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MATHEMATICS FOR TECHNICAL STUDENTS

BY

A. GEARY, M.A., M.Sc.

Head of the Mathematics Department, Northampton College of Advanced Technology

H. V. LOWRY, M.A.

Formerly Head of the Mathematics Department, Woolwich Polytechnic

AND

H. A. HAYDEN, D.Sc.

Formerly Head of the Mathematics Department, Battersea College of Technology

PART II

WITH DIAGRAMS

THE ENGLISH LANGUAGE BOOK SOCIETY

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PREFACE

This series is designed to provide the basic mathematical equipment for technical students. The range of work is that covered by national certificate courses in engineering, building and chemistry. Each of the three volumes will include one year's work, but the second and third will contain sufficient revision to allow for variations in the syllabuses of different technical institutions. It is hoped that the series may also prove useful in secondary schools with a technical side.

Short historical notes are included as they may help to develop in the student a further interest in the subject.

A. G.
H. V. L.
H. A. H.

AUTHORS' FOREWORD TO PART II

THIS volume is designed to cover the second year of a three years' national certificate course. It begins with a revision chapter in algebra, which, however, contains some harder examples than in Part I, and the extension of the use of symbols to include units.

Teachers may not wish to include the proofs of theorems given in Chapters VI and VII, but the results of these chapters are required for use later. Sufficient solid geometry has been introduced to enable the student to solve trigonometrical problems in three dimensions and harder problems in mensuration.

Although no previous knowledge of trigonometry is assumed in this volume, an easier introduction is provided by Chapters XVI and XVII of Part I.

The chapters on differentiation and integration are mainly graphical, but they include easy applications of the rules for differentiating and integrating a power of x.

The order of arrangement of the chapters is that which is most convenient for reference, but it is not intended that they should necessarily be read in that order. A suggested order of reading is as follows : Chapters I, IX (pp. 193–213), II, III, IV, VI, VII, VIII, IX (pp. 214–224), X, XI, V, XV, XVI, XII, XIII, XIV.

The authors have aimed at including all the work likely to be done in any technical institution in the second year of a course of this type, though it is not necessarily expected that any single institution would cover the whole of it in one year. If the whole volume cannot be covered the following chapters may be omitted : Chapters VIII, IX (pp. 214–224), XIII (pp. 292–303), XIV (pp. 320–340), XVI (pp. 384–398).

A. G.
H. V. L.
H. A. H.

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ALGEBRA

CHAPTER I

ALGEBRAIC PROCESSES, MAINLY REVISION

Powers of a number

The meanings of integral and fractional powers of a number, and the rules for multiplying and dividing powers of a number are summarized below.

If m is a positive integer (whole number),

a^m means $a \times a \times a \times a$ . . . . . m factors.

$\therefore a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a)$

$= a \times a \times a \times a \times a \times a \times a$ . . . . . 7 factors.

$= a^7 = a^{3+4}$.

In the same way, if m and n are any positive integers,

$$a^m \times a^n = a^{m+n}. \quad \text{(Rule I).}$$

Also $$\frac{a^6}{a^2} = \frac{a \times a \times a \times a \times a \times a}{a \times a}$$

$= a \times a \times a \times a$ . . . . . 4 factors.

$$\therefore \frac{a^6}{a^2} = a^4 = a^{6-2}.$$

In the same way, if m is a positive integer greater than n,

$$\frac{a^m}{a^n} = a^{m-n}. \quad \text{(Rule II).}$$

Again $(a^2)^3 = (a \times a) \times (a \times a) \times (a \times a)$

$= a \times a \times a \times a \times a \times a$ . . 6 factors.

$\therefore (a^2)^3 = a^6 = a^{2 \times 3}$.

In the same way, if m and r are positive integers,

$$(a^m)^r = a^{mr}. \quad \text{(Rule III).}$$

Fractional powers

If we assume that Rule III applies when m is a fraction,

$$(a^{\frac{2}{3}})^3 = a^{\frac{2}{3}\times 3} = a^2$$

$$\therefore\ a^{\frac{2}{3}} = \sqrt[3]{a^2}.$$

Also $$a^{\frac{2}{3}} = a^{2\times\frac{1}{3}} = (a^{\frac{1}{3}})^2 = (\sqrt[3]{a})^2.$$

In the same way

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p,$$

where $\sqrt[q]{a^p}$ means the number whose qth power is a^p, i.e. the qth root of a^p.

Fractional powers defined in this way obey Rules I and II. For instance if

$$x = a^{\frac{2}{3}} \times a^{\frac{1}{2}},$$

$$x^6 = (a^{\frac{2}{3}})^6 \times (a^{\frac{1}{2}})^6 = a^{\frac{2}{3}\times 6} \times a^{\frac{1}{2}\times 6} = a^4 \times a^3 = a^7$$

$$\therefore\ x = \sqrt[6]{a^7} = a^{\frac{7}{6}} = a^{\frac{2}{3}+\frac{1}{2}}$$

$$\therefore\ a^{\frac{2}{3}} \times a^{\frac{1}{2}} = a^{\frac{2}{3}+\frac{1}{2}}.$$

Hence the product is formed by using Rule I.

Negative powers and a^0

If we assume Rule I applies to negative indices as well as positive indices

$$a^3 \times a^{-2} = a^{3+(-2)} = a^{3-2} = a$$

$$\therefore\ a^{-2} = \frac{a}{a^3} = \frac{1}{a^2}.$$

Again $$a^3 \times a^{-\frac{2}{4}} = a^{3-\frac{2}{4}} = a^{\frac{3}{4}}$$

$$\therefore\ a^{-\frac{2}{4}} = \frac{a^{\frac{3}{4}}}{a^3} = \frac{1}{a^{\frac{2}{4}}}.$$

In the same way, if n is any number,

$$a^{-n} = \frac{1}{a^n}.$$

Also by Rule I, $a^0 = a^{2-2} = a^2 \times a^{-2} = \frac{a^2}{a^2}$

$$\therefore \underline{a^0 = 1.}$$

Powers of *ab*

$(4 \times 9)^{\frac{1}{2}} = 36^{\frac{1}{2}} = 6 = 2 \times 3 = 4^{\frac{1}{2}} \times 9^{\frac{1}{2}}$.

In the same way $(ab)^n = a^n \times b^n = a^n b^n$. This should not be confused with ab^n, which means $a \times b^n$.

Roots and Surds

$\sqrt[n]{a}$ stands for the positive number whose nth power is a. Since $(-1)(-1) = 1$, it follows that when n is even the nth power of $(-\sqrt[n]{a})$ is also a.

For instance, if $x^4 = 16$

$$x = \sqrt[4]{16} \text{ or } -\sqrt[4]{16} \qquad \text{i.e. } x = 2 \text{ or } -2.$$

A root can often be expressed in a simpler form by factorizing the number under the root sign.

Example.—Simplify (i) $\sqrt{128}$; (ii) $\sqrt[3]{189}$.

(i) $\sqrt{128} = \sqrt{64 \times 2} = \sqrt{64}\ \sqrt{2} = 8\sqrt{2}$.

(ii) $\sqrt[3]{189} = \sqrt[3]{27 \times 7} = \sqrt[3]{27}\ \sqrt[3]{7} = 3\sqrt[3]{7}$.

If a is not a perfect square $\sqrt{a}$ is called a surd. Thus $\sqrt{2}$ is a surd, but $\sqrt{16}$ is not. If a is not the nth power of a whole number or of a fraction $\sqrt[n]{a}$ is a surd.

The quickest way to calculate the reciprocal of a surd $\sqrt{a}$ is to multiply numerator and denominator by $\sqrt{a}$.

Example.—Express $\frac{1}{2\sqrt{3}}$ without a surd in the denominator and as a decimal.

$$\frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{2(\sqrt{3})^2} = \frac{\sqrt{3}}{2 \times 3} = \frac{\sqrt{3}}{6} \doteq \frac{1{\cdot}732}{6} \doteq 0{\cdot}289.$$

Examples.—Simplify (*a*) $\frac{a^4(b^2c^3)^3}{(a^2bc)^6}$; (*b*) $\frac{(\sqrt{2fp})^3}{4f^2p}$; (*c*) $\frac{k^{1\cdot9}}{k^{-0\cdot7}}$.

(*a*) $\frac{a^4(b^2c^3)^3}{(a^2bc)^6} = \frac{a^4b^{2\times3}c^{3\times3}}{a^{2\times6}b^6c^6} = \frac{a^4b^6c^9}{a^{12}b^6c^6} = \frac{c^{9-6}}{a^{12-4}} = \frac{c^3}{a^8}$.

(*b*) $\frac{(\sqrt{2fp})^3}{4f^2p} = \frac{(2^{\frac{1}{2}}f^{\frac{1}{2}}p^{\frac{1}{2}})^3}{4f^2p} = \frac{2^{\frac{3}{2}}f^{\frac{3}{2}}p^{\frac{3}{2}}}{4f^2p} = \frac{2^{\frac{3}{2}}p^{\frac{3}{2}-1}}{2^2.f^{2-\frac{3}{2}}}$

$= \frac{p^{\frac{1}{2}}}{2^{\frac{1}{2}}f^{\frac{1}{2}}}$ or $\sqrt{\frac{p}{2f}}$.

(*c*) $\frac{k^{1\cdot9}}{k^{-0\cdot7}} = k^{1\cdot9\times k^{0\cdot7}} = k^{1\cdot9+0\cdot7} = k^{2\cdot6}$.

Example.—From the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, find r in terms of π and V.

$$4\pi r^3 = 3V$$

$$\therefore r^3 = \frac{3V}{4\pi}$$

$$\therefore r = \sqrt[3]{\frac{3V}{4\pi}} \text{ or } \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}.$$

Exercise I

1. Simplify each of the following and find its value to three significant figures, given that $\sqrt{2} = 1\cdot414$, $\sqrt{3} = 1\cdot732$, $\sqrt{5} = 2\cdot236$, $\sqrt[3]{2} = 1\cdot260$.

(*a*) $\sqrt{27}$ (*b*) $\sqrt[3]{54}$ (*c*) $\sqrt{80}$ (*d*) $\sqrt{147}$ (*e*) $\sqrt{288}$.

2. Express without a surd in the denominator and also as a decimal:

(*a*) $\frac{1}{\sqrt{2}}$ (*b*) $\frac{1}{2\sqrt{5}}$ (*c*) $\frac{6}{\sqrt{3}}$ (*d*) $\frac{10}{\sqrt{5}}$.

3. Express with positive indices instead of root signs:

(*a*) $\sqrt{7}$ (*b*) $\sqrt[3]{3}$ (*c*) $\frac{1}{\sqrt[4]{2}}$ (*d*) $\sqrt{\frac{2}{3}}$ (*e*) $\sqrt{5^7}$.

4. Express with positive indices instead of root signs:

(*a*) $\sqrt{t}$ (*b*) $\sqrt{x^3}$ (*c*) $(\sqrt{y})^5$ (*d*) $\sqrt{4m}$ (*e*) $\sqrt[3]{x^2y^2}$.

5. Simplify:

(*a*) $\left(\frac{2}{a}\right)^2 \div \left(\frac{1}{2a}\right)^2$ (*b*) $\frac{(3x)^4}{3x^4}$ (*c*) $\frac{(xy^2)^3}{(xy^3)^2} \times \left(\frac{y}{x}\right)^4$.

6. Simplify:

(*a*) $\frac{\left(\frac{m}{2n}\right)^2}{2m^2n^2}$ (*b*) $\frac{\frac{3}{15}\pi r^5}{\frac{4}{3}\pi r^3}$ (*c*) $\frac{\frac{1}{2}\frac{w}{g}(\sqrt{2gh})^2}{Wh}$.

7. Which of $\frac{2p^2}{2p^3}$ and $\frac{(2p)^2}{(2p)^3}$ equals $\frac{1}{2p}$?

8. Which of $\frac{a^2x^2}{ax^3}$ and $\frac{ax^2}{(ax)^3}$ equals $\frac{a}{x}$?

9. Express as a power of x:

(*a*) $\sqrt{x^7}$ (*b*) $\sqrt[3]{\frac{1}{x^2}}$ (*c*) $\frac{x^2}{\sqrt{x^5}}$ (*d*) $\frac{x^{2\cdot7}}{\sqrt{x^{1\cdot8}}}$.

Simplify:

10. (*a*) $16^{\frac{1}{4}}$ (*b*) $y^{\frac{1}{5}} \times y^{\frac{3}{5}}$ (*c*) $\sqrt[3]{27a^2b}$ (*d*) $16a^{\frac{1}{2}} \times \left(\frac{a}{2}\right)^4$.

11. (*a*) $\left(\frac{27}{8}\right)^{-\frac{4}{3}}$ (*b*) $\frac{h^4}{8r^2} \times \left(\frac{2r}{h}\right)^4$ (*c*) $(81x)^{\frac{1}{4}} \times (8x)^{\frac{4}{3}}$

(*d*) $\frac{p^{1\cdot7} \times q^{-0\cdot6}}{p^{-0\cdot3} \times q^{0\cdot4}}$.

12. $\frac{p^{\frac{1}{3}} \times p^{\frac{1}{4}}}{p^{-\frac{1}{12}}}$ (*b*) $10a^{-\frac{5}{2}}(25a^5)^{\frac{1}{2}}$ (*c*) $\sqrt{\frac{a^5b^{-3}}{a^3b^{\frac{3}{4}}}}$.

13. (*a*) $(v^{2n})^{-\frac{1}{n}}$ (*b*) $\sqrt[3]{m^{\frac{3}{4}}n^{-\frac{9}{2}}}$ (*c*) $\frac{\sqrt{qp^{2n}r^n}}{3\left(\frac{p}{r}\right)^n}$.

Express without fractional or negative indices:

14. (*a*) $\sqrt{y^{-1}}$ (*b*) $\left(\frac{p}{q}\right)^{-3}$ (*c*) $\left(\frac{x}{y}\right)^{-\frac{1}{2}}$ (*d*) $ml^{-1}t^{-2}$.

15. (*a*) $\frac{(lt^{-1})^2}{lt^{-2}}$ (*b*) $x^{\frac{1}{2}}y^{-\frac{1}{2}}$ (*c*) $\frac{\sqrt{3p^{2k}q^{k-\frac{1}{2}}}}{(pq^{\frac{1}{2}})^{k+\frac{1}{2}}}$.

16. If $p_1v_1^n = p_2v_2^n$ prove $p_1v_1\left\{1-\left(\frac{p_2}{p_1}\right)^{1-\frac{1}{n}}\right\} = p_1v_1 - p_2v_2$.

17. If $y^4 = 16x$, express y in terms of x.

18. From the formula for the volume of a cylinder $V = \pi r^2 h$ express r in terms of V and h.

19. If $\frac{1}{m^3} = \frac{a^2}{b}$, prove $m = a^{-\frac{2}{3}}b^{\frac{1}{3}}$.

20. If $\pi E y d^4 = 8wl^4$, prove $d = l\left(\frac{8w}{\pi E y}\right)^{\frac{1}{4}}$.

Brackets and products

By putting an expression between two brackets we imply that the expression is to be treated as a whole ; thus $3(2x+4)$ means that each term of $(2x+4)$ is to be multiplied by 3.

$\therefore\ 3(2x+4) = 3\times 2x \ + \ 3\times 4 = 6x + 12.$

Example.—Remove the brackets from (*a*) $\frac{1}{3}h^2(3r-h)$; (*b*) $t^{\frac{1}{2}}(2\sqrt{t}-3t)$.

(*a*) $\frac{1}{3}h^2(3r-h) = \frac{1}{3}h^2\times 3r - \frac{1}{3}h^2\times h = h^2 - \frac{1}{3}h^3.$

(*b*) $t^{\frac{1}{2}}(2\sqrt{t}-3t) = t^{\frac{1}{2}}\times 2t^{\frac{1}{2}} - t^{\frac{1}{2}}\times 3t = 2t^{\frac{3}{2}} - 3t^{\frac{5}{2}}.$

Example.—Expand the products (*a*) $(p-2q)(p^2+pq)$; (*b*) $(2y-3)(y^2+4y-6)$.

(*a*) $\begin{aligned}(p-2q)(p^2+pq) &= p(p^2+pq) - 2q(p^2+pq)\\ &= p^3 + p^2q - 2p^2q - 2pq^2\\ &= p^3 - p^2q - 2pq^2.\end{aligned}$

(*b*) $\begin{aligned}(2y-3)(y^2+4y-6) &= 2y(y^2+4y-6) - 3(y^2+4y-6)\\ &= 2y^3 + 8y^2 - 12y - 3y^2 - 12y + 18\\ &= 2y^3 + 5y^2 - 24y + 18.\end{aligned}$

Any multiplication can be set out as in arithmetic. For instance, example (*b*) above can be set out as follows :

$$\begin{array}{ll} y^2+4y-6 & \\ \quad 2y-3 & \\ \hline 2y^3+8y^2-12y & \{\text{this line is } 2y(y^2+4y-6)\} \\ \quad -3y^2-12y+18 & \{\text{this line is } -3(y^2+4y-6)\} \\ \hline 2y^3+5y^2-24y+18 & \end{array}$$

Example.—Form the product $(t^{1\cdot6}+3)(2t^{1\cdot2}-t^{-0\cdot4})$.

$$\begin{aligned}\text{Expression} &= t^{1\cdot6}(2t^{1\cdot2}-t^{-0\cdot4})+3(2t^{1\cdot2}-t^{-0\cdot4})\\ &= t^{1\cdot6}\times 2t^{1\cdot2}-t^{1\cdot6}t^{-0\cdot4}+3\times 2t^{1\cdot2}-3t^{-0\cdot4}\\ &= 2t^{2\cdot8}-t^{1\cdot2}+6t^{1\cdot2}-3t^{-0\cdot4}\\ &= 2t^{2\cdot8}+5t^{1\cdot2}-3t^{-0\cdot4}.\end{aligned}$$

The effect of a $-$ sign in front of a bracket is to change all the signs of the terms within the bracket when it is removed.

Example.—Simplify $2p(r-s)-s(r-2p)-\frac{1}{2}r(4p-s)$.

$$\begin{aligned}\text{Expression} &= 2pr-2ps-sr+2sp-2rp+\tfrac{1}{2}rs\\ &= 2pr-2ps-rs+2ps-2pr+\tfrac{1}{2}rs.\\ &= -\tfrac{1}{2}rs.\end{aligned}$$

Some important products

$$(a+b)^2=(a+b)(a+b)=a^2+ab+ab+b^2$$

$$\therefore\ \underline{(a+b)^2=a^2+2ab+b^2} \quad \ldots\ \text{I.}$$

In the same way

$$\underline{(a-b)^2=a^2-2ab+b^2} \quad \ldots\ \text{II.}$$

$$\underline{(a+b)(a-b)=a^2-b^2} \quad \ldots\ \text{III.}$$

These results should be memorized. By using them the square of any expression containing two terms and the product of the sum and difference of two terms can be written down.

Examples.—Express without brackets (*a*) $(2x-3y)^2$; (*b*) $(5r-4s)(5r+4s)$; (*c*) $(2m^{0\cdot8}+m^{0\cdot2})^2$; (*d*) $(f+g-h)(f+g+h)$.

(*a*) $(2x-3y)^2=(2x)^2-2.2x.3y+(3y)^2$ [by II above]
$=4x^2-12xy+9y^2$.

(*b*) $(5r-4s)(5r+4s)=(5r)^2-(4s)^2$ [by III above]
$=25r^2-16s^2$.

(*c*) $(2m^{0\cdot8}+m^{0\cdot2})^2=(2m^{0\cdot8})^2+2.2m^{0\cdot8}m^{0\cdot2}+(m^{0\cdot2})^2$ [by I above]
$=4m^{1\cdot6}+4m+m^{0\cdot4}$

(*d*) $(f+g-h)(f+g+h)=(f+g)^2-h^2$ [by III with $a=f+g$, $b=h$]
$=f^2+2fg+g^2-h^2$.

An application of the formula for $(a+b)(a-b)$ occurs in calculating the values of expressions like $\dfrac{1}{5-3\sqrt{2}}$.

$(5-3\sqrt{2})(5+3\sqrt{2}) = 25-(3\sqrt{2})^2 = 25-18 = 7$, and hence does not contain the surd $\sqrt{2}$. Therefore, multiplying numerator and denominator of $\dfrac{1}{5-3\sqrt{2}}$ by $(5+3\sqrt{2})$, we get

$$\frac{1}{5-3\sqrt{2}} = \frac{5+3\sqrt{2}}{(5-3\sqrt{2})(5+3\sqrt{2})} = \frac{5+3\sqrt{2}}{7} \simeq \frac{9{\cdot}242}{7} = 1{\cdot}320.$$

Simplification of expressions containing products

$(p-2q)(p-3q)$ means that the expressions $(p-2q)$ and $(p-3q)$ are to be multiplied and, since multiplication takes precedence over addition or subtraction, the product must be found first and the resulting terms added or subtracted.

Example.—Simplify $(3p+q)^2-(p-2q)(p-3q)-(p-q)^2$.

Expression

$$\begin{aligned}
&= 9p^2+6pq+q^2-(p^2-5pq+6q^2)-(p^2-2pq+q^2)\\
&= 9p^2+6pq+q^2-p^2+5pq-6q^2-p^2+2pq-q^2\\
&= (9-1-1)p^2+(6+5+2)pq+(1-6-1)q^2\\
&= 7p^2+13pq-6q^2.
\end{aligned}$$

Addition and subtraction of fractions

Find the L.C.M. of the denominators of the fractions and express each fraction with this L.C.M. as denominator. Then all the fractions have a common denominator and the numerators can be added or subtracted. Remember that the bar of a fraction is equivalent to a bracket.

Example.—Simplify $\dfrac{6r+5}{6s}-\dfrac{2rs+r}{2s^2}$.

$$\begin{aligned}
6s &= 2\times 3\times s\\
2s^2 &= 2\times s\times s.
\end{aligned}$$

$\therefore$ L.C.M. $= 2 \times 3 \times s^2 = 6s^2$.

$$\therefore \text{ expression} = \frac{s(6r+5)}{6s^2} - \frac{3(2rs+r)}{6s^2} = \frac{s(6r+5) - 3(2rs+r)}{6s^2}$$

$$= \frac{6rs + 5s - 6rs - 3r}{6s^2} = \frac{5s - 3r}{6s^2}.$$

Exercise II

1. Remove brackets from :

   (*a*) $2(x^2 - 4x)$ (*b*) $\mathrm{M}(2\mathrm{M} + 3\mathrm{N})$ (*c*) $2\left(\omega t + \frac{2\pi}{3}\right)$

   (*d*) $a^2(a - h)$.

2. Add :

   (*a*) $x^2 + 2x - 1$, $3x - 4$, $2x^2 + 5$.
   (*b*) $\theta + \alpha$, $2\alpha - \theta$, $3\theta + 4\alpha$.
   (*c*) $y^2 - a^2$, $4y^2 + ay$, $y^2 + ay + 3a^2$.
   (*d*) $r_1 l_1 + r_2 l_2$, $r_1 l_1 - r_3 l_3$, $-r_2 l_2 + r_3 l_3$.

3. Add :

   (*a*) $x^2 + ax + bx + ab$, $a^2 - ab - ax$, $b^2 - bx$.
   (*b*) $p^2 + 2pq + q^2$, $p^2 - 2pq + q^2$, $2(p^2 - q^2)$.
   (*c*) $3(m^2 - 2mn - 3n^2)$, $4(n^2 + mn - m^2)$.

4. Subtract :

   (*a*) $m + 2n$ from $3m + 4n$.
   (*b*) $4x - 3y$ from $x + 2y$.
   (*c*) $a^2 - 2ab + b^2$ from $a^2 + 2ab + b^2$.
   (*d*) $nt + 30$ from $2nt - 60$.
   (*e*) $ab + 2b^2 + ca$ from $a^2 + ab + b^2$.

5. Remove the brackets and simplify :

   (*a*) $m - n + 2(3m + 2n)$.
   (*b*) $4(p - q) - 2(p + q)$.
   (*c*) $2(x^2 + 2ax + a^2) - (x^2 - a^2)$.
   (*d*) $2(t^3 + 1{\cdot}4t^2 - 2{\cdot}7t) - 4(0{\cdot}5t^3 - t^2 + 1{\cdot}3t)$.

6. Remove the brackets and simplify :

   (*a*) $3a^2(a - h) - a(a^2 - h^2) - 2ah(a + h)$.
   (*b*) $2(1 - \cos\theta) - 3(2 - 3\cos\theta)$.
   (*c*) $2(1 + \sin \mathrm{A}) - 3(2 \sin \mathrm{A} + \cos \mathrm{A}) + 4(\sin \mathrm{A} - \cos \mathrm{A})$.
   (*d*) $\sin x(1 - \cos x) - \cos x(1 - \sin x)$.

7. Remove the brackets and simplify:

(*a*) $t^7(1-5t)+t^6(t-t^2+1)$.

(*b*) $2a+b-c-\{a-2(b-c)\}+\{3a-(b+4c)\}$.

(*c*) $2[1-x-\{3-2(1-x)\}]$.

(*d*) $3\{5\cos x-2(\sin x+\cos x)\}-6(\cos x-\sin x)$.

8. Multiply:

(*a*) $2x-3$ by $3x+4$ (*b*) l^2+m^2 by $l-2m$

(*c*) $4-x$ by $2-3x$ (*d*) p^2-pq by $2p+q$.

9. Multiply:

(*a*) $2x^2-3x+4$ by $2x-4$ (*b*) x^2-x-7 by x^2+2x+3

(*c*) $p+lm$ by $l+mp$ (*d*) $1-e+e^2$ by $1+e+e^2$.

10. Multiply by picking out like products:

(*a*) $(x+1)(x+4)$ (*b*) $(a+2b)(a-3b)$

(*c*) $(p+q)(p-4q)$ (*d*) $(2y-7)(y+5)$

(*e*) $(2x^2+x-1)(3x+2)$ (*f*) $(l^2-lm)(l+2m)$.

11. Expand the following squares:

(*a*) $(1+x)^2$ (*b*) $(2y+3)^2$ (*c*) $(p-2q)^2$ (*d*) $(2ar-5s)^2$.

12. Express without brackets:

(*a*) $(m+n)(m-n)$ (*b*) $\left(\frac{r}{s}+1\right)\left(\frac{r}{s}-1\right)$

(*c*) $(a+b+c)(a-b-c)$.

13. Evaluate by expressing each number as a sum or difference as in (*a*):

(*a*) $169^2=170^2-2.170+1$ (*b*) 134^2 (*c*) 98^2 (*d*) 47^2.

14. Simplify

(*a*) $\frac{6}{a}-\frac{1}{2a}-\frac{13}{3a}$ (*b*) $\frac{2(x+y)}{7}-\frac{x+4y}{14}$

(*c*) $\frac{a+b}{ab}-\frac{a-b}{a^2}$ (*d*) $\frac{6xy+1}{y^2}-\frac{2(3x+1)}{y}$.

15. Express without a surd in the denominator and also as a decimal:

(*a*) $\frac{1}{\sqrt{2}+1}$ (*b*) $\frac{25}{4-\sqrt{6}}$ (*c*) $\frac{6}{5-\sqrt{7}}$.

16. Find the value of $20-\sqrt{399}$ given that $\sqrt{399}=19{\cdot}975$. Also by writing $20-\sqrt{399}=\frac{(20-\sqrt{399})(20+\sqrt{399})}{20+\sqrt{399}}$ find its value to four sig. fig. (The student will notice that the first method gives two sig. fig. only, whereas the second gives four sig. fig.)

17. By the method of Question 16 find $17 - \sqrt{288}$ to four sig. fig.

Simplify:

18. (*a*) $x(x+y)+y(x+y)$ (*b*) $(a+b)^2-(a-b)^2$.

19. (*a*) $(x+y)(x-y)-x(x+2y)$ (*b*) $(1+\sin\theta)(1-\sin\theta)$.

20. (*a*) $(2+\sin x)^2+(2-\sin x)^2$ (*b*) $\frac{1}{2}w(l-x)^2-\frac{1}{2}wl(l-x)$.

21. (*a*) $(2x+1)(x-1)-(3x+2)(x-2)$

(*b*) $x(x-1)(x-2)-(5-3x)(x+1)$.

22. By multiplying $(a+b)^2$ by $(a+b)$ prove that

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3,$$

and hence find $(2y+3)^3$ in powers of y.

23. Find a formula for $(a-b)^3$ by the method used in Question 22, and use it to expand $(k-1)^3$ in powers of k.

24. It can be shown that the volume V cu. ft. of a segment of a sphere of radius r ft. and height h ft. is given by

$$V=\tfrac{2}{3}\pi r^3-\pi r^2(r-h)+\tfrac{1}{3}\pi(r-h)^3$$

Prove that $V=\frac{1}{3}\pi h^2(3r-h)$.

25. If $A=akk'+k+k'$, $B=ak'+1$, $C=ak+1$, $D=a$, prove that $BC-AD=1$.

Equations

Generally, if an equation contains only one letter or symbol, that letter must have a definite numerical value or values. For instance, if $8x=5$, x must be $\frac{5}{8}$, and if $(x-1)(x+2)=0$; x must be 1 or -2. Neither of these equations is a true statement for any other value of x. The numerical values which the unknown letter must have, are called the "roots" of the equation and the process of finding them is called "solving the equation." The *degree of an equation is the highest power of the unknown letter that occurs in it.* The equations $2x+1=7$, $t^2-t-4=0$, $p^4=16$ are of the first, second and fourth degrees in x, t and p respectively.

An equation or formula may contain several letters, and then any particular letter can only have a certain value or values in terms of the other letters. For instance, if $v^2=2gh$, the

equations $g = \frac{v^2}{2h}$ and $h = \frac{v^2}{2g}$ express g and h in terms of the other two letters. The process of solving an equation to find a particular letter in terms of other letters is called making that letter the subject of the formula.

Simple equations

An equation or formula in which the unknown letter occurs to the first degree is called a simple equation.

Corresponding examples of solving an equation and of changing the subject of a formula are shown below side by side.

Example.—

Solve the equation :	Make x the subject of the formula :
$2(x-13) = 10 - 3(x+2)$	$2(x-b) = 5a - 3(x+a)$
$2x - 26 = 10 - 3x - 6$	$2x - 2b = 5a - 3x - 3a$
$\therefore\ 2x + 3x = 26 + 10 - 6$	$\therefore\ 2x + 3x = 5a - 3a + 2b$
$\therefore\ 5x = 30$	$\therefore\ 5x = 2a + 2b$
$\therefore\ x = 6$	$\therefore\ x = \frac{2a+2b}{5}$.

Example.—

Solve the equation :	Make t the subject of the formula :
$\frac{t+1}{3} - \frac{5t+1}{12} = \frac{t}{2}$	$\frac{t+p}{q} - \frac{5t+p}{4q} = \frac{t}{2}$
Multiply by 12 the L.C.M. of the denominators :	Multiply by $4q$ the L.C.M. of the denominators :
$4(t+1) - (5t+1) = 6t$	$4(t+p) - (5t+p) = 2qt$
$\therefore\ 4t + 4 - 5t - 1 = 6t$	$\therefore\ 4t + 4p - 5t - p = 2qt$
$\therefore\ 4t - 5t - 6t = 1 - 4$	$\therefore\ 4t - 5t - 2qt = -4p + p$
$\therefore\ -7t = -3$	$\therefore\ -(1+2q)t = -3p$
$\therefore\ t = \frac{-3}{-7} = \frac{3}{7}$.	$\therefore\ t = \frac{-3p}{-(1+2q)} = \frac{3p}{1+2q}$.

Equations involving surds

If one term of an equation contains a square root, write the equation with that term on one side and then square both sides.

Example.—Make x the subject of the formula :

$$\sqrt{\frac{x+a}{2r}} - \frac{a}{r} = 0.$$

Adding $\frac{a}{r}$ to both sides :

$$\sqrt{\frac{x+a}{2r}} = \frac{a}{r}.$$

Squaring both sides :

$$\frac{x+a}{2r} = \frac{a^2}{r^2}.$$

Multiplying throughout by $2r$:

$$x + a = \frac{2ra^2}{r^2} = \frac{2a^2}{r}$$

$$\therefore\ x = \frac{2a^2}{r} - a.$$

Identities

Some equations are true for every value of the letters. For instance, if we remove the bracket in the equation

$$x + 2(x-2) = 3x - 4,$$

we get

$$x + 2x - 4 = 3x - 4$$

i.e. $$3x - 4 = 3x - 4.$$

Since $3x = 3x$ and $4 = 4$ whatever value x has, the equation is true for all values of x.

An equation which is true for every value of the unknown letter or letters is called an "identity." We sometimes write $\equiv$ for " is identically equal to " instead of $=$ in identities. For instance, $(a+b)^2 \equiv a^2 + 2ab + b^2$.

Example.—Prove $(ab-cd)^2+(bc+ad)^2 \equiv (a^2+c^2)(b^2+d^2)$.

$$\begin{aligned}(ab-cd)^2+(bc+ad)^2 &\equiv a^2b^2-2abcd+c^2d^2+b^2c^2+2abcd+a^2d^2\\ &\equiv a^2b^2+a^2d^2+c^2b^2+c^2d^2\\ &\equiv a^2(b^2+d^2)+c^2(b^2+d^2)\\ &\equiv (a^2+c^2)(b^2+d^2).\end{aligned}$$

Problems involving simple equations and formulæ

To solve a problem in which one unknown quantity occurs we denote that quantity by some letter, often the first letter of the quantity, for instance, r for radius, A for area, etc. Then we form an equation in this letter from the given facts.

Example.—A hollow iron pipe 2 ft. long and 560 lb. wt. per cu. ft. weighs 2·65 lb. wt. It is $\frac{1}{10}$th in. thick. Find its internal radius.

If the internal radius is r in., the outside radius is $(r+\frac{1}{10})$ in. and the area of a cross section of the pipe is

$$\begin{aligned}\pi\{(r+\tfrac{1}{10})^2-r^2\} \text{ sq. in.} &= \frac{\pi}{144}\left(r^2+\frac{r}{5}+\frac{1}{100}-r^2\right) \text{ sq. ft.}\\ &= \frac{\pi}{144}\left(\frac{r}{5}+\frac{1}{100}\right) \text{ sq. ft.}\end{aligned}$$

Therefore the volume of iron used is :

$$\frac{2\pi}{144}\left(\frac{r}{5}+\frac{1}{100}\right) \text{ cu. ft.}$$

And hence its weight is :

$$\frac{560\times 2\pi}{144}\left(\frac{r}{5}+\frac{1}{100}\right) \text{ lb. wt.}$$

$$\therefore \frac{560\times 2\pi}{144}\left(\frac{r}{5}+\frac{1}{100}\right)=2{\cdot}65$$

$$\therefore \frac{70\pi}{9}\left(\frac{r}{5}+\frac{1}{100}\right)=2{\cdot}65$$

$$\therefore \frac{r}{5}+\frac{1}{100}=\frac{9\times 2\cdot 65}{70\pi}=0\cdot 1085$$

$$\therefore \frac{r}{5}\simeq 0\cdot 0985$$

$$\therefore \text{radius}\simeq 0\cdot 49 \text{ in. to } \tfrac{1}{100}\text{th in.}$$

Example.—What weight of liquid of specific gravity s_1 must be mixed with n kg. of a liquid of specific gravity s_2 to give a mixture of specific gravity s ?

Let the weight be x kg. Then the volumes of the two liquids are :

$$\frac{1000x}{s_1} \text{ and } \frac{1000n}{s_2} \text{ cu. cm.}$$

$$\therefore \text{Volume of mixture}=\left(\frac{1000x}{s_1}+\frac{1000n}{s_2}\right) \text{ cu. cm.}$$

$$\text{Weight of mixture}=1000(x+n) \text{ gm.}$$

$$\therefore s=\frac{1000\,(x+n)}{\left(\dfrac{1000x}{s_1}+\dfrac{1000n}{s_2}\right)}=\frac{x+n}{\dfrac{x}{s_1}+\dfrac{n}{s_2}}$$

Multiplying by $\dfrac{x}{s_1}+\dfrac{n}{s_2}$:

$$\frac{sx}{s_1}+\frac{sn}{s_2}=x+n$$

$$\therefore ss_2x+ss_1n=s_1s_2x+s_1s_2n$$

$$\therefore (ss_2-s_1s_2)x=(s_1s_2-ss_1)n$$

$$\therefore x=\frac{(s_1s_2-ss_1)n}{ss_2-s_1s_2}$$

$$\therefore \text{weight of liquid}=\frac{(s_1s_2-ss_1)n}{ss_2-s_1s_2} \text{ kg.}$$

Exercise III

1. Solve the equations :

(*a*) $2(x+1)=5x-7$ (*b*) $\dfrac{m-2}{5}=m-4$

(*c*) $\dfrac{y}{3}=\dfrac{2y-1}{6}+\dfrac{y-1}{4}$ (*d*) $\dfrac{1}{2t}=\dfrac{1}{3}$.

(*e*) $(t+2)^2=(t-2)^2+6$ (*f*) $\dfrac{x-1}{x+2}=\dfrac{1}{4}$.

2. Express the letter in brackets in terms of the other letters :

(*a*) $3p-q-r=2(p+q+r)$ [p] (*b*) $\frac{1}{2}(p+q)=p-q$ [q]

(*c*) $\dfrac{\mathrm{R}-2r}{\mathrm{R}}=\dfrac{3}{5}$ [R] (*d*) $\mathrm{S}=2\pi rh+\pi r^2$ [h]

(*e*) $y=\dfrac{4\mathrm{W}l^3}{\mathrm{E}bd^3}$ [E] (*f*) $\dfrac{1}{u}+\dfrac{1}{v}=\dfrac{1}{f}$ [v].

3. Solve the equations :

(*a*) $2\sqrt{x}=1-\sqrt{x}$ (*b*) $\dfrac{1}{x}+\dfrac{1}{2x}=\dfrac{3}{4}$

(*c*) $\sqrt{1-t}=\frac{1}{3}$ (*d*) $\sqrt{\dfrac{1+2m}{1-2m}}=0{\cdot}7$.

4. Express the letter in brackets in terms of the other letters :

(*a*) $\mathrm{D}=\sqrt{\dfrac{3h}{2}}$ [h] (*b*) $\mathrm{A}=p+\dfrac{prt}{100}$ [p]

(*c*) $\mathrm{T}=2\pi\sqrt{\dfrac{l}{g}}$ [g] (*d*) $\mathrm{V}=\pi r^2(h-\frac{1}{3}r)$ [h]

(*e*) $f=\dfrac{1}{2\pi\sqrt{\mathrm{LC}}}$ [C] (*f*) $\sqrt{r^2-(r-h)^2}=y$ [r]

5. If $2s=a+b+c$, express $b+c-a$ in terms of s and a and prove that $\dfrac{(b+c-a)(c+a-b)}{(a+b+c)(a+b-c)}=\dfrac{(s-a)(s-b)}{s(s-c)}$.

6. From the equation $k+kmx=m-x$ express (*a*) k, (*b*) m, (*c*) x in terms of the other letters.

7. If two shafts of diameters d_1 and d_2 with a belt of thickness t round them make n_1 and n_2 revolutions per minute,

$$\frac{n_1}{n_2}=\frac{d_2+t}{d_1+t};$$

make t the subject of this formula.

8. Make θ_m the subject of the formula $\dfrac{\theta_m-x}{\theta_m}=\dfrac{y}{\theta_1}$.

9. In calorimetry the following equation occurs:

$$\mathrm{M}s_1(t-32)+m_1s_2(t-32)+m_2(t-32)$$
$$=Ms_1(T-32)+m_1s_2(t_0-32)+m_2(t_0-32)$$

Prove $$\mathrm{T}=\frac{(m_1s_2+m_2)(t-t_0)}{\mathrm{M}s_1}+t.$$

10. If $\dfrac{1}{a}=\dfrac{2}{\mathrm{R}}-\dfrac{1}{x}$ which of the following is the right value of x?

(*a*) $2a-\mathrm{R}$ (*b*) $\dfrac{2a-\mathrm{R}}{a\mathrm{R}}$ (*c*) $\dfrac{a\mathrm{R}}{2a-\mathrm{R}}$ (*d*) $\dfrac{\mathrm{R}}{2}-a$ (*e*) $\dfrac{a\mathrm{R}}{\mathrm{R}-2a}$.

11. If $\omega=2\pi f$ and $\omega\mathrm{L}=\dfrac{1}{\omega\mathrm{C}}$ express f in terms of π, L and C.

12. Make W the subject of the formula $\mathrm{N}=188\sqrt{\dfrac{\mathrm{EI}}{\mathrm{W}l^4}}$.

13. If $\mathrm{T}_c=\mathrm{M}+\sqrt{\mathrm{M}^2+\mathrm{T}^2}$ express M in terms of T and T_c [write the equation $\mathrm{T}_c-\mathrm{M}=\sqrt{\mathrm{M}^2+\mathrm{T}^2}$ and square both sides].

14. Make ω^2 the subject of the formula $\dfrac{\mathrm{R}_1}{\mathrm{R}_2}=\dfrac{\sqrt{\mathrm{R}_3{}^2+\mathrm{L}_3{}^2\omega^2}}{\sqrt{\mathrm{R}_4{}^2+\mathrm{L}_4{}^2\omega^2}}$.

15. A point divides a line 12 in. long into two parts so that one part is $\frac{1}{2}$ in. longer than the other. Find the lengths of the parts.

16. A lens is in the form of a segment of a sphere. It is 0·2 in. thick and the radius of the plane face is 3 in. Find the radius of the curved surface.

17. A cyclist A is riding at 12 m.p.h. Another cyclist, B, starts $\frac{1}{4}$ mile behind A and rides at 15 m.p.h. How long will he take to overtake A?

18. Find two numbers whose sum is 65 so that the sum of a third of one number and a quarter of the other is 18.

19. A man buys 1000 oranges at 10 for 2*s.* 0*d.* and sells some at 60% profit and the remainder at 50%. How many must he sell at the higher price to make a profit of £5 12*s.* 0*d.*

20. Find what resistance must be put in parallel with a resistance of 24 ohms to produce a resistance of 8 ohms ?

21. How many pounds of glycerine of specific gravity 1·260 must be added to 20 gallons of water to make the specific gravity of the mixture 1·1 ? [1 gallon of water weighs 10 lbs.]

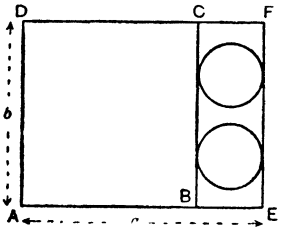

FIG. 1.

