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NUCLEAR THEORY

by

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PREFACE

Although a completely satisfactory theory of nuclei has still to come into being, there does exist a widely accepted theoretical approach to the understanding of nuclear phenomena. Most workers in the field start from the same broad interpretations of the qualitative features of the data and most use the same, or closely related, methods of analysis. These basic ideas and techniques of nuclear theory constitute the principal subject matter of this book. However, some departures from generally accepted views may occur as a consequence of the need to select material for presentation, the selection being unavoidably influenced by the taste and judgment of the author.

The treatment of the subject has been governed largely by the author's teaching experience. He has found that graduate students, especially those preparing for experimental work in nuclear physics, respond well to a detailed treatment of theoretical questions, so it seemed best to treat rather completely a coherent set of topics of general interest. The desire to keep the length of the book within bounds has therefore resulted in the neglect of a number of interesting matters, among them such topics as nuclear fission, the liquid drop model, the theory of angular correlations of radiations from nuclei, forbidden beta transitions, and so on. Despite these gaps, it is hoped that the reader will be able to sense the spirit and learn the basic methods of theoretical nuclear physics.

Particular emphasis is placed on the two-body problem because of its intrinsic interest and importance, and for other reasons, too. For one, the experimental facts are, on the whole, susceptible to simple analysis. Hence these facts are discussed in detail although it has been impossible to keep the discussion exactly up to date. The detailed analysis is to be taken as a consistent approach to the interpretation of the data rather than as the final word on the subject.

Another reason for emphasizing the two-body problem is that it introduces simple examples of many of the analytical techniques applicable throughout nuclear physics. Familiarity with this discussion should therefore help greatly in the understanding of the theory of complex nuclei (Part III). In the treatment of complex nuclei no attempt is made to present the great and growing mass of factual material in detail. The reader is assumed to be familiar with the basic facts and definitions found in elementary textbooks. Other detailed information can best be obtained from current review articles and reference books.

The qualitative as well as the quantitative arguments presented here are based on nonrelativistic quantum mechanics, and some knowledge of that subject is a prerequisite for even a casual reading of the book. However, the author has found that many students have not considered it unduly burdensome to attend a course covering selected parts of this material concurrently with a (second semester) course in quantum mechanics.

Remarks concerning the selection of the isotopic spin notation seem to be called for because the subject is the center of some controversy. There are many objections to the term *isotopic spin*, some based on the fact that "isotopic" is a misleading word, others based on the fact that the term "spin" is misleading. At about the time the author had completed Part II, a move to replace "isotopic" by "isobaric" was initiated at the 1952 conference on nuclear physics at the University of Pittsburgh. This change was so widely accepted that the new term is used in some recent review articles. However, we found that the change did not meet the most serious objection to the terminology, namely, that very awkward phrasing was sometimes necessary. Such a phrase as "a spin singlet, isotopic spin triplet, D-state" is both awkward and confusing. The author therefore leaped at a suggestion of Fermi that either "isotopic" or "isobaric" should be abbreviated to "i-." Hence the terms "i-spin," "i-singlet," and so on. Although this solution to the problem has disadvantages (such as the tendency to use I instead of T for the associated quantum number), it seemed reasonable when looked upon as an abbreviation. The reader can read the term "i-" as isotopic or isobaric according to his taste.

Since the number and variety of contributions to nuclear physics is so large, no attempt has been made to give detailed references to all original papers relating to the topics discussed. The sources of the experimental data that have been used are, of course, indicated. A theoretical paper is usually referred to under either of two conditions: if it is the original source of a basic idea treated in detail or if it contains the complete analysis of a problem too complicated to be treated thoroughly in the text. Several excellent bibliographies are available to the reader who has an interest in the history of the subject. The most up-to-date appears to be the list of references given by Blatt and Weisskopf in their book *Theoretical Nuclear Physics*.

It is no accident that this book has been written in Madison. The active program of experimental nuclear physics built up here by R. G. Herb provides an atmosphere most conducive to contemplation of the subject. Many members of the experimental group have contributed to the work, both by the questions they have raised and by the information they have provided. The local theorists have also been of great assistance. In particular, the author has been fortunate to have the critical advice and help of J. L. Powell, whose deep understanding of the intricacies of scatter-

ing theory has been especially useful. Aid of major proportions has been provided by W. G. Holladay who, as a graduate student, played the role of a guinea pig by reading the material without previous knowledge of the subject. He suggested changes, checked equations, and eliminated numerous dangling participles.

The book could not have been completed in a reasonable length of time without the help of Mrs. June Powell, who took the first draft of Parts I and II by dictation and prepared most of the final manuscript of the entire book. This would not have been possible but for the financial assistance of the Publisher, who has proved to be most cooperative in every respect.

Mention should also be made of the indirect support provided by the Atomic Energy Commission and the Wisconsin Alumni Research Foundation who, by their mutual support of the author's research program, have given him an opportunity to extend his knowledge of many of the topics discussed here.

R. G. S.

Madison, Wisconsin
May, 1953

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Part I

THE NUCLEAR FORCE PROBLEM

CHAPTER 1

INTRODUCTORY REMARKS

1–1 Objectives. There are in nature just three known types of fundamental interactions which correspond to observable "forces." These are the forces of gravitation, of electromagnetism, and the specific nuclear force. Gravitational forces are responsible for the retention of astronomical bodies in their orbits. Electrostatic forces lead, among other things, to the structure of atoms and their chemical behavior. The specific nuclear force plays the same role in atomic nuclei: it binds together neutrons and protons to form more or less stable configurations.

The immediate objective of theoretical nuclear physics is the investigation, through the interpretation of experiments, of the specific nuclear interaction. A quantitative description of this interaction, such as might be given by means of a potential function, should be the end result.

Any attempt to give quantitative form to an interaction brings up another problem, the theoretical mechanical scheme appropriate to the system. The question is, What is the nature of the mathematical setting within which the analysis of a dynamical problem is to be made? The form of the gravitational potential could be established only within a scheme of mechanics which defined the concept of potential. Thus Newtonian mechanics was required as a precursor to quantitative formulation of the Law of Gravitation. The same can be said of the interaction between two electric charges, which was determined through experiments to be given by the coulomb potential.

On the basis of experience in atomic physics it seems reasonable to start with the assumption that neutrons and protons are structureless particles subject to ordinary Schroedinger quantum mechanics. This is not likely to be the ultimate form of the theory, since it cannot be extended into the domain of very high energies where the conditions of relativistic covariance become manifest, but it is generally assumed that the Schroedinger quantum mechanics provides a reasonable low-energy approximation to the correct theory. Since our attention will be focused on low-energy problems, the methods of ordinary quantum mechanics will be used as a basis of discussion. It should always be kept in mind that the low-energy limit of the theory may take a different form;* the history of atomic physics provides good evidence that the form of mechanics is not ordained by earlier experience.

* As a matter of fact, those forms of meson field theory which seem most promising at present do not lead unambiguously to a nonrelativistic approximation of the form of the Schroedinger equation.

We start, then, with the assumption that neutrons and protons exist as elementary particles within nuclei, and that the states of the nucleus are determined by a corresponding Schroedinger equation. The interactions therefore appear in the form of a potential. To make progress, the potential is taken to be of the simplest possible form, namely, a sum of two-body interactions. Then investigation of the two-body nuclear systems should provide the most direct information concerning the potential. Accordingly, great emphasis is placed on the two-body problem. Ultimate contradiction between the properties of the interaction established by means of the two-body systems (deuteron and proton-proton system) and the properties of heavier nuclei may lead us to abandon the hypothesis of two-body interactions. However, at present no clear-cut contradiction has been established, so this hypothesis is made, with reservations, throughout the text.

The ultimate objective is a complete theory (usually assumed to be a field theory, built in analogy with quantum electrodynamics) of interacting nuclear matter. With this in mind, the opportunity to detect deviations from the simple theory must not be overlooked. In the domain of ordinary nuclear physics, effects such as the velocity dependence of nuclear forces, many-body interactions, changes in nucleon structure produced by interactions, and outright contradictions of the simple theory would yield clues concerning the ultimate theory.

Since it is generally believed that the meson field produces the nuclear interactions, the study of meson reactions provides a direct attack on the fundamental questions. At present that sphere of action is undergoing rapid development, but correlation of the results with the nuclear force problem is still in too primitive a stage to lead to definite conclusions. Although we shall give limited attention to the meson problem, the most recent periodicals provide the only up-to-date information on the subject.

In addition to the fundamental processes, there are a number of nuclear phenomena which are susceptible to theoretical analysis but which do not lead directly to information concerning nuclear forces. Nevertheless, these are of interest for the indirect information that they provide. They are the external phenomena associated with complex nuclei, such as nuclear scattering of neutrons and protons, reaction cross sections, gamma emission and absorption, and radioactivity. Although in general the exact solution of a many-body problem is impossible, we are still able to provide a very good theoretical description of these phenomena, insofar as they involve the internal nuclear structure only through such parameters as energy, angular momentum, or parity of the nuclear states. Occasionally, in a favorable situation, even more detailed properties of the wave functions are implied. This description therefore provides a means of systematizing our information concerning complex nuclei. The situation is very similar to that in chemistry when the valence theory was found to be very

successful and could be used to systematize information concerning chemical reactions. A fundamental understanding of valence theory came only much later, with the advent of quantum mechanics.

The nuclear states are, of course, in some way determined by the interactions, so systematic data concerning energy states can contribute to the fundamental problem by providing special properties of the interaction. However, the aforementioned difficulty with the many-body problem makes it clear that some simplification of the problem is required to obtain an interpretation in terms of interactions. The simplifying assumptions usually are classed as a nuclear "model." Insofar as conclusions concerning the interaction may be dependent upon the validity of the model, they are always subject to doubt.

Investigations of the external aspects of nuclei have raised a number of interesting problems concerning nuclear structure and nuclear forces. Under proper analysis they may culminate in as enlightening a development as did the analysis of atomic spectra. Therefore the theoretical basis of the analysis of external phenomena must be understood in terms of their relationship to the internal structure if we are to keep open every possible avenue of attack on the interaction problem.

1-2 The mechanics of nuclear physics. Quantum mechanics has been introduced as the most primitive mechanical scheme to be considered in treating nuclear problems because certain ideas concerning orders of magnitude are already fixed in our minds. One alternative might otherwise have been a Newtonian description. The requirement for validity of Newtonian mechanics is that the de Broglie wavelengths of nuclear particles be short compared with nuclear dimensions. Then it is possible to form wave packets which move as particles move in classical mechanics.

The dimensions of nuclei are of the order of magnitude of R_0, the classical electron radius:*

$$R_0 = e^2/m_0c^2 = 2.818 \times 10^{-13} \text{ cm.} \tag{1-1}$$

The de Broglie wavelength is given by

$$\lambda\!\!\!^{-} = (\hbar/Mc)\sqrt{Mc^2/2T}$$

for a particle of mass M and kinetic energy T. Since the Compton wavelength of a neutron or proton is

$$\hbar/Mc = 2.103 \times 10^{-14} \text{ cm}, \tag{1-2}$$

* The constants given without other reference are taken from a preliminary report on atomic constants to the National Research Council by J. W. Du Mond and E. R. Cohen.

while the rest energy is

$$Mc^2 = 938 \text{ Mev}, \tag{1-3}$$

the condition that one wavelength fit conveniently into a nucleus, $2\pi\lambda\!\!\!^{-} \approx 2R_0$, is

$$T = 25 \text{ Mev}. \tag{1-4}$$

Thus we see that a nuclear particle of energy less than some 100 Mev will undergo severe diffraction on striking the nucleus and that wave mechanics will therefore play an important role in the description of any such process. Furthermore, nuclear particles which are bound within the nucleus must have kinetic energies not much less than 25 Mev. The average potential acting on a particle must then be at least of this order to cause binding. Only if the attractive potential is much greater than this will it be possible to apply a Newtonian description to any of the nuclear particles. It happens that the potential is just of the order of 25 Mev, so the wave nature of matter is essential for the internal as well as the external description of nuclei.

Another interesting point is that at the very low energy of $\frac{1}{40}$ ev, which corresponds to thermal energies, the wavelength of a neutron is 1.8×10^{-8} cm. Thus for such low energies diffraction will occur on an atomic scale; the wave nature of neutrons can be established by means of the Bragg scattering of thermal neutrons from crystal lattices.

In the treatment of many atomic and electronic phenomena, extension of the theory to the relativistic domain is essential. To what extent must the Special Theory of Relativity be brought into the consideration of nuclear problems? The answer is given by considering the ratio v^2/c^2, where v is the particle velocity. If this ratio is very small compared with unity, it is expected that relativistic effects will be correspondingly small. An estimate can be made on the basis of the nonrelativistic relationship

$$v^2/c^2 = 2T/Mc^2. \tag{1-5}$$

Since the order of magnitude of T is 25 Mev, we find that

$$v^2/c^2 \approx \tfrac{1}{20},$$

so relativistic corrections of the order of five percent are to be expected. For most aspects of nuclear theory the theoretical and experimental uncertainties are sufficiently great so that effects of this magnitude can be ignored.

Aside from occasional parenthetic remarks, the treatment given in subsequent chapters will be entirely nonrelativistic. However, the relativistic corrections offer very interesting material for investigation, and we take this opportunity to discuss briefly the nature of the information they provide. Consider the analogy with the theory of the electron. An electron at rest produces an electrostatic field, but in motion it also produces a

magnetic field. The magnetic field is a relativistic correction to the electric field and, as such, it is velocity-dependent. Similar velocity-dependent corrections to the nuclear interactions are to be expected, and these relativistic effects offer a source of information concerning the nuclear field, just as the fundamental magnetic laws of force provided the basis for the development of the theory of the electromagnetic field.

It is of particular interest that first-order relativistic corrections to nuclear problems can be handled by a natural extension of the Schroedinger nonrelativistic, nonfield theoretic treatment. This point has been emphasized by Breit, who has shown that experimental detection of the corrections would provide information relevant to the nuclear field theory. In addition to this approach by small corrections, the relativistic and field theoretic aspects of nuclear physics are being attacked by the more vigorous methods of high-energy reactions, particularly those leading to meson production. Results so obtained are still subject to many uncertainties in interpretation. A direct connection with the nuclear force problem undoubtedly exists but has not yet been established because of the multitude of difficulties inherent in meson field theories.

1-3 The nuclear particles. Neutrons and protons are treated, for the most part, as elementary particles. Their properties, as far as their nuclear behavior is concerned, are very similar, and it is convenient to have a generic term serving as a name for either particle: the "nucleon." The nucleon may be viewed as a particle having two different states, one with the properties of the proton, the other with those of the neutron.

The known properties of a nucleon are charge, mass, spin, statistics, magnetic moment, and half-life for beta decay. The charge of the proton is positive and equal in magnitude to the charge of the electron. The charge of the neutron is zero. The masses of the nuclear particles are usually given with respect to a scale in which the oxygen atom (O^{16}) has a mass of 16 units. On this scale the equivalence of mass and energy is expressed by the statement that one atomic mass unit (amu) equals 931 Mev. The masses of interest are as follows:*

H atom	1.008142 amu,
proton	1.007579,
neutron	1.008982.

The difference in mass between the neutron and the hydrogen atom is 782 kev. This means that the neutron is unstable against emission of an electron (beta activity) to form a hydrogen atom.

The spin of both proton and neutron is one-half. Both satisfy the Pauli exclusion principle; that is, the wave function of any system has the prop-

* Li, Whaling, Fowler, and Lauritsen, *Phys. Rev.* **83,** 512 (1951).

erty that it is antisymmetric for interchange of any pair of neutrons or any pair of protons.

The magnetic moments are of particular interest because they have been measured so precisely. Numerical values are given in units of the nuclear magneton $e\hbar/2Mc$, M being the proton mass. We note that

$$e\hbar/2Mc \approx 5.05 \times 10^{-24} \text{ erg/gauss}. \tag{1-6}$$

The values of the magnetic moments in units of nuclear magnetons are

proton	2.79255,
neutron	−1.91280.

The minus sign expresses the fact that the direction of the magnetic moment of the neutron is opposite to that of the spin.

A rough value of the half-life for beta decay of the neutron is 13 minutes. The proton is stable against any of the known nuclear decay processes.

The neutron and proton are the simplest of the nuclei, and it would seem best to have an understanding of their properties as listed above before attempting an investigation of more complex nuclei. Since the nucleon has spin one-half, it is tempting to assume that the correct form of quantum mechanics required to describe its properties is the Dirac relativistic theory, which has been so successfully applied to the electron. However, according to the simplest form of the Dirac theory, the magnetic moment of a proton would be unity and that of the neutron would vanish. The observed magnetic moments can be accounted for within the scope of the Dirac equation only by means of a slight generalization of the original theory, a generalization based on the observation of Pauli that an arbitrary anomalous moment can be introduced into the Dirac theory simply by adding a term to the Hamiltonian. However, the completely arbitrary multiplicative factor determining the Pauli moment introduces into the theory a quantity of the dimensions of a length, a fundamental length which would be in no way related to the other fundamental constants of physics. Most physicists do not attach such great importance to the magnetic moments. Therefore, this approach, although it is useful for purposes of calculation, is generally looked upon as a stop-gap procedure. The preference is to interpret the anomalous magnetic moments of the nucleons as a manifestation of the structure of nucleons associated with their interaction with the meson fields, so they should be related directly to other, more fundamental, physical constants such as the strength of the coupling between nucleon and meson. Up to the present time, no completely satisfactory, detailed quantitative theory of the magnetic moment has been developed.

Another implication of the Dirac theory of protons is that there must exist in nature a negatively charged proton (antiproton). The energy re-

quired to produce proton-antiproton pairs is several times the rest energy of the proton, i.e., several billion electron volts. The ultimate production of such particles by the bevatrons presently under construction would provide a very happy confirmation of the guess that Dirac theory lies at the root of nuclear structure.

CHAPTER 2

QUALITATIVE DISCUSSION OF NUCLEAR FORCES

Completely convincing evidence for the existence of nuclear forces is given by the existence of stable nuclei, since the nuclei would otherwise be shattered by the coulomb repulsion between protons. From other general characteristics of stable nuclei it is possible to establish more detailed qualitative features of the interactions between nucleons. The following specific assumptions are made in formulating these ideas:

(1) Nuclear forces are well represented by two-body interactions.
(2) These interactions can be described in terms of a potential inserted into the Schroedinger equation.
(3) Effects of relativity and specific manifestations of field theory are unimportant.

There are three examples of two-body potentials to be discussed: the potential acting between neutron and proton, the potential acting between any pair of protons, and the potential acting between any pair of neutrons. These potentials, or at least their qualitative aspects, are simply denoted by n-p, p-p, and n-n, respectively. Note that p-p involves, among other things, the coulomb repulsion. However, it will turn out that for a given pair of particles this contribution is small, so that the distinction between the total p-p interaction and the specifically nuclear p-p interaction is not of great significance. The type of question which we can hope to answer on qualitative grounds is that of the relative strengths of the three interactions aside from the difference introduced by the coulomb potential.

2–1 Consequences of $A \approx 2Z$. An early and interesting observation concerning atomic nuclei was the fact that the atomic mass number A is approximately equal to twice the atomic number Z. This relationship is particularly accurate for the light and intermediate nuclei; for heavier nuclei the value of A increases more rapidly than $2Z$. Since A is equal to the total number of nucleons in the nucleus and Z is the total number of protons, the number of neutrons is about equal to the number of protons for the light nuclei, but for the heavier nuclei the neutron number increases more rapidly. Expressed differently, it seems that the light nuclei prefer to add nucleons in n-p pairs. The addition of a large number of neutrons or a large number of protons apparently does not lead to a stable nucleus.

Thus the interaction provides a strong attraction between neutrons and protons. There may be comparable interactions between like particles,

but these must not be so great as to stabilize nuclei consisting only of like particles, nor so different as to favor one type of nucleon over the other. We conclude, then, that*

(a) *Either* $\text{n-n} \approx \text{p-p} \approx \text{n-p}$
or $\text{n-p} \gg \text{n-n}$ *and* $\text{n-p} \gg \text{p-p}$.

2-2 The shape of the potential. Nuclear interactions certainly do not extend to very large distances beyond the nuclear radius. Any such long-range nuclear force would manifest itself through a modification of molecular behavior. From this fact it can immediately be seen that the nuclear potential must show a much stronger tendency to vanish at large distances than does the coulomb potential. For let us suppose that the potential does have the form q^2/r. We have seen that within nuclei the average potential is of the order of 25 Mev or greater. Thus

$$q^2/R_0 \approx 25 \text{ Mev},$$

from which it follows that

$$q^2/e^2 \approx 50.$$

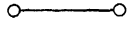
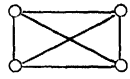
Hence the magnitude of the forces within nuclei is such that the $1/r$ potential would be much stronger than the coulomb potential due to the electric charge, and consequently it would play a prominent role in molecular problems. One might consider a power law which converges much more rapidly, such as $1/r^n$ for large n. However, the difficulty immediately arises that for an attractive potential of this form with $n \geq 2$, the Schroedinger equation is not susceptible to solution. To avoid this difficulty it would be necessary to cut off the potential at small distances.

The introduction of a cut-off in the potential implies that we must consider at least two parameters for the description of the interaction. The simple inverse power laws involve only one parameter, the "strength" or the "charge." The second parameter, in this case, would be the distance at which cut-off occurs. A more direct procedure is to assume that the potential involves two such parameters in a simple and straightforward way. The parameters are taken to be the strength V_0, which might, for example, be the value of the potential at the origin, and the range α, the distance beyond which the potential rapidly goes to zero. Note that the potential is taken to be zero for infinite separation of the nucleons.

The following argument, due to Wigner, gives us some indication of the magnitudes of these parameters on the basis of the observed binding energies of the very lightest nuclei, H^2, H^3, He^3, He^4. The energy values are shown in the second column of Table 2 1.

* The qualitative properties of interactions established in this chapter are labeled consecutively (a), (b), etc.

TABLE 2–1

Nucleus	Binding Energy (Mev)	Bond Diagram	No. of Bonds	Energy per Bond (Mev)
H^2	2.2		1	2.2
H^3, He^3	8		3	2.7
He^4	28		6	4.7

It is tempting to analyze these binding energies in terms of a simple chemical bond model; that is, to assume that the binding energy is simply proportional to the number of bonds. The third column of Table 2-1 gives a schematic diagram of the bonding picture, the fourth column gives the number of bonds, and the fifth column the binding energy per bond. The striking fact is that the binding energy per bond is not nearly constant, so that such a simple model fails completely.

This increase in binding energy per bond can be understood in terms of a potential which is narrow and deep. Consider a potential well of the shape indicated by Fig. 2–1. The deuteron problem then involves the

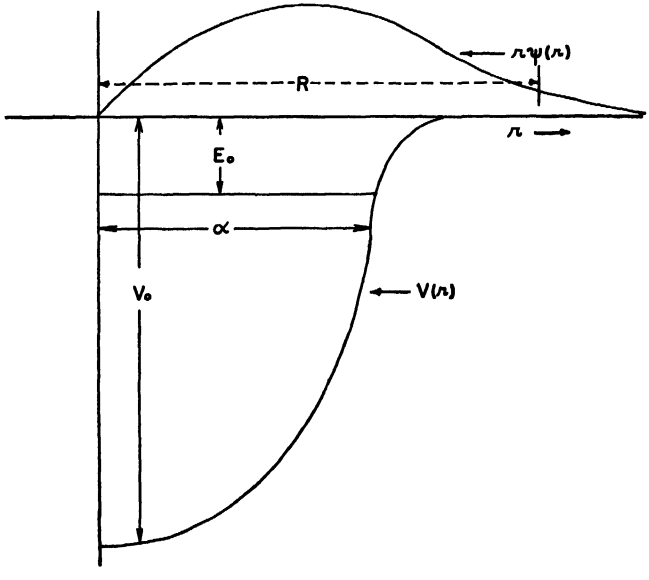


FIG. 2–1. The shape of the two-parameter potential and the corresponding ground state radial wave function.

motion of a particle of mass $M/2$ (the reduced mass) under the influence of this potential. A small range, α, would tend to constrain the particle to a small region of space. However, as one can see from the uncertainty principle, a large value of the average kinetic energy is thereby implied. The deuteron will then be bound only if the potential is deep enough. Now let us consider the situation in which the binding energy E_0 is small compared with the depth of the potential. The wave function of the almost-free particle then extends well beyond the range of the force, as indicated in the figure. By assuming that this is the situation in the deuteron, it is now possible to establish qualitatively that the binding energy per bond will increase on going to the three- and four-body nuclei.

The kinetic energy T of a particle in the nucleus is roughly proportional to the inverse square of the radius R of the region to which the particle is confined:

$$T \sim 1/R^2.$$

Here, R may be defined as the radius within which the wave function differs appreciably from zero. Furthermore, the average potential energy per particle, V, is proportional to the number of bonds, b, and inversely proportional to the confining volume. The latter statement is a direct result of the fact that the potential acts over a distance small compared with R (see Fig. 2-1), so that the average potential is reduced by the ratio α^3/R^3 of the active volume to the total volume. We find, then,

$$V \sim b/R^3.$$

For two nuclei containing different numbers of particles, the ratio of the kinetic energies per particle is

$$T_2/T_1 = R_1^2/R_2^2,$$

while the ratio of the potential energies per particle is

$$V_2/V_1 = b_2R_1^3/b_1R_2^3.$$

As the number of bonds is increased, the wave function is pulled into the potential well, so that R is decreased as long as it is greater than α. Thus for $b_2 > b_1$, we have the inequality

$$(V_2/V_1)/(T_2/T_1) = R_1b_2/R_2b_1 > 1,$$

and the potential energy per particle increases at a greater rate than the kinetic energy per particle. This establishes the increase in binding energy per bond with an increasing number of particles. The essential point in this argument is that the wave function of the deuteron extends well beyond the range of forces, as it must if the kinetic energy and potential energy in the deuteron are approximately equal.

To obtain a rough estimate of the deuteron kinetic energy, it seems reasonable to assume that just half of the wave function is contained in the potential well, that is, to characterize the function by a wavelength λ, with

$$2\pi\lambda = 4\alpha.$$

Then the kinetic energy is

$$T \approx \pi^2\hbar^2/4M\alpha^2. \tag{2-1}$$

According to the above argument the depth of the potential is roughly the same:

$$V_0\alpha^2 \approx \pi^2\hbar^2/4M. \tag{2-2}$$

Thus there appears a connection between the range and depth of the potential such that greater depth corresponds to shorter range.

We have already obtained a rough estimate of the average potential acting on a nucleon in Section 1–2. The argument presented there implies that

$$V_0 \gtrsim \pi^2\hbar^2/2MR_0^2,$$

from which it follows that

$$\alpha \lesssim \tfrac{1}{2}\sqrt{2}\,R_0 = 2.0 \times 10^{-13}\text{ cm}. \tag{2-3}$$

Note that the factor one-half is just our rather arbitrary choice of the ratio of α to the radius of the deuteron function. According to our arguments, this must be less than one. It follows that

(b) *The range of nuclear forces is probably somewhat less than nuclear dimensions.*

An assumption which seems essential to the success of this entire analysis is that bonds are formed between pairs of nucleons of all types and that these bonds are of the same magnitude. The first of the alternatives in property (a) of Section 2–1 should then be selected. However, this point will not be pressed, since more direct evidence is forthcoming.

2–3 Saturation of nuclear forces. The description of the binding processes of very light nuclei, outlined in the previous section, can now be extended to nuclei of mass greater than the alpha-particle. Eventually, as the mass number is increased, the one-particle wave function will be pulled into the potential well, so that the binding energy per bond should cease to increase. It might then be expected that the simple bond picture would be applicable and that the binding energy of a nucleus would increase in proportion to the number of bonds. Since the number of bonds is equal to $A(A-1)/2$, the binding energy per nucleon for large A would seem to be proportional to A. Furthermore, the increase in binding would lead to a reduction in the volume per nucleon of the nucleus for increasing A.

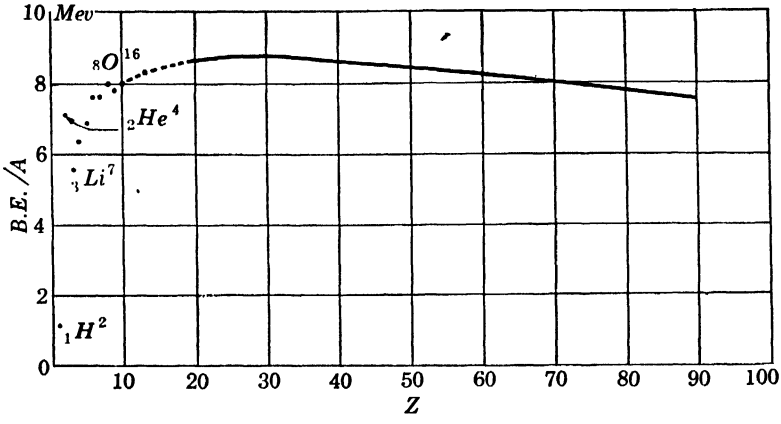


FIG. 2-2. Binding energy per nucleon for stable atoms. Compare Table 8-2. (Taken from Bitter, *Nuclear Physics*. Addison-Wesley, 1950.)

Neither of these conclusions is in accord with the facts. The binding energy per nucleon is shown in Fig. 2-2 as a function of atomic number. There are fluctuations associated with individual nuclei, but the trend is well represented by the smooth curve. Apparently the trend of the binding energy curve can be described by the statement that the binding energy per nucleon is essentially constant, rather than proportional to A. The rapid increase at very low A is related to the complications discussed in the previous section. The deviation of the smooth curve from a constant value, that is, the slow rise for Z less than 30 and the slow decrease for Z greater than 30, can be given a plausible explanation even if we assume that the dominating trend would hold the binding energy per particle constant. The smooth rise at the early part of the curve may be ascribed to the relative importance of surface effects for smaller nuclei. The contribution of the surface to the binding energy is somewhat less than the contribution of the interior of the nucleus. Since the surface-to-volume ratio decreases with increasing size, the effect of the surface should become less important, with the result that the average binding energy should increase.

The decrease in binding at the late end of the curve can be associated with the coulomb repulsion between protons. We know that the coulomb energy per proton is proportional to the number of other protons in the nucleus, $Z - 1$. Now if the specific nuclear binding is independent of Z, the coulomb term is of increasing relative importance with increasing Z. Since it has the form of a repulsion, the result will be a decrease in binding for the heavy nuclei. In spite of the fact that the coulomb interaction between a *pair* of nucleons is small compared with the specific nuclear force, the effect becomes large, since it involves all the protons.

Figure 2–2 indicates that the binding energy per particle has about the value corresponding to the alpha-particle. Therefore it can be said that the average number of bonds per nucleon in a nucleus is the same as that in the alpha-particle; on the average, a nucleon forms at most three bonds. This limiting of the number of bonds that can be formed by a nucleon is referred to as the *saturation* property of nuclear forces.

Further evidence for saturation is given by the observations on nuclear size. It is found that the nuclear radius R is proportional to $A^{1/3}$; that is, the nuclear volume is proportional to the number of nucleons. Unsaturated bonding of nucleons would lead to a *decrease* in the volume per nucleon with increasing numbers of particles, while saturated bonding implies that the constraints acting on a nucleon are essentially independent of the number of nucleons, so the volume per particle is constant.

Since the binding energy per particle is constant and the volume per particle is also constant, the saturation property can be characterized by the proportionality of binding energy to volume; the binding energy per unit volume is independent of mass number. In this form, it becomes apparent that the phenomenon of saturation is very familiar in atomic physics. Examples are the binding energy of liquids and solids, the homopolar chemical bond, and the polar chemical bond; in each of these cases the binding energy is proportional to the mass of material. The atomic saturation phenomena can be understood in terms of atomic structure. They depend in some detailed way on the electronic configurations of the atoms and molecules. However, if nuclei are to be understood as composites of structureless nucleons, we are forced to ascribe the saturation property to some peculiarity of the interaction potential. The result is that our concept of potential must be generalized, and we shall find that there is no classical analog (i.e., one that fits into the scheme of Newtonian mechanics) to the potential that will be used. The loss of a simple concept of potential is probably an indication that nuclear interactions will only find their ultimate explanation in a field theory. We may conclude that the nucleon is not really an elementary particle but that it has structure, and the introduction of an unconventional type of potential is necessary only as a result of our attempt to incorporate effects due to structure into the potential concept.

Without trying to formulate the required potential quantitatively, it is still possible to establish the desired saturation property by means of a qualitative condition on the interaction. For this purpose it is helpful to think in terms of a very simple nuclear model, namely, the independent particle model,* wherein each nucleon is assumed to move in the average field of all other nucleons. The wave function of the nucleus as a whole will then be a properly antisymmetrized product of one-particle functions.

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* A much more detailed discussion of this model is given in Chapter 8.

Each one-particle state will also be assumed to involve a single orbital state (space function) and spin state. Although this simple model is not expected to be a very good approximation, it is still of great use in formulating the saturation property as well as other properties of nuclei. Even if the model is quite poor, it should be kept in mind that the correct nuclear wave function can always be expanded in terms of functions of the type provided by the model. Therefore general properties established on the basis of the simple model may carry over to the correct wave function.

The saturation property can now be ascribed to the following characteristic of the nuclear interaction:

(c) *Nucleons attract each other strongly only if they are in the same orbital state.*

According to the Pauli principle, only two neutrons and two protons will be found in the same orbital state. Therefore strong attraction will occur between only four particles at a time. The binding energy of a nucleus will be given roughly by the sum of the binding energies of the sets of four nucleons contained in it. This means, of course, that for a heavy nucleus the binding is proportional to the number of particles, and that the alpha-particle is the saturated unit.

There must be some attraction between nucleons in different orbital states, otherwise the nucleus would separate into its constituent alpha-particles. All that is required for our purpose is that the binding between nucleons in the same orbital state should predominate.

The investigation of high-energy neutron-proton and proton-proton scattering provides convincing evidence for the existence of a property of the interaction akin to property (c). However, there are also indications that the degree to which (c) holds, i.e., the magnitude of the difference between interactions in the same and in other orbital states, is not great enough. Therefore some other property of the interaction may be required to complete the saturation process. There are a number of possible explanations. One is the "hard core" theory, an analog to the explanation of saturation in solids and liquids. The potential is taken to be repulsive at small distances, so that only a few nucleons can be brought within the range of attraction simultaneously. Another possibility is that many-body forces of such a form as to provide saturation are responsible for nuclear binding. But it may turn out that saturation can only be completely understood by detailed reference to the meson field. However, only the property (c) is readily susceptible to detailed analysis, and discussion of the other possibilities has been generally avoided.

2-4 Structure of the chart of stable nuclei. Those light nuclei which happen to be stable are indicated by the gray blocks on the excerpts from the chart of nuclides which are reproduced in Figs. 2-4, 2-5, and 2-6.

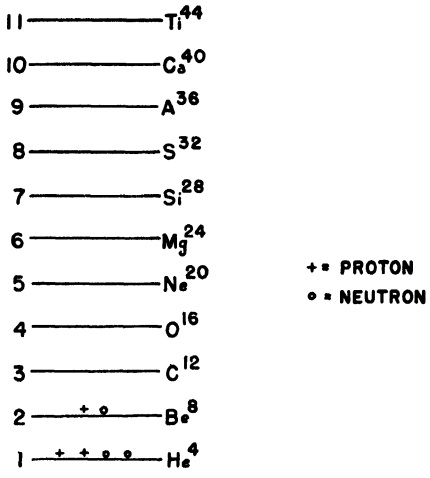

FIG. 2–3. Schematic diagram of one-particle orbital levels in a nucleus. Each level is labeled by the nucleus that is obtained when that level, and every one below it, is filled.

This chart consists of a plot of the number of protons (Z) as a function of the number of neutrons ($A - Z$). Examination leads to the conclusion that the selection of stable light nuclei shows certain definite regularities, and it is our purpose to discuss these regularities with reference to their relationship to any interesting properties of the nuclear interactions.

Again it is useful to think in terms of the independent particle model which suggests the diagram of horizontal lines set forth in Fig. 2–3, each line representing an orbital state of a single nucleon on some sort of energy scale. It is to be kept in mind that this representation is schematic, and that in particular the position of the energy level depends both on the number of nucleons in it and on the number of nucleons in the levels below. According to the Pauli principle, at most four nucleons can occupy each level. Since we are here interested only in the ground state of the nucleus, we shall always assume that there are four nucleons in every level below the first unfilled level. In Fig. 2–3 nucleons are shown filling the levels as they would in the nucleus Li^6. For convenience, each level has been assigned a number; it is also labeled by the nucleus obtained when it is occupied by four nucleons.

Note that He^4 corresponds to having the lowest state filled. The addition of a neutron to this system would produce He^5, or the addition of a proton would produce Li^5. Examination of the nuclear chart shows that neither of these nuclei exists, presumably because the single nucleon in orbital state number 2 does not feel a strong attraction from the nucleons in orbital state number 1. This is in accordance with property (c).

Li^6 to O^{17}. The part of the chart between Li^6 and O^{17} is reproduced in Fig. 2–4. A very great regularity is immediately apparent. Li^6 is formed by the addition of a neutron-proton pair in state number 2, as indicated in Fig. 2–3. The attraction between the pair in the same state provides the necessary stabilization of the nucleus. On the other hand, He^6, which would be formed by adding two neutrons in state number 2, is unstable against beta emission, and Be^6, which is formed by the addition of two protons, is completely unstable. The difference between He^6 and Be^6 may be ascribed to coulomb forces. However, the difference between He^6 and

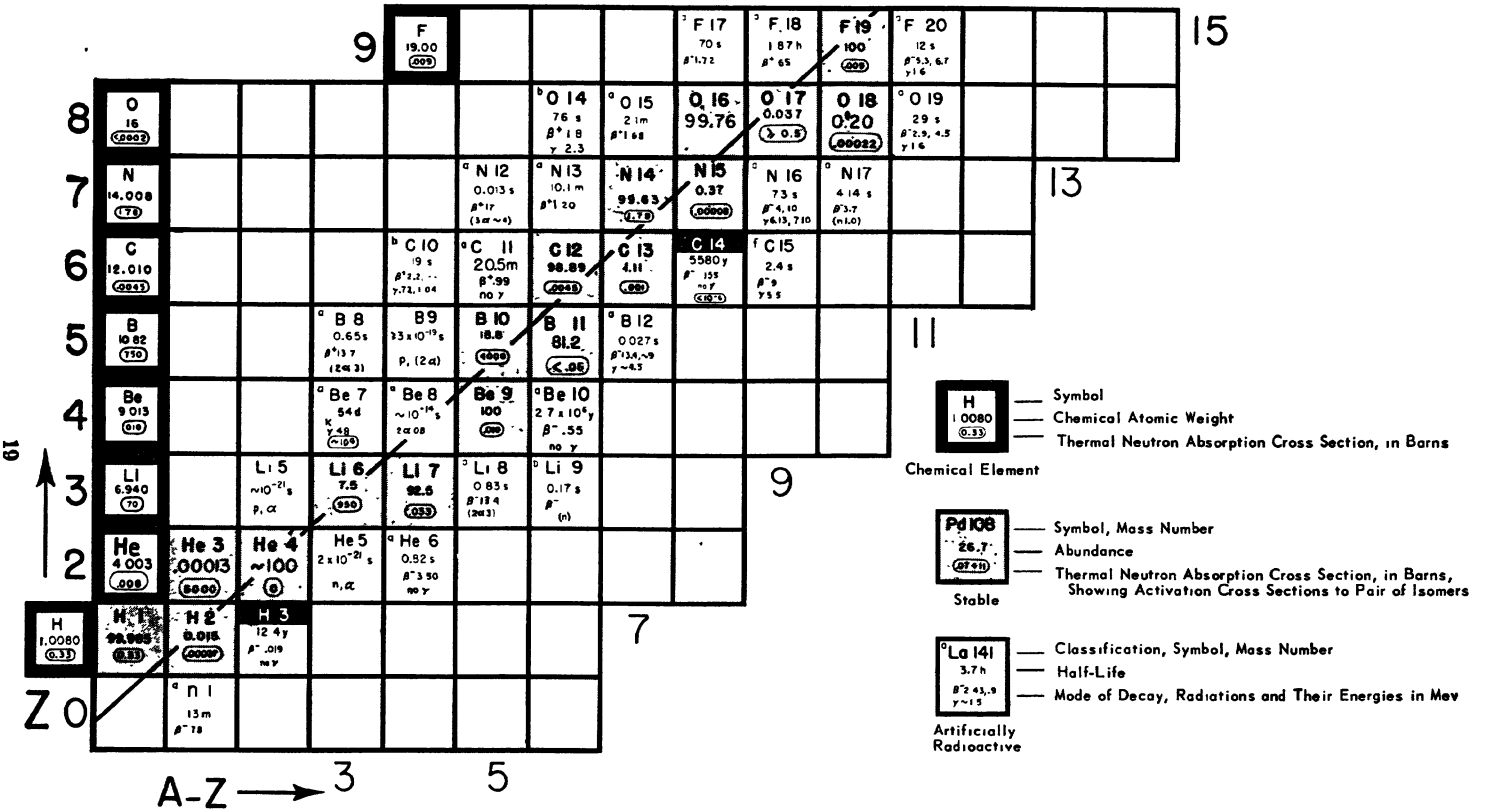

FIG. 2–4. Extract from nuclear chart, illustrating regularities in the selection of stable isotopes between Li^6 and O^{17}. (Figures 2–4, 2–5, and 2–6 courtesy of the General Electric Company, from their *Chart of the Nuclides*, 1952.)

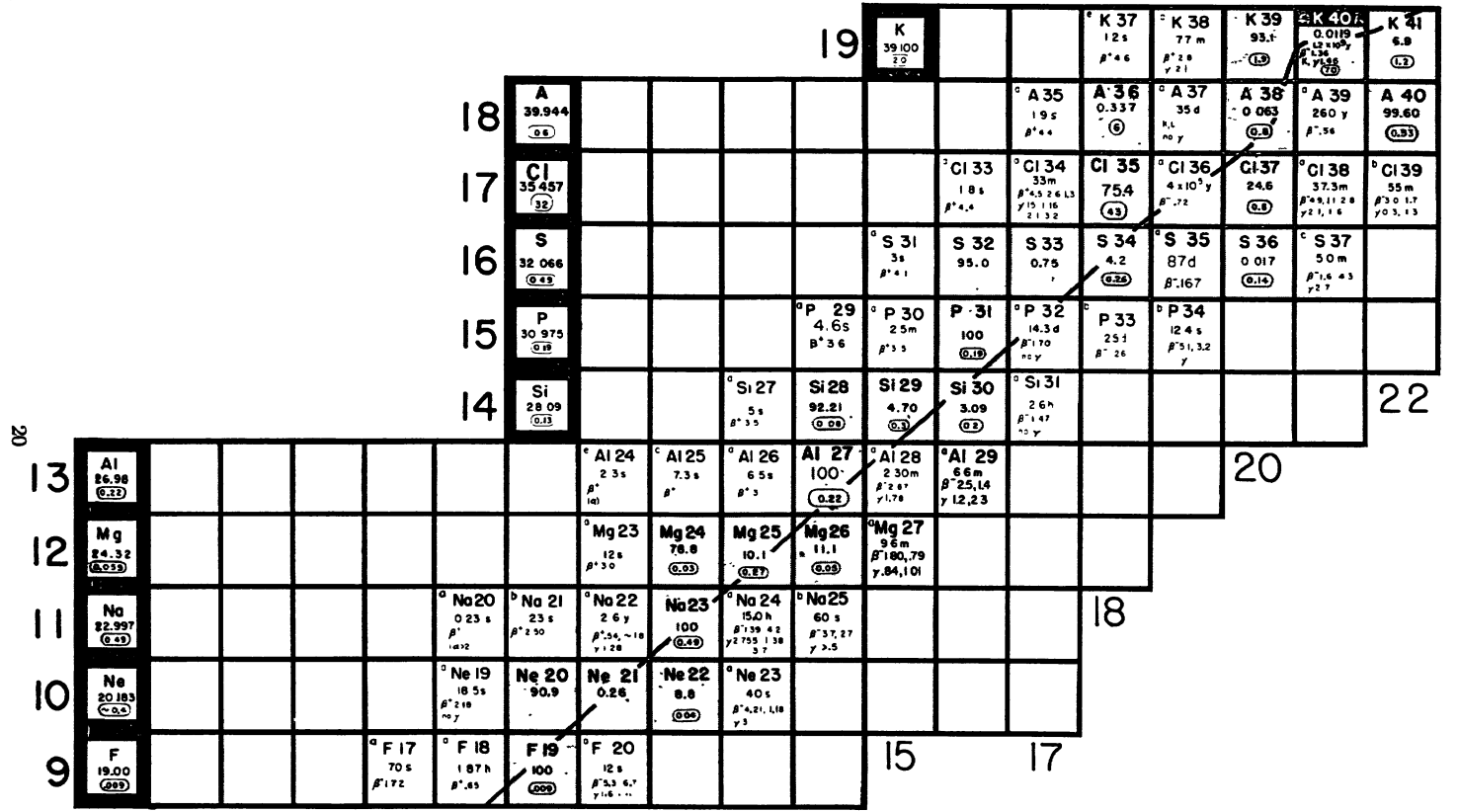

FIG. 2–5. Extract from nuclear chart, illustrating regularities in the selection of stable isotopes between O^{18} and A^{36}.

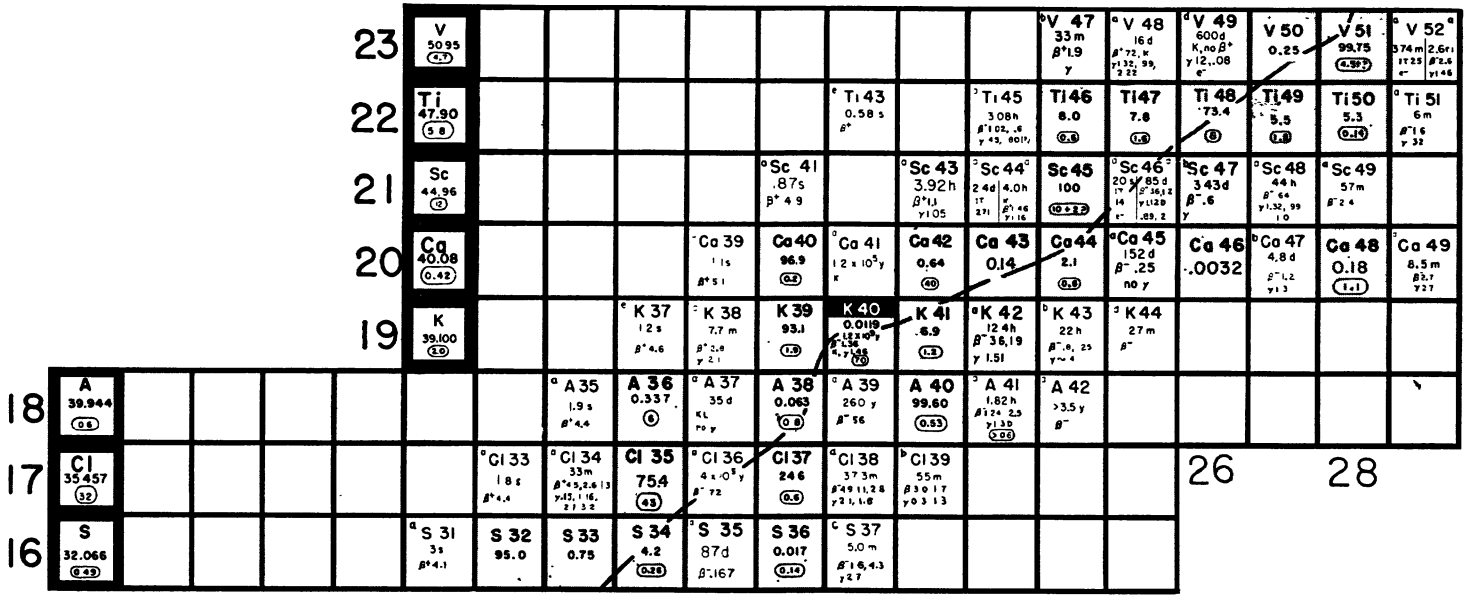

FIG. 2–6. Extract from nuclear chart, illustrating regularities in the selection of stable isotopes beyond S^{36}.

Li^6 gives us further information concerning the relative strengths of the n-p and n-n interactions, namely,

(d) *Either* n-p > n-n *or the interaction is spin-dependent.*

The first of these alternatives will clearly provide relative stability to Li^6. The second requires further explanation. It is to be remembered that two nucleons in the same orbital state must have opposite spins if the nucleons are both of the same kind, that is, either both neutrons or both protons. On the other hand, a neutron and proton may have parallel spins in the same orbit. Now if the nuclear interaction is spin-dependent in such a way that nucleons with parallel spins attract each other more strongly than nucleons with antiparallel spins, it would follow that a neutron-proton pair in a given orbital state could be more strongly bound than a neutron-neutron or proton-proton pair.

The next stable nucleus is obtained by adding a neutron to form Li^7. The stability relative to Be^7 is due to the fact that the coulomb energy difference is greater than the n-H mass difference. The next step is to add a proton to form Be^8. The fact that Be^8 is not stable might seem to be a contradiction. However, the instability corresponds to the unusually great stability of the alpha-particle, so that the nucleus divides into two alpha-particles spontaneously. The nucleus would be stable against any beta process, so that one can say that it does not require further attention. We now come to level number 3 and find that the addition of one neutron to form Be^9 leads to a stable situation. This gives us another qualitative characteristic of the interaction, namely,

(e) *There is some attraction between nucleons in different states.*

No new information is provided by the structure of the chart up to O^{17}, since it can be understood merely by a repetition of the arguments presented above.

O^{18} to A^{36}. This section of the chart is reproduced as Fig. 2–5. At O^{17} there is just one neutron in level number 5. According to the chart, the next stable nucleus is O^{18}, which indicates a marked change from the previous pattern. The second neutron in level 5 is more strongly bound than the first proton. Apparently the coulomb energy has become large enough to overcome the difference in the n-p and n-n interactions. Since the coulomb energy of such a light nucleus cannot be very large, we conclude that

(f) *Either the* n-n (*and therefore the* p-p) *interaction is comparable to the* n-p, *or the spin antiparallel interaction is comparable to the spin parallel,*

although in each case the former is somewhat smaller. The alternatives correspond to the alternatives cited in property (d) above. The conclusion concerning the p-p interaction is obtained from property (a).

The next two stable elements are formed by adding two protons in succession to level 5, as one would expect. Level 6 starts with a neutron, as before, and repeats the above pattern on up to A^{36}. Note that as a result of the fact that neutrons add in pairs and protons add in pairs, we have the general rule that

There are no stable odd-odd nuclei above N^{14}.

The term "odd" here refers to the number of neutrons or the number of protons.

S^{36} *on up.* The part of the chart in the neighborhood of S^{36} is presented as Fig. 2–6. At this point two effects come into play: first, the coulomb energy has increased to the point where two neutrons in level 10 may, as a consequence of their mutual attraction, have almost as low an energy as two protons in level 9. This is the effect which leads in general to an increase in the number of neutrons over the number of protons in heavy nuclei. The other effect has to do with the fact that 20 is a so-called *magic number;* 20 nucleons appear to form an especially stable configuration, usually called a *closed shell.* Although this is not the place for a discussion of magic numbers, we must recognize that they play a role at this particular point in the nuclear chart. The nuclei A^{36} and S^{36} are both stable. S^{36} has its last two neutrons in level 10 and its last two protons in level 8, while A^{36} has its last two neutrons in level 9 and its last two protons in level 9. A^{36} is therefore the well-behaved member of the family. The stability of S^{36} can be attributed either to the fact that 20 neutrons form an especially stable configuration or that the coulomb repulsion of the protons favors the addition of two neutrons in spite of the smaller number of bonds formed by the neutrons. The coulomb effect is clearly established independently of the closed shell effect only when we reach Ti^{48}. The magic number 20 has no bearing on the relative instability of the isobar Cr^{48}, while this is easily understood as a coulomb effect.

Note that at $A = 36$ we have our first example of an *isobaric* pair, a pair of stable nuclei of the same atomic weight but of different atomic number. Such pairs become common from this point on because of the favorable competition of neutron pairs with proton pairs. If two isobars are distinguished by a difference of just one unit of charge, the less stable member of the pair will beta decay into the more stable one. Hence stable pairs of neighboring isobars do not exist. The commonly occurring isobars differ by two units of charge, although they also have different binding energies, because the only process whereby one can be transformed into the other is double beta emission, which is a very improbable process, i.e., a process requiring an extremely long time.

Finally, we note that the odd-odd rule cited above is strengthened in this region because the last neutron occupies a different state from the

last proton, and the attraction between neutron-proton pairs, according to property (c), is greatly reduced.

2–5 Conclusion. The results of the qualitative discussion of nuclear interactions in this chapter may be summarized as follows:

A. The hypothesized two-body nuclear potential requires at least two parameters for its description. These are the strength and the range.

B. The range of the potential is somewhat less than the nuclear radius.

C. A reasonable working hypothesis is to assume that n-n = p-p = n-p, aside from the coulomb effect. The nuclear forces are then said to be "charge-independent."

D. Statement C requires that the interaction be assumed spin-dependent. Nucleons with parallel spins are more strongly attracted.

E. There is a saturation phenomenon. A qualitatively successful working hypothesis is that the attraction between nucleons in the same orbital state is considerably greater than that between those in different orbital states.

Further confirmation or contradiction of these statements must now be sought in the more detailed properties of the simple nuclear systems.

Part II

THE TWO-BODY PROBLEM

CHAPTER 3

GROUND STATE OF THE DEUTERON

The notion that nuclear forces can be described by a two-body interaction between nucleons has the consequence that everything about them can, in principle, be learned from a study of the two-body nuclear systems, the proton-proton, neutron-neutron, and neutron-proton. Of the three, only the last certainly forms a stable unit, the deuteron. Because abundant information is available concerning the ground state of the deuteron, we shall subject it to very close scrutiny, with particular reference to the nature of the n-p interaction. We shall find that an interaction capable of accounting for the ground state data must contain more parameters than the minimum of two indicated in Section 2.5. At the same time, these parameters are not completely determined by ground-state data alone.

3–1 Properties of the deuteron. The known properties of the ground state of the deuteron are as follows:

Total angular momentum (in units of $\hbar$)	$= 1$,
Binding energy	$= 2.226$ Mev,
Magnetic moment	$= 0.8573\ e\hbar/2Mc$,
Quadrupole moment	$= 2.77 \times 10^{-27} e\ \text{cm}^2$.

An auxiliary quantity which is of great use is

$$k_g = \sqrt{-ME_g/\hbar^2}, \tag{3–1}$$

which turns out to be

$$k_g = 2.316 \times 10^{12}\ \text{cm}^{-1} \approx \tfrac{2}{3}R_0^{-1},$$

to a very good approximation. Here $(-E_g)$ is the binding energy. The quantity $1/k_g \approx 1.5R_0$ is often referred to as the "radius of the deuteron," since it is the distance, outside of the range of forces, over which the radial wave function drops by the factor $1/e$.

Our first problem will be the analysis of the implications of the binding energy. Quantum mechanics provides a perfectly straightforward method for calculating the binding energy if the potential is given. Since our objective is the determination of the potential, a number of possibilities must be tried, and the two parameters in the potential must be adjusted to fit the binding energy. The binding energy provides only one condition on the parameters; it will be formulated as a relationship between the range and depth for a given shape of the potential.

At the outset, a central potential will be assumed, and it will be characterized by just the two parameters, the range α and the depth V_0. V_0 can conveniently be replaced by a parameter K, defined by the relationship

$$V_0 = \hbar^2 K^2 / M. \tag{3-2}$$

K then has the dimensions of a reciprocal length.

Analytical forms of the nuclear potential which have been treated extensively and are considered to give a good quantitative fit to various nuclear data are the following:

Yukawa,

$$V(r) = -\frac{\hbar^2 K^2}{M} \frac{e^{-r/\alpha}}{(r/\alpha)}. \tag{3-3}$$

Exponential,

$$V(r) = -\frac{\hbar^2 K^2}{M} e^{-r/\alpha}. \tag{3-4}$$

Gauss,

$$V(r) = -\frac{\hbar^2 K^2}{M} e^{-(r/\alpha)^2}. \tag{3-5}$$

Square well,

$$V(r) = -\frac{\hbar^2 K^2}{M}, \quad r < \alpha, \tag{3-6}$$

$$= 0, \qquad r > \alpha.$$

With the square-well potential, it is required in addition that a condition on the wave function be imposed at $r = \alpha$, an extra requirement arising directly from the nonphysical discontinuity in the potential. Since the discontinuous potential is useful only as an approximate representation of a smooth potential, it should be treated as the mathematical limit of a continuous function. Such a limiting process would lead to continuity of the wave function and its first derivative at $r = \alpha$. This condition is therefore imposed on the solution of the Schroedinger equation.

The Yukawa potential is probably the most interesting because it is intimately related to current concepts of a nuclear field theory. However, it is not susceptible to explicit analytic treatment. The Schroedinger equation must be solved either by numerical methods or by relatively elaborate approximate methods which obscure the simple quantitative features of the problem. For that reason, the square-well potential, which can be treated explicitly, will be used here most often as the basis of detailed calculations. It must be kept in mind that this choice is made only for purposes of exposition and not because physical significance is attached to this particular form of potential. As a matter of fact, the treatment of low-energy phenomena is bound to be rather insensitive to the shape of the potential, so that any choice of shape is useful for developing an insight into the essential physical features of the problem. This lack of

sensitivity to shape is a direct consequence of the large ratio of wavelength of the eigenfunction to range of the forces, a condition shown, in Section 2-2, to be satisfied by the ground state function. The wave function is capable of "seeing" only those effects which extend over a distance somewhat longer than a wavelength.

We shall find that none of the potentials, Eqs. (3-3) to (3-6), as they stand, is adequate for a complete description of the ground state of the deuteron; the forces must be taken to be noncentral. Therefore a detailed treatment of any of the central potentials is useful only to gain some understanding of the simpler aspects of the problem.

3-2 Binding energy. Square well. If the n-p interaction is described by the square-well potential, Eq. (3-6), the wave function of the ground state is a solution of the Schroedinger equation

$$\begin{aligned} \nabla^2\psi - (k_g^2 - K^2)\psi = 0, \quad r < a, \\ \nabla^2\psi - k_g^2\psi = 0, \quad r > a, \end{aligned} \tag{3-7}$$

where the variable r is the n-p distance. Use has been made of the fact that the reduced mass for the deuteron is just $M/2$. The boundary conditions are

$$\psi(0) \text{ finite}, \tag{3-8}$$

$$\psi \to 0 \quad \text{as} \quad r \to \infty, \tag{3-9}$$

$$\psi(a - 0) = \psi(a + 0), \tag{3-10}$$

$$\left(\frac{\partial\psi}{\partial r}\right)_{a-0} = \left(\frac{\partial\psi}{\partial r}\right)_{a+0}. \tag{3-11}$$

For convenience, we shall write

$$\kappa_g^2 = K^2 - k_g^2. \tag{3-12}$$

Then the kinetic energy inside the well is just $\hbar^2\kappa_g^2/M$.

As a result of the assumption of central symmetry, the solutions of the equation can be separated according to

$$\psi_{n,l,m} = f_{nl}(r)Y_l^m(\theta,\varphi), \tag{3-13}$$

where r, θ, and φ are the spherical polar coordinates for the system, n is a principal quantum number, l is the orbital angular momentum, and m is the magnetic quantum number. Y_l^m is, of course, the spherical harmonic. The solution of interest here corresponds to the lowest eigenvalue of the energy, since E_g is the energy of the ground state. For a central potential it can be shown* that this solution must have $l = 0$, $n = 1$, where n is

* See Appendix 1.

the number of radial nodes, including the one at infinity. In the language of the atomic spectroscopist, we then refer to the ground state as an S state.

By writing

$$f_{10}(r) = u(r)/r, \tag{3-14}$$

and making use of the fact that Y_0^0 is constant, we can reduce Eq. (3-7) to

$$\begin{aligned} u'' + \kappa_g^2 u &= 0, \quad r < \alpha, \\ u'' - k_g^2 u &= 0, \quad r > \alpha, \end{aligned} \tag{3-15}$$

with the boundary conditions

$$u(0) = 0, \tag{3-16}$$

$$u(\infty) = 0, \tag{3-17}$$

$$u(\alpha - 0) = u(\alpha + 0), \tag{3-18}$$

$$u'(\alpha - 0) = u'(\alpha + 0). \tag{3-19}$$

The solution of these equations is given by

$$\begin{aligned} u &= A_g \sin \kappa_g r, \quad r < \alpha, \\ u &= B_g e^{-k_g r}, \quad r > \alpha, \end{aligned} \tag{3-20}$$

where the constants A_g and B_g are to be determined by one boundary condition and a normalization condition. By combining the boundary conditions (3-18) and (3-19) to form

$$\left(\frac{u'}{u}\right)_{\alpha-0} = \left(\frac{u'}{u}\right)_{\alpha+0}, \tag{3-21}$$

we find immediately the relationship

$$\kappa_g \cot \kappa_g \alpha = -k_g. \tag{3-22}$$

In the more familiar problems of quantum mechanics the constants in the potential are assumed to be given, and Eq. (3-22) can then be solved for the value of k_g. Each possible solution provides a value of E_g according to Eq. (3-1) or, more directly,

$$E_g = -\hbar^2 k_g^2 / M. \tag{3-23}$$

In this way the energy levels of the system corresponding to S states can be determined. However, the situation is reversed in the problem under consideration. The binding energy of the deuteron is known; therefore a value of k_g for the ground state is given. The parameters in the potential are not known. Equation (3-22) provides a relationship between these parameters for the fixed value of k_g. This is the range-depth relationship

referred to before, given explicitly for the case at hand. A comparison with our earlier rough estimate of the relationship Eq. (2-2) is now possible. There, the assumption was made that the kinetic and the potential energy were virtually the same in magnitude, so the binding energy could be treated as a small quantity; that is,

$$\kappa_g \gg k_g.$$

Then Eq. (3-22) becomes, roughly,

$$\cot K\alpha \approx 0,$$

which yields

$$K\alpha \approx \frac{\pi}{2}, \tag{3-24}$$

or

$$\frac{\hbar^2 K^2}{M}\alpha^2 \approx \frac{\pi^2}{4}\frac{\hbar^2}{M},$$

which is identical with Eq. (2-2), as can be seen by reference to Eq. (3-2). No matter what the shape of the potential, the range-depth relationship will be roughly of the same form,

$$V_0\alpha^2 \approx \text{const},$$

as long as the depth V_0 is large compared with the binding energy.

The exact relationship between range and depth for the square-well potential is easily obtained by rewriting Eq. (3-22) in the form

$$k_g\alpha = \frac{k_g}{\kappa_g}\arctan\left(-\frac{\kappa_g}{k_g}\right), \tag{3-25}$$

where κ_g/k_g is directly related to the depth by Eq. (3-12) or its equivalent

$$\frac{\kappa_g}{k_g} = \left[\frac{V_0}{|E_g|} - 1\right]^{1/2}. \tag{3-26}$$

Equation (3-25) has multiple solutions for the range α, but only the smallest value is consistent with the interpretation of E_g as the ground state energy. Therefore the arctangent lies in the second quadrant. We note also that Eq. (3-25) then places $\kappa_g\alpha$ in the second quadrant,

$$\frac{\pi}{2} \leq \kappa_g\alpha \leq \pi. \tag{3-27}$$

The solution of Eq. (3-25) is shown in Fig. 3-1. This typical information provides us with some orientation concerning the relationship between depth and range, but of course another datum is needed in order to decide on a value for the range.

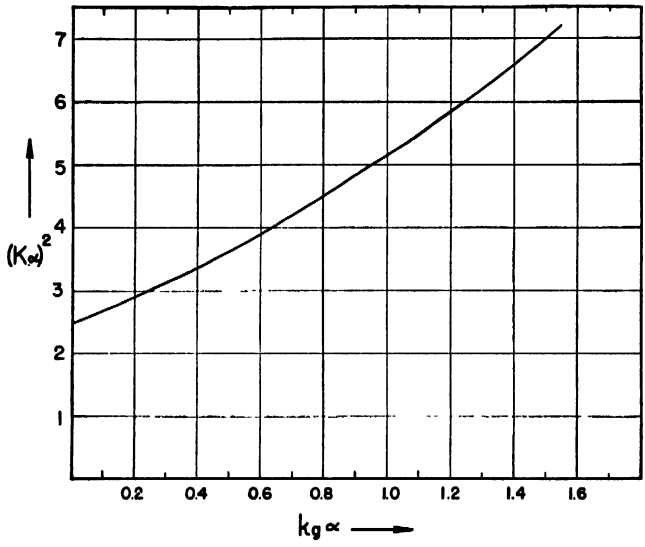

FIG. 3 1. Range-depth relation for a square well. The range is α, the depth of the potential is $\hbar^2K^2/M$. The binding energy of the deutron is $\hbar^2k_g^2/M$.

It is to be remembered that this is merely an illustrative calculation, and that the wave function of the deuteron may be quite different in its analytic form from that obtained here. However, for many purposes the square-well wave function will provide a useful basis for illustrative analysis of deuteron problems.

It is convenient for future purposes to normalize the square-well wave functions, Eq. (3–20), by the condition

$$\int_0^\infty u^2(r)\,dr = 1.$$

Now the relationship between the constants A_g and B_g is determined by the continuity condition, Eq. (3–18), which requires that

$$A_g \sin \kappa_g\alpha = B_g e^{-k_\kappa\alpha}.$$

But we can write

$$\sin \kappa_g\alpha = \pm(1 + \cot^2 \kappa_g\alpha)^{-\frac{1}{2}}$$

$$= \kappa_g/K,$$

according to Eq. (3–22). Use has been made here of the fact that $\kappa_g\alpha$ lies in the second quadrant, as indicated by Eq. (3–27). Thus

$$B_g e^{-k_g\alpha} = \frac{\kappa_g}{K} A_g,$$

and the normalization condition takes the form

$$A_g^2 \left[\int_0^\alpha \sin^2 \kappa_g r \, dr + \frac{\kappa_g^2}{K^2} e^{2k_g\alpha} \int_\alpha^\infty e^{-2k_g r} \, dr \right] = 1.$$

Evaluation of the elementary integrals yields the final result:

$$A_g = \left[\frac{2k_g}{1 + k_g\alpha} \right]^{1/2},$$
$$B_g = \frac{\kappa_g}{K} \left[\frac{2k_g}{1 + k_g\alpha} \right]^{1/2} e^{k_g\alpha}. \tag{3-28}$$

3-3 Binding energy. Other central potentials. Although the treatment of the square-well potential is useful because of the simplicity of the analysis, an analytically continuous potential would fit more naturally into our physical concepts. The smooth potentials, with the exception of the exponential,* Eq. (3-4), lead to a Schroedinger equation soluble only by numerical or approximation methods. Again attention is given only to the S-state. The differential equation for the radial function is

$$u'' + [W(r) - k_g^2]u = 0, \tag{3-29}$$

where $W(r) = K^2\alpha r^{-1}e^{-r/\alpha}$, for the Yukawa potential, Eq. (3-3), and so on. The boundary conditions are

$$u(0) = 0, \tag{3-30}$$

$$u(\infty) = 0. \tag{3-31}$$

There is, of course, no condition equivalent to Eq. (3-18) or (3-19).

Since a simple analytic solution of Eq. (3-29) has not been given, a straightforward attack by numerical integration can be attempted. This involves a step-by-step integration of the equation from $r = 0$ to $r = \infty$. The object of the integration is to obtain a range-depth relationship, but in the numerical integration values must be assumed for both of these parameters. Equation (3-31), taken along with the condition that $W(r)$ be short range, leads to the asymptotic form of $u(r)$:

$$u(r)_{r\to\infty} \to e^{-k_g r}.$$

Starting with this form at some finite but very large distance, the numerical integration can be carried into the origin. Since the parameters K and α have been chosen arbitrarily, the value of $u(0)$ will not satisfy the condition of Eq. (3-30). After intelligent observation of the calculated value of $u(0)$, a new choice of K (for the same α) is made, the procedure repeated,

* *Cf.* H. A. Bethe and R. F. Bacher, *Rev. Mod. Phys.* **8**, 82 (1936).

TABLE 3–1

Range-depth relation for the Yukawa potential, Eq. (3–3), as obtained by numerical integration [Sachs and Goeppert-Mayer, *Phys. Rev.* **53**, 991 (1938)]

α (10^{-13} cm)	0.62	1.18	1.54	1.93
$\hbar^2K^2/M$ (Mev)	217	68	43	30
αk_g	0.14	0.28	0.36	0.45
$K^2\alpha^2$	2.00	2.30	2.50	2.70

TABLE 3–2

Range-depth relation for the Yukawa potential obtained by a first iteration of the variational equation

$\xi = \alpha k_g$	0.14	0.28	0.36	0.45
γ	1.68	1.85	1.95	2.05
$\eta^{(1/2)} \approx K^2\alpha^2$	2.00	2.30	2.47	2.67

and, with luck, $u(0)$ will be closer to zero. A number of repetitions will determine that value of K, for a given α, which leads to $u(0) = 0$ within the desired approximation. For each value of α, this procedure must be repeated to obtain the corresponding K. Results obtained in this way for the Yukawa potential are given in Table 3–1.

The construction of the relationship between K and α by such a method is very tedious. Although some short-cuts in the procedure might be found, it would be much better to have an analytical approximation method capable of rapid convergence. The analytical method is particularly desirable for the practical problem of obtaining answers quickly when a readjustment of the shape of the potential, or some one of its other properties, is necessitated by new data.

One analytical approach is provided by the usual variational method of quantum mechanics. If H is the Hamiltonian of a system, then the expression

$$\mathcal{E} = \int \varphi^* H \varphi \, d^3r \Big/ \int |\varphi|^2 \, d^3r$$

is larger than or equal to the ground state energy E_g for *any* choice of the function φ, and the function ψ which minimizes $\mathcal{E}$ is the correct wave function. A good guess as to the form of φ will lead to a value of $\mathcal{E}$ near to E_g and, if several parameters are left free in the function φ, they may be chosen in such a way as to minimize $\mathcal{E}$, thereby improving the approximation to E_g. Although this method can be used, it puts the emphasis in the wrong place, namely, on approximating the binding energy. The binding energy is known and the parameters in the potential are to be determined.

Thomas* has pointed out that a variational treatment of the strength of the potential for given range and binding energy is possible, and that this variational method is very easily iterated to yield better wave functions and a more accurate value of the depth.

To apply the Thomas method to Eq. (3-29), it is rewritten as

$$\left(\frac{d^2}{d\rho^2} - k_g^2\alpha^2\right) u = -\eta B u,$$

where

$$\rho = r/\alpha,$$

$$\eta = K^2\alpha^2,$$

and

$$B(\rho) = W(r)/K^2.$$

A simplification in writing is provided by introducing the operator

$$A = -\frac{d^2}{d\rho^2} + k_g^2\alpha^2.$$

Then

$$Au = \eta B u. \tag{3-32}$$

Note that $B > 0$ for an attractive potential. A variational equivalent of Eq. (3-32) is

$$\eta = \text{minimum}\,\frac{(v,Av)}{(v,Bv)},$$

where v is a variational function and η is the lowest of the characteristic values associated with Eq. (3-32). The scalar product is used here in the sense of quantum mechanics, i.e., $(v,Av) = \int_0^\infty vAv\,d\rho$.

Another variational equation is obtained by rewriting Eq. (3-32) in the form

$$Bu = \eta BA^{-1}Bu,$$

namely,

$$\eta = \text{minimum}\,\frac{(v,Bv)}{(v,BA^{-1}Bv)}.$$

The iteration procedure is as follows. A guess at the shape of the wave function is made, but one or more parameters are left free, so as to make maximum use of the variational properties of η. Call the function $u^{(0)}$. Then a sequence of approximations to the wave function is given by

$$u^{(n+1)} = A^{-1}Bu^{(n)}. \tag{3-33}$$

* L. H. Thomas, *Phys. Rev.* **51**, 202 (1937). See also Temple and Bickley, *Rayleigh's Principle*, Oxford University Press (1933), especially p. 76 ff. This method has been particularly emphasized by Schwinger in unpublished lectures.

The corresponding approximations to η are defined by

$$\eta^{(n)} = \frac{(u^{(n)}, Au^{(n)})}{(u^{(n)}, Bu^{(n)})}$$

and, from the other variational principle,

$$\eta^{(n+1/2)} = \frac{(u^{(n)}, Bu^{(n)})}{(u^{(n)}, BA^{-1}Bu^{(n)})}.$$

Both of these are positive and larger than η. It is convenient to introduce the notation

$$(u^{(m)}, Bu^{(n)}) = (m,n).$$

Then, from Eq. (3-33), we find

$$\eta^{(n)} = \frac{(n,n-1)}{(n,n)}$$

and

$$\eta^{(n+1/2)} = \frac{(n,n)}{(n,n+1)}.$$

The sequence $\eta^{(0)}$, $\eta^{(1/2)}$, $\eta^{(1)}$, $\cdots$ can be shown to converge from above to η, where $\eta^{(n)}$ is a function of the parameters in $u^{(0)}$ and, at any desired stage, these parameters can be chosen so as to minimize the corresponding $\eta^{(n)}$. The larger n, the more accurate the resulting approximation to η.

Let us apply this method to the Yukawa potential,

$$B = \frac{e^{-\rho}}{\rho}.$$

A good guess at the wave function is

$$u^{(0)} = \rho e^{-\beta\rho},$$

where β is a parameter to be fixed, eventually, by variation. The next approximation to the wave function is

$$u^{(1)} = A^{-1}Bu^{(0)},$$

which satisfies the differential equation

$$Au^{(1)} = Bu^{(0)},$$

or

$$-\frac{d^2}{d\rho^2}u^{(1)} + k_s^2\alpha^2 u^{(1)} = e^{-(\beta+1)\rho}.$$

The solution of this equation, satisfying the conditions of Eqs. (3-30) and (3-31), is the Hulthén* function:

* *Ark. Mat. Ast. Fys.* **28,** No. 5.

$$u^{(1)} = \frac{1}{\gamma^2 - \xi^2}(e^{-\xi\rho} - e^{-\gamma\rho}), \tag{3-34}$$

where

$$\xi = k_g\alpha$$

and

$$\gamma = 1 + \beta.$$

The corresponding approximations to η are

$$\eta^{(0)} = \frac{(u^{(0)}, Au^{(0)})}{(u^{(0)}, Bu^{(0)})}$$

$$= -\int_0^\infty \rho e^{-2\beta\rho}(\rho\beta^2 - 2\beta - \xi^2\rho)\, d\rho \Big/ \int_0^\infty \rho e^{-(2\beta+1)\rho}\, d\rho$$

$$\eta^{(0)} = \frac{(2\beta + 1)^2}{4\beta^3}(\beta^2 + \xi^2);$$

and

$$\eta^{(12)} = \frac{(0,0)}{(0,1)}$$

$$= \int_0^\infty \rho e^{-(2\beta+1)\rho}\, d\rho \Big/ \int_0^\infty e^{-(\beta+1)\rho} \frac{1}{\gamma^2 - \xi^2}(e^{-\xi\rho} - e^{-\gamma\rho})\, d\rho$$

or

$$\eta^{(12)} = \frac{2\gamma(\gamma + \xi)^2}{(2\gamma - 1)^2}; \tag{3-35}$$

and, finally,*

$$\eta^{(1)} = \frac{(1,0)}{(1,1)} = \frac{(0,1)}{(1,1)} = \frac{(\gamma - \xi)^2}{2\gamma} \Big/ \ln \frac{(1 + \gamma + \xi)^2}{(1 + 2\gamma)(1 + 2\xi)}.$$

The value of the variational parameter β, or γ, is best determined by minimizing $\eta^{(1)}$. However, much simpler and sufficiently accurate results are obtained by minimizing $\eta^{(12)}$. The condition is

$$\frac{d\eta^{(12)}}{d\gamma} = 0,$$

which can be reduced to

$$\gamma^2 - \left(\xi + \frac{3}{2}\right)\gamma - \frac{\xi}{2} = 0,$$

* Evaluation of the integral (1,1) is easily accomplished by differentiating with respect to ξ and integrating the resulting differential equation subject to the condition that $(1,1) = 0$ for $\xi = \gamma$.

if use is made of the fact that γ and ξ are positive. Thus

$$\gamma = \tfrac{1}{2}(\xi + \tfrac{3}{2}) + \tfrac{1}{2}[\xi^2 + 5\xi + \tfrac{9}{4}]^{1/2}. \tag{3-36}$$

Insertion of γ from Eq. (3-36) into Eq. (3-35) provides an approximate relationship between η and ξ which is the desired connection between range and depth, since $\eta = K^2\alpha^2$ and $\xi = k_g\alpha$. Results thereby obtained are presented in Table 3-2 for those values of ξ given in Table 3-1. The close agreement with the direct numerical integrations shows the accuracy of this very simple method.

Although an accurate value of η is easily obtained, it does not follow that the corresponding wave function $u^{(1)}$ is also accurate. Actually, the point-by-point behavior of $u^{(1)}$ may be quite different from the true solution even if $\eta^{(1_2)}$ is very close to the correct value. Only in the mean is the wave function closely approximated; but even then the approximation is not as accurate as the corresponding value of η. For the problem at hand, accuracy in the mean is quite good enough, since the deuteron wave function is very smooth, and any reasonable trial function would also be smooth.

3-4 Magnetic moment of the deuteron. The treatment of the ground state of the deuteron has, up to this point, taken no account of the requirement that the total angular momentum has the value one. We have supposed that the ground state is an S-state, which means that the orbital angular momentum is zero. The total angular momentum is then presumably the sum of the intrinsic spins of the two nucleons, which must be combined to form a triplet. Then the ground state of the deuteron has the spectroscopic designation $^3S_1$. Although no specification of the spin dependence of the interaction has been given, it is implied here that the triplet state lies lower than the singlet state. The potential whose properties we have established in the last two sections is the triplet potential. The singlet potential must be somewhat weaker (if it is attractive) for the $^1S_0$ state to lie above the $^3S_1$ state. This provides a nice confirmation of the spin dependence of the interaction suggested by the arguments of Chapter 2.

A sensitive measure of the distribution of angular momentum between spin and orbital motion is provided by the magnetic moment. The precisely measured value of the deuteron moment therefore offers a rather direct test of the assignment $^3S_1$ to the ground state.

Calculation of the magnetic moment of an *atomic* system is a straightforward matter. We merely add the vectors corresponding to spin and orbital magnetic moments of the electrons to form the total magnetic moment operator. The simplest assumption that can be made about the magnetic moments of nuclei is that they are to be obtained in the same way, that is, by taking the vector sum of intrinsic neutron and proton

moments plus any orbital moments produced by the protons. It is clear from the outset that this assumption of additivity may lead us into error. We have noted in Section 1–3 that the magnetic moments of the nucleons must be related in some way to the nature of the meson field with which they interact. If that is the case, then when two or more nucleons are brought into interaction with one another, the meson field will be altered so that the intrinsic moments may very well be altered. We will find evidence of such a change in nuclei heavier than the deuteron.* For the present it will be assumed that the deuteron moment can be accounted for entirely by the addition of the moments of the free particles.

In terms of the Pauli spin operator, $\boldsymbol{\sigma}_p$, the intrinsic magnetic moment operator of a proton is given by $\mathfrak{M}_p = \mu_p \boldsymbol{\sigma}_p$, if μ_p is the numerical value of the proton moment. Similarly, the intrinsic moment operator for the neutron is $\mathfrak{M}_n = \mu_n \boldsymbol{\sigma}_n$. Since only the proton contributes an orbital magnetic moment, the magnetic moment of the deuteron, assuming additivity, is

$$\mathfrak{M} = \mu_p \boldsymbol{\sigma}_p + \mu_n \boldsymbol{\sigma}_n + \tfrac{1}{2}\mathbf{L}, \tag{3–36}$$

in units of nuclear magnetons, where $\mathbf{L}$ is the total orbital angular momentum operator of the deuteron and $\frac{1}{2}\mathbf{L}$ is that part assigned to the proton. Equation (3–36) can be conveniently rewritten in the form

$$\mathfrak{M} = \tfrac{1}{2}[(\mu_p + \mu_n + \tfrac{1}{2})(\mathbf{S} + \mathbf{L}) + (\mu_p + \mu_n - \tfrac{1}{2})(\mathbf{S} - \mathbf{L}) + (\mu_p - \mu_n)(\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_n)], \tag{3–37}$$

where

$$\mathbf{S} = \tfrac{1}{2}(\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_n) \tag{3–38}$$

is the total spin angular momentum operator.

The term "the magnetic moment of the deuteron" is used for the expectation value of the z-component of the operator $\mathfrak{M}$ taken in a state for which the total angular momentum has its maximum projection along the z-axis. The calculation of the expectation value can be carried out by means of the usual vector rule of atomic physics. If we denote by $\mathbf{J}$ the total angular momentum of the nucleus,

$$\mathbf{J} = \mathbf{S} + \mathbf{L}, \tag{3–39}$$

and by J_z the projection of $\mathbf{J}$ on the z-axis, the magnetic moment of the deuteron is given by the expectation value†

$$\langle \mathfrak{M}_z \rangle = \langle (\mathfrak{M} \cdot \mathbf{J}) J_z \rangle / j(j+1),$$

* See Sections 9–3 and 9–6.

† The angular brackets $\langle\ \rangle$ will be used to denote expectation value.

where j is the angular momentum quantum number of the state in question. Since $J_z = j$, we have

$$\langle \mathfrak{M}_z \rangle = \frac{1}{j+1} \langle (\mathfrak{M} \cdot \mathbf{J}) \rangle,$$

or, upon introducing Eq. (3-37) and Eq. (3-39),

$$\langle \mathfrak{M}_z \rangle = \frac{j}{2} \left\{ \left(\mu_p + \mu_n + \frac{1}{2} \right) + \frac{1}{j(j+1)} \left(\mu_p + \mu_n - \frac{1}{2} \right) \langle \mathbf{S}^2 - \mathbf{L}^2 \rangle + \frac{1}{j(j+1)} (\mu_p - \mu_n) \langle (\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_n) \cdot \mathbf{J} \rangle \right\}. \quad (3\text{-}40)$$

If the ground state of the deuteron can be described in terms of a single orbital angular momentum quantum number l, and spin quantum number s (Russell-Saunders coupling), then for that state

$$\mathbf{L}^2 = l(l+1),$$

$$\mathbf{S}^2 = s(s+1).$$

The only possible values of s are $s = 0$ or $s = 1$, corresponding to singlets or triplets, respectively. Since the wave function of a triplet state is symmetric in spin and the wave function of a singlet state is antisymmetric in spin, the antisymmetric operator, $\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_n$, converts a singlet into the orthogonal triplet function and a triplet into a singlet, so its expectation value vanishes in either a singlet or a triplet state. Thus for Russell-Saunders coupling the magnetic moment of the deuteron is given by

$$\langle \mathfrak{M}_z \rangle = \frac{j}{2} \left\{ \left(\mu_p + \mu_n + \frac{1}{2} \right) + \left(\mu_p + \mu_n - \frac{1}{2} \right) \frac{s(s+1) - l(l+1)}{j(j+1)} \right\}. \quad (3\text{-}41)$$

In a $^3S_1$ state, $j = 1$, $l = 0$, and $s = 1$. Therefore the corresponding magnetic moment of the deuteron would be

$$\langle \mathfrak{M}_z \rangle = \mu_p + \mu_n = 0.880;$$

namely, just the sum of neutron and proton intrinsic moments, as one would expect when the spins are parallel and the orbital angular momentum is zero. This value is to be compared with the experimental value 0.857. The discrepancy $\Delta\mathfrak{M} = 0.022$ is well outside of any experimental error.

There are a number of possible sources of error in the theory which might account for the discrepancy:

(1) The ground state is not a $^3S_1$ state.
(2) The moments are not really additive.
(3) Relativistic corrections to the magnetic moment are important

Of these three possibilities, the first is subject to the simplest analysis. Let us consider what other assignment we can make for the ground state of the deuteron. The spin state is limited to a singlet or triplet. The total angular momentum must be one. Therefore, according to the vector rule, the orbital angular momentum can be, at most, two; hence consideration is limited to the $^3S_1$, $^1P_1$, $^3P_1$, and $^3D_1$ states. According to Eq. (3-41), the magnetic moments for the P- and D-states are

$$^1P_1: \quad \langle \mathfrak{M}_z \rangle = \tfrac{1}{2},$$

$$^3P_1: \quad \langle \mathfrak{M}_z \rangle = \tfrac{1}{2}(\mu_p + \mu_n + \tfrac{1}{2}) = 0.689,$$

$$^3D_1: \quad \langle \mathfrak{M}_z \rangle = \tfrac{1}{2}(\mu_p + \mu_n + \tfrac{1}{2}) - (\mu_p + \mu_n - \tfrac{1}{2}) = 0.310,$$

therefore each of these states would lead to an even greater discrepancy in the magnetic moment.

The next logical step is to abandon the assumption of Russell-Saunders coupling and consider mixtures of the four states. Since the S- and D-functions have even parity and the P-functions have odd parity, a mixture of the P-states with the others cannot be considered. In order to reduce to a minimum the departure from the simple model, let us assume that the wave function is a mixture of a $^3S_1$ state and a $^3D_1$ state:

$$\psi = a\psi_S + b\psi_D,$$

where ψ_S is the $^3S_1$ and ψ_D is the $^3D_1$ function. Now the expectation value of the magnetic moment is given by

$$\langle \mathfrak{M}_z \rangle = \tfrac{1}{2}\int (a^*\psi_S^* + b^*\psi_D^*)\{(\mu_p + \mu_n + \tfrac{1}{2}) + \tfrac{1}{2}(\mu_p + \mu_n - \tfrac{1}{2})(\mathbf{S}^2 - \mathbf{L}^2)\}(a\psi_S + b\psi_D),$$

where the $\int$ denotes a sum over spin variables and an integral over the space coordinates. The antisymmetric term in Eq. (3–40) has been left out because it vanishes as long as there is no mixture of singlet and triplet states.* The operators $\mathbf{S}^2$ and $\mathbf{L}^2$ are diagonal matrices in a representation specifying spin and orbital angular momentum, hence they produce no cross terms between ψ_S and ψ_D. The result then is

$$\begin{aligned} \langle \mathfrak{M}_z \rangle &= |a|^2 \langle \mathfrak{M}_z \rangle_S + |b|^2 \langle \mathfrak{M}_z \rangle_D \\ &= 0.879|a|^2 + 0.310|b|^2. \end{aligned} \tag{3–42}$$

The quantities $|a|^2$ and $|b|^2$ are simply the probabilities for finding the system in the S- or the D-state, and they are subject to the normalization condition

$$|a|^2 + |b|^2 = 1.$$

* Even for a mixture of the $^1P_1$ and $^3P_1$ states this term vanishes for the deuteron, as can be seen from the theorem on self-mirror nuclei, Section 9–3.

There is left then one parameter, the D-state probability $|b|^2$, which can now be determined from the observed magnetic moment. The result is $|b|^2 = 0.039$. Thus the system has a probability of about four percent of being found in the D-state if this is the correct interpretation of the magnetic moment.

We shall find in the next section corroborative evidence for such a ground state wave function of the deuteron. However, the very simple calculation which has been carried out here, although it appears to be well founded, may be in error. Still to be kept in mind are the other two possibilities mentioned above, that the moments are not really additive and that relativistic corrections to the moments may be important. Although there is some reason to believe that the question of additivity will not cause trouble in the deuteron,* a serious difficulty arises concerning relativistic corrections. We estimated in Section 1-2 that relativistic corrections might amount to some five percent. The magnetic moments are measured with a precision much greater than five percent, consequently relativistic corrections of that order, if they exist, may be very important. As a matter of fact, the magnetic moment discrepancy of 0.022 is less than five percent of the total moment, 0.857. Several attempts† have been made to calculate the first-order relativistic corrections to the magnetic moments of the deuteron. It turns out that the correction, $\Delta_R\mathfrak{M}$, is somewhat smaller than the discrepancy $\Delta\mathfrak{M}$ but still appreciable. Furthermore, as one might expect, the magnitude and the sign of $\Delta_R\mathfrak{M}$ depend on the nature of the field theory and on the nature of the interaction of the nuclear field with nucleons, therefore the answer to the problem is ambiguous. Introduction of $\Delta_R\mathfrak{M}$ into Eq. (3–42) leads to a new estimate of $|b|^2$, but the ambiguity in the relativistic correction is mirrored by a corresponding ambiguity in the D-state probability. On this basis a D-state probability anywhere between 2.5 and 6 percent would be consistent with one or another form of the relativistic theory. We are then left in doubt concerning the interpretation of the discrepancy and the corresponding value of $|b|^2$. On the other hand, the fact that the relativistic correction to the magnetic moment is sensitive to the field theory may eventually prove to be very useful. If an independent determination of the D-state probability were possible, then any remaining magnetic moment anomaly might be interpreted as $\Delta_R\mathfrak{M}$, and its magnitude would provide information concerning the nuclear field theory. The quadrupole moment offers a chance to arrive at the D-state probability, but we shall find that the determination is not definite enough to clearly establish the relativistic magnetic moment term.

* See Section 9–2.

† R. G. Sachs, *Phys. Rev.* **72,** 91 (1947); G. Breit and I. Bloch, *Phys. Rev.* **72,** 135 (1947). For correction to the latter work see Breit and Thaler, *Phys. Rev.* **88,** 1214 (1952).

3–5 The quadrupole moment of the deuteron. The concept of a quadrupole moment arises most naturally in a classical discussion of the external potential produced by a cluster of electric charges when the dimensions of the cluster are small compared with the distance between its center and the point at which the potential is measured. If the charge on the kth particle is denoted by e_k and its position vector by $\mathbf{r}_k$, the potential at a point $\mathbf{R}$ is given by

$$\varphi(\mathbf{R}) = \sum_k \frac{e_k}{|\mathbf{R} - \mathbf{r}_k|}. \tag{3–43}$$

For an origin of coordinates within the cluster, the condition on the size of the cluster becomes $r_k \ll R$ for every k. The denominator in Eq. (3–43) can then be expanded in powers of r_k/R. The result up to second order takes the form

$$\varphi = \frac{\mathcal{E}}{R} + \frac{(\mathbf{D}\cdot\mathbf{R})}{R^3} + \frac{1}{2}\frac{Q'}{R^3} + \cdots, \tag{3–44}$$

where

$$\mathcal{E} = \sum_k e_k,$$

$$\mathbf{D} = \sum_k e_k \mathbf{r}_k, \tag{3–45}$$

$$Q' = \sum_k \frac{e_k}{R^2}[3(\mathbf{R}\cdot\mathbf{r}_k)^2 - r_k^2 R^2]. \tag{3–46}$$

The first of these quantities, $\mathcal{E}$, is the total charge of the system; the second, $\mathbf{D}$, is the electric dipole moment, and the third, Q', is the electric quadrupole moment.

In this form the definition of the quadrupole moment is evidently of a qualitatively different character from the definition of the monopole and dipole moments. $\mathcal{E}$ and $\mathbf{D}$ depend only on the charge distribution, while Q' involves the direction of $\mathbf{R}$ explicitly. Furthermore, $\mathcal{E}$ is a scalar under space rotations and $\mathbf{D}$ a vector, while the transformation properties of the part of Q' which is independent of $\mathbf{R}$ are not made clear. All three contributions to the potential, Eq. (3–44), are invariant under rotations, but the invariance of only the first two is made explicit by the form in which they are defined. A more satisfactorily defined quadrupole moment would clearly depend only on the charge distribution and have specified transformation properties under rotations. The second requirement implies that the quadrupole moment should be defined as a tensor of given rank, and rank two is strongly suggested by the fact that $\mathcal{E}$ is a tensor of rank zero and $\mathbf{D}$ a tensor of rank one.

Now the quadrupole moment involves a sum of quantities of the type

$$q(k) = z_k^2 - \tfrac{1}{3}r_k^2,$$

where z_k is the coordinate taken with respect to **R** as the z-axis. Reference to this particular choice of coordinates can be eliminated by noting that the quantity q is an element of the traceless symmetric tensor of the second rank*

$$q_{\alpha\beta}(k) = x_\alpha(k)x_\beta(k) - \tfrac{1}{3}r_k^2\delta_{\alpha\beta}, \tag{3–47}$$

where we have now introduced the convenient notation $x_1(k)$, $x_2(k)$, $x_3(k)$ in place of x_k, y_k, z_k. The definition of $\delta_{\alpha\beta}$ is, as usual,

$$\delta_{\alpha\alpha} = 1,$$

$$\delta_{\alpha\beta} = 0, \quad \alpha \neq \beta.$$

The close relationship between the tensor elements $q_{\alpha\beta}$ and the five D-functions will be of considerable interest. There are just five linearly independent elements of the tensor and each is a linear combination of the spherical harmonics $r^2Y_2^m$, the arguments of Y_2^m being the angular position of the vector $\mathbf{r} = (x_1,x_2,x_3)$. Thus the $q_{\alpha\beta}$ provide a useful substitute for the usual D-functions.

The vector **R** has components X_1, X_2, X_3, from which another symmetric tensor can be constructed, namely, the tensor $X_\alpha X_\beta$. The scalar product of the two tensors, $\sum_{\alpha,\beta} q_{\alpha\beta}X_\alpha X_\beta$, is just the explicit invariant† which is required to express the quadrupole term in the potential:

$$Q' = 3\sum_k e_k \sum_{\alpha\beta} q_{\alpha\beta}(k)\frac{X_\alpha X_\beta}{R^2}.$$

This suggests that the quadrupole moment tensor be defined by

$$Q_{\alpha\beta} = \sum_k e_k[3x_\alpha(k)x_\beta(k) - r_k^2\delta_{\alpha\beta}], \tag{3–48}$$

which is a symmetric tensor with the property that its trace is zero, i.e.,

$$\sum_\alpha Q_{\alpha\alpha} = 0. \tag{3–49}$$

Then the quadrupole term in the potential has the explicitly invariant form

$$\sum_{\alpha,\beta}\frac{Q_{\alpha\beta}X_\alpha X_\beta}{2R^5},$$

and $Q_{\alpha\beta}$ depends only on the charge distribution in the cluster.

* An entity, $T_{\alpha\beta}$, with two indices α and β running through the values 1, 2, 3 corresponding to the three coordinate axes, is defined as a tensor of the second rank in three dimensions if, under a rotation of axes, $T_{\alpha\beta}$ undergoes the same linear transformation as the products $x_\alpha x'_\beta$ of the components of two position vectors **r** and **r′**. The trace of a tensor is the sum of its diagonal elements.

† The invariance of the scalar product $\sum_{\alpha\beta} T_{\alpha\beta}R_{\alpha\beta}$ of two tensors follows directly from the transformation properties of the tensors.

In the formalism of quantum mechanics, the elements of the quadrupole tensor become operators obtained by treating the particle coordinates as operators. The quantity referred to as "the quadrupole moment" of a nucleus is an expectation value of such an operator in the nuclear ground state. It can be shown, most easily by the methods of group theory, that the expectation values of the various elements of a traceless symmetric tensor in a state of given total angular momentum are completely determined within a common multiplicative factor. The determination of this single factor is then the object of a measurement of the quadrupole moment.

For the deuteron, Eq. (3–48) gives

$$Q_{\alpha\beta} = e[3x_\alpha(p)x_\beta(p) - r_p^2\delta_{\alpha\beta}],$$

since the neutron carries no charge. The proton position vector $\mathbf{r}_p$ is measured from the center of mass, but the deuteron wave function is expressed in terms of the neutron-proton distance

$$\mathbf{r} = 2\mathbf{r}_p.$$

Therefore it is more convenient to write

$$Q_{\alpha\beta} = \frac{e}{4}(3x_\alpha x_\beta - r^2\delta_{\alpha\beta}),$$

where $\mathbf{r} = (x_1,x_2,x_3)$. The interesting quantities are the expectation values

$$\langle Q_{\alpha\beta}\rangle = \sum_{\text{sp}} \int |\psi_1^m|^2 Q_{\alpha\beta}\, d^3r, \tag{3–50}$$

where $\sum_{\text{sp}}$ denotes the sum over the spin variables of the neutron and proton, and ψ_1^m is the ground state wave function of the deuteron with total angular momentum 1 and magnetic quantum number m. Of the five independent quantities given by Eq. (3–50), those with $\alpha \neq \beta$ vanish, since $Q_{\alpha\beta}$, $\alpha \neq \beta$, is a combination of spherical harmonics Y_2^M with $M = \pm 1$ or $M = \pm 2$, while $|\psi_1^m|^2$ is invariant under rotations about the z-axis.* Furthermore, Q_{11} and Q_{22} are linear combinations of $Y_2^{\pm 2}$ and Y_2^0, the coefficient of Y_2^0 being the same for both Q_{11} and Q_{22}. Since the expectation value of $Y_2^{\pm 2}$ vanishes by the above argument, it follows that

$$\langle Q_{11}\rangle = \langle Q_{22}\rangle. \tag{3–51}$$

Finally, Eq. (3–49) has the consequence

$$\sum_\alpha \langle Q_{\alpha\alpha}\rangle = 0,$$

* An integral of the form $\int \psi_j^{m*} Q \psi_j^\mu\, d^3r$ must vanish unless the values of m and μ are such that some part of the integrand is invariant under a rotation about the z-axis (or any other rotation), since the rotation merely introduces a change in the dummy integration variables.

therefore $$\langle Q_{11} \rangle = \langle Q_{22} \rangle = -\tfrac{1}{2}\langle Q_{33} \rangle, \tag{3-52}$$

and only the one quantity $\langle Q_{33} \rangle$ is required to specify the quadrupole moment, as indicated by the general statements set forth in the previous paragraph.

This simple result can easily be understood in terms of a classical model. $|\psi_1^m|^2$ may be interpreted as a classical charge density, for which the quadrupole moment is given by Eq. (3–50). Because the state is specified by a definite magnetic quantum number, the density has axial symmetry about the z-axis. Thus the quadrupole ellipsoid, the ellipsoid associated with the quadrupole tensor, has the z-axis as one of its principal axes. Since the coordinate axes coincide with the principal axes, $\langle Q_{\alpha\beta} \rangle = 0$ for $\alpha \neq \beta$. Further, the axial symmetry establishes Eq. (3 51) and, of course, Eq. (3 52) still follows from the vanishing of the trace.

Although $\langle Q_{33} \rangle$ depends on the magnetic quantum number m of the state in question, this dependence is completely determined by symmetry considerations, so the value need be given for just one value of m. By definition, "the quadrupole moment of the deuteron" is the expectation value of Q_{33} in the state of maximum magnetic quantum number, $m = 1$. It will hereafter be denoted by Q, which is found from Eq. (3–50) to be

$$Q = \frac{e}{4} \sum_{sp} \int |\psi_1^1|^2 (3z^2 - r^2)\, d^3r. \tag{3-53}$$

Equation (3–53) makes it clear that the quadrupole moment is associated with the shape of the charge distribution in the deuteron. If $|\psi|^2$ has spherical symmetry, the average value of $\cos^2\theta$ is $\frac{1}{3}$; and the quadrupole moment vanishes. Another way to obtain the same result is to note that the quadrupole moment operator is simply the D-function, Y_2^0, and that a spherically symmetric $|\psi|^2$ is an S-function. The vanishing of the quadrupole moment is then a consequence of the orthogonality of the S- and D-functions. For other than spherical distributions, the sign of the quadrupole moment is determined by the nature of the distortion of the charge distribution from spherical symmetry. If the charge is extended in the z-direction, the quadrupole moment is positive, whereas if the charge distribution is flattened in the z-direction, the quadrupole moment is negative. The fact that the deuteron quadrupole moment is positive indicates that the charge distribution is elongated along the direction of the total angular momentum.

Our assumption that the ground state of the deuteron is an S-function leads immediately to a spherically symmetric charge distribution and therefore to a vanishing quadrupole moment. We see then that the assumption of a pure S-function leads us into a definite contradiction. There would be a quadrupole moment in a P-state, since the product of two P-functions contains a D-function; the integral of the square of a P-func-

tion times the quadrupole moment operator would not vanish. It is also clear that there would be a quadrupole moment in a pure D-state. Furthermore, for a mixture of an S- and D-state the square of the wave function would contain terms with the symmetry of a D-function. Therefore one could obtain in this way a contribution to the moment. It is of particular interest to note that in a mixture of S- and D-functions the cross term between the two functions has the symmetry of a D-function and would contribute to the moment; this contribution would be proportional to the *amplitude* of the D-function rather than the D-state probability. Since the four percent D-state probability deduced from the magnetic moment would imply a 20 percent amplitude, we see that the contribution of the cross term can be quite appreciable.

As an example, let us calculate the quadrupole moment in the $^1P_1$ state. The wave function is simply a product of a P-function and a singlet spin function,

$$\psi_1^m = \varphi_1^m \chi_0,$$

where m is the magnetic quantum number. If the spin function is taken to be normalized, the normalization condition on the space function becomes simply

$$\int |\, \varphi_1^m(\mathbf{r}) \,|^2 \, d^3r = 1. \tag{3-54}$$

and the expectation value of the quadrupole moment is

$$\langle Q_{33} \rangle = \frac{e}{4} \int (3z^2 - r^2) |\, \varphi_1^m(\mathbf{r}) \,|^2 \, d^3r$$

$$= \frac{e}{4} (3\langle r^2 \cos^2 \theta \rangle - \langle r^2 \rangle).$$

The expectation value is to be taken in the state $m = 1$, for which the wave function is

$$\varphi_1^1(\mathbf{r}) = Ce^{i\phi} \sin \theta f_P(r).$$

If we normalize the radial function $f_P(r)$ so that

$$\int_0^\infty f_P^2(r) r^2 \, dr = 1, \tag{3-55}$$

the constant is determined by the normalization condition, Eq. (3-54), to be $C = \sqrt{3/8\pi}$. Then the expectation value of $r^2 \cos^2 \theta$ is

$$\langle z^2 \rangle = \frac{3}{8\pi} \int_0^{2\pi} \int_{-1}^{1} \int_0^\infty r^2 \cos^2 \theta \sin^2 \theta f_P^2(r) r^2 \, dr \, d(\cos \theta) \, d\phi$$

$$= \tfrac{1}{5} \int_0^\infty r^2 f_P^2(r) r^2 \, dr,$$

or $$\langle z^2 \rangle = \tfrac{1}{5}\langle r^2 \rangle.$$

Since the expectation value of z^2 is less than $\frac{1}{3}$ of the expectation value of r^2, the charge distribution is flattened and the quadrupole moment is negative:

$$\langle Q_{33} \rangle = -\frac{e}{10}\langle r^2 \rangle.$$

Therefore a $^1P_1$ state would not provide a quadrupole moment of the correct sign.

Now consider the $^3P_1$ function. This function is a linear combination of the products of the three P-functions $\varphi_1^m(r)$ with the three triplet functions $\chi_1^m(\sigma_n^z, \sigma_p^z)$. A simple way of forming the necessary linear combination is to note that the function φ_1^1 is proportional to $x + iy$, the function φ_1^{-1} is proportional to $x - iy$, and the function φ_1^0 is proportional to z. Thus the three functions φ_1^m are linear combinations of the components of the vector $\mathbf{r}$. Similarly, for any state of unit angular momentum the three independent wave functions transform under rotations like these linear combinations of the components of a three-dimensional vector. Therefore the triplet functions may be treated as the components of a vector $\boldsymbol{\chi}$, and the wave function ϕ_1^m of the state with total angular momentum $j = 1$ as components of a vector $\boldsymbol{\phi}$. The only linear combination of products of the φ_1^m and χ_1^m which can form the vector $\boldsymbol{\phi}$ is

$$\boldsymbol{\phi} = [\mathbf{r} \times \boldsymbol{\chi}]\frac{f_P(r)}{r},$$

where $f_P(r)$ is now the $^3P_1$ radial function. The $m = 1$ state is therefore given by

$$\begin{aligned}\phi_1^1 &= \phi_1^x + i\phi_1^y \\ &= \{(y\chi^z - z\chi^y) + i(z\chi^x - x\chi^z)\}\frac{f_P(r)}{r}.\end{aligned}$$

The expectation value of z^2 is given by

$$\langle z^2 \rangle = \sum_{sp}\int |\,\phi_1^1\,|^2 z^2\, d^3r \Big/ \sum_{sp}\int |\,\phi_1^1\,|^2\, d^3r, \tag{3-56}$$

where the normalization integral is

$$\begin{aligned}\sum_{sp}\int |\,\phi_1^1\,|^2\, d^3r = \sum_{sp}\int \frac{f_P^2}{r^2}\{&|\,y\chi^z - z\chi^y\,|^2 + |\,z\chi^x - x\chi^z\,|^2 \\ &+ i[(y\chi^z - z\chi^y)^*(z\chi^x - x\chi^z) - (z\chi^x - x\chi^z)^*(y\chi^z - z\chi^y)]\}\, d^3r.\end{aligned}$$

The last term in the square brackets vanishes because it involves either the angular integral of products such as xz or yz, which are odd functions,

or the sum over spins of products $\chi^{r*}\chi^{y}$, which vanish because of their orthogonality. Since the spin functions are orthonormal, we find

$$\sum_{\text{sp}} \int | \phi_1^1 |^2 \, d^3r = \int \frac{f_P^2}{r^2} (x^2 + y^2 + 2z^2) \, d^3r = \frac{16\pi}{3},$$

if the radial function is normalized according to the condition Eq. (3-55). A similar calculation yields

$$\sum_{\text{sp}} \int | \phi_1^1 |^2 z^2 \, d^3r = \frac{32\pi}{15} \langle r^2 \rangle,$$

and Eq. (3-56) becomes

$$\langle z^2 \rangle = \tfrac{2}{5} \langle r^2 \rangle.$$

In this case $\langle z^2 \rangle$ is larger than $\frac{1}{3}\langle r^2 \rangle$, so the corresponding quadrupole moment is positive:

$$Q = \frac{e}{20} \langle r^2 \rangle.$$

Although the result has the correct sign, it would in all probability be too large on the basis of any reasonable estimate of $\langle r^2 \rangle$. If we take as an estimate $\langle r^2 \rangle = R_0^2$, Q turns out to be $3.9 \times 10^{-27} e$ cm^2, which is larger than the observed value. Yet this estimate would seem to be, if anything, too low, since the radius of the deuteron has been seen to be $1.5R_0$. Therefore it is likely that in neither a $^3P_1$ nor a $^1P_1$ state would the quadrupole moment have the right value. However, a mixture of these two states would yield a quadrupole moment between the two values and could therefore certainly account for the observed moment of the deuteron. But on the basis of the magnetic moment, it seems very unlikely that this is the correct explanation of the deuteron quadrupole moment. Therefore we consider in detail the mixture of the S- and D-functions so strongly suggested by our analysis of the magnetic moment.

In order to construct a $^3D_1$ function, ψ_1^m, for the deuteron, we proceed in a manner similar to that used for constructing the $^3P_1$ function.* The conditions that the $^3D_1$ function must satisfy are the following:

(1) It is a linear combination of products of Y_2^m and χ_1^m.
(2) The three functions ψ_1^m have the same transformation properties under rotations as the functions χ_1^m.

The tensor

$$T_{\alpha\beta} = 3r^{-2}(x_\alpha x_\beta - \tfrac{1}{3} r^2 \delta_{\alpha\beta}) \qquad (3\text{-}57)$$

again provides a convenient set of functions. It has been pointed out before that the five independent elements of this tensor are linearly inde-

* The method here is that of Rarita and Schwinger, *Phys. Rev.* **59**, 436 (1941).

pendent linear combinations of the Y_2^m. Therefore the wave function must be a linear combination of the products of the $T_{\alpha\beta}$ with the spin functions. If $O_{\alpha\beta}$ is a tensor depending only on the spin operators of the neutron and proton, then the expression

$$\Psi_1^m = g(r) \sum_{\alpha\beta} T_{\alpha\beta} O_{\alpha\beta} \chi_1^m$$

is a satisfactory function. It transforms under rotations like χ_1^m because the product of the two tensors appears in an invariant combination. Now the only tensor operator in the spins leading to a nonvanishing Ψ_1^m is*

$$O_{\alpha\beta} = \sigma_p^\alpha \sigma_n^\beta.$$

Therefore the ${}^3D_1$ function has the form

$$\Psi_1^m = g(r) \sum_{\alpha\beta} 3r^{-2}(x_\alpha x_\beta - \tfrac{1}{3} r^2 \delta_{\alpha\beta}) \sigma_p^\alpha \sigma_n^\beta \chi_1^m$$

or

$$\Psi_1^m = g(r) S_{np} \chi_1^m, \tag{3-58}$$

where

$$S_{np} = \frac{3(\boldsymbol{\sigma}_n \cdot \mathbf{r})(\boldsymbol{\sigma}_p \cdot \mathbf{r})}{r^2} - (\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p). \tag{3-59}$$

The invariance of S_{np} under rotations is apparent in this form, and it is clear that the wave function satisfies all of the conditions required of a ${}^3D_1$ function. Note that $\mathbf{r}$ is the neutron-proton distance.

Normalization of the D-function requires that

$$\int g^2(r) (S_{np} \chi_1^m, S_{np} \chi_1^m)\, d^3r = 1,$$

where the scalar product is a condensed notation for the spin sum.† Since the operator S_{np} is hermitian in the spin variables, we can write

$$(S_{np} \chi_1^m, S_{np} \chi_1^m) = (\chi_1^m, S_{np}^2 \chi_1^m).$$

From the fact that the square of the total spin

$$\mathbf{S}^2 = \tfrac{1}{4}[\boldsymbol{\sigma}_n^2 + \boldsymbol{\sigma}_p^2 + 2(\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p)]$$

is diagonal, as are the operators $\boldsymbol{\sigma}_n^2 = \boldsymbol{\sigma}_p^2 = 3$, it follows that $(\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p)$ is diagonal and has the characteristic value 1 in the triplet state and -3 in the singlet state. Since both χ_1^m and $S_{np} \chi_1^m$ are triplet states, we can write

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* Another possible tensor would be $\delta_{\alpha\beta}$, but that leads to $\psi_1^m = 0$ because the trace of $T_{\alpha\beta}$ vanishes. All other tensors are antisymmetric, so their scalar product with the symmetric tensor $T_{\alpha\beta}$ vanishes.

