direct method of solving linear systems has recently been published by Purcell [62]. The following geometrical interpretation is due to Motzkin (unpublished). Interpret the equations as representing n hyperplanes $H^1, \dots, H^n$ in which the solution point is known to lie. Now select $n + 1$ points $p_1^{n+1}, \dots, p_{n+1}^{n+1}$ (subject to certain restrictions which will not be stated). Use one of these, say p_{n+1}^{n+1} , as a center of projection and project the other p_i^{n+1} ($i = 1, \dots, n$) into H^n . Call the resulting points $p_1^n, \dots, p_n^n$. In the same way, use one of these, say p_n^n , as a center of projection to obtain the points $p_1^{n-1}, \dots, p_{n-1}^{n-1}$. The latter must lie in $H^{n-1} \cap H^n$. Continuing in this way we obtain $p_1^1 \in \bigcap_{i=1}^n H^i$; i.e., p_1^1 solves the given system.

If the original points are chosen with many zero components, the number of multiplications is close to that for Gaussian elimination. The process has the advantage of never altering the original matrix, and the operations are easily coded.

M. R. Hestenes [29] has extended the Purcell idea into a convenient code for obtaining the inverse of a matrix A . Suppose A has row vectors $r_1, \dots, r_n$. The inverse sought has column vectors $c_1, \dots, c_n$, with the property that $(r_i, c_j) = \delta_{ij}$ for all i, j . Hestenes starts with $c_j^0 = r_j$, and proceeds to orthonormalize the c_j^0 until finally $(r_i, c_j) = \delta_{ij}$. The process seems to work well; for details, see his paper. Gaussian elimination can be interpreted as a special case of the Hestenes inversion method. The choice $c_j^0 = r_j$ requires more multiplications than elimination but seems to reduce round-off error.

For large and sparse matrices, it is often possible, in principle, to solve systems by direct methods by first decomposing the matrix into a product LDU . Since A has mostly zeros, so will L and U , and one can predict where these zeros will be. Hence L and U can be held in a reasonable storage space and used to solve a linear system. However, this method has not been explored very much, and iterative methods are ordinarily preferred for solving systems with sparse matrices. For a survey of some of these, see an article by the author [14]. The oldest and commonest iterative processes are two, each of which has a variety of names. To explain these in their simplest forms, let the problem be to solve $Ax = b$. Suppose the matrix A is written as the sum of three matrices $L + D + U$. Here D has the diagonal elements of A ; L has the below-diagonal elements; and U has the above-diagonal elements. In both iterations, one starts with a first approximate solution vector x^1 . In the *total-step* (or ordinary, or Jacobi) iteration, for $k = 1, 2, \dots$, one obtains x^{k+1} by solving the system

$$Lx^k + Dx^{k+1} + Ux^k = b.$$

In the *cyclic single-step* (or Seidel, or Liebmann) iteration, for $k = 1, 2, \dots$, one obtains x^{k+1} by solving the system

$$Lx^{k+1} + Dx^{k+1} + Ux^k = b.$$

Many papers have been written which give sufficient conditions for the convergence of these two iterations and compare their convergence. Such problems have interested mathematicians in England, Germany, the Soviet Union, Switzerland, and elsewhere. Although there may be little to choose between the two methods for desk computing, on automatic digital computers the cyclic single-step method appears decidedly preferable from a storage stand-point. This is because it permits immediate replacement of a stored value by its new value, and the old value can be forgotten. Moreover, in most (but not all) problems where comparisons have been made, the cyclic single-step methods converge more rapidly than the total-step methods. The cyclic single-step iteration is exceedingly easy to code, and uses almost no storage except what is needed for the elements of A and x . Where it converges fast enough, it is an ideal computing scheme.

A great many variations of the cyclic single-step method have been proposed. Ostrowski [57] gives a quite complete convergence treatment of many such variations when A is a positive definite matrix. Strangely, for positive definite A , the convergence of the cyclic single-step method was apparently never proved rigorously until Schmeidler's proof [67] in 1947. This illustrates the comparative inattention given to numerical problems by mathematicians over the years.

Real understanding of the cyclic single-step iteration for symmetric matrices A seems to come by combining matrix methods with a study of the change of value of the quadratic form (x^k, Ax^k) , as $k \rightarrow \infty$.

There are interesting problems of extending most methods for non-singular systems $Ax = b$ to the case of singular A . Most methods go over reasonably well to singular A —better than one might suspect.

When A is not symmetric, the convergence of the cyclic single-step method becomes quite doubtful, and there is need for a more general iteration. Here a method of Kaczmarz [33] looks fairly inviting, and it has been exploited somewhat. In this, one again interprets the system as defining the point common to n hyperplanes $H^1, \dots, H^n$. Starting from an arbitrary point p , drop p orthogonally onto H^1 . Then drop the resulting point orthogonally in cyclic order onto H^2, H^3 , etc. It is trivial to prove that the moving point p converges to some solution of $Ax = b$, provided only that the system is consistent.

When one of these iterations converges, it ordinarily converges too

slowly, and there is need for simple processes to *accelerate* the convergence. A good deal of attention has been devoted to this problem [14].

An exposition and imaginative development of the total-step methods is due to Stiefel [72]. To solve a linear system $Ax = b$, Stiefel considers iterations of form $x^{k+1} = x^k + \Delta x^k$, where $\Delta x^k = (1/q_k)r^k$ and $r^k = b - Ax^k$. Then $r^n = R_n(A)b$, where $R_n(\lambda) = \prod_{k=0}^{n-1} (1 - \lambda/q_k)$ is a polynomial with $R_n(0) = 1$.

Stiefel poses the following question of choosing a best *strategy*: Suppose the eigenvalues of A are known to lie in a given real interval $[a, b]$. How shall the polynomial $R_n(\lambda)$ of degree at most n be selected so as to minimize $\int_a^b R_n(\lambda)^2 d\rho(\lambda)$, where $d\rho(\lambda)$ is a given nonnegative mass distribution? Since one does not know the eigenvalues, such minimization is about all that one can demand in smallness of $R_n(\lambda)$. Stiefel solves the minimization problem in terms of polynomials $P_k(x)$ orthogonal over $[a, b]$ with respect to the distribution $d\rho(\lambda)$, and shows various ways to construct the x^k . He points out, however, that the intermediate residuals r^k may become very large, so that the process may be unstable.

Stiefel now shows how to modify his process so that each intermediate r^k also corresponds to a best strategy, thus restoring stability. The change is to put $\Delta x^k = (1/q_k)(r^k + p_k \Delta x^{k-1})$, where p_k, q_k are numbers occurring in the three-term recurrence for the $P_k(\lambda)$. He shows that various choices of the distribution $d\rho(\lambda)$ lead to various known algorithms for solving linear systems, including the conjugate-gradient method of Lanczos, Hestenes, and Stiefel.

The unmodified algorithm which Stiefel described is often called the Richardson method. It has been studied also by Young, who has certain recommendations for dealing with the instability.

For further discussion of linear systems, see the material on relaxation methods in Art. 12.

In 1950 Faddeeva published the most elegant book [12] on the numerical methods of linear algebra known to the author. One chapter gives an excellent introduction to the algebra and analysis of matrices. The second chapter treats the solution of equations by elimination and simple iterative methods, while the third chapter discusses eigenvalue calculations. There are numerical examples and nice proofs.

F. R. Gantmaher has written a superb book [20] on matrices. Numerical analysts will like its development of matrix algebra with the applications really in mind. There is a mechanical interpretation of Gaussian elimination. The chapter on functions of matrices deals generously with systems of differential equations. There is a substantial chapter on the

geometrical theory of linear divisors, and one on linear operators on unitary spaces. Hankel forms $\sum s_{i+k} x_i x_k$ are treated after Hermitian forms. The last 200 pages are devoted to special problems, such as singular pencils of matrices, stochastic and oscillation matrices, the Routh-Hurwitz problem, and systems of linear time-dependent differential equations. Throughout the book there are numerical examples.

Gantmaher's material on oscillation matrices was taken from an earlier book [21] on the subject. *Completely nonnegative (positive)* matrices are those for which every minor determinant is nonnegative (positive). An *oscillation* matrix A is characterized by being completely nonnegative and having some integral power A^q ($q > 0$) completely positive. Oscillation matrices have numerous applications in elasticity, and have interesting theoretical properties. For example, their eigenvalues are positive and distinct, while the corresponding k th eigenvector has precisely k sign changes among its components.

11. Solving matrix eigenvalue problems

Of the various nonlinear algebraic equation systems, we treat here only the eigenvalue problem for finite matrices. This fascinating problem is to find some or all of the eigenvalues λ and perhaps also the corresponding eigenvectors x such that $Ax = \lambda x$. Occasionally one also wants the adjoint eigenvectors y such that $A^*y = \lambda y$, where A^* denotes the conjugate transpose of A . As with linear systems, there are many different types and sizes of matrices A that may occur. Let us confine our attention to the small dense matrices and the large sparse ones. If A is Hermitian, it simplifies computation, because all the λ are then real, and each y can be chosen to equal the corresponding x .

Present methods for computing eigenvalues fall into three main classes: (1) data condensation methods; (2) global iterations; (3) local iterations. In the *data-condensation* methods, one follows two separate algorithms. In the first, a finite number of simple transformations are applied to put the relevant information of A into a compact form. In some cases, the *characteristic polynomial* methods, the compact form consists of the coefficients of the characteristic polynomial $f_A(\lambda)$. These may be found, for example, by Danilevskii's similarity transformations, changing A to its companion matrix, in which the coefficients of f_A form the last column. References for this and other methods can be found in [4].* An old idea due to Leverrier is to compute the trace of A^k ($k = 0, \dots, n$), and put this into Newton's identities to determine the coefficients of f_A . This has been modified by Souriau, and others, to eliminate some of the steps.

* More accessible bibliographies on matrices and on numerical analysis in general are those of Householder [31, 32], and the author in [58].

Methods due to Krylov, Luzin, and Duncan determine the linear dependence between $v, Av, \dots, A^n v$, for some vector v , and, hence, the characteristic polynomial is found. Two or three authors have suggested similarity transformations to change A into Hessenberg's "almost triangular" form in which

$$a_{ij} = \begin{cases} 0 & (i - j > 1), \\ -1 & (i - j = 1), \end{cases}$$

from which one can then evaluate the characteristic polynomial by a recurrence. Or one can go on from Hessenberg's form with some similarity transformations to put A into the companion-matrix form. Givens (unpublished), in particular, obtains the Hessenberg form by unitary transformations on A .

Data-condensation methods which ignore the characteristic polynomial seem usually to reduce A by similarity transformations to a *tridiagonal* (or Jacobi) form T , in which all $t_{ij} = 0$ for $|i - j| > 1$. When A is real and symmetric, Givens [23] has shown that one can do this with orthogonal rotations of two-dimensional coordinate spaces, the rotations used by Jacobi for another purpose a century earlier. A road to the tridiagonal form of an arbitrary matrix is the *biorthogonalization algorithm* of Lanczos [39], which he called "minimized iterations." This interesting algorithm has inspired more research papers in the 1950's than any other in numerical analysis which this author can think of.* Let A have order n . Given almost arbitrary column vectors x_1, y_1 , pick α_1 such that $(y_1, x_2) = (y_2, x_1) = 0$, where $x_2 = Ax_1 - \alpha_1 x_1$ and $y_2 = A^* y_1 - \bar{\alpha}_1 y_1$. Then, for $k = 2, 3, \dots, n$, pick α_k, β_{k-1} such that $(y_k, x_{k+1}) = (y_{k-1}, x_{k+1}) = (y_{k+1}, x_k) = (y_{k+1}, x_{k-1}) = 0$, where

$$x_{k+1} = Ax_k - \alpha_k x_k - \beta_{k-1} x_{k-1},$$

$$y_{k+1} = A^* y_k - \bar{\alpha}_k y_k - \bar{\beta}_{k-1} y_{k-1}.$$

Ordinarily one obtains nonzero vectors x_k, y_k until $x_{n+1} = y_{n+1} = 0$. When this occurs, the $\{x_k\}$ and $\{y_k\}$ are biorthogonal, and A is similar to the tridiagonal matrix

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ 1 & \alpha_2 & \beta_2 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \\ & & & \alpha_{n-1} & \beta_{n-1} \\ & & & 1 & \alpha_n \end{bmatrix}.$$

* Papers by Stiefel, Hestenes, Stesin, Karush, Rutishauser, Bauer, etc.

The properties of this algorithm have been studied in detail by Rutishauser [64]. A third method of getting a tridiagonal form of a general matrix is first to get A into the Hessenberg almost triangular form, either as Hessenberg did it or by unitary transformations, as Givens proposes. Then follow with row and column similarity operations which get rid of the elements above the tridiagonal elements [68].

The second algorithm of the data-condensation methods is one for localizing the eigenvalues from the compact form. Whenever the compact form is f_A , we have to apply an algorithm for obtaining the zeros of a given polynomial. This problem needs a separate study, and we shall not discuss it, except to mention a promising recent machine method by Muller [55]. For matrices of large order, the characteristic polynomial method surely stands or falls on its ability to produce the coefficients accurately enough, and on the possession of a good root-solving routine. Some authors seem to ignore these factors.

When the compact form of A is a tridiagonal matrix T , the second algorithm is one for getting the eigenvalues of T . Here, again, there is a wealth of choice. For one thing, it is easy to evaluate $\det(T - \lambda I)$ for very many values of λ by a simple recurrence. The condensation of the data had, indeed, the very purpose of enabling us to do a great deal of computing with T , which is similar to A , but much more compact. When the eigenvalues of A (and, hence, of T) are known to be real, scanning should suffice to locate them very readily. If necessary, it can be aided by the fact that the successive minors of $\det(T - \lambda I)$ form a Sturm sequence. These minors are found in the recurrence by which $\det(T - \lambda I)$ is most rapidly computed [23]. When the λ are complex, systematic scanning may still be quite rapid enough to locate the eigenvalues. If not, one can always go to a polynomial solver which, like Muller's, uses the values of the polynomial for various arguments, but which does not require the coefficients of the powers of λ . One then evaluates $\det(T - \lambda I)$ for specific λ as needed. Finally, there is the interesting progressive form of Rutishauser's quotient-difference algorithm [64]. This yields the eigenvalues as limits of elements in a certain tableau which looks like a difference table. Rutishauser's algorithm has the advantage that complex eigenvalues of a real matrix can be found without having to use operations with complex numbers. Karush and, later, Stesin have considered the applicability of the Lanczos biorthogonalization algorithm to functional operators. Hestenes and Stiefel have used the closely related conjugate gradient method to solve linear systems.

A *global iteration* for finding eigenvalues consists of transformations by which one finds a sequence of n -tuples of complex numbers, such that the k th n -tuple converges, as $k \rightarrow \infty$, to the n -tuple of eigenvalues of the

given matrix A . The essence of the word “global” here is that all n eigenvalues are found simultaneously. The prototype of these is the original process of Jacobi, applicable only to a Hermitian matrix (a_{ij}) . Here one scans the off-diagonal elements to find the maximum value of $|a_{ij}|$, and then rotates a corresponding i, j -coordinate plane until a_{ij} becomes 0. The process is iterated until all the off-diagonal elements are effectively 0. The process can be proved to converge; its beauty is that $\sum_{i,j} |a_{ij}|^2$ is an invariant of the rotations, so that there is no problem of numbers growing too large.

For electronic computers, scanning all the off-diagonal elements for $\max |a_{ij}|$ is slow and awkward. Most codes avoid this by rotating all the off-diagonal elements to zero in a cyclic order (the “cyclic Jacobi process”), and repeating this to convergence. The cyclic Jacobi process also seems to work well, and the author has never heard of its failing to converge. On the other hand, until recently he knew of no proof that showed it would always converge, and this was an important gap in the theory.* Experimental data on convergence are given by Gregory [26]. When the order of the matrix is too large, the Jacobi process and its cyclic modification are too slow. This motivated the Givens tridiagonalization method referred to earlier.

The Jacobi process is so inviting in principle that one wants to extend it to nonsymmetric matrices. An attempt to do so has been reported by Greenstadt [24]. The basic theorem, due to Schur, states that an arbitrary matrix A can be reduced to upper triangular form by a transformation UAU^* , where U is an appropriately chosen unitary matrix. Again, the attempt is made to build up U by successive (complex) rotations of coordinate 2-planes. However, it is not easy to be sure that $\sum_{i>j} |a_{ij}|^2$ decreases at each step, and no general method has been proposed. It appears to the author that making this method work would be an important contribution. Certainly, finding eigenvalues of complex matrices A of orders 10 to 25 is a very frequent computing problem worthy of a simple universal code.

A *local iteration* aims to get one eigenvalue λ_1 , and generally also the corresponding eigenvector x_1 , at one time. There seem to be two main algorithms here. One is the *power method*—the repeated premultiplication of some vector x by the matrix A . This has been developed into a complete code by Wilkinson [84]. After $A^k x$ has converged to an eigenvector x_1 of A , Wilkinson *deflates* the matrix A to a new matrix (some-

* The gap has been filled by G. E. Forsythe and P. Henrici, The cyclic Jacobi method for computing the principal values of a complex matrix (to appear).

times of lower order) in which λ_1 is no longer present. There is a large world literature on this method. Less-known papers include those by Lopšić [45], Semendyaev [69], and a general deflation method by Feller and the author [13].

Another local iteration was proposed by Wielandt. In one of various forms it amounts to selecting an approximation λ to an eigenvalue, and then approximately solving the bilinear homogeneous system $(A - \lambda I)x = 0$ for a nonzero vector x . One then uses this x to improve the value of λ , and continues until λ is really close to an eigenvalue. The essential part is the solution of a singular or almost singular system, an important subject. An alternative is to solve the system $(A - \lambda I)y^{k+1} = y^k$ for $k = 0, 1, \dots$. Since $y^k = (A - \lambda I)^{-k}y^0$, ordinarily y^k converges to an eigenvector x_1 of A belonging to the eigenvalue λ_1 closest to λ .

We have attempted to suggest how many different methods there are for solving the eigenvalue problem. For dense symmetric matrices, the situation is well in hand: the Jacobi method and Givens' tridiagonalization method make very effective codes. For small, dense nonsymmetric matrices, there is no method of choice, but, at present, almost all the methods described in this article are being used somewhere or other. None seems really satisfactory. Schur's theorem offers an attractive direction for research.

We have said little about eigenvalues of large, sparse matrices, but the matter is discussed somewhat in Art. 12. For these, the local iterations seem to offer the only feasible approach, because of storage considerations. Simplified reactor calculations often amount to determining an eigenvalue of a large, sparse matrix with terms representing complicated boundary and interface conditions, and there is a great deal of interest in good methods here.

Besides the question of finding algorithms to locate eigenvalues, there is the matter of bounding the errors in the computed eigenvalues and eigenvectors. For Hermitian matrices, a powerful tool has been provided by Wielandt [35]. Suppose for the Hermitian matrix A one knows vectors v_1 such that $v_1 = Av_0, \dots, v_p = Av_{p-1}$. Wielandt solves the problem of specifying "minimal inclusion intervals" of the real axis in which the eigenvalues of A must lie. It is known that there is no easy extension of this work to arbitrary matrices A , but it would be desirable to have some bounds in the general case.

12. Difference methods for Laplace's equation

The general subject of numerical methods for solving differential equations and systems of differential equations, ordinary and partial, is so vast as to defy the merest possibility of a survey here. The most

popular practical method to solve differential equations is to reduce them to difference equations by confining the range of the independent variables to a network of lattice points. It is far from certain that this is numerically the best approach, but its employment makes it easy to devise algorithms which can be carried out. For difficult problems, this is a dominating advantage over competing functional approaches. We shall here confine our attention to difference methods, while remarking that methods like Ritz's have received a more satisfactory theoretical treatment.

L. F. Richardson divided difference-equation problems into marching problems and jury problems. In *marching problems*, the difference equations can be solved recursively from some initial point set outwards. In *jury problems*, the difference equations must be so solved as to satisfy a "jury" of boundary values all around. This basic dichotomy characterizes machine solution of differential equations. In both types of problems there are three basic questions:

1. How shall we set up the difference equations?
2. How shall we solve the difference equations?
3. How accurately does the solution of the difference equations represent the solution of the differential equation?

In marching problems it is usually trivial to set up reasonable equations and a recurrence to solve them. The problem is to do these in such a way that the solution is *stable*, i.e., accidental errors are not so magnified as to obscure the answers sought. The theoretical problem is to define stability carefully, and to take a realistic account of the effect of round-off errors [40].

In jury problems the solution is defined implicitly, and a recursive solution cannot get started. There are ordinarily too many points of the network to consider an elimination or other direct solution of the difference equations, even when they are linear. The usual approach is, therefore, an iterative solution. The difficult problem of stability is then replaced by the difficult problem of *rapidity*: Is the iteration going to converge rapidly enough to the solution of the difference equations?

For both marching and jury problems, question 3 is likely to be exceedingly difficult to answer. This is basically because the space of functions over the network domain is not directly comparable with the space of functions over the continuous domain of the differential equations.

The dichotomy between marching problems and jury problems does not prevent certain equations from sharing both aspects. For example, heat flow is usually a jury problem in the space variables and a marching

problem in time. And let us be clear that the above considerations apply to both ordinary and partial differential equations. In fact, the distinction between marching and jury problems is likely, at least in numerical problems, to be more important than the number of independent variables involved.

Just to give a sample of recent research, let us confine ourselves to two simple linear jury problems. These are both set on a finite region R in the x, y -plane bounded by a simple closed curve C , and both deal with the Laplacian operator $\Delta u \equiv u_{xx} + u_{yy}$, where the subscripts denote partial derivatives:

I. *The first boundary-value (or Dirichlet) problem:* Given the values of u on C , determine u in R such that $\Delta u = 0$ in R .

II. *The first (or membrane) eigenvalue problem:* Determine some eigenvalues λ^k and corresponding functions $u^k(x, y)$ such that, for all k involved, $\Delta u^k + \lambda^k u^k = 0$ in R , and $u^k(x, y) = 0$ on C .

Question 1 for either I or II is how to set up a network and the corresponding difference equations. Unless R has a very special shape suggesting other networks, it is usual to take a square grid. This has the advantage of yielding simple formulas and of mapping easily on the two-dimensional cell arrays of the usual computer memories. Let h be the mesh constant. For points of the network far interior to C , the usual approximation replaces Δ by the difference operator Δ_h defined by

$$h^2 \Delta_h v(x, y) = v(x + h, y) + v(x - h, y) \\ + v(x, y + h) + v(x, y - h) - 4v(x, y).$$

The main problem under question 1 is what to do about replacing Δ for points (x, y) of the network so near to C that one of the four neighbors $(x \pm h, y \pm h)$ lies outside C . To insure convergence, as $h \rightarrow 0$, of the solution $u_h(x, y)$ of the difference problems to the corresponding solution $u(x, y)$ of the continuous problems, it is sufficient to be very crude, and simply move C to the nearest *net polygon*. This was the approach of Courant and his collaborators in their classical paper [7]. Although this approach solves the problem in principle, it makes u_h too inaccurate an approximation to u for values of h which are easily handled in a machine.

The next approximation is recommended by Collatz [6]. It amounts to a linear interpolation of $u(x, y)$ between points of the network and points of C . This is ordinarily considered very satisfactory. The most delicate approximation consistent with the approximation Δ_h seems to be due to Mikeladze [59]. In this, one approximates u_{xx} near C by twice the second divided x -difference of $u(x, y)$ for three unequally spaced argu-

ments—two from the network and the third from C . One similarly approximates u_{nn} , and then adds the two to get the approximation $\Delta_h u$ to Δu . A good deal of material on all these matters can be found in the comprehensive reference book by Kantorovič and Krylov [34]. A survey by Wasow [80] shows that the Mikeladze approximation may be required if one intends to carry out Richardson's program of accelerating the convergence of u_h to u , as $h \rightarrow 0$.

Having selected difference equations over some network R_h , one now faces question 2 of how to solve them with a digital computer. Here there are two main subquestions. First, how shall one actually compute and store $u_h(x, y)$ and all the coefficients, especially when C is curved and there are many exceptional coefficients to deal with? Second, what sort of iterative process shall one use to solve the equations, once they are in the computer?

The first subquestion does not seem to have had any very general treatment, and each problem appears to have been treated for itself. This is an important area for research. There is a large and growing literature on methods of iteration, however. In 1918 Liebmann recommended using what we called the "cyclic single-step method" in Art. 10. One simply runs through the points (x, y) of the network R_h in a cyclic order, and replaces each value of $u(x, y)$ by the average of its four neighbors (if one is solving problem I). Now pencil-and-paper computers have devised a large number of improvements of the Liebmann process [19], but few of these are suitable for automatic computers. One of them, however, is suitable, namely, *overrelaxation*. Instead of moving $u(x, y)$ from its present value to the average of its neighbors, pencil-and-paper computers find that it pays to overrelax, i.e., to move u somewhat farther.

In a remarkable paper, Young [85] * investigates for problem I (and for more general self-adjoint problems) the advantage of using equal overrelaxation at each stage of the cyclic process. This becomes both a beautiful and practical theory. Suppose the points of the network can be divided into two sets, S and T , so that the difference equations provide for no "mutual coupling" between points of S , nor between points of T . Then the system is said to have *property A*; our problems I and II have this property for a square grid. When property A holds, and when the single-step method is done in a suitable cyclic order, Young tells how to select the optimum overrelaxation factor, and gives a good deal of information about it. This has led to its adoption for a great many codes. Even when property A does not hold, experience shows that such systematic overrelaxation can be very useful. It would be valuable to have a

* This had been Young's doctoral dissertation at Harvard, 1950.

theoretical investigation of how well Young's theory applies without property A.*

Other suggestions for relaxing a large elliptic problem have come from a group at the Humble Oil and Refining Company, coordinated mathematically by J. Douglas, Jr. Since a large problem involves so many data, one of the major difficulties is one of *data-handling*. The data have to be moved back and forth a great many times from the high-speed storage to magnetic drums or tapes. Since this motion wastes time, it pays to minimize it. One suggestion [60] we call the *method of lines*, by analogy with related processes discussed frequently in the Soviet literature. Whenever three lines of data are brought into the high-speed memory, as seems essential in any relaxation process with Δ_h , this method of lines precisely solves the difference equations on the central line, assuming the data of the other two lines to be fixed. This can be done reasonably easily in many ways, since the equations of this subsystem are in tridiagonal form. The extra computing time at each line often pays off in reducing the time required for the total solution. In theory, one can consider running the lines horizontally for one cycle, and vertically for another, but the linear arrangement of data on most intermediate storage devices makes this impractical.

There is some recent Russian research on the round-off error in the solution of Laplace's equation over a finite grid [1, 47].

A useful approach to solving problem I, both for computing and for theory, is to consider the solution as the asymptotic steady state of the time dependent problem

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in } R; \quad u \text{ prescribed on } C.$$

This point of view is most directly applicable to the total-step methods of solving problem I.

The numerical solution of problem II has been treated less than that of problem I. There is some discussion in a survey by the author [17]. Setting up the difference equations seems to cause no new difficulties. Solving them for the lowest eigenvalue λ^1 means solving a singular system of equations for its nontrivial solution. With problem II, this system is symmetric semidefinite, or may be made so by an artifice. The relaxation solution for such a problem goes just as well as for problem I, according to the author's experience with SWAC.

* A very substantial contribution to this question has just been made by W. Kahan, *Gauss-Seidel methods of solving large systems of linear equations*, Thesis, University of Toronto, 1958.

A harder problem is how to compute the higher eigenvalues λ^k ($k > 1$), since the operator $\Delta - \lambda^k I$ is indefinite. Probably an iteration of the Kaczmarz type will be required; see Art. 10. The author knows of no experience here, except with systems so small that they are treated as dense, filed matrices; see Art. 11.

Let us now consider question 3, the closeness of the difference equation solution to the solution of the continuous problem. For the boundary-value problem I, the main estimates follow a line started by Gerschgorin (Geršgorin) [22], a recent synthesis of which has been given by Wasow [80]. In the basic form, the result states that, if the Collatz interpolation is used at the boundary,

$$|u_h(x, y) - u(x, y)| \leq k_1 M_4 h^2 + k_2 M_2 h^2,$$

where k_i are constants independent of u or h , M_i is an upper bound for the modulus of all i th partial derivatives of u in R , and h is the mesh constant. Now it is logically bad that the estimate should depend on M_i , since u is the unknown quantity one is seeking to estimate. But, far worse, $M_4 = \infty$ unless both C and the boundary data are very smooth indeed. In a recent paper, Volkov [78] has given bounds for $|u_h - u|$ in terms of the constants in Hölder conditions satisfied by the unknown function u and its derivatives. For other Soviet work in this area, see the references in Volkov's paper and an important paper by Lyusternik [48].

A few American papers have analyzed the precise *degree of convergence* (i.e., the exponent r such that $|u_h - u| \sim h^r$ as $h \rightarrow 0$) for the very special case of a unit square R and $h = 1/N$. Walsh and Young [79] show that for continuous boundary values the degree of convergence may be arbitrarily slow. But, for boundary values which are piecewise continuous and of bounded variation, one has $|u_h - u| = O(h)$. These results are very special, of course, but they show how difficult the error estimates must be for a general C .

Bounding $|\lambda^k - \lambda_h^k|$ and $|u^k - u_h^k|$ for the eigenvalue problem II is even more difficult than for problem I. This is partly because for problem II we no longer have available the *maximum principle*, according to which a function harmonic in R assumes its maximum modulus on the boundary of R . A recent paper by Saul'ev [66] surveys the history of this problem since 1926, and contributes a general error estimate under the assumption that C is representable by functions with seven continuous derivatives.

In practical problems involving Laplace's operator, one often sees regions with corners; sometimes the corners are smoothed with arcs of circles. Therefore, practical error estimates should deal with regions

bounded by curves C which are, say, piecewise analytic. This should provide many problems for investigation. A basis for such investigations has been made by Lewy [44] and by Lehman [42].

A few recent papers [28, 82, 17] have succeeded in giving one-sided bounds from below (or asymptotic bounds) for the eigenvalues λ^k by use of quantities connected with the finite-difference approximations u_h^k and λ_h^k . There is ample room for more work.

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LINEAR PARTIAL
DIFFERENTIAL
EQUATIONS

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INTRODUCTION

This report deals primarily with advances in this field since 1953, with due reference to older work which provides the essential background. Even in this relatively short time span, the new results have been so voluminous, and have changed so radically our basic points of view, that it has been necessary to limit our treatment to a few topics, thus omitting much work of great importance. We have therefore confined our attention to linear differential equations, and within this domain we had to forego a description of the work of Riesz [574], Garding [261], [262], Leray [408], Petrovsky [510], Sobolev [622], Fourés-Bruhat [230], and Douglis [165] on hyperbolic equations, the special methods due to Bers [40] and Bergman [30] for treating two-dimensional problems (we hope the report of Bers (Vol. III, *Surveys in Applied Mathematics*) will partially fill this gap), and much important work on special equations such as the Laplace equation, which has as yet no counterpart in the general theory, and the work of Protter [556], Germain and Bader [284], and others on equations of mixed type. We have chosen to concentrate on methods which are applicable to wide classes of equations, rather than on methods which are adapted to special equations, even though the latter often yield more detailed results. Unfortunately, we have had to omit a discussion of numerical methods, especially difference equation techniques and variational methods, which are also very important for the general theory (see Morrey [462]). We have tried in most cases to state the results in the form of definite theorems for the sake of easy reference for those who wish to use them. Frequently, however, the method is more important than the specific results, and a sketch of the argument was necessary to bring out the underlying idea. In a few cases we have quoted unpublished results which we learned by oral communication and for which we are indebted to friends such as Browder, Calderón, and Hörmander. Occasionally we have included some of our own unpublished work. We hope to treat the subject more completely with full details in a book now in preparation.

PARTIAL DIFFERENTIAL EQUATIONS IN THE COMPLEX DOMAIN

THE CAUCHY PROBLEM

1. Transformation to normal form

The fundamental problem in the theory of partial differential equations is the Cauchy problem, which is important for its own sake, for the role it plays in the classification of differential equations, and for its applications to the study of other problems. For systems of first-order equations it can be formulated as follows: given M functions $f_1, \dots, f_M$ of the $N + M + NM$ variables $x_1, \dots, x_N, u_1, \dots, u_M, p_{11}, \dots, p_{MN}$, and an $N - 1$ dimensional surface $S: s(x_1, \dots, x_N) = 0$ in the space of the x -variables, and M functions $\phi_1, \dots, \phi_M$ defined on S , to find functions $u_1, \dots, u_M$ of $x_1, \dots, x_N$ defined in a neighborhood U of S which satisfy

$$(1) \quad f_j \left(x_1, \dots, x_N, u_1, \dots, u_M, \frac{\partial u_1}{\partial x_1}, \dots, \frac{\partial u_M}{\partial x_N} \right) = 0 \quad (1 \leq j \leq M)$$

in U and

$$(2) \quad u_j = \phi_j \quad \text{on } S$$

For example, the problem of finding an electro-magnetic field $\mathbf{E}$ and $\mathbf{B}$ in vacuum from the initial field at $t = 0$ is the Cauchy problem for Maxwell's equations

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \quad \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = 0$$

(with suitable choice of units), which is a special case of (1) and (2) with $M = 6$, $N = 4$, and where S is the plane $t = x_0 = 0$.

In this chapter we consider only the case where all functions involved are analytic and the variables are regarded as complex.

We consider a portion of S where $\nabla s \neq 0$, and we can, without loss of generality, assume that $(\partial s)/(\partial x_N) \neq 0$ there. If we make the transformation of variables

$$\begin{aligned} y_j &= x_j & (j < N) \\ y_N &= s(x_1, \dots, x_N), & u_j(x_1, \dots, x_N) = v_j(y_1, \dots, y_N) \\ q_{jk} &= \frac{\partial v_j}{\partial y_k}, & p_{jk} = \frac{\partial u_j}{\partial x_k} = q_{jk} + q_{jN} \frac{\partial s}{\partial x_k} \quad (k < N) \\ p_{jN} &= q_{jN} \frac{\partial s}{\partial x_N} \end{aligned}$$

then the problem takes the form

$$\begin{aligned} g_j(y_1, \dots, y_{MN}) &= 0 \quad \text{in } U \\ v_j(y_1, \dots, y_{N-1}, 0) &= \phi_j(y_1, \dots, y_{N-1}) \end{aligned}$$

where $g_j(y_1, \dots, y_{MN})$ represents $f_j(x_1, \dots, p_{MN})$ after the change of coordinates. These equations can be solved for $q_{1N}, \dots, q_{MN}$ in a neighborhood of a solution $(y_1^0, \dots, y_{MN}^0)$ providing that

$$(3) \quad h\left(x_1, \dots, p_{MN}, \frac{\partial s}{\partial x_1}, \dots, \frac{\partial s}{\partial x_N}\right) = \det(h_{jk}) \neq 0$$

where

$$h_{jk}(x_1, \dots, p_{MN}, \xi_1, \dots, \xi_N) = \sum_{\nu=1}^N \frac{\partial f_j}{\partial p_{k\nu}} \xi_\nu$$

The function $h(x_1, \dots, \xi_N)$, which is a homogeneous polynomial in $\xi_1, \dots, \xi_N$ of degree M , is called the *characteristic form* of the system (1). The planes

$$\sum_{j=1}^N \xi_j (x_j - x_j^0) = 0$$

where $(\xi_1, \dots, \xi_N)$ ranges over the solutions of $h(x_1^0, \dots, p_{MN}^0, \xi_1, \dots, \xi_N) = 0$, have as an envelope a cone $C(x_1^0, \dots, p_{MN}^0)$ with vertex at $(x_1^0, \dots, x_N^0)$, which is called the *characteristic cone* of (1) at the latter point. The condition (3) means geometrically that the surface S is not tangent to the characteristic cone at the point $(x_1, \dots, x_N)$. Note that, in general, the characteristic cone at $(x_1, \dots, x_N)$ depends also on the values of $u_1, \dots, p_{MN}$; only in the case of *linear* equations does the characteristic cone at a point depend only on the point.

In general, the problem (1), (2) has no solution if (3) is not satisfied. For example, the problem

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0, \quad u(x_1, x_2) = x_1 \quad \text{for} \quad x_2 = x_1^2$$

has the unique solution $u = \{1 - [1 - 4(x_1 - x_2)]^{1/2}\}/2$ in a neighborhood of the origin, and this has a singularity at the point $(\frac{1}{2}, \frac{1}{4})$ where $S: x_2 - x_1^2 = 0$ is tangent to the characteristic "cone," which in this case reduces to the line $x_1 - x_2 = \frac{1}{4}$. There is no solution of this equation which satisfies $u = x_1$ for $x_2 - x_1 = 0$, a curve which is everywhere tangent to the characteristic cone. The solution which satisfies $u = x_1$ on $x_2 - mx_1 = 0$ ($m \neq 1$) is $u = (x_1 - x_2)/(1 - m)$, which blows up as $m \rightarrow 1$.

In a neighborhood of a point where S is not tangent to the characteristic, the problem can be transformed into one of the form

$$(4) \quad \frac{\partial u_j}{\partial t} = F_j \left(t, x_1, \dots, x_N, u_1, \dots, u_M, \frac{\partial u_1}{\partial x_1}, \dots, \frac{\partial u_M}{\partial x_N} \right) \quad (1 \leq j \leq M)$$

$$(5) \quad u_j(0, x_1, \dots, x_N) = \phi_j(x_1, \dots, x_N) \quad (1 \leq j \leq M)$$

For convenience we have denoted by t the variable which plays the role of y_N above, and have made some other minor changes in notation. We can normalize the problem further by setting $u_j = \phi_j + v_j$, and can subject the domains of variation of $t, u_1, \dots, u_M, p_{11}, \dots, p_{MN}$ to a change in scale.

2. Basic existence theorems

We may assume, then, that $\phi_j \equiv 0$ ($1 \leq j \leq M$), and that F_j is analytic in the domain $\Omega_1: |t| < 1, |u_j| < 1, \sum_{k=1}^N |p_{jk}| < 1$ ($1 \leq j \leq M$), and $(x_1, \dots, x_M) \in \Omega$, where Ω is a given domain in the space C_N of N complex variables, and that $|F_j| \leq A$ in Ω_1 for $1 \leq j < M$. It is useful for some purposes to introduce the quantity $\epsilon = \sup |F_j(t, x_1, \dots, x_N, 0, \dots, 0)|$ for $1 \leq j \leq M, |t| < 1, (x_1, \dots, x_N) \in \Omega$, which is a measure of how good $u_j \equiv 0$ is as an approximate solution of (4), (5).

We set $u_j(t, x_1, \dots, x_N) = \sum_0^\infty u_{jk}(x_1, \dots, x_N)t^k$, and obtain recursion formulas for the coefficients u_{jk} , which shows that there is a unique such formal power series solution. By using estimates for the growth of the derivative of an analytic function in terms of the growth of the function

and an easy generalization of Bernstein's theorem on the derivative of a polynomial (see Bernstein [38]), we can obtain bounds on u_{jk} from which we deduce the convergence in a certain domain. Let $\delta(x)$ be the distance from the point $x = (x_1, \dots, x_N)$ to the boundary of Ω , where the distance between two points x and y is measured by $\|x - y\| = \max_{1 \leq j \leq N} |x_j - y_j|$. We obtain:

Theorem 1. Under the above hypotheses, if $\delta(x) \leq B$ for $x \in \Omega$, then (4), (5) has a unique solution analytic in the domain $\Omega_2: \delta(x) > 4e^2(1+B)(1+A+\epsilon)|t|$, and in Ω_2 we have

$$|u_j(t, x)| \leq 4\epsilon|t|, \quad \sum_{k=1}^N |\partial u_j / \partial x_k| \leq (4e\epsilon|t|)/\delta(x)$$

We can obtain more precise results for linear equations. Let

$$(6) \quad F_j = A_j(t, x) + \sum_{k=1}^N B_{jk}(t, x)u_k + \sum_{m=1}^N \sum_{n=1}^N C_{j,m,n}(t, x)p_{mn}$$

where $|A_j(t, x)| \leq a$, $\sum_k |B_{jk}(t, x)| \leq b$, $\sum_m \max_n |C_{j,m,n}(t, x)| \leq c$ for $1 \leq j \leq M$, $|t| < 1$, $x \in \Omega$. We have

Theorem 2. If u_j is the solution of (4), (5) with $\phi_j \equiv 0$ and F_j satisfies (6), then u is analytic in $\Omega_3: x \in \Omega$, $|t| < \delta(x)/(2\delta(x) + ce)$, and

$$|u_j(t, x)| \leq \delta(x)(a/b)\{(1 - z/z_0)^{-bz_0} - 1\}$$

where $z = |t|/\delta(x)$ and $z_0 = (\delta(x) + ce)^{-1}$.

Corollary 2a. If A_j , B_{jk} , and C_{jmn} are analytic for all values of t and x , and if the C_{jmn} are functions of t alone, then u is analytic for all t and x .

The fact that the domain of analyticity of u can be estimated in terms of the coefficients C_{jmn} alone was noticed by Schauder [596] and is used both by him and Leray [408] in their treatment of nonlinear hyperbolic equations. The facts expressed in the corollary, in the case where the C_{jmn} are constants, were remarked by John [352].

Even in the case of linear differential equations these existence theorems are local, i.e., they show that the solution exists for sufficiently small t . If the coefficients C_{jmn} depend on x , then we can expect nothing better. For example, the solution of the problem

$$u_t = x^2 u_x, \quad u(0, x) = x$$

is $u = x/(1 - xt)$ which is singular on the hyperbola $xt = 1$, although the coefficients are analytic for all t and x .

In general, nonlinearities in the equation force the existence of singularities in the solution. For example, the solution of the problem

$$u_t = F(u_x), \quad u(0, x) = x^2/2$$

where F is an entire function and $F''(0) \neq 0$ has a singularity in the region $|t| \leq 1/|F''(0)|$.

THE LERAY-FANTAPPIÉ OPERATIONAL CALCULUS

1. Preliminaries

While the solution of the Cauchy problem, indicated above in the form of power series in t , is theoretically completely satisfactory, it is difficult to extract from this solution certain desirable information concerning its properties, such as dependence on the equations, the initial values, and the surface S , and the behavior of the solution in the large. Hence it is important to obtain more "explicit" solutions in some convenient form. For wide classes of linear equations, Leray [410] has recently found such formulas by means of a generalization of the Laplace transformation related to some work of Fantappié [196]. Laplace had already introduced this method for the solution of ordinary differential equations (see Ince [343], p. 438). We shall explain Leray's new operational calculus for the case of two independent variables, x_1, x_2 , in order to avoid an extensive digression into the geometry and topology of the space of several complex variables.

We shall consider equations of the form

$$(7) \quad a_j \left(x, \frac{\partial}{\partial x} \right) \mathbf{u} = \Sigma a_{jk} \left(x, \frac{\partial}{\partial x} \right) u_k = v_j(x), \quad (1 \leq j \leq M)$$

where $x = (x_1, x_2)$, and $\partial/\partial x$ is the symbolic vector $[\partial/\partial x_1, \partial/\partial x_2]$, and $\mathbf{u}$ and $\mathbf{v}$ are M dimensional vectors whose components are analytic functions of x , and

$$a_{jk}(x, \xi) = B_{jk}(x) + \sum_{n=1}^2 C_{jkn}(x) \xi_n$$

for $1 \leq j, k \leq M$, and the coefficients B_{jk} and C_{jkn} are analytic functions of x . We are given a "surface" i.e., an analytic curve

$$S: s(x) = x_2 - \Psi(x_1) = 0$$

and seek a solution of (7) such that $\mathbf{u} = 0$ on S . The condition that S be characteristic is that

$$h(x, \xi) = \det \left(\sum_{n=1}^2 C_{jkn}(x) \xi_n \right) = 0$$

for $\xi_1 = -\Psi'(x_1)$, $\xi_2 = 1$. We shall assume that $\Psi(0) = \Psi'(0) = 0$ and that $h(0, 0; 0, 1) = \det(C_{jk2}(0)) \neq 0$, so that S is not characteristic at the origin, and therefore also in some neighborhood of the origin. Let $\|x\| = (|x_1|^2 + |x_2|^2)^{1/2}$ denote the length of the vector $x = (x_1, x_2)$. Then there are positive constants ρ_1 , ρ , and a such that both $h(x, \xi) \neq 0$ and $h_1(x, \xi) = \det(a_{jk}(x, \xi)) \neq 0$ in the domain

$$\Omega(\rho_1, \rho; a): \|x\| < a, \quad \|\xi\| > \rho, \quad \rho_1 |\xi_1|^2 < |\xi_2|^2$$

in the space C_4 of the 4 complex variables ξ_1, ξ_2, x_1, x_2 . We can furthermore choose the constant a so small and another positive constant r_1 such that $r_1 \rho_1 < 1$ and that the tangent vector $[1, \Psi'(x_1)]$ to S at any point in the sphere $\|x\| < a$ lies outside the cone $r_1 |y_1|^2 < |y_2|^2$, and such that, if z is any point in the sphere $\|z\| < a$ and y is a vector in this cone, then the complex line $z + ty$ intersects S in a unique point in the sphere $\|x\| < a$. The unique t such that $z + ty \in S$ will be denoted by $q(y, z)$. This function is analytic in both y and z and is homogeneous of degree -1 in y .

Leray's transforms are integrals over a set γ of 4 real dimensions defined by

$$\gamma: \alpha_1 |\eta_1|^2 \leq |\eta_2|^2, \quad y_1 = -\alpha_1 \bar{\eta}_1, \quad y_2 = \bar{\eta}_2, \quad y + z \in S$$

$$\xi = \tau \eta, \quad x = z + ty, \quad |\tau| = \alpha_2, \quad 0 \leq t \leq 1$$

where α_2 is chosen so that $\|\xi\| > \rho$ on γ and α_1 is chosen so that $\rho_1 < \alpha_1 < 1/r_1$. Here z is a given point such that $\|z\| < a$ and $z \notin S$. The set γ can also be described by

$$\xi = \tau \eta, \quad x = z + ty, \quad y_2 = \bar{\eta}_2 = q(-\alpha_1 \sigma, 1; z)$$

$$\eta_1 = \eta_2 \sigma, \quad y_1 = -\alpha_1 \bar{\eta}_1, \quad |\sigma| \leq 1/\alpha_1^{1/2}, \quad |\tau| = \alpha_2, \quad 0 \leq t \leq 1$$

which shows that γ is topologically the product of a disk, a circle, and a closed interval. Leray defines his transform by

$$J(f) = (2\pi i)^{-2} \int_{\gamma} f(\xi, x) \exp(\xi \cdot (z - x)) d\xi_1 d\xi_2 dx_1 dx_2$$

where $\xi \cdot x = \sum \xi_j x_j$ is the usual scalar product; clearly $J(f)$ is an analytic function of z for $\|z\| < a$.