22. The figure shows a rectangular sheet of metal AEFD. The part ABCD is rolled into a cylinder of length AD and the circular discs cut from BEFC are used as ends of the cylinder. If the volume of the cylinder is V, show that,

$$a = 2(\pi + 1)\sqrt{\frac{V}{\pi b}}.$$

23. If a point on a thermometer is the same number on the Fahrenheit and Centigrade scales, find that number.

24. In a certain district electricity is charged for at two different rates, either (1) 5% of the rateable value of the house + 1*d.* a unit, or (2) 8*d.* a unit. For a house of rateable value £84 find the least number of units that must be consumed for it to pay the householder to go on the first rate. Find also the least number for a house of rateable value £*n*.

25. Two ships 500 sea miles apart are steaming towards each other at 30 and 20 knots. To what must the speed of the latter ship be increased in order that it may meet the former 2 hrs. earlier than it would have done ? Also find what speed is necessary if the ships are initially *n* miles apart.

26. A U tube with both ends open contains mercury to within 10 cm. of the ends of the tube. Water is poured into one arm until the water is 4 cm. below its end. Find how far the mercury rises in the other arm ? (sp. gr. mercury = 13·6.)

27. A car averaged 19 miles per gallon of petrol for 500 miles. How many miles per gallon must it average for the next 500 miles in order to increase the average over the whole 1000 miles to 21 miles per gallon ?

28. There are two rates of charging for electricity in a certain district : (*a*) ordinary rate :—light 8*d.* per unit, power 2*d.*. per

unit; (*b*) tariff rate:—5% of rateable value and 1*d.* per unit for light and power. If a householder has a house of rateable value £60 find the total cost of 100 units for lighting and 500 units for power, under both schemes. If he actually uses 80 units for lighting what is the least number of units he must use for power in order that it shall pay him to go on the tariff rate?

Extended use of symbols. Units

In Part I symbols were used to represent numbers only, but, as we shall see below, the scope of a formula is greatly enlarged by supposing that each symbol in it represents not only the number which measures the quantity, but the unit in which the quantity is measured as well. Consider the formula $V=abc$, which gives the volume V cu. in. of a rectangular box of which the internal dimensions are a in., b in. and c in. This formula is clearly true if the internal dimensions are a cm., b cm. and c cm. and the volume is V cu. cm. In fact it is true whatever unit of length is used, provided the unit of volume is the volume of a cube whose edge is the unit of length. Because the formula is true whatever unit of length is used, it is useful to understand a, b, c to represent not only numbers of inches, centimetres, etc., but to include the unit. For instance, if the dimensions of a box are 3 ft. by 2 ft. by $1\frac{1}{2}$ ft. we write $a=3$ ft., $b=2$ ft. and $c=1\frac{1}{2}$ ft., then

$$\begin{aligned} V &= 3 \text{ ft.} \times 2 \text{ ft.} \times 1\tfrac{1}{2} \text{ ft.} \\ &= (3 \times 2 \times 1\tfrac{1}{2}) \text{ ft.} \times \text{ft.} \times \text{ft.} \\ &= 9 \text{ (ft.)}^3, \end{aligned}$$

where $(\text{ft.})^3$ is written as short for ft. × ft. × ft. We see that $(\text{ft.})^3$ is a convenient shorthand for 1 cubic foot. It does not mean that we can multiply a foot by a foot; this is as impossible as multiplying an apple by an apple.

If we make a the subject of the formula, we get:

$$a=\frac{V}{bc}.$$

Now suppose $V = 60$ cu. in., $b = 4$ in., $c = 5$ in. Then

$$a = \frac{60 \text{ (in.)}^3}{4 \text{ in.} \times 5 \text{ in.}} = \frac{60}{20} \frac{\text{(in.)}^3}{\text{(in.)}^2} = 3 \text{ in.}$$

This shows that the correct unit for the answer is obtained by treating the units as if they obeyed the index rules in the same way that numbers and symbols do.

If a car travels 200 ft. in 5 sec. its average speed is $\frac{200}{5}$ ft. per sec., i.e. 40 ft. per sec. We write this as $\frac{200 \text{ ft.}}{5 \text{ sec.}} = 40 \frac{\text{ft.}}{\text{sec.}}$ In the same way an acceleration of 3 ft. per sec. per sec. is written $3 \frac{\text{ft.}}{\text{sec.}^2}$.

Example.—The distance s fallen by a body under gravity in time t is given by $s = \frac{1}{2}gt^2$. Taking $g = 32$ ft. per sec. per sec., find the distance fallen in $\frac{1}{2}$ sec., putting the units in the formula.

Since $g = 32 \frac{\text{ft.}}{\text{sec.}^2}$,

$$s = \tfrac{1}{2} \cdot 32 \frac{\text{ft.}}{\text{sec.}^2} \cdot (\tfrac{1}{2} \text{ sec.})^2$$

$$= \frac{32}{2 \times 4} \frac{\text{ft.}}{\cancel{\text{sec.}^2}} \cdot \cancel{\text{sec.}^2}.$$

$$= 4 \text{ ft.}$$

When, from experience, we are sure of the unit of the answer it is unnecessary to put in the units in each line, as in the above example, but everyone should be able to check the units of an answer in this way.

Example.—Express a pressure of 1 lb. wt. per sq. in. in grams wt. per sq. cm.

$$1 \text{ lb. wt. per sq. in.} = \frac{1 \text{ lb. wt.}}{\text{in.}^2} = \frac{454 \text{ gm. wt.}}{(2{\cdot}54 \text{ cm.})^2},$$

$$= \frac{454 \text{ gm. wt.}}{6{\cdot}45 \text{ cm.}^2}$$

$$= 70{\cdot}4 \text{ gm. wt. per sq. cm.}$$

Example.—Young's modulus of elasticity for a metal bar is given by

$$E = \frac{4Wl^3}{bd^3y}$$

where l = length, b = breadth, d = depth of the bar, and y is the deflection of one end due to a load W when the other end is clamped horizontally.

If b, d, y, l are measured in inches and W in lb. wt., in what unit is E measured ? Also construct a formula to give E in $\frac{\text{ton. wt.}}{\text{ft.}^2}$ when W is in cwt., l in ft. and b, d, y in inches.

Writing the units inches and lb. wt. alongside the symbols

$$E = \frac{4W \text{ lb. wt. } (l \text{ in.})^3}{(b \text{ in.})\,(d \text{ in.})^3\,(y \text{ in.})} = \frac{4Wl^3}{bd^3y} \frac{\text{lb. wt. (in.)}^3}{(\text{in.})^5} = \frac{4Wl^3}{bd^3y} \frac{\text{lb. wt.}}{(\text{in.})^2}.$$

This shows that the unit for E is 1 lb. wt. per sq. in.

If W is in cwt., l in ft., b, d, y in inches.

$$E = \frac{4W \text{ cwt. } (l \text{ ft.})^3}{bd^3y\,(\text{in.})^5} = \frac{4W \cdot \frac{\text{ton wt.}}{2240}\, l^3(\text{ft.})^3}{bd^3y\left(\frac{\text{ft.}}{12}\right)^5}$$

$$\therefore E = \frac{4 \times 12^5}{2240} \frac{Wl^3}{bd^3y} \frac{\text{ton wt.}}{(\text{ft.})^2}.$$

Thus we can say that for this set of mixed units :

$$E = \frac{4 \times 12^5}{2240} \frac{Wl^3}{bd^3y}.$$

Example.—Ohm's Law $E = RI$, which relates the electromotive force E, resistance R and the current I in a wire, is true when E, R, I are measured in the practical units, volt, ohm and

ampère, and when they are measured in C.G.S. units. Assuming 1 ampère $= \frac{1}{10}$th C.G.S. unit of current, 1 volt $= 10^8$ C.G.S. units of electromotive force, find the number of C.G.S. units of resistance in 1 ohm.

Since $E = RI$ is true in practical units

$$1 \text{ volt} = 1 \text{ ohm} \times 1 \text{ ampère}$$

$$\therefore 1 \text{ ohm} = \frac{1 \text{ volt}}{1 \text{ ampère}}$$

$$= \frac{10^8 \text{ C.G.S. units of E.M.F.}}{\frac{1}{10} \text{ C.G.S. unit of current}}$$

$$= 10^9 \text{ C.G.S. units of resistance}$$

(since 1 C.G.S. unit of E.M.F. = 1 C.G.S. unit of resistance × 1 C.G.S. unit of current).

Exercise IV

1. If t is a time in seconds, s a distance in feet, v a velocity in ft. per sec., f an acceleration in ft. per sec. per sec., find the units of each of the following and state what kind of a quantity each is:

(*a*) ft (*b*) $\frac{1}{2}ft^2$ (*c*) $\sqrt{fs}$ (*d*) $\frac{v^2}{f}$.

2. If R is a resistance in ohms, i a current in ampères, V a potential difference in volts, find the units of each of the following in their simplest form, assuming 1 volt = 1 ohm × 1 ampère and 1 watt = 1 volt × 1 ampère.

(*a*) $\frac{V}{R}$ (*b*) Ri^2 (*c*) $\frac{V^2}{R}$.

3. If u and v are velocities in feet per second, s a distance in feet, and t a time in seconds, find in which of the formulæ

(*a*) $v^2 = u^2 + 2ft$ (*b*) $v^2 = u^2 + 2fs$

the units of all the terms are the same. Which formula do you think is correct?

4. If W is in lb. wt., v in ft./sec., g in ft./sec.^{2}, in what unit is kinetic energy $\frac{1}{2}\frac{Wv^2}{g}$ measured?

5. The weight of a mass of one pound gives it an acceleration of nearly 32·19 ft./sec.^{2} in London. What is the relation between 1 lb. wt., 1 lb. mass and 1 ft./sec.^{2} ?

6. The absolute unit of force (a poundal) is the force which gives the standard mass 1 lb. an acceleration of 1 ft./sec.^{2} What is the absolute unit in terms of lb. mass, ft. and sec. ? In the C.G.S. system the absolute unit is the dyne, a force which gives a mass of 1 gram an acceleration of 1 cm./sec.^{2} What is the relation between a dyne, a gram, a centimetre and a second ? What is the relation between the weight of a kilogram and a dyne ?

7. Taking $g = 32{\cdot}19$ ft./sec.^{2}, 1 lb. mass $= 453{\cdot}6$ grams, 1 ft. $= 30{\cdot}48$ cm. and 1 dyne $=$ 1 gram. cm./sec.^{2}, express a force of 1 lb. wt. in dynes.

8. The formula for the tension of a wire when it is stretched a length x is $T = (EAx)/l$, where A is the area of the cross-section of the wire, l is the length of the wire and E is Young's Modulus of elasticity for the material of which the wire is made. If l and x are in inches, A is in square inches and T in lb. wt., in what units is E measured ?

9. If a body moves in a circle of radius a with velocity v it has an acceleration $\frac{v^2}{a}$ towards the centre. By taking v in ft./sec. and a in ft. show that $\frac{v^2}{a}$ has the same units as an acceleration.

10. If the potential difference between the plates of a condenser having a capacity C is V when the charge on a plate is Q, $Q = CV$, when C is in farads, V in volts and Q in coulombs. Taking R to be a resistance in ohms, i a current in ampères, and assuming 1 ampère $=$ 1 coulomb/sec. show that :

(*a*) $\frac{Q^2}{Ct}$ is in watts. (*b*) $\frac{CV}{i}$ is a time in seconds.

(*c*) RCi is a charge in coulombs.

Functions

The current produced in a given wire of resistance 10 ohms by an electromotive force E volts is $\frac{E}{10}$ ampères. Because the current behaves or functions according to this relationship with E, it is called a function of the voltage E. If the current

is denoted by the letter i, then the formula $i = \frac{E}{10}$ expresses the functional relationship of i with E.

In general, any quantity, whose value depends on the value of a variable quantity x, is called a function of x. $\frac{1}{2}x^2$, $3\sin^2 t$, 2^n are functions of x, t and n respectively. Functions like $x^{\frac{1}{2}}$, $(1-p^2)^4$, $\frac{2r+1}{2r-1}$ are called algebraic functions; functions formed by using the trigonometrical ratios such as $\tan \theta°$, $3\cos(2t$ radians), $\sin^2 \alpha$, are called trigonometric functions; functions like 2^{-x}, $(1{\cdot}83)^{3x}$ are called exponential functions, because the index or exponent is the variable part in them. An equation relating two variable quantities tells us what function one quantity is of the other; for example, $y = 5x^2$, $C = \frac{5}{9}(F-32)$, $R = \frac{r+3}{r-3}$, show what function y is of x, C of F and R of r respectively. To see what function one quantity is of another it may be necessary to change the subject of the formula. For instance, to find what function F is of C from the formula $C = \frac{5}{9}(F-32)$ we put the equation in the form $F = 32 + \frac{9}{5}C$.

A quantity may be a function of more than one other variable quantity. For instance, the volume of a cylinder

GRAPH OF THE FUNCTION x^2

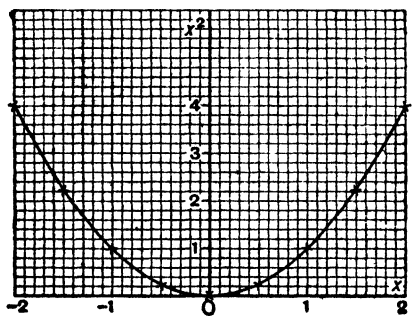

FIG. 2.

depends on both its height, h, and its radius, r; the volume is $\pi r^2 h$, and it is a function of both r and h. The deflection of the middle point of a beam supported at its ends at the same level is a function of (1) its length, (2) its breadth, (3) its depth, (4) its weight per unit length, (5) its elasticity, making five variable quantities in all.

The graph of the equation $y = x^2$ may also be considered as the graph of the function x^2, and when we are thinking of it in this way we label the vertical axis "x^2" instead of y, as in Fig. 2.

Exercise V

Express:

1. The electromotive force between two points in a wire of resistance 6 ohms as a function of the current i ampères.

2. The area of a square as a function of the length x of a diagonal.

3. The distance dropped by a body under gravity as a function of the time taken, t sec. (take $g = 32$ ft./sec.^{2}).

4. The area of a circular ring of width 1 in. as a function of the radius r in. of the inside of the ring.

5. The telephone bill of a subscriber as a function of the number of calls, n, in a quarter, if the charge for a telephone is £2 per quarter, the first 50 calls are free, and the charge for the remainder is 1d. a call.

6. The volume of a cube as a function of the area A of a face.

7. If $y = \frac{1}{2x} - 3$ express x as a function of y.

8. If $m = \frac{n+1}{n-1}$ show that n is the same form of function of m as m is of n.

9. If $y = x^2 - 1$ and $x = 1 - \frac{1}{2}u$, express y as a function of u.

10. Write down the relationship between the area A of a circle and its radius r. What function is r of A?

11. The pressure p, the volume v and the temperature t of a gas are related by the equation $pv = k(273 + t)$ where k is a fixed number. Express t as a function of p and v.

12. Express the length of the hypotenuse of a right-angled triangle as a function of the lengths a and b of the other two sides.

13. If the height of a cylinder is twice its radius, express its volume as a function of (1) its height, (2) its radius.

14. In a triangle ABC the angle B is 90°. If AB and BC have lengths a and b, express the perpendicular distance of B from AC as a function of a and b.

15. The impedance Z of a certain electric circuit is given by $Z = \dfrac{1}{\sqrt{\dfrac{1}{R^2} + \omega^2 C^2}}$. Express ω as a function of Z, R and C.

16. If $xy + 3x = 4y + 5$, express (a) x as a function of y, (b) y as a function of x.

17. If $\dfrac{k}{v} - \dfrac{1}{u} = \dfrac{k-1}{r}$, express k as a function of the other letters.

Simultaneous equations

The following example shows the two methods of solving simultaneously equations of the first degree.

Example.—Solve the equations :

$$4x - 3y = 7 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$6x - 7y = -2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

1st Method.—Eliminate a letter, say x, by multiplying (1) and (2) by numbers which will make the coefficients of x the same in both equations. These numbers can be 6 and 4, which make the coefficients of x both 24, but since the L.C.M. of 6 and 4 is 12 it will do if (1) is multiplied by $\frac{12}{4} = 3$, and (2) by $\frac{12}{6} = 2$. Then :

$$12x - 9y = 21$$
$$12x - 14y = -4$$

Subtracting $$5y = 25$$
$$\therefore\ y = 5$$

Substituting this value in (1) :

$$4x = 3y + 7 = 15 + 7 = 22$$
$$\therefore\ x = 5\tfrac{1}{2}.$$

2nd Method.—Make x the subject of equation (1):

$$4x = 3y + 7$$
$$\therefore\ x = \frac{3y+7}{4}.$$

Substitute this value in equation (2):

$$6\left(\frac{3y+7}{4}\right) - 7y = -2$$
$$\therefore\ \frac{3(3y+7)}{2} - 7y = -2$$
$$\therefore\ 9y + 21 - 14y = -4$$
$$\therefore\ -5y = -25$$
$$\therefore\ y = 5,$$

Whence $x = 5\frac{1}{2}$ as before.

When the coefficients of x and y contain three or more figures it is better to make the coefficients of x equal in the equations by dividing each equation by the coefficient of x in it.

Example.—Solve the equations:

$$4{\cdot}97x + 63{\cdot}1y = 26{\cdot}9, \quad 27{\cdot}3x - 15{\cdot}2y = 65{\cdot}4.$$

Dividing by 4·97 and 27·3 respectively:

$$x + 12{\cdot}70y = 5{\cdot}412$$
$$x - 0{\cdot}557y = 2{\cdot}396$$
$$\therefore\ \text{substracting, } 13{\cdot}26y = 3{\cdot}016$$
$$\therefore\ y = \frac{3{\cdot}016}{13{\cdot}26} = 0{\cdot}2275.$$
$$\therefore\ x = 5{\cdot}412 - 12{\cdot}70y = 5{\cdot}412 - 12{\cdot}70 \times 0{\cdot}2275 = 2{\cdot}521.$$

Ans. $x = 2{\cdot}52$, $y = 0{\cdot}228$.

Either of the methods used above may be used to eliminate a letter from two formulæ.

Example.—If two weights, W lb. wt. and w lb. wt. are connected by a string which passes over a light smoothly-running

pulley, their acceleration $f\frac{\text{ft.}}{\text{sec.}^2}$ and the tension of the string T lb. wt. are given by

$$\frac{Wf}{g} = W - T \text{ and } \frac{wf}{g} = T - w.$$

Find T in terms of W, w and g.

To eliminate f multiply the first equation by w and the second by W.

$$\frac{Wwf}{g} = Ww - Tw$$

$$\frac{Wwf}{g} = TW - Ww$$

Subtracting $0 = 2Ww - Tw - TW$

$$\therefore TW + Tw = 2Ww$$

$$\therefore T(W + w) = 2Ww$$

$$\therefore T = \frac{2Ww}{W + w}.$$

On the other hand, if we use the substitution method, we make f the subject of the first formula $\frac{Wf}{g} = W - T$, by multiplying by $\frac{g}{W}$. This gives

$$f = \frac{g}{W}(W - T)$$

Substituting this value in the second formula,

$$\frac{w}{g} \cdot \frac{g}{W}(W - T) = T - w$$

Multiplying by W,

$$w(W - T) = W(T - w)$$

$$\therefore wW - wT = WT - Ww$$

Whence, as above, $$T = \frac{2Ww}{W + w}.$$

Exercise VI

Solve the simultaneous equations :

1. $3x + 6y = 11$
   $14x - y = 3.$

2. $2x - 3y = 43$
   $3x + 5y = 18.$

3. $26x + 8y = 9$
   $2y - 6x = 2.$

4. $9x + 14y = 5$
   $12x + 21y = 7.$

Find the roots of the following equations to 3 significant figures.

5. $0{\cdot}05x + 0{\cdot}2y = 4{\cdot}5$
   $0{\cdot}3x + 0{\cdot}04y = 5{\cdot}5$

6. $14{\cdot}71x - 1{\cdot}96y = 20{\cdot}43$
   $2{\cdot}95x + 8{\cdot}24y = 16{\cdot}6.$

7. $1023x - 47y = 515$
   $147x + 212y = -100.$

8. $8{\cdot}14x + 0{\cdot}92y = 10{\cdot}07$
   $2{\cdot}93x + 12{\cdot}12y = 18{\cdot}95.$

9. Find p and q if $p - \frac{1}{2}q = \frac{1}{3}p + q = 7.$

10. The effort P lb. wt. required to raise a load W lb. wt. by means of a differential pulley block is given by $P = aW + b$. It is found in an experiment that $P = 1{\cdot}71$ when $W = 15$, and $P = 4{\cdot}36$ when $W = 75$. Find a and b.

11. Calculate the values of m and c, if $y = mx + c$ given that when $x = 6$, $y = 10$, and when $x = 15$, $y = 37$.

12. If, in the circuit shown, the currents in the two parts are x and y ampères,

$$4(x + y) + 6x = 15$$

and $$6x - 12y = 0.$$

Find x and y.

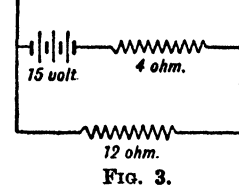

FIG. 3.

13. If in the circuit in Question 12 the battery has a voltage E and the resistances are R, r and s instead of 4, 6 and 12,

$$R(x + y) + rx = E$$
$$rx - sy = 0$$

Find x and y in terms of the other letters.

14. Fig. 4 shows the forces on a 20-lb. wt. when the weight is just being pushed up the plane by a horizontal force P lb. wt., and the coefficient of friction is 0·35. By resolving parallel

and perpendicular to the plane, the following equations are obtained:

$$P \cos 30° = 0{\cdot}35N + 20 \sin 30°$$
$$P \sin 30° + 20 \cos 30° = N.$$

Solve these equations for P and N.

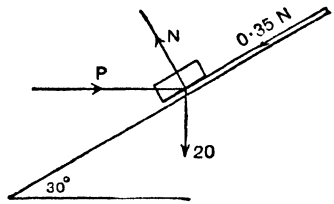

FIG. 4.

15. In a problem in mechanics the following equations occur:

$$\frac{\sqrt{3}}{2} T_1 - \tfrac{1}{2} T_2 = 50$$

$$\tfrac{1}{2} T_1 + \frac{\sqrt{3}}{2} T_2 = 100.$$

Find T_1 and T_2.

16. If $e = kDG = kd(G + R)$, find k in terms of e, D, d and R, but not G.

17. In calculating the strength of a reinforced concrete column the following equations occur: $f = mF = \dfrac{L - L_c}{a}$, $F = \dfrac{L_c}{A}$; prove that $f = \dfrac{mL}{am + A}$.

18. If $f = \dfrac{1}{2\pi\sqrt{LC}}$ prove that $2\pi f L_1 - \dfrac{1}{2\pi f C_1} = \dfrac{L_1 C_1 - LC}{C_1\sqrt{LC}}$.

19. If $\omega = 2\pi f$ and $\omega L = \dfrac{1}{\omega C}$, express f in terms of π, L and C.

20. If $B = \dfrac{4\pi IN}{10l}$ and $L = \dfrac{4\pi N^2 a 10^{-9}}{l}$, prove that $\tfrac{1}{2} LI^2 = \dfrac{B^2 a l 10^{-7}}{8\pi}$.

21. If $A = akk' + k + k'$, $B = ak' + 1$, $C = ak + 1$, express a, k and k' in terms of A, B and C.

22. If $p_x = \frac{16M}{\pi d^3}$, $p_y = \frac{16T}{\pi d^3}$ and $p_1 = \frac{1}{2}\{p_x + \sqrt{p_x^2 + p_y^2}\}$, prove that $p_1 = \frac{8}{\pi d^3}\{M + \sqrt{M^2 + T^2}\}$.

23. If $V^2 T_1^2 = d^2 + a_1^2$ and $V^2 T_2^2 = d^2 + a_2^2$, express V^2 and d^2 in terms of T_1, T_2, a_1 and a_2.

24. If $\frac{x}{h} = \frac{1}{\sqrt{3}d}$, $\frac{1}{d} = \frac{y}{\sqrt{3}}$ and $hy - x = 1$, express h in terms of d.

Graphs

To plot the graph of an equation such as

$$y = 20 - 5x^2 + \frac{40}{x+5}$$

for a given range of values of x, say $x = -3$ to $x = 4$, make a table of the values of each term of y for each of the selected values of x and then add the columns to obtain the values of y.

x ..	-3	-2	-1	0	1	2	3	4
20 ..	20	20	20	20	20	20	20	20
$-5x^2$ ..	-45	-20	-5	0	-5	-20	-45	-80
$\frac{40}{x+5}$ ..	20	13·3	10	8	6·7	5·7	5	4·4
Sum $= y$	-5	13·3	25	28	21·7	5·7	-20	$-55·6$

A graph is plotted from this table in Fig. 5. Such a graph can be used for the following purposes :

(*a*) to read off the value of y for any value of x between -3 and 4, or to read off the values of x between -3 and 4, which make y have any value between $-55·6$ and $+28·1$. This is called interpolation. For example, we find that when $x = 1·5$, $y \simeq 14·75$ at A, and when $y = 10$, $x \simeq -2·2$ or $1·8$ at B and C.

(*b*) to find any maximum or minimum values y may have in the given range of values of x.

In Fig. 5, y has a maximum value 28·1 at $x \simeq -0·2$ at D.

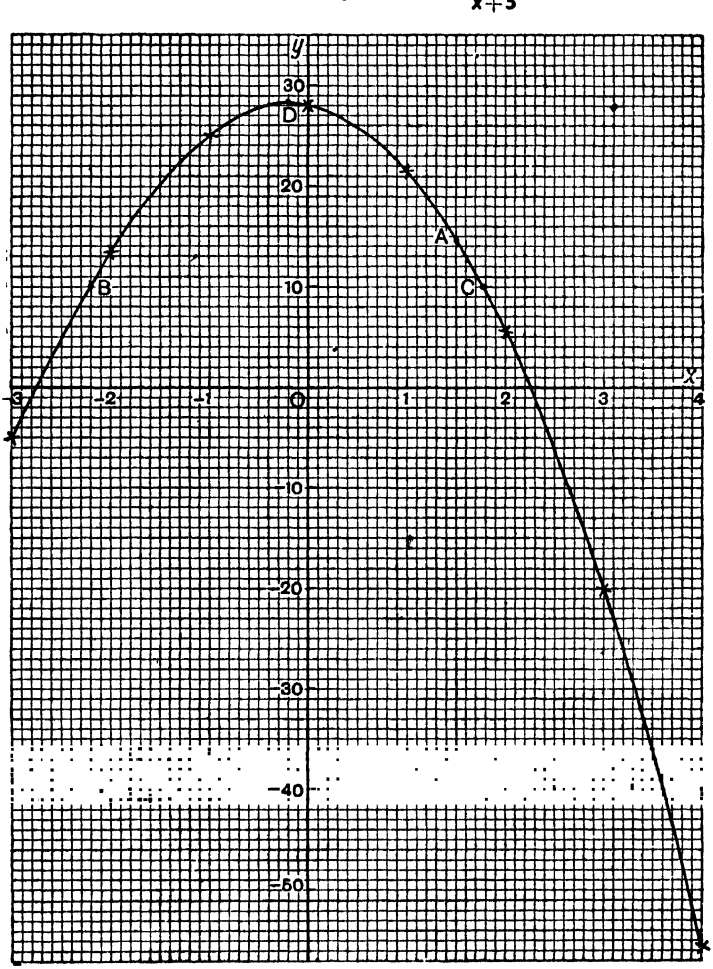
GRAPH OF $y=20-5x^3+\frac{40}{x+5}$
y
30
D
20
A
B
10
C
X
-3
-2
-1
O
1
2
3
4
-10
-20
-30
-40
-50

FIG. 5.

To obtain any one of these values more accurately the part of the graph near the corresponding point has to be plotted to a much larger scale. For instance, to find the maximum value of y its values at $x = -0{\cdot}3$, $-0{\cdot}2$ and $-0{\cdot}1$ are tabulated in addition to the value at $x = 0$, which has already been found.

x ..	$-0{\cdot}3$	$-0{\cdot}2$	$-0{\cdot}1$	0
20 ..	20	20	20	
$-5x^2$ ..	$-0{\cdot}45$	$-0{\cdot}20$	$-0{\cdot}05$	
$\frac{40}{x+5}$ ..	8·511	8·333	8·163	
y ..	28·061	28·133	28·113	28

A graph is plotted from this table in Fig. 6, and it shows that y has a maximum value 28·14, at $x \simeq -0{\cdot}17$. It should be noted that the maximum value can be found far more accurately than the corresponding value of x; moreover, $y \simeq 28$ from $x = -0{\cdot}3$ to $x = 0$ so that any value of x in this range makes y very nearly equal to its maximum.

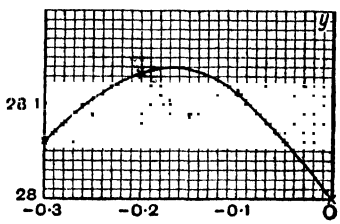

FIG. 6.

Example.—The time of a small oscillation, T sec., of a rod of length 120 cm. swinging in a vertical plane about an axis at h cm. from its middle point is given by

$$T = 2\pi\sqrt{\frac{h^2 + 1200}{gh}},$$

where $g = 981$. Draw a graph to show the variation of T with h from $h = 0$ to $h = 60$. Find from it the value of h for which $T = 2$, and the minimum value of T.

Putting $g = 981$,

$$T = \frac{2\pi}{\sqrt{981}}\sqrt{\frac{h^2 + 1200}{h}} \simeq 0{\cdot}2\sqrt{h + \frac{1200}{h}}.$$

GRAPH TO SHOW THE TIME OF OSCILLATION OF A ROD

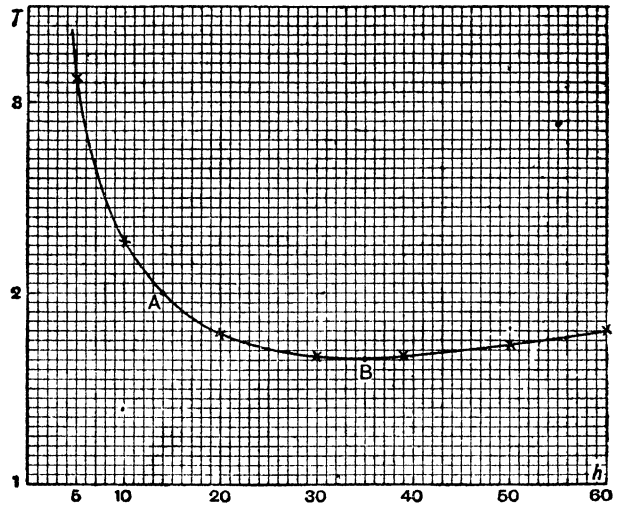

FIG. 7.

The table below shows how T is calculated from h. The values of h are taken at intervals of 10 cm., except the first value, which is taken at $h=5$ because T is infinite when h is zero.

h	5	10	20	30	40	50	60
$\frac{1200}{h}$	240	120	60	40	30	24	20
$h+\frac{1200}{h}$	245	130	80	70	70	74	80
$\sqrt{h+\frac{1200}{h}}$	15·65	11·40	8·94	8·37	8·37	8·60	8·94
T	3·13	2·28	1·79	1·67	1·67	1·72	1·79

Fig. 7 shows the graph of T against h. It shows that $T=2$ when $h \simeq 14$, and that the minimum value of T is approximately 1·66 and it occurs at $h \simeq 35$.

Straight line graphs. Gradient of a line

Consider the graph of $y = 2x + 3$. The table below gives the values of y from $x = -3$ to $x = 3$.

x ..	..	-3	-2	-1	0	1	2	3
y ..	..	-3	-1	1	3	5	7	9

GRAPH OF $y=2x+3$

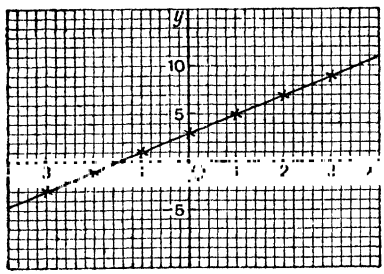

FIG. 8.

GRAPH OF $y=2x+3$ WITH THE SAME SCALE ON BOTH AXES

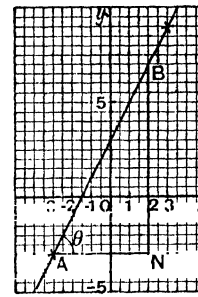

FIG. 9.

Thus y increases by 2 whenever x increases by 1. Hence the graph of the equation is a straight line as shown in Fig. 8. $y = 2x + 3$ is called the equation of the line. The ratio (increase of y) : (increase of x) is 2 whatever pair of points is taken on the line. For instance, from the table above, two points on the line are A given by $x = -3$, $y = -3$, called the point $(-3, -3)$, and B given by $x = 2$, $y = 7$, called the point $(2, 7)$. From A to B the increase of y is $7 - (-3) = 10$, and the increase of x is $2 - (-3) = 5$.

$$\therefore \frac{\text{increase of } y}{\text{increase of } x} = \frac{10}{5} = 2.$$

In mathematics this ratio is called the *gradient of the line.*

Fig. 9 shows the graph of $y = 2x + 3$, using the same scale on both axes.

When the scales on the two axes are the same the gradient is the ratio of the actual lengths of the vertical rise NB to the horizontal distance AN. If the line makes an angle θ with the horizontal, $\tan\theta = \frac{NB}{AN}$, and so :

Gradient of the line $= \tan\theta$, *when the scales on the axes are the same.*

In Fig. 9, $\tan\theta = 2$, and hence $\theta \simeq 63° 26'$.

In general the graph of $y = ax + b$, where a and b are given numbers, is a straight line of gradient a. Since $y = b$ when $x = 0$, the constant b is the intercept on Oy measured from O to the point where the line cuts Oy. If the line cuts the y axis below O, then b is negative.

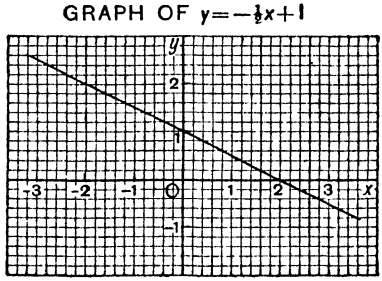

FIG. 10.

If a is negative the gradient of the line is still a. For instance, in the equation $y = -\frac{1}{2}x + 1$, when x increases by 1, y decreases by $\frac{1}{2}$, so we say that y increases by $-\frac{1}{2}$.

$$\therefore \frac{\text{increase of } y}{\text{increase of } x} = \frac{-\frac{1}{2}}{1} = -\frac{1}{2}.$$

Hence the gradient of the line is $-\frac{1}{2}$. The graph of this line is shown in Fig. 10. It is clear that lines with a positive gradient slope upwards to the right and lines with a negative gradient slope downwards to the right.

Any equation like $5x+4y=8$, in which the terms in x and y are of the first degree, has a straight line graph, for it can be written

$$4y=8-5x$$

$$\text{i.e. } y=2-1{\cdot}25x,$$

which is the straight line of gradient $-1{\cdot}25$ shown in Fig. 11.

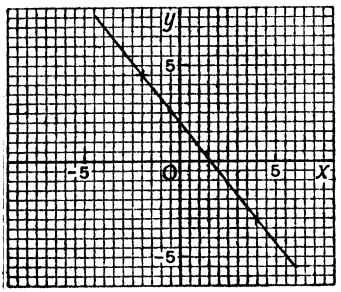

FIG. 11.

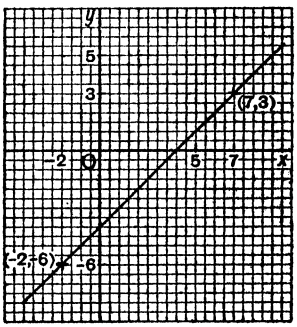

FIG. 12.

Equation of the line through two points

If the graph of $y=ax+b$ passes through the two points $(-2,\ -6)$, $(7,\ 3)$ [Fig. 12], then $y=-6$ when $x=-2$ and $y=3$ when $x=7$.

$$\therefore\ -6=-2a+b$$

and

$$3=7a+b$$

Subtracting

$$-9=-9a$$

$$\therefore\ a=1 \text{ and } b=3-7a=3-7=-4.$$

Hence the required equation is $y=x-4$.

Exercise VII

Draw the graphs of the following equations for the given range of values and use the graphs to answer the questions :

1. $y = 12x^2 - 18x - 7$ from $x = -2$ to 4. Find the minimum value of y and the values of x which make $y = 10$.

2. $R = 1 + 6v - 2v^3$ from $v = -2$ to 2. Find the maximum and minimum values of R and the values of v which make $R = 0$.

3. $y = x^3 + 4x - 8$ from $x = -2$ to 3. Find to 3 sig. fig. the values of x which make $y = 0$ and $y = -10$ respectively.

4. $t = m^2 + \frac{7}{m}$ from $m = 0$ to 4. Find to 3 sig. fig. the minimum value of t and the value of m for which t is a minimum.

5. $z = y^3 - 40y$ from $y = -5$ to 5. Find the maximum and minimum values of z. If P is a point on the graph and PQ is bisected by the origin, prove that Q is also a point on the graph.

6. $y = \frac{10}{x^2 + 4} - x$ from -2 to 3. Find to 3 sig. fig. the values of x which make $y = 0$ and 2 respectively.

7. $p = 8\sqrt{r} - 3r$ from $r = 0$ to 5. Find the maximum value of p to 3 sig. fig. Has p a graph for negative values of r ?

8. $y = 4x^3 - 3x^2 - 14x + 4$ from $x = -2$ to 3. Find the values of x which make $y = 0$, giving the larger positive one to 3 sig. fig.

9. $y = \frac{2}{3}\sqrt{9 - x^2}$ and $y = -\frac{2}{3}\sqrt{9 - x^2}$ from $x = -3$ to $+3$. Show that these two graphs together make up the graph of $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

10. $y = \frac{1}{x}$ from $x = -4$ to $+4$, taking values of x at intervals of $\frac{1}{4}$. Calculate the values of y when $x = -0{\cdot}01$, $-0{\cdot}001$, $0{\cdot}001$, $0{\cdot}01$. What happens to the graph as x increases through the value 0 ?

11. $y = \sin\theta^\circ + 2\cos\theta^\circ$ from $\theta = 0$ to $\theta = 90$. Find the maximum value of y to 3 sig. fig., and the value of θ (to the nearest degree) which makes $y = 1{\cdot}5$.

Plot the graphs of the following functions :

12. $2x^2$ from $x = -3$ to 3. **13.** $\frac{1}{x^2}$ from $x = -3$ to 3.

14. $t(6-t)$ from $t=-2$ to 8. **15.** sin (θ radians) from $\theta=0$ to 2π.

16. $4m^3-16m^2$ from $m=-2$ to 2. **17.** 2^x from $x=-3$ to 3.

18. $\log_{10}x$ from $x=\frac{1}{4}$ to 4. **19.** $\tan x^\circ$ from $x=0$ to 180.

Draw the straight lines which are the graphs of the following equations and state the gradient of each line and the intercept it makes on the vertical (y, C or V) axis.

20. $y=5-3x$. **21.** $2y+x=8$. **22.** $3y-4x=7$.

23. $5y=10x-12$. **24.** $C=\frac{5}{9}(F-32)$. **25.** $V=200-0{\cdot}8I$.

Find the equations of the straight lines joining the following pairs of points :

26. (2, 3), (8, 11). **27.** (6, 4), (1, 12). **28.** (−2, 5), (3, 0).

29. When a beam is clamped horizontally at one end the deflection y of the other end, which is distant l from the clamp, is given by $y=\frac{Wl^3}{3EI}$, where W is the load hung from that end. Draw a graph of y from $l=10$ to $l=100$ when $W=2$, $E=20\times10^6$, $I=0{\cdot}0012$.

30. Two wires containing resistances 10 ohms and R ohms are placed in parallel and a voltage V is applied to their ends. If the current through the resistance R ohms is 15 ampères, $V=\frac{150R}{R+10}$. Draw a graph of V against R from R = 1 to R = 50.

31. The formula $H=\frac{7{\cdot}35V}{V+10}-0{\cdot}13$ has been used to find the percentage, H, of hydrogen in coal when the percentage of volatile matter is V. V ranges from 10 to 50. Draw a graph of H against V for this range of values. From it find the value of V which makes H = 5 and check your answer by calculation.

32. Draw the graphs of the equations $\frac{x}{3}+\frac{y}{4}=1$, and $y=x^2-1$ on the same axes and find the co-ordinates of the points where they intersect.

33. For an L.M.S. train of one locomotive and twelve coaches the resistance r lb. wt. per ton at a speed of V m.p.h. is given by

$$r=5{\cdot}25+0{\cdot}0513V+0{\cdot}00162V^2.$$

Draw a graph of r from V = 0 to V = 80. Determine at what speed $r=10$ and at what speed r is double its value at 30 m.p.h.

34. For temperatures between 700° C. and 3000° C. the resistivity of tungsten, r microhms per cubic cm. at $t°$ C., is given by $r = 4{\cdot}053 + 0{\cdot}0275t + 0{\cdot}00000175t^2$. Draw a graph of r against t from $t = 700$ to $t = 3000$ and find what value of t makes $r = 47$.

35. A load of 200 lb. wt. is suspended by two equal ropes, length l ft., from two points 6 ft. apart at the same level. The tension T lb. wt. in each rope is then given by $T = \frac{100l}{\sqrt{l^2 - 9}}$. Plot a graph of T from $l = 4$ to $l = 10$. For what value of l is T equal to 150 ?

36. For a certain material the Brinell hardness number H is given in terms of the diameter of indentation d by the formula $H = \frac{W}{\frac{\pi D}{2}(D - \sqrt{D^2 - d^2})}$. It can be shown that this formula is approximately the same as $H = \frac{W}{\pi}\left(\frac{4}{d^2} - \frac{1}{D^2}\right)$ when d/D is small. Draw graphs of both these formulæ from $d = 1$ to 9 when $W = 3000$ and $D = 10$.

37. When a square coil of 4 turns, each side 40 cm., carries a current of 12 ampères per turn, the magnetic force H at a point on the diagonal at x cm. from the centre is given by

$$H = \frac{19{\cdot}2}{800 - x^2}(20\sqrt{2} + \sqrt{800 + x^2}).$$

Draw a graph to show how H varies as x varies from -25 to 25.

CHAPTER II

FACTORS AND FRACTIONS

Factors

The first step in factorizing an expression is to find the highest common factor of its terms.