† That is, $(\chi_1^m, O\chi_1^m) = \sum_{sp} \chi_1^{m*} O \chi_1^m$, where O is any operator involving the spins.

$$(\chi_1^m, S_{np}^2 \chi_1^m) = \left(\chi_1^m, \left[\frac{3(\boldsymbol{\sigma}_n \cdot \mathbf{r})(\boldsymbol{\sigma}_p \cdot \mathbf{r})}{r^2} - 1\right]^2 \chi_1^m\right)$$

$$= \left(\chi_1^m, \left[\frac{9(\boldsymbol{\sigma}_n \cdot \mathbf{r})^2(\boldsymbol{\sigma}_p \cdot \mathbf{r})^2}{r^4} - 6\frac{(\boldsymbol{\sigma}_n \cdot \mathbf{r})(\boldsymbol{\sigma}_p \cdot \mathbf{r})}{r^2} + 1\right] \chi_1^m\right).$$

But the square of any component of a Pauli spin operator is unity:

$$\frac{(\boldsymbol{\sigma}_n \cdot \mathbf{r})^2}{r^2} = \frac{(\boldsymbol{\sigma}_p \cdot \mathbf{r})^2}{r^2} = 1,$$

so this expression reduces to

$$(\chi_1^m, S_{np}^2 \chi_1^m) = (\chi_1^m, [8 - 2S_{np}]\chi_1^m), \tag{3-60}$$

and the normalization condition becomes

$$\int g^2(r)(\chi_1^m, [8 - 2S_{np}]\chi_1^m)\, d^3r = 1.$$

Upon making use of the fact that $S_{np}\chi_1^m$ is a D-function which is orthogonal to the S-function χ_1^m, the normalization condition reduces to

$$32\pi \int_0^\infty g^2(r) r^2\, dr = 1, \tag{3-61}$$

since the spin function is already normalized, i.e.,

$$(\chi_1^m, \chi_1^m) = 1.$$

The S-function has the form $f(r)\chi_1^m$, where $f(r)$ is the radial function normalized according to the condition

$$4\pi \int_0^\infty f^2(r) r^2\, dr = 1. \tag{3-62}$$

It is convenient to introduce in place of the radial functions new functions

$$u(r) = arf(r),$$

$$w(r) = brg(r),$$

where a and b are the amplitudes of the S- and D-functions, respectively, in the ground state of the deuteron. Then the wave function is

$$\psi_1^m = \left[\frac{u(r)}{r} + \frac{w(r)}{r} S_{np}\right] \chi_1^m, \tag{3-63}$$

which is normalized in such a way that

$$4\pi \left\{\int_0^\infty u^2(r)\, dr + 8\int_0^\infty w^2(r)\, dr\right\} = 1, \tag{3-64}$$

according to Eqs. (3–61) and (3–62). Furthermore, the D-state probability is given by

$$|\, b \,|^2 = 32\pi \int_0^\infty w^2(r)\, dr. \tag{3–65}$$

The quadrupole moment is now to be obtained in the state Ψ_1^1 of the set given by Eq. (3–63). It is

$$\begin{aligned} Q &= \frac{e}{4} \sum_{np} \int |\, \psi_1^1 \,|^2 (3\cos^2\theta - 1) r^2\, d^3r \\ &= \frac{e}{4} \int (3\cos^2\theta - 1)([u + wS_{np}]\chi_1^1, [u + wS_{np}]\chi_1^1)\, d^3r \\ &= \frac{e}{4} \int (3\cos^2\theta - 1)\{u^2 + 2uw(\chi_1^1, S_{np}\chi_1^1) + w^2(\chi_1^1, S_{np}^2\chi_1^1)\}\, d^3r. \end{aligned}$$

The u^2 term integrates to zero, since it represents the quadrupole moment in an S-state. Also, we can make use of Eq. (3–60) to obtain

$$Q = \frac{e}{2} \int_0^\infty r^2\, dr\, (uw - w^2) \int d\Omega\, (3\cos^2\theta - 1)(\chi_1^1, S_{np}\chi_1^1),$$

because the term proportional to $8w^2$ also represents an S-state so it, too, vanishes. It is to be remembered that S_{np} is a linear combination of the spherical harmonics Y_2^m and that the Y_2^m are orthogonal to each other for different values of m. The factor $(3\cos^2\theta - 1)$ is just Y_2^0, so only the Y_2^0 term in S_{np} will contribute to this integral. Thus Q reduces to

$$Q = \frac{e}{2} \int_0^\infty r^2\, dr\, (uw - w^2) \int d\Omega\, (3\cos^2\theta - 1)^2 (\chi_1^1, \sigma_n^z \sigma_p^z \chi_1^1).$$

Since χ_1^1 is a characteristic function of both σ_n^z and σ_p^z with characteristic value 1, this expression becomes

$$Q = \frac{8\pi}{5} e \int_0^\infty (uw - w^2) r^2\, dr. \tag{3–66}$$

It is clearly possible for Q to be positive if the S and D wave functions have the same relative phase. The larger term is expected to be the interference term proportional to uw. Q does not depend directly either on the D-state probability or on $\langle r^2 \rangle$, but rather on the detailed behavior of the functions u and w. Since the shape of the radial functions depends intimately on the shape and quantitative properties of the potential, it is not possible to give a simple interpretation of the quadrupole moment such as was possible for the magnetic moment. Unfortunately, Q does not provide the information concerning the D-state probability which

would resolve the difficulty concerning the relativistic correction to the magnetic moment.

Since the mixture of S- and D-states is clearly capable of accounting for both the quadrupole moment and the magnetic moment, the assumption will henceforth be made that this is the proper description of the ground state of the deuteron. In order to analyze further the meaning of the quadrupole moment, it is necessary to have quantitative information concerning the potential responsible for mixing the S- and D-functions. Therefore we now turn to a determination of the form of potential required for this purpose and to a description of the quantitative features of the potential to be determined from the measured quadrupole moment.

3–6 The tensor coupling. The probable existence of a mixture of S- and D-functions in the ground state of the deuteron is contrary to our original assumption of a central neutron-proton potential. The mixture can only be produced by an interaction depending in some way on a direction in space. Only the spin vectors $\boldsymbol{\sigma}_p$ and $\boldsymbol{\sigma}_n$ are available to fix a direction, therefore the interaction is of the nature of a spin-orbit coupling. A coupling U of rather special form is required to satisfy the condition that an eigenfunction of the Hamiltonian be a mixture of S- and D-terms. The matrix of the Hamiltonian is not in diagonal form in a representation specifying the orbital angular momentum; some of the off-diagonal elements, in particular that between the ${}^3S_1$ and ${}^3D_1$ states, are different from zero. Terms in the Hamiltonian other than U make no contribution to the nondiagonal matrix elements, so U must have the property that the angular integral $\int Y_0^0 U Y_2^m \, d\Omega$ does not vanish. Since Y_0^0 is independent of angle and Y_2^m is orthogonal to all spherical harmonics except Y_2^m, it follows that U must have the angular dependence of the spherical harmonic Y_2^m, or U must be a linear combination of the $T_{\alpha\beta}$ given by Eq. (3–57).

The spin-orbit coupling must depend on the vectors $\boldsymbol{\sigma}_n$ and $\boldsymbol{\sigma}_p$ in such a way that U is invariant under rotations of the coordinate axes. The arguments presented in the previous section then lead to the conclusion that U must be proportional to S_{np}, or

$$U = V_t(r) S_{np}, \tag{3–67}$$

where $V_t(r)$ gives the dependence on the neutron-proton distance. Because it is a combination of the $T_{\alpha\beta}$, this potential is usually called the *tensor interaction.*

The form of the tensor interaction is familiar in classical physics as the coupling between two dipoles if $\boldsymbol{\sigma}_n$ and $\boldsymbol{\sigma}_p$ are interpreted as the dipole moment vectors. This does not imply that the tensor interaction arises through the coupling of the intrinsic magnetic moments of the neutron and proton. For we shall find that the strength of the coupling is far

greater than could ever be accounted for by electromagnetic effects. The electromagnetic term is overwhelmed by nuclear forces.

The introduction of the tensor potential into the Schroedinger equation greatly complicates the problem of solution. To set up the calculation, let us assume that in addition to the tensor interaction there is a central interaction $V_c(r)$, and write

$$\begin{aligned} V_c(r) &= -\frac{\hbar^2}{M} W_c(r), \\ V_t(r) &= -\frac{\hbar^2\gamma}{M} W_t(r), \end{aligned} \tag{3-68}$$

where the functions W_c and W_t have the same strength, and γ is the ratio of the strength of the tensor term to that of the central term. Their ranges will be denoted by α_c and α_t. The Schroedinger equation takes the form

$$\nabla^2\psi + [W_c(r) + \gamma W_t(r)S_{np}]\psi = k_g^2\psi.$$

Now if $\mathbf{L}$ is the orbital angular momentum operator in units of $\hbar$, the Laplacian can be written as

$$\nabla^2\psi = \frac{1}{r}\frac{d^2}{dr^2}(r\psi) - \frac{(\mathbf{L})^2}{r^2}\psi.$$

Since $(\mathbf{L})^2$ has the characteristic values $l(l+1)$, it multiplies an S-function by zero, and when it operates on the D-function, it simply multiplies it by 6. Therefore, from the form, Eq. (3-63), of the wave function, we find

$$\nabla^2\psi_1^m = \frac{1}{r}(u'' + w''S_{np})\chi_1^m - \frac{6}{r^3}wS_{np}\chi_1^m.$$

Furthermore,

$$\begin{aligned} S_{np}\psi_1^m &= \frac{1}{r}(uS_{np} + wS_{np}^2)\chi_1^m \\ &= \frac{1}{r}[uS_{np} + w(8 - 2S_{np})]\chi_1^m, \end{aligned}$$

as can be seen from the derivation of Eq. (3-60). When these results are combined, the Schroedinger equation becomes

$$\begin{aligned} &[u'' + W_c u + 8\gamma W_t w - k_g^2 u]\chi_1^m \\ &\quad + \left[w'' - \frac{6}{r^2}w + W_c w + \gamma W_t u - 2\gamma W_t w - k_g^2 w\right] S_{np}\chi_1^m = 0. \end{aligned}$$

But the functions χ_1^m and $S_{np}\chi_1^m$, being S- and D-functions, respectively, are linearly independent, and their coefficients must vanish:

$$u'' + [W_c(r) - k_g^2]u + 8\gamma W_t(r)w = 0, \tag{3-69}$$

$$w'' + \left[W_c(r) - 2\gamma W_t(r) - \frac{6}{r^2} - k_g^2\right] w + \gamma W_t(r)u = 0. \tag{3-70}$$

These coupled radial equations determine the S radial function u, and the D radial function w. The coupling between the equations, i.e., the fact that the differential equation for u depends on w and conversely, has the consequence that an analytic solution cannot be given even for a square-well potential. All solutions that have been obtained involve the use of some numerical procedure.

In analogy to the determination of a range-depth relationship for a central potential, the radial equations now provide a relationship between the four parameters α_c, α_t, γ, and the common depth of the potentials W_c and W_t. Given the corresponding solutions u and w, the quadrupole moment Q can be calculated from Eq. (3-66). The requirement that Q have the observed value then places another condition on the four parameters. These two conditions are the only relationships between the parameters provided by the ground state data. Unfortunately, use of the magnetic moment is excluded by the uncertainty in the relativistic correction.

Numerical solution of Eqs. (3-69) and (3-70) for square-well potentials with $\alpha_c = \alpha_t$ has been given by Rarita and Schwinger.* Probably the most useful numerical results are those obtained for Yukawa potentials by Feshbach and Schwinger,† who have applied a generalization of the Thomas variational-iterational method outlined in Section 3-3. Their table of the values of depth and of Q as functions of the three parameters α_c, α_t, and γ is reproduced in Appendix 2. By interpolating to the best recent value of Q, one of the three parameters can be eliminated. We are then left with two unknowns which are to be determined from data other than those provided by the ground state.

In every numerical treatment of the problem it has been assumed that the tensor potential has the same shape as the central. There is no *a priori* justification for the assumption, except its simplicity. As a matter of fact, the meson theories indicate a difference in shape between the potentials. However, it is very difficult, on the basis of available experimental data, to distinguish a difference in shape from a difference in range, so the four-parameter potential is adequate for most purposes.

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* *Phys. Rev.* **59**, 436 (1941).

† *Phys. Rev.* **84**, 194 (1951).

CHAPTER 4

EXCITED STATES OF THE DEUTERON

The ground state of the deuteron provides, as we have seen, only limited information concerning the n-p interaction. Little can be said concerning the functional form, otherwise known as the *shape*, of either V_c or V_t, and if we make an assumption about the shape, only two relationships are provided between the four parameters involved. A promising source of further information would be the excited states of the deuteron, either the bound excited states if any exist, or the states in the continuum. The properties of any bound excited states would presumably help to fix the parameters in the potential, but it turns out that there is no such state. The nearest approximation to one is the singlet S-state, which happens to be slightly unstable against dissociation. The fact that there are no other bound states might in itself provide further information concerning the parameters in the potential, a possibility which is investigated in the next section.

States of positive energy are always available for study. The cross sections for neutron-proton scattering, neutron-proton capture, and the photodisintegration of the deuteron comprise the information available concerning these continuum states, but only the first of these will be subjected to analysis in this chapter. The other two cross sections will be considered in Sections 6–4 and 6–5.

4–1 Triplet states. A discussion of bound excited states should be based on the tensor interaction, but that would require an elaborate numerical treatment. Determination of the higher eigenvalues of the energy for a central square-well potential is quite simple and should provide a qualitative indication of the properties of the more complex interactions. Returning, then, to the square well with the range-depth relationship fixed by the ground state binding energy, let us suppose that there is a solution of the Schroedinger equation, Eq. (3–7), corresponding to a bound excited state of energy

$$E_e = -\frac{\hbar^2}{M} k_e^2. \tag{4–1}$$

The state in question would be a triplet, since the potential is prescribed by the triplet ground state.

Consider first the possible existence of an excited S-state. The treatment is then identical to that given for the ground state, and the excitation

energy is the next root above k_g of Eq. (3–22). Since this root has been denoted by k_e,

$$\kappa_e \cot \kappa_e \alpha = -k_e,$$

where

$$\kappa_e^2 = K^2 - k_e^2.$$

According to Eq. (3-27), $\kappa_g \alpha$ lies in the second quadrant; therefore $\kappa_e \alpha$ lies in the fourth quadrant:

$$\frac{3\pi}{2} \leq \kappa_e \alpha \leq 2\pi.$$

Now the condition $\kappa_g \alpha \leq \pi$ is equivalent to

$$K^2\alpha^2 - \pi^2 \leq k_g^2 \alpha^2,$$

while $\kappa_e \alpha \geq 3\pi/2$ leads to

$$K^2\alpha^2 - \pi^2 \geq \tfrac{5}{1}\pi^2 + k_e^2\alpha^2.$$

Comparison of these inequalities results in

$$k_g^2 \alpha^2 \geq \tfrac{5}{4}\pi^2$$

or

$$k_g \alpha > \pi.$$

From the value for k_g given by Eq. (3-1), it follows that

$$\alpha > \frac{3\pi}{2} R_0.$$

This exceeds by a factor of seven the upper limit on the range imposed by Eq. (2-3). Therefore, there should be no bound excited $^3S_1$ state.

If we turn now to states of higher orbital angular momentum, consideration can be limited to the lowest P-state, in view of the general theorem* that, for ordinary central potentials, the lowest states of successively higher orbital angular momenta have successively higher energies. A state of orbital angular momentum l has the radial equation

$$\begin{aligned} u'' + \left[\kappa_e^2 - \frac{l(l+1)}{r^2}\right] u = 0, \quad r < \alpha, \\ u'' - \left[k_e^2 + \frac{l(l+1)}{r^2}\right] u = 0, \quad r > \alpha, \end{aligned} \tag{4-2}$$

in place of Eq. (3–15). The boundary conditions, Eqs. (3–16) to (3–19), must still be satisfied. For $l = 1$ the solution is

* See Appendix 1.

$$u = A_e\left(\frac{\sin \kappa_e r}{\kappa_e r} - \cos \kappa_e r\right), \quad r < \alpha,$$
$$u = B_e\left(\frac{e^{-k_e r}}{k_e r} + e^{-k_e r}\right), \quad r > \alpha. \tag{4-3}$$

The continuity condition, Eq. (3–21), can be manipulated into the form

$$\kappa_e \alpha \cot \kappa_e \alpha = 1 + \left(\frac{\kappa_e}{k_e}\right)^2 (1 + k_e \alpha).$$

Now the function $x \cot x$ is less than 1 for all values of x in the first quadrant except $x = 0$. Therefore, the value of $\kappa_e \alpha$ cannot lie in the first quadrant. Because the cotangent is positive, the next choice is the third quadrant, so we have

$$\pi \leq \kappa_e \alpha \leq \frac{3\pi}{2}.$$

It follows that, if a bound P-level exists,

$$K^2\alpha^2 \geq \pi^2 + k_e^2\alpha^2$$

or

$$K^2\alpha^2 > \pi^2.$$

Then from $\kappa_g \alpha \leq \pi$, we have

$$K^2\alpha^2 - k_g^2\alpha^2 \leq \pi^2 < K^2\alpha^2$$

or

$$\frac{K^2}{k_g^2} - 1 \leq \frac{\pi^2}{k_g^2\alpha^2} < \frac{K^2}{k_g^2}.$$

But in Section 2–2 we found that the binding energy must be small compared with the potential energy, or $(k_g^2/K^2) \ll 1$. Therefore, the excited P-state exists only if

$$\frac{\pi^2}{k_g^2\alpha^2} \approx \frac{K^2}{k_g^2}$$

or

$$K\alpha \approx \pi.$$

But from Fig. 3–1 and the arguments leading thereto, we see that it is not possible, with a reasonable range, to fit the ground state data with such a value of $K\alpha$; $K\alpha = \pi$ corresponds to $\alpha = \infty$. Again we are led into a contradiction by the assumption of the existence of an excited state. Apparently the ground state condition is sufficient to guarantee the absence of excited triplet states if cognizance is taken of the other qualitative restrictions that have been placed on the potential. Therefore nothing new is learned from the nonexistence of excited states, although it does provide some reassurance concerning the consistency of the conditions we have imposed on the interaction.

4–2 The Majorana interaction. The analysis of excited states of the deuteron in Section 4–1 makes use of an ordinary potential, one that is independent of the state of the system. But quite the opposite property has been attributed to the nuclear interactions in order to account for saturation; the interaction between nucleons is supposed to depend on the orbital states which they occupy. In the deuteron ground state the two nucleons may be said to be in the same state, but that is not necessarily the case in the excited states. Therefore the form of the potential should differ from one state to another, in contradiction to our earlier assumption. Certainly the nature of this dependence is not such as to change the principal conclusion of Section 4–1, since the overlooked effect is one that tends to increase the energies of bound excited states. Nevertheless, some attention must be given to the quantitative formulation of an interaction with the desired property.

If two particles are in the same orbital state, interchanging their orbits will not change the wave function of the system. On the other hand, the wave function will undergo a change if the orbital states of the two particles differ. This suggests the introduction of the *space exchange operator*, P_{jk}, which interchanges the coordinates of the jth and kth particles of a nucleus. If the wave function of the nucleus is $\psi(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_A)$, then

$$P_{12}\psi(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_A) = \psi(\mathbf{r}_2,\mathbf{r}_1,\cdots,\mathbf{r}_A), \tag{4–4}$$

etc. Particles 1 and 2 are in the same orbital state if

$$P_{12}\psi = \psi.$$

To obtain the corresponding result when these two particles are in different states, it is convenient to analyze ψ into its symmetric and antisymmetric parts:

$$\psi = \psi_s + \psi_a,$$

where

$$\psi_s = \tfrac{1}{2}[\psi + P_{12}\psi]$$

and

$$\psi_a = \tfrac{1}{2}[\psi - P_{12}\psi],$$

so that

$$P_{12}\psi_s = \psi_s$$

and

$$P_{12}\psi_a = -\psi_a.$$

Then

$$P_{12}\psi = \psi_s - \psi_a,$$

and the presence of ψ_a expresses the fact that the particles are not in the same state. A similar treatment of the wave function is obviously possible for any other pair of particles.

The exchange operator provides a very convenient device for formulation of an interaction having the characteristic suggested by saturation. Define a "potential" V by*

$$V = \tfrac{1}{2} \sum_{j,k}{}' J(\mathbf{r}_j - \mathbf{r}_k) P_{jk}, \tag{4-5}$$

where J is a simple attractive potential function. Although a potential of this form has no meaning in classical mechanics, it causes no difficulty in quantum mechanics, since the interaction term in the Schroedinger equation contains the product $V\psi$. The exchange operator acts directly on the wave function and in such a way as to provide the desired property. For if two nucleons are in the same orbital state, the operator leaves the wave function unchanged, and the potential acting between the particles is simply given by J. But for two nucleons in different states, the antisymmetric part of the wave function realizes a potential of the opposite sign. Thus the full attraction produced by J is felt only by nucleons in the same state; those in different states feel in part an attraction and in part a repulsion, the net result being less attraction.

The space exchange potential is often referred to as the *Majorana* potential. An operator exchanging both space and spin variables is called a *Heisenberg* potential, and the product of the two, which exchanges only spin variables, is the *Bartlett* potential. To complete the picture and facilitate understanding of the older literature, we note that an ordinary potential acting between two nucleons is sometimes referred to as a *Wigner* potential. Of these the Majorana potential offers the best chance to obtain saturation with the alpha-particle as the saturated unit.†

It has been remarked that there is no classical analog to the Majorana potential. In fact, the motion of a pair of nucleons may be quite irrational even under the limiting conditions for which the correspondence principle should obtain. For example, consider the relationship between velocity and momentum operators. If T is the kinetic energy and H the Hamiltonian,

$$H = T + V,$$

the velocity is given by

$$\dot{\mathbf{r}}_k = \frac{i}{\hbar}[H,\mathbf{r}_k], \tag{4-6}$$

where the square bracket denotes the commutator. Since T has the usual form,

$$\frac{i}{\hbar}[T,\mathbf{r}_k] = \frac{\mathbf{p}_k}{M}.$$

* The prime on the summation indicates that the terms with $j = k$ are to be omitted.

† A detailed discussion of the contributions of the various exchange potentials to saturation is presented in Section 8–8. See also Section 8–4.

But now the exchange potential does not commute with the coordinate $\mathbf{r}_k$; rather, it satisfies the relationship

$$P_{jk}\mathbf{r}_k = \mathbf{r}_j P_{jk}.$$

Therefore,

$$\frac{i}{\hbar}[V,\mathbf{r}_k] = \frac{i}{\hbar}\sum_j \mathbf{r}_{jk} J(\mathbf{r}_{jk}) P_{jk},$$

and the velocity operator becomes

$$\dot{\mathbf{r}}_k = \frac{\mathbf{p}_k}{M} + \frac{i}{\hbar}\sum_j \mathbf{r}_{jk} J(\mathbf{r}_{jk}) P_{jk}. \tag{4-7}$$

This suggests that the classical analog of the exchange potential is a momentum-dependent interaction, with the property that the classical equation

$$\dot{\mathbf{r}}_k = \frac{\partial H}{\partial \mathbf{p}_k}$$

leads to a velocity-momentum relationship of the form Eq. (4-7). A similar analysis of the other Hamiltonian equation,

$$\dot{\mathbf{p}}_k = -\frac{\partial H}{\partial \mathbf{r}_k},$$

leads to the same conclusion. Explicit formulation of the exchange potential as a velocity-dependent potential has been given by Wheeler;* it requires the introduction of a transcendental function of the relative momenta as a factor in the interaction.

A somewhat deeper insight into the physical meaning of the exchange potential can be obtained by considering the electric current and charge density. The interpretation of ψ is such that the charge density is defined as

$$\rho(\mathbf{r}) = \sum_{k=1}^{A} e_k \int |\psi|^2_{r_k = r} d^{3(A-1)}r, \tag{4-8}$$

where e_k is the charge on particle k, and the integration in each term includes the $3(A-1)$ dimensional configuration space of all particles other than the kth. A sum over all spins is also implied.

The current density, $\mathbf{S}$, must be defined in such a way that the equation of continuity,

$$\operatorname{div}\mathbf{S} + \frac{\partial \rho}{\partial t} = 0,$$

* J. A. Wheeler, *Phys. Rev.* **50**, 643 (1936).

is satisfied. For ordinary forces the current is given by

$$\mathbf{S}_0(\mathbf{r}) = \frac{\hbar}{2Mi}\sum_k e_k \int \{\psi^* \operatorname{grad}_k \psi - \psi \operatorname{grad}_k \psi^*\}_{r_k = r} d^{3(A-1)}r, \quad (4\text{–}9)$$

which satisfies the equation of continuity by virtue of the time-dependent Schroedinger equation. However, if the potential in the Schroedinger equation involves a space exchange operator, $\mathbf{S}_0$ does not satisfy this equation. But a satisfactory current $\mathbf{S}$ can be obtained by merely adding a quantity $\mathbf{S}_x$ to $\mathbf{S}_0$:

$$\mathbf{S} = \mathbf{S}_0 + \mathbf{S}_x.$$

$\mathbf{S}_x$ will be referred to as the *space exchange current.*

The expression for $\mathbf{S}_x$ is not unique, since any vector operator expressible as the curl of a vector field can be added to the current without modifying the equation of continuity. However, there is a rather straightforward prescription* for defining $\mathbf{S}_x$ which has the result that $\mathbf{S}_x$ is the expectation value

$$\mathbf{S}_x(\mathbf{r}) = -\frac{ie}{\hbar}\left\langle \sum_{\nu,\pi} \mathbf{r}_{\pi\nu}\left[\int_0^1 d\alpha\, \delta(\mathbf{r} - \mathbf{r}_\pi - \alpha\mathbf{r}_{\pi\nu})\right] J(\mathbf{r}_{\pi\nu}) P_{\pi\nu}\right\rangle \quad (4\text{–}10)$$

in the state ψ. The indices π and ν are labels for protons and neutrons, respectively, and $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac δ-function. Note that contributions to the current come only from points lying on the line connecting a neutron and a proton; the current flows on filaments connecting unlike particles. A strongly suggested interpretation is that the electric charge switches back and forth between a neutron-proton pair, thereby producing a current along the line between them.

The same interpretation of the exchange potential will be suggested by the analysis of high-energy n-p scattering. If the kinetic energy of the neutron is much greater than the interaction potential, it would be expected to pass through the proton virtually undeflected. However, an exchange potential leads to strong back scattering of the neutron in the center of mass system; the proton moves off in the forward direction. An exchange of charge leads to a simple explanation of the result: the forward going proton is simply the original neutron which has picked up the charge of the scattering center.

A proton capable of losing its charge to a neutron certainly would not seem to be a structureless particle, so the exchange potential may be interpreted as a simple device for handling effects due to nucleon structure. The irrational mechanical behavior of the nucleons produced by the potential is merely the result of treating a structured system as a single particle. The saturation of nuclear forces is then a phenomenon associated with

* R. G. Sachs, *Phys. Rev.* **74,** 433 (1948). This particular form is due to E. N. Adams, II, *Phys. Rev.* **81,** 1 (1951).

nucleon structure, and it is indeed fortunate that there is a possibility of handling the effect by means of Schroedinger mechanics with the help of the exchange potential.

Returning now to the original objective of this chapter, let us consider the influence of the Majorana potential on our analysis of the excited states of the deuteron. The Schroedinger equation takes the form

$$\nabla^2\psi + WP\psi - k^2\psi = 0, \tag{4-11}$$

where $W(\mathbf{r}) = -MJ(\mathbf{r})/\hbar^2$, and P denotes the interchange of neutron and proton coordinates. Since $\mathbf{r} = \mathbf{r}_p - \mathbf{r}_n$,

$$P\psi(\mathbf{r}) = \psi(-\mathbf{r}),$$

and the exchange operation is equivalent to inversion through the origin. Therefore, P can be replaced by 1 for the even states, by -1 for the odd states. In particular, the assignment of even parity to the ground state indicates that the potential is repulsive for all odd states. Since the functions of odd orbital angular momenta, P, F, etc., have odd parity, they feel only a repulsion, so no bound states of this kind would be possible. The argument of Section 4-1 against bound P-states is therefore strengthened, while the discussion of even states is unchanged.

4–3 Singlet states. In Section 2–5 we came to the conclusion that the nucleon-nucleon interaction must be spin dependent, and the evidence for a tensor interaction certainly lends support to this notion. The spin dependence bespeaks a separate analysis for the bound singlet states of the deuteron. The ground state, being a triplet, does not establish any of the parameters in the potential appropriate to singlet states. But we do know that singlet states must lie above the lowest triplet and that the difference between singlet and triplet potentials is not very large. Therefore, it might be expected that the lowest $^1S_0$ state is bound, in which case it would be the only bound excited state of the system.

Consider first the influence of the tensor interaction on a singlet state, χ_0. The treatment is facilitated by use of the relationship

$$(\boldsymbol{\sigma}_n + \boldsymbol{\sigma}_p)\chi_0 = 0,$$

which will now be established. According to the general theorem that the spin functions form a complete set, we can write

$$(\boldsymbol{\sigma}_n + \boldsymbol{\sigma}_p)\chi_0 = \mathbf{c}_0\chi_0 + \sum_m \mathbf{c}_{1m}\chi_1^m.$$

The operator $\boldsymbol{\sigma}_n + \boldsymbol{\sigma}_p$ is symmetric for interchange of neutron and proton spin coordinates, while the singlet function χ_0 is antisymmetric, and the triplet functions χ_1^m are symmetric. Therefore, all the coefficients $\mathbf{c}_{1m}$ must vanish, and

$$(\boldsymbol{\sigma}_n + \boldsymbol{\sigma}_p)\chi_0 = \mathbf{c}_0\chi_0.$$

Taking the scalar product with $\sigma_n + \sigma_p$, we find

$$(\sigma_n + \sigma_p)^2 \chi_0 = c_0^2 \chi_0.$$

But $(\sigma_n + \sigma_p)^2$ is proportional to the square of the total spin angular momentum, which is zero in the singlet state. Therefore,

$$c_0 = 0,$$

which establishes the desired result. The relationship can be rewritten in the form

$$\sigma_n \chi_0 = -\sigma_p \chi_0. \tag{4-12}$$

In the singlet state the tensor term in the Schroedinger equation is proportional to

$$S_{np}\chi_0 = \left[\frac{3(\sigma_p \cdot \mathbf{r})(\sigma_n \cdot \mathbf{r})}{r^2} - (\sigma_p \cdot \sigma_n)\right] \chi_0.$$

Now we have seen that in the singlet state $(\sigma_n \cdot \sigma_p) = -3$. This, combined with Eq. (4–12), yields the result

$$S_{np}\chi_0 = \left[\frac{-3(\sigma_p \cdot \mathbf{r})^2}{r^2} + 3\right] \chi_0.$$

But the square of any component of the spin operator is unity:

$$\frac{(\sigma_p \cdot \mathbf{r})^2}{r^2} = 1.$$

Therefore,

$$S_{np}\chi_0 = 0;$$

the tensor interaction vanishes in the singlet state.

Introduction of the tensor interaction then provides a difference between singlet and triplet states, and this difference is in the right direction because the conditions on the ground state of the deuteron are such that the tensor interaction is an attractive potential. The singlet state is more weakly bound than the triplet state. However, it is by no means certain that the only difference between the singlet and triplet interactions is due to the tensor force. Direct information concerning a singlet state is required in order to establish the nature of the singlet potential. To allow for all exigencies we may put the central potential in the spin-dependent form

$$V_c = \frac{3 + (\sigma_n \cdot \sigma_p)}{4} {}^3V + \frac{1 - (\sigma_n \cdot \sigma_p)}{4} {}^1V. \tag{4-13}$$

Since in a triplet state $(\sigma_n \cdot \sigma_p) = 1$, the triplet central interaction is simply $V_c = {}^3V$, while in the singlet state, $(\sigma_n \cdot \sigma_p) = -3$ leads to $V_c = {}^1V$. Thus Eq. (4–13) provides a central interaction which can have any required distinct forms in the triplet and singlet states.

4–4 Low-energy neutron-proton scattering. The scattering of slow neutrons (up to 5 Mev) by protons provides a most fruitful source of information concerning the n-p interaction. The experiment is performed by bombarding a target of hydrogen-containing matter with a beam of neutrons. Although the target protons are in a state of thermal agitation for which they have no well-defined velocity, this causes no difficulty for neutron energies considerably greater than kT because the protons are, for all practical purposes, at rest. The corrections due to proton motion for very slow neutrons will be established in Sections 5–1 and 5–2.

Calculations are always carried out in the center of mass coordinate system, while observations are made in a frame of reference attached to the proton, the laboratory system. The connections between these two coordinate systems can easily be obtained by making use of the laws of conservation of energy and of momentum. If, for simplicity, we take the neutron and proton masses to be identical, then the relationship between the energy E_L of the neutron in the laboratory system and the relative energy E in the center of mass system is simply

$$E = \tfrac{1}{2}E_L. \tag{4–14}$$

Furthermore, if θ_L is the angle of scattering of the neutron from its original direction measured in the laboratory system, then the corresponding angle in the center of mass system is

$$\theta = 2\theta_L. \tag{4–15}$$

Finally, since the proton direction is just opposite to the neutron direction in the center of mass system, the angle at which the proton is scattered in the laboratory system is

$$\theta'_L = \frac{\pi}{2} - \theta_L. \tag{4–16}$$

The neutron and proton make an angle of 90° with respect to each ther.

In preparation for a calculation of the scattering cross section, some of the salient facts concerning the treatment of scattering problems are reviewed here, primarily to establish a convenient notation. The straightforward analysis of a scattering problem involving a short-range potential makes use of the method of partial waves.* If the energy in the center of mass system is

$$E = \frac{\hbar^2 k^2}{M}, \tag{4–17}$$

* See Mott and Massey, *The Theory of Atomic Collisions*, 2nd edition, Oxford (1949), Ch. 2.

and the interaction is a central Majorana potential

$$V(r) = -\frac{\hbar^2}{M} W(r)P,$$

then the radial equation takes the form

$$u_l'' + \left[k^2 + (-1)^l W(r) - \frac{l(l+1)}{r^2}\right] u_l = 0, \tag{4-18}$$

where

$$f_l(r) = u_l(r)/r$$

is the radial function of a state with orbital angular momentum l. For large r, the asymptotic behavior of the function u_l is

$$u_l \xrightarrow[r \to \infty]{} \frac{1}{k} \sin\left(kr + \delta_l - \frac{l\pi}{2}\right). \tag{4-19}$$

The phase shift δ_l is determined by the differential equation (4-18), and the scattering cross section can be expressed in terms of all the phases.

The cross section is directly related to the asymptotic behavior of the wave function, which is given by

$$\psi \xrightarrow[r \to \infty]{} e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{e^{ikr}}{r} f(\theta), \tag{4-20}$$

if the vector $\mathbf{k}$ has the magnitude k and the direction of incidence of the beam, and if θ is the polar angle measured with respect to an axis in the direction of $\mathbf{k}$. Then the differential cross section for scattering by angle θ into solid angle $d\Omega$ is

$$\sigma(\theta)\, d\Omega = |f(\theta)|^2\, d\Omega. \tag{4-21}$$

$f(\theta)$ may be expressed in terms of the phase shifts of the partial waves by

$$f(\theta) = -\frac{1}{2ik} \Sigma_l (2l+1)(e^{2i\delta_l} - 1) P_l(\cos\theta), \tag{4-22}$$

when the Legendre polynomials $P_l(x)$ are normalized so that

$$P_l(1) = 1,$$

$$P_l(-1) = (-1)^l.$$

The advantage of the method of partial waves at low energies is that for small velocities the angular momentum is small if the particle is within the range of the potential. Therefore only the waves of low angular momentum are scattered for low energy, and the lower the energy the smaller the number of partial waves that must be considered. Thus at very low

energies only the phase shift δ_0 will be appreciably different from zero, and the essential contribution to the scattering will be provided by the S-wave. Then

$$f(\theta) \approx f_0(k) = -\frac{1}{k} e^{i\delta_0} \sin \delta_0 \tag{4–23}$$

provides an adequate estimate of the cross section, namely,

$$\sigma_0(\theta) = |f_0|^2 = \frac{1}{k^2} \sin^2 \delta_0.$$

The angular distribution is isotropic, since we are dealing here with an S-wave. The total cross section is therefore given by

$$\sigma_0 = \int \sigma_0(\theta)\, d\Omega = \frac{4\pi \sin^2 \delta_0}{k^2}. \tag{4–24}$$

The calculation of the cross section for very low energy requires only a determination of the one phase shift δ_0.

To prepare the ground for an estimate of the correction due to P-wave scattering, let us assume that δ_1 is small, but not zero. Then

$$f(\theta) \approx f_0 - \frac{3\delta_1}{k} \cos \theta$$

and the cross section is

$$\sigma(\theta) = |f_0|^2 - 2 \operatorname{Re} (f_0) \frac{3\delta_1}{k} \cos \theta, \tag{4–25}$$

where Re denotes the "real part of." Just the contribution of the interference between the S- and P-waves has been included here. This term might be observed in the angular distribution of the scattering, but it does not contribute to the total cross section because of the orthogonality of the S- and P-functions. Integration over angles to obtain the total cross section leads to a second-order correction if second-order contributions are added to Eq. (4–25):

$$\sigma = \sigma_0 + \frac{12\pi}{k^2} \delta_1^2.$$

When the interference effect described by Eq. (4–25) is small, this effect of the P-wave will usually be very much smaller.

For the most part, we shall consider neutrons of quite low energy, therefore the limit $k \to 0$ is of particular interest. In this limit the cross section is determined by $f_0(0)$, a quantity called *the scattering length* and denoted by the letter a. The zero energy total cross section is then just $4\pi a^2$. A geometric interpretation of the scattering length can be given if

we note that for very small k the differential equation, Eq. (4–18), for $l = 0$ reduces to

$$u_0'' \approx 0,$$

outside the range of forces. The asymptotic solution is a linear function of r whose constants can be obtained by writing Eq. (4–20) in the form

$$r\psi = re^{i\mathbf{k}\cdot\mathbf{r}} - f_0 e^{ikr}, \quad r \gg \alpha,$$

and setting $k = 0$. The result is

$$u_0 = r - a, \quad r \gg \alpha. \tag{4–26}$$

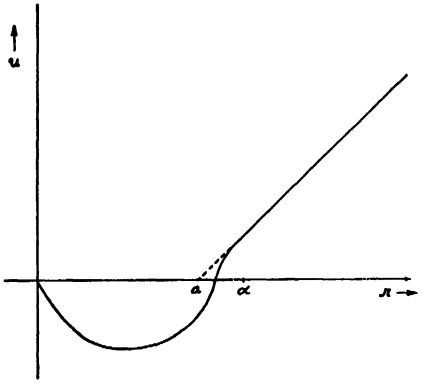

FIG. 4–1. The radial S-function for zero energy. The range of the potential is α, the scattering length is a.

The appearance of the function u_0 is displayed in Fig. 4–1. Extrapolation of the asymptotic straight line, Eq. (4–26), to the axis shows that a is just the intercept with the r-axis. That a must be real follows from the fact that only real quantities appear in the differential equation for u_0, Eq. (4–18).

The existence of a finite scattering length a implies that the phase shift must be proportional to k for small energies, since

$$a = \lim_{k \to 0} \left[-\frac{1}{k} e^{i\delta_0} \sin \delta_0 \right]. \tag{4–27}$$

If a is to be finite, δ_0 must have the property* $\sin \delta_0 \sim k$ or $\delta_0 \sim k$. The constant of proportionality can be determined from Eq. (4–27); since $e^{i\delta_0} \to 1$,

$$\delta_0 \underset{k \to 0}{\sim} -ak. \tag{4–28}$$

The scattering length has two possible signs corresponding to the signs of δ_0. It will be found that the sign is of considerable interest and that it can be determined by means of an interference experiment.

The dependence of the sign of the scattering length on the depth of the potential is illustrated in Fig. 4–2, which shows the wave function u as a function of distance for a sequence of potential depths starting from zero (labeled 0) and increasing with increasing label. The influence of the potential is to cause the function to curve toward the axis within the range α;

* δ_0 could be placed in the second or third quadrant, $\delta_0 = \pi - ak$, but a choice is always possible, so we elect the convention that δ_0 have the smallest absolute value consistent with the radial equation.

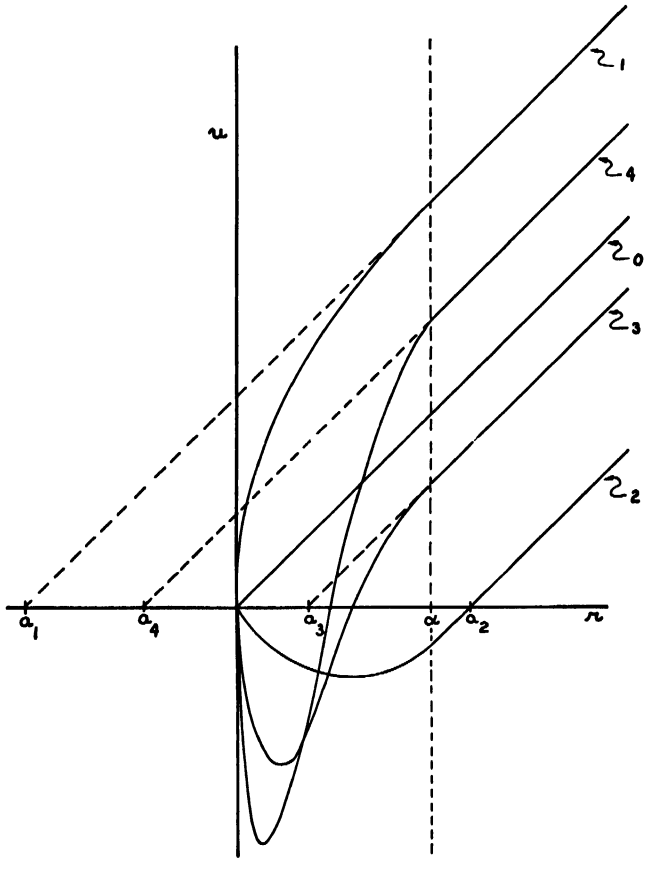

FIG. 4–2. Illustration of the dependence of scattering length on depth of the potential. The radial S-wave functions, u, are labeled in order of increasing depth, with 0 as zero depth. As the depth increases, the intercept of the extrapolated curve (broken line) moves from right to left and therefore as it moves from a_1 to a_2, a goes to $-\infty$, then jumps to $+\infty$.

the deeper the potential, the greater the curvature. In the absence of an interaction the function is a straight line passing through the origin. A very weak potential results in an intercept a which is negative (curve 1). As the potential is deepened, the negative a increases in magnitude until it reaches $-\infty$. Then a slight increase in depth introduces a node in the wave function near $+\infty$ which is identical with the intercept a (curve 2). Further deepening of the potential causes the node to move in (curve 3) and, at some point, the sign of the scattering length again changes (curve 4). Another node appears with increasing curvature, at which time the sign shifts even again. In this way it can be seen that the relative sign of the scattering length for two potentials is closely correlated with their relative depths. The case which will be of particular interest to us is comparison

of the ^3S and ^1S potentials for the deuteron. The scattering length will turn out to be positive for ^3S, but the ^1S potential is sufficiently shallow relative to the ^3S to lead to a scattering length of the opposite sign.

4–5 Low energy n-p scattering. Square well. Neutron-proton scattering at low energies is insensitive to the shape of the potential, since a neutron of wavelength long compared with the range of forces cannot "see" details within that range. Therefore the scattering cross section is related to certain average properties of the interaction; they may then be treated as the quantities measured by a low-energy scattering experiment. Although the simplest analytical procedure is to develop the theory directly in terms of these quantities, some of the physical features of the problem are thereby obscured, so a direct calculation of the scattering by a square well will be performed first.

The radial equation, Eq. (4 18), for the S-wave produced by a square well potential W of depth K^2 is

$$\begin{aligned} u'' + \kappa^2 u &= 0, \quad r < \alpha, \\ u'' + k^2 u &= 0, \quad r > \alpha, \end{aligned} \tag{4–29}$$

where

$$\kappa^2 = K^2 + k^2, \tag{4–30}$$

k^2 being proportional to the energy, as indicated by Eq. (4–17). The wave function is subject to the boundary conditions, Eqs. (3–16), (3–18), and (3–19), but the behavior at infinity is determined by the asymptotic form, Eq. (4–19). The solution is

$$\begin{aligned} u &= A \sin \kappa r, \qquad r < \alpha, \\ u &= \frac{1}{k} \sin (kr + \delta_0), \quad r > \alpha, \end{aligned} \tag{4–31}$$

and the constant A is fixed by the continuity condition, Eq. (3–18). The condition that the logarithmic derivative be continuous at the boundary yields an equation for the phase δ_0,

$$\kappa \cot \kappa\alpha = k \cot (k\alpha + \delta_0). \tag{4–32}$$

There is some ambiguity in the choice of a solution for this equation, since any integral multiple of π can be added to δ_0; the choice, which is entirely a matter of convention, is made in such a way as to give the smallest absolute value for δ_0.

Now $\sin \delta_0$ can be written as

$$\sin [(k\alpha + \delta_0) - k\alpha] = \frac{\pm 1}{\sqrt{1 + \cot^2 (k\alpha + \delta_0)}} \{\cos k\alpha - \cot (k\alpha + \delta_0) \sin k\alpha\}.$$

The leading factor is just $\sin(k\alpha + \delta_0)$, which takes the positive sign when $(k\alpha + \delta_0)$ lies in either of the first two quadrants, the minus sign otherwise. From Eq. (4–32) it follows that

$$\sin \delta_0 = \frac{\pm \sin k\alpha}{\sqrt{k^2 + \kappa^2 \cot^2 \kappa\alpha}} (k \cot k\alpha - \kappa \cot \kappa\alpha). \tag{4–33}$$

The total S-scattering cross section is then, according to Eq. (4–24),

$$\sigma_0 = \frac{4\pi}{k^2} \frac{\sin^2 k\alpha}{k^2 + \kappa^2 \cot^2 \kappa\alpha} (k \cot k\alpha - \kappa \cot \kappa\alpha)^2. \tag{4–34}$$

Consider first the behavior of the cross section in the limit of very small energy. Although it can be obtained directly from Eq. (4–34) by setting $k = 0$, some very useful relationships are obtained by back-tracking and working directly with the scattering length, a. The close connection between a and the phase shift δ_0 is indicated by Eq. (4–28). Equation (4–32) can be simplified by use of this connection and the equation

$$\lim_{k \to 0} k \cot (k\alpha - ka) = \frac{1}{\alpha - a},$$

whence, for $k = 0$,

$$K \cot K\alpha = \frac{1}{\alpha - a} \tag{4–35}$$

or

$$a = \alpha \left(1 - \frac{\tan K\alpha}{K\alpha}\right). \tag{4–36}$$

The zero energy cross section is $4\pi a^2$, just the value obtained by setting $k = 0$ in Eq. (4–34). Note that the behavior of the scattering length as a function of the depth of the potential is in accord with the qualitative discussion at the end of Section 4–4.

We can now proceed to evaluate the scattering length and the cross section at zero energy for the triplet potential on the supposition that the range-depth relation is fixed by the binding energy of the deuteron. The resulting dependence of scattering length on range is illustrated in Fig. 4–3, which corresponds directly to the range-depth relationship of Fig. 3–1. Since $K\alpha$ and $\kappa_g\alpha$ differ only slightly for any reasonable value of the range, $K\alpha$ also lies in the second quadrant, with the consequence that a is positive. The wave function is similar to curve 2 or curve 3 of Fig. 4–2.

According to Fig. 4–3, the value of a is close to $2R_0$ for any acceptable choice of the range. This leads to the estimate

$$\sigma_0 = 4\pi a^2 \approx 4 \text{ barns}, \tag{4–37}$$

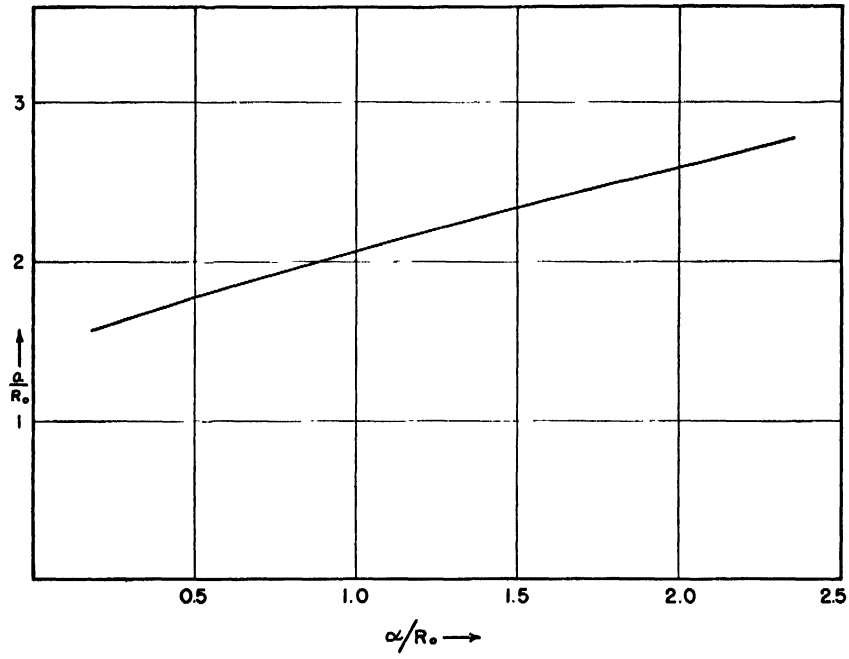

FIG. 4–3. Scattering length as a function of range for a square well.

one barn being 10^{-24}cm^2. The observed neutron-proton scattering cross section at very low energy is about 20 barns, and it seems that no manipulation of the triplet potential consistent with other data can possibly account for the observed scattering. It was first pointed out by Wigner that the large difference between the triplet scattering and the observed scattering offers a sensitive source of information concerning the singlet neutron-proton interaction. The total cross section is a composite of the triplet cross section σ_t and the singlet cross section σ_s, namely,

$$\sigma = \tfrac{3}{4}\sigma_t + \tfrac{1}{4}\sigma_s.$$

On the assumption that the singlet scattering accounts for the difference between triplet and observed scattering, the cross section is given by

$$\sigma_s = (80 - 3\sigma_t) \text{ barns.}$$

The estimate, Eq. (4–37), for σ_t yields

$$\sigma_s \approx 68 \text{ barns.} \tag{4–38}$$

A cross section of this magnitude suggests a resonance phenomenon such as would be expected if the $^1S_0$ state of the deuteron were very near to the energy at which the scattering is observed, namely, close to zero energy. It will be found that this is the case.

The estimate of σ_s provides a value of the singlet scattering length a_s, namely,

$$a_s = \pm\sqrt{\sigma_s/4\pi} \approx \pm 8R_0. \tag{4-39}$$

If the singlet potential is assumed to be a square well of depth K_s^2 and range α_s, Eq. (4-35) can be applied to the singlet scattering length with the result that $|K_s\alpha_s \cot K_s\alpha_s| \lesssim \frac{1}{8}$, since α_s is presumably somewhat less than R_0. Therefore $K_s\alpha_s$ must be very nearly equal to $\pi/2$:

$$K_s\alpha_s = \frac{\pi}{2} + \beta,$$

where β is a small quantity. Then

$$\tan K_s\alpha_s \approx -\frac{1}{\beta},$$

and if terms of order β^2 are neglected, Eq. (4-36) leads to

$$a_s \approx \left(1 + \frac{2}{\pi\beta}\right)\alpha_s$$

or

$$\beta = \frac{2}{\pi}\frac{\alpha_s}{a_s - \alpha_s},$$

whence

$$K_s\alpha_s \approx \frac{\pi}{2} + \frac{2}{\pi}\frac{\alpha_s}{a_s - \alpha_s}. \tag{4-40}$$

The estimate of a_s given by Eq. (4-39) yields

$$K_s\alpha_s \approx \frac{\pi}{2} \pm \frac{2}{\pi}\frac{\alpha_s}{8R_0 \mp \alpha_s}.$$

The controlling sign has been made explicit here; $K_s\alpha_s$ is greater than $\pi/2$ for positive scattering length, less than $\pi/2$ for negative scattering length. The change in sign of the scattering length corresponds to a passage from $-\infty$ to $+\infty$, as in the transition from curve 1 to curve 2 in Fig. 4-2. Thus a tremendous change in scattering length is associated with a small change in the singlet potential; a_s provides a very sensitive measure of the depth of the singlet interaction.

The physical meaning of the sign of a_s can best be brought out by supposing that the $^1S_0$ state of the deuteron is bound, with binding energy

$$E_s = -\frac{\hbar^2}{M}k_s^2.$$

Then k_s is determined by the analog of Eq. (3–22), namely,

$$\kappa_s \cot \kappa_s \alpha_s = -k_s,$$

where

$$\kappa_s^2 = K_s^2 - k_s^2.$$

The value

$$K_s \alpha_s = \frac{\pi}{2}$$

corresponds to zero binding energy, $k_s = 0$. The slightly larger value of $K_s\alpha_s$ associated with a positive scattering length leads to a finite binding energy that can easily be determined from Eq. (4–35),

$$k_s \approx -K_s \cot K_s \alpha_s = \frac{1}{a_s - \alpha_s}.$$

An estimate of the ratio k_s^2/k_g^2 of the singlet binding to triplet binding may be obtained by inserting $a_s \approx 8R_0$:

$$\frac{E_s}{E_g} \approx \frac{1}{(8k_g R_0 - \alpha_s k_g)^2}.$$

The values $k_g R_0 = \frac{2}{3}$, $\alpha_s k_g \approx \frac{1}{3}$ lead to

$$E_s/E_g \approx \tfrac{1}{25},$$

so the energy of the singlet state would be about 100 kev if it were stable.

A negative scattering length, on the other hand, implies $K_s\alpha_s < \pi/2$, with the result that the potential is too shallow to lead to a bound singlet state. But the behavior of the system can be characterized by a state with negative binding energy, the *virtual* state of the deuteron. It, too, must be close to zero energy. The fact that the singlet state lies close to zero energy, whether it be above or below, makes reasonable our interpretation of the large singlet scattering as a resonance phenomenon.

It is of some interest to remark that for the Yukawa potential, Eq. (3–3), zero binding energy corresponds to the range-depth relation*

$$K_s^2 \alpha_s^2 = 1.683. \tag{4–41}$$

A larger value leads to a bound singlet state, a smaller value to a virtual state.

The sign of the scattering length cannot be determined by a measurement of the zero energy cross section, which depends only on a_s^2. In principle, it could be discovered by a very precise measurement of the energy dependence of the cross section, but that does not turn out to be a practical

* Sachs and Goeppert-Mayer, *Phys. Rev.* **53**, 991 (1938). This result can easily be obtained by the Thomas iteration method, Section 3–3.

procedure. A more successful method depends on the interference between the waves scattered from the two protons in the H_2 molecule, since that depends directly on the sign of a_s. This method, which involves a separate determination of the orthohydrogen and parahydrogen scattering cross sections, was the first to lead to a determination of the sign. It will be discussed in Chapter 5. The result is a negative a_s, corresponding to a virtual singlet state.

Further information concerning the singlet potential is to be obtained by consideration of the energy dependence of the cross section. Since the singlet scattering is associated with a resonance, it should decrease rather rapidly with increasing energy, while the triplet cross section is expected to vary but little. This behavior can be established by inserting the appropriate singlet and triplet parameters in Eq. (4-34). However, a much simpler and more general procedure for determining the energy dependence is developed in Section 4–6, and a detailed discussion of the energy dependence of S-scattering will be deferred to that section.

Since only the S-wave is considered, it is necessary to obtain an estimate of the importance of P-scattering, which is expected to be small as long as

$$k\alpha \ll 1.$$

For definiteness, the condition

$$k\alpha \leq \tfrac{1}{3}$$

may be imposed. Then, from $k_g R_0 = \frac{2}{3}$,

$$\frac{E}{|E_g|} = \frac{k^2}{k_g^2} \leq \frac{1}{4}\left(\frac{R_0}{\alpha}\right)^2.$$

For $R_0/\alpha \approx 2$,

$$E \leq |E_g|.$$

This is the energy condition in the center of mass system. According to Eq. (4–14), the restriction on the laboratory (neutron) energy is

$$E_L \leq 2E_g = 4.5 \text{ Mev}.$$

The smallness of the P-scattering is still to be established. Equation (4–18) may be taken as the basis of discussion, with $W(r)$ a square well. The use of the Majorana potential does not have an appreciable influence* as long as we are interested only in the conditions under which P-scattering is negligible. The P-wave phase shift, δ_1, is determined by the radial equations

* However, a mixture of the Majorana and ordinary potentials of the form $(1 + P)$, the *Serber potential*, would lead to no P-scattering. The importance of this potential is brought out in Section 6–1.

$$u_p'' - \frac{2}{r^2}u_p - K^2 u_p + k^2 u_p = 0, \quad r < \alpha,$$
$$u_p'' - \frac{2}{r^2}u_p + k^2 u_p = 0, \quad r > \alpha, \tag{1-42}$$

whose solution is subject to the boundary conditions, Eqs. (3–16), (3–18), and (3–19). The asymptotic solution takes the form

$$u_p \underset{r \to \infty}{\longrightarrow} -\frac{1}{k}\cos(kr + \delta_1).$$

In view of the condition $E_L \leq 4.5$ Mev, the quantity

$$\kappa_p^2 = K^2 - k^2 \tag{4-43}$$

is positive, so the solutions are

$$u_p = A_p\left[\frac{\sinh \kappa_p r}{\kappa_p r} - \cosh \kappa_p r\right], \qquad r < \alpha,$$
$$u_p = \frac{1}{k}\left[\frac{\sin(kr + \delta_1)}{kr} - \cos(kr + \delta_1)\right], \quad r > \alpha. \tag{4-44}$$

The phase shift δ_1 is determined by the continuity condition on the logarithmic derivative, which can be manipulated into the form

$$1 - k\alpha \cot(k\alpha + \delta_1) = \left(\frac{k}{\kappa_p}\right)^2 (\kappa_p \alpha \coth \kappa_p \alpha - 1).$$

It turns out that δ_1 is proportional to $(k\alpha)^3$, so we write

$$\delta_1 = \gamma(k\alpha)^3.$$

Then the expansion of the second term on the left in powers of $(k\alpha)^2$, to first order, is

$$k\alpha \cot(k\alpha + \delta_1) = 1 - (\tfrac{1}{3} + \gamma)k^2\alpha^2,$$

and the equation for δ_1 becomes

$$\frac{1}{3} + \gamma = \frac{1}{(\kappa_p \alpha)^2}(\kappa_p \alpha \coth \kappa_p \alpha - 1).$$

Also $\kappa_p \alpha$ can be expanded in powers of k^2, but to be consistent only the leading term is to be carried. Therefore,

$$\frac{1}{3} + \gamma = \frac{1}{(K\alpha)^2}(K\alpha \coth K\alpha - 1).$$

For both the singlet and triplet potentials

$$K\alpha \gtrsim \pi/2$$

and coth $K\alpha$, which has the value 1.09 at $K\alpha = \pi/2$, approaches the value 1 asymptotically for increasing $K\alpha$. Therefore for all practical purposes we can replace it by unity within the range of interest, whence

$$\frac{1}{3} + \gamma \approx \frac{1}{(K\alpha)^2}(K\alpha - 1).$$

The corresponding expression for δ_1 is

$$\delta_1 = \left[-\frac{1}{3} + \frac{1}{(K\alpha)^2}(K\alpha - 1) \right] (k\alpha)^3.$$

The P-wave correction to the cross section, according to Eq. (4-25), is

$$\Delta\sigma = 2 \operatorname{Re}(f_0\alpha) \left[1 - \frac{3}{(K\alpha)^2}(K\alpha - 1) \right] (k\alpha)^2 \cos\theta.$$

The value relative to the S cross section is

$$\frac{\Delta\sigma}{\sigma} \approx 2\frac{\alpha}{a}\left[1 - \frac{3}{(K\alpha)^2}(K\alpha - 1) \right] (k\alpha)^2 \cos\theta, \tag{4-45}$$

where use has been made of the fact that f_0 can be expanded in powers of k^2 and only the leading term, namely a, is to be included. For the singlet state, we have the estimate $a_s \approx -8R_0$, while for the triplet $a_t \approx 2R_0$; for the singlet $K\alpha \approx \pi/2$, while for the triplet we may take $K\alpha \approx 3\pi/4$. Hence the coefficient in front of $(k\alpha)^2$ in Eq. (4-45) is less than 1 in every case. Since the P-wave correction is of the same order as $(k\alpha)^2$, it will amount to less than ten percent in the differential cross section for energies less than 5 Mev. The contribution of the P-waves to the total cross section is, of course, much smaller.

The relationship $\delta_1 \approx \gamma(k\alpha)^3$ is a special case of the more general low-energy condition

$$\delta_l \approx \gamma_l(k\alpha)^{2l+1},$$

which expresses the effectiveness of the centrifugal potential in reducing the scattering. To establish the general relationship, it is only necessary to consider the nature of the radial wave function outside the range of forces. The function can be written as

$$u_l = rj_l(kr)\cos\delta_l + rn_l(kr)\sin\delta_l, \quad r > \alpha,$$

where $j_l(kr)$ is the regular function having the property

$$j_l(x) = \frac{l!2^l}{(2l+1)!}x^l, \quad x \ll 1,$$

while $n_l(kr)$ is the irregular function having the property

$$n_l(x) = \frac{(2l)!}{2^l l!}\frac{1}{x^{l+1}}, \quad x \ll 1.$$

Furthermore, the asymptotic behavior of each function is

$$j_l(kr) \underset{r \to \infty}{\longrightarrow} \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2}\right),$$

$$n_l(kr) \underset{r \to \infty}{\longrightarrow} \frac{1}{kr}\cos\left(kr - \frac{l\pi}{2}\right),$$

which is in accord with Eq. (4–19). The phase shift is determined by the condition that the logarithmic derivative be continuous at the boundary $r = \alpha$. Because the function inside the well is nearly independent of energy for low energies, the inner logarithmic derivative can be treated as a constant,* C_l:

$$\left(\frac{1}{u_l}\frac{du_l}{dr}\right)_{r=\alpha-0} = C_l.$$

Outside the well, the function can be approximated by its behavior in the limit of small x, since $k\alpha \ll 1$ for small energies. Then

$$\left(\frac{1}{u_l}\frac{du_l}{dr}\right)_{r=\alpha+0} = \frac{1}{\alpha}\frac{(l+1)\Gamma_l(k\alpha)^{2l+1}\cos\delta_l - l\sin\delta_l}{\Gamma_l(k\alpha)^{2l+1}\cos\delta_l + \sin\delta_l},$$

where

$$\Gamma_l = \frac{1}{(2l+1)}\left[\frac{2^l l!}{(2l)!}\right]^2.$$

If we make use of the fact that δ_l will turn out to be small, of order $(k\alpha)^{2l+1}$, the continuity condition becomes

$$\frac{(l+1)\Gamma_l(k\alpha)^{2l+1} - l\delta_l}{\Gamma_l(k\alpha)^{2l+1} + \delta_l} = \alpha C_l.$$

Hence the relationship $\delta_l = \gamma_l(k\alpha)^{2l+1}$ is confirmed, and

$$\gamma_l = \Gamma_l\frac{l+1-\alpha C_l}{l+\alpha C_l}.$$

The agreement with values obtained in the special cases $l = 0$ and $l = 1$ can easily be verified by a direct calculation of C_l.

* If there happens to be a level at zero energy, C_l cannot be treated as constant. For a more detailed discussion see Mott and Massey, *The Theory of Atomic Collisions*, Oxford, 2nd edition (1949), p. 35.

Although this result indicates that the D-wave contributes very little to the low-energy scattering by a central potential, a somewhat different result might be expected for a tensor interaction. Actually the D-wave is not coupled to the S-wave outside the range of the tensor force, so the essential features of the above argument are unchanged. The amplitude of the scattered D-wave is proportional to $(k\alpha)^5$, thus the D-scattering can be ignored at low energy. However, the D-wave still has its effect because the generation of the D-wave inside the well alters the S-wave phase shift and thereby affects the scattering. A quantitative discussion of this effect is presented in the next section.

4–6 n-p Scattering. Effective range theory. This section is devoted to a general treatment of the n-p scattering, a treatment involving no specific assumption concerning the potential except that it has a finite range. It is anticipated that only average properties of the potential are required to determine the cross section as long as the wavelength is long compared with the range, a condition which must also be satisfied in order that treatment of only the S-wave be adequate. An appropriate treatment of the problem under the specified condition was given first by Breit,* for proton-proton as well as neutron-proton scattering, but for illustrative purposes the treatment of Bethe† is adequate and simpler.

Consider the scattering at two energies E_1 and E_2, and set $E_j = \hbar^2 k_j^2/M$. The radial equations for the S-waves then become

$$u_j'' + (k_j^2 + W)u_j = 0, \tag{4–46}$$

with $j = 1, 2$. Here the subscript 0, indicating $l = 0$, has been dropped, since most of the considerations of this section concern only the S-wave. It is convenient to introduce the auxiliary functions v_j which satisfy the auxiliary equation

$$v_j'' + k_j^2 v_j = 0,$$

and have the same asymptotic form as u_j,

$$\lim_{r \to \infty} [u_j(r) - v_j(r)] = 0.$$

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* A thorough discussion of the theory is given by Breit, *Rev. Mod. Phys.* **23,** 238 (1951). Considerations of this sort have been emphasized particularly by Schwinger, who has treated the problem by a variational method. See Blatt and Jackson, *Phys. Rev.* **76,** 18 (1949).

† *Phys. Rev.* **76,** 38 (1949). See also F. C. Barker and R. E. Peierls, *Phys. Rev.* **75,** 312 (1949), and G. F. Chew and M. L. Goldberger, *Phys. Rev.* **75,** 1637 (1949).

Multiplying Eq. (4–46) by u_2 for $j = 1$ and by u_1 for $j = 2$, subtracting, and integrating, we obtain the equation

$$u_2u_1' - u_1u_2'\Big|_0^\infty = (k_2^2 - k_1^2)\int_0^\infty u_1u_2\,dr.$$

Similarly,

$$v_2v_1' - v_1v_2'\Big|_0^\infty = (k_2^2 - k_1^2)\int_0^\infty v_1v_2\,dr,$$

and subtraction of one of these equations from the other leads to

$$v_1(0)v_2'(0) - v_2(0)v_1'(0) = (k_2^2 - k_1^2)\int_0^\infty (v_1v_2 - u_1u_2)\,dr,$$

since $u_j(0) = 0$. A further simplification ensues if the conventional normalization of the asymptotic function, Eq. (4–19), is abandoned in favor of

$$v_j(0) = 1,$$

whence

$$v_2'(0) - v_1'(0) = (k_2^2 - k_1^2)\int_0^\infty (v_1v_2 - u_1u_2)\,dr.$$

Since v_j is completely specified in terms of the S-wave phase shifts δ_1 and δ_2 for the two energies by

$$v_j = \frac{\sin(k_jr + \delta_j)}{\sin\delta_j},$$

its derivative is

$$v_j'(0) = k_j\cot\delta_j, \tag{4–47}$$

so

$$k_2\cot\delta_2 - k_1\cot\delta_1 = (k_2^2 - k_1^2)\int_0^\infty (v_1v_2 - u_1u_2)\,dr. \tag{4–18}$$

Now take E_1 and E_2 to be very close to some common energy $E = \hbar^2k^2/M$. Then Eq. (4–48) is equivalent to

$$\frac{d}{d(k^2)}(k\cot\delta) = \int_0^\infty (v^2 - u^2)\,dr,$$

where v and u are solutions associated with energy E. This relationship provides the coefficient of the second term in an expansion of $k\cot\delta$ as a power series in the energy:

$$k\cot\delta = (k\cot\delta)_{k=0} + \tfrac{1}{2}r_0k^2 + \cdots,$$

where

$$\frac{1}{2}r_0 = \int_0^\infty (v^2 - u^2)_{k=0}\,dr. \tag{4–49}$$

The quantity r_0 is called the *effective range;* it is one of the average properties of the potential to be determined from the scattering.

The evaluation of $k \cot \delta$ at $k = 0$ is facilitated by making use of Eq. (4–47). Since we know from Eq. (4–26) that the asymptotic solution at $k = 0$ is proportional to $r - a$, the function v, with $v(0) = 1$, is

$$v = 1 - \frac{r}{a}. \tag{4-50}$$

Therefore

$$[v'(0)]_{k=0} = -\frac{1}{a},$$

and the expansion of $k \cot \delta$ is

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2} r_0 k^2. \tag{4-51}$$

The total S-scattering cross section is given by

$$\sigma = \frac{4\pi}{k^2} \sin^2 \delta$$

or

$$\sigma = \frac{4\pi}{k^2 + \left[\frac{1}{2} r_0 k^2 - \frac{1}{a}\right]^2}. \tag{4-52}$$

This provides a complete description of the energy dependence of σ for moderate energies in terms of the two constants a and r_0. We have already seen that the magnitude of a can be determined from the zero energy scattering. In principle, the sign of the scattering length and the magnitude of r_0 can be determined from the scattering cross section at other values of the energy, but up to the present that has not proved to be a completely practical procedure.

Only one free parameter appeared in our discussion of the triplet scattering by a square well, whereas here there are two. The difference is the result of using, in the earlier treatment, the relationship between range and depth provided by the deuteron binding energy. In the more general treatment it is again possible to eliminate one of the parameters, that is, to obtain a relationship between a_t and r_{0t}, the triplet scattering length and triplet effective range. This is accomplished by extending Eq. (4–51) into the domain of negative energies, i.e., to the ground state, by replacing k^2 by $-k_0^2$. The result is

$$v_g'(0) = -\frac{1}{a_t} - \frac{1}{2} r_{0t} k_g^2$$

if Eq. (4–47) is used to replace $k \cot \delta$. The subscript g here refers to the

ground state, hence the auxiliary function normalized to unity at the origin is just

$$v_g = e^{-k_g r}.$$

Therefore

$$v_g'(0) = -k_g,$$

whence

$$k_g = \frac{1}{a_t} + \frac{1}{2} r_{0t} k_g^2,$$

which yields the desired connection between triplet effective range and triplet scattering length:

$$r_{0t} = \frac{2}{k_g^2}\left(k_g - \frac{1}{a_t}\right). \tag{4 53}$$

No corresponding information relating the effective range and scattering length is available for the singlet state. A determination of the scattering length a_s is possible, and that provides a depth-range relationship for a given potential. But in order to obtain the effective range a measurement of the scattering cross section at an energy of about 2 Mev is required. Available values of the cross section are not sufficiently accurate to provide a completely reliable value of r_{0s}.*

The form of the cross section as a function of energy for the square-well potential can now be determined. We had in Section 4–5 the estimate

$$a_t \approx 2R_0,$$

which yields

$$r_{0t} \approx \frac{1}{2k_g} = \frac{3}{4} R_0 \tag{4–54}$$

on the basis of Eq. (4–53). The triplet cross section can thereby be put in the form

$$\sigma_t = \frac{4\pi/k_g^2}{E/|E_g| + (\frac{1}{16})(E/|E_g| - 3)^2}. \tag{4–55}$$

The coefficient $4\pi/k_g^2$ is $\frac{9}{4}$ barns, so at $E = 0$, $\sigma_t = 4$ barns, as it must since that is a direct consequence of our estimate of a_t. At $E = |E_g|$, namely at 2.2 Mev, $\sigma_t \approx \frac{9}{5}$ barns, while at $E = 2|E_g|$ it is $\frac{2}{1}$ barns. Hence the triplet cross section is a rather slowly decreasing function of the energy within the range under discussion.

* Three independent experimental determinations of this cross section by Hafner, Hornyak, Falk, Snow, and Coor, *Phys. Rev.* **89,** 204 (1953), Storrs and Frisch, *Phys. Rev.* **90,** 339 (1953), and Fields, Adair, Becker, and Darden, *Phys. Rev.* **91,** 441 (1953) now appear to give a consistent value of r_{0s} which is also consistent with the value deduced from proton-proton scattering in Section 6–2 (communicated privately by R. K. Adair, June 1953).

The energy dependence of the singlet cross section for a square-well potential requires that we evaluate r_{0s} directly from our knowledge that the functions u and v are

$$v = 1 - \frac{r}{a_s}$$

and

$$u = A_s \sin K_s r, \quad \text{for} \quad r < \alpha.$$

The constant A_s satisfies the continuity condition

$$A_s \sin K_s \alpha_s = 1 - \frac{\alpha_s}{a_s}.$$

We can also make use of the equation

$$K_s \cot K_s \alpha_s = \frac{1}{\alpha_s - a_s}.$$

The effective range is defined as the integral

$$r_{0s} = 2\int_0^{\alpha} \left[\left(1 - \frac{r}{a_s}\right)^2 - A_s^2 \sin^2 K_s r \right] dr,$$

whence, by direct calculation,

$$r_{0s} = \alpha_s \left(1 - \frac{1}{3}\frac{\alpha_s^2}{a_s^2} - \frac{1}{K_s^2 \alpha_s^2}\frac{\alpha_s}{a_s}\right). \tag{4-56}$$

Note that for $a_s \to \pm\infty$, i.e., for the singlet binding energy exactly zero, $r_{0s} = \alpha_s$. Thus in the special case of a square-well potential having such a depth that the binding vanishes, the effective range equals the actual range.

In order to determine the energy dependence of the cross section, we use the estimate $a_s = -8R_0$, which is so large that to a very good approximation

$$r_{0s} = \alpha_s. \tag{4-57}$$

The rough estimate $\alpha_s \approx \frac{1}{2}R_0$ then leads to the cross section

$$\sigma_s \approx \frac{4\pi/k_s^2}{E/|E_s| + (\frac{1}{4})(\frac{3}{8} + E/3|E_s|)^2}. \tag{4-58}$$

This approximation yields a value of 64 barns at $E = 0$, at $E = |E_s|$ it is about 2 barns, while at $E = 2|E_s|$ it is about 1 barn. Therefore the singlet cross section falls very rapidly from its large value at zero energy to a value of the same order as that of the triplet cross section at about 2 Mev, a result which is in accordance with our interpretation of the singlet

scattering as a resonance phenomenon. However, it is to be noted that the resonance is a very broad one indeed, so broad, in fact, that no resonance peak appears at the position of the virtual level.

Similar computations are necessary to determine the range and depth of a potential having any specified shape from the observed values of scattering length and effective range. The required calculations have been carried out by Blatt and Jackson* for all of the conventional central potentials. Their results are shown in Figs. 4–4 and 4–5. Since adequate data for a direct determination of the triplet effective range have not been available up to the present time, the deuteron binding energy is usually used to fix r_{0t} by means of Eq. (4–53). No such result is available to evaluate r_{0s},

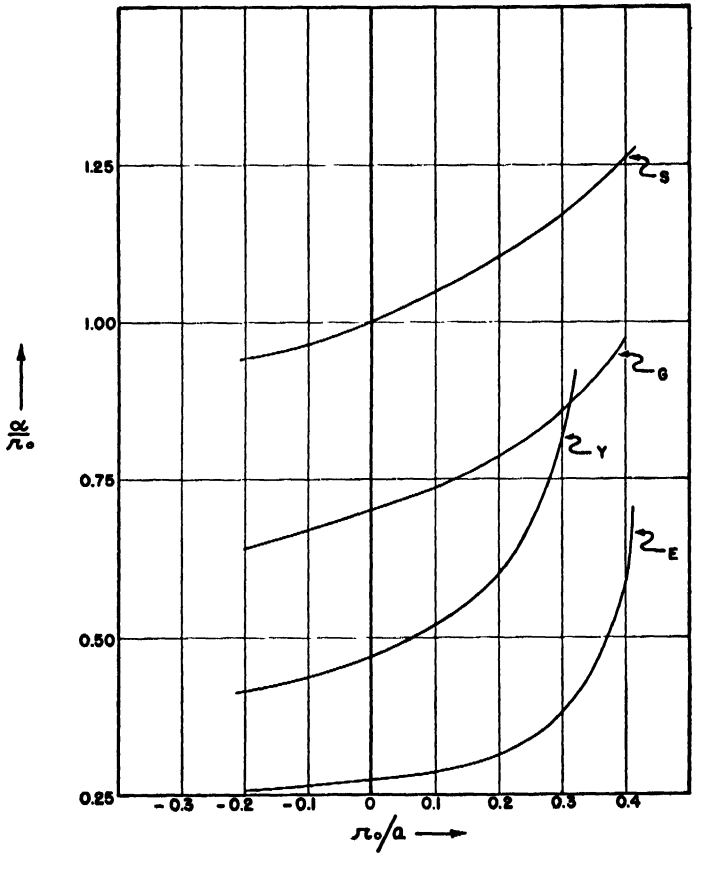

FIG. 4–4. Relationship between the scattering length, effective range, and range of the square well (S), Gauss potential (G), Yukawa potential (Y), and the exponential (E). [Adapted from Blatt and Jackson, *Phys. Rev.* **76**, 18 (1949).]

* *Phys. Rev.* **76**, 18 (1949).

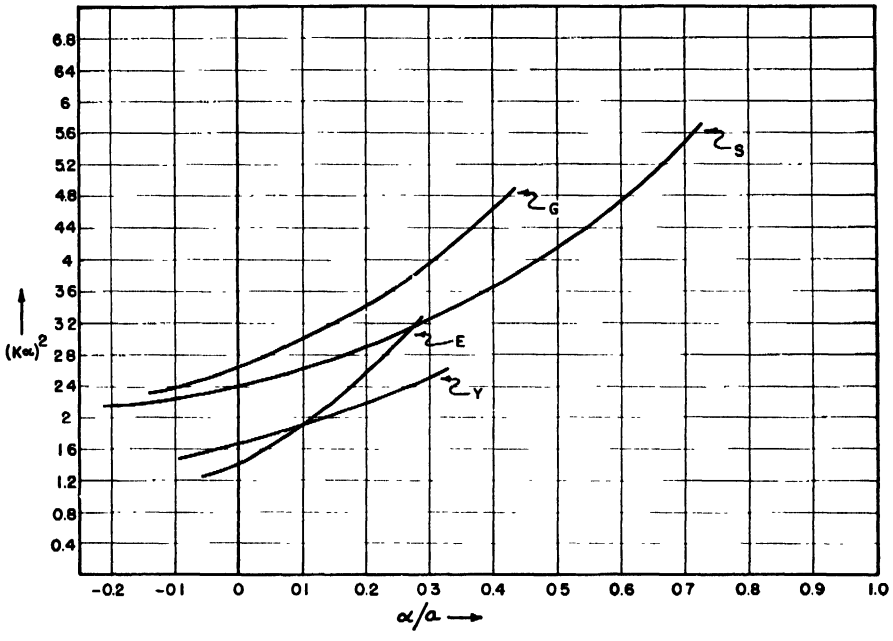

FIG. 4.5. Relationship between the scattering length, the range, and the depth for potentials of several forms. Source and notation same as Fig. 4–4.

but the value is often assumed to be the same as that obtained from proton-proton scattering. This procedure is highly questionable, since one of the most primitive objectives of the study of nuclear forces is to establish the relationship, if any exists, between neutron-proton and proton-proton interactions. The matter can be clarified by a very precise measurement of the n-p scattering at an energy of the order of 2 Mev.

The effective range theory is a result of carrying only the first two terms in an expansion of $k \cot \delta$ in powers of k^2. Since there are two parameters, a and r_0, in this approximation, any reasonable two-parameter potential can be adjusted to fit the data. However, an experimentally determined value of the coefficient of the k^4 term in the series could not be met so easily by any potential when the range and depth are fixed by determination of a and r_0. This term therefore provides information concerning the shape of the potential. To draw firm conclusions concerning the shape, an analysis of the influence of tensor forces and P-waves is required, but that involves more effort than is warranted by the data at present. Blatt and Jackson discuss the shape-dependent term, but only with reference to the S-scattering by a central potential.

We have developed the concept of effective range for scattering by a central potential and now, since there is convincing evidence for the exist-

ence of a tensor interaction, some consideration must be given to the influence of the noncentral potential on the interpretation of r_0. We are concerned here only with the triplet effective range, r_{0t}, since the tensor force vanishes in a singlet state. It has already been pointed out in Section 4-5 that S-scattering is the important effect at these energies, whatever the interaction, therefore *an* effective range can certainly be determined from the data. But the relationship between the effective range and the parameters in the potential no longer has the simple form indicated by Eq. (4–49), which is the basis of Figs. 4-4 and 4-5.

Analysis of the problem is simple when we realize that the wave function for the scattering problem can be expanded in characteristic functions of the total angular momentum. Since only the $^3S_1$ scattering is important, the relevant contribution to the wave function is the term with even parity and $J = 1$. We have established that this function must be of the form Eq. (3–63),

$$\psi_1^M = r^{-1}[u(r) + w(r)S_{np}]\chi_1^M,$$

where ψ_1^M is now a solution of the Schroedinger equation for positive energy. The appropriate modifications of Eqs. (3–69) and (3–70) are

$$u'' + [W_c(r) + k^2]u + 8\gamma W_t(r)w = 0, \tag{4–59}$$

$$w'' + \left[W_c(r) - 2\gamma W_t(r) - \frac{6}{r^2} + k^2\right]w + \gamma W_t(r)u = 0. \tag{4–60}$$

The first of these equations may be introduced in place of Eq. (4–46) in the analysis leading to Eq. (4-48), and we now obtain

$$k_2 \cot \delta_2 - k_1 \cot \delta_1$$

$$= (k_2^2 - k_1^2)\int_0^\infty (v_1 v_2 - u_1 u_2)\, dr + 8\gamma \int_0^\infty W_t(r)(w_1 u_2 - w_2 u_1)\, dr. \tag{4–61}$$

Here w_1 corresponds to energy k_1^2, w_2 to energy k_2^2.

This equation differs from the form obtained for central forces by the addition of the last term. To eliminate that undesirable appendage, some consideration must be given to Eq. (4–60). It is clear* that there are two linearly independent solutions, (u^α, w^α) and (u^β, w^β), of Eqs. (4-59) and (4-60) for given energy. The α solution is taken to be that which contributes primarily to S-scattering, so its phase shift $\delta \equiv \delta^\alpha$ is proportional to k at low energies. Then δ^β is expected to be essentially the D-wave phase shift proportional to k^5. Since the functions $u^{\alpha,\beta}$ have been assumed to have the usual asymptotic form characteristic of effective range theory, the asymptotic form of $w^{\alpha,\beta}$ is

* For a detailed discussion see Rohrlich and Eisenstein, *Phys. Rev.* **75**, 705 (1949).

$$w^{\alpha,\beta} \xrightarrow[r \to \infty]{} \frac{\eta^{\alpha,\beta}}{\sin \delta^{\alpha,\beta}} \sin (kr + \delta^{\alpha,\beta}),$$

where the coefficient $\eta^{\alpha,\beta}$ measures the amount of D-wave relative to the S-wave. The condition

$$\eta^{\alpha}\eta^{\beta} = -\tfrac{1}{8} \tag{4–62}$$

can easily be established by means of the generalized Wronskian condition

$$u^{\alpha}u^{\beta\prime} - u^{\beta}u^{\alpha\prime} + 8(w^{\alpha}w^{\beta\prime} - w^{\beta}w^{\alpha\prime}) = 0, \tag{4–63}$$

which follows from Eqs. (4–59) and (4–60).

The energy dependence of η^{α} and η^{β} is most easily established by considering a square-well (tensor) potential and making use of the continuity of the logarithmic derivative at the boundary, as we did at the end of Section 4–5. Since the solution of the scattering problem involves a linear combination of (u^{α},w^{α}) and (u^{β},w^{β}), the corresponding linear combination of $d \ln u/dr$ must be continuous, as must be the same combination of $d \ln w/dr$. From the condition on $d \ln u/dr$ it is found that the coefficient of u^{β} is proportional to k^4 (if u^{α} has unit coefficient). Then the condition on $d \ln w/dr$ yields an energy dependence of η^{α} of the form $\eta^{\alpha} \sim k^2$, if Eq. (4–62) is taken into account. Because the coefficient of u^{β} is so small, the low energy S-scattering is determined by δ^{α} and η^{α}, which we now denote simply by δ and η.

An equation analogous to Eq. (4–61) can be obtained from Eq. (4–60) by the usual procedure if an auxiliary function, $x(r)$, is defined by

$$x(r) = \frac{\eta}{\sin \delta} \sin (kr + \delta) = \eta v(r).$$

Then,

$$\eta_1\eta_2(k_2 \cot \delta_2 - k_1 \cot \delta_1)$$

$$= (k_2^2 - k_1^2)\int_0^{\infty} (\eta_1\eta_2 v_1 v_2 - w_1 w_2)\, dr - \gamma \int_0^{\infty} W_t(r)(w_1 u_2 - w_2 u_1)\, dr.$$

If this is multiplied by 8 and added to Eq. (4–61), there results

$$(1 + 8\eta_1\eta_2)(k_2 \cot \delta_2 - k_1 \cot \delta_1)$$

$$= (k_2^2 - k_1^2)\int_0^{\infty} [(1 + 8\eta_1\eta_2)v_1 v_2 - (u_1 u_2 + 8w_1 w_2)]\, dr.$$

Repetition of our earlier procedure for obtaining the effective range leads to

$$\frac{d}{d(k^2)} k \cot \delta = \frac{1}{1 + 8\eta^2} \int_0^{\infty} [y^2 - (u^2 + 8w^2)]\, dr,$$

where the new auxiliary function y is

$$y = \sqrt{1 + 8\eta^2}\, v. \tag{4-64}$$

Since $\eta^2 \sim k^4$ it can be neglected in the factor before the above integral* for the energy range under consideration. Repetition of the reasoning that led to Eq. (4–49) therefore provides the following expression for the triplet effective range:

$$\tfrac{1}{2}r_{0t} = \int_0^\infty (y^2 - u^2 - 8w^2)_{k=0}\, dr. \tag{4-65}$$

The relationship, Eq. (4–53), between binding energy and effective range is still valid and, in the same approximation, Eq. (4–65) can be replaced by

$$\tfrac{1}{2}r_{0t} = \int_0^\infty (y_g^2 - u_g^2 - 8w_g^2)\, dr, \tag{4-66}$$

where u_g and w_g are ground state functions, and y_g is the corresponding auxiliary function, $y_g = \sqrt{1 + 8\eta_g^2}\, e^{-k_g r}$. Values of r_{0t} obtained by Feshbach and Schwinger for Yukawa potentials by means of Eq. (4–66) are given in Appendix 2.

* But not in the integrand, since that must vanish outside the range of forces.

CHAPTER 5

S-SCATTERING BY BOUND NUCLEI

The discussion of neutron-proton scattering in Chapter 4 has been idealized by treating the proton as though it were free and at rest. Actually, the target protons are bound in some molecule or lattice so that they cannot recoil freely. Furthermore, the proton kinetic energy is different from zero; it is certainly not much less than kT, where T is the target temperature, and it may be effectively larger if the zero point vibrational energy of the molecule or lattice is large. Finally, the treatment of individual protons as independent scatterers will be erroneous when interference can occur between the waves scattered from different protons in the target material. All of these effects are important only at neutron energies comparable to or less than thermal energies. If the neutron energy is much greater, say of the order of 1 kev, the protons may be treated as free, at rest, and independent: the first because the recoil energy is much greater than the molecular binding, the second because the relative velocity is not appreciably affected by the proton thermal energy, and the third because the wavelength is too short for interference to be significant.

Our concern in this chapter will be the influence of the three effects on the scattering of very slow neutrons. Much of the treatment is independent of the details of the interaction, so the results are applicable to the scattering of slow neutrons by nuclei other than the proton. Therefore, wherever possible, answers will be given in a form suitable for discussion of any target nucleus. The parameters that will appear are the scattering length a, the mass M_n, and the total angular momentum (spin) I, of the nucleus.

5–1 The chemical binding effect. At first sight the binding of the target nucleus in a molecule seems to present a major analytical difficulty, since the determination of the neutron scattering involves the solution of a many-body problem. The nuclear interaction is strong, so no simple perturbation method would seem to be applicable. However, the fact that the range of nuclear forces is so very small suggests an approximation method first introduced by Fermi.* The method proceeds by substituting for the interaction an auxiliary potential of greater range and smaller depth than the true potential, but one that produces the same scattering. Because

* *Ricerca Scient.* **7,** 13 (1936).

the depth can be made quite small, scattering by the new potential is susceptible to a standard perturbation treatment which requires only a knowledge of the molecular wave functions.

The quantitative argument establishing the validity of this procedure is presented in two steps. First it will be shown that a long-range auxiliary potential, capable of giving the correct scattering length, exists for a *free* scattering center. Then we shall demonstrate that the scattering from the bound nucleus is, to a very good approximation, determined by the scattering length of the free nucleus. From this it follows that the auxiliary potential can be used in the treatment of scattering by the molecule.

The perturbation method to be used is equivalent to the Born approximation, which is expected to provide a good approximation at low energy when*

$$\begin{gathered} |W_0|\alpha^2 \ll 1, \\ k\alpha \ll 1, \end{gathered} \tag{5-1}$$

if W_0 is an appropriate average value of the strength of the potential in units of $\hbar^2/M$. The fact that $\sqrt{|W_0|}\,\alpha \gtrsim \pi/2$ for the n-p system shows that this approximation is certainly not directly applicable. However, for a free nucleus, the scattering of slow neutrons is determined entirely by the scattering length, so an auxiliary potential of strength $\mathfrak{W}_0$ and range $\mathfrak{a}$, which leads to the same scattering length, will do just as well. Then if $|\mathfrak{W}_0|\mathfrak{a}^2 \ll 1$ and $k\mathfrak{a} \ll 1$, the Born approximation can be applied to the auxiliary potential. To show that these conditions can be satisfied, we make use of the equation†

$$f(\theta) = \frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int e^{-i(\boldsymbol{\kappa}\cdot\mathbf{r})}V(\mathbf{r})\psi(\mathbf{r})\,d^3r, \tag{5-2}$$

where f is the function of scattering angle appearing in the asymptotic function [Eq. (4–20)], ψ is the wave function of the system, $\boldsymbol{\kappa}$ is the propagation vector of the scattered wave, $V(\mathbf{r})$ is the potential, and μ is the reduced mass of the neutron plus scattering nucleus. The scattering length a is defined as the value of f for zero energy:

$$a = \frac{1}{2\pi}\frac{\mu}{\hbar^2}\int V(\mathbf{r})\varphi(\mathbf{r})\,d^3r, \tag{5-3}$$

where φ is the zero energy wave function. Hence

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* See Schiff, *Quantum Mechanics*, McGraw-Hill (1949), p. 168.

† See Mott and Massey, *The Theory of Atomic Collisions*, 2nd edition, Oxford Univ. Press (1941), p. 116.

$$a = -\tfrac{2}{3}\mu M^{-1} W_0 \alpha^3, \tag{5-4}$$

if W_0 is the appropriately defined average depth of the potential.

When $V(\mathbf{r})$ is replaced by the auxiliary potential, the condition that the same scattering length be obtained can be put in the form

$$a = -\tfrac{2}{3}\mu M^{-1} \mathfrak{W}_0 \mathfrak{a}^3. \tag{5-5}$$

Equations (5-4) and (5-5) combine to give

$$\frac{\mathfrak{W}_0 \mathfrak{a}^2}{W_0 \alpha^2} = \frac{\alpha}{\mathfrak{a}}, \tag{5-6}$$

from which it follows that a large value of $\mathfrak{a}$ will lead to a sufficiently small value of $|\mathfrak{W}_0|\mathfrak{a}^2$ to satisfy the Born condition. Note that $\mathfrak{a}$ must be very small compared with $1/k = \lambda$, although it is large compared with α. Both conditions can be met as a consequence of the very small value of $k\alpha$ for a thermal neutron, namely, $k\alpha \approx 10^{-5}$.

The Born approximation is the result of replacing ψ in Eq. (5-2) by the unscattered plane wave. For the auxiliary potential, $\mathfrak{V}(\mathbf{r})$, Eq. (5-3) is therefore to be replaced by

$$a = \frac{1}{2\pi}\frac{\mu}{\hbar^2}\int \mathfrak{V}(\mathbf{r})\, d^3r, \tag{5-7}$$

so that $\mathfrak{W}_0$ is just the unweighted average strength of the potential. It is convenient to note that, *in the Born approximation*, the auxiliary potential can be taken to be

$$\mathfrak{V}(\mathbf{r}) = \frac{2\pi\hbar^2}{\mu} a\delta(\mathbf{r} - \mathbf{r}'), \tag{5-8}$$

since insertion of that potential into Eq. (5-7) clearly yields the correct result. This procedure can be followed *only after* the Born approximation has been justified, and only in the first Born approximation.

The use of the auxiliary potential for treatment of the scattering by a bound nucleus can be justified if the potential is subjected to the further restriction that the range $\mathfrak{a}$ be small compared with the amplitude A of zero point vibration of a nucleus in the molecule:

$$\mathfrak{a} \ll A. \tag{5-9}$$

Then we shall find that the wave function of the system at distances well beyond $\mathfrak{a}$ from any nucleus depends on the nuclear interaction only through the scattering length. The auxiliary potential therefore yields the same asymptotic function for the molecular system, and consequently the same scattering cross section, as the true potential.

The Schroedinger equation for the entire system of molecule plus neutron is

$$-\left(\frac{1}{2}\nabla_r^2+\sum_j\frac{M}{2M_j}\nabla_j^2\right)\psi(\mathbf{r},\mathbf{R}_1,\mathbf{R}_2,\cdots)+U(\mathbf{R}_1,\mathbf{R}_2,\cdots)\psi$$
$$-\sum_j W_j(\mathbf{r}-\mathbf{R}_j)\psi=E\psi, \quad (5\text{–}10)$$

where $\mathbf{r}$ is the neutron coordinate and $\mathbf{R}_j$ is the coordinate of the jth nucleus of mass M_j; U is the molecular potential, $-W_j$ is the nuclear potential, and E is the total energy of the system, all measured in units of $\hbar^2/M$. Let us consider the behavior of solutions of this equation when $|\mathbf{r}-\mathbf{R}_1| \lesssim \mathfrak{a}$, that is, when the neutron is in interaction with the nucleus numbered 1. Then it is appropriate to introduce the coordinates

$$\begin{aligned}\boldsymbol{\rho}_1&=\mathbf{r}-\mathbf{R}_1,\\ \mathbf{R}_1'&=\frac{M\mathbf{r}+M_1\mathbf{R}_1}{M_1'},\end{aligned} \quad (5\text{–}11)$$

with

$$M_1'=M+M_1.$$

Since the spacing between nuclei in the molecule is very much greater than the range of the auxiliary potential, all potentials W_j except W_1 vanish, and the Schroedinger equation becomes

$$-\left(\frac{M}{2\mu_1}\nabla_\rho^2+\frac{M}{2M_1'}\nabla_1'^2+\frac{M}{2M_2}\nabla_2^2+\cdots\right)\psi(\boldsymbol{\rho}_1,\mathbf{R}_1',\mathbf{R}_2,\cdots)$$
$$+U\left(\mathbf{R}_1'-\frac{M}{M_1'}\boldsymbol{\rho}_1,\mathbf{R}_2,\cdots\right)\psi-W_1(\boldsymbol{\rho}_1)\psi=E\psi,\ \rho_1\lesssim\mathfrak{a}, \quad (5\text{–}12)$$

where $(1/\mu_1)=(1/M)+(1/M_1)$. Now U varies extremely little for variations of $\boldsymbol{\rho}_1$ over distances of the order of $\mathfrak{a}$ as long as $\mathfrak{a}$ satisfies the condition of Eq. (5–9). Thus it is a very good approximation to drop the $\boldsymbol{\rho}_1$ dependence from the molecular potential. Then the equation separates if ψ is written as the product

$$\psi(\mathbf{r},\mathbf{R}_1,\mathbf{R}_2,\cdots)=\varphi_1(\boldsymbol{\rho}_1)\Phi_1(\mathbf{R}_1',\mathbf{R}_2,\cdots),\quad \rho_1\lesssim\mathfrak{a},$$

and φ_1 is a solution of

$$-\frac{M}{2\mu_1}\nabla_\rho^2\varphi_1-W_1(\boldsymbol{\rho}_1)\varphi_1=0, \quad (5\text{–}13)$$

while Φ_1 is determined by

$$-\left(\frac{M}{2M_1'}\nabla_1'^2+\sum_{j\neq1}\frac{M}{2M_j}\nabla_j^2\right)\Phi_1(\mathbf{R}_1',\mathbf{R}_2,\cdots)+U(\mathbf{R}_1',\mathbf{R}_2,\cdots)\Phi_1=E\Phi_1. \quad (5\text{–}14)$$

The function φ_1 is the zero energy wave function of a neutron in interaction with the free nucleus of type number 1, while Φ_1 is the Schroedinger function for an isotopic molecule, i.e., one differing from the original molecule by the substitution of a nucleus of mass $M_1' = M_1 + M$ for the nucleus of mass M_1.*

The same procedure applies to any one of the nuclei in the molecule, so the wave function in the neighborhood of the jth nucleus has the form

$$\psi(\mathbf{r},\mathbf{R}_1,\mathbf{R}_2,\cdots) = \varphi_j(\boldsymbol{\rho}_j)\Phi_j(\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_j',\cdots), \quad \rho_j \lesssim \mathfrak{a}. \qquad (5\text{–}15)$$

On the other hand, when every $\rho_j > \mathfrak{a}$, the nuclear interaction drops out of the differential equation determining ψ. The asymptotic wave function, for which $\rho_j \gg \mathfrak{a}$, is therefore influenced by the interaction only through its connection with the "inner" functions given by Eq. (5–15). But the connection condition involves the interaction through the function $\varphi_j(\boldsymbol{\rho}_j)$ and then only for $\rho_j \gtrsim \mathfrak{a}$ where the form $\varphi_j = 1 - a_j/\rho_j$ may be used. Since the auxiliary potential has been defined to yield the correct scattering length a_j, it provides the correct connection condition and therefore the correct asymptotic wave function. Consequently the scattering cross section, which depends only on the asymptotic wave function, may be calculated by means of the auxiliary potential.

The condition $\mathfrak{a} \ll A$ is somewhat more stringent than the low-energy condition $\mathfrak{a} \ll \lambda$, since A, the amplitude of molecular vibration, is of the order of one-tenth of the λ for a thermal neutron. Nevertheless, the condition for applicability of perturbation theory, $\mathfrak{a} \gg \alpha$, can still be easily satisfied, since $A \approx 10^1\alpha$. Thus introduction of the Fermi δ-function potential, Eq. (5–8), as a perturbation in the treatment of the scattering of slow neutrons by bound nuclei yields a good approximation to the solution of the scattering problem.†

A very simple application of the general theorem just established concerns the ratio of the cross section for the scattering of extremely slow neutrons to the cross section for scattering of fast neutrons. The slow neutron energy is taken to be too small to excite vibrations or rotations of

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* The fact that E is not, in general, an eigenvalue of the energy for the isotopic molecule might appear to lead to a contradiction. However, the energy eigenvalues are not determined only by the differential equation; the boundary condition is essential. Since Eqs. (5–13) and (5–14) are valid over only a limited region of space, the boundary condition at infinity, which normally leads to a restriction on the energy values, does not apply here. It is to be replaced by a continuity condition on the wave function in the neighborhood of $\rho_1 = \mathfrak{a}$, which does not restrict the energy, since a continuum of energies is accessible to the system as a whole.

† A more detailed discussion of this problem is given along with estimates of the corrections by G. Breit, *Phys. Rev.* **71**, 215 (1947). See also B. A. Lippmann and J. Schwinger, *Phys. Rev.* **79**, 469 (1950).

the molecule.* Therefore, the molecule recoils from the collision as a point mass M_M equal to the total molecular mass. The fast neutron energy may be 1 kev or more, enough so that molecular bonds will be easily broken and the scattering nucleus, of mass M_n, recoils freely. In the first case, the effective reduced mass is

$$\frac{1}{\mu_M} = \frac{1}{M} + \frac{1}{M_M},$$

while in the second it is

$$\frac{1}{\mu_n} = \frac{1}{M} + \frac{1}{M_n}.$$

Since the auxiliary potential responsible for the scattering is the same in both cases, we see from Eq. (5–7) that the ratio of the scattering length a_b of the (in effect) strongly bound nucleus to the scattering length a_f of the free nucleus is

$$\frac{a_b}{a_f} = \frac{M_M}{M_n}\frac{M_n + M}{M_M + M}. \tag{5–16}$$

In the special case of a solid or liquid target, $M_M \to \infty$, so

$$\frac{a_b}{a_f} = 1 + \frac{M}{M_n}.$$

If the scattering nucleus is a proton bound in paraffin, for example, then

$$\frac{a_b}{a_f} = 2,$$

and the corresponding ratio of cross sections is

$$\frac{\sigma_b}{\sigma_f} = 4.$$

This is the "famous Fermi factor of four." The success of the theory is indicated in Fig. 5–1, a curve of the total cross section for transmission of low-energy neutrons through paraffin. The low-energy cross section is very close to four times the free proton cross section of 20 barns which is obtained above 100 ev.

For neutron energies between the extremes in which the nucleus can be treated as bound or as free, a much more detailed discussion is required.

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* It also must be assumed that the target temperature is so low that *inelastic collisions of the second kind*, i.e., those giving up rotational or vibrational thermal energy to the neutron, do not play an important role.

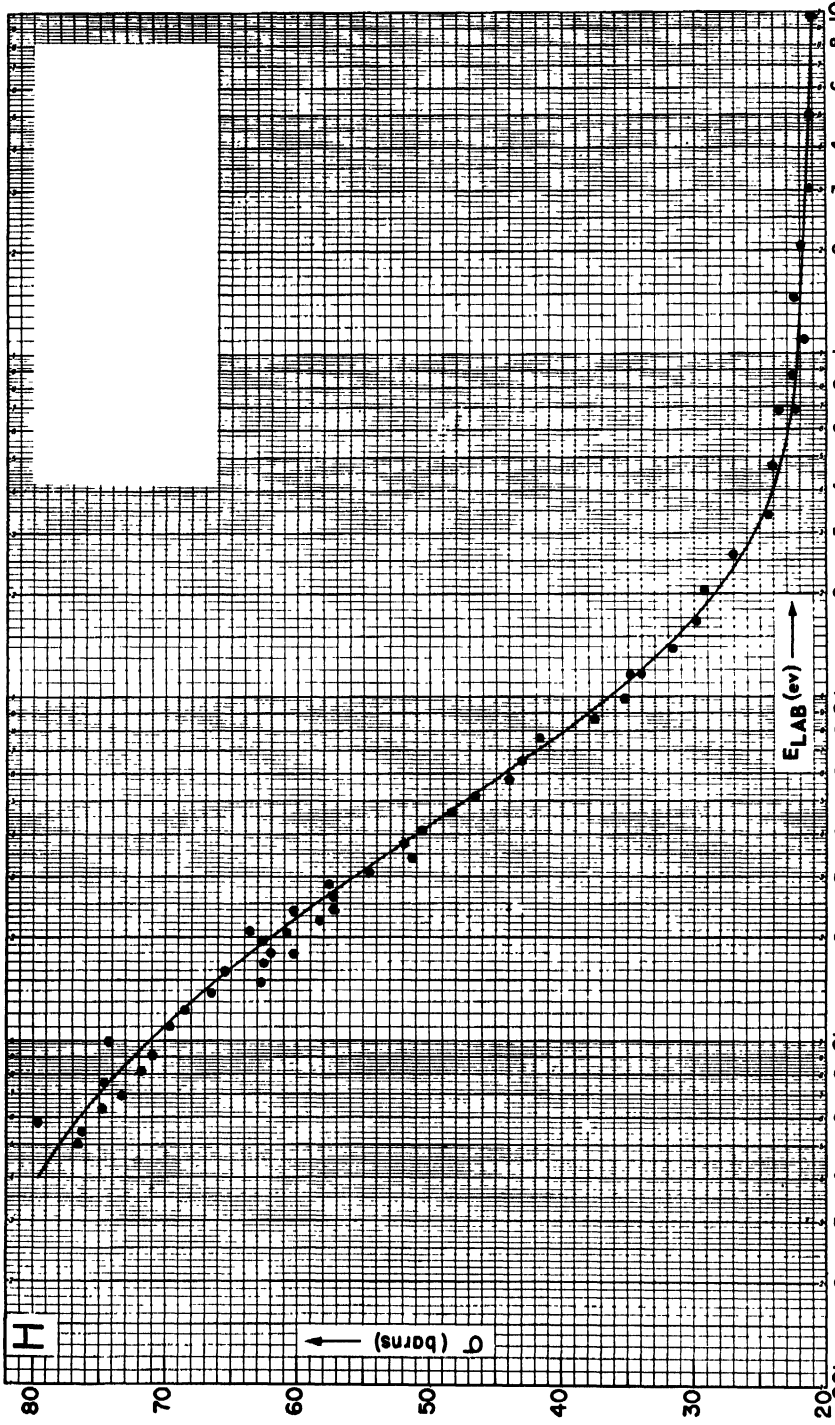

FIG. 5-1. The scattering of slow neutrons by protons in paraffin. [From Rainwater, Havens, Dunning, and Wu, *Phys. Rev.* **73**, 733 (1948).]

The interpretation of the cross section of solids in the intermediate energy region is complicated both by interference effects and by transfer of energy between the neutron and lattice vibrations. For gas molecules there is an important temperature effect in the low-energy region and, in addition, transfer of energy to rotations and vibrations of the molecule can occur. The importance of the Fermi theorem established in this section is that all of these problems can be treated to a good approximation by introducing the δ-function potential, Eq. (5–8), as a first-order perturbation on the vibrational and rotational motion of the molecular system.*

5–2 Effect of target motion. The scattering of a neutron from a gas target is a function of the temperature of the gas as well as its molecular structure. This is a consequence of the fact that the scattering depends on the relative velocity of the neutron and target nucleus, and the relative velocity in turn depends on the velocity of the molecule due to thermal agitation. The temperature effect is sometimes referred to in the literature as the "Doppler effect," although it is in no way related to the effect of that name which arises in studies of wave motion.

Calculation of the dependence on temperature is simple for a monatomic gas or for a polyatomic gas under certain special conditions. The simplest case will be treated here, namely, that for which only the translational degrees of freedom of the molecule are important. The results are then applicable either to monatomic gases or to molecular gases at such low temperatures that only the lowest rotational state occurs with appreciable probability. The neutron energy must also be small enough to avoid excitation of the higher rotational states by the collision.

The scattering cross section, σ_{CM}, in the center of mass system of the neutron and scattering center is the quantity of theoretical interest, since it depends directly on the interaction between the particles. On the other hand, the quantity measured experimentally is the number of neutrons scattered per unit time divided by the flux of neutrons in the incident beam. Since the number of neutrons scattered per unit time is independent of the frame of reference, it may be calculated in the CM system. There, the current incident on the scattering center is proportional to the relative velocity $|\mathbf{v} - \mathbf{V}|$, where $\mathbf{v}$ is the neutron velocity and $\mathbf{V}$ the velocity of the mass point, both taken in the laboratory frame. Consequently, the number scattered per unit time is proportional to $|\mathbf{v} - \mathbf{V}|\sigma_{CM}$. But the beam flux of neutrons in the laboratory system is proportional to v, so the observed cross section is

$$\sigma = \frac{|\mathbf{v} - \mathbf{V}|}{v}\sigma_{CM}$$

* A very thorough treatment of the general problem on this basis is given by G. Placzek, *Phys. Rev.* **86**, 377 (1952).

or

$$\sigma = \sqrt{1 + \zeta^2 - 2\zeta \cos\theta}\, \sigma_{CM},$$

where $\zeta = V/v$ and θ is the angle between $\mathbf{v}$ and $\mathbf{V}$. Since all directions of the velocity $\mathbf{V}$ occur in the gas with equal probability, an average over angles is to be taken for comparison with an observed cross section. If the angle average is denoted by $\bar{\sigma}$, we have

$$\bar{\sigma} = \frac{\sigma_{CM}}{2}\int_{-1}^{1} \sqrt{1 + \zeta^2 - 2\zeta \cos\theta}\, d(\cos\theta).$$

The integral is simplified by the substitution

$$\eta = \sqrt{1 + \zeta^2 - 2\zeta \cos\theta},$$

where η is always to be taken positive. Then

$$\bar{\sigma} = -\frac{\sigma_{CM}}{2\zeta}\int_{1+\zeta}^{|1-\zeta|} \eta^2\, d\eta$$

$$= \frac{\sigma_{CM}}{6\zeta}[(1 + \zeta)^3 - |1 - \zeta|^3],$$

whence we obtain for $\zeta < 1$,

$$\bar{\sigma} = \sigma_{CM}(1 + \tfrac{1}{3}\zeta^2), \tag{5-17}$$

while for $\zeta > 1$,

$$\bar{\sigma} = \frac{\sigma_{CM}}{\zeta}(\tfrac{1}{3} + \zeta^2). \tag{5-18}$$

The observed cross section is to be calculated by averaging $\bar{\sigma}$ over the molecular velocities V. These are weighted according to a Maxwell distribution, hence

$$\sigma_{obs} = \frac{\displaystyle\int_0^\infty \bar{\sigma} e^{-\beta\zeta^2}\zeta^2\, d\zeta}{\displaystyle\int_0^\infty e^{-\beta\zeta^2}\zeta^2\, d\zeta},$$

where $\beta = (M_M v^2/2kT) = (M_M/M)(E_L/kT)$ if E_L is the laboratory energy of the neutron. When $\bar{\sigma}$ is taken from Eqs. (5-17) and (5-18), straightforward integration yields the result

$$\sigma_{obs} = \sigma_{CM}\left[\frac{e^{-\beta}}{\sqrt{\pi\beta}} + \left(1 + \frac{1}{2\beta}\right)\Phi(\sqrt{\beta})\right], \tag{5-19}$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-w^2}\, dw$$

is the *Error Function* which is tabulated in a number of places.*

According to Eq. (5–19), the observed cross section approaches σ_{CM} for large values of β. On the other hand, it diverges as β approaches zero. The latter result is simply a consequence of the fact that zero energy neutrons are struck by moving molecules and thereupon scattered in spite of the absence of an apparent (laboratory) neutron current. For this reason the scattering by gases appears to increase indefinitely at low energies, as shown in Fig. 5–2. Detailed analysis of Fig. 5–2 cannot be made on the basis of the simple equation (5–19), since the data were taken with the target at room temperature, a temperature at which many rotational and vibrational states of H_2 are excited. The range of neutron energies also extends beyond the limits of validity of our simple treatment.

The analysis of a problem of this kind can be simplified in that region of neutron energies and temperatures for which many rotational states are excited. The rotations may then be treated classically, with the result that the molecule scatters like a mass point whose mass depends on the direction of the collision. This can be seen, for example, in the scattering by the H_2 molecule: when the collision occurs along the axis of the molecule, both protons recoil, but a collision perpendicular to the axis causes only the struck proton to recoil. It has been shown by Sachs and Teller† that this type of problem can be handled simply by treating the mass of the scattering center as a tensor. Then the effect of rotational recoil of the molecule on the scattering is just a reduced mass-tensor effect similar to the reduced mass effect discussed at the end of Section 5–1. An extension of the mass-tensor method to include vibrations (by means of the δ-function potential) has been given by Messiah.‡ He presents a theoretical analysis of the entire curve, Fig. 5–2.

5–3 Scattering of neutrons by ortho- and parahydrogen. We have seen in Section 4–5 that the sign of the neutron scattering length may be of considerable interest, since it is closely related to the magnitude of the interaction. This sign cannot be determined by measuring the scattering from a single nucleus because that depends only on the square of the scattering length. Therefore resort must be taken to studies of the interference between waves scattered from several nuclei. In this connection,

* See, for example, Jahnke-Emde, *Table of Functions*, Dover Publications (1943), p. 23 ff.

† *Phys. Rev.* **60**, 18 (1941).

‡ *Phys. Rev.* **84**, 204 (1951).

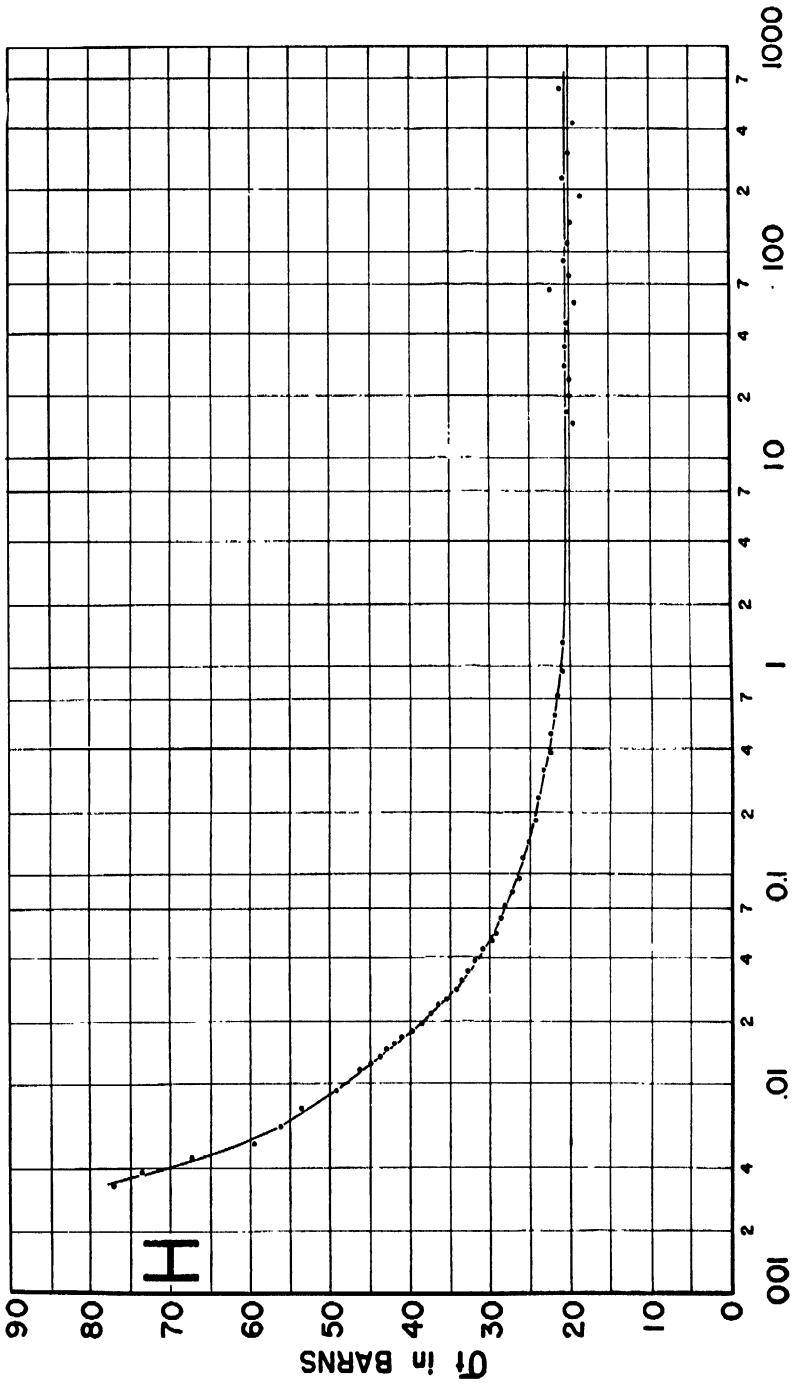

FIG. 5-2. The scattering of slow neutrons by protons in H_2 gas at room temperature. [From E. Melkonian, *Phys. Rev.* **76**, 1750 (1949).]

the scattering from H_2 gas of neutrons whose wavelength is comparable to, or larger than, the interatomic spacing is of particular interest. Waves scattered from the two protons interfere with one another, much as the radiations emerging from the slits in a Young's double slit experiment interfere with one another. The interference term in the scattered intensity depends on the relative spin orientations of the two protons because the n-p interaction is spin-dependent, and the consequent effect on the cross section makes possible a determination of the sign of the singlet scattering length.