We apply this transform to functions of *polynomial growth* with respect to ξ in any domain of the type $\Omega(\rho_1, \rho; a)$; i.e., we assume that there is an integer $m(f)$, positive, negative, or zero, and a constant $A(\alpha_1, x)$ such that

$$|f(\xi, x)| \leq A(\alpha_1, x) \|\xi\|^{-m(f)}$$

in the domain $\Omega_1(\alpha_1, \rho): \|\xi\| > \rho, \alpha_1 |\xi_1|^2 < |\xi_2|^2$, for every x in $\|x\| < a$,

and $\alpha_1 > \rho_1$. The integral $J(f)$ is independent of α_1 and α_2 for f in this class if $m(f) \geq 2$. We say that f has a zero of order $s(f)$ on S if f and all its derivatives with respect to x of order $< s(f)$ are identically zero on S .

This transform has the properties:

$$(8a) \quad J(\xi_k f) = J\left(\frac{\partial f}{\partial x_k}\right) \quad \text{if} \quad m(f) > 2, \quad s(f) > 0$$

$$(8b) \quad z_k J(f) = J\left(\left(x_k - \frac{\partial}{\partial \xi_k}\right)f\right) \quad \text{if} \quad m(f) > 1$$

$$(8c) \quad \frac{\partial}{\partial z_k} J(f) = J(\xi_k f) \quad \text{if} \quad m(f) > 2$$

$$(8d) \quad s(J(f)) \geq m(f) + s(f)$$

If $m(f) < 2$, we choose any polynomial $b(\xi)$ of degree $\geq 2 - m(f)$ such that $b \neq 0$ in $\Omega(\rho_1, \rho; a)$, and set

$$J(f) = b\left(\frac{\partial}{\partial z}\right) J(b(\xi)^{-1}f)$$

For example, we may take $b(\xi) = \xi_2^k$, where $k \geq 2 - m(f)$. The value of $J(f)$ is independent of the choice of b . The above properties remain valid without any restriction on $m(f)$. We have, furthermore,

$$(8e) \quad J(f) = J(g) \quad \text{if} \quad g(\xi, x) = f(L_1 \xi, Lx), \quad \text{and} \quad (L_1 \xi) \cdot Lx = \xi \cdot x$$

$$(8f) \quad J(f(x)) = f(z)$$

If f is homogeneous in ξ of degree -1 , the formula for $J(f)$ can be simplified down to a double integral:

$$J(f) = \frac{1}{2\pi i} \int_{|\sigma|=\alpha_1^{-1}} \int_q^0 \sigma^{-1} f(\sigma, 1; z + \lambda[-\sigma^{-1}, 1]) d\lambda d\sigma$$

where $q = q(-\sigma^{-1}, 1; z)$.

In order to solve the problem (7) when B_{jk} and C_{jkn} are polynomials and u_j is to be 0 on S , we set $u_j(z) = J(f_j(\xi, x))$, which yields

$$J\left(\sum_k a_{jk} \left(x - \frac{\partial}{\partial \xi}, \xi\right) f_k\right) = v_j(z) = J(v_j(x))$$

so that, if the functions f_j are solutions of

$$(9) \quad \sum_{k=1}^M a_{jk} \left(x - \frac{\partial}{\partial \xi}, \xi\right) f_k = v_j(x)$$

of polynomial growth with respect to ξ , and if $m(f_j) \geq 1$, then u_j is the

desired solution of the Cauchy problem. If the system (7) has constant coefficients, and $h(x, \xi) = h(0, \xi) \neq 0$ in $\Omega(\rho_1, \rho; a)$, then

$$\mathbf{f}(\xi, x) = a(\xi)^{-1} \mathbf{v}(x)$$

where $a(\xi)^{-1}$ is the inverse of the matrix $(a_{jk}(\xi)) = a(\xi)$. If $B_{jk} \equiv 0$ for all j, k , then $\mathbf{f}$ is homogeneous of degree -1 in ξ , so that the simpler formula for $J(\mathbf{f})$ applies.

If the coefficients of (7) are not constant, then it is usually difficult to obtain solutions of (9) having the desired growth properties. Leray introduces a more general transform which may sometimes be easier to work with. We consider the domain

$$\Omega_2(\rho_1, \rho, c; a): (\xi, x) \in \Omega(\rho_1, \rho; a), \quad |\lambda + \xi \cdot x| < c \|\xi\|$$

in the space of the five complex variables λ, ξ , and x , and we operate on functions of polynomial growth with respect to ξ in this domain. The integrations are performed over the 5-dimensional set γ_1 defined by

$$\gamma_1: (\xi, x) \in \gamma, \quad \lambda = -\mu(\xi \cdot z) - (1 - \mu)(\xi \cdot x), \quad 0 \leq \mu \leq 1,$$

For $m(f) > 2$ we define

$$(10) \quad J(f) = (2\pi i)^{-2} \int_{\gamma_1} f(\lambda, \xi, x) \exp(\lambda + \xi \cdot z) d\lambda d\xi_1 d\xi_2 dx_1 dx_2$$

If f is independent of λ , then this coincides with the previous definition of $J(f)$. We introduce the operator V :

$$(Vf)(\xi, x) = f(-\xi \cdot x, \xi, x)$$

This transform has the following properties:

$$(11a) \quad J\left(\frac{\partial f}{\partial x_k}\right) = J(\xi_k Vf) \quad \text{if} \quad m(f) > 2, \quad s(f) > 0$$

$$(11b) \quad \frac{\partial}{\partial z_k} J(f) = J(\xi_k f) \quad \text{if} \quad m(f) > 3$$

$$(11c) \quad J(f) = J\left(\left(V - \frac{\partial}{\partial \lambda}\right)f\right) \quad \text{if} \quad m(f) > 2$$

$$(11d) \quad z_k J(f) = J\left(\left(x_k V - \frac{\partial}{\partial \xi_k}\right)f\right) \quad \text{if} \quad m(f) > 2$$

$$(11e) \quad s(J(f)) \geq m(f) + s(f)$$

As before, we extend the definition of $J(f)$ for $m(f) \leq 2$. If f has a zero of order $\geq m - 1$ on $\lambda + \xi \cdot x = 0$, and $P(z_0, z)$ is a polynomial of degree m in the 3 variables z_0, z_1, z_2 , and $p(z_0, z)$ is the principal part, i.e., the homogeneous part of the highest degree of P , then

$$p(1, z) J(f) = J \left(P \left(-\frac{\partial}{\partial \lambda}, -\frac{\partial}{\partial \xi} \right) f - p(-1, -x) V \left(\frac{\partial}{\partial \lambda} \right)^{m-1} f \right)$$

The general transform J enables us to reduce the general Cauchy problem for an equation of order m with polynomial coefficients of degree $\leq \mu$ to a special Cauchy problem for an equation of order $\leq \mu$ and polynomial coefficients of degree m .

Let $P(\xi, x_0, x)$ be a polynomial of degree m in the vector ξ and of degree μ in the other three variables, and let p be its principal part with respect to x_0, x_1 , and x_2 . Let B be its principal part with respect to ξ . We assume that $B(\xi, 1, x) \neq 0$ in the domain $\Omega_1(\rho_1, 0)$. Let w be the solution of the Cauchy problem in some domain $\Omega_2(\rho_1, \rho, c; a)$:

$$(12) \quad P \left(\xi, \frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \xi} \right) w = 0, \quad V \left(\frac{\partial}{\partial \lambda} \right)^k w = 0 \quad \text{for } k < \mu - 1$$

$$V \left(\frac{\partial}{\partial \lambda} \right)^k w = (-1)^{\mu-1} / p(\xi, 1, x) \quad \text{for } k = \mu - 1$$

Then $u(z) = J(w(\lambda, \xi, x) v(x))$ is the solution of the Cauchy problem

$$(13) \quad P \left(\frac{\partial}{\partial x}, 1, x \right) u = v, \quad s(u) \geq m$$

In this last equation, a term $\xi_1^{\alpha_1} \xi^{\alpha_2} x_1^{\alpha_3} x_2^{\alpha_4}$ in $P(\xi, 1, x)$ is to correspond to the operator

$$\frac{\partial^{\alpha_1 + \alpha_2}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} (x_3^{\alpha_3} x_2^{\alpha_4} u)$$

Note that the transform J depends only on the surface S , whereas the function w depends only on the polynomial P , so that it is easy to separate the influences of these data upon the solution. If $\mu = 1$, then (13) is of first order and its solution can be reduced to that of ordinary differential equations.

For example, consider the equation

$$a \left(\frac{\partial}{\partial x}, x \right) u = x_2 \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = v(x_1, x_2)$$

which arises in the theory of transonic flow. If we take $P(\xi, x_0, x) =$

$\xi_1^2 x_2 + \xi_2^2 x_0$, we obtain the equation

$$\xi_1^2 \frac{\partial w}{\partial \xi_2} + \xi_2^2 \frac{\partial w}{\partial \lambda} = 0$$

with the Cauchy data

$$w = 1/P(\xi, 1, x) \quad \text{for} \quad \lambda + \xi \cdot x = 0$$

The function w is analytic in the domain $\Omega_2 = \Omega_2(12, 1; (\frac{1}{3}, \frac{2}{3})^2, 11)$ and is given by the formula

$$w = 1/(x_2 \xi_1^2 + \eta^2)$$

where η is the unique solution of the cubic equation

$$\xi_2^3 - \eta^3 = 3\xi_1^2(\lambda + x_1 \xi_1 + x_2 \eta)$$

in the circle $|\eta - \xi_2| < (3/8)|\xi_2|$; actually η satisfies $|\eta - \xi_2| < |\lambda + x_1 \xi_1 + x_2 \xi_2|/3$ in the domain Ω_2 . The kernel w satisfies $|w(\lambda, \xi, x)| \leq 1/\|\xi\|^2$.

Furthermore, in this problem the above statements still hold if the condition $\|x\| < 1$ is replaced by the weaker condition $|x_2| < 1$.

EQUATIONS OF HIGHER ORDER

The Cauchy problem for equations of higher order can be reduced to that for systems of the first order. It is convenient to introduce a convenient notation for the formulation of this problem. Let

$$\alpha = [\alpha_0, \alpha_1, \dots, \alpha_N]$$

be an $(N+1)$ -dimensional vector with nonnegative integral components. With each such vector we associate the operator

$$D^\alpha = \prod_{j=0}^N D_j^{\alpha_j}, \quad D_j = \frac{\partial}{\partial x_j}, \quad x_0 = t$$

and the *order* of this operator is $|\alpha| = \sum_0^N \alpha_j$. We consider systems of the form

$$(11) \quad D_0^{n_j} u_j = f_j(t, x, u, \dots), \quad 1 \leq j \leq M$$

with Cauchy data given on the hyperplane $t = 0$:

$$(15) \quad D_0^k u_j(0, x) = \phi_{jk}(x), \quad 0 \leq k < n_j$$

Here x denotes the point $(x_1, \dots, x_N)$ in the space C_N of N complex variables, and u denotes the point $(u_1, \dots, u_M)$ in C_M . We assume that the functions f_j are analytic in t, x, u , and all $D^\alpha u_k$ for $|\alpha| \leq n_k$, $\alpha_0 < n_k$,

and that the functions ϕ_{jk} are analytic in x . We introduce new unknowns $u_{j,\alpha}$, where $|\alpha| < n_j$, and consider the system

$$(16a) \quad D_0 u_{j,\alpha} = u_{j,\alpha_0+1,\alpha_1,\dots,\alpha_N}, \quad |\alpha| < n_j - 1$$

$$(16b) \quad D_0 u_{j,\alpha} = D_k u_{j,\alpha_0+1,0,\dots,\alpha_k-1,\alpha_{k+1},\dots,\alpha_N}, \\ \alpha_0 < n_j - 1, \quad |\alpha| = n_j - 1$$

$$(16c) \quad D_0 u_{j,\alpha} = f_j(t, x, u_{1,0}, \dots), \quad |\alpha| = n_j - 1 \quad \alpha_0 = n_j - 1$$

with the initial conditions

$$(16d) \quad u_{j,\alpha}(0, x) = \prod_{k=1}^N D_k^{\alpha_k} \phi_j \alpha_0$$

The problems (14), (15) and (16a-d) are equivalent. Note, however, that while every solution of (14) yields a solution of (16a-c), only solutions of the latter which satisfy initial conditions of the form (16d) arise from solutions of (14).

As an example of this reduction, consider the wave equation in two special dimensions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

$$u(0, x_1, x_2) = \phi_0(x_1, x_2), \quad u_t(0, x_1, x_2) = \phi_1(x_1, x_2)$$

We introduce the new unknowns u_{000} , u_{001} , u_{010} , and u_{100} and form the system

$$\frac{\partial u_{000}}{\partial t} = u_{100}, \quad \frac{\partial u_{010}}{\partial t} = \frac{\partial u_{100}}{\partial x_1}, \quad \frac{\partial u_{001}}{\partial t} = \frac{\partial u_{100}}{\partial x_2}, \\ \frac{\partial u_{100}}{\partial t} = \frac{\partial u_{010}}{\partial x_1} + \frac{\partial u_{001}}{\partial x_2}$$

with the initial conditions

$$u_{000} = \phi_0, \quad u_{100} = \phi_1, \quad u_{010} = \frac{\partial \phi_0}{\partial x_1}, \quad u_{001} = \frac{\partial \phi_0}{\partial x_2}$$

for $t = 0$.

The system (14), (15), is said to be in *normal form*. With the use of the transformation on p. 47 we can reduce the Cauchy problem for a system of higher order equations with data given on an analytic hypersurface to one in which the data are given on a hyperplane, and, if a certain determinant is different from zero, then the system can be put in normal form. Consider the system

$$f_j(x, u, \dots, D^\alpha u_k, \dots) = 0, \quad 1 \leq j \leq M$$

where the functions f_j are analytic in the variables $x \in C_N, u \in C_M$, and all derivatives $p_{k\alpha} = D^\alpha u_k$ with $|\alpha| \leq n_k, 1 \leq k \leq M$. (Now $\alpha = [\alpha_1, \dots, \alpha_N]$ ranges over N -dimensional vectors with nonnegative integral components, and n_k is the highest order of the derivatives of u_k which occur in the equations.) The functions u_k and their normal derivatives of order $< n_k$ are given on an analytic hypersurface $S: s(x) = 0$, where s is analytic and $\nabla s \neq 0$ in some domain. Then this Cauchy problem can be put in normal form, provided that the characteristic form

$$h(x, p, \xi) = \det \left(\sum_{|\alpha|=n_k} \frac{\partial f_j}{\partial p_{k,\alpha}} \xi^\alpha \right)$$

is $\neq 0$ for $\xi = \nabla s$. Here $\xi^\alpha = \prod_1^N \xi_j^{\alpha_j}$ is the monomial corresponding to the differential operator D^α . For example, the characteristic form of the single linear differential equation

$$Q(x, D)u - b(x) = \sum_{|\alpha| \leq m} q_\alpha(x) D^\alpha u - b(x) = 0$$

where the coefficients are given analytic functions of x , is the *principal part* of Q :

$$h(x, \xi) = q(x, \xi) = \sum_{|\alpha|=m} q_\alpha(x) \xi^\alpha$$

In the study of partial differential equations in the real domain and in the large, the most important step is their classification by means of the characteristic form.

The basic existence theorems, in the above general form, no longer hold if the restrictions on (14), (15) are relaxed. The restrictions on the orders of the derivatives $D^\alpha u_j$ which appear on the right-hand side are essential. For example, if f is analytic in a neighborhood of the origin and $f''(0) \neq 0$, then the equation

$$u_t = x^3 u_{xx} + f(x)$$

has no solution analytic in a neighborhood of the origin. The heat equation, which has constant coefficients, with Cauchy data given at $t = 0$:

$$u_t = u_{xx}, \quad u(0, x) = \phi(x)$$

has no solution analytic in a neighborhood of the origin unless ϕ is entire and satisfies a growth condition of the form $|\phi(x)| \leq \exp(K|x|^2)$ for all sufficiently large $|x|$. The restriction to analyticity of the functions

occurring in the problem is also essential. Thus the Cauchy problem for the Laplace equation:

$$u_{tt} = -u_{xx}, \quad u(0, x) = \phi_0(x), \quad u_t(0, x) = \phi_1(x) = 0$$

has no solution even in a half-neighborhood of the origin: $t \geq 0$, $x^2 + t^2 \leq r^2$, unless ϕ_0 is analytic. Even for equations with constant coefficients, the Cauchy-Kovalevski existence theorem is essentially local. Consider the Cauchy-Riemann equations:

$$\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} = 0, \quad \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0$$

with the Cauchy data

$$u_1 = x_1, \quad u_2 = -x_2$$

for $s(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$. The characteristic form is $h = \xi_1^2 + \xi_2^2$, and $h(\nabla s) = 4$ on the manifold S . But the solution is

$$u_1 = \frac{x_1}{x_1^2 + x_2^2}, \quad u_2 = -\frac{x_2}{x_1^2 + x_2^2}$$

which is singular at the origin.

We must omit for lack of space the theories of Cartan and Riquier (see Cartan [116], Kähler [358], Schouten and van der Kulk [602], Ritt [575]) of other problems besides the Cauchy problem. So far these theories are only local theories applying to problems where all functions involved are analytic. At present, only special problems have been treated in the large and in the real domain without assumptions of analyticity (see for example, Gårding [262], [263], Ingersoll [345], [346], Meltzer [439]). We must also forego a description of the work of Ritt and his pupils (see Ritt [575]) on the algebraic aspects of partial differential equations. It is quite probable that this work has important implications for the analytic theory, but so far no one has established such a connection. There is also some interesting work by Salekhov and Fridlender [588] and Pucci [560] on differential equations on the domain of infinitely differentiable functions.

GENERAL THEORY IN THE REAL DOMAIN

UNIQUENESS AND DOMAINS OF DEPENDENCE FOR THE CAUCHY PROBLEM

Let us now turn to a discussion of the Cauchy problem in the real domain. We begin with a system of linear equations (6) with analytic coefficients. John [351] has given an extension of a method due to Holmgren which leads to powerful uniqueness theorems. Let the operators L and L^* be defined by

$$(L\mathbf{u})_j = \frac{\partial u_j}{\partial t} - \sum_{k=1}^M B_{jk} u_k - \sum_{k=1}^M \sum_{n=1}^N C_{jkn} \frac{\partial u_k}{\partial x_n}$$

$$(L^*\mathbf{v})_k = \frac{\partial v_k}{\partial t} - \sum_{j=1}^M v_j B_{jk} - \sum_{j=1}^M \sum_{n=1}^N \frac{\partial}{\partial x_n} (v_j C_{jkn})$$

which give the components of the vectors $L\mathbf{u}$ and $L^*\mathbf{v}$. Let $S_\lambda: s(t, x, \lambda) = 0$ be a one-parameter family of surfaces, where s is analytic in all its variables, $s(t, x, 0) \equiv t$, $\nabla s \neq 0$, and

$$h(t, x, \nabla s) = \det \left(\delta_{jk} \frac{\partial s}{\partial t} - \sum_{n=1}^N C_{jkn} \frac{\partial s}{\partial x_n} \right) \neq 0$$

for $0 \leq \lambda \leq a$, and (t, x) in a neighborhood U of a region R in the hyperplane $t = 0$. We assume that the boundaries of all the surfaces S_λ are the same $(N - 1)$ -dimensional manifold lying in the region R . Let $V_{\lambda\mu}$ be the domain bounded by the surfaces S_λ and S_μ for $0 \leq \lambda < \mu \leq a$. By Green's theorem applied to the region $V_{\lambda\mu}$, we obtain for any solution

$\mathbf{u}$ of class C^1 of (6) and $\mathbf{v}$ of $L^*\mathbf{v} = 0$ the equation

$$J(\lambda) = \int_{S_\lambda} M(\mathbf{u}, \mathbf{v}, \xi) dA - \int_{V_{0\lambda}} (\mathbf{v} \cdot \mathbf{A}) dV = J(\mu)$$

where

$$M(\mathbf{u}, \mathbf{v}, \xi) = \sum_{k=1}^M u_k \left(v_k \xi_0 - \sum_{j=1}^M \sum_{n=1}^N v_j C_{jkn} \xi_n \right)$$

and the vector $\xi = [\xi_0, \dots, \xi_N]$ is the unit exterior normal to S_λ , $\xi = \nabla s / \|\nabla s\| = \nabla s / ((\partial s / \partial t)^2 + \sum_{\mathbf{i}} (\partial s / \partial x_k)^2)^{1/2}$. Thus the integrals $J(\lambda)$ are constant and can be evaluated from

$$J(0) = \int_{S_0} M(\mathbf{u}, \mathbf{v}, \xi) dA = \int_{S_0} \left(\sum_{j=1}^M \phi_j v_j \right) dx_1, \dots, dx_N$$

which depends only on the known functions ϕ_k . Let $\mathbf{q} = [q_1, \dots, q_M]$ be an arbitrary vector whose components are analytic in a neighborhood of the surface S_λ . We choose $\mathbf{v}$ as the solution of the Cauchy problem for $L^*\mathbf{v} = 0$ with data given on S_λ which satisfy

$$v_k \xi_0 - \sum_{j=1}^M \sum_{n=1}^N v_j C_{jkn} \xi_n = q_k, \quad 1 \leq k \leq M$$

The solution is uniquely determined since $h(t, x, \xi) \neq 0$. Thus, if $\mathbf{v}$ exists in the region $V_{0\lambda}$, then the integral $I(\lambda) = \int_{S_\lambda} \mathbf{u} \cdot \mathbf{q} dA$ can be com-

puted from $\mathbf{A}$ and the Cauchy data on R . Now we recall from Theorem 2 that the domain of existence of the solution of a Cauchy problem for a linear system depends only on the coefficients C_{jkn} of the highest order derivatives. Hence we find that there exists a constant b such that the solution of the Cauchy problem with data given on S_λ exists in the region $V_{\lambda-b, \lambda+b}$ for $0 \leq \lambda \leq a$. The constant b depends only on C_{jkn} and s , and not on $\mathbf{q}$. Therefore the integral $I(\lambda)$ can be computed for $0 \leq \lambda \leq b$. Since $\mathbf{q}$ is an arbitrary function analytic on S_λ , the solution $\mathbf{u}$ is uniquely determined on S_λ . For example, we could construct a complete orthonormal system of the vectors on S with polynomial components by the Gram-Schmidt process and compute $\mathbf{u}$ from its Fourier series. The argument can be repeated for the regions $V_{b, 2b}$, $V_{2b, 3b}$, etc., and so $\mathbf{u}$ can be determined in the whole region V_{0a} .

Thus $\mathbf{u}$ is uniquely determined at all points which can be reached by deforming S_0 analytically, keeping its boundary fixed, without $h(t, x, \xi)$ ever equaling zero, i.e., without S_λ ever having a characteristic direction. Furthermore, we have, in principle, a method of actually calculating $\mathbf{u}$. The analyticity of the coefficients B_{jk} and C_{jkn} was used only to insure the

existence of sufficiently many solutions of $L^*\mathbf{v} = 0$. If L^* belongs to a class of operators such that the solution of the Cauchy problem with data on S_λ exists for a complete set of vectors $\mathbf{q}$, then the same conclusion holds.

The question of characterizing the set $\Omega(S_0)$ of all points attainable by deformations of the above type of the surface S_0 is a geometrical problem. The simplest case is that in which $h \neq 0$ for all real ξ and all points (t, x) in a domain Ω . Then $\Omega(S_0) = \Omega$, no matter how small S_0 is. Hence the values of $\mathbf{u}$ on an arbitrarily small region of the plane $t = 0$ uniquely determine $\mathbf{u}$ throughout Ω and, therefore, also on the rest of the hyperplane $t = 0$.

These considerations apply to systems of arbitrary order. If the characteristic form $h \neq 0$ for all real $\xi \neq 0$ and all points x in a domain Ω , then we say that the system is *elliptic* in Ω . We have the result:

Theorem 3. The solution u of a linear elliptic system of equations with analytic coefficients in a domain Ω is uniquely determined by its Cauchy data on an arbitrarily small open set on an analytic hypersurface in Ω .

Corollary 3a. If a solution of a homogeneous linear elliptic system of equations with coefficients analytic in a domain Ω vanishes in an arbitrarily small neighborhood of a point in Ω , then it vanishes identically.

Until recently, corresponding theorems for systems with nonanalytic coefficients were known only for 2 dimensions (see Carleman [115], Bers [40]) and for special types of second-order equations in higher dimensions (Müller [473]). Aronszajn [15] has now proved such uniqueness theorems for general second-order equations, and his proof has been simplified by Pederson. In unpublished work Calderón [102] has proved these theorems for equations and systems of arbitrary order with some minor exceptional cases. These results have important implications for the differential geometry of smooth manifolds (see Spencer [629]).

As another example, we can take S_0 to be a closed convex domain with analytic boundary in the hyperplane $t = 0$, and consider the cone γ formed by joining a point $P = (t_0, x^0)$, $t_0^0 > 0$, to every point of S_0 . If $h(t, x, \xi) \neq 0$ for every point (t, x) in γ and every real vector ξ such that $\xi_0(t - t_0) + \sum \xi_k(x_k - x_k^0) \geq 0$ for all $(t, x) \in \gamma$, then $\gamma \subset \Omega(S_0)$. Again these remarks also apply to systems of higher order. We thus obtain a generalization of the familiar phenomenon that the solution u of the wave equation

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \sum_{k=1}^3 \frac{\partial^2 u}{\partial x_k^2} = 0, \quad u(0, x) = \phi_0(x), \quad u_t(0, x) = \phi_1(x)$$

at the point (t_0, x^0) , $t_0 > 0$, depends only on the values of ϕ_0 and ϕ_1 at points in the retrograde light cone

$$c^2(t - t_0)^2 - \sum_{k=1}^3 (x_k - x_k^0)^2 \geq 0, \quad t_0 - t \geq 0$$

In the case of an elliptic system, we saw that every point of $t = 0$ was accessible by deformations of the above type of an arbitrarily small piece S_0 . Hence the Cauchy data on $t = 0$ are not arbitrary; the values of ϕ in an arbitrarily small domain in $t = 0$ determine ϕ on the whole hyperplane. This is an instance of a quite general phenomenon, which also holds for arbitrary linear systems with analytic coefficients.

Theorem 4. Consider a linear system of equations with analytic coefficients and an analytic hypersurface S . If x^0 is a point on S such that in the space of the vectors ξ there is a line through the point ξ^0 , the unit normal to S at x^0 , which does not intersect the characteristic cone $h(x^0, \xi) = 0$ at x^0 , then in every open subset R of S containing x^0 , there are two open subsets R_1 and R_2 such that the Cauchy data in R_1 uniquely determine the Cauchy data in R_2 .

For example, consider the wave equation and the "time-like" hyperplane $S: x_1 = 0$. There are obviously lines through the normal vector $\xi^0 = [0, 1, 0, 0]$ which do not intersect the characteristic cone $h(x, \xi) = c^{-2}\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0$. Hence the Cauchy data on S cannot be arbitrarily prescribed. Similarly, the Cauchy problem for the ultrahyperbolic equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2} = 0$$

cannot be solved, in general, with nonanalytic data given on any hyperplane (see Hadamard [308] p. 347–357, Asgeirsson [21]).

For an interesting example of a Cauchy problem for which the solution is not unique, see Myškis [478], Pliš [542].

PRELIMINARIES ON FUNCTION SPACES

1. Testing spaces

Modern methods for the study of partial differential equations are based on the techniques of functional analysis, i.e., the theory of operators on function spaces. We shall now describe some of the important function spaces used in partial differential equations.

If A is any set in the N -dimensional Euclidean space E_N , then by $C^\infty(A)$ we mean the space of complex-valued functions ϕ which have

continuous partial derivatives of all orders in an open set containing A . If $\{\phi_n\}$ is a sequence in $C^\infty(A)$, then we say that $\phi_n \rightarrow 0$ if and only if there is an open set $\Omega \supset A$ such that $D^\alpha \phi_n \rightarrow 0$ uniformly on Ω , for each α . If Ω is a domain in E_N , then by $C_0^\infty(\Omega)$ we mean the subspace of $C^\infty(\Omega)$ consisting of those ϕ such that there is a bounded, closed subset $\Lambda(\phi)$ of Ω outside of which $\phi \equiv 0$. A sequence $\{\phi_n\}$ in $C_0^\infty(\Omega)$ is said to converge to 0 if $\phi_n \rightarrow 0$ in $C^\infty(\Omega)$ and there is a bounded, closed subset Λ of Ω such that $\Lambda(\phi_n) \subset \Lambda$ for all n . If $\Omega = E_N$, it is sometimes more convenient to work with the space $C_1^\infty(E_N)$ of all $\phi \in C^\infty(E_N)$ such that for every $k \geq 0$ and every $\alpha \geq 0$, $\|x\|^k D^\alpha \phi(x)$ is bounded in E_N . The sequence $\{\phi_n\}$ is said to converge to 0 in $C_1^\infty(E_N)$ if $\|x\|^k D^\alpha \phi_n \rightarrow 0$ uniformly in E_N for each $k \geq 0$ and $\alpha \geq 0$. These spaces are all *topological linear vector spaces*. (See Schwartz [603].)

The spaces $C_1^\infty(E_N)$ and $C_0^\infty(E_N)$ are examples of *testing spaces* (Gelfand and Šilov [277]). A testing space $\mathcal{S}$ is a linear subspace of $C_1^\infty(E_N)$, together with a definition of convergence of sequences (not necessarily the same as that of $C_1^\infty(E_N)$), satisfying the following conditions:

- (17a) if $\phi_m - \phi_n \rightarrow 0$ in $\mathcal{S}$, then there is a $\phi \in \mathcal{S}$ such that $\phi_n - \phi \rightarrow 0$;
- (17b) if $\phi \in \mathcal{S}$, $y \in E_N$, and λ is real and $\neq 0$, then $x_j \phi$, $D_j \phi$, $\phi(x + y)$, $\exp(i(x \cdot y))\phi(x)$, and $\phi(\lambda x) \in \mathcal{S}$;
- (17c) These operations and the algebraic operations $\beta\phi$ and $\phi_1 + \phi_2$ are continuous on $\mathcal{S}$.

Besides the two spaces $C_1^\infty(E_N)$ and $C_0^\infty(E_N)$, emphasized by Schwartz, other useful testing spaces are:

1. The space $\mathcal{K}_p(p > 1)$ of $\phi \in C^\infty(E_N)$ such that for $\alpha \geq 0$ there are constants $C(\alpha) > 0$ and $C_1(\alpha)$ such that $|D^\alpha \phi(x)| \leq C_1(\alpha) \exp(-C(\alpha) \|x\|^p)$ for all $x \in E_N$. We say that $\phi_n \rightarrow 0$ if $D^\alpha \phi_n \rightarrow 0$ uniformly in E_N for each $\alpha \geq 0$, and if there exists constants $C(\alpha) > 0$ and $C_1(\alpha)$ such that $|D^\alpha \phi_n(x)| \leq C_1(\alpha) \exp(-C(\alpha) \|x\|^p)$ for all n and all $x \in E_N$.
2. The space $\mathcal{Z}^p(p \geq 1)$ of entire functions ϕ on E_N which, on E_N , are in $C_1^\infty(E_N)$ and such that for any polynomial P there are constants $C_1(P)$ and $C(P)$ such that

$$I(P\phi; y) = \int_{E_N} |P(x + iy) \phi(x + iy)|^2 dV_x \leq C_1(P) \exp(C(P) \|y\|^p)$$

for all $y \in E_N$. We say that $\phi_n \rightarrow 0$ in $\mathcal{Z}^p$ if $\phi_n \rightarrow 0$ uniformly in every bounded domain, $I(P\phi_n, y) \rightarrow 0$ for each P and y , and if there are constants $C(P)$ and $C_1(P)$ such that $I(P\phi_n, y) \leq C_1(P) \exp(C(P) \|y\|^p)$, for all n and all $y \in E_N$.

3. The space Z_q^p ($p > 1, p \geq q > 0$) of entire functions ϕ on C_N which, on E_N , are in $C_1^\infty(E_N)$ and such that there are constants C_1, C_2, C_3 , and $C_4 > 0$ such that $|\phi(z)| \leq C_1 \exp(C_2 \|z\|^p)$ for all $z \in C_N$, and $|\phi(x)| \leq C_3 \exp(-C_4 \|x\|^q)$ for all $x \in E_N$. We say that $\phi_n \rightarrow 0$ in Z_q^p if $\phi_n \rightarrow 0$ uniformly in every bounded domain in C_N and if there exist constants $C_1, C_2, C_3, C_4 > 0$ such that $|\phi_n(z)| \leq C_1 \exp(C_2 \|z\|^p)$ for all $z \in C_N$ and $|\phi_n(x)| \leq C_3 \exp(-C_4 \|x\|^q)$ for all $x \in E_N$, and all n .

If $\mathcal{S}$ is a testing space, then by $\mathcal{S}(M_1 \times M_2)$ we mean the set of all $M_1 \times M_2$ matrix valued functions $\phi = (\phi_{jk}(x))$ such that $\phi_{jk} \in \mathcal{S}$ for $1 \leq j \leq M_1, 1 \leq k \leq M_2$.

In these and in most other useful cases the testing space $\mathcal{S}$ is dense in $C_1^\infty(E_N)$ with respect to the topology of the latter space.

2. Dual spaces and distributions

If S is a topological linear vector space, then we can define its *dual* space S^* as the space of linear functionals on S . A linear functional s^* on S is a continuous linear scalar-valued function on S :

$$s^*(\lambda_1 \phi_1 + \lambda_2 \phi_2) = \lambda_1 s^*(\phi_1) + \lambda_2 s^*(\phi_2)$$

$$s^*(\lim_{n \rightarrow \infty} \phi_n) = \lim_{n \rightarrow \infty} s^*(\phi_n)$$

for any scalars λ_1, λ_2 and elements $\phi_1, \phi_2 \in S$, and for every convergent sequence $\{\phi_n\}$ in S . The space S^* is also a topological linear vector space with the natural topology $s_n^* \rightarrow s^*$ if and only if $s_n^*(\phi) \rightarrow s^*(\phi)$ for all $\phi \in S$. The elements of S^* , where $\mathcal{S}$ is a testing space, are called *distributions*. These were introduced essentially by Bochner [58], but were exploited systematically for the first time by Schwartz [603]. For expositions of the theory of distributions see Schwartz [603], Korevaar [381], Temple [645], Halperin [310], Friedman [237], Gelfand and Šilov [278], Gelfand and Šapiro [276]. A related theory with certain advantages was developed independently by Mikusiński [443] (see Erdélyi lecture notes).

If $\mathcal{S}$ is a testing space, then the function f is said to be a *multiplier* of $\mathcal{S}$ if $f\phi \in \mathcal{S}$ for all $\phi \in \mathcal{S}$, and $f\phi_n \rightarrow 0$ whenever $\phi_n \rightarrow 0$ in $\mathcal{S}$. The set of multipliers of $\mathcal{S}$ forms a ring, denoted by $\mathfrak{M}(\mathcal{S})$. If $f \in \mathfrak{M}(\mathcal{S})$ and $T \in S^*$, then fT is the distribution defined by

$$(fT)(\phi) = T(f\phi)$$

If $P(\xi)$ is a polynomial of E_N and $D = [D_1, \dots, D_N]$ denotes the sym-

bolie vector with components $D_j = \partial/\partial x_j$, then we define the differential operator $P(D)$ on S^* by

$$(P(D)T)(\phi) = T(P(-D)\phi)$$

This formula is suggested by the special case where

$$T\phi = \int_{E_N} \Psi \phi \, dV_x = [\Psi]\phi$$

where $\Psi \in C_0^\infty(E_N)$. Then $P(D)[\Psi] = [P(D)\Psi]$ by Green's formula.

The *Dirac delta-distribution* δ_{x_0} is defined by

$$\delta_{x_0}(\phi) = \phi(x_0)$$

If $A(x, D)u = \Sigma D^\alpha(a_{\alpha\beta}(x)D^\beta u)$ is a differential operator with coefficients $a_{\alpha\beta} \in \mathfrak{K}(S)$, then a *fundamental solution* F_{x_0} of A at the point x_0 is a distribution $F_{x_0} \in S^*$ such that

$$(18) \quad A'(x, D)F_{x_0} = \Sigma(-1)^{|\alpha|+|\beta|} D^\beta(a_{\alpha\beta}(x)D^\alpha F_{x_0}) = \delta_{x_0}$$

If F_{x_0} is such a fundamental solution, then to compute $\phi(x_0)$, where ϕ is the solution in S of $A(x, D)\phi = \Psi$, we merely form

$$F_{x_0}(\Psi) = \delta_{x_0}(\phi) = \phi(x_0)$$

Thus the problem of solving a partial-differential equation under these conditions can be reduced to the determination of a fundamental solution. This concept goes back essentially to Fredholm [234] and Zeilon [734], and was first formulated in this generality by Schwartz [603].

In some cases the determination of a solution of (18) leads to difficulties because of the singularity of the right-hand side. In such cases we may represent δ_{x_0} as a limit, in the distribution sense, of smooth functions. For example, if we can determine a solution $[F(x_0, x; t)]$ in S^* of the equation

$$A'(x, D) F(x_0, x; t) = K(x - x_0, t)$$

where

$$K(x, t) = (4\pi t)^{-N/2} \exp(-\|x\|^2/4t), \quad t > 0$$

then the solution in S of $A(x, D)\phi = \Psi$ can be represented in the form

$$\phi(x) = \lim_{t \rightarrow 0} \int_{E_N} F(x, y, t) \Psi(y) \, dV_y$$

Representations of this kind have been used by John [352], [356] and others. The Fourier and Laplace transform techniques described in the next chapter can be regarded as special cases of this general method.

Suppose $T_1, T_2 \in \mathcal{S}^*$, and T_2 has the property that for each $\phi \in \mathcal{S}$ the function $\Psi(y) = T_2(\phi(x+y))$ is, as a function of y , also in $\mathcal{S}$. Then we define

$$(T_2 * T_1)\phi = (T_1 * T_2)\phi = T_1(\Psi_1) = T_{1,y}(T_{2,x}(\phi(x+y)))$$

where the subscripts on the right indicate the variables with respect to which the distributions operate. The distribution $T_1 * T_2$ is called the *convolution* of T_1 and T_2 . Let $\mathcal{G}(\mathcal{S}^*)$ denote the class of all $T \in \mathcal{S}^*$ such that $T_y(\phi(x+y)) \in \mathcal{S}$ for all $\phi \in \mathcal{S}$. If $T_1, \dots, T_k \in \mathcal{S}^*$ and all but one are in $\mathcal{G}(\mathcal{S}^*)$, then the convolution $T_1 * \dots * T_k$ is defined and is independent of the order and the grouping of the factors. If $T_j = [\Psi_j]$, where $\Psi_j \in C_0^\infty(E_N)$, $j = 1, 2$, then

$$T_1 * T_2 = [\Psi_1 * \Psi_2]$$

where

$$(\Psi_1 * \Psi_2)(x) = \int_{E_N} \Psi_1(x-y) \Psi_2(y) dV_y$$

The Dirac delta is in $\mathcal{G}(\mathcal{S}^*)$ for every testing space $\mathcal{S}$, and $\delta_0 * T = T * \delta_0 = T$ for all $T \in \mathcal{S}^*$. If $P(\xi)$ is a polynomial on E_N and F is a fundamental solution of $P(D)$ in $\mathcal{S}$ at the origin, then the solution in $\mathcal{S}^*$ of $P(-D)T = S$ is $T = F * S$ if for $S \in \mathcal{G}(\mathcal{S}^*)$. Here we have used the trivial formula

$$Q(D)(T_1 * T_2) = (Q(D)T_1) * T_2 = T_1 * Q(D)T_2$$

for any polynomial Q , if one of T_1, T_2 is in $\mathcal{G}(\mathcal{S}^*)$.

3. Banach and Hilbert spaces

If S is a topological linear vector space, then a *norm* N on S is a real valued function on S satisfying:

- (a) $N(\phi) \geq 0$ for all $\phi \in S$, $N(\phi) = 0$ if and only if $\phi = 0$;
- (b) $N(\lambda\phi) = |\lambda|N(\phi)$ for every scalar λ and $\phi \in S$;
- (c) $N(\phi_1 + \phi_2) \leq N(\phi_1) + N(\phi_2)$ for $\phi_1, \phi_2 \in S$.

A sequence $\{\phi_n\}$ in S is said to converge to ϕ , $\lim_{n \rightarrow \infty} \phi_n = \phi$, if $N(\phi_n - \phi) \rightarrow 0$. The space S is *complete* with respect to the norm N if every sequence $\{\phi_n\}$ in S such that $\lim_{m,n \rightarrow \infty} N(\phi_m - \phi_n) = 0$ is convergent. A complete normed linear vector space is called a *Banach* space (see Banach [23]). If S is a normed linear vector space, then there is an essentially unique Banach space S_1 with a norm N_1 such that $S \subset S_1$, $N(\phi) = N_1(\phi)$ for all $\phi \in S$, and S is dense in S_1 . The space S_1 is called the *completion* of S with respect to the norm N .

The simplest kind of norm arises from a *scalar product*. The scalar valued function (ϕ_1, ϕ_2) on S is called a scalar product if:

(a) $(\phi_1, \lambda_2\phi_2 + \lambda_3\phi_3) = \lambda_2(\phi_1, \phi_2) + \lambda_3(\phi_1, \phi_3)$ for λ_2, λ_3 scalars and $\phi_1, \phi_2, \phi_3 \in S$;

(b) $(\phi_1, \phi_2) = \overline{(\phi_2, \phi_1)}$ for $\phi_1, \phi_2 \in S$;

(c) $(\phi, \phi) \geq 0$ for $\phi \in S$, and $(\phi, \phi) = 0$ if and only if $\phi = 0$.

The space S is called a pre-Hilbert space with respect to this scalar product. The *Hilbert norm* on S is defined by $N(\phi) = \|\phi\| = +(\phi, \phi)^{1/2}$. If the space S is complete with respect to the Hilbert norm, then it is called a Hilbert space. For more details on Banach and Hilbert spaces see Stone [632], Halmos [309], Riesz and Nagy [572].

For example, the completion of the space $C_0^\infty(\Omega)$ with respect to the norm

$$M_{m,\Omega}(\phi) = \sup_{|\alpha| \leq m, x \in \Omega} |D^\alpha \phi(x)|$$

is the space $C_0^m(\Omega)$ of functions f of class C^m in $\bar{\Omega}$ such that $D^\alpha f(x) = 0$ for $x \in \partial\Omega$, the boundary of Ω , and $|\alpha| \leq m$. Similarly, the completion of $C^\infty(\bar{\Omega})$ with respect to $M_{m,\Omega}$ is the space $C^m(\bar{\Omega})$ of functions of class C^m on $\bar{\Omega}$. The completions of $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ with respect to the norm

$$L_{p,\Omega}(\phi) = \left\{ \int_\Omega |\phi(x)|^p dV_x \right\}^{1/p}$$

where $1 \leq p < +\infty$, coincide and can be identified with the space $\mathfrak{L}_p(\Omega)$ of measurable functions f on Ω such that $|f|^p$ is Lebesgue integrable. The special case $\mathfrak{L}_2(\Omega)$ is a Hilbert space with respect to the scalar product

$$(f, g)_\Omega = \int_\Omega \overline{f(x)} g(x) dV_x$$

4. Linear operators

If S_1 and S_2 are Banach spaces with norms N_1 and N_2 , then a *linear operator* L on S_1 to S_2 is a function defined on a linear manifold $\mathfrak{D}_L \subset S_1$ with values forming a linear manifold $\mathfrak{R}_L$ in S_2 such that for $\phi_1, \phi_2 \in \mathfrak{D}_L$ and λ_1, λ_2 arbitrary scalars

$$L(\lambda_1\phi_1 + \lambda_2\phi_2) = \lambda_1L\phi_1 + \lambda_2L\phi_2$$

The set $\mathfrak{D}_L$ is called the *domain* of L and $\mathfrak{R}_L$ is called the *range* of L . The operator L is said to be bounded if

$$N_{1,2}(L) = \sup_{\phi \in S_1, \phi \neq 0} (N_2(L\phi)/N_1(\phi)) < +\infty$$

and $N_{1,2}$ is called the norm of L . If $\mathfrak{D}_L = S_1$ and $N_{1,2}(L) < +\infty$, then L is called a *linear transformation* on S_1 to S_2 . The linear transformations on S_1 to S_2 form a Banach space, denoted by $S_2^{S_1}$, with the norm $N_{1,2}$.

Theorem 5 (Banach-Steinhaus). A linear operator L on S_1 to S_2 with $\mathfrak{D}_L$ dense in S_1 can be extended to a linear transformation on S_1 to S_2 if and only if $N_{1,2}(L) < +\infty$. In that case the extension is unique.

The operator L is said to be *closed* if for every sequence $\{\phi_n\}$ in $\mathfrak{D}_L$ such that both $\lim_{n \rightarrow \infty} \phi_n = \phi$ and $\lim_{n \rightarrow \infty} L\phi_n = \Psi$ exist, we have $\phi \in \mathfrak{D}_L$ and $L\phi = \Psi$. The operator L is *preclosed* if there is a closed operator $\bar{L}$ such that $\bar{L}\phi = L\phi$ on $\mathfrak{D}_L$, and $\mathfrak{D}_L$ is dense in $\mathfrak{D}_{\bar{L}}$. The operator $\bar{L}$, which is unique, is called the *closure* of L . It exists if and only if whenever $\{\phi_n\} \subset \mathfrak{D}_L$, $\phi_n \rightarrow 0$, and $\lim_{n \rightarrow \infty} L\phi_n = \Psi$, then $\Psi = 0$. If L is a preclosed operator on S_1 to S_2 with domain $\mathfrak{D}_L$, then $\bar{L}$ may be identified with the extension of L to a linear transformation on the completion of $\mathfrak{D}_L$ with respect to the norm $N_3(\phi) = (N_1(\phi)^2 + N_2(L\phi)^2)^{1/2}$.

Theorem 6 (closed graph theorem). A linear operator L on S_1 to S_2 with domain $\mathfrak{D}_L$ dense in S_1 can be extended to a closed linear operator on all of S_1 if and only if $N_{1,2}(L) < +\infty$.

If S_2 is the space of scalars, then $S_2^{S_1}$ is the *conjugate* space S_1^* of S_1 , and the norm $N_{1,2}(\phi^*) = N_1^*(\phi^*)$ makes S_1^* a Banach space. If S_1 and S_2 are arbitrary, then to every linear operator L on S_1 to S_2 there is associated in a natural way an operator L^* on S_2^* to S_1^* defined by

$$(L^*\phi^*)(\phi) = \phi^*(L\phi)$$

for $\phi \in \mathfrak{D}_L$, $\phi^* \in \mathfrak{D}_{L^*}$. This operator is called the *adjoint* of L . The domain of L^* is the set of $\phi^* \in S_1^*$ for which there is a constant C such that $|\phi^*(L\phi)| \leq CN_1(\phi)$ for all $\phi \in \mathfrak{D}_L$. If $\mathfrak{D}_L$ is dense in S_1 , then $L^*\phi^*$ is uniquely defined.

The following slight variants of theorems of Banach [23] are often useful.

Theorem 7a. If S_1 and S_2 are Banach spaces and L is a linear operator with dense domain in S_1 , and if there is a constant C such that $N_1(\phi) \leq CN_2(L\phi)$ for all $\phi \in \mathfrak{D}_L$, then $\mathfrak{R}_{L^*} = S_1^*$.