Example.—Factorize $4l^2x^2 - 2lx^3$.

H.C.F. of $4l^2x^2$ and $2lx^3$ is $2lx^2$.

$\therefore 4l^2x^2 - 2lx^3 = 2lx^2 \times 2l - 2lx^2 \times x$

$= 2lx^2(2l - x)$.

Example.—Factorize $(a+1)(2a+3)-(a+1)(b+3)$.

$(a+1)$ is a factor of both products and hence of the whole expression.

$$\therefore \text{ expression} = (a+1)\{(2a+3)-(b+3)\}$$
$$= (a+1)\{2a+3-b-3\}$$
$$= (a+1)(2a-b).$$

In many cases an expression is the product of factors, although its terms have no common factor. Such an expression can often be factorized by grouping its terms.

Example.—Factorize $a^2-2lm-2al+am$.

$$\text{Expression} = a^2-2al-2lm+am$$
$$= a(a-2l)+m(a-2l)$$
$$= (a+m)(a-2l).$$

Factors of a^2-b^2.

We have seen on p. 7 that

$$a^2-b^2=(a+b)(a-b).$$

This formula can be used to factorize the difference of any two squares.

Example.—Factorize $9x^2-4y^2$.

$$9x^2-4y^2=(3x)^2-(2y)^2$$
$$=(3x+2y)(3x-2y)$$

by putting $a=3x$ and $b=2y$ in the formula above.

Example.—Factorize l^4-m^4.

$$l^4-m^4=(l^2)^2-(m^2)^2=(l^2+m^2)(l^2-m^2)=(l^2+m^2)(l+m)(l-m).$$

The calculation of a^2-b^2 when a is very nearly equal to b.

Suppose a and b are 12·00544 and 12 respectively. Then all we can get using four figure tables of squares is:

$$a^2-b^2=(12\cdot01)^2-12^2=144\cdot2-144=0\cdot2.$$

This answer is not even correct to the one significant figure 2, and so we proceed as follows :

$$\begin{aligned} a^2 - b^2 &= (a-b)(a+b) \\ &= (12{\cdot}00544 - 12)(12{\cdot}00544 + 12) \\ &= 0{\cdot}00544 \times 24{\cdot}00544 \\ &\simeq 0{\cdot}00544 \times 24 \\ &\simeq 0{\cdot}131. \end{aligned}$$

The error in taking 24 instead of 24·00544 is less than 1 in 4000 and so will not affect the three figures 131. To get the same accuracy by squaring a and b it would be necessary to find a^2 to seven significant figures either from a larger table of squares or by ordinary multiplication.

Factors of x^2+px+q.

Let $x^2+8x+12$ be the product of $(x+a)$ and $(x+b)$. Since

$$(x+a)(x+b) = x^2 + (a+b)x + ab,$$

the product is $x^2+8x+12$, if

$$a+b=8, \text{ and } ab=12.$$

Therefore to find the two factors we have to find two numbers whose product is 12 and whose sum is 8. To do this we write out a list of the possible factors of 12, namely, 12×1, 6×2, 4×3 and pick out the pair whose sum is 8, in this case 6×2.

$$\therefore\ x^2+8x+12 = (x+6)(x+2).$$

This work can be set out in the following way :

$$x^2+8x+12 = \begin{matrix} x+12, & 6, & 4 \\ \times & & \\ x+1, & 2, & 3 \end{matrix}$$

Exercise VIII

Factorize :

1.	(a) a^2+ab.	(b) x^2-xy.	(c) a^2-h^2.
2.	(a) $2pq-3q^2$.	(b) p^2-q^2.	(c) $4l^2-m^2$.
3.	(a) x^2-9.	(b) $121-r^2$.	(c) 121^2-119^2.

4. (*a*) $9c^2-16r^2$. (*b*) $8k^2-200$. (*c*) $\frac{1}{4}a^2-r^2$.

5. (*a*) $p^2q^2-a^2b^2$. (*b*) $p^2q^2-p^2b^2$. (*c*) $20p^2q^2-45a^2b^2$.

6. (*a*) $(2x-1)^2-(x-2)^2$. (*b*) $(l-x)^2-(l-x)(l+x)$.

7. (*a*) l^4-16. (*b*) l^4-16l^2. (*c*) $(a+2r)^2-(a-r)^2$.

8. (*a*) $16x^4-81y^4$. (*b*) $(a+b+c)^2-(a+b-c)^2$.

9. (*a*) $\frac{t^2}{T^2}-1$; (*b*) $r^{2n}-1$; (*c*) $k^2-\frac{9}{m^2}$.

Calculate by using factors :

10. (*a*) 28^2-23^2. (*b*) 65^2-63^2. (*c*) $9{\cdot}87^2-9{\cdot}81^2$.

11. (*a*) 152^2-148^2. (*b*) 289^2-286^2. (*c*) 7594^2-7590^2.

12. Find the value of p^2-q^2 to 3 sig. fig. when $p=150{\cdot}0237$ and $q=150$.

13. Find the value of $\omega^2-\omega_0^2$ to 3 sig. fig. when $\omega_0=2\times10^5$ and $\omega-\omega_0=12{,}600$.

14. Two concentric circles have radii r_1 and r_2. If C is the length of the circumference of a circle whose radius is the mean of r_1 and r_2 and t is the distance between the circles, show that the area between them is Ct.

Factorize :

15. (*a*) $ax-2x-a+2$. (*b*) $l^2+lm+ln+mn$.

16. (*a*) $ka-kb+lb-la$. (*b*) $t^2-3st-3s+t$.

17. (*a*) $(a+b)^2-c^2$. (*b*) $a^2-2ab+b^2-c^2$.

18. (*a*) $a^2-b^2+2bc-c^2$. (*b*) $4k^2+4kl+l^2-9m^2$.

19. (*a*) x^2+5x+4. (*b*) x^2-5x+4. (*c*) x^2-3x-4.

20. (*a*) $x^2+23x+90$. (*b*) $x^2-23x+90$. (*c*) $x^2+13x-90$.

21. (*a*) $p^2+16p+63$. (*b*) $p^2-2p-63$. (*c*) $p^2+2p-63$.

22. (*a*) $y^2+8y+15$. (*b*) $y^2+2y-15$. (*c*) $y^2-2y-15$.

23. (*a*) x^2+3x-4. (*b*) $x^2+3x-28$. (*c*) $x^2-3x-28$.

24. (*a*) l^2+l-6. (*b*) l^2-l-6. (*c*) l^2-7l+6.

25. (*a*) $m^2-5m-14$. (*b*) $m^2+5m-14$. (*c*) $m^2-15m+14$.

26. (*a*) $p^2+3pq+2q^2$. (*b*) $p^2-3pq+2q^2$. (*c*) $p^2-pq-2q^2$.

27. (*a*) $l^2+lm-2m^2$. (*b*) $l^2-lm-2m^2$. (*c*) $l^2-lm-30m^2$.

28. (*a*) $p^2q^2-5pq+6$. (*b*) $p^2-5pq+6q^2$. (*c*) p^2q^2-pq-6.

Simplify, using factors :

29. $x^2(y-x)+x(y-x)^2$. 30. $(a-x)^2-(a^2-x^2)$.

31. $(l+n)(l^2-x^2)-(l-x)(l^2-n^2)$. 32. $(r+2t+1)^2-(r-3t+1)^2$.

Factorize :

33. (*a*) $1-\cos^2\theta$. (*b*) $\cos^2 A-\sin^2 A$.
(*c*) $(3\cos A+2\sin A)^2-(\cos A+\sin A)^2$.

34. (*a*) $4\tan^2\theta-1$. (*b*) $1+\sin\theta+2\cos\theta+2\sin\theta\cos\theta$.
(*c*) $\cos^4 x-\sin^4 x$.

Factors of ax^2+bx+c

$$(3x+2)(2x+1)=3\times 2x^2+(3\times 1+2\times 2)x+2\times 1$$
$$=6x^2+7x+2.$$

Now consider the reverse process of finding the factors of $6x^2+7x+2$.

Let $6x^2+7x+2=(ax+b)(cx+d)$
$$=acx^2+(ad+bc)x+bd.$$

The expression on the right-hand side is the same as the expression on the left if

$$ac=6,\ (ad+bc)=7 \text{ and } bd=2.$$

If we write the letters a, b, c, d in the form $\begin{matrix} a & \nearrow & b \\ c & \searrow & d \end{matrix}$ then the product of the two left-hand letters has to be 6, the product of the two right-hand letters has to be 2, and the sum of the cross products indicated by the arrow heads has to be 7. Now $6=6\times 1$ and 3×2, and $2=2\times 1$. Therefore, the scheme $\begin{matrix} a & \searrow & b \\ c & \nearrow & d \end{matrix}$ can be written in the following ways :

$$\begin{matrix} 6 & \nearrow & 2 \\ 1 & \searrow & 1 \end{matrix};\quad \begin{matrix} 6 & \nearrow & 1 \\ 1 & \searrow & 2 \end{matrix};\quad \begin{matrix} 3 & \nearrow & 2 \\ 2 & \searrow & 1 \end{matrix};\quad \begin{matrix} 3 & \nearrow & 1 \\ 2 & \searrow & 2 \end{matrix}$$

We examine each of these to see in which one the sum of the cross products is 7. In the first it is $6+2=8$, in the second, $12+1=13$, in the third, $3+4=7$. Hence the third arrangement is the correct one, that is, $a=3$, $b=2$, $c=2$ and $d=1$.

$\therefore\ 6x^2+7x+2=(3x+2)(2x+1)$.

The above schemes of factors can be written more compactly

$$\begin{matrix} 6, 3 \\ 1, 2 \end{matrix} \times \begin{matrix} 2, 1 \\ 1, 2 \end{matrix}$$

When some of the terms are negative we proceed in the same way except that in this case either b or d or both will be negative numbers.

Example.—Factorize $5t^2 - 8t - 4$.

Let $5t^2 - 8t - 4 = (at + b)(ct + d)$

$$= act^2 + (ad + bc)t + bd$$

Then $ac = 5$, $bd = -4$ and $ad + bc = -8$.

Therefore, a and c must be 5 and 1, and b and d can be any one of the combinations 4 and -1, -4 and 1, 1 and -4, -1 and 4, 2 and -2, -2 and 2. We write these :

$$\begin{matrix} 5 \\ 1 \end{matrix} \times \begin{matrix} 4, & -4, & 1, & -1, & 2, & -2. \\ -1, & 1, & -4, & 4, & -2, & 2. \end{matrix}$$

Working out the cross products and adding them we get in turn $-5 + 4 = -1$, $5 - 4 = 1$, $-20 + 1 = -19$, $20 - 1 = 19$, $-10 + 2 = -8$. The last one gives the correct coefficient of t. Hence $a = 5$, $c = 1$, $b = 2$, $d = -2$.

$\therefore 5t^2 - 8t - 4 = (5t + 2)(t - 2)$.

The factors of $5t^2 - 8th - 4h^2$ are found in the same way. For if $5t^2 - 8th - 4h^2 = (at + bh)(ct + dh)$, $ac = 5$, $bd = -4$, $ad + bc = -8$ as above.

$\therefore 5t^2 - 8th - 4h^2 = (5t + 2h)(t - 2h)$.

Division

We can often determine whether one expression divides exactly into another by factorizing the dividend. Thus

$$\frac{x^2 - x - 12}{x - 4} = \frac{(x - 4)(x + 3)}{x - 4} = x + 3$$

$\therefore x - 4$ divides exactly $x + 3$ times into $x^2 - x - 12$.

The division can however be carried out by the method used

in long division in arithmetic, after first arranging both dividend and divisor in descending powers of x.

$$\begin{array}{r|l} x-4 & x^2-x-12 \\ & x^2-4x \\ \hline & 3x-12 \\ & 3x-12 \\ \hline & 0 \end{array} \quad x+3$$

The first step is to divide x into x^2; the quotient is x and so we multiply the divisor by x, giving x^2-4x. Subtracting x^2-4x from the dividend the remainder is $3x-12$. This step shows that

$$\begin{aligned} x^2-x-12 &= x^2-4x+3x-12 \\ &= x(x-4)+3x-12. \end{aligned}$$

Now $x-4$ divides exactly 3 times into $3x-12$.

$$\begin{aligned} \therefore x^2-x-12 &= x(x-4)+3(x-4) \\ &= (x+3)(x-4), \end{aligned}$$

or $$\frac{x^2-x-12}{x-4} = x+3.$$

If the dividend is x^2-x-10, the same method gives,

$$\begin{array}{r|l} x-4 & x^2-x-10 \\ & x^2-4x \\ \hline & 3x-10 \\ & 3x-12 \\ \hline & 2 \end{array} \quad x+3.$$

Thus there is a remainder 2. The steps in the division show that:

$$\begin{aligned} x^2-x-10 &= x^2-4x+3x-10 \\ &= x(x-4)+3x-12+2 \\ &= x(x-4)+3(x-4)+12 \\ &= (x+3)(x-4)+2, \end{aligned}$$

or $$\frac{x^2-x-10}{x-4} = x+3+\frac{2}{x-4}.$$

Example.—Divide $4t^3+6t^2-7t+9$ by $2t^2-3t-1$.

$$\begin{array}{l|l|l} 2t^2-3t-1 & 4t^3+6t^2-7t+9 & 2t+6 \\ & \underline{4t^3-6t^2-2t} & \\ & 12t^2-5t+9 & \\ & \underline{12t^2-18t-6} & \\ & 13t+15 & \end{array}$$

We stop at this stage because $2t^2$ will not divide into $13t$ (without introducing negative powers of t). Hence the quotient is $2t+6$ and the remainder is $13t+15$. The division shows that:

$$4t^3+6t^2-7t+9=(2t^2-3t-1)(2t+6)+13t+15.$$

Factors of a^3-b^3 and of a^3+b^3

By division

$$\begin{array}{l|l|l} a-b & a^3-b^3 & a^2+ab+b^2 \\ & \underline{a^3-a^2b} & \\ & a^2b-b^3 & \\ & \underline{a^2b-ab^2} & \\ & ab^2-b^3 & \\ & \underline{ab^2-b^3} & \end{array}$$

$$\therefore\ a^3-b^3=(a-b)(a^2+ab+b^2).$$

Similarly, $a^3+b^3=(a+b)(a^2-ab+b^2)$.

Example.—Factorize $8x^3-27y^3$.

$$\begin{aligned} 8x^3-27y^3 &= (2x)^3-(3y)^3 \\ &= (2x-3y)\{(2x)^2+2x\times 3y+(3y)^2\} \\ &= (2x-3y)(4x^2+6xy+9y^2). \end{aligned}$$

Exercise IX

Factorize:

1. $2x^2+3x+1$.	**2.** $3t^2+7t+4$.	**3.** $3t^2+8t+4$.
4. $4y^2-7y-2$.	**5.** $2m^2-3mn-2n^2$.	**6.** $4x^2-15x-4$.
7. $6z^2-z-2$.	**8.** $3l^2-5l-2$.	**9.** $3+8k+5k^2$.

Divide by factorizing and by long division :

10. x^2-6x-7 by $x+1$. 11. $3y^2+5y+2$ by $3y+2$.

12. $a^2-7ab-8b^2$ by $a-8b$. 13. $8p^2-10pq-3q^2$ by $2p-3q$.

Find the quotient and remainder when :

14. x^3-2x-1 is divided by $x-1$.

15. $6x^3-7x^2+5x-7$ is divided by $2x-3$.

16. $4y^3+y^2+2y-2$ is divided by y^2-2.

17. t^3-2t+5 is divided by $3t-1$.

18. Prove $\dfrac{x^2+4x+5}{x+1}=x+3+\dfrac{2}{x+1}$.

19. Prove $\dfrac{y^2-7}{y+1}=y-1-\dfrac{6}{y+1}$.

Factorize :

20. y^3-1. 21. x^3-a^3. 22. $27+p^3$. 23. $20l^3+160$.

24. Divide $x-l$ into $x^3-2lx^2+l^3$. The deflection at a point of a beam is proportional to $x^4-2lx^2+l^3x$. Use your first result to factorize this expression.

25. Prove that $(\alpha^2+\beta^2)^2=(\alpha^2-\beta^2)^2+4\alpha^2\beta^2$.

If $\alpha^2-\beta^2=GR-LC\omega^2$ and $2\alpha\beta=(GL+RC)\omega$ prove that

$$\alpha^2+\beta^2=\sqrt{(R^2+L^2\omega^2)(G^2+C^2\omega^2)}.$$

26. Assuming that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, prove that the volume between two concentric spheres of radii R and r is $\frac{4}{3}\pi(R-r)(R^2+Rr+r^2)$ and show that, when R is nearly equal to r, this volume is nearly $4\pi r^2(R-r)$.

A spherical balloon has a thickness of 0·005 in. when its radius is 1 ft. Calculate the volume of the material of which it is made. How thick will it be when its radius is 5 ft. ?

Simplification of fractions

A fraction can often be simplified by finding the factors of the denominator and numerator and then dividing both the denominator and numerator by every factor common to both.

Example.—Simplify (i) $\dfrac{a^2bc}{abc^2}$; (ii) $\dfrac{2k^2-k}{k^2+3k}$; (iii) $\dfrac{x^2y+2xy}{x^2-4}$.

(i) $\dfrac{a^2bc}{abc^2}=\dfrac{a\cancel{(abc)}}{c\cancel{(abc)}}=\dfrac{a}{c}$.

(ii) $\dfrac{2k^2-k}{k^2+3k}=\dfrac{k(2k-1)}{k(k+3)}=\dfrac{2k-1}{k+3}$.

(iii) $\dfrac{x^2y+2xy}{x^2-4}=\dfrac{xy(x+2)}{(x+2)(x-2)}=\dfrac{xy}{x-2}$.

Example.—Simplify (i) $\dfrac{a^2-r^2}{a-2r}\div\dfrac{a+r}{a^2-2ar}$; (ii) $\dfrac{x^2-2x-3}{2x^2-x-3}$.

(i) $\dfrac{a^2-r^2}{a-2r}\div\dfrac{a+r}{a^2-2ar}=\dfrac{a^2-r^2}{a-2r}\times\dfrac{a^2-2ar}{a+r}$.

$$=\frac{(a+r)(a-r)a(a-2r)}{(a-2r)(a+r)}$$
$$=(a-r)a$$
$$=a^2-ar$$

(ii) $x^2-2x-3=(x-3)(x+1)$.

$2x^2-x-3=(2x-3)(x+1)$. $\begin{bmatrix}2\diagdown\diagup -3, & 3, & -1, & 1.\\ 1\diagup\diagdown 1, & -1, & 3, & -3.\end{bmatrix}$

$\therefore\ \dfrac{x^2-2x-3}{2x^2-x-3}=\dfrac{(x-3)(x+1)}{(2x-3)(x+1)}=\dfrac{x-3}{2x-3}$.

L.C.M. of more than two expressions

Example.—Find the L.C.M. of x^2-x-2, $2x^2-x-6$, $2x^2+5x+3$.

$$x^2-x-2=(x-2)(x+1)$$
$$2x^2-x-6=(2x+3)(x-2)$$
$$2x^2+5x+3=(2x+3)(x+1)$$

Hence the lowest common multiple into which each of the three expressions will divide is :

$$(x-2)(x+1)(2x+3).$$

Addition and subtraction of fractions

We have seen on p. 8 that fractions are added or subtracted by bringing them to a common denominator, which is the L.C.M. of their denominators. Some harder examples are given below.

Example.—Express $\frac{3}{x+1} - \frac{6}{x+2}$ as a single fraction.

L.C.M. of denominators is $(x+1)(x+2)$.

$$\therefore \frac{3}{x+1} - \frac{6}{x+2} = \frac{3(x+2)}{(x+1)(x+2)} - \frac{6(x+1)}{(x+1)(x+2)}$$

$$= \frac{3x+6-6x-6}{(x+1)(x+2)}$$

$$= -\frac{3x}{(x+1)(x+2)}.$$

Example.—Simplify $\frac{t+3}{t^2-t} - \frac{2t}{2t^2-3t+1}$.

$$t^2 - t = t(t-1),$$
$$2t^2 - 3t + 1 = (2t-1)(t-1).$$

$\therefore$ L.C.M. of denominators $= t(t-1)(2t-1)$.

$$\therefore \text{expression} = \frac{(2t-1)(t+3)}{t(t-1)(2t-1)} - \frac{2t^2}{t(t-1)(2t-1)}$$

$$= \frac{2t^2+5t-3-2t^2}{t(t-1)(2t-1)}$$

$$= \frac{5t-3}{t(t-1)(2t-1)}.$$

Example.—Simplify $\dfrac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{b^2}-\frac{1}{a^2}} \times \dfrac{\frac{1}{ab}+\frac{1}{b^2}}{2a}$

$$\text{Expression} = \frac{\frac{a-b}{ab}}{\frac{a^2-b^2}{a^2b^2}} \times \frac{\frac{b+a}{ab^2}}{2a}$$

$$= \frac{(a-b)a^2b^2}{ab(a^2-b^2)} \times \frac{(a+b)}{2a^2b^2}$$

$$= \frac{(a-b)(a+b)}{2ab(a^2-b^2)}$$

$$= \frac{1}{2ab}, \text{ since } a^2-b^2=(a+b)(a-b).$$

In an example of this type it is important to note the difference between the horizontal lines in the fraction $\dfrac{\dfrac{b+a}{ab^2}}{2a}$. This expression with the bottom line longer than the upper line means $\dfrac{b+a}{ab^2} \div 2a$, i.e. $\dfrac{b+a}{2a^2b^2}$, but the expression $\dfrac{b+a}{\dfrac{ab^2}{2a}}$ means $(b+a) \div \dfrac{ab^2}{2a}$, which is $\dfrac{2a(b+a)}{ab^2}$.

The same method is used in simplifying expressions containing fractional indices, but each term should be expressed without negative indices before carrying out the addition or subtraction.

Example.—Simplify $(r^2+a^2)^{-\frac{3}{2}} - 3r^2(r^2+a^2)^{-\frac{5}{2}}$.

$$\text{Expression} = \frac{1}{(r^2+a^2)^{\frac{3}{2}}} - \frac{3r^2}{(r^2+a^2)^{\frac{5}{2}}}.$$

Since $(r^2+a^2)^{\frac{5}{2}} = (r^2+a^2)^{\frac{3}{2}} \times (r^2+a^2)$, it is the L.C.M. of the denominators.

$$\therefore \text{ expression} = \frac{r^2+a^2}{(r^2+a^2)^{\frac{5}{2}}} - \frac{3r^2}{(r^2+a^2)^{\frac{5}{2}}}$$

$$= \frac{a^2-2r^2}{(r^2+a^2)^{\frac{5}{2}}}.$$

Equations involving fractions

Such equations are simplified by multiplying each term by the L.C.M. of the denominators.

Example.—Solve the equation $\frac{4}{x(x+3)} = \frac{1}{x^2+2x-3}$.

$$x^2+2x-3 = (x+3)(x-1).$$

$\therefore$ L.C.M. of denominators $= x(x+3)(x-1)$.

Multiplying each term of the equation by this L.C.M.,

$$\frac{4\cancel{(x+3)}(x-1)}{x\cancel{(x+3)}} = \frac{x\cancel{(x+3)}\cancel{(x-1)}}{\cancel{(x+3)}\cancel{(x-1)}}$$

$$\therefore 4(x-1) = x$$

$$\therefore 4x-4 = x$$

$$\therefore 3x = 4$$

$$\therefore x = \tfrac{4}{3}$$

Example.—Express r as the subject of the formula:

$$\frac{1}{x+r} + \frac{1}{y+r} = \frac{2}{x+y+r}.$$

Multiplying by the L.C.M., viz. $(x+r)(y+r)(x+y+r)$,

$$(y+r)(x+y+r) + (x+r)(x+y+r) = 2(x+r)(y+r)$$

$$\therefore xy+y^2+yr+xr+yr+r^2+x^2+xy+xr+xr+yr+r^2 = 2xy+2xr+2yr+2r^2$$

$$\therefore 3yr+3xr+x^2+y^2 = 2xr+2yr$$

$$\therefore xr+yr = -(x^2+y^2)$$

$$\therefore r(x+y) = -(x^2+y^2)$$

$$\therefore r = -\frac{x^2+y^2}{x+y}.$$

Exercise X

Simplify:

1. $\frac{ab^2x}{ax^2}$.
2. $\frac{16lmn^2}{12l^2m^2}$.
3. $\frac{\frac{8}{15}\pi r^5}{\frac{4}{5}\pi r^3}$.
4. $\frac{(6xr)^3}{(3xr^2)^2}$.
5. $\frac{x^2-xy}{x^2+xz}$.
6. $\frac{p^2-4}{4p-8}$.

7. $\dfrac{a^2-ab}{ac-bc}$.

8. $\dfrac{x^2-9}{x^2-2x-3}$.

9. $\dfrac{r^2-s^2}{(r+s)^2}$.

10. $\dfrac{p-q}{p^2+pq-2q^2}$.

11. $\dfrac{z^2-2z-3}{z^2+z-12}$.

12. $\dfrac{x^2-4x}{x^2+x-20}$.

13. $\dfrac{y^2-4}{y+3}\times\dfrac{y^2-9}{y-2}$.

14. $\dfrac{(ac-2c^2)^2}{a^2-ac-2c^2}\times\dfrac{a+c}{a^2-4c^2}$.

15. $\dfrac{x+3}{x+2}\div\dfrac{x+1}{x+2}$.

16. $\dfrac{r^2-x^2}{r^2x}\div\dfrac{rx+x^2}{r^3x^2}$.

17. $\dfrac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$.

18. $\left(\dfrac{2}{a}-\dfrac{3}{b}\right)\div\left(\dfrac{a}{2}-\dfrac{b}{3}\right)$.

19. $\dfrac{\frac{1}{r}+\frac{1}{s}}{\frac{1}{r}+s}\times\dfrac{r^2s^2-1}{r^2-s^2}$.

20. $\dfrac{(m-n)^2+4mn}{(m+1)^2-(n-1)^2}$.

21. $\dfrac{x-3}{2x^2-7x+3}$.

22. $\dfrac{3y^2+4y}{3y^2+y-4}$.

23. $\dfrac{m^2-2mn+n^2}{2m^2+mn-3n^2}$.

24. $\dfrac{2l^2-3l+1}{l^2-3l+2}$.

Express as single fractions:

25. $\dfrac{1}{x+3}-\dfrac{1}{x+4}$.

26. $\dfrac{1}{t}+\dfrac{2}{3-t}$.

27. $\dfrac{4}{3ab}-\dfrac{5}{6bc}$.

28. $\dfrac{m^2+2}{m^2+m}-\dfrac{m-2}{m}$.

29. $\dfrac{1}{l-1}-\dfrac{l+1}{l^2-l}$.

30. $\dfrac{3-y}{1-3y}-\dfrac{3+y}{1+3y}$.

31. $\dfrac{2a}{a+b}+\dfrac{2b}{a-b}+\dfrac{a^2-2b^2}{a^2-b^2}$.

32. $\dfrac{1}{y}-\dfrac{2}{y^2+2y}$.

33. $\dfrac{1}{(r-s)^2}-\dfrac{1}{(r+s)^2}$.

34. $\dfrac{5x+4}{x^2-2x}-\dfrac{7}{x-2}$.

35. $\dfrac{z}{2z^2-z-1}-\dfrac{1}{2z+1}$.

36. $\dfrac{p+q}{p-2q}-\dfrac{p-q}{p+2q}$.

37. $\dfrac{1}{(x-1)(x-2)} - \dfrac{1}{(x-2)(x+3)}$. **38.** $\dfrac{2}{x^2-x} - \dfrac{4}{2x^2-3x+1}$.

39. $\dfrac{1}{t^2-2t-15} - \dfrac{1}{3t^2+10t+3}$. **40.** $\dfrac{x^3+y^3}{x^4-y^4} - \dfrac{1}{x-y}$.

Simplify the following, if possible :

41. (*a*) $\dfrac{x^2+xy}{x+y}$. (*b*) $\dfrac{x^2+xy}{x^2+y^2}$.

42. (*a*) $\dfrac{a+2b}{a+b}$. (*b*) $\dfrac{2a+2b}{a+b}$.

43. (*a*) $\dfrac{m^2+n^2}{m+n}$. (*b*) $\dfrac{m^2+2mn+n^2}{m+n}$.

44. (*a*) $\dfrac{6mn}{3m^2n^2}$. (*b*) $\dfrac{6m+n}{3m^2+n^2}$.

45. (*a*) $\dfrac{k^2-l^2}{(k+l)^2}$. (*b*) $\dfrac{k^2-l^2}{k^2+l^2}$.

46. Using $\sin^2 A + \cos^2 A = 1$, prove :

$$\frac{(1+\sin A)^2}{\cos^2 A} = \frac{1+\sin A}{1-\sin A}.$$

Prove that :

47. $t^{-\frac{3}{2}} - 6t^{-\frac{5}{2}} + 9t^{-\frac{7}{2}} = (t-3)^2/t^{\frac{7}{2}}$.

48. $4(x+2)^{\frac{1}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + \frac{1}{5}(x+2)^{\frac{5}{2}} = \frac{1}{15}(3x^2-8x+32)\sqrt{x+2}$.

49. $(a-b)^{\frac{3}{2}} + 2b(a-b)^{\frac{1}{2}} + b^2(a-b)^{-\frac{1}{2}} = a^2/\sqrt{a-b}$.

Solve the equations :

50. $\dfrac{2}{x(x+3)} = \dfrac{1}{x(x+1)}$. **51.** $\dfrac{x+1}{(x-1)(x-2)} - \dfrac{x+2}{(x-2)(x+3)} = 0$.

52. $\dfrac{3}{t+1} - \dfrac{1}{t(t+1)} = \dfrac{1}{t}$. **53.** $\dfrac{p+2}{p-2} - \dfrac{p-2}{p+2} = \dfrac{16}{p^2-4}$.

54. Simplify $\frac{1}{2}b \cdot \dfrac{Wab}{a+b} \cdot \frac{2}{3}b \cdot + \frac{1}{2}a \cdot \dfrac{Wab}{a+b} \cdot (b + \frac{1}{3}a)$.

55. Two expressions for the same bending moment are :

$$\frac{wx^2}{2} - \frac{2wl(l-b)(x-a)}{2l-a-b} \text{ and } \frac{w(2l-x)^2}{2} - \frac{2wl(l-a)(2l-x-b)}{2l-a-b}.$$

Prove that these expressions are equal.

56. A frustum of a cone has end radii r_1 and r_2 and length h. Its volume V is given by $V = \frac{1}{3}\pi(r_2^2 h_2 - r_1^2 h_1)$, where $h_1/r_1 = h_2/r_2 = h/(r_2 - r_1)$. Prove $V = \frac{1}{3}\pi(r_2^2 + r_2 r_1 + r_1^2)h$. Is V equal to the volume of a cylinder of length h and radius equal to the mean of r_1 and r_2 ?

57. The radius of gyration k of a hollow cylinder of internal radius a and external radius b is given by $k^2 = \frac{\frac{1}{2}\pi(b^4 - a^4)}{\pi(b^2 - a^2)}$. Simplify this expression. Show that, if $(b - a)$ is small, $k \simeq a$.

58. If a resistance r is placed in parallel with resistances r_1 and r_2, which are in series, the combined resistance R is given by $\frac{1}{R} = \frac{1}{r} + \frac{1}{r_1 + r_2}$, but, if r is in series with r_1 and r_2 which are in parallel, $R = r + \frac{1}{\frac{1}{r_1} + \frac{1}{r_2}}$. Find R from each of these equations as a single fraction.

CHAPTER III

SOLUTION OF QUADRATIC AND OTHER EQUATIONS

Quadratic equations

If
$$x^2 - 3x + 2 = 0,$$
$$(x - 1)(x - 2) = 0$$
$$\therefore\ x - 1 = 0 \quad \text{or} \quad x - 2 = 0$$
$$\therefore\ x = 1 \quad \text{or} \quad x = 2$$
$$\therefore\ x = 1 \quad \text{or} \quad 2.$$

An equation like $x^2 - 3x + 2 = 0$, which is true for two values of the unknown letter x, that is, has two roots, is called a "quadratic equation." The simplest type of quadratic equation is one like $4x^2 - 9 = 0$, which has no term containing the first power of the unknown x. It can be solved in either of the following ways.

(i) Factorize $4x^2 - 9$, then

$$(2x+3)(2x-3)=0$$
$$\therefore\ 2x+3=0 \quad \text{or } 2x-3=0$$
$$\therefore\ 2x=-3, \text{ or } \quad 2x=3$$
$$\therefore\ x=-\tfrac{3}{2} \text{ or } \quad x=\tfrac{3}{2}$$
$$\therefore\ x=-\tfrac{3}{2} \text{ or } \tfrac{3}{2}.$$

(ii) Add 9 to both sides of the equation, then

$$4x^2-9+9=9$$
$$\therefore\ 4x^2=9$$
$$\therefore\ (2x)^2=3^2 \text{ or } (-3)^2.$$

Now take the square root of each side,

$$2x=3 \text{ or } 2x=-3$$
$$\therefore\ x=\tfrac{3}{2} \text{ or } -\tfrac{3}{2}.$$

± 3 is written for " $+3$ or -3 " and the two lines above are written :

$$2x=\pm 3 \qquad \therefore\ x=\pm\tfrac{3}{2}.$$

Solution of a quadratic equation by factors

Take all the terms of the equation to the left-hand side; then, if the left-hand side has easy factors, the roots of the equation are found by equating each factor to zero, as in (i) above.

Example.—Solve the equation $3x^2=2x+5$.

$$3x^2-2x-5=0$$
$$\therefore\ (3x-5)(x+1)=0$$
$$\therefore\ 3x-5=0 \text{ or } x+1=0$$
$$\therefore\ x=\tfrac{5}{3} \text{ or } -1.$$

Example.—Find t if $t^2-at+2bt-2ab=0$.

$$t(t-a)+2b(t-a)=0$$
$$\therefore (t-a)(t+2b)=0$$
$$\therefore\ t-a=0 \text{ or } t+2b=0$$
$$\therefore\ t=a \text{ or } -2b.$$

Solution of a quadratic equation by the "method of completing the square"

Since

$$(x+\tfrac{1}{2}p)^2 = x^2 \;+\; 2x\times\tfrac{1}{2}p \;+\; (\tfrac{1}{2}p)^2 = x^2+px+(\tfrac{1}{2}p)^2,$$

x^2+px is converted into a perfect square by adding $(\frac{1}{2}p)^2$. This is shown geometrically in Fig. 13, in which

x^2+px = area of square ABCD + area of rectangle BEFC + area of rectangle CHKD.

The addition of $(\frac{1}{2}p)^2$ to this area means the addition of the square CFGH, which makes up the total area of the square AEGK, which has an area $(x+\frac{1}{2}p)^2$.

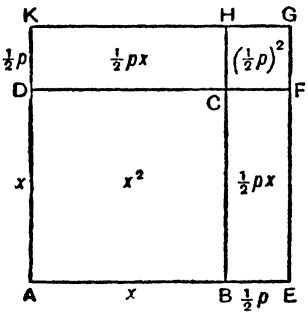

FIG. 13.

Example.—What must be added to x^2+3x to make the new expression a perfect square ?

Here $p=3$ and so $\left(\frac{p}{2}\right)^2 = (\frac{3}{2})^2 = \frac{9}{4}$. This makes the new expression

$$x^2+3x+(\tfrac{3}{2})^2 = (x+\tfrac{3}{2})^2.$$

Example.—Solve the equation $x^2+3x-5=0$.

$$x^2+3x=5$$

Adding $(\frac{3}{2})^2$ to both sides,

$$x^2+3x+(\tfrac{3}{2})^2=5+(\tfrac{3}{2})^2=5+\tfrac{9}{4}$$

$$\therefore\ (x+\tfrac{3}{2})^2=\tfrac{29}{4}$$

$$\therefore\ x+\tfrac{3}{2}=\pm\sqrt{\frac{29}{4}}=\pm\frac{\sqrt{29}}{2}\doteq\pm\frac{5\cdot385}{2}$$

$$\therefore\ x\doteq\frac{5\cdot385-3}{2}\doteq 1\cdot192 \text{ or } \frac{-5\cdot385-3}{2}\doteq -4\cdot192.$$

This example shows that to solve the equation $x^2+px+q=0$:

I. Write the equation in the form $x^2+px=-q$.

II. Add to both sides $(\frac{1}{2}p)^2$, that is the "square of half the coefficient of x." This makes the left-hand side the square of $(x+\frac{1}{2}p)$.

The above method of solution is called the "method of completing the square." It can be used in all cases whatever the values of p and q, whether positive or negative.

Example.—Solve the equation $y^2+4\cdot12=7\cdot66y$.

$$y^2-7\cdot66y=-4\cdot12.$$

Add $\left(\frac{7\cdot66}{2}\right)^2=3\cdot83^2$ to both sides,

$$y^2-7\cdot66y+\left(\frac{7\cdot66}{2}\right)^2=3\cdot83^2-4\cdot12=14\cdot67-4\cdot12$$

$$\therefore\ (y-3\cdot83)^2=10\cdot55=3\cdot25^2$$

$$\therefore\ y-3\cdot83=\pm3\cdot25$$

$$\therefore\ y=3\cdot83-3\cdot25=0\cdot58$$

$$\text{or}\quad y=3\cdot83+3\cdot25=7\cdot08.$$

If x^2 has a coefficient we divide the equation by it so as to make the coefficient of x^2 unity before completing the square.

Example.—Solve the equation $3x^2+x-7=0$.

Dividing by 3,

$$x^2+\tfrac{1}{3}x-\tfrac{7}{3}=0.$$

$$\therefore\ x^2+\tfrac{1}{3}x+(\tfrac{1}{6})^2=\tfrac{7}{3}+(\tfrac{1}{6})^2=\tfrac{7}{3}+\tfrac{1}{36}$$

$$\therefore (x+\tfrac{1}{6})^2 = \frac{84+1}{36} = \frac{85}{36}$$

$$\therefore x+\tfrac{1}{6} = \pm\sqrt{\frac{85}{36}} = \frac{\pm\sqrt{85}}{6} \simeq \pm\frac{9{\cdot}22}{6}$$

$$\therefore x \simeq \frac{9{\cdot}22-1}{6} = \frac{8{\cdot}22}{6} = 1{\cdot}37$$

$$\text{or} \quad x \simeq \frac{-9{\cdot}22-1}{6} = \frac{-10{\cdot}22}{6} \simeq -1{\cdot}70.$$

Example.—Make p the subject of the formula :

$$Lp^2+Rp+\frac{1}{C}=0.$$

$$p^2+\frac{R}{L}p+\frac{1}{LC}=0$$

$$\therefore p^2+\frac{R}{L}p = -\frac{1}{LC}$$

$$\therefore p^2+\frac{R}{L}p+\left(\frac{R}{2L}\right)^2 = -\frac{1}{LC}+\left(\frac{R}{2L}\right)^2 = -\frac{1}{LC}+\frac{R^2}{4L^2}$$

$$\therefore \left(p+\frac{R}{2L}\right)^2 = \frac{-4L+R^2C}{4L^2C}$$

$$\therefore p+\frac{R}{2L} = \pm\sqrt{\frac{R^2C-4L}{4L^2C}}$$

$$= \pm\frac{1}{2L}\sqrt{\frac{R^2C-4L}{C}}$$

$$= \pm\frac{1}{2L}\sqrt{R^2-\frac{4L}{C}}$$

$$\therefore p = \frac{-R\pm\sqrt{R^2-\frac{4L}{C}}}{2L}.$$

Solution of $ax^2+bx+c=0$

$ax^2+bx+c=0$ is the typical quadratic equation. To sum up, the methods of solving it are :

either I. Factorize ax^2+bx+c and equate each factor to zero;

or II. Divide by a ; take the term without x to the right-hand side ; add ($\frac{1}{2}$ coefficient of x)^2 to both sides ; take the square root of each side.

By using the second method we get :

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$$\therefore\ x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$\therefore\ x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\therefore\ \left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

$$\therefore\ x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

This may be remembered as a formula for the roots, but the student should not make use of it in this way until he has mastered the method of completing the square.

Example.—Solve, by rule, $3x^2-2x-6=0$.

Here $a=3,\ b=-2,\ c=-6$

$$\therefore x=\frac{-(-2)\pm\sqrt{(-2)^2-4.3(-6)}}{2.3}$$

$$=\frac{2\pm\sqrt{4+72}}{6}$$

$$= \frac{2 \pm \sqrt{76}}{6}$$

$$= \frac{2 \pm 8{\cdot}718}{6}$$

$$\therefore x = \frac{10{\cdot}718}{6} \simeq 1{\cdot}786$$

$$\text{or} \quad x = -\simeq \frac{6{\cdot}718}{6} \simeq -1{\cdot}120.$$

Factors of $ax^2 + bx + c$

Suppose $3x^2 - 2x - 6 = 3(x - a)(x - b)$, one factor being 3 so as to make the coefficient of x^2 equal to 3. Then $3x^2 - 2x - 6 = 0$ when $x = a$ or b. Hence a and b must be the roots found above, namely $\frac{1 + \sqrt{19}}{3}$ and $\frac{1 - \sqrt{19}}{3}$.

$$\therefore 3x^2 - 2x - 6 = 3\left(x - \frac{1 + \sqrt{19}}{3}\right)\left(x - \frac{1 - \sqrt{19}}{3}\right)$$

$$\text{or} \simeq 3(x - 1{\cdot}786)(x + 1{\cdot}120).$$

Graphical solution of a quadratic equation

A quadratic equation may also be solved graphically; but since the roots can be found much more accurately by algebra the graphical method should not generally be used. However, it is very good practice in drawing graphs to solve a quadratic equation by means of a graph and compare the answers with those found by calculation; moreover, the experience so gained is helpful when using graphs to solve harder equations which cannot be solved by using a formula. As an example we will solve graphically the equation $3x^2 - 2x - 6 = 0$, which has been solved by calculation in the example above.

Write $y = 3x^2 - 2x - 6$ and make a table of values of y for a range of values of x, say from -3 to 3.

x	..	-3	-2	-1	0	1	2	3
$3x^2$	..	27	12	3	0	3	12	27
$-2x$	..	6	4	2	0	-2	-4	-6
-6	..	-6	-6	-6	-6	-6	-6	-6
y	..	27	10	-1	-6	-5	2	15

GRAPH OF $y=3x^2-2x-6$

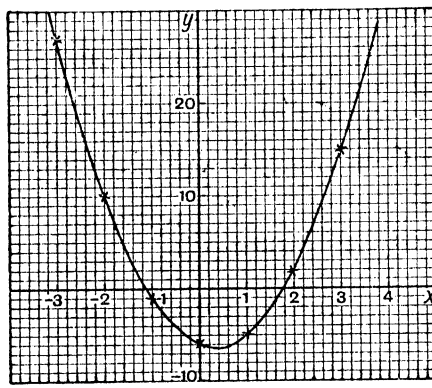

FIG. 14.