If the wavelength of the neutrons being scattered from H_2 is short compared with the molecular size, the interference pattern will alternate in phase as a function of the direction of scattering. The shorter the wavelength, the more rapid the alternation so, at very short wavelengths, the total cross section will contain no interference term; that term simply averages to zero. Then the molecular cross section is just twice the cross section of a single proton. On the other hand, at the opposite extreme of infinite wavelength, the distance between the scattering centers can be ignored, and the amplitudes (scattering lengths) of the waves add. The amplitude of the scattered wave is then just twice the amplitude of that scattered from a single proton, and the cross section, which is proportional to the square of the amplitude, is *four* times the proton cross section. However, the neutron energy is so low that the molecule as a whole recoils, and the proton cross section, according to Eq. (5–16), is $(\frac{4}{3})^2$ times the value for a free proton. Thus for H_2 the interference and recoil effects combine to yield an apparent slow neutron cross section per proton $\frac{3}{9}^2$ times greater than that of the free proton. This is the correct result (aside from the temperature effect) when the spins of both protons are parallel, but for antiparallel spins a more detailed analysis is required to take into account the spin dependence of the scattering length.

For this purpose, it is convenient to introduce an operator a which is equal to the singlet or triplet n-p scattering length when acting on the singlet or triplet n-p states, respectively. Since $(\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p) = 1$ in the triplet and $(\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p) = -3$ in the singlet state, the desired operator is just

$$a = \frac{a_s + 3a_t}{4} + \frac{a_t - a_s}{4}(\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p). \tag{5–20}$$

The spin states of the H_2 molecule are divided into two groups, that for which the proton spins form a triplet (orthohydrogen), and that for which they form a singlet (parahydrogen). Normal H_2 gas is a three-to-one mixture of ortho- and parahydrogen. If the spins of the two protons are denoted by $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, the spin state of the molecule is specified by the total spin

$$\mathbf{S} = \tfrac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2), \tag{5–21}$$

where

$$\mathbf{S}^2 = S(S+1),$$

and $S = 0$ for parahydrogen, $S = 1$ for orthohydrogen.

The scattering of neutrons of very long wavelength by H_2 is obtained by adding the scattering lengths of the two protons. Each of these is given by Eq. (5–20), with $\boldsymbol{\sigma}_p$ replaced by $\boldsymbol{\sigma}_1$ or $\boldsymbol{\sigma}_2$ as the case may be. However, the recoil effect is still to be taken into account. That is accomplished by replacing a_s and a_t by $a_s' = \frac{4}{3}a_s$ and by $a_t' = \frac{4}{3}a_t$, in accordance with Eq. (5–16). To avoid a complicated temperature dependence, we impose the further condition that the temperature of the gas be so low that the H_2 molecules are in their lowest rotational state. Then Eq. (5–19) provides an adequate description of the influence of molecular motion and σ_{CM} is to be determined directly from $\sigma_{CM} = 4\pi|\, a_M\,|^2$, where a_M is the molecular scattering length obtained by taking the sum of the a' values of the two protons. From Eqs. (5–20) and (5–21), we find

$$a_M = \frac{a_s' + 3a_t'}{2} + \frac{a_t' - a_s'}{2}(\boldsymbol{\sigma}_n \cdot \mathbf{S}). \tag{5–22}$$

The neutrons may be assumed to be unpolarized, so the average of σ_{CM} must be taken over both directions of neutron spin. We denote this average* by a bar. It is clear that $(\boldsymbol{\sigma}_n \cdot \mathbf{S}) = 0$, hence

$$\bar{\sigma}_{CM} = \pi[(a_s' + 3a_t')^2 + (a_t' - a_s')^2\overline{(\boldsymbol{\sigma}_n \cdot \mathbf{S})^2}].$$

The square of any component of $\boldsymbol{\sigma}_n$ is unity, therefore

$$(\boldsymbol{\sigma}_n \cdot \mathbf{S})^2 = \mathbf{S}^2 + \sigma_{nx}\sigma_{ny}S_xS_y + \sigma_{ny}\sigma_{nx}S_yS_x + \cdots,$$

and from the relationships

$$\sigma_{nx}\sigma_{ny} = i\sigma_{nz},$$

$$S_xS_y - S_yS_x = iS_z,$$

etc., it follows that

$$(\boldsymbol{\sigma}_n \cdot \mathbf{S})^2 = \mathbf{S}^2 - (\boldsymbol{\sigma}_n \cdot \mathbf{S}),$$

whence

$$\overline{(\boldsymbol{\sigma}_n \cdot \mathbf{S})^2} = S(S+1).$$

For orthohydrogen, the cross section is then

$$\sigma_{\text{ortho}} = \pi[(a_s' + 3a_t')^2 + 2(a_t' - a_s')^2], \tag{5–23}$$

while for parahydrogen it is

$$\sigma_{\text{para}} = \pi(a_s' + 3a_t')^2. \tag{5–24}$$

* The cross section actually is given by the spin-expectation value of $|\, a_M\,|^2$, i.e., $\Sigma_{sp}\chi^*|\, a_M\,|^2\chi$, where χ is the spin function of the entire system. The average over neutron spin states of any quantity linear in $\boldsymbol{\sigma}_n$ vanishes because $Tr\boldsymbol{\sigma}_n = 0$. A more detailed discussion of this matter is given in the next section.

It has been remarked by Teller* that a very sensitive test of the sign of a_s is provided by a measurement of the parahydrogen cross section. An indication of the degree of sensitivity can be obtained from our estimates $a_t \approx 2R_0$, $a_s \approx \pm 8R_0$, which lead to $a_s'/a_t' = a_s/a_t \approx \pm 4$. Since the ratio is luckily so near to 3, the parahydrogen cross section depends very strongly on the sign of a_s. The estimate gives a ratio $\frac{1}{9}$ of the cross section for positive a_s to that for negative a_s. The ratio of the ortho to the para cross section,

$$\frac{\sigma_{\text{ortho}}}{\sigma_{\text{para}}} = 1 + 2\frac{(1 - a_s/a_t)^2}{(3 + a_s/a_t)^2},$$

reflects this dependence on sign. For positive a_s it is close to 1, while for negative a_s it is about 50. Experimentally, $\sigma_{\text{ortho}}/\sigma_{\text{para}}$ is found to be much greater than 1, so the singlet scattering length is negative, a result we anticipated in Chapter 4.

Measurement of the ratio also provides a precise value of a_s/a_t, because the result depends sensitively on the small difference $(3 - |a_s/a_t|)$. The experimental value of a_s/a_t, when combined with the absolute value of the free proton cross section, $\sigma = \pi(a_s^2 + 3a_t^2)$, yields absolute values of both a_s and a_t. These are then independent of theory except insofar as temperature corrections and the like must be made. Recent experimental results† yield

$$\sigma_{\text{ortho}}/\sigma_{\text{para}} = 27.2,$$

which gives a ratio

$$a_s/a_t = -4.53.$$

The alternative solution for a_s/a_t, when combined with the total cross section, leads to much too large a value of a_t. The best value‡ of σ appears to be

$$\sigma = 20.36 \text{ barns} \pm 0.5\%, \tag{5-25}$$

which yields

$$a_t = 0.526 \times 10^{-12} \text{ cm} \approx 1.9R_0,$$

$$a_s = -2.38 \times 10^{-12} \text{ cm} \approx -8.5R_0.$$

The other solution of the quadratic equation is eliminated here because it corresponds to negative a_t. A more accurate determination will be discussed in Section 5–5.

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* *Phys. Rev.* **49,** 420 (1936). The detailed calculations are given by Schwinger and Teller, *Phys. Rev.* **52,** 286 (1937), Hamermesh and Schwinger, *Phys. Rev.* **71,** 678 (1947). See also B. A. Lippmann, *Phys. Rev.* **79,** 481 (1950).

† Sutton, Hall, Anderson, Bridge, DeWire, Lavatelli, Long, Snyder, Williams, *Phys. Rev.* **72,** 1147 (1947). Only the elastic part of σ_{ortho} has been used here, since just that is given by Eq. (5–23). Compare the remarks, below, on inelastic scattering.

‡ Melkonian, Rainwater, and Havens, *Phys. Rev.* **75,** 1295 (1949).

A few remarks concerning the method of separation of ortho- and parahydrogen may be appropriate at this point. The wave function of H_2 is antisymmetric for interchange of the two protons, since they obey Fermi statistics. This, with well-known properties of the electronic wave function of the H_2 molecule, establishes that only rotational states of even angular momentum can be attained by parahydrogen, while only states of odd angular momentum are accessible to orthohydrogen.* Since the rotational energy is proportional to the square of the angular momentum, the ortho and para states are separated in energy. In particular, the lowest energy state of zero angular momentum can only be occupied by parahydrogen. Therefore, if the hydrogen is at a temperature T for which kT is less than the spacing between the first two rotational states, the equilibrium state contains a large concentration of parahydrogen.†

The parahydrogen measurement is made on the low-temperature equilibrium state, but great care must be taken to determine the exact amount of orthohydrogen contaminant, since the ortho cross section is some thirty times the para cross section. A one percent contamination means that about twenty-five percent of the total scattering is due to orthohydrogen. Therefore an error in the measurement of the purity of the parahydrogen is greatly amplified as an error in the parahydrogen cross section.

Another effect must be taken into account to give a precise theoretical interpretation of the orthohydrogen cross section. Since the lowest energy state of orthohydrogen is higher than that of parahydrogen, the neutron can undergo an inelastic collision of the second kind, i.e., a collision in which ortho is converted to para with a transfer of energy to the neutron. However, for neutrons of infinite wavelength the appropriate scattering length is given by Eq. (5–22). Since this is diagonal in the total proton spin S of the molecule, such slow neutrons lead to no such conversion. On the other hand, for finite wavelengths the asymptotic wave function is the sum of the asymptotic waves scattered from the two protons

$$\psi \xrightarrow[r\to\infty]{} e^{i(\mathbf{k}\cdot\mathbf{r})}\chi - \frac{1}{r}\left[a_1' e^{ik|\mathbf{r}-\frac{1}{2}\mathbf{R}|} + a_2' e^{ik|\mathbf{r}+\frac{1}{2}\mathbf{R}|}\right]\chi,$$

where a_1' and a_2' are the scattering length operators of the two protons, χ is the spin function for all three particles, $\mathbf{R}$ is the distance between the protons, and $\mathbf{r}$ is measured from the center of mass of the molecule. Now for $kR \ll 1$,

$$\psi \xrightarrow[r\to\infty]{} e^{i(\mathbf{k}\cdot\mathbf{r})}\chi - \frac{e^{ikr}}{r}\left[a_1' + a_2' - \frac{ik}{2r}(a_1' - a_2')(\mathbf{r}\cdot\mathbf{R}) + \cdots\right]\chi.$$

* See, for example, G. Herzberg, *Molecular Spectra and Molecular Structure*, D. Van Nostrand, 2nd edition (1950), p. 128 ff.

† The rate of ortho-para conversion in pure H_2 is so slow that it is necessary to catalyze the process in order to obtain a reasonable concentration of parahydrogen.

In the scattered wave, the term proportional to $a_1' + a_2'$ is just that obtained in the infinite wavelength approximation. The next term, proportional to $a_1' - a_2'$, is capable of producing the conversion of ortho- to parahydrogen, since it contains the operator $\sigma_1 - \sigma_2$, which converts a triplet state into a singlet state. Note further that the added wave has the angular dependence of a P-wave; it can be interpreted as the P-scattering by the *molecule* which vanishes only when the neutron wavelength is large compared with molecular dimensions.

In concluding this discussion of ortho- and parahydrogen scattering, we should remark on one other important result that can be drawn from the very large ortho to para ratio. Schwinger* has called attention to the fact that the large ratio provides conclusive evidence for assigning to the spin of the neutron the value one-half. The only other acceptable value† of the spin, $\frac{3}{2}$, would lead to a ratio much closer to unity no matter what sign the scattering length happened to have.

5–4 The coherent scattering length. The scattering of slow neutrons from an array of nuclei such as is found in a molecule or crystal will lead to interference effects analogous to those discussed in the previous section. If only one isotope of a single nuclear species occurs in the scattering material, and if that nucleus has no spin, the total scattering of very slow neutrons from the array has a cross section $4\pi|\, Na\,|^2$, where N is the number of nuclei and a is the scattering length of a single nucleus. If several isotopes are present, but all have zero spin, the cross section is $4\pi|\, N_1 a(1) + N_2 a(2) + \cdots\,|^2$, where N_i is the number of nuclei of an isotopic type i with scattering length $a(i)$. This cross section can be rewritten as $4\pi|\, N\bar{a}\,|^2$, where $\bar{a}$ is the average scattering length of the mixture. Consideration of the scattering from isotopes of one kind with spin (total angular momentum) different from zero leads to a more complicated problem if the scattering length is spin-dependent. Then spin flips of individual nuclei may occur on scattering, and the associated scattered wave is not coherent with the waves radiating from other nuclei.‡ Since only part of the wave is

* *Phys. Rev.* **52**, 1250 (1937).

† Since the deuteron has spin one, and all data are quite consistent with the assignment of orbital angular momentum zero to at least part of the deuteron ground state.

‡ In this connection note that the scattered wave is a linear combination of products of spin functions; i.e., it is proportional to

$$\Phi = Na\varphi(1)\varphi(2)\cdots\varphi(N)\chi + a'|\varphi'(1)\varphi(2)\cdots \varphi(N)\chi' + \varphi(1)\varphi'(2)\cdots\varphi(N)\chi' + \cdots],$$

where $\varphi(i)$ is the initial spin state of the ith nucleus, χ is the initial spin state of the neutron, and $\varphi'(i)$, χ' are the corresponding spin-flipped states. Conservation of angular momentum requires that the neutron flip with the nucleus. The coefficients

(*Cont.*)

coherent, the scattering length of a single nucleus may be written as the sum of two terms,

$$a = a_c + a_i,$$

where a_c is the *coherent scattering length* and a_i is the incoherent scattering length. Interference terms in the scattering are produced only by a_c. In particular, for infinite wavelength the amplitudes a_c add, so the *coherent cross section* is $4\pi N^2 |a_c|^2$. The characteristic property of the coherent scattering is the proportionality to N^2. There is another, incoherent, term in the scattering cross section proportional to N, but N is so large for a macroscopic array of nuclei, such as a crystal, that the incoherent term is negligible. The scattering by such systems is therefore characterized by the coherent scattering length, a_c. For a system consisting of a mixture of isotopes, the appropriate quantity is the average coherent scattering length,

$$\bar{a}_c = \frac{1}{N} \sum_i N_i a_c(i). \tag{5-26}$$

Neutrons of very long wavelength have been used here as a basis for argument, since the discussion of the constructive interference is thereby greatly simplified. However, interference phenomena at shorter wavelengths, such as Bragg diffraction of neutrons from crystals, are also determined by the coherent scattering length. A Bragg spot is a point at which all scattered neutron waves arrive in phase; the amplitude of the wave is just the sum of the (coherent) amplitudes of waves scattered from all nuclei. Thus a_c is measurable by means of diffraction or other macroscopic interference phenomena, so it is of some importance to determine the connection between a_c and the scattering length a which is directly related to the interaction.

First let us consider the scattering of neutrons from a single nucleus of spin I. The combination of the nuclear spin with the neutron spin leads to two states of total angular momentum J, namely, $J = I \pm \frac{1}{2}$. Associated with each of the states there is a scattering length a_J. For example, in the n-p problem the values of J are 0 and 1, the corresponding a_J are the singlet and triplet scattering lengths. As in the n-p problem, we can introduce a scattering length operator a, which has the value a_J in a state of angular momentum J. Use is made here of the relationship

$$(\boldsymbol{\sigma}_n \cdot \mathbf{I}) = J(J+1) - I(I+1) - \tfrac{3}{4} \tag{5-27}$$

for the state in which J is specified. The operator $\mathbf{I}$ is the total angular

‡ continued

a and a' are the corresponding scattering lengths. The scattered intensity is proportional to the sum over all spin variables of $|\Phi|^2$. Interference terms between a and a', and those between the various spin-flipped states, vanish as a consequence of the orthogonality of $\varphi(i)$ and $\varphi'(i)$. Such terms are said to be *incoherent*.

momentum (spin) operator of the nucleus. For the two states $J = I \pm \frac{1}{2}$, Eq. (5–27) becomes

$$(\sigma_n \cdot \mathbf{I}) = I, \qquad J = I + \tfrac{1}{2},$$

$$(\sigma_n \cdot \mathbf{I}) = -(I + 1), \quad J = I - \tfrac{1}{2}.$$

Therefore the generalization of Eq. (5–20) is just

$$a = \frac{1}{2I + 1}[(I + 1)a_{I+\frac{1}{2}} + Ia_{I-\frac{1}{2}} + (a_{I+\frac{1}{2}} - a_{I-\frac{1}{2}})(\sigma_n \cdot \mathbf{I})]. \quad (5\text{–}28)$$

The wave function at a point of constructive interference of waves scattered from a large number N of nuclei is obtained by adding the waves scattered from each nucleus. If the initial spin state of all the nuclei and the neutron is Ξ and the scattering length operator of the ith bound nucleus is $a'(i)$,* then the wave function at the point in question is, for elastic scattering,

$$\psi_{\text{scatt}} = -\frac{e^{ikr}}{r} \Sigma_i a'(i)\Xi,$$

or, since each $a(i)$ is given by Eq. (5–28),

$$\psi_{\text{scatt}} = -\frac{e^{ikr}}{r} \frac{1}{2I + 1} \{N[(I + 1)a'_{I+\frac{1}{2}} + Ia'_{I-\frac{1}{2}}] + (a'_{I+\frac{1}{2}} - a'_{I-\frac{1}{2}})\Sigma_i(\sigma_n \cdot \mathbf{I}_i)\}\Xi.$$

Clearly, this can be rewritten in terms of

$$a_c = \sum_{J=I\pm\frac{1}{2}} (2J + 1)a_J \Big/ \sum_{J=I\pm\frac{1}{2}} (2J + 1) \quad (5\text{–}29)$$

as

$$\psi_{\text{scatt}} = -\frac{e^{ikr}}{r}\left\{Na'_c + \frac{1}{2I + 1}(a'_{I+\frac{1}{2}} - a'_{I-\frac{1}{2}})\Sigma_i(\sigma_n \cdot \mathbf{I}_i)\right\}\Xi. \quad (5\text{–}30)$$

It is now to be shown that, when the nuclear spins are randomly oriented, the coherent scattering length is the above-defined a_c. Note that the value of a_c so defined is just the average of a_J over all spin states.

The coefficient of $-e^{ikr}/r$ in Eq. (5–30) plays the role of a total scattering length, a_{total}, and the scattering cross section (for scattering to the point of constructive interference) is proportional to $|a_{\text{total}}|^2$. However, we are interested in the scattered intensity for any final spin orientation of the particles, so $|a_{\text{total}}|^2$ must be summed over all spin variables.

* The prime indicates that the scattering length refers to the bound nucleus.

Hence the cross section is proportional to the spin-expectation value of $|a_{\text{total}}|^2$:

$$\sigma \propto \left\langle \Xi, \left| Na_c' + \frac{1}{2I+1}(a_{I+\frac{1}{2}}' - a_{I-\frac{1}{2}}')\Sigma_i(\boldsymbol{\sigma}_n \cdot \mathbf{I}_i) \right|^2 \Xi \right\rangle.$$

Since the neutron beam is assumed to be unpolarized, the average of σ over initial states of the neutron is to be taken. Linear expressions in $\boldsymbol{\sigma}_n$ then vanish, and the average is found, as in Section 5–3, to be

$$\bar{\sigma} \propto N^2 a_c'^2 + \frac{1}{(2I+1)^2}(a_{I+\frac{1}{2}}' - a_{I-\frac{1}{2}}')^2 \langle (\Sigma_i \mathbf{I}_i)^2 \rangle,$$

where the expectation value is taken in the initial spin state of the scattering nuclei. Because the nuclear spins are randomly oriented, the products $(\mathbf{I}_i \cdot \mathbf{I}_j)$ appearing in the last term sum to zero and only the factor $\Sigma_i \mathbf{I}_i^2 = NI(I+1)$ remains. Hence

$$\bar{\sigma} \propto N^2 a_c'^2 + N \frac{I(I+1)}{(2I+1)^2}(a_{I+\frac{1}{2}}' - a_{I-\frac{1}{2}}')^2, \tag{5-31}$$

and the second term, which is proportional to N, contributes only to the incoherent scattering, leaving the coherent scattering to be completely determined by a_c'. Thus a_c' is the coherent scattering length.

It is of some interest to note that the coherent scattering length for the proton is

$$a_c'(\text{proton}) = \tfrac{1}{4}(3a_t + a_s),$$

which is just the combination that appears in the parahydrogen cross section. Therefore, any measurement of a_c' for the proton provides the same information as the parahydrogen measurement.

Crystal diffraction* of neutrons offers a direct method for determining the value of a_c'. Relative intensities of the Debye-Scherrer rings produced by a polyatomic crystal powder provide information concerning the relative signs of the coherent scattering lengths of the different nuclei.† Similar information is obtained from the transmission of neutrons through crystal powders.‡ Furthermore, the absolute values of the intensities of

* A detailed theoretical discussion will not be given here. Basically, the theory is very close to that of x-ray diffraction. Some important references to the theory are as follows: G. C. Wick, *Phys. Zeits.* **38,** 403 (1937); I. Pomerantschuk, *Phys. Zeits. d. Sowjetunion* **13,** 65 (1938); O. Halpern, M. Hamermesh, and M. H. Johnson, *Phys. Rev.* **59,** 981 (1941); R. Weinstock, *Phys. Rev.* **65,** 1 (1944); M. L. Goldberger and F. Seitz, *Phys. Rev.* **71,** 294 (1947); R. J. Finkelstein, *Phys. Rev.* **72,** 907 (1947); G. Placzek, B. R. A. Nijboer, and L. Van Hove, *Phys. Rev.* **82,** 392 (1951).

† E. Fermi and L. Marshall, *Phys. Rev.* **71,** 666 (1947); E. O. Wollan and C. G. Shull, *Phys. Rev.* **73,** 830 (1948).

‡ E. Fermi, W. J. Sturm, and R. G. Sachs, *Phys. Rev.* **71,** 589 (1947).

the Debye-Scherrer rings provide a numerical value for a_c. This method has been applied to crystals of NaH by Shull, Wollan, Morton, and Davidson* to obtain a value of a_c for the proton. They find

$$a_c \text{ (free proton)} = -(2.0 \pm 0.1) \times 10^{-13} \text{ cm},$$

which is in reasonable agreement with the parahydrogen result. The accuracy of the measurement is limited by the large background of incoherent scattering and by the diffuse scattering due to thermal vibrations of the crystal. The fact that the theoretical value of the temperature diffuse scattering is not well known also introduces the possibility of a systematic error in the evaluation of the results. A method which avoids these difficulties is the critical reflection experiment discussed in the next section.

5–5 Critical reflection of neutrons. A method for determining the coherent scattering length of the nuclei in a medium is to measure its index of neutron refraction, since refraction is the result of interference between the incident and scattered waves. This method has been used to fix the absolute signs of the scattering lengths of some nuclei and to obtain the very important coherent cross section of the proton. As a matter of fact, it provides what is probably the most accurate method for determining the latter quantity.

A convenient way to derive an expression for the index of refraction is to consider a neutron wave incident normally to a plane surface of the medium. Then refraction appears as a change in wavelength of the wave function within the medium. The apparent change in wavelength is a consequence of the interference between the incident wave and the waves elastically scattered in the forward direction by the various nuclei. The elastically scattered wave can be determined in Born approximation by use of the auxiliary potential:

$$\psi_{\text{scatt}} = -\frac{e^{ikr}}{r}\frac{1}{2\pi\hbar^2}\sum_j \mu_j \int e^{-i(\mathbf{k}'\cdot\mathbf{r})} \mathcal{V}_j(\mathbf{r} - \mathbf{R}_j) e^{i(\mathbf{k}\cdot\mathbf{r})}\, d^3r,$$

where $\mathcal{V}_j$ is the auxiliary potential associated with the jth nucleus and $\mathbf{k}'$ is the propagation vector of the scattered wave. But as a consequence of our interest only in forward elastic scattering, $\mathbf{k}' = \mathbf{k}$, whence†

$$\psi_{\text{scatt}}^{(\text{forward})} = -\frac{e^{ikr}}{r}\frac{M}{2\pi\hbar^2}\sum_j \int \mathcal{V}_j(\mathbf{r} - \mathbf{R}_j)\, d^3r. \tag{5–32}$$

* *Phys. Rev.* **73**, 842 (1948).

† The use of the neutron mass M in place of the reduced mass is based on the fact that we are discussing coherent phenomena in which the entire medium participates as a unit. Therefore the effective reduced mass is just M.

The integral appearing here is proportional to the scattering length, Eq. (5–7), so *any* auxiliary potential of sufficiently long range can be used to calculate the forward scattering as long as it leads to the correct scattering length. The limitation on range, Eq. (5–9), can be ignored for present purposes. The auxiliary potential may then be conveniently defined as a well of constant depth acting over the volume per nucleus where the volume is centered on the nucleus and the wells of neighboring nuclei adjoin. Within the volume, the strength $\mathcal{V}_0$ of the auxiliary potential is determined by Eq. (5–7) or its equivalent,

$$a_c' = \frac{1}{2\pi} \frac{M}{\hbar^2} \frac{\mathcal{V}_0}{\mathfrak{N}}, \tag{5–33}$$

where $\mathfrak{N}$ is the number of scattering centers per unit volume, and a_c' is their common coherent scattering length, corrected for the binding effect. The auxiliary potential for forward scattering due to all scattering centers is simply a well of depth $\mathcal{V}_0$ extending over the entire region occupied by matter.

The calculation of the incident plus forward scattered wave can now be accomplished directly; it is just the solution of the Schroedinger equation for the shallow potential well subject to the condition of continuity with the incident wave at the surface of the scattering medium.* Inside the matter, the interesting part of the wave function is a plane wave of kinetic energy k'^2, where

$$k'^2 = k^2 - \frac{2M}{\hbar^2} \mathcal{V}_0$$

or, by Eq. (5 33),

$$k'^2 = k^2 - 4\pi\mathfrak{N}a_c'.$$

The index of refraction, ν, can be defined as the ratio of the neutron wavelength outside the medium to that inside the medium, or

$$\nu = \frac{k'}{k} = \sqrt{1 - \frac{4\pi\mathfrak{N}a_c'}{k^2}}.$$

* The wave function obtained in this way is not, of course, the entire neutron function. In a crystal, for example, there are coherent waves produced by Bragg scattering, and inelastically scattered coherent waves (temperature diffuse scattering) which are not included. These do not appear here because the location of the scattering centers, which is unimportant for the forward scattering, has been lost in the process of averaging the potential. This loss is of no concern, since we are interested only in the forward scattering.

The quantity $4\pi\mathfrak{N}a_c/k^2$ is quite small (of order $4\pi ka_c$). A very good approximation is provided by the expansion

$$\nu = 1 - 2\pi \frac{\mathfrak{N}a_c'}{k^2}. \tag{5-34}$$

We see that the index of refraction is very close to 1; it is larger than 1 for negative scattering length and smaller than 1 for positive scattering length.

The most practical experimental method for obtaining the index of refraction is to determine the angle, θ_c, of critical reflection from a plane surface of the material. Critical reflection from the surface will occur only if the index of refraction is less than 1, thus only for a positive coherent scattering length. The value of θ_c may be obtained from Eq. (5-34) by means of Snell's law. If the angle is measured from the reflecting surface,

$$\cos\theta_c = 1 - 2\pi \frac{\mathfrak{N}a_c'}{k^2},$$

or, since $\theta_c \approx 0$,

$$\theta_c = \frac{2}{k}\sqrt{\pi\mathfrak{N}a_c'}. \tag{5-35}$$

When the scattering material is made up of a mixture of two nuclear species with coherent scattering lengths $a_c^{(1)}$ and $a_c^{(2)}$, the critical angle is

$$\theta_c = \frac{2}{k}\sqrt{\pi(\mathfrak{N}_1 a_c'^{(1)} + \mathfrak{N}_2 a_c'^{(2)})}, \tag{5-36}$$

where $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are the numbers per unit volume of each nuclear type.

An absolute measurement of the critical angle clearly provides a direct measurement of the coherent scattering length. Determination of the mere existence of critical scattering is a means of establishing that a coherent scattering length is positive. This was used by Fermi and Zinn* to fix the signs of the coherent scattering lengths of several elements. Their results can be combined with the measurements of relative sign by crystal diffraction [Section (5-4)] to obtain the signs for many elements.† It turns out that the scattering lengths of most nuclei are positive.

A direct method for determining the coherent scattering length of one nucleus in terms of that of another nucleus is suggested by Eq. (5-36). Measurement of the critical angle for a known mixture of the nuclei provides the answer. A modification of this method has been used by Hughes, Burgy, and Ringo‡ to obtain an accurate value of the coherent scattering

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* *Phys. Rev.* **70,** 103 (1946).

† E. Fermi and L. Marshall, *Phys. Rev.* **71,** 666 (1947).

‡ *Phys. Rev.* **77,** 291 (1950); *ibid.* **84,** 1160 (1951).

length of the proton in terms of that of carbon. They compare the critical angles at a given neutron energy for mirrors consisting of various liquid hydrocarbons. Since carbon and the proton have scattering lengths of the opposite sign, the hydrocarbons can be chosen to have an average coherent scattering length near zero. The square of θ_c is then a linear function of the carbon-to-hydrogen ratio, and only a slight extrapolation of the observed values is required to determine the ratio corresponding to $\theta_c = 0$. This gives directly and accurately the ratio of the proton scattering length to that of carbon. The latter value is known directly from the scattering cross section of the isotopically abundant C^{12}, which produces no incoherent scattering, since its spin is zero. The value obtained for the coherent scattering length of the bound proton in this way is $-(3.78 \pm 0.02) \times 10^{-13}$ cm. Since the scattering length of the bound proton is just twice that of the free proton, we have

$$\tfrac{1}{4}(3a_t + a_s) = -(1.89 \pm 0.01) \times 10^{-13} \text{ cm}.$$

When this is combined with the value of the scattering cross section, Eq. (5–25), the triplet and singlet scattering lengths of the proton are found to be

$$\begin{aligned} a_t &= 0.538 \times 10^{-12} \text{ cm}, \\ a_s &= -2.37 \times 10^{-12} \text{ cm}. \end{aligned} \tag{5–37}$$

CHAPTER 6

FURTHER CONSIDERATIONS OF THE TWO-BODY PROBLEM

The great variety of available low-energy data concerning the n-p system has led us to concentrate on them as a source of information concerning the nuclear interaction. However, these data do not resolve such important issues as the shape and exchange character of the interaction, its velocity dependence, or its charge dependence (is n-p = p-p = n-n?). To gain as deep an insight into these matters as possible, it is necessary to exploit all other sources of information concerning the two-body system, including high-energy neutron-proton scattering, proton-proton scattering at low and high energies, the capture of neutrons by protons, and its inverse, the photodisintegration of the deuteron.

For any such process the general features of the analysis at low energy are much the same as for the low-energy neutron-proton problem; only certain average properties of the interaction enter into the quantitative results. On the other hand, the high-energy reactions require a more detailed treatment than can be presented here. Consequently, we shall endeavor to indicate only qualitatively the way in which the high-energy cross sections are related to the relevant properties of the interaction. This qualitative discussion will be supplemented by a summary of conclusions drawn by detailed analysis from existing high-energy data.

6–1 Neutron-proton scattering at intermediate and high energies. We have found that for neutrons of laboratory energy less than some 5 Mev, only S-scattering need be considered, while at somewhat higher energies, the P-wave is expected to be of appreciable importance. As the energy is increased, higher and higher angular momenta come into play, the importance of the scattering of waves of higher l being governed by the ratio of the wavelength to the range of interaction. But the same ratio measures the sensitivity of the scattering to the shape of the potential: as the wavelength becomes short compared with the range, the detailed form of the surface of the potential well plays a greater and greater role in determining the pattern of scattering. Therefore the development of scattered waves of higher l indicates an increasing sensitivity of the scattering to the shape of the interaction. Furthermore, the scattering provides direct information concerning the exchange character of the interaction, since the Majorana potential alternates in sign between even and odd l values.

At about 5 Mev laboratory energy the interference of the P-wave with the S-wave is large enough to warrant consideration. Its contribution to the differential cross section is given by Eq. (4–25):

$$\Delta\sigma(\theta) = -\frac{6\,\mathrm{Re}\,(f_0)}{k}\,\delta_1\cos\theta, \tag{6-1}$$

where δ_1 is the P-wave phase shift. The angular distribution is unsymmetrical between the forward and backward directions, a behavior characteristic of the interference between waves of opposite parity. An idea of the dependence of $\Delta\sigma$ on the properties of the potential can be had from consideration of the square well, for which the P phase shift is calculated in Section 4–5 on the assumption that the potential is of the Majorana type:

$$\delta_1^x = -\frac{(k\alpha)^3}{3}\left[1 - \frac{3}{(K\alpha)^2}(K\alpha - 1)\right].$$

The corresponding result for ordinary forces is obtained by means of a similar straightforward analysis,

$$\delta_1^o = -\frac{(k\alpha)^3}{3}\left[1 + \frac{3}{(K\alpha)^2}(K\alpha\cot K\alpha - 1)\right].$$

A simplification of the latter result is obtained by use of Eq. (4–35), namely,

$$\delta_1^o = -\frac{(k\alpha)^3}{3}\left[1 - \frac{3}{(K\alpha)^2}\frac{a}{a-\alpha}\right], \tag{6-2}$$

where a is the appropriate scattering length.

Evaluation of the phase shift for the ${}^3P$ wave can now be made on the basis of the triplet scattering length, $a_t = 5.38 \times 10^{-13}$ cm [Eq. (5–37)]. The corresponding square-well range and depth are found from Eqs. (4–36) and (3–25) to be $\alpha = 2.05 \times 10^{-13}$ cm, or $\alpha/a_t = 0.381$, and $(K_t\alpha)^2 = 3.56$, whence the resulting phase shift for ordinary forces is

$${}^3\delta_1^o \approx 0.12(k\alpha)^3,$$

while for exchange forces it is

$${}^3\delta_1^x \approx -0.08(k\alpha)^3.$$

If the same range is taken for the singlet potential, $\alpha/a_s \approx -0.0865$, in terms of the observed singlet scattering length. From Eq. (4–40) we have, to good approximation, $K_s\alpha = \pi/2$. Thus the singlet phase shifts are

$${}^1\delta_1^o \approx 0.04(k\alpha)^3,$$

$${}^1\delta_1^x \approx -0.10(k\alpha)^3.$$

Only the sign of $\mathrm{Re}(f_0)$ remains to be determined in order to fix the sign of the interference term, Eq. (6 1). At zero energy, $f_0 = a$, so the sign is positive for the triplet state, negative for the singlet state. However, the phase of f_0 does, of course, depend on the energy, and it turns out that for the triplet state a reversal in sign occurs just at those energies for which P-scattering is expected to be important. A rough estimate of the point at which the sign changes can be obtained from the relationship (Section 4–6)

$$k \cot ({}^3\delta_0) = -\frac{1}{a_t} + \frac{1}{2} r_{0t} k^2.$$

According to Eq. (4–23), the real part of f_0 vanishes when ${}^3\delta_0 = \pi/2$, so the change in sign occurs when $\cot ({}^3\delta_0) = 0$ or, in the effective range approximation, when

$$k^2 = \frac{2}{r_{0t} a_t}.$$

The corresponding energy is

$$E = \frac{2}{(r_{0t} k_g)(a_t k_g)} E_g.$$

For $a_t \approx 2R_0$ and the value of r_{0t} given by Eq. (4–54), $k_g r_{0t} = \frac{1}{2}$, we find $E \approx 7$ Mev or $E_L \approx 14$ Mev. The estimate is rough because an extension of the effective range over a rather wide energy interval is involved, but the sign of $\mathrm{Re}\ (f_0)$ for the triplet scattering certainly changes in that region of energies for which the P-scattering starts to become important. Consequently, the S-P interference term is expected to be particularly small when the energy is in this neighborhood.

For singlet scattering, $\mathrm{Re}\ (f_0)$ is expected to be negative over quite a wide energy interval, since it is negative at zero energy. Therefore both the triplet and singlet S-P interference terms have the same sign on the high energy side of the point at which the triplet term reverses sign. The sign is such that ordinary forces lead to a slight predominance of forward scattering over back scattering, while exchange forces do just the opposite.

Although our estimates have been made for weak P-scattering, the qualitative dependence on the exchange character of the potential will carry through to higher energies. The reversal of the fore-and-aft asymmetry between exchange and ordinary forces is a direct consequence of the fact that the exchange potential is repulsive in the P-state, while the ordinary potential is attractive. Data on neutron-proton scattering at intermediate energies therefore could provide direct information concerning the exchange property of the interaction. Sufficiently precise data on the angular distribution would also yield a value of the range of the singlet potential for a given shape, but the required precision has not yet been attained.

Neutron-proton scattering data at intermediate energies (5 to 30 Mev laboratory energy) are sparse. At 14 Mev the work of Barschall and Taschek* shows no significant deviation from angular isotropy, although a five percent reduction in the scattering cross section at 90° (center of mass) cannot be ruled out. This result is confirmed by work of Baldwin** between 18 and 21 Mev, but at an energy of 28 Mev, Brolley, Coon, and Fowler† find definite evidence for an angular dependence of the scattering. The ratio of the scattering at 180° to that at 90° is given as 1.28 ± 0.10. The only information they provide concerning fore-and-aft asymmetry is the statement that the observed total cross section is not in disagreement with the value obtained on the assumption that the scattering between 0° and 90° is the same as is observed between 180° and 90°. However, a small amount of fore-and-aft asymmetry cannot be excluded. Unfortunately, precise angular distribution experiments are very difficult in this energy region because intense, well-collimated neutron beams are not available.

A more sensitive test of the shape and the exchange property of the potential is to be obtained by going to much higher energies. Calculation of the high-energy cross section involves the shape of the potential in every detail, and no simple general treatment of the problem can be given except at such high energies that the Born approximation is applicable. The condition for the validity of that approximation is‡

$$K^2\alpha/2k \ll 1,$$

where K^2 is the average strength of the potential. The above estimate $K^2\alpha^2 \approx 3.6$ for a square well leads to the condition $k\alpha \gg 1.8$ or

$$E \gg 3.2\frac{E_g}{(k_g\alpha)^2} \approx 30 \text{ Mev}.$$

Thus the laboratory value of the neutron energy must be much greater than 60 Mev, a condition which is not too well satisfied by neutrons in the energy range of interest, namely, 30 to 300 Mev. Consequently, the Born approximation certainly cannot be expected to yield accurate quantitative results. Nevertheless, it should provide a qualitative idea of the dependence of the scattering on the characteristics of the potential, and we shall use it with that in mind.

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* *Phys. Rev.* **75,** 1819 (1949).

** *Phys. Rev.* **83,** 495 (1951).

† *Phys. Rev.* **82,** 190 (1951).

‡ See Schiff, *Quantum Mechanics*, McGraw-Hill (1949), p. 168.

The Born formula for the differential cross section is

$$\sigma(\mathbf{k}') = \frac{1}{(4\pi)^2}\left|\int e^{-i\mathbf{k}'\cdot\mathbf{r}} W(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3r\right|^2, \tag{6-3}$$

where $\mathbf{k}$ is the propagation vector of the incident beam and $\mathbf{k}'$ that of the scattered beam in the direction of observation. W is the potential, including the appropriate exchange operator, measured in units of $\hbar^2/M$. Since the integral is just the $\mathbf{k} - \mathbf{k}'$ Fourier transform of W, it is apparent that the cross section is quite shape dependent. Any wrinkles on the surface of the potential well introduce Fourier components which then contribute to the high-energy scattering. The dependence on the exchange property of the potential can also be seen quite easily. Let us compare an ordinary and an exchange potential of the same shape:

$$W^x(\mathbf{r}) = W^o(\mathbf{r})P.$$

The cross section for ordinary forces is

$$\sigma^o(\mathbf{k}') = \frac{1}{(4\pi)^2}\left|\int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} W^o(\mathbf{r})\, d^3r\right|^2,$$

while for the exchange potential it is

$$\sigma^x(\mathbf{k}') = \frac{1}{(4\pi)^2}\left|\int e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} W^o(\mathbf{r})\, d^3r\right|^2.$$

Consequently,

$$\sigma^x(\mathbf{k}') = \sigma^o(-\mathbf{k}'),$$

so, in Born approximation, the forward scattering by the one potential equals the back scattering by the other.

At very high energies, such that the kinetic energy is great compared with the interaction potential, it is expected that a neutron will hardly be scattered. The cross section for ordinary forces should then show a strong maximum in the forward direction, whereupon it follows that an exchange potential should lead to strong back scattering. Since the recoiling proton goes forward in the latter case, the exchange potential can be said to cause an exchange of the electric charge from the proton to the neutron, which then continues on its way as a proton. This interpretation receives support from the analysis of the currents produced under the influence of exchange forces, as we have seen in Section 4–2.

To obtain more quantitative information concerning the cross section, a definite form of the potential function must be inserted into Eq. (6–3). The square well is now a very poor choice, since wild fluctuations are introduced into the Fourier transform by its most unphysical feature, the sharp edges. Let us consider, for the sake of definiteness, a central Yukawa potential without exchange. Then the integral

$$w(\mathbf{k} - \mathbf{k}') = \int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} W(\mathbf{r})\, d^3r$$

becomes

$$w(\mathbf{k} - \mathbf{k}') = K^2 \int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} e^{-r/\alpha}(\alpha/r)\, d^3r,$$

which is easily evaluated. The result is

$$w(\mathbf{k} - \mathbf{k}') = \frac{4\pi K^2\alpha^3}{4k^2\alpha^2 \sin^2(\theta/2) + 1}, \tag{6-4}$$

so the differential cross section has the form

$$\sigma(\theta) = \frac{\alpha^2(K\alpha)^4}{[4k^2\alpha^2 \sin^2(\theta/2) + 1]^2}. \tag{6-5}$$

The anticipated strong maximum in the forward direction is evident, and the sharpness of the peak is determined by the range α. In the limit $\alpha \to \infty$, the cross section goes over to the Rutherford formula if $K^2\alpha$ is required to be finite.

Some conception of the angular distribution is given by the ratio, in the center of mass system, of the forward to 90° scattering,

$$\frac{\sigma(0)}{\sigma(\pi/2)} = (2k^2\alpha^2 + 1)^2,$$

and the ratio of forward to backward scattering,

$$\frac{\sigma(0)}{\sigma(\pi)} = (4k^2\alpha^2 + 1)^2.$$

An appropriate value of α may be obtained from analysis of the observed value of the triplet scattering length:

$$\alpha = 1.18 \times 10^{-13} \text{ cm}.$$

As a basis for a sample calculation, we consider a laboratory energy of 90 Mev. Then

$$k^2\alpha^2 = 1.57,$$

whence

$$\frac{\sigma(0)}{\sigma(\pi/2)} = 17,$$

while

$$\frac{\sigma(0)}{\sigma(\pi)} = 53.$$

Corresponding results for an exchange potential may be obtained by interchanging fore and aft:

$$\frac{\sigma_x(\pi)}{\sigma_x(\pi/2)} = 17$$

and

$$\frac{\sigma_x(0)}{\sigma_x(\pi)} = \tfrac{1}{53}.$$

The relative angular distribution is the same for both singlet and triplet scattering if the singlet potential has the same range as the triplet.

The magnitude of the total cross section is obtained by integrating Eq. (6–5) over all angles,

$$\sigma = \frac{4\pi(K\alpha)^4\alpha^2}{4k^2\alpha^2 + 1}.$$

The appropriate triplet state value of $K\alpha$ can be found from Table 3–1, $K_t^2\alpha^2 = 2.30$ for our choice of range. The singlet value is given by Eq. (4–41), $K_s^2\alpha^2 = 1.68$. The corresponding total cross section is

$$\sigma = \tfrac{3}{4}\sigma_t + \tfrac{1}{4}\sigma_s \approx 11 \times 10^{-26}\ \text{cm}^2.$$

Relevant high energy n-p scattering cross sections have been given by Hadley, Kelly, Leith, Segré, Wiegand, and York* for 40- and 90-Mev neutrons, by Kelly, Leith, Segré, and Wiegand† for neutrons of 260 Mev energy, and by Wallace‡ for neutrons of energy 90 Mev. The total cross section at 90 Mev is found to be about 8×10^{-26} cm^2, in order of magnitude agreement with our estimate. At 260 Mev the observed cross section is 3.5×10^{-26} cm^2, which is to be compared with the theoretical estimate of 4.3×10^{-26} cm^2.

The experiments provide a good measure of the relative angular distribution, since the high-energy neutron beam is well collimated. The results are shown in Fig. 6–1. It is apparent that these results do not agree with what was expected for either ordinary or exchange forces. There is strong scattering both in the backward and the forward directions. In fact, although the measurements have not been carried to small angles, the data suggest fore-and-aft symmetry, which would imply that only waves of one parity are scattered. This behavior has suggested the introduction of a fifty-fifty mixture of ordinary and exchange interactions, $\frac{1}{2}(1 + P)$, which is referred to as the Serber potential. One can easily see, by appropriately modifying Eq. (4–11), that such an operator leads to a vanishing interaction in an odd state, so only waves of even orbital angular momentum are scattered.

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* *Phys. Rev.* **75,** 351 (1949).
† *Phys. Rev.* **79,** 96 (1950).
‡ *Phys. Rev.* **81,** 493 (1951).

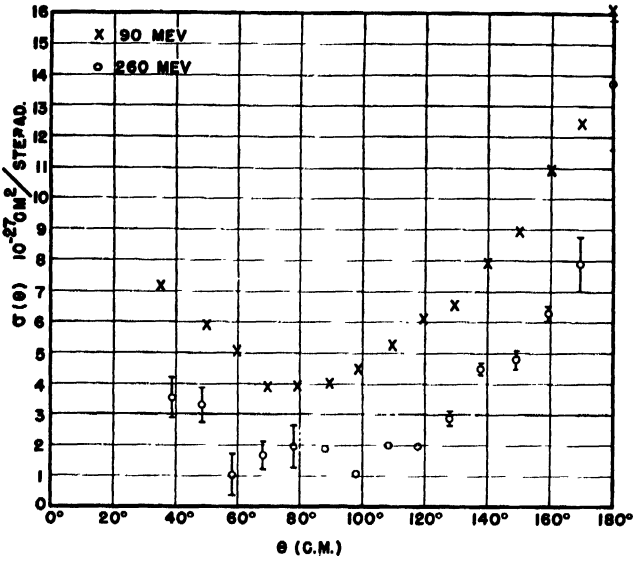

FIG. 6–1. Neutron-proton scattering at 90 and 260 Mev. [From Kelly, Leith, Segrè, and Wiegand, *Phys. Rev.* **79,** 96 (1950).]

The Serber exchange, when applied to a Yukawa potential, leads to the following modification of Eq. (6–4):

$$w^s(\mathbf{k} - \mathbf{k}') = 2\pi K^2\alpha^3\left(\frac{1}{4k^2\alpha^2\sin^2\theta/2 + 1} + \frac{1}{4k^2\alpha^2\cos^2\theta/2 + 1}\right).$$

Hence the Born approximation to the differential cross section becomes

$$\sigma(\theta) = \frac{1}{4}(K\alpha)^4\alpha^2\left[\frac{1}{4k^2\alpha^2\sin^2\theta/2 + 1} + \frac{1}{4k^2\alpha^2\cos^2\theta/2 + 1}\right]^2. \quad (6\text{–}6)$$

The resulting ratio of back scattering to 90° scattering is

$$\frac{\sigma(\pi)}{\sigma(\pi/2)} = \frac{(2k^2\alpha^2 + 1)^4}{(4k^2\alpha^2 + 1)^2}.$$

At 90 Mev this gives $\sigma(\pi)/\sigma(\pi/2) \approx 5.5$, which is in rather good agreement with the observed value of about 4, certainly in much better agreement than the value obtained for pure exchange. At 260 Mev our estimate is 28, while the experimental value is about 14. The agreement is again improved over the situation with pure exchange forces.

The total cross section resulting from integration of Eq. (6–6) is

$$\sigma^s = \sigma^o\left[\frac{1}{2} + \frac{1}{8k^2\alpha^2}\frac{1 + 4k^2\alpha^2}{1 + 2k^2\alpha^2}\ln(1 + 4k^2\alpha^2)\right]. \quad (6\text{–}7)$$

where σ^0 is the cross section obtained for ordinary (or pure exchange) forces. At 90 Mev the correction factor is about 0.78, so the estimated value of the cross section is about 8.6×10^{-26} cm^2, while at 260 Mev the factor is 0.65, which leads to a cross section of 2.8×10^{-26} cm^2. Both results are in fortuitously good agreement with the experimental values.

In order to draw from the data reliable conclusions concerning the interaction, a more careful treatment than can be provided by the Born approximation is required. The tensor potential must also be included in any such careful analysis. A program of this sort has been carried out by Christian and Hart,* who fix the parameters of the potential on the basis of the low-energy data and then seek to fit the high-energy results. They find (a) that the angular distribution indicates an exchange operator close to the Serber potential; (b) that to fit the angular distribution at 40 Mev as well as at the higher energies the potential must be assumed to have a long tail, similar to that associated with a Yukawa or exponential potential; (c) the calculated total cross section is always too high by some 30 percent.

It should be kept constantly in mind that static potentials may not provide an adequate basis for the analysis of experiments at these energies, which are comparable to that required for producing mesons.

6–2 Proton-proton scattering. Low energy. The specifically nuclear interaction between two protons is expected to be about the same as that between neutron and proton, on the basis of arguments presented in Chapter 2. At least the absence of any obvious contradiction allows us to assume that nature has arranged things in this manner, which is so satisfying to our sense of symmetry. However, even if the forces are identical, the dynamical behavior of the p-p system must be distinctly different from that of the n-p system because of the restrictions imposed on the p-p wave function by the Pauli principle. Two protons exist in the triplet state only if the parity is odd, and they exist in a singlet state only if the parity is even. This has the immediate consequence that the analog of the ^3S ground state of the deuteron cannot exist for the diproton (He^2). Since the ^1S state of the deuteron is virtual, the fact that the corresponding state of the diproton is unstable, as evidenced by the absence of a stable He^2 in nature, tends to confirm our notion of the equality of forces. Of course, the coulomb repulsion between protons would cause the He^2 state to be even higher in energy if everything else were equal.

The absence of a stable ground state means that the variety of information available concerning the diproton does not compare with that provided by the deuteron. Only scattering data are available and, even then,

* *Phys. Rev.* **77**, 441 (1950).

the states in which scattering can occur are limited by the exclusion principle. Low-energy scattering can provide information only about the singlet interaction, since the ^3S state is excluded. Nevertheless, of all the methods for obtaining direct information concerning the interaction between two nucleons, proton-proton scattering is by far the most precise, since the charge of the proton makes possible accurate energy and angle definition of the incident and scattered beam.

The theoretical analysis of low-energy p-p scattering has been given in great detail by Breit and his co-workers.* When the results are expressed directly in terms of the properties of a nuclear potential of given shape, they take on a rather complex appearance. However, at low energies only certain average properties of the nuclear interaction play a role in determining the cross section, just as is the case for n-p scattering. It is convenient to summarize the results in terms of these averages, which are analogous to the scattering length and effective range of the neutron-proton system. Formulation of the problem in the desired terms is contained implicitly in the early papers of Breit et al., but explicit use of the method came much later.† Before we go further into this question, some of the general principles underlying analysis of the scattering data must be considered.

For the proton-proton system, the radial equation analogous to Eq. (4–18) is

$$u_l'' + \left[k^2 - \frac{l(l+1)}{r^2} - \frac{2}{Dr} + W(r)\right] u_l = 0, \tag{6–8}$$

where $D = 2\hbar^2/Mc^2 = 5.76 \times 10^{-12}$ cm is the "Bohr radius" of the two-proton system. For the sake of convenience, the exchange character of the potential is assumed to be contained implicitly in W. An auxiliary equation is obtained by setting $W = 0$, namely,

$$v_l'' + \left[k^2 - \frac{l(l+1)}{r^2} - \frac{2}{Dr}\right] v_l = 0. \tag{6–9}$$

The function v_l is the radial function for pure coulomb scattering. Regular

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* Breit, Condon, and Present, *Phys. Rev.* **50,** 825 (1936); Breit, Thaxton, and Eisenbud, *Phys. Rev.* **55,** 1018 (1939); Hoisington, Share, and Breit, *Phys. Rev.* **56,** 884 (1939). For an up-to-date summary of data and theory see Jackson and Blatt, *Rev. Mod. Phys.* **22,** 77 (1950), and Yovits, Smith, Hull, Bengston, and Breit, *Phys. Rev.* **85,** 540 (1952).

† Landau and Smorodinsky, *J. Phys. Acad. Sci. U.S.S.R.*, **8,** 154 (1944); Smorodinsky, *ibid.* **8,** 219 (1944); **11,** 195 (1947); F. C. Barker and R. E. Peierls, *Phys. Rev.* **75,** 312 (1949); G. F. Chew and M. L. Goldberger, *Phys. Rev.* **75,** 1637 (1949); H. A. Bethe, *Phys. Rev.* **76,** 38 (1949). A treatment of the problem has been given by Schwinger in unpublished lectures; see Jackson and Blatt, *loc. cit.*

solutions, F_l, of Eq. (6–9) have the asymptotic behavior*

$$F_l \xrightarrow[r \to \infty]{} \frac{1}{k} \sin\left(kr - \frac{1}{2} l\pi + \eta_l - \frac{1}{kD} \ln 2kr\right), \tag{6–10}$$

where

$$e^{2i\eta_l} = \frac{\Gamma(l + 1 + i/kD)}{\Gamma(l + 1 - i/kD)}.$$

The irregular solutions, G_l, have the asymptotic behavior

$$G_l \xrightarrow[r \to \infty]{} \frac{1}{k} \cos\left(kr - \frac{l\pi}{2} + \eta_l - \frac{1}{kD} \ln 2kr\right). \tag{6–11}$$

Therefore, the standard asymptotic form of u_l is taken to be

$$u_l \xrightarrow[r \to \infty]{} F_l \cos \delta_l + G_l \sin \delta_l = \frac{1}{k} \sin\left(kr - \frac{l\pi}{2} + \eta_l + \delta_l - \frac{1}{kD} \ln 2kr\right). \tag{6–12}$$

The appearance of the logarithmic phase in these functions is a direct consequence of the fact that the coulomb potential manifests itself at extremely long distances, even in the asymptotic region. The phase η_l can be interpreted as the phase shift due to the coulomb potential (aside from the logarithmic phase) and δ_l is clearly the phase shift produced by the interaction W. Because of the long range of the coulomb potential, none of the phase shifts η_l may be treated as small, but the short-range nuclear potential leads to a rapidly converging sequence of phases δ_l as long as the collision energy is reasonably small. Thus at very low energies only δ_0 will be important, just as is the case for n-p scattering.

The asymptotic form of the total wave function can be shown to be

$$\psi \xrightarrow[r \to \infty]{} e^{i(\mathbf{k}\cdot\mathbf{r})} \exp\left[\frac{i}{kD} \ln (kr - \mathbf{k}\cdot\mathbf{r})\right] - f(\theta) \frac{e^{ikr}}{r} \exp\left[\frac{1}{ikD} \ln kr\right].$$

Again the long-range character of the coulomb potential manifests itself through logarithmic phase factors. The function $f(\theta)$ plays the same role as the corresponding function in the n-p problem, and the differential cross section is given by

$$\sigma(\theta) = |f(\theta)|^2. \tag{6–13}$$

Because of the identity of the two protons, a properly defined function $f(\theta)$ would take into account the symmetry of the wave function under

* For properties of the coulomb functions see Mott and Massey, *The Theory of Atomic Collisions*, 2nd ed. Oxford Press (1949), p. 45 ff. See also Yost, Wheeler, and Breit, *Phys. Rev.* **49**, 174 (1937).

interchange of the two protons. However, $f(\theta)$ is usually calculated without reference to the symmetry, and the appropriate combination is then used to calculate the scattering cross section. Since the interchange of the two protons is equivalent to replacing $\mathbf{r}$ by $-\mathbf{r}$, the correct combinations are simply

$$f(\theta) + f(\pi - \theta)$$

for the singlet state and

$$f(\theta) - f(\pi - \theta)$$

for the triplet state if $f(\theta)$ is the function that would be obtained for nonidentical particles. Thus the scattering of an unpolarized beam is given by

$$\begin{aligned}\sigma(\theta) &= \tfrac{1}{4}|f(\theta) + f(\pi - \theta)|^2 + \tfrac{3}{4}|f(\theta) - f(\pi - \theta)|^2 \\ &= |f(\theta)|^2 + |f(\pi - \theta)|^2 - \mathrm{Re}[f^*(\theta)f(\pi - \theta)]. \qquad (6\text{-}14)\end{aligned}$$

An expression for this $f(\theta)$ can be given in terms of the phases of the partial waves described above:

$$f(\theta) = -\frac{1}{2ik}\sum_{l=0}^{\infty}(2l + 1)[e^{2i(\eta_l + \delta_l)} - 1]P_l(\cos\theta). \qquad (6\text{-}15)$$

In the absence of a nuclear potential all $\delta_l = 0$, and we obtain the coulomb scattering function

$$f_C(\theta) = -\frac{1}{2ik}\sum_{l}(2l + 1)(e^{2i\eta_l} - 1)P_l(\cos\theta).$$

It can be shown that this is equivalent to

$$f_C(\theta) = -\frac{e^2}{4E}\frac{1}{\sin^2(\theta/2)}\exp\left[\frac{1}{ikD}\ln(1 - \cos\theta) + i\pi + 2i\eta_0\right],$$

which leads directly by substitution into Eq. (6-14) to the Mott formula for the coulomb scattering of two identical particles satisfying the Pauli principle:

$$\begin{aligned}\sigma_M(\theta) = \left(\frac{e^2}{4E}\right)^2\Bigg[&\mathrm{cosec}^4\frac{\theta}{2} + \sec^4\frac{\theta}{2} \\ &- \mathrm{cosec}^2\frac{\theta}{2}\sec^2\frac{\theta}{2}\cos\left\{\frac{2}{kD}\ln\left(\tan\frac{\theta}{2}\right)\right\}\Bigg]. \qquad (6\text{-}16)\end{aligned}$$

The first term is just the Rutherford cross section for the coulomb scattering of nonidentical particles. The second term is what one expects because no distinction can be made between the scattered and scattering particle after the collision, while the third is the interference term. The cross section is very strongly peaked in the forward and backward direc-

tions. As a matter of fact, the total cross section obtained by integrating this expression over all angles would be infinite because of the strong singularities at $\theta = 0$ and $\theta = \pi$. The infinite cross section is again a characteristic of the long range of the coulomb potential. In an actual physical situation the cross section would be cut off at very small angles because of the shielding of the coulomb potential produced at great distances from the scattering center by the electrons in the target molecule.

The observation of deviations from the Mott formula in proton-proton scattering provides direct evidence for the existence of a specific nuclear force acting between protons. Measured values of $\sigma(\theta) - \sigma_M(\theta)$ can be used to obtain the phase shifts δ_l which are directly related to the nuclear interaction. For that purpose it is convenient to separate the series, Eq. (6–15), into two terms, one characteristic of pure coulomb scattering and the other including the additional effect of nuclear scattering:

$$f(\theta) = f_C(\theta) + f_N(\theta).$$

Then all the nuclear effects are contained in

$$f_N(\theta) = -\frac{1}{2ik}\sum_l (2l+1)e^{2i\eta_l}[e^{2i\delta_l} - 1]P_l(\cos\theta),$$

a series which converges very rapidly. At very low energy, the coulomb barrier reduces the effect of the nuclear interaction so the δ_l are quite small. For an energy in the range 1 to 5 Mev the centrifugal barrier also takes effect, and it has about the same relative effect for different l as in the case of n-p scattering. Therefore it is expected that only δ_0 differs appreciably from zero at moderate energies, and $f_N(\theta)$ can be replaced by

$$f_0(\theta) = -\frac{1}{k}e^{2i\eta_0}e^{i\delta_0}\sin\delta_0,$$

a result which is to be compared with Eq. (4-23). Introduction of this approximation into Eq. (6-14) leads to the following expression for the S-wave differential cross section:

$$\sigma(\theta) - \sigma_M(\theta) = \left(\frac{e^2}{4E}\right)^2 \left\{-2kD\sin\delta_0\left[\operatorname{cosec}^2\frac{\theta}{2}\cos\left(\delta_0 + \frac{2}{kD}\ln\sin\frac{\theta}{2}\right)\right.\right.$$
$$\left.\left. + \sec^2\frac{\theta}{2}\cos\left(\delta_0 + \frac{2}{kD}\ln\cos\frac{\theta}{2}\right)\right] + 4k^2D^2\sin^2\delta_0\right\}. \quad (6\text{-}17)$$

A notable feature of this equation is the appearance of a linear term in $\sin\delta_0$ as a result of the interference between coulomb scattering and nuclear scattering. Because of this interference the experimental determination of the cross section leads directly to a value for the sign of the phase shift δ_0, in contrast to the situation for n-p scattering. A careful compari-

son of the values of $\sigma(\theta) - \sigma_M(\theta)$ observed below 5 Mev with the above equation makes possible a precise determination of the S-wave phase shift, δ_0. Such a comparison has been made for laboratory energies up to 4 Mev. The results are shown in Table 6-1.

TABLE 6-1

Values of the S-wave phase shift δ_0 for proton-proton scattering as a function of energy*

Energy (Mev)	δ_0 (degrees)	Energy (Mev)	δ_0 (degrees)
0.1765	5.78	0.867	29.32
0.2002	6.80	1.200	35.94
0.2259	7.82	1.390	38.76
0.2495	9.03	1.855	44.26
0.2753	10.06	2.425	48.33
0.2983	10.96	3.037	50.98
0.3214	11.82	3.527	52.52
0.670	24.68	3.899	53.31
0.776	27.12	4.203	53.85

* For energies up to 1.390 these results are taken from Jackson and Blatt, *loc. cit.* The other values were obtained by Hall and Powell, *Phys. Rev.* **90,** 912 (1953), from data due to Worthington, McGruer, and Findley, *Phys. Rev.* **90,** 899 (1953).

From the experimentally determined values of the phase shift as a function of energy it is possible to fix some parameters in a potential of given shape. At low energies the results are most easily formulated in terms of certain average properties of the potential analogous to the scattering length and effective range. This is accomplished by defining a function, f, of the phase shift which is the analog of k cot δ for neutron-proton scattering. Just as for the latter function, f is defined in such a way as to be, at low energies, a linear function of energy by virtue of the Schroedinger equation. To derive the essential relationships, we follow the method of Bethe, which is analogous to that used in Section 4-6.

Only the S-wave is of interest for the energies under consideration, so we denote the solution u_0 of Eq. (6-8) simply by u. The solution, $v_0(r)$, of Eq. (6-9) which is asymptotically equal to $u(r)$ is called the auxiliary function and simply denoted by $v(r)$. The functions u and v are also normalized in such a way that $v(0) = 1$. Then

$$v(r) = C_0[G_0(r) + F_0(r) \cot \delta],$$

where F_0 and G_0 are the regular and irregular coulomb functions whose asymptotic behavior is given by Eqs. (6-10) and (6-11). The constant C_0 is

$$C_0 = \left(\frac{2\pi/kD}{e^{2\pi/kD} - 1}\right)^{1/2}.$$

If the procedure of Section 4–6 is now applied to the differential equations for u and v, we find

$$[v_2'(r) - v_1'(r)]_{r=0} = (k_2^2 - k_1^2)\int_0^\infty (v_1 v_2 - u_1 u_2)\, dr, \tag{6–18}$$

where k_1^2 and k_2^2 represent two distinct energy values of the p-p system. The behavior of $v(r)$ for small argument can be shown to be

$$v \approx 1 + \frac{2r}{D}\left[\ln\frac{2r}{D} + \frac{1}{2}g\left(\frac{1}{kD}\right) + C - 1 + \frac{\pi\cot\delta}{e^{2\pi/kD} - 1}\right], \tag{6–19}$$

where $C = 0.577$ is Euler's constant, and g is a transcendental function given by

$$g(x) = -2\ln x + 2x^2\sum_{\nu=1}^{\infty}\nu^{-1}(\nu^2 + x^2)^{-1}.$$

Therefore

$$(v_2' - v_1')_{r=0} = \frac{1}{D}\left[g\left(\frac{1}{k_2 D}\right) - g\left(\frac{1}{k_1 D}\right) + \frac{2\pi\cot\delta_2}{e^{2\pi/k_2 D} - 1} - \frac{2\pi\cot\delta_1}{e^{2\pi/k_1 D} - 1}\right].$$

If we define the function f of the phase shift by

$$f = g\left(\frac{1}{kD}\right) + \frac{2\pi\cot\delta}{e^{2\pi/kD} - 1}, \tag{6–20}$$

Eq. (6–18) leads to the result

$$\frac{df}{d(k^2)} = D\int_0^\infty (v^2 - u^2)\, dr.$$

Thus if f is considered as a function, $f(k^2)$, of energy, an expansion about zero energy yields the result

$$f(k^2) = f(0) + \frac{D}{2}\rho_0 k^2, \tag{6–21}$$

where the "proton-proton effective range" ρ_0 is given by

$$\tfrac{1}{2}\rho_0 = \int_0^\infty (v^2 - u^2)_{k=0}\, dr. \tag{6–22}$$

It is clear that ρ_0 is very similar in its form to the neutron-proton effective range r_0, Eq. (4–49).

Those properties of the specific nuclear potential which can be determined from low-energy p-p scattering are characterized by the two param-

eters $f(0)$ and ρ_0. The former plays about the same role as the scattering length of the neutron-proton system, while the latter plays the role of an effective range. The experimental determination of these quantities can be used to fix the properties of a potential of given shape just as has been done for the neutron-proton system.

Very precise experimental results on p-p scattering are available up to 4 Mev.* On the basis of some of the data a determination of the constants has been made by Jackson and Blatt,† who give

$$f(0) = 8.66 \pm 0.05, \qquad \rho_0 = 2.65 \pm 0.07 \times 10^{-13} \text{ cm}. \tag{6-23}$$

These values could be used to determine the range and depth of a potential of given shape, which could in turn be compared with a singlet potential of the same shape appropriate to the neutron-proton system. By this procedure some conclusion concerning the equality of p-p and n-p forces should result. However, little is known about the singlet potential for the n-p system since the only accurate datum is the scattering length, which provides just one relationship between the parameters of the potential. It is possible to make a direct comparison of the singlet scattering length, a_s, of the n-p system and the apparent scattering length, a_p, of the p-p system, where a_p is the scattering length that would be obtained in the absence of the coulomb interaction. Unfortunately, the relationship between a_p and the experimentally determined quantity $f(0)$ depends to some extent on the shape of the nuclear potential. Therefore a clear-cut comparison of the p-p and n-p interaction is not possible on this basis. Nevertheless, we know from the discussion of Section 4-5 that, because the binding energy of the singlet state of the deuteron nearly vanishes, the value of a_s is a very sensitive function of the depth of the singlet potential, so any appreciable difference between the p-p and n-p interaction should show up as a very large difference in the values of a_p and a_s.

In order to derive the connection between a_p and $f(0)$, it is convenient to introduce the functions u_N and v_N, which are the radial and auxiliary functions that would be obtained at zero energy in the absence of a coulomb potential. If the n-p and p-p singlet interactions are identical, u_N and v_N are identical with the zero energy singlet functions u and v of Section 4-6. To make a clear distinction, we now denote the zero energy coulomb functions u and v with which we have been dealing in this section

* Herb, Kerst, Parkinson, and Plain, *Phys. Rev.* **55,** 998 (1939); Heydenburg, Hafstad, and Tuve, *Phys. Rev.* **56,** 1078 (1939); Blair, Freier, Lampi, Sleator, and Williams, *Phys. Rev.* **74,** 553 (1948); Worthington, McGruer, and Findley, *Phys. Rev.* **90,** 899 (1953).

† *Loc. cit.*

by u_C and v_C. By the usual manipulation of the differential equations for u_C, u_N, v_C, and v_N, we find

$$[u_N u_C' - u_C u_N']_{r \to \infty} = \frac{2}{D} \int_0^\infty u_N u_C \frac{dr}{r},$$

$$[v_N v_C' - v_C v_N']_b^\infty = \frac{2}{D} \int_b^\infty v_N v_C \frac{dr}{r}.$$

The lower limit, b, must be made explicit in the second equation because, as we shall see, a divergence is encountered for $b = 0$. The result of subtracting the second equation from the first can be put in the form

$$\lim_{b \to 0} [v_C'(b) - v_N'(b)] = \lim_{b \to 0} \frac{2}{D} \int_b^\infty (u_N u_C - v_N v_C) \frac{dr}{r}. \tag{6-24}$$

The scattering length a_p is defined in such a way that [compare Eq. (4–50)]

$$v_N = 1 - \frac{r}{a_p}; \tag{6-25}$$

consequently, $v_N' = -1/a_p$. Furthermore, Eqs. (6–19) and (6–20) can be combined to yield, for small r,

$$v_C(r) \approx 1 + \frac{2r}{D}\left[\ln \frac{2r}{D} + C - 1 + \frac{1}{2} f(0)\right] \tag{6-26}$$

so we can write

$$v_C'(b) = \frac{2}{D}\left[\ln \frac{2b}{D} + C + \frac{1}{2} f(0)\right]$$

Then Eq. (6–24) becomes

$$-\frac{1}{a_p} = \frac{2}{D}\left\{\frac{1}{2} f(0) + C + \lim_{b \to 0}\left[\ln \frac{2b}{D} - \int_b^\infty (u_N u_C - v_N v_C) \frac{dr}{r}\right]\right\}. \tag{6-27}$$

It can easily be seen that the logarithmically divergent term is cancelled by an equal term arising from the product $v_N v_C$ in the integral. Equation (6–27) gives the desired relationship between a_p and $f(0)$. The fact that this relationship is shape-dependent is clear from the appearance of the term $u_N u_C$ under the integral. However, estimates by Bethe* on a somewhat different basis indicate that the shape sensitivity is not very great as long as the parameters of the potential are adjusted to the observed values of $f(0)$ and ρ_0.

The integrand in Eq. (6–27) vanishes for distances large compared with the range of the forces. On the other hand, within the range α, the func-

* *Loc. cit.*

tions u_N and u_C are very nearly identical, while v_N and v_C differ but little. Therefore a reasonable estimate of the integral is

$$\int_b^\infty (u_N u_C - v_N v_C)\frac{dr}{r} \approx \int_b^\alpha (u_N^2 - v_N^2)\frac{dr}{r}. \tag{6-28}$$

This estimate should be particularly appropriate for a square well, so we take u_N to be the square-well function

$$u_N = A \sin Kr, \quad r < \alpha, \tag{6-29}$$

where (see Section 4-6)

$$A^2 = \left(1 - \frac{\alpha}{a_p}\right)^2 + \frac{1}{(K\alpha)^2}\left(\frac{\alpha}{a_p}\right)^2.$$

The introduction of the functions (6-29) and (6-25) into Eq. (6-28) leads to the estimate

$$\lim_{b\to 0}\int_b^\infty (u_N u_C - v_N v_C)\frac{dr}{r} \approx \left[\left(1 - \frac{\alpha}{a_p}\right)^2 + \frac{1}{(K\alpha)^2}\left(\frac{\alpha}{a_p}\right)^2\right] B(K\alpha) - \ln\frac{2\alpha}{D} + \lim_{b\to 0}\ln\frac{2b}{D} + 2\left(\frac{\alpha}{a_p}\right) - \frac{1}{2}\left(\frac{\alpha}{a_p}\right)^2,$$

if

$$B(K\alpha) = \tfrac{1}{2}[C + \ln(2K\alpha) - Ci(2K\alpha)],$$

where $Ci(x)$ is the cosine integral.* Then Eq. (6-27) may be put in the form

$$\left[\left(1 + \frac{1}{(K\alpha)^2}\right)B - \frac{1}{2}\right]\left(\frac{\alpha}{a_p}\right)^2 - 2\left(B - 1 + \frac{D}{4\alpha}\right)\left(\frac{\alpha}{a_p}\right) - \left[\frac{1}{2}f(0) + C + \ln\frac{2\alpha}{D} - B\right] = 0.$$

Because the quantity $D/4\alpha$ is considerably larger than unity, the quadratic term in this equation can be neglected, with the result that

$$a_p = -\frac{D}{2}\frac{1 + (B-1)4\alpha/D}{\frac{1}{2}f(0) + C + \ln(2\alpha/D) - B}. \tag{6-30}$$

To obtain estimates of $B(K\alpha)$ and α, we anticipate that the proton-proton potential is very nearly the same as the neutron-proton singlet potential. Then $K\alpha \approx \pi/2$ [Eq. (4-40)], whence

$$B \approx 0.823.$$

* See, for example, Jahnke-Emde, *Tables of Functions*, Dover (1943), p. 3.

Furthermore, the value of α would, according to Eq. (4–57), be very nearly given by

$$\alpha \approx \rho_0.$$

The experimental values of ρ_0 and $f(0)$, when inserted into Eq. (6–30), lead to the estimate

$$a_p \approx -1.6 \times 10^{-12} \text{ cm}. \tag{6–31}$$

The large value of a_p indicates the correctness of our basis for estimating $K\alpha$ and α. Although a_p is somewhat different from the value $a_s \approx -2.1 \times 10^{-12}$ cm, the difference has very little significance in view of the great sensitivity of the scattering length to the depth of the potential. As a matter of fact, it has been pointed out by Schwinger* that a difference between a_p and a_s of this order and in this direction can be expected on the basis of the magnetic interactions between the nucleons, which differ between the two systems because the neutron moment differs from the proton moment. For all practical purposes, we then conclude that the observed values of $f(0)$ and a_s are not in contradiction with the hypothesis of charge independence of the forces.

Another comparison of the two systems might be made on the basis of the effective range. In the absence of a measured value of r_{0s} for the n-p system it has become more or less traditional to assume that the forces are equal and that r_{0s} can be obtained directly from ρ_0. From the definition of ρ_0,

$$\tfrac{1}{2}\rho_0 = \int_0^\infty (v_C^2 - u_C^2)\, dr,$$

it is clear that the connection with r_0, given by

$$\tfrac{1}{2}r_0 = \int_0^\infty (v_N^2 - u_N^2)\, dr,$$

is very close indeed, and r_0 would be equal to r_{0s} if the p-p and n-p interactions were identical. In both integrals, contributions to the integrand occur only within the range of forces, and there the nuclear potential is so strong compared with the coulomb potential that u_C and u_N are essentially the same. Furthermore, v_C and v_N are very nearly equal within the range of forces, the difference being, at any point, no greater than one or two percent. Therefore, as a first rough approximation, one can simply assume that

$$\rho_0 = r_0.$$

However, this relationship is not valid within the limits of error of the ex-

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* *Phys. Rev.* **78,** 135 (1950).

perimental value of ρ_0. An estimate of the difference between ρ_0 and r_0 is given by

$$\rho_0 - r_0 \approx 2\int_0^\alpha (v_C^2 - v_N^2)\, dr,$$

where v_N is to be obtained from Eq. (6–25) and v_C from Eq. (6–26). The integrations involved are then elementary, and the constants needed may be obtained directly from the experimental values of $f(0)$ and ρ_0. The fact that the result depends directly on a choice of the range α indicates its essential shape dependence. For a square well, α may be taken equal to ρ_0, with the result* that $\rho_0 - r_0$ is about -0.1×10^{-13} cm. Therefore, if the neutron-proton and proton-proton forces are identical, r_{0s} may be as much as four percent larger than ρ_0.

Although the comparison of a_p with a_s and that of ρ_0 with r_{0s} provide a method for quickly estimating the situation concerning the charge dependence of nuclear forces, a more precise comparison directly in terms of the potentials is possible. The latter approach, which has the disadvantage that each potential shape must be treated separately and in detail, has been thoroughly expounded by Breit and his collaborators. Reference may be made to that work when precise answers are needed.

6–3 Proton-proton scattering. High energy. Just as in the case of neutron-proton scattering, it is expected that more information concerning the shape and exchange character of the potential can be obtained from high-energy p-p scattering than from the low-energy data. Furthermore, the coulomb potential plays a smaller and smaller part in the scattering as the energy is increased, and at very high energies a direct comparison of the n-p and p-p interactions should be possible without first subtracting the complicating coulomb effects. However, even if the interactions are the same, the identity of the two protons will lead to a marked difference in the behavior of the two systems. Since either the scattered or scattering proton may be detected, the angular distribution of the two-proton system is necessarily fore-and-aft symmetric, while such symmetry in the n-p system is indicative of a special exchange property of the potential. Furthermore, as we have already noted, two protons in a singlet state have even parity, while those in triplet states have odd parity. If the Serber exchange is assumed to act between protons on the grounds that it is suggested by the high-energy n-p data, then no scattering occurs in states of odd parity. Therefore there would be no triplet scattering of one proton by another.

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* The estimate of $\rho_0 - r_0$ is quite sensitive to the value of a_p. Since its determination from $f(0)$ is shape dependent, we have assumed that $a_p = a_s$ and have used the value of Eq. (5–37).

An estimate of high-energy scattering can again be made by the Born approximation. If the coulomb scattering, which is important only at very small angles, is ignored, the differential cross section calculated in this way for any potential with Serber exchange is just twice that for the n-p scattering calculated in the same approximation with the same potential. The factor two takes account of the detection of scattered and scattering particle in the special case of Serber exchange since, then, the angular distribution of the scattered particle already has fore-and-aft symmetry.

A central potential with the same range for singlet and triplet states leads, in Born approximation, to the same relative angular distribution in both states. Therefore the p-p angular distribution, except at small angles, is calculated in this case to be the same as the n-p. For a Yukawa potential the latter was found in Section 6–1 to have a rather marked angular dependence, corresponding to the ratio $\sigma(\pi)/\sigma(\pi/2) = 5.5$ and 28 at 90 Mev and 260 Mev, respectively. These ratios are somewhat reduced by using better approximations and including the tensor interaction. Whether or not the tensor force is to be included in the p-p scattering cannot be decided on the basis of n-p results. They do not fix the exchange dependence of the tensor interaction at all sharply. If the tensor force involves the Serber exchange operator, it cannot influence p-p scattering, since the exchange operator vanishes in the odd triplet states and the tensor interaction vanishes in the singlet states. However, a considerably different exchange dependence would be consistent with the n-p data, and that would introduce some triplet p-p scattering.

Since the long-range coulomb effect always makes its appearance at small angles, the correct value of $\sigma(\pi)$ would be quite different from that used as the basis for the above estimates. However, the ratios which are given indicate the angular behavior of the cross section for angles not too close to 0 or π. For similar reasons, a total cross section cannot be calculated directly from the n-p results, but an average differential cross section can be obtained by dividing the estimated n-p singlet total cross section by 2π. In this way an average differential cross section

$$\overline{\sigma(\theta)} = 8.4 \times 10^{-27} \text{ cm}^2/\text{sterad}$$

is obtained at 90 Mev, while at 340 Mev

$$\overline{\sigma(\theta)} = 2.7 \times 10^{-27} \text{ cm}^2/\text{sterad}.$$

Cross-section measurements have been made from 18 to 30 Mev,* at 75

* Panofsky and Fillmore, *Phys. Rev.* **79,** 57 (1950); Fillmore, *Phys. Rev.* **83,** 1252 (1951); Cork, Johnston, and Richman, *Phys. Rev.* **79,** 71 (1950); Cork, *Phys. Rev.* **80,** 321 (1950).

and 100 Mev,* and from 120 to 345 Mev.† The outstanding features of the results are the absence of an angular dependence of the differential cross section over a wide range of angles centered about 90° (CM system) and the absence of an energy dependence between 100 and 345 Mev. The magnitude of the constant differential cross section is about 4×10^{-27} cm^2/sterad. The lack of a dependence on energy is clearly inconsistent with the above estimates based on a central potential, although the qualitative features of the estimates should not depend greatly on the approximations used. It may be necessary to introduce‡ a singular potential, i.e., one that diverges more strongly than r^{-1} at the origin, in order to account for the independence of energy.

The angular dependence of the cross section is also a source of difficulty. We have seen that the angular distribution of the n-p and p-p scattering should be nearly the same if the interactions are the same, but they turn out to be quite different. This may mean that the n-p and p-p interactions are different, it may mean that both are singular in just such a way as to produce the observed difference in scattering, or it may mean that a static potential provides a very poor approximation to the interaction at these energies.

6-4 Photodisintegration of the deuteron. A standard method for the investigation of atomic or molecular structure is to make use of the interaction of the system with electromagnetic radiation, that is, to observe either its emission or its absorption spectrum. A corresponding study of the absorption or emission of gamma-rays by the neutron-proton system should lead to further information concerning the structure of the deuteron. Since the deuteron has no bound excited states, the phenomena to be considered are the photodisintegration of the deuteron and its inverse, the capture of neutrons by protons. Although, in principle, analysis of precise cross-section data for these processes could provide information concerning nuclear interactions, we shall find (Section 6-6) that there is sufficient uncertainty concerning the electromagnetic interactions of nucleons to prevent us from coming to very definite conclusions. Nevertheless, the data do confirm our general views concerning the n-p system, and with a little added precision they could give us a deeper insight into the electromag-

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* R. W. Birge, *Phys. Rev.* **80,** 490 (1950); Birge, Kruse, and Ramsey, *Phys. Rev.* **83,** 274 (1951).

† Chamberlain and Wiegand, *Phys. Rev.* **79,** 81 (1950); Chamberlain, Segré, and Wiegand, *Phys. Rev.* **83,** 923 (1951); C. L. Oxley and R. O. Schamberger, *Phys. Rev.* **85,** 416 (1952).

‡ Christian and Noyes, *Phys. Rev.* **79,** 85 (1950); Jastrow, *Phys. Rev.* **79,** 389 (1950); Case and Pais, *Phys. Rev.* **80,** 203 (1950); Goldfarb and Feldman, *Phys. Rev.* **88,** 1099 (1952); Noyes and Camnitz, *Phys. Rev.* **88,** 1206 (1952).

netic interactions. This would be of particular interest, because the electromagnetic interactions of nucleons are intimately related to the nature of the meson field.

The threshold for photodisintegration is just the binding energy of the deuteron, 2.226 Mev. As the gamma-ray energy increases above this threshold, the cross section must increase until such energies are attained that the wavelength of the outgoing particle is comparable to the dimensions of the deuteron. Then with increasing energy the matrix element for the process decreases and the associated cross section falls monotonically. The sharpness of the threshold depends strongly on the multipole character of the transition. If the ground state of the deuteron is taken to be a pure S-state, an outgoing S-wave can only be produced by a magnetic dipole transition. At low energy the cross section is then proportional to just the statistical weight of the final states, that is, to the square root of the kinetic energy of the disintegration products. The resulting infinite slope of the cross-section curve at threshold indicates that the magnetic dipole transition sets in very sharply. On the other hand, for the more familiar electric dipole transition, the outgoing wave is a P-function whose intensity in the neighborhood of the nucleus is inversely proportional to the square of the wavelength for long wavelength. Therefore, the electric dipole cross section increases as the $\frac{3}{2}$ power of the kinetic energy and has zero slope at the threshold. Thus this transition sets in very slowly, so that its threshold would be difficult to detect.

The calculation of the electric dipole cross section can be based on the familiar expression* for the transition probability for absorption of polarized radiation in the discrete spectrum:

$$w_{gf} = \frac{(2\pi)^2}{\hbar^2 c} \left| D_{gf} \right|^2 I(\omega), \tag{6-32}$$

where $I(\omega)\, d\omega$ is the incident intensity of polarized radiation, D is the component of the electric dipole moment in the direction of polarization of the radiation, and D_{gf} is its matrix element between initial and final states. In the case at hand the final states form a continuum, but by surrounding the system by an arbitrarily large enclosure the energy values may be made discrete. Then the required transition probability, dw, is the sum of w_{gf} over all final states in the energy interval between E and $E + dE$. The energy states produced by the photoprocess have $E = \hbar\omega - \left| E_g \right|$, where $\left| E_g \right|$ is the binding energy of the deuteron, so $dE = \hbar\, d\omega$. Thus, if the number of final energy states per unit energy interval is $\rho(E)$, the differential transition probability is given by

* For a general discussion of multipole transition probabilities see Section 9-1.

$$dw = \frac{(2\pi)^2}{\hbar c} \mid D_{gf} \mid^2 I(\omega)\, d\omega \rho(E),$$

where now the matrix element of the dipole moment is taken between the ground state of the deuteron denoted by the subscript g, and the final state in the continuum with energy E. Since the number of photons incident on the deuteron per unit area per unit time is just $I(\omega)\, d\omega/\hbar\omega$, the cross section for the electric dipole process is

$$\sigma_e = \frac{(2\pi)^2 \omega}{c} \mid D_{gf} \mid^2 \rho(E). \tag{6–33}$$

For the sake of simplicity, let us treat the ground state of the deuteron as a pure ^3S state. Then an electric dipole transition produces a pure ^3P state. $\rho(E)$ is the density of these P-states, and the matrix element D_{gf} is to be taken between the ground ^3S state and the continuum ^3P state. The P-functions that must be used to calculate the matrix element are the solutions of the Schroedinger equation which we have already considered in connection with the scattering problem. However, the actual continuum functions cannot be used here, since the matrix element in Eq. (6 33) is to be calculated with normalized wave functions. We have chosen to approximate the continuum states by a set of discrete states obtained by imposing the condition that the wave functions vanish on the surface of a sphere of extremely large radius L, which is centered on the deuteron. Then it will be found that in the limit $L \to \infty$ the cross section is independent of L, so the result is the same as that which would be obtained directly from continuum functions.

The radial P-function near the surface of the sphere can be found from the asymptotic P-wave of the scattering problem, namely,

$$u_p(r) \xrightarrow[r \to L]{} \frac{B}{k} \sin\left(kr + \delta_1 - \frac{\pi}{2}\right),$$

where B is a normalization constant and $\delta_1(k)$ is the phase shift characteristic of the scattered P-wave. The condition that this function vanish at L is then

$$kL + \delta_1(k) - \frac{\pi}{2} = n\pi,$$

where n is an integer. To each integer n there corresponds just one state, so the number of states with values of k between k and $k + dk$ is

$$dn = \frac{1}{\pi}\left(L + \frac{d\delta_1}{dk}\right) dk.$$

But for very large L the finite quantity $d\delta_1/dk$ may be neglected, so the number of states per unit energy interval is, in effect,

$$\rho(E) = \frac{dn}{dE} = \frac{L}{\pi}\frac{dk}{dE}.$$

Since $E = \hbar^2k^2/M$,

$$\rho(E) = \frac{L}{2\pi\hbar}\sqrt{\frac{M}{E}}. \tag{6-34}$$

The asymptotic form of the wave function can also be used to obtain the normalization constant B, which is determined by the condition

$$\int_0^L u_p^2\, dr = 1.$$

The only important contributions to the integral arise from distances comparable to L, so we can write, for very large L,

$$\frac{B^2}{k^2}\int_0^L \cos^2(kr + \delta_l)\, dr = 1,$$

from which it follows that

$$B^2 = 2k^2/L.$$

It is convenient to define a matrix element, D'_{gf}, with respect to the continuum function having the usual asymptotic form

$$u \xrightarrow[r\to\infty]{} \frac{1}{k}\sin\left(kr + \delta_l - \frac{\pi}{2}\right). \tag{6-35}$$

Then the above evaluation of B indicates that the matrix element D_{gf} is related to D'_{gf} by

$$D_{gf} = \sqrt{\frac{2}{L}}\, kD'_{gf}.$$

When this result is combined with Eq. (6-34) for the density of states and inserted into Eq. (6-33), the cross section is found to be

$$\sigma_e = \frac{4\pi M k\omega}{\hbar^2 c}\,|\,D'_{gf}\,|^2, \tag{6-36}$$

which is now in a form independent of the length L.

The determination of the matrix element happens to be particularly simple* for an ordinary central potential, a result that will serve to illus-

* J. F. Marshall and E. Guth, *Phys. Rev.* **78,** 738 (1950). They also present a detailed analysis of the high energy cross section for various central potentials, including exchange, as does L. I. Schiff, *Phys. Rev.* **78,** 733 (1950).

trate the kind of behavior found even in more complicated situations. Let us denote the unit polarization vector by $\mathbf{u}$. Then the electric dipole moment operator of the deuteron is

$$D = \frac{e}{2}(\mathbf{u}\cdot\mathbf{r}),$$

where $\mathbf{r}$ is, as usual, the neutron-proton separation. For ordinary forces described by a potential V, the equations of motion for the system may be put in the operational form

$$\ddot{\mathbf{r}}_{gf} = -\frac{2}{M}(\text{grad } V)_{gf}.$$

It is to be remembered that such a relation does not obtain if the potential V includes an exchange operator (see Section 4-2). We can now make use of the Heisenberg relation

$$\ddot{\mathbf{r}}_{gf} = -\left(\frac{E_f - E_g}{\hbar}\right)^2 \mathbf{r}_{gf},$$

or, in our particular case,

$$\ddot{\mathbf{r}}_{gf} = -\omega^2 \mathbf{r}_{gf}.$$

Then

$$\mathbf{r}_{gf} = \frac{2}{M\omega^2}(\text{grad } V)_{gf},$$

so the matrix element of the dipole moment is

$$D_{gf} = \frac{e}{M\omega^2}(\mathbf{u}\cdot\text{grad } V)_{gf},$$

which becomes for a central potential

$$D_{gf} = \frac{e}{M\omega^2}\left[\frac{(\mathbf{u}\cdot\mathbf{r})}{r}\frac{dV}{dr}\right]_{gf}.$$

If the wave functions of the ground state and final state are now introduced explicitly, the matrix element takes the form

$$D_{gf} = \frac{1}{\sqrt{4\pi}}\frac{e}{M\omega^2}\int u_g(r)\left[\frac{(\mathbf{u}\cdot\mathbf{r})}{r}\frac{dV}{dr}\right]u_p(r)\,Y_1^m(\theta,\varphi)\,dr\,d(\cos\theta)\,d\varphi,$$

where m is the magnetic quantum number of the outgoing particles. It is convenient to choose the z-axis along the direction of polarization $\mathbf{u}$, since then only the state with $m = 0$ is produced in the transition and the matrix element reduces to

$$D_{gf} = \frac{1}{\sqrt{3}}\frac{e}{M\omega^2}\int_0^\infty u_g(r)u_p(r)\frac{dV}{dr}\,dr. \qquad (6\text{-}37)$$

This result, although quite simple, becomes even simpler for the special case that the potential is a square well. Then

$$\frac{dV}{dr} = \frac{\hbar^2 K^2}{M}\,\delta(r - \alpha),$$

since

$$\int_{\alpha-0}^{\alpha+0} \frac{dV}{dr}\,dr = V(\alpha + 0) - V(\alpha - 0) = \frac{\hbar^2 K^2}{M}.$$

Hence the matrix element is simply given by

$$D_{gf} = \frac{1}{\sqrt{3}}\,\frac{e\hbar^2 K_t^2}{M^2\omega^2}\,u_g(\alpha)u_p(\alpha), \tag{6–38}$$

where the triplet depth K_t has been introduced because only the triplet states are involved in the transition.

When u_p is normalized as in Eq. (6–35), its form for a square well may be taken directly from Eq. (4–44), whence

$$u_p(\alpha) = \frac{1}{k}\left[\frac{\sin(k\alpha + \delta_1)}{k\alpha} - \cos(k\alpha + \delta_1)\right].$$

For low energies, corresponding to photon energies within a few Mev of threshold, the determination of the phase shift δ_1 has been carried out in connection with the problem of P-scattering. We found in Section 4–5 that

$$u_p(\alpha) \approx \frac{1}{3}k\alpha^2 \cos\delta_1 + \frac{1}{k^2\alpha}\sin\delta_1,$$

if $k\alpha \ll 1$. Also, δ_1 is small and proportional to $(k\alpha)^3$, so we write

$$u_p \approx \left[\frac{1}{3} + \frac{\delta_1}{(k\alpha)^3}\right]k\alpha^2, \tag{6–39}$$

where the coefficient of $k\alpha^2$ is essentially constant. When the approximate value of $\delta_1/(k\alpha)^3$ given by Eq. (6–2) for a square-well potential without exchange is introduced into Eq. (6–39), we find

$$u_p = \frac{k}{K_t^2}\,\frac{1}{1 - \alpha/a_t}, \tag{6–40}$$

where K_t^2 is the depth parameter for the triplet well and a_t is the triplet scattering length.

The function $u_g(\alpha)$ may be taken directly from Eq. (3–28):

$$u_g(\alpha) = \frac{\kappa_g}{K_t}\sqrt{\frac{2k_g}{1 + k_g\alpha}}.$$

Then

$$|D'_{gf}|^2 = \frac{2}{3}\frac{e^2\hbar^4 k_g k^2}{M^4\omega^4}\frac{1-k_g^2/K_t^2}{1+k_g\alpha}\left(1-\frac{\alpha}{a_t}\right)^{-2}.$$

When this is inserted into the cross-section formula, Eq. (6–36), we obtain

$$\sigma_e = \frac{8\pi}{3}R_0^2\frac{\hbar c}{e^2}\frac{m}{M}\left(\frac{mc^2}{\hbar\omega}\right)\left(\frac{|E_g|}{\hbar\omega}\right)^{1/2}\left(1-\frac{|E_g|}{\hbar\omega}\right)^{3/2}\frac{1-k_g^2/K_t^2}{1+k_g\alpha}\left(1-\frac{\alpha}{a_t}\right)^{-2}. \tag{6–41}$$

Numerical values of α/a_t and $K_t\alpha$ may be found in Section 6–1. Their introduction into the cross-section formula leads to the final result

$$\sigma_e = 8.2\times10^{-26}\left(\frac{mc^2}{\hbar\omega}\right)\left(\frac{|E_g|}{\hbar\omega}\right)^{1/2}\left(1-\frac{|E_g|}{\hbar\omega}\right)^{3/2}\text{cm}^2. \tag{6–42}$$

It is to be noted that the proportionality of the cross section to the $\frac{3}{2}$ power of the energy of the outgoing particles is just what was anticipated for an electric dipole transition. At a gamma-ray energy about 2.2 Mev above threshold this formula leads to a cross section of approximately 2.3×10^{-27} cm^2, in good agreement with experiment.*

It has been demonstrated by Bethe and Longmire† that the theoretical result depends on the interaction only through the effective range and binding energy, for small photon energies. On the other hand, the high energy cross section is explicitly shape dependent, as is indicated by Eq. (6–37). The factor u_p, as well as dV/dr, depends on the shape of the potential at such energies that the P-wave phase shift is appreciable. Equation (6–37) is applicable only to ordinary central potentials. Exchange or noncentral potentials must be treated in detail to calculate the cross section for energies above 10 Mev.

The angular distribution of the photoneutron or photoproton is of some interest, since we will find that it provides a means for discriminating between electric dipole and magnetic dipole transitions. The form of the angular distribution can be calculated simply from the angular behavior of the wave function in the final state, which is, as we know, a P-function. The intensity observed in any direction is proportional to the square of the wave function evaluated at the corresponding angle. The incident gamma-ray beam is presumed to be unpolarized, therefore the outgoing intensity is to be obtained by averaging over both polarizations of the photon. The evaluation of the matrix element for a particular direction of polarization led to the conclusion that only the outgoing function Y_1^0 is

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* Barnes, Carver, Stafford and Wilkinson, *Phys. Rev.* **86,** 359 (1952); Wilkinson, *ibid.*, p. 373.

† *Phys. Rev.* **77,** 647 (1950).

produced in the transition if the axis of quantization is taken parallel to the polarization vector. This choice of axis is inappropriate for our present purpose, since we want to average over both directions of polarization. It is more convenient to have the z-axis along the direction of propagation of the photon. Since the final state function, $Y_1^0(\theta',\varphi')$, in the original reference frame is proportional to the cosine of the angle between the direction of polarization and the direction of observation, the radiated intensity in the new reference system is proportional to $\sin^2\theta\cos^2\varphi$ for x-polarization, and to $\sin^2\theta\sin^2\varphi$ for y-polarization. It follows that the average of the intensity over the two directions of polarization is simply proportional to $\sin^2\theta$. Therefore the differential cross section for photodisintegration into the direction θ with respect to the incident beam is

$$\sigma_e(\theta) = \frac{3}{8\pi}\sigma_e \sin^2\theta. \tag{6–43}$$

The $\sin^2\theta$ dependence of the cross section is characteristic of electric dipole transitions in the absence of noncentral forces. The D-function introduced into the ground state by noncentral forces leads to such a mixture of the ${}^3P_0$, ${}^3P_1$, ${}^3P_2$ states that the cross section includes a small isotropic term.* The tensor potential acting on the outgoing P-wave has a similar effect. On the other hand, if the tensor interaction has the character of a Serber exchange, it vanishes in the P-states, and the latter effect does not occur.†

It will be found in the next section that the magnetic dipole transitions play quite an important role in the low energy photoprocesses. Since this transition produces a ${}^1S$ state from the ${}^3S$ ground state, the corresponding angular distribution must be isotropic. Therefore the differential cross section, including both the magnetic and electric effects, will have the form

$$\sigma(\theta) = A + B\sin^2\theta, \tag{6–44}$$

where the isotropic term A is largely magnetic but may be partly due to the tensor force contribution to the electric dipole transition. The coefficient B is determined primarily by the electric dipole process.

6–5 Capture of neutrons by protons. In the capture of a neutron by a proton to form a deuteron, the excess energy is liberated as electromagnetic radiation, hence this process is just the inverse of photodisintegration by gamma radiation of that energy. The capture cross section is measured accurately only for very slow neutrons, the reason being that then the neutron spends sufficient time in the neighborhood of the proton for the

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* W. Rarita and J. Schwinger, *Phys. Rev.* **59,** 436 and 556 (1941).

† A discussion of the influence of small deviations from the Serber exchange is given by N. Austern, *Phys. Rev.* **85,** 283 (1952).

capture to occur with a measurable probability. Therefore we shall concentrate our attention on the process for thermal neutrons, that is, on a neutron-proton relative energy of about $\frac{1}{80}$ ev. The experimental value of the thermal capture cross section is*

$$\sigma_c = 0.329 \pm 0.004 \text{ barn.} \tag{6 45}$$

For such slow particles the capture from the S-state is expected to be more probable than capture from the P, D, or other states, since only the S-function has an appreciable amplitude within the range of the forces. However, capture from the S-state can occur only by means of a magnetic dipole transition, and then only from the ^1S state; the spins must turn over in order to produce the variation in the magnetic moment required to cause emission of radiation. But the spin flip will not occur unless there are forces acting to reorient the spins, i.e., not unless the interaction is spin dependent. Therefore the existence of a rather large capture cross section provides additional evidence for the spin dependence of the interaction, and quantitative analysis of the cross section should yield further information concerning the singlet potential.

In view of the fact that the capture process is just the inverse of photodisintegration, a rather intimate connection between the capture cross section, σ_c, and the photodisintegration cross section, σ_p, is to be expected. The relationship between σ_c and σ_p can be obtained from a general principle, *the principle of detailed balance*. To derive the principle, we consider the connection between two transitions, the one from state A to state B of a system, and the other from state B to state A. Since the transitions of interest proceed by means of the weak electromagnetic interaction, the analysis of the transitions may be based on perturbation theory. It is convenient to deal with plane waves to describe the incident beam, so both states A and B will be so treated. The plane waves may be normalized to a cubic enclosure of large volume V if a periodic boundary condition is imposed in order to ensure the existence of an incident current. The transition probability for $A \rightarrow B$ is given by

$$w_{A \rightarrow B} = \frac{2\pi}{\hbar} | H_{BA} |^2 \rho_B,$$

where H_{BA} is the matrix element of the interaction responsible for the transition, and ρ_B is the density of final (plane wave) states of type B. For the inverse reaction, the transition probability is

$$w_{B \rightarrow A} = \frac{2\pi}{\hbar} | H_{AB} |^2 \rho_A.$$

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* Hamermesh, Ringo, and Wexler, *Phys. Rev.* **90**, 603 (1953). See also Harris, *et al.*, *Phys. Rev.* **91**, 125 (1953).

The interaction H is a hermitian operator, whence

$$| H_{AB} |^2 = | H_{BA} |^2,$$

so the ratio of transition probabilities is simply

$$\frac{w_{A \to B}}{w_{B \to A}} = \frac{\rho_B}{\rho_A}.$$

The cross section is related to the transition probability by

$$\sigma_{A \to B} = \frac{w_{A \to B}}{S_A},$$

where S_A is the incident current *per state* of the system A. Therefore

$$\frac{\sigma_{A \to B}}{\sigma_{B \to A}} = \frac{\rho_B S_B}{\rho_A S_A}.$$

Since the current per state for a plane wave is v/V, where v is the velocity* of the incident particle, the ratio of cross sections is

$$\frac{\sigma_{A \to B}}{\sigma_{B \to A}} = \frac{\rho_B v_B}{\rho_A v_A}.$$

Since the equilibrium current in state A is proportional to $\rho_A v_A$ and consequently $\rho_A v_A \sigma_{A \to B}$ is the relative number of transitions $A \to B$ per unit time, this is the principle of detailed balance in the form appropriate to problems of statistical mechanics: in equilibrium the rate of transition from $A \to B$ is equal to the rate from $B \to A$. The result can be put in another useful form by noting that the density of final states for the plane waves is proportional to the volume in momentum space per unit energy:

$$\frac{\rho_B}{\rho_A} = \frac{p_B^2 dp_B/dE_B}{p_A^2 dp_A/dE_A}.$$

The relationship $dp/dE = 1/v$ then leads to the result

$$\frac{\sigma_{A \to B}}{\sigma_{B \to A}} = \frac{p_B^2}{p_A^2}. \tag{6-46}$$

The observed cross sections are not those described above for each state A and B, but they are the average over the initial states of polarization of

* This is valid even for the photon, since the incident current is just equal to the Poynting vector divided by $\hbar\omega$, and the Poynting vector is the energy density $\hbar\omega N_\omega \, d\omega/V$ times the velocity of light. Here $N_\omega \, d\omega$ is the number of photons in the group of states between ω and $\omega + d\omega$, so the current per state is simply c/V.

the system and the sum over the final states of polarization. If this summing and averaging process is denoted by a bar,

$$\bar{\sigma}_{A \to B} = \frac{1}{g_A} \sum_{A,B} \sigma_{A \to B},$$

and

$$\bar{\sigma}_{B \to A} = \frac{1}{g_B} \sum_{A,B} \sigma_{B \to A},$$

where g_A is the number of states of polarization of type A, and the sum is taken over all possible substates of polarization, both in the initial and final states. We find then that

$$g_A p_A^2 \bar{\sigma}_{A \to B} = \sum_{A,B} p_A^2 \sigma_{A \to B}$$

and

$$g_B p_B^2 \bar{\sigma}_{B \to A} = \sum_{A,B} p_B^2 \sigma_{B \to A},$$

which, when combined with Eq. (6–46), has the consequence

$$\frac{\bar{\sigma}_{A \to B}}{\bar{\sigma}_{B \to A}} = \frac{g_B p_B^2}{g_A p_A^2}. \tag{6–47}$$

This is the principle of detailed balance in its most useful form for our purposes.

Let us take the process $A \to B$ to be neutron-proton capture. Then the photodisintegration of the deuteron is the transition $B \to A$. State A consists of a free proton and a free neutron, each of which has two states of polarization, so that $g_A = 2 \times 2 = 4$. A deuteron and a photon are the constituents of state B, and the photon has two states of polarization while the deuteron has three, corresponding to the three orientations of its total angular momentum. Therefore $g_B = 6$. The deuteron may be taken to be initially at rest, whereby p_B is fixed as the momentum of the photon $\hbar\omega/c$, while $p_A = \hbar k$, if $E = \hbar^2 k^2/M$ is the kinetic energy of dissociation of the deuteron in the center of mass system.* According to Eq. (6–47) the capture cross section σ_c is given in terms of the photodisintegration cross section σ_p by

$$\sigma_c = \frac{3}{2} \frac{(\hbar\omega)^2}{(\hbar c k)^2} \sigma_p,$$

which can be expressed in terms of the energy $E = \hbar\omega - |E_g|$ as

$$\sigma_c = \frac{3}{2} \left(\frac{\hbar\omega}{Mc^2} \right) \left(\frac{\hbar\omega}{E} \right) \sigma_p. \tag{6–48}$$

* The recoil due to the momentum of the photon is neglected here.

Let us now apply this result to the special case of electric dipole capture. Then the electric dipole photodisintegration cross section given by Eq. (6–42) is the appropriate value of σ_p, whence

$$\sigma_c(\text{e.d.}) = 6.6 \times 10^{-29} \left(\frac{|E_g|}{\hbar\omega}\right)^{1/2} \left(\frac{E}{\hbar\omega}\right)^{1/2} \text{cm}^2. \tag{6–49}$$

The proportionality of the cross section to the square root of the energy is characteristic of capture from a P-state. The fact that the cross section vanishes at zero energy is simply due to the centrifugal barrier which prevents the P-function from having an appreciable value within the range of the forces. At thermal energy, say at $E = \frac{1}{80}$ ev., the cross section is of the order of 5×10^{-9} barn, which is completely negligible as compared with the observed thermal capture cross section.

Only the capture from the S-state can lead to a cross section of the correct order of magnitude. Then there is no centrifugal barrier to reduce the wave function, with the result that the photodisintegration cross section is proportional to the square root of the kinetic energy E, as we have already remarked in the previous section. According to Eq. (6–48), the capture cross section is then proportional to $1/\sqrt{E}$ or $1/v$, for capture from the S-state. This "$1/v$ law" is characteristic for capture from the S-state by any nucleus; it is a simple consequence of the fact that the matrix element for the transition is essentially independent of E at low energies.

The selection rule for a magnetic dipole transition is

$$\Delta j = \pm 1, 0; \text{ no},$$

where the "no" means that there is no change in parity. Therefore, as we have remarked before, capture from the continuum S-state to the $^3S_1$ ground state of the deuteron via such a transition is possible. Furthermore, it turns out that only the $^1S_0 \rightarrow {}^3S_1$ transition can occur.

The cross section for the inverse process, magnetic dipole photodisintegration, can be obtained simply by replacing the electric dipole moment in Eq. (6–36) by the magnetic dipole moment:

$$\sigma_m = \frac{4\pi M k\omega}{\hbar^2 c} |\mathfrak{M}'_{gf}|^2, \tag{6–50}$$

where $\mathfrak{M} = (\boldsymbol{\nu}\cdot\mathfrak{M})$ if $\boldsymbol{\nu}$ is the unit vector normal to the direction of polarization $\mathbf{u}$ and to the direction of propagation of the photon. The vector $\mathfrak{M}$ is just the magnetic moment operator of the deuteron given by Eq. (3-37), or its equivalent,

$$\mathfrak{M} = \frac{e\hbar}{2Mc}\left\{\frac{1}{2}\mathbf{L} + (\mu_p + \mu_n)\mathbf{S} + \frac{1}{2}(\mu_p - \mu_n)(\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_n)\right\}.$$

Since both the initial and final states are S-functions, the matrix element of $\mathbf{L}$ vanishes. Furthermore, the matrix element of $\mathbf{S}$ vanishes between the triplet and singlet states because $\mathbf{S}$ is diagonal in this representation. The matrix element of $\mathbf{S}$ also vanishes between the two $^3S_1$ states, since the radial functions, being solutions of the same Schroedinger equation for different energies, are orthogonal.* Therefore only the last term contributes to the transition, and that can lead just to a singlet-triplet transition, since it is antisymmetric for interchange of the neutron and proton.

The initial state is

$$\psi_g = \frac{1}{\sqrt{4\pi}} \frac{u_g}{r} \chi_1^m,$$

where χ_1^m is the triplet spin function, and the final state is

$$\psi_f = \frac{1}{\sqrt{4\pi}} \frac{u_s}{r} \chi_0.$$

Here u_g is the radial function for the ground state of the deuteron, while u_s is the continuum singlet state function with the appropriate energy. The matrix element is

$$\mathfrak{M}'_{gf} = \int_0^\infty u_g u_s \, dr \, (\chi_1^m, (\boldsymbol{\nu} \cdot \mathfrak{M}) \chi_0).$$

If the expression for $\mathfrak{M}$ is now inserted and cognizance is taken of the selection rules established above, the matrix element becomes

$$\mathfrak{M}'_{gf} = C(\chi_1^m, (\boldsymbol{\nu} \cdot \boldsymbol{\sigma}_{pn}) \chi_0),$$

where

$$C = \frac{1}{2} (\mu_p - \mu_n) \frac{e\hbar}{2Mc} \int_0^\infty u_g u_s \, dr,$$

and $\boldsymbol{\sigma}_{pn} = \boldsymbol{\sigma}_p - \boldsymbol{\sigma}_n$. The quantity of interest is the average of the cross section over all initial states of polarization of the deuteron, that is, over the three possible m-values, $m = 1, 0, -1$. It is proportional to

$$\frac{1}{3} \sum_m |\mathfrak{M}'_{gf}|^2 = \frac{C^2}{3} \sum_m (\chi_0, (\boldsymbol{\nu} \cdot \boldsymbol{\sigma}_{pn}) \chi_1^m)(\chi_1^m, (\boldsymbol{\nu} \cdot \boldsymbol{\sigma}_{pn}) \chi_0).$$

Evaluation of the sum can be simplified by noting that

$$(\chi_0, (\boldsymbol{\nu} \cdot \boldsymbol{\sigma}_{pn}) \chi_0) = 0,$$

since the antisymmetric operator, $\boldsymbol{\sigma}_{pn}$, converts a singlet state into the

* This special selection rule is characteristic of magnetic dipole transitions. It is a consequence of the lack of any radial dependence in the magnetic moment operator.

orthogonal triplet state. Therefore the sum can be extended to include not only all of the triplet states, but also the singlet state, with the result that closure can be invoked to obtain

$$\frac{1}{3}\sum_m |\mathfrak{M}'_{gf}|^2 = \frac{C^2}{3}(\chi_0,(\boldsymbol{\nu}\cdot\boldsymbol{\sigma}_{pn})^2\chi_0).$$

We also use the relationship [Eq. (4–12)]

$$\boldsymbol{\sigma}_n\chi_0 = -\boldsymbol{\sigma}_p\chi_0$$

which, when combined with the fact that the square of any component of $\boldsymbol{\sigma}_p$ is 1, leads to the final result

$$\tfrac{1}{3}\sum_m |\mathfrak{M}'_{gf}|^2 = \tfrac{4}{3}C^2.$$

The cross section $\bar{\sigma}_m$ is to be obtained by averaging over the initial states of polarization of the photon also, but since our result is independent of the polarization, it is not thereby altered; nor is it altered by summing over final states, since only the one singlet state is produced. Consequently, the cross section is

$$\bar{\sigma}_m = \frac{16\pi}{3}\frac{Mk\omega}{\hbar^2 c}C^2.$$

Substitution for C in terms of the dimensionless integral,

$$J = \left(\frac{mc^2}{e^2}\right)^{3/2}\int_0^\infty u_g u_s\, dr,$$

leads to the result

$$\bar{\sigma}_m = \frac{\pi}{3}R_0^2\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{m}{M}\right)^{1/2}\left(\frac{\hbar\omega}{mc^2}\right)\left(\frac{E}{mc^2}\right)^{1/2}(\mu_p-\mu_n)^2J^2. \qquad (6\text{–}51)$$

The corresponding magnetic capture cross section, which is in effect the total capture cross section at low energies, is obtained by means of Eq. (6-48):

$$\sigma_c = \frac{\pi}{2}R_0^2\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{m}{M}\right)^{3/2}\left(\frac{\hbar\omega}{mc^2}\right)^3\left(\frac{mc^2}{E}\right)^{1/2}(\mu_p-\mu_n)^2J^2. \qquad (6\text{–}52)$$

Note here that the essential dependence on the energy of the incoming particle is given by $1/\sqrt{E}$; this is the anticipated $1/v$ law for capture from the S-state.