Theorem 7b. If $\mathfrak{R}_{L^*} = S_1^*$, then there is a C such that $N_1(\phi) \leq CN_2(L\phi)$ for all $\phi \in \mathfrak{D}_L$, and $\overline{\mathfrak{R}_L}$ is the set of $\phi_2 \in S_2$ such that $\phi_2^*(\phi_2) = 0$ for all ϕ_2^* such that $L^*\phi_2^* = 0$.

Theorem 8a. If S_1 and S_2 are Banach spaces and L is a preclosed operator on S_1 to S_2 with dense domain, and if $\mathfrak{R}_L = S_2$, then there is a constant C such that $N_2^*(\phi_2^*) \leq CN_1^*(L^*\phi_1^*)$ for $\phi_1^* \in \mathfrak{D}_{L^*}$, and $\mathfrak{R}_{L^*}$ is the set of $\phi_1^* \in S_1^*$ such that $\phi_1^*(\phi_1) = 0$ for all ϕ_1 such that $L\phi_1 = 0$.

Theorem 8b. If there is constant C such that

$$N_2^*(\phi_2^*) \leq CN_1^*(L^*\phi_2^*)$$

for all $\phi_2^* \in \mathfrak{D}_{L^*}$, then $\mathfrak{R}_L = S_2$.

Theorem 9. (Hörmander [337].) If L is a closed operator with dense domain on a Hilbert space S to itself, and if there is a constant C such that $N^*(\phi^*) \leq CN^*(L^*\phi^*)$ for all $\phi^* \in S^*$, then there is a bounded linear operator L_1 on S to S such that $LL_1 = I$.

A frequently recurring situation is that of a pre-Hilbert space S with a bilinear form $(\phi, \Psi)_A$ defined on S , i.e., a scalar-valued function such that

$$(\phi, \lambda_1\Psi_1 + \lambda_2\Psi_2)_A = \lambda_1(\phi, \Psi_1)_A + \lambda_2(\phi, \Psi_2)_A$$

$$(\lambda_1\phi_1 + \lambda_2\phi_2, \Psi)_A = \overline{\lambda_1}(\phi_1, \Psi)_A + \overline{\lambda_2}(\phi_2, \Psi)_A$$

for arbitrary scalars λ_1, λ_2 and arbitrary elements $\phi, \phi_1, \phi_2, \Psi, \Psi_1, \Psi_2 \in S$. We assume that

$$N(\phi) = \sup_{\|\Psi\|=1} |(\Psi, \phi)_A| < +\infty$$

for all $\phi \in S$, and $N(\phi) > 0$ for $\phi \neq 0$. Let $\mathfrak{H}$ be the completion of S with respect to the Hilbert space norm, and let $\mathfrak{D}_A$ be the completion of S with respect to N , which can easily be shown to be another Hilbert space norm. Then there is a bounded linear operator A on $\mathfrak{D}_A$ to $\mathfrak{H}$ such that $(\Psi, \phi)_A = (\Psi, A\phi)$ for $\phi, \Psi \in S$. Let $\mathfrak{R}_A$ be the closure in $\mathfrak{H}$ of the set of elements of the form $A\phi$, $\phi \in S$. If l is a linear functional on $\mathfrak{D}_A$, then there is a unique $u \in \mathfrak{R}_A$ such that $l(\phi) = (u, \phi)_A$ for all $\phi \in S$, and we have $\|u\| = N^*(l)$. This element u can be characterized by the variational principle

$$\|u\| = \sup_{\substack{l(\phi)=1 \\ \phi \in S}} \inf_{\substack{(\Psi, \phi)_A=1 \\ \Psi \in S}} \|\Psi\| = \max_{\substack{l(\phi)=1 \\ \phi \in \mathfrak{D}_A}} \min_{\substack{(\Psi, \phi)_A=1 \\ \Psi \in \mathfrak{H}}} \|\Psi\| = \sup_{\phi \in S, \phi \neq 0} |l(\phi)|/N(\phi)$$

the extrema being uniquely attained for $\Psi = A\phi = u/\|u\|^2$. This can be used both for numerical computation and for comparison theorems. If $|(\phi, \Psi)_A| \leq C\|\phi\|\|\Psi\|$ for $\phi, \Psi \in S$, and $N(\phi) \geq C_1\|\phi\|$ for all $\phi \in S$ and some $C_1 > 0$, then $\mathfrak{R}_A = \mathfrak{H}$, and the representation problem has a unique solution $u \in \mathfrak{H}$. If $(\phi, \Psi)_K$ is a scalar product on S such that $\|\phi\|_K \leq N(\phi)$ for all $\phi \in S$, then, if ϕ is given, there is a unique $u \in \mathfrak{R}_A$ such that $(\phi, \Psi)_K = (u, \Psi)_A$ for all $\Psi \in S$, and $\|u\| \leq C\|\phi\|_K$.

Conversely if, for every $\phi \in \mathcal{S}_K$, the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_K$, there is a u in $\mathcal{H}$ such that $(\phi, \Psi)_K = (u, \Psi)_A$ for all $\Psi \in \mathcal{S}$, then there is a constant C such that $\|\phi\|_K \leq CN(\phi)$ for all $\phi \in \mathcal{S}$. For this u to be the only element in $\mathcal{H}$ with this property, it is necessary and sufficient that $\mathcal{R}_A = \mathcal{H}$.

These theorems enable us to reduce the problem of proving the existence of solutions to certain types of equations to that of proving *a priori* estimates of solutions of related equations on the assumption that the latter solutions exist.

5. Applications to differential equations

We shall now show what kind of information about a differential equation is needed in order to apply these abstract theories. We consider a formal differential operator

$$P(x, X)u = \sum X^\alpha (p_{\alpha, \beta}(x) X^\beta u)$$

where $X = [X_1, \dots, X_N]$ is the symbolic vector with components $X_j = i^{-1}D_j = i^{-1}\partial/\partial x_j$, and all but a finite number of the coefficients $p_{\alpha, \beta}$ are identically zero. The operator P is said to be proper in the domain $\Omega \subset E_N$ if $p_{\alpha, \beta} \in C^{|\alpha|}(\Omega)$ for all α, β . Associated with P is the bilinear form

$$(\phi, \Psi)_P = \sum (X^\alpha \phi, p_{\alpha, \beta} X^\beta \Psi)_\Omega$$

which is well defined for ϕ or $\Psi \in C_0^\infty(\Omega)$ and $p_{\alpha, \beta}$ bounded and measurable on Ω . The order of P is the largest value of $|\alpha| + |\beta|$ for which $p_{\alpha, \beta}(x) \not\equiv 0$. The formal adjoint $\tilde{P}$ of P is the operator

$$\tilde{P}(x, X)u = \sum X^\beta \overline{(p_{\alpha, \beta}(x) X^\alpha u)}$$

which satisfies the equation

$$(\phi, \Psi)_P = \overline{(\Psi, \phi)_{\tilde{P}}}$$

If P is proper and ϕ or $\Psi \in C_0^\infty(\Omega)$, then $(\phi, \Psi)_P = (\phi, P\Psi)_\Omega$. If P is proper, then the adjoint P^* of P in $\mathcal{L}_2(\Omega)$ is defined for all $f \in \mathcal{L}_2(\Omega)$ such that for some constant C , $|(f, P\phi)_\Omega| \leq C \|\phi\|$ for all $\phi \in C_0^\infty(\Omega)$, and then P^*f is the unique $g \in \mathcal{L}_2(\Omega)$ such that $(f, \phi)_P = (g, \phi)_\Omega$ for all $\phi \in C_0^\infty(\Omega)$. If m is the order of P and $f \in C^m(\Omega)$ and $\tilde{P}$ is proper, then $P^*f = \tilde{P}f$. If $\tilde{P}$ is proper, then a function $u \in \mathcal{L}_2(\Omega)$ satisfying the equation $\tilde{P}^*u = f$ is called a *weak solution* of $Pu = f$.

Theorem 10. (Hörmander [337].) The equation $\tilde{P}^*u = f$ has at least one solution in $\mathcal{L}_2(\Omega)$ for every $f \in \mathcal{L}_2(\Omega)$ if and only if there is a constant C such that $L_2(\phi) \leq CL_2(\tilde{P}\phi)$ for every $\phi \in C_0^\infty(\Omega)$.

Thus the existence of weak solutions of $Pu = f$ is reduced to the problem of finding an *a priori estimate* for solutions of $\tilde{P}v = g$ with $v \in C_0^\infty(\Omega)$.

If Q is a formal differential operator such that $Q = \tilde{Q}$ and $\|\phi\|_Q^2 = (\phi, \phi)_Q > 0$ for all $\phi \in C_0^\infty(\Omega)$ except $\phi \equiv 0$, then $C_0^\infty(\Omega)$ is a pre-Hilbert space with respect to the scalar product $(\phi, \Psi)_Q$, and the completion, denoted by $\mathfrak{H}_0(\Omega; Q)$ is a Hilbert space of functions which, together with a certain number of their derivatives, vanish on the boundary of Ω , in a certain generalized sense. A Q of this type is said to be positive definite on $C_0^\infty(\Omega)$. Similarly, we may define the concept of a positive definite operator Q on $C^\infty(\bar{\Omega})$, and the space $\mathfrak{H}(\Omega; Q)$ as the completion of $C^\infty(\bar{\Omega})$ with respect to the corresponding norm. The functions in the spaces $\mathfrak{H}_0(\Omega; Q)$ and $\mathfrak{H}(\Omega; Q)$ have derivatives of certain orders in a certain generalized sense.

Theorem 11. If Q is a positive definite operator on $C_0^\infty(\Omega)$, then there is, for every $f \in \mathfrak{L}_2(\Omega)$, a $u \in \mathfrak{H}_0(\Omega, Q)$ such that

$$(19) \quad (f, \phi)_\Omega = (u, \phi)_Q$$

for all $\phi \in C_0^\infty(\Omega)$ if and only if there is a constant C such that

$$(20) \quad L_2(\phi) \leq C \|\phi\|_Q$$

for all $\phi \in C_0^\infty(\Omega)$. If (20) is satisfied, then u is unique.

If P is a formal differential operator on Ω and Q satisfies (20), then there is, for every $f \in \mathfrak{L}_2(\Omega)$, a $u \in \mathfrak{H}_0(\Omega; Q)$ such that

$$(21) \quad (f, \phi)_\Omega = (u, \phi)_{\tilde{P}}$$

for all $\phi \in C_0^\infty(\Omega)$ if and only if there is a constant $C_1 > 0$ such that

$$\sup_{\Psi \in C_0^\infty(\Omega), \|\Psi\|_Q = 1} |(\Psi, \phi)_{\tilde{P}}| \geq C_1 L_2(\phi)$$

for all $\phi \in C_0^\infty(\Omega)$. In particular, this condition is satisfied if there is a $C_1 > 0$ such that

$$(22) \quad |(\phi, \phi)_{\tilde{P}}| \geq C_1 L_2(\phi) \|\phi\|_Q$$

for all $\phi \in C_0^\infty(\Omega)$. If (22) is satisfied and there is a constant C_2 such that $|(\phi, \Psi)_{\tilde{P}}| \leq C_2 \|\phi\|_Q \|\Psi\|_Q$, then the u in (21) is unique.

There is, for every $f \in \mathfrak{H}_0(\Omega; Q)$, a $u \in \mathfrak{H}_0(\Omega, Q)$ such that

$$(f, \phi)_Q = (u, \phi)_{\tilde{P}}$$

for all $\phi \in C_0^\infty(\Omega)$ if and only if there is a constant C_1 such

$$\sup_{\phi \in C_0^\infty(\Omega), \|\phi\|_Q = 1} |(\phi, \Psi)_{\tilde{P}}| \geq C_1 \|\Psi\|_Q$$

for all $\Psi \in C_0^\infty(\Omega)$. In particular, this is true if $|(\phi, \phi)_{\tilde{P}}| \geq C_1 \|\phi\|_Q^2$ for all $\phi \in C_0^\infty(\Omega)$.

Theorem 12. If $\tilde{P}$ is a proper differential operator on Ω , then there is, for every bounded completely additive set function μ on Ω , such a function ν which satisfies

$$(23) \quad \int_{\Omega} \phi d\mu = \int_{\Omega} \tilde{P}\phi d\nu$$

for all $\phi \in C_0^\infty(\Omega)$ if and only if there is a constant C such that

$$\max_{x \in \Omega} |\phi(x)| \leq C \max_{x \in \Omega} |\tilde{P}\phi(x)|$$

for every $\phi \in C_0^\infty(\Omega)$. For a given $x_0 \in \Omega$, there is a bounded completely additive set function γ on Ω such that

$$\phi(x_0) = \int_{\Omega} \tilde{P}\phi(x) d\gamma$$

for all $\phi \in C_0^\infty(\Omega)$ if and only if there is a constant C such that

$$|\phi(x_0)| \leq C \max_{x \in \Omega} |\tilde{P}\phi(x)|$$

for all $\phi \in C_0^\infty(\Omega)$.

A solution of (21) is not only a solution of $Pu = f$ in a certain generalized sense, but also satisfies homogeneous boundary conditions in a generalized sense. In order to pass from such generalized solutions to ordinary solutions, we must prove *regularity theorems* which show that the generalized solutions have suitable differentiability properties, and we must prove that they have suitable behavior near the boundary of Ω which implies *assumption of boundary values* in the classical sense. A related problem is that of characterizing the elements of such spaces as $\mathcal{H}_0(\Omega; Q)$, that is, of identifying these elements with ordinary functions having suitable smoothness properties and suitable behavior near the boundary of Ω . This is equivalent to the question of what proper differential operators on Ω can be extended to closed operators on $\mathcal{H}_0(\Omega; Q)$ to suitable other Banach spaces.

Theorem 13. If Q is a positive definite operator on $C_0^\infty(\Omega)$ and P is a proper differential operator on Ω , then P can be extended to a closed operator on $\mathcal{H}_0(\Omega; Q)$ to $\mathcal{L}_2(\Omega)$ if and only if there is a constant C such that

$$L_2(P\phi) \leq C \|\phi\|_Q$$

for all $\phi \in C_0^\infty(\Omega)$. The operator P can be extended to a closed operator

on $\mathfrak{C}_0(\Omega; Q)$ to $C_0^0(\Omega)$ if and only if there is a constant C such that

$$\max_{x \in \Omega} |P\phi| \leq C \|\phi\|_Q$$

for all $\phi \in C^\infty(\Omega)$.

In the first case, for every $f \in \mathfrak{L}_2(\Omega)$, there is a $u \in \mathfrak{C}_0(\Omega; Q)$ such that

$$(f, P\phi)_\Omega = (u, \phi)_Q$$

for every $\phi \in C_0^\infty(\Omega)$. In the second case, for every bounded completely additive set function μ on Ω , there is a $u \in \mathfrak{C}_0(\Omega; Q)$ such that

$$\int_\Omega P\phi \, d\mu = (u, \phi)_Q$$

for all $\phi \in C_0^\infty(\Omega)$. If $\bar{P}$ is the closure of P on $\mathfrak{C}_0(\Omega; Q)$ to $C_0^0(\Omega)$, then $\bar{P}u(x) \rightarrow 0$ as x approaches the boundary of Ω for every $u \in \mathfrak{C}_0(\Omega; Q)$.

One important method of proving regularity theorems is to find an operator M such that the range of M^* consists of smooth functions and such that $J^* = (PM - I)^*$ is also a smoothing operator. For then the equation $P^*u = f$ is equivalent to $u + J^*u = M^*f$ for $f \in \mathfrak{D}_{M^*}$, and the properties of u may often be read off from this equation. Similarly, if there is a one-parameter family of operators $U(t)$ such that $U(0) = I$ and $U'(t) - PU(t) = V(t)$, and if $U(t)$ and $V(t)$ satisfy suitable conditions, then the equation $P^*u = f$ can be transformed into

$$u = U(t)^*u - \int_0^t V(\tau)^*u \, d\tau - \int_0^t U(\tau)^*f \, d\tau$$

which often gives quite precise information on the smoothness properties of u .

The idea of studying generalized solutions was introduced in the case of the Laplace equation by Zaremba [732], [733], and Wiener [713], and was put on an abstract basis by Nikodym [484]. They were first exploited systematically by Sobolev [623], [626] and Friedrichs [238], [241], who showed how to use *a priori* estimates of the above types to prove existence and regularity theorems. More recently, these methods have been further developed by Gårding [267], Browder [81], Višik [660], Lax and Milgram [405]. The last authors proved a general existence theorem under a condition slightly stronger than (22).

As we mentioned before, to look for a solution in some space of the type $\mathfrak{C}_0(\Omega; Q)$ amounts to prescribing linear homogeneous boundary conditions of a special type in a certain generalized sense. Aronszajn [11], Višik [663], Lions [423], Hörmander [337], Lions and Schwartz [427] have given various formulations of other boundary-value problems in a

function space setting. Unfortunately these involve more technical machinery so that we must omit a description of these ideas.

A linear transformation L on a Banach space S_1 to a Banach space S_2 is said to be *compact* (sometimes called *completely continuous*) if it transforms bounded sets into compact sets. A set $\mathfrak{A}$ in a Banach space S is said to be *compact* if for every $\epsilon > 0$, there is a number $\mathfrak{N}(\epsilon)$ such that there are not more than $\mathfrak{N}(\epsilon)$ elements $\phi_1, \dots, \phi_k \in \mathfrak{A}$ such that $\|\phi_i - \phi_j\| > \epsilon$ for $i \neq j$. This condition is also necessary and sufficient for the closure of $\mathfrak{A}$ in the completion of a normed linear vector space to be compact. A linear operator L on a dense domain $\mathfrak{D}_L$ on one normed linear vector space S_1 to another S_2 has a compact closure on S_1 to S_2 if and only if there is, for every $\epsilon > 0$, a number $\mathfrak{N}(\epsilon)$ such that there are not more than $\mathfrak{N}(\epsilon)$ elements $\phi_1, \dots, \phi_k \in \mathfrak{D}_L$ such that $N_1(\phi_j) \leq 1$ ($1 \leq j \leq k$) and $N_2(L(\phi_i - \phi_j)) > \epsilon$ for $i \neq j$.

There is a very complete theory of equations of the form $f - \lambda Lf = g$, where L is a compact transformation, due to Riesz [569] and Hildebrandt [319] (see Banach [23], Riesz and Nagy [572]), which generalizes the qualitative part of the Fredholm theory of integral equations. For example, the eigenvalues of L are discrete and have no finite limit point. Their distribution can be estimated in terms of the number $\mathfrak{N}(\epsilon)$, which measures the degree of compactness of the transform of the unit sphere of S_1 by L . Estimates for $\mathfrak{N}(\epsilon)$ have been obtained in some cases by Erdős and Rosenbloom [189] and Kolmogorov [376].

In the theory of partial-differential equations, the following theorem of Rellich [563] is of great importance.

Theorem 14. Let $Q_k = (-\Delta)^k = (\sum_1^N X_j^2)^k$, where Δ is the Laplace operator in E_N , and let Ω be a bounded domain in E_N . Then the set of $u \in \mathfrak{H}_0(\Omega; Q_k)$ such that $\|u\|_{Q_n} \leq 1$ is compact for $k < n$.

Erdős and Rosenbloom have estimated $\mathfrak{N}(\epsilon)$ in this case.

If Ω is a bounded domain in E_N , then we can find, for each $\delta > 0$, a set of functions $\phi_1, \dots, \phi_K \in C_0^\infty(E_N)$ such that $\phi_j \geq 0$ in E_N , $\phi_j \equiv 0$ outside a domain Ω_j of diameter less than δ , and $\sum \phi_j^2 \equiv 1$ in Ω . The set $\{\phi_j^2\}$ is called a *partition of unity* on $\bar{\Omega}$. Choose a point x_j in Ω_j , and set $v_j = \phi_j v$ so that $v = \sum \phi_j v_j$. It is convenient to introduce the operator $F_{\beta,j} = [X^\beta, \phi_j]$ defined by

$$F_{\beta,j} v = X^\beta(\phi_j v) - \phi_j(X^\beta v)$$

which is an operator of order $|\beta| - 1$. Then the equation $P(x, X)u = f$ implies that

$$(24) \quad \sum_{k=1}^K P_{jk} v_k = \phi_j f = f_j$$

where

$$P_{jk} = \delta_{jk} P(x_j, X) - \sum_{\alpha, \beta} X^\alpha c_{\alpha\beta j} X^\beta \phi_k \\ + \sum_{\alpha, \beta} (F_{\alpha j} p_{\alpha\beta} X^\beta \phi_k + X^\alpha p_{\alpha\beta}(x_j) F_{\beta j} \phi_k)$$

and

$$c_{\alpha, \beta, j}(x) = \phi_j(x)(p_{\alpha\beta}(x) - p_{\alpha\beta}(x_j))$$

Thus the equation $Pv = f$ leads to a system (24), where P_{jk} is the sum of an operator with constant coefficients, an operator with small coefficients, and an operator of lower order. Consequently the theory of partial-differential equations with continuous coefficients can be reduced to the study of the perturbation of operators with constant coefficients by operators of the latter types. In a wide class of important cases, an operator with small coefficients can be transformed into an operator of small norm in a suitable function space and, by Rellich's lemma, an operator of lower order can be transformed into a compact operator, so that the Riesz theory can be applied. Furthermore, in these cases, the system (24) is fully equivalent to the original equation $Pv = f$. Thus a detailed study of equations with constant coefficients is fundamental for the general theory, both for the solution of problems with variable coefficients as here indicated, and for the insight it gives in suggesting the type of generalizations to be sought for. The Fourier and Laplace transformations, and various other operational calculi, are very powerful tools in the theory of equations with constant coefficients.

FOURIER AND LAPLACE TRANSFORMS. OTHER OPERATIONAL CALCULI

1. Fourier transforms

If $f \in C_0^\infty(E_N)$, then its Fourier transform is defined by

$$(\mathfrak{F}f)(\xi) = \int_{E_N} f(x) \exp(-2\pi i(x \cdot \xi)) dV_x$$

and $\mathfrak{F}f \in C_1^\infty(E_N)$ also. This transform has the following properties:

- (25) (a) If P is a polynomial on E_N , then $\mathfrak{F}(P(X)f) = P(2\pi\xi)\mathfrak{F}f$; if $\Xi = [\Xi_1, \dots, \Xi_N]$, $\Xi_j = i^{-1} \partial/\partial \xi_j$, then $P(\Xi)\mathfrak{F}f = \mathfrak{F}(P(-2\pi x)f)$.

- (b) $\mathfrak{F}(f_1 * f_2) = (\mathfrak{F}f_1)(\mathfrak{F}f_2)$, where

$$(f_1 * f_2)(x) = \int_{E_N} f_1(x - y) f_2(y) dV_y;$$

$$\mathfrak{F}(f_1 f_2) = (\mathfrak{F}f_1) * (\mathfrak{F}f_2).$$

- (c) For $k \in E_N$, $(\mathfrak{F}f(x+k))(\xi) = \exp(2\pi i(\xi \cdot k))(\mathfrak{F}f)(\xi)$;
 $(\mathfrak{F}(\exp(-2\pi i(x \cdot k))f(x)))(\xi) = (\mathfrak{F}f)(\xi + k)$.
- (d) If p is a nonsingular linear transformation on E_N to E_N and q is the transformation such that $(q\xi) \cdot (px) \equiv \xi \cdot x$, then
 $(\mathfrak{F}(px))(\xi) = |\det p|^{-1}(\mathfrak{F}f)(q\xi)$.
- (e) If $f(x) = f_1(x^{(1)})f_2(x^{(2)})$, where $x^{(1)} = (x_1, \dots, x_m) \in E'_m$,
and $x^{(2)} = (x_{m+1}, \dots, x_N) \in E_{N-m}$, then $(\mathfrak{F}f)(\xi)$
 $= (\mathfrak{F}f_1)(\xi^{(1)}) (\mathfrak{F}f_2)(\xi^{(2)})$.
- (f) if $f(x) = g(\|x\|)$, then

$$(\mathfrak{F}f)(\xi) = 2\pi \|\xi\|^{-k} \int_0^\infty g(r) J_k(2\pi r \|\xi\|) r^{N/2} dr$$

where J_k is the Bessel function of order k and $k = (N-2)/2$.

- (g) If $a > 0$, then $\mathfrak{F}(\exp(-a\|x\|^2))$
 $= (\pi/a)^{N/2} \exp(-\pi^2 \|\xi\|^2/a)$.
- (h) $L_2(\mathfrak{F}f) = L_2(f)$; $(\mathfrak{F}f_1, \mathfrak{F}f_2)_{E_N} = (f_1, f_2)_{E_N}$.
- (i) $(\mathfrak{F}(\mathfrak{F}f))(-x) = f(x)$.
- (j) If $1 \leq p \leq 2$, $p+q = pq$, there is a constant C_p such that
 $L_q(\mathfrak{F}f) \leq C_p L_p(f)$ (see Titchmarsh [647]).

These properties hold for much wider classes of functions than $C_1^\infty(E_N)$. We shall use them freely without entering into details about the exact conditions for validity.

If $\mathcal{S}$ is a testing space, then we denote by $\mathfrak{F}\mathcal{S}$ the set of all Fourier transforms of elements of $\mathcal{S}$. The Fourier transform of a testing space is again a testing space, with the natural topology that $\mathfrak{F}\phi_n \rightarrow 0$ in $\mathfrak{F}\mathcal{S}$ if and only if $\phi_n \rightarrow 0$ in $\mathcal{S}$.

The following relations hold between the testing spaces introduced previously:

$$(26) \quad \mathfrak{F}C_1^\infty(E_N) = C_1^\infty(E_N); \mathfrak{F}C_0^\infty(E_N) = \mathcal{Z}^1; \text{ if } 1 < p < +\infty, p+q = pq, \text{ then } \mathfrak{F}\mathcal{K}_p = \mathcal{Z}^q \text{ and } \mathfrak{F}\mathcal{Z}_p^p = \mathcal{Z}_q^q.$$

The corresponding relations for $\mathcal{Z}_q^p$, $q < p$, are more complicated and the complete answer is not known. For some applications it is important to formulate the known results more precisely.

Theorem 15. If $\phi \in \mathcal{Z}_q^p$, $p > 1$, $q > 1$, then $\Psi = \mathfrak{F}\phi \in \mathcal{Z}_s^r$, $r = q/(q-1)$, $s = q/(p-1)$. If $|\phi(z)| \leq C_1 \exp(C_2 \|z\|^p)$ for all $z \in C_N$, $|\phi(x)| \leq C_3 \exp(-C_4 \|x\|^q)$ for all $x \in E_N$, $C_4 > 0$, then

$$|\Psi(\xi)| \leq C_5 C_4^{-N/q} \exp(-C_6 (\|\xi\|/C_7)^s)$$

for all $\xi \in E_N$, where $C_7 = \max((C_2 + C_4)^{1/p}, (C_2 + C_4)C_4^{-1/s})$, and

C_5 and C_6 depend only on p, q, C_1, C_3 , and N . For $\zeta = \xi + i\eta$, $\xi, \eta \in E_N$,

$$|\Psi(\zeta)| \leq C_8 C_4^{-N/q} \exp(C_9 C_4^{-1/q} \|\eta\|^r)$$

where C_8 and C_9 depend only on C_3, q , and N . If $q = 1$, then $\Psi(\zeta)$ is analytic in the tube $\|\eta\| < C_4/(2\pi)$, whereas if $0 < q < 1$, then

$$|\Xi^\alpha \Psi(\xi)| \leq C_{10} C_4^{-N/q} (2\pi C_4^{-1/q})^{|\alpha|} \Gamma((N + |\alpha|)/q)$$

where C_{10} depends only on C_3, q , and N .

This result was stated for $N = 1$ in a special case, with a sketch of a proof, by Šilov [613].

If $\mathcal{S}$ is a testing space and $T \in \mathcal{S}^*$, then the Fourier transform of T is in $(\mathfrak{F}\mathcal{S})^*$, and is defined by

$$(\mathfrak{F}T)(\mathfrak{F}\phi) = T(\phi(-x))$$

For example, we find that for the Dirac delta we have

$$\mathfrak{F}\delta_{x_0} = [\exp(-2\pi i(x_0 \cdot \xi))]$$

It is easy to extend most of (25) to distributions.

2. Laplace transforms

The Laplace transform of a function $f \in C_0^\infty(E_N)$ is defined by

$$(\mathfrak{L}f)(\zeta) = \int_{E_N} f(x) \exp(-2\pi(x \cdot \zeta)) dV_x$$

where $\zeta = \xi + i\eta$, $\xi, \eta \in E_N$, so that

$$(\mathfrak{L}f)(\zeta) = (\mathfrak{F}(f(x) \exp(-2\pi(\xi, x))))(\eta)$$

The inversion formula is

$$f(x) = \int_{E_N} (\mathfrak{L}f)(\zeta) \exp(2\pi(x \cdot \zeta)) dV_\eta$$

The properties of the Laplace transform can easily be obtained from those of the Fourier transform:

- (27) (a) If P is a polynomial, $P(\partial/\partial\zeta)\mathfrak{L}f = \mathfrak{L}(P(-2\pi x)f(x))$,
 $\mathfrak{L}P(\partial/\partial x)f = P(2\pi\zeta)\mathfrak{L}f$.
 (b) If $k \in E_N$, then $(\mathfrak{L}(f(x+k)))(\zeta) = \exp(2\pi(k \cdot \zeta))(\mathfrak{L}f)(\zeta)$;
 $(\mathfrak{L}(f(x) \exp(2\pi(x \cdot k))))(\zeta) = (\mathfrak{L}f)(\zeta - k)$.
 (c) If p and q are as in (25d), then $(\mathfrak{L}f(px))(\zeta)$
 $= |\det p|^{-1}(\mathfrak{L}f)(q\zeta)$.
 (d) $\mathfrak{L}(f^*g) = (\mathfrak{L}f)(\mathfrak{L}g)$; $\mathfrak{L}(fg) = (\mathfrak{L}f)^*(\mathfrak{L}g)$.

- (c) $L_2(\mathfrak{L}f(\xi + i\eta)) = L_2(f(x) \exp(-2\pi(x \cdot \xi)))$ for $\xi \in E_N$, where the norm on the left is taken with respect to η .

Let Ω be a convex domain in E_N and let $F_1(\Omega)$ be the set of functions f on E_N such that $f(x) \exp(-2\pi(x \cdot \xi)) \in \mathfrak{L}_2(E_N)$ for all $\xi \in \Omega$. Let $\mathfrak{A}_2(\Omega)$ be the class of functions $F(\zeta)$ which are analytic in the tube $\mathfrak{I}(\Omega)$, consisting of all ζ such that $\xi \in \Omega$, and such that, as a function of η , $F(\xi + i\eta) \in \mathfrak{L}_2(E_N)$ for all $\xi \in \Omega$. The basic theorem is:

Theorem 16. (Bochner and Martin [60], Leray [408].) The operator $\mathfrak{L}$ maps $F_1(\Omega)$ in a one-to-one manner onto $\mathfrak{A}_2(\Omega)$, and

$$L_{2,\eta}((\mathfrak{L}f)(\xi + i\eta)) = L_{2,x}(f(x) \exp(-2\pi(x \cdot \xi)))$$

for all $\xi \in \Omega$.

Corollary 16a. Let $E(\Omega)$ be the set of all sums of a finite number of distributions of the form $X^\alpha[f]$, where $f \in F_1(\Omega)$, and $\mathfrak{B}_2(\Omega)$ be the set of finite sums of products of polynomials in ζ by functions in $\mathfrak{A}_2(\Omega)$. Then $\mathfrak{L}$ maps $E(\Omega)$ in a one-to-one manner onto $\mathfrak{B}_2(\Omega)$.

Here we have used the concept of the Laplace transform of a distribution (Schwartz [604], Garnir [272]). If T is a distribution on a testing space $\mathfrak{S}$ such that $\exp(-2\pi(\xi \cdot x))T = T_\xi \in \mathfrak{S}^*$ for $\xi \in \Omega$, then we define the Laplace transform of T by $\mathfrak{L}T(\xi) = \mathfrak{F}T_\xi$, so that for $\phi \in \mathfrak{S}$, $\mathfrak{L}T$ is the function on Ω to $(\mathfrak{F}\mathfrak{S})^*$ defined by

$$(\mathfrak{L}T)(\mathfrak{F}\phi) = T_\xi(\phi(-x)) = T(\exp(-2\pi(\xi \cdot x)) \phi(-x))$$

If P is a polynomial on E_N , and u is in the testing space $\mathfrak{S}$, and $P(X)u = f$, then $\mathfrak{F}f = P(2\pi\xi)\mathfrak{F}u$, so that

$$u = \mathfrak{F}^{-1}(P(2\pi\xi)^{-1}\mathfrak{F}f) = \mathfrak{F}^{-1}(P(2\pi\xi)^{-1})^*f = E^*f$$

where

$$\begin{aligned} E(x) &= \mathfrak{F}(P(2\pi\xi)^{-1})(-x) = \int_{E_N} P(2\pi\xi)^{-1} \exp(2\pi i(x \cdot \xi)) dV_\xi \\ &= (2\pi)^{-N} \int_{E_N} P(\xi)^{-1} \exp(i(x \cdot \xi)) dV_\xi \end{aligned}$$

The first formula for u is rigorously true for $u \in \mathfrak{S}$. If $P(\xi)^{-1} \in \mathfrak{L}_p(E_N)$ for some p , $1 \leq p \leq 2$, then $E(x)$ exists in $\mathfrak{L}_q(E_N)$, where $q = p/(p-1)$, and the other representation formulas are also true in the usual sense. If $P(\xi) \neq 0$ for all $\xi \in E_N$, then E is a distribution, in general, and u can be obtained from the formula

$$u = (1 - \Delta)^m (E_m^* f)$$

$$E_m(x) = (\mathfrak{F}(P(2\pi\xi)^{-1}(1 + 4\pi^2 \|\xi\|^2)^{-m}))(-x)$$

for m sufficiently large. If P has real zeros, then the formulas for E are usually meaningless as they stand, and must be interpreted by some device.

Similarly, if u and f are in $F_1(\Omega)$, where Ω is a convex domain in E_N , and

$$(28) \quad P\left(\frac{\partial}{\partial x}\right)u = f$$

then

$$u = \mathfrak{L}^{-1}(P(2\pi\zeta)^{-1}\mathfrak{L}f) = E^*f$$

where $E = \mathfrak{L}^{-1}(P(2\pi\zeta)^{-1})$. If $P(2\pi\zeta)^{-1}$ is bounded in the tube $\mathfrak{I}(\Omega)$, then the operator

$$P_\Omega^{-1}f = \mathfrak{L}^{-1}(P(2\pi\zeta)^{-1}\mathfrak{L}f)$$

maps $F_1(\Omega)$ into itself, and yields the unique solution in $F_1(\Omega)$ of (28) for $f \in F_1(\Omega)$. To every convex domain Ω such that $P(2\pi\zeta)^{-1}$ is bounded in $\mathfrak{I}(\Omega)$, there corresponds an inverse operator to the differential operator $P(\partial/\partial x)$. The study of the maximal domains Ω with this property is the basis of Leray's theory of hyperbolic equations (Leray [408]). Of course, E is again, in general, a distribution. If $P(2\pi\zeta) \neq 0$ in $\mathfrak{I}(\Omega)$, we may find a polynomial Q such that $(Q(2\pi\zeta)P(2\pi\zeta))^{-1} \in \mathfrak{A}_2(\Omega)$, and then

$$E^*f = Q\left(\frac{\partial}{\partial x}\right)(E_1^*f), \quad E_1 = \mathfrak{L}^{-1}(Q(2\pi\zeta)^{-1}P(2\pi\zeta)^{-1})$$

For further information on Fourier and Laplace transforms, see Titchmarsh [647], Bochner and Chandrasekharan [59], Sneddon [619], Doetsch [156], Widder [710], Van der Pol and Bremmer [658], Friedman [237].

3. Systems of partial-differential equations and potentials

If u and f are functions on E_N to E_M and $P(\xi)$ is an $M \times M$ matrix with polynomial elements, then the above formulas still apply, with the obvious reinterpretations. In many problems it is useful to introduce the matrix Q such that

$$P(\xi)Q(\xi) = Q(\xi)P(\xi) = D(\xi)I$$

where $D(\xi) = \det P(\xi)$ and I is the identity matrix. If u is in the testing space $\mathfrak{S}(M \times 1)$ and $P(X)u = f$, then

$$\begin{aligned} u &= \mathfrak{F}^{-1}(Q(2\pi\xi)D(2\pi\xi)^{-1}\mathfrak{F}f) \\ &= Q(X) \cdot \Psi \end{aligned}$$

where

$$\Psi = \mathfrak{F}^{-1}(D(2\pi\xi)^{-1}\mathfrak{F}f)$$

which may be called the *potential* of the vector field f , satisfies the system

$$D(X)\Psi = f$$

in which each scalar equation contains only one unknown scalar-valued function. These formal considerations can be justified rigorously for a wide class of systems, so that every solution of such a system can be represented in terms of a potential satisfying the diagonalized system. It is clearly sufficient to find a matrix Q such that PQ is diagonal, and this is more convenient in many problems. If ϕ is a fundamental solution for the operator $D(X)$, then the matrix Φ defined by

$$\Phi\gamma = Q(X)(\phi\gamma)$$

where γ ranges over $\mathcal{L}_M$, is a fundamental solution for the operator $P(X)$.

For example, in the case of Maxwell's equations for a homogeneous dissipative medium, with units chosen so that the velocity of light $c = (\mu\epsilon)^{-1/2} = 1$, we have

$$P\left(\frac{\partial}{\partial x}\right)[\mathbf{E}, \mathbf{B}] = \left[\frac{\partial \mathbf{E}}{\partial x_4} + \lambda \mathbf{E} - \nabla \times \mathbf{B}, \frac{\partial \mathbf{B}}{\partial x_4} + \nabla \times \mathbf{E} \right] = [\mathbf{F}, \mathbf{G}]$$

where $x_4 = t$ is the time coordinate, $\lambda = \sigma/\epsilon$, $\mathbf{F} = -\mathbf{J}/\epsilon$, and $\mathbf{G}$ may be thought of as a "magnetic current," ordinarily set equal to zero. If ξ denotes the 3-dimensional vector (ξ_1, ξ_2, ξ_3) , then the matrix $Q(\xi)$ such that

$$Q(\xi)[\alpha, \beta] = [(\xi_4^2 + \lambda\xi_4)\alpha - \xi(\xi \cdot \alpha) + \xi_4(\xi \times \beta), (\xi_4^2 + \lambda\xi_4)\beta - \xi(\xi \cdot \beta) - (\xi_4 + \lambda)(\xi \times \alpha)]$$

satisfies

$$P(\xi)Q(\xi) = p(\xi)A(\xi), \quad p(\xi) = \xi_1^2 - \|\xi\|^2 + \lambda\xi_4$$

$$A(\xi)[\alpha, \beta] = [(\xi_4 + \lambda)\alpha, \xi_4\beta]$$

If $[\mathbf{Y}, \mathbf{Z}]$ is a solution of

$$p\left(\frac{\partial}{\partial x}\right)A\left(\frac{\partial}{\partial x}\right)[\mathbf{Y}, \mathbf{Z}] = [\mathbf{F}, \mathbf{G}]$$

then $Q(\partial/\partial x)[\mathbf{Y}, \mathbf{Z}]$ is a solution of Maxwell's equations. The vectors $\mathbf{Y}$ and $\mathbf{Z}$ are essentially the Hertz electrical and magnetic potentials. If we combine these remarks with the known formula for the solution of the Cauchy problem for the wave equation (Hadamard [308], Riesz [574], Baker and Copson [22]), we obtain the explicit formula for the solution of the Cauchy problem for Maxwell's equations, found independently by Durand [173] and James [349].

The theory of monochromatic waves leads to the system (Bateman [27])

$$P\left(\frac{\partial}{\partial x}\right)\mathbf{u} = \nabla \times \mathbf{u} + \lambda \mathbf{u} = \mathbf{f}, \quad \lambda \neq 0$$

The matrix Q , defined by

$$Q(\xi)\alpha = \lambda\alpha + \lambda^{-1}\xi(\xi \cdot \alpha) - \xi \times \alpha$$

satisfies

$$PQ = QP = (\lambda^2 + \|\xi\|^2)I$$

so that the solutions of the Helmholtz equation

$$(\lambda^2 + \Delta)\Psi = \mathbf{f}$$

can be converted into solutions of $P\mathbf{u} = \mathbf{f}$ by the formula

$$\mathbf{u} = Q\left(\frac{\partial}{\partial x}\right)\Psi = \lambda\Psi + \lambda^{-1}\nabla(\nabla \cdot \Psi) - \nabla \times \Psi$$

The matrix Φ which, operating on any constant vector γ , produces

$$\Phi(x)\gamma = -Q\left(\frac{\partial}{\partial x}\right)(\phi\gamma), \quad \phi(x) = \exp(i\lambda r)/(4\pi r), \quad r = \|x\|$$

is a fundamental solution for the operator P . In the theory of this equation, "Maxwellian" potentials of the form

$$\mathbf{v}(x) = \int \Phi(x - y) \mathbf{f}(y) dV_y$$

play the same role as Newtonian potentials in the theory of the Laplace equation.

In some problems it is convenient to perform a transformation only with respect to some of the variables. Consider an equation of the form

$$\sum_{k=0}^m A_k\left(t, \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}\right)^k u = f(t, x)$$

where $A_k(t, \xi)$ is a polynomial in ξ with coefficients which may depend on t . Let $U(t, \xi)$ be the Laplace transform of u with respect to $x \in E_N$. Then U satisfies the ordinary differential equation

$$\sum_{k=0}^m A_k(t, 2\pi\zeta) \frac{d^k U}{dt^k} = \mathfrak{L}f$$

with ζ as a parameter. Even if the coefficients A_k are independent of t ,

it may be advantageous to use this approach. Thus the solution of

$$(29) \quad \frac{\partial^2 u}{\partial t^2} + A_0 \left(\frac{\partial}{\partial x} \right) u = f, \quad u(0, x) = u(1, x) = 0, \quad (x \in E_N)$$

is

$$u(t, x) = \mathcal{L}^{-1} \left(\int_0^1 G(t, \tau; A_0(2\pi\zeta)^{-1/2}) (\mathcal{L}f)(\tau, \zeta) d\tau \right)$$

where

$$G(t, \tau; \lambda) = \sin(\lambda t) \sin(\lambda(1 - \tau)) / \lambda \sin \lambda, \quad 0 \leq t \leq \tau \leq 1$$

$$G(t, \tau; \lambda) = G(\tau, t; \lambda) \quad \text{for} \quad 0 \leq \tau \leq t \leq 1$$

If there is a convex domain Ω in E_N such that $|A_0(2\pi\zeta) - n^2\pi^2|$ is bounded from below for $n = 1, 2, \dots$, and all ζ in the tube $\mathfrak{J}(\Omega)$, then there is a unique solution $u \in F_1(\Omega)$ of (29) for every $f \in F_1(\Omega)$.

4. General remarks on operational calculi

There is a well-known operational calculus for normal operators and, in particular, for Hermitian operators in Hilbert space (see Stone [632], von Neumann [480], Murray [476], Halmos [309], Nagy [479], Riesz and Nagy [572], Akhiezer and Glazman [3]). Dunford [171] and others (see Hille [320], Dunford and Schwartz [172]) have developed a powerful generalization to a class of operators in Banach spaces. Other operational calculi are due to Fantappi  (see L vy [419]), Waelbroeck [668], Arens and Calder n [5]). The essential problem is to define $f(T)$, where T is a linear operator on a linear vector space S to itself and f is a function defined in some domain of scalars. If f is a polynomial with scalar coefficients, then $f(T)$ is well defined, and

$$(fg)(T) = f(T)g(T), \quad (f+g)(T) = f(T) + g(T)$$

$$h(T) = f(g(T)) \quad \text{if} \quad h(\lambda) = f(g(\lambda))$$

where f and g are polynomials. An identity between polynomials in a scalar variable carries over to polynomials in T . The object of an operational calculus is to assign a meaning to $f(T)$ for a wider class of functions in such a way as to preserve these properties and, usually, some limit relations as well.

In most cases where an operational calculus has been developed rigorously, the operators $f(T)$ actually have the desired properties when acting on a wider class of functions than is included in the rigorous theory. Often a further direct analysis establishes these results with greater generality and sharpness. Similarly, heuristic arguments based on the assumption that the operators $f(T)$ exist and have the desired

properties often lead to the discovery of results which can be established directly, in spite of the fact that the existing operational calculi are insufficient for this purpose.

For example, the operator $\exp (tL)$ is the solution of the differential equation

$$U'(t) - LU(t) = 0, \quad U(0) = I$$

The identity

$$\Gamma(z)a^{-z} = \int_0^\infty \exp (-at)t^{z-1} dt$$

valid for $\Re a > 0$, $\Re z > 0$, leads to the equation

$$(-L)^{-z} = \Gamma(z)^{-1} \int_0^\infty t^{z-1} U(t) dt$$

Similarly, the identity

$$(s - a)^{-1} = \int_0^\infty \exp (at) \exp (-st) dt$$

for $\Re(s - a) > 0$, leads to

$$(sI - L)^{-1} = \int_0^\infty \exp (-st) U(t) dt$$

thus expressing the solution of $-Lu + su = f$ as the Laplace transform of the solution of the equation

$$\frac{\partial u}{\partial t} = Lu, \quad U(0) = f$$

Thus, if $Lu = -\partial u/\partial x$, then $(U(t)f)(x) = f(x - t)$, and

$$((-L)^{-z}f)(x) = \Gamma(z)^{-1} \int_{-\infty}^x (x - y)^{z-1} f(y) dy$$

which is one useful definition of fractional integration (Weyl [704], Riesz [573]). If $L = \Delta$ in E_N , then $U(t)f$ is the solution of the heat equation:

$$(U(t)f)(x) = \int_{E_N} K(t, x - y) f(y) dV_y,$$

$$K(t, x) = (4\pi t)^{-N/2} \exp (-\|x\|^2/4t)$$

Then

$$((- \Delta)^{-z}f)(x) = C_N(z) \int_{E_N} \|x - y\|^{2z-N} f(y) dV_y$$

where

$$C_N(z) = \Gamma((N/2) - z)/4^z \pi^{N/2} \Gamma(z), \quad 0 < \Re z < N/2$$

which is the *Riesz potential* for the elliptic operator $-\Delta$ (Riesz [573]).

Furthermore, we have

$$((sI - \Delta)^{-z}f)(x) = \int_{E_N} \phi(s, z, \|x - y\|) f(y) dV_y$$

where

$$\phi(s, z, r) = \Gamma(z)^{-1} 2(4\pi)^{-N/2} (r^2/4s)^{N/2} K_p(s^{1/2}r)$$

where $p = z - N/2$, and K_p is the modified Bessel function of the third kind (Erdélyi [187]).