From this table it is clear that $y = 0$ between $x = -2$ and -1 and again between 1 and 2, so that the range chosen includes both roots of the equation.

The graph shows that the two values of x which make $y = 0$ are approximately $-1{\cdot}15$ and $1{\cdot}75$.

If we wish to find either root more accurately, we have to tabulate the values of y for a set of values x near the root, and draw a graph from this new table to a much larger scale. In the present example it is certain that the positive root lies

between 1·7 and 1·8, so we tabulate the values of y for $x = 1·7$, 1·75, 1·8 say.

x ..	1·7	1·75	1·8
$3x^2$ ..	8·670	9·189	9·720
$-2x$ ..	−3·4	−3·5	−3·6
−6 ..	−6	−6	−6
Sum $= y$ ..	−0·730	−0·311	0·120

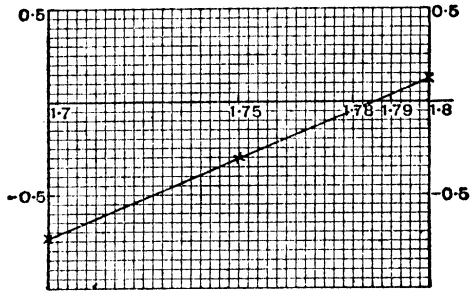

FIG. 15.

The graph drawn from this table is shown in Fig. 15, and it shows that $y = 0$ at 1·786 nearly, which agrees with the

TYPICAL PARABOLAS

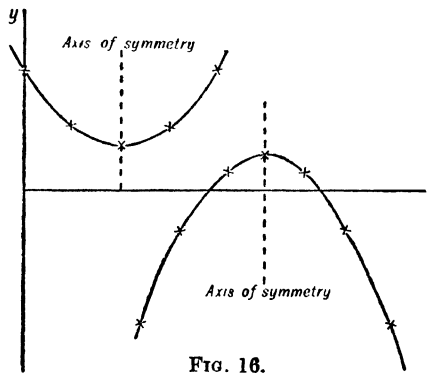

FIG. 16.

value 1·786 found by the formula. This graph is almost a straight line ; and so only a very small error is made in taking it to be a straight line.

Whatever the values of a, b and c, the graph of $y = ax^2 + bx + c$ is of a similar shape to the graph in Fig. 14, but it is the other way up when a is negative. It is symmetrical about the vertical line through the lowest, or highest point. Two typical graphs are shown in Fig. 16. Such curves are called " parabolas."

Equations of some other curves

The circle.

Fig. 17 shows a point (x, y) at a distance a from the point $(0, 0)$. If the scales on the axes of x and y are the same, it follows by Pythagoras' theorem that $x^2 + y^2 = a^2$. This equation is satisfied wherever (x, y) is on the circle. Hence the circle is the graph of the equation. We call the equation " the equation of the circle." Since $y^2 = a^2 - x^2$, $y = \pm\sqrt{a^2 - x^2}$. This shows algebraically what is obvious from the figure, namely, that for any value of x there are two points on the circle given by equal and opposite values of y.

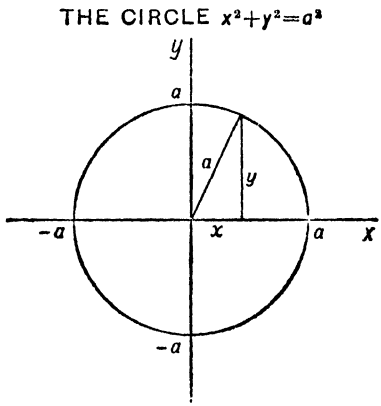

FIG. 17.

The ellipse.

Let the ordinate at every point on the circle above be reduced in the ratio b/a. Then the locus of the points formed in this way is called an ellipse. If Y is the ordinate of a point P on the circle, and y the ordinate of the point Q found from P by reducing Y in the ratio b/a, $y = bY/a$ or $Y = ay/b$. But, since (x, Y) is on the circle, $x^2 + Y^2 = a^2$:

$$\therefore\ x^2 + \frac{a^2y^2}{b^2} = a^2$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is therefore the equation of the ellipse.

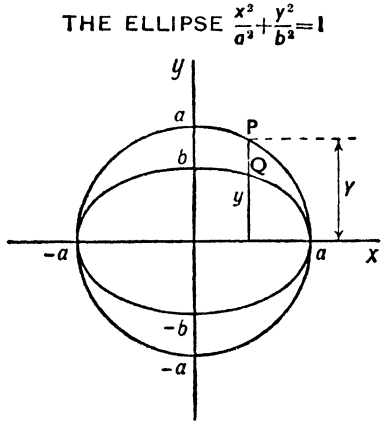

FIG. 18.

The hyperbola.

The graph of the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is as shown in Fig. 19. It is called a hyperbola.

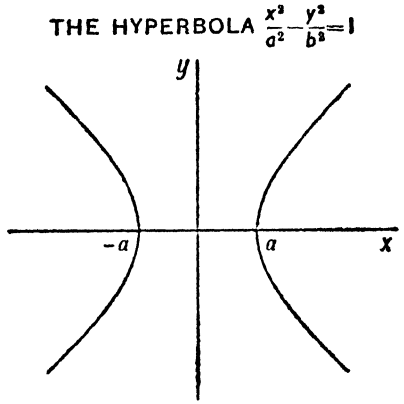

FIG. 19.

Conic sections.

The circle, ellipse, hyperbola and parabola are all sections of a double cone by different planes. Fig. 20 shows elliptic, hyperbolic and parabolic sections.

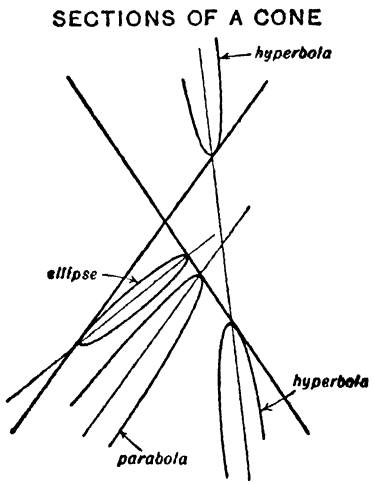

FIG. 20.

Exercise XI

Solve the equations:

1. $2x^2 = 3$. 2. $4 = \frac{1}{9y^2}$. 3. $\frac{r}{4} = \frac{5}{r}$. 4. $\frac{4}{(t-1)^2} = 9$.

Solve, by using factors:

5. $x^2 - 3x + 2 = 0$.
6. $y^2 - 6y + 5 = 0$.
7. $2r^2 - r - 3 = 0$.
8. $12m^2 + 29m - 8 = 0$.
9. $4x^2 + 8x + 3 = 0$.
10. $5(l^2 - 1) = 2l(1 - l)$.
11. $t^2 = t + 6$.
12. $x(x + 9) = 22$.
13. $\frac{1}{z+1} - \frac{4}{z} = \frac{1}{z+3}$.
14. $\frac{k+6}{k-4} = \frac{1}{k}$.

Solve by using factors and by completing the square:

15. $x^2 - 4x - 5 = 0$.
16. $n^2 + 6n + 8 = 0$.
17. $y^2 - 5y - 24 = 0$.
18. $r^2 + 9r + 8 = 0$.

19. If $x^2 + 6x + a$ is a perfect square, what must the value of a be?

20. If $x^2 - \frac{2}{3}x + a$ is a perfect square what must the value of a be?

21–30. Solve questions 5–14 by completing the square.

Solve by completing the square, and also by using a formula:

31. $x^2 + x = 4$.
32. $2x^2 + 5x + 1 = 0$.
33. $3t^2 + 4t - 9 = 0$.
34. $4v^2 = v + 1$.
35. $(k + 1)^2 + k^2 = 10k$.
36. $1{\cdot}6x^2 - 6{\cdot}2x - 2{\cdot}7 = 0$.
37. $\sqrt{2}y^2 - (1 + 2\sqrt{2})y - 1 = 0$.
38. $0{\cdot}84t^2 + 2{\cdot}15t + 0{\cdot}18 = 0$.

39. If $x^2 - 2ax = b^2 - a^2$ prove that $x = a + b$ or $a - b$.

40. Solve the equation $lh = h^2 + k^2$ for h and show that the sum of the roots is l.

41. If $p^2 - (p_x + p_y)p - q^2 + p_x p_y = 0$, prove that

$$p = \tfrac{1}{2}\{(p_x + p_y) \pm \sqrt{(p_x - p_y)^2 + 4q^2}\}.$$

Solve the following equations graphically and by calculation and compare your answers:

42. $x^2 - 6x + 3 = 0$.
43. $2t^2 + 5t - 1 = 0$.
44. $m^2 - 5m - 7 = 0$.
45. $r^2 + 20r - 50 = 0$.

46. Solve the equation $cx^2+ax-b=0$.

47. Find the values of x which make $\frac{1}{x}+\frac{1}{c-x}=\frac{1}{f}$.

48. Make (i) r, (ii) t, the subject of the formula:

$$W=\pi sl\{(r+t)^2-r^2\}.$$

49. If $4x^2+9y^2=36$, show that $y=\pm\frac{2}{3}\sqrt{9-x^2}$. Make a table of values of y from $x=-3$ to 3, and hence draw the graph of y against x. What is the name of this curve?

50. Draw the graphs of the circle $x^2+y^2=9$ and the ellipse $x^2+4y^2=9$ using the same axes for both graphs. Prove that the ellipse is obtained from the circle by halving each ordinate.

Draw the graphs of the following equations and state the name of each curve.

51. $y=4x(3-x)$. **52.** $\frac{1}{4}x^2-\frac{1}{2}y^2=1$. **53.** $\frac{1}{4}x^2+\frac{1}{2}y^2=1$.

54. $y^2=3x$. **55.** $y=1-6x+4x^2$. **56.** $y^2-3x^2=4$.

57. Draw the graphs of $x^2-y^2=3$ and $y^2-x^2=3$ using the same axes for both graphs.

Problems involving quadratic equations

Example.—A cylindrical cup is made of 35 sq. in. of thin sheet metal. If its height is 4 in., find its radius to the nearest $\frac{1}{100}$th in.

If the radius is r in., then the area of the curved surface is $2\pi r\times 4$ sq. in. and the area of the base is πr^2 sq. in.

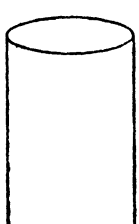

FIG. 21.

$$\therefore\ \pi r^2+8\pi r=35$$

$$\therefore\ r^2+8r=\frac{35}{\pi}=11{\cdot}14$$

$$\therefore\ (r+4)^2=11{\cdot}14+16=27{\cdot}14$$

$$\therefore\ r+4=\pm\sqrt{27{\cdot}14}=\pm 5{\cdot}21$$

$$\therefore\ r=1{\cdot}21,$$

the negative value being clearly an impossible one for the radius of the cup.

Example.—Solve the same problem as in the above example, given that the area of the metal is S sq. in. and the height of the cup is h in.

If the radius is r in., the area of the curved surface is $2\pi rh$ and the area of the base is πr^2 :

$$\therefore \pi r^2 + 2\pi rh = S$$

$$\therefore r^2 + 2rh = \frac{S}{\pi}$$

$$\therefore (r+h)^2 = \frac{S}{\pi} + h^2$$

$$\therefore r + h = \pm\sqrt{\frac{S}{\pi} + h^2}$$

$$\therefore r = \sqrt{\frac{S}{\pi} + h^2} - h,$$

the other root being negative.

Example.—If a stone is thrown vertically up under gravity at u ft./sec. its height s ft. after t sec. is given by $s = ut - \frac{1}{2}gt^2$ (neglecting air resistance). Given that $u = 60$, $g = 32$, find when the stone is 40 ft. high.

Substituting the given values :

$$40 = 60t - 16t^2$$

$$\therefore 16t^2 - 60t = -40$$

$$\therefore t^2 - \tfrac{60}{16}t = -\tfrac{40}{16}$$

$$\therefore t^2 - \tfrac{15}{4}t = -\tfrac{40}{16}$$

$$\therefore (t - \tfrac{15}{8})^2 = -\tfrac{40}{16} + \tfrac{225}{64} = \frac{-160+225}{64} = \tfrac{65}{64}$$

$$\therefore t - \tfrac{15}{8} \simeq \pm\frac{8{\cdot}06}{8}$$

$$\therefore t \simeq \frac{23{\cdot}06}{8} \text{ or } \frac{6{\cdot}94}{8}$$

$$\therefore t \simeq 2{\cdot}88 \text{ or } 0{\cdot}87.$$

In this example both the answers have a meaning; the smaller answer, 0·87, means that the stone takes 0·87 sec. to get to a height of 40 ft. on the way up, and $t = 2{\cdot}88$ means

that the stone is again at a height of 40 ft. at a time 2·88 sec. after it was thrown up, that is after it has risen to its highest position and fallen back again.

It should be noted from these two examples that in a practical problem either one or both roots of the quadratic equation may apply to the problem.

Exercise XII

1. From a rectangular sheet of metal 17 in. long, 10 in. wide, a strip x in. wide is cut off all round and the area of the remainder is the same as that of a square of side 12 in. Find x.

2. A sheet of metal is to be cut in the shape shown, so as to have an area of 30 sq. in. If it is 10 in. long find the radius of the semicircle.

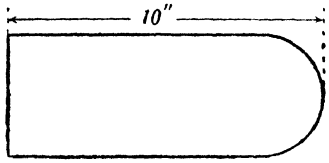

FIG. 22.

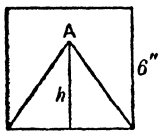

FIG. 23.

3. If a body is thrown vertically upwards under gravity at 100 ft./sec. its height s ft. after t sec. is given by $s = 100t - 16t^2$. At what time is it (*a*) 136 ft. high, (*b*) 100 ft. high.

4. If a beam of length l ft. is clamped horizontally at each end, the bending moment at x ft. from one end is $\frac{w}{6EI}(6x^2 - 6lx + l^2)$. Show that the bending moment is zero at the points given by $x = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right)l$. Find the values of x to the nearest inch if $l = 12$.

5. From a square with sides of length 6 in. (Fig. 23) an isosceles triangle, which has a side of the square as base and vertex at A, is cut out. If the centroid of the remaining area is at A prove that $h^2 - 18h + 54 = 0$. Find h from this equation.

6. A room is 12 ft. by 10 ft. Find x so that when the length and breadth of the room are each increased by x ft. the area of the room is increased 50%. Also find x if the length and breadth are initially a ft. and b ft. and the area is increased r%.

7. If the bob of a conical pendulum or governor in which the string or rod has a length l ft. describes a circle at v ft./sec. the depth h ft. of the bob beneath the point of suspension is given by $\frac{v^2}{g} = \frac{l^2 - h^2}{h}$. Find h if $l = \frac{1}{2}$, $v = 2$, $g = 32$. Also find a general expression for h in terms of g, v and l.

8. Find the radius of a solid cylinder 10 in. long if its total surface area is 600 sq. in.

9. A force W lb. wt. stretches a spring whose unstretched length is l ft. by a ft. If one end of the spring is fixed and a weight W lb. wt. attached to the other end is made to describe a horizontal circle at v ft./sec., the string is stretched x ft. where

$$\frac{v^2}{g(l+x)} = \frac{x}{a}.$$

If $g = 32$, $l = 2$, $a = \frac{1}{10}$, find (a) v when $x = \frac{1}{5}$, (b) x when $v = 20$.

10. If a rod of length l in. is supported at two points a in. from each end, its ends are horizontal if $\frac{wl}{4}\left(\frac{l}{2} - a\right)^2 = \frac{wl^3}{48}$. Find a in terms of l.

11. The impedance Z ohms of a circuit containing a resistance R ohms, inductance L henries, capacity C farads, when the frequency of the oscillations is n per sec., is given by

$$Z = \sqrt{R^2 + \left(2\pi nL - \frac{1}{2\pi nC}\right)^2}.$$

Make L the subject of this formula. If $n = 50$, $R = 15$, $C = 10^{-4}$, show that there are two values of L which make $Z = 20$, but only one value which will make $Z = 100$. Find these values.

12. If a train goes from one station to another d ft. away in t sec. by accelerating at f ft./sec.^{2}, then travelling at a uniform speed v ft./sec. and finally coming to rest with a deceleration of f' ft./sec.^{2}, $v^2\left(\frac{1}{f} + \frac{1}{f'}\right) - 2tv + 2d = 0$. Calculate the value of v if $f = \frac{1}{2}$, $f' = 1$, $t = 180$ and the distance between the stations is 1 mile. Find a general expression for v in terms of f, f', t and d.

13. If a train of length l ft. passes over a bridge the maximum bending moment of the bridge is greatest when the front of the train has reached a point x ft. from one end given by $w_t x^2 = 2lw_b\left(\frac{l}{2} - x\right)$, where w_t and w_b are the weights per unit length of train and bridge respectively. Find x.

Equal roots and imaginary roots

Example.—Draw the graph of $y = x^2 - 4x$ and use it to solve the equations $x^2 - 4x = 2$, $x^2 - 4x + 2 = 0$, $x^2 - 4x + 4 = 0$, $x^2 - 4x + 6 = 0$.

The graph in Fig. 24 is drawn from the following table :

x ..	−2	−1	0	1	2	3	4	5
x^2 ..	4	1	0	1	4	9	16	25
$-4x$ ..	8	4	0	−4	−8	−12	−16	−20
y ..	12	5	0	−3	−4	−3	0	5

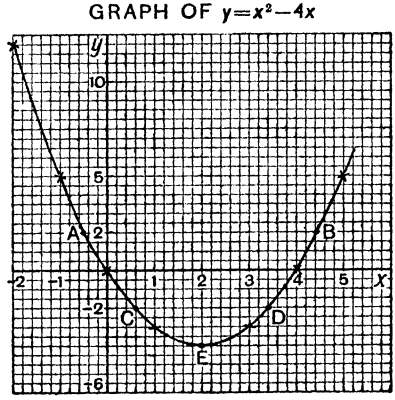

FIG. 24.

$y = 2$, that is $x^2 - 4x = 2$, at the points A and B at which $x \simeq -0{\cdot}4$ and $4{\cdot}5$.

$\therefore$ the roots of $x^2 - 4x = 2$ are $x \simeq -0{\cdot}4$ or $4{\cdot}5$.

By completing the square, these roots are $2 \pm \sqrt{6}$, or $-0{\cdot}4495$ and $4{\cdot}4495$.

In the same way the equation $x^2 - 4x + 2 = 0$ can be written $x^2 - 4x = -2$, and so the roots of the equation are given by the values of x at C and D where $y = -2$. These are $0{\cdot}6$ and $3{\cdot}4$ nearly. $x^2 - 4x + 4 = 0$ can be written $x^2 - 4x = -4$,

and $y = -4$ only at the one point E at which $x = 2$. Although there is only one point giving this value of y it is clear that if, starting at CD, we draw horizontal lines lower and lower they will cut the curve in *two points* which gradually approach closer and closer together. For this reason, when the points actually come together at $x = 2$, we say that the equation has two roots both equal to 2, instead of saying that it has one root.

If $x^2 - 4x + 6 = 0$, then $x^2 - 4x = -6$, and there are no points on the graph of $y = x^2 - 4x$ at which $y = -6$. Hence this equation has no roots which are ordinary numbers like $3\frac{1}{2}$, 6·8, etc. If, however, we solve the equation by completing the square, we get :

$$x^2 - 4x + 4 = 4 - 6 = -2.$$

$$\therefore (x-2)^2 = -2$$

$$\therefore x - 2 = \pm\sqrt{-2} = \pm\sqrt{2}\,\sqrt{-1}$$

$$\therefore x - 2 = \pm 1{\cdot}41\sqrt{-1}$$

$$\therefore x = 2 + 1{\cdot}41\sqrt{-1} \text{ or } 2 - 1{\cdot}41\sqrt{-1}.$$

We shall see later that use is made of numbers like these in practical problems in physics and engineering, but for the present we merely note that the roots of a quadratic equation are not always numbers like $\frac{2}{3}$, 3·75, -6, which can be marked on a graph.

Solution of equations of higher degree

Generally such equations have to be solved graphically. However, we can sometimes solve them by using factors.

Example.—Solve the equation $x^3 - 8x - 8 = 0$.

The factors of 8 are ± 1, ± 2, ± 4, ± 8, so we try these values of x to see if any one is a root.

$x = 1$ makes $x^3 - 8x - 8 = 1 - 8 - 8$; which is not 0.

$x = -1$ makes $x^3 - 8x - 8 = -1 + 8 - 8$, which is not 0.

$x = 2$ makes $x^3 - 8x - 8 = 8 - 16 - 8$, which is not 0.

$x = -2$ makes $x^3 - 8x - 8 = -8 + 16 - 8$, which *does* equal 0.

Hence $x = -2$ is a root of the equation. This suggests that $(x+2)$ is a factor of $(x^3 - 8x - 8)$, so we try this by long division:

$$\begin{array}{r|l} x+2 & x^3 - 8x - 8 \mid x^2 - 2x - 4 \\ & \underline{x^3 + 2x^2} \\ & -2x^2 - 8x - 8 \\ & \underline{-2x^2 - 4x} \\ & -4x - 8 \\ & \underline{-4x - 8} \end{array}$$

$$\therefore x^3 - 8x - 8 = (x+2)(x^2 - 2x - 4)$$

$$\therefore x^3 - 8x - 8 = 0 \text{ when}$$

$$x + 2 = 0 \quad \text{or} \quad x^2 - 2x - 4 = 0$$

$$\therefore x = -2 \quad x = 1 \pm \sqrt{5}$$

$$\therefore x = -2 \text{ or } 1 + \sqrt{5} \text{ or } 1 - \sqrt{5}$$

Hence $x = -2$ or $3{\cdot}236$ or $-1{\cdot}236$ (approx.).

Graphical solution of an equation

If an equation cannot be solved by using factors we solve it graphically as in the following example.

Example.—Solve the equation $x^3 - 3x^2 - 9x + 10 = 0$.

Write $y = x^3 - 3x^2 - 9x + 10$; then the required values of x make $y = 0$, which means that they are given by the values of x at the points where the graph of $y = x^3 - 3x^2 - 9x + 10$ cuts the axis $x'0x$. The graph is plotted in Fig. 25 from the following table.

x ..	-3	-2	-1	0	1	2	3	4	5
x^3 ..	-27	-8	-1	0	1	8	27	64	125
$-3x^2$ ..	-27	-12	-3	0	-3	-12	-27	-48	-75
$-9x$ ..	27	18	9	0	-9	-18	-27	-36	-45
$+10$ ..	10	10	10	10	10	10	10	10	10
y = sum	-17	8	15	10	-1	-12	-17	-10	15

GRAPH OF $y=x^3-3x^2-9x+10$

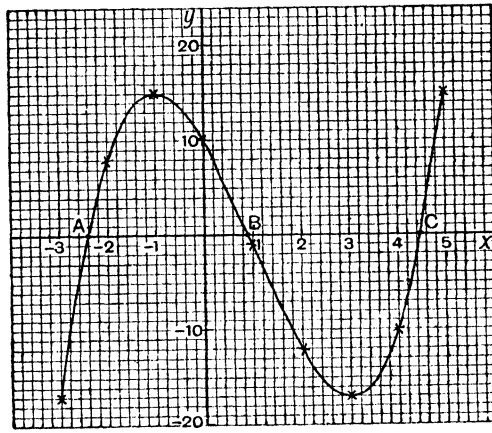

FIG. 25.

The graph cuts $x'0x$ at A, B, C, and the values of x at these points are $-2{\cdot}4$, $0{\cdot}9$, $4{\cdot}5$. Hence, $y=0$, when $x=-2{\cdot}4$ or $0{\cdot}9$ or $4{\cdot}5$ approximately.

The equation can, however, be solved more quickly as follows. Write it:

$$x^3-3x^2=9x-10.$$

Plot the graphs of $y_1=x^3-3x^2$, and $y_2=9x-10$. This is done in Fig. 26 from the tables below:

$y_1=x^3-3x^2$.

x	..	-3	-2	-1	0	1	2	3	4	5
x^3	..	-27	-8	-1	0	1	8	27	64	125
$-3x^2$	..	-27	-12	-3	0	-3	-12	-27	-48	-75
y_1	..	-54	-20	-4	0	-2	-4	0	16	50

$y_2=9x-10$

x	..	-3	0	5
y_2	..	-37	-10	35

These graphs intersect in the points P, Q, R. At each of these points the ordinates are equal, that is $y_1 = y_2$, and hence $x^3 - 3x^2 = 9x - 10$.

Therefore, the values of x at P, Q, R, which are nearly $-2{\cdot}4$, $0{\cdot}9$, $4{\cdot}5$ are the roots of the equation.

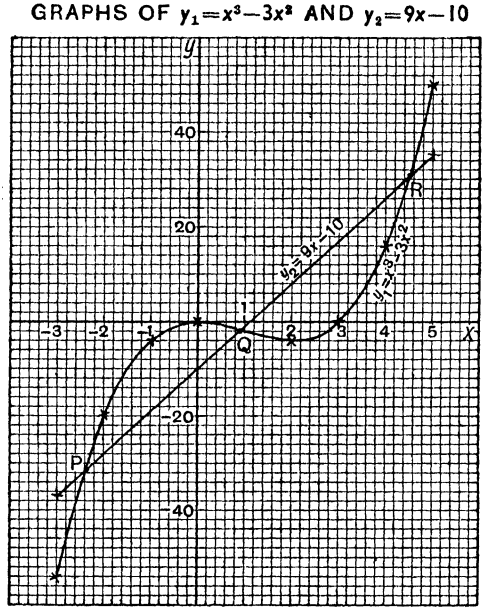

FIG. 26.

To find any one of these roots more accurately we plot the two graphs to a much larger scale near the point of intersection as on p. 63.

The reason this method is quicker than the first is that the graph of $y_2 = 9x - 10$ is a straight line, and so can be drawn by plotting two points only (though three should be plotted to ensure accuracy).

Exercise XIII

1. Draw the graph of $y = x^2 - 8x$ and use it to find the roots of the equations (*a*) $x^2 - 8x = 4$, (*b*) $x^2 - 8x + 16 = 0$. Check your answers by calculation.

2. Show that the equation $x^2 + 6x + 10 = 0$ has no real roots. What value must a have for the equation $x^2 + 6x + a = 0$ to have equal roots ?

3. Show that $x = 1$ is a root of the equation $x^3 + 2x^2 - x - 2 = 0$ and find the other roots.

4. One of the numbers 1, 2, 3, is a root of the equation $x^3 + 5x^2 - 2x - 24 = 0$. Find which it is, and find the other roots of the equation.

5. The point of maximum deflection of a beam is at a distance x from one end where $\frac{x^4}{24} - \frac{l^2x^2}{12} + \frac{7l^4}{360} = 0$. Find x in terms of l.

6. Find the three values of x which make $x^3 - 16x = 10$: (*a*) by drawing the graph of $y = x^3 - 16x - 10$ from $x = -4$ to 5; (*b*) by drawing the graphs of $y = x^3$ and $y = 16x + 10$ from $x = -4$ to 5.

7. Show by a graph that the equation $2x^3 + 9x - 8 = 0$ has one root. Find its value to 3 sig. fig.

8. The volume V of a spherical segment of height h of a sphere of radius r is given by $V = \frac{1}{3}\pi h^2(3r - h)$. If $r = 10$ find h so that the volume of the segment is a quarter of the volume of the sphere. [h is the distance a sphere of specific gravity $\frac{1}{4}$ is immersed when it is floating in water.]

9. If a beam 12 ft. long of uniform cross-section is clamped horizontally at one end and loaded at the free end, the deflection at a point x ft. from the clamped end is half the deflection at the other end when $x^3 - 36x^2 + 1728 = 0$. Find the value of x to the nearest inch.

CHAPTER IV

LOGARITHMS

The name logarithm is derived from the Greek words $\lambda o\gamma o\varsigma$ = reckoning, $\alpha\rho\iota\theta\mu o\varsigma$ = number. Although there had been several attempts during the sixteenth century to find some way of replacing multiplication by addition, John Napier, of Merchiston (near Edinburgh), was the first person to construct a table of [illegible]. This table, which was published in 1614, was a table of seven-figure logarithms of the sines of angles at every minute from 0° to 90°. The base used for the table was not 10, but $\frac{1}{e}$, where e is a number, nearly 2·7183, which has since become of increasing importance in applications of mathematics.

Henry Briggs, professor of geometry at Gresham College, London, was so interested in Napier's discovery that he went to stay with him in 1615 and 1616 to discuss the calculation of logarithm tables. They both had the plan of making a table of logarithms to base 10, but Napier died in April, 1617. In the same year Briggs published a table of logarithms to base 10 of all numbers from 1 to 1000, and in 1624 he published a more comprehensive table which contained the logarithms of all numbers from 1 to 20,000 and from 90,000 to 100,000 to fourteen decimal places. Thus Briggs's larger table was not complete, but it was completed as soon after as 1628 by Adrian Vlacq, of Gouda in Holland, who published an eight-figure table of the logarithms to base 10 of all numbers from 1 to 100,000, and most of the tables published since have been based upon it.

In 1620 Jobst Bürgi, a Swiss mathematician, published a table of anti-logarithms in Prague. It is generally believed that the calculation of this table was quite independent of the work of Napier and Briggs.

Edmund Gunter was the first person to realize (in 1620)

that sliding rules with logarithmic scales could be used for multiplication, but the first slide rule, of the type we know to-day, was made by Robert Bissaker in 1654; it is in the Science Museum, South Kensington.

The graph of 2^x

The two following tables show the values of 2^x when $x = 0$, $\frac{1}{2}$, 1, . . . 3, and when $x = -\frac{1}{2}$, -1, . . . -3.

x	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
2^x	1	$\sqrt{2}=1{\cdot}414$	2	$2\sqrt{2}=2{\cdot}828$	4	$4\sqrt{2}=5{\cdot}657$	8

x	$-\frac{1}{2}$	-1	$-1\frac{1}{2}$	-2	$-2\frac{1}{2}$	-3
2^x	$\frac{1}{\sqrt{2}}=0{\cdot}707$	$\frac{1}{2}=0{\cdot}5$	$\frac{1}{2\sqrt{2}}=0{\cdot}354$	$\frac{1}{2^2}=0{\cdot}25$	$\frac{1}{4\sqrt{2}}=0{\cdot}177$	$\frac{1}{2^3}=0{\cdot}125$

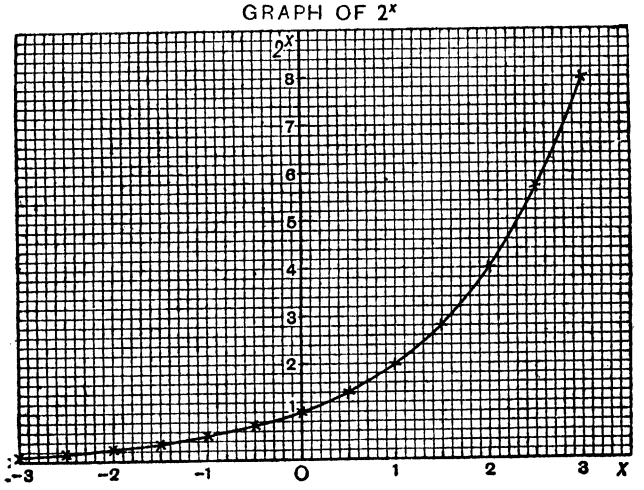

FIG. 27.

From the tables the points in Fig. 27 are plotted and joined by a smooth curve. It can easily be verified that any point obtained by giving another value to x, say $\frac{3}{4}$, lies on the curve. The student should find the values of $2^{\frac{1}{4}}$, $2^{\frac{3}{4}}$, $2^{\frac{5}{4}}$, etc., by finding the square roots of $2^{\frac{1}{2}}$, $2^{\frac{2}{2}}$, $2^{\frac{5}{2}}$, and verify that the points so found lie on the graph. It is very important to notice that as x varies the values of 2^x do give points lying on a smooth curve. Thus $2^{0\cdot1}$ is slightly larger than 2^0, which is 1, and $2^{2\cdot4}$ is slightly smaller than $2^{2\cdot5}$, which is 5·657.

The graph of $y = a^x$, where a is a number greater than 1, is similar in shape to the graph of $y = 2^x$. The student should draw the graphs of $y = 2^x$, $y = 3^x$, $y = 4^x$, using the same axes and scales for each graph.

Logarithms

If $y = 2^x$ *we call* x *the logarithm of* y *to the base* 2, *and write this* $x = log_2 y$. This clearly means that $x = \log_2 (2^x)$.

From the graph of $y = 2^x$ we find that, when $y = 0\cdot8$, $x \simeq -0\cdot32$, in other words, $0\cdot8 \simeq 2^{-0\cdot32}$ or $\log_2 0\cdot8 \simeq -0\cdot32$. In the same way, $3\cdot6 \simeq 2^{1\cdot85}$, or $\log_2 3\cdot6 \simeq 1\cdot85$:

$$\therefore 3\cdot6 \times 0\cdot8 = 2^{1\cdot85} \times 2^{-0\cdot32} = 2^{1\cdot85-0\cdot32},$$

which means that

$$\log_2 (3\cdot6 \times 0\cdot8) = 1\cdot85 - 0\cdot32 = \log_2 3\cdot6 + \log_2 0\cdot8.$$

This shows that a product can be found by addition of [illegible]; multiplication is replaced by addition. Herein lies the importance of logarithms.

Since 2^x is positive whether x is positive or negative, we cannot find any value of x to make 2^x negative ; in other words, a negative number has no logarithm.

Similarly, if $y = a^x$, where a is any positive number, we call x the logarithm of y to the base a and write this $x = \log_a y$.

Logarithms to the base 10

By the definition above :

$$\log_{10} 10^n = n$$

$$\therefore \log_{10} 10 = \log_{10} 10^1 = 1$$

$$\log_{10} 100 = \log_{10} 10^2 = 2$$

$$\log_{10} 1000 = \log_{10} 10^3 = 3$$

Similarly :

$$\log_{10} \left(\frac{1}{10}\right) = \log_{10} 10^{-1} = -1$$

$$\log_{10} \left(\frac{1}{10^2}\right) = \log_{10} 10^{-2} = -2$$

and so on.

Therefore :

$$\log_{10} 25 = \log_{10} (10 \times 2{\cdot}5) = \log_{10} 10 + \log_{10} 2{\cdot}5$$
$$= 1 + 0{\cdot}3979 = 1{\cdot}3979$$

$$\log_{10} 250 = \log_{10} (10^2 \times 2{\cdot}5) = \log_{10} 10^2 + \log_{10} 2{\cdot}5$$
$$= 2 + 0{\cdot}3979 = 2{\cdot}3979$$

Similarly :

$$\log_{10} 0{\cdot}25 = \log_{10} (10^{-1} \times 2{\cdot}5) = \log_{10} 10^{-1} + \log_{10} 2{\cdot}5$$
$$= -1 + 0{\cdot}3979 = \bar{1}{\cdot}3979$$

$$\log_{10} 0{\cdot}025 = \log_{10} (10^{-2} \times 2{\cdot}5) = \log_{10} 10^{-2} + \log_{10} 2{\cdot}5$$
$$= -2 + 0{\cdot}3979 = \bar{2}{\cdot}3979$$

Thus, using the base 10, the logarithms of all numbers formed by the same digits have the same positive decimal part or *mantissa*, the whole number part or *characteristic* being determined only by the position of the decimal point. This is not true for any other base, and this is the reason that logarithms to base 10 are used in nearly all calculations. We shall see later that there are a few problems in which it is convenient to use the number e as base. When using the base 10 the suffix indicating the base is usually omitted.

Rules of logarithms

Let a and b be any two positive numbers and let

$$a = 10^m, \quad b = 10^n.$$

Then,

$$\log a = m, \quad \log b = n$$

$$\therefore \log ab = \log (10^m \times 10^n) = \log 10^{m+n} = m + n$$

$$\therefore \underline{\log ab = \log a + \log b.}$$

In the same way,

$$\log \frac{a}{b} = \log \left(\frac{10^m}{10^n}\right) = \log 10^{m-n} = m - n$$

$$\therefore \underline{\log \frac{a}{b} = \log a - \log b.}$$

Also, if r is any number, positive or negative,

$$\log a^r = \log (10^m)^r = \log 10^{mr} = mr$$

$$\therefore \underline{\log a^r = r \log a.}$$

By using the formulæ underlined above with a table of logarithms we can calculate products, quotients and powers by addition and subtraction.

In carrying out a calculation it is often helpful to give a name, say x, to the number to be calculated, and then to write down an equation for $\log x$ before actually tabulating the logarithm.

Example.—Calculate $\dfrac{27{\cdot}12 \times (0{\cdot}7065)^2}{0{\cdot}004729 \times (987{\cdot}6)^{\frac{1}{3}}}$.

If this number is x,

$$\log x = \log (\text{numerator}) - \log (\text{denominator})$$

$$= \{\log 27{\cdot}12 + \log (0{\cdot}7065)^2\} - \{\log 0{\cdot}004729 + \log (987{\cdot}6)^{\frac{1}{3}}\}$$

$$= \{\log 27{\cdot}12 + 2 \log 0{\cdot}7065\} - \{\log 0{\cdot}004729 + \tfrac{1}{3} \log 987{\cdot}6\}$$

This equation shows us what logarithms have to be tabulated. The actual calculation is shown below.

Number	*Logarithm*	
27·12	1·4333	
$(0{\cdot}7065)^2$	$2 \times \bar{1}{\cdot}8491 = \bar{1}{\cdot}6982$	
	1·1315	1·1315
0·004729	$\bar{3}{\cdot}6747$	
$(987{\cdot}6)^{\frac{1}{3}}$	$\frac{1}{3}(2{\cdot}9916) = 0{\cdot}9982$	
	$\bar{2}{\cdot}6729$	$\bar{2}{\cdot}6729$
287·5		2·4586

Ans. 287·5.

Example.—Find the value of $(0{\cdot}009162)^{\frac{1}{4}}$.

If $x = (0{\cdot}009162)^{\frac{1}{4}}$

$\log x = \frac{1}{4} \log 0{\cdot}009162 = \frac{1}{4}(\bar{3}{\cdot}9620)$.

Therefore, because $\bar{3} = -3 = -4 + 1 = \bar{4} + 1$,

$\log x = \frac{1}{4}(\bar{4} + 1{\cdot}9620) = \bar{1} + 0{\cdot}4905 = \bar{1}{\cdot}4905$

$\therefore x = 0{\cdot}3094$.

Example.—Find the value of $(0{\cdot}6173)^{3{\cdot}15}$.

If $x = (0{\cdot}6173)^{3{\cdot}15}$

$\log x = 3{\cdot}15 \log 0{\cdot}6173 = 3{\cdot}15\ (\bar{1}{\cdot}7905)$.

Now $\bar{1}{\cdot}7905 = -1 + 0{\cdot}7905 = -0{\cdot}2095$.

$\therefore \log x = -3{\cdot}15 \times 0{\cdot}2095$.

$= -0{\cdot}6600$

$= -1 + 0{\cdot}3400$

$= \bar{1}{\cdot}3400$

$\therefore x = 0{\cdot}2188$.

No.	*Log*
3·15	0·4983
0·2095	$\bar{1}{\cdot}3212$
0·6600	$\bar{1}{\cdot}8195$

Example.—Calculate the values of $(1-t)e^{-t}$ where $e=2{\cdot}7183$, when $t=0{\cdot}6124$ and when $t=1{\cdot}7329$.

Calling the given expression y, when $t=0{\cdot}6124$,

$$y=0{\cdot}3876\times 2{\cdot}7183^{-0{\cdot}6124}$$

$\therefore \log y=\log 0{\cdot}3876-0{\cdot}6124\log 2{\cdot}7183$

$=\bar{1}{\cdot}5884-0{\cdot}6124\times 0{\cdot}4343$

$=\bar{1}{\cdot}5884-0{\cdot}2661$

$=\bar{1}{\cdot}3223$

$\therefore y=0{\cdot}2100.$

No.	*Log*
0·6124	$\bar{1}$·7871
0·4343	$\bar{1}$·6378
0·2661	$\bar{1}$·4249

When $t=1{\cdot}7329$,

$$y=-0{\cdot}7329\times 2{\cdot}7183^{-1{\cdot}7329}.$$

Since y is now a negative number we cannot take logs of this equation as it stands, so we write :

$$z=0{\cdot}7329\times 2{\cdot}7183^{-1{\cdot}7329}$$

Then $\log z=\log 0{\cdot}7329-1{\cdot}7329\log 2{\cdot}7183$

$=\bar{1}{\cdot}8650-1{\cdot}7329\times 0{\cdot}4343$

$=\bar{1}{\cdot}8650-0{\cdot}7526$

$=\bar{1}{\cdot}1124$

$\therefore z=0{\cdot}1295$

$\therefore y=-z=-0{\cdot}1295$

No.	*Log*
1·7329	0·2388
0·4343	$\bar{1}$·6378
0·7526	1·8766

Example.—Make a table of the values of $8{\cdot}43\times 2{\cdot}6^{-x}$ for $x=-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$, and hence draw the graph of $y=8{\cdot}43\times 2{\cdot}6^{-x}$. Use the graph to find approximately (i) the value of y when $x=-0{\cdot}85$, (ii) the value of x when $y=40$.

By the rules of logarithms :

$$\log y=\log 8{\cdot}43-x\log 2{\cdot}6$$
$$=0{\cdot}9258-0{\cdot}4150x.$$

From this equation the following table is constructed to give $\log y$, and then y is found from the [illegible] tables :

x	-3	-2	-1	0	1	2	3
log. 8·43 ..	0·9258	0·9258	0·9258	0·9258	0·9258	0·9258	0·9258
$-0{\cdot}4150x$ ..	1·2450	0·8300	0·4150	0	$-0{\cdot}4150$	$-0{\cdot}8300$	$-1{\cdot}2450$
log y	2·1708	1·7558	1·3408	0·9258	0·5108	0·0958	$\bar{1}$·6808
y	148·2	56·99	21·92	8·429	3·242	1·247	0·480

The graph is shown in Fig. 28, and from it we find:

(i) $y \simeq 18{\cdot}7$ when $x = -0{\cdot}85$ at the point A.