Evaluation of the integral J can be made to a very good approximation in terms of the singlet and triplet effective ranges and scattering lengths.* For this purpose it is necessary to introduce renormalized wave functions

* H. A. Bethe and C. Longmire, *Phys. Rev.* **77**, 647 (1950).

having the asymptotic behavior characteristic of effective range theory. If the renormalized functions are denoted by $\bar{u}_g$ and $\bar{u}_s$, we have the relationships

$$u_g = A\bar{u}_g$$

and

$$u_s = \frac{\sin \delta_s}{k}\bar{u}_s,$$

where $\delta_s(k)$ is the phase shift for singlet scattering. The normalization constant A is to be obtained from the condition

$$A^2\int_0^\infty \bar{u}_g^2\, dr = 1,$$

wherein the integral can be evaluated by noting that

$$\int_0^\infty \bar{u}_g^2\, dr = \int_0^\infty v_g^2\, dr - \int_0^\infty (v_g^2 - \bar{u}_g^2)\, dr.$$

If v_g is the auxiliary function of effective range theory,

$$v_g = e^{-k_g r},$$

the second term is just the expression for one-half the effective range,* and the integral involved in the first term can be evaluated directly. The result is

$$\int_0^\infty \bar{u}_g^2 dr = \frac{1}{2k_g} - \frac{1}{2}r_{0t},$$

whence

$$A^2 = \frac{2k_g}{1 - k_g r_{0t}},$$

and

$$\int_0^\infty u_g u_s\, dr = \frac{\sin \delta_s}{k}\sqrt{\frac{2k_g}{1 - k_g r_{0t}}}\int_0^\infty \bar{u}_g\bar{u}_s\, dr.$$

But we can write the identity

$$\int_0^\infty \bar{u}_g\bar{u}_s\, dr - \int_0^\infty v_g v_s\, dr = -\tfrac{1}{2}\int_0^\infty (v_g^2 - \bar{u}_g^2)\, dr - \tfrac{1}{2}\int_0^\infty (v_s^2 - \bar{u}_s^2)\, dr$$

$$+ \tfrac{1}{2}\int_0^\infty [(v_g - v_s)^2 - (\bar{u}_g - \bar{u}_s)^2]\, dr. \quad (6\text{–}53)$$

* Note that $\bar{u}_g$ and v_g are the ground state rather than the zero energy functions, so there is a small shape-dependent error in the evaluation of the integral.

Contributions to the last integral arise only within the range of the forces, and within that range the functions v_g and v_s as well as $\bar{u}_g$ and $\bar{u}_s$ are very nearly equal. Thus each term in the integral is small, and the difference between them is even smaller. The result is that this last integral may be neglected in very good approximation. When the other integrals are expressed in terms of the triplet and singlet effective ranges, we find

$$\int_0^\infty \bar{u}_g \bar{u}_s \, dr = \int_0^\infty v_g v_s \, dr - \tfrac{1}{4}(r_{0t} + r_{0s}).$$

The function v_g is given above, and the function v_s is

$$v_s = \frac{\sin (kr + \delta_s)}{\sin \delta_s},$$

so the integral of their product is found to be

$$\int_0^\infty v_g v_s \, dr = \frac{k_g + k \cot \delta_s}{k_g^2 + k^2}.$$

$k \cot \delta_s$ may be expressed directly in terms of the scattering length and effective range [Eq. (4–51)] with the result

$$\int_0^\infty v_g v_s \, dr = \frac{1}{k_g} \frac{1 - (1/k_g a_s) + \frac{1}{2} k_g r_{0s} (k/k_g)^2}{1 + (k/k_g)^2},$$

leading to the final expression for J:

$$J = \left(\frac{mc^2}{e^2}\right)^{\frac{1}{2}} \sqrt{\frac{2k_g}{1 - k_g r_{0t}}} \frac{\sin \delta_s}{k k_g} \times \left\{ \frac{1 - (1/k_g a_s) + \frac{1}{2} k_g r_{0s} (k/k_g)^2}{1 + (k/k_g)^2} - \frac{1}{4} k_g (r_{0t} + r_{0s}) \right\}. \quad (6\text{–}54)$$

Since they are proportional to J^2, it is evident that the capture and photodisintegration cross sections are directly proportional to the singlet scattering cross section $\sigma_s = 4\pi \sin^2 \delta_s / k^2$.

Capture only at very low energies is to be considered, therefore terms proportional to $(k/k_g)^2$ in J may be neglected. We also may set $\hbar\omega = |E_g|$. Insertion of (6–54) into Eq. (6–52) then leads to the result

$$\frac{\sigma_c}{\sigma_s} = 4.75 \times 10^{-7} \frac{1}{1 - k_g r_{0t}} \left[1 - \frac{1}{k_g a_s} - \frac{1}{4} k_g (r_{0t} + r_{0s}) \right]^2 \left(\frac{mc^2}{E}\right)^{\frac{1}{2}}. \quad (6\text{–}55)$$

A numerical value of the coefficient of $(mc^2/E)^{\frac{1}{2}}$ is to be obtained by

introducing the best available estimates of r_{0t}, a_s, etc. From Eq. (4–53), with the value of a_t given by Eq. (5–37), we find

$$k_g r_{0t} = 0.395.$$

Equation (5–37) also provides a_s:

$$1/k_g a_s = -0.182.$$

An estimate of r_{0s} can be made on the assumption that the neutron-proton and proton-proton interactions are identical. Then $r_{0s} \approx \rho_0$ and, from Eq. (6–23),

$$k_g r_{0s} \approx 0.613.$$

These figures lead to the ratio

$$\frac{\sigma_c}{\sigma_s} = 6.78 \times 10^{-7} \left(\frac{mc^2}{E}\right)^{1/2}. \tag{6–56}$$

For thermal neutrons,

$$\frac{\sigma_c}{\sigma_s} = 4.34 \times 10^{-3}.$$

The value of σ_s is given directly in terms of a_s, whence it is found to be 70.5 barns, and the thermal value of σ_c turns out to be*

$$\sigma_c = 0.306 \text{ barn}.$$

The fact that the calculated capture cross section is in rather good agreement with the observed value would seem to indicate that the choice of singlet effective range on the assumption that the n-p interaction is the same as the p-p interaction is at least approximately correct. As a matter of fact, it can be seen from Eq. (6–55) that the capture cross section could, in principle, be used to determine the singlet effective range. However, the sensitivity to that parameter is not very great An accurate determination by this method would require a good evaluation of the neglected shape-dependent contributions, such as the last term in Eq. (6–53). A much more serious limitation on this method of determining r_{0s} is due to a flaw in the assumption that the magnetic moment operator is made up additively of the moments of the nucleons. We shall see (Section 9–3) that there is a somewhat uncertain contribution to the matrix element of the magnetic moment due to nonadditivity which results in a correction to the cross section of the order of 4 percent. The uncertainty in that con-

* No correction for the velocity of the protons need be made here as a consequence of the $1/v$ law. In Section 5–2 we saw that the observed cross section is $\sigma = |\mathbf{v} - \mathbf{V}|\sigma_{CM}/v$, but $\sigma_{CM} = \text{const}/|\mathbf{v} - \mathbf{V}|$, so $\sigma = \text{const}/v$, which is just what would be obtained for a proton at rest.

tribution is reflected by a corresponding uncertainty in the value of r_{0s} obtained from the capture cross section.*

A final remark concerning the photodisintegration process seems appropriate here. From the energy dependence of the magnetic and electric cross sections we see that the ratio B/A of the coefficients in Eq. (6–44) is proportional to the energy of the disintegration products. Although the electric term vanishes at threshold, its contribution increases linearly with energy relative to the magnetic term at small energy, so it rapidly becomes the more important term as the energy is increased. At energies of the order of several Mev the magnetic term amounts to only a few percent of the whole. The same can be said of the isotropic term produced by the tensor interaction, and therefore the characteristic electric dipole angular distribution proportional to $\sin^2\theta$ is expected to predominate at appreciable energies.

6–6 Summarizing remarks on the two-body problem. Many and various conclusions concerning the two-nucleon interaction have been drawn from the phenomena discussed up to this point. In this section we shall attempt to draw together and summarize those facts which are of general importance.

The most detailed quantitative results are obtained from the low-energy data. The values of the scattering length, effective range, and other low-energy parameters are as follows:

n-p scattering lengths [Eq. (5–37)]:

$$a_t = 0.538 \times 10^{-12} \text{ cm},$$

$$a_s = -2.37 \times 10^{-12} \text{ cm}.$$

The radius of the deuteron is a convenient unit in which to express a:

$$k_g a_t = 1.25,$$

$$k_g a_s = -5.48.$$

n-p triplet effective range:

This may be calculated directly from Eq. (4–53) by using the above value of a_t. We find

$$k_g r_{0t} = 0.395$$

* Shape-dependent corrections do not alter appreciably the theoretical value of σ_c (N. Austern, in press). The discrepancy with the experimental value given by Eq. (6–45) seems to be outside of the experimental error. It may be interpreted as evidence for the existence of the nonadditivity effect.

or

$$r_{0t} = 1.70 \times 10^{-13} \text{ cm}.$$

Only preliminary measurements of r_{0s} are available.* The results seem to be in accord with the value estimated below on the basis of the proton-proton data.

p-p parameters:

The low energy proton-proton scattering parameters are given by Eq. (6-23):

$$f(0) = 8.66,$$

$$\rho_0 = 2.65 \times 10^{-13} \text{ cm}.$$

A corresponding singlet scattering length for p-p scattering can be estimated from these by means of Eq. (6 30). We found

$$a_p = -1.6 \times 10^{-12} \text{ cm},$$

and concluded that this compares well with the neutron-proton value of a_s, since the singlet scattering length is extremely sensitive to the depth of the interaction. The quantity ρ_0 is closely related to the singlet effective range, r_{0s}, for the neutron-proton system, although the two are not expected to be exactly the same even if the specific nuclear interaction is charge independent.

The range and depth parameters of a central n-p potential of given shape may now be determined from relationships such as those presented in Figs. 4-4 and 4-5. For a square well, the triplet parameters may be obtained directly from the scattering length by combining Eq. (3-25) with Eq. (4-36). The result is

$$\alpha_t = 2.05 \times 10^{-13} \text{ cm},$$

$$(K_t\alpha_t)^2 = 3.56.$$

The parameters may be obtained for a square-well singlet potential only if both the scattering length and effective range are known accurately. From the scattering length, we find [Eq. (4-40)]

$$K_s\alpha_s \approx \frac{\pi}{2}.$$

If the value ρ_0 based on p-p scattering is used for r_{0s}, an estimate of the difference between singlet range and singlet effective range can be made by means of Eq. (4-56):

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* Fields, Adair, Becker, and Darden, *Phys. Rev.* **91,** 441 (1953); Storrs and Frisch, *Phys. Rev.* **90,** 339 (1953); Hafner, Hornyak, Falk, Snow, and Coor, *Phys. Rev.* **89,** 204 (1953).

$$\alpha_s = r_{0s}(1 - 0.04).$$

From this result it is evident that the singlet and triplet square-well potentials would have somewhat different ranges:

$$\frac{\alpha_s}{\alpha_t} = 1.41.$$

However, for a Yukawa potential a problem of this sort does not arise. Let us consider a potential including the tensor interaction and treat the range, α_T, of the tensor interaction as independent of the range, α_c, of the central interaction. The numerical calculations carried out by Feshbach and Schwinger concern just such a potential; their results are tabulated in Appendix 2. From their table based on a quadrupole moment $Q = 2.766 \times 10^{-27} e$ cm^2, the sets of parameters found to be in reasonable agreement with the value of r_{0t} are given in Table 6–2. From this table one fact is clear, the tensor range must be greater than the central range if the shape of the potential is correctly described by a Yukawa function.*

TABLE 6–2

Values of the Yukawa tensor interaction constants that are consistent with the observed binding energy, quadrupole moment, and triplet effective range of the deuteron. See Appendix 2 for definitions.

$k_g\alpha_c$	$k_g\alpha_T$	γ	$K^2\alpha_c^2$	$\lvert b \rvert^2$	r_{0t} (10^{-13} cm)
0.313	0.320	1.786	1.12	0.042	1.71
0.275	0.355	0.836	1.34	0.038	1.71
0.256	0.355	0.734	1.33	0.039	1.68
0.224	0.640	0.079	1.92	0.021	1.68
0.224	0.492	0.170	1.75	0.028	1.68

If we again take the evaluation of r_{0s} from proton-proton scattering, an estimate can be made of the range of the singlet Yukawa potential. Here the tensor interaction plays no role, of course. From Fig. 4–4 the value

$$\alpha_s = 1.19 \times 10^{-13} \text{ cm},$$

or

$$k_g\alpha_s = 0.27$$

is obtained. According to Table 6–2, the same value of the range of the triplet central potential would be consistent with the other data concerning the triplet state. The range-depth relation for the singlet potential is given approximately by Eq. (4–41):

* Compare Section 8–3.

$$K_s^2\alpha_s^2 = 1.68.$$

To conclude the discussion of the low-energy data, it is of interest to record the results of the direct analysis by Hall and Powell* of the data in terms of the parameters of an appropriate charge-independent potential. The analysis is based on the properties of the deuteron ground state, the n-p scattering lengths, and the p-p scattering up to 4.3 Mev. A Yukawa potential of the form

$$V = -V_0\alpha_c r^{-1}\{(1 - \tfrac{1}{2}g + \tfrac{1}{2}g\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)e^{-r/\alpha_c} + \gamma(\alpha_T/\alpha_c)S_{12}e^{-r/\alpha_T}\}(1 - \epsilon + \epsilon P) \quad (6\text{-}57)$$

was considered. Here S_{12} is the tensor operator and P is the space exchange operator. The appearance of a definite amount of exchange is a consequence of the fact that, for the p-p scattering, a fair estimate of the P-wave phase shift can be obtained. The values of the parameters which lead to a good fit with the data when inserted into Eq. (6-57) are

$$\begin{aligned} \alpha_c &= 1.176 \times 10^{-13} \text{ cm}, \\ V_0 &= 39.83 \text{ Mev}, \\ \alpha_T &= 1.529 \times 10^{-13} \text{ cm}, \\ \gamma &= 0.8481, \\ g &= -0.0834, \\ \epsilon &= 0.62. \end{aligned} \quad (6\text{-}58)$$

It should be remarked that the above value of α_c is very nearly that obtained by setting $r_{0s} = \rho_0$. This indicates that the difference between r_{0s} and ρ_0 is very small for a Yukawa potential, while for a square well it was found in Section 6-2 to be of the order of four percent. Evidently the quantity $r_{0s} - \rho_0$ is rather sensitive to the shape of the potential, and therefore ρ_0 does not provide a very good basis for the direct comparison of p-p and n-p interactions.

Information obtained from the high-energy scattering data is much less quantitative. Two important features of the interaction suggested by the data are that the shape of the potential is such that it has a long tail, as for a Yukawa or exponential function, and that the exchange dependence does not differ greatly from the Serber exchange. The first of these conclusions is acceptable, in fact one might say that it is desirable, in view of the fact that the meson theory seems certain to lead to such a tail on the

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* *Phys. Rev.* **90,** 912 (1953). See also Hoisington, Share, and Breit, *Phys. Rev.* **56,** 884 (1939) and Yovits, Smith, Hull, Bengston, and Breit, *Phys. Rev.* **85,** 540 (1952).

potential. The second result, which is supported by the evidence for a small P-wave phase shift in low energy p-p scattering, is a source of serious difficulty. The exchange potential was originally hypothesized in order to account for the saturation of nuclear forces, but the Serber mixture introduces much too much ordinary potential to be consistent with the saturation property.* Therefore some other provision must be made for saturation if the analysis of the high-energy scattering data is to be accepted.

Another difficulty is the apparent lack of equality of the neutron-proton and the proton-proton interactions at high energy. No evidence of such a charge dependence of the specific nuclear force is given by the low-energy data. The high-energy p-p scattering also shows a rather strange angular and energy dependence.

There are a number of directions in which explanations, or rather excuses, for these phenomena may be sought. First of all, consideration should be given to the influence on the high-energy data of relativistic corrections. An offhand estimate would indicate that these corrections might amount to 30 percent at 300 Mev. It does not seem likely that corrections of that magnitude could account either for the fore-and-aft symmetry which suggests the Serber exchange or for the qualitative difference between the n-p and p-p systems. However, velocity-dependent terms of the same form but much larger than the relativistic corrections might occur as a direct contribution to the nonrelativistic interaction. This is strongly suggested by the success of the spin-orbit coupling scheme in heavier nuclei.† Unfortunately, a sufficiently complete analysis of the influence of such an interaction on the high-energy scattering has not been made, so no final judgment concerning its effect can be formed at present.

The difficulty concerning the exchange potential might mean that nuclear interactions are a many-body phenomenon. Since saturation is a many-body effect, it could then be ascribed to more-than-two-body interactions. However, the situation resulting from this assumption is so complicated and ambiguous that more definite evidence for it would seem to be required to justify extensive theoretical work in that direction.

Possibly the analysis of high-energy data in terms of a static potential, or even a simple velocity-dependent potential, is doomed to failure. After all, we know that for energies in the neighborhood of those being considered here, mesons are produced by nucleon-nucleon collisions. Therefore it is expected that the mesonic structure of the nucleons may play a great role in the scattering. One might say that the meson clouds are highly polarized in the high-energy collision. Since the cloud is presumably responsible for the interaction between nucleons, its polarization

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* See Section 8–8.

† See Section 8–7.

would be expected to lead to a marked change in the apparent interaction. An effect of a similar kind might be expected if the nucleon structure is so complicated that it has metastable excited states. The existence of these so-called nucleon isobars would certainly be expected to have a profound effect on the scattering for energies comparable to the excitation energy of an isobar.*

The ideal procedure for settling some of these matters would probably be to make measurements for energies below about 30 Mev, where polarization and velocity effects can be expected to be small, with sufficient precision to determine accurately the parameters in the static potential. Then the direct comparison between the high-energy scattering calculated for this well-defined potential and the observed scattering would shed some light on the energy-dependent effects.

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* A very interesting summary of the two-nucleon interaction problem has been presented by Breit and Hull, *Am. J. Phys.* **21,** 184 (1952).

CHAPTER 7

REMARKS ON THE MESON THEORY OF NUCLEAR FORCES

No completely satisfactory fundamental theory of nuclear forces exists at the present time. All of the theories that seem to warrant serious consideration have been built upon Yukawa's original meson theory, which was based on an analogy with electrodynamics. A feature common to every form of the theory is that the quanta of the field responsible for nuclear forces are particles of mass some 200 or 300 times the electron mass. These quanta must interact strongly with nucleons if they are to account for the nuclear forces. The subsequent discovery of just such particles, the π-mesons or *pions*, has lent great support to at least the qualitative aspects of this approach to the problem. Detailed application of the theory leads to very serious quantitative difficulties, but one holds to the hope that they may be due to the inadequacy of the approximations used in making the calculations. On the other hand, there is always the possibility that the qualitative successes of the theory are fortuitous, that the structure of existing theories is incorrect.

A quantized, relativistic formulation of the field theory is required to understand the problem in all of its most fundamental aspects. However, it is possible to obtain an insight into the structure of the theory and, in particular, to understand most of those qualitative features which seem to be correct, by limiting consideration to a classical (unquantized), nonrelativistic field. That very limited description will be given here.*

Before going into a discussion of the meson theory, it is convenient to introduce a device, known as the *isotopic spin* or *i-spin*,† which not only simplifies the description of the field theory but is also useful for the treatment of problems in nuclear spectroscopy and beta-decay.

7–1 I-spin and the generalized Pauli principle. The nucleon may be treated as a single entity having two states, the proton and the neutron. Then the wave function of the nucleon must be parametrized by a variable

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* A general discussion of quantized field theories with applications to the nuclear force problem is given by G. Wentzel, *Quantentheorie der Wellenfelder*, Franz Deuticke (1943). English translation, *Interscience* (1949).

† We use the abbreviations "*i*-spin," "*i*-triplet," "*i*-singlet," etc., in place of the more awkward, conventional terms "isotopic spin," "isotopic spin triplet," etc. The concept was first introduced by W. Heisenberg in his fundamental work on nuclear structure; *Zs. f. Phys.* **77**, 1 (1932).

indicating with which of the two states we have to deal. Let us denote the variable by τ, where $\tau = +1$ refers to the proton and $\tau = -1$ to the neutron.* Any property of a nucleon having a different value for neutron and proton may be expressed explicitly in terms of the quantity τ. For example, the charge is $(e/2)(1 + \tau)$, while the mass of the nucleon is $\frac{1}{2}(M_p + M_n) + \frac{1}{2}(M_p - M_n)\tau$, where M_p and M_n are proton and neutron masses, respectively.

The identity of the nucleon is buried in the wave function, which now depends on τ as well as on the other nucleon variables, $\mathbf{r}$ and σ_z. The wave function may, for example, be a product of a space-spin function and a function $\xi(\tau)$. There are just two linearly independent functions $\xi(\tau)$, and they may be taken to be $\xi^+(\tau)$ and $\xi^-(\tau)$, where

$$\begin{aligned} \xi^+(1) &= 1, & \xi^+(-1) &= 0, \\ \xi^-(1) &= 0, & \xi^-(-1) &= 1. \end{aligned} \tag{7-1}$$

Thus ξ^+ is the state in which the nucleon is known to be a proton, and ξ^- is that in which the nucleon is known to be a neutron.

The functions $\xi^{\pm}(\tau)$ are formally identical with the ordinary spin functions $\chi^{\pm}(\sigma_z)$. Just as for the spin functions, the τ-functions can be written as the two-dimensional column vectors

$$\xi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{7-2}$$

and these orthonormal vectors define a two-dimensional space wherein τ may be considered to be an operator analogous to σ_z. The functions $\xi^{\pm}$ are called *isotopic spin* or *i-spin* functions, since the isotopic identity of a nucleus of given Z is fixed by the sum of the τ-values for all nucleons in the nucleus. Now we know that for ordinary spin the complete set of operators in the two-dimensional spin space is formed by the components σ_x, σ_y, and σ_z of the spin vector $\boldsymbol{\sigma}$. Therefore a complete set of operators in the space of the i-spin functions is τ_1, τ_2, τ_3, where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7-3}$$

The operator τ_3 is identical with what we have been calling τ; the characteristic values $\tau_3 = \pm 1$ correspond to proton and neutron states, respectively. The collection of the three operators

$$\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$$

* The sign of τ has been reversed here over the more customary usage. However, the present convention has some advantages in applications to meson theories.

is called the *i-spin*. The components of this vector refer to axes in an abstract three-dimensional space, which we shall call *charge space*. The interrelationship between the charge space and the two-dimensional space defined by the i-spin functions is clearly the same as that between configuration space and the space of the spin functions. In particular, a unitary transformation in either spin space induces the same rotation of axes in the corresponding three-dimensional space. This fact turns out to be useful in spite of the absence of a simple physical meaning for the concept of a rotation in charge space.

The operators τ_1 and τ_2 can be given a very useful physical interpretation. If that were not possible, no purpose would be served by introducing them. The linear combinations

$$\tau_\pm = \tfrac{1}{2}(\tau_1 \pm i\tau_2) \tag{7-4}$$

have the matrix form

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and operation on the spin functions (7–2) results in the equations

$$\begin{aligned} \tau_+\xi^+ &= 0, & \tau_+\xi^- &= \xi^+, \\ \tau_-\xi^+ &= \xi^-, & \tau_-\xi^- &= 0. \end{aligned} \tag{7-5}$$

Thus τ_+ is an operator which annihilates a proton state and converts a neutron state into a proton state. Similarly, τ_- converts a proton to a neutron and annihilates a neutron state. Now such processes as conversion of a proton to a neutron certainly occur. For example, the negative beta-decay of a nucleus involves the conversion of one neutron into a proton, while positive beta-decay involves the conversion of a proton into a neutron. Therefore it is clear that the τ_+ and τ_- operators would be useful for the description of beta-processes. Furthermore, the possible existence of exchange forces suggests that proton and neutron are interconverted one into the other, and the τ_+ and τ_- operators provide a means for giving a fundamental description of that process.

In a system composed of several nucleons, the identity of each nucleon can be buried in the wave function, Ψ, by expressing it as a function of the τ_3 variables. Then Ψ is a function of the five coordinates of each nucleon,

$$\Psi = \Psi(\mathbf{r}_1,\sigma_z(1),\tau_3(1);\mathbf{r}_2,\sigma_z(2),\tau_3(2);\cdots),$$

and all nucleons are to be treated as identical. The interactions between nucleons must be written in a form that is symmetric for the interchange of any pair, even for those interactions that distinguish between neutron and proton. For example, the coulomb interaction would be given in the form

$$\frac{e^2}{8}\sum_{j\neq k}\frac{(1+\tau_3(j))(1+\tau_3(k))}{r_{jk}},$$

which is clearly symmetric for interchange of any pair of particles j and k. The identity of the nucleons leads us to investigate the problem of the statistics of the nucleon.* We know that systems of identical particles are always found to be separated into two classes: those satisfying the conditions of Fermi-Dirac statistics and those satisfying the conditions of Bose-Einstein statistics. Both a system of protons and a system of neutrons belong to the former class. Although there is no *a priori* necessity that the nucleon should belong to either class, we wish to establish that as a result of the physical possibility of converting a neutron to a proton or a proton into a neutron, the nucleon system as a whole *must* obey Fermi-Dirac statistics.

The point can most easily be illustrated by considering a simple two-nucleon system. If the system consists of a neutron and proton, there are two possible i-spin states, namely, $\xi^+(1)\xi^-(2)$ and $\xi^-(1)\xi^+(2)$. We could alternatively introduce the linear combinations

$$\begin{aligned}\xi_1^0 &= \frac{1}{\sqrt{2}}[\xi^+(1)\xi^-(2)+\xi^-(1)\xi^+(2)],\\ \xi_0 &= \frac{1}{\sqrt{2}}[\xi^+(1)\xi^-(2)-\xi^-(1)\xi^+(2)],\end{aligned}\tag{7-6}$$

which are the analog of the triplet and singlet wave functions χ_1^0 and χ_0. The i-triplet and i-singlet functions will, in general, be expected to have different energies, since they are not converted into each other by an interchange of the nucleons.† Only in the case of an accidental degeneracy of the i-triplet and i-singlet would the function $\xi^+(1)\xi^-(2)$ form a stationary state of the system. In general, then, the functions (7-6) provide the best basis for discussion of the two-nucleon system. The other members of the i-triplet state are the symmetric functions

$$\xi_1^1 = \xi^+(1)\xi^+(2)$$

and

$$\xi_1^{-1} = \xi^-(1)\xi^-(2),$$

the first being a two-proton state and the second a two-neutron state.

* Cassen and Condon, *Phys. Rev.* **50**, 846 (1936).

† Since the Hamiltonian is symmetric, functions which are transformed into each other by a permutation must be solutions for the same energy, i.e., they must form a set of degenerate states.

They are labeled by the third component, $T_3 = 1, 0, -1$, of the total i-spin

$$\mathbf{T} = \tfrac{1}{2}(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2).$$

Let us momentarily limit attention to a continuum ^3S state of the deuteron. There are two such functions:

$$\Psi_1 = \psi_s(r)\chi_1\xi_1^0 \tag{7-7}$$

and

$$\Psi_2 = \psi_s(r)\chi_1\xi_0, \tag{7-8}$$

since ξ_1^0 and ξ_0 are both neutron-proton states. The first function is symmetric for interchange of all coordinates, and the second function is antisymmetric. Now in a continuum state the neutron can undergo beta-decay to become a proton. Since the interaction responsible for the beta-process is, like all interactions, symmetric for interchange of the nucleons, the state (7-7) can go over only to a symmetric state, while the state (7-8) produces an antisymmetric state. Since two protons necessarily have a symmetric i-spin function, the wave function produced by decay of (7-7) would be symmetric in the space and spin coordinates, a wave function which cannot occur because two protons are subject to the Pauli principle. This difficulty does not arise if the symmetric wave function (7-7) does not occur in nature.

When the argument is broadened to include many nucleons, we conclude that the nucleon wave function must be antisymmetric for the interchange of any pair of nucleons. For, if the wave function contains a term which is symmetric under the interchange of a particular pair of (unlike) nucleons, a violation of the Pauli principle for particles of the same i-spin would result from a beta-decay or any other process causing one member of the pair to change its i-spin. The requirement of total antisymmetry of the nucleon function will be referred to as the *generalized Pauli principle.*

As a consequence of the generalized Pauli principle, all the exchange operators can be expressed as spin and i-spin operators. The following well-known properties of the triplet spin function, χ_1^m, and the singlet spin function, χ_0, are used for this purpose:

$$(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\chi_1^m = \chi_1^m,$$

$$(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\chi_0 = -3\chi_0.$$

We note that the consequent relationships,

$$\tfrac{1}{2}(1 + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\chi_1^m = \chi_1^m,$$

$$\tfrac{1}{2}(1 + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\chi_0 = -\chi_0,$$

are exactly the properties of the spin exchange operator P_{12}^{σ} which interchanges the spin variables of the two nucleons:

$$P^{\sigma}_{12}\chi_1^m = \chi_1^m,$$

$$P^{\sigma}_{12}\chi_0 = -\chi_0,$$

since the triplet function is symmetric and the singlet function is antisymmetric. Because any function of the spin variables $\sigma_z(1)$ and $\sigma_z(2)$ can be written as a linear combination of the singlet and triplet functions, it follows that the spin exchange operator is identical with the spin operator:

$$P^{\sigma}_{12} = \tfrac{1}{2}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2). \tag{7-9}$$

The same analysis can be carried through for the i-spin exchange operator P^{τ}_{12}, representing exchange of the i-spin variables. We find

$$P^{\tau}_{12} = \tfrac{1}{2}(1 + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2). \tag{7-10}$$

If the wave function is subjected first to P^{σ}_{12}, then to P^{τ}_{12}, and finally to the Majorana operator, P^{r}_{12}, which exchanges the space coordinates, the result is equivalent to an interchange of the nucleons. According to the generalized Pauli principle the wave function must therefore change sign:

$$P^{r}_{12}P^{\tau}_{12}P^{\sigma}_{12}\Psi = -\Psi.$$

Or, since an exchange operator is equal to its inverse,

$$P^{r}_{12}\Psi = -P^{\sigma}_{12}P^{\tau}_{12}\Psi. \tag{7-11}$$

Thus as long as we limit consideration to the admissible wave functions, i.e., those that satisfy the generalized Pauli principle, we can write

$$P^{r}_{12} = -P^{\sigma}_{12}P^{\tau}_{12}. \tag{7-12}$$

By use of Eqs. (7-9) and (7-10), the Majorana exchange operator can therefore be expressed directly in terms of the spin and i-spin operators of the system:

$$P^{r}_{12} = -\tfrac{1}{4}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(1 + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2), \tag{7-13}$$

and this is the form in which an exchange operator is most likely to appear in a fundamental theory of nuclear forces. Note that Eq. (7-13) is not a general operator equation but refers only to that part of the operator acting within the class of totally antisymmetric functions.

7-2 The neutral scalar field. An attempt to construct a theory of the field responsible for nuclear forces must start from some preconceived notion of the form the theory is to take. Since our experience with force fields other than the nuclear field is limited to the laws of gravitation and electromagnetism, it is natural that the initial attempt at a nuclear theory should be modeled after them. Thus we try to construct equations as close as possible to those of electrodynamics which describe a field having the essential short-range character of the nuclear interaction. Only the

static interactions will be considered here, so it will be sufficient to work with an analogy to classical electrostatics.

It is well known that the electrostatic field may be described in terms of a potential, $\varphi(\mathbf{r})$, whose source is the charge density $\rho(\mathbf{r})$. The Poisson equation,

$$\nabla^2\varphi = -4\pi\rho(\mathbf{r}),$$

determines the connection between φ and ρ. The solution of this equation can be put in the familiar form

$$\varphi = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\, d^3r'.$$

The total electrostatic energy of the charge distribution is given by

$$W = \tfrac{1}{2}\int \rho(\mathbf{r})\varphi(\mathbf{r})\, d^3r \tag{7-14}$$

or

$$W = \frac{1}{2}\iint \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\, d^3r'\, d^3r.$$

To formulate the theory of interacting point charges, we consider first a charge distribution consisting of two non-overlapping clusters centered at points $\mathbf{r}_1$ and $\mathbf{r}_2$. If ρ_1 represents the charge density in the neighborhood of the point $\mathbf{r}_1$ and ρ_2 the charge density in the neighborhood of $\mathbf{r}_2$, then the potential due only to the first cluster is

$$\varphi_1 = \int \frac{\rho_1(\mathbf{r}')\, d^3r'}{|\mathbf{r}' - \mathbf{r}|},$$

and the potential due to the second cluster is given by a similar expression. The energy W of the system may be divided into three parts: the self-energy of the first cluster, the self-energy of the second cluster, and the interaction between the clusters. Only the third term, representing the interaction, is of interest to us. According to Eq. (7–14), it is given by

$$V = \tfrac{1}{2}\int \rho_2(\mathbf{r})\varphi_1(\mathbf{r})\, d^3r + \tfrac{1}{2}\int \rho_1(\mathbf{r})\varphi_2(\mathbf{r})\, d^3r,$$

and from the form of φ_i and ρ_i it is evident that

$$V = \int \rho_2(\mathbf{r})\varphi_1(\mathbf{r})\, d^3r.$$

Let us now consider the case in which the charge clusters are simply point charges e_1 and e_2 located at the points $\mathbf{r}_1$ and $\mathbf{r}_2$. Then the charge density has the form of a δ-function:

$$\rho_i(\mathbf{r}) = e_i\delta(\mathbf{r} - \mathbf{r}_i).$$

Consequently,

$$\varphi_i = \frac{e_i}{|\mathbf{r}_i - \mathbf{r}|},$$

and the interaction energy is

$$V = e_2 \int \delta(\mathbf{r} - \mathbf{r}_2)\varphi_1(\mathbf{r})\, d^3r,$$

which reduces immediately to the well-known coulomb potential

$$V = \frac{e_1 e_2}{|\mathbf{r}_1 - \mathbf{r}_2|}.$$

This very elementary discussion of the electrostatic potential can now be made the basis for an analogous theory of the short-range nuclear interaction. The requirement that the field have a definite range implies that a characteristic length must appear explicitly in the theory. The length, denoted here by $1/\mu$, is expected to be closely related to the range of nuclear forces. Presumably, the potential function of the nuclear field is determined by a differential equation similar to the Poisson equation but involving explicitly the quantity μ. If the source of the field is a "charge" density $\eta(\mathbf{r})$, the simplest imaginable equation satisfying these requirements is

$$\nabla^2\varphi - \mu^2\varphi = -4\pi\eta(\mathbf{r}).$$

The solution of the equation is given by

$$\varphi = \int \frac{e^{-\mu|\mathbf{r}' - \mathbf{r}|}}{|\mathbf{r}' - \mathbf{r}|}\eta(\mathbf{r}')\, d^3r',$$

as can easily be shown by inserting it directly into the differential equation and making use of the fact that

$$\nabla^2 \frac{1}{|\mathbf{r}' - \mathbf{r}|} = -4\pi\delta(\mathbf{r}' - \mathbf{r}).$$

Now the "nucleostatic" energy of the system can be defined by analogy with the electrostatic energy given by Eq. (7-14), namely,*

$$W = -\tfrac{1}{2}\int \eta(\mathbf{r})\varphi(\mathbf{r})\, d^3r.$$

For two isolated clusters of nuclear matter we can define an η_1 and an η_2 such that the potential due to a single cluster is simply

$$\varphi_i = \int \eta_i(\mathbf{r}') \frac{e^{-\mu|\mathbf{r}' - \mathbf{r}|}}{|\mathbf{r}' - \mathbf{r}|}\, d^3r'. \tag{7-15}$$

* The choice of sign here is governed by the fact that we are dealing with a scalar field rather than the vector field of electrodynamics. Compare G. Wentzel, *op. cit.*, p. 38 (German edition), p. 39 (English edition).

Again the energy divides into three parts, of which only the interaction between the clusters is of interest. That interaction becomes

$$V = -\int \eta_2(\mathbf{r})\varphi_1(\mathbf{r})\, d^3r. \tag{7 16}$$

If now the clusters at $\mathbf{r}_1$ and $\mathbf{r}_2$ are taken to be point nucleons, the densities η_1 and η_2 may be expressed as δ-functions. The two nucleons may be assumed to have the same nucleonic charge, in which case

$$\eta_i = g\delta(\mathbf{r} - \mathbf{r}_i),$$

with the result that

$$\varphi_i = g\frac{e^{-\mu|\mathbf{r}_1 - \mathbf{r}|}}{|\mathbf{r}_1 - \mathbf{r}|} \tag{7–17}$$

and

$$V = -g^2\frac{e^{-\mu|\mathbf{r}_1 - \mathbf{r}_2|}}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{7–18}$$

This has just the form of the Yukawa potential, Eq. (3-3). It is to be noted that in the simple theory just outlined neither exchange force nor tensor interaction appears in the potential. Those potentials can be obtained by introducing fields of a considerably more complicated structure, as we shall see in the next section.

The order of magnitude of the charge constant g and the reciprocal length μ may be obtained from the available information concerning the two-nucleon interaction. We note that g has the dimensions of electric charge, so that $g^2/\hbar c$ is a dimensionless constant which provides a measure of the nuclear force. Comparison of Eq. (7–18) with Eq. (3-3) leads to the relationships

$$\mu = 1/\alpha$$

and

$$\frac{g^2}{\hbar c} = K^2\alpha^2\frac{\hbar\mu}{Mc}. \tag{7–19}$$

An appropriate choice for the range of the Yukawa potential has been found to be

$$\alpha = 1.18 \times 10^{-13} \text{ cm},$$

and if our estimate of the strength of the interaction is based on the neutron-proton singlet potential, we have

$$K^2\alpha^2 = 1.68,$$

from which we find

$$g^2/\hbar c \approx \tfrac{1}{3}. \tag{7–20}$$

Comparison of this quantity with the corresponding fundamental constant of electrodynamics, $e^2/\hbar c = 1/137$, indicates that the nuclear charge is

rather large compared with the fundamental electric charge. Investigation of the electrodynamic properties of elementary particles is usually based on an expansion in powers of the small quantity $e^2/\hbar c$. Clearly, the analogous expansion of the nuclear interaction cannot be expected to converge nearly so well. This fact, which is a direct consequence of the rather strong coupling between nucleons and their field, is one reason why no reliable approximation to the quantitative properties of the meson field has been found.

The dynamical aspects of the field as well as the static can be based on analogy with the electromagnetic field. We know, for example, that the radiation field, when quantized, is to be described in terms of elementary particles, the photons. Their properties are implied by the equations for the radiation field in the absence of matter:

$$\Box\varphi = 0,$$

$$\Box\mathbf{A} = 0,$$

where $\mathbf{A}$ is the vector potential and $\Box$ is the D'Alembertian operator

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

The corresponding generalization of the equation for the nuclear field in the absence of matter would be

$$\Box\varphi - \mu^2\varphi = 0. \tag{7-21}$$

Whether or not other components of the field must be included, as is the case for the electromagnetic field (the components $\mathbf{A}$), is a matter that can only be settled on the basis of experience. The electromagnetic field is referred to as a *vector field*, since a four-vector $(\mathbf{A}, \varphi)$ is required for its description. The existence of the $\mathbf{A}$ part of the field is implied by the existence of magnetic fields which in turn are produced by moving charges. In order to determine directly the corresponding properties of the nuclear field, information concerning the velocity-dependent terms in the nuclear interaction is required. Since no such information is available, the field could turn out to be a scalar, vector, or tensor field of higher rank. There is also the possibility that the field has opposite parity, in which case we speak of pseudoscalar, pseudovector, etc., fields. For the sake of simplicity, we limit consideration here to scalar and pseudoscalar fields. As a matter of fact, direct measurements* on the pion indicate that it has spin zero and

* Panofsky, Aamodt, and Hadley, *Phys. Rev.* **81,** 565 (1951); Brueckner, Serber, and Watson, *Phys. Rev.* **81,** 575 (1951); Cartwright, Richman, Whitehead, and Wilcox, *Phys. Rev.* **81,** 652 (1951); Crawford, Crowe, and Stevenson, *Phys. Rev.* **82,** 97 (1951); R. Durbin, H. Loar, and J. Steinberger, *Phys. Rev.* **83,** 646 (1951); *Phys. Rev.* **84,** 581 (1951); D. L. Clark, A. Roberts, and R. Wilson, *Phys. Rev.* **83,** 649 (1951).

that its wave function in an S-state has odd parity, results which strongly suggest that the nuclear field is pseudoscalar. Since the scalar fields have only one component, only the one equation, Eq. (7–21), is required for their determination.

If φ is considered to be the wave function of a particle, Eq. (7–21) (the Klein-Gordon equation) is just the relativistic generalization of the Schroedinger equation for a free particle of mass $\mu_0 = \hbar\mu/c$. This can be seen by introducing a plane wave for φ:

$$\varphi = e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}.$$

Then the differential equation reduces to

$$\omega^2 = c^2(k^2 + \mu^2),$$

and, if we set $E = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$, $\mu_0 c = \hbar\mu$,

$$E^2 = c^2(p^2 + \mu_0^2 c^2),$$

which is just the relativistic energy-momentum relationship for a free particle of mass μ_0.

From the estimate of μ based on the range of nuclear forces, μ_0 turns out to be 326 electron masses. Thus the theory suggests the existence of a particle whose mass is intermediate between the electron and proton masses. This particle is presumably to be identified with the pion produced in energetic collisions between nucleons, although the mass of the pion is found to be 276 m.* The spin of the quantum of a field is intimately related to the transformation properties of the field. That is the reason for assigning to the field the transformation properties of a pseudoscalar on the basis of the observed spin and parity of the pion.

The qualitative interpretation of the interaction between nucleons can now be given in terms of the quanta of the field, the pions. Each nucleon acts as a source and sink for pions, i.e., it is capable of emitting and absorbing them. When two nucleons are brought into close proximity, a pion emitted by one is absorbed by the other, with the result that the energy of the system is lowered and the two nucleons are attracted to each other. In the quantized formulation of the theory this process of emission and absorption appears explicitly. The fact that exchange forces do not come out of the theory formulated here indicates that the pions involved are neutral; the exchange of charged pions would entail an exchange of charge and therefore exchange forces. The absence of a pion charge is simply the consequence of our assumption that the field function φ is real. In the next section it will be seen that with a complex function a charge may be introduced, and that the exchange of charged pions does indeed lead to exchange forces.

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* W. H. Barkas, F. M. Smith, and E. Gardner, *Phys. Rev.* **82,** 102 (1951).

7-3 Charged and pseudoscalar fields. The source density considered above was a simple space function, but the concept can be extended to include functions of operators such as $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, since the source always acts as an operator on the nucleon wave function. As a simple example, let us consider the field produced by a source density of the form

$$\eta_k(\mathbf{r}) = g\tau_-(k)\delta(\mathbf{r} - \mathbf{r}_k),$$

where $\boldsymbol{\tau}(k) = \boldsymbol{\tau}_k$ is the i-spin operator of the nucleon at the point $\mathbf{r}_k$. Now we have seen [Eq. (7-5)] that τ_- vanishes in a neutron state, while it converts a proton state into a neutron state. Therefore, in this case, a neutron does not act as a source of a field, while a proton does. The fact that the proton is converted to a neutron in producing the field implies that the field carries unit positive charge; the associated pion is positive. The hermitian conjugate source density

$$\eta_k^*(\mathbf{r}) = g\tau_+(k)\delta(\mathbf{r} - \mathbf{r}_k)$$

acts on the neutron to provide a source of negative charge, so the conjugate field has negative pions associated with it.

The field functions are to be obtained by introducing the source density operator into Eq. (7-15), whence

$$\varphi_k(\mathbf{r}) = g\tau_-(k)Y(|\mathbf{r} - \mathbf{r}_k|)$$

and

$$\varphi_k^*(\mathbf{r}) = g\tau_+(k)Y(|\mathbf{r} - \mathbf{r}_k|),$$

where

$$Y(x) = \frac{e^{-\mu x}}{x}.$$

In order to have a hermitian interaction energy, Eq. (7-16) is replaced by

$$V = -\int [\eta_2^*(\mathbf{r})\varphi_1(\mathbf{r}) + \eta_1^*(\mathbf{r})\varphi_2(\mathbf{r})]\, d^3r,$$

so the interaction takes the form

$$V = -g^2[\tau_+(1)\tau_-(2) + \tau_-(1)\tau_+(2)]Y(r_{12}).$$

Conservation of charge is guaranteed by the form of this operator. If it acts on a nucleon function for which particle 1 is a neutron and particle 2 a proton, $\tau_-(1)\tau_+(2)$ vanishes, while $\tau_+(1)\tau_-(2)$ shifts the charge, via the pion field, from the proton to the neutron. We see here a detailed mechanism for shifting the charge back and forth between neutron and proton; the kind of process that we have associated with exchange potentials.

When the operators $\tau_\pm$ are expressed in terms of τ_1 and τ_2 the interaction takes the form

$$V = -\frac{g^2}{2}[\tau_1(1)\tau_1(2) + \tau_2(1)\tau_2(2)]Y(r_{12}), \tag{7-22}$$

from which it follows that the interaction is invariant under rotations about the 3-axis in charge space. Invariance under such rotations is equivalent to the requirement of charge conservation, as can be seen by analogy with the rotations in ordinary three-dimensional space. There, invariance of the interaction under rotations about the z-axis implies that the z-component of the total angular momentum is conserved. The quantity in charge space corresponding to the z-component of the angular momentum is T_3, the 3-component of total i-spin. And the total charge of the system is just eT_3, so the conservation of T_3 implied by invariance under rotations about the 3-axis is equivalent to charge conservation.

The operator $[\tau_+(1)\tau_-(2) + \tau_-(1)\tau_+(2)]$ vanishes when applied to a two-neutron or a two-proton state, therefore the interaction (7–22) is charge dependent. Charge independence can be attained by introducing, in addition to the complex field φ, a real field φ^0, having the source density

$$\eta_i^0 = g_0\tau_3(i)\delta(\mathbf{r} - \mathbf{r}_i).$$

Note that this source is different from the source of the neutral field considered in Section 7–2 in that it has opposite signs for neutron and proton. The corresponding contribution to the interaction is easily found to be $-g_0^2\tau_3(1)\tau_3(2)V(r_{12})$. The choice

$$g_0^2 = \tfrac{1}{2}g^2$$

then leads to the charge-independent form of the total interaction:

$$V = -\tfrac{1}{2}g^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)V(r_{12}). \tag{7–23}$$

From the fact that $(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2) = 1$ in the i-triplet state, it follows that the interaction is attractive for a pair of neutrons, a pair of protons, or a neutron and proton in any of their mutually accessible states. However, in the i-singlet state, which concerns only the n-p system, $(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2) = -3$, and there the interaction is three times stronger, and repulsive. According to the generalized Pauli principle, this would then be the situation in any neutron-proton $^3S_1$ state, in contradiction to the evidence concerning the ground state of the deuteron.

An interesting feature of the potential (7–23) is its invariance under all rotations in charge space, a considerably more restrictive condition than the invariance for rotations about the 3-axis. It implies, among other things, that the n-n, p-p, and n-p interactions are all the same. The following interpretation of this property, which will be referred to as *charge invariance*, is suggested: The functions φ and φ^* correspond to positive and negative pions, respectively, while φ^0 corresponds to a neutral pion. Thus φ, φ^*, and φ^0 may be considered as the $t_3 = 1$, -1, and 0 wave functions of a pion whose total i-spin is $t = 1$. The wave function then forms a vector $\boldsymbol{\varphi}$ in charge space in the same sense that a triplet function may be treated as a vector in configuration space. The interaction of the field

with the nucleon has been taken to be of the invariant form $g(\boldsymbol{\tau}_1 \cdot \boldsymbol{\varphi})$, which necessarily results in charge invariance of the consequent nucleon-nucleon interaction. We might remark that the less restricted condition n-n = p-p, that would hold even if the choice $g_0^2 = \frac{1}{2}g^2$ were dropped, corresponds to invariance under a change of sign of the 3-component of each i-spin. This transformation can be accomplished by means of a reflection in a plane containing the 3-axis, since the i-spin vectors (like the angular momentum) are pseudovectors.

The above consideration of the charged field illustrates that the theory leads to charge exchange interactions if the source density involves the i-spin operators in a reasonable way. The source density may also depend upon the ordinary spin, as it must if spin exchange or tensor interactions are to occur. For the sake of simplicity, let us consider a dependence on only the ordinary spin, in which case the field is neutral. If a scalar field is to be produced, the source density must necessarily be invariant under rotations, as one can see from Eq. (7-15). The source therefore involves the spin vectors in an invariant combination such as

$$\eta_i(\mathbf{r}) = g(\boldsymbol{\sigma}_i \cdot \operatorname{grad}_i)\delta(\mathbf{r} - \mathbf{r}_i).$$

Although this source density is a rotational invariant, it changes sign on inversion, since $\boldsymbol{\sigma}$ is a pseudovector while the gradient is an ordinary vector. Therefore η_i is a pseudoscalar, and the field it generates is a pseudoscalar field. When η_i is inserted into Eq. (7-15), an integration by parts leads to a field function of the form

$$\varphi_i = g(\boldsymbol{\sigma}_i \cdot \operatorname{grad}_i) Y(|\mathbf{r} - \mathbf{r}_i|). \tag{7-24}$$

This may be inserted into Eq. (7-16) to obtain the interaction

$$V = -g^2(\boldsymbol{\sigma}_2 \cdot \operatorname{grad}_2)(\boldsymbol{\sigma}_1 \cdot \operatorname{grad}_1) Y(r_{12}). \tag{7-25}$$

The result of the indicated differentiations is

$$V = g^2 \left\{ \frac{\mu^2}{3} (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + \left(\frac{1}{r_{12}^2} + \frac{\mu}{r_{12}} + \frac{\mu^2}{3} \right) S_{12} \right\} Y(r_{12}), \tag{7-26}$$

where S_{12} is the familiar tensor operator

$$S_{12} = \frac{3(\boldsymbol{\sigma}_1 \cdot \mathbf{r}_{12})(\boldsymbol{\sigma}_2 \cdot \mathbf{r}_{12})}{r_{12}^2} - (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2).$$

Therefore the pseudoscalar theory leads directly to a tensor interaction. It so happens that both the central and the tensor triplet interactions are repulsive in this particular form of the theory, which is, of course, contrary to the information we have concerning the potential.

A simple modification of the pseudoscalar theory is obtained if, instead of neutral mesons, charged mesons are introduced. The charge invariant

theory would lead to an interaction differing from (7–26) merely by a factor $(\tau_1 \cdot \tau_2)/2$, so the interaction would be attractive in both the ${}^3S_1$ and ${}^1S_0$ states of the two-nucleon system, and it would be stronger in the triplet state because of the tensor term. Although this form of the theory seems to satisfy, at least qualitatively, all the conditions imposed by experimental results, there is another difficulty with the potential that cannot be taken care of very easily. It has to do with the fact that the potential diverges as $1/r^3$ at the origin, and is attractive if the tensor interaction is attractive. The Schroedinger equation containing such a potential admits of no bound solutions. Therefore the theory certainly cannot be applied as it stands to the deuteron problem. Possibly the difficulty has to do with our use of the static approximation, since the $1/r^3$ potential implies that the deuteron kinetic energy is very large for small n-p separations. Therefore velocity-dependent corrections to the interaction become very important in just the troublesome region, and they may be such as to alleviate the difficulty. What would be needed then is a consistent dynamical theory of the nuclear field, rather than a static approximation, in order to describe even so simple a system as the deuteron.

The theory of nuclear forces outlined here leads to a rather definite picture of the structure of a single nucleon. An isolated nucleon cannot be in a stationary state because of its interaction with the meson field. The stationary states of the system have the property that there is a finite probability for finding one or more pions in the neighborhood of the nucleon. The function $\varphi(\mathbf{r})$ can be considered as the wave function of one such pion. In the neutral theory, only neutral pions occur, while in the charge invariant theory, both neutral and charged pions occur. A single charged pion emitted by a proton must be positive (leaving behind a neutron), while that emitted by a neutron is negative. In the scalar theories, the functions φ are evidently [see Eq. (7–17)] S-functions, so the pions do not contribute to the magnetic moment of the system. On the other hand, Eq. (7–24) shows that in the pseudoscalar theory the pions are in P-states. Therefore in a charged theory, a contribution to the magnetic moment of the nucleon would arise from its associated pion field, and the contribution can be large since the "pion magneton" is some seven times the nuclear magneton. Although quantitative calculations (based on an expansion in powers of $g^2/\hbar c$) have not led to correct numerical values, the view is widely held that this is the qualitatively correct explanation of the anomalous nucleon moments.

It seems unlikely that attempts to account for the interactions between nucleons will meet with much success until a quantitative account can be given of the properties of a single nucleon.

Part III

COMPLEX NUCLEI

CHAPTER 8

STRUCTURE OF COMPLEX NUCLEI

On the face of it, the complex nuclei would seem to be so prolific a source of data as to furnish answers to all unsettled questions concerning nuclear interactions. Unfortunately, that is not the case, because the determination of interactions from observations on the dynamical behavior of a complex system requires a detailed knowledge of the solutions of the many-body problem, solutions which have been given only in very crude approximation. Nevertheless, some important aspects of the data on complex nuclei are susceptible to rather exact theoretical analysis, namely, those which depend primarily on broad physical principles. Because their treatment can be given in general terms, these aspects of the many-body problem will be the main concern of Part III.

8–1 Introductory remarks. The application of general principles to the data on complex nuclei provides a *description* of properties of the nuclei which does not in itself constitute an understanding of those properties, at least, not an understanding at the deep level of nucleon-nucleon interactions. But the accumulation of descriptive information can lead to, or suggest strongly, special characteristics of the interactions, as we have already seen in the interpretation of the saturation phenomenon through exchange forces. The first step in the interpretation of accumulated data is to search for systematic features of the nuclear properties. Much progress along these lines has been made in recent years. However, even when a thorough description of nuclear systematics is given, the direct interpretation in terms of interactions is not easily obtained because of the difficulties associated with the solution of the many-body problem. In order to make progress in this direction, we are forced to use some *nuclear model*, that is, some guess as to the structure of the nuclear wave function. Then the implications of various assumptions concerning the interactions can be calculated within the framework of the model and compared with experiment. However, conclusions arrived at by this path are subject to all the doubts associated with the given model.

This chapter is devoted to a description of those models which seem to be useful for the discussion of fundamental questions, and to a discussion of some of their implications concerning nuclear energy states and nuclear interactions. A possible approach to the problem of constructing a nuclear model is to start from the Schroedinger equation containing what we know about the forces, and to attempt to find an approximate method of solution which is applicable to the many-body problem. In principle, this requires that we justify the approximations used and have some estimate of the

errors made. It is the ideal approach from the point of view of pure theoretical physics. At the same time, it is an approach which has not been carried through successfully except for rather special problems.

An alternative procedure depends on the data to lead to some suggestion concerning nuclear structure. For example, the observation that alpha-particles leave a nucleus in alpha-decay suggests that the nucleus is composed of alpha-particles as a basic unit. One might call this the extreme phenomenological approach. The most widely accepted procedure is actually a combination of the theoretical and phenomenological points of view. One tries to build a model while keeping in mind both the theoretical equations of motion of the system and the available facts about the behavior of nuclei. The relative emphasis of the two points of view depends very much on the complexity of the nucleus as well as the particular property under consideration. In the treatment of the simpler systems such as very light nuclei there is some hope of carrying out the theoretical program, whereas in heavy nuclei the trend is toward the empirical point of view.

The earliest of the nuclear models is the above-mentioned alpha-particle model. The detailed development of this model has followed an analogy with molecular structure. Nuclei composed of an integral number of alpha-particles are assumed to consist of a framework of alphas whose geometry is based on considerations of closest packing. Other nuclei are constructed by adding the extra individual nucleons in orbits about the framework, just as the valence electrons in a molecule occupy orbits which extend throughout the structure. Energy states of the nucleus are associated with rotational, vibrational, and nucleonic excitation.

One weakness of the model is that the amplitude of zero-point vibration of the alphas must be comparable to their spacing, since the particles are closely packed in the nucleus and the alpha-alpha binding is weak. Hence there is a marked overlap of the internal wave functions of the alpha-particles, with the consequence that they lose their identity; a constant interchange of nucleons between the overlapping particles must occur. Apart from this difficulty, the model is flexible enough to account for some properties of nuclei, such as their spins and magnetic moments. But it provides little insight into the fundamental problem of nuclear interactions. Most of the binding energy of the nucleus is implicitly contained in the structural unit, the alpha-particle, so the nuclear forces play only an incidental role in the dynamics of the model.

The same limitation applies even more strongly to another of the early nuclear models, Bohr's liquid drop model of heavy nuclei. In this model, use is made of the fact that for a sufficiently complex system the density of states may be so great as to allow an approximate classical treatment of the motion of each nucleon. Then an appropriate analog is the macroscopic system consisting of a very cohesive medium in which there is a

rapid interchange of energy between particles. This is just the situation in liquids, and one can conceive of the nucleus as a liquid sphere with nucleons playing the role of molecules. The energy states of the system correspond to surface and volume waves in the medium; hence they are characterized by the surface tension, compressibility, and density of the nuclear matter. These quantities are related to the nuclear forces, but in a most indirect fashion, so, of all the models, the Bohr model provides the least direct knowledge of the interactions between nucleons. But it is useful in that it provides a means for describing the trends of many nuclear properties in terms of just a few parameters such as the surface tension and compressibility. Examples of such properties are the dependence of binding energy on atomic mass and atomic number, level densities, and the rate of interchange of energy between particles. Only the average behavior of these properties can be obtained from the model, since it offers no means of accounting for the detailed differences between neighboring nuclei.

Although the alpha model and liquid drop model serve many useful purposes, we shall direct our attention to other models, those which are more or less directly concerned with the nuclear force problem, since that problem is our main concern here. But a detailed discussion of any model requires a procedure for classifying nuclear states, so we first turn to this, the basic problem of nuclear spectroscopy.

8-2 Nuclear multiplets. Every nuclear model is subject to restrictions having the nature of conservation laws. The state of the system must be a characteristic state of certain operators such as the total angular momentum, under which condition we say that the angular momentum, J, is a "good" quantum number. Other good quantum numbers are the z-component, M, of the total angular momentum, the three components of the linear momentum of the center of mass, and the parity of the state. All of these conservation laws are the result of general invariance properties of the interactions: invariance under rotations of the coordinate axes (angular momentum), invariance for translation of the system as a whole (momentum of CM), and invariance for inversion through the center of mass (parity). Another symmetry property generally ascribed to interactions is their invariance under reversal of the sense of time. Some consequences of the time-reversal property are treated in Appendix 3.

There is often reason to assume that the interaction is subject to more restrictive invariance conditions, in which case additional good quantum numbers are available to classify the states of the nuclear system. Since there usually exist corrections to such an interaction, corrections which do not satisfy the stronger invariance conditions, the associated quantum numbers are good only in the same approximation that these, presumably small, correction terms are neglected. Introduction of the corrections

leads to a mixing of states belonging to different values of these quantum numbers, but as long as the amount of admixture is small, the classification of the state by the approximate quantum number is a useful concept.

This procedure for classifying states is quite familiar in atomic and molecular spectroscopy, and the methods used there can be carried over to the nuclear problem if the concepts are extended to include the i-spin variable. But before we go into the implications of i-spin, some of the more familiar facts concerning the classification of levels by spin and orbital angular momentum will be reviewed.

To demonstrate the connection between symmetry of the interaction and associated quantum numbers, let us assume that the interactions contain no spin-orbit coupling. Then the Hamiltonian of the system is invariant under rotations of the coordinate vectors alone and under rotations of the spin vectors alone. Consequently, the total orbital angular momentum and the total spin angular momentum are separately conserved, and L, M_L and S, M_S are good quantum numbers. Then not only is J a good quantum number, but all values of J obtainable from given L and S by the vector rule, $J = L + S, L + S - 1, \cdots, |L - S|$, belong to the same energy state. Hence there is a notable accidental degeneracy. Such increased degeneracy is usually associated with an increase in the number of good quantum numbers, and the splitting of the degeneracy provides a measure of the extent to which the associated symmetry condition fails; small splitting means that the interactions not endowed with the required symmetry are weak, and that the associated quantum number is quite good.

The condition that there is no spin-orbit interaction does not exclude spin-spin coupling, but it is worth noting that the only two-body interactions satisfying the condition are one that is independent of the spin and one that is proportional to $(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)$. The space dependence is also strongly limited, particularly if just static interactions are involved. Then the additional requirement of translational invariance implies that the potential can only be a function of $|\mathbf{r}_i - \mathbf{r}_j| = r_{ij}$.

A spin-orbit coupling of the Russell-Saunders type will split the degeneracy in J. The quantum numbers M_L and M_S are no longer good, but L and S can still be used to classify the states. Such a coupling would be proportional to $(\mathbf{L} \cdot \mathbf{S})$, where $\mathbf{L}$ is the total orbital angular momentum operator and $\mathbf{S}$ the total spin angular momentum operator. Note that this coupling is not a static interaction; it is linear in the momenta of the nucleons.

On the basis of our experience with the deuteron, serious deviations from Russell-Saunders coupling are to be expected. The tensor interaction will, in general, not permit L or S to be treated as good quantum numbers. Nevertheless, it may be possible to treat the tensor potential as a weak coupling in many cases, so the classification of states in terms of L and S

at least provides a point of departure for the description of a group of states.

Considerations very similar to the above, but in general simpler, are applicable to the i-spin of a nuclear system. We define the i-spin vector $\mathbf{T}$ as

$$\mathbf{T} = \tfrac{1}{2} \sum_j \boldsymbol{\tau}_j, \tag{8-1}$$

where $\boldsymbol{\tau}_j = [\tau_1(j), \tau_2(j), \tau_3(j)]$ is the i-spin operator of the jth nucleon. Now there is no *a priori* reason to expect that the vector $\mathbf{T}$ will be conserved, but it is necessary that the component T_3 be conserved in the stationary state of a nucleus. This follows from the fact that

$$T_3 = \tfrac{1}{2} \sum_j \tau_3(j),$$

so its characteristic values are

$$t = \tfrac{1}{2}(Z - N), \tag{8-2}$$

where Z and N are the (fixed) number of protons and neutrons in the nucleus. Conservation of T_3 is merely a formal expression of the conservation of charge. To have a better understanding of the implications of the conservation law for T_3, we again resort to the analogy between $\mathbf{T}$ and the angular momentum $\mathbf{J}$. The z-component, J_z, of $\mathbf{J}$ is a good quantum number for any system with circular symmetry, i.e., as long as the interactions are invariant for rotations about the z-axis. For example, the quantum number m of an atom can be specified if the atom is in a uniform magnetic or electric field parallel to the z-axis, although J itself is not a good quantum number. The corresponding condition must always be satisfied in *charge* space; the interactions must be invariant under rotations about the 3-axis in order that charge be conserved.

This symmetry principle leads to strong restrictions on the possible forms of the interactions insofar as their i-spin dependence is concerned. Let us limit attention to two-body interactions of the form

$$\sum_{i,j} V(\mathbf{r}_{ij}, \boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j, \boldsymbol{\tau}_i, \boldsymbol{\tau}_j) = \sum_{i,j} V_{ij}.$$

Then V_{ij} must be proportional to a linear combination of the six operators with circular symmetry:

$$\begin{gathered} 1 \\ \boldsymbol{\tau}_1(i)\boldsymbol{\tau}_1(j) + \boldsymbol{\tau}_2(i)\boldsymbol{\tau}_2(j), \\ \boldsymbol{\tau}_3(i)\boldsymbol{\tau}_3(j), \\ \tfrac{1}{2}[\boldsymbol{\tau}_3(i) \pm \boldsymbol{\tau}_3(j)], \\ [\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j]_3. \end{gathered} \tag{8-3}$$

Further conditions on the interactions are introduced by such requirements as n-n = n-p or the more restrictive conditions of charge inde-

pendence. To see how these arise, the values of the above operators when applied to wave functions describing a neutron-proton pair, a proton-proton pair, or a neutron-neutron pair are listed in Table 8–1. Certain linear

TABLE 8–1

Two-body interactions compatible with the requirements of charge conservation

Designation	Interaction	n-p	p-p	n-n
A	1	1	1	1
B	$\frac{1}{2}[1 + \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j]$	P^τ	P^τ	P^τ
C	$\frac{1}{2}[\tau_3(i) + \tau_3(j)]$	0	1	−1
D	$\frac{1}{2}[\tau_3(i) - \tau_3(j)]$	±1	0	0
E	$\tau_3(i)\tau_3(j)$	−1	1	1
F	$\frac{1}{4}[\tau_3(i) - \tau_3(j)][1 + \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j]$	$\pm P^\tau$	0	0

combinations of the interactions (8–3) have been taken for purposes of simplification. For example, it is clearly convenient to use the linear combination B since, according to Eq. (7–10), this expression is simply the i-spin exchange operator P^τ. Furthermore, it is possible to show from the commutation relations for the $\boldsymbol{\tau}$ operators (which are identical with those for $\boldsymbol{\sigma}$ operators) that

$$[\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j]_3 = \frac{i}{2}[\tau_3(i) - \tau_3(j)](1 + \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j), \tag{8–4}$$

so the operator labeled F in Table 8–1 is simply $-(i/2)[\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j]_3$. It is of some interest to note that the generalized Pauli principle implies that $P^\tau = -P^\sigma P^r$, which is just the Heisenberg space-spin exchange operator.

We found in Section 2–1 that the condition n-n = p-p should be very nearly satisfied in the approximation that coulomb forces are negligible. This clearly eliminates C as a possible interaction. Actually, a stronger statement can be made, namely, that the binding energy of the nucleus with N neutrons and Z protons differs from that of its *mirror nucleus*, with Z neutrons and N protons, by an amount comparable to the difference in coulomb energy. There also seems to be good evidence that the differences in the low excited states of mirror nuclei can be accounted for by just coulomb effects. Therefore the nuclear interactions have, to a very good approximation, the property that they are invariant when neutrons are changed to protons and protons to neutrons, that is, invariance under the transformation

$$\tau_3(i) \rightarrow -\tau_3(i)$$

for all i. Not only interaction C, but also interactions D and F are excluded by this condition. To see more directly why the D and F terms are eliminated, we note that they couple a neutron only to the protons in

a nucleus and a proton only to neutrons, and that the sign of the coupling is opposite in the two cases. Thus a single odd neutron would be influenced oppositely from a single odd proton (as in the mirror nuclei Li^7, Be^7), and the mirror property would be violated.

The more stringent condition of charge independence, n-n = p-p = n-p, eliminates interaction E as well. Then we are left with just A and B, both of them having the property of charge invariance, i.e., invariance under rotations in charge space. This suggests that in place of the charge independence criterion, we adopt its equivalent, charge invariance, even if many-body interactions are involved. Then the operator $\mathbf{T}^2$ can be assigned a good quantum number, $T(T+1)$. Actually, we know that T is not exactly a good quantum number because the coulomb forces and the neutron-proton mass difference will yield different energies for the different values of T_3. But as long as these effects are small, they can be treated as weak perturbations acting to split the i-multiplet into its $(2T+1)$ different states. Thus these effects are analogous to the influence on a state of given J of a uniform magnetic field, that is to say, to the Zeeman effect. Each value of T_3 for given T (and mass number A) specifies a given isobar, and these isobars have common energy levels in the approximation of charge invariance. The above-mentioned splitting is a shift in the position of the level of one isobar with respect to another, and only one isobar in the multiplet is expected to be stable. The other members of the i-multiplet will decay by beta-emission or K-capture into the stable isobar.

If now we return to the classification of nuclear states in terms of orbital and spin angular momentum and add to this the specification of the i-spin multiplet, we see that the nuclear Russell-Saunders scheme requires the specification of the three quantum numbers L, S, and T. Therefore an appropriate spectroscopic notation would be, let us say, $^{1,3}S$ for a state which is an i-singlet, spin triplet, with $L = 0$. The generalization of this notation is obvious. It must be kept in mind that the generalized Pauli principle leads to a restriction of the possible states of a nuclear system in the same way that the Pauli principle restricts atomic energy terms. For example, we have already seen that in the two-nucleon system the only S-states that can be obtained are the $^{1,3}S$ and $^{3,1}S$, the first being just a deuteron and the second being either the n-p, the p-p, or the n-n system in a singlet state. Other allowed states of the two-nucleon system are $^{1,1}P$, $^{3,3}P$, $^{1,3}D$, $^{3,1}D$, etc. Although the selection of the allowed levels is simple for the two-nucleon system, the problem is a great deal more complex than the corresponding atomic problem for the system of more than two particles. The required general principles for identifying the allowed states have been given in fundamental papers on nuclear spectroscopy by Wigner* and by Hund.†

* *Phys. Rev.* **51**, 106 (1937).

† *Zs. f. Phys.* **105**, 202 (1937).

8–3 Structure of H^3 and He^3. The nuclear systems next to the deuteron in order of simplicity are the three-body nuclei H^3 and He^3. Their study forms the logical sequel to investigation of the deuteron in a program devoted to the nuclear force problem. In principle, the supplementary information concerning nuclear interactions provided by the data on these nuclei should make it possible either to gain further knowledge of the parameters in the potential or to draw the conclusion that the assumption of a static two-body potential is erroneous. Actually, the analytical difficulties associated with the three-body problem are so great that the interpretation of the data is too uncertain for definite conclusions of that kind to be reached.

Most theoretical attention has been given to the ground state, in particular to the binding energy and magnetic moment. We shall consider in detail those features of the ground state which are fixed by general principles, since they serve so well to illustrate principles applicable to many problems. Discussion of the binding energy will be limited to a summary of methods and results. Detailed treatment of the magnetic moment is deferred to Section 9–3.

It seems reasonable to treat the two nuclei, H^3 and He^3, as a single entity. This presumes that n-n and p-p forces are identical and that the influence of the coulomb repulsion between the protons in He^3 is negligible. It further assumes that the neutron-proton mass difference has little influence on the wave function. Then the mirror nuclei, H^3 and He^3, can be described by a single wave function, $\psi(1,2,3)$, where, let us say, the labels 1 and 2 refer to the two neutrons in H^3 and to the two protons in He^3. From the point of view of the i-spin formalism, this fact is brought out quite clearly, since H^3 and He^3 are simply two states of the same system, those with $t = -\frac{1}{2}$ and $t = +\frac{1}{2}$, respectively.

The available experimental information concerning the ground state of the three-body system includes the total angular momentum J, the binding energy E_B, and the magnetic moments $\mu(H^3)$ and $\mu(He^3)$. They are*

$$J = \tfrac{1}{2},$$

$$E_B(H^3) = 8.48 \text{ Mev},$$

$$E_B(He^3) = 7.72 \text{ Mev},$$

$$\mu(H^3)/\mu_p = 1.0664,$$

$$\mu(He^3)/\mu_n = 1.110,$$

where μ_p and μ_n are the magnetic moments of the proton and neutron. The value $J = \frac{1}{2}$ implies that the system can have no static moment of

* Taken from National Bureau of Standards Circular 499 (1950), except the binding energies, which are given by Li, Whaling, Fowler, and C. C. Lauritsen, *Phys. Rev.* **83**, 512 (1951).

higher order than the magnetic dipole moment; in particular, the quadrupole moments must vanish. The small difference between the binding energies of the two nuclei may be ascribed to the coulomb energy of He^3.

One more quantum number, the total i-spin, T, of the ground state of the system can be specified on the basis of somewhat less direct data. There are only two possible values of T, $T = \frac{1}{2}$ or $T = \frac{3}{2}$, since the system consists of three particles, each one of i-spin $\frac{1}{2}$. Now if T is a good quantum number, that is, if the principle of charge invariance can be applied to the system, then the ground state must have

$$T = \tfrac{1}{2}.$$

The alternative possibility, $T = \frac{3}{2}$, would imply that the states with $t = \pm\frac{3}{2}$ have about the same energy as the ground state of the three-body system. This would mean that Li^3 ($t = \frac{3}{2}$) and n^3 ($t = -\frac{3}{2}$) are sufficiently stable to be observed as beta-active nuclei. Since they are not observed in nature, we draw the conclusion that $T = \frac{1}{2}$. It must be kept in mind that deviations from charge invariance will introduce a small admixture of the $T = \frac{3}{2}$ state in the ground state. But for most practical purposes it seems reasonable to ignore this contribution.

Since the spin system can only form doublets ($S = \frac{1}{2}$) or quartets ($S = \frac{3}{2}$), the total orbital angular momenta L which can be combined with S to form $J = \frac{1}{2}$ are $L = 0,1,2$. The ground state may then be any one of, or a linear combination of, the following terms: ${}^{2,2}S_{\frac{1}{2}}$, ${}^{2,2}P_{\frac{1}{2}}$, ${}^{2,4}P_{\frac{1}{2}}$, ${}^{2,4}D_{\frac{1}{2}}$. It seems most reasonable to start with the assumption that the ground state is an S-state, since this is what would be expected on the basis of ordinary central potentials. In general, we shall hold to the view that the ${}^{2,2}S_{\frac{1}{2}}$ term predominates in the wave function.

The form of the accessible three-body wave functions, insofar as their spin and angular dependences are concerned, can be found by following a procedure very similar to that used in the discussion of the deuteron functions, Section 3–5. We start by writing down a very simple ${}^2S_{\frac{1}{2}}$ function. Multiplication of this function by any scalar operator then provides another function with $J = \frac{1}{2}$. The desired value of the orbital angular momentum is obtained by choosing an operator with the transformation properties of an S-, P-, or D-function under space rotations. The space variables with which we work are

$$\mathbf{r} = \mathbf{r}_{12}$$

and

$$\boldsymbol{\rho} = (\mathbf{r}_{13} + \mathbf{r}_{23}),$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and $\mathbf{r}_i$, $i = 1, 2, 3$, is the space coordinate of the ith nucleon. Particles 1 and 2 are given a special role just for convenience; introduction of the i-spin notation will place every nucleon on the same footing.

Let us consider first the spin dependence of the ^2S functions. There are two such functions to be taken into account. Let $\chi_0(1,2)$ be the singlet function in nucleons 1 and 2. Then if $\chi^m_{\frac{1}{2}}(3)$, $m = \pm\frac{1}{2}$, is the spin function of nucleon 3, one of the doublet functions is simply

$$\varphi^m = \chi_0(1,2)\chi^m_{\frac{1}{2}}(3). \tag{8-5}$$

If we introduce the operator

$$\boldsymbol{\sigma}_{12} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \tag{8-6}$$

and note that according to Eq. (4-12)

$$\boldsymbol{\sigma}_1\varphi^m = -\boldsymbol{\sigma}_2\varphi^m,$$

we find

$$\boldsymbol{\sigma}_1\varphi^m = \tfrac{1}{2}\boldsymbol{\sigma}_{12}\varphi^m. \tag{8-7}$$

Therefore the other doublet function given by

$$\bar{\varphi}^m = (\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_3)\varphi^m \tag{8-8}$$

completes the set of ^2S spin functions. Doublet functions linearly independent of φ^m and $\bar{\varphi}^m$ can be constructed only for states of orbital angular momentum different from zero. These additional possibilities will be illustrated below.

Since use will be made of the generalized Pauli principle, it is of some interest to consider the effect of permuting the particle labels in the spin functions. The singlet function is antisymmetric, so the effect of the spin exchange operator P^σ_{12} is

$$\begin{aligned} P^\sigma_{12}\varphi^m &= -\varphi^m, \\ P^\sigma_{12}\bar{\varphi}^m &= \bar{\varphi}^m. \end{aligned} \tag{8-9}$$

The only other permutation that need be considered is P^σ_{13}, since the other interchange can be obtained as a product of the two: $P_{23} = P_{12}P_{13}P_{12}$. By making use of the relationship [see Eq. (7-9)]

$$P^\sigma_{13} = \tfrac{1}{2}(1 + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3) \tag{8-10}$$

and Eq. (8-7), we find

$$P^\sigma_{13}\varphi^m = \tfrac{1}{2}\varphi^m + \tfrac{1}{4}\bar{\varphi}^m. \tag{8-11a}$$

Furthermore, application of P^σ_{13} to Eq. (8-11a) results in

$$P^\sigma_{13}\bar{\varphi}^m = 3\varphi^m - \tfrac{1}{2}\bar{\varphi}^m. \tag{8-11b}$$

The symmetry properties of φ^m and $\bar{\varphi}^m$ exhibited here may be used to define an *intermediate* symmetry class of functions of three variables. It can be established that functions of three variables may be assigned to one of three symmetry classes, the class of symmetric functions, the class of antisymmetric functions, and the set of two functions having the above

intermediate property. The doublet functions for three particles of spin $\frac{1}{2}$ always belong to the intermediate symmetry class. On the other hand, the quartet functions are symmetric. The absence of an antisymmetric spin function is simply a result of the fact that the spin variable takes on only two values; hence antisymmetry in more than two variables would imply that the function vanishes.

The $T = \frac{1}{2}$, or i-doublet functions, also belong to the intermediate symmetry class for permutations P^r_{ij}. If $t = \pm\frac{1}{2}$ denotes the value of T_3, one i-doublet function is

$$\eta^t = \xi_0(1,2)\xi^t_{\frac{1}{2}}(3), \tag{8-12}$$

where $\xi_0(1,2)$ is the i-singlet function of nucleons 1 and 2. The other is

$$\bar{\eta}^t = (\boldsymbol{\tau}_{12}\cdot\boldsymbol{\tau}_3)\eta^t. \tag{8-13}$$

Attention is now to be given to the "radial" function of the S-state. It must be a rotationally invariant function of $\mathbf{r}$ and $\boldsymbol{\rho}$, hence it depends only on r^2, ρ^2, and $(\mathbf{r}\cdot\boldsymbol{\rho})$. The radial functions can also be classified according to the three symmetry classes. From the definitions of $\mathbf{r}$ and $\boldsymbol{\rho}$, it follows that

$$\begin{aligned} P^r_{13}\mathbf{r} &= \tfrac{1}{2}(\mathbf{r} - \boldsymbol{\rho}), \\ P^r_{13}\boldsymbol{\rho} &= -\tfrac{1}{2}(3\mathbf{r} + \boldsymbol{\rho}). \end{aligned} \tag{8-14}$$

Thus S-functions of the intermediate class are

$$\begin{aligned} R &= (\mathbf{r}\cdot\boldsymbol{\rho}), \\ \bar{R} &= (\rho^2 - 3r^2). \end{aligned} \tag{8-15}$$

There is no antisymmetric S-function, but there are, of course, many symmetric functions.

Application of the Pauli principle requires that some attention be given to the symmetry properties of products of spin, i-spin, and space functions. All such products fall into the same three symmetry classes under simultaneous permutations of the space, spin, and i-spin variables. Consider two sets of functions, F_s, F_a, F, $\bar{F}$ and G_s, G_a, G, $\bar{G}$ which belong to the indicated symmetry classes. Then the products F_sG_s and F_aG_a are clearly symmetric. Another symmetric product function is

$$H_s = 12FG + \bar{F}\bar{G}, \tag{8-16a}$$

as can be seen by direct application of P_{12} and P_{13} to the factors. In addition to the antisymmetric products F_sG_a and F_aG_s, we have

$$H_a = F\bar{G} - \bar{F}G. \tag{8-16b}$$

Finally, the intermediate symmetry class of products is given by

$$H = F\bar{G} + \bar{F}G,$$
$$\bar{H} = 12FG - \bar{F}\bar{G}. \tag{8–16c}$$

Let us turn now to the construction of ${}^{2,2}S_{1/2}$ functions satisfying the Pauli principle. First consider a symmetric radial function f_1. Then the antisymmetric product of doublet spin and i-spin functions is required. According to Eq. (8–16b) the desired function is

$$\psi_1^{m,t} = [\varphi^m\bar{\eta}^t - \bar{\varphi}^m\eta^t]f_1. \tag{8–17}$$

The only other ${}^{2,2}S$ function admitted by the Pauli principle is that obtained by using the radial functions R and $\bar{R}$, Eq. (8–15), as factors of a symmetric function f_2. First we form the spin-i-spin functions of intermediate symmetry:

$$\Phi = \varphi^m\bar{\eta}^t + \bar{\varphi}^m\eta^t,$$
$$\bar{\Phi} = 12\varphi^m\eta^t - \bar{\varphi}^m\bar{\eta}^t,$$

and then these are combined with R and $\bar{R}$ to form the antisymmetric function

$$\psi_2^{m,t} = [(\mathbf{r}\cdot\boldsymbol{\rho})(12\varphi^m\eta^t - \bar{\varphi}^m\bar{\eta}^t) - (\rho^2 - 3r^2)(\varphi^m\bar{\eta}^t + \bar{\varphi}^m\eta^t)]f_2. \tag{8–18}$$

A linear combination of the functions $\psi_1^{m,t}$ and $\psi_2^{m,t}$ is the only possible form for the ${}^{2,2}S$ state of the three-body system.

Some admixture of states other than ${}^{2,2}S_{1/2}$ is to be expected as a consequence of the presence of the tensor interaction. Any or all of the even parity ${}^2P$, ${}^4P$, and ${}^4D$ states may contribute, and it is therefore of some interest to describe the structure of these functions as we have the ${}^2S$ function. The condition of even parity implies that the space dependence of all the functions will be given by products of polynomials of even degree in $\mathbf{r}$ and $\boldsymbol{\rho}$ with symmetric radial S-functions, f_i. Thus the P-functions must be proportional to $\mathbf{r} \times \boldsymbol{\rho}$. The expression

$${}^2P = i\boldsymbol{\sigma}_3\cdot\mathbf{r} \times \boldsymbol{\rho}\varphi^m \tag{8–19a}$$

is clearly a ${}^2P_{1/2}$ function which is antisymmetric for interchange of the spins of nucleons 1 and 2. Since the doublets belong to the intermediate symmetry class, another doublet function can be obtained by applying the operator P^{σ}_{13}, given by Eq. (8–10). This leads directly* to the function

$${}^2\bar{P} = (i\boldsymbol{\sigma}_{12} + \boldsymbol{\sigma}_{12} \times \boldsymbol{\sigma}_3)\cdot\mathbf{r} \times \boldsymbol{\rho}\varphi^m. \tag{8–19b}$$

Now $\mathbf{r} \times \boldsymbol{\rho}$ is an antisymmetric function under interchange of the space

* Operations of the type $\frac{1}{2}(1 + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3)\cdot{}^2P$ can be performed rapidly by using the equation $(\boldsymbol{\sigma}\cdot\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B}) + i(\boldsymbol{\sigma}\cdot\mathbf{A} \times \mathbf{B})$ which follows from the commutation relations for $\boldsymbol{\sigma}$. In this connection it is usually convenient to replace $\boldsymbol{\sigma}_{12}\varphi^m$ by $2\boldsymbol{\sigma}_1\varphi^m$, in accordance with Eq. (8–7).

variables, so a totally antisymmetric function may be constructed by applying Eq. (8-16a) to the spin functions,

$$\psi_3^{m,t} = ([12i\sigma_3\eta^t + (i\sigma_{12} + \sigma_{12} \times \sigma_3)\bar{\eta}^t]\cdot \mathbf{r} \times \boldsymbol{\rho})\varphi^m f_3. \tag{8-20}$$

Another ^2P term can be formed by again using the functions R and $\bar{R}$. We need only combine R, $\bar{R}$ with η^t, $\bar{\eta}^t$ to form the pair of functions H and $\bar{H}$ in accordance with Eq. (8-16c), and then combine these with ^2P, $^2\bar{P}$ in accordance with Eq. (8-16a):

$$\begin{aligned}\psi_4^{m,t} = (\{&12[(\mathbf{r}\cdot\boldsymbol{\rho})\bar{\eta}^t + (\rho^2 - 3r^2)\eta^t]i\sigma_3 \\ &+ [12(\mathbf{r}\cdot\boldsymbol{\rho})\eta^t - (\rho^2 - 3r^2)\bar{\eta}^t](i\sigma_{12} + \sigma_{12} \times \sigma_3)\}\cdot \mathbf{r} \times \boldsymbol{\rho})\varphi^m f_4.\end{aligned} \tag{8-21}$$

The $^{2,2}P_{1/2}$ function of even parity is necessarily a linear combination of ψ_3 and ψ_4.

The $^4P_{1/2}$ function is symmetric in the spin variables, and therefore can only be a linear combination of $(\sigma_{12}\cdot\mathbf{r} \times \boldsymbol{\rho})\varphi^m$ and $(\sigma_{12} \times \sigma_3\cdot\mathbf{r} \times \boldsymbol{\rho})\varphi^m$. The appropriate coefficients can be determined by the condition of symmetry under P_{13}^σ, whence

$$^4P = (i\sigma_{12} - \tfrac{1}{2}\sigma_{12} \times \sigma_3)\cdot \mathbf{r} \times \boldsymbol{\rho}\varphi^m.$$

Since this function is totally antisymmetric, it must be combined with the symmetric product of η^t, $\bar{\eta}^t$ with R, $\bar{R}$. Hence the $^{2,4}P$ function is

$$\psi_5^{m,t} = [12(\mathbf{r}\cdot\boldsymbol{\rho})\eta^t + (\rho^2 - 3r^2)\bar{\eta}^t]([i\sigma_{12} - \tfrac{1}{2}\sigma_{12} \times \sigma_3]\cdot \mathbf{r} \times \boldsymbol{\rho})\varphi^m f_5. \tag{8-22}$$

Formation of the D-functions requires the use of products such as $(\mathbf{A}\cdot\mathbf{r})(\mathbf{B}\cdot\mathbf{r})$, $(\mathbf{A}\cdot\boldsymbol{\rho})(\mathbf{B}\cdot\boldsymbol{\rho})$ and $(\mathbf{A}\cdot\mathbf{r})(\mathbf{B}\cdot\boldsymbol{\rho})$, where $\mathbf{A}$ and $\mathbf{B}$ are spin operators. That the combination

$$D_s = (\mathbf{A}\cdot\boldsymbol{\rho})(\mathbf{B}\cdot\boldsymbol{\rho}) + 3(\mathbf{A}\cdot\mathbf{r})(\mathbf{B}\cdot\mathbf{r}) \tag{8-23}$$

is symmetric under space permutations follows immediately from Eq. (8-14). This equation also shows that the functions of intermediate space symmetry are

$$\begin{aligned} D &= \tfrac{1}{2}[(\mathbf{A}\cdot\mathbf{r})(\mathbf{B}\cdot\boldsymbol{\rho}) + (\mathbf{A}\cdot\boldsymbol{\rho})(\mathbf{B}\cdot\mathbf{r})], \\ \bar{D} &= (\mathbf{A}\cdot\boldsymbol{\rho})(\mathbf{B}\cdot\boldsymbol{\rho}) - 3(\mathbf{A}\cdot\mathbf{r})(\mathbf{B}\cdot\mathbf{r}). \end{aligned} \tag{8-24}$$

The corresponding antisymmetric function is the P-function

$$(\mathbf{A} \times \mathbf{B}\cdot\mathbf{r} \times \boldsymbol{\rho}).$$

By reasoning as in Section 3-5, we find that there are three $^4D_{1/2}$ functions; the space-symmetric function

$$^4D_s = [(\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\boldsymbol{\rho}) + 3(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\mathbf{r}) - \tfrac{1}{3}(\rho^2 + 3r^2)(\sigma_{12}\cdot\sigma_3)]\varphi^m$$

and the space-intermediate functions

$$^4D = \tfrac{1}{2}[(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\boldsymbol{\rho}) + (\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\mathbf{r}) - \tfrac{2}{3}(\mathbf{r}\cdot\boldsymbol{\rho})(\sigma_{12}\cdot\sigma_3)]\varphi^m,$$

$$^4\overline{D} = [(\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\boldsymbol{\rho}) - 3(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\mathbf{r}) - \tfrac{1}{3}(\rho^2 - 3r^2)(\sigma_{12}\cdot\sigma_3)]\varphi^m.$$

Since they are quartets, these functions are necessarily spin-symmetric; hence ^4D_s occurs only in the combination

$$\psi_6^{m,t} = [(\mathbf{r}\cdot\boldsymbol{\rho})\bar{\eta}^t - (\rho^2 - 3r^2)\eta^t]$$

$$\times [(\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\boldsymbol{\rho}) + 3(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\mathbf{r}) - \tfrac{1}{3}(\rho^2 + 3r^2)(\sigma_{12}\cdot\sigma_3)]\varphi^m f_6. \quad (8\text{–}25)$$

Two functions satisfying the Pauli principle can be obtained from ^4D and $^4\overline{D}$, namely,

$$\psi_7^{m,t} = \{\tfrac{1}{2}[(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\boldsymbol{\rho}) + (\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\mathbf{r}) - \tfrac{2}{3}(\mathbf{r}\cdot\boldsymbol{\rho})(\sigma_{12}\cdot\sigma_3)]\bar{\eta}^t$$

$$- [(\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\boldsymbol{\rho}) - 3(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\mathbf{r}) - \tfrac{1}{3}(\rho^2 - 3r^2)(\sigma_{12}\cdot\sigma_3)]\eta^t\}\varphi^m f_7, \quad (8\text{–}26)$$

and

$$\psi_8^{m,t} = \{\tfrac{1}{2}[(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\boldsymbol{\rho}) + (\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\mathbf{r}) - \tfrac{2}{3}(\mathbf{r}\cdot\boldsymbol{\rho})(\sigma_{12}\cdot\sigma_3)]$$

$$\times [12(\mathbf{r}\cdot\boldsymbol{\rho})\eta^t - (\rho^2 - 3r^2)\bar{\eta}^t]$$

$$- [(\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\boldsymbol{\rho}) - 3(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\mathbf{r}) - \tfrac{1}{3}(\rho^2 - 3r^2)(\sigma_{12}\cdot\sigma_3)]$$

$$\times [(\mathbf{r}\cdot\boldsymbol{\rho})\bar{\eta}^t + (\rho^2 - 3r^2)\eta^t]\}\varphi^m f_8. \quad (8\text{–}27)$$

This completes the description of the terms that can be included in the ground state of the three-body system if the specification $T = \frac{1}{2}$ is a good approximation. A similar analysis yields the $T = \frac{3}{2}$ functions that can be admixed with the $T = \frac{1}{2}$ states by coulomb effects and the like, but we shall persist in the view that such terms can be ignored.

It is instructive to consider a given one of the isobaric pair of the three-body nuclei, say the triton, H^3. Then by choosing particles 1 and 2 to be neutrons and nucleon 3 to be the proton, the function η^t is made to vanish, so the triton functions are of the form*

$$^2S: \quad \psi_1^m = \varphi^m f_1,$$

$$\psi_2^m = [(\mathbf{r}\cdot\boldsymbol{\rho})(\sigma_{12}\cdot\sigma_3) + (\rho^2 - 3r^2)]\varphi^m f_2;$$

$$^2P: \quad \psi_3^m = ([i\sigma_{12} + \sigma_{12}\times\sigma_3]\cdot\mathbf{r}\times\boldsymbol{\rho})\varphi^m f_3,$$

$$\psi_4^m = ([12(\mathbf{r}\cdot\boldsymbol{\rho})i\sigma_3 - (\rho^2 - 3r^2)(i\sigma_{12} + \sigma_{12}\times\sigma_3)]\cdot\mathbf{r}\times\boldsymbol{\rho})\varphi^m f_4;$$

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* Triton functions of substantially this form were first constructed by Gerjuoy and Schwinger, *Phys. Rev.* **61,** 138 (1942). However, they did not specify T, hence their D-functions were not orthogonal. Furthermore, they describe four, instead of three, ^4D functions, but these are not linearly independent. Some of the states have been classified according to symmetry properties by M. Verde, *Helv. Phys. Acta,* **22,** 340 (1949), *ibid.* **23,** 453 (1950). See also R. E. Clapp, *Phys. Rev.* **76,** 873 (1949).

$$
\begin{aligned}
{}^4P\text{:}\quad \psi_5^m &= (\rho^2 - 3r^2)([i\sigma_{12} - \tfrac{1}{2}\sigma_{12} \times \sigma_3]\cdot \mathbf{r} \times \boldsymbol{\rho})\varphi^m f_5;\\
{}^4D\text{:}\quad \psi_6^m &= (\mathbf{r}\cdot\boldsymbol{\rho})[(\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\boldsymbol{\rho}) + 3(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\mathbf{r})\\
&\quad - \tfrac{1}{3}(\rho^2 + 3r^2)(\sigma_{12}\cdot\sigma_3)]\varphi^m f_6,\\
\psi_7^m &= [(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\boldsymbol{\rho}) + (\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\mathbf{r}) - \tfrac{2}{3}(\mathbf{r}\cdot\boldsymbol{\rho})(\sigma_{12}\cdot\sigma_3)]\varphi^m f_7,\\
\psi_8^m &= \{\tfrac{1}{2}(\rho^2 - 3r^2)[(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\boldsymbol{\rho}) + (\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\mathbf{r})]\\
&\quad + (\mathbf{r}\cdot\boldsymbol{\rho})[(\sigma_{12}\cdot\boldsymbol{\rho})(\sigma_3\cdot\boldsymbol{\rho}) - 3(\sigma_{12}\cdot\mathbf{r})(\sigma_3\cdot\mathbf{r})]\\
&\quad - \tfrac{2}{3}(\rho^2 - 3r^2)(\mathbf{r}\cdot\boldsymbol{\rho})(\sigma_{12}\cdot\sigma_3)\}\varphi^m f_8.
\end{aligned}
$$

The ground state function of the triton is a linear combination,

$$\psi^m = \sum_i a_i \psi_i^m, \tag{8-28}$$

of these eight functions. The coefficients a_i can be shown to be real by means of a time-reversal argument, as is demonstrated in Appendix 3. Reasonable guesses can be made concerning certain features of the wave functions. For example, it seems reasonable to assume that the predominant term in the ground state function will be symmetric for the interchange of the space coordinates of any pair of nucleons. Two rather compelling arguments are available to substantiate this view. First of all, the Majorana potential will always favor such a state. Furthermore, the function with the smallest number of nodes is expected to have the lowest kinetic energy, and a high degree of symmetry usually implies that the number of nodes is a minimum.

The only space symmetric state is ψ_1^m, so we are led to the view that not only is the ground state predominantly ^2S, but of the two ^2S states it is predominantly ψ_1^m. A somewhat more direct physical interpretation of the difference between the functions ψ_1^m and ψ_2^m can be made in terms of the states of the individual nucleons. If each nucleon were assigned a given orbital angular momentum, it would be expected to have $l = 0$ in the lowest state of the system; that would correspond to an s^3 configuration. The next state of the same parity would belong to an sp^2 configuration, and the dp^2 configuration might follow that. Clearly, the s^3 term is of the form ψ_1^m, while ψ_2^m includes an sp^2 term, since it contains the products of two p-functions. Classification of all ψ_i in this manner leads to the following assignments for the highest occurring configurations:

$$
\begin{aligned}
&\psi_1^m \sim s^3, && \psi_5^m \sim dp^2 \text{ and } sd^2,\\
&\psi_2^m \sim sp^2, && \psi_6^m \sim dp^2 \text{ and } sd^2,\\
&\psi_3^m \sim sp^2, && \psi_7^m \sim sp^2 \text{ and } s^2d,\\
&\psi_4^m \sim dp^2 \text{ and } sd^2, && \psi_8^m \sim dp^2 \text{ and } sd^2.
\end{aligned}
\tag{8-29}
$$

These assignments involve the assumption that the functions f_i belong to the s^3 configuration. They could also involve many higher configurations, presumably at some sacrifice in energy.

The tensor interaction couples the ^4D states directly to the ^2S state, so some admixture of D-functions is to be expected. On the basis of the assignments (8–29), a mixture of ψ_7^m with ψ_1^m would seem probable. P-functions arise only through coupling with the D-function, so their amplitudes are expected to be small compared with the amplitude of the D-function. The assignments (8–29) indicate that ψ_3^m is the choice candidate for this role. The S-function ψ_2^m might also be brought in by coupling with the D-state.

The determination of the functions f_i and their relative amplitudes is to be made on the basis of a differential equation which is obtained by inserting (8–28) into the Schroedinger equation for the three-body system. Since the f_i are functions of three variables, for example ρ^2, r^2, and $(\mathbf{r}\cdot\boldsymbol{\rho})$, this procedure will result in a set of coupled partial differential equations in three variables. The only definite information we have concerning the functions is that they are symmetric for interchange of any pair of particles. Rather than attempt to solve this complicated differential equation, the usual procedure* has been to make use of the variational method for determining the binding energy. One of the forms of two-body potentials which seems adequate to account for all of the data on the two-nucleon system is introduced as the potential in the variational energy integral. A trial function f_i involving just a few parameters is then introduced into this integral and the variation is carried out with respect to the parameters. Under all conditions the binding energy so obtained should be smaller than the observed binding.

Let us consider the case of central forces only. Then it is expected that the function ψ_1 will be adequate for the description of the ground state. If the two-body potential is taken to have a Yukawa shape, a very successful form of function is the one parameter Hulthén-like function†

$$f_1 = e^{-(\beta/2)(r_{12}+r_{13}+r_{23})}. \tag{8–30}$$

This function matches into the potential so well that when the range 1.18×10^{-13} cm for the central Yukawa potential is used and the other parameters in the potential are chosen to fit the ground state and singlet scattering data of the deuteron, a binding energy nearly twice as large as the observed binding is obtained for the three-body system.‡ This clearly

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* See, however, S. I. Rubinow, *Phys. Rev.* **86,** 388 (1952).

† Huang, Fröhlich, and Sneddon, *Proc. Roy. Soc.* **A191,** 61 (1947).

‡ Avery and Adams, *Phys. Rev.* **75,** 1106 (1949).

means that the two-body central Yukawa potential cannot be a correct form for the interaction. The difficulty occurs only because the tensor interaction has still to be taken into account.

Tensor forces are expected to mix the function ψ_7 with the function ψ_1. On this assumption, a variational calculation was carried out by Gerjuoy and Schwinger using a square well shape for both the central and tensor interactions.* The parameters in the well were fixed on the basis of calculations on the deuteron under the assumption that the central and tensor potentials both had a range of 2.8×10^{-13} cm. Maximum binding was obtained with a wave function containing† about 4 percent ^4D state of the type ψ_7.

When a potential of the Yukawa shape is used in this problem, results of a considerably more definite character are obtained. Our information on the two-body problem indicates that the range of the central potential should be about 1.18×10^{-13} cm. If it is assumed that the tensor range is the same, we see from Appendix 2 that the data on the deuteron would indicate a strength of the tensor relative to the central interaction of $\gamma = 10$. Thus the tensor interaction would predominate. Avery and Adams‡ found that this had the consequence that the triton would not be bound because the tensor interaction is not nearly as effective in binding the system as is the central interaction. In order to obtain binding for the three-body system, they found that it was necessary to take a tensor interaction of longer range, thereby reducing its relative strength as determined from the deuteron data. With a ratio of tensor to central range of 1.33 and both functions f_1 and f_7 chosen to be of the form Eq. (8–30), with β replaced by β_S and β_D, respectively, it was found possible to adjust the values of the two parameters β_S and β_D so as to obtain 60 percent of the binding. In this case the probability of the D-state was found to be 2.6 percent, and the values of the parameters β to be

$$\begin{aligned} \beta_S^{-1} &= 1.04 \times 10^{-13} \text{ cm}, \\ \beta_D^{-1} &= 0.54 \times 10^{-13} \text{ cm}. \end{aligned} \tag{8–31}$$

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* Gerjuoy and Schwinger, *Phys. Rev.* **61,** 138 (1942). See also Feshbach and Rarita, *Phys. Rev.* **75,** 1384 (1949); Tsi-Ming Hu and Kung-Ngow Hsu, *Phys. Rev.* **78,** 633 (1950).

† Since the tensor interaction is not really weak, it is important to be sure that there is not an appreciable admixture of P-functions. A variational calculation using a Gaussian potential led to the conclusion that the functions ψ_3 and ψ_5 always tend to raise the energy of the system; therefore the neglect of the P-functions appears to be justified. M. Goeppert-Mayer and R. G. Sachs, *Phys. Rev.* **73,** 185 (1948).

‡ *Loc. cit.*

The quality of these results has been confirmed by the more detailed and accurate calculations of Pease and Feshbach.* They have used a more elaborate trial function including ψ_1, ψ_6, ψ_7, and ψ_8. A number of choices of the parameters in a potential of the form Eq. (6–57) were considered, and these choices include a selection very close** to Eq. (6–58). For that selection, the calculated (variational) binding energy is about 89 percent of the experimental value. Hence we may conclude that the Yukawa potential with parameters given by Eq. (6–58) can account for the binding energy of the three-body system as well as the low-energy data on the two-body system. The result suggests that many-body interactions are weak compared with the two-body interactions,† a view which is supported by the fact that the observed binding energy of the alpha-particle also seems to be concordant‡ with the value calculated for the same two-body potential.

A test of our assumption that the predominant term in the wave function of the ground state is ψ_1 is provided by the magnetic moments. In the state ψ_1 the moment of H^3 should clearly be equal to that of the proton, and the moment of He^3 should be equal to that of the neutron. Our assumption is almost confirmed by the observation that the magnetic moments differ from the above values only by a few percent. However, it can be shown that an admixture of the ^4D state will reduce the magnitude of the moment so as to widen the discrepancy between the experimental and theoretical values. It turns out that this discrepancy is significant, in that it constitutes good evidence for the existence of meson currents in nuclei. Further discussion of this matter is deferred to Section 9–3, since it is closely related to the general problem of the interaction of electromagnetic radiation with nuclei.

8–4 The Wigner coupling scheme. The wave functions of any complex nucleus could, in principle, be described as a general linear combination of states of given symmetry, just as has been done for H^3 and He^3. However, the manifold of states is so great that the attainment of complete generality would require more than a reasonable effort. Therefore it is

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* *Phys. Rev.* **88,** 945 (1952).

** The exchange dependence was not included.

† This is not to be taken as a firm conclusion, since the result is sensitive to the shape of the potential. Thomas [*Phys. Rev.* **47,** 903 (1935)] has shown that a very short-range potential is capable of giving arbitrarily large binding for the triton even if it is fitted to the deuteron ground state. The divergence of the Yukawa potential at the origin therefore should be expected to cause the Yukawa potential to be very effective in binding the three-body system as compared with the two-body system.

‡ J. Irving, *Phys. Rev.* **87,** 519 (1952).

desirable, just as for complex atoms, to introduce some simplifying assumption, known as a "coupling scheme," concerning the interactions, an assumption designed to reduce the number of terms in the expansion of the wave function. The coupling scheme implies that the nuclear state is characterized by certain quantum numbers in addition to the total angular momentum, parity, and energy. For example, in Russell-Saunders coupling, L and S are good quantum numbers. Even when departures from the coupling scheme are taken into account, an expansion of the nuclear wave function in terms of states characterized by the new quantum numbers will converge rapidly if the assumed approximation is valid.

In a fundamental paper* on nuclear spectroscopy, Wigner has presented a classification of nuclear states which is based on a coupling scheme closely related to, but more restricted than, the Russell-Saunders scheme. His assumption is that the predominant interactions are charge-independent central potentials of either the ordinary or Majorana type. This limitation on the interaction results in a degeneracy between the usual Russell-Saunders multiplets of different S and different T, a degeneracy which is subsequently removed by introducing the (small) spin-spin coupling. The entire group of levels consisting of the almost degenerate multiplets is referred to as a supermultiplet.

This classification of levels is important for many reasons. It provides a means for systematizing the data on groups of levels. It suggests important selection rules for radiative transitions, beta transitions, and nuclear reactions. It also provides a semiempirical method for determining the mass differences between neighboring nuclei of intermediate mass and, in addition, a method for predicting trends in the behavior of the low-lying excited states of some of these nuclei. The correlation of the data with these various theoretical predictions clearly provides a test of Wigner's basic assumption, that the size of the splitting of the supermultiplet is not too large.

Wigner's zero order Hamiltonian contains no explicit spin or i-spin dependence; nevertheless the energy values do depend to some extent on these quantities because of the Pauli principle. The effectiveness of the space potential depends on the space symmetry of the wave function, and functions of different space symmetry are associated with functions of different spin-i-spin symmetry. Hence there exists a separation in energy of spin-i-spin states of differing symmetry. Majorana forces, which will be assumed to predominate in the zero order Hamiltonian, give a particular emphasis to this effect, the more symmetric the space function, the more attractive the potential.

Because the Hamiltonian is assumed to be independent of spin (both varieties), there are four states of a given nucleon to be treated on an equal

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* *Phys. Rev.* **51**, 106 (1937). See also F. Hund, *loc. cit.*

footing. These are the states for which each of the τ_3 and σ_z operators takes on its two possible values. A nuclear wave function can be classified by the number of nucleons in the four possible states. Thus we let μ_1, μ_2, μ_3, and μ_4 be the number of nucleons with (σ_z,τ_3) equal to (1,1), (1,−1), (−1,1), (−1,−1), respectively. Since the number of nucleons in the nucleus is A, the μ_i satisfy the condition

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = A.$$

Now each of the degenerate states in the supermultiplet will be described by a different set of μ_i values. For example, the state $(\mu_1,\mu_2,\mu_3,\mu_4) = (A,0,0,0)$ is degenerate with $(0,A,0,0)$, $(0,0,A,0)$, and $(0,0,0,A)$. These four states are clearly spin-symmetric; they are degenerate with all other spin-symmetric states, and the entire group forms one supermultiplet. States of different spin-i-spin symmetry will have a different energy, hence they will define another supermultiplet.

Without attempting a detailed construction of the states of different symmetry, for which reference should be made to Wigner's paper, we wish to provide a system for naming the various supermultiplets, each one of which has a given symmetry. This is accomplished by simply specifying the *largest* set of values of $(\mu_1,\mu_2,\mu_3,\mu_4)$ that occurs in the group. A set $(\mu_1',\mu_2',\mu_3',\mu_4')$ is defined to be larger than $(\mu_1,\mu_2,\mu_3,\mu_4)$ if the following conditions are satisfied: either $\mu_1' > \mu_1$ or $\mu_1' = \mu_1$ and $\mu_2' > \mu_2$, or $\mu_1' = \mu_1$, $\mu_2' = \mu_2$, $\mu_3' > \mu_3$. If one is given all sets $(\mu_1,\mu_2,\mu_3,\mu_4)$ belonging to a given supermultiplet, then the largest set $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ can be selected. The important point is that the set λ_i completely characterizes the supermultiplet. For example, the symmetric functions belong to the supermultiplet $\lambda_1 = A$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, as we have seen above. It is important to note that each of the μ_i runs through the same set of values within a given supermultiplet.

The quantum numbers M_S and t associated with the z-component S_z of the total spin and the 3-component of $\mathbf{T}$ may be expressed directly in terms of the μ_i by means of the relationships

$$M_S = \tfrac{1}{2}(\mu_1 + \mu_2 - \mu_3 - \mu_4), \tag{8–32}$$

$$t = \tfrac{1}{2}(\mu_1 - \mu_2 + \mu_3 - \mu_4). \tag{8–33}$$

The latter quantity is just the isotopic number $\frac{1}{2}(Z - N)$. The μ_i serve to define one other quantum number which we denote by y:

$$y = \tfrac{1}{2}(\mu_1 - \mu_2 - \mu_3 + \mu_4). \tag{8–34}$$

From the definition of the μ_i, y is found to be just the quantum number associated with the difference $S_{zp} - S_{zn}$ of proton and neutron total spin values. The fact that such a quantity can be specified is a direct consequence of our assumption that the Hamiltonian is independent of the i-spin and spin operators.

The μ_i values for a given state are determined completely by the four quantum numbers A, M_S, t, and y by the above equations, and conversely. A corresponding set of quantum numbers may be used to replace those specifying the supermultiplet, i.e., the λ_i. We define

$$\begin{aligned} Q &= \tfrac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4), \\ Q' &= \tfrac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4), \\ Q'' &= \tfrac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4), \end{aligned} \tag{8-35}$$

and we have, of course,

$$A = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4. \tag{8-36}$$

The quantity Q is the largest value of M_S, of t, or of y that occurs in the supermultiplet. Furthermore if, for example, $t = Q$ (i.e., $\mu_1 = \lambda_1$, $\mu_3 = \lambda_2$, $\mu_2 = \lambda_3$, $\mu_4 = \lambda_4$), then Q' is the largest value of M_S or y occurring in this part of the supermultiplet. The set (Q,Q',Q'') plays the same role as the total spin quantum number S in an ordinary multiplet. There, the value of S is the largest value of M_S attained within the multiplet. For given A, the supermultiplet is characterized by (Q,Q',Q''), just as the ordinary multiplet is characterized by S.

We have seen that the symmetric spin state has $\lambda_1 = A$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$ or $Q = Q' = Q'' = \frac{1}{2}A$. But this state must have a totally antisymmetric space function, so it would not be expected to be stable if the Majorana potential predominates. Everything else being equal, the state of least spin-symmetry would be the most stable. This is the state in which the λ_i are, as nearly as possible, all equal. The nucleons are then distributed uniformly over all possible i-spin and spin states as required for maximum antisymmetry of the spin function. We conclude that, in particular, the ground state of a nucleus belongs to the supermultiplet whose λ_i are as nearly as possible equal.

Consider, for example, the "$4n$ nuclei," namely, those whose mass number A is an integral multiple of 4, $A = 4n$. The ground states of these nuclei should belong to the supermultiplet $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = A/4$, according to the above argument. Hence $(Q,Q',Q'') = (0,0,0)$, and the nuclei have spin quantum number zero and zero i-spin. The latter condition implies that the number of neutrons equals the number of protons. Thus our conclusion is that the stable $4n$ nuclei are the alpha-particle nuclei, and that these have $S = 0$, a conclusion which seems to be correct for A not too large.* Heavier nuclei violate the condition of charge in-

* The alpha-particle nuclei are the only stable $4n$ nuclei up to and including S^{32}. However, the assignment $S = 0$ might be in doubt, since only the total angular momentum of the nucleus is observed.

variance which has been imposed on the Hamiltonian, because the coulomb interaction becomes important.

The ground states of the $4n + 1$ nuclei, i.e., those for which $A = 4n + 1$, have $\lambda_1 - 1 = \lambda_2 = \lambda_3 = \lambda_4 = (A - 1)/4$. Hence the supermultiplet is $(Q,Q',Q'') = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$, and therefore both S and T are $\frac{1}{2}$. Furthermore, if $M_S = \frac{1}{2}$ and $t = \frac{1}{2}$, we must have $y = \frac{1}{2}$, in which case $S_{zn} = 0$; the neutron spins are paired, as are all but one of the proton spins. On the other hand, for $A = 4n - 1$, the designation is $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 + 1 = (A + 1)/4$ or $(Q,Q',Q'') = (\frac{1}{2},\frac{1}{2},-\frac{1}{2})$, whence for $M_S = \frac{1}{2}$, $t = \frac{1}{2}$, we have $y = -\frac{1}{2}$, and the neutron contributes the unpaired spin.