If $U(t)$ is the solution of

$$U''(t) + t^{-1}k U'(t) + LU(t) = 0, \quad U(0) = I$$

then

$$U(t) = F_k(tL^{1/2}) = U_k(t)$$

where

$$\begin{aligned} F_k(z) &= \Gamma((k+1)/2) \sum_0^\infty \frac{(-z^2/4)^m}{m! \Gamma(m+1+(k-1)/2)} \\ &= \Gamma((k+1)/2) (z/2)^{(1-k)/2} J_{(k-1)/2}(z) \end{aligned}$$

for $k \neq -1, -3, -5, \dots$. We can read off the properties of $U_k(t)$ from those of Bessel functions (Erdélyi [187]). For example, Sonine's first integral formula

$$2^\rho \Gamma(\rho+1) J_{\rho+\mu+1}(z) = z^{\rho+1} \int_0^{\pi/2} J_\rho(z \sin \theta) (\sin \theta)^{\mu+1} (\cos \theta)^{2\mu+1} d\theta$$

valid for $\Re \rho > -1$, $\Re \mu > -1$, leads to

$$\begin{aligned} U_{k+\alpha}(t) &= (\Gamma(\alpha/2) \Gamma((k+1)/2))^{-1} 2^{1-\alpha/2} \Gamma((k+\alpha+1)/2) \\ &\quad \int_0^{\pi/2} U_k(t \sin \theta) (\sin \theta)^k (\cos \theta)^{\alpha-1} d\theta \end{aligned}$$

for $\Re k > -1$, $\Re \alpha > 0$, which shows that, if $U_k(t)$ is known for one value, k_0 , in the half-plane $\Re k > -1$, then it can be determined for $\Re k > \Re k_0$. Let

$$\tilde{U}_k(s) = s^{(k-1)/2} U_k(s^{1/2})$$

$$(J^\alpha f)(s) = \Gamma(\alpha)^{-1} \int_0^s (s-\sigma)^{\alpha-1} f(\sigma) d\sigma$$

the latter being a Riemann-Liouville fractional integral. Then

$$\tilde{U}_{k+\alpha} = \Gamma((k+1)/2)^{-1} \Gamma((k+\alpha+1)/2) J^{\alpha/2} \tilde{U}_k$$

which can be used for the evaluation of U_k for other values of k ; for example, by analytic continuation with respect to α (Riesz [573]), or by

the formula $J^{-1} = (d/ds)$, so that

$$\tilde{U}_{k-2m} = \frac{2^m}{(k-1)(k-3) \cdots (k+1-2m)} \left(\frac{d}{ds} \right)^m \tilde{U}_k$$

for any positive integer m . If $L = -\Delta$ in E_N , then

$$(U_{N-1}(t)f)(x) = (\omega_N t^{N-1})^{-1} \int_{\partial S(x,t)} f(y) dA_y$$

here ω_N is the area of the boundary of the unit sphere in E_N , and $\partial S(x, t)$ is the boundary of the sphere $S(x, t)$ with center at x and radius t . The above formulas then yield many of the recent results of Weinstein and his associates (Weinstein [696], Diaz and Weinberger [152], Blum [55], Davis [131]) on the Euler-Poisson-Darboux equation. (See also Rosenbloom [579], Friedman [236], Lions [426], Żitomirski [738]). In particular, the case $k = 0$ yields the solution of the wave equation. The operators $U_k(t)$ were used for the proof of differentiability theorems for solutions of elliptic equations by John [355], [356].

5. Theory of semigroups

We shall describe now the operational calculus arising in the theory of semigroups because of its special importance in the theory of partial differential equations. A number of recent developments have shown this to be a very powerful tool, which is certainly far from being fully exploited at present. (See Hille [320], Yosida [717], [731], Feller [204], Phillips [517], Lions [423], Lax [403], Rosenbloom [580].)

A semigroup is a one-parameter family $T(t)$ of continuous linear operators on a linear topological space S to itself such that $T(t_1 + t_2) = T(t_1)T(t_2)$ for $t_1, t_2 > 0$, and $T(t)u$ is a continuous function of t for $t > 0$ and $u \in S$. The *infinitesimal operator* is defined by

$$T'(0)u = \lim_{t \rightarrow 0} t^{-1}(T(t) - I)u$$

whenever this limit exists. The operator $T'(0)$ need not be continuous nor closed. Its closure, when it exists, is called the *infinitesimal generator* of the semigroup. If U is the infinitesimal generator of the semigroup, then $u(t) = T(t)u_0 \in \mathfrak{D}_U$ for $t > 0$ and $u'(t) = Uu(t)$. We shall consider only semigroups such that $\mathfrak{D}_U$ is dense in S . If $u_0 \in \mathfrak{D}_U$, we have $T(t)u_0 \rightarrow u_0$ as $t \rightarrow 0$. If this is satisfied for all $u_0 \in S$, then we shall say that the semigroup is of class (C_0) . In that case $\mathfrak{D}_U$ is closed. Formally $T(t) = \exp(tU)$, so that the problem of determining a semigroup generated by a given operator is equivalent to that of defining the exponential function of the operator. The remark on p. 83 shows

that $(\lambda I - U)^{-1} = R(\lambda; U)$ is the Laplace transform of $T(t)$, so that we can expect an intimate connection between the semigroup generated by an operator U and its *resolvent* $R(\lambda; U)$.

The following theorem was obtained in 1953 by Feller [208], Miyadera [459], and Phillips [516]:

Theorem 17. A necessary and sufficient condition that a closed operator U on a Banach space S generate a semigroup of class (C_0) is that $\mathfrak{D}_U$ be dense in S and that there exist constants B and ω such that

$$N(R(\lambda; U)^n) \leq B(\lambda - \omega)^{-n}$$

for all $\lambda > \omega$ and all $n > 0$. In this case, $N(T(t)) \leq B \exp(\omega t)$ for $t > 0$.

Corollary 17a. If U is a closed operator on a Banach space S with dense domain, and if $\lambda N(R(\lambda; U)) \leq 1$ for all $\lambda > 0$, then U generates a semigroup of class C_0 and $N(T(t)) \leq 1$ for $t > 0$.

This important corollary was discovered by Hille [320] and Yosida [717] in 1948.

The condition (C_0) can be relaxed in several ways. A semigroup $T(t)$ is said to be of class $(1, A)$ if: (1) $\lim_{t \rightarrow t_0} N(T(t) - T(t_0)) = 0$ for $t_0 > 0$; (2)

$\int_0^1 N(T(t)) dt < +\infty$; and (3) $\lim_{\lambda \rightarrow 0} \lambda \int_0^\infty \exp(-\lambda t) T(t) u dt = u$ for all $u \in S$.

Theorem 18. (Phillips [516].) A necessary and sufficient condition that a closed operator U on a Banach space S generate a semigroup of class $(1, A)$ is that: (1) $\mathfrak{D}_U$ is dense in S ; (2) $N(R(\lambda; U)) = O(\lambda^{-1})$ as $\lambda \rightarrow +\infty$; and (3) there exists constants B and ω such that for $n > 0$

$$\int_0^\infty \lambda^{n-1} N(R(\lambda + \omega; U)^{n+1}) d\lambda \leq B/n$$

Corollary 18. If U satisfies (1) and (4) $N(R(\sigma + \tau; U)) \leq B/(\sigma - \omega + |\tau|^\beta)$ for $\sigma = \Re \lambda > \omega$ and some $\beta > 1/2$, then U generates a semigroup of class $(1, A)$.

Phillips also obtained important results on the perturbation of the infinitesimal generator of a semigroup. Related results in a somewhat different direction were obtained by Rosenbloom [580].

If $V_1, \dots, V_n$ are operators such that $V_k - (I + U \Delta t_k) = o(\Delta t_k)$ where $\sum \Delta t_k = t$, and t is given, then we may expect that under quite general conditions

$$\lim V_1 V_2 \cdots V_n = T(t)$$

For example, if $T(t)$ and all finite products of the V 's are uniformly bounded operators, and if the above asymptotic relation holds uniformly on some dense domain in S , then this process of "product integration" yields a construction for $T(t)$. This method was introduced for ordinary differential equations by Schlesinger [600], and was generalized to Banach spaces in the case where U is a bounded operator by Birkhoff [49]. A result of the above general type was obtained by Lachterman [393], who applied it to the heat equation. In particular, we may choose $V_k = (\Delta t_k)^{-1} R((\Delta t_k)^{-1}; U)$. This was done for a class of parabolic equations by Rothe [581], and was used by Yosida [717] in the general case to establish Corollary 17a.

GENERAL THEORY OF EQUATIONS WITH CONSTANT COEFFICIENTS

THE WORK OF EHRENPREIS, MALGRANGE, AND HÖRMANDER

1. General existence theorems

We shall now give an account of recent work by Ehrenpreis [175], Malgrange [435], and Hörmander [337] on equations with constant coefficients. The work of the first two authors and some of the results of the latter are couched in the language of the theory of distributions, but as is usually the case in concrete applications, one can extract from their proofs much sharper results by Banach and Hilbert space methods. The basic step is to obtain, by the use of Fourier and Laplace transforms, *a priori* estimates of the kind indicated in Theorems 10 and 11.

If P is a polynomial of degree m on E_N , then we call its homogeneous part of degree m its *principal part* and denote it by $p(\xi)$. We begin with the following theorem which is due to Ehrenpreis for $q = +\infty$, Malgrange for $q = 1$, and Hörmander for $q = 2$.

Theorem 19. Let P be a polynomial of degree m on E_N with principal part p , and suppose that $\eta \in E_N$, $\|\eta\| = 1$, $p(\eta) \neq 0$. Let $f \in C_1^\infty(E_N)$ be such that $f(x) \cosh(2\pi(x \cdot \eta)) \in \mathfrak{L}_1(E_N)$. Then for $1 \leq q \leq \infty$, we have

$$L_q(\Im f) \leq \pi^{-m} |p(\eta)|^{-1} \{L_q(\eta; \Im P(X)f) L_q(-\eta; \Im P(X)f)\}^{1/2}$$

where

$$L_q(\eta; F) = \left\{ \int_{E_N} |F(\xi + i\eta)|^q dV_\xi \right\}^{1/q}, \quad 1 \leq q < +\infty$$

$$L_\infty(\eta; F) = \sup_{\xi \in E_N} |F(\xi + i\eta)|$$

and

$$L_q(F) = L_q(0; F)$$

Proof. If $\phi(z)$ is an analytic function of the complex variable $z = x + iy$ in the strip $|y| < 1$ and continuous in $|y| \leq 1$ and is $O(\exp(\exp(\epsilon|x|)))$ there for arbitrary $\epsilon > 0$, then $\log h_q(y; \phi)$ is a convex function of y for $|y| \leq 1$, and in particular, $h_q(0; \phi) \leq (h_q(1; \phi)h_q(-1; \phi))^{1/2}$, where

$$h_q(y; \phi) = \left(\int_{-\infty}^{\infty} |\phi(x + iy)|^q dy \right)^{1/q}$$

(See Gabriel [256].) We apply this to the function $A(z) F(z)/G(z)$, where $A(z)$ is a polynomial of degree m , $A(z) = a_m z^m + \dots$, $F(z)$ is analytic and satisfies the above growth condition in the strip, and $G(z) = \Pi g(z; z_k)$, the product being taken over all zeros z_k of A in the strip; and $g(z; z_k)$ is a function which maps the strip conformally onto the unit circle $|w| \leq 1$ and the point z_k into 0:

$$g(z; z_k) = \frac{\Psi(z) - \Psi(z_k)}{\Psi(z) + \Psi(\bar{z}_k)} = (z - z_k) \gamma(z; z_k), \quad \Psi(z) = \exp(\pi z/2)$$

We have $h_q(\pm 1; \phi) = h_q(\pm 1; AF)$, while $h_q(0; \phi) \geq |a_m|^{2-m} h_q(0; F)$, since, if x is real and $|y_k| < 1$, then $|\gamma(x; z_k)| \leq 2$, whereas if $|y_k| \geq 1$, then $|x - z_k| \geq 1 > 1/2$. Hence, for arbitrary $\lambda > 0$,

$$2h_q(0; F)^q \leq (2^m/|a_m|)^q (\lambda h_q(1; AF)^q + \lambda^{-1} h_q(-1; AF)^q)$$

We choose a coordinate system such that $\eta = [1, 0, 0, \dots]$ and apply this inequality to $F(z) = (\mathfrak{F}f)(z, \xi_2, \dots, \xi_N)$ and $A(z) = P(2\pi z, 2\pi\xi_2, \dots)$, then integrate with respect to $\xi_2, \dots, \xi_N$, and minimize the right-hand side as a function of λ .

Corollary 19a. If $f \in C_0^\infty(\Omega)$, where Ω is a domain in E_N such that $R = \sup_{x \in \Omega} |x_1| < +\infty$, and if P is a polynomial of degree m on E_N , and $p(1, 0, \dots, 0) = p_0 \neq 0$, then $L_q(\mathfrak{F}f) \leq (e^{2\pi R}/\pi^m |p_0|) L_q(\mathfrak{F}P(X)f)$.

This follows from a theorem of Plancherel and Pólya [525] (see Boas [57] *).

Corollary 19b. Let $Q_k(X) = (1 + \sum_{j=1}^N X_j^2)^k = (1 - \Delta)^k$. Then, under the hypotheses of Theorem 18, for $k > N/(2r)$, where $r = q/(q-1)$, there is a constant C such that

$$M_0(f) \leq C \{L_q(\eta; \mathfrak{F}Q_k(X) P(X)f) L_q(-\eta; \mathfrak{F}Q_k(X) P(X)f)\}^{1/2}$$

for all $f \in C_1^\infty(E_N)$ such that $f(x) \cosh(2\pi(x \cdot \eta)) \in \mathfrak{L}_1(E_N)$. If Ω and P

* If the strip $|y| < 1$ in the above proof is replaced by $|y| < m(2\pi R)$, then we obtain this inequality with the coefficient $|p_0|^{-1}(2eR/m)^m$.

satisfy the hypotheses of the first corollary, and $f \in C_0^\infty(\Omega)$, then

$$M_0(f; \Omega) \leq C \exp(2\pi R) L_q(\mathfrak{F}Q_k(X) P(X)f)$$

Application of Theorems 7 and 8 yield:

Corollary 19c. If $2 \leq q \leq +\infty$ and $k > N/(2r)$, $r = q/(q-1)$, then for every completely additive set function μ on E_N , there is a function $\phi \in \mathfrak{L}_q(E_N)$ such that

$$(30) \quad \int_{E_N} f d\mu = \int_{E_N} (Q_k(X) P(X)f) \phi(x) \cosh(2\pi(x \cdot \eta)) dV_x$$

for all $f \in C_1^\infty(\Omega)$ such that $\cosh(2\pi(x \cdot \eta)) Q_k(X) P(X)f \in \mathfrak{L}_q(E_N)$. If $k \geq [(N+2)/2]$, then there is for every such μ a bounded continuous ϕ which satisfies (30) for all $f \in C_1^\infty(E_N)$ such that $\cosh(2\pi(x \cdot \eta)) Q_k(X) P(X)f \in \mathfrak{L}_1(E_N)$. If Ω and P satisfy the hypotheses of Corollary 19a and $2 \leq q \leq +\infty$ and $k > N/(2r)$, then there is, for every completely additive set function μ on Ω a function $\phi \in \mathfrak{L}_q(\Omega)$ such that

$$(31) \quad \int f d\mu = \int_{\Omega} \phi(Q_k(X) P(X)f) dV$$

for all $f \in C_0^\infty(\Omega)$.

Let Ω' be the set of points in E_N of the form $x - y$, where $x, y \in \Omega$, and let Ω'' be the set of points of the form $x + y - z$, where $x, y, z \in \Omega$. Let ϕ be a function which satisfies (31) for the domain Ω' and $\Psi = \delta_0$, the Dirac delta with singularity at the origin. Let $B_\Omega(x) = \phi(-x)$ for $x \in \Omega'$ and $B_\Omega(x) = 0$ otherwise. Then

$$f = Q_k(X)(B_\Omega * P(X)f)$$

for all $f \in C_0^\infty(\Omega)$. If $g \in C_0^\infty(\Omega)$ and $f = Q_k(X)(B_\Omega * g)$, then $P(X)f = g$ and $f \in C_0^\infty(\Omega'')$.

Corollary 19d. Let $\{\Omega_n\}$ be a sequence of bounded domains such that $E_N = \bigcup \Omega_n$ and no point of E_N belongs to more than a finite number of the domains Ω_n . Let $\phi_n \in C_0^\infty(\Omega_n)$ for all n and $\sum \phi_n(x) = 1$ for all $x \in E_N$. Then for $g \in C_0^\infty(E_N)$, the function $f = \sum Q_k(X)(B_{\Omega_n} * (\phi_n g)) \in C^\infty(E_N)$ and $P(X)f = g$.

Corollary 19e. If Ω and P satisfy the hypotheses of 19a, then for every $\Psi \in \mathfrak{L}_2(\Omega)$, there is a $\phi \in \mathfrak{L}_2(\Omega)$ such that

$$(\Psi, f)_\Omega = (u, P(X)f)_\Omega$$

for all $f \in C_0^\infty(\Omega)$.

This corollary can be made more precise by the use of Theorem 11. Let $Q(X) = \tilde{P}(X) P(X)$. Then by Theorem 13, the operator $P(X)$ can

be extended to a closed operator $\bar{P}$ on $\mathfrak{H}_0(\Omega; Q)$ to $\mathfrak{H}_0(\Omega; Q)$. Then by Theorem 11 we obtain

Corollary 19f. If $f \in \mathfrak{L}_2(\Omega)$ and Ω and P satisfy the hypotheses of 19a, then there is a unique $u \in \mathfrak{H}_0(\Omega; \tilde{P}P)$ such that

$$(f, \phi)_\Omega = (\bar{P}u, P(X)\phi)_\Omega$$

for all $\phi \in C_0^\infty(\Omega)$.

Hörmander's inequality (Corollary 19a, $q = 2$) can be easily generalized.

Corollary 19g. If Q is a nonnegative polynomial of degree m on E_N with principal part $q(\xi)$, and $q_0 = q(1, 0, \dots, 0) \neq 0$, and if $R = \sup_{x \in \Omega} |x_1| < +\infty$, then

$$L_2(f) \leq (\exp(2\pi R)/\pi^m q_0) \|f\|_Q$$

for all $f \in C_0^\infty(\Omega)$.

Corollary 19h. If P is a polynomial such that $\Re P(\xi) \geq 0$ for all $\xi \in E_N$, and $Q = (P + \bar{P})/2$, and Q and Ω satisfy the hypotheses of 19g, then for all $f \in \mathfrak{L}_2(\Omega)$ there is a $u \in \mathfrak{H}_0(\Omega; Q)$ such that

$$(f, \phi)_\Omega = (u, \tilde{P}(X)\phi)_\Omega$$

for all $\phi \in C_0^\infty(\Omega)$.

All of these results can easily be generalized to systems of equations by the methods of the preceding section. Fuglede [250] has some important unpublished results in this direction.

The proof of Theorem 19 suggests an explicit formula for a fundamental solution.

Theorem 20. Let P be a polynomial of degree m with principal part p and suppose that $p(1, 0, \dots, 0) \neq 0$. For $\xi^{(2)} = [\xi_2, \dots, \xi_N] \in E_{N-1}$ let $G(z, \xi^{(2)} = \Pi g(z; z_k)$, the product being taken over all zeros of $P(2\pi z, 2\pi \xi^{(2)})$ in the strip $|y| < 1$. Let

$$\beta(x_1, \xi^{(2)}, \lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp(-2\pi i x_1 \sigma) d\sigma}{G(\sigma + \lambda i, \xi^{(2)})(\sigma + \lambda i - \xi_1)(\sigma + \lambda i - 2i)}$$

and let

$$\begin{aligned} B_k(x) = & \int_{E_N} \frac{G(\xi_1, \xi^{(2)})(\xi_1 - 2i)}{P(2\pi \xi)Q_k(2\pi \xi)} \{ \beta(-x_1, \xi^{(2)}, 1) \\ & \times \exp(2\pi x_1 + 2\pi i(x^{(2)} \cdot \xi^{(2)})) \\ & - \beta(-x_1, \xi^{(2)}, -1) \exp(2\pi x_1 + 2\pi i(x^{(2)} \cdot \xi^{(2)})) \} dV_\xi \end{aligned}$$

Then $B_k(x)/\cosh(2\pi x_1)$ is a bounded continuous function on E_N for $k \geq [(N+3)/2]$, and

$$f(0) = \int_{E_N} B_k(-x) (Q_k(X) P(X)f)(x) dVx$$

for all $f \in C_1^\infty(E_N)$ such that $\cosh(2\pi x_1)Q_k(X)P(X)f \in \mathfrak{L}_1(E_N)$ and more generally

$$f = Q_k(X)(B_k * P(X)f)$$

Other general formulas for fundamental solutions were found independently by Trèves [652] and Hörmander [339]. The latter has shown the existence of fundamental solutions with important smoothness properties. John [352] has given an explicit formula for a fundamental solution for an arbitrary linear elliptic operator with analytic coefficients. Other formulas for fundamental solutions for special classes of operators were given by Fredholm [234], Zeilon [734], Herglotz [316], Riesz [574], Gårding [261], [262], Leray [408], de Rham [567], and Fourés-Bruhat [233].

2. General regularity theorems

We turn now to the question of when a differential operator $K(X)$, where K is a polynomial, can be extended to a closed operator on a space of the type $\mathfrak{H}_0(\Omega, Q)$. The problem is essentially to translate the criteria of Theorems 3 and 13 into more applicable form. It is easy to prove.

Theorem 21. If K and Q are polynomials and $Q \geq 0$ on E_N , then $K(X)$ can be extended to a closed operator on $\mathfrak{H}_0(E_N, Q)$ if and only if there is a constant C such that $|K(\xi)|^2 \leq CQ(\xi)$ for all $\xi \in E_N$. This condition is sufficient for $K(X)$ to be extendable to a closed operator on $\mathfrak{H}_0(\Omega, Q)$ to $\mathfrak{L}_2(\Omega)$ for an arbitrary domain $\Omega \subset E_N$.

In his thesis [337] Hörmander gave a complete solution of this problem for a bounded domain in the case where $Q = \tilde{P}P$. In a recent letter to the author, he gave the solution of the general problem. The solution is conveniently expressed in terms of his concept of relative strength of differential operators. He says that P is weaker than Q if there is a bounded domain Ω in E_N and a constant C such that $L_2(P(X))\phi \leq CL_2(Q(X)\phi)$ for every $\phi \in C_0^\infty(\Omega)$. We introduce the polynomial

$$P^\sharp(\xi) = \sum_{\alpha} \tilde{P}^{(\alpha)}(\xi) P^{(\alpha)}(\xi)$$

where

$$P^{(\alpha)}(\xi) = (\Xi^\alpha P)(\xi)$$

Theorem 22. The polynomial P is weaker than Q if and only if $|P(\xi)|^2/Q^\sharp(\xi)$ is bounded in E_N , and then for any bounded domain $\Omega \in E_N$ and any $1 \leq q \leq +\infty$, $L_q(\mathfrak{F}P(X)\phi) \leq CL_q(\mathfrak{F}Q(X)\phi)$ for all $\phi \in C_0^\infty(\Omega)$. If Q is nonnegative on E_N , then there is a bounded domain Ω and a constant C such that $|(\phi, P(X)\phi)_\Omega| \leq C \|\phi\|_Q^2$ for all $\phi \in C_0^\infty(\Omega)$ if and only if P is weaker than Q , and then this condition holds for all bounded domains. If Q is nonnegative on E_N and K is a polynomial on E_N , then the operator $K(X)$ can be extended to a closed operator on $\mathfrak{H}_0(\Omega; Q)$ to $\mathfrak{L}_2(\Omega)$ for some bounded domain if and only if $\tilde{K}K$ is weaker than Q , i.e., $|K(\xi)|^4/Q^\sharp(\xi)$ is bounded on E_N , and then this is true for every bounded domain.

A major step in the proof of this theorem is

Theorem 23. If P and Ω satisfy the hypotheses of 19a, then

$$L_q(\mathfrak{F}(\Xi_1^k P(X)f)) \leq (2^k m!/(m-k)!) \exp(2\pi R) L_q(\mathfrak{F}P(X)f)$$

$$L_q(\mathfrak{F}P(X + \lambda\eta)f) \leq (1 + 2|\lambda|)^m \exp(2\pi R) L_q(\mathfrak{F}P(X)f)$$

for $1 \leq q \leq +\infty$, all $f \in C_0^\infty(\Omega)$, $\eta = [1, 0, \dots, 0]$. If Ω is bounded and $a = \sup_{x \in \Omega} \|x\|$, then

$$L_q(\mathfrak{F}P^{(\alpha)}(X)f) \leq (m! 2^{|\alpha|} \exp(2\pi Na)/(m - |\alpha|)!) L_q(\mathfrak{F}P(X)f)$$

$$L_q(\mathfrak{F}P(X + \eta)f) \leq (1 + 2\|\eta\|)^m \exp(2\pi Na) L_q(\mathfrak{F}P(X)f)$$

for $f \in C_0^\infty(\Omega)$, all $\eta \in E_N$. Thus $P^{(\alpha)}$ and $P(\xi + \eta)$ are weaker than P for $\alpha \geq 0$, all $\eta \in E_N$.

In the special case $q = 2$, Hörmander proved a sharper result: Under the hypotheses of 19a, $L_2(\Xi_1^k P(X)f) \leq (2R)^k (m!/(m-k)!) L_2(P(X)f)$ and

$$L_2(P(X + \lambda\eta)f) \leq (1 + 2R|\lambda|)^m L_2(P(X)f)$$

for all $f \in C_0^\infty(\Omega)$, $\eta = [1, 0, \dots, 0]$.

The condition that $|P(\xi)|^2/Q^\sharp(\xi)$ be bounded brings into play the terms in Q of lower degree. Thus, in the case of the Schrödinger equation for a free particle $Q(\xi) = \xi_N - \sum_{k=1}^{N-1} \xi_k^2$, and P is weaker than Q if and only if $P(\xi) = a_1 + \sum_{j=1}^{N-1} a_j \xi_j + a_N Q(\xi)$. In the case of the heat equation $Q(\xi) = i\xi_N - \sum_{k=1}^{N-1} \xi_k^2$, and P is weaker than Q if and only if $P(\xi) = a_N \xi_N + P_1(\xi)$, where P_1 is a polynomial of degree ≤ 2 in $\xi_1, \dots, \xi_{N-1}$. Here $a_0, a_1, \dots, a_N$ are arbitrary constants.

Theorem 24. If P is a polynomial of degree m on E_N , then P is stronger than any polynomial of degree $< m$ if and only if there is no point $\xi \neq 0$ in E_N at which $\nabla p(\xi) = 0$.

Theorem 21 can be generalized:

Theorem 25. If $1 \leq q \leq 2$ and K and Q are polynomials on E_N and $Q \geq 0$ on E_N , and if $|K|/\sqrt{Q} \in \mathfrak{L}_r(E_N)$, $r = 2q/(2 - q)$, then $K(X)$ can be extended to a closed operator on $\mathfrak{H}_0(\Omega; Q)$ to $\mathfrak{L}_s(\Omega)$ $s = q/(q - 1)$, for every domain $\Omega \subset E_N$. If Ω and Q satisfy the hypothesis of 19g, then the condition $|K|/(1 + Q)^{1/2} \in \mathfrak{L}_r(E_N)$ is sufficient. In the case $q = 1$, $s = +\infty$, $r = 2$, these conditions are sufficient for $K(X)$ to be extendable to a closed operator on $\mathfrak{H}_0(\Omega, Q)$ to $C_0^0(\Omega)$.

This fairly trivial result can be sharpened considerably, as in Theorem 22.

Theorem 26. Under the hypotheses of Theorem 25, if Ω is a bounded domain and $|K|^4/Q^{\sharp}(\xi) \in \mathfrak{L}_{r/4}(E_N)$, then $K(X)$ can be extended to a closed operator on $\mathfrak{H}_0(\Omega; Q)$ to $\mathfrak{L}_s(\Omega)$. If $q = 1$, $r = 2$, this condition is necessary and sufficient that $K(X)$ be extendable to a closed operator on $\mathfrak{H}_0(\Omega; Q)$ to $C_0^0(\Omega)$.

If we apply these results to the operator $Q(X) = (\sum_1^N X_j^2)^n = (-\Delta)^n$, $\mathfrak{H}_0^n(\Omega) = \mathfrak{H}_0(\Omega; (-\Delta)^n)$, we obtain

Corollary 26a. If $n > k + (N/2)$ and Ω is bounded in some direction, then every element of $\mathfrak{H}_0^n(\Omega)$ is equal almost everywhere to an element of $C_0^k(\Omega)$. If $1 \leq q \leq 2$ and $n > k + N(q^{-1} - 2^{-1})$, and K is a polynomial of degree k , then $K(X)$ can be extended to a closed operator on $\mathfrak{H}_0^n(\Omega)$ to $\mathfrak{L}_s(\Omega)$.

This is part of an important lemma of Sobolev [624].

We now turn to the problem of the compactness of certain operators.

Theorem 27. If K and Q are polynomials and Ω is a bounded domain in E_N and $Q \geq 0$ in E_N , then the operator $K(X)$ can be extended to a compact operator on $\mathfrak{H}_0(\Omega; Q)$ to $\mathfrak{L}_2(\Omega)$ if $|K(\xi)|/(1 + Q(\xi))^{1/2} \rightarrow 0$ as $\|\xi\| \rightarrow \infty$.

Theorem 28. (Hörmander [337].) Under the same hypotheses, $K(X)$ can be extended to a compact operator $\mathfrak{H}_0(\Omega; Q)$ to $\mathfrak{L}_2(\Omega)$ if and only if $|K|(\xi)^4/Q^{\sharp}(\xi) \rightarrow 0$ as $\|\xi\| \rightarrow \infty$.

We say that P is a *complete polynomial* if the only $\eta \in E_N$ such that $P(\xi + t\eta) = P(\xi)$ for all t is $\eta = 0$.

Corollary 28a. Let P be a polynomial and Ω be a bounded domain in E_N . Consider $P(X)$ as an operator on $\mathfrak{L}_2(\Omega)$ to itself with dense domain $C_0^\infty(\Omega)$, and let R_P be the closure of the range of $P(X)$. Then $P(X)^{-1}$ can be extended to a compact operator on R_P to $\mathfrak{L}_2(\Omega)$ if and only if P is a complete polynomial.

If we consider the operator $P(X)$ in this way, then $u \in \mathfrak{D}_{P*}$ if and only if $u \in \mathfrak{L}_2(\Omega)$ and there is a $v \in \mathfrak{L}_2(\Omega)$ such that

$$(u, P(X)\phi)_\Omega = (v, \phi)_\Omega$$

for all $\phi \in C_0^\infty(\Omega)$.

Theorem 29. If $u \in \mathfrak{D}_{P*}$ and $u \equiv 0$ outside a compact subset of Ω , then $u \in \mathfrak{H}_0(\Omega; \bar{P}P)$. If Ω is bounded and there exists a finite number of domains Ω_j such that $\bar{\Omega} \subset U\Omega_j$, and for every j and $\epsilon > 0$ there is a vector y with $\|y\| < \epsilon$ such that $x + y \in \Omega$ for all $x \in \bar{\Omega} \cap \Omega_j$, then P^* is the closure of $P(X)$, considered as an operator on $\mathfrak{L}_2(\Omega)$ to itself with domain $C_0^\infty(\Omega)$.

The last statement is an unpublished result of Hörmander. For second-order elliptic operators with variable coefficients it has been announced without proof by Birman [51].

An operator $P(X)$ is said to be of *local type* on Ω if, for any $u \in \mathfrak{D}_{P*}(\Omega)$ and any $\phi \in C_0^\infty(\Omega)$, the function $\phi u \in \mathfrak{D}_{P*}(\Omega)$.

Theorem 30. If P is a complete polynomial, then each of the following conditions is equivalent to $P(X)$ being of local type:

- (a) For an arbitrary constant A there is a constant $B(A)$ such that $P(\xi + i\eta) \neq 0$ for $\xi, \eta \in E_N$, $\|\xi\| > B(A)$, $\|\eta\| < A$.
- (b) For every $\eta \in E_N$, $P(\xi + \eta)/P(\xi) \rightarrow 1$ as $\|\xi\| \rightarrow \infty$, $\xi \in E_N$.
- (c) For every $\alpha > 0$, $P^{(\alpha)}(\xi)/P(\xi) \rightarrow 0$ as $\|\xi\| \rightarrow \infty$, $\xi \in E_N$.
- (d) For every $\eta \in E_N$, $|P(\xi + i\eta)| \rightarrow \infty$ as $\|\xi\| \rightarrow \infty$, and the convergence is uniform for $\|\eta\| \leq A$, A arbitrary.
- (e) For an arbitrary constant $A > 0$, there is a constant C such that

$$P^\#(\xi) = \sum |P^{(\alpha)}(\xi + i\eta)|^2 \leq C(1 + |P(\xi + i\eta)|^2)$$

for $\|\eta\| \leq A$ and all $\xi \in E_N$.

Hörmander proves this by constructing a fundamental solution with special smoothness and growth properties.

Theorem 31. If $P(X)$ is of local type and $u \in C_0^\infty(E_N)$, set

$$E(u) = \int_{E_{N-1}} d\xi_2 \cdots d\xi_N \int_C \frac{(\mathfrak{F}u)(-z_1, -\xi_2, \dots, -\xi_N)}{P(2\pi z_1, 2\pi \xi_2, \dots, 2\pi \xi_N)} dz_1$$

where C consists of the real axis with a certain interval $(-a, a)$ replaced by a semicircle in the lower half-plane. Then

$$E(P(-X)u) = u(0)$$

for $u \in C_0^\infty(E_N)$. There is a function $E(x) \in C^\infty$ for $x \neq 0$ such that if $u \in C_0^\infty(\Omega)$ and $0 \notin \bar{\Omega}$, then $E(u) = \int_{E_N} E(x) u(x) dV_x$. There is a constant $b > 0$ such that for every bounded domain Ω for which $0 \notin \bar{\Omega}$ there is a constant $C = C(\Omega)$ such that

$$|D^\alpha E(x)| \leq C^{|\alpha|} \Gamma(|\alpha|/b)$$

for $x \in \Omega$, $\alpha \geq 0$.

Hörmander also gives the following result on the smoothness of weak solutions of equations with operators of local type.

Theorem 32. If P is a complete polynomial of local type, then there is for each n an integer $k(n)$ such that if $u \in \mathfrak{D}_{(P^k)*}(\Omega)$, then u is equal almost everywhere to a function in $C^n(\Omega)$. If $u \in \mathfrak{D}_{(P^k)*}(\Omega)$ for all k , then $u \in C^\infty(\Omega)$. In particular, if $P(X)*u = 0$, then $u \in C^\infty(\Omega)$.

This last result is a generalization of a lemma of Zaremba [732], often attributed to Weyl [705]. It can be made more precise.

Theorem 33. For $y \in E_N$, let

$$\rho(y) = \limsup_{\|y \cdot \xi\| \rightarrow \infty} \left(\sup_{P(\xi)=0} \frac{\log \|y \cdot \xi\|}{\log \|\xi - \xi\|} \right)$$

If P is a complete polynomial of local type and $P(X)*u = 0$ in Ω , then for every compact subset $A \subset \Omega$, there is a constant C such that

$$M_0((y \cdot X)^n u(x); A) \leq C^n \Gamma(n\rho(y)), \quad n \geq 0$$

If $\rho(y) \leq 1$, then $u(x + ty)$ is analytic in t for each $x \in \Omega$.

If the principal part $p(\xi)$ of $P(\xi)$ is different from 0 for all $\xi \neq 0$ in E_N , then P is of *elliptic type*. An elliptic operator is always of local type. If $P(\xi) = Q(\xi)^{2k} + R(\xi)$, Q is a real polynomial of degree m , $k \geq 2$, R is a polynomial of degree $2km - 2(k - 1)$, and $r(\xi) > 0$ for every $\xi \in E_N$ such that $\xi \neq 0$, $q(\xi) = 0$, then P is of local type. If P is of local type

and $P_1(\xi)/P(\xi) \rightarrow 0$ as $\|\xi\| \rightarrow +\infty$, then $P + \lambda P_1$ is of local type for every constant λ . Thus there is a wide class of operators of local type.

Theorem 34. If P is a complete polynomial of local type, then $P^* = \tilde{P}$, where $P(X)$ is considered as an operator on $\mathfrak{L}_2(\Omega)$ to itself with domain $C^\infty(\Omega)$.

THE METHOD OF GELFAND AND ŠILOV

In this section we report on some recent work of Gelfand and Šilov [277], Šilov [613], Borok [61], Kostyučenko and Šilov [385], and Lyantse [430]. It turns out, on the basis of their work, that instead of working with some universal testing space as in Schwartz [603], it is preferable to choose a testing space adapted to the specific problem under consideration.

As we remarked before, if $\mathfrak{S}$ is a testing space, then by $\mathfrak{S}(M_1 \times M_2)$ we mean the set of $M_1 \times M_2$ matrix-valued functions $\phi(x) = (\phi_{jk}(x))$, where $\phi_{jk} \in \mathfrak{S}$ for $1 \leq j \leq M_1$, $1 \leq k \leq M_2$. In particular, $\mathfrak{S}(M \times 1)$ is the space of M -dimensional column vectors with components in $\mathfrak{S}$. The dual of $\mathfrak{S}(M \times 1)$ is $(\mathfrak{S}(M \times 1))^* = \mathfrak{S}^*(1 \times M)$, the space of M -dimensional row vectors with components in $\mathfrak{S}^*$.

Consider a system of the form

$$(32) \quad u_t = P(X)u, \quad u(0, x) = u_0(x), \quad x \in E_N, \quad t \geq 0$$

where u is an M -dimensional vector, $u = [u_1, \dots, u_M]$, and $P(\zeta)$ is an $M \times M$ matrix-valued polynomial. The Fourier transform of (32) is

$$(33) \quad U_t = P(2\pi\xi)U, \quad U(0, \xi) = U_0(\xi)$$

whose solution is

$$U(t, \xi) = \exp(tP(2\pi\xi)) U_0(\xi) = K(t, \xi) U_0(\xi)$$

The matrix $K(t, \zeta)$ is an entire function of t and ζ , and is of order $\leq m$ in ζ for fixed $t > 0$, where m is the degree of P . These formal considerations can be interpreted in various ways.

Theorem 35. If the elements of $K(t, \xi)$ are in $\mathfrak{M}(\mathfrak{S})$ for $0 \leq t \leq t_0$, then (33) has a unique solution in $\mathfrak{S}^*(M \times 1)$ for every $U_0 \in \mathfrak{S}^*(M \times 1)$ and $0 \leq t \leq t_0$. This solution depends continuously on U_0 in the sense of the topology of $\mathfrak{S}^*(M \times 1)$, i.e., if $U_{0n} \rightarrow U_{0\infty}$ in $\mathfrak{S}^*(M \times 1)$, then $U_n(t, \xi) = K(t, \xi)U_{0n}(\xi) \rightarrow U_\infty(t, \xi) = K(t, \xi)U_{0\infty}(\xi)$ in $\mathfrak{S}^*(M \times 1)$ for $0 \leq t \leq t_0$.

The uniqueness may fail if the elements of $K(t, \xi)$ are not in $\mathfrak{M}(\mathfrak{S})$. Thus if $N = 1$, $P(\xi) = -\xi^2/(4\pi^2)$, $\mathfrak{S} = \mathbb{Z}^p$, $1 < p < 2$, then $K(t, \xi) = \exp(-t\xi^2)$ is not a multiplier of $\mathfrak{S}$. If f is any function in $C_1^\infty(E_1)$ such

that $|f^{(n)}(t)| \leq n^{kn}$ for all t and all $n \geq 0$, where k is a constant and $1 < k < 2/p$, then the element $F(t)$ in S^* defined by

$$F(t)(\phi) = \Sigma(-1)^n \phi^{(2n)}(0) f^{(n)}(t)/(2n)!$$

is a solution of $F'(t) = -\xi^2 F(t)$, i.e., for every $\phi(\xi) \in \mathcal{S}$, we have

$$\frac{d}{dt}(F(t)(\phi)) = F(t)(-\xi^2 \phi)$$

If, as is possible (Carleman [111]), we choose f so that $f^{(n)}(0) = 0$ for $n \geq 0$, and $f(t) \neq 0$, then $F(0) = 0$, $F(t) \not\equiv 0$.

Now equation (33) on $S^*(M \times 1)$ is equivalent to (32) on $(\mathcal{FS})^*(M \times 1)$. We thus obtain

Corollary 35a. Under the hypotheses of Theorem 35, the equation (32) has a unique solution in $(\mathcal{FS})^*(M \times 1)$ for every $u_0 \in (\mathcal{FS})^*(M \times 1)$. This solution depends continuously on u_0 in the topology of $(\mathcal{FS})^*(M \times 1)$.

As we shall see, this is a strong uniqueness theorem but a weak existence theorem, for, if $\mathcal{S}$ is any of the common testing spaces, S^* is a very large space, so that a solution in S^* may be a very pathological sort of object. These results show that we must study the matrix K carefully and choose the testing space for (32) and (33) accordingly.

A first crude result is

Theorem 36. If P is a polynomial of degree m , then $K_{jk}(t, \xi)$ is a multiplier of Z_r^r for $r > m$, where $K(t, \xi) = (K_{jk}(t, \xi))$. Let $\mu = m/(m-1)$. If $q < \mu$, then equation (32) has a unique solution in $Z_q^{q*}(M \times 1)$ for every $u_0 \in Z_q^{q*}(M \times 1)$, and this solution depends continuously on u_0 . In particular, if $u(t, x) \in C^m(E_N)(1 \times M)$ for $0 \leq t \leq t_0$ and satisfies (32) there with $u_0(x) = 0$, and if for $\epsilon > 0$ there is a constant $A(\epsilon)$ such that

$$\int_{E_N} \|X^\alpha u(t, x)\| \exp(-\epsilon \|x\|^q) dV_x \leq A(\epsilon) < +\infty$$

for $0 \leq t \leq t_0$, $|\alpha| \leq m$, then $u(t, x) = 0$ for $0 \leq t \leq t_0$. Or if $u(t, x)$ is a measurable $1 \times M$ matrix function and $u(0, x) = 0$, and $\|u(t, x)\| \exp(-\epsilon \|x\|^q) \in \mathcal{L}_1(E_N)$ for $0 \leq t \leq t_0$, and if

$$\frac{d}{dt} \int_{E_N} u(t, x) \phi(x) dV_x = \int_{E_N} u(t, x) P(-X) \phi(x) dV_x$$

for $0 \leq t \leq t_0$, all $\phi \in Z_q^q(M \times 1)$, then $u(t, x) = 0$ almost everywhere in E_N for $0 \leq t \leq t_0$.

For finer results we must introduce the functions

$$\begin{aligned}\Lambda(\zeta) &= \max \Re \lambda_j, & \text{where} & \quad \det (\lambda_j I - P(\zeta)) = 0 \\ \Lambda_1(R) &= \max \Lambda(\xi) & \text{for} & \quad \|\xi\| = R, \quad \xi \in E_N \\ \Lambda_2(R) &= \max \Lambda(\zeta) & \text{for} & \quad \|\zeta\| = R, \quad \zeta \in C_N\end{aligned}$$

and the constant $A = \max \|P(\zeta)\|$ for $\|\zeta\| = 1, \zeta \in C_N$. Then there exist constants C and ν such that

$$(34) \quad \exp(t\Lambda(2\pi\zeta)) \leq \|K(t, \zeta)\| \leq C(1 + At\|\zeta\|^m)^{\nu-1} \exp(t\Lambda(2\pi\zeta)) \text{ for } t \geq 0$$

The constant ν can always be taken equal to $1 + n$, where n is the degree of the minimal equation of $P(2\pi\zeta)$. If the projection operators of C_N onto the eigenspaces of $P(2\pi\zeta)$ have uniformly bounded norms, then ν may be taken equal to the maximum multiplicity of the eigenvalues of $P(2\pi\zeta)$. If $P(2\pi\zeta)$ is a normal matrix, then $\nu = 1$. Now by a theorem of Seidenberg [605] (see Hörmander [337] pp. 225–226) the functions $\Lambda_j(R)$ are algebraic for R sufficiently large, and so, by Puiseux expansion (see Bliss [54]), there exist rational numbers b and m_0 such that $\lim_{R \rightarrow \infty} \Lambda_1(R)R^{-b} = a_1 \neq 0$ exists if $\Lambda_1(r) \not\equiv 0$ and $\lim_{R \rightarrow \infty} \Lambda_2(R)R^{-m_0} = a_2 > 0$ exists. By the Phragmen-Lindelöf principle (Boas [57]), it follows that $b \leq m_0 \leq m$. For example, the equation ($M = N = 1$)

$$\frac{\partial^2 f}{\partial t^2} = -\frac{\partial^4 f}{\partial x^4}, \quad f(0, x) = f_0(x), \quad f_t(0, x) = f_1(x)$$

is equivalent to

$$\frac{\partial u_1}{\partial t} = u_2, \quad \frac{\partial u_2}{\partial t} = -\frac{\partial^4 u_1}{\partial x^4}, \quad u(0, x) = [f_0(x), f_1(x)]$$

and

$$\begin{aligned}P(\xi) &= \begin{pmatrix} 0 & 1 \\ -\xi^4 & 0 \end{pmatrix} \\ K(t, \xi) &= I \cos((2\pi\xi)^2 t) + P(2\pi\xi) \frac{\sin((2\pi\xi)^2 t)}{(2\pi\xi)^2}\end{aligned}$$

$\Lambda_1(R) = 0, m_0 = 2, m = 4$. In the equation

$$f_{tt} = i^{-1}f_x - f_{xxxx}$$

$b = 1, m_0 = 2, m = 4$.

We immediately see that in Theorem 36, m can be replaced by m_0 .

Theorem 37. The elements of $K(t, \xi)$ are multipliers of $\mathcal{Z}^p$ if and only if $\Lambda(\xi) \leq C_1 + C_2 \|\eta\|^p$ for some constants C_1 and C_2 and all $\xi \in C_N$. In that case, if $p > 1$, then (32) has a unique solution in $\mathcal{K}_q^*(M \times 1)$,

$q = p/(p - 1)$, for every $u_0 \in \mathcal{K}_q^*(M \times 1)$. If $p = 1$, then (32) has a unique solution in $(C_0^\infty(E_N))^*(M \times 1)$ for every u_0 in this space. These solutions depend continuously on u_0 .

The argument of Theorem 35 can be turned in a different way.

Theorem 38. If (32) has a solution in $\mathcal{S}(M \times 1)$ for every $u_0 \in \mathcal{S}(M \times 1)$, then the elements K_{jk} of K are multipliers of $\mathfrak{FS}$ for $t \geq 0$, and, conversely, if this is true, then (32) has a unique solution in $\mathcal{S}(M \times 1)$ for every $u_0 \in \mathcal{S}(M \times 1)$ and this solution depends continuously on u_0 .