(ii) $x \simeq -1{\cdot}65$ when $y = 40$ at the point B.

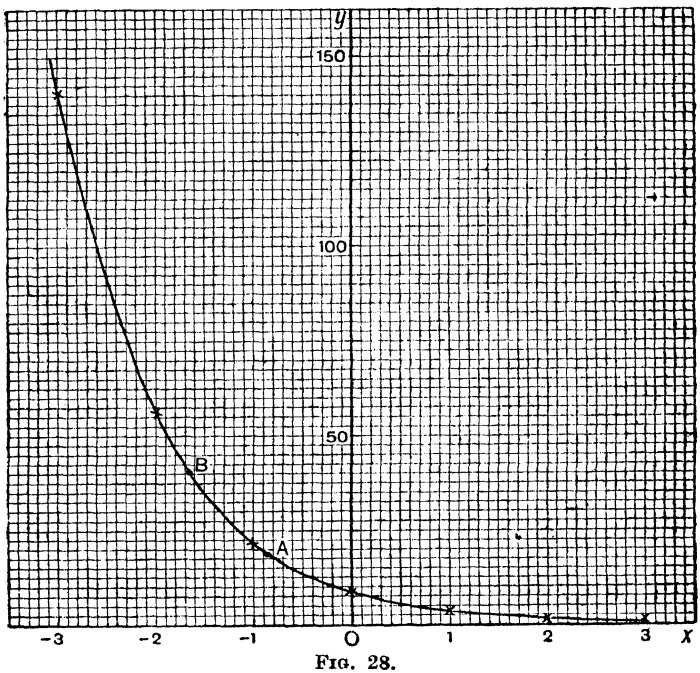

FIG. 28.

Exercise XIV

Calculate the values of:

1. $\dfrac{237{\cdot}9 \times 6{\cdot}248}{9832}$.

2. $\dfrac{0{\cdot}2791}{0{\cdot}04125}$.

3. $\dfrac{0{\cdot}9328 \times 4{\cdot}197^3}{0{\cdot}2196^3}$.

4. $\dfrac{\sqrt{7{\cdot}916} \times 43{\cdot}21}{1760}$.

5. $\dfrac{\sqrt[3]{178{\cdot}2}}{\sqrt{278{\cdot}1}}$.

6. $\dfrac{46{\cdot}19 \times 0{\cdot}001753}{\sqrt[3]{195{\cdot}7}}$.

7. $\left(\dfrac{47{\cdot}23 \times 0{\cdot}1972}{2{\cdot}456}\right)^{\frac{2}{3}}$.

8. $\sqrt{\dfrac{47{\cdot}23 \times 0{\cdot}1792}{2{\cdot}4256}}$.

9. $0{\cdot}735^{\frac{1}{5}}$.

10. $0\ 0009^{\frac{1}{5}}$.

11. $432{\cdot}7^{1{\cdot}427}$.

12. $0{\cdot}6459^{2{\cdot}56}$.

13. $83{\cdot}9^{-1{\cdot}2}$.

14. $0{\cdot}8731^{-4}$.

15. $\dfrac{0{\cdot}0437}{(0{\cdot}9436)^{\frac{1}{3}}}$.

16. $0{\cdot}9712 \times 8{\cdot}7^{0{\cdot}431}$.

17. $\sqrt{\sin 81° 15'}$.

18. $\dfrac{\text{Cos } 23° 4' \cos 41° 52'}{\cos 35° 6'}$.

19. $\dfrac{145{\cdot}2 \sin 59° 10'}{\sin 75° 39'}$.

20. $\dfrac{\tan 37° 23'}{\tan 64° 19'}$.

21. Draw a graph of $y = 3^x$ from $x = -2$ to 2 using values of x at intervals of $\frac{1}{2}$. From your graph read off the values of $\log_3 6$ and $\log_3 0{\cdot}5$.

22. Draw a graph of $y = 10^x$ from $x = 0$ to 1 using values of x at intervals of $\frac{1}{4}$ and a table of square roots. From it read off the values of $10^{0{\cdot}3}$, $10^{0{\cdot}7}$, $\log_{10} 2$, $\log_{10} 6$, $\log_{10} 9$. Deduce the values of $\log_{10} 0{\cdot}2$, $\log_{10} 600$ and $\log_{10} 0{\cdot}09$.

Prove that:

23. $\log \dfrac{p}{q} = \log p - \log q$.

24. $\log \dfrac{1}{k} = -\log k$.

25. $\log \sqrt[n]{\dfrac{a}{b}} = \dfrac{\log a - \log b}{n}$.

26. $\log \left(\dfrac{1}{r}\right)^{-n} = n \log r$.

27. Calculate the values of $50x^{0{\cdot}8}$ when x is $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, 2, 3 and 4, and draw a graph of this function from $x = 0$ to 4. From the graph find at what values of x the function has the values 40 and 120 respectively.

28. Calculate the values of $4{\cdot}75 \times (0{\cdot}943)^x$ when x is -2, -1, to 1, 2 and 3, and draw a graph of $y = 4{\cdot}75 \times (0{\cdot}943)^x$ from $x = -2$ 0, 3. From the graph find the value of y when $x = 1{\cdot}7$ and the value of x when $y = 4$.

29. Using logarithms to base 10 calculate the values of e^x, where $e = 2{\cdot}7183$, when x is -2, -1, 0, 1, 2, and 3. Deduce from these the values of e^{-x} and $\frac{1}{2}(e^x + e^{-x})$, and draw the graphs

of these three functions on the same axes from $x = -2$ to 3. Find the values of x which make the three functions each equal to 6.

30. Draw a graph of $y = 0{\cdot}004x^{5{\cdot}6}$ from $x = 4$ to $x = 6$, and find from it the value of x [to 3 sig. fig.] which makes y equal to 35.

Evaluation of technical formulæ

In calculating the value of some quantity from a formula it is advisable to keep to the symbols as long as possible. There is considerable saving in writing one symbol instead of a number of four figures, but this is not the only advantage, for it is easier to trace mistakes and for any one else to follow the argument if symbols are adhered to. For the same reason, one should not generally write 3·1416 for π, or 1·4142 for $\sqrt{2}$ in a formula. The following examples illustrate the evaluation of several formulæ from engineering and physics.

In any of the following examples a slide rule can be used for every step involving multiplication or division provided that only two significant figures are required in the answer.

Example.—The inductance, L henrys, of a closely-wound coil on a cylindrical former of diameter d ft. and length l ft. is given by :

$$\mathrm{L} = \frac{k\pi^2 d^2 n^2}{l} \times 10^{-9},$$

where n is the number of turns, and k is a constant depending on the leakage of magnetic flux. Find the value of L for a coil of 28 turns wound on a former of diameter 7 in. and length 15 in., if $k = 0{\cdot}8$.

Expressing the diameter and length in feet :

$$d = \tfrac{7}{12} = 0{\cdot}5833 \text{ and } l = \tfrac{15}{12} = 1{\cdot}25.$$

Taking logs and omitting the factor 10^{-9}, which can be inserted later,

$$\begin{aligned}\log\left(\frac{k\pi^2 d^2 n^2}{l}\right) &= \log \pi^2 d^2 n^2 + \log k - \log l \\ &= 2 \log \pi dn + \log k - \log l \\ &= 2 (\log \pi + \log d + \log n) + \log k - \log l.\end{aligned}$$

No.			*Log*
π	..	..	0·4971
d	..	..	$\bar{1}$·7659
n	..	..	1·4472
πdn	..	..	1·7102
$(\pi dn)^2$		..	3·4204
k	..	..	$\bar{1}$·9031
			3·3235
l	..	..	0·0969
1685	..	..	3·2266

$$\therefore \frac{k\pi^2 d^2 n^2}{l} = 1685$$

$$\therefore L \simeq 1685 \times 10^{-9} = 1{\cdot}685 \times 10^{-6}.$$

Example.—The period T of small oscillation of a simple pendulum of length l is given by :

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Calculate the value of g obtained from an experiment in which it is found that the period of oscillation of a pendulum 46 cm. long is 1·36 sec.

First, g should be made the subject of the formula. This can be done before or after taking logs. To make it the subject, first square the given equation :

$$T^2 = \frac{4\pi^2 l}{g}$$

$$\therefore gT^2 = 4\pi^2 l$$

$$\therefore g = \frac{4\pi^2 l}{T^2}$$

$$\therefore \log g = \log 4 + \log\pi^2 + \log l - \log T^2.$$

Alternatively, taking logs of the given equation:

$$\log T = \log 2 + \log \pi + \tfrac{1}{2}\log l - \tfrac{1}{2}\log g$$
$$\therefore \tfrac{1}{2}\log g = \log 2 + \log \pi + \tfrac{1}{2}\log l - \log T$$
$$\therefore \log g = 2\log 2 + 2\log \pi + \log l - 2\log T.$$

which is the same value of log g as before, since $2 \log 2 = \log 4$, etc. The calculation is set out below:

No.		*Log*
4		0·6021
π^2	$2 \times 0{\cdot}4971 =$	0·9942
l		1·6628
		3·2591
T^2	$2 \times 0{\cdot}1335 =$	0·2670
g		2·9921

$$\therefore g = 981{\cdot}9.$$

Note that since $g = \dfrac{4\pi^2 l}{T^2}$ the unit of g is 1 $\dfrac{\text{cm.}}{\text{sec.}^2}$.

Example.—When a volume v_1 of hydrogen is compressed adiabatically to a volume v_2 the relationship between the volumes and the initial and final pressures p_1 and p_2 is $v_2 = v_1\left(\dfrac{p_1}{p_2}\right)^{\frac{1}{1{\cdot}4074}}$. Calculate v_2 if $v_1 = 3{\cdot}45$ ft.^{3}, $p_1 = 15$ lb./in.^{2} and $p_2 = 200$ lb./in.^{2}.

Since the ratio p_1/p_2 is independent of the units of p_1 and p_2,

$$v_2 = 3{\cdot}45\left(\frac{15}{200}\right)^{\frac{1}{1{\cdot}4074}}$$

gives the volume in cubic feet.

$$\therefore \log v_2 = \log 3{\cdot}45 + \frac{1}{1{\cdot}4074}(\log 15 - \log 200)$$
$$= 0{\cdot}5378 + \frac{1}{1{\cdot}4074}(1{\cdot}1761 - 2{\cdot}3010)$$

$$= 0{\cdot}5378 - \frac{1{\cdot}125}{1{\cdot}4074}$$

No.	*Log*
1·125	0·0511
1·4074	0·1483
0·7995	$\bar{1}$·9028

$$= 0{\cdot}5378 - 0{\cdot}7995$$

$$= -1 + (1{\cdot}5378 - 0{\cdot}7995)$$

$$= \bar{1}{\cdot}7383$$

$$\therefore v_2 = 0{\cdot}5474.$$

The use of indices and logarithms in changing the subject of a formula

Suppose y is to be made the subject of the formula $y^{0{\cdot}3} = 2$. To do this $y^{0{\cdot}3}$ must be raised to such a power that the result is y. This power is $\frac{1}{0{\cdot}3}$ because,

$$(y^{0{\cdot}3})^{\frac{1}{0{\cdot}3}} = y^{0{\cdot}3 \times \frac{1}{0{\cdot}3}} = y^1 = y.$$

Hence, raising both sides of the equation $y^{0{\cdot}3} = 2$ to the power $\frac{1}{0{\cdot}3}$,

$$y = 2^{\frac{1}{0{\cdot}3}} = 2^{\frac{10}{3}} \text{ or } \sqrt[3]{2^{10}}.$$

$$\therefore y = \sqrt[3]{1024} \simeq 10{\cdot}08.$$

This method can be used to make y the subject of any formula which gives the value of y^n, where n is any number. Sometimes logarithms are required to express y in its simplest form.

Example.—The volume of water Q cu. ft. discharged per minute through a certain V-shaped notch was found by experiment to be given by

$$Q = 2{\cdot}72\ H^{2{\cdot}53}.$$

Make H the subject of this formula.

Dividing by 2·72, $H^{2{\cdot}53} = \frac{Q}{2{\cdot}72}$.

Raising both sides to the power $\frac{1}{2\cdot53}$,

$$H^{2\cdot53\times\frac{1}{2\cdot53}} = \left(\frac{Q}{2\cdot72}\right)^{\frac{1}{2\cdot53}}$$

$$\therefore H = \left(\frac{Q}{2\cdot72}\right)^{\frac{1}{2\cdot53}}.$$

H is the subject of this formula, but the formula is in a more convenient form if we express H in the form aQ^n.

Since $\frac{1}{2\cdot53} = 0\cdot3953$ and $\frac{1}{2\cdot72} = 0\cdot3676$,

$$H = (0\cdot3676Q)^{0\cdot3953} = (0\cdot3676)^{0\cdot3953}Q^{0\cdot3953}.$$

To calculate the numerical coefficient on the right-hand side, let

$$a = (0\cdot3676)^{0\cdot3953}$$

$$\log a = 0\cdot3953 \log 0\cdot3676$$

$$= 0\cdot3953 \times \bar{1}\cdot5654$$

$$= 0\cdot3953\ (-0\cdot4346)$$

$$= -(0\cdot3953 \times 0\cdot4346)$$

$$= -0\cdot1718$$

$$= \bar{1}\cdot8282$$

$$\therefore a = 0\cdot6733$$

No.	*Log*
0·3953	$\bar{1}$·5969
0·4346	$\bar{1}$·6381
0·1718	$\bar{1}$·2350

Hence $H = 0\cdot6733\ Q^{0\cdot3953}$.

Solution of equations using logarithms

If an unknown quantity occurs in an index we can sometimes find its value by taking logs.

Example.—Find x if $2^x = 0\cdot09173$.

Since the logarithm of 2^x is $x \log 2$, by taking logarithms we get :

$$x \log 2 = \log 0\cdot09173$$

$\therefore x(0{\cdot}3010) = \bar{2}{\cdot}9625 = -2 + 0{\cdot}9625$
$= -1{\cdot}0375$
$\therefore x = -\dfrac{1{\cdot}0375}{0{\cdot}3010}$
$= -3{\cdot}446.$

No.	*Log*
1·0375	0·0160
0·3010	$\bar{1}$·4786
3·446	0·5374

Note that before dividing $\bar{2}{\cdot}9625$ by 0·3010 we write $\bar{2}{\cdot}9625$ as a negative number.

Exercise XV

1. The capacity C of a condenser of n plates is given by:

$$C = \frac{kA(n-1)}{4\pi d \times 9 \times 10^{11}}.$$

Find C if $k = 5{\cdot}42$, $A = 35$, $n = 19$ and $d = 0{\cdot}008$.

2. The whirling speed, N rev. per min., of a shaft is given by:

$$N = 30\pi\sqrt{\frac{12gEI}{wl^4}}.$$

Find N, if $\pi = 3{\cdot}142$, $g = 32{\cdot}2$, $E = 30 \times 10^6$, $w = 0{\cdot}28$, $l = 24$ and $I = 0{\cdot}0564$.

3. If a current of frequency n per sec. passes through a wire of resistance R ohm and inductance L henrys, the impedance Z is given by $Z = \sqrt{R^2 + 4\pi^2 n^2 L^2}$. Find the value of Z when $R = 28{\cdot}7$, $L = 0{\cdot}00462$, $n = 400$.

4. If £P is invested at r% compound interest it amounts after n years to £A where $A = P\left(1 + \frac{r}{100}\right)^n$. Find A, if $P = 250$, $r = 4$ and $n = 12$.

5. If $y = \dfrac{Wl^3}{4Ebd^3}$, find E when $y = 0{\cdot}685$ cm., $W = 20{,}000$ gm., $l = 57{\cdot}7$ cm., $b = 2{\cdot}54$ cm., $d = 0{\cdot}635$ cm. State the units of E.

6. The ratio of the tensions T_1 and T_2 of the parts of a belt on opposite sides of a grooved pulley is given by $\dfrac{T_1}{T_2} = e^{(\mu\pi/\sin\alpha)}$. Find T_1, if $T_2 = 150$, $e = 2{\cdot}718$, $\mu = 0{\cdot}3$, $\pi = 3{\cdot}142$, and $\alpha = 20°$.

7. In the construction of an isochronous governor the proportional change in angular velocity due to a change in the angle which the arms make with the vertical from α to β is given by :

$$p=\sqrt{\frac{\cos\alpha}{\cos^3\beta}-\frac{\tan^3\beta}{\tan\alpha}}.$$

Find p, if $\alpha=36°$ and $\beta=30°$.

8. The maximum current I ampères which can flow in a cable without causing a rise in temperature of more than 20° F. is given by $I=2{\cdot}6\ A^{0{\cdot}82}$, where A is the total cross-sectional area of all the wires forming the cable in 1000ths of a sq. in. Find the total current that a cable with 19 wires of diameter 0·064 in. can transmit without the temperature rising more than 20° F. Also calculate the least number of wires of diameter 0·083 in. which must be used to carry a current of 200 ampères under the same temperature condition.

9. Steinmetz law for the area of a hysteresis loop is $W=\eta B^x$. Find η if $W=4300$, $B=11{,}900$, $x=1{\cdot}6$.

10. The quantity of water flowing over a weir is calculated from the formula $Q=3{\cdot}10\ L^{1{\cdot}02}\ H^{1{\cdot}47}$. Find Q, if $L=18$, $H=10$.

11. The impedance of a circuit is given by :

$$Z=\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2},$$

where $\omega=2\pi n$. Find Z if $R=15$, $L=0{\cdot}1$, $n=50$, $C=80\times10^{-6}$. {Use square and reciprocal tables}.

12. The percentage efficiency of a petrol engine is given by:

$$E=100\left[1-\left(\frac{1}{R}\right)^{0{\cdot}25}\right],$$

where R is the expansion ratio. Make a table of the values of E for values of R from 2 to 18 at intervals of 4. Draw a graph of E against R.

13. The relation between the pressure p and volume v of a gas expanding adiabatically is given by $pv^{1{\cdot}41}=C$. Find C if $p=2000$ when $v=5$. Calculate the values of p when $v=2{\cdot}5$, 7·5, 10, 12·5, and draw a graph of p against v from $v=2{\cdot}5$ to $v=12{\cdot}5$. From it estimate the value of v when $p=1000$.

14. At t sec. after its temperature is 30° C. the temperature of a cooling body, T° C., is given by $T=30e^{-kt}$, where $e=2{\cdot}718$ and k is a constant. If $T=24$ when $t=20$ show that $T=30(0{\cdot}8)^{0{\cdot}05t}$. Calculate the values of T at intervals of 20 sec. from $t=0$ to 120, and draw a graph of T against t.

15. The cutting speed of a tool, V ft./min., and the life of the tool, t min., are related by the formula $Vt^{\frac{1}{3}} = C$, where C is a fixed number. It is found that for a certain tool $t = 40$ when $V = 110$. Calculate the value of C and draw a graph of V against t from $t = 20$ to $t = 200$. Estimate at what speed the tool will last for 128 min.

16. If $T_2 = T_1\left(\frac{v_1}{v_2}\right)^{\gamma-1}$ find the value of T_2 when $T_1 = 3466$, $v_1/v_2 = 1/5$, and $\gamma = 1{\cdot}4$.

Make the letter in brackets after each formula the subject of the formula:

17. $N = 188\sqrt{\frac{EI}{wl^4}}$ (l). **18.** $vt^{\frac{1}{3}} = C$ (t).

19. $T_2 = T_1\left(\frac{v_1}{v_2}\right)^{\gamma-1}$ (v_1). **20.** $I = 2{\cdot}6A^{0{\cdot}32}$ (A).

21. $E = 100\left\{1 - \left(\frac{1}{R}\right)^{0{\cdot}25}\right\}$ (R). **22.** $pv^n = C$ (n).

23. $Q = 3{\cdot}10L^{1{\cdot}02}H^{1{\cdot}47}$ (L). **24.** $A = P\left(1 + \frac{r}{100}\right)^n$ (r).

25. The power P watts dissipated per sq. cm. from the cooling surface of a cylindrical wire of radius r cm. is given by $P = \frac{pi^2}{2\pi^2 r^3}$ where i ampères is the current and ρ ohm/cm.^{3} the resistivity of the wire. Make (a) i, (b) r, the subject of this formula. If for a certain material $\rho = 1{\cdot}8 \times 10^{-6}$, calculate the values of r required for $i = 1$, 2, 5 and 10, when $P = \frac{1}{10}$.

26. If $pv^{\frac{16}{15}} = a$, and $pv = b$, express p and v in terms of a and b. Calculate the value of p if $a = 490$ and $b = 429$.

Logarithms to any base; change of base

To find the logarithm of a number y to base a we have to find x so that $y = a^x$. Taking logs of this equation to base 10:

$$\log_{10} y = \log_{10} (a^x) = x \log_{10} a$$

$$\therefore\ x = \frac{\log_{10} y}{\log_{10} a}.$$

that is $$\log_a y = \frac{\log_{10} y}{\log_{10} a}.$$

It follows that

$$\log_y a = \frac{\log_{10} a}{\log_{10} y} = \frac{1}{\log_a y}.$$

Example.—Find $\log_2 3$.

$$\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2} = \frac{0{\cdot}4771}{0{\cdot}3010} = 1{\cdot}585.$$

The above formulæ can be remembered easily in the following way. Remove the word *log* and write a line between each number and the base. The equations are then still true for the resulting fractions. Thus the last equation on p. 94 becomes:

$$\frac{y}{a} = \frac{y/10}{a/10}.$$

The base *e*

There is one base that is commonly used as well as the base 10. This is the base *e*, which is nearly 2·71828. Logarithms to base *e* are called "natural" logarithms or "Napierian" logarithms after Napier. The reason why the number *e* is of importance in mathematics is that there are considerable simplifications in some of the more advanced work if numbers are expressed as powers of the base *e* rather than of the base 10.

To four decimal places

$$\log_{10} e = 0{\cdot}4343;$$

to six significant figures

$$\log_{10} e = 0{\cdot}434294$$

and therefore $\log_e 10 = \dfrac{1}{0{\cdot}434294} = 2{\cdot}3026.$

Hence

$$\log_e a = \log_{10} a \div \log_{10} e = \log_{10} a \times \log_e 10 = 2{\cdot}3026 \times \log_{10} a.$$

In words, *to convert logarithms to base* 10 *into logarithms to base* e *multiply by* 2·3026.

Example.—Express 5 as a power of e.

If $5 = e^x$, then $\log_{10} 5 = x \log_{10} e$.

$\therefore x \times 0{\cdot}4343 = 0{\cdot}6990$

$\therefore x = \dfrac{0{\cdot}6990}{0{\cdot}4343} = 1{\cdot}610$. Hence $5 = e^{1{\cdot}610}$.

In some books tables of logarithms to base e are given. The log of a number to base e can then be read directly from the tables provided the number lies within the range covered by tables. Most books give the logs of numbers from 1 to 5, and also $\log_e 10$, $\log_e 20$, etc. . . . $\log_e 100$. If a number does not lie between 1 and 5, express it as a product or a quotient of a number between 1 and 5 and one of the numbers 2, 10, 20, etc.

Example.—Find $\log_e 764{\cdot}8$ using logs to base 10 and from tables of logs to base e.

Using logs to base 10

$$\log_e 764{\cdot}8 = 2{\cdot}3026 \log_{10} 764{\cdot}8$$
$$= 2{\cdot}3026 \times 2{\cdot}8836$$
$$= 6{\cdot}6398$$

Using logs to base e

$$\log_e 764{\cdot}8 = \log_e (200 \times 3{\cdot}824)$$
$$= \log_e 2 + \log_e 100 + \log_e 3{\cdot}824$$
$$= \log_e 2 + 2 \log_e 10 + \log_e 3{\cdot}824$$
$$= 6{\cdot}6396.$$

No.	*Log*
2	0·6931
100	4·6052
3·824	1·3413
	6·6396

Example.—The capacity of a cable which consists of a conducting cylindrical core of diameter d surrounded by a coaxal conducting sheath of external diameter D, is $\dfrac{K}{2 \log_e \left(\dfrac{D}{d}\right)}$ electrostatic units of capacity per cm. where K is the dielectric constant for the insulator between the core and the sheath. Given that 1 microfarad $= 9 \times 10^5$ electrostatic units and 1 in. $= 2{\cdot}54$ cm.

show that the capacity of the cable is $\frac{0{\cdot}03883K}{\log_{10}\left(\frac{D}{d}\right)}$ microfarads per mile. Calculate this capacity when $d = 0{\cdot}064$, $D = 0{\cdot}3875$ and $K = 3{\cdot}2$.

Capacity of 1 mile

= capacity of $5280 \times 12 \times 2{\cdot}54$ cm.

$$= \frac{5280 \times 12 \times 2{\cdot}54K}{2 \log_e \left(\frac{D}{d}\right)} \text{ electrostatic units}$$

$$= \frac{5280 \times 12 \times 2{\cdot}54K}{2 \times 9 \times 10^5 \log_e \left(\frac{D}{d}\right)} \text{ microfarads}$$

$$= \frac{5280 \times 12 \times 2{\cdot}54K}{2 \times 9 \times 10^5 \times 2{\cdot}3026 \log_{10} \frac{D}{d}} \text{ microfarads}$$

$$= \frac{0{\cdot}03883K}{\log_{10}\left(\frac{D}{d}\right)} \text{ microfarads.}$$

No.	*Log*	
5280	3·7226	
12	1·0792	
2·54	0·4048	
	5·2066	5·2066
18×10^5	6·2553	
2·3026	0·3622	
	6·6175	6·6175
0·03883		$\bar{2}$·5891

When $K = 3{\cdot}2$, $d = 0{\cdot}064$, $D = 0{\cdot}3875$,

$$\text{capacity} = \frac{0{\cdot}03883 \times 3{\cdot}2}{\log\left(\frac{0{\cdot}3875}{0{\cdot}064}\right)}$$

$$= \frac{0{\cdot}03883 \times 3{\cdot}2}{\log 0{\cdot}3875 - \log 0{\cdot}064}$$

$$= \frac{0{\cdot}03883 \times 3{\cdot}2}{0{\cdot}7821}$$

$$= 0{\cdot}1588.$$

No.	*Log*
0·3875	$\bar{1}$·5883
0·064	$\bar{2}$·8062
	0·7821

No.	*Log*
0·03883	$\bar{2}$·5891
3·2	0·5051
	$\bar{1}$·0942
0·7821	$\bar{1}$·8933
	2009

Hence the capacity is 0·159 microfarads per mile.

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Exercise XVI

Find the values of:

1. $\log_3 9$. 2. $\log_{10} 150$. 3. $\log_2 0{\cdot}1$. 4. $\log_4 \sqrt[3]{2}$.

5. Show that $\log a/\log b$ has the same value whatever base is used for the logarithms.

6. Find the values of $\log_e 3{\cdot}815$ and $\log_e 42{\cdot}6$ by using a table of Napierian logarithms. Also calculate their values by using a table of ordinary logarithms.

7. The self-inductance L henrys per mile of parallel conductors radius r in., d in. apart, is given by $L = 0{\cdot}000644\left\{\log_e \frac{d}{r} + \frac{1}{4}\right\}$. Calculate the self-inductance of 50 miles of wire if the diameter of the conductors is 0·5 cm., and they are 50 cm. apart.

8. If $n = \dfrac{\log (p_2/p_1)}{\log (v_2/v_1)}$, find n if $p_1 = 100$, $p_2 = 15$, $v_1 = 4{\cdot}89$ and $v_2 = 24{\cdot}4$. Show that the value of n is the same whatever base is used for the logarithms.

9. The change of entropy of a quantity of air during a compression is $C_v\left(\frac{\gamma - n}{n - 1}\right)\log_e\left(\frac{T_1}{T_2}\right)$. Calculate the value of this expression given that $C_v = 0{\cdot}169$, $\gamma = 1{\cdot}408$, $n = 1{\cdot}15$, $T_1 = 189{\cdot}3$ and $T_2 = 144$.

CHAPTER V

VARIATION, LINEAR LAWS

Ratio. Homogeneous equations

In the equation $6x^2 + 4xy + 6y - 3 = 0$, the term $6x^2$ is of the second degree in x, the term $4xy$ is of the first degree in x and the first degree in y, that is, of the second degree altogether, $6y$ is of the first degree and -3 is of no degree.

An equation like $x^2 - 3xy - 9y^2 = 0$ in which all the terms are of the same degree is called a *homogeneous* equation.

Although it is impossible to find the numerical values of two unknown quantities from one equation, ***the ratio of***

the two unknown quantities can be found from a homogeneous equation.

Example.—Find the ratio of y to x if $2x+8y=16y-5x$.

Re-arranging the terms

$$16y-8y=2x+5x$$

$$\therefore\ 8y=7x.$$

Dividing by $8x$,

$$\frac{y}{x}=\frac{7}{8}.$$

Example.—Find the ratio of p to q if $p^2-2pq-4q^2=0$.

Dividing by q^2,

$$\frac{p^2}{q^2}-\frac{2p}{q}-4=0.$$

Writing x for $\frac{p}{q}$,

$$x^2-2x-4=0$$

$$\therefore\ x=1\pm\sqrt{5}$$

$$\therefore\ \frac{p}{q}=1\pm\sqrt{5}.$$

Proportion

If a, b, c are proportional to p, q, r,

$$\frac{a}{p}=\frac{b}{q}=\frac{c}{r}.$$

In some problems involving proportion it is useful to use a single letter for each of the equal ratios.

Example.—The lengths of the sides a, b, c of a triangle ABC are proportional to 7, 6 and 5. Find the value of $\frac{b^2+c^2-a^2}{2bc}$

[this ratio is cos A, see p. 308.]

We are given that $a:7=b:6=c:5$.

Write each ratio equal to k. Then,

$$\frac{a}{7}=\frac{b}{6}=\frac{c}{5}=k$$

$$\therefore\ a=7k,\ b=6k,\ c=5k.$$

$$\therefore\ \frac{b^2+c^2-a^2}{2bc}=\frac{36k^2+25k^2-49k^2}{2\,.\,6k\,.\,5k}$$

$$=\frac{12k^2}{60k^2}=\frac{1}{5}.$$

It should be noted that it is possible to find the numerical value of the ratio of two expressions only if all the terms in each of the expressions are of the same degree. For the example above, if the fraction had been $\frac{b^2-a}{2c}$, we should get

$$\frac{b^2-a}{2c}=\frac{36k^2-7k}{10k}=\frac{36k-7}{10}.$$

As the fraction contains k, which is unknown, it is impossible to find its numerical value.

Variation

The expression "y varies as x", which is written $y \propto x$, means $y=kx$, where k is a fixed number, for every value of x. To distinguish this kind of variation from others we say that y varies directly as x.

Other simple laws are obtained when y varies as some other expression containing x such as x^2, $1/x$, $\log x$. If $y \propto x^2$, then $y=kx^2$ is the law connecting y and x.

$y \propto 1/x$ is given the special name of "inverse variation"; we say "y varies inversely as x." Thus y varies inversely as $\sqrt{x}$ means $y \propto 1/\sqrt{x}$, or $y=k/\sqrt{x}$.

Example.—Assuming that at a constant temperature the pressure, p lb. wt. per sq. ft., of a gas varies inversely as its volume, v cu. ft. (Boyle's Law), find the law connecting p and v if $p=20$ when $v=4$. Also find p when $v=15$, and v when $p=30$.

Since p varies inversely as v, $p = k/v$.
But $p = 20$ when $v = 4$.
$\therefore\ 20 = k/4 \qquad \therefore\ k = 80.$
$\therefore\ p = 80/v.$
When $v = 15$, $p = 80/15 = 16/3 = 5\frac{1}{3}$.
When $p = 30$, $80/v = 30 \qquad \therefore\ v = 80/30 = 2\frac{2}{3}$.

Graphs of variation

It is important to know the shapes of the graphs of different kinds of variation. The graph of direct variation $y = kx$ is a straight line of gradient k through (0, 0).

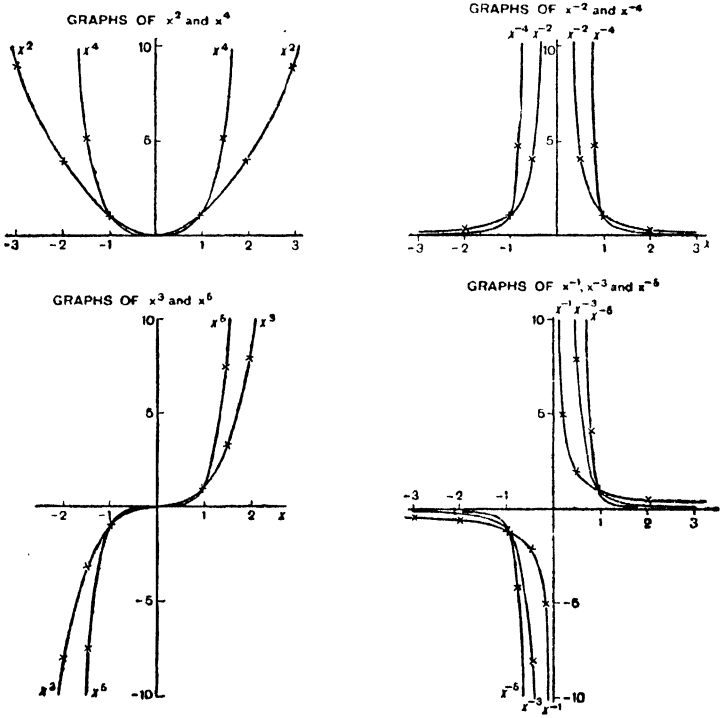


Fig. 29.

If $y \propto x^n$, $y = kx^n$ and the graph of this equation is of the same shape as the graph of $y = x^n$, because it is obtained by multiplying each ordinate of $y = x^n$ by k. Hence it is sufficient if we know the shape of the graph of $y = x^n$ for different values of n.

At the top of Fig. 29 the graphs of $y = x^2$ and $y = x^4$, and the graphs of $y = \frac{1}{x^2}$ and $y = \frac{1}{x^4}$ are shown. Since in all these graphs y has the same value if the sign of x is changed, the graphs are symmetrical about the vertical axis.

At the bottom of Fig. 29 the graphs of $y = x^3$ and $y = x^5$, and the graphs of $y = \frac{1}{x}$, $y = \frac{1}{x^3}$, $y = \frac{1}{x^5}$ are shown. In each of these graphs, if the sign of x is changed y keeps the same numerical value, but changes sign. For instance, if P and Q are the points on $y = x^3$ given by $x = +a$ and $x = -a$, as in Fig. 30, PQ is bisected at O. When this is so a graph is said to be symmetrical about the origin O.

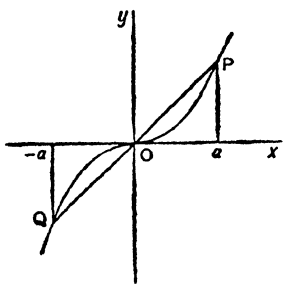


FIG. 30.

Since $x^{2\cdot3}$ lies between x^2 and x^3 for every positive value of x, the graph of $y = x^{2\cdot3}$ for positive values of x lies between the graphs of $y = x^2$ and $y = x^3$. Between $x = 0$ and 1, $y = x^2$ comes above $y = x^3$ and $y = x^{2\cdot3}$ lies between them. When $x > 1$, $y = x^2$ is below $y = x^3$ and $y = x^{2\cdot3}$ is between them again. The three graphs are shown in Fig. 31.

When we expect that values of y and x obtained from an experiment are such that y varies as some power of x, by plotting y against x we can see if the graph looks like one of the forms of the graph of x^n in Fig. 29, but the fact that the shapes just look similar is not a proof that $y \propto x^n$. We shall see on p. 115 how we can test whether a relationship of this type really exists.

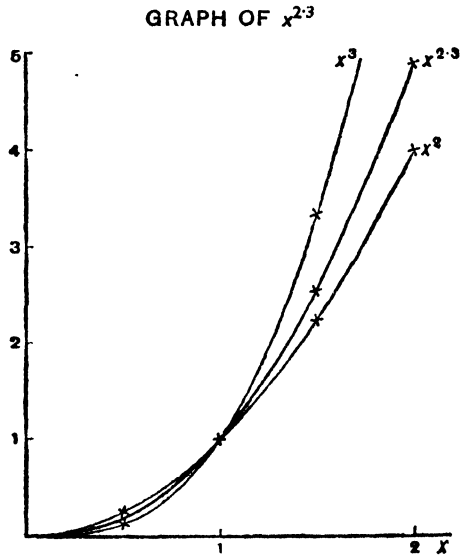


FIG. 31.

Joint Variation

If the volume of a cylinder of radius r ft. and height h ft. is V cu. ft., $V = \pi r^2 h$,

$$\therefore h = \frac{V}{\pi r^2}.$$

Now, if r is kept fixed and V varied, $(1/\pi r^2)$ is constant, and hence $h \propto V$. Fig. 32 shows three cylinders of the same radius but different volumes; in this figure $h \propto V$.

If, however, V is kept fixed and r varied, $\frac{V}{\pi}$ is constant and $h \propto \frac{1}{r^2}$. Fig. 33 shows three cylinders of the same volume but with different radii; in this figure $h \propto \frac{1}{r^2}$.

Thus we may say that h varies directly as V and inversely as the square of r. This is called "joint variation," and we may say that h varies jointly as V and $1/r^2$.

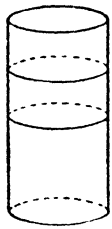


FIG. 32.

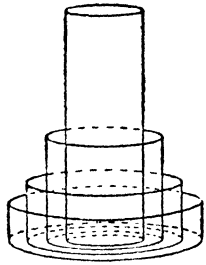


FIG. 33.

In general, if the value of z depends on x and y, and if $z \propto x$ when y is constant and $z \propto y$ when x is constant, then $z \propto xy$ when x and y both vary, which means that the law relating z, x and y is $z = kxy$, where k is a constant.

Example.—When a given beam is placed on two supports A and B, l in. apart, and a load W lb. wt. is hung from the middle point of the beam, the deflection y in. produced by the load varies jointly as W and l^3.

In an experiment it is found that $y = 0{\cdot}35$ when $W = 5$ and $l = 40$. Find the relation between y, W and l, and from it calculate the deflection produced by a load 8 lb. wt. when $l = 60$.

Since $y \propto W$, and $y \propto l^3$,

$$\therefore y \propto Wl^3,$$

that is $y = kWl^3$, where k is constant.

Substituting $y = 0{\cdot}35$, $W = 5$, $l = 40$

$$0{\cdot}35 = k \times 5 \times 40^3$$

$$\therefore k = \frac{0{\cdot}35}{5 \times 40^3} = \frac{0{\cdot}07}{40^3} = 1{\cdot}094 \times 10^{-6}$$

$$\therefore y = 1{\cdot}094 \times 10^{-6} Wl^3.$$

When $W = 8$ and $l = 60$,

$$y = 1{\cdot}094 \times 10^{-6} \times 8 \times 60^3 = 1{\cdot}89.$$

Therefore the deflection is 1·89 in.

Example.—It is known that the electrical resistance, R ohms, of a wire of given material varies directly as its length and inversely as the area of its cross section. Find the relation between the resistance of a wire, its length l cm. and its weight W gm. wt.

Let the area of the cross-section be A sq. cm. Then $R \propto l$ and $R \propto \frac{1}{A}$.

$$\therefore R = \frac{kl}{A}.$$

But the weight of the wire varies as its volume lA cu. cm.

$$\therefore W = k_1 l A$$

$$\text{whence } A = \frac{W}{k_1 l}.$$

Substituting this value of A in the value of R.

$$R = kl \Big/ \left(\frac{W}{k_1 l}\right) = \frac{k k_1 l^2}{W}.$$

$$\therefore R = \frac{K l^2}{W},$$

where K is a constant for a wire of given material.

Exercise XVII

1. Find the ratio of r_1 to r_2 if $l_1/r_1 = l/(r_2 - r_1)$.

2. Find the ratio of p to q if $3(p + 2q) = 5(p - q)$.

3. Find the ratio of m to n if $(m^2 - n^2)/(m^2 + n^2) = 2/3$.

4. If six equal circles are drawn so that each touches two others and they all touch two concentric circles of radii R and r, show that $(R - r)/(R + r) = \sin 30°$. Hence find the ratio of R to r.

5. If in the figure in Question 4, there are n circles touching the two concentric circles, show that $(R-r)/(R+r)=\sin\frac{\pi}{n}$, and find the ratio of R to r.

6. If a circle of radius b and centre C rolls on a fixed circle of centre A and radius a, the angular velocity ω of the rolling circle and Ω the angular velocity of AC are related by the equation $a\Omega=b(\omega-\Omega)$. Find the ratio of Ω to ω.

7. If $\frac{R_1}{R_2}=\sqrt{\frac{R_1^2+R_3^2}{R_2^2+R_4^2}}$ prove that $R_1 : R_2 = R_3 : R_4$.

8. The acceleration f of two masses m_1 and m_2 connected by a string over a smoothly-running pulley is given by $f=\left(\frac{m_1-m_2}{m_1+m_2}\right)g$. Find the ratio of m_1 to m_2 in terms of f and g.

9. If there is no current in the galvanometer arm of a Wheatstone bridge the resistances of the other four wires and the currents in them are related by the equations $R_1i_1=R_4i_2$, $R_2i_1=R_3i_2$. Prove that $R_1 : R_2 = R_4 : R_3$.

10. If l, m and n are proportional to 3, 4 and 6, calculate the values of

(i) $\frac{l+m+n}{l+m-n}$, (ii) $\frac{lm-n^2}{mn-l^2}$, (iii) $\frac{n^4}{l^2m^2}$.

11. If a, b, c are proportional to $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{5}{4}$ find the values of

(i) $\frac{a+b}{b+c}$, (ii) $\frac{c^2-a^2}{c^2-b^2}$,

and show that it is impossible to find the values of $\frac{c^2}{a}$ and $\frac{a+c}{ab-c^2}$.

12. The air resistance R lb. wt. to the motion of a certain locomotive and 12 coaches varies as the square of the velocity V m.p.h. If the resistance is 710 lb. wt. when the speed of the train is 30 m.p.h., find R in terms of V. Draw a graph to show the values of R from V = 0 to 120. From the graph read off the values of R when V = 48, 86 and 112, and the value of V when R = 8000.

13. If water flows out of a tank through a tap in the bottom the volume V cu. ft. which flows out per minute varies as the square root of the depth x ft. of water left in the tank. If V = 6·5 when $x=6$, find the relation between V and x. How fast is the water flowing out when it is 2 ft. deep? Draw a graph of V against

x from $x=0$ to 6. From it read off the value of V when $x=4{\cdot}7$ and the value of x when $V=4{\cdot}2$.