When $A = 4n + 2$, the ground state has $\lambda_1 = \lambda_2 = \lambda_3 + 1 = \lambda_4 + 1 = (A + 2)/4$, so $(Q,Q',Q'') = (1,0,0)$. Here, states with $S = 1$ and $S = 0$ are degenerate, as are states with $T = 1$ and $T = 0$. When $S = 1$, then $T = 0$, and conversely. The odd-odd nuclei, H^2, Li^6, B^{10}, N^{14} belong to this group. Any sort of spin-orbit or spin-spin coupling will split the degeneracy between $S = 1$ and $S = 0$, and we know that such a coupling occurs in the two-body system, with the consequence that $S = 1$ lies lower than $S = 0$. The ground state then has $S = 1$, $T = 0$, as in the deuteron, but the $S = 0$, $T = 1$ multiplet should not be far above, according to our assumptions.

The lowest supermultiplet generally contains a set of states belonging to a group of isobars. The ground state of the stable member of the group is determined by the splitting of the supermultiplet caused by spin-spin, spin-orbit, and coulomb coupling. A convenient delineation of the effect on the supermultiplet of these corrections to the Hamiltonian is given in Figs. 8–1(a) and 8–1(b). The former figure illustrates, on an arbitrary energy scale, the influence of spin-spin coupling; the supermultiplet is simply split into its constituent multiplets of given S and T. Each level is labeled by the appropriate S-value, and a scale of t-values is provided so

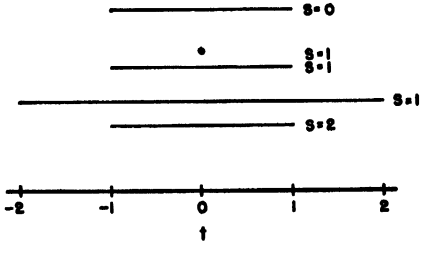

FIG. 8–1(a). Schematic diagram of the splitting of the (2,1,1) supermultiplet. The quantity t is a characteristic value of the operator T_3.

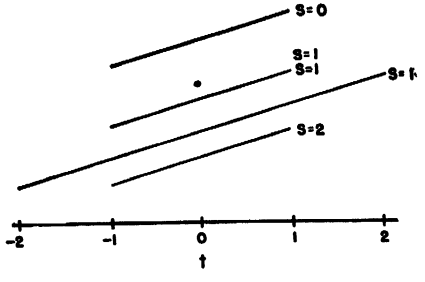

FIG. 8 1(b). Schematic diagram of the influence of the coulomb interaction on the (2,1,1) supermultiplet. The energy increases with t because the number of protons increases.

that the T of a given level is determined by the length of the line in the diagram. This form of the plot is particularly useful for illustrating the influence of the coulomb forces. These favor the most negative value of t, so the effect is simply to tilt the lines as indicated in Fig. 8–1(b). It can be seen here how the coulomb energy can cause a reordering of levels of different S.

An unrealistic degeneracy is still implied by Fig. 8–1(b), since no account has been taken of the spin-orbit coupling. This involves the total orbital angular momentum L of the ground state, which can only be specified on the basis of a more detailed nuclear model than we wish to invoke at this point. Even when the value of L is given, the nature of the spin-orbit coupling comes into question. The importance of this coupling is that it selects from the various possibilities $J = |L - S|, \cdots, L + S$ for the total angular momentum the value of J for the ground state, a quantity which has been measured for many stable nuclei. However, these matters may just as well be left until we consider a more detailed nuclear model, one that prescribes a value of L for the ground state.

Although a theory capable of determining nuclear J-values and the relative binding energies of neighboring isobars requires knowledge of the deviations from Wigner coupling, the supermultiplet approximation still sheds light on the relative behavior of neighboring stable nuclei. We have seen that there is an essential periodicity in the supermultiplet assigned to the ground state if it is considered as a function of A. The period is, of course, 4; as we pass from one $4n$ nucleus to another, (Q,Q',Q'') takes on the values (0,0,0), $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, (1,0,0), $(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$, (0,0,0), and so on. The periodicity is just a consequence of the fact that there are exactly four spin states. Because of the Pauli principle, this should be reflected in the behavior of calculated binding energies, a result that agrees nicely with the observed binding of nuclei having $A < 36$. The binding energy per nucleon is tabulated for the stable light nuclei in Table 8–2. The period of 4 is quite evident either as maxima or as changes in the rate of increase of binding at the $4n$ nuclei. This observation could lend credence to the alpha-particle model, but we see here that any attractive, spin-independent potential may be quite capable of accounting for the periodicity without reference to a detailed model.

A demonstration of the expected behavior of the theoretical binding energy curve has been provided by Wigner in his very general treatment of the binding of the fairly complex nuclei,* i.e., nuclei of mass greater than 16. Because of the complexity, he assumes that the expectation values of the potential and kinetic energies will, on the whole, be smooth functions of A. Then the characteristic differences in the energies of neighboring nuclei depend only on the nature of the supermultiplet to which

* *Phys. Rev.* **51**, 947 (1937).

TABLE 8–2

Binding energy per nucleon of stable light nuclei. Taken from A. H Wapstra [*Phys. Rev.* **84**, 837 (1951)].

Nucleus	E_B/nucleon	Nucleus	E_B/nucleon
H^2	1.11 Mev	Na^{23}	8.11 Mev
H^3	2.83	Mg^{24}	8.26
He^3	2.57	Mg^{25}	8.22
He^4	7.07	Mg^{26}	8.33
He^5	5.5	Al^{27}	8.33
Li^6	5.33	Si^{28}	8.45
Li^7	5.60	Si^{29}	8.45
Be^8	7.06	Si^{30}	8.52
Be^9	6.46	P^{31}	8.48
B^{10}	6.47	S^{32}	8.49
B^{11}	6.93	S^{33}	8.49
C^{12}	7.68	S^{34}	8.58
C^{13}	7.47	Cl^{35}	8.51
N^{14}	7.47	A^{36}	8.51
N^{15}	7.70	S^{36}	8.56
O^{16}	7.98	Cl^{37}	8.56
O^{17}	7.75	A^{38}	8.62
O^{18}	7.76	K^{39}	8.56
F^{19}	7.77	Ca^{40}	8.55
Ne^{20}	8.03	A^{40}	8.59
Ne^{21}	7.97	K^{41}	8.58
Ne^{22}	8.09	Ca^{42}	8.61

they belong. If the coulomb energy is included, Wigner finds that the nuclear binding has the form

$$E_B = -E_0(A) - \Xi' L(A) - 1.2(mc^2/A^{\frac{1}{3}})(A + t - 1)t - (50Q^2/A)mc^2, \tag{8–37}$$

where E_0 and L are smooth functions of A, the former being negative, the latter positive. The quantity Ξ' is defined by

$$\Xi' = \tfrac{1}{2}\{(Q + 2)^2 + (Q' + 1)^2 + Q''^2\}, \tag{8–38}$$

if the ground state belongs to the (Q,Q',Q'') supermultiplet. The third term in Eq. (8–37) is Wigner's estimate of the coulomb energy, and the last term is his estimate of the change of kinetic energy which would be produced by changing neutrons into protons.* The scallops in the bind-

* The kinetic energy is estimated on the basis of free waves confined to the nuclear volume. The kinetic energy then depends on the relative depths of the neutron and proton Fermi lakes.

ing energy curve are clearly indicated by the values of Ξ' for the four supermultiplets:

$$(Q,Q',Q'') = (0,0,0), \qquad \Xi' = 2.5,$$
$$(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}), \qquad 4.375,$$
$$(1,0,0), \qquad 5,$$
$$(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}), \qquad 4.375.$$

Note that the large jump occurs between the $4n$ and $4n \pm 1$ nuclei, therefore alpha-particle nuclei are particularly stable.

Equation (8–37) may be used to calculate the masses of nuclei once the smooth functions $E_0(A)$ and $L(A)$ have been determined empirically on the basis of a few known masses. Such a semiempirical program has been carried through by Barkas* with some success. There certainly has appeared no clear-cut contradiction to the gross features of the Wigner coupling scheme, a fact which may be taken as an indication that our ideas on nuclear forces have at least some qualitative validity. However, there do exist important deviations from the detailed predictions of the coupling scheme, as we shall see in Section 8–6. These deviations are associated with a strong spin-orbit interaction, an interaction which necessarily fails to have the symmetry properties required for Wigner coupling.

8–5 The independent particle model. Russell-Saunders coupling. The models on which detailed calculations of properties of nuclei heavier than He^4 are based have been developed primarily for the ease with which the calculations can be made. Their selection is largely governed by experience in molecular or atomic physics rather than by any legitimate theoretical argument concerning the validity of the approximation involved. The most widely used and most successful of them is the independent particle model, which in the older literature is often referred to as the Hartree model. It is based on an analogy with Hartree's method for treating the structure of atoms. Each nucleon is assumed to move in a fixed potential well which is assumed to represent the average of the interactions of all nucleons with a given nucleon. Then one can assign definite states to a single nucleon, and the nucleus is built up by filling each such state with nucleons in accordance with the Pauli principle (compare Section 2–4).

The Hartree model is quite successful in the treatment of atomic structure, and for good reason. In the atom, the principal part of the potential

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* *Phys. Rev.* **55,** 691 (1939). Barkas provides a table of masses based on the data available at the time. Application of Eq. (8–37) to more recent data is made by Collins, Nier, and Johnson, *Phys. Rev.* **86,** 408 (1952).

acting on the electron is due to a fixed center, the nucleus, and the smaller electron-electron interactions can be replaced in good part by their average because of the long-range nature of the coulomb potential. Clearly, no such theoretical justification applies to the nuclear problem. There is no fixed center to produce a dominant central potential. And if we maintain our view that nuclear interactions can be described by two-body forces, these forces have short range, so that a given nucleon feels the potential due to only a few neighboring nucleons at any given instant. The nucleon is subjected to a series of billiard ball-like collisions in its motion through the nucleus; one can hardly hope to construct an average potential which will give an accurate representation of such an effect. This lack of justification has not acted as a deterrent to those who felt the need to calculate properties of nuclei. The model has been widely used, and it is just as well that it has, for it is much more successful than one would expect on the basis of the above argument.

The implication of the independent-particle model is that the wave functions of the nucleus may be written as an antisymmetric linear combination of products of one-particle functions. Since the problem has central symmetry, the most reasonable choice of the one-particle space functions is

$$\phi_{nl}^{m}(\mathbf{r}) = f_{nl}(r)Y_l^m,$$

where the quantum numbers l and m have their usual meaning and the index n denotes a quantum number fixing the energy. We shall define n as the number of nodes in the radial function, exclusive of that at $r = 0$ (compare Appendix 1). The nuclear function is then a linear combination of the products

$$\phi_q(\mathbf{r}_1)\phi_{q'}(\mathbf{r}_2)\phi_{q''}(\mathbf{r}_3)\cdots, \tag{8-39}$$

where q is a symbol for the triplet (n,l,m) and $\mathbf{r}_i$ is the coordinate vector of the ith nucleon. A specification of the form of the product function is given by indicating the *configuration* of the nucleons. If four nucleons are in the state $n = 1$, $l = 0$, as might be expected in the ground state of the alpha-particle, we speak of a $(1s)^4$ configuration. The ground state of Li^6 might be a $(1s)^4(1p)^2$ configuration, in which case the product (8-39) contains four factors with $n = 1$, $l = 0$ and two with $n = 1$, $l = 1$.

The order in which the levels are filled with increasing nuclear mass is determined by the order of the energy levels associated with given n and l. Now n has been defined to have the same order as the energy for given l, but the dependence of the energy on l is determined by the form of the potential well in which the nucleon is moving. The selection of a form is somewhat arbitrary, since the connection between this auxiliary central potential and the direct nucleon-nucleon interactions is so remote. One fact at our disposal is the apparent sharpness of the nuclear boundary; the nuclear density is virtually constant within its radius and drops off very

rapidly beyond. This suggests that the auxiliary potential should have steep walls at a distance comparable to the nuclear radius. One could, for example, use a square well. Another possibility is the oscillator potential, since the wave functions of a harmonic oscillator drop off with extreme rapidity outside the potential well. Since the auxiliary potential is only to be used as a means for fixing the order of the levels, the exact form of the potential makes little difference. All steep-walled potentials give about the same results, at least for the low-lying levels. Hence we use that which gives the simplest ordering scheme, the oscillator potential. The level order is shown in Table 8–3. The characteristic accidental degeneracies

TABLE 8–3

The levels of the isotropic oscillator. The energy is $(N + \frac{3}{2})\hbar\omega$ and n is the number of nodes of the radial function, excepting that at $r = 0$.

N	l	n	Designation	Degeneracy	Total No. of Protons or Neutrons
0	0	1	$1s$	1	2
1	1	1	$1p$	3	8
2	2	1	$1d$	6	20
	0	2	$2s$		
3	3	1	$1f$	10	40
	1	2	$2p$		
4	4	1	$1g$	15	70
	2	2	$2d$		
	0	3	$3s$		
5	5	1	$1h$	21	112
	3	2	$2f$		
	1	3	$3p$		
6	6	1	$1i$	28	168
	4	2	$2g$		
	2	3	$3d$		
	0	4	$4s$		

occur there, but it is implied that a more realistic potential would split these degeneracies, although the splitting is presumed to be small compared with the spacing of other levels.

Since four nucleons, a neutron and a proton of each spin, can occupy each of the states, the $1s$ level is filled at the alpha-particle. According to Table 8–3, the next levels to be filled are the $1p$ levels, so the lowest configurations of nuclei between $A = 5$ and $A = 16$ are of the form $(1s)^4 (1p)^\nu$ with $\nu = 1, 2, \cdots, 12$. The $1p$ level is filled at $\nu = 12$, which corresponds to O^{16}. Beyond this, either the $1d$ or $2s$ level starts to fill, and configurations of the form $(1s)^4 (1p)^{12} (1d)^\nu$ might be expected to occur.

The quantum numbers m, m', m'', $\cdots$ in the product function (8–39) can usually be assigned a number of sets of values, with the consequence that there is a corresponding degree of degeneracy associated with the configuration. Only when there are $4(2l + 1)$ nucleons in the highest state of the configuration is there no degeneracy, and then we speak of a *complete shell.** Actually, this definition is not adequate for the heavier nuclei, since the coulomb potential raises the proton states with respect to the neutron states. Then the neutron and proton configurations are specified separately. If l and l' correspond to the last occupied proton and neutron states, respectively, a complete shell occurs when there are just $2(2l + 1)$ protons and $2(2l' + 1)$ neutrons in these states. The last column of Table 8–3 indicates the total number of neutrons or protons required to complete all shells up to and including that designated by the quantum number N.

Destruction of the degeneracy of the incomplete shells occurs through the action of the nucleon-nucleon potential. The effect is conveniently treated on the basis of some particular coupling scheme. If, for example, a scheme of the Russell-Saunders type is applicable, the degenerate product functions (8–39) are linearly combined to form characteristic functions of the total orbital angular momentum L. For a given spin and i-spin state, or for a given supermultiplet, some of the values of L arrived at in this manner are excluded by the Pauli principle, just as is the case in an atomic system. The selection of the state of lowest energy for the remaining values of L involves the properties of the interactions and can only be made on the basis of a detailed calculation. This procedure can be illustrated for the $(1s)^4\ (1p)^\nu$ configurations which have been treated in detail by Feenberg and Wigner† and by Feenberg and Phillips.‡ They have classified the states of the configurations in terms of the supermultiplets and allowed L-values. In Table 8–4 are listed the admissible terms corresponding to the supermultiplet of lowest energy. Also, the average potential energies of the $(1p)^\nu$ nucleons for each one of the four two-body interactions, $J(r_{ij})$, $J(r_{ij})P^r_{ij}$, $J(r_{ij})P^\tau_{ij}$, $J(r_{ij})P^\sigma_{ij}$, are listed in the last four columns of the table. The quantities Λ and K in terms of which these potential energies are expressed are

$$\Lambda = \iint J(r_{12}) f^2(r_1) f^2(r_2) (x_1/r_1)^2 (x_2/r_2)^2 \, d^3r_1 \, d^3r_2 \qquad (8\text{–}40)$$

* We use the term *complete* rather than "closed" shell because the concept of a closed shell has taken on a somewhat different meaning in nuclear physics. See Section 8–6.

† *Phys. Rev.* **51,** 95 (1937). See also F. Hund, *loc. cit.*

‡ *Phys. Rev.* **51,** 597 (1937). For classification of the terms arising from the d^ν shell, see H. A. Jahn, *Proc. Roy. Soc.* **A201,** 516 (1950), and for those from the f^ν shell, see B. H. Flowers, *Proc. Roy. Soc.* **A210,** 497 (1952).

and

$$K = \iint J(r_{12}) f^2(r_1) f^2(r_2) (x_1 y_1 / r_1^2)(x_2 y_2 / r_2^2)\, d^3 r_1\, d^3 r_2, \qquad (8\text{–}41)$$

where $f(r) = f_{11}(r)$ is the radial p-function with one node and xyz are the cartesian coordinates of a nucleon. The labels 1 and 2 refer to any pair of p nucleons.

It will be noted that the ordinary interaction and the Majorana interaction cause no splitting of the supermultiplet; that is, each term in the supermultiplet associated with given L has the same average energy for these potentials. This is in accord with the fact that these two interactions lead to Wigner coupling. On the other hand, the interaction P_{ij}^{σ} is a spin-spin coupling which separates the terms of different S within the supermultiplet. This is clearly illustrated by the last two columns of the table.

Evaluation of the integrals Λ and K requires a knowledge of the radial p-functions $f(r)$. These could be taken to be the functions characteristic of the auxiliary potential, in our case the radial functions of the isotropic harmonic oscillator. However, that would be a somewhat more literal interpretation of the model than would seem to be justified. An alternative procedure is to introduce functions $f(r)$ of reasonable shape which contain a number of free parameters. These parameters are then chosen so as to minimize the total energy, in accordance with the usual variational method. Binding energies obtained in this way are much too small, a result which indicates that the independent particle functions are not a very good approximation, or that the assumed potential is incomplete.

The order of the terms in a given supermultiplet can be fixed on a more qualitative basis. The integral Λ is clearly negative for an attractive potential. That K is also negative follows from the fact that it represents the difference in energy between a space-symmetric two-nucleon function and a space-antisymmetric function. Clearly, the symmetric state leads to a lower potential energy, since the average distance between the nucleons is lower than in the antisymmetric state. This information is sufficient to fix the order of the levels for given S. To determine whether the singlets or triplets are lower, use can be made of either one of the spin-spin coupling terms after it is normalized so as to give the known splitting in the deuteron. It then follows that the triplets are the lower terms, and the ground state terms are found to be those indicated in Table 8–5.

The observed total angular momenta (spins) of the nuclei are also listed in Table 8–5. It is to be noted that, with the exception of B^{10}, these J-values are consistent with the assigned lowest terms. Evidently the B^{10} spin can be understood within the limits of this model only by assuming that some perturbation, such as the tensor interaction, causes the $^{1,3}S_1$ state to cross over one of the low $^{1,3}D$ states listed for B^{10} in Table 8–4.

TABLE 8–4

Matrix elements and quantum numbers within the group of low terms

Configuration	(Q,Q',Q'')	State	Nucleus	Average Potential Energy			
				P^r	1	P^rP^σ	P^σ
p^2	(1,0,0)	$^{1,3}S$	Li^6	$\Lambda + 2K$	$\Lambda + 2K$	$\Lambda + 2K$	$\Lambda + 2K$
		$^{1,3}D$		$\Lambda - K$	$\Lambda - K$	$\Lambda - K$	$\Lambda - K$
		$^{3,1}S$	He^6, Li^6, Be^6	$\Lambda + 2K$	$\Lambda + 2K$	$-\Lambda - 2K$	$-\Lambda - 2K$
		$^{3,1}D$		$\Lambda - K$	$\Lambda - K$	$-\Lambda + K$	$-\Lambda + K$
p^3	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$^{2,2}P$	Li^7, Be^7	$3\Lambda + 2K$	$3\Lambda + 2K$	0	0
		$^{2,2}F$		$3\Lambda - 3K$	$3\Lambda - 3K$	0	0
p^4	(0,0,0)	$^{1,1}S$	Be^8	$6\Lambda + 4K$	$6\Lambda + 4K$	0	0
		$^{1,1}D$		$6\Lambda + K$	$6\Lambda + K$	0	0
		$^{1,1}G$		$6\Lambda - 6K$	$6\Lambda - 6K$	0	0
p^5	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$^{2,2}P$	Be^9, B^9	$5\Lambda + 9K$	$10\Lambda - 6K$	$-2\Lambda + 6K$	$2\Lambda - 6K$
		$^{2,2}D$		$5\Lambda + 7K$	$10\Lambda - 8K$	$-2\Lambda + 6K$	$2\Lambda - 6K$
		$^{2,2}F$		$5\Lambda + 4K$	$10\Lambda - 11K$	$-2\Lambda + 6K$	$2\Lambda - 6K$
		$^{2,2}G$		5Λ	$10\Lambda - 15K$	$-2\Lambda + 6K$	$2\Lambda - 6K$
p^6	(1,0,0)	$^{1,3}S$	B^{10}	$5\Lambda + 16K$	$15\Lambda - 14K$	$-3\Lambda + 29K/2$	$5\Lambda - 19K/2$
		$^{1,3}D_{\mathrm{I}}$		$5\Lambda + 13K$	$15\Lambda - 17K$	$-3\Lambda + 13K$	$5\Lambda - 11K$
		$^{1,3}D_{\mathrm{II}}$		$5\Lambda + 13K$	$15\Lambda - 17K$	$-3\Lambda + 53K/5$	$5\Lambda - 67K/5$

		$^{1,3}F$		$5\Lambda + 10K$	$15\Lambda - 20K$	$-3\Lambda + 10K$	$5\Lambda - 14K$
		$^{1,3}G$		$5\Lambda + 6K$	$15\Lambda - 24K$	$-3\Lambda + 12K$	$5\Lambda - 12K$
		$^{3,1}S$	Be^{10}, B^{10}, C^{10}	$5\Lambda + 16K$	$15\Lambda - 14K$	$-5\Lambda + 19K$ 2	$3\Lambda - 29K$ 2
		$^{3,1}D_{\mathrm{I}}$		$5\Lambda + 13K$	$15\Lambda - 17K$	$-5\Lambda + 11K$	$3\Lambda - 13K$
		$^{3,1}D_{\mathrm{II}}$		$5\Lambda + 13K$	$15\Lambda - 17K$	$-5\Lambda + 67K$ 5	$3\Lambda - 53K$ 5
		$^{3,1}F$		$5\Lambda + 10K$	$15\Lambda - 20K$	$-5\Lambda + 14K$	$3\Lambda - 10K$
		$^{3,1}G$		$5\Lambda + 6K$	$15\Lambda - 24K$	$-5\Lambda + 12K$	$3\Lambda - 12K$
p^7	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$^{2,2}P$	B^{11}, C^{11}	$6\Lambda + 21K$	$21\Lambda - 24K$	$-6\Lambda + 18K$	$6\Lambda - 18K$
		$^{2,2}D$		$6\Lambda + 19K$	$21\Lambda - 26K$	$-6\Lambda + 18K$	$6\Lambda - 18K$
		$^{2,2}F$		$6\Lambda + 16K$	$21\Lambda - 29K$	$-6\Lambda + 18K$	$6\Lambda - 18K$
		$^{2,2}G$		$6\Lambda + 12K$	$21\Lambda - 33K$	$-6\Lambda + 18K$	$6\Lambda - 18K$
p^8	(0,0,0)	$^{1,1}S$	C^{12}	$8\Lambda + 28K$	$28\Lambda - 32K$	$-8\Lambda + 24K$	$8\Lambda - 24K$
		$^{1,1}D$		$8\Lambda + 25K$	$28\Lambda - 35K$	$-8\Lambda + 24K$	$8\Lambda - 24K$
		$^{1,1}G$		$8\Lambda + 18K$	$28\Lambda - 42K$	$-8\Lambda + 24K$	$8\Lambda - 24K$
p^9	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$^{2,2}P$	C^{13}, N^{13}	$6\Lambda + 38K$	$36\Lambda - 52K$	$-12\Lambda + 36K$	$12\Lambda - 36K$
		$^{2,2}F$		$6\Lambda + 33K$	$36\Lambda - 57K$	$-12\Lambda + 36K$	$12\Lambda - 36K$
p^{10}	(1,0,0)	$^{1,3}S$	N^{14}	$5\Lambda + 50K$	$45\Lambda - 70K$	$-15\Lambda + 50K$	$17\Lambda - 46K$
		$^{1,3}D$		$5\Lambda + 47K$	$45\Lambda - 73K$	$-15\Lambda + 47K$	$17\Lambda - 49K$
		$^{3,1}S$	C^{14}, N^{14}, O^{14}	$5\Lambda + 50K$	$45\Lambda - 70K$	$-17\Lambda + 46K$	$15\Lambda - 50K$
		$^{3,1}D$		$5\Lambda + 47K$	$45\Lambda - 73K$	$-17\Lambda + 49K$	$15\Lambda - 47K$
p^{11}	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$^{2,2}P$	N^{15}, O^{15}	$5\Lambda + 60K$	$55\Lambda - 90K$	$-20\Lambda + 60K$	$20\Lambda - 60K$
p^{12}	(0,0,0)	$^{1,1}S$	O^{16}	$6\Lambda + 72K$	$66\Lambda - 108K$	$-24\Lambda + 72K$	$24\Lambda - 72K$

TABLE 8-5

Lowest terms of $(1s)^4(1p)^p$ configuration

Nucleus	p-configuration	Lowest Term	Observed* J
Li^6	p^2	$^{1,3}S_1$	1
Li^7	p^3	$^{2,2}P_{1/2,3/2}$	$\frac{3}{2}$
Be^9	p^5	$^{2,2}P_{1/2,3/2}$	$(\frac{3}{2})$
B^{10}	p^6	$^{1,3}S_1$	3
B^{11}	p^7	$^{2,2}P_{1/2,3/2}$	$\frac{3}{2}$
C^{12}	p^8	$^{1,1}S_0$	0
C^{13}	p^9	$^{2,2}P_{1/2,3/2}$	$\frac{1}{2}$
C^{14}	p^{10}	$^{3,1}S_0$	0
N^{14}	p^{10}	$^{1,3}S_1$	1
N^{15}	p^{11}	$^{2,2}P_{1/2,3/2}$	$\frac{1}{2}$
O^{16}	p^{12}	$^{1,1}S_0$	0

* Taken from J. E. Mack, *Rev. Mod. Phys.* **22,** 64 (1950).

An anomaly is also presented by B^{11}, since all of the other J-values appear to be selected on the basis of inverted multiplets, i.e., the ground state has the higher J-value in the first half of the shell, while it has the lower value for the second half of the complete shell. The irregular behavior of B^{11} could again be interpreted as an indication that the ground state is a D-function.

The occurrence of the crossing over of states of different L suggests that the spin-orbit coupling plays an appreciably more important role than is consistent with Russell-Saunders coupling. States of different L may be mixed to a considerable extent; for example, in B^{11} one might find that the $P_{3/2}$ and $D_{3/2}$ states couple strongly, with the consequence that a mixture of these two lies lower than the $P_{1/2}$ state. Admixtures of this kind should not come as a surprise, since we believe that the tensor interaction has a strength comparable to that of the central potential. Further evidence for the importance of some type of spin-orbit interaction will be found in the success of the Mayer-Jensen coupling scheme, therefore the application of Wigner coupling should be limited to the broad features of the supermultiplet, exclusive of the fine structure.

8-6 The independent-particle model. Mayer-Jensen coupling. Literal interpretation of the independent-particle model leads immediately to the conclusion that special properties should be displayed by nuclei having the requisite number of either neutrons or protons to just form a complete shell. Such a nucleus would be reluctant to add a nucleon to the completed shell, since that nucleon must occupy a higher state and hence is more weakly bound. Consequently, nuclei should show a high degree of stability with respect to the number of nucleons of a given kind when that

number has the complete shell values listed in the last column of Table 8–3. This means that there should occur a larger number of stable isotopes with $Z = 8, 20$, etc., than for other values of Z. Similarly, the number of stable isotones* with $N = 8, 20$, etc., should be especially large. Furthermore, these nuclei should tend to be particularly stable; the dependence of the binding energy per nucleon on N or Z should show a change in slope at a point corresponding to a completed shell.

For the shells ending in 8 and 20 nucleons these predictions are, on the whole, borne out by the facts. According to Table 8–2, O^{16}, which has a complete shell in both neutrons and protons, is particularly stable. This can be established by noting that the drop in binding energy in going from a $4n$ to a $4n + 1$ nucleus tends to decrease with increasing mass, a tendency which is reversed at O^{16}. Furthermore, we note that oxygen is the first of the light nuclei to add two neutrons in succession, rather than a neutron and then a proton (compare Section 2–4). Similarly, the fact that S^{36} is stable can be attributed to the stability of 20 neutrons. This could be ascribed to the coulomb effect, as we have seen in Section 2–4, but then A^{39} would be expected to be stable with respect to K^{39}, which it is not. And Ca^{40} would not be stable, whereas it is, with a binding energy comparable to that of A^{40}. The shell structure would clearly lead to large binding for Ca^{40}, since it has a complete shell in both neutrons and protons. Finally, we note that there are six stable isotopes of Ca, including Ca^{48} which has eight more neutrons than protons, so the 20-proton configuration must be very stable indeed.

An interpretation of the complete shells at 8 and 20 nucleons as *closed* shells, that is, as configurations which are especially stable and especially reluctant to increase their number, would seem to be completely justified. But the properties of nuclei with N or Z equal to 40, 70, and so on, lend no support to the view that these completed shells are closed. On the other hand, closed shells do seem to occur for nuclei beyond Ca^{40}, but at neutron and proton numbers equal to 50, 82, or 126. Evidence for the existence of these "magic numbers" comes in many different forms. A thorough exposition of the evidence based on the number of stable isotopes and isotones, on the abundances of the nuclei, and so on, is given in the classic paper of Goeppert-Mayer.† Additional evidence based on nuclear cross sections, isomerism, nuclear moments, etc., has been accumulating since that writing, and we shall discuss these matters in their proper places. There is no question that the nucleon numbers 50, 82, and 126 play a very special role in nuclear structure.

It is natural to first raise the question whether any simple modification of the auxiliary central potential could lead to a reordering of the levels

* Nuclei having the same number of neutrons.

† *Phys. Rev.* **74**, 235 (1948). W. Elsasser is credited with the first observation of this property; *J. de phys. et rad.* **5**, 625 (1934).

that would establish the magic numbers as the number of nucleons in a complete shell. However, care must be taken not to violate the theorem* that for any central potential, the state (n,l) lies higher than (n,l') if $l > l'$. This seems to preclude the possibility of developing, on the basis of a central auxiliary potential, a reordering scheme that can account for all the magic numbers.**

A successful reordering principle applicable to the independent-particle model was suggested simultaneously by Goeppert-Mayer† and by Haxel, Jensen, and Suess.‡ They suggested that the spin-orbit coupling plays a role second in importance only to the auxiliary potential in determining the level order. Associated with the wave functions ϕ_{nl}^m of the single nucleon are two classes of states, those with total angular momentum $j = l + \frac{1}{2}$ and those with $j = l - \frac{1}{2}$. Russell-Saunders coupling implies that these have very nearly the same energy. The new scheme proposes just the opposite: that these states are split appreciably by coupling of the nucleon spin with its own orbit, that the splitting increases with increasing l, and that the $j = l + \frac{1}{2}$ term is the lower (inverted doublet). Then the closed shells are not built on the $2l + 1$-fold degeneracy of the ϕ_{nl}^m, but rather on the $2j + 1$-fold degeneracy of the states ψ_{nlj}^μ of total angular momentum j. Here‡‡

$$\psi_{nlj}^\mu = (2j)^{-\frac{1}{2}}\{(j+\mu)^{\frac{1}{2}}\phi_{nl}^{\mu-\frac{1}{2}}\chi_{\frac{1}{2}}^{+} + (j-\mu)^{\frac{1}{2}}\phi_{nl}^{\mu+\frac{1}{2}}\chi_{\frac{1}{2}}^{-}\} \qquad (8\text{–}42a)$$

for $j = l + \frac{1}{2}$, and

$$\psi_{nlj}^\mu = (2j+2)^{-\frac{1}{2}}\{-(j-\mu+1)^{\frac{1}{2}}\phi_{nl}^{\mu-\frac{1}{2}}\chi_{\frac{1}{2}}^{+} + (j+\mu+1)^{\frac{1}{2}}\phi_{nl}^{\mu+\frac{1}{2}}\chi_{\frac{1}{2}}^{-}\} \qquad (8\text{–}42b)$$

for $j = l - \frac{1}{2}$, where $\chi_{\frac{1}{2}}^{\pm}$ are the nucleon spin functions. The independent particle functions are products of single nucleon functions ψ_{nlj}^μ, so the configurations are given in terms of the number of $s_{\frac{1}{2}}$, $p_{\frac{3}{2}}$, $p_{\frac{1}{2}}$, etc., nucleons.

The $p_{\frac{3}{2}} - p_{\frac{1}{2}}$ splitting is presumably quite small, so the $(1p)$ closed shell occurs, just as before, at O^{16}. The configuration of neutrons and protons is $(1s_{\frac{1}{2}})^2\ (1p_{\frac{3}{2}})^4\ (1p_{\frac{1}{2}})^2$, which has the same wave function as the $(1s)^2\ (1p)^6$ configuration of Russell-Saunders coupling. In the next shell,

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* See Appendix 1.

** Feenberg succeeded in accounting for the closed shell at 50 by such a scheme, but he was forced to assume that the $2s$ level is higher than the $2d$, to account for 82, and a similar violation of the theorem occurred for 126. Feenberg and Hammack, *Phys. Rev.* **75,** 1877 (1949).

† *Phys. Rev.* **75,** 1969 (1949).

‡ *Phys. Rev.* **75,** 1766 (1949).

‡‡ See, for example, Condon and Shortley, *The Theory of Atomic Spectra*, Cambridge University Press (1951), p. 56 ff.

the $1d_{\frac{5}{2}}$ state lies below the $1d_{\frac{3}{2}}$, but not to such an extent as to modify our conclusions concerning the closed shell at 20 nucleons. The configuration is $(1s_{\frac{1}{2}})^2\ (1p_{\frac{3}{2}})^4\ (1p_{\frac{1}{2}})^2\ (1d_{\frac{5}{2}})^6\ (2s_{\frac{1}{2}})^2\ (1d_{\frac{3}{2}})^4$, and it has the same wave function as the $(1s)^2\ (1p)^6\ (1d)^{10}\ (2s)^2$ configuration. The next shell is markedly affected because the $1g_{\frac{9}{2}}$ state is added to it as a result of the large depression of that level caused by spin-orbit coupling. Thus the closed shell at 50 has the configuration (20) $(1f_{\frac{7}{2}})^8\ (2p_{\frac{3}{2}})^4\ (2p_{\frac{1}{2}})^2\ (1f_{\frac{5}{2}})^6\ (1g_{\frac{9}{2}})^{10}$, where (20) is an abbreviation for the configuration of the closed shell at 20. The next shells are formed in like manner. The new grouping of levels is indicated* in Table 8–6, which is to be compared with Table 8–3.

TABLE 8-6

The independent-particle levels in the Mayer-Jensen coupling scheme

Shell No.	l	n	j	Designation	Degeneracy	Total No. to Complete Shell
0	0	1	$\frac{1}{2}$	$1s_{\frac{1}{2}}$	2	2
1	1	1	$\frac{3}{2}$	$1p_{\frac{3}{2}}$	6	8
	1	1	$\frac{1}{2}$	$1p_{\frac{1}{2}}$		
2	2	1	$\frac{5}{2}$	$1d_{\frac{5}{2}}$	12	20
	0	2	$\frac{1}{2}$	$2s_{\frac{1}{2}}$		
	2	1	$\frac{3}{2}$	$1d_{\frac{3}{2}}$		
3	3	1	$\frac{7}{2}$	$1f_{\frac{7}{2}}$	30	50
	1	2	$\frac{3}{2}$	$2p_{\frac{3}{2}}$		
	1	2	$\frac{1}{2}$	$2p_{\frac{1}{2}}$		
	3	1	$\frac{5}{2}$	$1f_{\frac{5}{2}}$		
	4	1	$\frac{9}{2}$	$1g_{\frac{9}{2}}$		
4	2	2	$\frac{5}{2}$	$2d_{\frac{5}{2}}$	32	82
	0	3	$\frac{1}{2}$	$3s_{\frac{1}{2}}$		
	2	2	$\frac{3}{2}$	$2d_{\frac{3}{2}}$		
	4	1	$\frac{7}{2}$	$1g_{\frac{7}{2}}$		
	5	1	$\frac{11}{2}$	$1h_{\frac{11}{2}}$		
5	3	2	$\frac{7}{2}$	$2f_{\frac{7}{2}}$	44	126
	1	3	$\frac{3}{2}$	$3p_{\frac{3}{2}}$		
	1	3	$\frac{1}{2}$	$3p_{\frac{1}{2}}$		
	3	2	$\frac{5}{2}$	$2f_{\frac{5}{2}}$		
	5	1	$\frac{9}{2}$	$1h_{\frac{9}{2}}$		
	6	1	$\frac{13}{2}$	$1i_{\frac{13}{2}}$		

* The existence of other closed shells, or at least semiclosed shells, is suggested by the model; for example, the $f_{\frac{7}{2}}$ level should be somewhere between shell number 2 and shell number 3, so 28 might be a magic number. There is some evidence for this and other such subshells, but we wish to avoid so detailed a discussion of these matters.

The question now arises as to the implications of this specification of independent nucleon functions for systems which do *not* constitute closed shells. There are two problems: the order of the levels within the shell, and the effect of j-j coupling, i.e., coupling between like nucleons of given j. The order of the levels is given rather arbitrarily in Table 8–6. It is based on the assumption that the spin-orbit splitting is symmetrical about the position of the oscillator levels. Rearrangement of this order can be expected to occur as a result of the splitting of the accidental oscillator degeneracy, therefore this decision may be made in such a way as to obtain the best fit to observations on a given nucleus.*

A remarkably successful scheme for the j-j coupling has been suggested by Goeppert-Mayer** and by Haxel, Jensen, and Suess.† They point out that almost every observed nuclear spin and magnetic moment†† can be accounted for by the assumption that when the number of neutrons or protons in a shell of given l and j is even, they couple together to form a state of total angular momentum zero, and that when the number is odd, the last nucleon is simply added to such a zero angular momentum state. To show that this coupling scheme is reasonable for the present model, Goeppert-Mayer calculates‡ the expectation value of the proton-proton (or neutron-neutron) interaction in proton (or neutron) states formed by taking antisymmetrized products of ψ^{α}_{nlj} for $j \leq \frac{7}{2}$. No account is taken of the interaction between unlike nucleons. Goeppert-Mayer uses a δ-function potential to simplify the calculations, and finds that the state prescribed by her coupling scheme does, indeed, lead to the lowest value of the energy. The use of a δ-function for the potential is equivalent to assuming that the range of forces is very small compared with the nuclear radius. Thus her result should be, at best, applicable only to heavy nuclei. Hence the few cases of minor violations of the coupling scheme that occur in light nuclei have been attributed to the effect of finite range of the forces.‡‡

For comparison with the results of Russell-Saunders coupling given in Table 8–4, a list of terms admitted by the Mayer-Jensen coupling scheme for the $1p$ shell is given in Table 8–7. In every case of a unique prediction of J, there is agreement with observation. A notable example of the success of the scheme is offered by B^{11}, for which the Russell-Saunders coupling

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* A specification of the configuration of every nucleus on the basis of presently available information is given by P. F. A. Klinkenberg, *Rev. Mod. Phys.* **24,** 63 (1952). These very tentative assignments are reproduced in Appendix 5.

** *Phys. Rev.* **78,** 16 (1950).

† *Z. f. Phys.* **128,** 295 (1950).

†† See, however, Section 9–3.

‡ *Phys. Rev.* **78,** 22 (1950).

‡‡ D. Kurath, *Phys. Rev.* **80,** 98 (1950).

TABLE 8–7

Lowest terms of $(1s)^4\,(1p)^r$ configuration according to Mayer-Jensen coupling

Nucleus	p-configuration Neutrons	Protons	Allowed J	Observed J
Li^6	$p_{3/2}$	$p_{3/2}$	1,3	1
Li^7	$p_{3/2}^2$	$p_{3/2}$	$\frac{3}{2}$	$\frac{3}{2}$
Be^9	$p_{3/2}^3$	$p_{3/2}^2$	$\frac{3}{2}$	$(\frac{3}{2})$
B^{10}	$p_{3/2}^3$	$p_{3/2}^3$	1,3	3
B^{11}	$p_{3/2}^4$	$p_{3/2}^3$	$\frac{3}{2}$	$\frac{3}{2}$
C^{12}	$p_{3/2}^4$	$p_{3/2}^4$	0	0
C^{13}	$p_{3/2}^4 p_{1/2}$	$p_{3/2}^4$	$\frac{1}{2}$	$\frac{1}{2}$
C^{14}	$p_{3/2}^4 p_{1/2}^2$	$p_{3/2}^4$	0	0
N^{14}	$p_{3/2}^4 p_{1/2}$	$p_{3/2}^4 p_{1/2}$	1	1
N^{15}	$p_{3/2}^4 p_{1/2}^2$	$p_{3/2}^4 p_{1/2}$	$\frac{1}{2}$	$\frac{1}{2}$
O^{16}	$p_{3/2}^4 p_{1/2}^2$	$p_{3/2}^4 p_{1/2}^2$	0	0

in its most straightforward interpretation led to a predicted J of $\frac{1}{2}$ instead of the observed $\frac{3}{2}$.

The model does not yield a unique value for J for the odd-odd nuclei because both the neutron shell and the proton shell have a nonvanishing angular momentum. Therefore some additional prescription is required to establish which of the various possible J-values lies lowest. Let us consider Li^6, which has a single $p_{3/2}$ neutron and a single $p_{3/2}$ proton. On the assumption of charge invariance, this is to be treated as a $p_{3/2}^2$ configuration of nucleons which admits of two possible T-values, $T = 0$ and $T = 1$. According to the Pauli principle, the first must be associated with a symmetric space and spin function, the second with an antisymmetric function. It can be argued that the symmetric function will be the lower, since the Majorana potential favors a space-symmetric state and the spin-spin coupling leads to a lowering of the symmetric triplet state with respect to the singlet state. Hence the ground state is assumed to have $T = 0$, a result which agrees with the conclusion reached on the basis of Russell-Saunders coupling. Now the states Ψ_J^M of given J may be expressed directly in terms of the products of the functions ψ_{njl}^μ for the two nucleons. For the sake of simplicity, let us denote the $j = \frac{3}{2}$, $l = 1$ functions of the kth nucleon by ψ_k^μ. Then the Ψ_J^M are as follows:*

* Condon and Shortley, *loc. cit.*

$$\Psi_0^0 = -\frac{i}{2}\sum_{\mu}(-1)^{\mu}\psi_1^{\mu}\psi_2^{-\mu}, \tag{8-43a}$$

$$\begin{aligned}\Psi_1^M = \frac{1}{2\sqrt{10}}\{&\sqrt{3(1+M)(2-M)(3-M)}\,\psi_1^{3/2}\psi_2^{M-1/2} \\ &- (1+3M)\sqrt{2-M}\,\psi_1^{1/2}\psi_2^{M-1/2} - (1-3M)\sqrt{2+M}\,\psi_1^{-1/2}\psi_2^{M+1/2} \\ &+ \sqrt{3(2+M)(3+M)(1-M)}\,\psi_1^{-3/2}\psi_2^{M+3/2}\},\end{aligned} \tag{8-43b}$$

$$\begin{aligned}\Psi_2^M = \frac{1}{6\sqrt{2}}\{&-\sqrt{3(1+M)(2+M)(3-M)}\,\psi_1^{3/2}\psi_2^{M-1/2} \\ &- 3(1-M)\sqrt{2+M}\,\psi_1^{1/2}\psi_2^{M-1/2} + 3(1+M)\sqrt{2-M}\,\psi_1^{-1/2}\psi_2^{M+1/2} \\ &+ \sqrt{3(3+M)(1-M)(2-M)}\,\psi_1^{-3/2}\psi_2^{M+3/2}\},\end{aligned} \tag{8-43c}$$

$$\begin{aligned}\Psi_3^M = \frac{1}{2\sqrt{30}}\{&\sqrt{(1+M)(2+M)(3+M)}\,\psi_1^{3/2}\psi_2^{M-1/2} \\ &+ \sqrt{3(2+M)(3+M)(3-M)}\,\psi_1^{1/2}\psi_2^{M-1/2} \\ &+ \sqrt{3(3+M)(2-M)(3-M)}\,\psi_1^{-1/2}\psi_2^{M+1/2} \\ &+ \sqrt{(1-M)(2-M)(3-M)}\,\psi_1^{-3/2}\psi_2^{M+3/2}\}.\end{aligned} \tag{8-43d}$$

We note that the $J = 0$ and $J = 2$ functions are antisymmetric for interchange of the nucleons, so they are excluded for $T' = 0$. It is for this reason that the states of even J have not been included in Table 8-7 for any of the odd-odd nuclei.

The selection between the values $J = 1$ and $J = 3$ for either Li^6 or B^{10} requires further analysis. To that end, let us expand the Li^6 functions Ψ_1^M and Ψ_3^M in a series of Russell-Saunders terms. The available values of S are $S = 0$ and $S = 1$ and, because the configuration is p^2, the L-values are $L = 0,1,2$. The associated ^3S, ^1S, ^3P, etc., functions are uniquely determined in this particular case. If the $\phi_{11}^m(k)$ of the kth nucleon are abbreviated to ϕ_k^m, the orbital functions $\Phi_L^{M_L}$ are*

* These results may be established by noting that the $L = 0$ function has the form $(\mathbf{r}_1 \cdot \mathbf{r}_2)$ (a scalar), the $L = 1$ functions are the components of $\mathbf{r}_1 \times \mathbf{r}_2$, and the $L = 2$ functions are $x_i(1)x_j(2) - \frac{1}{3}r^2\delta_{ij}$. They may also be obtained from Condon and Shortley, *loc. cit.*

$$\Phi_0^0 = -\frac{1}{\sqrt{3}}\sum_m (-1)^m \phi_1^m \phi_2^{-m},$$

$$\Phi_1^{M_L} = \tfrac{1}{2}\{-\sqrt{(1+M_L)(2-M_L)}\,\phi_1^1\phi_2^{M_L-1} + \sqrt{2}\,M_L\phi_1^0\phi_2^{M_L} + \sqrt{(1-M_L)(2-M_L)}\,\phi_1^{-1}\phi_2^{M_L+1}\}, \tag{8-44}$$

$$\Phi_2^{M_L} = \frac{1}{2\sqrt{3}}\{\sqrt{(1+M_L)(2+M_L)}\,\phi_1^1\phi_2^{M_L-1} + \sqrt{2(2-M_L)(2+M_L)}\,\phi_1^0\phi_2^{M_L} + \sqrt{(1-M_L)(2-M_L)}\,\phi_1^{-1}\phi_2^{M_L+1}\}.$$

The $L = 0$ and $L = 2$ functions are symmetric, and the $L = 1$ antisymmetric; hence the restriction $T = 0$ limits attention to the ^3S, ^3D, and ^1P functions. Examination of Eqs. (8–42), (8–43), and (8–44) leads to the relationships

$$\Psi_1^M = \tfrac{1}{3}\{\sqrt{\tfrac{10}{3}}\,{}^{1,3}S_1 - \sqrt{\tfrac{2}{3}}\,{}^{1,3}D_1 - \sqrt{5}\,{}^{1,1}P_1\}, \tag{8-45a}$$

$$\Psi_3^M = {}^{1,3}D_3, \tag{8-45b}$$

where ${}^{1,3}D_1$ is a symbol for the $J = 1$ combination of $\Phi_2^{M_L}$ with the i-singlet and triplet functions, and so on.

A decision between $J = 1$ and $J = 3$ might be reached by now giving consideration to the Wigner coupling scheme. As can be seen in Table 8–4, the lowest supermultiplet of Li^6 leads to either a ^3S or a ^3D term, but the ^1P term belongs to the next supermultiplet. An estimate of the energy differences introduced by a central Majorana potential* is 2.0 Mev for the ^3D-^3S separation and 8.3 Mev for the ^1P-^3S separation. These figures provide us with a basis for assessing the energy difference between the functions (8–45a) and (8–45b). The former contains the ^1P function with a probability of $\frac{5}{9}$ and the ^3D with probability $\frac{2}{27}$; hence the energy difference between Ψ_1^M and the pure ^3S state should be about $\frac{5}{9} \times 8.3 + \frac{2}{27} \times 2.0 = 4.7$ Mev. The Ψ_3^M state, on the other hand, has, on the same basis, an energy only 2.0 Mev higher than the ^3S state. Hence we

* The average values of the energies are given in Table 8–4 in terms of A and K. Evaluation of these integrals can be made on the basis of the central potential of the deuteron problem. The estimates given here are obtained from calculations by A. M. Feingold, Thesis, Princeton University (1952). A reduction in the energy differences would result from the assumption that the central potential is in part an ordinary potential.

conclude that the supplemented j-j coupling scheme would lead to $J = 3$ rather than the observed $J = 1$ for the ground state of Li^6. This apparent contradiction could have to do with the fact that the dissociation of Li^6 into $He^4 + H^2$ requires an energy of only 1.5 Mev. Since the dissociated state at low energy is predominantly a $J = 1$ state, it may perturb the $J = 1$ bound state enough to force it to a position below the $J = 3$ state. Another indication that Eq. (8–45a) does not provide a correct wave function for Li^6 is given by the observed magnetic moment, which is quite different from what would be expected for this function (see Section 9-3).

B^{10} may be subjected to a similar analysis by noting that just four nucleons are required to fill a $p_{3/2}$ shell, so B^{10} lacks one proton in the $p_{3/2}$ proton shell and one neutron in the $p_{3/2}$ neutron shell. Therefore one can speak of adding a neutron and a proton hole to the completed $p_{3/2}$ shell.* The wave functions of the holes have the same form as the Li^6 functions, Eq. (8-43), so the expansions, Eq. (8–45), are again applicable to the $J = 1$ and $J = 3$ states. The difference in energy between these states produced by the Majorana interaction (between the holes) is exactly what has been estimated for Li^6 except for the influence on the integrals A and K of a somewhat larger nuclear radius. Thus the ground state should have** $J = 3$, as it does. In this case there is no nearby continuum as there is in Li^6 to cause a reordering of the levels.

These examples of the odd-odd nuclei serve to illustrate the most complicated and unfavorable aspects of application of the Mayer-Jensen scheme. For other nuclei, and that means the great majority of stable nuclei, the model yields a unique prediction of J after the configuration has been assigned. In particular, the spin of each even-even nucleus is found to be zero in the ground state. The fact that every measurement of the J-value of an even-even nucleus results in $J = 0$ is not easily explained on the basis of any other reasonably well-established model.

In this coupling scheme, many properties of the odd-even nuclei are governed by the wave function of the last odd nucleon. Hence many calculations, such as the determination of the magnetic moment (see Section 9–3), are very simple. This simplicity does not obtain for the calculation of properties, such as the interaction energy, involving a coupling between pairs of nucleons. Then a knowledge of the wave functions of the core composed of an even number of nucleons is required. The terms arising from the functions have been classified by Flowers,† who also calculates

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* For a general discussion of hole theory, see W. Heisenberg, *Ann. der Phys.* **10,** 888 (1931).

** Compare E. Feenberg, *Phys. Rev.* **76,** 1275 (1949).

† *Proc. Roy. Soc.* **A212,** 248 (1952).

the splitting between terms on the basis of a short-range interaction. A detailed calculation of the splitting of the p_j^n configurations in terms of the integrals A and K of Section 8–5 is given by Kurath.*

As we consider other nuclear properties in later chapters, the evidence supporting the Mayer-Jensen coupling scheme will be found to be quite convincing. In spite of this success, we must be very cautious about interpreting the model literally. Most of the evidence is of such a nature as to corroborate the rules for determining the angular momenta and parities of the levels, but there is relatively little information concerning the detailed structure of the wave functions. We have already seen that the short range of the nucleon-nucleon interactions makes the independent particle model an implausible basis for constructing wave functions. A correct nuclear function would be expected to show a high degree of correlation between nucleon pairs. If this "correct" function is expanded in a series of independent particle functions, terms describing many different configurations should appear with appreciable amplitude. Possibly this is the true situation, in which case the success of the Mayer-Jensen coupling scheme simply indicates that it is a method for determining the most important single term in the expanded wave function. The function could still contain many other configurations with a high probability, but the total angular momentum and the other symmetry properties of the states would be fixed in accordance with the coupling scheme.

The possibility that the independent particle model actually provides accurate wave functions would necessitate a fundamental revision of our ideas about nuclear forces. The elimination of violent fluctuations in the interactions between pairs could be accomplished by the introduction of many-body forces. That possibility has also been mentioned in connection with the saturation problem (Section 6–6) and it certainly is not to be excluded. However, much more evidence is needed to justify a step which would invalidate most of our thought processes concerning the many-body problem. What is required is information concerning the purity of the independent particle functions, and the acquisition of such information is one of the important objectives of the present-day physics of complex nuclei. Although from time to time we shall note the existence of some evidence concerning this question, the story is far from complete.

8–7 The spin-orbit coupling. The success of the Mayer-Jensen coupling scheme suggests that spin-orbit coupling plays a major role among the interactions responsible for nuclear binding. We are faced with the problem of fitting an appropriate coupling into a description of nucleon-nucleon potentials. The most reasonable procedure would be to ascribe the effects

* *Phys. Rev.* **88**, 804 (1952).

in question to the tensor interaction. This procedure has not been the usual one for the simple reason that the problem of determining the splitting of levels produced by the tensor potential in complex nuclei is so complicated.* Therefore attempts to account for the spin-orbit coupling have concentrated around the simpler forms of coupling either on the grounds that the average effect of the tensor interaction can be so described or on the assumption that another, velocity-dependent term occurs in the nucleon-nucleon interaction.

There exist substantial arguments to support the view that velocity-dependent spin-orbit terms occur in the interaction. Such terms appear in the Hamiltonian of atomic electrons due to the Larmor magnetic coupling and the Thomas coupling.** Both of these terms can be considered as relativistic corrections of order v^2/c^2 to the Hamiltonian. Similar corrections to nuclear interactions are to be expected,† but the form of the correction depends very much on the transformation properties of the nuclear force field. A scalar field theory leads only to the Thomas type of spin-orbit coupling,†† but a vector theory, which is the analog of the electromagnetic theory, would be expected to contain a Larmor-like term. A general discussion of the v^2/c^2 corrections has been given by Breit.‡ He shows that the spin-orbit coupling for a nucleon moving in a (scalar) central potential $J(r)$ has the form $(\hbar^2/4M^2c^2)(\mathbf{l}\cdot\boldsymbol{\sigma})r^{-1}(dJ/dr)$, where $\boldsymbol{\sigma}$ is the spin operator and $\mathbf{l}$ is the (dimensionless) orbital angular momentum operator of the nucleon. This is, as anticipated, of the order of (T/Mc^2) times the central potential if T is the kinetic energy, so it amounts to a three percent correction. If it were the only spin-orbit coupling term, the Russell-Saunders scheme would be applicable‡‡ to nuclear problems. Of course, a correct formulation of the spin-orbit coupling would relate it directly to the nucleon-nucleon interaction but, as Breit has shown, this does not introduce a substantial change in the order of magnitude of the effect. Hence we conclude that the relativistic spin-orbit term in the Hamiltonian is not large enough to account for the success of the Mayer-Jensen coupling scheme.

The magnitude of the tensor interaction is certainly large enough to admit it as a serious contender for this role. However, the angular de-

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* However, see J. Keilson, *Phys. Rev.* **82,** 759 (1951).

** See, for example, Pauling and Goudsmit, *Structure of Line Spectra*, McGraw-Hill, N. Y. (1930), pp. 58–60.

† D. R. Inglis, *Phys. Rev.* **50,** 783 (1936).

†† W. H. Furry, *Phys. Rev.* **50,** 784 (1936).

‡ *Phys. Rev.* **51,** 248 and 778 (1937); *ibid.* **53,** 153 (1938).

‡‡ Application of the Thomas term as a perturbation has been made to the $(1s)^4(1p)^r$ configurations by Bethe and Rose [*Phys. Rev.* **51,** 205 and 993 (1936)] to determine the nuclear spins and magnetic moments for Russell-Saunders coupling.

pendence is so complicated as to discourage speculation concerning its influence. A detailed treatment by Feingold* of the influence of tensor forces on the lowest configurations of Li^6 and Li^7 shows that a large admixture of higher configurations is produced. The calculated splitting of the lowest multiplet is in rough agreement with observation, as are the calculated magnetic moments. This result suggests that the tensor force may indeed be responsible for all the important spin-orbit coupling effects. However, until a simple procedure for treating the interaction is developed, there will be a tendency to ascribe the coupling to a different interaction of more elementary form.

One can always make the assumption that the nuclear forces include a spin-orbit term other than those which arise naturally from considerations of static potentials. After all, the only justification we can give for limiting attention to static potentials and their relativistic corrections is that they present the simplest analog to electromagnetic interactions. We might consider "potentials" which are velocity dependent in the lowest approximation of relativity theory, i.e., when $1/c \to 0$. It has been remarked that the exchange potential is of this form, in fact, that it involves an infinite series of powers of the nucleon velocity. The spin-orbit coupling, on the other hand, might be just a linear function of the velocity. The number of such two-body potentials is quite limited because the interactions are subject to the conditions of invariance under rotations, inversions, and, presumably, time-reversal. They must be symmetric under the interchange of two nucleons if use is made of the i-spin notation. The six interactions which satisfy these conditions have been described by Eisenbud and Wigner.† In terms of the coordinate difference $\mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k$ and the momentum difference $\mathbf{p}_{jk} = \mathbf{p}_j - \mathbf{p}_k$ of the jth and kth nucleons, they have the form

$$U = \tfrac{1}{4} \sum_{j,k} \{(\mathbf{r}_{jk} \times \mathbf{p}_{jk} \cdot \mathbf{S}_{jk}) T_{jk} + \text{hermitian conjugate}\} J(r_{jk}), \quad (8\text{–}46)$$

where, in the six different cases,

$$\mathbf{S}_{jk} = (\boldsymbol{\sigma}_j + \boldsymbol{\sigma}_k), \quad \begin{cases} T_{jk} = \tfrac{1}{2}[\tau_3(j) + \tau_3(k)], & (8\text{–}47a) \\ T_{jk} = \tfrac{1}{2}[1 + \tau_3(j)\tau_3(k)], & (8\text{–}47b) \\ T_{jk} = \tfrac{1}{2}[1 + \boldsymbol{\tau}_j \cdot \boldsymbol{\tau}_k], & (8\text{–}47c) \\ T_{jk} = \tfrac{1}{2}[1 - \tau_3(j)\tau_3(k)], & (8\text{–}47d) \end{cases}$$

$$\mathbf{S}_{jk} = (\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_k), \quad T_{jk} = \tfrac{1}{2}[\tau_3(j) - \tau_3(k)], \quad (8\text{–}47e)$$

$$\mathbf{S}_{jk} = [\boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_k], \quad T_{jk} = \tfrac{1}{2}[\boldsymbol{\tau}_j \times \boldsymbol{\tau}_k]_3. \quad (8\text{–}47f)$$

* A. M. Feingold, *loc. cit.*

† *Proc. Nat. Acad. Sci.* **27**, 281 (1941). The implications of the time-reversal argument are discussed in Appendix 3.

It is to be noted that all of the six allowed forms of i-spin dependences, as listed in Table 8–1, occur here. According to the arguments presented in Section 8–2, the terms (8–47a), (8–47e), and (8–47f) do not satisfy the mirror condition. They imply that the spin-orbit splitting has the opposite sign for an odd neutron than for an odd proton. In that case the Mayer-Jensen scheme would lead to quite a different behavior for odd N nuclei as compared with odd Z nuclei. Hence these terms can be excluded as possibilities for accounting for the principal part of the spin-orbit coupling.*

Another test for the potentials (8–46) is provided by the electromagnetic interactions of nuclei, since the electromagnetic properties of a system are strongly affected by the presence of velocity-dependent terms in the Hamiltonian, as we shall see in Section 9–2. It turns out that the term (8–47d) would lead to a contribution to the deuteron magnetic moment, the contribution being of the order of 0.25 nuclear magnetons if (8–47d) is large enough to account for Mayer-Jensen coupling. Since we have been able to give a good account of the deuteron moment without reference to this term, it must be excluded. Then we are left with (8–47b) and (8–47c) as the only two-body interactions linear in the velocity which could account for the large coupling. Of these, just (8–47c) is charge invariant, so it seems reasonable to limit attention to that form of the coupling:

$$U = \tfrac{1}{8} \sum_{j,k} (\mathbf{r}_{jk} \times \mathbf{p}_{jk} \cdot [\boldsymbol{\sigma}_j + \boldsymbol{\sigma}_k])[1 + \boldsymbol{\tau}_j \cdot \boldsymbol{\tau}_k] J(r_{jk}). \tag{8–48}$$

One might hope to detect the presence of the term (8–48) in the Hamiltonian by observations on the two-body system. There, the potential becomes

$$U = (\mathbf{L} \cdot \mathbf{S}) P^{\tau} J(r), \tag{8–49}$$

where $\mathbf{L}$ is the orbital and $\mathbf{S}$ the total spin angular momentum operator. Clearly, U vanishes for the S-state, and that is the state for which the most detailed information is available. Only the high-energy scattering is affected by the presence of this term. Unfortunately, the theoretical discussion of high-energy scattering is already complicated by the tensor interaction and the uncertainty in the exchange character of the central potential. A precise treatment of the influence of U on the scattering has not been given,† so no definite decision as to its existence can be made at the time of this writing.

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* Compare Blanchard and Avery, *Phys. Rev.* **81,** 35 (1951).

† Case and Pais, *Phys. Rev.* **80,** 203 (1950), have calculated the very high energy scattering for the interaction (8–49) in first Born approximation. But they find that $J(r)$ must have a strong singularity at the origin to account for observations, and in this case the approximation is quite unreliable. [See Jost and Pais, *Phys. Rev.* **82,** 840 (1951).]

8-8 Saturation conditions. It is appropriate to complete our discussion of the ground states of complex nuclei with some remarks concerning the relationship between exchange forces and the saturation property. The use of an exchange potential was suggested on the grounds that it would lead to saturation, but only qualitative arguments have been presented here in support of that supposition. We have given no indication of the amount of ordinary potential that can be tolerated in the presence of exchange, although the high-energy neutron-proton scattering has established that that question is of major importance.

Under certain conditions it is possible to establish an upper limit on the tolerable amount of ordinary potential.* The argument is based directly on the variational property of the Schroedinger equation. We know that the expectation value of the energy operator in a state described by *any* properly antisymmetrized wave function cannot be less than the energy of the lowest state of the system. Now we shall show that, for a particular wave function, an attractive ordinary potential introduces a term in the average energy proportional to $-A^2$, where A is the number of nucleons. This energy, and hence the binding energy, can therefore be made arbitrarily great in magnitude by choosing A to be sufficiently large. Thus nuclei of arbitrarily large A would be stable and their binding energy would increase at least as rapidly as A^2 (rather than the observed increase as A). This contradiction of the facts can be avoided by introducing a sufficiently strong exchange potential, since this leads to a term in the average energy proportional to A^2, and positive. The minimum amount of exchange potential which will provide the necessary cancellation is then a lower limit on the strength of the potential.

As an illustration, we first present the argument for the simple case of a central two-body potential containing a linear combination of the four exchange operators. The potential is written as

$$V = \tfrac{1}{2}\Sigma'_{ij}[g_0 J_0(r_{ij}) + g_r J_r(r_{ij})P^r_{ij} + g_\sigma J_\sigma(r_{ij})P^\sigma_{ij} + g_\tau J_\tau(r_{ij})P^\sigma_{ij}P^\tau_{ij}], \quad (8\text{–}50)$$

where the potentials $J_\lambda(r)$ are assumed to be everywhere attractive and normalized so that $\lim_{r\to 0}[J_\lambda(r)/J_\mu(r)] = 1$. The constants g, g_r, g_σ, and g_τ are then a measure of the relative strengths of the various terms, and are assumed to satisfy the condition

$$g_0 + g_r + g_\sigma + g_\tau = 1. \quad (8\text{–}51)$$

* The methods used here are substantially those presented by Breit and Feenberg, *Phys. Rev.* **50,** 850 (1936). See also Kemmer, *Nature* **140,** 192 (1937); Feenberg, *Phys. Rev* **52,** 667 (1937); Volkoff, *Phys. Rev.* **62,** 134 (1942); Gerjuoy, *Phys. Rev.* **77,** 568 (1950); H. Kanazawa, *Prog. Theo. Phys.* **6,** 126 (1951).

We choose as a wave function an independent-particle function made up of products of one-nucleon functions

$$\psi_k(j) = \varphi_k(\mathbf{r}_j)\chi_s(j)\eta_t(j), \tag{8–52}$$

where the functions χ_s and η_t are spin and i-spin functions, so s and t can each take on the two values $\pm\frac{1}{2}$. Specification of k fixes the values of s and t as well as the nature of the space function $\varphi_k(\mathbf{r})$.

It is well known* that in a state consisting of an antisymmetrized product of functions $\psi_k(j)$, the expectation value of any two-particle operator of the form $V = \frac{1}{2}\Sigma'_{ij}V_{ij}$ can be written as

$$\langle V\rangle = \tfrac{1}{2}\sum_{k,k'}{}'\sum^{\sigma}\sum^{\tau}\iint \psi_k^*(1)\psi_{k'}^*(2)\,V_{12}(1 - P_{12})\psi_k(1)\psi_{k'}(2)\,d^3r_1\,d^3r_2, \tag{8–53}$$

where the operator P_{12} indicates exchange of the particles labelled 1 and 2. The symbol $\sum^{\sigma}$ denotes the sum over spin variables of nucleons 1 and 2, while $\sum^{\tau}$ denotes the sum over i-spin variables. Let us apply (8–53) to Eq. (8–50):

$$\langle V\rangle = g_0\langle V_0\rangle + g_r\langle V_r\rangle + g_\sigma\langle V_\sigma\rangle + g_\tau\langle V_\tau\rangle, \tag{8–54}$$

with

$$\langle V_0\rangle = \tfrac{1}{2}\sum_{k,k'}{}'\sum^{\sigma}\sum^{\tau}\iint \psi_k^*(1)\psi_{k'}^*(2)\{\psi_k(1)\psi_{k'}(2) - \psi_k(2)\psi_{k'}(1)\} \times J_0(r_{12})\,d^3r_1\,d^3r_2, \tag{8–55a}$$

$$\langle V_r\rangle = \tfrac{1}{2}\sum_{k,k'}{}'\sum^{\sigma}\sum^{\tau}\iint \psi_k^*(1)\psi_{k'}^*(2)P_{12}^r\{\psi_k(1)\psi_{k'}(2) - \psi_k(2)\psi_{k'}(1)\} \times J_r(r_{12})\,d^3r_1\,d^3r_2, \tag{8–55b}$$

$$\langle V_\sigma\rangle = \tfrac{1}{2}\sum_{k,k'}{}'\sum^{\sigma}\sum^{\tau}\iint \psi_k^*(1)\psi_{k'}^*(2)P_{12}^\sigma\{\psi_k(1)\psi_{k'}(2) - \psi_k(2)\psi_{k'}(1)\} \times J_\sigma(r_{12})\,d^3r_1\,d^3r_2, \tag{8–55c}$$

$$\langle V_\tau\rangle = \tfrac{1}{2}\sum_{k,k'}{}'\sum^{\sigma}\sum^{\tau}\iint \psi_k^*(1)\psi_{k'}^*(2)P_{12}^\sigma P_{12}^r\{\psi_k(1)\psi_{k'}(2) - \psi_k(2)\psi_{k'}(1)\} \times J_\tau(r_{12})\,d^3r_1\,d^3r_2. \tag{8–55d}$$

The functions ψ_k are assumed to have the form (8–52), and the shape of the space function $\varphi_k(\mathbf{r})$ may be chosen arbitrarily as far as the variational property is concerned. Let us therefore choose φ_k to be a small normalized wave packet such that different packets do not overlap but all packets lie

* See, for example, Mott and Sneddon, *Wave Mechanics and Its Applications*, Oxford (1948), Chap. VI.

within a fixed volume whose largest dimension is smaller than the range of any one of the potentials $J_\lambda(r)$. Then the overlap terms in the integrals may be set equal to zero and the $J_\lambda(r)$ may be replaced by their common average,* $\bar{J}$, over the region occupied by the packets:

$$\langle V_0\rangle = \frac{A(A-1)}{2}\bar{J}, \tag{8-56a}$$

$$\langle V_r\rangle = -\frac{\bar{J}}{2}\sum_{k,k'}{}'\sum^{\sigma}\sum^{\tau}\chi_s^*(1)\chi_{s'}^*(2)\chi_s(2)\chi_{s'}(1)\eta_t^*(1)\eta_{t'}^*(2)\eta_t(2)\eta_{t'}(1), \tag{8-56b}$$

$$\langle V_\sigma\rangle = \frac{\bar{J}}{2}\sum_{k,k'}{}'\sum^{\sigma}\chi_s^*(1)\chi_{s'}^*(2)\chi_s(2)\chi_{s'}(1), \tag{8-56c}$$

$$\langle V_\tau\rangle = -\frac{\bar{J}}{2}\sum_{k,k'}{}'\sum^{\tau}\eta_t^*(1)\eta_{t'}^*(2)\eta_t(2)\eta_{t'}(1). \tag{8-56d}$$

Use has been made here of the fact that the functions φ_k, χ_s, and η_t are normalized.

The spin and i-spin functions may now be chosen in any convenient manner. We consider these distinct cases:

(a) Equal numbers of one-nucleon states occur with each of the four values $(s,t) = (\frac{1}{2},\frac{1}{2}), (\frac{1}{2},-\frac{1}{2}), (-\frac{1}{2},\frac{1}{2}), (-\frac{1}{2},-\frac{1}{2})$.
(b) All states have $t = \frac{1}{2}$ and equal numbers have $s = \pm\frac{1}{2}$.
(c) All states have $s = \frac{1}{2}$ and equal numbers have $t = \pm\frac{1}{2}$.

Since the spin sums vanish when $s \neq s'$ or when $t \neq t'$, we find in case (a) that $\langle V_r\rangle$ receives contributions from just one-fourth of the terms, while for $\langle V_\sigma\rangle$ and $\langle V_\tau\rangle$ one-half of the terms are different from zero. Thus

$$\langle V\rangle = \frac{A(A-1)}{2}\bar{J}\left[g_0 - \frac{1}{4}g_r + \frac{1}{2}g_\sigma - \frac{1}{2}g_\tau\right], \tag{8-57a}$$

in case (a). In case (b) the terms $\langle V_r\rangle$ and $\langle V_\sigma\rangle$ have half as many non-vanishing terms as $\langle V_0\rangle$ and $\langle V_\tau\rangle$, so

$$\langle V\rangle = \frac{A(A-1)}{2}\bar{J}\left[g_0 - \frac{1}{2}g_r + \frac{1}{2}g_\sigma - g_\tau\right]. \tag{8-57b}$$

Finally, in case (c),

$$\langle V\rangle = \frac{A(A-1)}{2}\bar{J}\left[g_0 - \frac{1}{2}g_r + g_\sigma - \frac{1}{2}g_\tau\right]. \tag{8-57c}$$

* Although $\bar{J}$ will be treated as if it were independent of A, it would be a monotonic increasing function of A for a Yukawa potential or any attractive potential that diverges at the origin. This would only serve to strengthen the saturation arguments given below.

Since J is negative, the contribution of $\langle V \rangle$ to the binding energy increases as A^2 and is positive as long as the coefficient in the square brackets is positive. Since we know that the binding is greater when calculated for the correct wave function, the nonexistence of stable nuclei of great mass imposes the following necessary conditions* on the coefficients g:

$$g_0 - \tfrac{1}{4}g_r + \tfrac{1}{2}g_\sigma - \tfrac{1}{2}g_\tau \leq 0, \tag{8–58a}$$

$$g_0 - \tfrac{1}{2}g_r + \tfrac{1}{2}g_\sigma - g_\tau \leq 0, \tag{8–58b}$$

$$g_0 - \tfrac{1}{2}g_r + g_\sigma - \tfrac{1}{2}g_\tau \leq 0. \tag{8–58c}$$

The coefficients are also subject to the relationship Eq. (8–51).

To complete the argument, a few remarks are needed concerning the kinetic energy, since that reduces the binding. We know that the shape of the wave packet can always be chosen in such a way as to produce a mean momentum inversely proportional to the radius of the packet, or a kinetic energy per nucleon proportional to $v^{-2/3}$, if v is the volume of the packet. Since the range of forces is fixed, the volume of the packet must be in inverse proportion to A in order to fulfill the conditions imposed on the packet for large A. Thus the kinetic energy per nucleon is proportional to $A^{2/3}$ and the total kinetic energy to $A^{5/3}$. Hence the potential energy term overwhelms the kinetic energy at sufficiently large A and the conclusion stands as stated.

The condition (8–58a) means, for example, that if $g_\sigma = g_\tau = 0$, the strength of the ordinary potential must be less than one-fourth the strength of the Majorana exchange potential. This is a necessary condition, but we have not established it as a sufficient condition. Remarkably enough, it *is* a sufficient condition in this special case, as Wigner has shown.† Since the potential in question satisfies the requirements of Wigner coupling, the lowest state of the system can be assigned to a definite supermultiplet. By limiting attention to functions of this class, Wigner establishes the existence of a lower bound for the expectation value of the potential energy, and the lower bound is proportional to A if

$$g_0 - \tfrac{1}{4}g_r \leq 0. \tag{8–59}$$

Hence, under this condition, the binding energy can increase no more rapidly than the number of nucleons. This is just the saturation condition. Wigner's argument therefore shows that the Majorana potential is

* The coulomb interaction causes a slight modification of these conditions but, since the coulomb energy is also proportional to A^2, no substantial change is introduced as long as e^2/α is small compared with J. Here α is the range of the potential of shortest range.

† *Proc. Nat. Acad. Sc.* **22,** 662 (1936). A limitation on the shape of $J(r)$ is required for the proof, but the allowed forms are quite inclusive.

capable of producing saturation. Furthermore, the condition (8–59) is both necessary and sufficient for that purpose. Wigner produces a similar argument for the case $g_\sigma = g_\tau = 0$, in which circumstance

$$g_0 - \tfrac{1}{2}g_\tau \leq 0 \tag{8–60}$$

is a necessary and sufficient condition for saturation. Note that the spin-dependent Heisenberg potential is just twice as effective as the Majorana potential in producing saturation. Although the sufficiency of the conditions (8–58) has been established only for these two special cases, the fact that it can be established at all suggests that none of the conditions is exceedingly weak compared with the sufficient conditions.

Since tensor forces seem to occur in great strength, it is important to determine the extent to which they modify the saturation conditions.* Let us consider the effect of adding to the potential (8–50), the tensor term†

$$V^t = \tfrac{1}{2}{\sum_{ij}}'[\gamma_0 J_0^t(r_{ij}) + \gamma_\tau J_\tau^t(r_{ij})P_{ij}^\tau]S_{ij}, \tag{8–61}$$

where

$$S_{ij} = \frac{3(\boldsymbol{\sigma}_i\cdot\mathbf{r}_{ij})(\boldsymbol{\sigma}_j\cdot\mathbf{r}_{ij})}{r_{ij}^2} - (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j).$$

The shape functions $J_\lambda^t(r)$ are also assumed to satisfy the condition $\lim_{r\to 0}[J_\lambda^t(r)/J_\mu(r)] = 1$. Furthermore,

$$\gamma = \gamma_0 + \gamma_\tau, \tag{8–62}$$

where γ is the ratio of the tensor strength to central strength. The equivalent of Eqs. (8–56) for the tensor interaction is

$$\langle V_0^t\rangle = \tfrac{1}{2}\bar{J}{\sum_{k,k'}}'\sum^\sigma \chi_s^*(1)\chi_{s'}^*(2)S_{12}(k,k')\chi_s(1)\chi_{s'}(2), \tag{8–63a}$$

$$\langle V_\tau^t\rangle = -\tfrac{1}{2}\bar{J}{\sum_{k,k'}}'\sum^\sigma\sum^\tau \chi_s^*(1)\chi_{s'}^*(2)S_{12}(k,k')$$
$$\times \chi_s(2)\chi_{s'}(1)\eta_t^*(1)\eta_{t'}^*(2)\eta_t(2)\eta_{t'}(1). \tag{8–63b}$$

Here, the notation $S_{12}(k,k')$ indicates that $\mathbf{r}_{12}$ is replaced by the vector $\mathbf{r}_{kk'}$ connecting the centers of the packets $\varphi_k,\varphi_{k'}$.

Of the three cases treated above, case (c) is the easiest to discuss because, when all the spins are parallel, the spin sums are simply replaced by $(3\cos^2\theta_{kk'} - 1)$, where $\theta_{kk'}$ is the angle between $\mathbf{r}_{kk'}$ and the axis of quan-

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* A proof that tensor forces alone cannot lead to saturation has been given by Volkoff, *loc. cit.*

† Since the tensor potential vanishes in singlet states, the introduction of spin exchange operators leads, in effect, to no new terms in V^t.

tization. The appearance of this quantity makes necessary a decision concerning the relative orientations of the various packets. Two extreme cases are to be distinguished:

Case (c′): all packets lie within a long ellipsoid along the axis of quantization. Then $(3\cos^2\theta_{kk'} - 1) \approx 2$.

Case (c″): all packets lie on a flat ellipsoid normal to the axis. In this case $(3\cos^2\theta_{kk'} - 1) \approx -1$.

The rest of the argument goes as before, and when the tensor terms are added to the central potential terms and use is made of Eq. (8–62), the following necessary conditions for saturation replace Eq. (8–58c):

$$g_0 - \tfrac{1}{2}g_\tau + g_\sigma - \tfrac{1}{2}g_\tau - 3\gamma_r + 2\gamma \leq 0, \tag{8–64}$$

$$g_0 - \tfrac{1}{2}g_r + g_\sigma - \tfrac{1}{2}g_\tau + \tfrac{3}{2}\gamma_r - \gamma \leq 0. \tag{8–65}$$

In cases (a) and (b), half of the spins are parallel to a given one, and half are antiparallel. For parallel spins, the spin sums again lead to the factor $(3\cos^2\theta_{kk'} - 1)$. For antiparallel spins, Eqs. (8–63a) and (8–63b) require separate treatment, but for both the spin sums lead to the factor $-(3\cos^2\theta_{kk'} - 1)$. Thus, as long as the two spin directions have the same spatial distribution, the contribution of the tensor term vanishes, so it seems reasonable to conclude that the tensor interaction will introduce no saturation problem for antiparallel spins, and the conditions (8–58a), (8–58b) remain unaffected by its presence.

To complete the discussion of the tensor interaction it is necessary to include a case (d) for which both the spins and i-spins are all parallel. Then one finds the additional necessary condition for saturation*

$$g_0 - g_r + g_\sigma - g_\tau + 2\gamma_r - \gamma \leq 0. \tag{8–66}$$

The inequalities (8–58a), (8–58b), (8–64), (8–65), and (8–66) are necessary conditions for saturation in the presence of tensor forces.

It is of interest to apply the conditions to several special cases. If there are no central potentials, Eqs. (8–64), (8–65), and (8–66) reduce to

$$\gamma_r/\gamma \geq \tfrac{2}{3},$$

$$\gamma_r/\gamma \leq \tfrac{2}{3},$$

$$\gamma_r/\gamma \leq \tfrac{1}{2},$$

respectively. Since these conditions are mutually exclusive, we conclude that a very large value of γ, i.e., a situation in which the central interaction is quite small, is not consistent with the saturation requirement.

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* The other condition obtained in this way is taken care of by Eq. (8–64) as long as $\gamma \geq 0$, by virtue of the relationship (8–51). For $\gamma = 0$, Eq. (8–66) is weaker than Eq. (8–65) so it, too, can be ignored.

A quantitative formulation of this result can be obtained by considering the most favorable situation for saturation, that is, a central potential entirely of the Majorana or Heisenberg type, $g_r + g_\tau = 1$, $g_0 = g_\sigma = 0$. Then Eqs. (8–64) (8–65), and (8–66) yield

$$\tfrac{1}{3}(2\gamma - \tfrac{1}{2}) \leq \gamma_r \leq \begin{cases} \tfrac{1}{3}(2\gamma + 1), \\ \tfrac{1}{2}(\gamma + 1). \end{cases} \tag{8–67}$$

This set of conditions requires that $(2\gamma - \frac{1}{2}) \leq \frac{3}{2}(\gamma + 1)$ or

$$\gamma \leq 4, \tag{8–68}$$

which is the desired upper limit on the strength of the tensor potential. Equation (8–68) provides a third indication that, for potentials of the Yukawa shape, the range of the tensor interaction must be greater than that of the central interaction. Otherwise the values of γ which are in accord with the properties of the ground state of the deuteron, and correspond to the best estimates of the range of the central potential, exceed the limit (8–68), as will be seen on examination of the second table of Appendix 2.

Equation (8–67) can also be used to place limits on the exchange dependence of the tensor interaction. If attention is limited to positive values of γ_0 and γ_r, we find that a lower limit is placed on γ_r/γ as long as $\gamma > \frac{1}{4}$. There is a significant upper limit on γ_r/γ only if $\gamma > 1$. These conditions can be applied to the results given in Table 6–2. For that purpose we note that γ is defined differently in the table; that value must be multiplied by α_T/α_c to yield our present value. The conditions placed on γ_r/γ for the five cases listed in the table are presented in Table 8–8.

TABLE 8–8

Limits on the exchange dependence of the tensor interaction for the cases listed in Table 6–2

$k_0\alpha_c$	α_T/α_c	Condition on γ_r/γ
0.313	1.02	$0.58 \leq \gamma_r/\gamma \leq 0.77$
0.275	1.29	$0.51 \leq \gamma_r/\gamma \leq 0.96$
0.256	1.39	$0.50 \leq \gamma_r/\gamma \leq 0.99$
0.224	2.86	....
0.224	2.20	$0.22 \leq \gamma_r/\gamma$

Finally, we come to the important question: Is the Serber exchange operator, or an operator of nearly that form, consistent with the saturation property? To secure an answer to this question, use is made of the properties of the $^1S_0$ state of the deuteron. The relationship between the singlet and triplet potentials takes the form

$$g_0 + g_r - g_\sigma - g_\tau = h^2, \tag{8–69}$$

where h^2 is the ratio of the strength of the singlet potential to the strength of the *central* triplet potential. If K^2 is the depth and α_c the range of the central potential, Eq. (4–41) leads to the result

$$h^2 = 1.68/K^2\alpha_c^2 \tag{8–70}$$

for a Yukawa potential, while, for a square well,

$$h^2 \approx (\pi/2K\alpha_c)^2,$$

according to Eq. (4–40).

The strength of the nonexchange potentials is measured by $g_0 + g_\sigma$, while that of the space-exchange potentials is measured by $g_r + g_\tau$. The Serber exchange potential is obtained by setting the two equal. Let us denote their ratio by β:

$$g_r + g_\tau = \beta(g_0 + g_\sigma). \tag{8–71}$$

Now Eqs. (8–51), (8–69), and (8–71) can be solved for g_r, g_τ, and g_σ in terms of g_0, h^2, and β. The result is

$$\begin{aligned} g_\sigma &= \frac{1}{1+\beta} - g_0, \\ g_r &= \tfrac{1}{2}(h^2 + 1) - g_0, \\ g_\tau &= \frac{\beta}{1+\beta} - \frac{1}{2}(h^2 + 1) + g_0. \end{aligned} \tag{8–72}$$

These results may now be inserted into the inequalities (8–58a), (8–58b), (8–64), (8–65), and (8–66) to yield:

$$\beta \geq (5 + h^2 + 2g_0)/(3 - h^2 - 2g_0), \tag{8–73a}$$

$$\beta \geq (3 + h^2)/(3 - h^2), \tag{8–73b}$$

$$\frac{\beta - 2}{\beta + 1} \geq 2(2\gamma - 3\gamma_r), \tag{8–73c}$$

$$\frac{\beta - 2}{\beta + 1} \geq -(2\gamma - 3\gamma_r), \tag{8–73d}$$

$$\frac{\beta - 1}{\beta + 1} \geq -(\gamma - 2\gamma_r). \tag{8–73e}$$

Since the right side of (8–73d) has the opposite sign from (8–73c), one of the two must be positive unless both vanish. Hence

$$\beta \geq 2. \tag{8–74}$$

This is the only definite conclusion offered by the last three inequalities in the absence of information concerning γ_r/γ.

An evaluation of the conditions (8–73a) and (8–73b) requires an estimate of h^2. This can be obtained from Eq. (8–70) on the basis of the values of K^2 tabulated in Appendix 2 for a Yukawa potential. These values are such that

$$h^2 \geq 0.8,$$

so that Eqs. (8–73a) and (8–73b) become

$$\beta \geq (5.8 + 2g_0)/(2.2 - 2g_0)$$

and

$$\beta \geq 1.7.$$

Because of (8–74), the second condition may be ignored. The first condition is stronger than (8–74) for $g_0 > -0.23$, but otherwise (8–74) may be taken as the limit on the amount of nonspace-exchange potential. Thus we conclude that under the most favorable conditions, at least 67 percent of the central potential must contain a space-exchange operator, and that situation can occur only in the unlikely circumstance that $g_0 < -0.23$. The Serber potential is clearly excluded by this result, since it is a fifty-fifty mixture of ordinary and exchange forces.

In addition to the above general remarks concerning them, the conditions (8–58a), (8–58b), (8–64), (8–65), and (8–66) can be used to test any specific potential. For example, the Yukawa potential, Eqs. (6–57) and (6–58), violates the first three of these necessary conditions for saturation, hence it cannot provide a complete description of nuclear forces. This result is particularly interesting because it has been indicated that the potential is capable of accounting for the binding energies of the three- and four-body systems.

8–9 Excited states of complex nuclei. From the experimental point of view, excited states of nuclei fall into two quite distinct classes. The bound excited states constitute one class, the virtual states the other. Bound states are the analog of the familiar excited states of an atomic system; their observation is usually accomplished by the detection of radiation emitted in a transition between states or by means of an inelastic scattering process. On the other hand, the existence of virtual states is manifest through the processes of resonance reaction or resonance scattering, processes which occur when a nuclear particle of given energy impinges on a nucleus. The existence of an abundance of easily observed, discrete states of the latter kind seems to be a characteristic of nuclear systems.

On the whole, no distinction between the two classes of excited states is to be made insofar as a discussion of nuclear structure is concerned. Every

state is characterized by certain quantum numbers such as the total angular momentum and parity. Charge invariance should be applicable in the same degree to both sets of states, so T is expected to be as good a quantum number for the one set as for the other. There is some quantitative difference between the two classes of levels. The position of a virtual level may be affected differently from that of a bound level by coulomb forces because the one involves unbound coulomb waves, the other bound coulomb waves.* Furthermore, we must account, in terms of nuclear structure, for the near stability of the virtual levels, since they must be nearly stable to form a discrete spectrum.

The implications of charge invariance are quite clear-cut. Each level of a nucleus is characterized by some value, T, of the total i-spin, and a level of given T occurs in $2T + 1$ isobars. This fact provides, on the one hand, a method for determining T, and on the other hand, a test of the charge-invariance hypothesis. A search can be made for levels in isobars that correspond to a level in a given nucleus. These corresponding levels cannot be expected to have exactly the same energy, since the coulomb interaction causes the energy to increase with increasing t, while the neutron-proton mass difference leads to a decrease in energy, but at a different rate. However, an estimate of the energy differences can easily be made, and we can thereby hope to identify corresponding levels in isobars.† If no corresponding levels are found for the level in a nucleus with $N = Z$, then it can be concluded that $T = 0$. The occurrence of corresponding levels with $N = Z + 2$ and $N = Z - 2$, but for no other isobars, implies that $T = 1$. The absence of any correspondence between levels would imply that the assumption of charge invariance is faulty. The occurrence of a correspondence is clearly illustrated for the observed levels of the nuclei with $A = 14$, as shown in Fig. 8–2. The ground level of N^{14} has $T = 0$ since, after being corrected for the coulomb energy and n-p mass difference, the ground states of O^{14} and C^{14} are well above the N^{14} state. But the 2.32 Mev level of N^{14} corresponds closely to these ground states; hence we surmise that it has $T = 1$. Further light would be shed on this question by a determination of the parities and J-values of the levels, since these must be the same for corresponding levels.

The lower levels of the odd A nuclei usually offer a test of the mirror property rather than the more general property of charge invariance. The minimum T is $T = \frac{1}{2}$, so at least a mirror pair with $t = \pm\frac{1}{2}$ should occur

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* J. B. Ehrman, *Phys. Rev.* **81,** 412 (1951); R. G. Thomas, *Phys. Rev.* **88,** 1109 (1952).

† The coulomb effect is expected to be about the same for all the low-lying levels, since no great change of charge distribution should occur in going from one of these levels to another. However, some care must be taken in applying this rule, since there is the above-mentioned difference between the coulomb effect on a bound level and that on a virtual level.

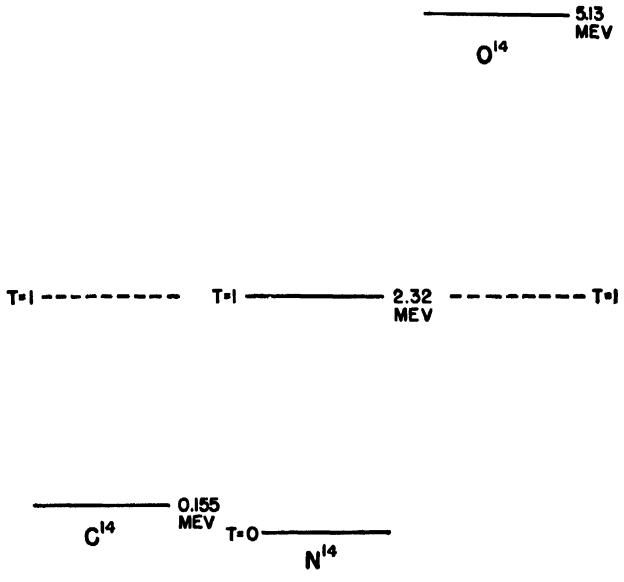

FIG. 8-2. Illustration of the correspondence between levels of isobaric nuclei of even A. The actual positions of the levels are shown by the solid lines,* while the dashed lines indicate the positions after corrections have been made for the coulomb repulsion and the n–p mass difference.†

for every level. The existence of a correspondence between the levels of mirror nuclei provides evidence for the mirror property of the interactions, that is, the invariance of the interactions for a rotation by π about either the 1- or 2-axis in charge space. The detection of a correspondence between levels of the four isobars with $N = Z \pm 3$, $N = Z \pm 1$ would indicate $T = \frac{3}{2}$ and would constitute further evidence for charge invariance. An example of the observed correspondence between the levels of mirror nuclei is illustrated in Fig. 8–3.

The prediction of the positions and properties of excited states requires the use of a model, and a severe test of a model may result from a comparison of the predicted and observed states. In particular, it should be possible to draw some conclusion concerning the coupling scheme from the properties of the several lowest states, since each coupling scheme suggests a characteristic pattern for these states. Consider, for example, the case of Wigner coupling. The supermultiplet to which the ground state is assigned is quite definite. But the supermultiplet is presumed to be split by a small perturbation and the ground state is just the lowest of the terms in the group. Since the coupling scheme requires that this splitting be

* F. Ajzenberg and T. Lauritsen, *Rev. Mod. Phys.* **24**, 321 (1952).
† T. Lauritsen, *Ann. Rev. of Nuc. Sc.* **1**, 67 (1952).

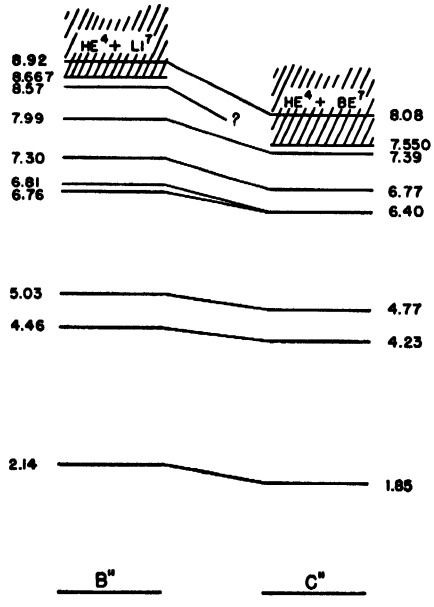

FIG. 8-3. Illustration of the mirror principle for odd A. The positions of the levels are given relative to the ground state in each case.* The shift in position of corresponding levels is presumed to be due to the closer proximity of the continuum in C^{11}. The close correspondence between the relative probabilities for production of these levels by the $B^{10}(d,p)B^{11}$ and $B^{10}(d,n)C^{11}$ reactions indicates that they are true mirror levels.†

small compared with the spacing between other levels, the first few excited states of the system are members of the lowest supermultiplet. Therefore their spin, i-spin, and the space symmetry of the wave function are fixed by the coupling scheme, and these qualities are subject to experimental test.

More definite statements can be made if attention is focused on the independent particle model. Then the lowest group of levels belongs to a given configuration. In the Wigner coupling scheme, differences in energy arise through differences in symmetry and through spin-spin coupling, as indicated in Table 8–4. A further splitting of the levels would be caused by spin-orbit coupling. Thus the lowest state of Li^7 would be ${}^{2,2}P_{3/2}$, the next state ${}^{2,2}P_{1/2}$, the third and fourth states ${}^{2,2}F_{7/2,5/2}$. Just how far this analysis could be carried depends on the position of the next supermultiplet. Levels of the lower supermultiplet which come close to or cross over the bottom level of the higher supermultiplet can be expected to lose their identity since, for them, the Wigner coupling condition is not satisfied. The basic theoretical information needed to check this point can be obtained from an analysis of the second supermultiplet given in the paper by Feenberg and Phillips.‡

Mayer-Jensen coupling leads to a somewhat different selection of levels. The occurrence of a low level may be associated with any one of three sources. First of all, the combining of the angular momenta of an even number of like nucleons to form angular momenta different from zero will lead to states other than the ground state. The energies of these states depend on the strength of the j-j coupling. There are indications§ that this coupling is rather weak, therefore the levels in question may occur

* Ajzenberg and Lauritsen, *loc. cit.*
† V. R. Johnson, *Phys. Rev.* **86**, 306 (1952).
‡ *Loc. cit.*
§ M. Goldhaber and A. W. Sunyar, *Phys. Rev.* **83**, 906 (1951).

near the ground state. Other low states belong to those configurations which would form the ground state if there were not the tendency of single nucleon states having large l and j to cross over the states of low l. For example, when the number of nucleons is between 32 and 50, levels formed by placing an odd nucleon in the $p_{1/2}$, $f_{5/2}$, or $g_{9/2}$ state are close together. Thus, for odd A nuclei, states differing greatly in angular momentum may occur near the ground state. An excellent confirmation of this conclusion is offered by observations on isomeric transitions, as will be discussed in detail in Section 9–6.

The third source of excitation is the spin-orbit coupling. The ground state of an odd A nucleus is, according to the Mayer-Jensen scheme, a member of a doublet. In the light nuclei, the other member of the doublet should not be too distant from the ground state, but in the heavier nuclei the doublet splitting is large, possibly of the order of 2 Mev.* In a light nucleus like Li^7 the ground state has the neutron configuration $(p^2_{3/2})_0$, of total (neutron) angular momentum zero, while the odd proton is in the $p_{3/2}$ state. The first excited state may be produced by changing the proton to the $p_{1/2}$ state, in which case $J = \frac{1}{2}$. On the other hand, if the j-j coupling between neutrons is small enough, the first excited state might be produced by recoupling the neutrons to form the $(p^2_{3/2})_2$ configuration. This is the only other possibility allowed by the Pauli principle, as can be seen from Eqs. (8–43). Then the coupling between neutrons and protons must be invoked to determine the order of the states $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$.

As an example of the behavior of a heavier nucleus, we might consider Sr^{87}, since that has 49 neutrons, just one short of a closed shell. The ground state is a $g_{9/2}$ state; there is a $g_{9/2}$ hole in the closed shell. But the close competition between the $g_{9/2}$ shell and the $p_{1/2}$ shell has the consequence that the $p_{1/2}$ level is only 394 kev above the ground state.†

In general, it is difficult to predict the order of the levels without having more specific information concerning the magnitudes of the spin-orbit coupling, j-j coupling between like particles, and the j-j coupling between unlike particles. A better idea of the form of an adequate auxiliary potential is also required, since this provides the starting point for giving a level order. In the present state of affairs the procedure has been to correlate observations on the low levels with the shell model, rather than to attempt detailed predictions of what the level order should be.

The higher states of excitation of a nucleus offer an even more formidable problem because the number of available degrees of freedom increases with increasing energy. This can be seen clearly on the basis of the independent-particle model; configurations other than the ground state configuration may be obtained by radial excitation (increasing n) or by increasing l. And one or more nucleons may be promoted in this way. Then for each con-

* M. G. Mayer, *Phys. Rev.* **78**, 16 (1950).

† See Goldhaber and Sunyar, *loc. cit.*

figuration there are many states corresponding to the various values of the total angular momentum. Clearly, the complexity of this description increases with increasing energy. It also increases with increasing number of nucleons. The increase in complexity means that the spacing between levels decreases, since one configuration involving excitation of a number of nucleons can be expected to have an energy close to that of another, similar configuration. The net result is then that the number of excited states increases rapidly with increasing A, and the spacing between levels decreases with increasing energy.

The independent particle model is even less likely to be applicable to the higher excited states than it is to the lower states, since the small spacing in energy between configurations will lead to an increase in the mixing of configurations. However, the number and distribution of states will be substantially the same at high excitation for any model, since a mixing of functions does not change the number of linearly independent functions. It will merely cause some reordering of energy values, a reordering that is not likely to change our qualitative conclusions concerning the density of states.*

There is a property of the highly excited states which does depend strongly on the model, and it provides us with a very helpful means for deciding between various models. The property in question concerns the virtual states of nuclei. By a virtual state is meant a state of the system with energy greater than the energy of dissociation into one or more sets of nuclear products. Hence a virtual state is expected to be unstable. The degree of instability is very closely related to the nuclear model, hence it is the property which is to be considered.

Consider, for example, states of the nucleus (A,Z) having sufficient energy to allow the emission of a single neutron. In the pure independent-particle model many such states are stable as a result of the possibility of exciting two or more nucleons to a total energy greater than the 6 to 8 Mev required to dissociate a neutron. Excitation of a single neutron to that energy would, of course, result in dissociation; the state would be part of a continuum. But as long as each nucleon in a state obtained by multinucleon excitation has less than that energy, the system is perfectly stable in the pure form of the model. The influence of any perturbation which mixes different configurations would be likely to cause instability of the state by mixing in some of the continuum neutron function. The degree of instability is then a direct measure of the amount of mixing of the two configurations.

The level in question can be studied by bombarding the nucleus $(A - 1,Z)$ with a neutron. According to the pure independent-particle model, the neutron will merely undergo scattering by the auxiliary poten-

* See H. A. Bethe, *Rev. Mod. Phys.* **9**, 69 (1937), for estimates of the density of states based on the statistical model and liquid drop model.

tial. But if there is any mixing of the almost stable level formed by many-nucleon excitation with the free neutron state then, when the neutron has an energy close to that level, it may be captured, with the consequent formation of the excited state of the "compound nucleus" (A,Z). The subsequent behavior of the system will depend on the relative probabilities for the various processes whereby it can lose its excitation energy, processes such as emission of gamma radiation, emission of a neutron, or emission of some other nuclear particle. This example should make it clear that virtual nuclear levels may be investigated by means of nuclear reactions which show resonances at the positions of the levels. However, it must be emphasized that *not every* virtual level will necessarily be observed this way. An appreciable coupling between the level and the free-particle system is required to allow detection of the presence of the level. Thus one may obtain entirely the wrong impression of the density of virtual levels if it is assumed to equal the density of observed resonance levels.

The reaction processes are of such importance in the study of complex nuclei that a great deal of attention has been directed to the determination of the form of the resonance curve and the interpretation thereof. These matters are the concern of Chapter 10, wherein the question of the purity of the independent-particle functions will again be discussed.

The characteristics of the bound states of complex nuclei are determined by means of observations on emitted beta or gamma radiations, or by certain nuclear reaction processes such as inelastic scattering or the stripping process. Just as in the case of nuclear reactions, the interpretation of radiative processes requires an elaborate analytical machinery. Chapters 9 and 11 are largely devoted to the development of that machinery for the electromagnetic and beta transitions, respectively.

CHAPTER 9

ELECTROMAGNETIC INTERACTIONS OF NUCLEI

From experience with atomic problems, we know that the electromagnetic interactions of a system provide a remarkably useful tool for investigating its structure. The reason for the success of this method is, in part, that the essential features of electromagnetic theory are well understood. Another very important reason is the small magnitude of the interaction between material systems and the electromagnetic field; it is of the order of $e^2/\hbar c = \frac{1}{137}$, so a system may be probed for information via this interaction without being greatly disturbed. The smallness of the interaction also has the consequence that simplification of the calculations by an expansion in powers of the electromagnetic field variables is a safe procedure for the treatment of most problems.

The investigation of nuclei by means of electromagnetic fields leads to two somewhat different kinds of information. Static magnetic or electric fields are used to determine magnetic moments and electric quadrupole moments, which are properties of the nuclear ground states. Dynamical electromagnetic fields, on the other hand, cause transitions between states of the nucleus, therefore they yield information concerning the relative properties of different nuclear states. This information takes the form of a transition probability or radiative lifetime if both of the involved states are bound, and it takes the form of a photodisintegration or capture cross section if one of the nuclear states happens to lie in the continuum.

The primary concern of this chapter is to develop the methods whereby nuclear moments or radiative transition probabilities may be interpreted in terms of nuclear structure.

9–1 Multipole moments and transitions. Selection rules. Radiative transitions in atomic systems are usually classified as "allowed" or "forbidden." This distinction is possible because the linear dimension (R) of the atom is very small compared with the wavelength (c/ω) of the radiation. Hence an expansion of the interaction with the electromagnetic field in powers of ($\omega R/c$) leads to a rapidly converging series. The leading term in the series is referred to as the electric dipole term; it is responsible for all allowed transitions. The intensity of a forbidden transition is smaller than that of an allowed transition by some power of $(\omega R/c)^2$, so it is negligibly small. Since intensity of the radiation is the observable of atomic and molecular spectroscopy, the forbidden transitions are rarely observed, as their name indicates.

The quantity $(\omega R/c)$ is small for most of the interesting nuclear transitions too, but the forbidden transitions nevertheless play an extremely important role in nuclear spectroscopy. One reason for this difference is the simple fact that at nuclear energies the observed quantity is transition rate rather than intensity. It is far easier to observe a low transition rate directly than to observe it by an intensity measurement. Of course, other reasons also exist; for example, there is the fact that forbidden transitions frequently occur between the very lowest nuclear states, so that their occurrence is not obscured by overlapping allowed transitions. It must be remembered, too, that collisions rapidly remove an atomic system from a state which would otherwise be long-lived, while the nucleus is thoroughly protected from such external influences.

Since the forbidden transitions play an important role in nuclear problems, it is necessary to treat them in some detail. This means that the interaction of the nucleus with the electromagnetic field is to be developed in powers of $(\omega R/c)$ and that each term in the series is to be examined to determine under just what physical conditions it becomes important. The classification by powers of $(\omega R/c)$ turns out not to be quite sufficient; the terms must also be classified by their parity. Then each term can be associated with either an electric or a magnetic multipole moment, the order of the moment being determined by the power of $(\omega R/c)$.

The actual multipole moments are operators depending on the coordinate and momentum operators of the individual nucleons in the nucleus. Since all relevant matrix elements of these operators are limited in magnitude by the nuclear dimension R, an expansion in powers of $(\mathbf{r}_i \cdot \boldsymbol{\omega}/c)$ is the equivalent of the above-mentioned expansion if $\mathbf{r}_i$ is the displacement of the ith nucleon from the center of mass of the nucleus. To illustrate how this expansion leads in a natural way to the multipole moment operators, let us consider the special case of a spinless charged particle moving under the influence of an ordinary potential. We know that the transition probability is in this case proportional to the absolute square of the matrix element of the interaction $e(\mathbf{A}(\mathbf{r}) \cdot \mathbf{p})$, where $\mathbf{r}$ and $\mathbf{p}$ are the coordinate and momentum operators of the particle, e is its charge, and $\mathbf{A}$ is the vector potential of the plane wave describing the photon emitted in the transition:

$$\mathbf{A}(\mathbf{r}) \sim \mathbf{u} \exp\,(-i\boldsymbol{\omega} \cdot \mathbf{r}/c),$$

if $\mathbf{u}$ is the unit polarization vector and $\boldsymbol{\omega}/c$ is the propagation vector of the photon. The matrix element is taken between an initial state (i) of the particle and a final state (f), the states having energy E_i and E_f, respectively. Then we also have

$$\hbar\omega = E_i - E_f, \tag{9-1}$$

if attention is limited to emission of the photon.

As long as the wavelength of the radiation is large compared with the linear extent of the wave functions describing states i and f, the effective interaction

$$I = e \exp(-i\boldsymbol{\omega}\cdot\mathbf{r}/c)(\mathbf{u}\cdot\mathbf{p})$$

may be expanded in powers of $(\boldsymbol{\omega}\cdot\mathbf{r})/c$. For our illustrative purpose, attention is restricted to the first two terms in the expansion:

$$I = e(\mathbf{u}\cdot\mathbf{p}) - ie(\boldsymbol{\omega}\cdot\mathbf{r})(\mathbf{u}\cdot\mathbf{p})/c + \cdots. \tag{9-2}$$

The matrix element of the second term is quite small compared with the first in normal circumstances, therefore only the contribution of the first is considered unless it happens to vanish. The relationship $\mathbf{p} = M\dot{\mathbf{r}}$ shows that the first term is proportional to the time derivative of $D = e(\mathbf{u}\cdot\mathbf{r})$, the $\mathbf{u}$-component of the electric dipole moment. Thus the leading term in the interaction

$$I_1 = M\dot{D} \tag{9-3}$$

is the electric dipole term. As a consequence of the Heisenberg relation

$$\langle i|\, \dot{D}\, |f\rangle = (i/\hbar)(E_i - E_f)\langle i|\, D\, |f\rangle \tag{9-4}$$

and of Eq. (9–1), the dipole term in the interaction may also be taken to be

$$I_1 = i\omega MD. \tag{9-5}$$

The selection rules for the electric dipole transition are well known. They are statements of the relationship which must hold between the angular momentum and parity of the initial and final states in order that the matrix element of (9–5) be different from zero. If the states i and f are not so related, the transition is said to be forbidden, and the determination of its rate requires that the second or higher terms in the series Eq. (9–2) be considered. The second term may be written as

$$I_2 = -(ie/2c)\{[(\boldsymbol{\omega}\cdot\mathbf{r})(\mathbf{u}\cdot\mathbf{p}) + (\mathbf{u}\cdot\mathbf{r})(\boldsymbol{\omega}\cdot\mathbf{p})] + [(\boldsymbol{\omega}\cdot\mathbf{r})(\mathbf{u}\cdot\mathbf{p}) - (\mathbf{u}\cdot\mathbf{r})(\boldsymbol{\omega}\cdot\mathbf{p})]\}. \tag{9-6}$$

If use is again made of $\mathbf{p} = M\dot{\mathbf{r}}$, the first term of I_2 is found to be simply $(-i\omega M/6c)\dot{Q}_{\omega u}$, where

$$Q_{\omega u} = 3e(\boldsymbol{\omega}\cdot\mathbf{r})(\mathbf{u}\cdot\mathbf{r})/\omega. \tag{9-7}$$

Since $\mathbf{u}$ is orthogonal to $\boldsymbol{\omega}$, $Q_{\omega u}$ is clearly an off-diagonal element of the electric quadrupole moment tensor Eq. (3–48). It is a linear combination of the spherical harmonics Y_2^m, so it will have a nonvanishing matrix element only when the magnitude of the angular momentum change is less than 3 and when the parity does not change. Thus, if $\Delta J = \pm 2$ and the parities of states i and f are the same, the emission of a photon will occur by means of an electric quadrupole transition.

The second part of I_2 can be rewritten as

$$I_2^m = -i\omega M\mathfrak{M}, \tag{9-8}$$

where

$$\mathfrak{M} = (e\hbar/2Mc)(\boldsymbol{\omega} \times \mathbf{u}\cdot\mathbf{L})/\omega$$

if $\mathbf{L}$ is the orbital angular momentum operator expressed in units of $\hbar$. Hence $\mathfrak{M}$ is simply the component in the direction $\boldsymbol{\omega} \times \mathbf{u}$ of the orbital magnetic moment. The corresponding emission of a photon occurs via a magnetic dipole transition. In this case the selection rule is $\Delta J = 0, \pm 1$, *no* (no change of parity) since the components of $\mathbf{L}$ behave like the P-functions under rotations but are invariant under inversion. Hence the distinction from the electric dipole selection rule lies in the parity condition. If states i and f do not differ in parity and $\Delta J = 0, \pm 1$, the transition is forbidden in the usual sense, but it can occur with a relatively long lifetime by emission of a mixture of magnetic dipole and electric quadrupole radiation.

The distinction between the magnetic dipole and electric quadrupole transition is of considerable interest. Note that both terms evidently arise in the same order of $(\boldsymbol{\omega}\cdot\mathbf{r})/c$. However, the magnetic dipole interaction, Eq. (9–8), has the same form and frequency dependence as the electric *dipole* interaction, Eq. (9–5). The ratio of the two is not of order $(\boldsymbol{\omega}\cdot\mathbf{r})/c$ but, rather, of the order $\hbar/Mcr$, a quantity which is independent of frequency. As a matter of fact, $\hbar/Mcr \approx v/c$, where v is the velocity of the particle, and we see that the difference in rate between magnetic and electric transitions is measured by the magnitude of relativistic corrections to the motion of the particle. On the other hand, the electric quadrupole term is smaller than the electric dipole by the amount $(\boldsymbol{\omega}\cdot\mathbf{r})/c$. Since the latter quantity is smaller than v/c for most nuclear transitions in which we will be interested, the magnetic dipole transition will usually be much stronger than the electric quadrupole transition when both are allowed by the selection rules.

The same qualitative relationships hold between the electric and magnetic multipole moments of any order. The 2^l-pole magnetic moment arises in conjunction with the 2^{l+1}-pole electric moment insofar as the original expansion (9–2) is concerned, but it leads to a transition probability having the same frequency dependence as the 2^l-pole electric moment. The magnetic and electric 2^l-pole moments differ in magnitude by a factor of the order v/c. Selection rules for the two transitions differ only insofar as the parity is concerned; they always require opposite parity changes.

With this simple illustration in mind, we now turn to a general treatment of the multipole transitions in a nucleus.* Specific assumptions concerning

* This treatment follows that given by Sachs and Austern, *Phys. Rev.* **81,** 705 (1951), but some of the details are omitted here for the sake of simplicity.

the interaction between the nucleons and electromagnetic field must be avoided, since the probable existence of exchange currents suggests that the complete form of the interaction is still to be discovered. However, it is assumed that the interaction, and the dynamics of the nucleus, can be completely described in terms of nucleon position and spin variables. The possibility that the role of the mesons may need to be expressed explicitly is simply not taken into account. Therefore our formulation will certainly fail at very high photon energies, values of the energy at which pion photoproduction is known to occur. But for low photon energies we can at least hope that the meson effects are adiabatic, i.e., that the mesons move so fast compared with the nucleons that a good approximation is obtained by averaging over their motion. In that case, only the nucleon variables would remain explicit in the dynamical description of the system.