If we combine this with Theorem 35, we obtain

Corollary 38a. If (32) has a solution in $\mathcal{S}(M \times 1)$ for every $u_0 \in \mathcal{S}(M \times 1)$ then it has a solution in $\mathcal{S}^*(M \times 1)$ for every u_0 in this space, and this solution depends continuously on u_0 .

Let us take $\mathcal{S} = C_1^\infty(E_N)$ so that $\mathfrak{FS} = \mathcal{S}$. Then $K_{jk}(t, \xi)$ is a multiplier of $\mathfrak{FS}$ for a fixed $t > 0$ and $1 \leq j, k \leq M$ if and only if $b \leq 0$ or $b > 0$, $a_1 < 0$, so that $\Lambda(\xi)$ is bounded above for $\xi \in E_N$. Now $\|K(t, \xi)\| \leq C(t)(1 + \|\xi\|^2)^r$, where $C(t)$ is a polynomial in t , and $r \leq m(\nu - 1)/2$. Then $(1 + 4\pi^2 \|\xi\|^2)^{-k} K(t, \xi) = \mathfrak{F}G_k(t, x)$, where G_k is a bounded continuous matrix-valued function on E_N for $k > r + (N/2)$.

Hence if $Q_k(X) = (1 + \sum_{j=1}^N X_j^2)^k = (1 - \Delta)^k$, then

$$(35) \quad u(t, x) = Q_k(X)(\iota_k^* u_0) = Q_k(X) \int_{E_N} G_k(t, x - y) u_0(y) dV_y$$

for $u_0 \in C_1^\infty(E_N)$.

Corollary 38b. If (32) has a solution in $C_1^\infty(E_N)(M \times 1)$ for every $u_0 \in C_1^\infty(E_N)(M \times 1)$, then $\Lambda(\xi)$ is bounded above for $\xi \in E_N$. In that case, (32) has a unique solution in $C_1^\infty(E_N)(M \times 1)$ for every u_0 in this space and this solution depends continuously on u_0 . For $k > r + (N/2)$, there is a bounded continuous function $G_k(t, x)$ such that this solution is given by (35). If $\gamma > N/2$, then there is a constant $C_1(t)$ such that

$$\|u(t, x)\| \leq C_1(t) M_0((1 + \|x\|^2)^\gamma \|Q_k(X) u_0(x)\|; E_N)$$

for $t > 0$. The number r can always be chosen as $m(\nu - 1)/2$.

Now let us choose $\mathcal{S} = C_0^\infty(M \times 1)$ so that $\mathfrak{FS} = \mathcal{Z}^1$. Then $K_{jk}(t, \xi)$ is a multiplier of $\mathfrak{FS}$ for $1 \leq j, k \leq M$ if and only if there exist constants C_1 and C_2 such that $\Lambda(\xi + i\eta) \leq C_1 + C_2 \|\eta\|$ for $\xi, \eta \in E_N$. By a generalization of a classical theorem of Paley and Wiener [495] (see Schwartz [603], Gelfand and Šilov [277]), there is an $M \times M$ matrix-valued continuous function $G_k(t, x)$, for $k > r + (N/2)$ such that $u(t, x)$ is represented by (35), and $G_k(t, x) = 0$ for $\|x\| \geq C_2 t$. This is essen-

tially a result due to Gårding [262] for the case $M = 1$, Šilov [613] for the case $N = 1$, and Borok [61] for the general case. Gårding treated the general case in an unpublished manuscript.

If $P(X)'$ is the transpose of $P(X)$ and G_k is the function appearing in (35), then the solution of the equation

$$(36) \quad v_t = P(-X)'v, \quad v(0, x) = v_0(x)$$

in $C_1^\infty(E_N)(M \times 1)$, for given v_0 in this space, is

$$v(t, x) = Q_k(X) \int_{E_N} (G_k(y - x, t))' v_0(y) dV_y$$

We can now solve (32) in the space $(C_1^\infty(E_N))^*(M \times 1)$. If $u(t)$ is a solution of (32) in $(C_1^\infty(E_N))^*(M \times 1)$ and $w(t, x) \in C_1^\infty(E_N)(M \times 1)$ and is differentiable in t , then

$$\frac{d}{d\tau} (u(\tau))' w(\tau) = u(\tau)'(w_\tau + P(-X)'w)$$

so that if $w(\tau, x) = v(t - \tau, x)$ is the solution of (36), then $(u(\tau))' w(\tau)$ is constant, and

$$(37) \quad (u(t))'(v_0) = (u_0)'(Q_k(X) \int_{E_N} (G_k(y - x, t))' v_0(y) dV_y)$$

which is equivalent to (35), the convolution being interpreted in the sense of the theory of distributions. We thus obtain

Corollary 38c. If $\Lambda(\xi)$ is bounded above for $\xi \in E_N$, and $u_0 \in C^{2k}(E_N)$ and if there exist constants C_1 and ρ such that $\|X^\alpha u_0\| \leq C(1 + \|x\|)^\rho$ for all $x \in E_N$, $|\alpha| \leq 2k$, then there exists a unique distribution $u(t)$ in $(C_1^\infty(E_N))^*(M \times 1)$ which satisfies (32), and this distribution satisfies (37) for all $v_0 \in C_1^\infty(E_N)(M \times 1)$. This solution $u(t, x)$ is given by (35).

We can treat the case of $(C_0^\infty(E_N))^*(M \times 1)$ in a similar fashion.

We shall now discuss various classes of differential equations in further detail, guided by the general theories developed in this chapter.

EQUATIONS OF PARABOLIC TYPE

EQUATIONS WITH CONSTANT COEFFICIENTS

1. The Cauchy problem for homogeneous equations

We consider a system of equations of the form (32). We say that (32) is *parabolic* if $\Lambda(\xi) \rightarrow -\infty$ as $\|\xi\| \rightarrow \infty$, $\xi \in E_N$. Hence the number b , such that $\lim_{R \rightarrow \infty} \Lambda_1(R)R^{-b} = a_1 \neq 0$ exists, must be positive and $a_1 < 0$. We can show that b must be an even integer and $2m_0$ must be an integer. It follows from (34) that there exist constants $C_0, C_1, C_2 > 0$ such that $\|K(t, \xi)\| \leq C_0 t^{-\gamma} \exp((C_1 - C_2 \|\xi\|^b)t)$ for $t > 0$, $\xi \in E_N$, where $\gamma = (m - b)(\nu - 1)/b$, so that for $t > 0$, $K(t, \xi)$ is in the class $Z_b^{m_0}(M \times M)$. Let $K(t, \xi) = \mathfrak{F}G(t, x)$. The function G is the *fundamental* solution of (32),

$$G_t = P(X)G, \quad G(0, x) = \delta_0$$

Application of Theorem 15 yields

Theorem 39. For $t > 0$, $x \in E_N$, we have

$$\|G(t, x)\| \leq C_3 t^{-\gamma - (N/b)} \exp(C_4 t - C_5 t^{1-s} \|x\|^s)$$

where $s = b/(m_0 - 1)$ and $C_3, C_4, C_5 > 0$ are constants. As a function of x , $G(t, x)$ is an entire function of class Z_s^r , $r = b/(b - 1)$, for $t > 0$. Furthermore, if $\beta \geq 0$, then

$$\|D^\beta G(t, x)\| \leq C_3(\beta) t^{-\gamma - N/b - |\beta|/b} \exp(C_4 t - C_5 t^{1-s} \|x\|^s)$$

for $t > 0$, $x \in E_N$.

Special cases of this theorem were obtained by Petrovski [506], Ladyženskaya [394], Eidelman [185], and Šilov [613].

The following elementary properties of G are easy to prove:

$$\begin{aligned} \int_{E_N} G(t, x) dV_x &= \exp (tP(0)) \quad \text{for} \quad t > 0 \\ \int_{E_N} G(t, x-y) G(\tau, y) dV_y &= G(t+\tau, x) \quad \text{for} \quad t, \tau > 0 \\ \mathfrak{L}_p(\|x\|^k D^\beta G(t, x)) &\leq C_6(k, \beta, p)t^\sigma \\ \sigma &= (k + Np^{-1})(1 - s^{-1}) - \gamma - (N + |\beta|)/b \end{aligned}$$

If $U_0(\xi) = (\mathfrak{F}u_0)(\xi)$ and $(1 + \|\xi\|^2)^k U_0 \in \mathfrak{L}_2(E_N)$ for some $k \geq b\gamma/4$, then $\lim_{t \rightarrow 0} u(t, x) = u_0(x)$ in $\mathfrak{L}_2(E_N)$. If $(1 + \|\xi\|^2)^k U_0 \in \mathfrak{L}_2(E_N)$ for some $k \geq (b\gamma/2) + (N/4)$, then $u(t, x) \rightarrow u_0(x)$ uniformly as $t \rightarrow 0$.

By the use of the detailed information on G contained in Theorem 39, we can obtain quite precise results on the solutions of (32).

Theorem 40. If $\|u_0(x)\| \exp(-B\|x\|^c) \in \mathfrak{L}_1(E_N)$ and $c < s$, then the function

$$(38) \quad u(t, x) = \int_{E_N} G(t, x-y) u_0(y) dV_y = (T_t u_0)(x)$$

exists and satisfies (32) for all $t > 0$. If $c = s > 1$, then (38) exists and satisfies (32) for $0 < t < (C_5/B)^{1/(s-1)}$. If $c = s = 1$, and $B < C_5$, then (38) exists for all $t > 0$. If $1 < c < s$ and $\beta > B$, then $\|u(t, x)\| \exp(-\beta\|x\|^c) \in \mathfrak{L}_1(E_N)$ for all $t > 0$, while, if $0 \leq c \leq 1$, $c < s$, this is also true for $\beta = B$. If $c = s > 1$, then this is true for $\beta > B$ in an interval $0 < t < t_0(\beta, s)$. Also, under the indicated conditions,

$$\|u(t, x)\| \leq C_7 t^{-\gamma-(N/b)} \exp(C_8\|x\|^c)$$

for all $x \in E_N$. If $s > 1$ and $u_0 \in C^k$ for some $k \geq m_0(\gamma + (N/b)) - N$ at the point x_0 , then $\lim_{t \rightarrow 0} u(t, x) = u_0(x_0)$ as $t \rightarrow 0$, $x \rightarrow x_0$. If $s \leq 1$ and $\phi \in \mathcal{K}(E_N; Q_k)$, $Q_k(\xi) = (1 + \|\xi\|^2)^k$, for some $k > b\gamma + (N/2)$, then

$$\lim_{t \rightarrow 0} \int_{E_N} u(t, x) \phi(x) dV_x = \int_{E_N} u_0(x) \phi(x) dV_x$$

If $b = m_0 = m$, then $\lim_{t \rightarrow 0} u(t, x) = u_0(x)$ at every point in the Lebesgue set of u_0 (see Titchmarsh [648], Saks [587]) and, in particular, for almost all x .

This yields a fairly strong existence and uniqueness theorem.

Theorem 41. If $s > 1$ and $u(t, x)$ satisfies (32) for $0 < t < t_0$, and if there exists constants B and C such that

$$(39) \quad \int_{E_N} \|u(t, x)\| \exp(-B \|x\|^s) dV_x \leq C$$

for $0 < t < t_0$, and if there is a function g , integrable over every bounded domain in E_N , such that $\|u(t, x)\| \leq g(x)$ for $0 < t \leq \epsilon$, all $x \in E_N$, and if $u(t, x) \rightarrow u_0(x)$ as $t \rightarrow 0$ for almost all x , then u is given by (38) for $0 < t < (C_5/B)^{1/(s-1)}$. If $u_0(x) \equiv 0$, then $u(t, x) \equiv 0$ in $0 < t < t_0$. If u is a solution in $0 < t < t_0$ which satisfies (39), then there is a bounded completely additive set function μ on E_N to E_M such that

$$(40) \quad u(t, x) = \int_{E_N} G(t, x - y) \exp(B \|y\|^s) d\mu_y$$

and, conversely, any solution of the form (40) satisfies (39) in every interval $0 < \delta \leq t < (C_5/B)^{1/(s-1)}$.

For the case of the heat equation, $u_t = \Delta u$, this result was obtained independently by Rosenbloom [576] in 1946 and Fulks [251] in 1949. Weaker results were obtained previously by Tykhonov [653], Widder [709], and Pollard [543].* In this case $C_5 = 1/4$. For the case $N = 1$ and a finite rectangle, analogous results were obtained independently by Fulks [251] and Hartman and Wintner [313]. For the case $b = m_0 = m$, $M = 1$, slightly weaker results were obtained by Ladyženskaya [394], for $M > 1$ by Eidelman, and for the general case $b > 0$, $M \geq 1$, by Gelfand and Šilov [277]. See also Rosenbloom [578].

The representation (38) is, in general, valid only in the interval $0 < t < (C_5/B)^{1/(s-1)}$, even if (39) holds for all $t \geq 0$. For example, in the case of the heat equation with $N = 1$, the solution

$$u(t, x) = (1 - 4t \exp(\pi i/4))^{-1/2} \exp(x^2 \exp(\pi i/4)/(1 - 4t \exp(\pi i/4)))$$

satisfies (39) for $t \geq 0$, $B > \sqrt{2}$, but the representation (38) is valid only for $0 < t < 1/4\sqrt{2}$ and diverges for $t \geq 1/4\sqrt{2}$. This example is essentially due to Blackman [53].

In the case of the heat equation, the uniqueness theorem can be essentially sharpened: if (39) and $u(t, x) \rightarrow 0$ as $t \rightarrow 0$ for all $x \in E_N$, then $u(t, x) = 0$ (Rosenbloom [578]). For the case of the heat equation and certain other equations, condition (39) can be weakened considerably in the uniqueness theorem. (Täcklind [641], Rosenbloom [578].)

Results have been obtained on the behavior of solutions of the form (40) as $t \rightarrow 0$. For the case of the heat equation, see Rosenbloom [576], [578].

* In recent unpublished work Widder and Rosenbloom, jointly, and Gehring have obtained very precise results on the representation of solutions of the heat equation.

For the case $M = N = 1$, $P(\xi) = -\xi^m$, m even, $m > 2$, Pólya has, in his diary, obtained an asymptotic expansion for $G(t, x)$ which shows that, as $t \rightarrow 0$, $G(t, x)$ changes sign infinitely often. If $m = 4$, this expansion is

$$G(t, x) \sim \pi^{-1}(x/t)^{1/2} \exp(-3 \times 2^{1/2} \rho) \sum_0^{\infty} \alpha_k (2/27)^{(1+4k)/6}$$

$$\rho^{-k-1/2} \cos(3^{3/4} \times 2^{1/4} \rho - \pi(1 + 4k)/6)$$

where

$$\rho = (x^4/t)^{1/2}, \quad \alpha_k = (2/(2k)!) \int_0^{\infty} s^{4k} H_k(2s/\sqrt{6}) \exp(-s^2) ds$$

and H_k is the k th Hermite polynomial. This result is obtained by the saddle-point method (Perron [502]). Such asymptotic expansions would be valuable for the more detailed investigation of the general equation.

Theorems 39–41 indicate that the behavior of the solution of (32) as $t \rightarrow 0$ is markedly different according as $s = b/(m_0 - 1) > 1$ or $s \leq 1$. In the former case, $G(t, x) \rightarrow 0$ very rapidly as $t \rightarrow 0$ for $\|x\| \geq \delta > 0$, so that $G(t, x)$ really behaves like the Dirac δ_0 for small t , and the limiting behavior of G^*u_0 as $t \rightarrow 0$ depends only on the continuity properties of u_0 at each point. If $s \leq 1$, then $G^*u_0 \rightarrow u_0$ as $t \rightarrow 0$ for sufficiently smooth u_0 by virtue of a rapidly oscillating factor in $G(t, x)$, resembling the Dirichlet kernel in the theory of Fourier series. Thus, if $N = M = 1$, m even, $m > 2$, and $P(\xi) = -\xi^2 - i\xi^m$, then $b = 2$, $m_0 = m$, $s = 2/(m - 1) < 1$, and

$$G(t, x) = (2\pi(m - 1))^{-1/2} (mtx^{m-2})^{-1/2(m-1)} \times \exp(i(\lambda - \pi/4))(1 + o(t^{1/2(m-1)}))$$

where

$$\lambda = (m - 1)(x^m/mt)^{1/(m-1)}/m$$

as $t \rightarrow 0$, for $x \neq 0$, and

$$G(t, 0) = \pi^{-1} \Gamma(1 + m^{-1}) \exp(-\pi i/(2m)) t^{-1/m} (1 + o(t^{1-s}))$$

2. Nonhomogeneous equations

Let us now consider the equation

$$(41) \quad u_t = P(X)u + f(t, x), \quad t > 0, \quad x \in E_N; u(0, x) = u_0(x)$$

whose solution, under quite general conditions, is given by

(42)

$$\begin{aligned} u(t, x) &= \int_{E_N} G(t, x - y) u_0(y) dV_y + \int_0^t \int_{E_N} G(t - \tau, x - y) f(\tau, y) dV_y d\tau \\ &= v(t, x) + w(t, x) \end{aligned}$$

Here $v(t, x)$ is the solution of (32) and w is the solution of (41) with $u_0 = 0$, and has the form

$$w(t, x) = \int_0^t W(t - \tau, x; \tau) d\tau$$

where $W(t, x; \tau)$ is the solution of (32) with $u_0(x) = f(\tau, x)$. We have already given the main properties of v , so it only remains to study the function w . For the sake of simplicity, we shall assume that $s > 1$.

Theorem 42. If for some

$$k > m_0(\sigma - 1) - N, \sigma = b^{-1}\{N + (m - b)(\nu - 1)\}$$

there is a polynomial $Q(t, x)$ in x such that

$$\|f(t, x) - Q(t, x)\| \leq A \|x - x_0\|^k$$

for $0 < t < t_0$ and x in a neighborhood of x_0 , and if for some constant B

$$(43) \quad \int_{E_N} \|f(t, x)\| \exp(-B \|x\|^s) dV_x \leq M_1 < +\infty$$

for $0 < t < t_0$, then $\lim_{t \rightarrow 0} w(t, x_0) = 0$.

For $\epsilon > 0$, let

$$w_\epsilon(t, x) = \int_0^{t-\epsilon} W(t - \tau, x; \tau) d\tau$$

If f satisfies (43) for $0 < t < t_0$, then $w_\epsilon(t, x)$ exists and $\in C^\infty(E_N)$ for $\epsilon \leq t < \min(t_0, t_1)$, where $t_1 = (C_5/B)^{1/(s-1)}$. The existence and properties of $w(t, x)$ and its derivatives can be studied by examining the behavior of w_ϵ as $\epsilon \rightarrow 0$.

Theorem 43. Suppose f satisfies (43) and for some polynomial Q in x let

$$\omega(h; x_0) = \sup \|f(\tau, x) - Q(\tau, x)\| \quad \text{for} \quad \|x - x_0\| \leq h, \quad t - \theta \leq \tau \leq t$$

and let β be a vector with nonnegative integral components. Then $D^\beta w(t, x_0)$ exists for:

- | | | | |
|-----|-------------------------------|----|---|
| (a) | $b(\sigma - 1) + \beta > 0$ | if | $\omega(r; x_0)r^{N-1-m_0(\sigma-1+ \beta /b)} \in \mathfrak{L}_1[0, \delta]$ |
| (b) | $b(\sigma - 1) + \beta = 0$ | if | $\omega(r; x_0)r^{N-1} \log r \in \mathfrak{L}_1[0, \delta]$ |
| (c) | $b(\sigma - 1) + \beta < 0$ | if | $\omega(r; x_0)r^{N-1} \in \mathfrak{L}_1[0, \delta]$ |

for some $\delta > 0$. If $\omega_1(h; k) = \max \omega(h; x)$ for $\|x - x_0\| \leq k$, then the corresponding conditions on $\omega_1(r; k)$ are sufficient for $D^\beta w(t, x)$ to be continuous in a neighborhood of x_0 . If these conditions are satisfied for $|\beta| = m$, then $w(t, x)$ satisfies (41) at $x = x_0$.

Corollary 43a. If f is bounded in a neighborhood of the point (t, x_0) , and $b = m$, then $w \in C^{m-1}$ in this neighborhood. If $b = m$ and $\omega_1(r; k) = \max \|f(\tau, x_1) - f(\tau, x_2)\|$ for $\|x_j - x_0\| \leq k$, $\|x_1 - x_2\| \leq r$, $t - \epsilon \leq \tau \leq t$, and if $\omega_1(r; k)/r \in \mathfrak{L}_1([0, \delta])$ for some $\delta > 0$, then $w(t, x) \in C^m$ at x_0 and satisfies (41). If $\omega_1(r; k) \leq Br^\lambda$, $0 < \lambda \leq 1$, then $D^\beta w \in \text{Lip}(\lambda)$ for $|\beta| = m$, $0 < \lambda < 1$, and $\|D^\beta w(t, x_1) - D^\beta w(t, x_2)\| \leq O(\|x_1 - x_2\| (1 + \log^+(1/\|x_1 - x_2\|)))$ for $|\beta| = m$, $\lambda = 1$. If $f \in \mathfrak{L}^q(E_N)$ uniformly for $0 \leq t \leq t_0$, and $|\alpha| + 1 < m - N/q$, then $D^\alpha w \in \text{Lip } 1$. If $|\alpha| + 1 = m - N/q$, then $D^\alpha w$ has a modulus of continuity $\leq O(r(\log^+(1/r) + 1))$, whereas, if $|\alpha| < m - N/q < |\alpha| + 1$, then $D^\alpha w \in \text{Lip}(m - (N/q) - |\alpha|)$.

These results are obtained by combining Theorem 39 with Young's inequality (Hardy, Littlewood, and Pólya [311]).

Corollary 43b. If $b = m = 2$, $1 < q < +\infty$, then there is a constant A_q such that

$$L_q(\|D^\alpha w(t, x)\|) \leq A_q \max_{0 \leq \tau \leq t} L_{q, E_N}(f(\tau, x)) \quad \text{for } t > 0, \quad |\alpha| = 2$$

This requires the deep methods of Calderón and Zygmund [103].

The fundamental solution corresponding to the operator $P(X) - \lambda I$, where I is the identity matrix, is $G_\lambda(t, x) = \exp(-\lambda t) G(t, x)$. If $f \in C^k(E_N)$ for some $k \geq m_0\sigma - N$, and if $f(x) \exp(-B\|x\|^c) \in \mathfrak{L}_1(E_N)$ for some $c < \min(1, s)$, and $\Re \lambda > C$, then

$$(Pf)(x) = \int_0^\infty \exp(-\lambda t) (T_t f)(x) dt$$

exists for all $x \in E_N$. Consider the equation

$$u_t = P(X)u - \lambda u + f(x), \quad u(0, x) = u_0(x)$$

Under the above conditions, if $u_0(x) \exp(-B\|x\|^c) \in \mathfrak{L}_1(E_N)$ for some $c < \min(1, s)$, then $u(t, x) \rightarrow (Pf)(x)$ uniformly on every bounded set as $t \rightarrow \infty$. If $m_0 - b = \gamma = 0$, then Pf is the unique solution of

$$P(X)u - \lambda u = -f$$

such that $u(x) \exp(-B\|x\|^c) \in \mathfrak{L}_1(E_N)$. The approach of solutions of parabolic equations to a steady state and their connection with elliptic equations were also investigated by Eidelman [183]. Petrovski [511] described some interesting results of Bruk in this connection, but as far as we know this work is still unpublished.

PARABOLIC EQUATIONS WITH VARIABLE COEFFICIENTS

I. Perturbation and parametrix methods

The general theory of parabolic equations with variable coefficients is still virgin territory. Let us begin with a system of the form

$$(44) \quad u_t = P(X)u + Q(t, x, X)u, \quad u(0, x) = u_0(x)$$

where $Q(t, x, \zeta) = \sum_{|\alpha| \leq m} q_\alpha(t, x) \zeta^\alpha$, and where the system (32) is parabolic.

We can transform this equation into an integral equation of Volterra type in various ways. Let $v(t, x)$ be the solution of (32), and let $G(t, x)$ be the fundamental solution of (32). Then by the discussion of equation (41) we obtain

$$(45) \quad u(t, x) = v(t, x) + \int_0^t \int_{E_N} G(t - \tau, x - y) Q(\tau, y, Y) u(\tau, y) dV_y d\tau$$

where $Y_j = i^{-1} \partial / \partial y_j$. If u and its derivatives and the q_α and their derivatives satisfy certain mild growth conditions, then we may apply Green's formula and obtain an equation of the form

$$(46) \quad u(t, x) = v(t, x) + \int_0^t \int_{E_N} K(t, \tau, x, y) u(\tau, y) dV_y d\tau$$

where

$$K(t, \tau, x, y) = \sum_{|\alpha| \leq m} (-Y)^\alpha (G(t - \tau, x - y) q_\alpha(\tau, y))$$

Alternatively, we may set

$$u(t, x) = v(t, x) + \int_0^t \int_{E_N} G(t - \tau, x, y) \rho(\tau, y) dV_y d\tau$$

where ρ is an unknown density, and obtain

$$(47) \quad \rho(t, x) = \rho_0(t, x) + Q(t, x, X) \int_0^t \int_{E_N} G(t - \tau, x - y) \rho(\tau, y) dV_y d\tau$$

where

$$\rho_0(t, x) = Q(t, X) v(t, x)$$

If $Q(t, x, \zeta)$ is in a suitable sense "small," then these integral equations can be solved by successive approximations. Equation (47) has the advantage over (46) of requiring fewer assumptions about the coefficients of Q ; in particular, no conditions on the derivatives of q_α are needed. On the other hand, (46) can be treated directly as an equation on a suitable Banach space, which yields immediately important smoothness and growth properties of the solution of (44).

If in (32) $b = m_0 = m$, then some results can be obtained easily. If q_α is bounded and in Lip k , $0 < k < 1$, for all α , and if

$$\sum_{|\alpha|=m} \sup \|q_\alpha(t, x)\| = B(Q)$$

is sufficiently small, then (45) and (47) can be solved by successive approximations in a sufficiently small time interval, if $u_0 \in C^m(E_N)$ and $X^\alpha u_0 \in \text{Lip } k$ for $|\alpha| = m$. If $B(Q) = 0$, i.e., if Q is an operator of order less than m , then the solution exists in the whole time interval in which the other conditions are satisfied. The solution can be expressed in the form

$$u(t, x) = \int_{E_N} G_1(t, x, y) u_0(y) dV_y$$

where G_1 is the solution of the integro-differential equation

$$G_1(t, x, y) = G(t, x - y) + \int_0^t \int_{E_N} G(t - \tau, x - z) Q(\tau, z, X) G_1(\tau, z, y) dV_z d\tau$$

and can be expressed in the form

$$G_1(t, x, y) = G(t, x - y) + \int_0^t \int_{E_N} G(t - \tau, x - z) W(z, y; \tau, 0) dV_z d\tau$$

where $W(x, y; t, \tau)$ is the solution of the integral equation

$$W(x, y; t, \tau) = Q(t, x, X) G(t - \tau, x - y) + Q(t, x, X) \int_\tau^t \int_{E_N} G(t - \sigma, x - z) W(z, y; \sigma, \tau) dV_z d\sigma$$

The properties of G_1 can be read off from these relations.

This method was used by Korn [382] in the treatment of elliptic equations of the second order and, for this case, was refined by Serrin [608].

By the use of the device on pages 74–75 we can reduce the theory of a very general class of equations to that of (44) with “small” Q . We shall not enter into the details here. Another approach to the equation

$$(48) \quad u_t = P(t, x, X)u, \quad u(0, x) = u_0(x)$$

where for each $t_0 > 0$ and $x_0 \in E_N$, the equation $u_t = P(t_0, x_0, X)u$ is parabolic, is by the methods of page 73. We guess at the solution of (48) in the form

$$u(t, x) = \int_{E_N} U(t, 0; x, y) u_0(y) dV_y + \int_0^t \int_{E_N} U(t, \tau; x, y) \rho(\tau, y) dV_y d\tau$$

where $U(t, \tau; x, y)$ is a kernel such that

$$\lim_{\tau \rightarrow t-} \int_{E_N} U(t, \tau; x, y) f(y) dV_y = f(x)$$

for all continuous functions f , and such that for $\tau < t$, U is of class $C^m(E_N)$ in x and C^1 in t . Then ρ satisfies the integral equation

$$(49) \quad \rho(t, x) = \int_{E_N} V(t, 0; x, y) u_0(y) dV_y - \int_0^t \int_{E_N} V(t, \tau; x, y) \rho(\tau, y) dV_y d\tau$$

where

$$V(t, \tau; x, y) = U_t - P(t, x, X) U(t, \tau; x, y)$$

If, with respect to some suitable norm,

$$\int_0^t N(V) d\tau < +\infty$$

then this integral equation can be solved for ρ by successive approximations. For example, we may take

$$U(t, \tau; x, y) = G(t - \tau, x - y; P(\tau, y, \xi))$$

the fundamental solution corresponding to the operator $P(\tau, y, X)$ with constant coefficients. Then

$$V(t, \tau; x, y) = (P(t, y, X) - P(t, x, X)) U(t, \tau; x, y)$$

If (48) is *uniformly parabolic* in the sense that there exist constants $C_1, C > 0$, and A such that

$$\|P(t, x, \xi)\| \leq A(1 + \|\xi\|^m) \quad \text{and} \quad \Lambda(\xi; t, x) \leq C_1 - C \|\xi\|^m$$

for all $t > 0$, $x \in E_N$, and $\xi \in E_N$, where $\Lambda(\xi; t, x)$ is the maximum of the real parts of the eigenvalues of $P(t, x, \xi)$, and if the coefficients of P are bounded and measurable, and the coefficients of the derivatives of order m satisfy a condition of the form

$$\|q(t, x) - q(\tau, y)\| \leq B(|t - \tau|^{k/m} + \|x - y\|^k)$$

for some $k > 0$, then (49) has a unique solution for arbitrary bounded measurable u_0 . It is easy to obtain mild sufficient conditions for the solution of (49) to yield a solution of (48). This is an instance of the *parametrix method* originated by Levi [412] and further developed by

Hilbert [318]. For the case $m = 2$, it was applied by many authors, including Feller [199], Dressel [167], Yosida [729], Minakshisundaram [447], Ito [347], Milgram and Rosenbloom [446]. The underlying general theory is due to Rosenbloom [580], who made essential use of the work of Hille and Tamarkin [329] on integral equations of Volterra type. For other treatments of these and related problems see Fortet [229], Cameron [108], [109], Eidelman [185], and Slobodetski [615].

Suppose that the operator $\tilde{P}$ is proper. Let $\tilde{P}_1(t, x, X)v = (\tilde{P}'v)'$, where v' denotes the transpose of the matrix v . If u is a solution of (48) and $v(\tau, y)$ is a solution of the adjoint equation

$$(50) \quad v_\tau + P_1(\tau, y, Y)v = V$$

and if u and v satisfy suitable conditions at infinity, then

$$\frac{\partial}{\partial \tau} \int_{EN} \bar{v}(\tau, y) u(\tau, y) dV_y = \int_{EN} \overline{V(\tau, y)} u(\tau, y) dV$$

so that

$$\begin{aligned} \int_{EN} \overline{v(t, y)} u(t, y) dV_y &= \int_{EN} \overline{v(0, y)} u_0(y) dV_y \\ &\quad + \int_0^t \int_{EN} \overline{V(\tau, y)} u(\tau, y) dV_y d\tau \end{aligned}$$

This identity, and the corresponding identity for integrals over a bounded domain Ω , can be applied in various ways. If, for a sufficiently large class of functions $v_0(y)$, there is a solution of (50) with $V \equiv 0$ satisfying the desired growth conditions such that $v(t, y) = v_0(y)$, then $u(t, y)$ is uniquely determined by $u_0(y)$. On the other hand, if (50) has a solution $v(\tau, y; t, x)$ with $V \equiv 0$ such that

$$(51) \quad \lim_{\tau \rightarrow t-} \int_{EN} \overline{v(\tau, y; t, x)} f(y) dV_y = f(x)$$

for all sufficiently smooth functions f , then we obtain a representation of the solution of (48) in the form

$$u(t, x) = \int_{EN} \overline{v(0, y; t, x)} u_0(y) dV_y$$

The function $G_P(t, x; \tau, y) = \overline{v(\tau, y; t, x)}$ is called the *fundamental solution* of (48). Its properties are important for the detailed study of the solutions of (48). If, however, $v(\tau, y; t, x)$ is a function satisfying (50) and (51) such that $V(\tau, y) = V(\tau, y; t, x)$ is not too singular as $\tau \rightarrow t-$,

then the basic identity yields an integral equation for u :

$$(52) \quad u(t, x) = \int_{E_N} \overline{v(0, y; t, x)} u_0(y) dV_y \\ + \int_0^t \int_{E_N} \overline{V(\tau, y; t, x)} u(\tau, y) dV_y d\tau$$

from which u may be determined. Such integral equations were used in the case $m = 2$ by Gevrey [285] and Milgram and Rosenbloom [446]. For example, if

$$\int_{E_N} \|v(0, y; t, x)\| dV_x, \quad \int_0^t \int_{E_N} \|V(\tau, y; t, x)\| dV_y d\tau$$

and $u_0(y)$ are uniformly bounded, then (52) has a unique bounded solution. This equation can be used to obtain regularity properties of the solution of (48).

Finally, we may use the process of product integration to construct the solution of (48). Assume that this equation is uniformly parabolic. Let $\Psi_k(E, x)$ be a completely additive set function of the Borel set $E \subset E_N$ for all $x \in E_N$ and for $1 \leq k \leq n$, and let $|\Psi_k|(E, x)$ be its total variation (see Saks [587]). Suppose that for every polynomial q of degree $\leq m$ we have the relation

$$\int_{E_N} q(y) \Psi_k(dy, x) = q(x) + (P(t_k, x, X) q(x)) \Delta t_k + o(\Delta t_k)$$

where $t_k = \Delta t_1 + \dots + \Delta t_k$, and the term $o(\Delta t_k)/\Delta t_k$ approaches zero uniformly with respect to x and k , and that

$$\int_{\|y-x\| \geq \epsilon} \|x-y\|^m |\Psi_k|(dy, x) = o(\Delta t_k)$$

uniformly with respect to x and k for every $\epsilon > 0$. Let

$$u_1(x) = \int_{E_N} u_0(y) \Psi_1(dy, x), \quad u_{k+1}(x) = \int_{E_N} u_k(y) \Psi_k(dy, x)$$

for $1 \leq k < n$. Then $u_n(x) \rightarrow u(t, x)$ as $\max \Delta t_k \rightarrow 0$, $\Sigma \Delta t_k \rightarrow t$. This was proved by Petrovski [504] and Kolmogorov [374] for k the case $M = N = 1$, $m = 2$ (see Khinčin [369]). In this case the problem is equivalent to the approximation of continuous stochastic processes by discrete processes (Khinčin [369], Doob [157]), and, in the case of the heat equation, reduces to the central limit theorem in the theory of probability (Feller [198], Bergström [37]). Many numerical methods for solving

partial differential equations of parabolic type, such as difference equation techniques, reduce to special cases of this procedure. For the case $M = N = 1$, $m = 2$, John [354] has given a very general discussion of difference equation methods (see also Kamynin [360]). For boundary value problems for the heat equation, these methods were discussed by Courant, Friedrichs, and Lewy [128], Laasonen [392], Juncosa and Young [357], Leutert [411], and Douglas [161].

2. Applications of semigroup theory

For the case $M = N = 1$, $m = 2$, Hille [328], Feller [204], and Yosida [723] have made a thorough study of the application of the theory of semigroups to the theory of parabolic equations, and Yosida [730] has studied a class of problems for $N > 1$, $m = 2$. Reference should also be made to the work of Spencer [630] and Gaffney [258] using Hilbert space methods.

Consider the equations ($N = M = 1$, $m = 2$)

$$(53) \quad u_t = au_{xx} + bu_x, \quad u(0, x) = u_0(x)$$

$$(54) \quad v_t = \frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial x} (av) - bv \right\} \quad v(0, x) = v_0(x)$$

where $a = a(x)$ and $b = b(x)$ are continuous and $a(x) > 0$ in some interval $r_1 < x < r_2$, where $-\infty \leq r_1 < r_2 \leq +\infty$. The function

$$W(x) = \exp \left(- \int_{x_0}^x b(s) a(s)^{-1} ds \right)$$

where x_0 is a fixed point in (r_1, r_2) , plays a decisive role in the study of these equations. For $j = 1, 2$, we classify the boundary r_j as follows:

- (a) r_j is *regular* if $W(x) \in \mathfrak{L}_1(x_0, r_j)$, $a(x)^{-1}W(x)^{-1} \in \mathfrak{L}_1(x_0, r_j)$;
- (b) r_j is an *exit boundary* if $a^{-1}W^{-1} \notin \mathfrak{L}_1(x_0, r_j)$,

$$W(x) \int_{x_0}^x a(s)^{-1}W(s)^{-1} ds \in \mathfrak{L}_1(x_0, r_j);$$

- (c) r_j is an *entrance boundary* if $a^{-1}W^{-1} \in \mathfrak{L}_1(x_0, r_j)$,

$$a^{-1}W^{-1} \int_{x_0}^x W(s) ds \in \mathfrak{L}_1(x_0, r_j);$$

- (d) r_j is a *natural boundary* in all other cases.

For the heat equation, where $a \equiv 1$, $b \equiv 0$, r_j is natural if it is infinite and regular if it is finite. If $a(x) = \alpha x$, $b(x) = \beta x + \gamma$, where α, β , and γ are constants, we obtain an equation arising in genetics (Feller [202]). If $r_j = 0$, then it is an exit boundary if $\gamma \leq 0$, regular for $0 < \gamma < \alpha$,

and an entrance boundary of $\gamma \geq \alpha$. The heat equation in polar coordinates, in which $a(x) \equiv 1$, $b(x) \equiv x^{-1}$, has regular boundaries at 0 and infinity.

At a regular boundary we can prescribe a boundary condition of the form

$$q_j \lim_{x \rightarrow r_j} u(t, x) + p_j(-1)^j \lim_{x \rightarrow r_j} W(x)^{-1} u_x(t, x) = 0$$

for (53) and of the form

$$q_j \lim_{x \rightarrow r_j} W(x) a(x) v(t, x) + p_j(-1)^j \lim_{x \rightarrow r_j} \{(av)_x - bv\} = 0$$

for (54), where q_j and p_j are constants. If both boundaries are regular, then $\sup |u(t, x)|$ and $\int_{r_1}^{r_2} |v(t, x)| dx$ are nonincreasing as functions of t if and only if $p_j \geq 0$ and $q_j \geq 0$, $j = 1, 2$, and under these conditions the solutions for nonnegative initial values are nonnegative. If $q_1 = q_2 = 0$, then $\int_{r_1}^{r_2} |v(t, x)| dx$ is constant.

At a natural boundary, the solution $u(t, x)$ can be required to be continuous while $v(t, x)$ can be required to be in $\mathcal{L}_1(x_0, r_j)$. If both boundaries are natural, then the solution is uniquely determined by these requirements.

If r_1 is a natural boundary and r_2 is an exit boundary, then for (53) we can impose a condition of the form

$$(55) \quad p_2 u(t, r_2) - p_1 u(t, r_1) = \tau \int_{r_1}^{r_2} u(t, x) dp(x) \\ - \sigma \lim_{x \rightarrow r_j} (a(x) u_{xx} + b(x) u_x)$$

where p_1, p_2, τ, σ are constants and $p(x)$ is a nondecreasing function and $p_j \geq 0$, $\tau \geq 0$, $\sigma \geq 0$, $p_1 + \tau \leq p_2$, $p(r_1+) = 0$, $p(r_2-) = 1$. If $\sigma > 0$, then there is a unique solution of (53) satisfying (55) for arbitrary $u_0 \in C[r_1, r_2]$, while for $\sigma = 0$, the solution satisfying (55) exists and is unique if u_0 satisfies (55). The solution of (54) is uniquely determined by the condition that $v(t, x) \in \mathcal{L}_1(r_1, r_2)$. Note that (55) is a *global* condition involving not only the boundary values of u but also its values throughout the interval.

For a further discussion, including various mixed cases, see Feller [204]. Hille [328] has studied in further detail the cases of two entrance boundaries and two regular boundaries, giving important results on the expression of solutions in terms of eigen-function expansions. Both authors apply their results to the theory of probability.

3. Boundary value problems

Boundary value problems for second-order parabolic equations were discussed in considerable detail by Gevrey [285], Siddiqi [610], Rothe [581], Amundsen [4], Müntz [475], and Barrar [25]. Müntz' treatment of the heat equation follows the classical model for the Laplace equation and can be applied with little change to the general parabolic equation of the second order. The heat equation was discussed by Sternberg [631], Petrovski [505], and Doob [160], using the analogue for the heat equation of Perron's method of subharmonic functions (see Carathéodory [110], Brelot [66]), and the last two authors used very successfully the connections with the theory of Brownian motion and probability theory. This method depends very strongly on the maximum principle for second-order parabolic equations, which has been proved in a very general and sharp form by Nirenberg [488] (see also Hopf [331]).

Theorem 44. Let

$$P(t, x; \tau, \xi) = \Sigma a_{jk} \xi_j \xi_k + \Sigma b_{jk} \tau_j \tau_k + i(\Sigma c_j \xi_j + \Sigma \gamma_j \tau_j)$$

where the coefficients are continuous in some domain Ω , in the space E_{N+M} of the variables $(t, x) = (t_1, \dots, t_M, x_1, \dots, x_N)$. We assume that $\Sigma a_{jk} \xi_j \xi_k$ is negative definite for $(t, x) \in \Omega$, but that $\Sigma b_{jk} \tau_j \tau_k$ is only non-positive for $(t, x) \in \Omega$ and $\tau_1, \dots, \tau_M$ real, and that the coefficients c_j and γ_j are real. If $u \in C^2(\Omega)$ and $P(t, x; T, X)u \geq 0$ in Ω , where $X_j = i^{-1} \partial/\partial x_j$, $T_j = i^{-1} \partial/\partial t_j$, and if $u(t, x) \leq u(t_0, x_0)$ for all (t, x) in Ω , where (t_0, x_0) is some point in Ω , then $u(t_0, x) \equiv u(t_0, x_0)$ for all x in the same component as x_0 in $\Omega(t_0)$, which is the set of x such that $(t_0, x) \in \Omega$.

Corollary 44a. If $a(t, x)$ is continuous and nonpositive in Ω , and if $P(t, x; T, X)u + au \equiv 0$ in Ω and $u(t, x) \leq u(t_0, x_0)$ in Ω where $u(t_0, x_0) > 0$, then $u(t_0, x) = u(t_0, x_0)$ for all x in the same component of $\Omega(t_0)$ as x_0 .

Corollary 44b. Let $M = 1$ and $\gamma_1 < 0$ and $b_{11} = 0$ in Ω . Let $S(t_0, x_0)$ be the set of all points $(t, x) \in \Omega$ which can be connected to (t_0, x_0) in Ω by a simple curve along which t is nondecreasing. If $P(t, x; T, X)u \geq 0$ in $S(t_0, x_0)$ and $u(t, x) \leq u(t_0, x_0)$, and $(t_0, x_0) \in \Omega$, then $u(t, x) = u(t_0, x_0)$ in $S(t_0, x_0)$.

Corollary 44c. Under the hypotheses of Corollary 44b, if $a(t, x)$ is continuous and nonpositive in Ω , and if $P(t, x; T, X)u + au = 0$ in Ω , and $u(t, x) \leq u(t_0, x_0)$ in $S(t_0, x_0)$, where $(t_0, x_0) \in \Omega$ and $u(t_0, x_0) > 0$, then $u(t, x) = u(t_0, x_0)$ in $S(t_0, x_0)$.

It is probable that there exists a positive constant $c = c(t, x, t_0, x_0, P, \Omega)$ such that, if $P(t, x; T, X) = 0$ and $u \geq 0$ in Ω , then $u(t, x) \leq cu(t_0, x_0)$ for $(t, x) \in S(t_0, x_0)$, $t < t_0$. This is an analogue of Harnack's inequality for harmonic functions. A strong result of this type has recently been established by Serrin [607] for elliptic equations. For the case of the heat equation

$$Pu = \Delta u - u_t = 0$$

in the whole space, Resch [565] proved that if u is a nonnegative solution in $0 < t \leq t_0$, $x \in E_N$, then

$$u(t, x) \leq (t_0/t)^{N/2} \exp(\|x - x_0\|^2/4(t_0 - t)) u(t_0, x_0)$$

for $0 < t < t_0$. Similar inequalities can be proved for other regions but even in the simple case of the semi-infinite rod, $N = 1$, $\Omega = \{(t, x) | x > 0, t > 0\}$, the dependence of c on the points (t, x) and (t_0, x_0) is quite complicated. If such a result is proved for a class of equations, then the methods of Martin [438] and BreLOT [73] for the study of harmonic functions can be applied.

Boundary value problems for general parabolic equations have been investigated recently by Lax and Milgram [405], Lyantse [431], Browder [79], Višik [665], Ito [348], and Lions [423]. Milgram and Lax and Lyantse studied equations of the form

(56)

$$u_t = -P(x, X)u = - \sum_{|\alpha| \leq k} \sum_{|\beta| \leq k} X^\alpha (p_{\alpha, \beta}(x) X^\beta u) \text{ for } t > 0, x \in \Omega$$

$$u(0, x) = u_0(x) \text{ for } x \in \Omega, \quad u(t, x) \in \mathcal{H}_0^k(\Omega) \text{ for } t > 0$$

where P is uniformly elliptic in the bounded domain Ω in E_N in the sense that

$$\Re \left(\sum_{|\alpha| = |\beta| = k} p_{\alpha, \beta}(x) \xi^{\alpha + \beta} \right) \geq c \|\xi\|^{2k}$$

for some $c > 0$ and all $x \in \Omega$, all $\xi \in E_N$. It is assumed that the coefficients $p_{\alpha, \beta}$ are bounded and measurable in Ω , and that for $|\alpha| = |\beta| = k$, $p_{\alpha, \beta}$ is continuous in $\bar{\Omega}$. According to the theory of semigroups, the properties of (56) can be deduced from those of the elliptic equation

$$(57) \quad \lambda u + P(x, X)u = f, \quad u \in \mathcal{H}_0^k(\Omega), \quad f \in \mathcal{L}_2(\Omega)$$

As we shall see in the next chapter, there is a constant ω_0 and a positive constant h such that

$$\Re(u, u)_P \geq h \|u\|_k^2 - \omega_0 \|u\|_0^2 \quad (\text{all } u \in \mathcal{H}_0^k(\Omega))$$

where $(u, v)_k = (u, v)_{Q_k}$, $Q_k = (1 - \Delta)^k$. Hence, by Theorem 11, if

$\Re \lambda > \omega_0$, then the resolvent $R(\lambda; \tilde{P}^*)$ is a bounded operator on $\mathcal{L}_2(\Omega)$ to $\mathcal{H}_0^k(\Omega)$. Lax and Milgram proved that the Hille-Yosida conditions are satisfied, so that by Corollary 17a, $\tilde{P}^*$ generates a semigroup of class (C_0) on $\mathcal{H}_0^k(\Omega)$. The regularity properties of the weak solutions of (56) will be discussed below. Milgram [445] has generalized these results to certain classes of equations with time-dependent coefficients. The method of Kato [364] can be applied to this type of problem. Browder [79] developed a theory of parabolic equations on the basis of his work on the eigenfunctions of elliptic equations. This extends the work of Minakshisundaram [447] on second-order equations. Ito [348] treated quite general boundary value problems for second-order equations.