14. If $h=\frac{glr}{v^2}$ in what ratio is h decreased if r is halved and v is doubled ?

15. If y varies inversely as $\sqrt{x}$ in what ratio is y changed if x is multiplied by n ?

16. For a given voltage the current in a wire varies inversely as its resistance. If the current is 10 ampères for a resistance of 2·5 ohms find the relationship between the current and resistance, and draw a graph of this relation for resistances from 0·5 to 5 ohms.

17. In experiments on the wave resistance of ships it is found that in order to produce similar behaviour in a model and a ship the ratio (length)/(velocity)^2 must be the same for the model and the ship. Express the velocity v_m of the model in terms of the velocity v of the ship and the lengths l and l_m of the ship and model respectively.

18. If z varies jointly as x and y state the formula relating x, y and z with a constant of variation. Find this constant if $z=10$ when $x=5$ and $y=4$. Find z when $x=0{\cdot}2$, $y=40$, and find x when $z=16$, $y=15$.

19. x varies jointly as f and as the square of t. Express this as a formula. If $x=64$ when $f=32$ and $t=4$ find the constant of variation. Find x when $f=3$ and $t=10$, and find t when $x=100$, $f=2$. Also make t the subject of the formula.

20. The volume V of a cone varies as its height h and as the square of its base radius r. It is found experimentally that $V\simeq 100$ cu. in. when $h=6$ in., $r=4$ in. Find the formula connecting V, h and r. If the height of the cone is doubled and the base radius is trebled in what ratio is the volume of the cone increased ?

21. The pressure of a gas p lb./ft.^{2} varies directly as its absolute temperature T° when its volume is constant, and inversely as its volume v cu. ft. when its temperature is constant. Express this by means of a formula with a constant of variation k. Find k for 1 lb. hydrogen if $T=303$ when $p=20{,}000$ and $v=35$. From the formula find T when $p=10{,}000$ and $v=5$, and find v when $T=100$ and $p=5000$.

22. The resistance R ohms of a wire of given material varies directly as its length l cm. when its diameter d cm. is constant, and inversely as the square of its diameter when its length is constant.

Write down the formula connecting R, l and d. Find the constant of variation if R = 2·5 for a wire 100 cm. long, 0·015 cm. diameter. From the formula (a) calculate R for a wire 150 cm. long, 0·02 cm. diameter, (b) find what length of wire of 0·1 cm. diameter will have a resistance of 3·4 ohms.

Express each of the following statements as a formula with a constant of variation k.

23. The moment of inertia I of a cone about its axis varies as its height h and as the fourth power of its base radius r.

24. The torque N lb. wt. ft. required to twist one end of a wire of radius r in. and length l in. through θ radians when the other end is fixed, varies directly as l and as θ and varies as the fourth power of r.

25. The bending moment M at the middle point of a beam supported at its ends varies directly as its weight W and as its length l.

26. The exposure E necessary for a photograph varies as the square of the stop f used and inversely as the speed s of the plate.

27. The self-inductance L of a solenoid varies as the square of the number of turns n, as the area of the cross-section A and inversely as the length l.

28. The period T of small horizontal oscillations of a body hung by a vertical wire varies as the square root of the length l of the wire and inversely as the square of the diameter d of the wire.

29. The capacitance C farads of a parallel plate condenser of n plates varies directly as $(n-1)$ and as A sq. cm. the area of a plate, and inversely as d cm. the distance between the plates. If the capacitance is 424 micromicrofarads for a condenser with 9 plates each of 9 sq. cm. and 1 mm. apart, find the formula connecting C, n, A and d, and from it calculate the capacitance of a condenser with 11 plates each of 50 sq. cm. with 2 mm. between the plates.

30. The force F required to stretch a wire of given material varies directly as the area of its cross-section A, directly as the distance x by which the wire is extended and inversely as l the natural length of the wire. It is found that a force of 50 lb. wt. is required to stretch a wire 12 ft. long, 0·02 in. radius through a distance 0·15 in. Find the law connecting F, A, x and l. From

it calculate the force required to stretch a wire 6 ft. long, 0·03 in. radius, made of the same material, through a distance 0·05 in.

31. For ships of the same shape the wave resistance varies as $l^{3+k}v^{2-2k}$ where l is a linear dimension of the ship (e.g. its length) and v is its velocity. Show that if the linear dimensions of a ship are n times those of a model and the velocity of the ship is $\sqrt{n}$ times the velocity of the model, then the resistance to the motion of the ship is n^3 times the resistance to the model.

32. The resistance R of a wire varies as its length l and inversely as the square of its diameter d. The weight W of the wire varies as its length and as the square of its diameter. Find the formula connecting R, W and l. The resistance of a bronze wire used for a telephone line is 22 ohms per mile and the weight of the wire is 80 lb. per mile; find the resistance of a line of 8 miles using wire of the same material, but weighing 120 lb. per mile.

33. A circuit has a variable condenser and a fixed induction coil. The resonance frequency f of the circuit varies inversely as the square root of the capacity C of the condenser. The capacity of the condenser varies inversely as the distance d between its plates. How does the resonance frequency depend on d?

34. If lamps of given candle-power are constructed to work at a definite efficiency for all values of the voltage V, the length of the filament l, its diameter d, its resistance R and the current i through it are related to one another and to V in the following way: $i \propto \frac{1}{V}$, $d^3 \propto i^2$, $d \propto 1/l$, $R \propto l/d^2$. Prove that $R \propto V^2$, $d \propto V^{-2/3}$ and $l \propto V^{2/3}$.

Sketch roughly, both for positive and negative values of the variables, the graphs of the following equations.

35. $y = 2x^3$. **36.** $y = -2x^3$. **37.** $y = 4/x^2$. **38.** $S = 16t^2$.

39. $pv = 50$. **40.** $y^2 = 4x$. **41.** $v^2 = 20h$. **42.** $Rd^2 = 100$.

Indicate by a rough sketch the graph of y against x when:

43. $y \propto x^3$. **44.** $y \propto \frac{1}{x}$. **45.** $y \propto x^{1\cdot5}$.

46. $y \propto x^{0\cdot7}$. **47.** $y \propto x^{-1\cdot2}$.

Laws containing two constants

The laws or equations of variation which we have used so far have each contained only one constant and in each case

that constant has been found by substituting some special pair of values of the variable quantities in the epuation.

In many cases physical and mechanical quantities are related by more complicated laws containing two or more constants. The simplest law involving two constants is the equation $y = ax + b$ where x and y are the variables and a and b the constants. a and b can be found from the pair of simultaneous equations obtained by substituting in this equation two pairs of values of x and y. Because the graph of $y = ax + b$ is a straight line of gradient a (pp. 35 and 36) this law is sometimes called a " straight line law " or " linear law." It includes the equation of direct variation $y = kx$ as a special case in which the graph passes through the origin.

Other simple laws with two constants are obtained by replacing x by x^2, $\frac{1}{x}$ or some other expression in the equation $y = ax + b$. For instance, the law $y = ax^3 + b$ is obtained by replacing x by x^3. The graph of this law is not of course a straight line if we plot y against x, but it is a straight line if we plot y against x^3. The constants in any two-constant law are found by substituting two pairs of values of x and y in the equation.

Example.—The resistance to the motion of a motor car is R lb. wt. at a speed of v m.p.h. and R is related to v by the equation $R = av^2 + b$. If $R = 53$ when $v = 10$, and $R = 65$ when $v = 20$ find the law connecting R and v, and also find R when $v = 70$.

Substituting the given values of R and v,

$$53 = 100a + b$$

$$65 = 400a + b$$

Subtracting $12 = 300a$ $\quad \therefore a = 0{\cdot}04.$

Substituting in the first equation

$$53 = 100 \times 0{\cdot}04 + b$$

$$\therefore b = 53 - 4 = 49.$$

Hence the required law is

$$R = 0{\cdot}04v^2 + 49.$$

When $v = 70$,

$$R = (0{\cdot}04 \times 4900) + 49 = 245.$$

Another common type of law containing two constants occurs when one variable quantity y varies as some unknown power of another variable quantity x. Suppose $y \propto x^n$, then $y = kx^n$, where both the constants k and n are unknown.

To find these constants we have to take logarithms. The method used is shown in the following example.

Example.—During the expansion of a gas without loss of heat the pressure and volume are related by the equation $pv^n = c$, where n and c are constants for a particular gas. If $p = 50$ when $v = 20$ and $p = 280$ when $v = 5{\cdot}14$ find the values of n and c, and state the relation between p and v.

Since
$$pv^n = c$$
$$\log (pv^n) = \log c$$
$$\therefore \log p + n \log v = \log c.$$

Substituting the values of p and v,

$$\log 50 + n \log 20 = \log c$$
$$\log 280 + n \log 5{\cdot}14 = \log c.$$

Hence,
$$1{\cdot}6990 + 1{\cdot}3010n = \log c$$
$$2{\cdot}4472 + 0{\cdot}7110n = \log c.$$

Subtracting,

$$-0{\cdot}7482 + 0{\cdot}5900n = 0$$
$$\therefore n = \frac{0{\cdot}7484}{0{\cdot}5900} \simeq 1{\cdot}268.$$

Substituting,	*No.*	*Log*
$1{\cdot}6990 + 1{\cdot}3010 \times 1{\cdot}268 = \log c$	1·3010	0·1142
$\therefore \log c = 1{\cdot}6990 + 1{\cdot}649$	1·268	0·1031
$= 3{\cdot}348$		
$\therefore c \simeq 2228.$	1·649	0·2173

Hence the law connecting p and v is approximately

$$pv^{1{\cdot}27} = 2230.$$

Straight line law from experimental values

If it is thought that the values of x and y obtained from an experiment are related by a straight line law $y = ax + b$, this conjecture can be tested by plotting the values of y against the corresponding values of x. If the points lie nearly on a straight line then we are justified in concluding that the relationship between x and y is represented very nearly by the equation $y = ax + b$. The graph of this equation is taken to be a straight line drawn as evenly as possible between the points, and the constants a and b are found by substituting the values of x and y at two points *on the line*, which are generally not points given by the experimental values. Alternatively a can be found from the gradient of the line, and b, being the value of y at $x = 0$, can be read directly from the graph *provided that the line $x = 0$ comes on the graph paper.*

Example.—In an engine test the values of the indicated horse power I, brake horse power B and steam consumption S lb. per hour were found to be :

I ..	1·197	4·17	5·85	7·96	8·91	10·77	14·68
B ..	0	2·91	4·55	6·32	7·45	9·05	12·3
S ..	51·6	171·8	216	316	351	410	564

Show that the relations between B and I and between S and I are both approximately linear laws. Find approximations to each of these laws.

The graphs of B against I and S against I are plotted from the above table in Fig. 34 and Fig. 35. The points in each figure lie nearly on a straight line, showing that in both figures a linear law is a very good approximation to the relation between the variables. In Fig. 34 let $B = a + bI$. At the point C, $I = 2$, $B = 1$ and at D, $I = 12$, $B = 10$.

$$\therefore\ 10 = a + 12b,$$
$$1 = a + 2b,$$

whence $9 = 10b$. $\quad\therefore\ b = 0{\cdot}9$ and $a = 1 - 1{\cdot}8 = -0{\cdot}8$.

Hence $B = -0{\cdot}8 + 0{\cdot}9I$.

GRAPH OF BRAKE HORSE POWER B AGAINST INDICATED HORSE POWER I

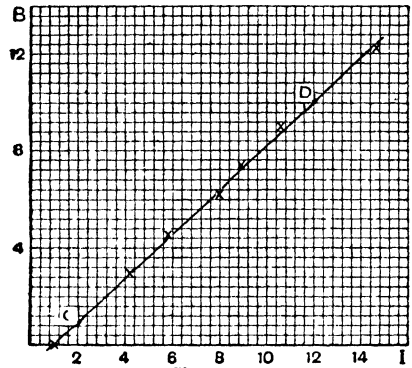


FIG. 34.

In Fig. 35 let $S = a + bI$. The points E, $I = 2$, $S = 80$, and F, $I = 12$, $S = 460$, lie on the line.

$\therefore 460 = a + 12b,$

$80 = a + 2b,$

whence $380 = 10b$. $\therefore b = 38$, $a = 80 - 76 = 4$.

Hence $S = 4 + 38I$.

GRAPH OF STEAM CONSUMPTION S LP/HR AGAINST INDICATED HORSE POWER I

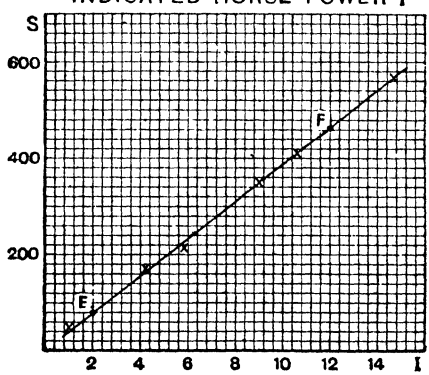


FIG. 35.

Laws which can be converted to straight line laws

The equation $xy = ax^2 + b$ becomes $Y = aX + b$ if we write $xy = Y$ and $x^2 = X$. The graph of Y against X is then a straight line and this means that, if *we plot xy against* x^2, *the graph of* $xy = ax^2 + b$ *is a straight line.* It is, of course, a curve when y is plotted against x.

It is therefore possible to test whether x and y are related by a law of the form $xy = ax^2 + b$ by plotting xy against x^2 and seeing whether the points lie on a straight line or not. If they do then the approximate values of a and b can be found in the same way as before, by drawing the straight line which lies most evenly between the points and calculating a and b by substituting the co-ordinates of two points in the equation remembering that these co-ordinates are values of xy and x^2, not of y and x.

In the same way any two-constant law which can be put in the form $Y = aX + b$ where X and Y stand for expressions containing x and y gives a straight line graph if we plot Y against X.

Example.—In the following table w watts is the iron loss in a dynamo due to hysteresis when f is the frequency of the current. Show that the relationship between w and f is of the form $w = af + bf^2$ and find the approximate values of a and b.

w	..	45	62	76	90	120
f	..	23	30	35·5	40	49·7

Dividing by f the given equation becomes

$$\frac{w}{f} = a + bf$$

which has a straight line graph if w/f is plotted against f. The values of w/f are tabulated below and the points in Fig. 36 are plotted from this table.

f	..	23	30	35·5	40	49·7
$\frac{w}{f}$	..	1·957	2·067	2·141	2·230	2·414

These points lie nearly on the straight line AB, thus showing that the assumed law is the relation between $\frac{w}{f}$ and f. The point A is given by $f=20$, $\frac{w}{f}=1{\cdot}905$; and B by $f=50$, $\frac{w}{f}=2{\cdot}39$.

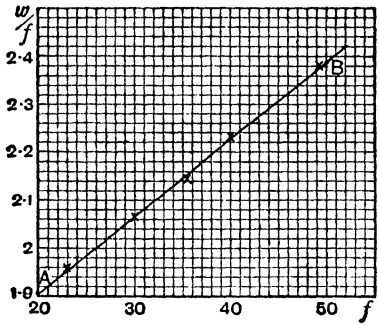


Fig. 36.

Substituting these values,

$$2{\cdot}39 = a + 50b,$$
$$1{\cdot}905 = a + 20b,$$
$$\therefore\ 0{\cdot}485 = 30b$$
$$\therefore\ b = 0{\cdot}0162, \text{ and } a = 1{\cdot}905 - 0{\cdot}324 = 1{\cdot}581.$$

Hence the law relating w and f is :

$$\frac{w}{f} = 1{\cdot}581 + 0{\cdot}0162f, \text{ or } w = 1{\cdot}581f + 0{\cdot}0162f^2.$$

y varying as a power of x

If $y \propto x^n$, $y = kx^n$ and hence taking logarithms as on p. 111,

$$\log y = \log k + n \log x.$$

If in this equation we write X for $\log x$, Y for $\log y$ and a for $\log k$, we get $Y = a + nX$ which has a straight line graph if Y is plotted against X, that is $\log y$ plotted against $\log x$.

Conversely, if log y plotted against log x gives a straight line graph, y varies as some power of x and the actual power n and the constant of variation k can be found by substituting the values of log y and log x at two points on the line in the equation $\log y = \log k + n \log x$.

Example.—The following table gives the quantity of water Q cu. ft. per sec. which flows through a V-shaped notch when the depth of the water is H ft. Show that Q varies as a power of H and find approximately the law connecting Q and H.

H ..	0·166	0·209	0·270	0·338	0·392	0·473	0·550
Q ..	0·0293	0·0509	0·0987	0·171	0·258	0·410	0·602

If $Q = kH^n$, $\log Q = \log k + n \log H$. Hence we tabulate the values of log Q and log H.

log H ..	$\bar{1}$·220	$\bar{1}$·320	$\bar{1}$·431	$\bar{1}$·529	$\bar{1}$·593	$\bar{1}$·675	$\bar{1}$·740
log Q ..	$\bar{2}$·467	$\bar{2}$·707	$\bar{2}$·994	$\bar{1}$·233	$\bar{1}$·412	$\bar{1}$·613	$\bar{1}$·780

Points are plotted from this table in Fig. 37. Remember that $\bar{2}$·467 means $-2 + 0{\cdot}467$ so that if the origin is taken at

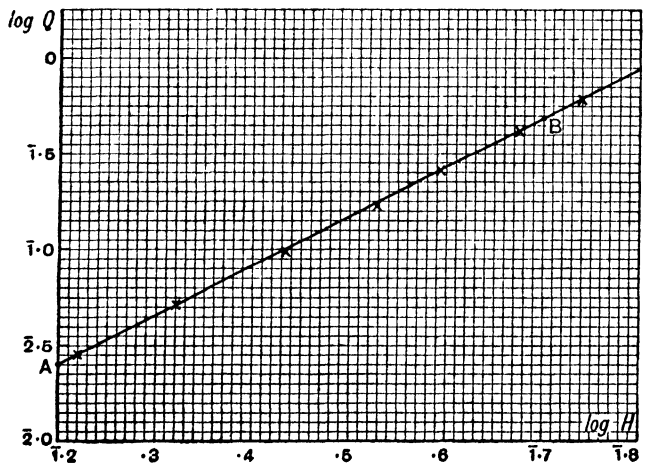


FIG. 37.

-2 on the vertical axis $\bar{2}{\cdot}467$ is 0·467 above it; $\bar{1}$ is 1 above the origin because $\bar{1} = -1 = -2+1$.

Since the points lie very nearly on a straight line, Q does vary as a power of H. At the point A on the line,

$$\log H = \bar{1}{\cdot}2 = -0{\cdot}8, \log Q = \bar{2}{\cdot}40 = -1{\cdot}60,$$

and at the point B,

$$\log H = \bar{1}{\cdot}7 = -0{\cdot}3, \log Q = \bar{1}{\cdot}69 = -0{\cdot}31.$$

Substituting these values in the equation

$$\log Q = \log k + n \log H,$$

we get

$$-1{\cdot}60 = \log k - 0{\cdot}8n,$$
$$-0{\cdot}31 = \log k - 0{\cdot}3n.$$

$$\therefore 1{\cdot}29 = 0{\cdot}5n \text{ and } n = \frac{1{\cdot}29}{0{\cdot}5} = 2{\cdot}58$$

$$\therefore \log k = -0{\cdot}31 + 0{\cdot}3 \times 2{\cdot}58 = -0{\cdot}31 + 0{\cdot}774 = 0{\cdot}463$$
$$\therefore k \simeq 2{\cdot}9.$$

Hence the law connecting Q and H is $Q = 2{\cdot}9\ H^{2{\cdot}58}$.

Exercise XVIII

1. If $y = axy + bx^2$ and $y = 25$ when $x = 2$, $y = 50$ when $x = 6$, find a and b. Also calculate the value of y when $x = 10$.

2. If u cm. and v cm. are the distances of an object and its image from a lens, $\frac{1}{u} = \frac{a}{v} + b$, where a and b are constants. In an experiment it was found that $v = 21{\cdot}45$ when $u = 300$ and $v = 24{\cdot}25$ when $u = 100$. Find the law connecting u and v. Find also the focal length, f cm., of the lens which is given by $f = 1/b$.

3. If a body moves s ft. in t sec. with constant acceleration, $s = ut + \frac{1}{2}ft^2$, where u and f are constant. If $s = 600$ when $t = 24$ and $s = 3000$ when $t = 60$, find u and f.

4. If $y = ax^n$ and $y = 35{\cdot}2$ when $x = 290$, $y = 77{\cdot}3$ when $x = 370$, find the values of a and n. Also calculate the value of y when $x = 410$.

5. If T sec. is the period of oscillation of a weight W lb., which is hanging by a vertical spring, T varies as W^n. It is found in an

experiment that $T = 0{\cdot}471$ when $W = 2$ and $T = 0{\cdot}745$ when $W = 5$. Find the law connecting T and W. What must the weight be for the period of oscillation to be 1 sec.?

6. When a body cools the temperature, $\theta°$ C., of the body above the surrounding air is related to the time t min. that the body has been cooling by the equation $\log \theta = a + bt$. In an experiment $\theta = 27$ when $t = 4$ and $\theta = 5{\cdot}3$ when $t = 20$. Find a and b and hence show that $\theta = 40{\cdot}6 \times 10^{-0.0442t}$.

Verify that the given variables are related by a law of the given form. Find the approximate value of the constants in the law. Write out the law and check it for one value of the independent variable.

7. R ohms is the resistance of a conductor at $t°$ C. Law: $R = R_0(1 + at)$.

t ..	2	4	6	8	10	12
R ..	100·9	101·5	102·5	103·1	104	104·8

Also express R in the form $R = k\{1 + b(t - 2)\}$.

8. p atmospheres is the osmotic pressure of a solution at $t°$ C. Law: $p = at + b$.

p ..	24·8	25·3	25·7	26·2	26·6	27·0
t ..	0	5	10	15	20	25

9. W lb. wt. is the weight of one hundred $\frac{1}{4}$-in. Whitworth hexagon screws of length l in. Law: $W = al + b$.

l ..	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$
W ..	1·75	1·82	1·89	1·95	2·02	2·15	2·29	2·42	2·56	2·69	2·83

10. B tons wt. is the breaking load of No. 4/37 wire ropes of diameter d in. Law: $B = ad^2 + b$.

d ..	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{3}{8}$
B ..	3·7	7·9	11·8	16·6	22·1	28·5	35·6	43·6	55·1

11. l in. is the gauge length between two marks on a mild steel rod before a breaking test and e is the percentage elongation of the length l when the rod breaks. Law: $e = \frac{a}{l} + b$. [Plot e against $\frac{1}{l}$.]

l ..	2	4	6	8	10	12	14
e ..	64·2	41·6	34·3	30·4	28·2	26 8	25·6

12. W lb. wt. is the load applied to a bar at a distance x in. from a fulcrum in an experiment on moments. Law: $W = \frac{a}{x} + b$.

x	8	10	12	14	16	18
W	48	38·2	30·8	26·1	22	19·4

13. l cm. is the length of the simple pendulum which has the same period of oscillation as a rigid body oscillating about a horizontal axis h cm. from its centre of gravity. Law: $l = ah + \frac{b}{h}$.

h ..	5	7·5	10	12·5	15	20	25
l ..	50·6	37·5	32·4	30·3	29·9	31·3	34·0

14. H is the horse-power absorbed in drilling cast iron with a drill of diameter d in. Law: $H = a\sqrt{d} + \frac{b}{\sqrt{d}}$. [Plot $H\sqrt{d}$ against d.]

d	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
H ..	1·58	1·34	1·27	1·26	1·29	1·34	1·39	1·46

15. f lb./in.^{2} is the buckling stress of a strut in which the ratio of length to radius is x. Law: $f = \frac{a}{b + x^2}$. [Plot $\frac{1}{f}$ against x^2.]

x ..	10	20	40	60	80	100	120
f/100 ..	494	475	412	338	270	214	171

16. v cu. ft. is the volume of a quantity of steam when its pressure is p lb./in.^{2}. Law: $pv^n = C$. [Plot $\log p$ against $\log v$.]

p ..	10	25	50	100	200	300	400
v ..	5·36	2·75	1·66	1·01	0·60	0·49	0·36

17. t sec. is the time taken for a circular disc mounted on an axle to roll down an inclined plane 6 ft. long when the upper end of the plane is h ft. above the lower end. Law: $t = ah^n$.

h ..	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	2
t ..	22·6	15·9	11·3	9·2	8·0	6·5	5·6

18. I candles is the luminosity of a metal filament lamp at V volts. Law: $I = kV^n$.

I	21·3	35·5	56·3	89·1	128·8	186
V	70	80	90	100	110	120

GEOMETRY

CHAPTER VI

SIMILAR FIGURES

Some facts about parallel lines. Proportional division.

Exercise.—Mark three points A, B, C on a straight line at equal distances apart, and through these points draw any three parallel straight lines, as in Fig. 38. Now draw a number of other straight lines cutting the three parallel lines in points such as D, E, F; L, M, N, etc.

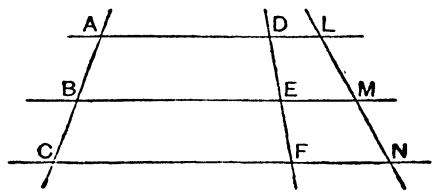


FIG. 38.

Measure DE and EF; LM and MN; etc. What do you notice?

The student will find that $DE = EF$; $LM = MN$, etc. We may express this result as follows:

Theorem.—If three parallel lines cut off equal lengths on any one transversal they also cut off equal lengths on any other transversal.

We shall now *prove* this theorem.

Draw DH and EK parallel to the line ABC (Fig. 39). Then $ADHB$ is a parallelogram and so $DH = AB$. Also, $BEKC$ is a parallelogram, and so $EK = BC$.

But $AB = BC$, whence $DH = EK$.

Thus, in $\triangle$'s DHE and EKF,

$$DH = EK,$$
$$\widehat{\alpha} = \widehat{\beta} \text{ (since } BE \text{ is parallel to } CF)$$
$$\widehat{\gamma} = \widehat{\delta} \text{ (since } DH \text{ is parallel to } EK)$$
$$\therefore \triangle DHE \equiv \triangle EKF \text{ (two } \angle\text{'s and a side).}$$

Hence $DE = EF$.

It is easily seen that this theorem is also true if we have more than three parallel lines. For example, in Fig. 40, if $AB = BC = CD = DE$ then it follows that $A'B' = B'C' = C'D' = D'E'$.

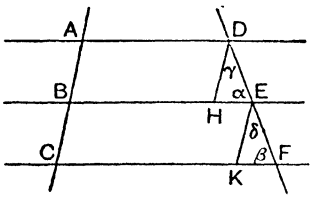


FIG. 39.

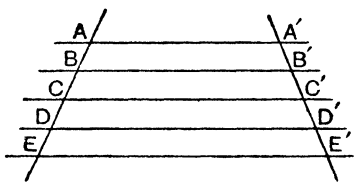


FIG. 40.

Construction.—To divide a given straight line into any number of equal parts, without measurement or calculation.

Suppose we wish to divide the line AB into five equal parts. Through A draw any line AC of indefinite length, and *with a pair of compasses* mark off on AC five equal lengths AD, DE, EF, FG, GH, of any convenient size (Fig. 41).

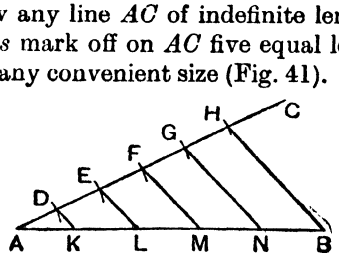


FIG. 41.

Join the last point, H, to B, and through D, E, F, G draw straight lines parallel to HB (by means of a ruler and set square). The points K, L, M, N, in which the lines cut AB, are the required points of division.

We have here divided AB into five equal parts, but the method of construction is the same whatever the number of parts.

Exercise.—Mark off three points A, B, C on a straight line such that AB is twice the length of BC. Through A, B, C draw any three parallel straight lines (Fig. 42). Now draw a number of other straight lines cutting these parallels in points such as D, E, F; L, M, N, etc.

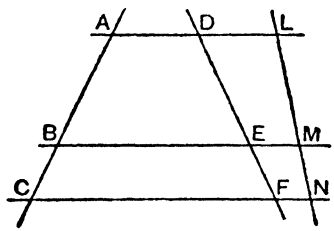


FIG. 42.

Measure DE and EF, LM and MN, etc., and find the ratios $\frac{DE}{EF}$, $\frac{LM}{MN}$, etc. What do you notice?

Exercise.—Repeat the previous construction making AB three times the length of BC. What do you notice about the ratios $\frac{DE}{EF}$, $\frac{LM}{MN}$, etc., in this case?

Exercise.—Repeat the construction, making the ratio $\frac{AB}{BC} = \frac{2}{5}$, and state your conclusion.

The student will find that, in every case, the ratios $\frac{DE}{EF}$, $\frac{LM}{MN}$, etc. are each equal to the ratio $\frac{AB}{BC}$.

We therefore conclude the following:

Theorem.—The ratios of the intercepts made by three given parallel lines on all transversals are the same.

To *prove* this, suppose that $\frac{AB}{BC}=\frac{m}{n}$, where m and n are whole numbers. Divide AB into m equal parts, and BC into n equal parts; then all the $m+n$ parts into which AC is divided are equal. Through the points of division draw lines parallel to AD, as shown by the dotted lines in Fig. 43. These lines cut DF into $m+n$ equal parts, of which DE contains m and EF contains n. (In Fig. 43 $m=7$ and $n=5$.)

$$\therefore \frac{DE}{EF}=\frac{m}{n} \qquad \therefore \frac{AB}{BC}=\frac{DE}{EF}.$$

In Fig. 43 DEF is any transversal. If we draw it to pass through A, we obtain Fig. 44, in which D coincides with A. The preceding theorem now tells us that, if BE is parallel to the side CF of a triangle ACF, then $\frac{AB}{BC}=\frac{AE}{EF}$.

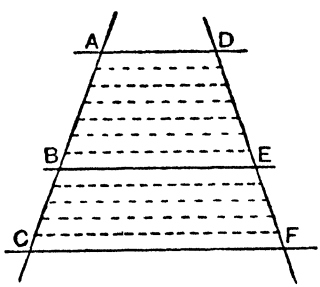


Fig. 43.

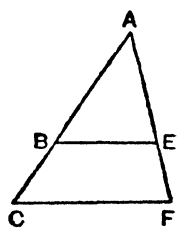


Fig. 44.

Thus we have the following:

Theorem.—If a line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.

The converse of this theorem is also true, viz.:

Theorem.—If a line divides two sides of a triangle proportionally, it is parallel to the third side.

To prove this, suppose we are given that DE divides the sides AB, AC of the $\triangle ABC$ proportionally; i.e. that $\frac{AD}{DB} = \frac{AE}{EC}$.

We have to prove that DE is parallel to BC.

If it is not, draw the line DF, through D, parallel to BC.

Then, from the previous theorem, $\frac{AD}{DB} = \frac{AF}{FC}$.

Hence $\frac{AE}{EC} = \frac{AF}{FC}$, and therefore E coincides with F. But DF was drawn parallel to BC; therefore DE is parallel to BC.

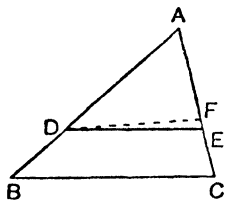


Fig. 45.

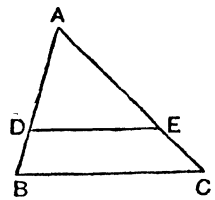


Fig. 46.

As *particular cases* of these two theorems we have the following:

(1) The line through the mid-point of one side of a triangle parallel to the base bisects the other side.

(2) The line joining the mid-points of two sides of a triangle is parallel to the third side.

Note.—There is another way of stating the result expressed in the theorem at the foot of p. 123. If DE is parallel to the side BC of a triangle ABC (Fig. 46) we proved there that

$$\frac{AD}{DB} = \frac{AE}{EC}$$

$$\therefore \frac{DB}{AD} = \frac{EC}{AE}$$

$$\therefore 1+\frac{DB}{AD}=1+\frac{EC}{AE}$$

$$\therefore \frac{AD+DB}{AD}=\frac{AE+EC}{AE}$$

i.e. $$\frac{AB}{AD}=\frac{AC}{AE} \quad \text{or} \quad \frac{AD}{AB}=\frac{AE}{AC}.$$

This form of the result is often useful.

Internal and external division

Suppose that A, B are two points on a line, 5 in. apart.

Mark the point P, between A and B, 3 in. from A, and the point Q, in AB produced, 15 in. from A (Fig. 47).

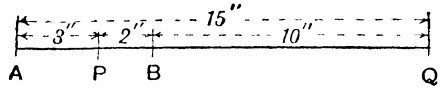


FIG. 47.

Then $PB=2$ in., and $QB=10$ in. and hence $\frac{AP}{PB}=\frac{3}{2}$ and $\frac{AQ}{QB}=\frac{15}{10}=\frac{3}{2}$.

The two points P and Q both divide AB in the ratio 3 : 2. We say that "P divides AB internally in the ratio 3 : 2," and that "Q divides AB externally in the ratio 3 : 2."

If we take *any* ratio, we shall find that there are always two points which divide AB in that ratio, one of the points dividing AB internally, the other externally.

Example.—A, B are two points 12 cm. apart. Find, by calculation, the positions of the two points which divide AB in the ratio 3 : 5.

Let P be the point of internal division, and suppose $AP=x$ cm. Then $PB=(12-x)$ cm.

$$\therefore \frac{x}{12-x}=\frac{3}{5}$$

$$5x = 3(12 - x)$$
$$= 36 - 3x$$
$$8x = 36$$
$$\therefore x = \tfrac{36}{8} = \tfrac{9}{2} = 4{\cdot}5$$

$\therefore$ P is 4·5 cm. from A, between A and B.

[Check: $AP = 4{\cdot}5$ cm. $\therefore$ $PB = 7{\cdot}5$ cm. $\therefore \dfrac{AP}{PB} = \dfrac{4{\cdot}5}{7{\cdot}5} = \tfrac{9}{15} = \tfrac{3}{5}$.]

Let Q be the point of external division, and let $AQ = y$ cm. Since $\dfrac{AQ}{QB} = \tfrac{3}{5}$, AQ must be less than QB, and hence Q must be nearer to A than to B. Thus Q must lie outside AB beyond A (see Fig. 48).

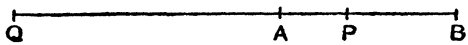


FIG. 48.

$$\therefore QB = (y + 12) \text{ cm.}$$
$$\therefore \frac{y}{y + 12} = \tfrac{3}{5}$$
$$5y = 3(y + 12)$$
$$= 3y + 36$$
$$2y = 36$$
$$\therefore y = 18.$$

$\therefore$ Q is 18 cm. from A, outside AB.

[Check: $AQ = 18$ cm., $\therefore$ $QB = 30$ cm., $\therefore \dfrac{AQ}{QB} = \tfrac{18}{30} = \tfrac{3}{5}$.]

Exercise.—Verify that the theorems on pp. 123 (at foot) and 124 are true whether the line divides the sides internally or externally.

A *graphical* construction for dividing a line in a given ratio is given below.

Construction.—To divide a given straight line in a given ratio, internally or externally, without measurement or calculation.

Suppose, for example, we wish to divide the line AB in the ratio 7 : 2.

(i) *Internally.*—Draw any line AC through A, and with a pair of compasses mark off seven equal lengths from A, terminating at D (Fig. 49). From D mark off another two of these equal lengths, terminating at E. Join E to B, and draw DP parallel to EB. Then $\frac{AP}{PB}=\frac{AD}{DE}=\frac{7}{2}$.

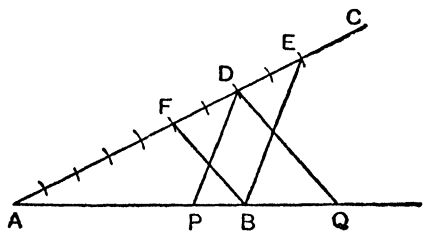


FIG. 49.

(ii) *Externally.*—In this case mark off the two equal lengths from D *backwards* towards A, terminating at F. Join F to B, and draw DQ parallel to FB. Then $\frac{AQ}{QB}=\frac{AD}{DF}=\frac{7}{2}$.

The above constructions apply whatever the ratio $m : n$ in which it is required to divide AB. If m is less than n, however, the point F will be on the opposite side of A to D, as in Fig. 50.

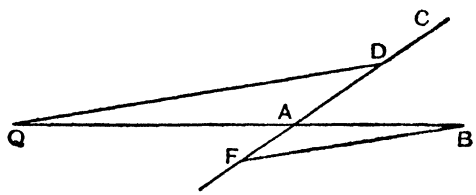


FIG. 50.

Definition.—If a, b, c, d are four magnitudes such that $\frac{a}{b}=\frac{c}{d}$, then d is called the *fourth proportional* to a, b, c.

Construction.—To find, graphically, the fourth proportional to three given lengths.

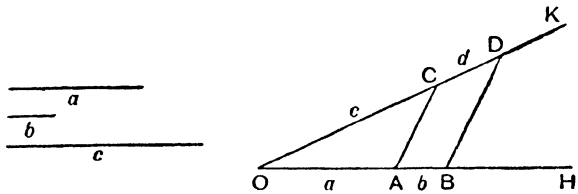


FIG. 51.

Let a, b, c be the three given lengths.

Draw any two intersecting lines OH, OK.

Along OH mark off $OA=a$, $AB=b$, and along OK mark off $OC=c$.

Join AC and, through B, draw BD parallel to AC, cutting OK in D.

Then CD is the fourth proportional to a, b, c.

The proof is left to the student.

Theorem.—The internal and external bisectors of an angle of a triangle divide the opposite side, internally and externally, in the ratio of the sides containing the angle.

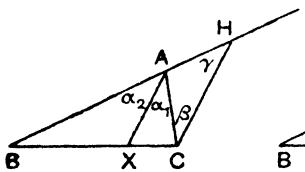


FIG. 52 (*a*).

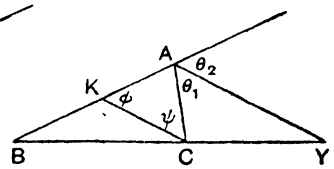


FIG. 52 (*b*).

Let ABC be the triangle and AX, AY the internal and external bisectors of the angle A.

In Fig. 52 (a), draw CH parallel to XA cutting BA produced in H.

Then

$\widehat{\beta} = \widehat{\alpha_1}$ (alternate angles)

$\widehat{\gamma} = \widehat{\alpha_2}$ (corresponding angles)

But $\widehat{\alpha_1} = \widehat{\alpha_2}$

$\therefore \widehat{\beta} = \widehat{\gamma}$

$\therefore \triangle ACH$ is isosceles

$\therefore AC = AH$.

Since AX is parallel to HC,

$$\frac{BX}{XC} = \frac{BA}{AH}$$

$$\therefore \frac{BX}{XC} = \frac{BA}{AC}$$

In Fig. 52 (b), draw CK parallel to YA cutting BA in K.

Then

$\widehat{\psi} = \widehat{\theta_1}$ (alternate angles)

$\widehat{\phi} = \widehat{\theta_2}$ (corresponding angles)

But $\widehat{\theta_1} = \widehat{\theta_2}$

$\therefore \widehat{\psi} = \widehat{\phi}$

$\therefore \triangle AKC$ is isosceles.

$\therefore AC = AK$.

Since AY is parallel to KC,

$$\frac{BY}{YC} = \frac{BA}{AK}$$ (see exercise, p. 126.)

$$\therefore \frac{BY}{YC} = \frac{BA}{AC}$$

Exercise XIX

1. Draw a line about 4 in. long, and divide it into seven equal parts by a graphical construction.

2. Draw a line about 5 in. long, and divide it *graphically* in the ratio 4 : 3 (i) internally, (ii) externally.

3. Draw a line AB of length 2·6 in. Find, by a graphical construction, the points in AB (produced if necessary) such that $AP = 3PB$. Check by calculation.

4. Find, *graphically*, the fourth proportional to 3, 7, 2. Check by calculation.

5. [If $a : b = b : x$, then x is called the *third proportional* to a and b.] Find, graphically, the third proportional to 4·5 and 3·1.

6. If D, E, F are the mid-points of the sides of the triangle ABC (Fig. 53), prove that the four triangles shown are all congruent to one another.

7. Deduce, from Question **6**, the following theorem: The line joining the mid-points of two sides of a triangle is equal in length to half the third side.

8. Find a construction for drawing a triangle when only the mid-points of its sides are given.

9. If $ABCD$ is any quadrilateral (Fig. 54) and P, Q, R, S, are the mid-points of the sides taken in order, prove that $PQRS$ is a parallelogram. [Hint.—Draw the diagonals AC, BD.]

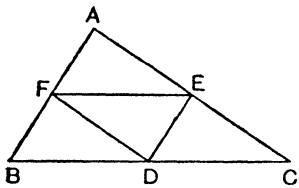


FIG. 53.

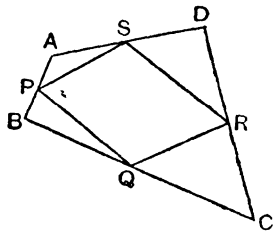


FIG. 54.

10. A ladder, 18 ft. long, rests over a wall which is 6 ft. high, the foot of the ladder being on the ground and its upper end resting against the side of a house. If the foot of the ladder is 8 ft. from the bottom of the wall, find the distance between the wall and the house. Verify by a scale drawing.

11. In Fig. 55 $ABCD$ is a trapezium, and M, N are the mid-points of AD, BC. Prove that MN is parallel to AB and DC. [Compare the proof of the theorem on p. 124.]

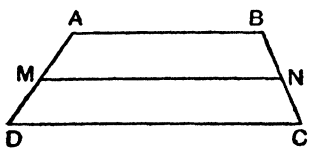


FIG. 55.

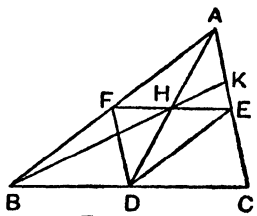


FIG. 56.

12. In Fig. 56 D, E, F are the mid-points of the sides of the triangle ABC. The lines AD, EF intersect in H and the line BH cuts AC in K.

Prove that (i) H is the mid-point of AD; (ii) K is a point of trisection of AC.

[Hint for (ii): Draw the line through D parallel to BK.]

13. P is any point in the side AB of a triangle ABC (Fig. 57). PQ is parallel to BC, and QR is parallel to AB. Prove that $\frac{BR}{RC} = \frac{AP}{PB}$.