In the presence of a vector potential $\mathbf{A}(\mathbf{r})$ due to external sources, the Hamiltonian of the system is denoted by $H\{\mathbf{A}\}$, where the curly bracket is meant to indicate that H is not a simple function of $\mathbf{A}$, but that it has a more complicated dependence on the field.* We shall take H to be the Hamiltonian of internal motion of the nucleus, so that it contains only internal nucleon variables such as the displacement $\mathbf{r}_j$ of the jth nucleon from the center of mass. This has the consequence that the coordinate of the center of mass commutes with all nucleon variables appearing in H, hence it may be treated as a number rather than as an operator for our purposes. In this sense the origin of coordinates may be taken to coincide with the center of mass. If account is taken of the motion of the center of mass, it is such as to conserve the total linear momentum of the nucleus and photon.

Because the coupling with the electromagnetic field is weak, it is very useful to introduce the expansion

$$H\{\mathbf{A}\} = H_0 + H_1\{\mathbf{A}\} + \frac{1}{2!} H_2\{\mathbf{A}\} + \cdots, \tag{9–9}$$

in which H_1 depends linearly on $\mathbf{A}$, H_2 quadratically, and so on. H_0 is the nuclear Hamiltonian in the absence of an electromagnetic field. Certain restrictions on the form of $H_n\{\mathbf{A}\}$ exist as a consequence of the requirement that the currents flowing within the nucleus must satisfy an equation of continuity. These restrictions are easily formulated by making use of the general property of gauge invariance of the Hamiltonian, a property that is closely related to the equation of continuity.† If $G(\mathbf{r})$ is an arbitrary

* A function of $\mathbf{A}$ would be a function of the point $\mathbf{r}$, while H depends on the values of $\mathbf{A}$ at a number of specific points such as the positions of the protons. In this case H is said to be a "functional" of $\mathbf{A}$.

† For a discussion of the role of gauge invariance in quantum mechanics, see H. Weyl, *The Theory of Groups and Quantum Mechanics*, Methuen and Co. Ltd., London (1931), p. 100.

space function and $\mathbf{F}(\mathbf{r})$ an arbitrary vector field, the gauge property of $H\{\mathbf{A}\}$ may be expressed by the equation

$$H\{\mathbf{F} + \text{grad } G\} = e^{iD/\hbar c} H\{\mathbf{F}\} e^{-iD/\hbar c}, \tag{9–10}$$

where

$$D = \Sigma_j e_j G(\mathbf{r}_j), \tag{9–11}$$

e_j being the charge of the jth nucleon. Since H involves the momenta of the nucleons, the expression on the right side of Eq. (9–10) amounts to a complicated unitary transformation of $H\{\mathbf{F}\}$. The transformation can be expressed as an infinite series of commutators by means of the relationship*

$$e^{iD/\hbar c} H e^{-iD/\hbar c} = H + \frac{i}{\hbar c}[D,H] + \frac{1}{2!}\left(\frac{i}{\hbar c}\right)^2 [D,[D,H]] + \cdots,$$

where $[D,H]$ denotes the usual commutator $[DH - HD]$. If this result is now applied to Eq. (9–10) in the special case $\mathbf{F} = 0$, there results

$$H\{\text{grad } G\} = H_0 + \frac{i}{\hbar c}[D,H_0] + \frac{1}{2!}\left(\frac{i}{\hbar c}\right)^2 [D,[D,H_0]] + \cdots.$$

This may be compared with the series (9–9) which, in the present case, reads

$$H\{\text{grad } G\} = H_0 + H_1\{\text{grad } G\} + \frac{1}{2!} H_2\{\text{grad } G\} + \cdots.$$

The results of the comparison are the equations

$$H_1\{\text{grad } G\} = \frac{i}{\hbar c}[D,H_0], \tag{9–12}$$

$$H_2\{\text{grad } G\} = \left(\frac{i}{\hbar c}\right)^2 [D,[D,H_0]], \tag{9–13}$$

and similar relationships for the higher $H_n\{\text{grad } G\}$.

Only the term $H_1\{\mathbf{A}\}$ plays a role in the determination of the transition probability for the emission or absorption of a single photon. Since H_1 is linear in $\mathbf{A}$, Eq. (9–12) may be generalized to

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* A simple proof of this relationship can be obtained by expanding $O(\alpha) = e^{\alpha f} H e^{-\alpha f}$ in powers of α: $O(\alpha) = O(0) + \alpha\left(\frac{dO}{d\alpha}\right)_{\alpha=0} + \frac{\alpha^2}{2!}\left(\frac{d^2O}{d\alpha^2}\right)_{\alpha=0} + \cdots$, where α is a number which commutes with H. Since $\left(\frac{dO}{d\alpha}\right)_{\alpha=0} = [f,H]$, etc., we have $O(\alpha) = H + \alpha[f,H] + \frac{\alpha^2}{2!}[f,[f,H]] + \cdots$.

$$H_1\{\mathbf{F} + \text{grad } G\} = H_1\{\mathbf{F}\} + \frac{i}{\hbar c}[D,H_0].$$

Furthermore, use may be made of the Heisenberg relationship for the time derivative to replace this equation by

$$H_1\{\mathbf{F} + \text{grad } G\} = H_1\{\mathbf{F}\} - \dot{D}/c. \tag{9–14}$$

The transition probability for the emission of radiation is proportional to the square of the matrix element of $H_1\{\mathbf{A}\}$ between an initial state consisting of the nucleus in state i and a final state consisting of the nucleus in state f plus a photon of given momentum and polarization. The contribution of the radiation field to the matrix element is taken into account* when the vector $\mathbf{A}$ appearing in the interaction is replaced by a properly normalized plane wave $\mathbf{A}_{u,\omega}$ describing the emitted photon of polarization $\mathbf{u}$ and propagation vector $\boldsymbol{\omega}/c$. If the wave is normalized to a large enclosure of volume V, it turns out that

$$\mathbf{A}_{u,\omega} = c(2\pi\hbar/\omega V)^{1/2}\mathbf{u} \exp(-i\boldsymbol{\omega}\cdot\mathbf{r}/c).$$

The transition probability is now given by the usual expression

$$dw = \frac{2\pi}{\hbar}|\langle i|H_1\{\mathbf{A}_{u,\omega}\}|f\rangle|^2 d\rho_f,$$

where the matrix element is taken with respect to the initial and final states of the nucleus. The density of final states, $d\rho_f$, is determined by the density of photon states:

$$d\rho_f = [\omega^2 V/(2\pi)^3\hbar c^3]\, d\Omega$$

if the propagation vector of the emitted photon lies in the element of solid angle $d\Omega$. Since H_1 depends linearly on $\mathbf{A}$, the constant coefficients of $\mathbf{A}_{u,\omega}$ may be extracted from the matrix element, with the result

$$dw = (\omega/2\pi\hbar c)|\langle i|Q|f\rangle|^2 d\Omega, \tag{9–15}$$

if Q is the operator

$$Q = H_1\{\mathbf{u} \exp(-i\boldsymbol{\omega}\cdot\mathbf{r}/c)\}. \tag{9–16}$$

Thus Q determines the rates of all transitions.

The multipole expansion is obtained by expanding the exponential in Eq. (9–16) in powers of $(\boldsymbol{\omega}\cdot\mathbf{r})/c$. The required form of the expansion is†

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* See W. Heitler, *The Quantum Theory of Radiation*, 2nd edition, Oxford (1947), p. 94.

† This expansion can easily be established by noting that $\mathbf{u} \exp(-i\mathbf{k}\cdot\mathbf{r}) = \text{grad}[(\mathbf{u}\cdot\mathbf{r})\int_0^1 \exp(-iy\mathbf{k}\cdot\mathbf{r})\,dy] + i[(\mathbf{u}\times\mathbf{k})\times\mathbf{r}]\int_0^1 y \exp(-iy\mathbf{k}\cdot\mathbf{r})\,dy$. Equation (9–17) is obtained when the exponentials are expanded and integrated term by term.

$$\mathbf{u} \exp(-i\boldsymbol{\omega}\cdot\mathbf{r}/c) = \sum_{l=1}^{\infty} (-i\omega/c)^{l-1}\{\text{grad } G_l + (i/c)[(\mathbf{u}\times\boldsymbol{\omega})\times\mathbf{r}]W_l\}, \tag{9-17}$$

with

$$G_l = (\mathbf{u}\cdot\mathbf{r})(\boldsymbol{\kappa}\cdot\mathbf{r})^{l-1}/l!, \tag{9-18}$$

and

$$W_l = l(\boldsymbol{\kappa}\cdot\mathbf{r})^{l-1}/(l+1)!, \tag{9-19}$$

where $\boldsymbol{\kappa} = \boldsymbol{\omega}/\omega$ is the unit vector in the direction of propagation. When this expansion is inserted into Eq. (9–16), we find

$$Q = \sum_{l=1}^{\infty} (-i\omega/c)^{l-1} H_1\{\text{grad } G_l + (i/c)[(\mathbf{u}\times\boldsymbol{\omega})\times\mathbf{r}]W_l\},$$

as a consequence of the linearity of H_1. Equation (9–14) then leads to the result

$$Q = (1/c)\sum_{l=1}^{\infty} (-i\omega/c)^{l-1}[iH_1\{[(\mathbf{u}\times\boldsymbol{\omega})\times\mathbf{r}]W_l\} - \dot{D}_l], \tag{9-20}$$

where, according to Eqs. (9–11) and (9–18),

$$D_l = (1/l!)\Sigma_j e_j(\mathbf{u}\cdot\mathbf{r}_j)(\boldsymbol{\kappa}\cdot\mathbf{r}_j)^{l-1}. \tag{9-21}$$

Clearly, for $l = 1$ this is just the $\mathbf{u}$-component of the electric dipole moment of the nucleus, while for $l = 2$ it is, within the multiplicative constant* $\frac{1}{6}$, the appropriate element of the electric quadrupole moment tensor, Eq. (3–48). We shall refer to D_l as the electric 2^l-pole moment operator of the system, since it is an element of a symmetric tensor of rank l and parity $(-1)^l$. Note that the interaction is proportional to the time derivative $\dot{D}_l$, just as was the case for our simple example treated at the beginning of the section. Use can again be made of the Heisenberg relation

$$\langle i|\,\dot{D}_l\,|f\rangle = i\omega\langle i|\,D_l\,|f\rangle$$

to calculate the matrix elements appearing in Eq. (9–15). For all practical purposes, Eq. (9–20) can therefore be replaced by

$$Q = \sum_{l=1}^{\infty} (-i\omega/c)^l(D_l + \mathfrak{M}_l), \tag{9-22}$$

where

$$\mathfrak{M}_l = -[l/(l+1)!]H_1\{[(\mathbf{u}\times\boldsymbol{\kappa})\times\mathbf{r}](\boldsymbol{\kappa}\cdot\mathbf{r})^{l-1}\}. \tag{9-23}$$

* It is conventional to use the normalization of Eq. (3–48) for defining static quadrupole moments, while the present normalization is much more natural for arbitrary l. Unfortunately, all the data on static quadrupole moments are tabulated on the basis of the unnatural, conventional definition, so we must maintain this inconsistency.

In the special case of a particle moving under the influence of an ordinary potential, it is easily shown that $\mathfrak{M}_1$ is the component in the direction $\mathbf{u} \times \boldsymbol{\kappa}$ of the usual magnetic dipole moment. In general,* $\mathfrak{M}_l$ is an element of a tensor of rank l, but it has the opposite parity from D_l, that is, it has the parity $(-1)^{l+1}$. Thus $\mathfrak{M}_l$ will be referred to as the magnetic 2^l-pole moment operator.

Because the series (9–22) converges rapidly, only the leading term need be considered in a calculation of the transition probability. The nature of that term is determined by the selection rules. Since D_l and $\mathfrak{M}_l$ are tensors of rank l, they transform under rotations like linear combinations of spherical harmonics of order l and lower. Thus their matrix elements between states i and f of the nucleus must vanish unless the difference, ΔJ, between the total angular momenta of these states has one of the values

$$\Delta J = \pm l, \pm(l-1), \cdots, \pm 1, \tag{9–24}$$

or

$$\Delta J = 0 \quad \text{for} \quad J \neq 0.$$

Furthermore, a nonvanishing matrix element occurs for D_l only if the parity change between i and f is $(-1)^l$, while the opposite parity change is required to give a nonvanishing contribution from $\mathfrak{M}_l$. The leading term in Eq. (9–22) capable of contributing to the matrix element is therefore determined by the angular momentum change in the transition. If $\Delta J = \lambda$, then either D_λ or $\mathfrak{M}_\lambda$ will be responsible for the transition, the choice between the two being governed by the parity change. Contributions to the transition can also arise from the D_l and $\mathfrak{M}_l$ with l larger than λ, but these contributions are so small as to be negligible. No contributions can arise from the D_l or $\mathfrak{M}_l$ with $l < \lambda$ because of the selection rules (9–21). In the special case $\Delta J = 0$, D_1 will be responsible for the transition if the parity changes, $\mathfrak{M}_1$, otherwise.

When $\Delta J = l$ and the parity change is $(-1)^l$, the emission of electric 2^l-pole radiation occurs with a transition probability which may be found from Eqs. (9–15) and (9–22) to be

$$dw_l(\text{elect}) = (\omega^{2l+1}/2\pi\hbar c^{2l+1}) \left| \langle i | D_l | f \rangle \right|^2 d\Omega. \tag{9–25}$$

If, on the other hand, the parity change is $(-1)^{l+1}$, magnetic 2^l-pole radiation is emitted at the rate

$$dw_l(\text{mag}) = (\omega^{2l+1}/2\pi\hbar c^{2l+1}) \left| \langle i | \mathfrak{M}_l | f \rangle \right|^2 d\Omega. \tag{9–26}$$

The magnitude of each of these rates depends on the structure of the nuclear wave functions and on the form of the operators D_l and $\mathfrak{M}_l$. The operators D_l are given explicitly by Eq. (9–21), but the form of $\mathfrak{M}_l$ is

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* For a proof that this statement is independent of the form of H_1, see Sachs and Austern, *loc. cit.*

implicit in Eq. (9–23). The interaction $H_1\{\mathbf{A}\}$ must be given before the $\mathfrak{M}_l$ can be determined. We consider here the special case of ordinary (velocity-independent) forces. Then $H\{\mathbf{A}\}$ has the form:

$$H\{\mathbf{A}\} = (4AM)^{-1}\sum_{j,k}\left[\mathbf{p}_j - \mathbf{p}_k - \frac{e_j}{c}\mathbf{A}_j + \frac{e_k}{c}\mathbf{A}_k\right]^2 + U - (e\hbar/2Mc)\sum_j \mu_j(\boldsymbol{\sigma}_j\cdot\operatorname{curl}\mathbf{A}_j), \quad (9\text{–}27)$$

where U is the potential, $\mathbf{p}_j$ is the momentum of the jth nucleon, $\mathbf{A}_j = \mathbf{A}(\mathbf{r}_j)$, $\mu_j = \mu_p$ for a proton and $\mu_j = \mu_n$ for a neutron, and A is, as usual, the number of nucleons. The term linear in $\mathbf{A}$ is

$$H_1\{\mathbf{A}\} = -(4AMc)^{-1}\sum_{j,k}\{([\mathbf{p}_j - \mathbf{p}_k]\cdot[e_j\mathbf{A}_j - e_k\mathbf{A}_k]) + ([e_j\mathbf{A}_j - e_k\mathbf{A}_k]\cdot[\mathbf{p}_j - \mathbf{p}_k])\} - (e\hbar/2Mc)\sum_j \mu_j(\boldsymbol{\sigma}_j\cdot\operatorname{curl}\mathbf{A}_j).$$

If this form of H_1 is inserted into Eq. (9–23), the magnetic multipole moment is found to be

$$\mathfrak{M}_l = (e\hbar/2Mc)\frac{1}{(l-1)!}\left\{\frac{1}{l+1}\sum_\pi [(\mathbf{u}\times\boldsymbol{\kappa}\cdot\mathbf{l}_\pi)(\boldsymbol{\kappa}\cdot\mathbf{r}_\pi)^{l-1} + (\boldsymbol{\kappa}\cdot\mathbf{r}_\pi)^{l-1}(\mathbf{u}\times\boldsymbol{\kappa}\cdot\mathbf{l}_\pi)] + \sum_j \mu_j(\mathbf{u}\times\boldsymbol{\kappa}\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\kappa}\cdot\mathbf{r}_j)^{l-1}\right\}. \quad (9\text{–}28)$$

Here, the index π denotes a proton variable and $\mathbf{l}_\pi$ is the orbital angular momentum operator of the πth proton.* This will be referred to as the *ordinary* moment, since it does not contain the complicating effects of exchange currents or velocity-dependent potentials. These will be discussed in the next section.

9–2 Exchange effects. If we accept the widely held view that nuclear forces are caused by interaction of the nucleons with mesons, the fact that the mesons are charged leads to the conclusion that there are meson currents in nuclei. The nature of these currents can, at least in principle, be investigated by means of a close study of the interaction of a nucleus with the electromagnetic field.† Conversely, any discussion of the electromagnetic properties of nuclei must take the possible existence of such currents

* Note that $\mathbf{l}_j = \mathbf{r}_j \times (\mathbf{p}_j - A^{-1}\Sigma_k\mathbf{p}_k)$ is the orbital angular momentum with respect to the center of mass.

† The direct approach from the point of view of a field theory of nuclear forces is taken by A. J. F. Siegert, *Phys. Rev.* **52,** 787 (1937); Lamb and Schiff, *Phys. Rev.* **53,** 651 (1938); Møller and Rosenfeld, *Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd.* **20,** No. 12 (1943); A. Pais, *ibid.*, No. 17 (1943). The phenomenological method is used here since it is simpler and allows much more freedom than any one form of the meson theory.

into account. This problem is important because it brings to light a method for quantitative study of the mechanism of nuclear forces, a method which is largely independent of direct measurements on the forces.

If they are to be detected, the meson currents must lead to effects other than those that can be accounted for by the currents associated with the motion of the nucleons. Such effects will be referred to as *exchange effects* because they are so strongly suggested by the notion that nuclear forces are caused by the exchange of mesons between nucleons. An exchange effect is expected to depend in some way on the interactions between nucleons, therefore, in general, it will not have the simple additive property which is usually associated with electric currents occurring in systems described by the Schroedinger equation. For example, the contribution of the exchange current to the magnetic moment would not be expected to satisfy the principle of additivity. These deviations from additivity are one powerful means for establishing the existence of the effect. However, it should be kept in mind that the discovery of a nonadditive effect does not in itself establish the existence of meson currents in nuclei since, as we shall see below, any velocity-dependent potential has associated with it a nonadditive (anomalous) current. Nevertheless, in either case the anomalous current is directly associated with the nuclear interaction, and a knowledge of its quantitative properties may help in the construction of a theory of nuclear forces.

When the problem of analyzing nuclear data in terms of interactions first arose, it was found necessary to introduce a number of assumptions concerning the form of the interaction. Similar assumptions must be made concerning the exchange effects, and they are made in the same form and in the same spirit. The simplest imaginable conditions are imposed until circumstances (i.e., nuclear data) force us to abandon them. The simplifying assumptions are as follows:

(1) Exchange effects can be described directly in terms of nucleon variables. This means that meson variables do not appear explicitly. Such would be the case in the adiabatic approximation mentioned in Section 9–1.

(2) Operators describing exchange effects involve only the coordinates and spins of the nucleons. This excludes velocity-dependent terms, in accord with the view that the potentials are velocity independent.

(3) Exchange effects are two-body effects. Operators representing exchange effects are, according to this assumption, to be expressed as a sum of operators involving just pairs of nucleons. This assumption would certainly have to be abandoned if a many-body nuclear potential were found to be required.

(4) Exchange effects have a short range. Since they are so intimately related to the interaction, the operator describing an exchange effect

between a pair of nucleons probably vanishes when the distance between the nucleons is appreciably greater than the range of nuclear forces.

(5) The exchange current has the property that it changes sign under the mirror operation on all i-spins. Since the mirror operation sends the deuteron into itself, this requirement guarantees that there will be no exchange contribution to the magnetic moment of the deuteron. If such a contribution exists, indications are that it is quite small.

Under these assumptions, only the Hamiltonian $H\{\mathbf{A}\}$ of the nucleus in the presence of the external field $\mathbf{A}(\mathbf{r})$ need be given in order to derive the exchange effects. $H\{\mathbf{A}\}$ will include a term of the form Eq. (9–27), which describes effects due to the additive (ordinary) currents, and terms of the form $\frac{1}{2}\Sigma'_{ij}U\{\mathbf{r}_{ij},\mathbf{A}\}$ which describe the influence on the interaction with the electromagnetic field when the displacement $\mathbf{r}_{ij}$ between a pair of nucleons is changed. The short-range character of the effect is established by taking U to have the appropriate range as a function of $\mathbf{r}_{ij}$.

We are now in a position to make some remarks concerning the possibility of observing these effects. First of all, we note that the magnetic multipole moment operators, Eq. (9 23), depend explicitly on the form of $H_1\{\mathbf{A}\}$, and consequently will contain exchange terms when $U_1\{\mathbf{r}_{ij},\mathbf{A}\}$, the linear part of $U\{\mathbf{r}_{ij},\mathbf{A}\}$, does not vanish. The exchange moments appear as operators added to the ordinary moment operator, Eq. (9–28). Therefore one test for the existence of a term U in the Hamiltonian would be to compare the observed magnetic dipole moments with those calculated on the basis of the ordinary moment operator. Another test would be provided by a comparison of the observed rates of magnetic multipole transitions with the rate predicted on the assumption of additivity. We shall find that application of these tests to the available data leads to the conclusion that U_1 is different from zero and even gives some indication of the quantitative form of the interaction U_1.

At first thought it appears likely that the electric moments would also be affected by exchange processes. However, the moment operators are given by Eq. (9-21) quite independently of the form of $H\{\mathbf{A}\}$. This result, that *the electric multipole moment operators are independent of exchange effects*, is known as the *Siegert theorem.** Its importance lies in the

* Siegert, *loc. cit.*, suggested this theorem for the dipole term on the basis of qualitative arguments of the sort given at the end of the section. He also tested it by the adiabatic approximation to the beta-decay theory of nuclear forces, the only available theory of the time. Later, Lamb and Schiff and Møller and Rosenfeld, *loc. cit.*, showed that some forms of the meson theory were, in adiabatic approximation, in accord with the theory. The general proof of the theorem as stated here was given by Sachs and Austern, *loc. cit.* Before Siegert's work, concern was expressed about the validity of a calculation of the electric dipole transition rate in, say, the photodisintegration of the deuteron when exchange forces were causing the electric charge to shift around. See Condon and Breit, *Phys. Rev.* **49**, 904 (1936).

fact that we can feel reasonably secure when making predictions concerning electric moments or electric multipole transition probabilities even in the absence of a full knowledge of the exchange effects. Any observed, clear-cut contradiction of this result would indicate that the coupling of a nucleus with the electromagnetic field does involve the meson coordinates explicitly.*

Although the Siegert theorem is clearly established by the formal analysis of Section 9–1, there may be some point in a somewhat more qualitative discussion of the matter. How does it happen that the electric moments are not affected by the nuclear interactions, while the magnetic moments may be very strongly affected? The answer lies in the fact that the electric moments depend on the charge distribution, while the magnetic moments are determined by both the charge and current distributions. To the extent that Schroedinger mechanics gives a good description of the nucleus, the charge density is uniquely defined in terms of $|\psi|^2$, therefore the relationship between charge distribution and wave function is independent of the interactions. But the current density must be defined in such a way that the equation of continuity is a consequence of the time-dependent Schroedinger equation. When it is recognized that the latter equation involves the interactions directly, it should not be surprising to find that the connection between current distribution and wave function is affected by the interactions.

The Siegert theorem does not imply that the rate of electric multipole transitions is unaffected by the presence of exchange forces and other interactions which correspond to charge exchange. On the contrary, such interactions have a profound influence on the wave functions which must be used to calculate the matrix element in Eq. (9–25). It might seem that the theorem is therefore virtually useless. This is not the case, because one is often willing to (in fact, usually must) guess the form of a nuclear wave function without making detailed reference to the interactions. The theorem then establishes that a direct test of the wave function can be made by comparing calculated and observed electric multipole moments or transition rates. The magnetic multipole moments provide no such test unless all of the exchange contributions are known at the outset.

The form of the Hamiltonian $H\{\mathbf{A}\}$ in the presence of the electromagnetic field remains to be determined. There is a standard prescription of the Schroedinger theory for determining $H\{\mathbf{A}\}$ from H_0, the Hamiltonian in the absence of the field, namely, to replace every momentum operator $\mathbf{p}_j$ which appears in H_0 by $\mathbf{p}_j - (e_j/c)\mathbf{A}(\mathbf{r}_j)$, where e_j is the charge on the jth particle. Equation (9–27) was obtained by just such a procedure. This prescription guarantees that the condition of gauge invariance, Eq. (9–10),

* Compare F. Villars, *Phys. Rev.* **86**, 476 (1952).

will be satisfied, and it is a first step in the construction of $H\{\mathbf{A}\}$. However, momentum-independent terms involving curl $\mathbf{A}$ may appear in $H\{\mathbf{A}\}$ without affecting the gauge condition, and these are not prescribed by the form of H_0. The existence of such terms can only be established by recourse to experiment. An example is the spin-exchange term which is discussed in the next section.

Any momentum-dependent terms appearing in H_0 must be modified in accordance with the standard prescription in order to maintain the gauge invariance. In particular, the occurrence of a velocity-dependent interaction in the Hamiltonian will require modification of the current density. This is the reason for the association of the space-exchange current, Eq. (4-10), with the space-exchange potential which, it is to be recalled, can be formulated as a velocity-dependent interaction. For the same reason, any of the spin-orbit coupling terms, Eqs. (8-47), lead to an addition to the current.* Therefore if such terms are responsible for the success of the Mayer-Jensen coupling scheme, then they might be detectable through the associated effect on the magnetic multipole moments.† It is to be noted that since these currents are directly associated with the proton motion, they do not necessarily satisfy the mirror condition (5) which we plan to impose on any moment not derived directly from H_0.

9-3 Nuclear magnetic moments. Although our discussion of the multipole moment operators has been based on the nonstatic radiative processes, the same operators determine the static electromagnetic properties of nuclei. The only difference is that the static moments are given by the diagonal matrix elements (expectation values) of the operators, while the dynamical effects depend on the nondiagonal elements. The static moments of interest, i.e., those which are readily observable, are the magnetic dipole and electric quadrupole moments of nuclei. The former play an exceptionally useful role in the study of nuclear structure because, for one thing, they are measured very exactly. The numerical values of magnetic moments are given with higher precision than any nuclear property other than the mass. On this basis alone, a concentrated effort to understand the theoretical implications of the moments would be justified. Such a program is further encouraged by the fact that magnetic moments are so simply related to the nuclear structure. We have noted such simple connections in the discussion of the deuteron moment, but it has also been found there that certain small corrections, for example the relativistic corrections, mar the beauty of the analysis. Exchange moments also

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* Blanchard, Avery and Sachs, *Phys. Rev.* **78,** 292 (1950).

† The contributions to magnetic multipole moments due to space-exchange and to spin-orbit coupling are obtained by Austern and Sachs, *Phys. Rev.* **81,** 710 (1951).

would appear as corrections but, as in all such cases, we try to take advantage of the new information provided by these corrections rather than mourn over lost simplicity.

The calculation of nuclear moments will be made first on the basis of the ordinary moment operator, Eq. (9–28). Subsequent comparison with experiment will put us in a position to discuss the need for introducing exchange terms. In the i-spin notation, the ordinary moment operator. measured in units of $e\hbar/2Mc$, takes the form

$$\mathfrak{M} = \sum_k \tfrac{1}{2}[1 + \tau_3(k)]\mathbf{l}_k + \mu_+\mathbf{S} + \tfrac{1}{2}\mu_- \sum_k \tau_3(k)\boldsymbol{\sigma}_k, \qquad (9\text{–}29)$$

where $\mathbf{l}_k$ is the orbital angular momentum operator of the kth nucleon, $\mathbf{S} = \frac{1}{2}\Sigma_k\boldsymbol{\sigma}_k$ is the *total* spin angular momentum operator, μ_+ is the sum of neutron and proton moments,

$$\mu_+ = 0.87975, \qquad (9\text{–}30)$$

and μ_- is the difference of the moments,

$$\mu_- = 4.70535. \qquad (9\text{–}31)$$

The numerical value of the moment is the expectation value $\langle\mathfrak{M}_z\rangle$ of the z-component of (9–29) calculated for that state of the system which has the maximum magnetic quantum number, $M = J$. The calculation is straightforward if the wave function is known, but there is not a general expression for the moment of a nucleus comparable in simplicity to the Landé g-factor for an atom because the nuclear moment depends on the distribution of angular momentum between the neutrons and protons. A nuclear model is therefore a prerequisite for the calculation of the moment.

Despite this complication, the nuclear moments satisfy one relationship which has exactly those simple features characteristic of the atomic moments.* This is a relationship between the moments of mirror nuclei, and it is valid only in the approximation that the forces have the mirror property. Mirror nuclei are defined as a pair of isobars having the values t and $-t$ for the third component of the i-spin. In other words, one has neutrons where the other has protons. If the forces have the mirror property, the wave functions of the two nuclei are closely related; in fact, the function ψ_{-t} can be obtained from ψ_t by changing the i-spin state of each nucleon in ψ_t to the state of opposite i-spin. But this means that the expectation value of any operator proportional to $\tau_3(k)$ has opposite signs in the two states, that is to say, in the pair of mirror nuclei. Hence the sum of the moments of the mirror nuclei has the property that there is a cancellation

* R. G. Sachs, *Phys. Rev.* **69**, 611 (1946).

between the contributions arising from terms proportional to $\tau_3(k)$ in Eq. (9–29). This sum is therefore

$$\langle \mathfrak{M}_z \rangle_t + \langle \mathfrak{M}_z \rangle_{-t} = \langle L_z + 2\mu_+ S_z \rangle,$$

where $\mathbf{L}$ is the total orbital angular momentum operator. The expectation value on the right side may be taken in either the state ψ_t or the state ψ_{-t}. The result can be rewritten in terms of the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ as

$$\langle \mathfrak{M}_z \rangle_t + \langle \mathfrak{M}_z \rangle_{-t} = \left(\frac{1}{2} + \mu_+\right) J + \frac{1}{J+1}\left(\frac{1}{2} - \mu_+\right) \langle \mathbf{L}^2 - \mathbf{S}^2 \rangle \quad (9\text{–}32)$$

by making use of the usual vector rule

$$\langle L_z - S_z \rangle_M = \langle (\mathbf{L} - \mathbf{S}) \cdot \mathbf{J} \rangle M / J(J+1).$$

It is to be noted that this mirror theorem is valid even if a large exchange moment must be included; at least it is valid as long as the exchange term satisfies the mirror condition (5) suggested in Section 9–2. For under that condition the exchange moment in state ψ_{-t} is equal and opposite to that in ψ_t, so the net contribution to the sum of the moments vanishes.

If the nuclear ground state can be properly fitted into the Russell-Saunders coupling scheme, the result, Eq. (9–32), is particularly simple, since the expectation values of $\mathbf{L}^2$ and $\mathbf{S}^2$ are then known to be just $L(L+1)$ and $S(S+1)$, respectively. Although we do not expect this coupling scheme to apply in general, it is always possible to expand the wave function in terms of the states ψ_{LS} of given L and S:

$$\psi = \sum_{L,S} C_{LS} \psi_{LS}. \quad (9\text{–}33)$$

Then Eq. (9–32) may be expressed directly in terms of the probability coefficients $|C_{LS}|^2$ for the occurrence of the various states. If the numerical value, Eq. (9–30), is inserted for μ_+, the result is

$$\langle \mathfrak{M}_z \rangle_t + \langle \mathfrak{M}_z \rangle_{-t} = 1.380J - \frac{0.380}{J+1} \sum_{L,S} [L(L+1) - S(S+1)] |C_{LS}|^2. \quad (9\text{–}34)$$

Equation (9–34) can be used to calculate the sum of the moments on the basis of any model which specifies the coefficients $|C_{LS}|^2$. An alternative procedure is to use the equation as a means for determining the probabilities $|C_{LS}|^2$ from measured moments. The latter procedure requires that a limitation be placed on the number of terms to appear in the expansion equation (9–33), therefore some reference must be made to a model even in that case. However, it is still a somewhat more flexible procedure than the direct calculation based on a specific model.

The mirror theorem is useful, of course, only if the magnetic moments of both members of the mirror pair are measured. This is the case for only a few nuclei. The simplest examples are the self-mirrored nuclei, the odd-odd nuclei with equal numbers of neutrons and protons. Since for these nuclei $\langle \mathfrak{M}_z \rangle_t = \langle \mathfrak{M}_z \rangle_{-t}$, Eq. (9–34) is a direct statement concerning the nuclear moment:

$$\langle \mathfrak{M}_z \rangle_0 = 0.690J - \frac{0.190}{J+1} \sum_{LS} [L(L+1) - S(S+1)] \mid C_{LS} \mid^2. \quad (9\text{–}35)$$

A particular example is presented by the deuteron, for which we already have noted that the magnetic moment depends only on the relative probabilities of the S- and D-states and that it contains no cross terms between the two states.

The observed magnetic moments of self-mirrored nuclei are given in Table 9–1. To analyze the moments of the nuclei heavier than the deuteron, some specification must be made of the Russell-Saunders terms to be considered. The selection of terms indicated in the fourth column of the table includes both those terms which correspond to Wigner coupling (taken from Table 8–4) and those which are suggested by the Mayer-Jensen coupling scheme. In column five of the table are listed the conditions on the term probabilities obtained by inserting the observed moments into Eq. (9–35). The theoretical values of $\mid C_{LS} \mid^2$ determined by the two coupling schemes are indicated in the next two columns of the table.

It is clear that the amplitudes of the states allowed by Wigner coupling can be adjusted to fit the observed moments exactly. However, examination of the conditions shows that an increasing amplitude of states of large L is required for the heavier nuclei. This is an indication of the increase in importance of the spin-orbit coupling with increasing mass. The Mayer-Jensen coupling is much more specific, and a direct comparison of that scheme with the experimental conditions of column 5 is possible. The corresponding theoretical relationship is shown in the last column of the table. Evidently the agreement is very poor for Li^6; the higher angular momentum states occur with a probability at least 34 percent greater than is compatible with the observed moment. The agreement improves rapidly with increasing mass.

The comparison with the Mayer-Jensen coupling scheme could, of course, be made directly on the basis of the magnetic moment. It should be emphasized that such a comparison can be quite misleading. The difficulty is that the magnetic moments of the self-mirrored nuclei are not very sensitive to the coefficients C_{LS}, i.e., they are not sensitive to the structure of the wave function. Hence a very large change in the wave function might be required to correct a small difference between observed and calculated moments. The case of B^{10} may be taken as an example. The calculated moment in Mayer-Jensen coupling is 1.880, only 4.5 percent larger than

TABLE 9–1

Analysis of magnetic moments of self-mirrored nuclei

Nucleus	J	Moment*	Terms Considered	Condition on† $\lvert C_{LS}\rvert^2$ Imposed by Obs. Moment	Theoretical values of $\lvert C_{LS}\rvert^2$ for: Wigner Coupling	Mayer-Jensen Coupling‡	Mayer-Jensen Coupling Compared with Column 5
H^2	1	0.857	$^3S_1$, $^3D_1$	$\lvert {}^3C_D\rvert^2 = 0.04$	Not fixed	....	....
Li^6	1	0.822	$^3S_1$, $^3D_1$, $^1P_1$	$\lvert {}^3C_D\rvert^2 + \frac{2}{3}\lvert {}^1C_P\rvert^2 = 0.10$	$\lvert {}^1C_P\rvert^2 = 0$	$\lvert {}^3C_D\rvert^2 = \frac{2}{27}$, $\lvert {}^1C_P\rvert^2 = \frac{5}{9}$	$\lvert {}^3C_D\rvert^2 + \frac{2}{3}\lvert {}^1C_P\rvert^2 = 0.44$
B^{10}	3	1.800	$^3D_3$, $^3F_3$, $^3G_3$	$\lvert {}^3C_F\rvert^2 + \frac{7}{3}\lvert {}^3C_G\rvert^2 = 0.28$	None fixed	$\lvert {}^3C_F\rvert^2 = \lvert {}^3C_G\rvert^2 = 0$	$\lvert {}^3C_F\rvert^2 + \frac{7}{3}\lvert {}^3C_G\rvert^2 = 0$
N^{14}	1	0.404	$^3S_1$, $^3D_1$, $^1P_1$	$\lvert {}^3C_D\rvert^2 + \frac{2}{3}\lvert {}^1C_P\rvert^2 = 0.83$	$\lvert {}^1C_P\rvert^2 = 0$	$\lvert {}^3C_D\rvert^2 = \frac{20}{27}$, $\lvert {}^1C_P\rvert^2 = \frac{2}{9}$	$\lvert {}^3C_D\rvert^2 + \frac{2}{3}\lvert {}^1C_P\rvert^2 = 0.89$
Na^{22}	3	1.746	$^3D_3$, $^3G_3$, $^1F_3$	$\lvert {}^3C_G\rvert^2 + \frac{4}{7}\lvert {}^1C_F\rvert^2 = 0.20$	....	$\lvert {}^3C_G\rvert^2 = \frac{4}{175}$, $\lvert {}^1C_F\rvert^2 = \frac{9}{25}$	$\lvert {}^3C_G\rvert^2 + \frac{4}{7}\lvert {}^1C_F\rvert^2 = 0.23$

* Taken from J. E. Mack, *Rev. Mod. Phys.* **22,** 64 (1950).

† The subscript L is replaced by the equivalent spectroscopic symbol and the subscript S is replaced by a superscript $(2S + 1)$ to facilitate interpretation of the conditions. One term is always eliminated by means of the normalization condition $\Sigma_{LS}\lvert C_{LS}\rvert^2 = 1$.

‡ In the case of Na^{22} the configuration of the odd neutron-proton pair has been taken to be $(d^2_{5/2})_3$, as suggested by the moment. See Appendix 5.

the observed moment. But the error in $|C_{LS}|^2$ is at least 12 percent [$\frac{3}{7} \times 0.28$] and may be as large as 28 percent. This means that the error in the wave function (measured by the probability *amplitudes*) is somewhere between 30 and 50 percent, therefore we cannot argue that the Mayer-Jensen wave function is a good one simply because it gives a magnetic moment very nearly equal to the observed moment.

An analysis of the above kind could also be applied to mirror pairs of odd-even nuclei, and the results would probably be even more enlightening on the issue of the validity of the Mayer-Jensen wave functions, since the prescribed functions are much simpler than those of the odd-odd nuclei. Unfortunately, magnetic moment measurements in mirror pairs are very difficult, since one of the isobars is always radioactive. The only pair for which the measurement has been made is H^3, He^3. Their moments will be discussed below, but not with the hope of testing the Mayer-Jensen scheme. Another pair for which measurement might be made is Be^7, Li^7, since the Be^7 lifetime is rather long (50 days). For the heavier mirror pairs the lifetimes are much shorter; by far the longest (20 min.) occurs for C^{11}, which is paired with B^{11}. Until such measurements are made, the observed moments will provide very little general information concerning the structure of the nuclear wave functions.

The great importance of the H^3, He^3 moments is that they provided the first direct indication of the occurrence of exchange currents in nuclei. The values of the moments are given in Section 8–3, where it was noted that they are in general agreement with the notion that the ground state may be described as predominantly $^2S_{1/2}$. We are now in a position to discuss this question in more detail on the basis of the mirror theorem, Eq. (9–34). The allowed terms are $^2S_{1/2}$, $^2P_{1/2}$, $^4P_{1/2}$, and $^4D_{1/2}$, as has been shown in Section 8–3. When the sum of the moments,

$$\mu(H^3) + \mu(He^3) = 0.8512,$$

is introduced into Eq. (9–34), there results a linear equation relating the probabilities of the various terms:

$$2|\,{}^2C_P\,|^2 - |\,{}^4C_P\,|^2 + 3|\,{}^4C_D\,|^2 = 0.114.$$

Values of $|\,{}^2C_P\,|^2$, $|\,{}^4C_P\,|^2$, and $|\,{}^4C_D\,|^2$ which are consistent with this relationship are indicated by the solid lines in Fig. 9–1. There are many such values, but if consideration is limited to just the ^2S and ^4D states, as suggested in Section 8–3, we find that the ^4D state probability is 3.8 percent, in reasonable agreement with the various calculations based on the two-body potential.

Let us turn now to consideration of the moment of one of the nuclei, H^3 for example. The calculation of the moment can be carried out for the wave function, Eq. (8–28). If only the ^2S state ψ_1 and the ^4D state ψ_7

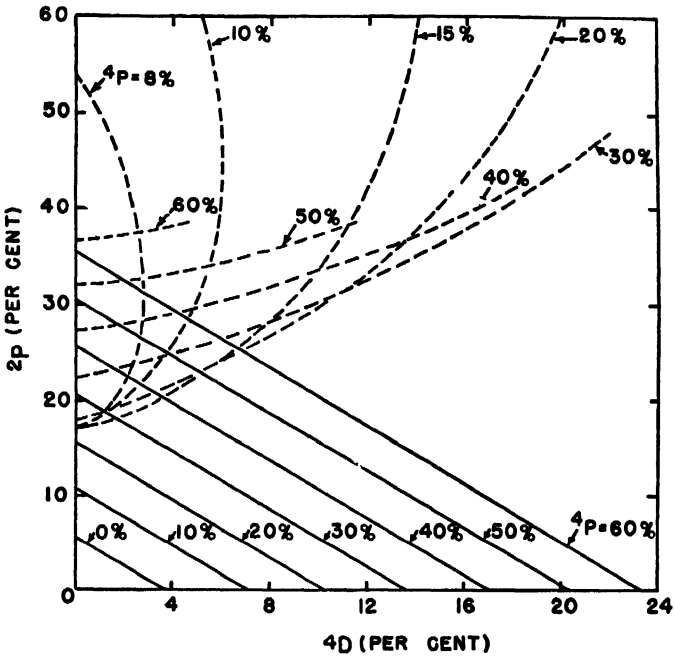

FIG. 9-1. Conditions on the ^2P, ^4P, and ^4D state probabilities for H^3 and He^3 obtained on the basis of observed magnetic moments. The solid lines represent the condition imposed by the sum of the moments and the dashed lines that imposed by the H^3 moment alone. A solution would correspond to an intersection of the pair of dashed and solid curves having a common ^4P state probability. No such intersection occurs. Curves taken from Avery and Sachs, *loc. cit.*

suggested by Eqs. (8–29) are considered, the calculation is particularly simple.* The result is

$$\mu(\mathrm{H}^3) = \mu_P - \tfrac{2}{3}|\,{}^4C_D\,|^2(\mu_n + 2\mu_p - \tfrac{1}{2}). \tag{9-36}$$

Since this moment is clearly less than μ_p, whereas the observed moment is larger than μ_p, some consideration must be given to terms in Eq. (8–28) other than ψ_1 and ψ_7. A calculation of the moment for a completely general function in terms of the coefficients a_k is a straightforward matter.† Cross terms between the various states contribute to the moment, so the phases of the a_j can be chosen in such a way as to obtain a moment larger than μ_p. The dashed curves in Fig. 9–1 show the values of $|\,{}^2C_P\,|^2$, $|\,{}^4C_P\,|^2$, and $|\,{}^4C_D\,|^2$ which are obtained by fitting the moment if reasonable assump-

* Sachs and Schwinger, *Phys. Rev.* **70,** 41 (1946).

† Sachs, *Phys. Rev.* **72,** 312 (1947). The four redundant D-functions of Gerjuoy and Schwinger were used in this calculation, so one should set $a_9 = 0$. The results can be further simplified by noting that the a_k are real in the notation of Section 8–3 (compare Appendix 3).

tions are made concerning the radial functions.* The intersections of the two sets of curves then indicate the values of the probabilities which are consistent with both the H^3 and the He^3 moment. No actual intersection occurs, but one almost appears for $|\,^2C_P\,|^2 \approx 0.20$, $|\,^4C_P\,|^2 \approx 0.20$, $|\,^2C_S\,|^2 \approx 0.60$. The failure to intersect could, in this case, be accounted for by relativistic corrections and small departures from the mirror property.

It is dangerous to say that anything is impossible in nuclear physics. Yet the suggestion that the ground state of the three-body system should consist of 40 percent P-state and little or no D-state seems so unreasonable that it is generally said to be impossible. Certainly none of the two-body potentials which have been considered leads to such a wave function, yet they lead to a function which agrees well with the sum of the H^3 and He^3 moments. Therefore we conclude that the description of the function as a simple mixture of $^2S_{1_2}$ and $^4D_{1_2}$ is correct, and that the difference between the observed moment and that given by Eq. (9–36) with $|\,^4C_D\,|^2 = 0.038$ is a moment "anomaly," due presumably to exchange effects.† As we shall see in Section 9–6, there is other evidence to support this view. The anomaly of H^3 turns out to be

$$\Delta\mu(H^3) = 0.27. \tag{9–37}$$

That of He^3 is found to be equal and opposite to $\Delta\mu(H^3)$ because the mirror theorem has been used to determine $|\,^4C_D\,|^2$. This is in accord with condition (5) of Section 9 2.

Since it is generally believed that the nuclear potential contains a space-exchange operator, the anomaly might be ascribed to the space-exchange current, Eq. (4–10), which must be associated with such a potential. However, the corresponding magnetic moment operator has zero expectation value in the $^2S_{1_2}$ state of H^3 and no matrix element between the ^2S and ^4D functions, so the only contribution is the expectation value in the ^4D state. This turns out‡ to be much smaller than (9 37) for any reasonable assumption about the magnitude of the exchange potential. We must therefore seek another source for the exchange current. In other words,

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* The radial functions come into the cross terms. By introducing a large oscillation in the radial functions, a small amount of ^2P state could yield the observed moment, but the required oscillations increase the kinetic energy tremendously. See Avery and Sachs, *Phys. Rev.* **74,** 1320 (1948).

† F. Villars, *Helv. Phys. Acta* **20,** 476 (1947), was the first to point out that the moment anomaly could be understood as due to meson exchange. He showed that the pseudoscalar meson theory leads to an exchange current of the right kind to account for the moment. However, the failure of the pseudoscalar theory in other areas, as mentioned in Chapter 7, leads one to doubt the validity of any quantitative estimates. Therefore Villars' results are used here only as a guide in formulating a phenomenological theory.

‡ Sachs, *Phys. Rev.* **74,** 433 (1948); Avery and Adams, *Phys. Rev.* **75,** 1106 (1949).

the Hamiltonian $H\{\mathbf{A}\}$ must contain a term in addition to those contributed by ordinary and space-exchange currents. There will be associated with this added term an exchange-moment operator, $\Delta\mathfrak{M}$, which is to be added to the sum of the ordinary moment and the space-exchange moment. If the view is maintained that $\Delta\mathfrak{M}$ is a two-body operator, as suggested in Section 9–2, then every reasonable form which is distinctly different from the space-exchange moment depends on the nucleon spins. Consequently, we shall refer to $\Delta\mathfrak{M}$ as the *spin-exchange moment* and its associated current as the *spin-exchange current.*

A spin-exchange moment subject to the five conditions stated in Section 9–2 is quite limited as to the form it can take. Any one of the four following forms is allowed and each is recommended by the fact that its expectation value does not vanish in the ^2S state, ψ_1, of the triton. Hence a relatively small strength of the moment is required to meet the condition Eq. (9–37). The four spin-exchange moments are:

$$\Delta_1\mathfrak{M} = \frac{e\hbar}{2Mc}\sum_{j,k}\tau_{jk}\boldsymbol{\sigma}_{jk}\Phi_1(r_{jk}), \tag{9–38a}$$

$$\Delta_2\mathfrak{M} = \frac{e\hbar}{2Mc}\sum_{j,k}\tau_{jk}\boldsymbol{\sigma}_{jk}\Phi_2(r_{jk})P^r_{jk}, \tag{9–38b}$$

$$\Delta_3\mathfrak{M} = 3\frac{e\hbar}{2Mc}\beta^2\sum_{j,k}\tau_{jk}(\boldsymbol{\sigma}_{jk}\cdot\mathbf{r}_{jk})\mathbf{r}_{jk}\Phi_3(r_{jk}), \tag{9–38c}$$

$$\Delta_4\mathfrak{M} = 3\frac{e\hbar}{2Mc}\beta^2\sum_{j,k}\tau_{jk}(\boldsymbol{\sigma}_{jk}\cdot\mathbf{r}_{jk})\mathbf{r}_{jk}\Phi_4(r_{jk})P^r_{jk}, \tag{9–38d}$$

where

$$\tau_{jk} = \tau_3(j) - \tau_3(k),$$

$$\boldsymbol{\sigma}_{jk} = \boldsymbol{\sigma}_j - \boldsymbol{\sigma}_k,$$

P^r_{jk} is a space-exchange operator and $\Phi_\alpha(r_{jk})$ is a function of range β^{-1}. For the Yukawa shape and a range $\beta^{-1} = 1.18 \times 10^{-13}$ cm, Φ_i must have a strength of about* 0.5 nuclear magneton to yield the moment anomaly. A linear combination of the four expressions (9–38a) to (9–38d) may, of course, be used to account for the anomaly.

The spin-exchange moment admits of a simple physical interpretation. The total magnetic moment operator of a nucleus is obtained by adding $\Delta\mathfrak{M}$ to the ordinary moment. Two terms which are linear in $\boldsymbol{\sigma}_j$ then appear in the moment, that introduced by the intrinsic spin moment of the free nucleon and that introduced by $\Delta\mathfrak{M}$. The part due to $\Delta\mathfrak{M}$ may

* Austern and Sachs, *loc. cit.*

therefore be interpreted as the change in the intrinsic moment of a given nucleon due to the proximity of other nucleons. Since the intrinsic moments of the free nucleons are presumed to be largely due to meson currents, it is not surprising to find that the moments are changed when the nucleons are close enough together to modify those currents.

The probable existence of spin-exchange currents must be kept in mind throughout our discussion of electromagnetic phemonena. Not only are the magnetic dipole moments affected, but all the magnetic multipoles also contain spin-exchange terms. However, the higher multipole corrections cannot be deduced from the measured dipole moments, since the dipole moment yields information about the interaction with a uniform magnetic field, while the other moments depend on the interaction with nonuniform fields.

If attention is now given to the magnetic moments of the other odd-even nuclei, which comprise the great majority of measured moments, it is found that they do not lend themselves to interpretation in such general terms. They can be calculated only for a particular model, and then only if the coupling scheme is completely specified. Because it is most specific, the independent particle model with Mayer-Jensen coupling is the only case that will be treated in detail here. The results so obtained are quite definite and are in rough agreement with experiment. The calculation will be performed on the basis of the ordinary moment, Eq. (9–29), but the corrections due to exchange effects will be taken up in connection with the comparison to the data.

In the Mayer-Jensen scheme, the odd nucleon in an odd-even nucleus carries all of the angular momentum if the recoil of the massive core is neglected. Therefore only this one nucleon contributes to the magnetic moment, and the operator (9–29) may be replaced by the one-particle operator

$$\mathfrak{M} = \tfrac{1}{2}(1 + \tau_3)1 + \tfrac{1}{2}(\mu_+ + \tau_3\mu_-)\sigma,$$

where the variables all refer to the single odd nucleon. For given orbital angular momentum l of the nucleon, and given total angular momentum j, the nuclear magnetic moment is, according to the usual vector rule,

$$\langle \mathfrak{M}_z \rangle = \frac{1}{2(j+1)} \left\{ \frac{1}{2}(1 + \tau_3) \left[j(j+1) + l(l+1) - \frac{3}{4} \right] + (\mu_+ + \tau_3\mu_-) \left[j(j+1) + \frac{3}{4} - l(l+1) \right] \right\}.$$

Just two values of j are admitted for given l, $j = l \pm \frac{1}{2}$, and the corresponding nuclear moments are

$$\langle \mathfrak{M}_z \rangle_{l+\frac{1}{2}} = \tfrac{1}{2}[(j - \tfrac{1}{2})(1 + \tau_3) + (\mu_+ + \tau_3\mu_-)],$$

and

$$\langle \mathfrak{M}_z \rangle_{l-\frac{1}{2}} = \frac{j}{2(j+1)} \left[\left(j + \frac{3}{2} \right) (1 + \tau_3) - (\mu_+ + \tau_3 \mu_-) \right]$$

For an odd proton nucleus, $\tau_3 = 1$, while for an odd neutron nucleus $\tau_3 = -1$. Hence the magnetic moments are

$$\langle \mathfrak{M}_z \rangle_{p,l+\frac{1}{2}} = (j - \tfrac{1}{2}) + \mu_p, \tag{9-39a}$$

$$\langle \mathfrak{M}_z \rangle_{p,l-\frac{1}{2}} = \frac{j}{j+1} \left[\left(j + \frac{3}{2} \right) - \mu_p \right], \tag{9-39b}$$

$$\langle \mathfrak{M}_z \rangle_{n,l+\frac{1}{2}} = \mu_n, \tag{9-39c}$$

$$\langle \mathfrak{M}_z \rangle_{n,l-\frac{1}{2}} = -\frac{j}{j+1} \mu_n. \tag{9-39d}$$

The magnetic moments obtained from these formulas are plotted as a function of J in Figs. 9–2 and 9–3, which are usually referred to as "Schmidt diagrams."* The solid curves in this diagram are meaningful only at points corresponding to half-integral values of j. Since j is the total nuclear angular momentum ($J = j$) in the Mayer-Jensen model, the curves may be used to assign a value of l to the odd nucleon for those odd-

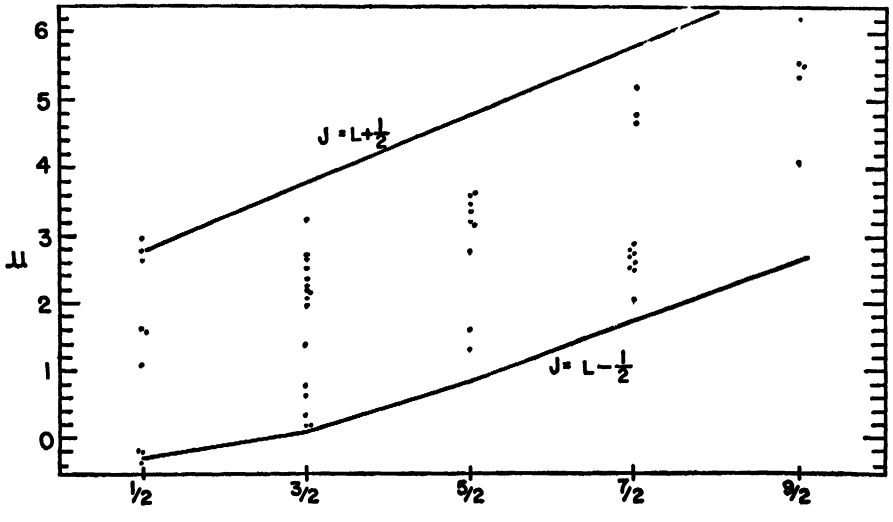

FIG. 9–2. Schmidt diagram of the magnetic moments of odd proton nuclei. The solid lines represent Eqs. (9–39a) and (9–39b). The points are experimental values. Diagram adapted from P. F. A. Klinkenberg, *Rev. Mod. Phys.* **24**, 63 (1952).

* T. Schmidt, *Zs. f. Phys.* **106**, 358 (1937).

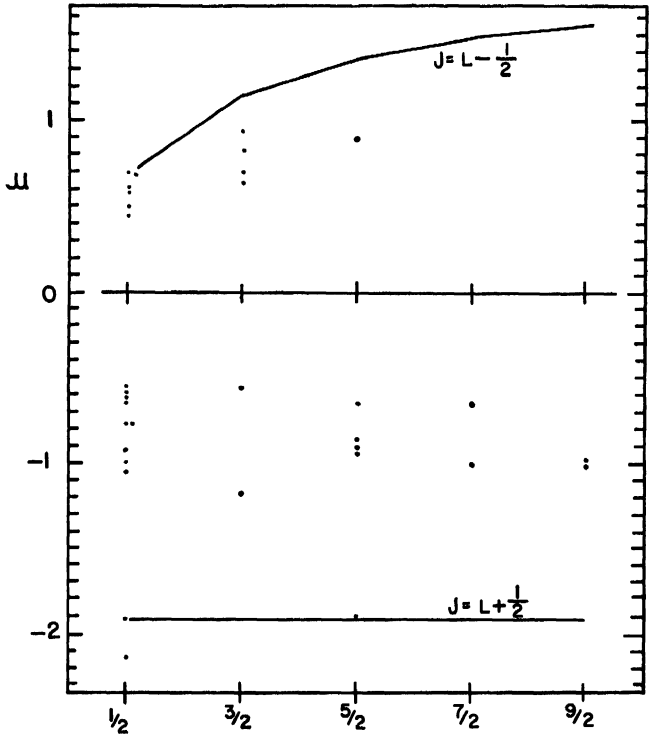

FIG. 9–3. Schmidt diagram of the magnetic moments of odd neutron nuclei. The solid lines represent Eqs. (9–39c) and (9–39d). The points are experimental values. Diagram adapted from Klinkenberg, *loc. cit.*

even nuclei whose spins and magnetic moments have been measured. But very few of the many experimental values, which are also plotted in Figs. 9–2 and 9–3, lie on the Schmidt lines. Therefore the assignment of the l-value is made on the basis of the closest Schmidt line, a procedure that leads to the values of l indicated in Appendix 5. They are very much in accord with Mayer-Jensen coupling. In fact, this was one of the original* arguments in favor of that scheme. However, it would probably be incorrect to draw the conclusion that the agreement establishes the validity of the Mayer-Jensen wave functions. The magnetic moments calculated on the basis of a quite different model, the model in which each nucleon in the nucleus contributes equally to the orbital angular momentum, deviate from the observed moments by amounts which are not much greater than the deviations from the Schmidt lines.† This is just an

* M. G. Mayer, *Phys. Rev.* **78,** 16 (1950); Haxel, Jensen, and Suess, *Zs. f. Phys.* **128,** 295 (1950).

† Margenau and Wigner, *Phys. Rev.* **58,** 103 (1939). See also E. Feenberg, *Phys. Rev.* **77,** 771 (1950).

example of what appears to be a general property of theoretical nuclear moments: any reasonable wave function having the correct total angular momentum seems to lead to a magnetic moment of about the correct magnitude. Therefore it is not possible to establish the form of the wave function on the basis of the magnetic moment alone.

The converse procedure is more profitable. Let us assume on other grounds that the Mayer-Jensen function gives the predominant term in the wave function but that there are additional contributions to the function which may be treated as corrections. Then the calculated magnetic moment will contain cross terms between the predominant function and the admixed functions. Since the cross terms are proportional to the amplitudes of admixture, they may be large even for a small probability of admixture, the probability being just the square of the amplitude. Thus the deviations from the Schmidt lines may be interpreted as an effect of admixed functions, in which case they provide a means for estimating the amount of admixture. This view is supported by the fact that the deviations fluctuate from one nucleus to another in an irregular fashion, just as the amplitudes of admixed functions would be expected to fluctuate. A more detailed statement would require knowledge of the form of the admixed function, but so little information is available to provide that information that no definite conclusion is possible. However, it is noteworthy that every estimate of the effect* brings the moments inside the Schmidt lines, since the fact that most of the observed moments lie between the Schmidt lines is the most outstanding systematic feature of the data.

Before any final conclusion concerning nuclear wave functions can be drawn from the data on magnetic moments, the possible influence of exchange currents on the calculated moment must be taken into account. There are two reasonably well-established contributions of this kind to be considered: the space-exchange moment associated with the space-exchange part of the potential, and the spin-exchange moment. Possibly, there also exists a velocity-dependent spin-orbit interaction between nucleons which would introduce another modification of the moment, as we have seen in Section 9–2. However, this third effect is not well enough established to warrant giving it further consideration here. Estimates of the contribution of the space-exchange moment have been made by Spruch,† who finds that it is, in general, small compared with the deviations from

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* One form of admixed function is suggested by the quadrupole moments, J. Rainwater, *Phys. Rev.* **79,** 432 (1950), Feenberg and Hammack, *Phys. Rev.* **81,** 285 (1951). Application to the magnetic moment problem has been made by A. Bohr, *Phys. Rev.* **81,** 134 (1951). Other forms of admixed function are considered in this connection by E. Feenberg, *loc. cit.*, J. P. Davidson, *Phys. Rev.* **85,** 432 (1952), and by M. Ross, *Phys. Rev.* **88,** 935 (1952).

† L. Spruch, *Phys. Rev.* **80,** 732 (1950).

the Schmidt lines. Estimates of the shift in the Schmidt lines introduced by a spin-exchange moment adjusted to fit the triton anomaly leads, for odd proton nuclei and odd neutron nuclei, to a shift of about the same size and sign as the triton and He^3 anomaly, respectively.* This shift is small compared with the observed deviations and is of opposite sign, since the H^3 and He^3 moments happen to have the unusual property that they lie outside the Schmidt lines.

There is the possibility that the main source of the deviations from the Schmidt lines is an exchange effect of some sort not considered here.† However, effects of this kind involve an average over many pairs of nucleons, so they should vary smoothly from nucleus to nucleus in contrast with the observed rather large fluctuations. As mentioned above, this seems to indicate that much of the deviation is due to a fluctuating amount of admixed wave function, although there is probably a background contribution due to exchange currents. In any case, the two contributing effects are not easily untangled, and we must seek elsewhere for confirmation of the existence of either effect by itself.

9–4 Nuclear quadrupole moments. Both the experimental and theoretical situations with regard to the static nuclear quadrupole moment, are quite different from what we found to be the case for the magnetic moments. The experimental values of the quadrupole moments are not very accurate, in spite of the fact that the experiments are done with high precision. The difficulty has to do with the indirect nature of the observations. They yield a value for the interaction of the nuclear quadrupole moment with the inhomogeneous electric field of an atom or molecules and therefore a calculated value of the inhomogeneous field goes into the determination of the moment. The main source of uncertainty in the observed moment is the uncertainty in the molecular or atomic wave function which is required to calculate this field. The theoretical determination of that wave function is very difficult indeed. Hence there are cases for which even the sign of the quadrupole moment is in doubt.

With regard to the interpretation of quadrupole moments in terms of nuclear structure, it is clear that they are far more sensitive to the details of the nuclear wave functions. Both the detailed radial and angular distributions of charge within the nucleus enter into the determination of the moment. Hence no simple relationship, such as the mirror theorem for magnetic moments, can be expected to be available to aid in the inter-

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* M. Ross, *loc. cit.*, presents a detailed discussion of the various possibilities for accounting for deviations from the Schmidt lines.

† H. Miyazawa, *Prog. Theor. Phys.* **6**, 263 (1951); A. de Shalit, *Helv. Phys. Acta* **24**, 296 (1951); F. Bloch, *Phys. Rev.* **83**, 1062 (1951).

pretation of these moments. Therefore it is necessary to use a detailed nuclear model for the theoretical derivation of quadrupole moments.

The theoretical moment, Q, is defined as the expectation value of Q_{33}, given by Eq. (3 48), in the ground state of the nucleus when the magnetic quantum number is equal to the total angular momentum J. The calculation of this quantity can be carried out in a straightforward fashion for the independent particle model if the coupling scheme is completely specified. The results for odd-even nuclei are particularly simple and particularly significant in the case of Mayer-Jensen coupling. Only the odd proton nuclei have a moment in this case, because the protons in the odd neutron nuclei have zero total angular momentum. The value of the moment is a function of the value of j and of the number of protons, ζ, which are included in the open shell with the odd proton. This function is derived in Appendix 4. It has the form

$$Q(\zeta) = -\frac{2j - 2\zeta + 1}{2(j+1)} e\langle r^2 \rangle, \tag{9–40}$$

where $\langle r^2 \rangle$ is the mean squared displacement of the odd proton from the center of mass. It is worth noting that Eq. (9–40) has the property that would be expected on the basis of hole theory: ζ proton holes in a closed shell yield a moment of equal magnitude but opposite in sign from that of ζ protons. In particular, the quadrupole moment of a half-filled shell vanishes. Equation (9–40) is plotted for the important j-values in Fig. 9–4. Note that these results are valid even if other shells are partly filled by an even number of nucleons, so long as the angular momenta of all but the last proton in the odd shell combine to the value zero.

Figure 9–4 indicates that the quadrupole moment has its largest magnitude, but is negative, when the first proton is added to the shell. As more protons are added (for odd Z only), the moment increases smoothly, passes through zero, and comes to its maximum (positive) value for a single hole in the closed shell. When another shell is started, the pattern repeats, and the moments should show a discontinuous jump from positive to negative, and then increase smoothly again. The actual behavior of the moment as a function of Z will be somewhat more confused because of the vacillation on the part of the odd proton in making its choice between open shells of different j-value. One way to obtain definite predictions of the behavior of the quadrupole moment is to accept the tentative assignment of configurations given in Appendix 5 on the basis of the observed nuclear spins, magnetic moments, and a considerable amount of guesswork. The result of using those configurations is indicated by the solid line in Fig. 9–5. The jagged character of the line is directly associated with the fact that states of two quite different j-values (such as the $p_{1/2}$ and $g_{9/2}$ states) are usually close competitors for the odd proton. However, the change in sign at each

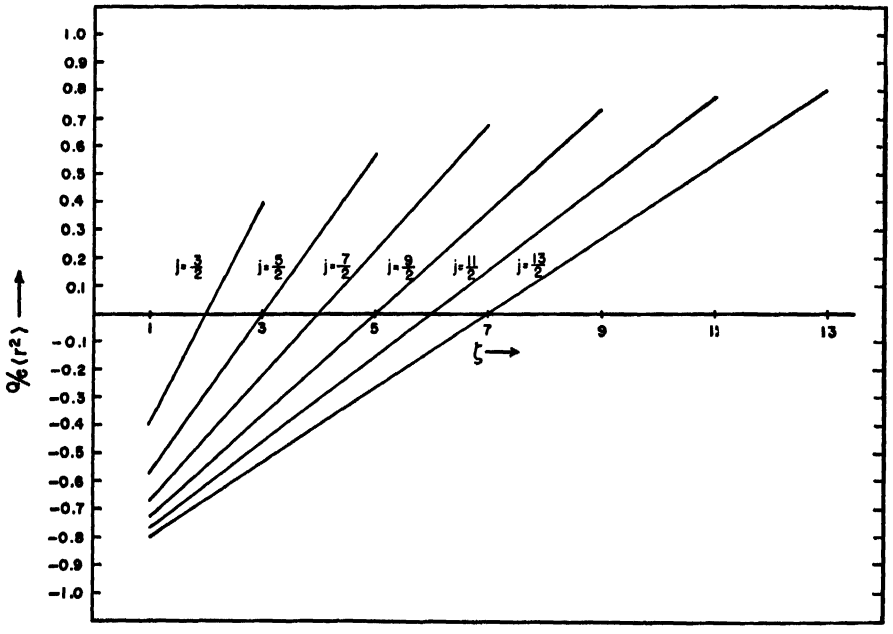

FIG. 9-4. Dependence of the quadrupole moment of an odd proton-even neutron nucleus on the number of protons (ζ) in a shell of given j-value according to the Mayer-Jensen scheme. Only points at odd integral values of ζ are significant.

magic number is due to a quite different cause, namely, a single proton outside a closed shell has a negative moment, while a single proton hole in a closed shell has a positive moment.

The fact that the observed nuclear quadrupole moments show a correlation with the magic numbers was first reported by T. Schmidt.* To illustrate this correlation, the experimental values of $Q/e(1.5 \times 10^{-13}A^{1/3}\text{ cm})^2$ are plotted as a function of the odd proton number in Fig. 9–5. Use is made here of the expectation that $\langle r^2\rangle$ will not differ greatly from the square of the nuclear radius, which is usually estimated at $1.5 \times 10^{-13}A^{1/3}$ cm. It is evident that Q tends to behave in the manner suggested by Mayer-Jensen coupling. However, in some cases, the values of Q are much larger than the predicted values. An explanation for this phenomenon has been suggested by Rainwater,† namely, that the odd proton polarizes the core by nuclear forces in such a way as to destroy its spherical symmetry.

* *Naturwiss.* **28,** 565 (1940). See also W. Gordy, *Phys. Rev.* **76,** 139 (1949), R. D. Hill, *ibid.*, 998, and Townes, Foley, and Low, *ibid.*, 1415.

† *Loc. cit.* See also Feenberg and Hammack, *loc. cit.*, A. Bohr, *loc. cit.*, and Van Wageningen and de Boer, *Physica* **18,** 369 (1952).

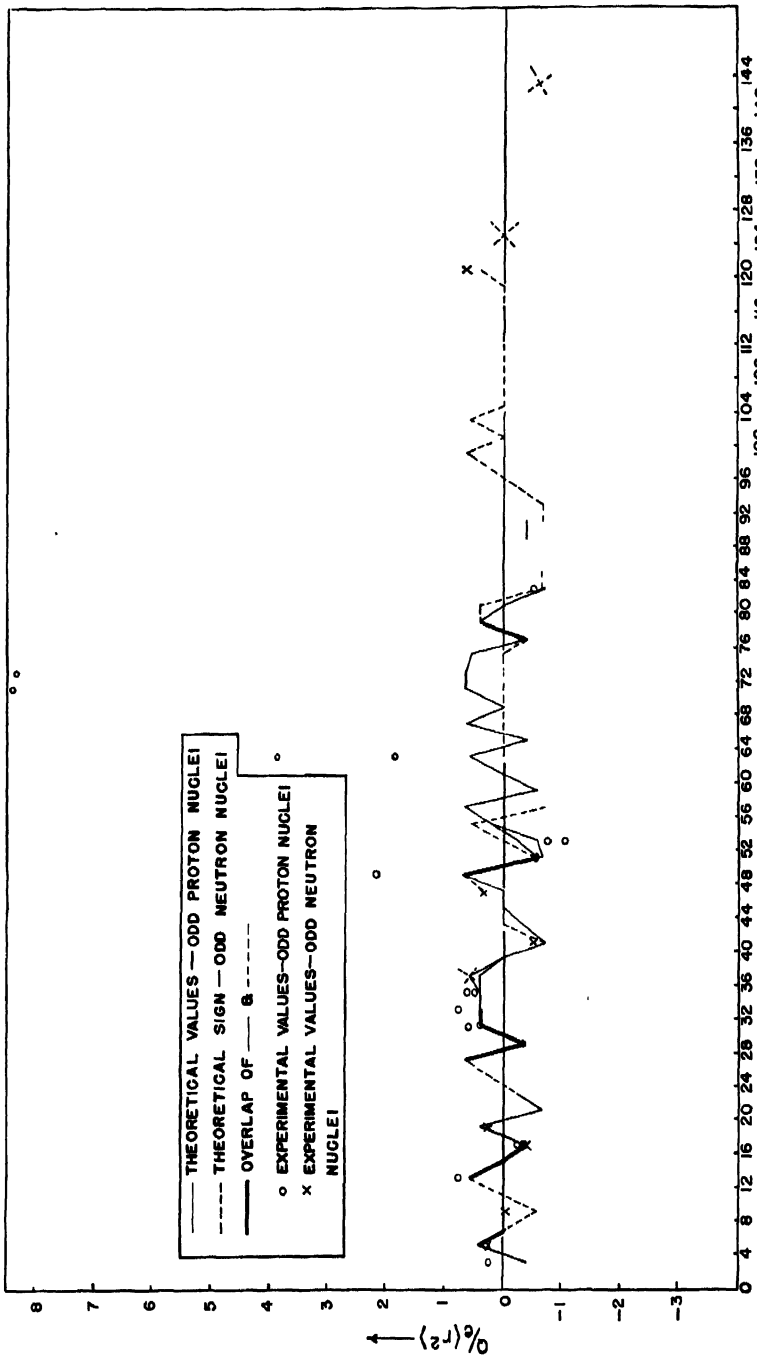

FIG. 9-5. Quadrupole moments of odd-even nuclei. The solid and dashed curves are based on Eq. (9-40), the values of ζ having been taken from the configurations of Appendix 5. These assignments are quite arbitrary, so discrepancies in sign between the curves and newly measured moments are usually to be taken as an indication of incorrect configurations rather than as a failure of Mayer-Jensen coupling.

Since the core contains a large number of protons, a small distortion is likely to provide a large contribution to the quadrupole moment. The distortion is assumed to have essentially the same angular distribution as the odd proton, therefore the resulting addition to Q presumably amounts to the multiplication of the theoretical values (9–40) by a factor depending on the polarizability of the core.

This explanation of the large quadrupole moments of the odd proton nuclei suggests that, although in the pure form of the model their moments would vanish, the odd neutron-even proton nuclei will have quadrupole moments which correlate with the neutron configuration. If an odd proton polarizes the core by nuclear forces, an odd neutron would be expected to do likewise. Therefore the charge distribution of the core would be expected to have a form similar to the space distribution of the neutron, and the quadrupole moment would again be expected to be a multiple of $Q(\zeta)$ given by Eq. (9–40), where ζ is now the (odd) number of neutrons in the shell of angular momentum j. The values of $Q(\zeta)$ based on the odd-neutron configurations tabulated in Appendix 5 are plotted as the broken line in Fig. 9-5. The few available experimental values are indicated in the same figure. The correlation is quite good. However, it is somewhat disturbing to find that for the light nuclei the odd neutron quadrupole moments are comparable in magnitude to the odd proton moments in spite of the fact that little distortion of the core is indicated for the latter group of nuclei.

It may appear that the successful prediction of the alternations in sign of the Q moments merely signifies that there is a correlation between Q and the values of nuclear spin and magnetic moment. This is not necessarily the case, since the spin and magnetic moment of a nucleus depend only on the state of the odd nucleon, while the sign of the quadrupole moment depends on the number of nucleons, ζ, which share the shell with the odd nucleon. Hence, there is some indication that the states of nucleons other than the last one have been properly assigned in Appendix 5. It is to be noted that a departure from the original assignments of Klinkenberg* (on which Appendix 5 is based) is required for the nuclei Yb^{173} and Hg^{201} in order that the quadrupole moments turn out to have the correct signs. All other configurations were assigned without reference to the quadrupole moments.

In spite of the rather remarkable success of the Mayer-Jensen model in this connection, the fact that core distortion is required to bring about quantitative agreement indicates again that the nuclear wave functions may depart appreciably from the pure independent particle model functions. The distortion of the core means that higher configurations are mixed into the wave function, and quite large admixtures are indicated by the large polarizations required to account for many of the observed

* *Rev. Mod. Phys.* **24**, 63 (1952).

moments. We have here another indication that the Mayer-Jensen function gives an excellent insight into the symmetry properties of the ground state, although it may constitute but a small part of the actual wave function.

9–5 Internal conversion. The dynamical aspects of the interaction of a nucleus with the electromagnetic field are manifest through the processes of emission and absorption of radiation. They are an important source of information concerning nuclear structure, since the rate of emission or absorption depends directly on the matrix elements of nuclear multipole moments, as indicated by Eqs. (9–25) and (9–26). However, the radiative process is not the only one that can lead to a transition between nuclear states; the transition may be accelerated as a result of the electromagnetic interaction between a nucleus and its associated atomic electrons. Thus the emission of radiation may be supplemented by the ejection of an atomic electron,* an event which is commonly called *internal conversion.* An understanding of the factors that govern the rate of internal conversion is essential because it is required to obtain a reliable estimate of the lifetime of a nuclear state. Even more important is the fact that the internal conversion coefficient, the number of electrons ejected per photon emitted, is sensitive to the multipolarity of the transition. Since the multipole order of the transition is directly related to the angular momentum and parity differences between the states involved, the measurement of internal conversion coefficients provides a powerful tool of nuclear spectroscopy.

Let us consider the interaction with the nucleus of an atomic electron at distance $\boldsymbol{\rho}$ from the center of mass of the nucleus. There is an electrostatic interaction between the electron and each proton, and there are electromagnetic interactions with the currents due to the protons and with the magnetic moments of both protons and neutrons. Attention will first be devoted to the electrostatic term for which the potential energy is

$$\varphi = -e^2 \sum_{\pi=1}^{Z} \frac{1}{|\boldsymbol{\rho} - \mathbf{r}_\pi|}, \tag{9–41}$$

where $\mathbf{r}_\pi$ is the coordinate of the πth proton with respect to the nuclear center of mass. Since for the atomic wave functions the probability that ρ is much greater than r_π is very large, use can be made of a multipole expansion of φ:

$$\varphi = -\frac{e^2}{\rho} \sum_{l,m} \sum_{\pi=1}^{Z} \left(\frac{r_\pi}{\rho}\right)^l \frac{4\pi}{2l+1} (-1)^m Y_l^m(\pi) Y_l^{-m}(e), \tag{9–42}$$

where $Y_l^m(\pi)$ and $Y_l^{-m}(e)$ are spherical harmonics in the proton and electron

* Taylor and Mott, *Proc. Roy. Soc.* **A142,** 215 (1933); Hebb and Uhlenbeck, *Physica* **5,** 605 (1938).

variables, respectively.* The leading term in the expansion is the usual coulomb potential $-Ze^2/\rho$ which determines the atomic wave function. The second term,

$$\varphi_1 = -\frac{e^2}{\rho^3}\sum_{\pi}(\boldsymbol{\rho}\cdot\mathbf{r}_{\pi}), \tag{9-43}$$

is proportional to the electric dipole moment of the nucleus and to the atomic dipole moment. Hence it has a nonvanishing matrix element only for an electric dipole transition of both systems. The result is that a nuclear electric dipole transition is accelerated by the interaction with every atomic electron whose ionization potential is smaller than the difference in energy between the nuclear states. And a similar effect occurs for the higher order electric multipole transitions through the influence of the higher terms in Eq. (9-42).

In addition to the direct effect of the electrostatic interaction, internal conversion is produced by the indirect interaction introduced by the radiation field. The usual methods of second order perturbation theory show that an effect of this kind will occur in two steps: the nucleus first emits a photon into an intermediate state and then the photon is absorbed by the atomic electron. All intermediate states produced by the interaction must be taken into account, so photons of every energy contribute to the process; hence the effect is *not* a simple photoelectric ejection of the electron by the radiation produced in the transition.

An alternative description of this process associates the radiative terms with the corrections to the direct interaction caused by the finite velocity of propagation of the field (retardation effects). Therefore the radiative terms amount to corrections of order v/c to the direct conversion process, if v is the electron velocity. For magnetic transitions no contribution arises from the electrostatic term, and the retardation effect is completely responsible for the conversion process.

The exact calculation of the conversion coefficient is rather complicated because it requires a knowledge of the correct wave function of the electron in the neighborhood of the nucleus. Tables of conversion coefficients† have been, and are being, computed by numerical methods, and these provide the best basis for interpretation of the experimental values. However, it is possible to obtain a qualitative idea of the trends in the behavior of the conversion coefficient with multipole order, energy, and atomic number; our efforts will be confined to that. But first it should be remarked

* See, e.g., Jahnke and Emde, *Tables of Functions*, Dover (1947), p. 107, ff. Use is made of the addition theorem p. 115 to express P_l in terms of the Y_l^m. Compare Eq. (3-44) *et seq.* with Eq. (9-42). The spherical harmonics are always defined as in Condon and Shortley, *op. cit.*

† Rose, Goertzel, Spinrad, Harr, and Strong, *Phys. Rev.* **83**, 79 (1951).

that the conversion coefficient is, to a very good approximation,* independent of the details of nuclear structure because the transition probabilities for *both* the radiation process and the conversion process are proportional to the square of the matrix element of the nuclear multipole moment.

For the sake of simplicity, detailed consideration is given only to the electric 2^l-pole conversion coefficient, and then for energy values such that the electron may be treated nonrelativistically. Under the latter condition, the retardation effects may be ignored; only the direct coupling φ need be considered. The transition probability for emission of an electron into the solid angle $d\Omega$ is then

$$dw_l(\text{elect. conv.}) = \frac{2\pi}{\hbar} |\langle i,I|\, \varphi_l \,|F,f\rangle|^2 \, d\rho_F,$$

if I and F are the initial and final electron states. The density of final states, $d\rho_F$, refers to electron states of type F in the continuum,

$$d\rho_F = \frac{Vm^{3/2}}{(2\pi\hbar)^3}(2\epsilon)^{1/2}\, d\Omega$$

for an electron of energy ϵ emitted into an enclosure of volume V.

To make possible a direct comparison between the term in φ_l and the electric moments D_l, we take the z-axis to be in the direction $\boldsymbol{\kappa}$ and consider a transition which produces the circular polarization $\mathbf{u} = 2^{-1/2}(\mathbf{u}_x + i\mathbf{u}_y)$. Then only the $\Sigma_\pi\, Y_l^1\,(\pi)$ part of φ has a nonvanishing matrix element. It is to be compared with the corresponding term in the expansion of D_l in spherical harmonics. That term has the form

$$\frac{2^l(l-1)!}{(2l)!}\sqrt{\frac{2\pi l(l+1)}{2l+1}}\sum_\pi r_\pi^l Y_l^1(\pi),$$

so

$$dw_l(\text{elect. conv.}) = \frac{2}{\pi}\frac{1}{l(l+1)(2l+1)}\left[\frac{(2l)!}{2^l(l-1)!}\right]^2 \times \frac{e^2 Vm^{3/2}(2\epsilon)^{1/2}}{\hbar^4} |\langle i|\, D_l \,|f\rangle\langle I\,|\rho^{-(l+1)}Y_l^{-1}(e)\,|F\rangle|^2\, d\Omega.$$

Since the corresponding transition probability for emission of the photon is given by Eq. (9–25), the conversion coefficient

$$\alpha_l = \frac{w_l(\text{elect. conv.})}{w_l(\text{elect.})}$$

* Tralli and Goertzel, *Phys. Rev.* **83**, 399 (1951).

is found (after averaging over initial states i) to be

$$\alpha_l = \frac{\pi^{-1}}{(2l+1)l(l+1)}\left[\frac{(2l)!}{2^l(l-1)!}\right]^2 \frac{e^2 V m^{3/2}}{\hbar^3}(2\epsilon)^{1/2}\left(\frac{c}{\omega}\right)^{2l+1} \times \int d\Omega \,|\langle I|\, \rho^{-(l+1)} Y_l^{-1}(e)\,|F\rangle|^2. \quad (9\text{–}44)$$

Those initial electron states of interest are the s-states belonging to the K or L shell of the atom, so the wave function has the form

$$\psi_I = (\pi a^3)^{-1/2} e^{-\rho/a},$$

where a is the radius of either the K or the L orbit, $a = a_0 = \hbar^2/Zme^2$ or $a = 2a_0$, respectively. Because the initial state is an s-function, the matrix element in Eq. (9–44) vanishes unless the ejected electron is in a state of orbital angular momentum l with $m = 1$. This is just a statement of the conservation of angular momentum for the system as a whole.

Let us first consider the case in which ϵ is much larger than the ionization potential. Then the electron wave function ψ_F is not strongly affected by the coulomb potential, and it may be replaced by a plane wave:

$$\psi_F = V^{-1/2} e^{i\mathbf{k}\cdot\boldsymbol{\rho}},$$

where $\mathbf{k}$ is the propagation vector of the emitted electron, $k = (2m\epsilon)^{1/2}/\hbar$. By making use of the standard expansion

$$e^{i\mathbf{k}\cdot\boldsymbol{\rho}} = \sum_l i^l (2l+1) P_l j_l(k\rho),$$

or, its equivalent,

$$e^{i\mathbf{k}\cdot\boldsymbol{\rho}} = 4\pi \sum_{l,m} i^l (-1)^m Y_l^{-m}(\mathbf{k}/k) Y_l^m(e) j_l(k\rho),$$

where j_l is the regular radial function for a free electron (compare the end of Section 4–5), we find that the matrix element is

$$\langle I|\, \rho^{-(l+1)} Y_l^{-1}(e)\,|F\rangle = \frac{4\pi i^l}{(\pi V a^3)^{1/2}} Y_l^{-1}(\mathbf{k}/k) \int_0^\infty e^{-\rho/a} \rho^{-(l-1)} j_l(k\rho)\, d\rho.$$

Because the energy of the emitted electron has been assumed to be large compared with the ionization potential, we have $ka \gg 1$. The radial integral in the matrix element may therefore be estimated as*

$$\lim_{a\to\infty} \int_0^\infty e^{-\rho/a} \rho^{-(l-1)} j_l(k\rho)\, d\rho = \frac{2^{l-1}(l-1)!}{(2l-1)!} k^{l-2}, \quad (9\text{–}45)$$

and the conversion coefficient, Eq. (9–44), takes the form

* See, e.g., G. N. Watson, *Theory of Bessel Functions*, Cambridge Univ. Press (1948), p. 391.

$$\alpha_l = \frac{16}{(2l+1)}\frac{l}{l+1}\frac{e^2m}{\hbar^2a^3}\left(\frac{c}{\omega}\right)^4\left(\frac{kc}{\omega}\right)^{2l-3}. \tag{9–46}$$

Under the stated conditions, it is a good approximation to take $\hbar\omega \approx \hbar^2k^2/2m$, and Eq. (9–46) may be replaced by the equation

$$\alpha_l = \frac{Z^3}{(2l+1)}\frac{l}{l+1}\left(\frac{e^2}{\hbar c}\right)^4\left(\frac{a_0}{a}\right)^3\left(\frac{2mc^2}{\hbar\omega}\right)^{l+\frac{1}{2}}.$$

Thus we see that the internal conversion coefficient increases with decreasing energy, and at a greater rate for higher order transitions. It also increases rather rapidly with atomic number.

Since the conversion process seems to make its largest contribution at low energy, it is of considerable interest to extend our considerations to the region of energies comparable to the ionization potential, ϵ_I, i.e., to the case $ka \ll 1$. Presumably, the increasing trend of Eq. (9–46) will reverse as the energy is lowered to values for which the coulomb force on the emitted electron cannot be ignored, since the conversion coefficient must vanish when $\hbar\omega \leq \epsilon_I$. For $ka \ll 1$ use can be made of the fact that the radial integral appearing in the matrix element is controlled by ψ_I. Therefore the radial function of the emitted electron may be replaced by its form for small argument* $[2^l l!/(2l+1)!](\rho/a_0)^l$. Hence the integral of interest is, in place of Eq. (9–45),

$$\frac{2^l l!}{(2l+1)!}\int_0^\infty e^{-\rho/a}\rho^{-(l-1)}(\rho/a_0)^l\,d\rho = \frac{2^l l!}{(2l+1)!}a^2a_0^{-l}.$$

Equation (9–46) is then replaced by

$$\alpha_l \approx \frac{16}{(2l+1)^3}\frac{l}{l+1}Z^{-1}\left(Z\frac{e^2}{\hbar c}\frac{mc^2}{\hbar\omega}\right)^{2l+1}ka,$$

so α_l approaches zero in the same manner as $k = \hbar^{-1}\sqrt{2m(\hbar\omega - \epsilon_I)}$, for $\hbar\omega \approx \epsilon_I$. Note that α_l is, in general, a more sensitive function of Z at low than at high energy.

When the nucleus undergoes a magnetic 2^l-pole transition, conversion is produced only by the radiative interaction, therefore the magnetic conversion coefficients, β_l, are of the order of $(v/c)^2$ times smaller than the α_l. This is just what would be expected on the basis of the ratio of electric to magnetic moments of the electron system, since $v/c \approx (e\hbar/mc)/e\rho$. Another

* See, e.g., Yost, Wheeler, and Breit, *Phys. Rev.* **49**, 174 (1936). In the absence of coulomb attraction, the function would be $j_l \approx [2^l l!/(2l+1)!](k\rho)^l$, but the large kinetic energy of an electron near the nucleus eliminates the influence of k, which measures the kinetic energy at large distances. The quantity k is replaced by $1/a_0$, where a_0 is the natural length of the coulomb problem.

difference between the electric and magnetic transitions is introduced by the requirement that parity be conserved. Consequently the emitted electron must have angular momentum $l - 1$. Both of these changes lead to a small difference in the magnitude of β_l and α_l, but the trend of β_l with l, energy, and Z does not differ in any very important way from the corresponding behavior of α_l.

It must be emphasized again that the conversion process occurs *in addition to* the radiative transition and not as a substitute for it. Consequently, the complete nuclear transition probability takes the form

$$W_l(\text{elect.}) = w_l(\text{elect.}) + w_l(\text{elect. conv.}) = (1 + \alpha_l)w_l(\text{elect.}),$$

or

$$W_l(\text{mag.}) = w_l(\text{mag.}) + w_l(\text{mag. conv.}) = (1 + \beta_l)w_l(\text{mag.}).$$

When the nucleus undergoes a transition for which $\Delta J = l$, either electric or magnetic multipole radiation is produced in conjunction with the emission of an electron. However, the radiative transition

$$J = 0 \rightarrow J = 0$$

is strictly forbidden for any multipole order. Nevertheless, the transition can occur by means of electron conversion,* so this special case warrants brief attention. No conversion coefficient is defined for the process (it would be infinite), but a comparison can be made between the rate of the $0 \rightarrow 0$ transition and the rate of a dipole transition.

The occurrence of this "electric monopole" phenomenon is a consequence of the finite size of the nucleus. The electron spends part of its time inside the nucleus, with the result that the expansion Eq. (9–42) is not valid. Instead, the proton-electron electrostatic coupling may be expanded in the form

$$\varphi = -e^2 \sum_{l,m} \sum_{\pi=1}^{Z} \frac{1}{r_\pi} \left(\frac{\rho}{r_\pi}\right)^l \frac{4\pi}{2l+1} (-1)^m Y_l^m(\pi) Y_l^{-m}(e),$$

when $\rho < r_\pi$. The matrix element of φ for a nuclear zero-zero transition receives a contribution only from the term $l = 0$ whether the electron is inside or outside the nucleus. Hence, in this case,

$$\langle i,I|\varphi|F,f\rangle = -e^2 \left\langle i \left| \sum_\pi \left\{ \int_{\rho<r_\pi} \psi_I^* \psi_F (r_\pi^{-1} - \rho^{-1})\, d^3\rho + \int \psi_I^* \psi_F \rho^{-1}\, d^3\rho \right\} \right| f \right\rangle.$$

The matrix element of the second term vanishes because the nuclear states i and f are described by orthogonal functions. Since all contributions to the first term arise for very small values of ρ, the functions ψ_I and ψ_F may be replaced by their limiting values for small argument. These values

* R. H. Fowler, *Proc. Roy. Soc.* **A129**, 1 (1930).

are $(\pi a^3)^{-\frac{1}{2}}$ and $V^{-\frac{1}{2}}$, respectively, the latter being a consequence of the fact that the electron is ejected into an s-state. Thus the matrix element takes the form

$$\langle i,I|\ \varphi\ |F,f\rangle = \frac{2\pi e^2}{3(\pi a^3 V)^{\frac{1}{2}}} \langle i|\ \sum_\pi r_\pi^2\ |f\rangle,$$

and the rate of the $0 \rightarrow 0$ transition is

$$W_0 = \tfrac{4}{9}\left(\frac{e^2}{\hbar c}\right)^2 \frac{mc^2}{\hbar} ka|\ \langle i|\ \sum_\pi (r_\pi/a)^2\ |f\rangle\ |^2.$$

The matrix element of $\Sigma_\pi\ (r_\pi/a)^2$ may be estimated at $(R/a)^2$, where R is the nuclear radius. The rate of the transition is quite low compared with an electric dipole transition, but it is comparable to that of a magnetic 2^4-pole transition if both have an energy of about 1 Mev.

The above-described $0 \rightarrow 0$ transition can occur only if the parity does not change. The alternative $0 \rightarrow 0$, *yes*, transition is strictly forbidden for the emission of a single electron or photon. However, it can take place by the emission of two photons, two electrons, or an electron and a photon.* It is a higher order process and therefore much slower than the $0 \rightarrow 0$, *no*, transition. This event is distinguished from the others by the fact that the energy is divided between the two emitted particles, and therefore the radiation appears as a weak continuum.

9-6 Nuclear isomerism. The occurrence of long-lived, low-lying excited states is rather common among nuclei of intermediate and large mass. These metastable states are usually referred to as *isomeric* states of the nucleus or, on occasion, just as *nuclear isomers*. They are detected by means of the radiations emitted in the transition to lower states. Observed lifetimes vary over wide limits, the upper limit being in the neighborhood of 10^8 years. To specify a lower limit, it is necessary to make the necessarily arbitrary decision as to what is meant by "long-lived" in order to distinguish an isomeric state from an ordinary excited state. That decision is usually made on a purely empirical basis; any state having a directly measurable lifetime is referred to as an isomeric state. This definition has the consequence that every excited state will ultimately fall into the category of isomers as techniques improve. However, at the present time, the lower limit occurs in the neighborhood of 10^{-9} sec except in a few special cases, such as that of Li^7, which has an excited state of observed† lifetime about 10^{-13} sec.

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* R. G. Sachs, *Phys. Rev.* **57,** 194 (1940); M. L. Goldberger, *Phys. Rev.* **73,** 1119 (1948).

† Bell and Elliot, *Phys. Rev.* **76,** 168 (1949).

The theoretical interpretation of the existence of isomers was first proposed by Weizsäcker.* He pointed out that the states would be long-lived if the transition to every lower state were necessarily a multipole transition of sufficiently high order. It can be seen from the selection rules, Eq. (9–24), that this is the case if there exists a large difference between the total angular momentum of the isomeric state and the angular momentum of every lower state.

To obtain an estimate of the lifetime for various multipole orders, we note that the matrix element of D_l, which appears in the electric multipole transition probability, is expected to be of the form

$$\langle i|\, D_l \,|f\rangle = \frac{e^*}{l!} R^l,$$

where R is the nuclear radius and e^* is a symbol (of the dimensions of electric charge) for the complicated integral occurring in the matrix element. Then the rate of an electric 2^l-pole transition is given by

$$W_l(\text{elect.}) = \frac{2}{(l!)^2}\,\omega\,\frac{\bar{e}^2}{\hbar c}\left(\frac{R\omega}{c}\right)^{2l}(1+\alpha_l),$$

if $\bar{e}^2$ is the average of $(e^*)^2$ over all directions of emission. Similarly, the matrix element of the ordinary magnetic multipole moment, Eq. (9–28), may be written as

$$\langle i|\, \mathfrak{M}_l \,|f\rangle = \frac{\mu^*}{(l-1)!}\,\frac{e\hbar}{2Mc}\,R^{l-1},$$

if μ^* is the effective magnetic moment. The rate of a magnetic 2^l-pole transition is then

$$W_l(\text{mag.}) = \frac{1}{2[(l-1)!]^2}\,\omega\bar{\mu}^2\,\frac{e^2}{\hbar c}\left(\frac{R\omega}{c}\right)^{2l}\left(\frac{\hbar}{RMc}\right)^2(1+\beta_l).$$

A reasonable estimate of the nuclear radius is $R \approx \frac{1}{2}(e^2/mc^2)A^{1/3}$, whence

$$R\omega/c \approx \frac{1}{137}\,\frac{\hbar\omega}{2mc^2}\,A^{1/3}. \tag{9–47}$$

Thus, even at $\hbar\omega = 1$ Mev, which is a very high energy for an isomeric transition, $R\omega/c$ is less than 1/20, and the decrease of transition probability with increasing l is quite rapid.

A numerical estimate of the lifetime can be made by assuming that $\bar{e} \approx e$, $\bar{\mu} \approx 1$. Then if use is made of Eq. (9–47), the half-life in each case becomes

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* *Naturwiss.* **24**, 813 (1936).

$$\ln 2/W_l(\text{elect.}) \approx 2(l!)^2 \frac{\left(137 \times \frac{2mc^2}{\hbar\omega}\right)^{2l+1}}{1+\alpha_l} A^{-2l/3} \times 10^{-22}\,\text{sec}, \quad (9\text{-}48a)$$

and

$$\ln 2/W_l(\text{mag.}) \approx 2\left(\frac{Ml}{137m}\right)^2 [(l-1)!]^2 \frac{\left(137 \times \frac{2mc^2}{\hbar\omega}\right)^{2l+1}}{1+\beta_l} A^{-2(l-1)/3} \times 10^{-22}\,\text{sec}. \quad (9\text{-}48b)$$

Values of these half-lives are presented for typical cases in Table 9-2, which was constructed with the aid of conversion coefficients taken from the table of Rose, *et al.** Although the estimates are very rough as to absolute value, they do correctly represent the trend as a function of photon energy and mass number, and they illustrate the great range of half-lives that can be accounted for in this way.

A reasonably exact calculation of the lifetime requires explicit use of a nuclear model, and the results are apt to be quite sensitive to the details of the model because the lth power of the coordinate of each proton occurs in the moment operator.† Consequently the identification of the multipole order of a transition is not easily made on the basis of a lifetime measurement. A much more reliable method for identifying the multipole order is provided by the internal conversion process. Conversion coefficients are essentially independent of nuclear structure and they are quite sensitive to multipole order. Therefore a comparison of the observed and calculated conversion coefficients gives a very direct means of identifying the transition. Or, if the measurement of the radiation intensity is particularly difficult, the ratio of the K conversion coefficient to the L conversion

TABLE 9-2

Estimated half-lives of multipole transitions for $A = 125$, $Z = 40$

Description	Energy (kev)	$l = 1$ (sec)	$l = 2$ (sec)	$l = 3$ (sec)	$l = 4$ (sec)	$l = 5$ (sec)
Electric	150	8×10^{-15}	8×10^{-10}	1×10^{-4}	2×10	3×10^{6}
Magnetic	150	3×10^{-11}	8×10^{-7}	4×10^{-1}	2×10^{3}	2×10^{8}
Electric	900	3×10^{-17}	1×10^{-13}	1×10^{-9}	2×10^{-5}	3×10^{-1}
Magnetic	900	1×10^{-13}	1×10^{-10}	4×10^{-7}	4×10^{-3}	8×10

* *Loc. cit.*

† A complete calculation of the multipole transition probabilities for the Mayer-Jensen model has been made by Stech, *Zs. f. Naturforschung* **7a**, 401 (1952). See also S. A. Moszkowski, *Phys. Rev.* **83**, 1071 (1951) and *Phys. Rev.* **89**, 474 (1953); and V. F. Weisskopf, *Phys. Rev.* **83**, 1073 (1951).

coefficient can be compared with theory. This is just the ratio of the intensities of the K and L conversion electrons. The two groups of electrons can be separated by means of the difference in their energy if the energy is not too high. The K/L ratio is quite sensitive to multipole order* at low energy, but for values of the energy large compared with the K ionization potential this sensitivity disappears. As a matter of fact, it can be seen from Eq. (9–46) that the ratio is just $(a_L/a_K)^3 = 8$ in the limiting case of negligible ionization potential.

Since the low energy cases are of particular interest, the K/L ratio provides a very useful spectroscopic device. Unfortunately, the L conversion coefficients have still to be calculated with sufficient accuracy to provide the theoretical information required for this purpose. However, Goldhaber and Sunyar† have constructed semiempirical curves of K/L ratios as a function of energy for given multipole order by making use of cases for which the observed K conversion coefficient provides the necessary identification. These curves have been used to identify multipolarity in many cases for which the only available conversion data are the K/L ratios.

Although, as we have remarked, the measured lifetime of a transition does not provide a dependable direct determination of multipole order, it can be used indirectly after a sufficiently large number of identifications have been made on other nuclei. The point is that although Eqs. (9–48) do not provide an accurate value of the magnitude of the lifetime, they certainly describe correctly the dependence of lifetime on energy and mass number. If $t_{\frac{1}{2}}$ is the observed lifetime, α the observed conversion coefficient, and $\lambda = l$ or $\lambda = l - 1$, for those transitions which are identified as electric or magnetic 2^l-pole, respectively, the points in a log-log plot of $A^{\frac{2}{3}\lambda}\, t_{\frac{1}{2}}(1 + \alpha)$ against energy should lie close to a straight line. Deviations from the straight line would be introduced by variations in the matrix element of the multipole moment from nucleus-to-nucleus, but these would appear to be quite small on such a plot because the lifetime is so sensitive to energy and multipolarity. When sufficient data are available to construct such curves for every interesting multipole order, it will be possible to make a reasonably safe identification of new transitions on the basis of lifetime by simply determining which curve on the lifetime plot lies closest to the observed point. The reliability of this procedure depends very much on the observed scatter about the straight line. It has proved to be particularly successful for M4 transitions.‡

The most notable aspect of the data on nuclear isomers is that there

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* Compare Hebb and Nelson, *Phys. Rev.* **58,** 486 (1940).

† *Phys. Rev.* **83,** 906 (1951).

‡ Goldhaber and Sunyar, *loc. cit.* We use here and henceforth the term "El transition" and "Ml transition" for electric and magnetic 2^l-pole transitions, respectively.

exist so many examples of isomeric states. Even double isomers, nuclei having two metastable excited states, are found to occur rather frequently. Evidently the occurrence of large angular momentum differences between the low-lying states is the rule rather than the exception. We remarked in Section 8–9 that large differences in angular momentum between the first few excited states are expected in the Mayer-Jensen coupling scheme. These differences arise as follows: when the number of nucleons in a nucleus is somewhat less than is required to form a closed shell, the odd nucleon may go into either the high angular momentum state which has been introduced into the shell grouping by spin-orbit coupling, or it may go into one of the low angular momentum states. For example, there is close competition between the $g_{9/2}$ state and the $p_{1/2}$ state before the shell is closed at 50 (compare Appendix 5). This means that the energy difference between these two states may be so small that they form the two lowest energy levels of the nucleus, which is just what is required to account for isomerism.

Such reasoning leads to the conclusion that isomers are most likely to occur in odd-even nuclei when the odd number of nucleons is great enough to partially fill a shell, since the first few states to be occupied in a given shell have angular momenta differing but little one from the other. That this is indeed the case is illustrated by Fig. 9-6, which shows the distribution of observed isomers.* Although other explanations of the distribution can undoubtedly be constructed, the correlation which occurs here certainly lends strong support to the view that the Mayer-Jensen coupling scheme gives the right sort of explanation of the magic numbers.

A more detailed evaluation of the Mayer-Jensen scheme can be made by determining the multipole orders of each of the isomeric transitions. The parity and angular momentum changes in the transitions can thereby be determined and compared with those suggested by Mayer-Jensen coupling for the first few states. This program has been carried out for the available data by Goldhaber and Hill,† with results that lend strong support to the coupling scheme. There are a few cases in which a mild readjustment of the *j-j* coupling pattern is required.‡ For example, some of the Kr isomers undergo E3 transitions which can occur only if the excited state belongs to a $(g^3_{9/2})_{7/2}$ configuration. In other words, the *j-j* coupling between pairs of like nucleons must be weak enough for the excited state formed by reorienting the j vectors in the ground state configuration to be lower than the state obtained by simple excitation of a single nucleon; the energy of $(g^3_{9/2})_{7/2}$ must be less than that of $(g^2_{9/2})_0 p_{1/2}$.

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* This distribution curve is taken from Goldhaber and Hill, *Rev. Mod. Phys.* **24**, 179 (1952).

† *Loc. cit.*

‡ See Goldhaber and Sunyar, *loc. cit.* Also compare the ground state assignments for Na^{23} and Mn^{55} given in Appendix 5.

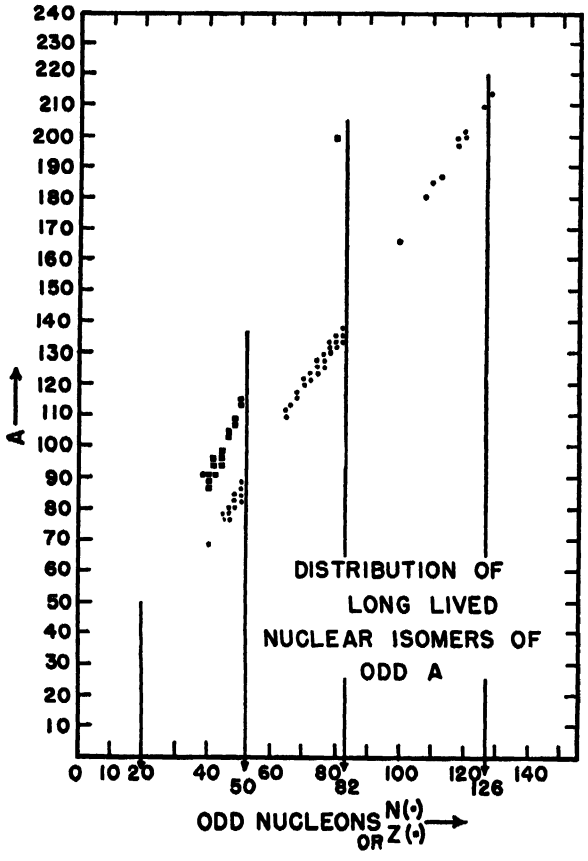

FIG. 9–6. Distribution of observed isomers. Adapted from Goldhaber and Hill, *loc. cit.*

It must again be emphasized that the general success of the Mayer-Jensen scheme does not imply at all that the wave functions are correctly given by the independent particle model. Only the total angular momentum and parity of the isomeric states are involved in determining the multipolarity of the transition. Thus the only definite information provided by the data under discussion is that the Mayer-Jensen coupling correctly prescribes these properties. Values of isomeric lifetimes calculated on the basis of the model are not in good agreement with observations,* and this is just the kind of failure that one would expect if the wave function contains a large admixture of terms other than that suggested by the model.

Another indication of admixture is the occurrence of electric transitions

* Stech, *loc. cit.;* Moszkowski, *loc. cit.;* Weisskopf, *loc. cit.*

in odd neutron nuclei. Goldhaber and Sunyar list many E3 transitions in such nuclei. If the transition were due entirely to a change in the state of the odd neutron, the rate would be very slow indeed, because the electric multipole moments vanish for the neutron. Only the recoil of the rest of the nucleus could provide the shift in electric charge required to produce the transition, and since the displacement of the core is $1/A$ times the displacement of the neutron, the electric multipole moment of the core is, according to Eq. (9–21), Z/A^l times smaller than the moment of a single odd proton outside the core. Thus the motion of the core leads to a transition probability $(Z/A^l)^2$ times smaller than the corresponding rate for an odd proton transition. But the observed lifetimes for electric transitions in the odd neutron nuclei with $A \geq 80$ are comparable to those for odd proton nuclei even for $l = 3$. The most reasonable explanation of this phenomenon would seem to be that part of the wave function of the excited state has a proton configuration different from that of the ground state. The transition probability would then be proportional to the square of the amplitude of the admixed state. If that is the case, the observed lifetime for the transition provides a direct measure of the probability of the admixed state. On the basis of the data mentioned above, a probability of 5 percent is rather common, so the amplitude of the admixed term would have to be about 0.20. This is the total amplitude of those states capable of contributing to the transition. Other states must also occur, and we are again led to the conclusion that large deviations from the wave functions of the Mayer-Jensen model are to be expected.

The data on nuclear isomers also offer some enlightenment concerning the existence of exchange currents. It has been mentioned before that there exist several double isomers. One of the transitions in most of these cases is of type M1, a fact which is particularly important because the M1 transitions are subject to a selection rule having no analog in transitions of any other type. The selection rule applies strictly only if there is no exchange current. In that case the rate of M1 emission is governed by the matrix element of the ordinary magnetic dipole moment, Eq. (9–28):

$$\mathfrak{M}_1 = \left(\mathbf{u} \times \boldsymbol{\kappa} \cdot \left[\sum_\pi (\mathbf{l}_\pi + \mu_p \boldsymbol{\sigma}_\pi) + \mu_n \sum_\nu \boldsymbol{\sigma}_\nu\right]\right).$$

If we consider states of a nucleus for which each nucleon has a specified l-value (orbital angular momentum), the operator $\mathfrak{M}_1$ clearly has no matrix element between those differing in l-value, even if the difference is limited to only one nucleon;* the operator $\mathbf{l}_\pi$ is diagonal in l, and $\boldsymbol{\sigma}_\pi$ or $\boldsymbol{\sigma}_\nu$ certainly cannot lead to a change in l. $\mathfrak{M}_1$ differs from the higher order multipole moments in this respect because it contains no space-dependent factor.

* Note also that the principal quantum numbers will not change because of the orthogonality of the corresponding radial functions.

The introduction of a factor of that kind immediately leads to matrix elements between states of different l. Therefore an exchange magnetic moment will cause a violation of the selection rule.

Among the observed M1 transitions there are some which can be described as a shift of the odd nucleon between the $d_{3/2}$ and $s_{1/2}$ states. This assignment is quite consistent with the order of levels suggested by Mayer-Jensen coupling, and it is consistent with the other data on the levels. But the transition violates the selection rule for ordinary M1 transitions. It might be suggested that we have here added evidence for a departure from the pure independent particle wave functions. However, the M1 selection rule is so exclusive that a very special choice of admixed states would be required to account for the transition. Even if those states occur, their amplitudes of admixture appear at least to the second power in the calculated lifetime, because the lifetime involves the square of the matrix element. This is to be contrasted with the situation for the static moments, which depend linearly on the amplitude of admixture. It turns out* that those states having the correct properties to produce the transition must have quite large amplitudes (0.50) to account for the observed lifetime in the best known case (Xe^{131}). Since there is every reason to expect that other terms would also occur in the wave function, this explanation of the M1 transition would seem to suggest that the Mayer-Jensen term appears with a very small amplitude in the wave function.

It may be more reasonable to ascribe the M1 transitions to the exchange currents, especially in view of the fact that there is already other evidence for their existence. The space-exchange current, Eq. (4–10), does not lead to the $d_{3/2}$-$s_{1/2}$ transition in a pure independent-particle model, but the spin-exchange moments, Eqs. (9–38c) and (9–38d), do. If the latter moments are adjusted to fit the anomalies in the H^3 and He^3 moments, they lead to a lifetime comparable in order of magnitude to the observed values for the transitions in question.†

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* M. Ross, *loc. cit.* Ross shows that for any reasonable selection of admixed terms it is necessary to assume that the matrix element arises from admixed terms in both the initial and final states, so the lifetime actually depends on the fourth power of the amplitude of admixture. He lists the existing data on these transitions and provides those estimates of lifetime due to exchange moments which are mentioned below.

† Graham and Bell, *Can. J. of Physics* **31,** 377 (1953), have measured a number of lifetimes of forbidden M1 transitions other than that of Xe^{131}, and their results seem to be in excellent agreement with the interpretation in terms of a spin-exchange current.

CHAPTER 10

NUCLEAR REACTIONS

Observation of the properties of virtual states of nuclei is usually accomplished by means of some nuclear reaction. A target nucleus of mass A' and atomic number Z' is bombarded by a nuclear particle of mass A'' and atomic number Z''. The events that occur during the collision of these two nuclei then refer to states in the continuum of the nucleus of mass $A = A' + A''$ and atomic number $Z = Z' + Z''$. Processes which may occur are elastic or inelastic scattering, radiative capture, or capture with the subsequent emission of a pair (or more) of product nuclei differing in either A or Z values from the original pair of nuclei. The cross sections for these processes show a marked change when the relative kinetic energy of the initial nuclei is nearly equal to the energy E_λ of a virtual state of the nucleus (A,Z). The energy E_λ is said to correspond to a *resonance* in the nuclear reaction, since the cross sections will usually show a strong maximum at or near E_λ.

The behavior of the cross section as a function of energy in the neighborhood of E_λ has for many years been assumed to have essentially the same form as the cross section of an atom for resonance radiation.* This formula, with appropriate modifications to take account of the special characteristics of nuclei, is known as the Breit-Wigner formula.** It is capable of providing a good fit to many observed cross-section curves. The fact that an analogy with the purely electromagnetic atomic phenomenon works well is somewhat surprising, since electromagnetic waves interact weakly with atoms while the interactions between nuclei are quite strong. Therefore many attempts have been made to provide a derivation of the Breit-Wigner formula that takes account of the strong nuclear interaction.† A really satisfactory treatment of the problem was finally provided by Wigner, Eisenbud, and Teichmann.‡ Theirs is the treatment to be fol-

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* See, e.g., P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford (1935), p. 203 ff.

** Breit and Wigner, *Phys. Rev.* **49,** 519 (1936).

† Kapur and Peierls, *Proc. Roy. Soc.* (London) **A166,** 277 (1938); Bethe and Placzek, *Phys. Rev.* **51,** 450 (1937); G. Breit, *Phys. Rev.* **58,** 506 and 1068 (1940), *ibid.* **69,** 472 (1946); Feshbach, Peaslee, and Weisskopf, *Phys. Rev.* **71,** 145 (1947).

‡ Wigner and Eisenbud, *Phys. Rev.* **72,** 29 (1947); Wigner, *Phys. Rev.* **73,** 1002 (1948); T. Teichmann, *Phys. Rev.* **77,** 506 (1950); Teichmann and Wigner, *Phys. Rev.* **87,** 123 (1952). See also Wigner, *Am. J. of Phys.* **17,** 99 (1949).

lowed here in the derivation of the resonance formula. However, the rigorous handling of the problem is preceded by a discussion of some of the qualitative features of the reactions.