4. The method of Višik

We shall present here an adaption of the method of Višik, based on the general theory on pages 69–70. Let $P(t, x, X)u = \sum_{|\alpha| \leq k} \sum_{|\beta| \leq k} X^\alpha (P_{\alpha, \beta}(t, x) X^\beta u)$ be an elliptic operator with bounded measurable coefficients in the set $\Omega_T: 0 \leq t \leq T, x \in \Omega$, such that for some $c > 0$

$$\Re \left(\sum_{|\alpha| = |\beta| = k} p_{\alpha, \beta}(t, x) \xi^{\alpha + \beta} \right) \geq c \|\xi\|^{2k}$$

for $(t, x) \in \Omega_T$, all $\xi \in E_N$, and such that $p_{\alpha, \beta}$ is continuous in $\bar{\Omega}_T$ for $|\alpha| = |\beta| = k$. By introducing the new unknown function $v = u \exp(-t(1 + \omega_0))$ in (56), we can transform that equation into one in which

$$\Re(u, u)_P \geq h \|u\|_k^2 + \|u\|_0^2, \quad h > 0$$

for all $u \in \mathcal{H}_0^k(\Omega)$. Let $(u, v)_Q = \{(u, v)_P + \overline{(v, u)_P}\}/2$, so that

$$\|u\|_Q^2 = (u, u)_Q \geq h \|u\|_k^2 + \|u\|_0^2, \quad 0 \leq t \leq T, \quad u \in \mathcal{H}_0^k(\Omega)$$

We consider the space $\mathcal{S} = \mathcal{S}([0, T], \Omega)$ of the functions $v(t, x) \in C^\infty(\Omega_T)$ such that $v \equiv 0$ for $0 \leq t \leq T$ and all x outside some compact subset of Ω . On $\mathcal{S}$ we introduce the scalar products

$$(u, v)_B = \int_0^T \{(u(t), v(t))_k + (u_t, v_t)_0\} dt$$

and

$$(u, v)_K = \{(u(0), v(0))_0 + (u(T), v(T))_0\}/2 + \int_0^T (u(t), v(t))_Q dt$$

and the bilinear form

$$(u, v)_A = (u(T), v(T))_0 + \int_0^T \{(u(t), v(t))_P - (u_t, v_t)_0\} dt$$

Then there is a constant C such that $|(u, v)_A| \leq C \|u\|_B \|v\|_K$ for $u, v \in \mathcal{S}$, and also $\Re(u, u)_A = \|u\|_K^2$, so that, if

$$N(v) = \sup_{\|u\|_K=1} |(u, v)_A|, \quad N^*(v) = \sup_{\|u\|_K=1} |(v, u)_A|$$

then

$$\|v\|_K < N(v) \leq C \|v\|_B, \quad \|v\|_K \leq N^*(v) \leq C \|v\|_B$$

for $v \in \mathcal{S}$. Let $\mathcal{S}_K$, $\mathcal{S}_N$, and $\mathcal{S}_B$ be the completions of $\mathcal{S}$ with respect to the corresponding norms. If l is a linear functional on $\mathcal{S}_N$, then there are elements u and $v \in \mathcal{S}_K$ such that $l(w) = (v, w)_A = \overline{(w, u)_A}$ for all $w \in \mathcal{S}_N$, and $\|v\|_K \leq N^*(l) = \sup |l(w)|/N(w)$, $\|u\|_K \leq N^*(l)$. In particular, if l is a linear functional on $\mathcal{S}_K$, then $\|u\|_K \leq \|l\|_K$, $\|v\|_K \leq \|l\|_K$.

For example, if $u_0 \in \mathcal{L}_2(\Omega)$ and $f \in \mathcal{L}_2(\Omega_T)$ are given and

$$l(w) = (u_0, w(0))_0 + \int_0^T (f(t), w(t))_0 dt$$

then there is a constant C_1 such that

$$|l(w)| \leq C_1 (\|u_0\|_0^2 + \int_0^T \|f(t)\|_0^2 dt)^{1/2} \|w\|_K$$

Hence there is a $u \in \mathcal{S}_K$ such that $l(w) = \overline{(w, u)_A}$ for all $w \in \mathcal{S}$ and

$$\|u\|_K \leq C_1 (\|u_0\|_0^2 + \int_0^T \|f(t)\|_0^2 dt)^{1/2}$$

Similarly, if $v_1 \in \mathcal{L}_2(\Omega)$, $g \in \mathcal{S}_K$, and

$$l_1(w) = (v_1, w(T))_0 + \int_0^T (g, w)_Q dt$$

then there is a $v \in \mathcal{S}_K$ such that $l_1(w) = (v, w)_A$ for all $w \in \mathcal{S}$, and

$$\|v\|_K \leq \{2\|v_1\|_0^2 + \int_0^T \|g\|_Q^2 dt\}^{1/2}$$

The function u is a weak solution of

(58)

$$u_t = -P(t, x, X)u + f, \quad u(0, x) = u_0(x), \quad u(t, x) \in \mathcal{H}_0^k(\Omega) \quad \text{for } t > 0$$

while the function v is a weak solution of

(59)

$$v_t = \tilde{P}(t, x, X)v - Q(t, x, X)g, \quad v(T) = v_1, \quad v(t, x) \in \mathcal{H}_0^k(\Omega) \quad \text{for } t > 0$$

If $u \in \mathcal{S}_K$ and $N(u) < +\infty$, and we set

$$u_n(t) = \int_0^T u(\tau) \phi_n(t - \tau) d\tau$$

where $\phi_n \in C^\infty$, $\phi_n = 0$ outside the interval $0 \leq t \leq 1/n$, $\phi_n(t) \geq 0$ for all n and all t , and $\int_0^T \phi_n(t) dt = 1$, then $\|u_n - u\|_K \rightarrow 0$ and

$$N(u_n) \leq C_2(\|u\|_K + N(u))$$

for some constant C_2 , which implies that the weak solution of (58) is unique in $\mathcal{S}_K$. Similarly, the weak solution of (59) is unique in $\mathcal{S}_K$, and if $w \in \mathcal{S}_K$, there is a unique $W \in \mathcal{S}_N$ such that $(\Psi, W)_A = (\Psi, w)_K$ for all $\Psi \in \mathcal{S}$, and there is a sequence $W_n \in \mathcal{S}$ such that $\|W_n - W\|_K \rightarrow 0$ and

$$|(\Psi, W_n)_A - (\Psi, w)_K| \leq \|\Psi\|_K/n$$

for all $\Psi \in \mathcal{S}$.

5. Variational principles

The solutions of (58) and (59) can be characterized by variational principles, which can be used both for numerical purposes and for comparison theorems. For $\lambda > 0$, let

$$N_\lambda(u) = \sup_{w \in \mathcal{S}, w \neq 0} |(w, u)_A| / \{\|w(0)\|_0^2 + \lambda \int_0^T \|w(t)\|_0^2 dt\}^{1/2}$$

If u is the weak solution of (58), then

$$N_\lambda(u) = \{\|u_0\|_0^2 + \lambda^{-1} \int_0^T \|f(t)\|_0^2 dt\}^{1/2}$$

Consequently, if v is the weak solution of (59), the identity

$$(v, u)_A = (v(0), u_0)_0 + \int_0^T (v, f)_0 dt = (v_1, u(T)) + \int_0^T (g, u)_Q dt$$

implies that

$$\{\|v(0)\|_0^2 + \lambda \int_0^T \|v(t)\|_0^2 dt\}^{1/2} = \sup_{u \in \mathcal{S}, u \neq 0} |(v, u)_A| / N_\lambda(u)$$

the maximum being attained when u is the weak solution of (58) with $u_0 = v(0)$ and $f = \lambda v$. The dual principle is based on the norm

$$N_\lambda^*(v) = \sup_{w \in \mathcal{S}, w \neq 0} |(v, w)_A| / \{\|w(T)\|_0^2 + \lambda \int_0^T \|w(t)\|_0^2 dt\}^{1/2}$$

and characterizes the solution u of (58) by means of

$$\{\|u(T)\|_0^2 + \lambda \int_0^T \|u(t)\|_0^2 dt\}^{1/2} \\ = \sup_{v \in S, v \neq 0} \Re\{(v(0), u_0) + \int_0^T (v, f)_0 dt\} / N_\lambda^*(v)$$

the maximum being attained when v is the weak solution Ψ of

$$\Psi_t = \tilde{P}(t, x, X)\Psi - \lambda u, \quad \Psi(T) = u(T), \quad \Psi(t, x) \in \mathcal{H}_0^k(\Omega) \quad \text{for } 0 \leq t < T$$

As an application, suppose $\bar{\Omega} \subset \Omega_1$ and E is the operation of extending a function defined on Ω by setting it equal to zero in $\Omega_1 - \Omega$. Let V be the weak solution in $S_K([0, T], \Omega_1)$ of (59) corresponding to the functions Ev_1 and Eg , and let $N_{\lambda,1}$ be the norm for Ω_1 corresponding to N_λ for Ω . Then for $u \in S([0, T], \Omega)$ we have $N_{\lambda,1}(Eu) = N_\lambda(u)$. Hence,

$$\|v(0)\|_0^2 + \lambda \int_0^T \|v(t)\|_0^2 dt \leq \|V(0)\|_0^2 + \lambda \int_0^T \|V(t)\|_0^2 dt$$

Now we let $\lambda \rightarrow 0$, and obtain $\|v(0)\|_0 \leq \|V(0)\|_0$. In a similar fashion we may prove that, if u_0 and f vanish outside Ω , then $\|u(T)\|_0$ is a nondecreasing function of the domain Ω .

By using the results of Sobolev [624] (see Corollary 26), we can weaken the conditions on f in (58). Višik did not investigate the question of whether the weak solutions of (58) and (59) are actual solutions. He did give an adaption of Galerkin's method (see Mikhlin [441], Kantorovič and Krylov [363]) to the computation of u , and also results on the approach of $u(t, x)$ to the solution of an elliptic equation as $t \rightarrow \infty$, provided that $P(t, x, \xi) \rightarrow P(\infty, x, \xi)$ exists. He also studied equations of the form

$$(60) \quad A(t)u_t + B(t)u = f(t), \quad u(0) = u_0$$

where $A(t)$ is an elliptic operator of order $m = 2k$ satisfying the above conditions on P , and $B(t)$ is an operator of order r with bounded measurable coefficients. If $r \leq m$, then (60) has weak solutions in $S_B([0, T], \Omega)$; in fact they can be obtained by a process of successive approximations. If $r = 2s > m$ and if $B(t)$ is an elliptic operator satisfying the conditions imposed on P , then (60) has weak solutions in the S_B - space corresponding to $\mathcal{H}_0^s(\Omega)$.

Unfortunately space does not permit a description of the important results of Lions [423].

6. Regularity properties

The author and Browder, in unpublished work, have recently obtained, independently, results on regularity properties of solutions of (58) by apparently different, though related, methods. First, we remark that, if $G(t, \tau)u_0$ is the unique function in the space $\mathcal{S}_K([\tau, T], \Omega)$ such that

$$(w(T), u(T))_0 + \int_{\tau}^T \{(w, u)_P - (w_t, u)_0\} dt = (w(\tau), u_0)_0$$

for all $w \in \mathcal{S}([\tau, T], \Omega)$, then the solution of (58) is given by the formula $u(t) = G(t, 0)u_0 + \int_0^t G(t, \tau) f(\tau) d\tau$. Hence it is sufficient to treat the case $f = 0$.

First we show that, if $u_\epsilon(t) = \epsilon^{-1} \int_0^T \phi((t - \tau)/\epsilon) u(\tau) d\tau$, where $\phi(t) = \pi^{-1/2} \exp(-t^2)$, then $\lim_{\epsilon \rightarrow 0} u_\epsilon(T) = u(t)$ exists for $0 < t < T$, and that

$$\|u(t)\|_0^2 + \int_0^t \|u(\tau)\|_Q^2 d\tau = \|u_0\|_0^2 \quad (0 < t \leq T)$$

from which it follows that $u(t)$ is continuous as a function of t in a sense of $\mathcal{L}_2(\Omega)$ and $u(t) \rightarrow u_0$ as $t \rightarrow 0$. Furthermore, if $w \in \mathcal{S}$, then

$$(w(t), u(t))_0 = (w(0), u_0)_0 + \int_0^t \{(w'(\tau), u(\tau)) - (w(\tau), u(\tau))_P\} d\tau$$

for $0 < t \leq T$. We set $w_\epsilon(\tau) = \phi(y) \overline{G(t - \tau + \epsilon, x - y; t_0, x_0)}$, where $G(t, x; t_0, x_0)$ is the fundamental solution of the equation

$$v_t = -p(t_0, x_0, X)v = p_0(X)v_0$$

with constant coefficients, and $p(t_0, x_0, \xi)$ is the principal part of $P(t_0, x_0, \xi)$, $\phi \in C_0^\infty(\Omega)$, x and x_0 are fixed points of Ω , and $0 < t_0 < T$. When we let $\epsilon \rightarrow 0$, we obtain almost everywhere in Ω .

$$\begin{aligned} (61) \quad \phi(x) u(t, x) &= \int_{\Omega} G(t, x - y; t_0, x_0) \phi(y) u_0(y) dV_y \\ &+ \int_0^t \overline{(G(t - \tau, x - y; t_0, x_0), \phi u(\tau))_{p_0} - (w_0(t - \tau), u(\tau))_P} d\tau \\ &= u_1(t, x) + u_2(t, x) \end{aligned}$$

If $\overline{\Omega_1} \subset \Omega$, choose ϕ so that $\phi \equiv 1$ in Ω_1 . Then this fundamental identity expresses u in Ω_1 as the sum of the solution of an equation with constant

coefficients taking on the initial values ϕu_0 and the solution of a non-homogeneous equation with constant coefficients, both of these operators being smoothing operators. Their properties can be read off from Theorems 40–44, by repeated use of Young's inequality. We begin with the property

$$\int_0^T \|u(\tau)\|_Q^2 d\tau \leq \|u_0\|_0^2$$

and successively obtain sharper results by a "bootstrap" method until we have the desired information. If $|\alpha| \leq k$, then $X^\alpha u(t, x)$ can be computed from (61) in Ω_1 and can be estimated on the assumption that

$$|p_{\alpha, \beta}(t, x) - p_{\alpha, \beta}(\tau, y)| \leq C(\|x - y\|^\nu + |t - \tau|^{\nu/m})$$

for some ν , $0 < \nu \leq 1$, and all $(t, x), (\tau, y) \in \Omega_T$, and all α, β such that $|\alpha| = |\beta| = k$. We obtain

$$\|u(t)\|_{k, \Omega_1} \leq \|u_1(t)\|_k + C_1 \int_0^t (t - \tau)^{\lambda-1} \|u(\tau)\|_{k, \Omega_1} d\tau$$

where $\lambda = \nu/m$, and C_1 is a certain constant. From $\|u_1(t)\|_k \leq C \|u_0\|_0 t^{-1/2}$, we obtain

$$\|u(t)\|_{k, \Omega_1} \leq C_2(\Omega_1) \|u_0\|_0 t^{-1/2}$$

Suppose that for some $q \geq 2$ we have already shown that

$$L_{q, \Omega_1}(X^\alpha u(t)) \leq C_{q, \alpha}(\Omega_1) \|u_0\|_0 t^{-s}, \quad s = s(q)$$

for $|\alpha| \leq k$ and $0 < t \leq T$. Then for $q^{-1} - r^{-1} < \nu/N$, we obtain

$$L_{r, \Omega_1}(X^\alpha u(t)) \leq C_\alpha(\Omega_1) \|u_0\|_0 t^{-s(r)}$$

where $s(r) = s(q) + (N/m)(q^{-1} - r^{-1})$. Hence $s(q) = (\frac{1}{2}) + (N/2)(m^{-1} - q^{-1})$. We apply this repeatedly until we reach a q such that $N < \nu q$, and, at the next step, we obtain the inequality for $r = \infty$. The argument also shows the continuity of $X^\alpha u(t)$ for $|\alpha| \leq k$. Thus we obtain

$$|X^\alpha u(t, x)| \leq C_\alpha(\Omega_1) \|u_0\|_0 t^{-(1/2) - (N/2m)}$$

for $|\alpha| \leq k, x \in \Omega_1$. One more application of the fundamental identity yields

$$(62) \quad |X^\alpha u(t, x)| \leq C_\alpha(\Omega_1) \|u_0\|_0 t^{-s}, \quad s = (2|\alpha| + N)/m$$

$$|X^\alpha u(t, x) - X^\alpha u_1(t, x)| \leq C_{\alpha'}(\Omega_1) \|u_0\|_0 t^{(\nu/m) - s}$$

for $|\alpha| \leq k, x \in \Omega_1, 0 < t \leq T$. If we assume that $p_{\alpha, \beta} \in C^r(\Omega)$ for $|\alpha| \leq k, |\beta| \leq k$, then we conclude by this reasoning that $u \in C^{k+r}(\Omega)$ for $0 < t \leq T$.

Hence $X^\alpha u(t, x)$ is a linear functional on $\mathfrak{L}_2(\Omega)$ for each fixed t, x , and $\alpha, |\alpha| \leq k$, so that there is a Green's function $G_P(t, x, y) = G_P(t, 0; x, y) \in \mathfrak{L}_2(\Omega)$ as a function of y such that

$$u(t, x) = \int_{\Omega} G_P(t, x, y) u_0(y) dV_y$$

and $G_P(t, x, y) \in C^k$ as a function of $x \in \Omega$ for $0 < t \leq T$. The operator $G(t, \tau)$ introduced on page 121 has a representation of the form

$$(G(t, \tau)u_0)(t, x) = \int_{\Omega} G_P(t, \tau; x, y) u_0(y) dV_y$$

and has the property that, if $0 \leq t_1 < t_2 < t_3 \leq T$, then

$$(63) \quad G(t_3, t_2) G(t_2, t_1) = G(t_3, t_1)$$

If v is the solution of (59) with $g = 0$, then

$$v(t, x) = \int_{\Omega} \overline{G(T, t; y, x)} v_1(y) dV_y$$

The fact that $\|u(t)\|_0$ is a nondecreasing domain function if u_0 vanishes outside Ω implies that, if $\Omega_1 \subset \Omega_2$ and $G_P(t, \tau; x, y, \Omega_j)$ is the Green's function for the operator P and the domain Ω_j , then

$$\begin{aligned} \int_{\Omega_1} |G_P(t, \tau; x, y, \Omega_1)|^2 dV_y &\leq \int_{\Omega_1} |G_P(t, \tau; x, y, \Omega_2)|^2 dV_y \\ &\leq \int_{\Omega_2} |G_P(t, \tau; x, y, \Omega_2)|^2 dV_y \end{aligned}$$

for $0 \leq \tau < t \leq T, x \in \Omega_1$. The middle term shows that for the solution of (58) we have

$$|u(t, x)| \leq C_0 \|u_0\|_0 t^{-N/(2m)}, \quad 0 < t \leq T$$

uniformly in Ω , and

$$\int_{\Omega} |G_P(t, x; \tau, y)|^2 dV_y \leq C_0^2 (t - \tau)^{-N/m}, \quad 0 \leq \tau < t \leq T, \quad x \in \Omega$$

and similarly for the L_2 -norm with respect to x . Consequently, we conclude from (62) and (63) that for $|\alpha| \leq k$

$$|X^\alpha G_P(t, x; 0, y)| \leq C_\alpha(\Omega_1) t^{-(N+|\alpha|)/m}$$

for $0 < t \leq T, x \in \Omega_1, y \in \Omega$. In a similar manner we can prove that

$$|G_P(t, x; 0, y) - G(t, x - y; t, x)| \leq C_0(\Omega_1) t^{(N-N)/m}$$

for $y \in \Omega, x \in \Omega_1, 0 < t \leq T$. If $P = \tilde{P}$ is independent of t , then

$G_P(t, x; \tau, y) = G_P(t - \tau, x; 0, y) = G_P(t - \tau, x, y) = \overline{G_P(t - \tau, y, x)}$. It follows that $G_P(t, x, x)$ is a nondecreasing domain function.

7. Barrar's *a priori* estimates

For second-order parabolic equations, Barrar [25] obtained *a priori* estimates of the solution and its derivatives from which he deduced very sharp existence theorems. He uses the method introduced by Schauder [595] for the treatment of elliptic equations. Let

$$P(t, x; \tau, \xi) = \sum_1^N a_{jk} \xi_j \xi_k + i \Sigma b_j \xi_j + c - i \tau$$

where (a_{jk}) is a negative definite matrix, and a_{jk} , b_j , and c are Hölder continuous in the bounded domain $\Omega \subset E_{N+1}$ and let f be Hölder continuous in Ω . Suppose that for every point $(t_0, x_0) \in \partial\Omega$ with $t_0 < \sup_{(t,x) \in \Omega} t$, there is a one-to-one mapping of the form

$$x_j = \phi_j(y_1, \dots, y_N, s), \quad 1 \leq j \leq N, \quad t = \phi_{N+1}(s), \quad \phi_{N+1}(s) > 0$$

of a neighborhood Ω_0 of (t_0, x_0) onto a neighborhood Ω_1 of the origin in E_{N+1} such that ϕ_j has Hölder continuous second derivatives and such that $\Omega_0 \cap \partial\Omega$ is mapped into one of the three sets:

$$y_1 = 0, \quad s = 0, \quad \text{or} \quad y_1 s = 0, \quad y_1 \geq 0, \quad s \geq 0$$

Then if g is continuous on $\partial\Omega$, there exists a unique solution of

$$P(t, x; T, X)u = f$$

in Ω which is continuous in $\bar{\Omega}$ and such that $u = g$ on $\partial\Omega$.

Let

$$M_{p,k}(u) = \sup_{P \in \Omega, |\alpha| = k} d(P)^{p+k} |X^\alpha u|$$

$$M_{p,k,\lambda}(u) = \sup_{P, Q \in \Omega, |\alpha| = k} d(P, Q)^{p+k+\lambda} (|X^\alpha u(P) - X^\alpha u(Q)| / \delta(P, Q)^\lambda)$$

where for $P = (t, x)$, $Q = (\tau, y)$, $\delta(P, Q)$ is the metric defined by

$$\delta(P, Q) = (\|x - y\|^2 + |t - \tau|)^{1/2}$$

and

$$d(P) = \min_{Q \in \partial\Omega} (\delta(P, Q)), \quad d(P, Q) = \min (d(P), d(Q))$$

and let

$$N_{p,k}(u) = \sum_{j=0}^k M_{p,j}(u)/j!, \quad N_{p,k,\lambda}(u) = N_{p,k}(u) + M_{p,k,\lambda}(u)$$

$$N_k'(u) = \sum_{j=0}^k M_{-j,j}(u)/j!, \quad N_{k,\lambda}'(u) = N_k'(u) + M_{-k-\lambda,k,\lambda}(u)$$

Then if $0 < \lambda < \mu < 1$, $a = |\det (a_{jk})|^{-1}$, we have

$$N_{0,2,\lambda}(u) \leq C(N'_{0,\lambda}(f) + N_0(g))$$

where C depends on $N'_{0,\mu}(a_{jk})$, $N'_{0,\lambda}(b_j)$, $N'_{0,\lambda}(c)$, $N_0(a)$, and Ω . If the functions ϕ_j appearing above satisfy $N'_{2,\mu}(\phi_j) < +\infty$, then

$$N'_{2,\lambda}(u) \leq C_1(N'_{0,\lambda}(f) + N'_{2,\lambda}(g))$$

where C_1 depends on the same quantities as C as well as on $N'_{2,\mu}(\phi_j)$.

Such *a priori* estimates are fundamental for the treatment of nonlinear problems.

Interesting work on problems of heat conduction with unknown moving boundaries was done by Datzev [129], Mann and Wolf [436], and Kolodner [378].

EQUATIONS OF ELLIPTIC TYPE

GENERAL THEORY

1. Classification

An equation is said to be of *elliptic* type if it has no real characteristics. If $P(x, \xi) = \sum P_\alpha(x) \xi^\alpha$ is a scalar polynomial of degree m , and $p(x, \xi) = \sum_{|\alpha|=m} p_\alpha(x) \xi^\alpha$ is its principal part, then the operator $P(x, X)$ is elliptic if $p(x, \xi) \neq 0$ for all real $\xi \neq 0$. If p has real coefficients, then m must be even and p is definite. If $P = (P_{ij}(x, \xi))$ is a matrix-valued polynomial, then P is elliptic if

$$p^\diamond(x, \xi) = \det(p_{ij}^\diamond(x, \xi)) \neq 0 \quad \text{for } \xi \in E_N, \quad \xi \neq 0$$

where $p_{ij}^\diamond$ is the homogeneous part of degree m_j of P_{ij} , and $m_j = \max_i \deg P_{ij}$. Višik [661] calls P *strongly elliptic* if $R(\bar{\eta} \cdot p(x, \xi) \eta) \neq 0$ for all $\xi \in E_N$, $\xi \neq 0$, and all $\eta \in C_N$, $\eta \neq 0$, where p is the principal part of P . Douglis and Nirenberg [166] introduced, by analogy with a class of hyperbolic systems studied by Leray [408], a class of systems which have similar properties to elliptic systems. Let $m_{ij} = \deg P_{ij}$. Suppose that integers $s_1, \dots, s_M, t_1, \dots, t_M$ are chosen such that $m_{ij} \leq s_i + t_j$ ($1 \leq i, j \leq M$), it being understood that if $s_i + t_j < 0$, then $p_{ij} = 0$. Let $p_{ij}^\square = p_{ij}$ if $m_{ij} = s_i + t_j$ and $p_{ij}^\square \equiv 0$ otherwise. Then Douglis and Nirenberg say that P is elliptic if $\det(p_{ij}^\square(x, \xi)) \neq 0$ for $\xi \in E_N$, $\xi \neq 0$. Let $q(x, \xi) = (q_{ij}(x, \xi))$ be the matrix of the cofactors of P_{ji} , so that $qp^\square = p^\square q = I \det p^\square$. Then q is a matrix such that qP and Pq are elliptic in the ordinary sense. This suggests that we call an operator P *weakly elliptic* if there exists an operator Q such that QP is elliptic. Now $QP - PQ$ is an operator of lower order than QP , so that PQ is also elliptic. Q can always be chosen so that the principal part of QP is a diagonal operator, or even a scalar operator. This concept of weak ellipticity has the advantage over the Douglis-Nirenberg concept that it

is invariant under changes of the coordinate systems in E_M and E_N and under nonsingular recombination of the equations. If u is a solution of $P(x, X)u = f$, then it is also a solution of the elliptic system $Q(x, X)P(x, X)u = Q(x, X)f$ if the coefficients of P and f are sufficiently often differentiable. Also, if v is a solution of the weakly elliptic system $Q(x, X)v = u$, then it is a solution of the strongly elliptic system $PQv = f$, so that the introduction of such potentials v may often yield valuable information. If P is weakly elliptic, and Q is chosen so that QP is elliptic and the principal part of QP is scalar, then the equation $QP(U + iV) = QF$, satisfied by the real and imaginary parts, U and V , of u , can be put in the form

$$\begin{aligned} p_1 U - p_2 V + C_{11} U + C_{12} V &= g_1 \\ p_2 U + p_1 V + C_{21} U + C_{22} V &= g_2 \end{aligned}$$

where C_{jk} and the scalars p_j are real operators, and the orders of C_{jk} are less than $\max(\deg p_1, \deg p_2)$, and $p_1(x, \xi)^2 + p_2(x, \xi)^2 \neq 0$ for all $\xi \in E_N$, $\xi \neq 0$, so that this equation is strongly elliptic. These transformations must be employed with care, however, since they have an effect on boundary conditions which is often difficult to keep track of, and it is sometimes difficult to obtain in this way results on the existence of solutions satisfying given boundary conditions.

2. John's fundamental solution

For elliptic operators with analytic coefficients John [352] has given an ingenious construction of a fundamental solution. This generalizes previous work on equations with constant coefficients by Somigliana [628], Fredholm [234], Zeilon [734], Levi [412], Herglotz [316], and Bureau [88]. Let $P(x, X) = \sum_{|\alpha| \leq m} p_\alpha(x) X^\alpha$ be an elliptic operator with coefficients analytic in Ω , where Ω is a bounded domain in E_N . Here $p_\alpha(x)$ is an $M \times M$ matrix. Let R be the portion of Ω contained in the half-space $(x - y) \cdot \xi < -r$, where $\xi \in E_N$, $\|\xi\| = 1$, y is a fixed point in Ω , and $0 < r < \delta(y) = \text{distance from } y \text{ to } \partial\Omega$. If u is a solution of $P(x, X)u = f$ in Ω and $v \in C^\infty(\bar{\Omega})(M \times M)$, then

$$\int_R (P_1(x, X)v)u \, dV = \int_R v f \, dV + \int_{\partial_R} M(v, u) \, dA$$

where

$$P_1(x, X) = \Sigma(-X)^\alpha (vp_\alpha(x))$$

and $M(v, u)$ is a bilinear form in v and u and their derivatives:

$$M(v, u) = i^{-1} \sum_\beta n_j M_{j,\beta}(v) X^\beta u$$

where $\mathbf{n} = [n_1, \dots, n_N]$ is the unit exterior normal to R , and the principal part of $M_{j,0}(v)$ is

$$(-1)^m m^{-1} \sum_{|\alpha|=m} \left(\left(\frac{\partial}{\partial x_j} X^\alpha \right) v \right) p_\alpha(x)$$

Set $v = Q_1(x, X) w = \Sigma(-X)^\alpha (w q_\alpha(x))$, where Q is a matrix such that the principal part of $Q(x, \xi) P(x, \xi)$ is a scalar matrix $p(x, \xi) I$, and $\deg p = n$. Then on S , the part of ∂R in the hyperplane $(x - y) \cdot \xi = -r$, the coefficient of u in $M(v, u)$ has the form

$$(-1)^n p(x, i^{-1} \xi) (\xi \cdot \nabla)^{n-1} w + M_1(w)$$

where $M_1(w)$ involves tangential derivatives of w on S and normal derivatives of order less than $n - 1$. We set $w = w(x, \xi, (y \cdot \xi) - r)$ equal to the solution of the Cauchy problem

$$P_1(x, X) Q_1(x, X) w = 0, \quad (\xi \cdot \nabla)^k w = 0 \quad (k < n - 1)$$

$$(\xi \cdot \nabla)^{n-1} w = -p(x, i\xi)^{-1} I, \quad \text{on } S$$

Then w exists and is analytic in Ω if Ω is sufficiently small. We thus obtain

$$\int_S u \, dA = \int_R v f \, dV + \int_B M(v, u) \, dA$$

where B is the rest of the boundary of R . Now we form the integral

$$J(y; g) = \int_{\|\xi\|=1} dA_\xi \int_0^\infty g(r) \, dr \int_S u(x) \, dA_x$$

where $g(r)$ is an arbitrary function integrable over every finite interval. Using the identity

$$\begin{aligned} \int_{\|\xi\|=1, (y-x) \cdot \xi > 0} g((y-x) \cdot \xi) \, dA_\xi \\ = \omega_{N-1} \int_0^1 (1-r^2)^{(N-3)/2} g(r \|y-x\|) \, dr \end{aligned}$$

we obtain

$$J(y; g) = \omega_{N-1} \int_\Omega u(x) \int_0^1 (1-r^2)^{(N-3)/2} g(r \|y-x\|) \, dr \, dV_x$$

If we set $g(r) = r^s$, $\Re s > -1$, $J(y; g) = J_s(y)$, then we find that

$$((-\Delta)^{-s} u)(y) = 4^{-s} \pi^{1/2-N} \Gamma((N/2) - s) J_{2s-N}(y) / \Gamma(s - (N-1)/2)$$

for $(N-1)/2 < \Re s < N/2$, where $(-\Delta)^{-s}$ is the Riesz potential (page 83). The function $((-\Delta)^{-s} u)(y)$ is by the formula on page 83 analytic for $0 < \Re s < N/2$, and, by the above formula, it is also ana-

lytic for $\Re s > (N-1)/2$ except possibly for poles at $s = (N/2) + k$, $k = 0, 1, \dots$. The function u can be obtained by analytic continuation down to $s = 0$ or by the formula

$$(-\Delta)^k ((-\Delta)^{-s} u)(y) = ((-\Delta)^{-(s-k)}) u(y)$$

for $k \geq 0$. Hence, if N is odd, we obtain

$$u(y) = -2^{-N} \pi^{1-N} (-\Delta)^{(N+1)/2} J_1(y)$$

Similarly, if N is even, we have

$$u(y) = -(-\Delta)^{N/2} \frac{\partial}{\partial s} (2^{-2s-N} \pi^{-N-1/2} \Gamma(1-s) J_{2s}(y) / \Gamma(s + \frac{1}{2})) \big|_{s=0}$$

Let

$$k(x, y, \xi, r) = Q_1(x, X) w(x, \xi, (y \cdot \xi) - r)$$

Then

$$\begin{aligned} \int_{\|\xi\|=1} dA_\xi \int_0^\infty g(r) \int_{\mathbb{R}^N} k(x, y, \xi, r) f(x) dV_x \\ = \int_\Omega \left(\int_{\|\xi\|=1, (y-x) \cdot \xi > 0} \int_0^{(y-x) \cdot \xi} g(r) k(x, y, \xi, r) dr dA_\xi \right) f(x) w dV_x \end{aligned}$$

Hence, if

$$K(x, y, s) = (4^{-s} \pi^{1/(2-N)} \Gamma((N/2) - s) / \Gamma(s - (N-1)/2))$$

$$\int_{\|\xi\|=1, (y-x) \cdot \xi > 0} \int_0^{(y-x) \cdot \xi} r^{2s-N} k(x, y, \xi, r) dr dA_\xi$$

then

$$((-\Delta)^{-s} u)(y) = \int_\Omega K(x, y, s) f(x) dV_x + \int_{\partial\Omega} M(K(x, y, s), u) dA_x$$

The kernel $K(x, y, s)$ has the properties

$$P_1(x, X) K(x, y, s) = 4^{-s} \pi^{-N/2} (\Gamma((N/2) - s) / \Gamma(s)) \|x - y\|^{2s-N}$$

$$(-\Delta_y)^{-s} K(x, y, s) = K(x, y, s - k)$$

and $K(x, y, s)$ is analytic for $\Re s > (N-1)/2$ except possibly for poles at $s = (N/2) + k$, $k = 0, 1, 2, \dots$. In particular, the analytic continuation $K(x, y, 0)$ is a fundamental solution for the operator $P(y, Y)$ for each fixed x , and can be represented in the form

$$K(x, y, 0) = (-\Delta_y)^{(N+1)/2} K(x, y, (N+1)/2), \quad N \text{ odd}$$

$$K(x, y, 0) = -(-\Delta_y)^{N/2} \frac{\partial}{\partial s} \{(N/2 - s) K(x, y, N/2 + s)\} \big|_{s=0},$$

N even

There are also other ways of continuing $K(x, y, s)$ analytically into the half-plane $\Re s < (N - 1)/2$.

The Cauchy data for w show that

$$w(x, \zeta, (y \cdot \zeta) - r) = (((x - y) \cdot \zeta) + r)^{n-1} w_1(x, \zeta, (y \cdot \zeta) - r)$$

where w_1 is analytic. Hence,

$$k(x, y, \zeta, r) = (((x - y) \cdot \zeta) + r)^{m-1} k_1(x, y, \zeta, r)$$

where k_1 is analytic:

$$k_1(x, y, \xi, r) = \sum_{\nu=0}^{\infty} (((x - y) \cdot \xi) + r)^{\nu} \mathcal{K}_{\nu}(x, y, \xi)$$

$$\mathcal{K}_0(x, y, \xi) = -q(x, i\xi)/(p(x, i\xi)(m - 1)!)$$

where q is the principal part of Q . A straightforward computation yields

$$K(x, y, s) = 2^{-N} \pi^{1-N} r^{2s-N+m} \csc(\pi(s - N/2)) \\ \sum_{\nu=0}^{\infty} \frac{(-1)^{m+\nu} r^{\nu} (m - 1 + \nu)!}{\Gamma(2s - N + m + \nu + 1)} K_{\nu}(x, y, \zeta, s)$$

$$K_{\nu}(x, y, \zeta, s) = \int_{\|\xi\|=1, \xi \cdot \zeta > 0} (\xi \cdot \zeta)^{2s-N+m+\nu} \mathcal{K}_{\nu}(x, y, -\xi) dA_{\xi}$$

where $r = \|x - y\|$, $\zeta = (x - y)/r$. This formula shows that $K(x, y, s) \sin(\pi(s - N/2))$ is an entire function of s for $x \neq y$, since $\Gamma(2s - N + m + \nu + 1)^{-1} K_{\nu}(x, y, \zeta, s)$ can be expressed as a Riemann-Liouville fractional integral of an analytic function (Riesz [573]). Thus for N odd, we obtain

$$K(x, y, 0) = r^{m-N} A(x, y, r, \zeta)$$

where A is analytic in all four arguments. If N is even, then

$$K(x, y, 0) = r^{m-N} (A(x, y, r, \zeta) + B(x, y, r, \zeta) \log r)$$

where A and B are analytic in all four arguments. For N odd, the main term in $K(x, y, 0)$ as $r \rightarrow 0$ is

$$r^{m-n} A(x, y, 0, \zeta) = \frac{(-1)^{m+1}}{2(2\pi)^{N-1}(m+1)!} (-\Delta_y)^{(N+1)/2} \\ \int_{\|\xi\|=1, (x-y) \cdot \xi > 0} ((x - y) \cdot \xi)^{m+1} \frac{q(x, -i\xi)}{p(x, -i\xi)} dA_{\xi}$$

This is the only term of $K(x, y, 0)$ in the case where $P(x, \xi)$ is a homogeneous polynomial in ξ with constant coefficients. We can obtain similar

formulas for N even. The coefficient $r^{m-N}B$ of $\log r$ is a regular solution of $P(y, Y)u = 0$ even at $x = y$, and can vanish identically only if $m < N$.

These formulas can be simplified in the case of operators with constant coefficients. The function w can be expressed explicitly as a contour integral, and we obtain

$$K(x, y, s) = \frac{1}{2(2\pi)^N \sin(\pi(s - N/2))} \int_C \int_{\|\xi\|=1, (y-x) \cdot \xi > 0} P(-\lambda\xi)^{-1} F(\lambda, (y-x) \cdot \xi, s) dA_\xi d\lambda$$

where

$$F(\lambda, t, s) = \frac{1}{\Gamma(2s - N + 1)} \int_0^t (t - \tau)^{2s-N} e^{-i\lambda\tau} d\tau$$

which is a simple Riemann-Liouville integral, and C is a sufficiently large circle. If $P(\xi)$ is homogeneous, then

$$K(x, y, s) = - \frac{i^m}{2(2\pi)^{N-1} \sin(\pi(s - N/2)) \Gamma(2s - N + m + 1)} \int_{\|\xi\|=1, (y-x) \cdot \xi > 0} P(\xi)^{-1} ((y-x) \cdot \xi)^{2s+m-N} dA_\xi$$

If $m \geq N$ and N is odd, then

$$K(x, y, 0) = - \frac{i^m (-1)^N}{2(2\pi)^{N-1} (m - N)!} \int_{\|\xi\|=1, (y-x) \cdot \xi > 0} P(\xi)^{-1} ((y-x) \cdot \xi)^{m-N} dA_\xi$$

If $m < N$ and N is odd, then

$$K(x, y, 0) = \frac{(-i)^m r^{m-N}}{2(2\pi)^{N-1}} \phi^{(N-m-1)}(0)$$

where $r = \|x - y\|$, $\zeta = (x - y)/r$, and

$$\phi(t) = (1 - t)^{(N-3)/2} \int_{\|\eta\|=1, \eta \cdot \zeta = 0} P(t\zeta + (1 - t^2)^{1/2}\eta)^{-1} dA_\eta$$

For N even, we use the formula

$$K(x, y, 0) = \pi^{-1} (-1)^{N/2} \frac{\partial}{\partial s} \{ (\sin(\pi(s - N/2))) K(x, y, s) \} |_{s=0}$$

which yields

$$K(x, y, 0) = \frac{2(-1)^{1+N/2} i^m}{(2\pi)^N (m-N)!} \int_{\|\xi\|=1, (y-x) \cdot \xi > 0} P(\xi)^{-1} ((y-x) \cdot \xi)^{m-N} \log((y-x) \cdot \xi) dA_\xi + K_1(x, y)$$

for $m \geq N$, where K_1 is a polynomial of degree $\leq m - N$. The case $m < N$ and N even can be treated similarly.

3. Strongly elliptic systems

We shall now describe the work of Gårding [267], Višik [661], and Browder [81] on strongly elliptic systems. Their method is a generalization of the procedure used by Zaremba [732], [733] for the Laplace equation and rediscovered later by Weyl [705]. The abstract basis of this method was already given by Nikodym [484]. See also Sobolev [623], Lax and Milgram [405], Lax [404], Nirenberg [489], Friedrichs [243], and Lions [423].

Let us consider a strongly elliptic system of order $m = 2k$ of the form

$$(64) \quad P(x, X)u = \sum_{|\alpha| \leq k} \sum_{|\beta| \leq k} X^\alpha p_{\alpha, \beta}(x) X^\beta u = f$$

We shall assume that the coefficients $p_{\alpha, \beta}(x)$ are bounded and measurable in the domain $\Omega \subset E_N$ and that for $|\alpha| = |\beta| = k$ $p_{\alpha, \beta}$ is continuous in $\bar{\Omega}$, and that there exists a constant $h > 0$ such that

$$\Re(\bar{\eta} \cdot p(x, \xi)\eta) \geq h \|\xi\|^m \|\eta\|^2$$

for all $\xi \in E_N$, all $\eta \in C_M$, and all $x \in \Omega$, where p is the principal part of P . If these conditions are satisfied, we shall say that P is *uniformly strongly elliptic* in Ω . The bilinear form associated with P has the property

$$|(v, u)_P| \leq A \|v\|_{k,0} \|u\|_{k,0}, \quad (\text{all } u, v \in C_0^\infty(\Omega))$$

where

$$(u, v)_k = (u, (-\Delta)^k v)_\Omega, \quad (u, v)_{k,0} = (u, v)_0 + (u, v)_k$$

and

$$A = \sum_{\alpha, \beta} \sup_{x \in \Omega} \|p_{\alpha, \beta}(x)\|$$

Suppose first that $p_{\alpha, \beta}$ is constant for $|\alpha|, |\beta| \leq k$, and that

$$\Re(\bar{\eta} \cdot P(\xi)\eta) \geq (h \|\xi\|^m + a) \|\eta\|^2$$

for all $\xi \in E_N$, $\eta \in C_M$, where a and h are positive constants. By using Fourier transforms, we obtain

$$\Re(u, u)_P \geq h_1 \|u\|_{k,0}^2, \quad h_1 = \min((2\pi)^m h, a)$$

for all $u \in C_0^\infty(\Omega)$. Hence, by Theorem 11, for every $f \in \mathfrak{L}_2(\Omega)$, there is a unique $u \in \mathfrak{H}_0^k(\Omega)$ such that $(f, \phi)_0 = (u, \phi)_P$ for all $\phi \in \mathfrak{H}_0^k(\Omega)$. We denote u by $G_P f$, and call G_P the *Green's operator* of Ω with respect to P . The operator G_P is a linear transformation on $\mathfrak{L}_2(\Omega)$ to $\mathfrak{H}_0^k(\Omega)$ and

$$\|G_P f\|_{k,0} \leq h_1^{-1} \|f\|_0, \quad \|G_P f\|_{k,0}^2 \leq h_1^{-1} \|f\|_0 \|G_P f\|_0$$

If $\phi \in C_0^\infty(\Omega)$, then $G_P P(X)\phi = \phi$. Since $(u, v)_P$ is a continuous bilinear form on $\mathfrak{H}_0^k(\Omega)$ it can be represented in the form $(u, v)_P = (u, L_P v)_{k,0}$, where L_P is a linear transformation on $\mathfrak{H}_0^k(\Omega)$ to itself. It follows easily that

$$(L_P)^* = L_{\bar{P}}, \quad (G_P)^* = G_{\bar{P}}, \quad (L_P G_P)^* = L_{\bar{P}} G_{\bar{P}}$$

where in the first and third relations the adjoint operators are considered in $\mathfrak{H}_0^k(\Omega)$, while in the second they are considered in $\mathfrak{L}_2(\Omega)$.

If Ω is bounded, then by Rellich's lemma (p. 74) G_P is a compact operator on $\mathfrak{L}_2(\Omega)$ to itself. It follows easily that G_P is also compact as an operator on $\mathfrak{L}_2(\Omega)$ to $\mathfrak{H}_0^k(\Omega)$, and *a fortiori* on $\mathfrak{H}_0^k(\Omega)$ to itself.

For any scalar λ , the weak solutions of $P(X)u + \lambda u = f$ in $\mathfrak{H}_0^k(\Omega)$ are the solutions of

$$(65) \quad u + \lambda G_P u = G_P f$$

for arbitrary $f \in \mathfrak{L}_2(\Omega)$. By the theory of F. Riesz [569] (see also Banach [23], Riesz and Nagy [572]), the operator $(I + \lambda G_P)^{-1}$ exists and is a bounded operator for all λ outside a certain denumerable set $\sigma(G_P)$, the *spectrum* of P . The only limit point of the set $\sigma(G_P)$ is the point at infinity. The point $\lambda \in \sigma(G_P)$ if and only if there is a $u \neq 0$ in $\mathfrak{L}_2(\Omega)$ such that $u + \lambda G_P u = 0$. The space of such u is finite-dimensional. The point $\lambda \in \sigma(G_P)$ if and only if $\lambda \in \sigma(G_{\bar{P}})$. If $\lambda \in \sigma(G_P)$, then (65) has a solution in $\mathfrak{L}_2(\Omega)$ if and only if $G_P f$ is orthogonal to all solutions of $v + \bar{\lambda} G_{\bar{P}} v = 0$, and this is equivalent to f being orthogonal to all solutions v of $\bar{P}(X)v + \bar{\lambda} v = 0$ in $\mathfrak{H}_0^k(\Omega)$. Note that if $f \in \mathfrak{L}_2(\Omega)$ and u is a solution of (65) in $\mathfrak{L}_2(\Omega)$, then actually $u = G_P(f - \lambda u) \in \mathfrak{H}_0^k(\Omega)$. If $\lambda \in \sigma(G_P)$ and $u + \lambda G_P u = 0$, $\|u\|_0 = 1$, then

$$1 = (u, u)_0 = (G_P u, u)_{\bar{P}} = -\lambda^{-1} (u, u)_{\bar{P}}$$

so that

$$-\Re \lambda \geq (2\pi)^m h \|u\|_k^2 \geq h(\pi/R)^m$$

by Theorem 23, where $2R$ is the minimum width of Ω . Hence $\sigma(G_P)$ lies

in the half-plane $\Re \lambda \leq -h(\pi/R)^m$. If $P = \tilde{P}$, then we can apply the theory of self-adjoint operators in Hilbert space (Riesz-Nagy [572]) and conclude that the eigenfunctions of G_P are complete in $\mathfrak{L}_2(\Omega)$.