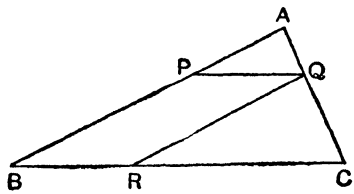


FIG. 57.

14. From the theorem on p. 128 devise a construction for dividing a line internally and externally in a given ratio.

15. Prove that the locus of a point which moves so that the ratio of its distances from two fixed points is constant, is a circle.

[Hint.—Use the theorem on p. 128.]

Similarity

Figures which are of the same shape but different in size are said to be *similar*.

When a photograph is enlarged any object in the picture retains its shape but is magnified in size. The object and its enlargement are similar figures.

A plan of a field is similar to the actual field. Two maps of a town, drawn on different scales, are similar.

We can also have similar solids. For example, in a model of a ship all the parts are reproduced to their proper shape, but on a smaller scale. The model is *similar* to the actual ship.

The principle of similarity is very important. Models are widely used for testing the behaviour of new designs of aeroplanes and ships in the experimental stages.

In similar figures it is clear that angles are unaltered, and although lengths are altered they are all increased or decreased in the same ratio, so that the *ratio* of the lengths of any two lines is unaltered.

A *polygon* is a figure bounded by any number of straight lines.

If two polygons are similar, they must have the same number of sides and the angles of one polygon must be equal to the angles of the other, *taken in the same order* round the figure. For example, in Fig. 58, $ABCDE$ and $A'B'C'D'E'$ are similar, and $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, etc.

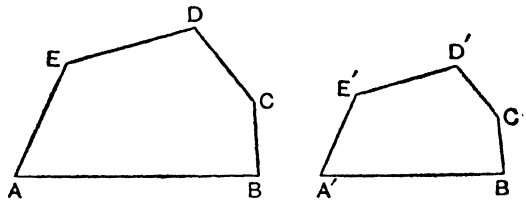


FIG. 58.

Also, since all lengths are altered in the same ratio in passing from one figure to the other, it follows that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'};$$

that is, the sides of one polygon must be proportional to the corresponding sides of the other polygon.

It is important to notice that two polygons may be equiangular (i.e. have their angles equal taken in the same order) and yet not be similar, since the angles alone do not determine the shape of a figure. This is seen in Fig. 59, where HK is parallel to CD; the figures $ABCD$ and $ABHK$ have the same angles but are obviously not of the same shape. Neither is it sufficient for similarity that the sides of the two figures should be proportional, since the lengths of the sides alone do not determine the shape of a polygon. We can see this

from Fig. 60, where $ABCD$ and A_1BCD_1 are two quadrilaterals whose sides are of the same lengths. (If we regard $ABCD$ as a frame made up of four rods pin-jointed at their ends, we can deform the frame into the shape A_1BCD_1 without altering the lengths of the rods.)

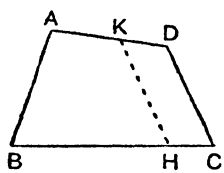


FIG. 59.

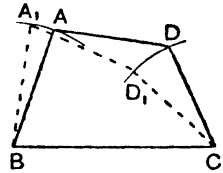


FIG. 60.

Thus, for two polygons of the same number of sides to be similar, *two* conditions must be satisfied, viz. :

(i) they must be equiangular.
(ii) their corresponding sides must be proportional in length.

Similar triangles

A triangle, which is the simplest of all polygons, is different from all other polygons in one important respect; namely, that its angles determine its shape, so that if two triangles are equiangular they are similar.

Thus if two triangles are equiangular their corresponding sides *are* proportional; the converse is also true, that if the sides of two triangles are proportional the triangles are equiangular.

These two statements are so important that we shall prove them, but the student is first advised to work the following exercises.

Exercise.—Draw two triangles of different sizes having their angles equal to 25°, 70°, 85°, and measure their sides. Compare the ratios of the pairs of corresponding sides of the two triangles.

Exercise.—Draw two triangles having sides of lengths 4 cm., 6 cm., 2·5 cm., and 8 cm., 12 cm., 5 cm. Measure the angles of the two triangles, and verify that their corresponding angles are equal.

[NOTE.—"Corresponding angles" here means "angles opposite corresponding sides."]

We shall now prove the two theorems stated above.

Theorem.—If two triangles are equiangular, their corresponding sides are proportional.

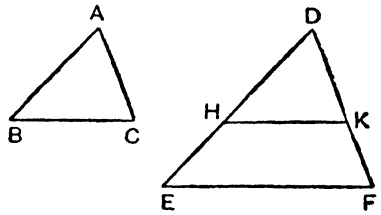


FIG. 61.

Let ABC, DEF be the triangles, with $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

We have to prove that $\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}$.

On DE mark off DH equal to AB, and on DF mark off DK equal to AC. Join HK.

Then in $\triangle$'s ABC and DHK,

$$AB = DH,\ AC = DK,\ \angle A = \angle D,$$

$\therefore \triangle ABC \equiv \triangle DHK$ (two sides and included angle)

$\therefore \angle DHK = \angle B = \angle E$.

$\therefore HK$ is parallel to EF

$\therefore \frac{DH}{DE} = \frac{DK}{DF}$ (p. 124)

$\therefore \frac{AB}{DE} = \frac{AC}{DF}$.

In the same way, by marking off lengths on ED and EF equal to BA and BC, we can prove that $\frac{AB}{DE} = \frac{BC}{EF}$.

Hence $$\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}.$$

Theorem.—If the sides of two triangles are proportional, their corresponding angles are equal.

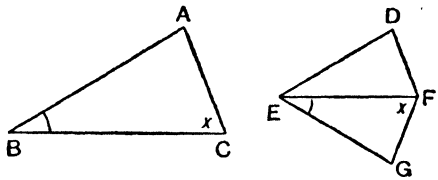


FIG. 62.

Let ABC, DEF be the triangles, in which

$$\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}.$$

We have to prove that $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

(NOTE.—Angles A and D are corresponding angles, since they are opposite the corresponding sides BC and EF.)

Draw a triangle EFG, on the opposite side of EF to the triangle DEF, having $\angle FEG = \angle B$ and $EFG = \angle C$. Then $\triangle$'s ABC and GEF are equiangular, and hence (from the previous theorem) $\frac{BC}{EF} = \frac{CA}{FG} = \frac{AB}{GE}$.

But $\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}$, and hence, by comparing these ratios, $FG = FD$ and $GE = DE$.

Thus the $\triangle$'s DEF and GEF have their three sides equal, and therefore they are congruent.

$\therefore \angle D = \angle G$, $\angle DEF = \angle FEG$ and $\angle DFE = \angle EFG$.

It follows that $\angle D = \angle A$, $\angle DEF = \angle B$ and $\angle DFE = \angle C$.

[*Note.*—In this proof the triangle *EFG* was drawn on the *opposite* side of *EF* to *D* merely in order to obtain a clear figure, since if it were drawn on the same side as *D* it would coincide with the triangle *EFD*.]

The following theorem will sometimes be found useful. Its proof is left as an exercise for the student, but the main steps of the proof are given.

Theorem.—If two triangles have one angle of one equal to one angle of the other and the sides containing those angles proportional, the triangles are similar.

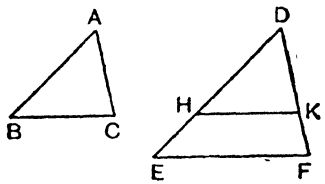


FIG. 63.

Suppose we are given $\triangle$'s *ABC*, *DEF* in which $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$.

Mark off $DH = AB$, $DK = AC$. Show that *HK* is parallel to *EF*. Then show that each of the triangles *ABC*, *DEF* is equiangular to $\triangle DHK$. It follows that the triangles *ABC*, *DEF* are themselves equiangular and therefore similar.

Diagonal scale

By means of a diagonal scale we can measure distances more accurately than would be possible on an ordinary scale. For example, it is possible to calibrate an ordinary scale, and to read it, to tenths of an inch with ease, but to calibrate to hundredths of an inch would require much greater care and expense, and, even if it were so calibrated, to read the divisions would not be easy without a magnifying lens. By means of a

diagonal scale we can measure distances to a hundredth of an inch. The scale is usually drawn on hardwood, ivory or celluloid. It is shown in Fig. 64.

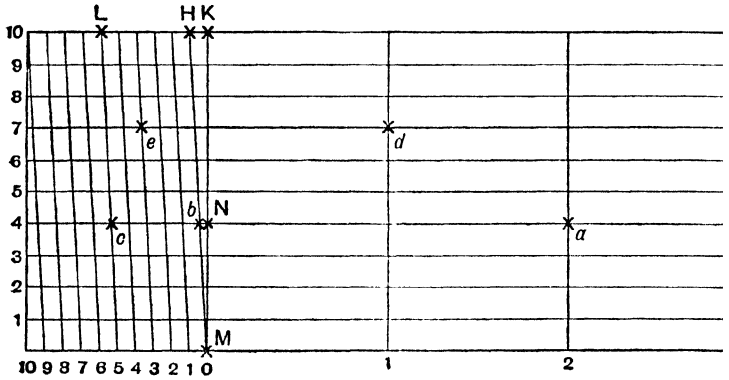


FIG. 64.

Each of the large divisions along the top and bottom of the scale = 1 in.

Each of the small divisions along the top and bottom of the scale = $\frac{1}{10}$ in.

The horizontal lines are at equal distances apart.

The triangles MbN, MHK are equiangular and therefore similar.

$$\therefore \frac{bN}{HK} = \frac{MN}{MK} = \tfrac{4}{10}.$$

$$\therefore bN = \tfrac{4}{10}HK = \tfrac{4}{10} \times \tfrac{1}{10} \text{ in.} = \tfrac{4}{100} \text{ in.}$$

$$\therefore ab = aN + Nb = (2 + \tfrac{4}{100}) \text{ in.} = 2{\cdot}04 \text{ in.}$$

It is clear that the oblique lines, such as bH and cL, are all parallel, and the lengths cut off between two consecutive oblique lines on all the horizontal lines are equal, being equal to $\frac{1}{10}$ in. Thus bc measures $5 \times \frac{1}{10}$ in., i.e. $\frac{5}{10}$ in.

Hence $ac = ab + bc = (2 + \frac{4}{100} + \frac{5}{10})$ in. $= 2{\cdot}54$ in.

To read off a length of 2·54 in. the procedure is as follows :

Place the point of a pencil on the figure 2 on the bottom scale to the *right* of *O*, and the point of another pencil on the figure 5 on the bottom scale to the *left* of *O*. Now move both pencils up the vertical, or oblique, lines through those points as far as the horizontal line through the point marked 4 on the vertical scale at the left. The distance between the two pencil points will then be 2·54. This length can then be transferred to paper if required by means of a pair of dividers. Similarly for any other length.

Exercise.—1. What is the length of *de* in Fig. 64?

2. Mark a distance 3·72 in. on the scale in Fig. 64.

Similar polygons

Construction.—To construct a polygon similar to a given polygon and on a given scale.

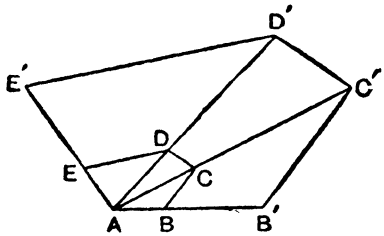


FIG. 65.

Suppose, for example, we wish to draw a polygon similar to *ABCDE* (Fig. 65) but having its sides three times the lengths of those of the given polygon.

Produce *AB* and mark off $AB' = 3 \cdot AB$.

Through *B'* draw *B'C'* parallel to *BC*, cutting *AC* produced in *C'*.

Through *C'* draw *C'D'* parallel to *CD*, cutting *AD* produced in *D'*.

Through *D'* draw *D'E'* parallel to *DE*, cutting *AE* produced in *E'*.

Then $AB'C'D'E'$ is the required polygon.

Proof: The two polygons are obviously equiangular since the sides of the second polygon have been drawn parallel to those of the first.

We have to prove, in addition, that their corresponding sides are proportional.

Since $B'C'$ is parallel to BC, $\triangle$'s ABC and $AB'C'$ are similar.

$$\therefore \frac{B'C'}{BC} = \frac{AB'}{AB} = \frac{3}{1}. \quad \text{Also } \frac{AC'}{AC} = \frac{AB'}{AB} = \frac{3}{1}.$$

Since $C'D'$ is parallel to CD, $\triangle$'s ACD and $AC'D'$ are similar.

$$\therefore \frac{C'D'}{CD} = \frac{AC'}{AC} = \frac{3}{1}. \quad \text{Also } \frac{AD'}{AD} = \frac{AC'}{AC} = \frac{3}{1}.$$

Finally, since $D'E'$ is parallel to DE, $\triangle$'s ADE and $AD'E'$ are similar.

$$\therefore \frac{D'E'}{DE} = \frac{AE'}{AE} = \frac{AD'}{AD} = \frac{3}{1}.$$

Thus each side of the polygon $AB'C'D'E'$ is three times the length of the corresponding side of the polygon $ABCDE$.

The method of construction is the same whatever the required ratio of the sides. Thus, if the sides of the new polygon are to be n times the length of those of the given polygon, we make $AB' = n \,.\, AB$ and then proceed as above. (n may be greater than, or less than, 1.)

An *alternative construction* is as follows:

Take any point O and join it to each of the vertices of the given polygon $ABCDE$.

(The point O may be inside or outside the given polygon; see Figs. 66 and 67.)

On OA, produced if necessary, mark off $OA' = n \,.\, OA$.

Through A' draw $A'B'$ parallel to AB cutting OB (produced if necessary) in B'. Through B' draw $B'C'$ parallel to BC cutting OC (produced if necessary) in C'; and so on.

Then $A'B'C'D'E'$ is the required polygon.

The proof is left as an exercise to the student. It is very like that for the previous construction; in fact, the student will notice that the previous construction is only the special case of that just given when O is taken at A.

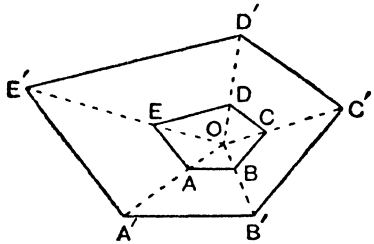


FIG. 66.

Figures, such as $ABCDE$ and $A'B'C'D'E'$ in Figs. 66 and 67, which are so placed that the lines joining corresponding points are concurrent are said to be "*in perspective.*" The point of concurrence, which, in Figs. 66 and 67, is the point O, is called the *centre of perspective.*

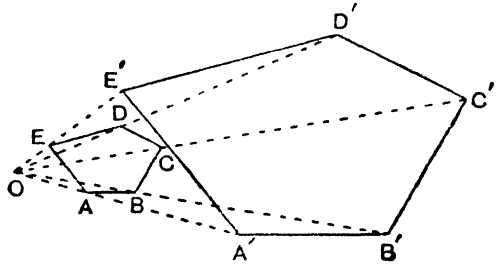


FIG. 67.

Figures in two different planes may be in perspective. For example, in the case of a pin-hole camera, the object and its image are in perspective, the centre of perspective being the pin-hole (Fig. 68).

NOTE.—If two figures are in perspective they are not necessarily similar. They will be so when they are in parallel planes, *or*, if in the same plane, when their corresponding sides are parallel.

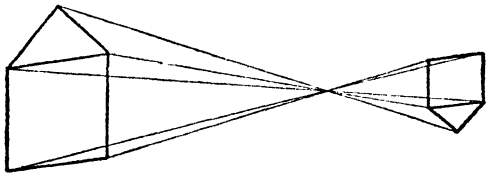


FIG. 68.

Similar irregular figures

We can construct a figure similar to a given figure, whatever its shape, on any desired scale by using the method of perspective.

For example, to draw a figure similar to that shown on the left in Fig. 69, but of one-quarter the scale, take any point O, join it to a point A of the figure and mark off $OA' = \frac{1}{4}OA$. Then join O to another point B of the figure and mark off $OB' = \frac{1}{4}OB$, and so on for a large number of points. Join up the points A', B', etc., by a freehand curve. The larger the number of points taken, of course, the better will be the resulting figure.

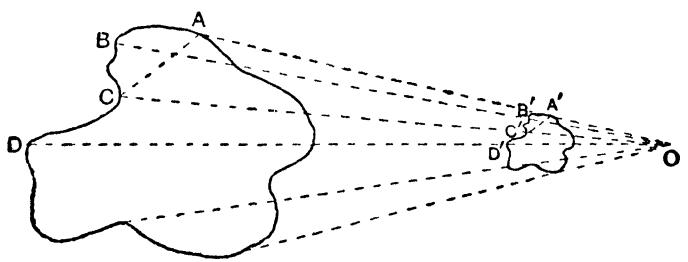


FIG. 69.

The length of any line in the second figure will be one-quarter that of the corresponding line in the first figure; for example, $\frac{A'C'}{AC} = \frac{OA'}{OA} = \frac{1}{4}$.

[Such constructions can be performed mechanically by means of an instrument called a *pantograph*, the theory of which is indicated in Exercise XX, Question 23 (p. 148).]

Areas of similar figures

Two rectangles whose sides are 2 in., 3 in., and 4 in., 6 in., respectively, are similar, since they are equiangular and the sides of the second rectangle are twice the lengths of the sides of the first rectangle. Their areas are 6 sq. in. and 24 sq. in., respectively, so that the area of the second is four times that of the first.

Generally, if the sides of the first rectangle are of lengths a and b, and those of the second na and nb, the ratio of the areas of the rectangles is $\frac{na \times nb}{a \times b} = \frac{n^2ab}{ab} = \frac{n^2}{1}$.

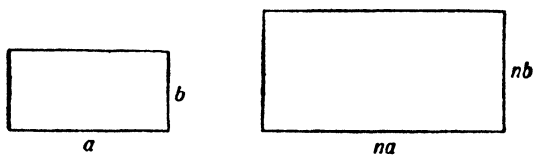


FIG. 70.

Thus the ratio of the areas of similar rectangles is equal to the *square* of the ratio of corresponding sides.

Any area, with an irregular or curved boundary, can be regarded as built up of rectangles. For if we inscribe rectangles in the area and then draw smaller and smaller rectangles in the spaces which remain near the boundary, as shown in Figs. 71 (*a*) or 71 (*b*), the rectangles will cover more and more of the area each time, and will tend ultimately to cover the whole area.

If the figure is enlarged, or reduced, a similar figure is obtained. Every line in the figure is enlarged, or reduced, in the same ratio. Fig. 71 (*b*) is an enlargement of Fig. 71 (*a*). If every line in Fig. 71 (*b*) is n times the length of the corresponding line in Fig. 71 (*a*) (e.g. $A'B' = n \,.\, AB$), the area of each of the rectangles in Fig. 71 (*b*) is n^2 times the area of the corresponding rectangle in Fig. 71 (*a*). Thus by adding up the rectangles we see that the total area in Fig. 71 (*b*) is n^2 times the total area in Fig. 71 (*a*).

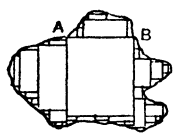


FIG. 71 (*a*).

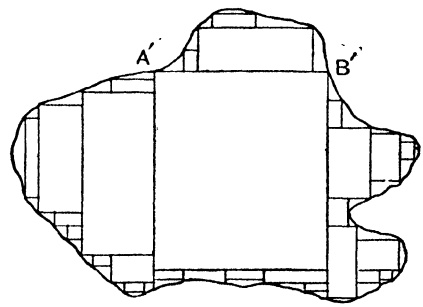


FIG. 71 (*b*).

Hence the ratio of the areas of two similar figures is equal to the square of the ratio of the lengths of corresponding lines in the two figures.

We usually express this briefly by saying that *the areas of similar figures are proportional to the squares of their corresponding linear dimensions.*

Example.—Two scale drawings of a machine are made, one on a scale of 2 ft. to 1 in., the other on a scale of 3 ft. to 1 in. If the area of a certain plate on the first drawing is 1·64 sq. in. what is its area on the second drawing ?

An actual length of 1 ft. is represented by $\frac{1}{2}$ in. on the first drawing and by $\frac{1}{3}$ in. on the second drawing.

$$\therefore \frac{\text{Area on second drawing}}{\text{Area on first drawing}} = \left(\frac{\frac{1}{3}\text{ in.}}{\frac{1}{2}\text{ in.}}\right)^2 = \left(\tfrac{2}{3}\right)^2 = \tfrac{4}{9}.$$

$\therefore$ Area on second drawing $= \frac{4}{9} \times$ area on first drawing.

$= \frac{4}{9} \times 1{\cdot}64$ sq. in.

$= 0{\cdot}73$ sq. in.

Exercise XX

1. State whether the following pairs of figures are similar or not (if not, give your reasons):

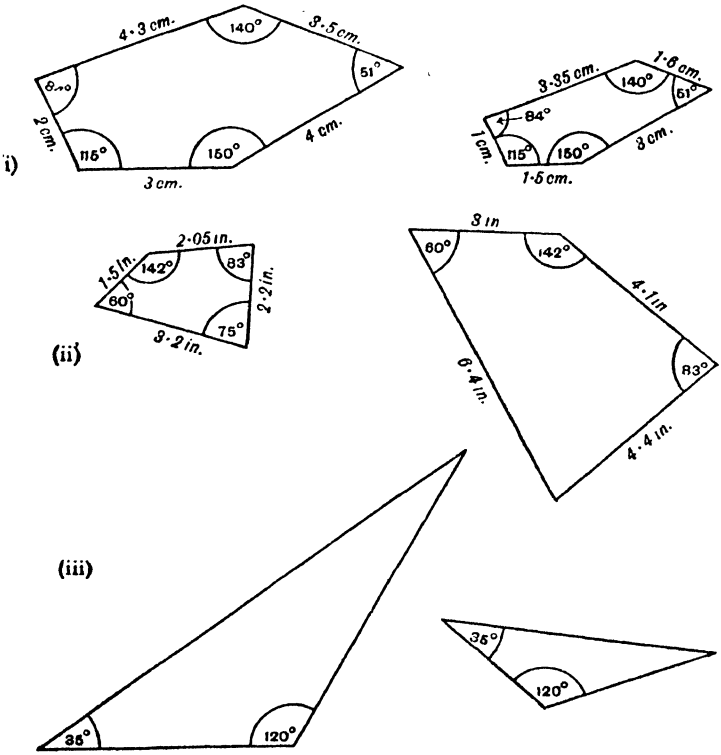


FIG. 72.

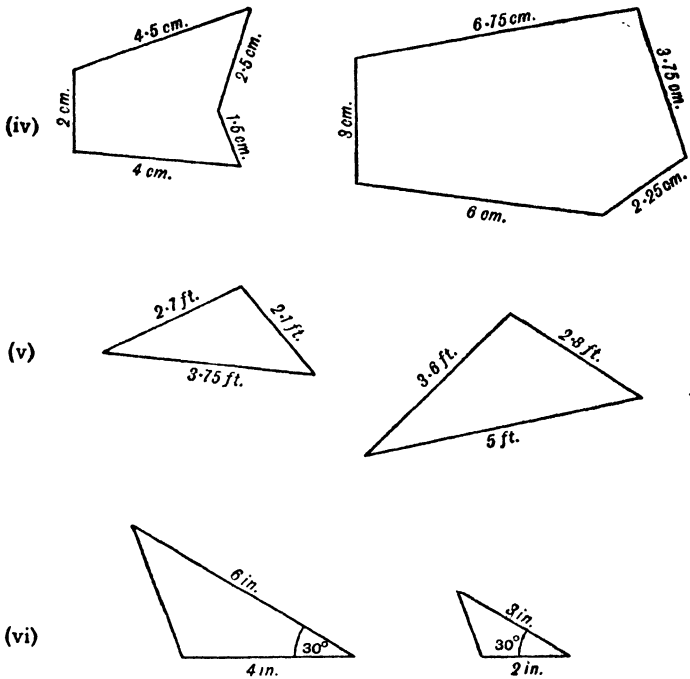


FIG. 72—*continued.*

2. The sides of a quadrilateral are 6 in., 4 in., 5 in., 7 in. Find the lengths of the sides of a similar quadrilateral whose shortest side measures 5 cm.

3. Fig. 73 represents a ground plan of a house, the lengths indicated being the lengths as measured on the plan. If the actual length of the side AB is 36 ft., find all the other lengths and the actual area of the site.

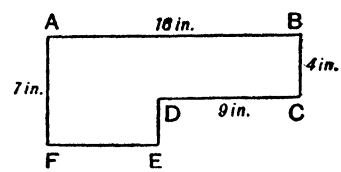


FIG. 73.

4. A locomotive engine, which runs on a track 4 ft. 8½ in. wide, is 24 ft. long and the diameter of its driving wheels is 5 ft. 6 in. If

a model is to be constructed to run on a track 3 in. wide, what must be its length and the diameter of its wheels ?

5. A road has a constant gradient. After travelling half a mile a man finds that he has risen 150 ft. How much farther must he travel before he has risen a total of 400 ft. ?

6. The shadow of a vertical pole 8 ft. high cast by the sun is 10 ft. long, and at the same time the shadow of a chimney stack is 95 ft. long. What is the height of the stack?

7. Find the length of the shadow of a man 5 ft. 10 in. high standing 9 ft. from the foot of a street lamp which is 17 ft. above the ground.

8. A halfpenny (diameter 1 in.) held at 9 ft. $7\frac{1}{2}$ in. from the eye just covers the moon. If the moon's diameter is 2160 miles, find its distance from the Earth.

9. The legs of a step-ladder are each 8 ft. long and they are connected by a rope 4 ft. long attached to each of the legs at a distance 2 ft. from the foot. Find the distance between the feet when the ladder is fully open.

10. In order to measure the height of a wireless mast a man erects a vertical pole 10 ft. high at a distance of 40 yds. from the mast. He then erects another pole 6 ft. high in such a position that the tops of the two poles and of the mast are in line. If the distance between the two poles is 8 ft., find the height of the mast.

Questions 11–14 are to be done on squared paper, using any convenient axes and scale :

11. Mark the points A, B, C whose co-ordinates are (0, 0), (1, 2), (3, 6) respectively. Prove that they lie in a straight line and find the ratio $AB : BC$.

12. Calculate the co-ordinates of the point P which divides BC in Question 11 internally in the ratio 3 : 1. Mark the point P in your figure and verify that $BP : PC = 3 : 1$ by measurement.

13. Draw the quadrilateral whose vertices, taken in order, are the points A (0, 0), B (3, 2), C (2, 3), D (− 1, 4). Draw a similar quadrilateral $PQRS$, with P, Q at the points (2, 4), (3·5, 5) and with the side PQ corresponding to AB. Read off the co-ordinates of R and S.

14. Draw a hexagon having its vertices at the points (0, 0), (0, 2), (5, 3), (3, 5), (1, 5), (− 1, 2). Draw a similar hexagon with the vertices corresponding to the first two points at (0, 0), (0, 3·2). Read off the co-ordinates of the remaining four vertices.

15. If ABC is a triangle with a right angle at A, and AD is perpendicular to BC (Fig. 74), prove that the triangles ABC, DBA and DAC are all similar. Deduce that $\frac{BD}{AD} = \frac{AD}{DC}$, and hence that $AD^2 = BD \cdot DC$.

[The *mean proportional* (or geometric mean) of two quantities a and b is defined as the quantity x, such that $\frac{a}{x} = \frac{x}{b}$, i.e. such that $x^2 = ab$, or $x = \sqrt{ab}$. Thus AD is the mean proportional of BD and DC.]

16. In Fig. 74 show that AB is the mean proportional of BC and BD, and that AC is the mean proportional of BC and CD.

17. From the results of Question 16 deduce a proof of Pythagoras's Theorem.

18. Prove that the line joining the mid-points of two sides of a triangle is equal in length to half the third side.

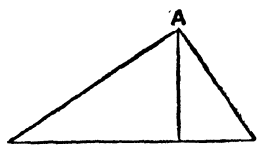


FIG. 74.

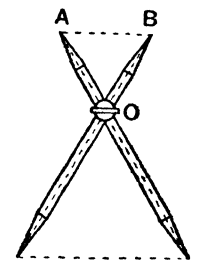


FIG. 75.

19. ABC is a triangle; X, Y, Z are the mid-points of the sides BC, CA, AB respectively. If AX and BY intersect in G, prove that $AG : GX = BG : GY = 2 : 1$. [*Hint.*—Join XY and use the result of Question 18.] Deduce that CZ also passes through G.

20. $ABCD$ is a trapezium in which AB and DC are the parallel sides. If the diagonals intersect in O, prove that $AO : OC = BO : OD$.

21. *Proportional compasses.*—In Fig. 75 AC, BD are two metal rods with pointed ends and slotted in the middle. A screw O can be moved along the slots and fixed in any desired position. Show that, for a fixed position of the screw, the ratio $AB : DC$ is constant whatever the angle between the legs.

22. If in Fig. 75, $AC = BD = 6$ in., and the screw is set at a distance of $2\frac{1}{2}$ in. from A and B, what is the ratio $DC : AB$?

Find also the position of the screw for which $DC = 3AB$.

23. *Pantograph.*—A pantograph is a device, in the form of a linkage, for copying any figure on an enlarged or reduced scale. One type of instrument is shown in Fig. 76. $OABC$ is a rhombus of four rods freely hinged at the corners. PQ, RQ are two rods freely hinged together at Q and to the original frame at P and R, such that $OPQR$ is also a rhombus. A fine steel point is fixed at Q (called the "tracing point") and a pencil at B (called the "scribing point").

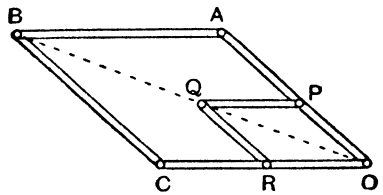


FIG. 76.

Since the diagonals of a rhombus bisect the angles, $\widehat{AOB} = \frac{1}{2}\widehat{AOC} = \widehat{POQ}$, and thus O, Q, B are in a straight line.

Prove that triangles OAB, OPQ are similar and hence that $\frac{OB}{OQ} = \frac{OA}{OP}$.

It follows that if O is kept fixed and Q traces out any given figure, the point B will trace out a similar figure.

[The tracing point and scribing point may obviously be interchanged.]

24. Fig. 77 shows another type of pantograph. AE, AB, CD, CF are four rods freely hinged at A, B, C, D, so that $AD = CB$

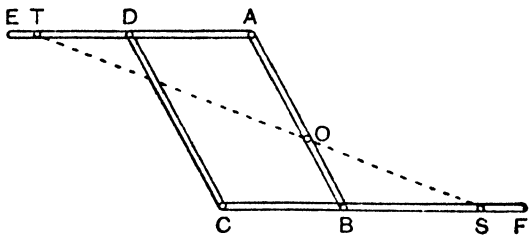


FIG. 77.

and $AB = DC$. The rods AE, CF, AB are slotted and the tracing point T, the scribing point S and a pivot O can be fixed by means of screws anywhere in those rods. T, O, S are fixed so that they lie in a straight line.

Prove that if O is fixed and T describes a given figure, S describes a similar figure.

If $AD = BC = 6$ in., $AB = 12$ in., $DT = 2$ in., and the scale of the new figure is required to be three times that of the figure traced out by T, find the distances of the pivot and the scribing point from B.

25. The sides of a rectangle are 5 in. and 8 in. long. What are the lengths of the sides of a similar rectangle of four times its area ?

26. On a certain map an area of 54 sq. ml. is represented by a rectangle 3 in. by $\frac{1}{2}$ in. What is the scale of the map ?

27. The county of Sussex covers an area of 2·53 sq. in. on a map drawn to a scale of 24 mls. to the inch. What area will it cover on a map having a scale of 4 mls. to the inch ?

28. A drawing of a certain machine part has the dimensions shown in Fig. 78. Find the area of the drawing. If the drawing is one-quarter actual size, what is the area of the actual part ?

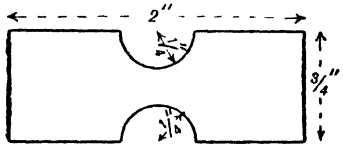


FIG. 78.

29. ABC and $A'B'C'$ are similar triangles (Fig. 79); AD is perpendicular to BC and $A'D'$ perpendicular to $B'C'$. Prove

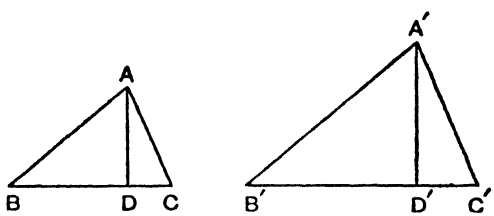


FIG. 79.

that $AD : A'D' = BC : B'C'$. Hence prove that area $\triangle ABC$: area $\triangle A'B'C' = BC^2 : B'C'^2$; i.e. that the areas of similar triangles are proportional to the squares on corresponding sides.

30. A plug fits into the top of a conical hole (Fig. 80). If the thickness of the plug is t and the diameters of its end faces are a and b, find the depth of the hole.

31. In Fig. 81 the depth of the hole is d and the dimensions of a plug which fits into the top are as shown. Prove that the diameter of the bottom of the hole is $a - \frac{d}{t}(a - b)$.

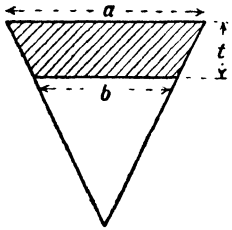


FIG. 80.

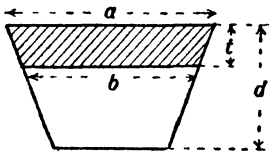


FIG. 81.

32. A pair of compasses rests over a cylinder which lies on a table (as in Fig. 82). If the distance BC between the feet of the compasses is 12 cm., and the length (AB or AC) of each leg is 10 cm., find the radius of the cylinder.

[Hint.—Prove that triangles AOE, ACD are similar.]

33. A cone rests over a sphere, as shown in Fig. 83, touching it along a circle of radius b. If each point of that circle is a distance a from the vertex of the cone, prove that the radius of the sphere is $\frac{ab}{\sqrt{a^2 - b^2}}$.

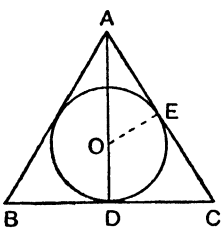


FIG. 82.

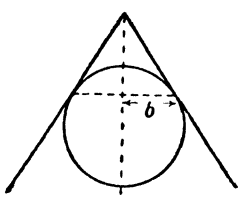


FIG. 83.

34. A line PQ, parallel to the side BC of a triangle ABC, cuts AB, AC in P, Q. A line through A cuts BC in D and PQ in R. Prove that $PR : RQ = BD : DC$.

35. The internal bisector of the angle A of a triangle ABC cuts BC in P and the circum-circle of the triangle in Q. Prove that $AB \cdot AC = AP \cdot AQ$.

36. A sphere of radius 3 in. is placed with its centre (C) 7 in. in front of a screen, and a point source of light (O) is held 1 ft. in front of the screen, so that OC is perpendicular to the screen. Find the radius of the circular shadow of the sphere thrown on to the screen.

CHAPTER VII

MISCELLANEOUS THEOREMS AND CONSTRUCTIONS

Intersecting chords of a circle

Theorem.—If two chords of a circle intersect the product of the segments of one chord is equal to the product of the segments of the other.

Let AB, CD be the two chords, intersecting in P. They may intersect inside the circle (as in Fig. 84 (a)) or outside the circle (as in Fig. 84 (b)).

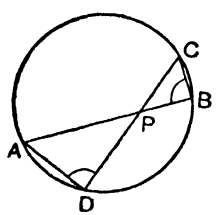


FIG. 84 (a).

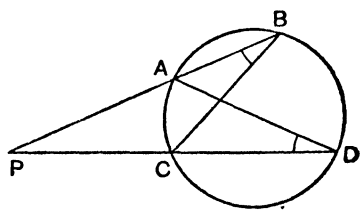


FIG. 84 (b).

PA, PB are the "segments" of the chord AB; PC, PD are the segments of the chord CD. Thus we have to prove that $PA \cdot PB = PC \cdot PD$.

Join AD and BC.

Then in the $\triangle$'s PAD and PCB,

$\angle PDA = \angle PBC$ (in the same segment, standing on arc AC).

$\angle APD = \angle CPB$ (vertically opposite angles in Fig. 84 (a), or same angle in Fig. 84 (b)).

Thus the triangles are equiangular and therefore similar.

Hence $$\frac{PA}{PC} = \frac{PD}{PB},$$

$$\therefore PA \,.\, PB = PC \,.\, PD.$$

Corollary.—The product of the segments of a chord drawn from a point outside a circle is equal to the square of the length of the tangent drawn from that point.

This is seen by swinging the chord PCD round P, as shown in Fig. 85, until the points C and D coincide in the point T, when PT will be a tangent to the circle. Thus

$$PA \,.\, PB = PC \,.\, PD = PC' \,.\, PD' = \ldots = PT^2.$$

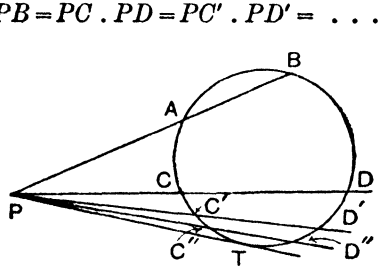


Fig. 85.

Exercise.—Prove the preceding corollary directly from Fig. 86. [Hint.—Prove the $\triangle$'s PAT, PTB similar, using the "alternate segment theorem" (Part I, p. 209).]

The *converse* of the above theorem is: If A, B, C, D are four points and P is the intersection of AB and CD (produced if necessary), then if $PA \,.\, PB = PC \,.\, PD$ the four points are concyclic.

The proof is left as an exercise for the student.

[Hint.—A circle can be drawn to pass through three given points. Let the circle through A, B, C cut CD in E. Then prove that D coincides with E.]

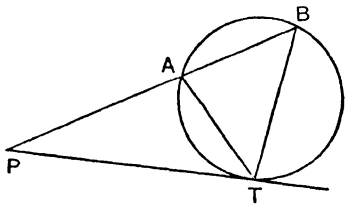


FIG. 86.

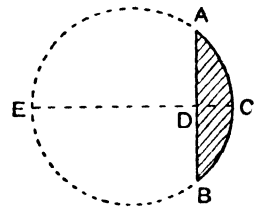


FIG. 87.

The *converse* of the corollary is : If A, B, T are three points on a circle and P is a point in AB produced, then if $PT^2 = PA \,.\, PB$ the line PT is a tangent to the circle.

The proof is left to the student.

Example.—A plano-convex lens is of thickness $\frac{1}{2}$ in. and the diameter of its plane face is 2 in. Find the radius of its spherical surface.

Let ACB represent a cross-section of the lens (Fig. 87), D the centre of the plane face and CE the diameter of the sphere through D. Then $AB = 2$ in. ; $DC = \frac{1}{2}$ in.

If the radius of the sphere is r in.,

$$ED = (2r - \tfrac{1}{2}) \text{ in.}$$

But $$ED \,.\, DC = AD \,.\, DB$$

$$\therefore (2r - \tfrac{1}{2}) \times \tfrac{1}{2} = 1 \times 1$$

$$r - \tfrac{1}{4} = 1$$

$$r = 1\tfrac{1}{4}$$

Hence the radius of the spherical surface is $1\frac{1}{4}$ in.

Length of chord of a circle

Let AB be any chord of a circle, and P, Q the mid-points of the minor and major arcs AB. Then PQ is a diameter and it bisects AB at right-angles at N (Fig. 88).

Let the chord $AB = c$, $PN = h$ and the diameter $= d$. Then

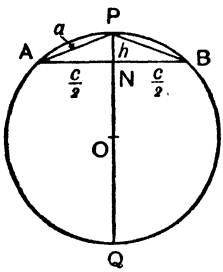


FIG. 88.

$$AN \cdot NB = PN \cdot NQ$$

$$\therefore \frac{c^2}{4} = h(d-h)$$

$$\therefore c^2 = 4h(d-h).$$

Let $AP = a$ (AP is the "chord of half the arc"). Then

$$a^2 = h^2 + \left(\frac{c}{2}\right)^2 = h^2 + \frac{c^2}{4}$$

$$= h^2 + h(d-h) = h^2 + dh - h^2$$

$$\therefore a^2 = dh.$$

Approximations to length of arc and area of segment

The length of an arc and the area of a segment are evaluated exactly, in terms of the angle subtended at the centre, in Chapter IX.

The following approximations are sometimes useful:

$$\text{Length of minor arc } AB \simeq \frac{8a - c}{3},$$

where, as above, a is the chord of half the arc and c is the chord of the whole arc.

[This is known as Huygens' approximation.]

$$\text{Area of minor segment cut off by } AB \simeq \frac{h^3}{2c} + \tfrac{2}{3}ch.$$

Huygen's approximation is fairly accurate, the error never exceeding 1·2%. For small arcs the accuracy is much better than that.

The approximation for the area is not so good, though it never exceeds 4%. It should be used only for small segments.

Mean proportional

The mean proportional to two quantities a and b has been defined on p. 147 as the quantity x such that $a : x = x : b$, i.e. such that $\frac{a}{x} = \frac{x}{b}$.

Hence $x^2 = ab$ and $x = \sqrt{ab}$.

[It is sometimes called the " geometric mean " of a and b.]

Construction.—To draw the mean proportional to two given lengths.

Set out the two given lengths AB, BC end to end in a straight line (Fig. 89).

Draw a semicircle on AC as diameter.

Through B draw the line BD perpendicular to AC, and let it cut the semicircle in D.

Then BD is the mean proportional to AB and BC.

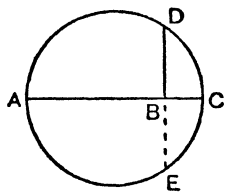


FIG. 89.

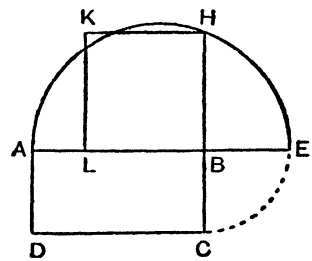


FIG. 90.

Proof.—Complete the circle on AC as diameter and produce DB to cut the circle in E.

$DB \cdot BE = AB \cdot BC$ (from the theorem on p. 151).

But $BE = DB$

$\therefore DB^2 = AB \cdot BC$.

As an alternative proof we may notice that $\angle ADC$ is a right-angle (angle in semicircle) and then use Exercise XX, Question 15.

Construction.—To draw a square equal in area to a given rectangle.