Although in most cases the information provided by a study of nuclear reactions concerns the virtual states of nuclei, certain reactions may be used to investigate the properties of bound states. For example, in inelastic scattering, the target nucleus (A',Z') is usually left in an excited bound state and therefore the reaction depends on the properties of the excited state. Another and important type of reaction has recently received much attention in this connection, namely, the *stripping reaction* which often occurs when the bombarding nucleus ("projectile") is the deuteron. A (d,n) or (d,p) reaction may occur with the formation of a bound state of nucleus $(A' + 1,Z' + 1)$ or $(A' + 1,Z')$, respectively, and the angular distribution of the emitted neutron or proton depends strongly on the properties of the state so formed.* This happens when the discrete virtual states of nucleus $(A' + 2,Z' + 1)$ do not play an important role in the reaction. Hence the Breit-Wigner formula is not applicable and a quite different formulation of the theory is required. A complete theoretical treatment of the problem has not been given at the time of this writing, so our attention will be directed to its qualitative features.

10–1 The compound nucleus. In our discussion of the virtual states, it has been tacitly assumed that such states represent a reasonably stable configuration. Since, by definition, a virtual state of a nucleus is always capable of dissociation, some account of the causes leading to stability, or rather to metastability, must be given. Bohr's† very graphic description of the nuclear reaction in terms of the compound nucleus leads to a nice qualitative explanation of this phenomenon.

The term "compound nucleus" refers to the nucleus (A,Z) formed by combining the projectile nucleus (A'',Z'') with the target nucleus. Since the interactions between nuclear particles are strong and of short range, it may be expected that, during the collision, the energy of the projectile is rapidly distributed among the many nucleons comprising the compound nucleus. Then no one nucleon has sufficient energy to be emitted. Eventually, a single nucleon or other nuclear entity may, by a statistical fluctuation, acquire enough energy to emerge from the compound nucleus. If the probability for the occurrence of such a fluctuation is small, the compound nucleus may have time to undergo a radiative transition to a bound state, in which case we speak of a radiative capture having taken place. On the other hand, if the dissipation of the energy of the original projectile takes place slowly, the projectile may be re-emitted, in which case we

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* S. T. Butler, *Proc. Roy. Soc.* (London) **A208,** 559 (1951).

† *Nature,* **137,** 344 (1936).

speak of elastic (or, possibly, inelastic) scattering. The relative probability of scattering, radiative capture, or capture with the emission of some other nuclear particle depends in a detailed way on the process of energy transfer, and any one of them may be predominant in a given situation.

When the chance that the compound nucleus will emit a particle or radiation is great, its energy state is not very well defined since, strictly speaking, a state of given energy must be a stationary state. This lack of definition of the energy of the virtual state can be expressed quantitatively as the width, Γ_λ, in energy, of the level at E_λ. The width Γ_λ may be defined as the range of energy in which the resonance cross section drops from its maximum value to half the maximum value. According to the usual interpretation of the complementary relationship between energy and time, it is expected that Γ_λ will be inversely proportional to the lifetime Δt_λ of the state of the compound nucleus:

$$\Gamma_\lambda \approx \hbar/\Delta t_\lambda. \tag{10-1}$$

In other words, Γ_λ provides a direct measure of the transition probability for the emission of a nuclear particle or photon from the virtual state. When several types of particle are emitted, a width can be assigned to each on the basis of its relative probability. For example, if emission of the original projectile particle (elastic scattering), inelastic scattering, alpha-particle emission, and the emission of gamma radiation can occur, we have

$$\Gamma_\lambda = \Gamma_{\lambda 0} + \Gamma_{\lambda i} + \Gamma_{\lambda \alpha} + \Gamma_{\lambda \gamma},$$

where the partial widths $\Gamma_{\lambda 0}$, $\Gamma_{\lambda i}$, etc., correspond to the indicated reactions.

One may well ask how the present description of the virtual states is related to the description presented in Section 8–9. There, we noted that stable virtual states may be formed in the independent particle model by multinucleon excitation. The connection between these states and the states of the continuum is provided by the interactions which mix independent particle functions. This process of mixing functions corresponds to the process referred to as interchange of energy in the Bohr model. If the functions are mixed strongly, as suggested by the Bohr description, the independent particle model gives a particularly poor representation of the wave function.

The amount of admixture is closely related to the partial widths, $\Gamma_{\lambda s}$, of a level. In the absence of any admixture, the incident particle can only be scattered by the auxiliary potential associated with the independent particle model. Thus the cross section would have the smooth dependence on energy characteristic of potential scattering; the "reaction" in this case would not involve a compound state, and its description at energy E

would be given by the wave function $\psi_p^0(E)$ associated with the auxiliary potential. Let us now suppose that a stable virtual state ψ_λ^0, corresponding to many-nucleon excitation, occurs at energy E_λ in the independent particle model. Then a small admixture of ψ_λ^0 into ψ_p^0 would lead to a disturbance of the scattering cross section in the neighborhood of energy E_λ. The width of the disturbance would be the natural width Γ_λ of the level. As the amount of admixture is increased, the partial width, $\Gamma_{\lambda 0}$, increases, so the level grows in width.

This latter phenomenon can probably be understood more easily by considering the influence of admixtures on the state ψ_λ^0. In the independent particle model, ψ_λ^0 is pure, and the only transition it can undergo is a radiative transition to a lower state. The total width due to such transitions is $\Gamma_{\lambda\gamma}$. The mixing of states makes possible the disintegration of ψ_λ^0 by other means. For example, a state ψ_α^0 corresponding to emission of an alpha-particle may be combined with ψ_λ^0 to form a true energy state of the system. Then the probability of emission of the alpha-particle, $\Gamma_{\lambda\alpha}$, would be a measure of the amount of admixture.

A schematic representation of the general situation can be obtained in terms of the various independent particle functions ψ_s^0 of the states that are mixed together to form a true energy state. These functions may include bound states corresponding to various configurations, or continuum states corresponding to decay into any one of the energetically possible dissociated modes. A different value of the index s is assigned to each such mode, and we let $s = \lambda$ refer to the original bound state of energy E_λ, and $s = 0$ to the outgoing part of the function describing potential scattering of the incident projectile. Because the wave functions involve mixtures of continuum functions, the energy spectrum is continuous. A truly stationary function associated with energy E will be denoted by $\psi(E)$. Then $\psi(E)$ is to be expressed as a linear combination of the ψ_s^0. If E_s denotes a characteristic value of the energy associated with the independent particle state s, the linear combination may be expressed as

$$\psi(E) = \sum_s \int B_s(E,E_s)\psi_s^0(E_s)\,dE_s + \psi_{\text{inc.}}(E), \tag{10–2}$$

where $\psi_{\text{inc.}}$ is the incident wave. The integral occurs, of course, only for those states s with a continuous spectrum E_s.

An insight into the meaning of the coefficients $B_s(E,E_s)$ can be obtained by restricting attention to the special case in which s takes on only the two values $s = \lambda$ and $s = 0$. Since the spectrum of E_λ is discrete (we are interested only in one value), $\psi(E)$ then takes the form

$$\psi(E) = B_\lambda(E)\psi_\lambda^0 + \int B_0(E,E')\psi_0^0(E')\,dE' + \psi_{\text{inc.}}(E).$$

The probability for finding the system in the bound state ψ_λ^0 is proportional to $|B_\lambda(E)|^2$ and this function of E may be expected* to have a strong maximum at $E = E_\lambda$, the width of the peak being about equal to $\Gamma_{\lambda 0}$. Thus it is $|B_\lambda(E)|^2$ that describes the "shape of the level" at E_λ.

When E is quite distant from E_λ, we expect that $B_\lambda(E) \approx 0$. Furthermore, the wave function is expected to be closely approximated by the potential scattered state $\psi_p^0(E) = \psi_{\text{inc.}}(E) + \psi_0^0(E)$, consequently the coefficients B_0 must have nearly the form

$$B_0(E,E') \approx \delta(E - E').$$

On the other hand, when $E \approx E_\lambda$, the delta-function is spread out, $B_0(E_\lambda,E')$ extends over a finite interval as a function of E'. The width of this interval is greater the greater the amount of admixture of states, i.e., the greater the deviation from the independent particle model. Now $|B_0(E,E')|^2$ measures the probability for finding the system in its dissociated state. Therefore the integral over E' of $|B_0(E_\lambda,E')|^2$ gives a measure of the probability for dissociation when the system is initially in the bound state, therefore it, too, is proportional to the partial width $\Gamma_{\lambda 0}$. This means that the amount of admixture of states and the natural width are closely related; the greater the amount of admixture of a given type of dissociated state, the greater the associated partial width.

There is a limit on the magnitude of $\Gamma_{\lambda 0}$, or, for that matter, on the magnitude of any partial width, $\Gamma_{\lambda s}$. This limit can be established by means of a normalization condition on the function $\psi(E)$ which is derived in Section 10–5. The widths are found there to satisfy the sum rule

$$\sum_s [\Gamma_{\lambda s}/\Gamma_p^l(E_\lambda)] = 1, \tag{10–3}$$

where the sum includes the values of s corresponding to every state of excitation and of relative orbital angular momentum l of a given pair of fragments. Although the energy is too low to produce most of these states, they may be mixed into the wave function, and $\Gamma_{\lambda s}$ is then meant to be an appropriate measure of the amount of admixture. The energy-dependent function $\Gamma_p^l(E)$ is the width† obtained by fitting the scattering cross section

* A convincing way to arrive at the conclusions stated here *et seq.* is to assume that the mixing of states is caused by a weak perturbation. The continuous spectrum E' should also be replaced by a discrete spectrum by introducing a large enclosure. Then there are a number of states ψ_p^0 which are almost degenerate with ψ_λ^0, and these are linearly combined by the perturbation. Note that the perturbation is to be assumed large compared with the spacing of levels E', so the limit of an infinite enclosure must be taken before letting the perturbation go to zero.

† Γ_p^l depends on the energy of the incident wave, not alone on the energy of the resonance, because the resonance is so broad. Compare Eq. (4–52), in which k^2 plays the role of $(\Gamma_p^0)^2$. The form of the energy dependence is governed by the orbital angular momentum of the scattered wave.

for a pure potential to a broad resonance. Note that the maximum of this cross section occurs at zero energy for S-scattering of neutral particles (the cross section decreases monotonically) but the imposition of a centrifugal or coulomb barrier shifts the peak to higher energy.*

Since all $\Gamma_{\lambda s}$ are positive, no one of them can be larger than Γ_p^l. In fact, the partial widths for dissociation associated with a true metastable state of the compound nucleus will be much smaller than Γ_p^l. For let us consider again the state of many-nucleon excitation ψ_λ^0. This complex function can have little average overlap with the relatively simple function describing *any* type of dissociation. The mixing of states is governed by the matrix element of some sort of interaction, and the lack of overlap implies that the matrix element is small, so $\Gamma_{\lambda 0} \ll \Gamma_p^l$.

Only when the functions are very well matched will $\Gamma_{\lambda 0}$ approach Γ_p^l, and this is the case only if the two functions are essentially the same, namely, when the resonance corresponds to simple potential scattering. Thus the observation of a resonance for which $\Gamma_{\lambda 0} \approx \Gamma_p^l$ implies that nearly pure potential scattering would occur for incident particles of type 0, that the amount of admixture of bound states into ψ_p^0 is quite small and that the associated "level" E_λ is just one of the peaks of the potential scattering curve. Thus we are led into a seeming contradiction: as the amount of admixture of bound states into ψ_p^0 increases $\Gamma_{\lambda 0}$ increases, but $\Gamma_{\lambda 0}$ can attain a really large value only if ψ_p^0 makes a large contribution to the wave function. However, in the two cases we are dealing with resonances of quite different characters. The small values of $\Gamma_{\lambda 0}$ concern "true" levels E_λ associated with complex states of the compound nucleus, while the large values concern the lump in the potential scattering curve which occurs at some other value of E (and is also assigned a label λ). The position in energy of the lump is determined by the properties of the potential and the orbital angular momentum of the dissociated system. If there were no mixing of the functions characteristic of the independent particle model, the only apparent resonances would be these lumps, and they would provide no information about the multitude of stable states formed by all the different combinations of many-nucleon excitation. But the mixing of functions causes the stable states to become metastable

* Compare the curves for potential scattering given by Mott and Massey, *The Theory of Atomic Collisions*, Oxford University Press, 2nd edition (1949), p. 33. It should be noted that potential scattering can always be analyzed into an infinite number of resonances (see Mott and Massey, p. 41 ff.) but these resonances are false and have nothing to do with the above-mentioned resonances, which actually appear as peaks in the cross-section curve. The false resonances are so wide and phased in such a way that they may lead to a perfectly smooth cross-section curve. Unfortunately, these false resonances appear in any detailed (many-level) resonance theory on the same footing as the true resonances which correspond to metastable states of the compound nucleus, and they are a constant source of confusion.

and to influence the cross section; and as the mixing increases the lumps are reduced in width (as is indicated by the sum rule) and obscured by the greatly increased number of observed resonances.

From this discussion it should be clear that the density in energy of observed resonance states may be smaller than the actual density of excited states of the nucleus. The latter quantity is comparatively independent of the nuclear model, since it involves essentially the counting of the number of wave functions which comprise a complete set. The density of observed states, on the other hand, is governed by the sensitivity of the method of observation. Only the states interacting with the incident wave in sufficient strength to cause a detectable change in the cross section will be noticed. The density of observable states can be roughly related to the widths of the detectable energy levels, and thus to the amount of mixing of states. To see this relationship, let us consider a situation in which the amount of mixing of states is large. Then for a given type, s, of dissociation of the compound nucleus, there will be contributions from many bound states λ. But the $\Gamma_{\lambda s}$ are proportional to the square of a mixing coefficient, so they must satisfy another normalization condition, which is shown in Section 10–5 to take the form

$$\sum_{\lambda} \Gamma_{\lambda s} \approx \Gamma_p'.$$

Presumably the amplitudes of the various admixed states are nearly the same, so all $\Gamma_{\lambda s}$ are of about the same magnitude. Therefore the sum rule implies that, on the average,

$$\Gamma_{\lambda s} \approx \frac{\Gamma_p'}{W} D, \tag{10–4}$$

where W is the spread in the E_λ values of the states ψ_λ^0 which are strongly mixed with ψ_s^0, and D is the mean spacing between these levels. Our conclusion is then that the observed width of a level will, on the average, be proportional to the spacing of observed levels. It is evident from the nature of the arguments used here that quite large deviations from the rule $\Gamma_{\lambda s} \sim D$ can be expected for individual resonances, especially in the case of slight mixing. Nevertheless it is expected that, on the whole, the widely spaced levels of light nuclei will have widths approaching Γ_p, while the widths of the levels of the more complex nuclei will be much smaller. The fact that levels of light nuclei should have such large widths is consistent with the notion that even in the independent particle model almost every excited state of a very light nucleus must lie in the continuum, hence correspond to potential scattering.

It should be remarked that our extensive use of the independent particle model is not an essential part of the qualitative arguments presented here, since any set of bound states ψ_λ^0 would lead to substantially the same con-

clusions. The separation of the bound from the dissociated states, however, is an essential step which has its counterpart in any theory that leads to isolated true resonances. The emphasis on the independent particle model comes, of course, because it is particularly interesting to know that the resonance phenomena may provide some test of the model.

We call attention to the fact that for the states of relatively high excitation considered here, the simple connection with the details of the coupling scheme in the independent particle model is lost. Only for the very light nuclei can we expect the few virtual states to be closely related to each other and to the ground state. In heavier nuclei, a large number of states occur between the lowest virtual state and the ground state, and there are many virtual states. These are certainly expected to be confused mixtures of independent particle functions, and little direct information concerning the coupling scheme is to be obtained from them.

The arguments presented up to this point have been qualitative, and lead to results which are still to be put in quantitative form. That is the object of the following four sections. In Section 10 2 we shall obtain the general connection between the asymptotic wave functions and the cross section. The form of the asymptotic function is determined by the condition that it tie in smoothly, at the nuclear surface, with the bound state functions of the compound nucleus. The fact that such a tie-in exists indicates that the functions are not truly bound, but a quite general definition of an almost bound function X_λ with energy states E_λ is given in Section 10–3. The connection conditions at the nuclear surface are then shown to lead to a many-level dispersion phenomenon in the cross section. For purposes of practical application it is important to isolate the effect of a single level, and this is accomplished in Section 10–4, wherein the approximations required to obtain the Breit-Wigner single-level formula are discussed. Finally, in Section 10–5, some attention is given to application and interpretation of the one-level formula.

10–2 The collision matrix. Quantitative results of observations on nuclear reactions are usually expressed in terms of a differential cross section $d\sigma$ for the emission of the products into a solid angle $d\Omega$. The connection between these data and the properties of the compound nucleus involves the wave function of the system for a state having the energy at which the observation is made. But only the asymptotic behavior of the wave function, that is, its behavior for large separation of reactants or products of the reaction, enters directly into the calculation of the cross section. In the asymptotic region of space, the specifically nuclear interactions between the reacting pair or between the products can be ignored. The influence of nuclear interactions is imposed on the wave function when the reactants or products are within the range of nuclear forces, and that corresponds to a very small region of the available space. Hence the way in

which the interactions act to determine the wave function is, in effect, to impose a boundary condition on the wave function for some small separation of the reactants. This boundary condition fixes the ratio of the two parts of the wave function known as the "regular" and "irregular" parts, and that ratio, in turn, fixes the cross section. Our first concern is the connection between this ratio, or rather these ratios, which are to be described in terms of a collision matrix U, and the cross section. We will conform, as far as possible, to the notation of Teichmann and Wigner.*

For the sake of simplicity, attention will be limited to the important case in which no more than two nuclear particles are produced in the reaction. Both particles may be charged, or one may be neutral. A common notation for the regular and irregular functions will be used for both situations; for the charged particles these are to be interpreted as the coulomb functions [compare Eqs. (6-10) and (6-11)], while for the charged-neutral combination they are to be interpreted as the free, spherical waves.

A remark on notation is needed here. A compound nucleus may be able to dissociate into any one of a number of different pairs of nuclei or any one of a number of excited states of a given pair, and it is necessary to identify these modes of disintegration. Furthermore, each member of the pair may have a nuclear spin, so either the orientation of each of the spins is to be specified, or they are to be combined to form a specified total internal angular momentum and associated magnetic quantum number. All of the internal properties of the nuclear pair will be combined into one index, which will appear as the subscript σ. In addition to the internal properties, the relative orbital angular momentum (l,m) will usually be prescribed. Then the asymptotic function for nuclei of type σ is proportional to the spherical harmonic $Y_l^m(\sigma)$, given as a function of the angular position of the vector connecting the two separated nuclei. It will often be convenient to lump the internal and external parameters into one index $s = (\sigma,l,m)$.

The distance between the fragments σ is denoted by either r_σ or its equivalent r_s. Similarly, the reduced mass is M_σ or M_s. The asymptotic radial functions of r_σ multiplied by r_σ are denoted by $F_l(k_\sigma r_\sigma) \equiv F_s$ for the regular function and $G_l(k_\sigma r_\sigma) \equiv G_s$ for the irregular function. These functions have the property that

$$E_s = G_s + iF_s \tag{10-5a}$$

represents an outgoing spherical wave in the asymptotic region, and

$$I_s = G_s - iF_s \tag{10-5b}$$

represents an incoming wave. In place of the normalization used in earlier

* *Loc. cit.*

chapters, we adopt the normalization* that I_s provides unit flux per unit solid angle of incoming and E_s unit flux of outgoing particles. Then the Wronskian condition on the radial functions is

$$F_s'G_s - G_s'F_s = M_\sigma/\hbar, \tag{10–6}$$

where the prime denotes differentiation with respect to r_σ.

Another bit of useful information is obtained by considering the wave function χ_σ that would describe the behavior of a pair of type σ in the absence of a nuclear interaction between the pair. An analysis of χ_σ into spherical waves leads to the well-known result†

$$\chi_\sigma = (4\pi/k_\sigma)^{1/2} r_\sigma^{-1} \sum_l i^l (2l+1)^{1/2} e^{i\eta_l} F_l(k_\sigma r_\sigma) Y_l^0(\sigma),$$

if the asymptotic kinetic energy is $\hbar^2 k_\sigma^2/2M_\sigma$ and if the momentum is directed along the positive z-axis. The quantity η_l is the coulomb phase shift (compare Section 6–2) if both members of the pair are charged, otherwise it vanishes. By means of Eqs. (10–5), the expansion of χ_σ may be written in terms of outgoing and incoming waves:

$$\chi_\sigma = -i(\pi/k_\sigma)^{1/2} r_\sigma^{-1} \sum_l i^l (2l+1)^{1/2} e^{i\eta_l} (E_s - I_s) Y_l^0(\sigma). \tag{10–7}$$

If one member of the pair σ is a neutron, χ_σ is just the incident plane wave with flux k_σ. On the other hand, if each member of the pair carries an electric charge, χ_σ describes the incident plus scattered coulomb wave.

The complete wave function in the asymptotic region is a linear combination of products of functions Ψ_σ, describing the internal behavior of both members of the pair σ, with external functions $I_s Y_l^m(\sigma)$ or $E_s Y_l^m(\sigma)$. If the products $r_\sigma^{-1}\Psi_\sigma Y_l^m(\sigma)$ are denoted by ψ_s, the asymptotic wave function may be written as

$$\varphi_\sigma = \sum_{l,m} \delta_{m0} A_{\sigma l} (\psi_s I_s - \sum_{s'} U_{ss'} \psi_{s'} E_{s'}) \tag{10–8}$$

for a situation in which the reaction is produced by the collision of the pair σ along a line parallel to the z-axis. The matrix U, with elements $U_{ss'}$, is called the *collision matrix*. It is determined, as we shall find below, by the condition that the asymptotic wave function φ_σ must tie in smoothly with the wave function of the compound nucleus. The coefficients $A_{\sigma l}$, on the other hand, are specified by the condition that the only incoming wave

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* The functions used here are obtained by multiplying the usual dimensionless functions by a factor $(M_s/\hbar k)^{1/2}$.

† See Mott and Massey, *The Theory of Atomic Collisions*, Oxford, 2nd edition (1949), p. 22 and p. 46.

be of the type χ_σ, Eq. (10–7). Thus the terms in I_s must drop out of the difference $\varphi_\sigma - \chi_\sigma\Psi_\sigma$, so

$$A_{\sigma l} = i^{l+1}(\pi/k_\sigma)^{1/2}(2l+1)^{1/2}e^{i\eta_l}.$$

Then the asymptotic outgoing wave has the form

$$\varphi_\sigma - (\chi_\sigma - C_\sigma)\Psi_\sigma = -i(\pi/k_\sigma)^{1/2}\sum_{l,s'} i^l(2l+1)^{1/2}e^{i\eta_l}(U-1)_{ss'}\psi_{s'}E_{s'} + C_\sigma\Psi_\sigma, \quad (10\text{–}9)$$

where C_σ describes the Rutherford scattered wave:*

$$C_\sigma = -\alpha(M_\sigma/\hbar)^{1/2}(2r_\sigma k_\sigma)^{-1}\operatorname{cosec}^2\tfrac{1}{2}\theta_\sigma \times \exp\{ik_\sigma r_\sigma + 2i\eta_0 - i\alpha\ln[k_\sigma r_\sigma(1-\cos\theta_\sigma)]\}.$$

Here,

$$\alpha = Z'Z''e^2M_\sigma/\hbar^2k_\sigma,$$

and Z' is the charge of the target nucleus, Z'' that of the projectile.

The differential cross section $d\sigma_{\sigma\sigma'}$ for a process initiated by reactants σ producing the pair σ' may now be obtained from Eq. (10–9). The incident current associated with χ_σ is k_σ, the emitted flux associated with $E_{\sigma'}$ is 1, and that associated with C_σ is

$$\pi|\,\rho_\sigma\,|^2k_\sigma^{-1},$$

where the amplitude for Rutherford scattering is

$$\rho_\sigma = -i\,(\alpha/2\sqrt{\pi}\,)\operatorname{cosec}^2(\tfrac{1}{2}\theta_\sigma)\exp[2i\eta_0 - i\alpha\ln\tfrac{1}{2}(1-\cos\theta_\sigma)]. \quad (10\text{–}10)$$

Therefore the differential cross section is†

$$d\sigma_{\sigma\sigma'} = (\pi/k_\sigma^2)\Big|\sum_{l,l'} i^{l-l'}(2l+1)^{1/2}e^{i\eta_l}e^{i\eta_{l'}}(U-1)_{ss'}Y_{l'}^{m'}(\sigma') + \delta_{\sigma\sigma'}\rho_\sigma\Big|^2 d\Omega_{\sigma'}. \quad (10\text{–}11)$$

We make use here of the fact that the Ψ_σ form an orthonormal set of functions of the internal variables. The sum over m' has been dropped because conservation of angular momentum requires that m' be equal to the difference between the magnetic quantum numbers of states σ and σ'.

In the case $\sigma \neq \sigma'$, that is, for any process *other* than elastic scattering, the expression (10–11) reduces to

$$d\sigma_{\sigma\sigma'} = (\pi/k_\sigma^2)\Big|\sum_{l,l'} i^{l-l'}(2l+1)^{1/2}e^{i\eta_l}e^{i\eta_{l'}}U_{ss'}Y_{l'}^{m'}(\sigma')\Big|^2 d\Omega_{\sigma'}. \quad (10\text{–}12)$$

The corresponding total cross section is

$$\sigma_{\sigma\sigma'} = (\pi/k_\sigma^2)\sum_{l'}\Big|\sum_{l} i^l e^{i\eta_l}(2l+1)^{1/2}U_{ss'}\Big|^2, \quad (10\text{–}13)$$

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* Mott and Massey, *op. cit.*, p. 48.

† Note that, asymptotically, $E_s \to i^{-l}e^{i\eta_l}e_\sigma(r)$, where $e_\sigma(r)$, which describes a spherical wave of unit flux, is independent of l.

since the $Y_{l'}^{m'}$ form an orthonormal set. We may also define a partial cross section for production of the state of orbital angular momentum l':

$$\sigma_{\sigma\sigma'}^{l'} = (\pi/k_\sigma^2)\Big|\sum_l i^l e^{i\eta_l}(2l+1)^{1/2}U_{ss'}\Big|^2. \tag{10-14}$$

Finally, the subpartial cross section for a reaction produced by an incident spherical wave of orbital angular momentum l is

$$\sigma_{ss'} = (\pi/k_\sigma^2)(2l+1)\,|\,U_{ss'}\,|^2. \tag{10-15}$$

Note that although $\sigma_{\sigma\sigma'} = \Sigma_{l'}\sigma_{\sigma\sigma'}^{l'}$, no such relationship exists between $\sigma_{\sigma\sigma'}^{l'}$ and $\sigma_{ss'}$, since Eq. (10–14) contains interference terms which are not described by $\sigma_{ss'}$. Nevertheless, $\sigma_{ss'}$ is a quantity of interest since, at low energy, only the partial wave with the smallest possible l will be large enough at the nucleus to give an appreciable contribution to the collision matrix.

Before we consider the relationship between U and the properties of the compound nucleus, several general characteristics of U should be given some attention. For one thing, it is of interest that it is a unitary matrix. This can be established by use of the fact that not only the expression (10–8), but any expression of the form

$$\varphi = \sum_s A_s(\psi_s I_s - \sum_{s'} U_{ss'}\psi_{s'}E_{s'}), \tag{10-16}$$

where the A_s are arbitrary constants, is a stationary solution of the Schroedinger equation in the asymptotic region, and every solution has this asymptotic form. The stationary property means that the total incoming flux described by the terms in I_s must be equal to the total outgoing flux described by terms proportional to $E_{s'}$. Thus

$$\sum_s |\,A_s\,|^2 = \sum_{s,s',s''} A_s^* A_{s'} U_{ss''}^* U_{s's''},$$

or, since the A_s are arbitrary,

$$\sum_{s''} U_{ss''}^* U_{s's''} = \delta_{ss'}.$$

This is equivalent to the condition that U be unitary:

$$UU^\dagger = 1, \tag{10-17}$$

where $U^\dagger$ denotes the hermitian conjugate of U.

Another important property of U is a direct result of the usual assumption that the Hamiltonian of the system is invariant under reversal of the sense of time. It is shown in Appendix 3 that, as a consequence of this assumption, there is associated with any stationary state φ of the system another state $K\varphi$ of the same energy. The operator K produces the conjugate complex of any constants or radial functions appearing in φ, and it

both multiplies by i^{2m} and changes the sign of m in any characteristic function of the angular momentum having magnetic quantum number m. Let us therefore consider a state φ having the value m_J for the z-component of the total angular momentum, and having the asymptotic form given by Eq. (10–16). Then the asymptotic form of $K\varphi$ is*

$$K\varphi \to i^{2m_J} \sum_s A_s^* (\psi_{-s} I_s^* - \sum_{s'} U_{ss'}^* \psi_{-s'} E_{s'}^*),$$

if the index $-s$ is defined as that derived from s by changing the signs of all magnetic quantum numbers. Now it is quite clear from their definitions, Eqs. (10–5), that $I_s^* = E_s$ and $E_s^* = I_s$, since F_s and G_s are real functions. Hence, if we set $A_s^* = -i^{-2m_J} U_{ss''}$, the new asymptotic solution is

$$K\varphi = \psi_{-s''} I_{s''} - \sum_s U_{ss''} \psi_{-s} E_s,$$

as a consequence of Eq. (10–17). But this function describes a state which must also be expressible in the form Eq. (10–16). The linear independence of the ψ_s therefore has the consequence that

$$U_{ss'} = U_{-s',-s}. \tag{10–18}$$

This result may be used to establish the principle of detailed balance for nuclear reactions. For example, the cross section for the reaction $s \to s'$ produced by an incoming spherical wave of angular momentum l is given by Eq. (10–15), while that for the transition $-s' \to -s$ is given by

$$\sigma_{-s',-s} = (\pi/k_\sigma^2)(2l' + 1) \left| U_{-s',-s} \right|^2.$$

According to Eq. (10–18), the two cross sections satisfy the equation

$$(2l + 1) k_\sigma^2 \sigma_{-s',-s} = (2l' + 1) k_\sigma^2 \sigma_{ss'}, \tag{10–19}$$

which is the principle of detailed balance for spherical waves. Note that the spins of the nuclei must be inverted in making the comparison.† However, the principle of detailed balance for the cross sections $\bar{\sigma}_{ss'}$ and $\bar{\sigma}_{s's}$ obtained by averaging over initial magnetic quantum numbers and summing over final magnetic quantum numbers has the more usual form

$$g_\sigma k_\sigma^2 \bar{\sigma}_{ss'} = g_{\sigma'} k_{\sigma'}^2 \bar{\sigma}_{s's}, \tag{10–20}$$

where g_σ is the degeneracy due to nuclear spins of the state Ψ_σ.

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* The possibility exists that the radial part of the internal functions Ψ_σ is complex. This would imply a degeneracy, therefore we assume that the index σ includes the quantum number specifying one of the degenerate states and that this state is so defined that the radial part of the function is real.

† Compare F. Coester, *Phys. Rev.* **84**, 1259 (1951). Also Blatt and Weisskopf, *Theoretical Nuclear Physics*, Wiley (1952), p. 528 ff.

10–3 Dispersion theory of nuclear reactions. The energy dependence of the cross section for a nuclear reaction may be described, in the neighborhood of a resonance, by a formula analogous to the dispersion formula for electromagnetic waves. The careful derivation of the cross-section formula is complicated by the many variables necessarily associated with the many-body problem. Nevertheless, Wigner and Eisenbud* have found it possible to derive the collision matrix describing the dispersion due to all levels by a method which is basically quite simple. A more difficult problem is to obtain from the general theory the behavior of the cross section in the neighborhood of a single resonance, the Breit-Wigner one-level formula. In this formula the influence of resonances other than the single important one is expressed in terms of a single parameter, the effective nuclear radius, so it provides the most practical means for analyzing the observed cross sections. A penetrating discussion of the conditions under which the Breit-Wigner theory can be expected to apply is given by Teichmann and Wigner.† Our discussion, which follows theirs very closely, is presented in the next section.

The collision matrix U can be calculated explicitly in terms of the solutions φ of the Schroedinger equation,

$$H\varphi = E\varphi, \tag{10–21}$$

for the system of A nucleons. These solutions are not known, but they can be written formally as an expansion in terms of a complete set of functions and, if the set is properly chosen, it turns out that U can be expressed as a known function of the expansion coefficients and of the characteristic values of the operator defining the set of functions. These expansion coefficients and characteristic values may then be treated as parameters to be determined by observations. They are so chosen as to be closely related to the resonance widths and energies.

The function φ depends on the $3A$-coordinates of the nucleons as well as on their spin variables, therefore it is necessary to describe the spatial properties of the function in a $3A$-dimensional space. If the origin of the coordinate system in this space is taken to be the center of mass of the system, the wave function of the compound nucleus will be concentrated in a volume of small average diameter centered on the origin, since no pair of nucleons is very far apart in that state. On the other hand, a nucleus which is dissociated into a pair of nuclei has a wave function extending over a $3A$-dimensional cylinder. The mean diameter of the cylinder is determined by the size of the product nuclei, and the position along the length is a measure of the separation of the nuclei. This cylinder in

* *Loc. cit.*
† *Loc. cit.*

the $3A$-dimensional space is usually referred to as a *channel*. Its angular orientation is determined by the nature of the product nuclei, consequently a number of channels with different orientations will be required to describe a reaction which occurs at such an energy as to yield several different products.

Let us consider a region surrounding the origin of the $3A$-dimensional space, the surface of the region being denoted by S. A conveniently selected complete set of orthonormal functions on the surface S is denoted by ψ_s. A useful complete set within the region bounded by S is given by the characteristic functions X_λ defined by the characteristic equation

$$HX_\lambda = E_\lambda X_\lambda, \tag{10-22}$$

where H is the Hamiltonian of the system, if the X_λ are subject to the condition of regularity and to the surface boundary conditions

$$\int_S \langle (\text{grad}_n X_\lambda + b_s X_\lambda), \psi_s \rangle \, dS = 0. \tag{10-23}$$

The angular brackets are used to denote a sum over all spin variables:

$$\langle \Psi, \Phi \rangle = \sum_{\text{spin}} \Psi^* \Phi.$$

The quantities b_s are constants and grad_n indicates the derivative normal to S.

It is convenient to normalize grad_n in such a fashion that it becomes equal to $\partial/\partial r_\sigma$. The connection with r_σ is established below in the discussion of the way in which S is to be defined. The surface integral is also renormalized, in this case so that it becomes the integral over the angles of $\mathbf{r}_\sigma$ and over a set of variables suitable for use as the internal variables of the two nuclear fragments whose distance apart is denoted by $\mathbf{r}_\sigma$. These are the reasons for the appearance of the factor MM_s^{-1} in the form of Green's theorem used below. The occurrence of the factor can be established by applying Green's theorem to functions of the type that can be written as a product of an internal function (having limited extent) with a function of $\mathbf{r}_\sigma$.

Although the physically meaningful solution of Eq. (10-21) is defined over the entire space, for points within the region bounded by S it can be expanded in terms of the complete set X_λ:

$$\varphi(\text{inside}) = \Sigma_\lambda X_\lambda \int_{\text{in}} \langle X_\lambda, \varphi \rangle \, d^{3A}r,$$

the integral being extended over the space within the surface S. Now from Eqs. (10-21) and (10-22) it follows that

$$\int_{\text{in}} \langle X_\lambda, \varphi \rangle \, d^{3A}r = (E_\lambda - E)^{-1} \int_{\text{in}} \{ \langle HX_\lambda, \varphi \rangle - \langle X_\lambda, H\varphi \rangle \} \, d^{3A}r. \tag{10-24}$$

If the potential energy terms, V, in H have the usual property*

$$\int_{\text{in}} \{\langle VX_\lambda,\varphi\rangle - \langle X_\lambda,V\varphi\rangle\}\, d^{3A}r = 0, \tag{10–25}$$

the kinetic energy terms which remain on the right side of Eq. (10–24) can be integrated by Green's theorem (in $3A$-dimensional space) to give

$$\int_{\text{in}}\langle X_\lambda,\varphi\rangle\, d^{3A}r = -(\hbar^2/2M)(E_\lambda - E)^{-1}\int_S MM_s^{-1}\{\langle \text{grad}_n\, X_\lambda,\varphi\rangle - \langle X_\lambda,\text{grad}_n\,\varphi\rangle\}\, dS.$$

Therefore the expansion of φ in the interior region is

$$\varphi = -(\hbar^2/2M)\Sigma_\lambda X_\lambda(E_\lambda - E)^{-1}\int_S MM_s^{-1}\{\langle \text{grad}_n\, X_\lambda,\varphi\rangle - \langle X_\lambda,\text{grad}_n\,\varphi\rangle\}\, dS. \tag{10–26}$$

The integral in Eq. (10–26) depends on the values of φ and $\text{grad}_n\varphi$ only for points on the surface S. Since the ψ_s form a complete set on S, these two functions may be expanded in terms of the ψ_s:

$$\begin{aligned}\varphi(S) &= \sum_s M_s^{1/2}\alpha_s\psi_s,\\ \text{grad}_n\,\varphi(S) &= \sum_s M_s^{1/2}\beta_s\psi_s.\end{aligned} \tag{10–27}$$

Then Eq. (10–26) becomes

$$\varphi = -(\hbar^2/2M)\Sigma_{\lambda,s}X_\lambda(E_\lambda - E)^{-1}\{\alpha_s\int_S MM_s^{-1/2}\langle \text{grad}_n\, X_\lambda,\psi_s\rangle\, dS - \beta_s\int_S MM_s^{-1/2}\langle X_\lambda,\psi_s\rangle\, dS\},$$

so the boundary condition Eq. (10–23) leads to the result

$$\varphi = 2^{-1/2}\hbar\sum_{\lambda,s}(E_\lambda - E)^{-1}(b_s\alpha_s + \beta_s)\gamma_{\lambda s}X_\lambda, \tag{10–28}$$

where

$$\gamma_{\lambda s} = 2^{-1/2}\hbar\int_S M_s^{-1/2}\langle X_\lambda,\psi_s\rangle\, dS. \tag{10–29}$$

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* In this case it is usually said that V is *self-adjoint*. Note that some forms of velocity-dependent interactions are not self-adjoint. However, exchange potentials do satisfy the condition (10–25), since the region of integration will always be chosen to be symmetric under interchange of nucleons. Furthermore, spin-orbit couplings that depend on the momenta only through the orbital angular momenta of the nucleons satisfy (10–25), since integration over all angular positions is implied (only the distances between nucleons are limited), and therefore use may be made of the fact that the operators are hermitian.

Furthermore, according to Eq. (10–27),

$$\alpha_s = \int_S M_s^{-\frac{1}{2}} \langle \psi_s, \varphi \rangle \, dS.$$

When φ is replaced by (10–28), this takes the form*

$$\alpha_s = \sum_t R_{st}(\beta_t + b_t \alpha_t),$$

where

$$R_{st} = \sum_\lambda \gamma_{\lambda s}\gamma_{\lambda t}/(E_\lambda - E).$$

It is convenient to work with the symmetric matrix R which has matrix elements R_{st}. If the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}_\lambda$ with components α_s, β_s, $\gamma_{\lambda s}$ are introduced, the above two equations read

$$\boldsymbol{\alpha} = R(\boldsymbol{\beta} + b\boldsymbol{\alpha}) \tag{10–30}$$

and

$$R = \sum_\lambda \boldsymbol{\gamma}_\lambda \times \boldsymbol{\gamma}_\lambda/(E_\lambda - E), \tag{10–31}$$

where $\boldsymbol{\gamma}_\lambda \times \boldsymbol{\gamma}_\lambda$ is the direct product† of the vector $\boldsymbol{\gamma}_\lambda$ with itself and b is a diagonal matrix with diagonal matrix elements b_s.

We now come to the problem of giving these formal manipulations some physical content. It is to be noted that the nature of the complete set of functions ψ_s has been left free to be chosen in any way that is convenient. This choice is now made in such a way as to cause the functions to correspond as closely as possible to the functions ψ_a of Section 10–2. To demonstrate the connection between these two sets of functions, let us consider the rather artificial situation in which S is a surface at infinity. The different physical conditions described by some points on such a surface may be classified as follows:

(1) A single nucleon infinitely distant from a nucleus of mass $A - 1$.

(2) A single deuteron infinitely distant from a nucleus of mass $A - 2$ and charge $Z - 1$.

(3) A single alpha-particle infinitely distant from a nucleus of mass $A - 4$ and charge $Z - 2$.

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(n) A pair of infinitely separated nuclei of masses A' and A'' and charges Z' and Z'', each in an arbitrary bound excited state.

* The phases of the wave functions can be chosen in such a way that the coefficients α_s, β_s, and $\gamma_{\lambda s}$ are real. This can be established by means of the time-reversal argument outlined in Appendix 3. R then turns out to be a real symmetric matrix.

† The direct product $\mathbf{A} \times \mathbf{B}$ of vectors with components A_s and B_t is the matrix whose elements are $(\mathbf{A} \times \mathbf{B})_{st} = A_s B_t$.

In case (1), ψ_s may be defined as the product of the nucleon spin function, its relative angular function Y_l^m, and the wave function of a given state of the remaining nucleus multiplied by r_s^{-1}. Case (2) allows a similar choice of product functions, the nucleon spin function being replaced by the ground state function of the deuteron. In case (3), ψ_s would involve the ground state function of the alpha-particle. Finally, case (n) leads to a function which is obtained by multiplying a function Y_l^m of the angular positions of the fragments by r_s^{-1} and by the product, Ψ_σ, of the internal (excited state) functions of the fragments. A complete set of functions on S can clearly be obtained in this way by including every angular function and every excited state of every possible pair of fragments. Furthermore, these correspond exactly to the ψ_s defined in Section 10–2.

This analysis suggests that S has the shape of a polyhedron,* each hyperplane corresponding to one particular type of fragmentation of the compound nucleus. Thus the cylinders referred to earlier as "channels" are normal to these planes. Of course, this detailed description has little significance as long as attention is limited to the surface at infinity; our real interest lies in the finite surface of the same shape, that is, the surface obtained by reducing the distances between fragments to a finite (but fixed) value. The reason for having introduced the infinite surface is that the internal wave functions Ψ_σ of the fragments require all space for their complete description. Only when unlimited space is available, i.e., only on the surface at infinity, do the above-defined functions ψ_s provide a complete orthonormal set on the surface. Nevertheless, the same functions provide a good approximation to the complete set on the finite polyhedron. Although the functions are then cut off at the edges of the polyhedral plane, the change in the functions is not necessarily significant. The functions on a given face of the polyhedron are large only within a distance corresponding to the size of each fragment. Beyond that distance from the center of the face the functions decay exponentially, so for a sufficiently large polyhedron, the functions are very small near the edges. Thus we can safely associate the functions ψ_s of Section 10–2 with the complete set on a large but finite surface S. This fact will be used to obtain a connection between the collision matrix U and the matrix R given by Eq. (10–31).

When a nuclear reaction is produced by a projectile having an energy in the region of easily discernible resonance effects, just a few modes of disintegration of the compound nucleus are likely to occur. For this reason, the collision matrix U has matrix elements referring only to several possible states Ψ_σ and $\Psi_{\sigma'}$. The number of states, and therefore the order

* Actually, we want to choose every face of this "polyhedron" in such a way that the distance between each pair of fragments remains constant over the face. Then the face is not a plane, but a quadratic surface in the $3A$-dimensional space. Nevertheless, we use the terms "polyhedron" and "plane" since they are simple and portray the essential physical features of the surface.

of the matrix, then increases with increasing energy. On the other hand, the order of the matrix R is independent of the energy; it has matrix elements referring to every pair of states Ψ_σ, $\Psi_{\sigma'}$ whether or not the energy is sufficient to cause dissociation into those states. If the energy is insufficient, the wave function simply dies off exponentially as the separation of the fragments is increased. The states characterized by such an exponential decay are referred to as "closed channels," while the others are described as "open channels."

To establish the connection between U and R it is necessary to pick out that part of R referring to open channels. The matrix R divides into four submatrices as follows:

$$R = \begin{pmatrix} R_{oo} & R_{oc} \\ R_{co} & R_{cc} \end{pmatrix},$$

where R_{oo} contains all elements connecting open channels, R_{oc} those connecting open with closed channels, and so on. Now R arises as a matrix relating the coefficients α_s and β_s which define the wave function and its derivative on the surface S. Since our attention will be directed to only those α_s and β_s referring to open channels, it is convenient to rewrite the important Eq. (10–30) in terms of a matrix $\mathfrak{R}$ which refers only to open channels,

$$\boldsymbol{\alpha}^o = \mathfrak{R}(\boldsymbol{\beta}^o + b^o\boldsymbol{\alpha}^o). \tag{10–32}$$

The connection between $\mathfrak{R}$ and R can be obtained by noting that

$$\beta_s^c = K_s\alpha_s^c$$

for closed channels, if K_s is the logarithmic derivative of the part of the wave function associated with ψ_s, the derivative being taken along the normal to the corresponding face of S. The energy-dependent quantity K_s is just the damping constant of the exponentially decaying wave function. If a diagonal matrix K having diagonal matrix elements K_s is now introduced, Eqs. (10–30) and (10–32) may be compared to obtain

$$\mathfrak{R} = R_{oo} + R_{oc}[(K + b^c)^{-1} - R_{cc}]^{-1}R_{co}. \tag{10–33}$$

Thus $\mathfrak{R}$ can, in principle, be obtained from Eq. (10–31), although its energy dependence is bound to be more complicated than that of R since even the order of $\mathfrak{R}$ changes with energy.

The elimination of all but the open channels from the discussion implies that our particular selection of the complete set of functions ψ_s is important only for the open channels. In the energy range of interest, only a few states ψ_s, corresponding to only two fragments, and those in or near their ground states, will belong to open channels. Since these functions drop off very rapidly beyond the center of the hyperplane (the highly

excited states drop off much more slowly), our choice of the functions is a good approximation for the open channels even if the surface S is rather small. It will be found advantageous to choose S to be as small as possible, but it must be large enough so that the fragments do not overlap appreciably. In other words, S must be at least so large that at its intersection with a given open channel, the wave function describing the fragments has its asymptotic form, Eq. (10–16).

The connection between U and $\mathfrak{R}$ can now be established by making use of the fact that the function obtained by extrapolating the asymptotic φ to the surface S is described both by Eq. (10–16) and by the open channel part of Eq. (10–27). It is convenient to replace (10–16) by the equivalent asymptotic function

$$\varphi = \sum_s B_s[F_s\psi_s + \sum_{s'} Q_{ss'}G_{s'}\psi_{s'}]. \tag{10–34}$$

The connection between Eqs. (10–34) and (10–16) can be established by means of Eqs. (10–5) and the definition of Q:

$$U = \frac{1 + iQ}{1 - iQ}. \tag{10–35}$$

In order that U be unitary, it is clear that Q must be hermitian.* The coefficient B_s is related to the A_s by

$$A_{s'} = (i/2) \sum_s B_s(1 - iQ)_{ss'}.$$

If this equation and Eq. (10–35) are inserted into Eq. (10–16), the result is Eq. (10–34).

When the expressions for $\varphi(S)$ and $\text{grad}_n\varphi(S)$ obtained from Eq. (10–34) are compared with Eqs. (10–27) (open channels only), and use is made of the linear independence of the ψ_s, the α_s^0 and β_s^0 are found to be related to the B_s by the equations

$$M_s^{-\frac{1}{2}}\alpha_s^0 = \sum_{s'} B_{s'}(F + QG)_{s's}$$

and

$$M_s^{-\frac{1}{2}}\beta_s^0 = \sum_{s'} B_{s'}(F' + QG' - a^{-1}F - Qa^{-1}G)_{s's},$$

where a_σ is the value of r_σ on the surface S and F, G, F', G', and a are diagonal matrices having diagonal elements $F_s(a_\sigma)$, $G_s(a_\sigma)$, $(dF_s/dr_\sigma)_{a_\sigma}$, $(dG_s/dr_\sigma)_{a_\sigma}$, and a_σ, respectively. The terms in a^{-1} appear in the normal derivative because the ψ_s are proportional to r_σ^{-1}. These expressions may

* The elements of Q are real if the phases are chosen so as to yield a real $\gamma_{\lambda s}$ (see footnote p. 293); then Q, and therefore U, is symmetric.

now be inserted into Eq. (10–32) and solved for Q. By making use of the fact that the B_s are arbitrary, we find

$$Q = \mu^{-\frac{1}{2}}[(F' + \bar{b}F)\mathcal{R} - F][G - (G' + \bar{b}G)\mathcal{R}]^{-1}\mu^{\frac{1}{2}}, \quad (10\text{–}36)$$

where

$$\bar{b} = b - a^{-1} \quad (10\text{–}37)$$

and μ is a diagonal matrix with diagonal elements $M_s/\hbar$. Equation (10–36) can be manipulated into the form given by Teichmann and Wigner by use of Eq. (10–6), which can now be written as

$$F'G - G'F = \mu. \quad (10\text{–}38)$$

Then

$$Q = -(F' + \bar{b}F)(G' + \bar{b}G)^{-1} + \mu^{\frac{1}{2}}[G(G' + \bar{b}G) - (G' + \bar{b}G)\mathcal{R}(G' + \bar{b}G)]^{-1}\mu^{\frac{1}{2}}, \quad (10\text{–}39)$$

which is clearly a real, symmetric matrix. This result establishes the relationship between Q and $\mathcal{R}$, and it can be used to express the collision matrix directly in terms of $\mathcal{R}$ by means of Eq. (10–35). In principle, the energy dependence of the collision matrix, and therefore that of the cross section, can be expressed via Eqs. (10 33) and (10–31) directly in terms of the energy-independent quantities $\gamma_{\lambda s}$ and E_λ. The $\gamma_{\lambda s}$ and E_λ may be considered as parameters to be determined from experiment, but their large number (infinite) makes this an impractical procedure. We must therefore take advantage of simplifications that can be justified by physical arguments to reduce the parametrization of the cross-section formula.

10–4 The Breit-Wigner one-level formula. The exact description of the cross section in terms of R clearly has the characteristic resonance behavior of a dispersion formula, since the function R of the energy given by Eq. (10–31) has poles at the points E_λ. There are an infinite number of such resonances and their complete description requires a knowledge of the infinitely many parameters $\gamma_{\lambda s}$. That the energy dependence of the cross section can be described in terms of so many parameters is not surprising, nor is the fact, in itself, very useful. However, one can hope that for values of the energy in the neighborhood of a given resonance, the net contributions of other resonances to the cross section can be expressed in terms of just a few parameters. Of course, this behavior can be anticipated only if the single resonance stands in reasonable isolation from other resonances.

The development of a single-level formula for the energy dependence of the cross section is based* on the fact that, in a sufficiently restricted region of energy, Q can be written as the sum of a term depending smoothly

* Teichmann and Wigner, *loc. cit.*

on the energy and a term that has a simple pole. It is to be expected that Q has poles closely corresponding to those of R, but the restriction on energy is essential if an energy dependence more complicated than that describable by a simple pole is to be avoided. The energy range is to be limited by the condition that the matrices F, F', G, G' do not change very much. Then Q may be expressed in the form

$$Q = \Omega(H_\nu - E)^{-1} + Q_\nu, \tag{10–40}$$

where Ω and Q_ν are smoothly varying functions of the energy E in a small range about the pole H_ν.

Since the matrix Q is singular at $E = H_\nu$, it is convenient to work with its inverse

$$P(E) = Q^{-1}, \tag{10–41}$$

which is well behaved in the neighborhood of H_ν. The various energy values H_ν may be determined as the solutions of the equation

$$\det P(H_\nu) = 0 \tag{10–42}$$

if the matrices Ω and Q_ν are regular for $E = H_\nu$. Because the determinant vanishes, there exist nontrivial solutions $\omega_{\nu s}$ to the set of linear homogeneous equations

$$\sum_s \omega_{\nu s} P_{st}(H_\nu) = 0. \tag{10–43}$$

The form Eq. (10–40) of Q may be established by expanding $P(E)$ in powers of $E - H_\nu$ for values of E near H_ν. The coefficient Ω then depends directly on the $\omega_{\nu s}$. The expansion is expected to converge if other H_ν values are remote from the one under consideration and if the functions F, F', G, and G' appearing in P vary but little with E over the range of interest. The latter condition implies the existence of rather severe restrictions on the choice of the surface S. These will be discussed below.

The matrix Q and hence the values of the H_ν are independent of the boundary conditions on the surface S, but the energy values E_λ depend explicitly on the quantities b_s. It is convenient to choose b in such a way that a given E_λ coincides with some one of the H_ν. The desired coincidence is established when $\bar{b}$ is given by

$$\bar{b} = -(G'G^{-1})_{E=H_\nu}. \tag{10–44}$$

This can be seen by considering the particular solution of $H\varphi = E\varphi$ having the asymptotic behavior Eq. (10–34) with coefficients B_s of the form

$$B_s = \sum_t \omega_{\nu t} P_{ts}(E).$$

Then, by use of Eq. (10–41), φ is found to have the asymptotic form

$$\varphi(E) = \sum_s \left\{ \sum_t \omega_{\nu t} P_{ts}(E) \psi_s F_s + \omega_{\nu s} \psi_s G_s \right\} \cdot$$

The function $\varphi_\nu = \varphi(H_\nu)$ has the asymptotic form

$$\varphi_\nu = \sum_s \omega_{\nu s}\psi_s G_s(H_\nu)$$

because the $\omega_{\nu s}$ satisfy Eq. (10–43). Consequently,

$$\int \langle(\text{grad}_n \varphi_\nu + b_s\varphi_\nu),\psi_s\rangle\, dS = \omega_{\nu s}(G_s' + \bar{b}_s G_s)_{E=H_\nu},$$

so the condition Eq. (10–44) results in the vanishing of the integral. Thus φ_ν is found to satisfy the same boundary conditions (and the same differential equation) as the X_λ, whence it follows that φ_ν is proportional to one of the X_λ and H_ν is equal to the corresponding E_λ.

This result suggests that the singularity of Q in the neighborhood of E_λ is produced by the term $\boldsymbol{\gamma}_\lambda \times \boldsymbol{\gamma}_\lambda(E_\lambda - E)^{-1}$ in R and that the rest of the terms in R produce only a smooth energy dependence in the limited region of energy under consideration. Since R contributes to Q through $\mathfrak{R}$, we assume that, for the special condition Eq. (10–44), the approximation

$$\mathfrak{R} = \boldsymbol{\gamma}_\lambda \times \boldsymbol{\gamma}_\lambda(E_\lambda - E)^{-1} + \mathfrak{R}_r,$$

where the remainder term $\mathfrak{R}_r$ is a smooth function of E, may be used in an energy range that is remote from all singularities in $\mathfrak{R}$ except E_λ.

The above form of $\mathfrak{R}$ is to be substituted into Eq. (10–36) for Q, but before making that substitution we note that G_s'/G_s is a very slowly varying function of the energy over the range of interest (widely spaced resonances occur only at low energy). In fact, for neutrons with $ka \ll 1$,

$$G_l'(ka)/G_l(ka) \approx -la^{-1},$$

which is constant in energy. Hence the condition Eq. (10–44) implies that $G' + \bar{b}G \approx 0$ throughout the energy range under consideration. Therefore Eq. (10–36) becomes

$$Q \approx \mu^{-\frac{1}{2}}(F' + \bar{b}F)\left[\frac{\boldsymbol{\gamma}_\lambda \times \boldsymbol{\gamma}_\lambda}{E_\lambda - E} + \mathfrak{R}_r\right] G^{-1}\mu^{\frac{1}{2}} - FG^{-1}.$$

The Wronskian Eq. (10–38) leads then to the following estimate of Q in the neighborhood of E_λ:

$$Q \approx \mathbf{g}_\lambda \times \mathbf{g}_\lambda(E_\lambda - E)^{-1} + Q_\lambda \tag{10–45}$$

with

$$Q_\lambda = \mu^{\frac{1}{2}}G^{-1}\mathfrak{R}_r G^{-1}\mu^{\frac{1}{2}} - FG^{-1}$$

and

$$\mathbf{g}_\lambda = \mu^{\frac{1}{2}}G^{-1}\boldsymbol{\gamma}_\lambda.$$

The quantity Q_λ is an expression for the accumulated effect of all resonances other than that at E_λ. Since these contributions to R are proportional to $\boldsymbol{\gamma}_\eta \times \boldsymbol{\gamma}_\eta$, one can argue that the nondiagonal elements of Q_λ are

very small because $\gamma_{\eta s}\gamma_{\eta t}$ is as likely to be negative as positive for a given E_η. Therefore a reasonable approximation to Q should be obtained if the diagonal part, $\bar{Q}_\lambda$, of Q_λ is substituted for Q_λ:

$$Q \approx \mathbf{g}_\lambda \times \mathbf{g}_\lambda (E_\lambda - E)^{-1} + \bar{Q}_\lambda.$$

Equation (10–35) provides the collision matrix corresponding to this form of Q:

$$U = \left[1 + i\bar{Q}_\lambda + i\frac{\mathbf{g}_\lambda \times \mathbf{g}_\lambda}{E_\lambda - E}\right]\left[1 - i\bar{Q}_\lambda - i\frac{\mathbf{g}_\lambda \times \mathbf{g}_\lambda}{E_\lambda - E}\right]^{-1},$$

which may be rewritten as

$$U = (1 - i\bar{Q}_\lambda)^{-1}\{(1 + i\bar{Q}_\lambda) + 2i(\mathbf{g}_\lambda \times \mathbf{g}_\lambda)(1 - i\bar{Q}_\lambda)^{-1}[E_\lambda - E - i(\mathbf{g}_\lambda \times \mathbf{g}_\lambda)(1 - i\bar{Q}_\lambda)^{-1}]^{-1}\}.$$

We may insert a factor

$$[E_\lambda - E - i\langle \mathbf{g}_\lambda, (1 - i\bar{Q}_\lambda)^{-1}\mathbf{g}_\lambda\rangle][E_\lambda - E - i\langle \mathbf{g}_\lambda, (1 - i\bar{Q}_\lambda)^{-1}\mathbf{g}_\lambda\rangle]^{-1} = 1$$

before the second term in the braces and make use of the fact that, for any vector $\mathbf{q}$ and any matrix W, the matrix product

$$(\mathbf{q} \times \mathbf{q})W(\mathbf{q} \times \mathbf{q}) = \langle \mathbf{q}, W\mathbf{q}\rangle(\mathbf{q} \times \mathbf{q}),$$

if the angular brackets denote a scalar product. Then

$$U = \frac{1 + i\bar{Q}_\lambda}{1 - i\bar{Q}_\lambda} + 2i\frac{(1 - i\bar{Q}_\lambda)^{-1}(\mathbf{g}_\lambda \times \mathbf{g}_\lambda)(1 - i\bar{Q}_\lambda)^{-1}}{E_\lambda - E - i\langle \mathbf{g}_\lambda, (1 - i\bar{Q}_\lambda)^{-1}\mathbf{g}_\lambda\rangle},$$

and the matrix elements of U are given by

$$U_{ss'} = e^{2i\zeta_s}\delta_{ss'} + i\left[\frac{\Gamma_{\lambda s}\Gamma_{\lambda s'}}{(E'_\lambda - E)^2 + \Gamma_\lambda^2/4}\right]^{1/2} e^{2i(\zeta_{ss'} + \zeta)}, \tag{10–46}$$

where

$$\Gamma_{\lambda s} = 2M_s\hbar^{-1}\gamma_{\lambda s}^2(1 + Q_{\lambda,ss}^2)^{-1}G_s^{-2}(a_\sigma), \tag{10–47a}$$

$$\Gamma_\lambda = \sum_s \Gamma_{\lambda s}, \tag{10–47b}$$

$$E'_\lambda = E_\lambda + \tfrac{1}{2}\sum_s Q_{\lambda,ss}\Gamma_{\lambda s}, \tag{10–47c}$$

$$\zeta_s = \arctan Q_{\lambda,ss}, \tag{10–47d}$$

$$\zeta_{ss'} = \tfrac{1}{2}(\zeta_s + \zeta_{s'}), \tag{10–47e}$$

and

$$\zeta = \frac{1}{2}\arctan\frac{\Gamma_\lambda}{2(E'_\lambda - E)}. \tag{10–47f}$$

The cross section for a reaction produced by an incident wave (s) of orbital angular momentum l is now found by means of Eq. (10–15) to be

$$\sigma_{ss'} = (2l+1)(\pi/k_\sigma^2)\frac{\Gamma_{\lambda s}\Gamma_{\lambda s'}}{(E'_\lambda - E)^2 + \Gamma_\lambda^2/4}, \tag{10–48}$$

which is the original form of the Breit-Wigner reaction cross section, wherein E'_λ was interpreted as the resonance energy, Γ_λ as the total width, and $\Gamma_{\lambda s}$ as the partial width for decay of the compound nucleus into channel s.

The elastic scattering cross section can be obtained from Eq. (10–11). If the possibility of a change of polarization during scattering is included and the new state of polarization is denoted by σ', the differential cross section is

$$d\sigma_{\sigma\sigma'} = (4\pi/k_\sigma^2)\Big|\sum_{l,l'} i^{l-l'}(2l+1)^{\frac{1}{2}}e^{i\eta_l}e^{i\eta_{l'}}\{\delta_{ss'}e^{i\zeta_s}\sin\zeta_s$$

$$+\frac{1}{2}\left[\frac{\Gamma_{\lambda s}\Gamma_{\lambda s'}}{(E'_\lambda - E)^2 + \Gamma_\lambda^2/4}\right]^{\frac{1}{2}}e^{2i\zeta_{ss'}}e^{2i\zeta}\}Y_{l'}^{m'}(\theta) - \frac{i}{2}\rho(\theta)\delta_{\sigma\sigma'}\Big|^2 d\Omega. \tag{10–49}$$

When the resonance term is set equal to zero, Eq. (10–49) just describes the scattering by a potential consisting of a coulomb term (if both particles are charged) and a short-range potential characterized by the phase shifts ζ_s. These are therefore referred to as the potential scattering terms, although the $\bar{Q}_\lambda$, and therefore the ζ_s, actually express the net effect of resonances other than that at E_λ as well as the effect of any potential which may actually exist. The total scattering consists of a potential scattering cross section, a resonance cross section, and a term due to interference between the two.

It is possible to make a close identification between the potential scattering terms in Eq. (10–49) and the scattering from a hard sphere of radius a_0. For hard-sphere scattering there are no reactions other than scattering, nor are there any resonances, so the wave function Eq. (10–34) takes the simple form

$$\varphi_0 = \sum_l B_l[F_l + Q'_{ll}G_l]\psi_s,$$

where Q' is the Q characteristic of this special problem. Since φ_0 must vanish at a_0, the value of Q'_{ll} is

$$Q'_{ll} = -F_l(a_0)/G_l(a_0).$$

Therefore, if in an actual scattering problem we determine an a_{0s} by the condition that

$$F_l(a_{0s})/G_l(a_{0s}) = -Q_{\lambda,ss}, \tag{10–50}$$

the phase shift ζ_s will be that due to scattering by a hard sphere of such a radius.

The value of a_{0s} determined from an observed $Q_{\lambda,ss}$ by Eq. (10–50) can be expected to be a function of both energy and l-value. However, there is reason to expect that this dependence is slight, that a_{0s} is closely related to the energy independent and l-independent quantity a_σ. This expectation is derived from the fact that the contributions to R of very remote resonances are very small, as indicated by Eq. (10–31). Therefore, if Q_λ is due entirely to such resonances, it is the Q-matrix corresponding to $R = 0$, or, equivalently, to $\mathfrak{R} = 0$. But, according to Eq. (10–36),

$$Q(\mathfrak{R} = 0) = -FG^{-1}.$$

Hence the off-diagonal elements of Q_λ nearly vanish, as anticipated above, and the diagonal elements are estimated to be

$$Q_{\lambda,ss} \approx -F_s(a_\sigma)/G_s(a_\sigma).$$

The accuracy of this approximation depends on the degree of isolation of the level E_λ; it increases with increasing distance of the neighboring levels, and, if the one-level approximation is very good, one may expect on the basis of Eq. (10–50) that

$$a_{0s} \approx a_\sigma. \tag{10–51}$$

The validity of the one-level approximation depends not only on the isolation of the level from other levels, but also on the assumption that $F_s(a_\sigma)$ and $G_s(a_\sigma)$ are slowly varying functions of the energy. Now these are essentially periodic functions of k_σ, and the period is roughly a_σ^{-1}. Therefore the requirement of slow variation is that a_σ be small. It is for this reason that we must choose the surface $\mathcal{S}$ to have as small a mean radius as possible. On the other hand, the surface must be large enough so that the asymptotic form of the wave function can be used at the intersection with an open channel. Furthermore, it must be large enough for the ψ_s, defined as they are, to approximate the open channel part of a complete set on the surface $\mathcal{S}$. It is not at all certain that all the conditions can be satisfied. Especially for charged particles there may be some difficulty because deviations from the coulomb potential may set in as soon as there is slight overlap between the reacting nuclei. In any case, it is quite clear that the short-range character of the nuclear forces is an essential feature of the theory and even a long tail on the nuclear interaction, such as occurs for the Yukawa potential, could lead to deviations from the theory. However, these deviations may be expected to require changes in the interpretation of the terms appearing in the cross-section formulas rather than changes in the form of the cross-section curve.

Within the limits set by the rather severe restrictions mentioned above, the size of $\mathcal{S}$ (hence of the a_σ) is undetermined. As a matter of fact, in the

many-level formulation of the theory there is no upper limit on s; the equations obtained for the cross section for each set a_σ appear to be different but are equivalent. As far as the application to experiments is concerned, it is to be noted that the a_σ do not appear explicitly in the cross-section formulas Eqs. (10–48) and (10–49), and a knowledge of a_σ is important only for the interpretation of the constants that are found to give the best fit of the one-level formula to the observed cross section.

The qualitative features of the cross-section formulas, Eqs. (10–48) and (10–49), agree with the expectations outlined in Section 10–1. There is a strong maximum at the resonance E'_λ, and the partial width $\Gamma_{\lambda s}$ of the resonance, as well as its magnitude, according to Eq. (10–29), is closely related to the amount by which the function X_λ of the compound nucleus is mixed with the surface function leading directly to dissociation. But the state of energy E'_λ was expected to be a metastable state of the compound nucleus while, in the present description, it has to do with the somewhat abstract auxiliary state X_λ. Since X_λ is defined only within the surface S, it is, in a sense, a state of the compound nucleus, but the boundary condition it satisfies has a form quite different from that expected of a bound state. This is as it should be, since the general theory must take into account every degree of stability. Strongly bound virtual states, such as those that arise in the independent particle model, can be described by the X_λ satisfying the boundary condition (10–23) if an appropriate choice is made of the coefficients b_s. On the other hand, the X_λ associated with potential scattering would not be analogous to a bound state. The theory in its present form is able to account for all situations between these extremes.

The appearance of the factor $\Gamma_{\lambda s}\Gamma_{\lambda s'}$ in the numerator of Eq. (10–48) has to do with the fact, suggested by Eq. (10–1), that both the probability of formation of the compound nucleus and the probability of disintegration via a given channel are proportional to the partial width. The appearance of $\Gamma_{\lambda s}\Gamma_{\lambda s'}$ as a factor in the resonance term of the scattering cross section may then be interpreted as an indication that that term describes a process of capture and re-emission of the incident particle. The interconnection between the width and rate of emission is indicated, but certainly not established, by these considerations. The matter is taken up more quantitatively in Section 10–6 where it is shown that, at least for narrow levels, the transition probability is given by $\hbar^{-1}\Gamma_{\lambda s}$.

The occurrence of radiative capture has not been included in the general discussion, although it is one of the simpler reactions to be taken into account. No general treatment of reactions including this effect seems to have been given. However, a very natural generalization of the reaction cross section may be used to describe the process. If the transition probability for photon emission from the state X_λ to any lower state is $\hbar^{-1}\Gamma_{\lambda\gamma}$,

the capture cross section would be expected* to have the same form as Eq. (10–48), with $\Gamma_{\lambda s'}$ replaced by $\Gamma_{\lambda\gamma}$. Then the total width appearing in the one-level formula for any reaction should be modified to include the gamma width $\Gamma_{\lambda\gamma}$, so Eq. (10–47b) is replaced by

$$\Gamma_\lambda = \sum_s \Gamma_{\lambda s} + \Gamma_{\lambda\gamma}. \tag{10–52}$$

The radiative process is then included in Eq. (10–48) if it is treated as though it simply corresponds to another channel $s' = \gamma$ for each allowed radiative transition.

10–5 Interpretation of the one-level formula. Almost all of the available information concerning virtual states of nuclei is obtained by observation of the resonance phenomenon in nuclear reactions. Conclusions concerning the properties of isolated virtual states are based directly on the one-level formula. We give attention here to some of those general features of this formula which are frequently used in its application to the data.

The isolation of the level is essential for reliable application of the one-level theory, therefore our considerations are limited to situations corresponding to a low density of energy levels. Since the level density increases rapidly with A, a reaction in which the target consists of a light nucleus is most likely to display isolated resonances. The level separation is also dependent upon the binding energy of the projectile nucleus in the target nucleus. If the projectile is a neutron or proton, this binding energy will lie between 6 Mev and 8 Mev on the average, but fairly large fluctuations will occur. Since the level density increases rapidly with total energy (relative kinetic energy plus binding), a fluctuation in binding may lead to an appreciable change in level density. An example is offered by the reaction of neutrons with Pb^{208}. For such a heavy nucleus the virtual levels would be expected to be very closely spaced, but the binding of the neutron is so small (4.0 Mev†) that a level density characteristic of a much lighter nucleus is observed.‡

Even for rather light nuclei, the energy at which the reaction is observed should not exceed several Mev if the levels are to be isolated. At such low

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* This procedure can certainly be justified for virtual states X_λ which are very stable, since then the transition probability for emission of radiation can be obtained by the application of ordinary perturbation theory. See, e.g., Dirac, *loc. cit.* However, for states of slight stability, i.e., those of large natural width, the extension of X_λ into the region outside S may make important, but not easily estimated, contributions to the transition probability. See D. C. Peaslee, *Phys. Rev.* **88,** 812 (1952).

† J. A. Harvey, *Phys. Rev.* **78,** 345 (1950).

‡ R. K. Adair, *Rev. Mod. Phys.* **22,** 249 (1950).

energies, only the part of the incoming wave having low orbital angular momentum makes an important contribution to the reaction. Therefore, in applying Eq. (10–48) or Eq. (10–49), attention can be limited to just one value of l, the lowest that is consistent with the condition of conservation of angular momentum. For the resonance term, l is determined by the total angular momentum J of the state X_λ of the compound nucleus and the spins of the reacting nuclei. On the other hand, $l = 0$ will give the dominant term in the potential scattering as long as the equivalent hard-sphere radius, a_0, is comparable to nuclear dimensions.

It is to be recalled that the labels σ and σ' refer to states of definite polarization of the reacting nuclei, hence the expressions (10–48) and (10–49) give the cross sections for transitions between states of specified polarization. On the other hand, the observed cross sections concern the transition between an unpolarized incident beam and every possible state of polarization of the products, unless a special effort is made to define the polarization. Thus our expressions for the cross section are usually to be averaged over initial states of polarization and summed over final states of polarization. The resulting total cross section will be denoted by $\bar{\sigma}_{ss'}$. To carry out the sum and average, it is convenient to define the label σ in terms of j, the resultant of the spins of the reacting pair of nuclei, and m_j, the associated magnetic quantum number. Then

$$\bar{\sigma}_{ss'} = g_\sigma^{-1} \sum_{j,j'} (2j + 1)\sigma'_{ss'}, \tag{10–53}$$

where

$$\sigma'_{ss'} = (2j + 1)^{-1} \sum_{m_j,m_{j'}} \sigma_{ss'}$$

and g_σ is the degree of degeneracy of the states σ. The advantage of this division of the sum into two stages is that the evaluation of $\sigma'_{ss'}$ can be made on the basis of the conservation of total angular momentum alone. The conservation law implies that incoming waves of a given J, m_J can produce only outgoing waves of the same J, m_J. Hence, if the collision matrix is adapted to the specification of total angular momentum, non-diagonal elements between two different values of J and m_J must vanish. And the fact that the wave functions are degenerate in m_J means that the collision matrix is independent of that quantity. In the J, m_J representation, the collision matrix may therefore be written as $U^J_{SS'}$ where the index S does not involve the quantum numbers m_j and m but is otherwise the same as s.

The determination of $U^J_{SS'}$ from $U_{ss'}$ involves the usual change of representation by the unitary transformation* $\langle j,l,J,m_J \mid j,l,m_j,m\rangle$. Hence

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* *Cf.*, e.g., Condon and Shortley, *The Theory of Atomic Spectra*, Cambridge (1951), pp. 76–77, for tables of these coefficients.

$$U_{ss'} = \sum_{J'} \langle j,l,m_j,m \mid j,l,J',m_J \rangle U^{J'}_{SS'} \langle j',l',J',m_J \mid j',l',m'_j,m' \rangle. \quad (10\text{–}54)$$

The unitary character of the transformation coefficients then leads to the result*

$$\sum_{m_j,m_j'} \mid U_{ss'} \mid^2 = (2l+1)^{-1} \sum_{J'} (2J'+1) \mid U^{J'}_{SS'} \mid^2. \quad (10\text{–}55)$$

In the new representation, the partial widths, phase shifts, and other quantities given by Eq. (10–47) refer to a state of definite angular momentum, and the angular momentum has just that value, J, associated with the state X_λ of the compound nucleus. Therefore, in the one-level approximation, only the one term with $J' = J$ in Eq. (10–55) is different from zero. Equation (10–48) may then be replaced by

$$\sigma'_{ss'} = \frac{\pi}{k_o^2} \frac{2J+1}{2j+1} \frac{\Gamma_{\lambda s} \Gamma_{\lambda s'}}{(E'_\lambda - E)^2 + \Gamma_\lambda^2/4}. \quad (10\text{–}56)$$

The corresponding $\bar{\sigma}_{ss'}$ can be obtained by means of Eq. (10–53) only if the dependence on j and j' is known.

A similar procedure can be followed to obtain the scattering cross section, but the general expression for the cross section is quite complicated as a result of the interference between potential and resonance scattering. In any particular case the desired expression can easily be obtained by direct application of the above procedure to Eq. (10–49).

Mention should be made of the fact that a total scattering cross section is only well-defined for neutron-induced reactions. For charged particle reactions, the Rutherford term leads to the usual divergence, at small angles, of the angular distribution. The measurements are normally made at one or more specified angles of scattering, and the comparison with theory is straightforward as long as the angle of observation is sufficiently well known.

Under good experimental conditions, the application of these results to observed resonances makes possible the determination of the resonance energy E'_λ, the angular momentum J, various partial widths $\Gamma_{\lambda s}$, and a hard-sphere radius, a_0, for s-wave potential scattering. These figures are

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* Note that only the one value, $m = 0$, of the orbital magnetic quantum number occurs in the incident wave. Use is made of the normalization condition

$$\Sigma_{m_j,m} \langle J,m_J \mid m_j,m \rangle \langle m_j,m \mid J,m_J \rangle = 1$$

in the form

$$\Sigma_m \{ \Sigma_{m_J,m_j} \langle J,m_J \mid m_j,m \rangle \langle m_j,m \mid J,m_J \rangle \} = 2J+1.$$

The quoted result is obtained by noting that the expression in the braces is independent of m. See, for example, Breit and Darling, *Phys. Rev.* **71**, 402 (1947).

usually quoted, to the extent that they are determined, in summaries of the data on isolated resonances.* The interpretation of E'_λ and J as the energy and angular momentum of a level of the compound nucleus is straightforward enough, but the interpretation of the partial widths is much less direct. The quantity $\Gamma_{\lambda s}$ depends both on the nature of the state X_λ and on the nature, energy, and angular momentum of the reacting particles denoted by S. According to Eq. (10–47a), the energy dependence of $\Gamma_{\lambda s}$ is determined essentially by the value of l, while the dependence on X_λ is contained in the related quantity $\gamma^2_{\lambda s}$, which is therefore called "the reduced width." As one can see from its definition, Eq. (10–29), for a system known to be in state X_λ, $\gamma^2_{\lambda s}$ is just the probability for finding the system within a shell of thickness $(2M_s)^{-1/2}\hbar$ on the hypersurface and in the state ψ_s. Therefore it gives the probability for the appearance of the fragments at the surface. The other factors in $\Gamma_{\lambda s}$ determine the probability for escape after arrival, and they depend only on superficial matters such as the nature of the potential barrier to be penetrated. Hence the reduced width is the quantity of more fundamental interest.

The evaluation of the reduced width for an observed value of $\Gamma_{\lambda s}$ can be made by means of Eq. (10–47a) if $Q_{\lambda,ss}$ and a_σ are known. Now $Q_{\lambda,ss}$ may be determined directly from the potential scattering, or we may make a reasonable guess, based on interpolation between known values for other nuclei, as to the value of the "nuclear radius" a_{0s}, whence $Q_{\lambda,ss}$ can be calculated. But the only available information on a_σ is that it is approximately the same as a_{0s}, as indicated by Eq. (10–51). It is therefore customary to assign the same values to a_σ and a_{0s} in order to calculate the reduced width. This procedure seems quite safe when the resonances are very widely separated, but rather large deviations might be expected when the separation between levels is only several times larger than the width.

If a single radius a_{0s} is used, Eq. (10–47a) may be rewritten as

$$\Gamma_{\lambda s} = 2M_s\hbar^{-1}\gamma^2_{\lambda s}[F_l^2(k_\sigma a_{0s}) + G_l^2(k_\sigma a_{0s})]^{-1}, \qquad (10\text{–}57)$$

where, it is to be remembered, F_l and G_l are the appropriate regular and irregular radial functions. For neutrons, these are the well-known functions, some of whose properties are given at the end of Section 4–5.†

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* Data summaries available at the time of this writing are Bureau of Standards Circular No. 499, U. S. Printing Office (1950), and supplements to this document. For neutron data, see R. K. Adair, *loc. cit.* A summary of energy-level data on light nuclei is given by Hornyak, Lauritsen, Morrison, and Fowler, *Rev. Mod. Phys.* **22,** 291 (1950) and by Ajzenberg and Lauritsen, *Rev. Mod. Phys.* **24,** 321 (1952).

† Note that $F_l = (M_s k/\hbar)^{1/2} r j_l$ and $G_l = (M_s k/\hbar)^{1/2} r n_l$, according to the normalization indicated by Eq. (10–6). The function $j_l(x) = (\pi/2x)^{1/2} J_{l+1/2}(x)$ and $n_l(x) = (\pi/2x)^{1/2} N_{l+1/2}(x)$, where J_n and N_n are the Bessel and Neumann functions.

Numerical values of the *transmission coefficient*, $2[F_l^2(ka_{0S}) + G_l^2(ka_{0S})]^{-1}$, can therefore be obtained directly from standard tables for neutrons of energy E having known values of ka_{0S}, where $k = (2M_sE)^{1/2}\hbar^{-1}$. Hence the "observed" reduced width may be obtained from an observed neutron width $\Gamma_{\lambda S}$ and from either an experimentally determined or an interpolated value of the nuclear radius a_{0S}. It is to be remembered that this procedure provides only an estimate of the reduced width, because Eq. (10-57) differs from the correct Eq. (10-47a) by a factor $G_l^2(ka_c)/G_l^2(ka_{0S})$. Since the irregular function may be a quite sensitive function of its argument, particularly for large l, the error in this estimate may be appreciable.

Particular interest attaches to the behavior of the cross sections and related quantities for relatively slow neutrons, i.e., those for which $ka_{0S} \ll 1$. Then we find from the formulas for small argument given at the end of Section 4–5 that

$$F_l \sim k^{l+1/2},$$

$$G_l \sim k^{-(l+1/2)}.$$

Therefore

$$\Gamma_{\lambda S} \sim k^{2l+1} \tag{10-58}$$

for neutrons at low energy. The increase of $\Gamma_{\lambda S}$ with increasing energy is due in part to the fact that the probability of emitting a neutron is proportional to the density of free neutron states, a quantity which increases with the neutron kinetic energy. The rapid decrease at high l for low energy is, of course, due to the fact that low-energy neutrons cannot penetrate the centrifugal barrier. Hence the reaction goes slowly, and the width is small. An interesting consequence of this result is that the absorption cross section, Eq. (10–56), for any very slow neutron initiated process must be due to the S-waves, hence it contains a factor proportional to k^{-1}. The product of the reaction will be an energetic photon, or some other particle of energy considerably greater than that of the slow neutron, therefore the dependence of $\Gamma_{\lambda S'}$ on neutron energy can be neglected. Except in the very special case that $E'_\lambda \approx 0$, the neutron absorption cross section is then proportional to v^{-1}, where v is the neutron velocity. This "$1/v$ law" is substantiated by many observations, and it is often used to separate the slow neutron scattering cross section (independent of v) from the absorption cross section.

Another interesting characteristic of the slow neutron process is the scattering length a. If there is no resonance near zero energy for neutrons with $l = 0$, the very low energy scattering is due entirely to the potential scattering term. From the definition of a_0 as the distance at which the wave function vanishes, Fig. 4–1 makes it evident that $a = a_0$ in this case. The scattering length is then positive, and that is just what is usually observed. On the other hand, if a scattering resonance (i.e., one for which

the total width is nearly equal to the neutron width) with $l = 0$ occurs at very low energy, the scattering length is found from Eq. (10–49) to be

$$a = a_0 - a_1 e^{2i\zeta},$$

where a_1 is a positive constant depending on the strength and position of the resonance. The relative sign of the two terms has been fixed by use of the low-energy limit of Eq. (10–50). The value of ζ is to be obtained from Eq. (10–47f) with $\Gamma_\lambda \approx 0$ (since $\Gamma_\lambda \approx \Gamma_{\lambda n} \sim k$). Thus $2\zeta = 0$ for $E < E'_\lambda$ and $2\zeta = \pi$ for $E > E'_\lambda$. Hence a negative scattering length may be obtained under the very special condition that there is a state of the compound nucleus just above the threshold for neutron emission. We find here, again, substantiation of the connection between the sign of the singlet neutron-proton scattering length and the position of the singlet state of the deuteron. Some methods for determining the sign of the scattering length for any target nucleus are discussed in Sections 5–4 and 5–5.

The determination of the reduced width for a *charged* particle reaction by means of Eq. (10–57) can be accomplished with the aid of tables of the coulomb functions.* These functions have the property that $|F_l^2 + G_l^2|$ increases rapidly with decreasing energy and decreasing a_{0s}; its reciprocal is usually called the transmission coefficient or *penetration factor* for the coulomb barrier. It has the effect that observed widths for, say, proton reactions are much smaller than those for neutron reactions under corresponding conditions. Again, as in the case of neutrons, the correct value of the penetration factor should be taken from Eq. (10–47a) rather than from the estimate Eq. (10–57). The irregular coulomb function G_l is extremely sensitive to the value of the argument in most cases of interest, consequently an estimate of the reduced width based on the application of Eq. (10–57) may be very much in error. Unfortunately, no other procedure for determining the penetration factor appears to be available at the present time.

In Section 10-1 we have surmised that a knowledge of the width of a level would provide some estimate of the degree to which the wave function may be described by the independent particle model. It should now be evident that the quantity of real interest in this connection is the reduced width, $\gamma_{\lambda s}^2$. The basis of our surmise was a sum rule which can now be established by means of the definition, Eq. (10–29), of the $\gamma_{\lambda s}$. Since the ψ_s form a complete orthonormal set, the reduced widths are subject to the general sum rule

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* Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, *Rev. Mod. Phys.* **23,** 147 (1951). *Tables of Coulomb Wave Functions,* National Bureau of Standards Mathematics Series 17, U. S. Government Printing Office, Washington (1952). Note that the functions F_l and G_l used here differ in normalization from those given in the tables: F_s (here) $= (M_s/\hbar k)^{1/2} F_s$ (table).

$$\sum_s (2M_s/\hbar^2)\gamma_{\lambda s}^2 = \int_{\mathcal{S}} \langle X_\lambda, X_\lambda \rangle \, d\mathcal{S}.$$

Every state s is included in the sum and the entire surface $\mathcal{S}$ is included in the integral. A less rigorous but more useful sum rule is obtained by noting that there is very little overlap between surface states associated with different faces of $\mathcal{S}$ for values of E_λ within the range of application of the one-level formula. Hence the equality is expected to hold if the sum is limited to the set c of channels normal to a given face $\mathcal{S}(c)$, and the integral is carried over just $\mathcal{S}(c)$:

$$\sum_{s(c)} \gamma_{\lambda s}^2 \approx (\hbar^2/2M_c) \int_{\mathcal{S}(c)} \langle X_\lambda, X_\lambda \rangle \, d\mathcal{S}. \tag{10–59}$$

The $s(c)$ run through values corresponding to every state of excitation, polarization, and orbital angular momentum of a pair of product nuclei whose channel orientation is denoted by c. An estimate of the integral appearing in (10–59) can be made by noting that the density $\langle X_\lambda, X_\lambda \rangle$ should have about the same average value on $\mathcal{S}(c)$ as it has throughout the region within $\mathcal{S}$ if the surface is so small that the product nuclei are nearly in contact when on $\mathcal{S}$. Since the functions X_λ are normalized over the volume within $\mathcal{S}$, the value of the integral should be roughly equal to the ratio of the area of $\mathcal{S}(c)$ to the total volume within $\mathcal{S}$, namely, $4\pi a_c^2/(4\pi a_c^3/3) = 3/a_c$, where a_c is the common value of a_σ for all the $s(c)$. Thus

$$\sum_{s(c)} \gamma_{\lambda s}^2 \approx 3\hbar^2/2M_c a_c. \tag{10–60}$$

This is the "first sum rule" of Teichmann and Wigner.

The connection with the independent particle model has to do with the form of the functions X_λ. If X_λ can be written as a product of one-particle functions it can be separated into a product of a radial function $u(r_s)$ and one of the surface functions ψ_s characteristic of channels c. Then the integral, Eq. (10–29), defining $\gamma_{\lambda s}$ will vanish for all but that one value of s, and the sum rule (10–60) reads

$$\gamma_{\lambda s}^2 \approx 3\hbar^2/2M_c a_c \tag{10–61}$$

for that particular s. In the situation described here, there is a definite λ associated with each state $s(c)$ of excitation or angular momentum of the nuclei. The scattering of a wave denoted by s corresponds to ordinary potential scattering, therefore it can be expected to show a single very broad maximum in the cross section, and (10–61) does, indeed, imply a very large width. If an observed reduced width approaches the value (10–61) it can be concluded that the corresponding X_λ separates into a product $u(r_s)\psi_s$ in good approximation or, in other words, that the interaction between the pair of nuclei is closely approximated by a simple potential, as one would expect if the independent particle model were valid.

Even when the independent particle model is a poor approximation, the functions X_λ can be expanded in terms of a complete set of product functions:*

$$X_\lambda = \sum_{n,s(c)} C_{sn}^\lambda \psi_s u_n(r_s). \tag{10-62}$$

Then

$$\gamma_{\lambda s} = (\hbar^2/2M_c)^{1/2} \sum_n C_{sn}^\lambda u_n(a_c).$$

If many terms n contribute to the expansion, it is reasonable to assume that the C_{sn}^λ will vary in an almost random fashion. Then in $\gamma_{\lambda s}^2$ the products of terms corresponding to different values of n will be as often negative as positive, hence their sum nearly vanishes and

$$\gamma_{\lambda s}^2 \approx (\hbar^2/2M_c)\,\overline{u^2} \sum_n |\, C_{sn}^\lambda \,|^2,$$

where $\overline{u^2}$ is the mean square value of $u_n(a_s)$, a quantity that can be estimated by means of the sum rule (10-60). Since the X_λ are normalized,

$$\sum_{n,s(c)} |\, C_{sn}^\lambda \,|^2 = 1, \tag{10-63}$$

so $\overline{u^2} \approx 3/a_c$. Therefore our estimate of the reduced width in the case of a very complex state is

$$\gamma_{\lambda s}^2 \approx (3\hbar^2/2M_c a_c) \sum_n |\, C_{sn}^\lambda \,|^2. \tag{10-64}$$

When many different values of s contribute to the sum in the normalization condition (10-63), the value of $\Sigma_n |\, C_{sn}^\lambda \,|^2$ is expected to be quite small, hence the reduced width associated with a complex state should be very much smaller than that given by Eq. (10-60) for a state as simple as would be suggested by the independent particle model.

While in the case of pure potential scattering a single value of λ, and therefore a single resonance, is associated with each s, there is no such simple correlation when the state X_λ is complex. On the basis of Eq. (10-62), it is expected that the number of values of λ would be very large for given s, so many resonances should be observed in a reaction initiated by the particles denoted by s. There are a corresponding number of nonvanishing coefficients C_{sn}^λ, all of the same order of magnitude. Let us say that C_{sn}^λ is distinct from zero for all E_λ within an energy interval W_{sn}. Since the coefficients are subject to the normalization condition

$$\sum_\lambda |\, C_{sn}^\lambda \,|^2 = 1,$$

* Note that the *same* set, u_n, of radial functions is used here for every s.

we find that

$$|C_{sn}^{\lambda}|^2 \approx D/W_{sn},$$

where D is the average level spacing. It then follows from Eq. (10–64) that

$$\gamma_{\lambda s}^2 \approx \tfrac{3}{2}\hbar^2 D/M_c a_c W_s, \tag{10–65}$$

where W_s^{-1} is the sum over n of W_{sn}^{-1}. W_s therefore corresponds approximately to the smallest of the W_{sn}. Since it represents the spread in energy associated with the analysis of the function $\psi_s u_n$ in terms of the X_λ, it can be expected to be of the order of the mean kinetic energy within the nucleus, hence roughly independent of the complexity of the nucleus. Thus, on the average, the reduced width should be about proportioned to D as is, indeed, the case.* Large fluctuations from this rule are, however, to be expected for any small group of levels.

These considerations make possible a rather direct interpretation of the data in either one of two extreme situations: if one observes a single level whose reduced width for a given process closely approaches the limit (10–61), the wave function for that level is probably a simple product. On the other hand, observation of many levels having much smaller widths would indicate an almost statistical mixture of complex terms in the wave function. It is difficult to interpret results that fit neither of these criteria. If $\frac{2}{3}M_s a_s \gamma_{\lambda s}^2 \hbar^{-2}$ is somewhat smaller than one, but not very small, it might be interpreted, on the basis of Eq. (10–64), as the probability of the occurrence in X_λ of a pure single particle state. However, this interpretation is correct only if there are many terms u_n contributing to X_λ, that is, only if the small deviation of X_λ from a single-particle state involves many terms, each with small amplitude. If just a few terms contribute, the interference between them could lead to quite large deviations from Eq. (10–64) and therefore from the interpretation of the reduced width as a measure of the purity of the product function.

An excellent example of the application of these ideas is the analysis of the scattering of protons on C^{12} by Jackson and Galonsky.† The observed‡ behavior of the differential cross section at several angles of scattering makes it possible to separate the potential scattering and to identify clearly

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* Compare the table of reduced widths given by Teichmann and Wigner, *loc. cit.* The important fact that the width should be proportional to the level spacing for complex states was suggested much earlier by Bohr and Kalckar, *Kg. Dansk. Acad.* (1939), on the basis of the liquid drop model, for which the states of the compound nucleus are truly complex. See also the discussion of level widths by H. A. Bethe, *Rev. Mod. Phys.* **9,** 69 (1937).

† *Phys. Rev.* **89,** 370 (1953).

‡ Jackson, Galonsky, Eppling, Hill, Goldberg and Cameron, *Phys. Rev.* **89,** 365 (1953).

three levels of N^{13} having the properties indicated in Table 10–1. In this case the agreement with the Mayer-Jensen scheme is rather remarkable. In the ground state of N^{13}, the odd neutron is expected to be in the $1p_{\frac{1}{2}}$ state. The next higher configurations are obtained by exciting the neutron to the $2s_{\frac{1}{2}}$ and the $1d_{\frac{5}{2}}$ states. Evidently these correspond to the first and last of the levels described in Table 10-1. The fact that the reduced widths are large is in accord with the assumption that they are one-particle levels. The other level does not fall naturally into the coupling scheme, since the formation of a $1p_{\frac{3}{2}}$ hole should require a large excitation energy. However, the state could be lowered by an interaction that mixes in other configurations and the existence of such a mixing is indicated by the very small reduced width.

TABLE 10-1

Properties of virtual states of N^{13}. Energies are measured from the ground state.

E'_λ	J	Parity	$2\gamma_\lambda^2 M_c a_c/3\hbar^2$
2.369	$\frac{1}{2}$	+	0.54
3.511	$\frac{3}{2}$	−	0.031
3.558	$\frac{5}{2}$	+	0.21

10–6 Alpha radioactivity. It is well known that nuclei of very large mass are likely to be unstable against the emission of an alpha-particle as a result of the rapid increase of the coulomb energy with Z at large Z. This means that even the ground states of such nuclei are virtual states although, to be sure, they are more stable than the virtual states of the compound nucleus occurring in the usual nuclear reaction. The process of radioactive decay can therefore be considered as a special example of a resonance reaction to which the foregoing theory is applicable. But because of the special experimental conditions associated with this problem, the measurable quantity is the lifetime of the state rather than a reaction cross section. The lifetime does not appear explicitly in our theoretical considerations since we have been dealing with stationary states, while the lifetime is related to the change of a state in time. Nevertheless, it seems reasonable to surmise on the basis of the qualitative equation (10–1) that the rate of emission is proportional to $\Gamma_{\lambda\alpha}\hbar^{-1}$, where $\Gamma_{\lambda\alpha}$ is the alpha-particle width of the radioactive state, λ. If such a connection can be made quantitative, it provides a direct method for calculating the lifetime by means of the stationary state theory.

To establish the desired result, we consider a thought experiment based on the idea that the virtual state could, in principle, be produced by inverting the process and bombarding the product nucleus (i.e., the nucleus remaining after alpha-decay) by alpha-particles of the correct energy. The cross section for the formation of the radioactive state λ followed by

re-emission of the alpha-particle is just the resonance part of the one-level scattering cross section, since the level is isolated from other levels of the compound nucleus in most cases of alpha activity. Furthermore, the total width of the level may be set equal to $\Gamma_{\lambda\alpha}$ because there must be no competing emission process if the lifetime of the state is to be determined by alpha-decay alone. The part of the cross section referring to absorption and re-emission of the alpha-particle is then obtained by appropriate specialization of Eq. (10–56):

$$\bar{\sigma}_{\alpha\alpha}(\text{res.}) = \frac{\pi}{k_\alpha^2}\frac{2J+1}{2j+1}\frac{\Gamma_{\lambda\alpha}^2}{(E'_\lambda - E)^2 + \Gamma_{\lambda\alpha}^2/4}.$$

The averaging indicated by Eq. (10-53) has been carried out by noting that the alpha-particle has zero spin, so the only value of j_σ is j, the spin of the product nucleus. It is assumed that the energy is low enough for all but the lowest admissible state of orbital angular momentum to be ignored.

If a product nucleus is placed in a bath of He at a temperature kT much greater than E_λ, there will be a finite probability for forming the radioactive nuclei. The rate at which these decay is then determined by $\bar{\sigma}_{\alpha\alpha}$(res.). This rate will be the same as that of decay of the radioactive nucleus, since the compound nucleus is actually a very stable system, and consequently even the rather extreme temperature implied here should not have an important effect on the decay process. The number of alphas emitted per unit time per product nucleus is

$$\int s_\alpha(E)\bar{\sigma}_{\alpha\alpha}(\text{res.})\,dE,$$

where $s_\alpha(E)\,dE$ is the incident flux of alphas in the energy interval dE. At the high temperature considered, classical statistics apply, therefore there is one alpha-particle per volume $(2\pi\hbar)^3$ in phase space. The particle density is therefore

$$\rho = M_\alpha k_\alpha (2\pi^2\hbar^2)^{-1}\,dE,$$

and the flux $v_\alpha\rho$ is

$$s_\alpha(E)\,dE = k_\alpha^2(2\pi^2\hbar)^{-1}\,dE.$$

The number of product nuclei per compound nucleus is $(2j+1)/(2J+1)$ at equilibrium, thus the decay rate per radioactive nucleus is

$$w = \frac{1}{2\pi\hbar}\Gamma_{\lambda\alpha}^2\int_{-\infty}^{\infty}\frac{dE}{(E'_\lambda - E)^2 + \Gamma_{\lambda\alpha}^2/4},$$

the range of integration having been extended on the grounds that the principal contribution to the integral arises from the small region of energy near E'_λ. Finally,

$$w = \Gamma_{\lambda\alpha}/\hbar,$$

so we see that the width of the level is indeed $\hbar$ times the transition rate.

If the approximation used to obtain Eq. (10–57) is applied here, the rate of decay of the radioactive nucleus may be expressed as a function of the effective alpha-particle radius, $a_{0\alpha}$, of the nucleus:

$$w = \frac{2M_\alpha}{\hbar^2} \frac{\gamma_{\lambda\alpha}^2}{F_l^2(k_\alpha a_{0\alpha}) + G_l^2(k_\alpha a_{0\alpha})}. \tag{10–66}$$

The factor $\gamma_{\lambda\alpha}^2$, the reduced width, is a measure of the probability for the arrival of the alpha-particle at the nuclear surface, as we have noted before, while the coefficient $[F_l^2 + G_l^2]^{-1}$ is again to be interpreted as the probability for penetration of the coulomb (and centrifugal) barrier, and its numerical values are to be obtained from the aforementioned tables of coulomb functions. The transmission coefficient is controlled by G_l for alpha-particles of not too high energy, and it increases exponentially with increasing energy. This fact is responsible for the empirical Geiger-Nuttall law, that the logarithm of the lifetime is an approximately linear function of the energy.

In order to estimate the value of $\gamma_{\lambda\alpha}^2$, we note that the alpha-particle can be expected to lose its identity upon entering the nucleus, and this implies that the expansion (10–62) of the wave function will contain many terms. Therefore the estimate (10–65) of the reduced width may be expected to apply. However, in the usual case of alpha-decay, the virtual state is either the ground level of the compound nucleus or a level close to it. The level spacing D is then comparable to the spacing characteristic of light nuclei, for which D/W_s is of the order unity. We are led, then, to the rough estimate

$$\gamma_{\lambda\alpha}^2 \approx \tfrac{3}{2}\hbar^2/M_\alpha a_{0\alpha},$$

and this is the value, Eq. (10–61), obtained under the quite different assumption that the motion of the alpha-particle within the nucleus is determined by a simple potential. This latter assumption was the basis for the original Gamow,* Condon, and Gurney† theory of alpha-decay.

Equation (10–66) now becomes

$$w \approx \frac{3}{a_{0\alpha}} [F_l^2(k_\alpha a_{0\alpha}) + G_l^2(k_\alpha a_{0\alpha})]^{-1}, \tag{10–67}$$

an equation which may be used to calculate $a_{0\alpha}$ from the observed energy and lifetime for alpha-decay. Because the transmission coefficient is a very sensitive function of $a_{0\alpha}$, the uncertainty in the estimate used here for the reduced width will have little effect upon the calculated alpha-

* *Zs. f. Phys.* **52,** 510 (1929).

† *Phys. Rev.* **33,** 127 (1929).

particle radii of nuclei. Lifetime formulas of the sort expressed by Eq. (10–67) have, for many years, been the basis for estimating the radii of alpha-active nuclei.*

10–7 Selection rules and related matters. A nuclear reaction may be considered as a transition between state s, the initial system, and state s', the product system. Since the transition occurs in the absence of any appreciable external forces, the total parity and angular momentum of state s' are identical with those of s. These simple selection rules are the only ones that must obtain in every circumstance. Even then it must be remembered that the incident wave consists of a mixture of states having values of s corresponding to both parities and a variety of angular momenta, so the selection rules lead to no definite conclusion concerning the final state although they do indicate that certain elements of the collision matrix vanish, namely, those that connect states s and s' of different parity and angular momentum.

The rules become much more useful in the event that the reaction occurs via a single state of the compound nucleus, that is, when the one-level formula applies. Then the parity and total angular momentum of the compound nucleus are fixed, and only those terms in the incoming (or outgoing) wave with that parity and angular momentum can contribute to the reaction. If Π is the parity and J the angular momentum of the compound state, π_1 and π_2 are the internal parities of the reacting nuclei, and j_1 and j_2 are their internal angular momenta (spins), then the relative orbital angular momentum l of an initial state that can contribute to the reaction is limited by the conditions

$$\Pi = (-1)^l \pi_1 \pi_2 \tag{10–68}$$

and

$$J = j_1 + j_2 + l, j_1 + j_2 + l - 1, j_1 + j_2 + l - 2, \cdots. \tag{10–69}$$

Similar conditions apply to the emitted wave.

For small values of the energy, i.e., for $ka_\sigma \ll 1$, the elements of the collision matrix are expected to decrease very rapidly with increasing l. According to Eq. (10–68), the admissible values of l differ by two units, and therefore only the lowest of the values of l compatible with (10–68) and (10–69) needs to be considered under usual conditions. This fact has been used to simplify the form of the one-level reaction cross section in the foregoing sections. However, it should be recognized that for values of ka_σ close to one, more than one l may play a role in the reaction. In that event, the contributions to the total reaction cross section are additive† but the

* See, for example, H. A. Bethe, *loc. cit.*; M. A. Preston, *Phys. Rev.* **71,** 865 (1947); Perlman and Ypsilantis, *Phys. Rev.* **79,** 30 (1950).

† Wigner and Eisenbud, *loc. cit.*

differential cross section is affected by interference between the various initial states of different l.

Besides the rules governing changes in angular momentum and parity, there are important restrictions on the form of the angular distribution of the products. For example, if the incident beam and target are unpolarized, the differential cross section is a function only of the angle θ between the direction of observation and the direction of incidence. That there is no dependence on the azimuthal angle, φ, follows from the absence of any specified direction normal to the beam with respect to which the angle φ could be determined. Of course, the introduction of polarization of the nuclear spins in a direction normal to the beam would provide the required fixed direction, so a notable dependence on φ might occur in that situation.

Another restriction is imposed on the angular distribution of the products of a reaction in the usual case that there is an upper limit on the values of l contributed by the incident beam. For example, if only the S-wave part of the incident beam contributes to the reaction, the emitted wave must have spherical symmetry. This result holds even if the emitted wave has $l' \neq 0$, as it can when the product nuclei have an internal angular momentum. The correctness of the statement is evident from the fact that an incident S-wave has no reference to a definite direction in space, and therefore can produce no directional effect on the products of the reaction.

In general, if l_0 is the largest value of l for which the collision matrix element is different from zero, the angular distribution of the products may contain any spherical harmonic up to Y_{2l_0} but no higher term. In other words:* *In a nuclear reaction produced by an unpolarized beam incident upon an unpolarized target, the angular distribution of the outgoing intensity cannot be more complicated than that of the incoming intensity.* Note that it is the angular distribution in intensity of the wave and not in the wave function to which reference is made here.

To establish the stated result, we consider, in place of the usual function Eq. (10–7), an incident wave containing an arbitrary mixture of Y_l^m but including just those values of l that contribute to the reaction. If the coefficients in this mixture are denoted by A_{lm}, the appropriate generalization of Eq. (10–12) for the reaction cross section takes the form

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* Attention was first called to this theorem by the conjecture of Critchfield and Teller, *Phys. Rev.* **60,** 10 (1941), although a proof was implicit in calculations of R. D. Myers, *Phys. Rev.* **54,** 361 (1938). A general proof along the lines presented here was given by Eisner, Sachs, and Wolfenstein, *Phys. Rev.* **72,** 680 (1947); **73,** 528 (1948). A somewhat different proof is given by C. N. Yang, *Phys. Rev.* **74,** 764 (1948), who calls attention to the fact that the theorem can be generalized to include emission of photons and angular correlations of two photons emitted in cascade, as well as angular correlations of the electron and neutrino in beta-decay.

$$d\sigma_{\sigma\sigma'} = (\pi/k_\sigma^2)\Big|\sum_{l,m,l'} i^{-l'} e^{i\eta_{l'}} A_{lm} U_{ss'} Y_{l'}^{m'}(\sigma')\Big|^2 d\Omega_{\sigma'}. \qquad (10\text{–}70)$$

The angular distribution therefore involves a linear combination of the products

$$A_{lm}^* A_{\lambda\mu} Y_{l'}^{m'*} Y_{l''}^{\mu'} = i^{2m'} A_{lm}^* A_{\lambda\mu} Y_{l'}^{-m'} Y_{l''}^{\mu'}, \qquad (10\text{–}71)$$

if the Y_l^m are defined in the usual way (as, for example, in Appendix 3). The values of m' and μ' are determined by the conservation conditions

$$m + m_j = m' + m_j',$$

$$\mu + m_j = \mu' + m_j',$$

where m_j and m_j' are the magnetic quantum numbers associated with the total nuclear spins of states σ and σ'. The products (10–71) therefore have the angular dependence

$$Y_{l'}^{-m+m_j'-m_j}(\sigma') Y_{l''}^{\mu+m_j-m_j'}(\sigma').$$

This product may be analyzed as a linear combination of spherical harmonics $Y_L^M(\sigma')$ with $L = l' + l''$, $l' + l'' - 1$, $\cdots$, $|l' - l''|$ and $M = \mu - m$. Our theorem concerns the maximum value of L appearing in this sum, and this can be determined by noting the range of M-values to be associated with every value of m and every state of polarization of the incident and emitted nuclei. If L_0 is the maximum L, then the range of M is $-L_0 \leq M \leq L_0$, and every such M must occur. But we have found that $M = \mu - m$, where $-l_0 \leq m \leq l_0$ and $-l_0 \leq \mu \leq l_0$, since m and μ are orbital magnetic quantum numbers of the incident wave. The range of M is therefore $-2l_0 \leq M \leq 2l_0$, from which the theorem follows, namely, $L_0 = 2l_0$. This in spite of the fact that l' may be larger than l_0.

The result holds for an unpolarized beam because the average of interference terms between states of different m_j then vanishes. If cross terms between states of different m_j occur among the products (10–71), the value of M is no longer limited to $\mu - m$. Thus the limitation on the angular distribution of reactions produced by a polarized beam or target is less restrictive than that for the unpolarized beam. A statement, with proof, of these more general conditions has been given by Blin-Stoyle.*

The angular distribution of the products is subject to even another general restriction if the reaction proceeds via a single state of the compound nucleus or via a set of states having the same parity. Since the outgoing wave function then contains only terms of the same parity as the compound nucleus, the corresponding flux (square of the wave function)

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* *Proc. Phys. Soc.* (London) **A64,** 700 (1951).

has even parity. Therefore, it is unchanged by the inversion $\mathbf{r} \rightarrow -\mathbf{r}$ of the coordinate system. Since inversion changes θ to $\pi - \theta$ and φ to $\pi + \varphi$, the invariance of the flux implies, for the differential cross section:

$$d\sigma(\pi - \theta, \pi + \varphi) = d\sigma(\theta, \varphi).$$

But for an unpolarized beam we have seen that $d\sigma$ is independent of φ, whence

$$d\sigma(\pi - \theta) = d\sigma(\theta),$$

therefore the distribution of the products has fore-and-aft symmetry in the center of mass system.

Deviations from fore-and-aft symmetry observed in a resonance reaction would imply that more than one level is contributing to the reaction and that levels of both parities are involved. The reason is that the emitted flux then contains cross terms between wave functions of opposite parity, hence it contains odd as well as even terms, and the symmetry is destroyed. Deviations also occur when the reaction does not proceed via a compound nucleus, as in the case of the stripping reactions to be discussed in the next section.

A simple example of application of many of the rules established here is offered by the reaction*

$$\mathrm{Li}^7(p,\alpha)\mathrm{He}^4.$$

The reaction is known to have a resonance at about 3 Mev, and this resonance is the subject of our discussion. We are dealing here with a very special situation, in that the two product nuclei are identical. Since they are Einstein-Bose particles, the alphas must be created in a state that is symmetric under interchange, namely, a state of even parity, hence their orbital angular momentum l' must be an even number. Furthermore, the angular momentum J of the compound nucleus equals l', since the spin of the alpha-particle is zero; consequently this resonance must be associated with a state having $J = 0$, 2, or some other even value.

The even parity of the final state implies even parity in the compound and initial states. Then if the parity of Li^7 is denoted by π_L (and that of the proton is defined to be even), Eq. (10–68) becomes

$$(-1)^l \pi_L = 1, \tag{10–72}$$

where l is the orbital angular momentum of the part of the incident wave responsible for the reaction. From the point of view of conservation of angular momentum, any value of l is admissible in the absence of further knowledge of J, since the spin of $\mathrm{Li}^7(\frac{3}{2})$ and that of the proton $(\frac{1}{2})$ can be

* Critchfield and Teller, *loc. cit.*

compounded to either $j = 1$ or $j = 2$. However if $\pi_L = 1$, Eq. (10–72) implies that l is even, and therefore the incident S-wave would be presumed to be the source of the reaction. Then $J = 2$ and the emitted particles are described by a D-wave. Nevertheless, our general theorem asserts that the angular distribution of the emitted particles is isotropic. On the other hand, if $\pi_L = -1$, l is odd, so the P-wave would be the source of the reaction. Then we may have $J = 0$ or $J = 2$, and in the latter case the angular distribution would contain a term proportional to $\cos^2\theta$ [i.e., $Y_2^0(\theta)$]. Examination of the angular distribution in this reaction has therefore made it possible to determine the parity of Li^7.

Since the compound nucleus has $J = 2$, a conclusion can be drawn only after some account has been taken of the contribution to the reaction of incident waves with higher l. If the parity were even, the incident D-wave could contribute to the reaction and that would allow an angular distribution of products containing both a $\cos^2\theta$ and a $\cos^4\theta$ term. For $\pi_L = -1$, the $\cos^4\theta$ term could appear, but then it would be due to the incident F-wave. Thus it should be much weaker for the odd parity than for the even parity case, while the $\cos^2\theta$ contribution should be much stronger relative to the isotropic term. The comparison of the observed intensity of these terms leads to the conclusion* that $\pi_L = -1$, as would be expected on the basis of any nuclear model we have considered.

In addition to the selection rules on angular momentum and parity in nuclear reactions, it seems very probable that, to a very good approximation, the nuclear i-spins will be subject to a conservation law. This supposition is based on the fact that the principle of charge invariance (or charge independence) seems to apply rather accurately to the nuclear forces. Therefore the total i-spin of the compound nucleus should be the same as that of the incoming and outgoing nuclei. For this reason, reactions which would involve changes in i-spin can be expected to be strongly suppressed.†

The effect of deviations from the principle of charge invariance would be to introduce a linear combination of states of different i-spin into the incoming and outgoing waves. Then, even if the reaction is forbidden to the dominant i-spin states, it could occur via the small admixed terms. But the cross section would be proportional to the flux associated with just these small terms. Hence the cross section for an i-spin-forbidden transition is proportional to the square of the interaction representing the deviation from charge invariance. This can be expected to be a very small effect indeed on the basis of our present ideas about nuclear forces.

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* Young, Ellet, and Plain, *Phys. Rev.* **58,** 498 (1940); Swartz, Rossi, Jennings, and Inglis, *Phys. Rev.* **65,** 80 (1944); Rubin, Fowler, and Lauritsen, *Phys. Rev.* **71,** 212 (1947); D. R. Inglis, *Phys. Rev.* **74,** 21 (1948).

† R. K. Adair, *Phys. Rev.* **87,** 1041 (1952).

10–8 Stripping reactions. Although the resonance theory gives a very successful account of observations on many nuclear reactions, there exist some for which an entirely different mechanism must be invoked to account for the data. Of these, the most notable are the (d,p) and (d,n) processes, i.e., those processes in which a deuteron is the projectile and a proton or neutron is one of the products. It is often observed that in a (d,p) or (d,n) reaction the emitted nucleon goes predominantly in the forward direction, in direct contradiction to the fore-and-aft symmetry expected on the basis of one-level resonance theory. The cross section shows strong resonances as a function of the energy of the *emitted* nucleon for fixed deuteron energy. These peaks occur at energies corresponding to formation of the various bound states of the product nucleus, the highest in energy being associated with production of the nucleus in its ground state. This aspect of the process would be in agreement with the formation of a compound nucleus except for the fact that there is an observed trend toward decreasing intensity of the emitted group with increasing energy of the group. If the process were to proceed via the compound nucleus, the intensity of each energy group would be proportional to the partial width associated with emission of that group, and the partial width increases with increasing energy because of the increased penetration factor and statistical weight of the final state [compare Eq. (10–58)]. Thus the observations appear to be inconsistent with resonance theory.