4. Regularity theorems

So far we have only proved the existence of weak solutions of (64). We must still show that the weak solutions are solutions in the classical sense. To do this, it is sufficient to show that they are of class $C^m(\Omega)$. Our method is based on an identity similar to (61). Suppose $f \in \mathfrak{L}_2(\Omega)$, $u \in \mathfrak{L}_1(\Omega)$, and $(u, \tilde{P}(X)\Psi)_0 = (f, \Psi)_0$ for all $\Psi \in C_0^\infty(\Omega)$. Let $G(t, x)$ be the fundamental solution of the parabolic system $v_t = -P(X)v$, let Ω_1 be a compact subdomain of Ω , and choose a $\phi \in C_0^\infty(\Omega)$ such that $\phi = 1$ on Ω_1 , and set $\Psi(y) = \phi(y) G(t, x - y)^* \eta$ where η is a constant vector in C_M , and A^* means the complex conjugate of the transpose of the matrix A . Then in Ω_1 we have $\Psi_t = -\tilde{P}(X)\Psi$, so that

$$\begin{aligned} (f, \Psi)_0 &= -(u, \Psi_t)_{0, \Omega_1} + (u, \tilde{P}(X)\Psi)_{0, \Omega - \Omega_1} \\ &= -\frac{d}{dt} (u, \Psi)_{0, \Omega_1} + (u, \tilde{P}(X)\Psi)_{0, \Omega - \Omega_1} \end{aligned}$$

Then for $0 < \epsilon < t$

$$\begin{aligned} (G(\epsilon, x - y) u(y), \eta)_{0, \Omega_1} \\ = (G(t, x - y) u(y), \eta)_{0, \Omega_1} - \int_\epsilon^t (u, \tilde{P}(Y) \Psi(\tau))_{0, \Omega - \Omega_1} d\tau \\ - \int_\epsilon^t (G(\tau, x - y) \phi(y) f(y), \eta)_0 d\tau \end{aligned}$$

and since η is arbitrary we obtain the identity

$$\begin{aligned} \int_{\Omega_1} G(\epsilon, x - y) u(y) dV_y &= \int_{\Omega_1} G(t, x - y) u(y) dV_y - \int_\epsilon^t u_1(\tau, x) d\tau \\ &\quad - \int_\epsilon^t \int_{\Omega} G(\tau, x - y) \phi(y) f(y) dV_y \end{aligned}$$

Now if

$$U(t, x) = \int_{\Omega_1} G(t, x - y) u(y) dV_y$$

then U is the solution of

$$U_t = -P(X)U, \quad U(0, x) = u(x) \text{ in } \Omega_1, \quad U(0, x) = 0 \text{ in } E_N - \Omega_1$$

and, by Theorem 41, $\lim_{t \rightarrow 0} U(t, x) = u(x)$ almost everywhere in Ω_1 , and

for $t > 0$, $U(t, x)$ is an entire function of x . Also

$$w(t, x) = \int_0^t \int_{\Omega} G(\tau, x - y) \phi(y) f(y) dV_y$$

in the solution

$$w_t = -P(X)w + g, \quad w(0, x) = 0$$

where $g = \phi f$ in Ω and $g = 0$ in $E_N - \Omega$. Finally, $\Psi(\tau)$ is of class C^∞ for $0 \leq \tau \leq t$, $x \in \Omega_1$, $y \in \Omega - \bar{\Omega}_1$, so that $u_1(\tau, x) \in C^\infty(\Omega_1)$ for $0 \leq \tau \leq t$. Hence for almost all $x \in \Omega_1$, we have

$$(66) \quad u(x) = U(t, x) - \int_0^t u_1(\tau, x) d\tau - w(t, x)$$

so that the properties of u can be read off from those of w . By Theorem 44, $w \in C^h(\Omega_1)$ if $f \in \mathcal{L}_q(\Omega_1)$ where $q > N/(m - h)$ and in particular, $w \in C^{m-1}(\Omega_1)$ if $q > N$. The derivatives of w of order h are in $\mathcal{L}_r(\Omega_1)$ if $f \in \mathcal{L}_q(\Omega_1)$ where $q^{-1} < r^{-1} + (m - h)/N$. At every point x where f satisfies a Dini condition, i.e., $\omega_1(r)/r \in \mathcal{L}_1[0, \delta]$, where $\omega_1(r) = \max_{\|y-x\| \leq r} \|f(y) - f(x)\|$, the function w has derivatives of order m and (64) is satisfied. If f satisfies a uniform Dini condition in Ω_1 , then w is of class $C^m(\Omega_1)$.

The regularity of weak solutions of elliptic equations was first proved by Zaremba [732] for the case of the Laplace equation and rediscovered by Weyl [705], and is sometimes called "Weyl's lemma." Generalizations are due to Gårding [264], Schwartz [603], Browder [81], Friedrichs [243], John [355], and Hörmander [337]. The first complete proof for equations with variable coefficients is due to Browder. Both Browder and Friedrichs use the latter's method of smoothing operators or mollifiers, but it seems that the above use of smoothing operators especially adapted to the equation under consideration yields the sharpest results. John's method is based on general properties of spherical means which also have no special relation to the given equation. Other proofs (e.g., Gårding) are based on the fundamental solution for elliptic operators.

The identity (66) can be used for many other purposes. For example, we may obtain *a priori* estimates of u and its derivatives in Ω_1 . If a sequence u_n of solutions of $P(X)u = 0$ is uniformly bounded in $\mathcal{L}_1(\Omega)$ and converges at a set of points which is dense in some sphere $S \subset \Omega$, then the sequence converges uniformly in every compact subset of Ω together with the derivatives of all orders.

If we take $P(X) = (-\Delta)^k = \left(\sum_{j=1}^N X_j^2\right)^k$, then w can be put in the form

$$w(t, x) = \Sigma P_{j1}(X) \int_0^t \int_{\Omega} G(\tau, x - y) \phi_j(y) P_{j2}(Y) u(y) dV_y d\tau$$

where the $P_{j\nu}$ are certain differential operators of order $\leq k$. Hence if $u \in \mathcal{H}^k(\Omega)$, then u coincides almost everywhere in Ω with a function of class $C^h(\Omega)$ if $k > h + (N/2)$, and for any compact subdomain Ω_1 of Ω , there is a constant $C_{h,k}(\Omega_1, \Omega)$ such that

$$M_h(u; \Omega_1) \leq C_{h,k}(\Omega_1, \Omega) \|u\|_{k,0,\Omega}$$

This is essentially Sobolev's important lemma [624]. It can also be proved by using the inequality of Sobolev, according to which

$$L_q((-\Delta)^{-z}u) \leq C(p, z) L_p(u)$$

if $0 < \Re z < N/(2p)$, $p > 1$, $q^{-1} = p^{-1} - 2(\Re z)/N$ (see Sobolev [624], Thorin [646]). If u vanishes outside a compact set and $u \in \mathcal{L}_p(E_N)$, then the fact that $(-\Delta)^{-z}u \in \mathcal{L}_r(E_N)$ for $r < q$ is a trivial consequence of Young's inequality. The case $r = q$ is much deeper.

5. Gårding's inequality

Let us now turn to equations with variable coefficients in a bounded domain Ω . Let $\{\phi_j^2\}$, $1 \leq j \leq K$, be a partition of unity in $\bar{\Omega}$, let Ω_j be the set where $\phi_j \neq 0$, let δ_j be the diameter of Ω_j , and let x_j be a point in Ω_j . We transform (64) as on pages 74–75 into a system of the form

$$(67) \quad \sum_{j=1}^K L_{hj}(x, X)v_j = f_h$$

where $v_j = \phi_j u$, $f_j = \phi_j f$, and

$$L_{hj}(x, X)w = \delta_{hj}p(x_h, X)w - \sum_{|\alpha| = |\beta| = k} X^\alpha (\phi_h(p_{\alpha,\beta}(x_h) - p_{\alpha,\beta}(x))\phi_j X^\beta w) + Q_{hj}(x, X)w$$

where Q_{hj} is an operator of order $< m$. We introduce in the space $\mathcal{S} = C_0^\infty(\Omega)(K \times 1)$ the scalar product

$$(v, w)_k = \sum_{j=1}^K (v_j, w_j)_k$$

If $(v, w)_L$ is the bilinear form on $\mathcal{S}$ associated with the operator L , then for all $v \in \mathcal{S}$ we have

$$\Re(v, v)_L \geq (h - \epsilon) \|v\|_k^2 - C \|v\|_{k-1} \|v\|_k$$

where $\epsilon = \max \| (p_{\alpha, \beta}(x_j) - p_{\alpha, \beta}(x)) \|$ for $x \in \Omega_j$, $|\alpha| = |\beta| = k$, and C is a certain constant. If $\max \delta_j$ is sufficiently small, so that $\epsilon \leq h/4$, and we use the logarithmic convexity of $\|v\|_k$ as a function of h , we find that

$$\Re(v, v)_L = \Re(v, v)_L \geq (h/2) \|v\|_k^2 - C' \|v\|_0^2$$

where C' is a certain constant. Since there exists a constant σ_k such that

$$\sigma_k^{-1} \|u\|_{k,0} \leq \|v\|_{k,0} \leq \sigma_k \|u\|_{k,0}$$

for $v_j = \phi_j u$, $1 \leq j \leq K$, then we obtain for $u \in C_0^\infty(\Omega)$

$$(68) \quad \Re(u, u)_P = \Re(u, u)_P \geq (h/2\sigma_k^2) \|u\|_k^2 - C'\sigma_0^2 \|u\|_0^2$$

The fundamental inequality was proved by Gårding [267] after having been an open problem for several years. It was anticipated by van Hove [656].

Let $L(\lambda) = L + \lambda I$, $P(\lambda) = P + \lambda I$. Then if $\Re \lambda \geq C' \max(\sigma_0^2, 1)$, we have for $u \in C_0^\infty(\Omega)$, $v \in \mathcal{S}$, the existence of a constant $h_1 > 0$ such that

$$\Re(v, v)_{L(\lambda)} \geq h_1 \|v\|_{k,0}^2, \quad \Re(u, u)_{P(\lambda)} \geq h_1 \|u\|_{k,0}^2$$

so that Theorem 11 can be applied as before to prove the existence of Green's operators $G_{P(\lambda)}$ and $G_{L(\lambda)}$ with the same properties as in the case of constant coefficients. It is easy to prove the equivalence of (64) and (67) when $f_j = \phi_j f$. Thus we obtain the discreteness of the spectra of P and L , the finite dimensionality of the eigenspaces, the Fredholm alternative, and the solvability of (64) and (67) if the diameter of Ω is sufficiently small and P is fixed. Note that if the oscillation of the coefficients of $p(x, X)$ in Ω is less than h , then we can take $K = 1$, and the consideration of (67) is unnecessary. For full details see Browder [81] and Višik [661].

It follows from Gårding's inequality and Theorem 10 that, if λ is not in the spectrum of P and l is any linear functional on $\mathcal{H}_0^k(\Omega)$, then there exists a unique $u \in \mathcal{H}_0^k(\Omega)$ such that $l(\phi) = (u, \phi)_{P(\lambda)}$ for all $\phi \in \mathcal{H}_0^k(\Omega)$. In particular, if $w \in C^k(\bar{\Omega})$,

$$|||w|||_k^2 = \sum_{h=0}^k \sum_{|\alpha|=h} \frac{h!}{\alpha!} \|X^\alpha w\|_0^2$$

then there exists a unique $u \in \mathcal{H}_0^k(\Omega)$ and a unique $v \in \mathcal{S}^k(\Omega)$, where $\mathcal{S}^k(\Omega)$ is the completion of $C^k(\bar{\Omega})$ with respect to the norm $|||w|||_k$, such that $w = u + v$

$$(w, \phi)_{P(\lambda)} = (u, \phi)_{P(\lambda)}, \quad (v, \phi)_{P(\lambda)} = 0$$

for all $\phi \in C_0^\infty(\Omega)$. That is, v is a weak solution of $P(\lambda)v = 0$ in Ω which

satisfies the same Dirichlet boundary conditions as w , in the sense that $w - v \in \mathcal{H}_0^k(\Omega)$. If

$$|(\phi, \phi)_{\tilde{P}(\lambda)}| \leq C \|\phi\|_k^2$$

and

$$|(w, \phi)_{\tilde{P}(\lambda)}| \leq B \|w\|_k \|\phi\|_k$$

for all $\phi \in C_0^\infty(\Omega)$, $w \in \mathcal{S}^k(\bar{\Omega})$, then $\|u\|_k \leq (B/C) \|w\|_k$ and $\|v\|_k \leq (1 + (A/C)) \|w\|_k$.

The existence of weak solutions of many other boundary value problems can be proved by similar methods (see Sobolev [626], Višik [663], Lions [423]).

6. The Douglis-Nirenberg estimates

We now turn to the question of the regularity properties of the weak solutions of (64) in $\mathcal{H}_0^k(\Omega)$. Such a function is also a weak solution of (58) with $u_0 = u$. We thus obtain

Theorem 45. If the coefficients of P are bounded and measurable, and $p_{\alpha,\beta} \in \text{Lip } \nu$, $0 < \nu \leq 1$, for $|\alpha| = |\beta| = k$, and $f \in \mathcal{L}_q(\Omega)$ for some $q > N/k$, and if u is a weak solution of (64) in $\mathcal{S}^k(\Omega)$, then $u \in C^k(\Omega_1)$ for every compact subdomain Ω_1 of Ω . If $\lambda = \min(\nu, k - N/q) < 1$ and $|\sigma| = k$, then $X^\sigma u \in \text{Lip } \lambda$ in Ω_1 , while, if $\lambda = 1$, then the modulus of continuity of $X^\sigma u$ in Ω_1 satisfies $\omega(r) = O(r(\log^+(1/r) + 1))$. If $p_{\alpha,\beta} \in C^h(\Omega_1)$ for $|\alpha| \leq k$, $|\beta| \leq k$, and $f \in \mathcal{L}_q(\Omega_1)$ for some $q > N/(k - h)$, $0 < h < k$, then $u \in C^{k+h}(\Omega_1)$. If f satisfies a Dini condition in Ω_1 and $p_{\alpha,\beta} \in C^k(\Omega_1)$ for $|\alpha| \leq k$, $|\beta| \leq k$, then $u \in C^m(\Omega_1)$.

We can also obtain *a priori* estimates for u and its derivatives in Ω_1 under the same hypotheses. Douglis and Nirenberg [166] have obtained such estimates for weakly elliptic equations, generalizing those of Schauder [594] for second-order equations. For second-order equations, Serrin [609] has sharpened Schauder's estimates.

To describe the results of Douglis and Nirenberg, we use the notations of page 124, with $\delta(P, Q)$ replaced by the ordinary distance $\|P - Q\|$ and $d(P)$ and $d(P, Q)$ modified accordingly. Suppose that the equation

$$P(x, X)u = \Sigma P_\alpha(x) X^\alpha u = f, \quad P_\alpha(x) = (p_{ij,\alpha}(x))$$

is elliptic in the sense of Douglis and Nirenberg and the constants s_i and t_j are normalized so that $\max s_i = 0$. If for some λ , $0 < \lambda < 1$, we have

$$N_{s_i+t_j-|\alpha|, -s_i, \lambda}(p_{ij,\alpha}) \leq K_1$$

for all i, j , and α , and

$$N_{s_i+t, -s_i, \lambda}(f_i) \leq K_2, \quad 1 \leq i \leq M, \quad t = \max t_j$$

and if P is uniformly weakly elliptic in Ω , then

$$N_{t-t_j, t_j, \lambda}(u_j) \leq K \left(\sum_i' M_{t-t_i}(u_i) + K_2 \right)$$

where Σ' denotes summation over those i for which $t_i > 0$, and K is a constant depending only on $N, M, s_1, \dots, t_M, \lambda, K_1$, and $K_3 = \inf(|p^\square(x, \xi)|, x \in \Omega, \|\xi\| = 1)$.

We have now shown that weak solutions of (64) exist in $\mathcal{H}_0^k(\Omega)$ and that if the coefficients of P and f are sufficiently smooth, then u is a bona-fide solution with all the desired derivatives. A function in $\mathcal{H}_0^k(\Omega)$ satisfies the Dirichlet boundary conditions: $X^\alpha u \rightarrow 0$ as $x \rightarrow \partial\Omega$ for $|\alpha| < k$, in a certain generalized sense. If α is a portion of $\partial\Omega$ of class C^1 , i.e., there is a coordinate system and a function $g(x_1, \dots, x_{N-1})$ of class C^1 such that α is represented by the equation $x_N = g(x_1, \dots, x_{N-1})$, then $X^\alpha u(x) \rightarrow 0$ in the $\mathcal{L}_2$ sense with respect to the variables $x_1, \dots, x_{N-1}$ as $x \rightarrow \alpha$. (Deny and Lions [135].) More precisely, if we make a change of coordinate system so that α becomes a portion β of the hyperplane $x_N = 0$, then $(X^\alpha u, BX^\alpha u)_\beta \rightarrow 0$ as $x_N \rightarrow 0$ for $|\alpha| < k$, where B is a certain nonnegative operator such that $B^2 = \sum_{j=1}^{N-1} X_j^2$ (Aronszajn [24]). We now consider the question of the assumption of the boundary values in the classical sense. This has been settled in a very precise manner for the Laplace equation by Wiener [712] and in two dimensions for strongly elliptic equations with sufficiently smooth coefficients in domains with smooth boundaries by Browder [78], and for second-order systems by Morrey [463]. The problem for strongly elliptic systems of any order, under strong smoothness assumptions, was solved by Nirenberg [489].

7. The assumption of boundary values

A basic tool is the lemma of Sobolev [624], now used in the sharp form which holds up to the boundary. Let S be an open set on the boundary of the unit sphere $\|\xi\| = 1$, and let $C_a(x, S)$ be the conical sector $x + r\xi$, $0 \leq r \leq a$, $\xi \in S$. We say that Ω has the *cone property* if there is a constant $b > 0$ such that for every $x \in \Omega$ there is a cone $C_a(x, S) \subset \Omega$ such that $a \geq b$ and $A(S) \geq b$; we say that Ω has the *strong cone property* if there is, furthermore, an $\epsilon > 0$ such that for every $x \in \Omega$ the set $C_a(x, S) - C_{a/2}(x, S)$ is contained in the set Ω_ϵ of all $y \in \Omega$ such that $d(y) > \epsilon$. If Ω has the cone property and $X^\alpha u \in L_q(\Omega)$ for $|\alpha| \leq k$, then $u \in C^h(\bar{\Omega})$ for $k > h + (N/q)$. If Ω has the strong cone property and $h = [k - 1 - (N/q)]$, then $X^\alpha u \in \text{Lip } \lambda$ for $\lambda \leq k - h - (N/q)$,

$\lambda < 1$, and $|\alpha| = h$. If Ω has the strong cone property and $k - (N/q) \leq h \leq k$, then $X^\alpha u \in \mathcal{L}_r(\Omega)$ for $r < \infty$, $r^{-1} > q^{-1} - (k - h)/N$. These results follow from an identity of the form

$$u(x) A(S) = u(t, x)$$

$$+ \int_0^t \int_{C_a(x, S)} \sum_{|\alpha|=k} \sum_{h=1}^k \|x - y\|^{h-k} (X^\alpha G(\tau, x - y)) B_{\alpha, h} u \, dV \, d\tau$$

where

$$u(t, x) = \int_{C_a(x, S)} G(t, x - y) u(y) \, dV_y$$

and G is the fundamental solution of the equation $v_t = -(-\Delta)^k v$, and $B_{\alpha, h}$ is a differential operator of order h with bounded coefficients which are of class C^∞ for $y \neq x$, and u is any function of class C^k which vanishes when $\|y - x\|$ is close to a .

Another essential lemma is an inequality of obscure origin proved by Ehrling [181] (see also Hadamard [307], Hardy, Littlewood, and Pólya [311], p. 194, Pitt and Halperin [524], Kolmogorov [375], Bang [24], Lax and Milgram [405], Nirenberg [489]). We shall say that Ω has the *third cone property* if it has the strong cone property with S a spherical cap, and if for all $y \in \Omega - \Omega_\epsilon$, the set of $x \in \partial\Omega$ such that $y \in C_a(x, S)$ has area between b and b^{-1} . If Ω has this third cone property, then for $0 < j < k$ there is a constant $C_{j, k}$ such that

$$(69) \quad \|u\|_j \leq C_{jk}(\epsilon^{k-j} \|u\|_k + \epsilon^{-j} \|u\|_0)$$

for $0 < \epsilon \leq 1$. Let us indicate a quick proof in the one-dimensional case. Let $\Psi(t) \in C^k$ for $0 \leq t \leq 1$, $\Psi^{(h)}(1) = 0$ for $h < k$, $\Psi^{(h)}(0) = (-1)^{j+1} \delta_{h, k-j-1}$ for $h < k$. Then

$$\int_0^1 \Psi^{(k)}(t) u(x + t\epsilon) \, dt = \epsilon^j u^{(j)}(x) + (-1)^h \epsilon^k \int_0^1 \Psi(t) u^{(k)}(x + t\epsilon) \, dt$$

It is now easy to estimate the norm of $u^{(j)}$. We then prove (69) for a cube by integration over cross sections. It is easy then to pass to a cone $C_a(x, S)$ when S is a spherical cap, and then to a general Ω with the third cone property. Actually we shall only use the case of a cube in what follows, and shall not use the full strength of this inequality. This inequality has many other applications.

Let $f \in C^1(\Omega)$, let $\mathcal{Q}$ be a portion of the boundary of Ω of class C^{m+1} , i.e., a neighborhood of $\mathcal{Q}$ in Ω can be transformed by a coordinate transformation of class C^{m+1} into the cube $\Omega_1: 0 \leq x_j \leq 1$, $1 \leq j \leq N$, in such a way that $\mathcal{Q}$ becomes the face $x_N = 0$. We assume that P is a strongly

elliptic operator with coefficients of class $C^1(\bar{\Omega})$ such that for some constant $a > 0$

$$\Re(\phi, \phi)_P \geq a \|\phi\|_k^2$$

for all $\phi \in C_0^\infty(\Omega)$. Let $\xi_j(h)$ be the difference operator

$$\xi_j(h)\phi = (\phi(x + h\delta_j) - \phi(x))/h$$

where δ_j is the vector with components δ_{jk} , $k = 1, \dots, N$. We choose a function $\Psi \in C^\infty(E_N)$ such that $\Psi = 1$ in a neighborhood of the set $x_N = 0$, $\epsilon \leq x_j \leq 1 - \epsilon$ ($1 \leq j \leq N$) and $\Psi = 0$ in a neighborhood of the other faces of Ω_1 . If u is a weak solution of (64) in $\mathcal{H}_0^k(\Omega)$, then $U = \Psi u \in \mathcal{H}_0^k(\Omega_1)$ and for $\phi \in C_0^\infty(\Omega_1)$

$$(U, \phi)_{\bar{P}} = (\Psi f, \phi)_0 + (u, \phi)_Q = (f_1, \phi)_0 + (u, \phi)_Q$$

where $(u, \phi)_Q$ is a bilinear form of order less than k in u . For $j < N$, $|h| < \epsilon$, $\xi_j(h)U$ is in $\mathcal{H}_0^k(\Omega_1)$, and we have

$$\begin{aligned} |(\xi_j(-h)U, \phi)_{\bar{P}}| &= |(U, \xi_j(h)\phi)_{\bar{P}} + (T_{-h}U, \phi)_{\bar{P}_j}| \\ &\leq |(f_1, \xi_j(h)\phi)_0| + |(u, \xi_j(h)\phi)_{\bar{P}}| + |(T_{-h}U, \phi)_{\bar{P}_j}| \leq C \|\phi\|_k \end{aligned}$$

where P_j is the operator whose coefficients are $\xi_j(-h)p_{\alpha,\beta}$, $T_{-h}U$ is $U(x - h\delta_j)$, and C is a certain constant. Hence $\|\xi_j(-h)U\|_k \leq C/a$ for $0 < |h| < \epsilon$. But since $\xi_j(-h)U \rightarrow iX_jU$ in $\mathcal{L}_2(\Omega_1)$ as $h \rightarrow 0$ and $\|\xi_j(-h)U\|_k$ is bounded, it follows that $X_jU \in \mathcal{H}_0^k(\Omega_1)$ for $j < N$. By repetition of this argument we find that, if $f \in C^p(\bar{\Omega}_1)$ and the coefficients of P are in $C^p(\bar{\Omega}_1)$ and α is of class C_{m+p} , then $X^\alpha U \in \mathcal{H}_0^k(\Omega_1)$ for $|\alpha| \leq p$, $\alpha_N = 0$, and $X^\alpha U \in \mathcal{L}_2(\Omega_1)$ for $|\alpha| \leq k + p$, $\alpha_N \leq k$.

Now if the coefficients of P are in $C^k(\bar{\Omega}_1)$ and $f \in C^1(\bar{\Omega}_1)$, then $u \in C^m(\Omega)$ and satisfies the equation (64) in Ω_1 in the ordinary sense. If $m = 2$, then the coefficient of X_N^2u is bounded away from zero and all the other terms are in $\mathcal{L}_2(\Omega_1)$, so that $X_N^2u \in \mathcal{L}_2(\Omega_1)$. If $m > 2$, then we can deduce from the equation (64) an equation of the form $X_N^kv = f_2$, where $v = Q_1U$ and Q_1 is a differential operator of order k such that the coefficient of X_N^kU in Q_1U is bounded away from zero in $\bar{\Omega}_1$ and $f_2 \in \mathcal{L}_2(\Omega_1)$. Since both v and X_N^kv are in $\mathcal{L}_2(\Omega_1)$, then by the lemma (69) above we conclude that $X_Nv \in \mathcal{L}_2(\Omega_1)$. But $X_Nv = c(x)X_N^{k+1}U + \dots$, where $c(x)$ is bounded away from zero in Ω_1 , and all the other terms are in $\mathcal{L}_2(\Omega)$, so that $X_N^{k+1}U \in \mathcal{L}_2(\Omega_1)$. We apply this result to X_jU , $1 \leq j < N$, which is the solution in $\mathcal{H}_0^k(\Omega_1)$ of the equation

$$P(x, X)X_jU + Q_2(x, X)u = f_3$$

where Q_2 is an operator of order m and $f_3 \in \mathcal{L}_2(\Omega_1)$, and infer that $X_N^{k+1}X_jU \in \mathcal{L}_2(\Omega_1)$. Now we can deduce the fact that $X_N^{k+2}U \in$

$\mathfrak{L}_2(\Omega_1)$ from the fact that $X_N^2 v \in \mathfrak{L}_2(\Omega_1)$. This argument can be repeated, with certain simplifications as soon as we reach $X_N^p U$ for $p > m$. Thus we find that if $\mathfrak{Q}$ is of class C^{3k+p-2} and the coefficients of P are in $C^{2k+p-2}(\Omega_1)$ and $f \in \mathfrak{S}^{p-1}(\Omega_1)$, then $U \in \mathfrak{S}^{k+p}(\Omega_1)$. In particular, if $p > (N/2) - 1$, then $U \in C_{k-1}(\Omega_1)$, so that $X^\alpha u(x) \rightarrow 0$ as $x \rightarrow \mathfrak{Q}$ for $|\alpha| < k$, and the boundary conditions are satisfied on $\mathfrak{Q}$ in the classical sense. Browder [82] subsequently found another proof, based on an inequality of Aronszajn [13], requiring weaker assumptions on f . For the case $m = 2$, Morrey [463] has obtained very sharp results.

8. Green's and Neumann functions. The Bergman kernel

Let $(u, v)_Q$ be a bilinear form on a subspace $\mathfrak{S}$ of $\mathfrak{S}^k(\Omega)$ containing $C_0^\infty(\Omega)$, and let (u, v) be a scalar product on $\mathfrak{S}$ with respect to which $\mathfrak{S}$ is a Hilbert space, and let $\|u\|$ be the corresponding norm. We assume that

$$|(u, v)_Q| \leq A \|u\| \|v\|, \quad |(u, u)_Q| \geq C \|u\|^2, \quad \| \|u\| \|k \leq C_1 \|u\|$$

for all $u, v \in \mathfrak{S}$, and that $(u, v)_Q = (u, v)_P$ if either u or $v \in C_0^\infty(\Omega)$, where P is a strongly elliptic operator on Ω with bounded measurable coefficients such that $p_{\alpha, \beta}$ is Hölder continuous for $|\alpha| = |\beta| = k$, and such that $\Re(u, u)_P \geq a \|u\|_k^2$ for some constant $a > 0$ and all $u \in C_0^\infty(\Omega)$.

For every $u \in \mathfrak{S}^k(\Omega)$, there exist unique elements $X_P u \in \mathfrak{H}_0^k(\Omega)$ and $Y_P u$ such that $u = X_P u + Y_P u$ and

$$(u, v)_P = (X_P u, v)_P, \quad (Y_P u, v)_P = 0$$

for all $v \in C_0^\infty(\Omega)$. The function $Y_P u$ is a weak solution of $\tilde{P}w = 0$ in Ω . An example of a form Q satisfying the above conditions with $\mathfrak{S} = \mathfrak{S}^k$, $\|u\| = \| \|u\| \|_k$ is

$$(u, v)_Q = (X_P u, X_P v)_P + (((Y_P u, Y_P v)))_k$$

The choice of Q and $\mathfrak{S}$ can be varied considerably according to the boundary value problem under investigation.

For every $f \in \mathfrak{L}_2(\Omega)$, the functionals $(f, u)_0$, $(f, X_P u)_0$, and $(Y_P u)(x)$ (fixed $x \in \Omega$) are linear functionals on $\mathfrak{S}$, and so, by pages 69–70, there exist uniquely determined elements $N_Q f$, $G_Q f$, and K_x in $\mathfrak{S}$ such that

$$(u, f)_0 = (u, N_Q f)_Q, \quad (X_P u, f)_0 = (u, G_Q f)_Q, \quad (Y_P u)(x) = \overline{(u, K_x)_Q}$$

for all $u \in \mathfrak{S}$. Since for $u \in C_0^\infty(\Omega)$ we have $(u, K_x)_Q = (u, K_x)_P = 0$, then K_x is a weak solution of $PK_x = 0$ in Ω , and so $K_x(y) = K(x, y)$ is a well-defined function of class $C^k(\Omega)$ for each $x \in \Omega$. It is called the *Bergman kernel* associated with the form Q and the space $\mathfrak{S}$. For special

spaces and second-order operators and for the operator $P = (-\Delta)^2$ it was introduced by Bergman [28] and was discussed in considerable detail by Bergman and Schiffer [36] (see also Aronszajn [6]). The operators N_Q and G_Q (as we shall see in a moment G_Q depends only on P) are the *Neumann* and *Green* operators associated with Q and S .

Now $(Y_P u, G_Q f)_Q = 0$, so that

$$(Y_P u, Y_P G_Q f)_Q = -(Y_P u, X_P G_Q f)_Q = -(Y_P u, X_P G_Q f)_P = 0$$

for all $u \in S$, from which we conclude, on setting $u = G_Q f$, that $Y_P G_Q f = 0$, so that $G_Q f \in \mathfrak{H}_0^k(\Omega)$. Hence $G_Q f$ coincides with the unique element $G_P f$ in $\mathfrak{H}_0^k(\Omega)$ such that $(u, f)_0 = (u, G_P f)_P$ for all $u \in C_0^\infty(\Omega)$. Thus $G_P f = G_Q f$ is the weak solution in $\mathfrak{H}_0^k(\Omega)$ of (64). Similarly, if $u \in \mathfrak{H}_0^k(\Omega)$, then $(u, f)_0 = (u, N_Q f)_Q = (u, N_Q f)_P$, so that $N_Q f$ is another weak solution in the space S of the same equation.

We remark that, if $(u, v)_{Q^*}$ is the bilinear form defined by

$$(u, v)_{Q^*} = \overline{(v, u)_Q}$$

then Q^* satisfies the same conditions as Q , and is associated with the operator $\tilde{P}$. Furthermore,

$$(N_Q^* u, v)_0 = (N_Q^* u, N_Q v)_Q = \overline{(N_Q v, N_Q^* u)_{Q^*}} = (u, N_Q v)_0$$

which implies that $N_Q^* = (N_Q)^*$ in $\mathfrak{L}_2(\Omega)$. We treat G_P similarly to show that $G_{\tilde{P}} = (G_P)^*$ in $\mathfrak{L}_2(\Omega)$.

By application of Theorem 45, we infer that, if $q \geq 2$, $q > N/k$, and $f \in \mathfrak{L}_q(\Omega)$, then $N_Q f \in C^k(\Omega)$ and, for $y \in \Omega$, there is a constant $C_1(y)$ such that

$$|(N_Q f)(y)| \leq C_1(y) L_q(f)$$

Hence there is a function $N_Q(x, y)$ of class $\mathfrak{L}_r(\Omega)$, $r = q/(q - 1)$, with respect to x such that

$$(N_Q v)(y) = \int_{\Omega} N_Q(x, y) v(x) dV_x$$

Similarly there exists a function $G_P(x, y) \in \mathfrak{L}_r(\Omega)$ with respect to x such that

$$(G_P f)(y) = \int_{\Omega} G_P(x, y) f(x) dV_x$$

For $x, y \in \Omega$, $x \neq y$, $N_Q(x, y)$ and $G_P(x, y)$ are, as functions of y , weak solutions of $Pw = 0$, and so are of class $C^k(\Omega - \{x\})$. Since

$$N_Q(x, y) = \overline{N_Q^*(y, x)}, \quad G_P(x, y) = \overline{G_{\tilde{P}}(y, x)}$$

these functions are also of class $C^k(\Omega - \{y\})$ as functions of x and are there weak solutions of $\bar{P}\bar{w} = 0$. Finally,

$$(v, Y_P u)_0 = (u, \int_{\Omega} v(x) K_x dV_x)_Q = \overline{(u, N_Q v - G_P v)_Q}$$

for all $u \in \mathfrak{S}$, so that

$$\int_{\Omega} K_Q(x, y) v(x) dV_x = \int_{\Omega} (N_Q(x, y) - G_P(x, y)) v(x) dV_x$$

for all $v \in \mathfrak{L}_Q(\Omega)$. Hence

$$K_Q(x, y) = N_Q(x, y) - G_P(x, y)$$

We can look upon this situation in another way, since

$$(u, v)_Q = (X_P u, X_P v)_Q + (Y_P u, Y_P v)_Q = (X_P u, X_P v)_P + (u, L v)$$

where L is a certain bounded linear operator on $\mathfrak{S}$ to $\mathfrak{S}$ such that $Lv = LY_P v$, so that Lv depends only on the values of v in an arbitrarily small neighborhood of $\partial\Omega$, and the range of L is a certain linear manifold $\mathfrak{M}$ in $\mathfrak{S}$. We may formulate the following boundary value problem: to find a weak solution v in $\mathfrak{S}$ of $Pv = f$ such that $Lv = g$, where f is a given element of $\mathfrak{L}_2(\Omega)$ and g is a given element of $\mathfrak{M}$. The solution is the unique $v \in \mathfrak{S}$ such that

$$(u, v)_Q = (X_P u, f)_0 + (Y_P u, g)$$

for all $u \in \mathfrak{S}$.

The operator S is called a *fundamental solution* of P in Ω if S is a bounded operator on $\mathfrak{L}_2(\Omega)$ to $\mathfrak{S}^k(\Omega)$ such that $(u, v) = (u, Sv)_P$ for all $u \in C_0^\infty(\Omega)$. By the same reasoning as before, we can show that

$$Sv = \int_{\Omega} S(x, y) v(x) dV_x$$

where $S(x, y) \in \mathfrak{L}_r(\Omega)$ as a function of x and is, as a function of y , a weak solution of $Pw = 0$ in $\Omega - \{x\}$, so that it is of class C^k in y for $y \neq x$. If the Bergman kernel $K_Q(x, y)$ and a fundamental solution are known, then N_Q and G_P can be constructed. Let $x \in \Omega$, and choose $\phi \in C_0^\infty(\Omega)$ such that $\phi = 1$ in a neighborhood of x . Then

$$G_P(x, y) = S(x, y) - \overline{(K_Q(z, y), (1 - \phi(z)) S(x, z))_{Q(z)}}$$

$$N_Q(x, y) = G_P(x, y) + K_Q(x, y)$$

where $Q(z)$ indicates that the operations are to be performed on $\overline{K_Q(z, y)}$ and $(1 - \phi(z)) S(x, z)$ are considered as functions of z .

9. Construction of solutions

So far our theory is based on the abstract existence Theorem 6, and gives no method for actually constructing solutions of the problems considered. One method of great generality is to begin with a countable dense set in $\mathcal{S}$ and to orthonormalize them over $\mathcal{L}_2(\Omega)$, thus obtaining a sequence $\{\phi_n\}$ in $\mathcal{S}$. We can now find linear combinations $\Psi_{j,n}$ of $\phi_1, \dots, \phi_n$ such that

$$(\Psi_{j,n}, \phi_h)_Q = (\phi_j, \phi_h)_0 = \delta_{jh} \quad \text{for } 1 \leq j, h \leq n$$

Then

$$N_Q u = \lim_{n \rightarrow \infty} \sum_1^n \phi_j (\Psi_{j,n}, u)_0$$

$$N_Q^* u = \lim_{n \rightarrow \infty} \sum_1^n \Psi_{j,n} (\phi_j, u)_0$$

A similar procedure can be applied to $\mathcal{H}_0^k(\Omega)$ for the construction of $G_P u$ and $G_{\bar{P}} u$. The errors can be estimated in terms of the degree to which one can approximate a function of known smoothness properties by linear combinations of $\phi_1, \dots, \phi_n$. For elliptic operators of second order these questions were studied in great detail by Kantorovič and Krylov [363], Indritz [344]. The work of Kharrik [368] will be very important in the investigation of the general problem.

If $Q = Q^*$, then we can take $(u, v)_Q$ as the scalar product in $\mathcal{S}$. Now we can characterize $G_P f$ as the solution of certain variational problems. It is the function in $\mathcal{H}_0^k(\Omega)$ which minimizes

$$J(u) = \|u\|_Q^2 - 2\Re(f, u)_0$$

and also maximizes $J_1(u) = \Re(f, u)_0 / \|u\|_Q$. The extrema are respectively $-\|G_P f\|_Q^2$ and $\|G_P f\|_Q$. It can also be determined by choosing a function $\phi_0 \in \mathcal{H}_0^k(\Omega)$ such that $(f, \phi_0)_0 = 1$ and finding the minimum of $\|\phi_0 - \Psi\|_Q$ as Ψ ranges over all elements of $\mathcal{H}_0^k(\Omega)$ such that $(f, \Psi)_0 = 0$. The minimum is $1/\|G_P f\|_Q$. Every trial function in each of these variational problems leads to an estimate of $\|G_P f\|_Q$ from below. The operator N_Q can be treated similarly.

The operators X_P and Y_P are also related to variational problems. Since $Q = Q^*$, $X_P = (X_P)^*$, $Y_P = (Y_P)^*$, so that these are both projection operators on the subspaces $\mathcal{H}_0^k(\Omega)$ and Σ_P , the space of weak solutions in $\mathcal{S}$ of the equation $Pw = 0$, and these subspaces are orthogonal complements in $\mathcal{S}$. Hence $X_P u$ is the element of $\mathcal{H}_0^k(\Omega)$ which minimizes $\|u - v\|_Q$ and $Y_P u$ is the element of Σ_P which minimizes the Q -distance from u . A boundary problem may be posed in the form: to find an

element $w \in \mathfrak{S}$ which is a weak solution of $Pw = Pv$ and such that $w - u \in \mathfrak{H}_0^k(\Omega)$, where u and v are given elements of $\mathfrak{S}$; in other words, to find a $w \in \mathfrak{S}$ such that $X_P w = X_P v$ and $Y_P w = Y_P u$. This w minimizes $\|v - W\|_Q$ on the manifold of elements W such that $Y_P W = Y_P u$, and minimizes $\|u - W\|_Q$ on the manifold of elements W such that $X_P W = X_P v$. If u is a weak solution in $\mathfrak{S}$ of $Pu = f$, then $X_P u = G_P f$, so that this procedure yields upper bounds for $\|G_P f\|_Q$.

Often a complete determination of w is more detailed information than we need for the applications. The desired information may frequently be summarized by the values of a few functionals of the form $(z, w)_Q$, where $X_P z = X_P Z$ and $Y_P z = Y_P \zeta$ and Z and ζ are known. The identities

$$\begin{aligned}(u, \zeta)_Q - (v, \zeta)_Q - (u, Z)_Q + (w, z)_Q \\ = (u - w, \zeta - z)_Q + (w - v, \zeta)_Q + (u, z - Z)_Q\end{aligned}$$

and

$$\begin{aligned}(v, Z)_Q - (v, \zeta)_Q - (u, Z)_Q + (w, z)_Q \\ = (w - v, z - Z)_Q + (w - u, Z)_Q + (v, z - \zeta)_Q\end{aligned}$$

yield the estimates

$$\begin{aligned}|(u, \zeta)_Q - (v, \zeta)_Q - (u, z)_Q + (w, z)_Q| &\leq g_1 \\ |(v, Z)_Q - (v, \zeta)_Q - (u, Z)_Q + (w, z)_Q| &\leq g_2\end{aligned}$$

where

$$g_1 = (\{\|u - v\|_Q^2 + \|u\|_Q^2\} \{\|\zeta - Z\|_Q^2 + \|\zeta\|_Q^2\})^{1/2}$$

$$\text{if} \quad \|u\|_Q \|\zeta\|_Q \leq \|u - v\|_Q \|Z - \zeta\|_Q$$

$$g_1 = \|u - v\|_Q \|\zeta\|_Q + \|Z - \zeta\|_Q \|u\|_Q$$

$$\text{if} \quad \|u - v\|_Q \|Z - \zeta\|_Q \leq \|u\|_Q \|\zeta\|_Q$$

and similarly for g_2 . By adding to u and ζ suitable elements of $\mathfrak{H}_0^k(\Omega)$ and to v and Z suitable elements of Σ_P , we can make g_1 arbitrarily small, and so can compute $(z, w)_Q$ to any desired degree of accuracy, the error at each stage being estimated in terms of known quantities. In particular, we can set $z = w$, and obtain approximations to $\|w\|_Q$. This method is essentially due to Diaz and Greenberg [140]. See also Diaz [137], Fichera [219], Friedrichs [238], Fichera and Picone [227], Pólya and Szegő [553].

10. Eigenvalue problems

In all the above variational problems, we can approximate the solution arbitrarily closely by exhausting the manifolds over which the

functions vary by finite-dimensional submanifolds. This is essentially the Rayleigh-Ritz procedure.

Let $(u, v)_R$ be a bilinear form on $\mathcal{S}$ such that $R = R^*$ and

$$|(u, v)_R| \leq C_1(\|u\|_Q \|v\|_{k-1} + \|v\|_Q \|u\|_{k-1})$$

Then there is a bounded self-adjoint operator B on $\mathcal{S}$ to $\mathcal{S}$ such that $(u, v)_R = (u, Bv)_Q$, and if Ω has the third cone property then B is a compact operator on $\mathcal{S}$. Hence there exists a sequence $\{\lambda_n\}$ of eigenvalues with no limit point except zero, and a corresponding complete orthonormal system of eigenfunctions ϕ_n such that $B\phi_n = \lambda_n\phi_n$, i.e.,

$$(u, \phi_n)_R - \lambda_n(u, \phi_n)_Q = 0$$

for all $u \in \mathcal{S}$. It is convenient to label the positive eigenvalues in descending order: $\lambda_1^+ \geq \lambda_2^+ \geq \dots$, each counted according to the dimensionality of its eigenspace, and, similarly, we arrange the negative eigenvalues in ascending order: $\lambda_1^- \leq \lambda_2^- \leq \dots$; if either of these sequences is finite, we adjoin to it infinitely many zeros. If $\mathfrak{M}$ is any subspace of $\mathcal{S}$ and $\Pi(\mathfrak{M})$ is the projection of $\mathcal{S}$ onto $\mathfrak{M}$, then $\Pi(\mathfrak{M}) B \Pi(\mathfrak{M})$ is a self-adjoint operator with eigenvalues $\lambda_n^+(\mathfrak{M})$ and $\lambda_n^-(\mathfrak{M})$ ordered as above.

These eigenvalues are determined by variational problems. Thus,

$$\begin{aligned} \lambda_1^+(\mathfrak{M}) &= \max_{u \in \mathfrak{M}, u \neq 0} \frac{(u, u)_R}{\|u\|_Q^2}, & \lambda_1^-(\mathfrak{M}) &= \min_{u \in \mathfrak{M}, u \neq 0} \frac{(u, u)_R}{\|u\|_Q^2} \\ \lambda_n^+(\mathfrak{M}) &= \min_{\dim(\mathfrak{M}') < n} \lambda_1^+(\mathfrak{M} \perp \mathfrak{M}'), & \lambda_n^-(\mathfrak{M}) &= \\ &= \max_{\dim(\mathfrak{M}') < n} \lambda_1^-(\mathfrak{M} \perp \mathfrak{M}') \end{aligned}$$

where $\mathfrak{M} \perp \mathfrak{M}'$ is the space of elements of $\mathfrak{M}$ which are orthogonal to $\mathfrak{M}'$. Of course, $\lambda_n^+ = \lambda_n^+(\mathcal{S})$, $\lambda_n^- = \lambda_n^-(\mathcal{S})$. These variational definitions of λ_n are essentially due to Fischer [228], and were especially emphasized by Courant [126]. They give rise to two important comparison principles which are of the greatest value for the estimation and numerical determination of the eigenvalues:

Theorem 46. If $\mathfrak{M} \subset \mathfrak{M}_1$, then $\lambda_n^+(\mathfrak{M}) \leq \lambda_n^+(\mathfrak{M}_1)$ and $\lambda_n^-(\mathfrak{M}) \geq \lambda_n^-(\mathfrak{M}_1)$.

Theorem 47. If Q_1 and R_1 are self-adjoint bilinear forms on $\mathcal{S}$ such that $\|u\|_{Q_1}^2 - \|u\|_Q^2$ and $(u, u)_R - (u, u)_{R_1}$ are nonnegative on $\mathcal{S}$, and $\lambda_{n,1}^+$, $\lambda_{n,1}^-$ are the eigenvalues of R_1 with respect to Q_1 , then $\lambda_{n,1}^+ \leq \lambda_n^+$. If $Q_1 - Q$ and $R_1 - R$ are nonnegative on $\mathcal{S}$, then $\lambda_{n,1}^- \geq \lambda_n^-$.