The failure of resonance theory for reactions involving deuterons results from a rather obvious deviation from the conditions that have been imposed in developing the theory. The diameter of the deuteron is quite large; the wave function is almost constant over a region some 10^{-12} cm in extent, and this distance is greater than the radius of any but the heaviest nuclei. Consequently, the deuteron has lost its identity on the $3A$ dimensional surface S if that surface has its usual, small diameter. Stated differently, the complete set of functions ψ_s could include the deuteron function only if S were quite remote from the origin. But that would require a radius a_D, characteristic of the deuteron channel, too large to meet the condition of the simple resonance theory that the radial functions $F_l(k_D a_D)$ and $G_l(k_D a_D)$ vary slowly with energy. Their variations are so rapid as to obscure the characteristic resonance aspects of the phenomenon. Thus, although the reaction can be described in terms of the general collision matrix, the approximations used to obtain the one-level resonance form of the collision matrix fail in this instance.

It has been pointed out by S. T. Butler* that the (d,p) and (d,n) reactions can be interpreted in a manner which, if correct, provides a useful device for investigating the states of the nuclei that are formed in the

* *Proc. Roy. Soc.* (London) **A208,** 559 (1951).

reaction. For the sake of definiteness, we shall limit our discussion of his interpretation to the (d,p) process. Butler notes that the large size of the deuteron allows the neutron to enter the target nucleus even though the proton is too far away to interact with the nucleus. If there are receptive states of the nucleus target-plus-neutron, the neutron may stick while the proton proceeds onward. Receptive states occur at energies corresponding to bound states of target-plus-neutron, and these are formed directly by simply picking the neutron out of the deuteron function. The process can happen only by means of that part of the function for which the proton has just the momentum required by energy and momentum conservation. The proton then proceeds with this momentum without being at any time in strong interaction with the target. This is referred to as a *stripping reaction* because the neutron is stripped out of the deuteron by the target nucleus.

It turns out that the characteristics of the stripping phenomenon are closely related to the properties of the resultant state of the product nucleus. For example, Butler has shown that the angular distribution of the freed proton has a characteristic dependence on the orbital angular momentum l of the captured neutron in the product nucleus. Since the value of l would be expected to be well defined for a low state in the independent particle model, a test of any particular ordering of levels may be obtained by observing the angular distribution resulting from stripping.* On the other hand, if the state involves a mixture of l-values for the added neutron, its parity can be determined and something can be said about its total angular momentum J. The parity of the product state is $(-1)^l$ times the parity of the target, therefore only one of the l-values need be detected to make this determination. The angular momentum is not fixed so sharply, but we do know that the angular momentum of the target must combine with l and the neutron spin to form J. For a target nucleus with zero spin, only states $J = l \pm \frac{1}{2}$ can be obtained by stripping.

The complete theoretical treatment of the stripping reaction is rather elaborate,† therefore we shall limit attention here to a description of those qualitative features of the process which make clear the connection between the angular distribution and the value of l. If the momentum of the incident deuteron (in the laboratory system) is denoted by $\hbar\mathbf{k}_D$, that of the product nucleus by $\hbar\mathbf{k}_N$, and that of the freed proton by $\hbar\mathbf{k}_p$, we have

$$\mathbf{k}_D = \mathbf{k}_N + \mathbf{k}_p. \tag{10–73}$$

* Bethe and Butler, *Phys. Rev.* **85,** 1045 (1952); Parkinson, Beach, and King, *Phys. Rev.* **87,** 387 (1952); King and Parkinson, *Phys. Rev.* **88,** 141 (1952); S. T. Butler, *Phys. Rev.* **88,** 685 (1952).

† In addition to Butler's work, *loc. cit.*, see Bhatia, Huang, Huby, and Newns, *Phil. Mag.* **43,** 485 (1952) and Daitch and French, *Phys. Rev.* **87,** 900 (1952).

Although $\mathbf{k}_N$ measures the momentum of the product nucleus, $\hbar\mathbf{k}_N$ must also be the momentum transferred to that nucleus by the neutron, since our assumption is that only the neutron interacts with the target. Thus the angular momentum of the added neutron is of the order

$$l \approx \mathbf{k}_N a = |\mathbf{k}_D - \mathbf{k}_p| a,$$

where a is the average radius of interaction. It will be found below that k_p and k_D are of the same order of magnitude, consequently $l = 0$ corresponds to $\mathbf{k}_p$ parallel to $\mathbf{k}_D$. To obtain values of l different from zero, the vector $\mathbf{k}_p$ must make an angle with $\mathbf{k}_D$ which is larger, the larger the value of l. If the fact that a is not sharply defined is taken into account, we see that the angular distribution of the proton has a maximum in the forward direction for $l = 0$, and that the maximum shifts to larger angles as the value of l is increased. This is the qualitative form of Butler's effect.*

To put the result into somewhat more quantitative form, a rough determination of the angular distribution can be made by assuming that the reaction occurs by simply picking out of a free deuteron function the neutron with momentum $\hbar\mathbf{k}_N$. The deuteron function has the form

$$\psi_D = \exp\left(\tfrac{1}{2}i\mathbf{k}_D\cdot[\mathbf{r}_n + \mathbf{r}_p]\right)\int g(\mathbf{k})\exp\left(i\mathbf{k}\cdot[\mathbf{r}_n - \mathbf{r}_p]\right)d^3k,$$

where $\mathbf{r}_n$ and $\mathbf{r}_p$ are neutron and proton coordinates and $g(\mathbf{k})$ is the Fourier transform of the internal wave function of the deuteron. The momenta of the neutron and proton are measured by $\mathbf{k}_n = \frac{1}{2}\mathbf{k}_D + \mathbf{k}$ and $\mathbf{k}_p = \frac{1}{2}\mathbf{k}_D - \mathbf{k}$, respectively, and Eq. (10-73) shows that $\mathbf{k}_n$ may be replaced by $\mathbf{k}_N$. Therefore we may write

$$\psi_D = \int g(\tfrac{1}{2}\mathbf{k}_D - \mathbf{k}_p)e^{i\mathbf{k}_p\cdot\mathbf{r}_p}e^{i\mathbf{k}_N\cdot\mathbf{r}_n}d^3k_p.$$

The contribution to a state in which the neutron has given orbital angular momentum can be obtained by introducing into this expression the expansion, Eq. (10-7):

$$e^{i\mathbf{k}_N\cdot\mathbf{r}_n} = (4\pi\hbar/Mk_N)^{\frac{1}{2}}r_n^{-1}\sum_l i^l(2l+1)^{\frac{1}{2}}Y_l^0(\theta_n)F_l(k_Nr_n),$$

where θ_n is the angle between $\mathbf{r}_n$ and $\mathbf{k}_N$. Then the probability for a neutron to arrive at the nuclear surface with orbital angular momentum l when the proton has linear momentum $\hbar\mathbf{k}_p$ is

$$P_l(\mathbf{k}_p) = 4\pi\hbar(2l+1)(Mk_Na^2)^{-1}[g(\tfrac{1}{2}\mathbf{k}_D - \mathbf{k}_p)F_l(k_Na)]^2, \quad (10\text{–}74)$$

* The argument presented here was devised by N. Austern, Butler, M. Ross, and R. R. Wilson in a discussion at Cornell University (1952). It was subsequently communicated to the author.

if a is the nuclear radius. For qualitative purposes, we make the reasonable assumption that the probability for capture of the neutron into the state of orbital angular momentum l is proportional to $P_l(\mathbf{k}_p)$, which then gives the relative intensity for emission of protons in the direction of $\mathbf{k}_p$.

The dependence on the direction of $\mathbf{k}_p$ can be obtained by means of the condition for energy conservation:

$$\tfrac{1}{2}k_D^2 = k_p^2 - q, \tag{10–75}$$

if the small recoil energy of the product nucleus is neglected. Here, $-\hbar^2 q/2M$ is the difference between the binding energy of the deuteron and the binding of the neutron in the product nucleus. Equation (10–75) can be used to obtain some idea of the angle dependence of the factor g^2 in Eq. (10–74) if we consider just the S-state of the deuteron. Then $g(\mathbf{k})$ is a function only of k, so $g(\frac{1}{2}\mathbf{k}_D - \mathbf{k}_p)$ may be expressed as a function of

$$\tfrac{1}{4}k_D^2 + k_p^2 - k_D k_p \cos\theta = \tfrac{3}{4}k_D^2 + q - k_D(\tfrac{1}{2}k_D^2 + q)^{1/2}\cos\theta,$$

according to Eq. (10–75). The angle of emission of the proton with respect to the deuteron beam is denoted here by θ. Because the form of the wave function of the deuteron is known fairly well, a reasonable estimate of $g(k)$ can be obtained.* It is a slowly varying function which decreases with increasing k. Hence the factor g^2 in $P_l(\mathbf{k}_p)$ decreases smoothly with increasing θ but has no other notable effect on the distribution.

By combining Eqs. (10–73) and (10–75), we find

$$k_N^2 = \tfrac{3}{2}k_D^2 + q - 2k_D(\tfrac{1}{2}k_D^2 + q)^{1/2}\cos\theta. \tag{10–76}$$

This gives the angle dependence of the important factor $k_N^{-1}a^{-2}F_l^2(k_N a) = \hbar^{-1}Mj_l^2(k_N a)$ in Eq. (10–74). Since $j_l(x)$ is just the regular radial function of a free wave of orbital angular momentum l, we know that it behaves (see the end of Section 4–5) as $2^l l! x^l/(2l+1)!$ for small x and as $x^{-1}\sin(x - \frac{1}{2}l\pi)$ for large x. Hence j_0^2 has a maximum at the origin and oscillates with decreasing amplitude, while j_l^2, for $l \neq 0$, vanishes at the origin, goes to a maximum, which is the more remote the larger the value of l, and then oscillates with decreasing amplitude. Although we have described the behavior as a function of x, the behavior as a function of θ is very similar for small q, since $\theta = 0$ leads to $k_N \approx 0$ in that case. Hence the angular position of the first maximum in $P_l(\mathbf{k}_p)$ depends on l in the manner we had guessed on purely qualitative grounds, and the function oscillates with decreasing amplitude beyond that maximum. For arbitrary values of q, the angular distribution plotted as a function of $\cos\theta$, between

* Butler uses the Hulthén function, Eq. (3–34), for which this behavior can be established.

$\cos\theta = 1$ and $\cos\theta = -1$, is determined by the slice between the values x_-^2 and x_+^2,

$$x_\pm = a[(\tfrac{1}{2}k_D^2 + q)^{1/2} \pm k_D],$$

of the universal function $j_l^2(x)$ plotted as a function of x^2. We see that this leads to a reduction in the total cross section with increasing q, since the slice moves to higher values of x^2 with increasing q. This accounts for the tendency of the emitted particles to show a decrease in intensity with increasing energy. The effect is supplemented by the factor g^2 in $P_l(\mathbf{k}_p)$, since it also has a smaller mean value for large q.

The determination of l from observations on a stripping reaction usually is accomplished by comparing the position of the first maximum in the angular distribution with the theoretical curves given by Butler in his original work on this subject. These curves were calculated for energies high enough to allow the neglect of coulomb effects. They may also be applied to (d,p) reactions at lower energy because the coulomb effects are not too important if only the neutron enters the target.* Theoretical results for the more complicated (d,n) reaction at low energy have yet to be obtained.

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* Compare Oppenheimer and Phillips, *Phys. Rev.* **48,** 500 (1935), who emphasize the advantage of avoiding penetration of the barrier by the deuteron if only the neutron enters the target. See also the discussion of the Oppenheimer-Phillips process by H. A. Bethe, *Phys. Rev.* **53,** 39 (1938).

CHAPTER 11

CONCERNING BETA DECAY

The phenomenon of beta-decay indicates that there exists a specific interaction between nucleons and electrons, an interaction which has every appearance of being as fundamental and distinct as the gravitational, electromagnetic, and nuclear forces. It is believed that this nucleon-electron interaction is just one expression of a very general property of particles of spin $\frac{1}{2}$, that the beta-decay of mu mesons and the capture of mu mesons by nucleons involve essentially the same interaction. The very breadth of the problem then makes it an interesting subject for investigation, and beta-active nuclei provide the most extensive source of information for that purpose. In spite of the existence of a large amount of data, the general form of the interaction has not been fixed at the time of this writing. Nevertheless, some of its features are well enough established to use the data as a source of spectroscopic information concerning nuclei. It is in this connection that the phenomenon plays an important role from the point of view of our principal objective, namely, the elucidation of the nuclear force problem.

Unfortunately, a complete interpretation of beta transitions requires a complete knowledge of the interaction, just as the interpretation of electromagnetic transitions rests on our understanding of the interaction of nucleons with photons. In the absence of the required knowledge, attempts are made to gain some insight into both the interaction and the nuclear structure problem by use of the beta-decay data. This procedure is quite fruitful because it turns out that there is one class of transitions for which the uncertainty in the interaction introduces little ambiguity into the interpretation, namely, the allowed transitions. On the other hand, many characteristics of the forbidden transitions depend directly on the details of the nucleon-electron coupling, and information concerning nuclear structure that is drawn from them is bound to be somewhat ambiguous. Compensation for this difficulty is afforded by the fact that these forbidden transitions present an opportunity to gain a deeper insight into the form of the interaction.

The coupling responsible for beta-decay also introduces a nucleon-nucleon force by means of the second order process of virtual beta emission and absorption. However, the effect is extremely small compared with other nuclear forces because the beta interaction is extremely weak. As

a matter of fact, the interaction is even small compared with the nucleon-photon coupling, as is evidenced by the fact that the smallest half-lives for beta-decay are much greater than the smallest lifetimes characteristic of electromagnetic transitions. The small value of the coupling has the advantage that the perturbation approximation is very reliable, and therefore no purely mathematical difficulty enters into the interpretation of the data. This is in sharp contrast to the prevailing situation for the meson theory of nuclear forces, which is overwhelmed with mathematical difficulties, as we have remarked in Chapter 7.

11-1 Form of the interaction. It was first suggested by Pauli that emission of the electron in beta-decay must be accompanied by emission of another particle, the neutrino. Presumably the neutrino has no charge, spin $\frac{1}{2}$, Fermi-Dirac statistics, and zero rest mass.* The following arguments lend support to Pauli's neutrino hypothesis:

(1) A beta-particle may be emitted with an energy less than the fixed energy difference between the initial and final nucleus. The neutrino is assumed to carry off the extra energy.

(2) The total number of nucleons does not change in beta-decay. Since every observation of nuclear statistics indicates that they are Fermi-Dirac for an odd number of nucleons, Einstein-Bose for an even number, the statistics must be conserved in the transition. Emission of the electron alone would lead to a change in the statistics, since it is a Fermi-Dirac particle. The emission of electron and neutrino does not violate the conservation condition.

(3) Every observed spin for nuclei containing an odd number of nucleons is half-integral, while the observations on even A nuclei yield integral spins. The conservation in A during beta-decay then implies that the angular momentum change is integral. This could not be the case for emission of a single electron, but the combination of neutrino and electron carries off integral angular momentum.

On the basis of these arguments we assume that the beta interaction is such as to lead to simultaneous emission of the electron and neutrino. Furthermore, it is assumed that the particles are *created*, just as an emitted photon is created by an excited atom through its interaction with the electromagnetic field. The alternative possibility that the electron-neutrino pair exists in the nucleus prior to emission can be excluded because nuclear magnetic moments are only 10^{-3} of the electron moment. It would be

* The mass has been shown by Langer and Moffat, *Phys. Rev.* **88**, 689 (1952), to be less than 0.05 percent of the electron mass.

difficult to establish a means for reducing the electron moment so drastically by merely binding it into the nuclear volume.*

The general features of an interaction which causes electron-neutrino creation have been developed by Fermi** on the basis of an analogy with the photon-producing property of the coupling of charged particles with the electromagnetic field. The important feature of the electromagnetic coupling is its proportionality to the vector potential **A**, which is to be interpreted as the wave function of the photon. Analogously, the beta interaction is taken to be proportional to the wave function of the electron and to the wave function of the neutrino.†

The interaction must conserve charge, so it also contains as a factor the operator leading to the required change in charge of a nucleon. Operators having the desired form are given in the i-spin notation by Eq. (7–4), τ_+ converts a neutron into a proton, as is required for β^- emission, and τ_- converts a proton into a neutron, as is required for β^+ emission. Therefore we assume that the complete nuclear Hamiltonian contains a term of the form

$$H^\beta = G \sum_j [\tau_+(j)\psi^*(j)\varphi(j) + \tau_-(j)\varphi^*(j)\psi(j)], \tag{11–1}$$

where $\varphi(j)$ and $\psi(j)$ are the neutrino and electron wave functions evaluated at the position of the jth nucleon.‡ G is a constant measuring the strength of the interaction. It has the dimensions of energy times volume if the wave functions φ and ψ are normalized in the usual fashion.

The rate of transition between nuclear states Ψ_i and Ψ_f is given by the familiar formula for the transition probability:

$$dw = 2\pi\hbar^{-1} |H_{if}^\beta|^2 \, d\rho_F, \tag{11–2}$$

where $d\rho_F$ is the density of final states for the electron-neutrino system

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* An often quoted argument against the existence of electrons in nuclei is based on the very large potential required to constrain the electron to so small a volume. However, this argument does not apply to the electron-neutrino pair since the existence of a large potential acting only on the pair (not on each particle) would be manifest only in a three-body collision of nucleus, electron, and neutrino.

** *Zeits. f. Physik* **88,** 161 (1934).

† This procedure is less arbitrary in the theory of quantized fields, used by Fermi, but the results are, of course, the same.

‡ ψ^* appears when the electron is emitted and ψ when it is absorbed. Positron emission corresponds to absorption of an electron from Dirac's sea of negative energy electrons. φ is actually the wave function of the *antineutrino*, the particle that bears the same relationship to the neutrino as the positron to the electron. The appearance of φ means absorption of an antineutrino or emission of a neutrino, and φ^* the converse. The fact that the neutrino has no charge means that there is little reason to distinguish it from the antineutrino.

when the electron energy lies between energy E and $E + dE$. Evaluation of the matrix element

$$H_{if}^{\beta} = \langle \Psi_i, H^{\beta} \Psi_f \rangle$$

is simplified by noting that the wavelengths of electron and neutrino are very long compared with nuclear dimensions, and therefore $\varphi(j)$ and $\psi(j)$ may be replaced, in a first approximation, by the constants $\varphi(0)$ and $\psi(0)$, i.e., the values of the wave functions at the origin. In this approximation, the matrix element for, say, electron emission is

$$H_{if}^{\beta} = 2G\varphi(0)\psi^*(0)\langle \Psi_i, T_+ \Psi_f \rangle,$$

where the vector $\mathbf{T}$ is the total i-spin:

$$\mathbf{T} = \tfrac{1}{2} \sum_j \boldsymbol{\tau}_j.$$

The dependence of the transition probability on nuclear structure is then contained entirely in the matrix element of $T_+ = \frac{1}{2}(T_1 + iT_2)$. The energy distribution of the emitted electrons, which will be discussed in the next section, is governed by the other factors which appear in the transition probability

$$dw = 8\pi\hbar^{-1}G^2 \left| \varphi(0) \right|^2 \left| \psi(0) \right|^2 \left| \langle \Psi_i, T_+ \Psi_f \rangle \right|^2 d\rho_f. \tag{11-3}$$

The corresponding expression for positron emission is obtained by substituting $T_- = \frac{1}{2}(T_1 - iT_2)$ for T_+. Since the matrix element of T_- vanishes for a nuclear transition involving an increase of charge, and the matrix element of T_+ vanishes for a decrease of charge, both electron and positron emission are included if T_+ is replaced by $T_+ - T_- = iT_2$ in Eq. (11-3). A general expression for the transition probability is therefore

$$dw = 8\pi\hbar^{-1}G^2 \left| \varphi(0) \right|^2 \left| \psi(0) \right|^2 \left| \langle \Psi_i, T_2 \Psi_f \rangle \right|^2 d\rho_f. \tag{11-4}$$

Equation (11-1) gives only one possible form of the interaction. One other form, depending on the spin operators of the nucleons, electrons, and neutrinos, can be constructed. Since the Hamiltonian must be rotationally invariant, the additional term, $H^{\beta'}$, involves a scalar product of the spin vectors of nucleons and light particles. Thus

$$H^{\beta'} = 2^{-\frac{1}{2}}G' \sum_j [\tau_+(j)\psi^*(j)\varphi(j)\boldsymbol{\sigma}(j)\cdot(\chi_e, \boldsymbol{\sigma}\chi_\nu) + \tau_-(j)\varphi^*(j)\psi(j)\boldsymbol{\sigma}(j)\cdot(\chi_\nu, \boldsymbol{\sigma}\chi_e)]. \tag{11-5}$$

Here $\boldsymbol{\sigma}(j)$ is the spin operator of the jth nucleon, χ_e and χ_ν are the spin functions of electron and neutrino, and the scalar product $(\chi_e, \boldsymbol{\sigma}\chi_\nu)$ indicates the usual spin sum. When this interaction is introduced into Eq. (11-2) and, as above, the functions ψ and φ are assumed constant, the corresponding transition probability becomes

$$dw' = \pi\hbar^{-1}G'^2 |\varphi(0)|^2 |\psi(0)|^2 (\chi_e,\sigma\chi_\nu)\cdot\langle\Psi_i, \sum_j \tau_2(j)\sigma(j)\Psi_f\rangle|^2 d\rho_F.$$

This expression is to be summed over the possible spin states χ_ν of the neutrino. Consequently, the cross terms appearing in the square of the scalar product are found to vanish, while terms such as

$$\sum_\nu (\chi_e,\sigma_x\chi_\nu)(\chi_\nu,\sigma_x\chi_e) = (\chi_e,\sigma_x^2\chi_e)$$

are equal to one. In effect, the transition probability therefore has the form

$$dw' = \pi\hbar^{-1}G'^2 |\varphi(0)|^2 |\psi(0)|^2 \sum_\alpha |\langle\Psi_i,\sum_j \tau_2(j)\sigma_\alpha(j)\Psi_f\rangle|^2 d\rho_F, \quad (11\text{–}6)$$

where α labels the x, y, z-components of σ.

In the event that both interactions (11–1) and (11–5) occur in the Hamiltonian, the transition probability involves cross terms between them. However, these terms play no role so long as the two polarizations of the emitted beta-particle are not distinguished. Then the transition probability is to be summed over electron spin states as well as neutrino spin states, with the consequence that the cross terms proportional to $(\chi_e,\sigma_\alpha\chi_e)$ vanish.* Hence the transition probability is simply the sum (11–4) plus twice (11–6), the factor two being just the number of electron spin states. Because the various nuclear spin states are not usually distinguished, we average over the initial spin states of the radioactive nucleus and sum over the final spin states to obtain the general expression for the transition probability:

$$dw = 2\pi\hbar^{-1}G'^2 |\varphi(0)|^2 |\psi(0)|^2 (2J_i+1)^{-1} d\rho_F$$
$$\times \sum_{m_i}\sum_{m_f} \{4K^2 |\langle\Psi_i,T_2\Psi_f\rangle|^2 + \sum_\alpha |\langle\Psi_i,\sum_j \tau_2(j)\sigma_\alpha(j)\Psi_f\rangle|^2\}, \quad (11\text{–}7)$$

where $K = G/G'$, J_i is the spin of the radioactive nucleus, m_i is the corresponding magnetic quantum number, and m_f is the magnetic quantum number of the final nucleus.

Two significant approximations have been introduced to obtain Eq. (11–7): the variation of the functions φ and ψ over the nucleus has been neglected, and relativistic considerations concerning the interaction have been ignored. The corrections associated with variation of the electron and neutrino functions over the nucleus can be taken into account by expanding the matrix element [containing the functions $\varphi(j)$ and $\psi(j)$] in powers of the ratios of the nuclear radius to the electron and neutrino wavelengths. This would correspond to the multipole expansion, Eq.

* To include the spin functions, we introduce the factor $2^{-\frac{1}{2}}(\chi_e,\chi_\nu)$ in the interaction (11–1), the factor $2^{-\frac{1}{2}}$ being introduced so that the sum over electron spin states still leads to Eq. (11–4).

(9–17), for the problem of photon emission. Since we have retained only the leading term in the expansion, Eq. (11–7) gives the probability for allowed transitions. The higher terms then correspond to forbidden transitions; their half-lives are much longer, so they are important only when the states Ψ_i and Ψ_f are such that a selection rule precludes occurrence of the allowed transition.

The forbidden transitions have a probability of the order of $(R/\lambda\!\!\!^{-})^2$ times that of an allowed transition, where R is the nuclear radius and $\lambda\!\!\!^{-}$ the wavelength of emitted neutrino or electron. Now $R \approx e^2/mc^2$ and, for a neutrino, $\lambda\!\!\!^{-} = \hbar/p = \hbar c/E_\nu$, since the mass is taken to be zero. Hence the ratio of forbidden to allowed transitions is of the order $(e^2/\hbar c)^2(E_\nu/mc^2)^2 \approx 137^{-2}$. On the other hand, the relativistic corrections to the interaction due to the nucleon motion can be expected to be at least of the order of the ratio of the nucleon kinetic energy to Mc^2. Since this quantity is about $\frac{1}{20}$, it would be incorrect to neglect the relativistic effects in a discussion of forbidden transitions. This situation also has its analog in the treatment of electromagnetic multipole transitions. The electric multipole moments depend on the static charge distribution; they therefore can be obtained in a nonrelativistic approximation. The magnetic multipole moments, on the other hand, are proportional to the nucleon velocity (divided by c), therefore they represent a relativistic correction. We have seen in Section 9–1 that for nuclear transitions the magnetic dipole contribution is usually larger than the electric quadrupole term, just as the above estimate indicates that the relativistic contribution to the first forbidden transition is more important than the term introduced by the variation of the neutrino function over the nucleus. Actually, the same numbers appear in both cases because the neutrino, being of mass zero, acts just like a photon as far as our qualitative argument is concerned.

The determination of the relativistic correction on theoretical grounds requires a full knowledge of the relativistic form of the interaction. Given the form, an expansion in powers of the ratio of nucleon kinetic energy to Mc^2 is a straightforward matter. Since the neutrino always and the electron usually emerges from the nucleus at relativistic energies, the relativistic expression for the interaction should be used as a basis for discussion of even the allowed transitions. A relativistic treatment of the light particles is possible, since it seems safe to assume that both the electron and neutrino are described by the Dirac equation. When we seek to replace Eqs. (11–1) and (11–5) by covariant expressions, it is necessary to consider bilinear forms in φ and ψ having definite transformation properties under Lorentz transformations, just as it was necessary to have, in the nonrelativistic case, definite transformation properties under space rotations and inversions. It is well known* that five different bilinear forms having the

* See, e.g., W. Pauli, *Handb. d. Physik*, **24**/1. Julius Springer, Berlin (1933), p. 221.

transformation properties of scalar, vector, tensor (second rank), pseudovector, and pseudoscalar under Lorentz transformations can be constructed in terms of the four component Dirac functions φ and ψ. The scalar product of each of these quantities with a nucleon operator having the same transformation properties then provides an invariant interaction term which can be added to the Hamiltonian. There are five such expressions, known, respectively, as the scalar, vector, tensor, etc. coupling terms.

The construction of the nucleon operator with known transformation properties is not so straightforward. A knowledge of the correct relativistic formulation of the theory of the nucleon is required, and we have seen that little direct information is available on that point. The usual procedure is to assume that the nucleon, being a particle of spin $\frac{1}{2}$, is also to be described by the Dirac equation. The operators having the desired transformation properties can thereupon be constructed in the same manner as for the light particles. Clearly, the nucleon factor in each term of the interaction then has the same form as the electron-neutrino factor.

The interaction obtained in this way contains five coupling constants, corresponding to the five operators having different transformation properties. However, when only the nonrelativistic approximation to the nucleon factor is included, and the variation over the nucleus of the radial parts $\varphi(r)$ and $\psi(r)$ of the Dirac functions is ignored, the transition probability takes on exactly the form* (11–7) except that $\psi(0)$ is customarily replaced by $\psi(R)$, where R is the nuclear radius.† This last substitution is required to overcome a troublesome divergence at the origin of the Dirac radial function in the coulomb field. Since the coulomb field acting on the electron fails at $r = R$, it seems reasonable to take $\psi(R)$ as the constant value within the nucleus.

The complete expression for the transition probability in the case of a forbidden transition can be obtained by expanding the nucleon part of the interaction in powers of v/c. The results of this expansion are given by E. Greuling‡ and reference should be made to that work for details. It is to be noted that these terms depend on all five coupling constants and on the explicit form of the operators used to express the nucleon contribution to the interaction. Thus the forbidden transitions are a promising source

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* M. Fierz, *Zeits. f. Phys.* **104,** 553 (1937), has pointed out that energy-dependent deviations from the form (11–7) could occur as a result of interference between certain of the interactions. However, the experimental data on the shape of the beta spectrum indicate that these terms vanish.

† For a more detailed treatment, see the general discussion of beta-decay by E. J. Konopinski, *Rev. Mod. Phys.* **15,** 209 (1943).

‡ *Phys. Rev.* **61,** 568 (1942).

of information concerning the structure of the interaction.* On the other hand, the rate of allowed transitions depends primarily on nuclear structure, the dependence on the nature of the interaction being contained just in the ratio K of the spin-independent to spin-dependent term.

11-2 Shape of the allowed spectrum. The probability that an electron be emitted with energy between E and $E + dE$ in an allowed beta transition, according to Eq. (11-7), is proportional to $|\varphi(0)|^2|\psi(0)|^2 d\rho_F$, where ψ is the wave function of an electron of energy E and φ is the wave function of a neutrino of energy $E_\nu = E_0 - E$, E_0 being the total energy of the nuclear transition. The density of final states $d\rho_F$, including all directions of emission of the two particles, is

$$d\rho_F = \tfrac{1}{4}V^2\pi^{-4}\hbar^{-6}c^{-3}(E_0 - E)^2p^2\,dp,$$

if p is the electron momentum and V the volume of an enclosure introduced, as usual, for purposes of normalization. Use is made here of the energy momentum relationship $E_\nu = cp_\nu$ for the zero-mass neutrino. The corresponding equation for the electron is

$$E^2 = m^2c^4 + p^2c^2,$$

whence we find

$$d\rho_F = \tfrac{1}{4}\pi^{-4}m^5c^4\hbar^{-6}V^2(W_0 - W)^2(W^2 - 1)^{\frac{1}{2}}W\,dW, \qquad (11\text{-}8)$$

where $W = E/mc^2$ and $W_0 = E_0/mc^2$. Note that the rest energy of the beta-particle is included in E, so that E_0 is measured by the difference of *nuclear*, rather than atomic, masses. The distribution function is proportional to $d\rho_F/dW$, which has the value zero for $W = 1$ (i.e., for an electron of zero kinetic energy), rises to a maximum, and decreases to the value zero at $W = W_0$, namely, at the maximum available energy.

The nucleus exerts no appreciable force on the neutrino, so φ may be taken to be a plane wave normalized to volume V. Then $|\varphi(0)|^2 = V^{-1}$. For beta-particles of large kinetic energy emitted from nuclei of low atomic number, the function ψ may also be taken to be a plane wave, in which case $|\psi(0)|^2 = V^{-1}$. The spectrum of the emitted electrons under these conditions is given by the function $d\rho_F/dW$. Therefore if the observed number of beta-particles of energy W is $N(W)$, a plot of $N^{\frac{1}{2}}(W)W^{-\frac{1}{2}}(W^2 - 1)^{-\frac{1}{4}}$ as a function of W should take the form of a straight line, the intercept with the W-axis being W_0. This is the simplest form of a "Kurie plot,"† a method of presenting the data designed to provide a simple comparison with the allowed spectrum.

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* See, in particular, R. Marshak, *Phys. Rev.* **70,** 980 (1946).

† Kurie, Richardson, and Paxton, *Phys. Rev.* **49,** 368 (1936).

We must expect that the spectrum will deviate from this simple form at small values of W since, even for light nuclei, the coulomb function of a low-energy electron near the nucleus is distinctly different from a plane wave. The effect on the spectrum is given by the absolute square of the ratio $\psi_{\text{coul}}(0)/\psi_{\text{plane}}(0)$, $\psi_{\text{coul}}(\mathbf{r})$ being the regular coulomb function of the electron. Since only the S-wave is different from zero at the origin, the ratio of the S partial waves may be used for this purpose. This ratio is given by the quantity C_0, introduced in Section 6–2,

$$C_0^2 = 2\pi(kD)^{-1}[\exp(2\pi/kD) - 1]^{-1},$$

where k is the electron wave number, $k = mc\hbar^{-1}(W^2 - 1)^{1/2}$ and $D = \pm\hbar^2/Zmc^2$, the sign being positive for positron emission, negative for electron emission. The low-energy end of the spectrum then has the form $2\pi(kD)^{-1}[\exp(2\pi/kD) - 1]^{-1}(d\rho_F/dW)$ for light nuclei. This expression has the property that the intensity of the positron spectrum decreases to zero with decreasing energy while, for an electron, the product of C_0^2 with $d\rho_F$ is proportional to $(W_0 - W)^2W$ and approaches a finite limit as $W \to 1$. This difference in the behavior of the positively and negatively charged beta-particles is just what one would expect as a consequence of the repulsion exerted on a positron by the nucleus and the attraction exerted on the electron. The former causes the appearance of a low-energy positron at the nucleus to be quite improbable, the latter causes the appearance of even a zero-energy electron to have a finite probability.

For nuclei of large Z, the coulomb functions must be used even for an electron whose energy lies in the relativistic region. Then the relativistic coulomb functions are needed and, as we have mentioned before, the matrix element is taken to be proportional to the value of the function at the nuclear surface rather than at the origin. The intensity distribution is then given by* $F(Z,W)(d\rho_F/dW)$, with

$$F(Z,W) = 2(2kR)^{2s-2}(1 + s)e^{-\pi W/kD}\left|\frac{\Gamma(s - iW/kD)}{\Gamma(2s + 1)}\right|^2, \quad (11\text{–}9)$$

where R is the nuclear radius and

$$s = [1 - (Ze^2/\hbar c)^2]^{1/2}. \quad (11\text{–}10)$$

The function $F(Z,W)$ has not been completely tabulated, but a very good approximation to it in the range of interest has been given in terms of elementary transcendental functions by H. Hall.†

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* Fermi, *loc. cit.*

† *Phys. Rev.* **79**, 745 (1950).

The shape of the spectrum* and the absolute intensity may now be obtained from Eqs. (11-7) and (11-8):

$$dw = (2\pi^3)^{-1}(mc^2/\hbar)g'^2 \left\{ K^2 \left| \int \mathbf{1} \right|^2 + \left| \int \boldsymbol{\sigma} \right|^2 \right\} F(Z,W)(W_0 - W)^2(W^2 - 1)^{\frac{1}{2}} W\, dW, \quad (11\text{-}11)$$

where g' is the dimensionless constant

$$g' = m^2 c G'/\hbar^3, \quad (11\text{-}12)$$

and the symbols $|\int \mathbf{1}|^2$ and $|\int \boldsymbol{\sigma}|^2$ are a conventional notation for the following quantities:

$$\left| \int \mathbf{1} \right|^2 = 4(2J_i + 1)^{-1} \sum_{m_i} \sum_{m_f} |\langle \Psi_i, T_2 \Psi_f \rangle|^2, \quad (11\text{-}13)$$

$$\left| \int \boldsymbol{\sigma} \right|^2 = (2J_i + 1)^{-1} \sum_{\alpha} \sum_{m_i} \sum_{m_f} |\langle \Psi_i, \sum_j \tau_2(j)\sigma_\alpha(j)\Psi_f \rangle|^2. \quad (11\text{-}14)$$

Since the characteristic time

$$\hbar/mc^2 = 1.28 \times 10^{-21} \text{ sec}$$

appearing in Eq. (11-11) is many powers of ten smaller than the typical lifetime for beta-decay, it is evident that the dimensionless constant g' is extremely small. In fact, we shall find below (Section 11-5) that g'^2 is of the order of 10^{-23}. A comparison with the corresponding constant, $e^2/\hbar c = \frac{1}{137}$, for photon emission shows that the beta process is due to a coupling many orders of magnitude weaker than that responsible for electromagnetic transitions.

11-3 Half-life for beta-decay and K-capture. The total rate of transition, w, of the nuclear system undergoing beta-decay is obtained by integrating dw over the complete energy spectrum. According to Eq. (11-11),

$$w = (2\pi^3)^{-1} g'^2 (mc^2/\hbar) \left\{ K^2 \left| \int \mathbf{1} \right|^2 + \left| \int \boldsymbol{\sigma} \right|^2 \right\} f(Z,W_0), \quad (11\text{-}15)$$

where

$$f(Z,W_0) = \int_1^{W_0} F(Z,W)(W_0 - W)^2(W^2 - 1)^{\frac{1}{2}} W\, dW. \quad (11\text{-}16)$$

* For a summary of recent experimental results on the shape of both the allowed and forbidden spectra, see Chien-Shiung Wu, *Rev. Mod. Phys.* **22**, 386 (1950).

Since the half-life for the transition is

$$t_{1/2} = \ln 2/w,$$

the relationship

$$f(Z,W_0)t_{1/2} = (2\pi^3 \ln 2)g'^{-2}(\hbar/mc^2)\left\{K^2\left|\int \mathbf{1}\right|^2 + \left|\int \boldsymbol{\sigma}\right|^2\right\}^{-1} \quad (11\text{–}17)$$

is equivalent to Eq. (11–15).

On the left side of Eq. (11–17) are quantities directly related to the data on beta-decay. Since $f(Z,W_0)$ is a known* function, the "ft-value" may be obtained directly from a measurement of the half-life and the energy, W_0, of the end point of the spectrum. On the other hand, the quantities appearing on the right side of the equation are of direct theoretical interest. The matrix elements relate to the structure of the nuclear states involved, the constant K indicates the relative importance of the two possible types of interaction, and g' measures the coupling. The current practice is to use the empirical ft-values to gain some insight into these three aspects of the problem.

Our expression for the half-life of a positron emitter is incomplete in one respect; it does not include the possibility of capture of an atomic electron. Since the probability is greatest for capture of a K-electron, this process is referred to as K-capture.† It can occur by means of the interaction (11–1) or (11–5) whenever positron emission is possible. The capture is, of course, accompanied by the emission of a neutrino. K-capture plays its most important role when the energy difference between the radioactive nucleus of charge Z and the product nucleus of charge $Z - 1$ is less than mc^2, i.e., when the atomic mass difference is less than $2m$. Then positron emission is energetically forbidden, so K-capture is the only effect contributing to the transition.

The calculation of the lifetime for K-capture follows the same procedure as that used for emission. The transition probability is given, in the same approximation, by Eq. (11–7), with the slight modification that the energy is fixed. Hence

$$w_K = 2\pi\hbar^{-1}G'^2|\,\varphi(0)\,|^2|\,\psi_K(0)\,|^2\rho_F\left\{K^2\left|\int \mathbf{1}\right|^2 + \left|\int \boldsymbol{\sigma}\right|^2\right\},$$

where $\psi_K(0)$ is the value of the wave function of the K-electron at the origin (or at R for the relativistic case), $\varphi(0) = V^{-1/2}$ as before, and ρ_F is

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* A rather complete set of curves is presented by Feenberg and Trigg, *Rev. Mod. Phys.* **22,** 399 (1950). Values used here are taken from unpublished curves prepared by S. Moszkowski.

† Yukawa and Sakata, *Proc. Phys. Math. Soc.* (Japan) **17,** 467 (1935); C. Møller, *Phys. Rev.* **51,** 84 (1937), *Physik. Z. Sowjetunion* **11,** 9 (1937); H. A. Bethe and R. F. Bacher, *Rev. Mod. Phys.* **8,** 82 (1936).

the density of final states for the neutrino:

$$\rho_F = (2\pi^2\hbar^3)^{-1}m^2cVW_\nu^2.$$

W_ν is the energy of the emitted neutrino in units of mc^2 which, for small Z, is the nuclear energy difference plus the self-energy of the annihilated electron:

$$W_\nu = W_0 + 1.$$

For large Z, the K ionization potential may be comparable to W_0, so*

$$W_\nu = W_0 + s,$$

where s is given by Eq. (11-10).

For small Z the wave function of the K-electron is given by

$$\varphi(r) = (\pi D^3)^{-\frac{1}{2}}e^{-r/D},$$

and therefore the transition probability has the value

$$w_K = \pi^{-2}g'^2(mc^2/\hbar)(Ze^2/\hbar c)^3(W_0 + 1)^2 \left\{K^2\left|\int \mathbf{1}\right|^2 + \left|\int \boldsymbol{\sigma}\right|^2\right\}. \quad (11\text{-}18)$$

In general, we can write in analogy with Eq. (11-15),

$$w_K = (2\pi^3)^{-1}g'^2(mc^2/\hbar)\left\{K^2\left|\int \mathbf{1}\right|^2 + \left|\int \boldsymbol{\sigma}\right|^2\right\} f_K(Z,W_0),$$

with

$$f_K(Z,W_0) = 2\pi(Ze^2/\hbar c)^{2s+1}(2mcR/\hbar)^{2s-2}(1 + s)(W_0 + s)^2/\Gamma(2s + 1).$$

Therefore the value $f_K t_{\frac{1}{2}}$ for K-capture is related to the matrix elements by Eq. (11-17). When Eq. (11-17) is applied to a positron emitter, the function f should be replaced by $f + f_K$ in order to include the contribution of K-capture. Both f_K and $f(\text{pos}) + f_K$ are tabulated by Feenberg and Trigg.†

Although the connection Eq. (11-17) between the ft-value and nuclear matrix elements is valid only for an allowed transition, an ft-value may be computed for any transition of known energy and half-life. This is a useful procedure because it provides an immediate orientation as to whether the transition is allowed or forbidden. Presumably, the ft-values of forbidden transitions are considerably larger than those of allowed transitions, therefore a comparison of the "degree of forbiddenness" of different transitions can be made on this basis. However, some care must be exercised in

* Note that here, as before, any excitation energy of the residual atom is treated as negligible.

† *Loc. cit.*

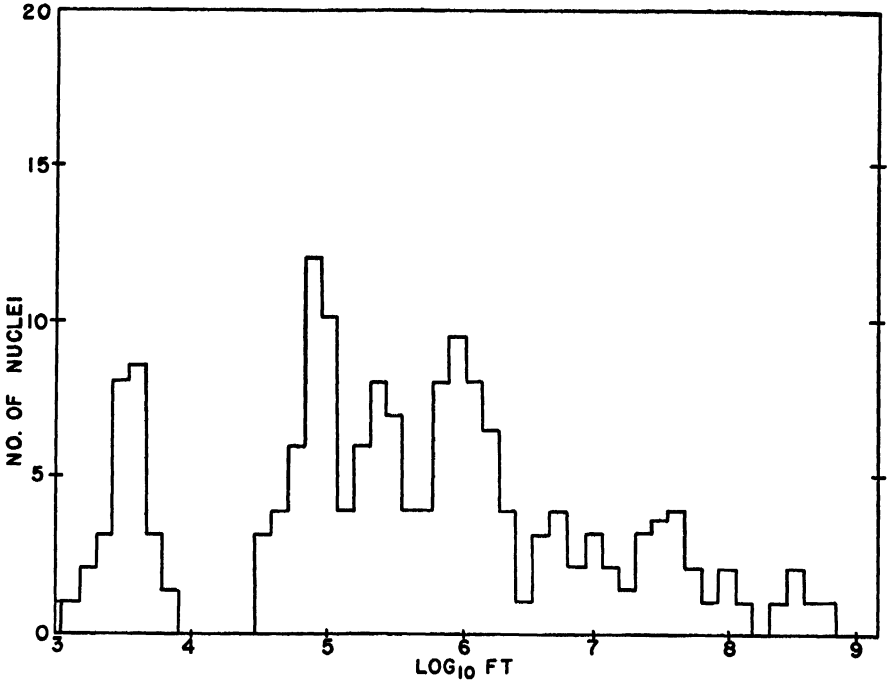

FIG. 11–1. Distribution of observed *ft*-values for transitions between ground states of odd nuclei. Adapted from Feenberg and Trigg, *loc. cit.*

doing so, since, even for allowed transitions, a considerable variation in the matrix elements is to be expected as a consequence of variations in nuclear structure. Thus the *ft*-values of allowed transitions vary over a considerable range. Nevertheless, the *ft*-values for observed transitions do divide into distinct groups, as indicated in Fig. 11–1, and it is usually possible to account for the separation by assigning to each group a definite order of forbiddenness.

11–4 Selection rules. The selection rules governing the allowed transitions fall into two classes. There are selection rules for the angular momentum and parity on the one hand, and selection rules for the *i*-spin on the other. The former are strictly valid, since they involve only the assumption of the isotropy of space. The latter depend on the assumption of charge independence of the forces, which, although it is only an approximation, seems to be a very good one. All of the selection rules are governed by the properties of the matrix elements symbolized by $\int \mathbf{1}$ and $\int \boldsymbol{\sigma}$. According to Eq. (11–13), no space- or spin-dependent operator appears in $\int \mathbf{1}$, so no change in total angular momentum, orbital angular momentum, spin angular momentum, or parity can be brought about by this term. Hence selection rules associated with $\int \mathbf{1}$ are

$$\Delta J = 0,\ \Delta m_J = 0,\ \Delta L = 0,\ \Delta m_L = 0,\ \Delta S = 0,\ \Delta m_S = 0,\ \text{no.} \quad (11\text{–}19)$$

These are known as the Fermi selection rules, since the only terms included by Fermi in his original treatment of beta-decay were of the form $\int 1$. The somewhat different selection rules associated with $\int \sigma$ are, on the other hand, known as the Gamow-Teller rules, since the proposal* that such a term be included in the interaction was first made by them. It is clear from Eq. (11–14) that $\int \sigma$ can lead to changes of one unit in the total nuclear spin quantum number S, but the orbital angular momentum and parity are left unchanged. Therefore the Gamow-Teller rules take the form

$$\Delta J = 0, \pm 1;\quad \Delta m_J = 0, \pm 1;\quad \Delta L = 0;$$

$$\Delta m_L = 0;\quad \Delta S = 0, \pm 1;\quad \Delta m_S = 0, \pm 1;\quad \text{no}, \quad (11\text{–}20a)$$

except that

$$J = 0 \rightarrow J = 0 \quad (11\text{–}20b)$$

is forbidden.

The selection rules on i-spin also distinguish between the two interactions. The component T_2 of $\mathbf{T}$ has no matrix elements connecting different values of T, just as a component of the angular momentum of a system has no nondiagonal matrix elements with respect to the angular momentum quantum number. Hence, for the Fermi interaction,

$$\Delta T = 0. \quad (11\text{–}21)$$

The corresponding Gamow-Teller selection rule is

$$\Delta T = 0, \pm 1, \quad (11\text{–}22)$$

since the operator $\Sigma_j \tau_2(j)\boldsymbol{\sigma}(j)$ has essentially the same properties with respect to T as it does with respect to the total spin S.

The distinction between Fermi and Gamow-Teller rules makes possible, at least in principle, a rather direct determination of the constant K. If the value of ΔJ or ΔT is determined to be ± 1 for a beta transition by some other means, such as the direct measurement of the nuclear spins, and the ft-value corresponds to an allowed transition, the transition is produced only by the Gamow-Teller interaction. A comparison with an allowed transition known to have $\Delta J = 0$ or $\Delta T = 0$, which is produced by both interactions, or a $J = 0 \rightarrow J = 0$ transition produced only by the Fermi interaction, then makes possible an evaluation of K. The accuracy of this determination depends, unfortunately, on our knowledge of the wave functions used to calculate the matrix elements. We shall find below that our knowledge of these functions is not sufficiently accurate in the cases of interest to lead to a completely reliable value of K.

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* *Phys. Rev.* **49**, 895 (1936).

An additional general selection rule would apply to allowed transitions if nuclear states were formed according to the very special conditions of Wigner coupling. Then each state of a nucleus belongs to a definite supermultiplet (Q,Q',Q'') and the selection rule in question pertains to the changes that can occur in these quantum numbers. Now both interactions, T_2 and $\mathbf{Y}_2 = \Sigma_j \tau_2(j)\boldsymbol{\sigma}(j)$, happen to be simple operators in the combined charge and spin space in which the proper functions associated with (Q,Q',Q'') are defined.* They play the same role as a component of the total spin operator $\mathbf{S}$ in ordinary spin space. Hence neither interaction can cause a transition between two different supermultiplets:

$$\Delta(Q,Q',Q'') = 0. \tag{11-23}$$

This rule implies, remarkably enough, that very few allowed beta transitions between nuclear ground states should occur in nature. The reason has to do with the fact that the ground state of a nucleus of isobaric number T_3 belongs to the supermultiplet with $Q = |\, T_3 \,|$, i.e., with the smallest possible value of Q. This is a consequence of the fact that levels with larger Q lie higher, according to the principles of Wigner coupling outlined in Section 8-4. If a nucleus with negative T_3 undergoes positron emission, $|\, T_3 \,|$ increases, therefore there must be a change in supermultiplet since Q is, by definition, the largest value of $|\, T_3 \,|$ in a given supermultiplet. Hence positron emission violates the rule (11-23). On the other hand, the splitting of the supermultiplet by the coulomb interaction is such that states of larger T_3 usually lie higher, and electron emission would not be expected to occur on energetic grounds.

Exceptions occur for nuclei with positive T_3 in the ground state. Then $|\, T_3 \,|$ decreases on positron emission, and the transition stays within the supermultiplet. This situation is expected to occur only for nuclei light enough so that a state of positive T_3 is sufficiently stable to be observed. Well-known examples are the light nuclei with $T_3 = \frac{1}{2}$, which transform to their mirror nuclei, $T_3 = -\frac{1}{2}$, by means of positron emission. An exception of a quite different sort occurs for certain nuclei having $T_3 = 0$. It may happen in this case that $Q = 0$ cannot occur, that $Q = 1$ is the lowest state because of conditions imposed by the Pauli principle. Then $T_3 = \pm 1$ belong to the same supermultiplet and a transition by positron emission would be in accord with Eq. (11-23). The important examples arise in the case of the $4n + 2$ nuclei which we have found to have, in the Wigner coupling scheme, ground states belonging to the (1,0,0) supermultiplet.

Actually, it is not expected that (11-23) will be a hard and fast rule because there is much evidence to indicate that Wigner coupling provides

* E. P. Wigner, *Phys. Rev.* **56**, 519 (1939).

at best a rough approximation to the structure of nuclei. Nuclear states probably contain a mixture of several supermultiplet functions. Each term in this mixture is still subject to the selection rule (11-23), but the conclusions drawn above concerning the very special circumstances required for an allowed transition cannot be expected to hold. However, it is likely that the supermultiplet suggested by Wigner coupling plays an important role in the nuclear ground state. Hence transitions of the two special classes described above, the $T_3 = \frac{1}{2}$ to $T_3 = -\frac{1}{2}$ mirror transition, and the $T_3 = 1$ to $T_3 = 0$ or $T_3 = 0$ to $T_3 = 1$ transition in the $4n + 2$ nuclei, will occur more readily than other allowed transitions. These are therefore referred to as the *favored* or *superallowed* transitions.

The distinction between favored and other allowed transitions can be understood on qualitative grounds. The matrix elements $\int \mathbf{1}$ and $\int \boldsymbol{\sigma}$ are just overlap integrals between two nuclear states which differ by the conversion of a neutron to a proton (or the converse) and, in some cases, by the inversion of the spin of the converted nucleon. Under the charge invariance hypothesis, the stationary states of the nuclei (A,Z) and $(A,Z \pm 1)$ are identical for states that are compatible, in both nuclei, with the Pauli principle. If it should happen that the states Ψ_i and Ψ_f involved in the transition are identical in this sense, their overlap would be large, and consequently the matrix element would be large. On the other hand, if different states of the two nuclei are involved, they would be expected to be very nearly orthogonal, and the matrix element would be small. The mirror transition $T_3 = \frac{1}{2}$ to $T_3 = -\frac{1}{2}$ clearly falls in the first category. The two nuclei differ in that one odd nucleon is a proton in the initial system and a neutron in the same state in the final system. These transitions are therefore favored. It should be evident from this argument that the result does not depend primarily on the validity of the Wigner coupling scheme, but rather on the mirror property of the nuclear forces.

The situation for the $4n + 2$ nuclei is somewhat more complicated. The transition may be assigned to one of the extra two nucleons. Let us say that in the initial state these are two neutrons in the same orbit and therefore in a singlet spin state. The corresponding i-spin state is a triplet. When one neutron is converted to a proton with emission of an electron, either an i-singlet (Gamow-Teller selection rule) or i-triplet may be formed. If both nucleons remain in the same orbit, the corresponding spin state must be triplet or singlet, respectively. In the latter case, the transition is certainly favored, since the states Ψ_i and Ψ_f overlap completely. On the other hand, the transition to the triplet-i-singlet will involve a change of wave function because the forces are spin-dependent. Hence the matrix element will be somewhat reduced. However, if the spin dependence is small, the matrix element will be close to its maximum value. For Wigner coupling there is no distinction between the two cases, since it implies the absence of any spin dependence of the forces.

The favored transition is exceptional because the ground states of the initial and final nuclei are usually described by functions having little overlap. For example, a light $4n$ nucleus consisting of $n - 1$ alphas plus three neutrons and a proton will decay by electron emission into the nucleus formed by n alpha-particles. One of the three neutrons in the initial nucleus is required by the Pauli principle to be in a state quite different from that of the proton into which it is converted by the beta process; hence the absence of overlap between the functions Ψ_i and Ψ_f. The effect is even more evident for the heavier nuclei which have a neutron excess (over the number of protons) of three or larger. From the point of view of the independent particle model, the last neutron occurs in a quite different orbit from the last proton, consequently the conversion of a neutron to a proton, or the converse, leads to a marked change in the wave function.

11–5 Special cases. The values of the coupling constant g' and the ratio, K, of Fermi to Gamow-Teller interactions can be obtained in principle from the observed ft-values of allowed transitions. For this purpose, the quantities $|\int \mathbf{1}|^2$ and $|\int \boldsymbol{\sigma}|^2$ must be calculated, and this requires some knowledge of the wave functions. Only very limited knowledge of the functions is needed to derive $|\int \mathbf{1}|^2$ since, as can be seen from Eq. (11-13), it depends only on the matrix elements of the operator T_2. We have observed that this matrix is diagonal in the total i-spin quantum number T because it behaves in every respect like an angular momentum operator. The analogy may be extended to obtain the numerical value of the matrix element between two different values of T_3. It is the same as that of the y-component of the angular momentum if T is given the role of the total angular momentum and T_3 the role of the magnetic quantum number. Thus, for example, in the mirror transition $T_3 = \frac{1}{2}$ to $T_3 = -\frac{1}{2}$ of the charge doublet $T = \frac{1}{2}$ the matrix element is $\frac{1}{2}$. Other values may be obtained with equal ease from any standard reference on angular momentum matrices.

This information is adequate for the computation of $\langle \Psi_i, T'_2 \Psi_f \rangle$ if Ψ_i and Ψ_f are known linear combinations of i-spin functions having given T'. But only one term appears in each function if T is a good quantum number, a condition which seems, on the basis of other evidence, to be fulfilled to rather good approximation. Hence the matrix elements $\langle \Psi_i, T'_2 \Psi_f \rangle$ are reasonably well known. Given the matrix elements, $|\int \mathbf{1}|^2$ may be obtained from Eq. (11-13). The calculation can be simplified by noting that $m_f = m_i$ and that $\langle \Psi_i, T'_2 \Psi_f \rangle$ is independent of m_i, so that

$$\left| \int \mathbf{1} \right|^2 = 4 |\langle \Psi_i, T_2 \Psi_f \rangle|^2.$$

Values of $|\int \mathbf{1}|^2$ are presented in Table 11-1 for two cases of interest.

TABLE 11-1

Values of $|\int \mathbf{1}|^2$ and $|\int \boldsymbol{\sigma}|^2$ for transitions within the indicated multiplets. Taken from Wigner, *Phys. Rev.* **56**, 519 (1939). Note, however, that Wigner gives $\frac{1}{4}$ the quantity tabulated here. J_i and J_f are the total angular momenta in initial and final states respectively, and L is the total orbital angular momentum in both.

T	$\lvert\int \mathbf{1}\rvert^2$	(Q,Q',Q'')	L	$J_i \rightarrow J_f$	$\lvert\int \boldsymbol{\sigma}\rvert^2$
$\frac{1}{2}$	1	$(\frac{1}{2},\frac{1}{2},\pm\frac{1}{2})$	0	$\frac{1}{2} \rightarrow \frac{1}{2}$	3
			1	$\frac{3}{2} \rightarrow \frac{3}{2}$ $\frac{3}{2} \rightarrow \frac{1}{2}$ $\frac{1}{2} \rightarrow \frac{3}{2}$ $\frac{1}{2} \rightarrow \frac{1}{2}$	$\frac{5}{3}$ $\frac{1}{3}$ $\frac{8}{3}$ 1 3
			2	$\frac{5}{2} \rightarrow \frac{5}{2}$ $\frac{5}{2} \rightarrow \frac{3}{2}$ $\frac{3}{2} \rightarrow \frac{5}{2}$ $\frac{3}{2} \rightarrow \frac{3}{2}$	$\frac{7}{5}$ 8 5 $\frac{12}{5}$ 3 5
1	2	(1,0,0)	0	$0 \rightarrow 1$ $1 \rightarrow 0$	6 2

A considerably more complicated situation prevails with regard to evaluation of $|\int \boldsymbol{\sigma}|^2$. Exact expressions for the matrix elements involved in Eq. (11-14) can be given only in terms of states that belong to a definite supermultiplet. Table 11-1 contains the corresponding values of $|\int \boldsymbol{\sigma}|^2$ as given by Wigner* for several cases of interest. Our knowledge of this quantity for a given transition then depends on information concerning the amount of admixture of supermultiplets and that information is, in general, lacking. There are a few very special cases which involve little admixture of supermultiplets, such as the neutron decay or the decay of H^3, and they offer the best chance to obtain a value of g'. For other nuclei the experimental determination of $|\int \boldsymbol{\sigma}|^2$ would be a source of information on the amount of mixing, that is, on the magnitude of the deviations from Wigner coupling. These deviations have the effect of increasing, and spreading out, the *ft*-values of the favored transitions and decreasing (from infinity) those of the unfavored transitions. The existence of the two groups of allowed transitions and the spread introduced by variations in the amount of mixing is adequately illustrated in Fig. 11-1. The group around $\log_{10} ft = 3.5$ is presumed to be favored, and that between $\log_{10} ft = 4.5$ and $\log_{10} ft = 6$ is presumed to be allowed but unfavored.

* *Loc. cit.*

We now turn to the discussion of a few special cases for which the deviations from Wigner coupling may be expected to be small, so that $|\int \boldsymbol{\sigma}|^2$ can be evaluated, or those for which we must make use of only the charge invariance property because a selection rule eliminates the Gamow-Teller term.

$n \rightarrow H$

The instability of the neutron against beta-decay is indicated by the n-H mass difference of 782 kev (Section 1–3). This is the best possible example of a system for which Wigner coupling is valid, since a single nucleon certainly is described by the $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ supermultiplet. J. M. Robson* has obtained the value 12.8 ± 2.5 minutes for the half-life. The corresponding *ft*-value is

$$(1300 \pm 250)\ \text{sec} = (1.01 \pm 0.20) \times 10^{24}\ \hbar/mc^2.$$

Both $|\int \mathbf{1}|^2$ and $|\int \boldsymbol{\sigma}|^2$ can easily be calculated directly in this case, or they may be taken from the first row of Table 11–1. Equation (11–17) then leads to the following condition on the constants g' and K:

$$[K^2 + 3]^{-1} = (2.35 \pm 0.46) \times 10^{22} g'^2. \tag{11–24}$$

$H^3 \rightarrow He^3$

In the approximation that only the wave function ψ_1, Eq. (8–17), occurs in the ground state, this transition is very similar to the n-p transition. In fact ψ_1 is a characteristic state of the $(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ supermultiplet with $L = 0$ and therefore, according to Table 11–1, the same values of $|\int \mathbf{1}|^2$ and $|\int \boldsymbol{\sigma}|^2$ apply here as for the neutron decay. The *ft*-value is†

$$(1014 \pm 20)\ \text{sec} = (7.94 \pm 0.16) \times 10^{23}\ \hbar/mc^2,$$

so

$$[K^2 + 3]^{-1} = (1.85 \pm .04) \times 10^{22}\ g'^2. \tag{11–25}$$

This result lies within the experimental error of the comparable condition Eq. (11–24). However, it must be kept in mind that the ground state of the three-body system probably contains functions other than ψ_1. The rather good agreement obtained here indicates that the deviations from ψ_1 are small, as we assumed in Section 8–3.

$He^6 \rightarrow Li^6$

Every known spin of an even-even nucleus in the ground state is zero, therefore the ground state of He^6 is assumed to have $J = 0$. The observed

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* *Phys. Rev.* **83**, 349 (1951).

† Langer and Moffat, *loc. cit.*

spin of Li^6 is $J = 1$, hence only the Gamow-Teller selection rules admit this transition as allowed. The observed* *ft*-value is

$$(815 \pm 70) \text{ sec} = (6.36 \pm 0.55) \times 10^{23} \hbar/mc^2.$$

Since this is of the order of the previously quoted values, the transition is presumed to be allowed and favored, and this is a strong indication that the Gamow-Teller term makes an important contribution to the interaction. Furthermore, it is an indication that Wigner coupling is a fair approximation, since deviations from Wigner coupling tend to reduce the value of $|\int \boldsymbol{\sigma}|^2$. According to Table 8–4, the predominant supermultiplet would be (1,0,0) and L would be expected to have the value $L = 0$. Then, from Table 11–1, $|\int \boldsymbol{\sigma}|^2 = 6$, so Eq. (11–17) reads

$$g'^2 = (1.13 \pm 0.09) \times 10^{-23}. \tag{11–26}$$

This result, combined with (11–25), yields a value of K^2:

$$K^2 = 1.79 \pm 0.49. \tag{11–27}$$

It is to be remembered that only experimental errors are indicated in Eqs. (11–26) and (11–27). The theoretical uncertainty in the wave functions may lead to an additional error. There is the already mentioned uncertainty in the three-body function and an even greater uncertainty in the six-body function. That the latter function is different from the pure Wigner form is clearly indicated by Table 9–1, which shows the conditions imposed on the wave function by the observed magnetic moment.

$C^{10} \rightarrow B^{10*}$; $O^{14} \rightarrow N^{14*}$

Again the initial states belong to even-even nuclei, hence they are presumed to have $J = 0$. The final states, which are observed to emit gamma radiation, are evidently the first excited states of the nuclei B^{10} and N^{14}. The *ft*-values are†

$$C^{10}: \quad (2270 \pm 200) \text{ sec} = (1.77 \pm 0.16) \times 10^{24} \hbar/mc^2,$$

$$O^{14}: \quad (3320 \pm 270) \text{ sec} = (2.59 \pm 0.21) \times 10^{24} \hbar/mc^2,$$

and both transitions are apparently allowed and favored. It seems reasonable to interpret these as transitions between the members of the *i*-multiplet $T = 1$, since the first excited state of either B^{10} or N^{14} would, according to Fig. 8–2, be expected to meet this description. If this is the correct description of the transition, each final state must have the same J-value as the initial state, namely, $J = 0$. Then the transition is Gamow-Teller for-

* Wu, Rustad, Perez-Mendez, and Lidofsky, *Phys. Rev.* **87,** 1140 (1952).

† Sherr, Muether and White, *Phys. Rev.* **75,** 282 (1949).

bidden, so it provides direct information on the strength of the Fermi interaction. The matrix element is

$$\left| \int 1 \right|^2 = 2,$$

whence the product g'^2K^2 is found to be

$$g'^2K^2 = (1.22 \pm 0.10) \times 10^{-23} \tag{11-28}$$

for C^{10} and

$$g'^2K^2 = (0.82 \pm 0.07) \times 10^{-23} \tag{11-29}$$

for O^{14}. The discrepancy between these values is not far enough outside the limits of error to be considered as significant. The agreement within the implied errors with Eqs. (11–26) and (11–27) would seem to lend support to the assumptions made throughout this discussion.*

It is quite clear that somewhat more precise experimental *ft*-values for the neutron and for the C^{10} and O^{14} nuclei would make possible a quite accurate determination of the constants g' and K. Then the experimental results on allowed transitions would provide direct information on the structure of the wave functions. For example, a deeper insight into the structure of the three-body wave functions would ensue from analysis of the *ft*-value associated with the decay of H^3.

In the absence of a precise determination of the coupling constants, most attempts at extracting information concerning nuclear wave functions from observed *ft*-values have been essentially qualitative. Some support for the general features of the Mayer-Jensen coupling scheme has been provided by such considerations.** But the quantitative results again suggest that the nuclear wave functions contain important terms in addition to the Mayer-Jensen independent particle functions.†

To avoid giving the impression that the subject of beta-decay is clearly understood except for minor quantitative details, we close this discussion with one more example.

$C^{14} \rightarrow N^{14}$

The spin of each of these nuclei has been measured directly. For‡ C^{14}, $J = 0$, while N^{14} has $J = 1$. Gamow-Teller selection rules therefore apply, so only the term $\left| \int \sigma \right|^2$ can contribute to the *ft*-value. According

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* See Konopinski and Langer, *Annual Review of Nuclear Science* **2,** 261 (1953), for a summary of similar information obtained from forbidden transitions.

** Feenberg and Hammack, *Phys. Rev.* **75,** 1877 (1949); Moszkowski and Mayer, *Rev. Mod. Phys.* **23,** 315 (1951); L. W. Nordheim, *ibid.*, 322.

† G. L. Trigg, *Phys. Rev.* **86,** 506 (1952).

‡ F. A. Jenkins, *Phys. Rev.* **74,** 355 (1948); A. Roberts, *Phys. Rev.* **73,** 1405 (1948).

to Fig. 8-2, the active state of C^{14} is a member of the $T = 1$ multiplet, which includes the ground state of O^{14} and the first excited state of N^{14}. Therefore it should lie below the first excited state of N^{14}, and this conclusion is in accord with the absence of any detectable gamma radiation following the beta-decay. The separation of the $T = 1$ levels from the $T = 0$ level indicates that there may be a considerable mixing of other supermultiplet terms with the (1,0,0) state, the one that would lie lowest according to the Wigner coupling scheme. Furthermore, substantial evidence that the state contains a mixture of terms having distinct L-values is provided by the magnetic moment of N^{14}, as can be seen by reference to Table 9–1. It follows that the value of $|\int \sigma|^2$ should be somewhat reduced from what would be expected for a favored transition, but it would still be expected to be of an order corresponding to an allowed and unfavored transitions. However, the observed* ft-value,

$$1.1 \times 10^9 = 8.6 \times 10^{29}\, \hbar/mc^2,$$

is some 10^4 or 10^5 times larger than that of an allowed, unfavored transition. No simple explanation of this fact has been forthcoming. It might result from an accidental selection of the mixing coefficients in the initial and final states. For example, if the $T = 0$ state belongs to a different supermultiplet or has a different L-value than the $T = 1$ state, the transition would be forbidden. In fact, it would be second forbidden should the two states have the same parity. However, an explanation of this kind requires that the mixing of supermultiplets or L-values in each of the states occur with a very small probability, because a mixture of the ground state supermultiplet (or L-value) in the excited state, or a mixture of the excited state supermultiplet in the ground state, would cause the transition to be allowed. To account for the large ft-value, mixtures of this kind must occur with probabilities smaller than some 10^{-2} or 10^{-3} percent, or a cancellation by interference between their contributions must occur to that order. The occurrence of a matching of terms to such a high order would seem to be most unlikely, but such a happening in a field so replete with uncertainties as is nuclear physics certainly cannot be said to be impossible.

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* Taken from the summary of ft-values given by A. M. Feingold, *Rev. Mod. Phys.* **23,** 10 (1951).

APPENDIX 1

Theorem on Ordering of Levels in a Central Potential

If a particle moves under the influence of a static central potential, the solutions of the Schroedinger equation separate into the product of a radial function and a function of given orbital angular momentum l. The theorem to be proved is that of two solutions with the same number of radial nodes, the one with the higher value of l has the higher energy.

If the central potential measured in units of $\hbar^2/2M$ is $V(r)$, then the effective potential acting on the particle is

$$V_l(r) = V(r) + \frac{l(l+1)}{r^2}.$$

The radial wave function, u_{nl}/r, is parametrized by the number of nodes in u_{nl}, excluding $r = 0$ but including that at ∞, and by the value of l. Unless there is an accidental degeneracy, a different energy value, E_{nl}, is associated with each such function. The radial functions and associated energies are determined by the equation

$$u''_{nl} + (E_{nl} - V_l)u_{nl} = 0$$

if the energy is also measured in units of $\hbar^2/2M$. Consider the solution for a given n but for two different l-values, l and λ. Then by multiplying the equation for u_{nl} by $u_{n\lambda}$ and that for $u_{n\lambda}$ by u_{nl}, subtracting, and integrating between two arbitrary points r and r', we find:

$$(E_{nl} - E_{n\lambda})\int_r^{r'} u_{nl}u_{n\lambda}\,dr = (u_{nl}u'_{n\lambda} - u_{n\lambda}u'_{nl})\Big|_r^{r'} + \int_r^{r'} (V_l - V_\lambda)u_{nl}u_{n\lambda}\,dr. \quad \text{(A1–1)}$$

For the sake of definiteness, let us assume that $\lambda > l$. Then, from the definition of V_l,

$$V_\lambda > V_l \quad \text{(A1–2)}$$

at every point r. If the first node of u_{nl} beyond $r = 0$ is denoted by r_{l0} and the last node before $r = \infty$ by $r_{l\infty}$, there are three possibilities for the positioning of the corresponding nodes in $u_{n\lambda}$:

(1) $r_{\lambda 0} < r_{l0}$,

(2) $r_{\lambda 0} > r_{l0}$ and $r_{\lambda\infty} > r_{l\infty}$,

(3) $r_{\lambda 0} > r_{l0}$ and $r_{\lambda\infty} < r_{l\infty}$.

Each of these cases is to be treated in turn.

CASE 1. In Eq. (A1–1) choose $r = 0$, $r' = r_{\lambda 0}$. Since $u(0) = 0$,

$$(E_{nl} - E_{n\lambda}) \int_0^{r_{\lambda 0}} u_{nl} u_{n\lambda}\, dr = u_{nl}(r_{\lambda 0}) u'_{n\lambda}(r_{\lambda 0}) + \int_0^{r_{\lambda 0}} (V_l - V_\lambda) u_{nl} u_{n\lambda}\, dr.$$

Let us take the (arbitrary) phases of the $u_{n\lambda}$ and u_{nl} to be such that both are positive between $r = 0$ and $r = r_{\lambda 0}$. Then $u'_{n\lambda}(r_{\lambda 0})$ is necessarily negative (the functions are single valued). Because of the condition (A1–2), the integrand of the second term on the right is negative over the entire range of integration. Therefore

$$(E_{nl} - E_{n\lambda}) \int_0^{r_{\lambda 0}} u_{nl} u_{n\lambda}\, dr < 0,$$

and, from the fact that the integrand is positive,

$$E_{n\lambda} > E_{nl}.$$

CASE 2. Here we choose $r = r_{\lambda\infty}$, $r' = \infty$ in Eq. (A1–1). Then

$$(E_{nl} - E_{n\lambda}) \int_{r_{\lambda\infty}}^{\infty} u_{nl} u_{n\lambda}\, dr = -u_{nl}(r_{\lambda\infty}) u'_{n\lambda}(r_{\lambda\infty}) + \int_{r_{\lambda\infty}}^{\infty} (V_l - V_\lambda) u_{nl} u_{n\lambda}\, dr.$$

The functions u_{nl} and $u_{n\lambda}$ are taken to be positive in the interval $r = r_{\lambda\infty}$ to $r = \infty$, whence it follows that $u'_{n\lambda}(r_{\lambda\infty}) > 0$. Repetition of the argument of Case 1 then leads to the result

$$E_{n\lambda} > E_{nl}.$$

CASE 3. Since $u_{n\lambda}$ has the same number of nodes as u_{nl}, there must be two nodes of $u_{n\lambda}$ between some adjacent pair of nodes of u_{nl}. Denote the positions of these nodes of $u_{n\lambda}$ by r_λ and r'_λ, $r'_\lambda > r_\lambda$. If we set $r = r_\lambda$, $r' = r'_\lambda$ in Eq. (A1–1),

$$(E_{nl} - E_{n\lambda}) \int_{r_\lambda}^{r'_\lambda} u_{nl} u_{n\lambda}\, dr$$
$$= u_{nl}(r'_\lambda) u'_{n\lambda}(r'_\lambda) - u_{nl}(r_\lambda) u'_{n\lambda}(r_\lambda) + \int_{r_\lambda}^{r_\lambda} (V_l - V_\lambda) u_{nl} u_{n\lambda}\, dr.$$

The phases are now chosen so as to have u_{nl} and $u_{n\lambda}$ positive in the interval $r = r_\lambda$ to $r = r'_\lambda$. Then $u'_{n\lambda}(r_\lambda) > 0$ and $u'_{n\lambda}(r'_\lambda) < 0$, and it is again found that

$$E_{n\lambda} > E_{nl}.$$

Thus, of two solutions with the same number n of radial nodes, that with the higher value of l has the higher energy.

APPENDIX 2

Feshbach-Schwinger Solution of the Deuteron Ground State Problem for a Yukawa Potential Including Tensor Interaction

Numerical values of the ground state properties of the deuteron and the neutron-proton triplet effective range as calculated by Feshbach and Schwinger* for a central and tensor interaction of the Yukawa shape are given below. The central potential is

$$V_c(r) = -\frac{\hbar^2 K^2}{M}\frac{e^{-r/\alpha_c}}{(r/\alpha_c)},$$

and the factor of S_{np} in the tensor potential is

$$V_t(r) = -\gamma\frac{\hbar^2 K^2}{M}\frac{e^{-r/\alpha_T}}{(r/\alpha_T)}.$$

The calculated quadrupole moment is Q, and $|b|^2$ is the D-state probability. The triplet effective range, calculated by means of Eq. (4–66), is tabulated as r_{0t}.

$k_g\alpha_T$	γ	$K^2\alpha_c^2$	Q/e (10^{-27} cm^2)	$\lvert b\rvert^2$	r_{0t} (10^{-13} cm)
		$k_g\alpha_c = 0.313$			
0.640	0.1140	2.1520	2.52	0.016	1.92
	0.1300	2.1016	2.78	0.019	1.96
	0.1350	2.0855	2.86	0.020	1.96
0.492	0.2400	2.0129	2.53	0.021	1.88
	0.2700	1.9496	2.74	0.024	1.93
	0.3000	1.8886	2.89	0.026	1.93
0.355_5	0.8550	1.5429	2.62	0.032	1.79
	0.9300	1.4784	2.72	0.035	1.79
	1.0000	1.4220	2.79	0.036	1.79
0.320	1.5600	1.2227	2.67	0.042	1.71
	1.7000	1.1575	2.74	0.042	1.71
	1.8400	1.0983	2.78	0.042	1.71
0.275	6.340	0.5214	2.65	0.054	1.52
	9.200	0.3757	2.72	0.054	1.52
	13.000	0.2737	2.77	0.054	1.52

* *Phys. Rev.* **84,** 194 (1951).

$k_g\alpha_T$	γ	$K^2\alpha_c^2$	Q/e (10^{-27} cm^2)	$\lvert b \rvert^2$	r_{0t} (10^{-13} cm)
		$k_g\alpha_c = 0.275$			
0.640	0.1000	2.0498	2.64	0.018	1.85
	0.1040	2.0347	2.71	0.019_5	1.85
	0.1080	2.0195	2.78	0.020_5	1.85
0.492	0.2239	1.8676	2.72	0.025	1.79
	0.2400	1.8277	2.83	0.027	1.82
	0.3000	1.6860	3.18	0.034	1.85
0.355_5	0.7107	1.4631	2.63	0.034	1.71
	0.7660	1.4070	2.71	0.036	1.71
	0.7820	1.3915	2.72	0.037	1.71
0.320	1.3392	1.1285	2.69	0.042	1.66
	1.4280	1.0818	2.73	0.041	1.66
	1.5600	1.0190	2.79	0.046	1.66
0.275	4.7300	0.5331	2.63	0.051	1.52
	5.8000	0.4470	2.67	0.053	1.52
	9.2000	0.2952	2.75	0.056	1.49
		$k_g\alpha_c = 0.256$			
0.640	0.0920	2.0031	2.67	0.020	1.79
	0.0978	1.9789	2.80	0.021	1.79
	0.1140	1.9100	3.08	0.025	1.84
0.492	0.1900	1.8616	2.64	0.024_5	1.74
	0.2100	1.8064	2.79	0.027	1.77
	0.2400	1.7260	3.00	0.030	1.79
0.355_5	0.7300	1.3297	2.76	0.039	1.68
	0 8100	1.2524	2.85	0.040	1.68
	0.8550	1.2125	2.89	0.040	1.71
0.320	1.2000	1.1009	2.69	0.043	1.63
	1.3392	1.0231	2.76	0.045	1.63
	1.5600	0.9192	2.85	0.046	1.63
0.275	6.2400	0.3692	2 70	0.055	1.49
	10.5000	0.2288	2.77	0.054	1.49
	13.0000	0.1871	2.79	0 052	1.49
		$k_g\alpha_c = 0.224$			
0.640	0.0740	1.9429	2.62	0.020	1.68
	0.0770	1.9279	2.71	0.021	1.68
	0.0850	1.8871	2.97	0.023_5	1.74
0.492	0.1550	1.7961	2.61	0.025	1.66
	0.1700	1.7461	2.77	0.028	1.68
	0.1900	1.6808	2.94	0.030	1.73

$k_g\alpha_T$	γ	$K^2\alpha_c^2$	Q/e (10^{-27} cm^2)	$\mid b \mid^2$	r_{0t} (10^{-13} cm)
		$k_g\alpha_c = 0.224$			
0.355_5	0.5800	1.2899	2.72	0.039	1.60
	0.6240	1.2370	2.81	0.040	1.63
	0.7107	1.1434	2.91	0.044	1.63
0.320	0.9500	1.0687	2.60	0.044	1.57
	1.1000	0.9681	2.75	0.046	1.57
	1.2000	0.9105	2.81	0.045	1.60
0.275	4.7300	0.3713	2.69	0.055	1.46
	5.8000	0.3087	2.72	0.054_5	1.49
	10.5000	0.1773	2.78	0.059	1.49

Interpolated values of γ, $K^2\alpha_c^2$, $\mid b \mid^2$, r_{0t} for $Q = 2.766 \times 10^{-27} e$ cm^2:

$k_g\alpha_c$	$k_g\alpha_T$	γ	$K^2\alpha_c^2$	$\mid b \mid^2$	r_{0t} (10^{-13} cm)
0.313	0.640	0.1291	2.1044	0.019	1.96
	0.492	0.2747	1.9400	0.024	1.93
	0.355_5	0.9744	1.4424	0.036	1.79
	0.320	1.7860	1.1209	0.042	1.71
	0.275	12.6421	0.2819	0.054	1.52
0.275	0.640	0.1072	2.0226	0.020	1.85
	0.492	0.2305	1.8512	0.026	1.79
	0.355_5	0.8359	1.3406	0.038	1.71
	0.320	1.5074	1.0431	0.043	1.66
	0.275	10.0816	0.2681	0.056	1.49
0.256	0.640	0.0962	1.9857	0.021	1.79
	0.492	0.2067	1.8153	0.027	1.77
	0.355_5	0.7344	1.3251	0.039	1.68
	0.320	1.3525	1.0164	0.045	1.63
	0.275	10.0684	0.2371	0.054	1.49
0.224	0.640	0.0788	1.9188	0.021	1.68
	0.492	0.1696	1.7475	0.028	1.68
	0.355_5	0.5985	1.2666	0.039	1.63
	0.320	1.1244	0.9537	0.046	1.57
	0.275	9.0980	0.2087	0.057	1.49

APPENDIX 3

The Time-Reversal Properties of the Schroedinger Equation

In 1932, Wigner presented* a very general discussion of the time-reversal properties of the Schroedinger equation. Since this material is not easily accessible, we present here some of the related arguments which are of importance in the treatment of problems in nuclear physics.

The fundamental assumption is that the positive sense of time cannot be distinguished from the negative sense of time by a mechanical system.† This means that the Hamiltonian, H, of the system is invariant under the transformation

$$t \rightarrow -t. \tag{A3-1}$$

The important problem is to determine the transformation

$$\psi(t) \rightarrow \psi'(t) = K\psi(t)$$

of the wave function associated with the time-reversal operation (A3-1). The invariance of H under time-reversal can then be expressed by the equation

$$KHK^{-1} = H. \tag{A3-2}$$

If we are given any solution $\psi(t)$ of the time-dependent Schroedinger equation, the function $\psi(-t)$ has many of the properties desired for ψ'. For example, if $\psi(t)$ describes a small wave packet moving with constant velocity, $\psi(-t)$ describes a wave packet moving with the opposite velocity. The time behavior of $\psi(-t)$ can be determined from the equation

$$-\frac{\hbar}{i}\frac{\partial\psi(t)}{\partial t} = H\psi(t) \tag{A3-3}$$

by simply replacing t by $-t$:

$$\frac{\hbar}{i}\frac{\partial\psi(-t)}{\partial t} = H\psi(-t). \tag{A3-4}$$

The difference in sign between Eqs. (A3-4) and (A3-3) may be associated with the coefficient i, which suggests that $\psi(-t)$ is related to $\psi^*(t)$, the

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* *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*, No. 32, p. 35 (1932).

† That is, for an isolated system. If one introduces external magnetic fields, for example, the sense of time is fixed by the currents producing the fields.

conjugate complex of $\psi(t)$. It therefore becomes convenient to introduce the operator K_0, which produces the conjugate complex of a function or operator. Then a quite general set of transformations having the desired characteristics is described by

$$K = UK_0, \tag{A3 5}$$

where U is a time-independent unitary transformation. Application of this K to both sides of Eq. (A3–3) leads to the result

$$\frac{\hbar}{i}\frac{\partial}{\partial t}[K\psi(t)] = KHK^{-1}[K\psi(t)].$$

But this equation has exactly the form of Eq. (A3–4) if U is chosen so as to satisfy the condition (A3–2), which expresses the invariance of the Hamiltonian. Under this condition the function

$$\psi'(t) = K\psi(t) \tag{A3–6}$$

is identical with $\psi(-t)$ if both satisfy the same initial condition.

The form of the unitary transformation U depends explicitly on the nature of the observables describing the system. The observables fall into two categories: those which are independent of the sense of time and those whose sign depends on the sense of time. The coordinate is an example of the former, which Wigner calls a "real" hermitian operator, and the momentum is an example of the latter, the "imaginary" hermitian operator. The requirement of time-reversibility is that the expectation value of a real hermitian operator must be invariant under the transformation (A3–6), while the expectation value of an imaginary operator must change sign. This guarantees that Eq. (A3–6) describes the reversal of time in the sense of classical mechanics.

Expressed analytically, the conditions take the form

$$\langle\psi',R\psi'\rangle = \langle\psi,R\psi\rangle,$$

if R is a real operator, and

$$\langle\psi',I\psi'\rangle = -\langle\psi,I\psi\rangle$$

if I is imaginary. Since ψ is an arbitrary function, these conditions are equivalent to

$$KRK^{-1} = R, \tag{A3–7}$$

$$KIK^{-1} = -I. \tag{A3–8}$$

Note that $K^{-1} = K_0^{-1}U^{-1} = K_0U^{-1}$. The conditions (A3–7) and (A3–8) are sufficient to determine U when they are applied to all independent observable operators of a system. It is to be remarked that Eq. (A3–2) is of the form Eq. (A3–7), as it should be.

If, for example, the system consists of a single spinless particle, the independent observables are the coordinate $\mathbf{r}$ and the momentum $\mathbf{p}$. The conditions on U are

$$K^{-1}\mathbf{r}K = \mathbf{r},$$

and

$$K^{-1}\mathbf{p}K = -\mathbf{p}.$$

If $\mathbf{r}$ is taken to be diagonal,

$$K_0\mathbf{r}K_0 = \mathbf{r},$$

since the characteristic values are real. On the other hand,

$$\mathbf{p} \equiv \frac{\hbar}{i}\,\mathrm{grad},$$

therefore

$$K_0\mathbf{p}K_0 = -\mathbf{p}.$$

Thus we can take $U = 1$ in this case. Note that this implies $H^* = H$, which will be the case when $\mathbf{r}$ and $\mathbf{p}$ are the only observables.

If the particle has spin one-half, the spin vector forms another independent observable; and it is an imaginary hermitian operator since it represents an angular momentum. If the usual representation of the spin operators is used,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we see that

$$K_0\sigma_xK_0 = \sigma_x, \quad K_0\sigma_yK_0 = -\sigma_y, \quad K_0\sigma_zK_0 = \sigma_z.$$

Thus the condition (A3-8) becomes in each case.

$$U\sigma_xU^{-1} = -\sigma_x,$$

$$U\sigma_yU^{-1} = \sigma_y,$$

$$U\sigma_zU^{-1} = -\sigma_z,$$

or, the equivalent,

$$U\sigma_x + \sigma_xU = 0,$$

$$U\sigma_y - \sigma_yU = 0,$$

$$U\sigma_z + \sigma_zU = 0.$$

Hence U commutes with σ_y and it anticommutes with σ_x and σ_z. This is just the property of σ_y itself, so

$$U = \sigma_y.$$

Thus U commutes with $\mathbf{r}$ and $\mathbf{p}$, and the time-reversal properties of those observables are preserved.

It is now easy to generalize this argument to obtain U for a system consisting of N particles of spin one-half. We find

$$U = \sigma_y(1)\sigma_y(2) \cdots \sigma_y(N),$$

where $\sigma_y(k)$ is the spin operator of the kth particle, which must be given in the conventional representation used above.

For a complex system having central symmetry a very general statement can be made about the effect of K on the wave functions. These functions may be taken to be functions of given total angular momentum, J,M. Then, if the representation is chosen properly,

$$K\psi_J^M = i^{2M}\psi_J^{-M}. \tag{A3-9}$$

To establish this result, the functions ψ_J^M are constructed by taking linear combinations of products of one-particle functions ψ_j^m. For a particle having no spin, the ψ_j^m are just the spherical harmonics $Y_l^{m_l}$. These are defined in accordance with the property*

$$K_0 Y_l^{m_l} = (-1)^{m_l} Y_l^{-m_l}, \tag{A3-10}$$

which is equivalent in form to Eq. (A3–9). For a spin one-half particle, the space functions are chosen in accordance with (A3–10), while the transformation of the spin function $\chi^{m_s}(m_s = \pm\frac{1}{2})$ is

$$\chi' = \sigma_y K_0 \chi^{m_s} = i^{2m_s}\chi^{-m_s},$$

if the usual representation in which the χ^{m_s} are the real numbers 1 or 0 is used. The ψ_j^m of a single particle of spin one-half is a linear combination of the products $Y_l^{m_l}\chi^{m_s}$, with $m = m_l + m_s$, $j = l \pm \frac{1}{2}$. The coefficients in this linear combination are the usual Clebsch-Gordan coefficients $(j,m,l,\frac{1}{2} \mid m_l,m_s,l,\frac{1}{2})$ which are given in standard references.† In the usual representation, the coefficients are real and have the general property

$$(J,-M,j_1,j_2 \mid -m_1,-m_2,j_1,j_2) = (-1)^{j_1+j_2-J}(J,M,j_1,j_2 \mid m_1,m_2,j_1,j_2).$$

Thus if we change from the conventional phase to write

$$\psi_j^m(k) = i^{l+\frac{1}{2}-j} \sum_{m_l+m_s=m} (j,m,l,\tfrac{1}{2} \mid m_l,m_s,l,\tfrac{1}{2}) Y_l^{m_l}(k)\chi^{m_s}(k),$$

* This choice can be made freely and it is that used, e.g., by Condon and Shortley, *The Theory of Atomic Spectra*, Cambridge Univ. Press (1951), p. 17 ff.

† In addition to Condon and Shortley, *loc. cit.*, see Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektern*, Friedr. Vieweg und Sohn, Braunschweig (1931), p. 198 ff., or V. D. Waerden, *Gruppentheoretische Methode in der Quantenmechanik*, Julius Springer, Berlin (1932), p. 68 ff.

we find

$$K\psi_j^m(k) = (-i)^{l+\frac{1}{2}-j} \sum_{m_l+m_s=m} (j,m,l,\tfrac{1}{2} \mid m_l,m_s,l,\tfrac{1}{2}) i^{2(m_l+m_s)} Y_l^{-m_l}\chi^{-m_s}$$

$$= i^{2m} i^{l+\frac{1}{2}-j} \sum_{-(m_l+m_s)=-m} (j,-m,l,\tfrac{1}{2} \mid -m_l,-m_s,l,\tfrac{1}{2}) Y_l^{-m_l}\chi^{-m_s}$$

$$= i^{2m}\psi_j^{-m}(k).$$

The same choice of phase can be used in combining j_1 and j_2, in combining that function with j_3, and so on. Then Eq. (A3–9) is obtained for the resultant function ψ_J^M.

The result (A3–9) is very useful for establishing reality conditions on the wave functions of stationary states. From the equation

$$H\psi = E\psi$$

and the fact that $KHK^{-1} = H$, we find that

$$H(K\psi) = E(K\psi),$$

so ψ and $K\psi$ are both solutions for the same energy. There is a degeneracy unless

$$K\psi = \psi.$$

Now if a nondegenerate ψ_J^M is a linear combination of functions of given (J,M) normalized as described above, the coefficients may all be chosen to be real. Let the different functions in the linear combination be denoted by $\psi_{J,\alpha}^M$, $\alpha = 1,2,3, \cdots$. Then

$$\psi_J^M = \sum_\alpha a_\alpha \psi_{J,\alpha}^M, \tag{A3–11}$$

and

$$K\psi_J^M = \sum_\alpha a_\alpha^* i^{2M} \psi_{J,\alpha}^{-M}.$$

But

$$K\psi_J^M = i^{2M}\psi_J^{-M},$$

so

$$\psi_J^{-M} = \sum_\alpha a_\alpha^* \psi_{J,\alpha}^{-M}. \tag{A3–12}$$

Thus either $a_\alpha^* = e^{i\eta}a_\alpha$, where the phase η is independent of α, or there are two distinct functions ψ_J^{-M}, one given by Eq. (A3–11) (replace M by $-M$) and the other by Eq. (A3–12).

Use was made of this result when it was asserted that the coefficients appearing in the nondegenerate, ground state triton function, Eq. (8–28), are real. That the phases of the functions ψ_1 to ψ_8 have been properly chosen can be demonstrated by applying the time-reversal operator directly to them. Since each spin operator changes sign, as does each factor i, we see that every one of the functions has the same behavior under time-reversal. Hence the coefficients must not change their relative values. They may only contain a common complex phase factor.

When nuclear problems are treated in the i-spin notation, the operators τ_k play the role of additional independent observables, so we must consider the conditions that they impose on U. The basic question is whether or not the operators τ_1, τ_2, and τ_3 of a single nucleon depend on the sense of time flow. Clearly τ_3 does not; the identity of a nucleon as a neutron or proton is independent of the choice of the direction of the positive time axis. This information is sufficient for our purpose. It means that

$$U\tau_3 U^{-1} = \tau_3,$$

since τ_3 has real matrix elements. Hence τ_3 is invariant under the transformation U, therefore U must correspond to a rotation about the 3-axis in charge space.

This result may be used to show how the condition of invariance of the Hamiltonian under time-reversal limits the form of terms such as the spin-orbit interactions, Eq. (8–46). We confine our attention to the two-body interactions proportional to one of the operators, Eq. (8–3). Under time-reversal, the transformations of four of these operators are

$$K1K^{-1} = 1,$$

$$K[\tau_3(i) \pm \tau_3(j)]K^{-1} = \tau_3(i) \pm \tau_3(j),$$

$$K\tau_3(i)\tau_3(j)K^{-1} = \tau_3(i)\tau_3(j),$$

since τ_3 has only real matrix elements and is unaffected by U. The operator $(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)$ is left unchanged by K_0 because i occurs linearly in both $\tau_2(i)$ and $\tau_2(j)$ and nowhere else. Thus

$$K(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)K^{-1} = (\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j),$$

since the scalar product is invariant under rotations. Finally, we consider the hermitian operator $[\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j]_3$. It contains $\tau_2(i)$ linearly in one term and $\tau_2(j)$ linearly in the other, so it changes sign under K_0. On the other hand, the fact that it is a 3-component of a vector means that it is invariant under the rotation U about the 3-axis. Hence

$$K[\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j]_3 K^{-1} = -[\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j]_3.$$

To test the time-reversal properties of the spin-orbit interactions, Eq. (8–46), we note that $(\boldsymbol{\sigma}_i \pm \boldsymbol{\sigma}_j) \cdot [\mathbf{r}_{ij} \times \mathbf{p}_{ij}]$ is invariant under time-reversal, since both $\boldsymbol{\sigma}$ and $\mathbf{p}$ change sign. Therefore $[\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j]_3$ cannot occur in combination with these factors. On the other hand, $[\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_j] \cdot [\mathbf{r}_{ij} \times \mathbf{p}_{ij}]$ changes sign on time-reversal, so it can occur only in combination with $[\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j]_3$. These rules have been used in constructing the list of spin-orbit couplings, Eqs. (8–47).

APPENDIX 4

Quadrupole Moments of Odd Z-Even N Nuclei for Mayer-Jensen Coupling

The expectation value of $Q_{33} = e\Sigma_k(3z_k^2 - r_k^2)$ is to be obtained for a state of ζ protons described by a properly antisymmetrized product of the functions ψ^{μ}_{nlj} given by Eqs. (8–42). Each of the functions has the same n, l, j value, but μ takes on the ζ values $\mu_1, \mu_2, \cdots, \mu_\zeta$. It is easily shown that the expectation value of any one-particle operator

$$O = \sum_k O(\mathbf{r}_k)$$

in the antisymmetric state

$$\Psi(\mu_1,\mu_2, \cdots, \mu_\zeta) = \frac{1}{\sqrt{\zeta!}} \sum_P (-1)^P P\psi^{\mu_1}(1)\psi^{\mu_2}(2) \cdots \psi^{\mu_\zeta}(\zeta)$$

[P permutes the particle variables which are designated by (1), (2), $\cdots$, (ζ)] is simply the sum of one-particle expectation values in the states ψ^{μ_i} over all i:

$$\langle \Psi(\mu_1,\mu_2,\cdots,\mu_\zeta),\, O\Psi(\mu_1,\mu_2,\cdots,\mu_\zeta)\rangle = \sum_{i=1}^{\zeta} \langle \psi^{\mu_i}(1),\, O(\mathbf{r}_1)\psi^{\mu_i}(1)\rangle. \quad \text{(A4–1)}$$

Thus we are interested in the expectation value

$$\langle \psi^{\mu}_{nlj}(\mathbf{r}),\, (3z^2 - r^2)\psi^{\mu}_{nlj}(r)\rangle.$$

When the $\psi^{\mu}_{nlj}(r)$ are expressed in terms of the spherical harmonics Y_l^m by means of Eqs. (8–42), we find, by use of the relationship*

$$\int Y_l^{m*} \cos^2\theta Y_l^m \, d\Omega = \frac{1}{2l+3}\left[\frac{2(l-m)(l+m)}{2l-1} + 1\right],$$

the formula

$$Q^{\mu} = \langle \psi^{\mu}_{nlj}(\mathbf{r}),(3z^2 - r^2)\psi^{\mu}_{nlj}(\mathbf{r})\rangle = \frac{j(j+1) - 3\mu^2}{2j(j+1)}\, e\langle r^2\rangle, \quad \text{(A4–2)}$$

where $\langle r^2\rangle$ is the mean square displacement in the state $\psi^{\mu}_{nlj}(\mathbf{r})$. Note that

$$Q^j = -\frac{2j-1}{2(j+1)}\, e\langle r^2\rangle. \quad \text{(A4–3)}$$

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* This result can be obtained by direct application of the expression for $\cos\theta Y_l^m$ which is given, for example, as Eq. (21) of Condon and Shortley, *Theory of Atomic Spectra*, Cambridge University Press (1951), p. 53.

Also the sum $\sum_{\mu=\frac{1}{2}}^{j} Q^{\mu}$ is of interest. For this we need $\sum_{\mu=\frac{1}{2}}^{j} \mu^2$, which can be obtained by means of the equation

$$\sum_{\mu=\frac{1}{2}}^{j} \mu^2 = \left[\frac{\partial^2}{\partial\alpha^2} \sum_{\mu=\frac{1}{2}}^{j} e^{\alpha\mu}\right]_{\alpha=0}$$

$$= \left[\frac{\partial^2}{\partial\alpha^2} \frac{e^{\alpha/2} - e^{(j+1)\alpha}}{1 - e^{\alpha}}\right]_{\alpha=0}$$

$$= \tfrac{1}{6} j(j+1)(2j+1).$$

Therefore

$$\sum_{\mu=\frac{1}{2}}^{j} Q^{\mu} = 0, \tag{A4-4}$$

as would be expected, since Q^{μ} is an even function of μ.

The evaluation of Q for a given number, ζ, of protons in the shell n, l, j now reduces to the problem of determining the appropriate linear combination of functions $\Psi(\mu_1,\mu_2, \cdots, \mu_\zeta)$. If $\zeta = 1$, we are simply concerned with the expectation value in the state ψ^j_{nlj}, that is, Q^j. Hence $Q(1) = Q^j$. For $\zeta = 3$, the states of two of the protons are to be combined to give total angular momentum zero. The third has $\mu = j$, since the total angular momentum is j. Equation (8–43a) indicates how the first two protons are to be paired off. The complete wave function is therefore

$$\Psi = \sqrt{\frac{2}{2j-1}} \sum_{\mu=\frac{1}{2}}^{j} (-1)^{\mu} \Psi(\mu,-\mu,j).$$

The determination of the normalization constant is the important point. This is accomplished by noting that each function in the series is normalized and orthogonal to the other functions and that each occurs with the same weight. However, the terms with $\mu = \pm j$ vanish identically because $\Psi(\mu_1,\mu_2,\mu_3)$ is totally antisymmetric. The value of Q for $\zeta = 3$ is therefore

$$Q(3) = \frac{2}{2j-1} \sum_{\mu=\frac{1}{2}}^{j-1} \langle \Psi(\mu,-\mu,j),\, Q_{33}\Psi(\mu,-\mu,j)\rangle. \tag{A4-5}$$

The cross terms vanish because Q_{33} is a one-particle operator, while any two functions $\Psi(\mu,-\mu,j)$ appearing in Ψ differ by a change in the μ-values of *two* protons. The evaluation of the expectation value in Eq. (A4–5) can be accomplished by means of Eq. (A4–1):

$$\langle \Psi(\mu,-\mu,j),\, Q_{33}\Psi(\mu,-\mu,j)\rangle = 2Q^{\mu} + Q^j,$$

since Q^μ is an even function of μ. Thus

$$Q(3) = Q^j + \frac{4}{2j-1}\sum_{\mu=\frac{1}{2}}^{j-1} Q^\mu.$$

But according to Eq. (A4–4),

$$\sum_{\mu=\frac{1}{2}}^{j-1} Q^\mu = -Q^j, \tag{A4-6}$$

so

$$Q(3) = \left(1 - \frac{4}{2j-1}\right) Q^j.$$

Consider as the last special case $\zeta = 5$. Here, the total angular momentum of four of the protons is to vanish. A wave function which accomplishes this end can be obtained by pairing the first two protons to angular momentum zero according to Eq. (8–43a), doing the same to the second pair of protons, and then combining the two pairs. Hence a properly antisymmetric function is

$$\Psi = \frac{2}{\sqrt{(2j-1)(2j-3)}} \sum_{\mu=\frac{1}{2}}^{j} \sum_{\mu'=\frac{1}{2}}^{j} (-1)^{\mu+\mu'} \Psi(\mu,-\mu,\mu',-\mu',j).$$

In this case the normalization constant is determined by noting that all terms have the same weight, but those with $\mu = j$, $\mu' = j$, and $\mu' = \mu$ vanish identically because $\Psi(\mu,-\mu,\mu',-\mu',j)$ is totally antisymmetric. Now we find

$$Q(5) = \frac{4}{(2j-1)(2j-3)} \sum_{\mu,\mu'}{}' \langle \Psi(\mu,-\mu,\mu',-\mu',j),\ Q_{33}\Psi(\mu,-\mu,\mu',-\mu',j)\rangle,$$

where both μ and μ' run through the values $\frac{1}{2}$ to $j-1$. The prime on the summation symbol implies that the term $\mu' = \mu$ does not occur. According to Eq. (A4–1),

$$\langle \Psi(\mu,-\mu,\mu',-\mu',j),\ Q_{33}\Psi(\mu,-\mu,\mu',-\mu',j)\rangle = 2Q^\mu + 2Q^{\mu'} + Q^j.$$

Therefore

$$Q(5) = Q^j + \frac{8}{(2j-1)(2j-3)}\left[\sum_{\mu,\mu'}(Q^\mu + Q^{\mu'}) - 2\sum_{\mu} Q^\mu\right],$$

and use of the relationship (A4–6) leads to the result

$$Q(5) = \left[1 - \frac{8}{2j-1}\right] Q^j.$$

It now should be clear that in the general case of $\zeta = 2\nu + 1$ protons,

$$Q(\zeta) = Q^j + \frac{2^{\nu+1}}{(2j-1)(2j-3)\cdots(2j-2\nu+1)} \sum_{\mu_1,\mu_2,\cdots\mu_\nu}{}' (Q^{\mu_1} + Q^{\mu_2} + \cdots Q^{\mu_\nu}),$$

where each μ_i runs through the values $\frac{1}{2}$ to $j-1$. The prime on this summation symbol indicates that no two μ_i values are to be set equal. Then, by use of Eq. (A4–4), it is easily shown that

$$Q(\zeta) = \left(1 - 2\frac{\zeta - 1}{2j-1}\right) Q^j.$$

Equation (9–40) is obtained by substituting for Q^j from Eq. (A4–3).

APPENDIX 5

Configurations of Odd-Even Nuclei for Mayer-Jensen Coupling

The following table gives a possible specification of independent nucleon configurations for odd-even nuclei in the Mayer-Jensen coupling scheme. These assignments are taken from Klinkenberg, *Rev. Mod. Phys.* **24,** 63 (1952), who based them on observed nuclear spins and magnetic moments. Many of the assignments are quite arbitrary; they are presented only as an indication of what one can hope to do with more data. The spin and magnetic moment of a nucleus can be used to fix the state of just the last odd nucleon, and then only if the model is taken quite literally. The states of other nucleons have been fixed by simply assuming that the number of nucleons in a shell should be, on the whole, an increasing function of the number of odd nucleons. Further information concerning the configuration of the inner nucleons can be obtained from observed values of the quadrupole moments, as in the cases of Yb^{173} and Hg^{201}, for which changes from Klinkenberg's original assignments have been made to account for the sign of the observed Q.

In some cases, Klinkenberg has chosen to follow the pattern suggested by the model even though the resulting nuclear spin is contrary to the observed spin. Such cases have the ground state assignment set in parentheses. Others set in parentheses are the $3p_{1/2}$ configurations of the odd neutron nuclei. These are contrary to the Mayer-Jensen coupling scheme; it would seem to be more consistent to assign several neutrons to the $3p_{3/2}$ shell (by raising them from lower shells if necessary) and to couple these together to form $J = \frac{1}{2}$. The quadrupole moments can give no insight into this matter, since they vanish whatever the configuration.

Odd-Proton Nuclei

N	Z	Atom	Filled Shells	$1s$	$1p_{3/2}$	$1p_{1/2}$	$1d_{5/2}$	$2s$	$1d_{3/2}$	Ground State
0	1	H	0	1						$s_{1/2}$
2	1	H		1						$s_{1/2}$
4	3	Li		2	1					$p_{3/2}$
6	5	B		2	3					$p_{3/2}$
8	7	N		2	4	1				$p_{1/2}$
10	9	F		2	4	2		1		$s_{1/2}$
12	11	Na	8				3			$(d_{5/2})^3_{3/2}$
14	13	Al					5			$d_{5/2}$
16	15	P					6	1		$s_{1/2}$
18	17	Cl					6	2	1	$d_{3/2}$
20	17	Cl					6	2	1	$d_{3/2}$
20	19	K					6	2	3	$d_{3/2}$
22	19	K					6	2	3	$d_{3/2}$

Odd-Proton Nuclei

N	Z	Atom	Filled Shells	$1f_{7/2}$	$2p_{3/2}$	$1f_{5/2}$	$2p_{1/2}$	$1g_{9/2}$	Ground State
24	21	Sc	20	1					$f_{7/2}$
28	23	V		3					$f_{7/2}$
30	25	Mn		5					$(f_{7/2})^5_{5/2}$
32	27	Co		7					$f_{7/2}$
34	29	Cu		8	1				$p_{3/2}$
36	29	Cu	28		1				$p_{3/2}$
38	31	Ga			3				$p_{3/2}$
40	31	Ga			3				$p_{3/2}$
42	33	As			3	2			$p_{3/2}$
44	35	Br			3	4			$p_{3/2}$
46	35	Br			3	4			$p_{3/2}$
48	37	Rb			4	5			$f_{5/2}$
50	37	Rb			3	6			$p_{3/2}$
50	39	Y			4	6	1		$p_{1/2}$
52	41	Nb			4	6	2	1	$g_{9/2}$
56	43	Tc			4	6	2	3	$g_{9/2}$
58	45	Rh			4	6	1	6	$p_{1/2}$
60	47	Ag			4	6	1	8	$p_{1/2}$
62	47	Ag			4	6	1	8	$p_{1/2}$
64	49	In			4	6	2	9	$g_{9/2}$
66	49	In			4	6	2	9	$g_{9/2}$

Odd-Proton Nuclei

N	Z	Atom	Filled Shells	$1g_{7/2}$	$2d_{5/2}$	$1h_{11/2}$	$2d_{3/2}$	$3s$	Ground State
70	51	Sb	50		1				$d_{5/2}$
72	51	Sb		1					$g_{7/2}$
74	53	I		2	1				$d_{5/2}$
76	53	I		3					$g_{7/2}$
78	55	Cs		5					$g_{7/2}$
80	55	Cs		5					$g_{7/2}$
82	55	Cs		5					$g_{7/2}$
82	57	La		7					$g_{7/2}$
82	59	Pr		8	1				$d_{5/2}$
86	61	Pm		8	3				$d_{5/2}$
88	63	Eu		8	5				$d_{5/2}$
90	63	Eu		8	5				$d_{5/2}$
94	65	Tb		8	6		1		$d_{3/2}$
98	67	Ho		7	6	4			$g_{7/2}$
100	69	Tm		8	6	4		1	$s_{1/2}$
104	71	Lu		7	6	8			$g_{7/2}$
108	73	Ta		7	6	10			$g_{7/2}$
110	75	Re		8	5	12			$d_{5/2}$
112	75	Re		8	5	12			$d_{5/2}$
114	77	Ir		8	6	12	1		$d_{3/2}$
116	77	Ir		8	6	12	1		$d_{3/2}$
118	79	Au		8	6	12	3		$d_{3/2}$
122	81	Tl		8	6	12	4	1	$s_{1/2}$
124	81	Tl		8	6	12	4	1	$s_{1/2}$
				$1h_{9/2}$	$2f_{7/2}$	$2f_{5/2}$ $1i_{13/2}$	$3p_{3/2}$	$3p_{1/2}$	
126	83	Bi	82	1					$h_{9/2}$
138	89	Ac		6			1		$p_{3/2}$
140	91	Pa		8			1		$p_{3/2}$
144	93	Np		10	1				$(f_{7/2})$

Odd-Neutron Nuclei

N	Z	Atom	Filled Shells	$1s$	$1p_{3/2}$	$1p_{1/2}$	$1d_{5/2}$	$2s$	$1d_{3/2}$	Ground State
1	0	n	0	1						$s_{1/2}$
1	2	He		1						$s_{1/2}$
5	4	Be		2	3					$p_{3/2}$
7	6	C		2	4	1				$p_{1/2}$
9	8	O		2	4	2	1			$d_{5/2}$
11	10	Ne	8				3			$d_{5/2}$
13	12	Mg					5			$d_{5/2}$
15	14	Si					6	1		$s_{1/2}$
17	16	S					6	2	1	$d_{3/2}$
19	16	S					6	2	3	$d_{3/2}$
				$1f_{7/2}$	$2p_{3/2}$	$1f_{5/2}$		$2p_{1/2}$	$1g_{9/2}$	
23	20	Ca	20	3						$f_{7/2}$
25	22	Ti		5						$f_{7/2}$
27	22	Ti		7						$f_{7/2}$
29	24	Cr		8	1					$p_{3/2}$
31	26	Fe	28		3					$p_{3/2}$
33	28	Ni			3	2				$p_{3/2}$
37	30	Zn			4	5				$f_{5/2}$
41	32	Ge			4	6		2	1	$g_{9/2}$
43	34	Se			4	6		1	4	$p_{1/2}$
47	36	Kr			4	6		2	7	$g_{9/2}$
49	38	Sr			4	6		2	9	$g_{9/2}$

Odd-Neutron Nuclei

N	Z	Atom	Filled Shells	$1g_{7/2}$	$2d_{5/2}$	$1h_{11/2}$	$2d_{3/2}$	$3s$	Ground State
51	40	Zr	50		1				$d_{5/2}$
53	42	Mo			3				$d_{5/2}$
55	42	Mo			5				$d_{5/2}$
55	44	Ru			5				$d_{5/2}$
57	44	Ru		1	6				$g_{7/2}$
59	46	Pd		3	6				$(g_{7/2})$
63	48	Cd		6	6			1	$s_{1/2}$
65	48	Cd		8	6			1	$s_{1/2}$
65	50	Sn		8	6			1	$s_{1/2}$
67	50	Sn		8	6	2		1	$s_{1/2}$
69	50	Sn		8	6	4		1	$s_{1/2}$
71	52	Te		8	6	6		1	$s_{1/2}$
73	52	Te		8	6	8		1	$s_{1/2}$
75	54	Xe		8	6	10		1	$s_{1/2}$
77	54	Xe		8	6	12	1		$d_{3/2}$
79	56	Ba		8	6	12	3		$d_{3/2}$
81	56	Ba		8	6	12	3	2	$d_{3/2}$

Odd-Neutron Nuclei

N	Z	Atom	Filled Shells	$1h_{9/2}$	$2f_{7/2}$	$2f_{5/2}$	$1i_{13/2}$	$3p_{3/2}$	$3p_{1/2}$	Ground State
83	60	Nd	82		1					$f_{7/2}$
85	60	Nd		2	1					$f_{7/2}$
85	62	Sm		2	1					$(f_{7/2})$
87	62	Sm		4	1					$(f_{7/2})$
91	64	Gd		8	1					$f_{7/2}$
93	64	Gd		10	1					$f_{7/2}$
95	66	Dy		10	3					$f_{7/2}$
97	66	Dy		10	5					$f_{7/2}$
99	68	Er		10	7					$f_{7/2}$
101	70	Yb		10	8				1	$(p_{1/2})$
103	70	Yb		10	6	5				$f_{5/2}$
105	72	Hf		10	8	4			1	$(p_{1/2})$
107	72	Hf		10	8	6			1	$(p_{1/2})$
109	74	W		10	8	6	2		1	$(p_{1/2})$
111	76	Os		10	8	6	4		1	$(p_{1/2})$
113	76	Os		10	8	6	6		1	$(p_{1/2})$
117	78	Pt		10	8	6	10		1	$(p_{1/2})$
119	80	Hg		10	8	6	12		1	$(p_{1/2})$
121	80	Hg		10	8	6	12	3		$p_{3/2}$
125	82	Pb		10	8	6	14	4	1	$(p_{1/2})$
					$2g_{9/2}$		$1i_{11/2}$	$3d_{5/2}$		
143	92	U	126		10		6	1		$d_{5/2}$

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