The first of these theorems is the basis of the Rayleigh-Ritz (see Collatz [124], Kantorovič and Krylov [363]) and the Weinstein (see [680]) methods for the estimation of eigenvalues. In the former one begins with a subspace $\mathfrak{M}_0$ (usually the null-space) for which the eigenvalues are known, and adjoins successively elements u_n which form an orthonormal basis for $S \perp \mathfrak{M}_0$. The eigenvalues $\lambda_n^+(\mathfrak{M}_h)$ and $\lambda_n^-(\mathfrak{M}_h)$, where $\mathfrak{M}_h$ is the space spanned by $\mathfrak{M}_0$ and $u_1, \dots, u_h$, can be computed by elementary algebraic processes, and yields sequences which approach λ_n^+ and λ_n^- from below and above respectively. In the Weinstein method one begins with a space $\mathfrak{M}_0$ containing S for which the solution of the eigenvalue problem is known, and chooses an orthonormal basis $\{u_h\}$ of $\mathfrak{M}_0 \perp S$. If $\mathfrak{M}_h$ is the orthogonal complement in $\mathfrak{M}_0$ of the space spanned by $u_1, \dots, u_h$, then the eigenvalues $\lambda_n^+(\mathfrak{M}_h)$ and $\lambda_n^-(\mathfrak{M}_h)$ can be computed, and constitute sequences which approach λ_n^+ and λ_n^- from above and below respectively. These methods were developed systematically by Aronszajn [8], [10], and applied to concrete problems by him and his associates [16], [20]. As is shown by the work of Pólya and Szegő [553], remarkably good results can already be obtained by an adroit choice of u_1 in each method. The application of the second theorem has not been as fully developed (see, however, Koehler [373]). The latter paper also gives a discussion of the approximation to eigenfunctions.

Besides the estimation of individual eigenvalues, it is also important to know their asymptotic distribution, for example, for the applications to statistical mechanics. For special equations results of this kind were obtained by Weyl [703], [706] and Courant [126] by the use of the above comparison theorems. In 1934, Carleman [112] introduced a very powerful method based on Tauberian theorems (see Hardy [310a]) for attacking this problem, and his method has been applied with great success by Pleijel [526], Gårding [265], Ehrling [181], Brownell [84], Browder [80], and Sandgren [589]. We shall illustrate this method with the special case $(u, v)_R = (u, v)_0$, $(u, v)_Q = (u, v)_P$ and $S = \mathfrak{C}_0^k(\Omega)$, and shall work with the numbers $\mu_n = \lambda_n^{-1}$. In this case all the eigenvalues are positive.

Let $P = \tilde{P}$ in (64), and let $G_P(t, x, y)$ be the Green's function for the domain Ω and the parabolic equation $u_t = -P(x, X)u$. Then if $\{\phi_n\}$ are the orthonormalized eigenfunctions, we have

$$G_P(t, x, y) = \sum \phi_n(x) \overline{\phi_n(y)} e^{-\mu_n t}$$

We already know that if $p(x, X)$ is the principal part of P and $G(t, x; x_0)$

is the fundamental solution of the equation $u_t = -p(x_0, X)u$, then

$$|G_P(t, x, y)| \leq Ct^{-\sigma}$$

for $0 < t, x, y \in \Omega$, where $\sigma = N/m$, and also that

$$|G_P(t, x, y) - G(t, x - y; x)| \leq C_0(\Omega_1)t^{(\nu/m)-\sigma}$$

for $t > 0, x \in \Omega_1, y \in \Omega$, where $\bar{\Omega}_1 \subset \Omega$. In particular, we apply the last inequality to the case $y = x$, and observe that

$$G(t, 0; x) = (2\pi)^{-N} \Gamma(\sigma + 1) V(x)t^{-\sigma}$$

where $V(x)$ is the volume of the set of $\xi \in E_N$ such that $p(x, \xi) \leq 1$. We apply the Hardy-Littlewood Tauberian theorem (see [710], p. 192) to the Dirichlet series $G_P(t, x, x)$ with nonnegative terms in the form essentially given by Korevaar [380], and obtain

$$(70) \quad \sum_{\mu_n \leq T} |\phi_n(x)|^2 = (2\pi)^{-N} V(x)T^\sigma + O(T^\sigma / \log T)$$

as $T \rightarrow \infty$. If we apply the same result to

$$\begin{aligned} \Sigma |\phi_n(x) + \lambda \phi_n(y)|^2 \exp(-\mu_n t) \\ = G_P(t, x, x) + 2\Re(\lambda G_P(t, x, y)) + G_P(t, y, y) \end{aligned}$$

where $|\lambda| = 1$, we obtain

$$\sum_{\mu_n \leq T} \phi_n(x) \overline{\phi_n(y)} = O(T^\sigma / \log T)$$

for $x \neq y$. So far these results apply for $x, y \in \Omega_1$, where $\bar{\Omega}_1 \subset \Omega$. But the above inequality shows that $t^\sigma |G_P(t, x, x) - G(t, 0; x)|$ is uniformly bounded in Ω and approaches zero pointwise as $t \rightarrow 0$, so that

$$\lim_{t \rightarrow 0} t^\sigma \int_{\Omega} (G_P(t, x, x) - G(t, 0; x)) dV_x = 0$$

This implies that

$$\sum_{\mu_n \leq T} 1 \sim (2\pi)^{-N} T^\sigma \int_{\Omega} V(x) dV_x$$

For the special case where P is the Laplace operator and the boundary of Ω is sufficiently smooth, Pleijel [539], Minakshisundaram [451], and Brownell [84] have given more precise results.

ELLIPTIC EQUATIONS OF SECOND ORDER

1. Maximum principle

For elliptic equations of second order the maximum principle provides a tool for which no analogue is known, except for the biharmonic equation in the plane (Miranda [454]), for higher order equations. (Analogues of the maximum principle are indeed known for the equation $\Delta^k u = 0$ (Nicolesco [481], [482]), but these have not been applied up to now to the solution of boundary value problems.) The maximum principle enables us to utilize order relations between functions, whereas in the study of higher order equations, we had to content ourselves with order relations between their absolute values.

Let

$$P(x, D)u = \Sigma a_{ij}(x)u_{ij} + \Sigma b_i u_i + cu, \quad u_i = D_i u, \quad u_{ij} = D_i D_j u$$

be an elliptic operator with real coefficients in a domain $\Omega \subset E_N$. We assume that there is a function $h(x) > 0$ in Ω such that

$$\Sigma a_{ij}(x)\xi_i \xi_j \geq h(x) \|\xi\|^2$$

for all ξ in E_N . We first give some simple consequences of the maximum principle.

If $c \leq 0$ in Ω and $Pu < 0$ in Ω and $u \geq -B$ on $\partial\Omega$, where $B \geq 0$, then $u \geq -B$ in Ω . If $c < 0$ in Ω and $Pu \leq 0$ in Ω and $u \geq -B$ on $\partial\Omega$, where $B \geq 0$, then $u \geq -B$ in Ω . If $c \leq 0$ in Ω and there is a w such that $w \geq 0$ on $\partial\Omega$, $Pw < 0$ in Ω , then the first statement still holds under the condition $Pu \leq 0$ in Ω .

A device due to Picard [518] enables us to reduce the general case to the case $c < 0$ in many cases. If $w > 0$ and $Pw < 0$ in Ω , set $u = wv$. Then

$$P_2 v = w^{-1}P(wv) = P_1 v + v(Pw/w)$$

$$P_1 v = \Sigma a_{ij}v_{ij} + \Sigma(h_i + w^{-1}a_{ij}w_j + w^{-1}a_{ji}w_j)v_i$$

so that the above results apply to the operator P_2 . Hence if $Pu \leq 0$ in Ω and $u \geq -Bw$ on $\partial\Omega$, where $B \geq 0$, then $u \geq -Bw$ in Ω . This result can be sharpened as follows: if $1 \geq w \geq a > 0$ and $Pw \leq -b < 0$ in Ω , and if $u \geq -\beta$ on $\partial\Omega$, $Pu \leq \alpha$ in Ω , then $u \geq -\max(0, (wb)^{-1}\alpha + a^{-1}\beta^+)$ in Ω , where $\alpha^+ = \max(\alpha, 0)$. The application of this result to the function $-u$ leads to the *a priori* estimate

$$(71) \quad |u(x)| \leq \max((ab)^{-1} \sup_{\Omega} |Pu|, a^{-1} (\sup_{\partial\Omega} |u|))$$

for $x \in \Omega$.

Under mild conditions on the coefficients of P , we can construct such functions w in sufficiently narrow domains. If $c \leq 0$, then the size of Ω is not restricted except by a finiteness condition. Suppose that Ω is contained in the strip $0 \leq x_1 \leq \delta$, and let $\beta_1 = \sup (b_1/a_{11})$, $\gamma_1 = \sup (c/a_{11})$. Then if

$$\gamma_1^+ \leq H = (4\delta^2)^{-1}(2 + |\beta_1|\delta) \exp(-2 - \delta\beta_1^+)$$

the function

$$w = B(A - \exp(-kx_1))$$

for a suitable choice of the constants A , B , and k , will satisfy $1 \geq w \geq \frac{1}{2}$, $Pw \leq -Ha_{11}(x) \leq -Hh(x)$ in Ω . For certain purposes it is convenient to have a function w which assumes its minimum on $\bar{\Omega}$ at a specific boundary point. This can be achieved by the following construction due to Hopf [331]. If Ω is contained in the spherical annulus $r \leq \|x\| \leq r + \delta$, and $\beta = \sup_{\Omega} \{h(x)^{-1} \sum_{i=1}^N (a_{ii}(x) + |b_i(x)|)\}$, $\gamma = \sup_{\Omega} h(x)^{-1} c(x)$, then there is a constant $H_1 = H_1(\beta, r, \delta) > 0$ such that, if $\gamma^+ \leq H_1$, we can choose the constants A and $k > 0$ such that $w = A(\exp(-kr^2) - \exp(-k\|x\|^2))$ has the properties: $0 < w < 1$ in Ω , $w = 0$ only on $\|x\| = r$, $Pw \leq -h(x)H_1$ in Ω .

In his important work on Harnack's inequality Serrin [607] introduced the following useful type of comparison function. Let Ω be the domain bounded by an ellipsoid and a plane defined by

$$\sigma \leq 1, \quad x_1 \geq a/2$$

where

$$\sigma^2 = a^{-2}x_1^2 + \alpha^{-2} \sum_2^N x_j^2, \quad 0 < \alpha \leq a$$

and set

$$w = f(\sigma)/f(\frac{1}{2}), \quad f(\sigma) = \exp(-k\sigma^2) - \exp(-k), \quad k > 0$$

Let

$$\beta = 2(a/\alpha)^2 \sup_{\Omega} h(x)^{-1} \{\sum a_{ii} + 2\alpha(\sum b_i(x)^2)^{1/2}\}$$

$$\Gamma = \sup_{\Omega} (-h(x)^{-1}c(x)),$$

and let γ be as above. Then k can be chosen so that

$$Pw \geq Ca^{-2}h(x)(1 + \beta + a(\Gamma^+)^{1/2}) \exp(-3(\beta + a(\Gamma^+)^{1/2})/4)$$

in Ω , where C is a numerical constant. If

$$\gamma^+ \leq (2a^2)^{-1}(\beta + 2) \exp(-3(\beta + 2)/4) = H_2$$

then k can be chosen so that $P(1 - w) \leq -h(x)H_2$. The function w

also satisfies $0 < w < 1$ in Ω , $w = 0$ on $\sigma = 1$, $w = 1$ in $\bar{\Omega}$ only at $x_1 = a/2$, $x_j = 0$ ($j > 1$).

In two dimensions, Bers and Nirenberg (see [47], [48]) have, by the use of peculiarly two-dimensional methods, such as conformal mapping, obtained very general results on the existence of solutions of (64), taking on prescribed boundary values under the assumption that the coefficients are bounded and measurable. Other authors (e.g., Serrin [607]) were able to obtain far-reaching results by using special topological properties of the plane. We shall omit the description of these methods here. It will be convenient in what follows to assume that $N > 2$, since some of the formulas must be modified slightly in two dimensions.

2. Laplace equation in a sphere

A fundamental tool in the general theory is the detailed study of the Laplace equation. We begin with the problem

$$(72) \quad \Delta u = -f \quad \text{in } S(0, R), \quad u = g \quad \text{on } \partial S(0, R)$$

The solution is classical and is given by the formula

$$u = G_0 f + K_0 g$$

where

$$(G_0 f)(x) = \int_{S(0, R)} G_0(x, y) f(y) dV_y$$

$$(K_0 g)(x) = \int_{\partial S(0, R)} K_0(x, y) g(y) dA_y$$

$$G_0(x, y) = \phi(x - y) - (\|x\|/R)^{2-N} \phi(x^* - y) = G_0(y, x)$$

$$\phi(z) = ((N-2)\omega_N)^{-1} \|z\|^{2-N}, \quad x^* = (R^2/\|x\|^2)x$$

$$K_0(x, y) = \omega_N^{-1} (\|y\|^2 - \|x\|^2) / \|y\| \|x - y\|^N$$

The function $G_0(x, y)$ is the Green's function for the Laplace operator and the sphere $S(0, R)$ and $K_0(x, y)$ is the *Poisson kernel*.

This solution has the properties

$$(73) \quad (a) \quad \|G_0 f\|_s \leq C_{p,s} R^\lambda \|f\|_p, \quad s^{-1} > p^{-1} - 2N^{-1} > 0, \\ \lambda = 2 + N(s^{-1} - p^{-1})$$

$$(b) \quad \|G_0 f\|_s \leq C_{p,s} R^\lambda \|f\|_p, \quad p = N/2, \quad 1 \leq s < \infty, \\ \lambda = N^2(2 + s)/(s(N + 2s(N - 2)))$$

$$(c) \quad |(G_0 f)(x)| \leq C_p (R - \|x\|)^\lambda \|f\|_p, \quad \lambda = 2 - N/p,$$

$$(N/2) < p < N$$

- (d) $|(G_0 f)(x)| \leq C_p (R(1 + |\log R|))^{\lambda} (R - \|x\|)(\log(eR/(R - \|x\|)))^{\lambda} \|f\|_p, \quad p = N, \quad \lambda = 1 - 1/N,$
- (e) $|(G_0 f)(x)| \leq C_p R^{\lambda} (R - \|x\|) \|f\|_p, \quad p > N,$
 $\lambda = 1 - N/p$
- (f) $\|X_j(G_0 f)\|_s \leq C_{p,s} R^{\lambda} \|f\|_p, \quad 1 \leq p < N,$
 $s^{-1} > p^{-1} - N^{-1}, \quad \lambda = 1 + N(s^{-1} - p^{-1}),$
- (g) $|X_j G_0 f(x)| \leq C_p R^{\lambda} \|f\|_p, \quad p > N, \quad \lambda = 1 - N/p$
- (h) $\|X^{\alpha} G_0 f\|_p \leq C_p \|f\|_p, \quad |\alpha| = 2, \quad 1 < p < \infty$
- (i) $\sum_{|\alpha| \leq 2} \|(R - \|x\|)^{k+|\alpha|} X^{\alpha} K_0 g\|_s \leq C_{p,s} R^{\lambda} \|g\|_q,$
 $\lambda = k + 1 - N(q^{-1} - s^{-1}) > 0, \quad s \geq q$
- (j) $|X^{\alpha} u(x)|$
 $\leq C_{p,q} \{R^{2-|\alpha|-N/p} (1 + (R/(R - R_0)))^{|\alpha| + (N-1)/q} \|f\|_p$
 $+ R^{-1/q} (R - R_0)^{-|\alpha| - (N-1)/q} \|u\|_q\}, \quad |\alpha| \leq 1,$
 $p > N, \quad \|x\| \leq R_0 < R$
- (k) $\|X^{\alpha} u\|_{p,S(0,R_0)}$
 $\leq C_{p,q} (1 + (R_0/R)^{N/p} (R/(R - R_0))^{2+(N-1)/q} \|f\|_p$
 $+ R^{-1/q} R_0^{(N-1)/p} (R - R_0)^{(1/p)-2-(N-1)/q} \|u\|_q,$
 $|\alpha| = 2, \quad p > N, \quad \|x\| \leq R_0 < R$
- (l) $|X_j(G_0 f)(x) - X_j(G_0 f)(y)|$
 $\leq C_p \|x - y\|^{\lambda} \|f\|_p, \quad \lambda = 1 - (N/p),$
 $N < p < +\infty$
- (m) $|X_j(G_0 f)(x) - X_j G_0 f(y)|$
 $\leq C_p \|x - y\| (1 + \log(2R/\|x - y\|)) \|f\|_p, \quad p = \infty$

All of these properties can be established by straightforward, but tedious, computations except for (h) and (k) which depend on the deep results of Calderón and Zygmund [103]. At any point at which f satisfies a Dini condition u actually has derivatives of the second order in the classical sense and satisfies equation (72). If $f \in \mathfrak{L}_p(S(0, R))$, then u has second derivatives almost everywhere which satisfy (72). If $g \in C^k(\partial S(0, R))$, then $K_0 g \in C^k(S(0, R))$ and takes on the boundary values g in the classical sense. If $g \in \mathfrak{L}_q(\partial S(0, R))$, then u approaches g almost everywhere on nontangential paths and also in the sense of $\mathfrak{L}_q$

on concentric spheres. We shall still consider u as the generalized solution of (72) in these latter cases.

3. Local existence theorems

Consider now the problem

$$(74) \quad Pu = \Delta u + \Sigma b_i u_i + cu = \Delta u + Qu = -f \\ \text{in } S(0, R), \quad u = g \quad \text{on } \partial S(0, R)$$

We assume that b_i and c are in $\mathcal{L}_r(S(0, R))$ and set $A_r(Q) = \Sigma \|b_i\|_r + \|c\|_r$, and that $c(x) \leq -\gamma < 0$ for $x \in S(0, R)$. This problem is equivalent to

$$u - G_0 Qu = G_0 f + K_0 g$$

For the sake of simplicity we assume that $r > N$. Choose any p , $N < p \leq r$. Consider the Banach space $\mathfrak{B} = C^1(\overline{S(0, R)})$ with the norm N_1' as defined on page 124, with the simplification that t no longer occurs. The transformation $G_0 Qu$ satisfies

$$N_1'(G_0 Qu) \leq C_p \|Qu\|_p \leq C_{p,r} A_r(Q) R^\lambda N_1'(u), \quad \lambda = N(p^{-1} - r^{-1})$$

and $G_0 Qu$ is, by (73h and 73m), a compact transformation of $\mathfrak{B}$ into itself. Hence (see Riesz and Nagy [572]) the equation $u - G_0 Qu = h$ has a solution for every $h \in \mathfrak{B}$ unless the homogeneous equation $u - G_0 Qu = 0$ has a nontrivial solution. But an easy extension of the maximum principle shows that the homogeneous equation has only the trivial solution. Therefore the operator $I - G_0 Q$ has a bounded inverse on $\mathfrak{B}$. If $C_{p,r} A_r(Q) R^\lambda < 1$, then the solution of $u - G_0 Qu = h$ can be obtained by successive approximations, and this is, in particular, true if $R < R_0(Q)$, that is, for sufficiently small spheres. The solution of (74) can be expressed in the form

$$u = G_P f + K_P g, \quad G_P f = (I - G_0 Q)^{-1} G_0 f, \quad K_P g = (I - G_0 Q)^{-1} K_0 g$$

Let $B(Q)$ be the norm of $(I - G_0 Q)^{-1}$ on $\mathfrak{B}$, so that

$$B(Q) \leq (1 - C_{p,r} A_r(Q) R^\lambda)^{-1}$$

if $R < R_0(Q)$. Then for $p > N$, we find that

$$N_1'(G_P f) \leq B(Q) N_1'(G_0 f) \leq B(Q) R^\sigma (1 + R) \|f\|_p, \quad \sigma = 1 - N/p$$

We can show that the above properties of G_0 hold also for G_P if $p > N$. Hence there is a *Green's function* $G_P(x, y) \in \mathcal{L}_s(S(0, R))$ for $1 \leq s < N/(N - 1)$, as a function of y , such that

$$(G_P f)(x) = \int_{S(0, R)} G_P(x, y) f(y) dV_y$$

The function $K_P g$ is at first defined only if $K_0 g \in \mathfrak{B}$. But by the maximum principle $|K_P g(x)| \leq \sup |g(y)|$. In order to estimate derivatives of $K_P g$, let $u = K_P g$ and set

$$U_\lambda(x) = \max |X_j u(y)| \quad \text{for } 1 \leq j \leq N, \quad \|y - x\| \leq \lambda(R - \|x\|)$$

where $0 < \lambda < 1$. Then if $0 < \lambda \leq \mu < 1$, we have

$$U_\lambda = \max_{S(0, R)} (R - \|x\|) U_\lambda(x) \leq U_\mu \leq (1 - \mu)^{-1} U_\lambda$$

If we apply (73j) to the sphere $S(x, \mu(R - \|x\|))$, we obtain an inequality of the form

$$U_\lambda \leq C_p \{A_r(Q)(\mu R)^{1-N/r}(1 + \mu/(\mu - \lambda))(1 - \mu)^{-1} U_\lambda \\ + (A_r(Q)(\mu R)^{1-N/r} R(1 + \mu/(\mu - \lambda)) + (\mu - \lambda)^{-1}) \|u\|_\infty\}$$

so that if $\mu = 2\lambda$, $\mu \leq \frac{1}{2}$, $6C_p A_r(Q)(\mu R)^{1-N/r} \leq \frac{1}{2}$, then we conclude that there is a constant $C = C(Q, p, r, R)$ such that

$$(R - \|x\|) |X_j u(x)| \leq C \|u\|_\infty$$

It now follows from (73k) that there is a constant C_p for $p > N$ such that

$$(R - R_0)^{2-1/p} \|X^\alpha u\|_{p, S(0, R_0)} \leq C_p \|u\|_\infty$$

for $0 < R_0 < R$, $|\alpha| = 2$. Therefore if $g_n \in C^1(\partial S(0, R))$ and $g_n \rightarrow g$ uniformly on $\partial S(0, R)$, then $K_P g_n$ converges to a function $K_P g$ which is of class $C^1(S(0, R_0))$ for every $R_0 < R$ and has derivatives of class $\mathfrak{L}_p$ for every $p < \infty$. This function $K_P g$ is continuous in the closed sphere, and satisfies (74) in the generalized sense. The value of $K_P g$ at any particular point $x \in S(0, R)$ is a linear functional on g in $C(\partial S(0, R))$, so that there is a measure $\mu_P(E, x)$ on $\partial S(0, R)$ such that

$$(K_P g)(x) = \int_{\partial S(0, R)} g(y) \mu_P(dy, x), \quad \mu_P(\partial S(0, R), x) \leq 1$$

If b_i is bounded in $S(0, R)$, $\sum b_i^2 \leq B^2$ then we can say more about the solution of (74). If $q > 1 + (B^2/4\gamma) = q_0$, and

$$\alpha_q(r; u) = (\omega_N r^{N-1})^{-1} \int_{\partial S(0, r)} |u|^q dA$$

and if $f \geq 0$, $g \geq 0$, then there is a constant $E = E(B, \gamma, q)$ such that

$$h(r) = r^{(N-1)/2} \alpha_q(r; u) + E \int_0^r (r - \rho) \rho^{(N-1)/2} \alpha_q(\rho; f) d\rho$$

is an increasing function of r .

A simple computation yields

$$\|u\|_q \leq C(R^{1/q} \|g\|_q + R^{2/q} \|f\|_q)$$

From this we can obtain estimates on $X^\alpha u$ for $|\alpha| \leq 2$ in $S(0, R)$ and thus show that if $g \in \mathcal{L}_q(\partial S(0, R))$, then $K_P g$ is defined and is of class $C^1(S(0, R))$ and has second derivatives in $\mathcal{L}_p(S(0, R_0))$ for $R_0 < R$, $p < +\infty$. This implies that there is a *Poisson kernel* $K_P(x, y)$ of class $\mathcal{L}_{q'}(\partial S(0, R))$ as a function of y such that

$$(K_P g)(x) = \int_{\partial S(0, R)} K_P(x, y) g(y) dA_y$$

We now turn to the problem

$$(75) \quad P_1 u = \Sigma a_{ij} u_{ij} + Qu = Pu + P_2 u = -f$$

in $S(0, R)$, $u = g$ on $\partial S(0, R)$

which is equivalent to

$$u - G_P P_2 u = G_P f + K_P g$$

Let $\mathfrak{B}_1$ be the completion of $C^2(\overline{S(0, R)})$ with respect to the norm

$$N_p''(u) = N_1'(u) + \sum_{|\alpha|=2} \|X^\alpha u\|_p$$

and let $\delta = \delta(P_2)$ be defined by

$$\delta(P_2) = \sup_{S(0, R)} |a_{ij}(x) - \delta_{ij}|, \quad 1 \leq i, j \leq N$$

We show that there is a constant $C_p(Q)$ such that

$$N_p''(G_P P_2 u) \leq C_p(Q) \delta N_p''(u), \quad p > N$$

and for $R < R_0(Q)$, $C_p(Q)$ can be estimated in terms of $A_r(Q)$. Hence, if $\delta < \delta_0(Q) = C_p(Q)^{-1}$, then the operator $I - G_P P_2$ has a bounded inverse on $\mathfrak{B}_1$ which can be computed by successive approximations. Therefore if $f \in \mathcal{L}_p(S(0, R))$, $p > N$, and $g \in C^2(\partial S(0, R))$, then there are operators G_{P_1} and K_{P_1} such that

$$u = G_{P_1} f + K_{P_1} g, \quad G_{P_1} = (I - G_P P_2)^{-1} G_P, \quad K_{P_1} = (I - G_P P_2)^{-1} K_P$$

and there is a constant $C_p(P_1)$ such that

$$N_p''(G_{P_1} f) \leq C_p(P_1) \|f\|_p$$

Consequently there is a Green's function $G_{P_1}(x, y) \in \mathcal{L}_s(S(0, R))$ as a function of y for $s < N/(N - 1)$ such that

$$(G_{P_1} f)(x) = \int_{S(0, R)} G_{P_1}(x, y) f(y) dV_y$$

In order to study $K_{P_1}g$ we need an *a priori* estimate of the solution of (75). For $x \in S(0, R)$, $0 < \lambda < 1$, set

$$U_{\lambda,p}(x) = (\omega_N \rho^N)^{-1/p} \sum_{|\alpha|=2} \|X^\alpha u\|_{p,S(x,\rho)}, \quad \rho = \lambda(R - \|x\|)$$

$$U_{\lambda,p,k} = \max_{S(0,R)} (R - \|x\|)^k U_{\lambda,p}(x)$$

Then if $0 < \lambda \leq \mu/2 < \frac{1}{2}$, we have

$$U_{\lambda,p}(x) \leq (\mu/\lambda)^{N/p} U_{\mu,p}(x), \quad U_{\mu,p,k} \leq C_{p,k} U_{\lambda,p,k}$$

Let x_0 be a point at which $(R - \|x\|)^k U_{\lambda,p}(x)$ attains its maximum, $U_{\lambda,p,k}$, with $k = 2 + N/q$. Then for $p > N$, we apply (73k) in the sphere $S(x_0, \mu(R - \|x_0\|))$ and find that, if μ is chosen so that $A_r(Q)(\mu R)^{1-N/r} < C_{p,q,r}$ and if $\delta < \delta_0(p, q, r)$, and $\lambda = \mu/2$, then we obtain the existence of a constant $C(P_1)$ such that

$$(76) \quad (R - \|x\|)^{2+N/q} U_{\lambda,p}(x) \leq C(P)(\|f\|_p + \|u\|_q),$$

$$N < p \leq \infty, \quad 1 \leq q \leq \infty$$

and that, if $A_r(Q)R^{1-N/r} \leq 1$, then $C(P)$ is less than a constant depending only on p, q, r . It follows easily from (73j) that

$$(77) \quad (R - \|x\|)^{|\alpha|+N/q} |X^\alpha u(x)| \leq C_\alpha(P)(\|f\|_p + \|u\|_q)$$

for $|\alpha| \leq 1$. If we apply this inequality in the sphere $S(x, (R - R_0)/2)$ where $\|x\| = (R + R_0)/2$ and $0 < R_0 < R$, and then the maximum principle, then we obtain for $u = K_{P_1}g$

$$(R - R_0)^{N/q}(R_0 - \|x\|)^{|\alpha|+N/q} |X^\alpha u(x)| \leq C_\alpha(P) \|u\|_{q,S(0,R)-S(0,R_0)}$$

$$(R - R_0)^{N/q}(R - \|x\|)^{2+N/q} U_{\lambda,p}(x) \leq C(P) \|u\|_{q,S(0,R)-S(0,R_0)}$$

If we apply the maximum principle, we see that $K_{P_1}g$ is defined for $g \in C(\partial S(0, R))$ and can be represented by a Poisson measure. If, for some $R_0 < R$, $a_{ij}(x) = \delta_{ij}$ for $R_0 < \|x\| < R$, and if $\Sigma b_i^2 \leq B^2 < +\infty$ in $S(0, R) - S(0, R_0)$, and if $q > q_0 = 1 + (B^2/4\gamma)$, then there is a Poisson kernel of class $\mathcal{L}_s(\partial S(0, R))$ as a function of y for $s = q/(q - 1)$ and

$$(R - R_0)^{(N-1/q)}(R_0 - \|x\|)^{|\alpha|+N/q} |(X^\alpha K_{P_1}g)(x)| \leq C_\alpha(P) \|g\|_q$$

for $\|x\| < R_0$.

Serrin [608] has shown more generally that, if $a_{ij}(x)$ satisfies a Dini condition, i.e., has a modulus of continuity $\omega(r)$ such that $\omega(r)/r \in \mathcal{L}_1[0, 1]$, then there is a Poisson kernel for P_1 . He showed this by the construction of an integral equation for $K_{P_1}(x, y)$ and a direct study of

its solvability. On the other hand, in [607] he gave an ingenious but elementary construction of two kernels $K_{\pm}(x, y)$ such that, if $A_{\infty}(Q) < +\infty$, then

$$(78) \quad (a) \quad P_1 K_+ \geq 0, \quad P_1 K_- \leq 0$$

$$(b) \quad \lim_{x \rightarrow x_0} (K_{\pm} g)(x) = \lim_{x \rightarrow x_0} \int_{\partial S(0, R)} K_{\pm}(x, y) g(y) dA_y = g(x_0),$$

$$g \in C(\partial S(0, R)), \quad x_0 \in \partial S(0, R)$$

$$(c) \quad C_1 K_0(x, y) \leq K_{\pm}(x, y) \leq C_2 K_0(x, y)$$

$$\text{for } \|x\| < R, \quad \|y\| = R$$

where C_1 and C_2 are positive constants. Hence, if $g \in C(\partial S(0, R))$, $g \geq 0$, we have

$$K_+ g \leq K_{P_1} g \leq K_- g$$

in $S(0, R)$, by the maximum principle. Also

$$C_1 \omega_N r^N \leq \int_{\partial S(0, r)} K_{\pm}(x, y) dA_x \leq C_2 \omega_N r^N$$

for $r < R$. Therefore

$$\alpha_q(r; K_{P_1} g) \leq \alpha_q(r; K_- g) \leq C_2^q \alpha_q(R, g)$$

so that

$$\|K_{P_1} g\|_q \leq C_2 \|g\|_q$$

for $1 \leq q \leq +\infty$. It follows from (76) and (77) that $K_{P_1} g$ is defined for $g \in \mathcal{L}_q(S(0, R))$ and is represented by a bounded Poisson kernel. This was used by Serrin to prove the Harnack inequality for nonnegative solutions of $P_1 u = 0$:

Theorem 48. If $P_1 u = 0$ in $S(0, R)$ and $u(x) \geq 0$ there, $u(0) = 1$, then

$$C_1 R^{N-2} (R - \|x\|)^{N-1} \leq u(x) \leq C_2 (R / (R - \|x\|))^{N-1}$$

where C_1 and C_2 are positive constants.

In the two-dimensional case, Serrin gave a proof with less explicit bounds under the assumption that the coefficients are merely bounded. Bers and Nirenberg [47] had previously given a much less elementary proof. Older work by Lichtenstein [420] and Feller [197] had given such results for equations with sufficiently smooth coefficients. The device used above to obtain (76) and (78) goes back to Korn [382] and was developed further by Hopf [332], Schauder [594], [595], Cacciopoli [96], [100], and Miranda [458], and is the basis of the generalization by Douglis and Nirenberg [166]. See also Giraud [297], Ladyženskaya [397].

4. Perron's method

Once we have established the local existence theorem we can apply the Perron method of subharmonic functions to construct solutions in the large. For the treatment of the Laplace equation by this method see Perron [503], Remak [564], Rado and Riesz [562], Whitney [708], Carathéodory [110], Ahlfors [2], Fubini [249], Wiener [712], [713], Brelot [66], [67]. The method was extended to more general equations, under various assumptions on the coefficients, by Püschel [561], Tautz [642], [643], and Oleinik [492], and even to certain classes of nonlinear equations by Simonov [614]. We shall indicate briefly the argument, following the exposition of Brelot as modified by some suggestions of Tautz.

Let Ω be a bounded domain in E_N , and let b, c be in $\mathfrak{L}_r(\Omega)$ for some $r > N$, and suppose that for every $x \in \bar{\Omega}$, there is an $R_1(x) \leq R_0(Q)$ such that in the ellipsoid $U(x, R_1)$ defined by $\sum A_{ij}(x)(y_i - x_i)(y_j - x_j) \leq R_1^2$, where $(A_{ij}(x))$ is the inverse matrix of $(a_{ij}(x))$, the coefficients a_{ij} satisfy the oscillation condition $|\sum (a_{ij}(y) - a_{ij}(x))\xi_i\xi_j| \leq \delta \sum a_{ij}(x)\xi_i\xi_j$ for all $\xi \in E_N$ and some $\delta < \delta_0(p, q, r)$. Note that this condition requires merely that a_{ij} has sufficiently small oscillation in the neighborhood of each point of $\bar{\Omega}$, and is less than a requirement of continuity. We continue to assume also that $c(x) \leq -\gamma < 0$ for some constant γ . Then for $x \in \bar{\Omega}$, $R < R_1(x)$, there exist operators $G_P(x, R)$ and $K_P(x, R)$ such that for $f \in \mathfrak{L}_p(U(x, R))$ and $g \in C(\partial U(x, R))$, the function $G_P(x, R)f + K_P(x, R)g$ is the unique solution of $Pu = -f$ in $U(x, R)$ satisfying $u = g$ on $\partial U(x, R)$.

We shall say that u is a *sub- P -function* in Ω if, for $x \in \Omega$, there is an $r_0(x) < R_1(x)$ such that $u(x) \leq (K_P(x, R)u)(x)$ for $R < r_0(x)$, and that u is a *super- P -function* on Ω if $-u$ is a *sub- P -function*. If u is both a super- and a *sub- P -function* in Ω , then $u \in C^1(\Omega)$ and has second derivatives almost everywhere in Ω which satisfy $Pu = 0$ and the second derivatives are in $\mathfrak{L}_p(\Omega_1)$ for $p < +\infty$ and every compact subdomain Ω_1 of Ω . If u is a sub- P -function and $R < R_1(x)$, then $u(y) \leq (K_P(x, R)u)(y)$ for $y \in U(x, R)$. A sub- P -function in Ω cannot have a positive maximum at an interior point. If $Pu \leq 0$ in Ω , then u is a super- P -function there. Thus a positive constant is a super- P -function. If $\mathfrak{F}$ is a family of sub- P -functions and $v(x) = \sup u(x)$, for $u \in \mathfrak{F}$, then v is a sub- P -function. If P is an operator for which a Harnack inequality holds in the neighborhoods $U(x, R)$, $R < R_1(x)$, then any family of P -functions in Ω which are uniformly bounded from below is equicontinuous and has equicontinuous derivatives in every compact subdomain of Ω .

Let f be in $\mathfrak{L}_p(\Omega_1)$ for every compact subdomain of Ω for some $p > N$, and let g be defined on $\partial\Omega$. We shall say that u is an *overfunction* of

the pair (f, g) in Ω if u is continuous in Ω , and for every $x_0 \in \partial\Omega$ we have $\liminf_{x \rightarrow x_0} u(x) \geq g(x_0)$, and if for $x \in \Omega$ there is an $r_0(x) < R_1(x)$ such that

$$(79) \quad u(x) \geq (G_P(x, R)f)(x) + (K_P(x, R)u)(x), \quad R < r_0(x)$$

We denote the class of overfunctions by $\mathfrak{F}_0(f, g, \Omega)$. Similarly, we say that u is an *underfunction* of (f, g) in Ω if $-u$ is an overfunction of $(-f, -g)$ in Ω , and we denote this class by $\mathfrak{F}_u(f, g, \Omega)$. If $u \in \mathfrak{F}_0(f, g, \Omega)$ and $v \in \mathfrak{F}_u(f, g, \Omega)$, then $u - v$ is a nonnegative super- P -function in Ω . If $u \in \mathfrak{F}_0(f, g, \Omega)$, then for $R < R_1(x)$ and $y \in U(x, R)$, we have

$$u(y) \geq (G_P(x, R)f)(y) + (K_P(x, R)u)(y)$$

Let

$$M^+(f, g) = \max_{\Omega} (\sup f(x)/|c(x)|, \sup_{\partial\Omega} g(x))$$

$$M^-(f, g) = -M^+(-f, -g)$$

$$M(f, g) = M^+(f, g) - M^-(f, g)$$

Then

$$M^+(f, g) \in \mathfrak{F}_0(f, g, \Omega), \quad M^-(f, g) \in \mathfrak{F}_u(f, g, \Omega)$$

and if u is an overfunction of (f, g) and attains a minimum in Ω , then this minimum is at least $\inf_{\Omega} f(x)/|c(x)|$.

We define the *Perron functions* of the pair (f, g) in Ω by $\underline{H}(f, g, \Omega)(x) = \sup u(x)$ for $u \in \mathfrak{F}_u(f, g, \Omega)$,

$$\overline{H}(f, g, \Omega) = -\underline{H}(-f, g, \Omega)$$

These functions exist and are finite if $M(f, g) < +\infty$. The main theorem of Perron remains valid. If $M(f, g) < \infty$, then $u = \underline{H}(f, g, \Omega)$ satisfies

$$(80) \quad u(y) = (G_P(x, R)f)(y) + (K_P(x, R)u)(y)$$

for $y \in U(x, R)$, $R < R_1(x)$, $x \in \Omega$. This is also true for $\overline{H}(f, g, \Omega)$, so that these functions are continuously differentiable in Ω , have second derivatives in $\mathcal{L}_p(\Omega_1)$ for $p \leq r$, and satisfy almost everywhere the equation $Pu = -f$.

If $\underline{H}(f, g, \Omega) = \overline{H}(f, g, \Omega)$, then we say that the pair (f, g) is *P-solvable* in Ω , and denote the common value by $H(f, g, \Omega)$, which corresponds to Wiener's generalized solution of the Dirichlet problem for the Laplace equation. It is easy to extend the elementary properties of the operators $\overline{H}$ and $\underline{H}$ to the above general class of operators. In particular, if u is a super- P -function in Ω and is continuous in $\overline{\Omega}$, then $(0, u)$ is *P-solvable* in Ω . If a pair (f, g) can be approximated by *P-solvable* pairs (f_n, g_n) in the sense that $M(f - f_n, g - g_n) \rightarrow 0$, then (f, g) is *P-solvable*.

Hence, if we can prove that $(0, g)$ is P -solvable in Ω whenever $g \in C^2(E_N)$, it follows that $(0, g)$ is P -solvable for $g \in C(\partial\Omega)$. Let $\Omega_1 \supset \bar{\Omega}$, and extend the definitions of the coefficients of P to Ω_1 in such a way that they continue to satisfy the above assumptions. Let $Pg = -f$, and set $u^+ = \underline{H}(f^+, g, \Omega_1)$. Then u^+ is a super- P -function in Ω_1 and is continuous there, while $u^- = u^+ - g = \underline{H}(-f^-, 0, \Omega_1)$ is also a superfunction in Ω_1 and is continuous there. Therefore $g = u^+ - u^-$ is the difference of two super- P -functions in $C(\bar{\Omega})$, so that $(0, g) = (0, u^+) - (0, u^-)$ is P -solvable in Ω .

If $M(f, 0) < \infty$, extend f to Ω_1 , and set $v = \underline{H}(f, 0, \Omega_1)$. Then $v \in C^1(\bar{\Omega})$ and $Pv = -f$ in Ω_1 , so that the pair (f, v) is P -solvable in Ω . But since the pair $(0, v)$ is also P -solvable in Ω , it follows that $(f, 0)$ is also P -solvable in Ω .

We say that a point $x_0 \in \partial\Omega$ is a P -regular boundary point if for every g such that $M(0, g) < +\infty$ and g is continuous at x_0 , we have

$$\lim_{x \rightarrow x_0} \bar{H}(0, g, \Omega)(x) = \lim_{x \rightarrow x_0} \underline{H}(0, g, \Omega) = g(x_0)$$

A barrier for x_0 and Ω in $U(x_0, R)$ is a function u which is a super- P -function in $\Omega \cap U(x_0, R)$ such that for some $h > 0$, $u(x) \geq h$ on $\Omega \cap \partial U(x_0, R)$ and $\lim_{x \rightarrow x_0} u(x) = 0$. A necessary and sufficient condition

that x_0 be a P -regular boundary point of Ω is that for all sufficiently small R there be a barrier for x_0 and Ω in $U(x_0, R)$. The proof, by the method of Tautz, depends on the fact that $(K_P(x, R)1)(y) \geq 1 - CA_R R^{2-N/r} > 0$ for $y \in U(x, R)$ and R sufficiently small. The domain Ω is regular for the operator P if every boundary point is P -regular.

It is easy to show that if R is sufficiently small then $U(x, R) - U(x, R_0)$ is P -regular for $0 < R_0 < R$. We can then show that x_0 is a P -regular point of Ω if there is an ellipsoid $U(x_1, R)$, $R < R_1(x_1)$, lying outside of Ω such that $x_0 \in \partial U(x_1, R)$.

Tautz showed that if $a_{ij} = \delta_{ij}$, then x_0 is a P -regular boundary point of Ω if and only if it is Δ -regular. Püschel and Oleinik proved similar results for equation with sufficiently smooth coefficients. Brelot [62], [63], [64], [72], studied this problem for operators of the form $Pu = \Delta u + cu$. The general problem seems to be unsolved.

5. Concluding remarks

It would be of great interest to extend to such general operators the methods and results of potential theory, such as those of Frostman [247], de la Vallée Poussin [655], and especially the theory of capacities (Choquet [121]) and the theory of positive solutions (Martin [438], Maria and Martin [437], Brelot [73], [76]) leading to the treatment of

the Dirichlet problem with multiple boundary points (Perkins [501], Green [305], and these papers of Brelot). For a survey of the present state of knowledge on these questions for the Laplace equation see Brelot [74], [76].

Unfortunately space does not permit a description of recent important work on the isolated singularities of solutions of elliptic equations (Hartman and Wintner [312], [314], Gilbarg and Serrin [289]). The analogue of the Phragmén-Lindelöf principle which is closely related to the theory of positive solutions has been studied by Krzyżanski [386], [389], and Serrin [606], [607], Gilbarg [288].

The theory of the eigenvalues and eigenfunctions for equations under these general assumptions does not seem to be developed. Of course, under slightly stronger assumptions the results of the previous section can be applied. For the Laplace operator, Brownell [85] has recently given very precise results on the asymptotic distribution of eigenvalues, obtaining in a certain generalized sense higher order terms in the formula (70). He has also studied [83] the spectrum of Schrödinger's equation. Criteria for the discreteness of the spectrum of the operator $\Delta u + c(x)u$ in E_N were given by Rellich [563] and Molčanov [460], who found important necessary and sufficient conditions. His results are based on the following lemma which is of considerable interest in itself. If Ω is a bounded convex domain in E_N which is symmetric with respect to the origin and $c(x) \leq -1$ in Ω , and λ_0 is the smallest eigenvalue of $Pu = \Delta u + cu$ for the Dirichlet problem $Pu + \lambda u = 0$ in Ω , $u = 0$ on $\partial\Omega$, while λ_1 is the smallest eigenvalue for the Neumann problem $Pu + \lambda u = 0$ in Ω , $\partial u / \partial n = 0$ on $\partial\Omega$, then

$$\lambda_0 \leq C\lambda_1, \quad C = 288(N^{-2} + (4d + N + 2)/d_1)^2$$

where $d = \max_{\Omega} \|x\|$, $d_1 = \min_{\partial\Omega} \|x\|$. The inequality $\lambda_1 \leq \lambda_0$ is trivial. See also Titchmarsh [651]. Other important work on the spectrum of an operator of this form, both in bounded and unbounded domains, and the corresponding eigenfunction expansions has been done by Ilyin [342], Povzner [555], Levitan [418], and Titchmarsh [649], [650].

Pólya and Szegő [553] have done very important work on the estimation of individual eigenvalues. Other results of this kind were obtained by Payne [496], Payne and Weinberger [498], Pólya [547], [548], Kornhauser and Stakgold [383], Schiffer and Szegő [599], Szegő [638], [640], Weinstock [702], Weinberger [679]. These authors show how eigenvalues and other functionals, with physical interpretations, of a given domain can be compared with geometrical quantities or with each other.

Garabedian and Schiffer [260] have studied the variation of these quantities with the domain.

Levinson [415], Oleinik [494], and Kamenomostskaya [359] have investigated the behavior of solutions of equations of the form

$$\epsilon \Sigma a_{ij} u_{ij} + Qu = f$$

as $\epsilon \rightarrow 0$. This is important in the theory of the boundary layer in fluid mechanics (Schlichting [601]), the JWKB method in quantum mechanics (Birkhoff [50]), and high-frequency electromagnetic waves (Kline [370], Keller [367]). Older work in this direction was done by Rothe [585], [586], Wasow [669], Friedrichs and Stoker [246].

Important work has been done recently on the boundary value problems arising in electromagnetic theory for the equation $\Delta u + cu = 0$, $c = \text{constant}$ and Maxwell's equations $\nabla \times \nabla \times \mathbf{E} - c^2 \mathbf{E} = 0$. The basic existence theorems were obtained by Kupradze [391], Müller [469], (see also Weyl [707], Müller [471]), and Saunders [592]. Saunders found an unexpectedly sharp uniqueness theorem for Maxwell's equations. For other work in this direction see Dolph and Barrar [26], Calderón [101]. Müller [470] proved important results on the behavior at infinity of solutions of $\Delta u + u = 0$. See also Wilcox [714]. It would be valuable to extend this work to Schrödinger's equation, which has not, up to now, been given a satisfactory mathematical treatment.

Mention should be made of Kupradze's work on the equations for elastic vibrations [391].

Finally, there has been a great deal of work recently on the theory of harmonic forms on Riemannian manifolds, with applications to many branches of mathematics. For an exposition of older work see Hodge [330], de Rham and Bidal [568], de Rham [566], Kodaira [371], de Rham and Kodaira [372]. Recent developments are due to Duff and Spencer [168], Gaffney [257], [259], Milgram and Rosenbloom [446], Connors [125], Friedrichs [244], Morrey and Eells [467], and Morrey [465], [466].

Clearly the theory of elliptic equations is in such a rapid state of development that no such brief survey as this can give more than an inkling of our knowledge at the moment.

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