Let $ABCD$ be the given rectangle (Fig. 90). Produce AB to E, making $BE = BC$.

$$\text{Area of rectangle} = AB \,.\, BC$$
$$= AB \,.\, BE.$$

$\therefore$ Side of required square $= \sqrt{AB \,.\, BE}$ = mean proportional to AB and BE. This is found by the previous construction. It is BH in Fig. 90, and $BHKL$ is the required square.

Construction.—To draw the tangents to a circle from a given point outside it.

Let P be the given point and O the centre of the given circle. Draw the circle on OP as diameter and let it cut the given circle in the points A and B. Joint PA and PB. Then PA and PB are the tangents from P to the given circle.

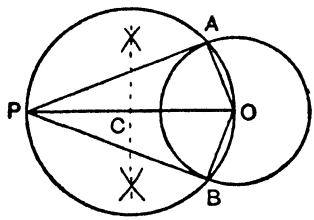


Fig. 91.

Proof.—Since the angle in a semicircle is a right angle, each of the angles PAO, PBO is a right angle. Thus PA is perpendicular to OA, and PB perpendicular to OB.

But OA, OB are radii of the given circle, and hence PA, PB are tangents to that circle.

It is clear from symmetry that the two tangents from any point to a circle are equal in length.

This can be easily proved from Fig. 91. For the $\triangle$'s PAO, PBO are each right-angled, have the common side PO, and have their sides AO, BO equal. They are therefore congruent, and hence $PA = PB$.

Common tangents of two circles

A line which touches each of two circles is called a common tangent to the two circles.

If two circles do not intersect they have four common tangents, as in Fig. 92 (*a*). *CD* and *EF* are called the *exterior* (or *direct*) common tangents; *HK* and *MN* are called the *interior* (or *transverse*) common tangents. If the two circles intersect they have only two common tangents, as in Fig. 92 (*b*).

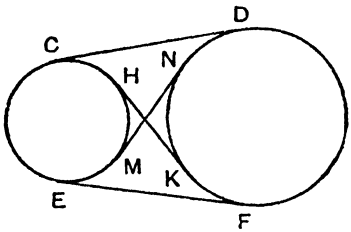


FIG. 92 (*a*).

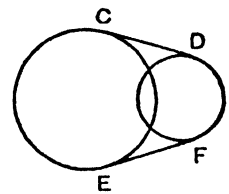


FIG. 92 (*b*).

When a belt passes over two pulleys the straight part of the belt is a common tangent to the two pulleys. In the case of an *open belt* (assuming there is no slackness) the tangents are exterior common tangents, and the two pulleys rotate in the same direction.

In the case of a *crossed belt* the tangents are interior common tangents, and the two pulleys rotate in opposite directions. It will be noticed from Fig. 92 (*a*) that the angle of lap is larger for a crossed belt than for an open one, and so there is less tendency to slip when the belt is crossed.

Construction.—To draw the common tangents to two circles.

(1) *Exterior common tangents.*

Let *A*, *B* be the centres of the two circles *CHE*, *DKF* (Fig. 93). Suppose that *B* is the centre of the larger circle.

With centre *B* draw a circle whose radius is equal to the *difference* of the radii of the given circles.

From *A* draw the tangents *AP*, *AQ* to that circle.

Join *BP* and *BQ*, and produce them to cut the larger circle in *D* and *F*.

Through A draw AC and AE parallel to BD and BF, and on the *same side* of AB as those radii.

Join CD and EF.

Then CD, EF are the exterior common tangents.

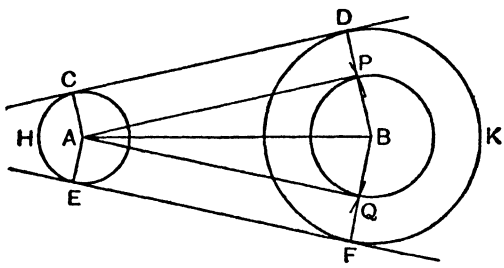


FIG. 93.

Proof.—Since BP is equal to the difference of the radii BD and AC, it follows that $PD = AC$. Thus PD is equal and parallel to AC, and hence $APDC$ is a parallelogram (Part I, p. 183).

Further, since AP is a tangent to the circle PQ, it is perpendicular to the radius BP and hence to PD. Thus $\angle APD$ is a right angle, and so the parallelogram $APDC$ is a rectangle.

CD is therefore perpendicular to AC and to BD, and hence it is a tangent to each of the given circles. [The construction is the same whether the circles intersect or not.]

(2) *Interior Common Tangents.*

In this case with centre B (the centre of the larger circle) draw a circle whose radius is equal to the *sum* of the radii of the given circles (Fig. 94).

From A draw the tangents AP, AQ to that circle.

Join BP and BQ, and suppose they cut the larger circle in D and E.

Through A draw AC parallel to DB but on the *opposite side* of AB, and AE parallel to FB on the opposite side of AB. Join CD and EF.

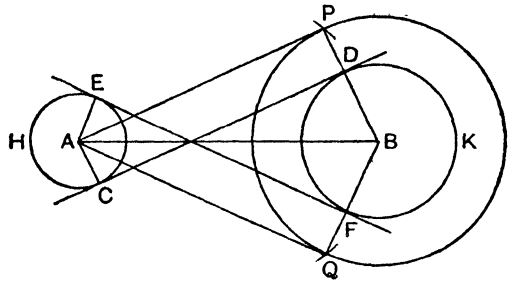


FIG. 94.

Thus CD, EF are the interior common tangents.
The proof is left to the student.

Medians of a triangle

A line joining a vertex of a triangle to the mid-point of the opposite side is called a *median*.

A triangle has three medians, one through each vertex.

Theorem.—The three medians of a triangle meet in a point, which trisects each of them.

Let Y be the mid-point of AC and Z the mid-point of AB, and let the medians BY, CZ intersect in G.

Join AG, and let AG produced cut BC in X. Then we have to prove that X is the mid-point of BC.

Draw BP parallel to ZC, cutting AG produced in P; join CP.

In $\triangle ABP$, Z is the mid-point of AB and ZG is parallel to BP; hence G is the mid-point of AP (Theorem, p. 124).

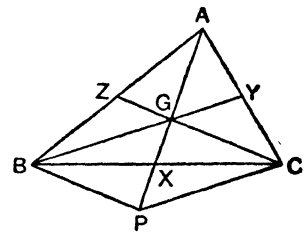


FIG. 95.

In $\triangle APC$, GY joins the mid-points of AP and AC; hence GY is parallel to PC, i.e. BG is parallel to PC.

$BGCP$ is a parallelogram.

X is the mid-point of BC. (Diagonals of a parallelogram bisect each other; Part I, p. 183.)

$\therefore$ AX is a median.

Hence the three medians meet in the point G.

Also $GX = \frac{1}{2}GP$. (Diagonals of a parallelogram bisect each other.)

But $GP = AG$ (since G is mid-point of AP; proved above).

$\therefore$ $GX = \frac{1}{2}AG$, i.e. $GX = \frac{1}{3}AX$.

Similarly it can be proved that $GY = \frac{1}{3}BY$ and $GZ = \frac{1}{3}CZ$.

Thus G is the point of trisection of each of the medians.

[For another proof, depending on similar triangles, see Exercise XX, Question 19.]

Centre of gravity, or centroid, of a triangular area

In Fig. 96 the triangular area ABC has been divided up into a number of thin strips by drawing lines parallel to the side BC. One such strip, PQ, is shown shaded. The centre of gravity of each strip, which may be regarded as a thin rod, is at its mid-point. Since the mid-points of all the strips lie on the median AX (see Exercise XX, Question 34) it follows that the centre of gravity of the whole area lies on AX.

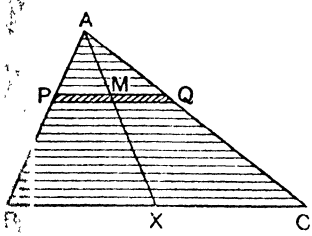


FIG. 96.

In the same way, by dividing up the triangle into thin strips parallel to AB or AC, it is seen that the centre of gravity also lies on each of the other medians. It must therefore lie at the point of intersection of the medians.

Hence the centre of gravity of a triangular area is the point of intersection, G, of the medians. The point G is usually called the *centroid* of the triangle.

and the inside width is 4 ft. 0 in. Find the outside

ct a line of length $\sqrt{6}$ in., by drawing the mean pro- in. and 3 in. Measure the length of your line, and h the value of $\sqrt{6}$ found from tables or by calculation.

$\sqrt{5}$ by a geometrical construction, and compare with given in your tables or found by calculation.

raw a rectangle of sides 2·6 in. and 1·5 in. Construct a equal in area to the rectangle, and measure its side.

. Draw a circle of radius $1\frac{1}{4}$ in. and mark a point 4 in. from centre. Draw the tangents from the point to the circle, and asure their lengths.

Also find their lengths by calculation.

17. Draw two circles of radii 2 cm. and 3·5 cm. with their centres 12 cm. apart. Draw the exterior common tangents and measure their lengths.

18. Draw the two circles in Question 17, and draw the interior common tangents. Measure their lengths.

19. Calculate the lengths of the four common tangents to the two circles in Question 17.

20. Two circles of radii a and b have their centres distant c apart, c being greater than $a+b$. Calculate the lengths of (i) the exterior common tangents, (ii) the interior common tangents.

CHAPTER VIII

SOLID GEOMETRY

Lines in space

If we draw any two lines in a plane (e.g. on a flat sheet of paper) they will intersect if produced far enough, provided they are not parallel.

Two lines in space, even if they are not parallel, i.e. not in the same direction, do not generally intersect. The student will see this fact if he takes a pencil in ea[illegible]nd and holds them in any positions at random.

If two lines in space do not intersect nor [illegible] say they are *skew* to each other. The following [illegible] of skew lines: (1) a line [illegible] ceiling of a room and [illegible] parallel to the first one, [illegible] floor, (2) the top edge [illegible] cover of a book and the [illegible] of the back cover, when the [illegible] partly open, (3) the diagonals [illegible] $B'D'$ of opposite faces of the [illegible] Fig. 104, (4) the lines AC and B[illegible] Fig. 104.

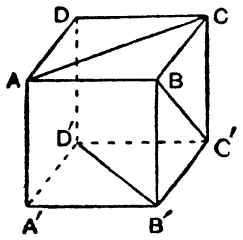


FIG. 104.

Since two lines which lie in the same plane either intersect or are parallel, two skew lines cannot lie in the same plane; that is, it is impossible to draw a plane to contain each of two skew lines.

Planes in space

Any two planes in space intersect in a line, unless they are parallel.

We call that line the *line of intersection* of the planes.

A line in space cuts any plane in a point, unless it is parallel to the plane.

Projection on a plane

If from any point outside a plane we draw the line perpendicular to the plane, the foot of the perpendicular is called the *projection* of that point on the plane.

If from every point of a curve, or figure of any sort in space, we draw perpendiculars down to a plane, the feet of the perpendiculars will lie on another curve or figure. This is called the projection of the original curve, or figure, on the plane.

[We sometimes speak of this projection as the *orthogonal projection* or *central projection* to distinguish it from other types of projection used in drawing.]

It is clear that the projection of a line on a plane is another line.

A *plan* is a projection on a horizontal plane; an *elevation* is a projection on a vertical plane. An object has a single plan, but may have various elevations according to the vertical plane on which it is projected; e.g. we speak of a side elevation, end elevation, etc.

Angle between a line and a plane

If M, N are the feet of the perpendiculars from P, Q on the plane $ABCD$ (Fig. 105), the line MN is the projection of the line PQ on the plane.

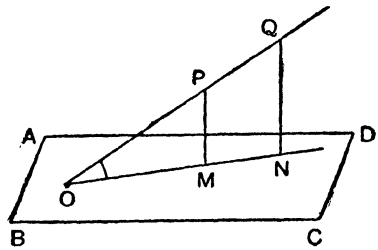


FIG. 105.

If the line PQ cuts the plane in O, the line MN passes through O.

The angle between a line and a plane is defined as being the angle between the line and its projection on the plane.

The angle between the line PQ and the plane $ABCD$ in Fig. 105 is $\angle POM$.

Angle between two planes

The angle between two planes, such as $ABCD$ and $ABEF$ in Fig. 106, is defined as follows: Take any point P in their line of intersection AB, and draw the lines PQ and PR, one in each of the planes, both lines being perpendicular to

AB. The angle between those lines, viz. $\angle QPR$, is defined to be the angle between the planes.

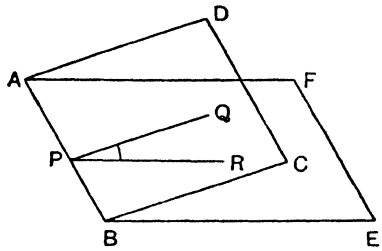


FIG. 106.

The student can make a model of Fig. 106 by drawing a line AB on a sheet of stiff paper, then drawing any line QPR perpendicular to AB on the paper, and finally folding the paper about AB.

Angle between two skew lines

The angle between two lines depends only on their directions.

The angle between two skew lines is therefore equal to the angle between two intersecting lines parallel to the given lines.

In practice the following is usually the most convenient way of defining the angle: Let AB, CD (Fig. 107) be two skew lines. Through any point P of AB draw PQ parallel

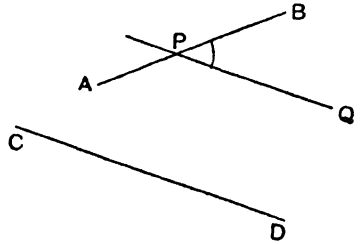


FIG. 107.

to CD. Then the angle between AB and CD is the angle between AB and PQ, i.e. the $\angle BPQ$.

Example.—What is the angle between the diagonals AC and $B'D'$ of opposite faces of the cube in Fig. 104 ?

AC is clearly parallel to $A'C'$. Thus the angle between AC and $B'D'$ is equal to the angle between $A'C'$ and $B'D'$, which is a right angle, since $A'B'C'D'$ is a square. Hence AC and $B'D'$ are perpendicular to each other.

Lines of greatest slope in a plane

If in Fig. 108 the plane $ABEF$ is horizontal, the lines of greatest slope (i.e. the steepest lines) in the plane $ABCD$ are those which are perpendicular to AB.

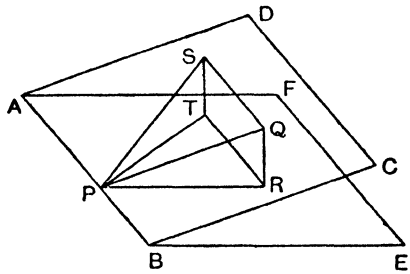


FIG. 108.

For let PQ be the line through P perpendicular to AB in the plane $ABCD$, and let PS be *any other* line through P in that plane. Let SQ be parallel to AB, and let R, T be the projections of Q, S on the horizontal plane. SQ is horizontal, since it is parallel to AB which is horizontal, and hence $RQ = TS$.

Also PQ is perpendicular to QS, since QS is parallel to BA, and hence PQ is shorter than PS.

Thus in moving from P to Q we rise the same vertical distance as in moving from P to S, but the distance travelled

in the first case is less than in the second. Hence PQ is steeper than PS.

It is clear that the inclination of a plane to the horizontal is the same as the inclination of its lines of greatest slope.

Direction of a line in space

Directions of lines on the ground, or in any horizontal plane, are given by compass-bearings, such as N.E. or S. 20° W. To indicate the direction of a line in space we need to specify two angles. For example, in describing the position of a star, we might say that it was seen in a direction N. 50° W. at an elevation of 42°. To locate the star from this information we should first turn our telescope through a *horizontal* angle of 50° from the North towards the West, and then raise it through an angle of 42° in the *vertical* plane. The angles are shown in Fig. 109; NOA is a horizontal plane, AOB is a vertical plane.

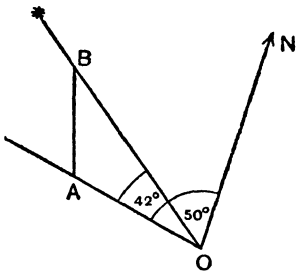


FIG. 109.

It is important to realize that angles measured by a compass are angles in a horizontal plane, so that when we say that a line, such as OB in Fig. 109, has a bearing of N. 50° W. we mean that its projection on a horizontal plane is in the direction N. 50° W. Thus all lines in the same vertical plane have the same compass-bearing.

Direction (or orientation) of a plane in space

A line which is perpendicular to a plane is called a *normal* to the plane. There is one normal through each point of a plane, and all the normals are parallel. Thus the direction of a plane (or its *orientation* as it is sometimes called) is known when the direction of its normals is known.

When we say that a wall (or, more correctly, one side of a wall) faces south, we mean that its normals (on that side) point south. If the wall is not vertical then its direction will be specified when we know, in addition, its inclination to the horizontal plane.

Thus if we say that a hill-side faces N. 70° E. at an inclination of 14°, we mean that it is inclined at 14° to the horizontal and that its normals bear N. 70° E.

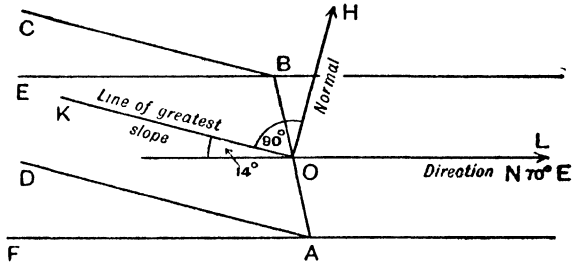


FIG. 110.

In Fig. 110, *ABCD* is the plane of the hill-side, *ABEF* a horizontal plane, *OH* a normal and *OK* the line of greatest slope through *O*. *KOHL* is a vertical plane.

Length of the projection of a line on a plane

Let *PQ* be a line of length *l* making an angle θ with a plane *ABCD* (Fig. 111), and let *pq* be its projection on the plane.

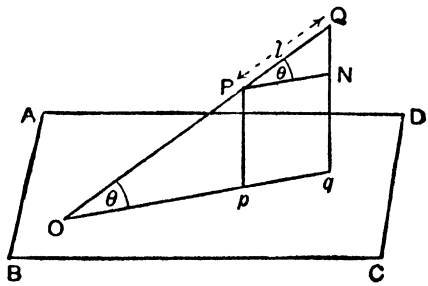


FIG. 111.

(The projections of points are often denoted by the corresponding small letters; thus here we have denoted the projection of the point P by the letter p, and the projection of Q by q.) Draw PN parallel to pq. Then $\angle QPN = \angle POp$ (by parallels), and $\angle PNQ$ is a right angle.

Also $PNqp$ is a rectangle. $\therefore pq = PN$.

Now in the right-angled triangle PNQ,

$$\frac{PN}{PQ} = \cos\theta$$

$$\therefore PN = PQ\cos\theta = l\cos\theta$$

$$\therefore pq = l\cos\theta.$$

Projection of an area

In Fig. 112 AB is the line of intersection of two planes X and Y, and $PQRS$ is a rectangle in the plane X, whose sides are parallel to AB and perpendicular to AB. The projection

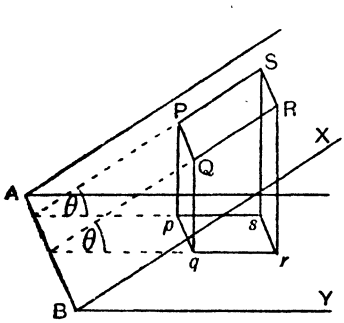


FIG. 112 (*a*).

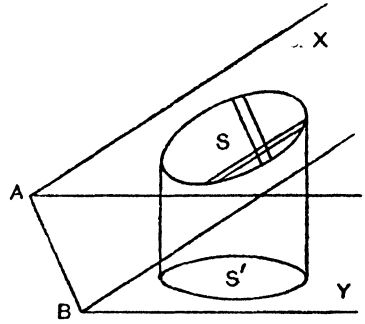


FIG. 112 (*b*).

of the rectangle on the plane Y is $pqrs$. It is clear that pq and sr are parallel to AB, while ps and qr are perpendicular to AB; thus $pqrs$ also is a rectangle.

If the angle between the planes is θ, then

$$pq = PQ,\ sr = SR,\ qr = QR\cos\theta,\ ps = PS\cos\theta.$$

$$\therefore \text{Area } pqrs = pq \times qr = PQ \times QR \cos\theta$$
$$= (\text{area } PQRS) \times \cos\theta.$$

Now suppose we consider *any* area S in the plane X (Fig. 112 (*b*)) and its projection S' on the plane Y.

As explained in Chapter VI (p. 142), we can regard the area S as built up of rectangles whose sides are parallel to AB and perpendicular to AB. When we project on to the plane Y, the area of each of these rectangles is reduced in the ratio $\cos\theta : 1$, and hence the sum of their areas is reduced in the same ratio. Thus

$$S' = S \cos\theta.$$

Reduction of lengths in projection

If, in Fig. 112, we regard Y as a horizontal plane, the lines of greatest slope in the plane X are the lines perpendicular to AB. Any line, in the plane X, oblique to AB is inclined to the plane Y at an angle less than θ, and the cosine of its angle of inclination is therefore greater than $\cos\theta$. Hence for such lines the ratio *length of projection : original length* is greater than $\cos\theta$, the ratio depending on the inclination of the line.

The lines of greatest slope, therefore, suffer the greatest reduction in length.

Thus, in projecting from one plane on to another, all lines parallel to the line of intersection of the planes are unaltered in length, all lines perpendicular to the line of intersection are reduced in the ratio $\cos\theta : 1$, and all other lines are reduced in some intermediate ratio.

The ellipse

If a circle in a plane X (Fig. 113) is projected on to another plane Y, the oval-shaped figure obtained is called an *ellipse*.

The centre of the circle, H, projects into a point O which is called the *centre* of the ellipse. Chords of the ellipse which pass through its centre are called *diameters* of the ellipse. Diameters of the circle project into diameters of the ellipse.

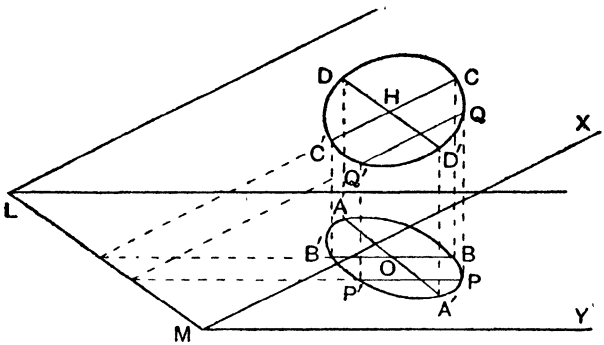


FIG. 113.

The diameter DD' parallel to LM, the line of intersection of the planes, is unaltered in length by projection ; the diameter CC' perpendicular to LM is reduced more than any other diameter. Hence their projections, AA' and BB', are the greatest and least diameters of the ellipse ; we call AA' the *major axis* and BB' the *minor axis* of the ellipse.

If the radius of the circle is a, and the angle between the planes is θ, $AA' = 2a$ and $BB' = 2a \cos \theta$.

Also, any chord QQ' of the circle which is perpendicular to LM projects into a chord PP' of the ellipse, such that $PP' = QQ' \cos \theta$.

In Fig. 114 the circle and the ellipse are drawn in the same plane with the diameters DD' and AA' superposed.*

The lengths of the major and minor axes of an ellipse are usually denoted by $2a$ and $2b$ respectively. Thus

$$2b = BB' = 2a \cos \theta$$

$$\therefore \cos \theta = \frac{b}{a}.$$

* We may regard Fig. 114 as obtained by projecting the circle $ACA'C'A$ on a plane through AA' inclined to the plane of the circle, and then rotating the second plane together with the projected figure (the ellipse) back into the plane of the circle.

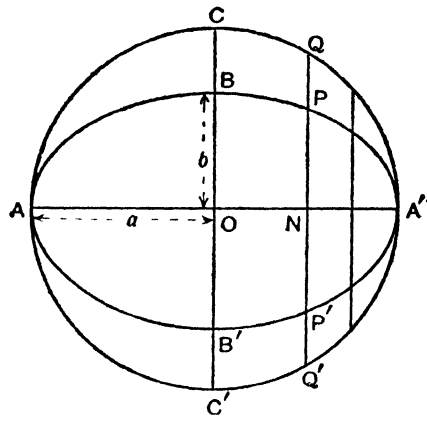


FIG. 114.

Hence an ellipse of major axis $2a$ and minor axis $2b$ may be regarded as obtained by projecting a circle of radius a on a plane inclined to the plane of the circle at an angle θ such that $\cos\theta = \frac{b}{a}$.

Since $PP' = QQ' \cos\theta$ and N is the mid-point of both QQ' and PP',

$$NP = NQ \cos\theta = \frac{b}{a} \times NQ.$$

Hence we have the following construction for drawing an ellipse with axes of lengths $2a$, $2b$:

Draw a circle of radius a, and any diameter AA'. From any point Q of the circle draw QN perpendicular to AA' and mark the point P on QN such that $NP = \frac{b}{a} \times NQ$. Do this for a large number of positions of Q, and join up, by a smooth curve, all the points such as P obtained in this way. The resulting curve will be the required ellipse.

The way in which this construction leads to the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has been shown on page 65.

There are various simple mechanical constructions for drawing an ellipse by means of a *trammel*. For these the student is referred to any book on geometrical drawing.

Area of an ellipse

$$\begin{aligned}\text{Area of ellipse} &= (\text{area of circle}) \times \cos\theta \\ &= \pi a^2 \times \frac{b}{a} \\ &= \pi ab.\end{aligned}$$

Example.—Find the area of a section of a circular cylinder of diameter d by a plane inclined at an angle α to the axis of the cylinder.

Fig. 115 shows the section, PQ, and also a cross-section, MN (i.e. a section by a plane perpendicular to the axis).

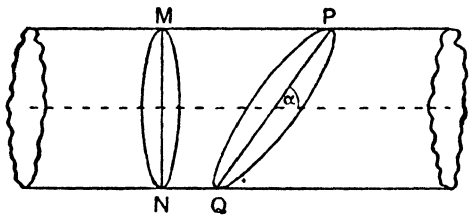


Fig. 115.

The section MN is the orthogonal projection of the section PQ on the plane MN.

The angle between the two planes is $90^\circ - \alpha$.

$$\begin{aligned}\therefore \text{Area of section } MN &= \text{area of section } PQ \times \cos(90^\circ - \alpha) \\ &= \text{area of section } PQ \times \sin\alpha.\end{aligned}$$

$$\begin{aligned}\therefore \text{Area of section } PQ &= \frac{\text{area of section } MN}{\sin\alpha} \\ &= \frac{\pi d^2}{4\sin\alpha}.\end{aligned}$$

Latitude and longitude

The Earth is very nearly a sphere and for most practical purposes may be regarded as such. Its radius is approximately 3960 miles.

If, in Fig. 116, P is any point on the Earth's surface and N, S are the North and South poles, the plane NPS cuts the sphere in a circle. The semicircle NPS, i.e. any semicircle on NS as diameter, is called a *meridian*. The angle between the plane of this meridian and the plane of the meridian

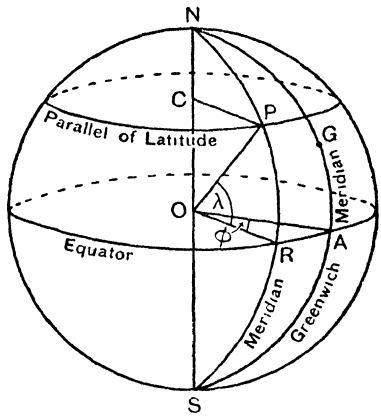


FIG. 116.

through Greenwich (the *Greenwich meridian*) is called the *longitude* of P. Longitude is reckoned East or West of Greenwich, up to 180° in each direction. All points on the same meridian have the same longitude, of course. In Fig. 116 the longitude of P is the angle ϕ (West).

The plane through P parallel to the plane of the Equator, i.e. perpendicular to the axis NS, cuts the sphere in a circle which is called a circle of latitude or, more generally, a *parallel of latitude*. The angle between OP and the plane of the Equator is called the *latitude* of P. Latitude is reckoned

north or south of the Equator, up to 90° in each direction. All points on the same parallel of latitude have the same latitude, of course. In Fig. 116 the latitude of P is λ (North).

The position of a point on the Earth's surface is determined by its latitude and longitude (usually abbreviated Lat. and Long.); for example New York is in Lat. 40° 20′ N., Long. 74° 0′ W.

Exercise XXII

1. A rectangular brick measures 9 in. × 4½ in. × 3 in. What is the distance between two opposite corners ?

2. If each side of the cube in Fig. 117 is of length a, find (i) the length of the diagonal AC', (ii) the angle between AC' and the face $A'B'C'D'$, (iii) the angle between AC' and AD.

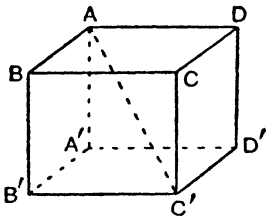


Fig. 117.

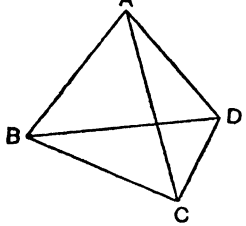


Fig. 118.

3. A vertical mast is strengthened by three stay-wires, each 35 ft. long, attached to a point of the mast 28 ft. above the ground, the wires being arranged symmetrically around the mast. Find (i) the angle each wire makes with the ground, (ii) the distance of the foot of each wire from the mast, (iii) the distance between the feet of the wires. [Assume the wires are straight and neglect the thickness of the mast.]

4. Fig. 118 shows a *regular tetrahedron* (i.e. a triangular pyramid whose six edges are all equal, so that all its faces are equilateral triangles). Find (i) the angle between the edge AB and the face BCD, (ii) the angle between any two faces (e.g. ABC and BCD).

5. Prove that the figure obtained by joining the mid-points of consecutive sides of a skew quadrilateral (i.e. a quadrilateral not in one plane) is a parallelogram.

6. A board in the shape of an equilateral triangle is placed in the corner of a room, with one edge resting on the floor and one against each of the walls. Find the inclination of the board to the floor.

7. A door, 6 ft. 6 in. high and 2 ft. 6 in. wide, originally shut, is opened through an angle of 30°. Find the angle between a diagonal of the door and the wall.

8. A 60° set square is placed on a table and is then rotated about its longest side through an angle of 45°. What are then the inclinations of the other two sides to the table ?

9. A pyramid has a square base of side 6 in. and its vertical height is 8 in. Find (i) the angle between a sloping edge and the base, (ii) the angle between a sloping face and the base.

10. Fig. 119 shows a *hipped roof*. The *ridge* AB is horizontal, and so is the plane $CDEF$ (the plane of the *eaves*). AC, AF, BD, BE are *the hip rafters*. If $AB = 15$ ft., $CD = FE = 20$ ft., $DE = CF = 14$ ft. ($CDEF$ being a rectangle), and the ridge is 5 ft. (vertically)

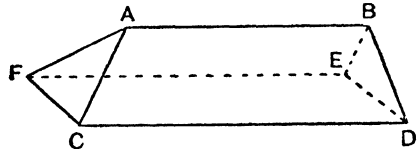


FIG. 119.

above the plane of the eaves, find (i) the area of the roof, (ii) the length of the hip rafters, (iii) the slope (i.e. inclination to the horizontal) of the hip rafters, (iv) the slopes of the roof-faces. [Hint.—Draw the perpendicular from A to the plane of the eaves.]

11. The ridges of two sloping roofs meet at right angles ; if the roofs are both inclined at 45° to the horizontal, find the inclination of their line of intersection.

[Hint.—Draw a plan of the roofs.]

12. A line of greatest slope on a hill-side has a gradient of 1 (vertical) in 10 (up the plane). What is the gradient of a road which makes an angle of 20° with the lines of greatest slope ?

13. A hill-side, which may be regarded as a plane, faces due south and is inclined at 20° to the horizontal. Find the inclination of a road on the hill-side in the direction N.E.

14. A prism whose cross-section is an equilateral triangle of side 4 in. is cut by a plane inclined at 60° to the edges of the prism. Find the area of the section.

15. A circular disc of diameter 1 ft. is held at right-angles to the sun's rays when the sun's altitude (i.e. elevation) is 55°. What is the area of the shadow on the ground ?

16. Fig. 120 shows the side-elevation and front-elevation of a semicircular dormer window. Find the area cut away from the main roof.

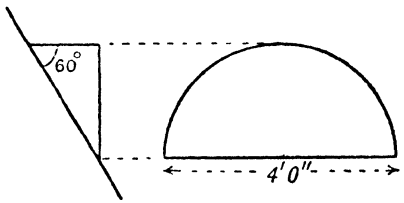


FIG. 120.

17. Find the diameter of a circle whose area is equal to that of an ellipse of axes 10 in. and 6 in.

18. A nautical mile is the length of arc on the Equator which subtends an angle of 1 minute at the centre of the Earth. Find the length of a nautical mile in feet.

19. Calculate the distance of Greenwich (Lat. 51° 29′ N.) from (i) the Equator, (ii) the North Pole.

20. Find the length of the Arctic Circle (parallel of Latitude $66\frac{1}{2}°$).

Mensuration. Volumes and surfaces

We shall assume the formulæ for the volumes and surfaces of the following simple solids.

Some of these formulæ have been demonstrated in Part I, Chap. XV; the remainder are proved in Chap. XVI of this Part or in Part III.

Prism.—Volume = (area of base) × (perpendicular distance between parallel ends).

Cylinder.—This may be regarded as a special case of a prism. For a circular cylinder, if the radius of the base $= r$, and height $= h$, then area of base $= \pi r^2$.

$$\therefore \text{ volume} = \pi r^2 h.$$

Area of curved surface = (circumference of base) × height

$$= 2\pi r h.$$

Pyramid. — Volume = $\frac{1}{3}$(area of base) × (perpendicular height).

Cone.—This may be regarded as a special case of a pyramid. If the radius of the base $= r$, and height $= h$, area of base $= \pi r^2$.

$$\therefore \text{ volume} = \tfrac{1}{3}\pi r^2 h.$$

If l is the length of the slant height,

area of curved surface = $\frac{1}{2}$ (circumference of base) × (slant height)

$$= \tfrac{1}{2} \times 2\pi r \times l$$

$$= \pi r l.$$

Sphere.—If the radius $= r$,

$$\text{volume} = \tfrac{4}{3}\pi r^3;$$

$$\text{surface area} = 4\pi r^2.$$

Volumes of similar solids

Similar solids have been defined in Chap. VI as solids which are of the same shape but different in size.

Of two similar solids, one may be regarded as an enlargement of the other.

One of the simplest solids is a rectangular block, shaped like a brick. It is sometimes called a *cuboid.*

Fig. 121 shows two similar rectangular blocks. If the edges of one of the blocks are of lengths a, b, c, the corresponding edges of the other block are equal multiples of those lengths; say na, nb, nc.

The volume of the first block is $a \times b \times c$, i.e. abc; the volume of the second block is $na \times nb \times nc$, i.e. n^3abc. Hence the

ratio of the volumes of the blocks is $\frac{n^3abc}{abc} = \frac{n^3}{1}$. Thus the ratio of the volumes of similar rectangular blocks is equal to the *cube* of the ratio of corresponding edges.

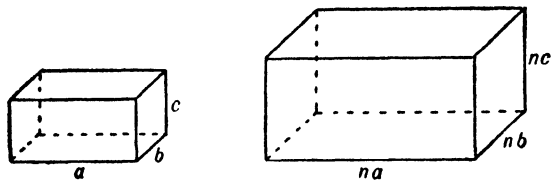


FIG. 121.

We have already seen (p. 142) how any plane area may be regarded as built up of rectangles. In much the same way any solid, whatever its shape, may be regarded as built up of rectangular blocks.

If the solid is enlarged, or reduced, every line in it is enlarged, or reduced, in the same ratio. If this ratio is $n : 1$, the volume of each of the inscribed rectangular blocks is increased in the ratio $n^3 : 1$, and hence the sum of the volumes of the blocks also is increased in the ratio $n^3 : 1$. Thus the ratio of the volumes of two similar solids is equal to the cube of the ratio of the lengths of corresponding lines in the two solids.

We usually express this in the form :

The volumes of similar solids are proportional to the cubes of their corresponding linear dimensions.

Thus, for example, if the linear dimensions of a body are doubled its volume is increased eight-fold.

Example.—Two iron castings are made to similar patterns, one being 3 ft. 6 in. long, the other 5 ft. 4 in. long. If the smaller casting weighs 14 cwt., what is the weight of the larger one ?

Since the castings are of the same material, their weights are proportional to their volumes.

$$\frac{\text{Volume of larger casting}}{\text{Volume of smaller casting}} = \left(\frac{5 \text{ ft. } 4 \text{ in.}}{3 \text{ ft. } 6 \text{ in.}}\right)^3$$

$$= \left(\frac{64 \text{ in.}}{42 \text{ in.}}\right)^3$$

$$= (\tfrac{3}{2}\tfrac{2}{1})^3 = (1{\cdot}524)^3 \simeq 3{\cdot}54.$$

$$\therefore \frac{\text{Wt. of larger casting}}{\text{Wt. of smaller casting}} = 3{\cdot}54$$

$$\therefore \text{Wt. of larger casting} = 3{\cdot}54 \times 14 \text{ cwt.}$$

$$= 49{\cdot}56 \text{ cwt.}$$

$$\simeq 2 \text{ tons } 9\tfrac{1}{2} \text{ cwt.}$$

Areas of surfaces of similar solids

If a solid bounded by plane faces is enlarged so that every length in it is doubled, the area of each face is increased fourfold and hence the total surface area of the solid also is increased fourfold.

More generally, if each length is increased or decreased in the ratio $n : 1$, the total surface area increases, or decreases, in the ratio $n^2 : 1$.

This is true also for solids bounded by curved surfaces; that is, *the surface-areas of similar solids are proportional to the squares of their corresponding linear dimensions.*

As an example, we might verify this for the case of a cylinder by using the formula for its surface area. If the radius of the cylinder is r and its height h,

$$\text{area of curved surface} = 2\pi rh,$$

$$\text{area of each end} = \pi r^2.$$

$$\therefore \text{Total surface area} = 2\pi rh + 2\pi r^2.$$

For a similar cylinder, whose radius and height are nr and nh,

$$\text{total surface area} = 2\pi nr \,.\, nh + 2\pi(nr)^2$$

$$= 2\pi n^2 rh + 2\pi n^2 r^2$$

$$= n^2(2\pi rh + 2\pi r^2).$$

Hence the ratio of their surface areas is $n^2 : 1$.

Example.—A manufacturer makes three types of motor-car headlamps, all of the same pattern but of different sizes. The diameters of the glasses for the three types are 12 in., 8 in., 6 in. If it costs 5 shillings to chromium plate the largest size, how much does it cost for the other sizes ? (Assume the same thickness of plating in each case.)

The areas of the surfaces to be plated are proportional to the squares of corresponding linear dimensions, i.e. are in the ratios $12^2 : 8^2 : 6^2$, i.e. 144 : 64 : 36, i.e. 36 : 16 : 9.

Hence the cost of plating the second size lamp

$$= \tfrac{16}{36} \times 5 \text{ shillings} = \tfrac{20}{9} \text{ shillings} \simeq 2s.\ 2\tfrac{1}{2}d.,$$

and the cost of plating the smallest size lamp

$$= \tfrac{9}{36} \times 5 \text{ shillings} = \tfrac{5}{4} \text{ shillings} = 1s.\ 3d.$$

Frustum of a pyramid

By a *frustum* of a pyramid is meant a portion of a pyramid cut off between two planes parallel to the base. In Fig. 122

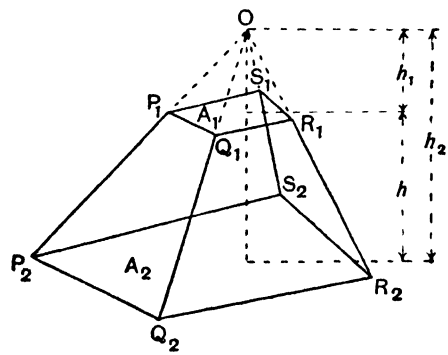


FIG. 122.

the solid between the planes $P_1Q_1R_1S_1$, and $P_2Q_2R_2S_2$ is a frustum of a pyramid.

Volume of frustum = volume of pyramid $OP_2Q_2R_2S_2$ – volume of pyramid $OP_1Q_1R_1S_1$.

Let the volumes of the pyramids $OP_1Q_1R_1S_1$, $OP_2Q_2R_2S_2$ be V_1, V_2 and their perpendicular heights h_1, h_2, and the areas of their bases A_1, A_2.

Let the volume of the frustum be V and the perpendicular distance between its parallel ends be h.

Then $h = h_2 - h_1$, and $V = V_2 - V_1$.

The pyramids $OP_1Q_1R_1S_1$, $OP_2Q_2R_2S_2$ are similar solids, and hence $\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$.

$$\therefore V = V_2 - V_1 = V_2\left(1 - \frac{V_1}{V_2}\right) = V_2\left(1 - \frac{{h_1}^3}{{h_2}^3}\right)$$

$$= \frac{V_2}{{h_2}^3}({h_2}^3 - {h_1}^3) = \frac{V_2}{{h_2}^3}(h_2 - h_1)({h_2}^2 + h_2h_1 + {h_1}^2)$$

$$= \tfrac{1}{3}\frac{A_2h_2}{{h_2}^3} \cdot h({h_1}^2 + h_1h_2 + {h_2}^2)$$

$$= \tfrac{1}{3}A_2h\left(\frac{{h_1}^2}{{h_2}^2} + \frac{h_1}{h_2} + 1\right).$$

The bases $P_1Q_1R_1S_1$, $P_2Q_2R_2S_2$ are similar figures, since they are in parallel planes and in perspective (from O). Hence

$$\frac{A_1}{A_2} = \left(\frac{h_1}{h_2}\right)^2$$

$$\therefore V = \tfrac{1}{3}A_2h\left(\frac{A_1}{A_2} + \sqrt{\frac{A_1}{A_2}} + 1\right)$$

$$= \tfrac{1}{3}h(A_1 + \sqrt{A_1A_2} + A_2).$$

Frustum of a cone.

Since a circle may be regarded as the limit of an inscribed polygon when the number of sides increases indefinitely, a cone may be regarded as the limiting case of a pyramid when the base becomes a circle.

Thus the previous formula is true also for a frustum of a cone, shown in Fig. 123. It may be expressed in another form as follows.

If the radii of the circular ends are r_1 and r_2, $A_1 = \pi r_1^2$, $A_2 = \pi r_2^2$.

$\therefore$ V = vol. of frustum of cone $= \frac{1}{3}h(\pi r_1^2 + \sqrt{\pi^2 r_1^2 r_2^2} + \pi r_2^2)$

$= \frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$.

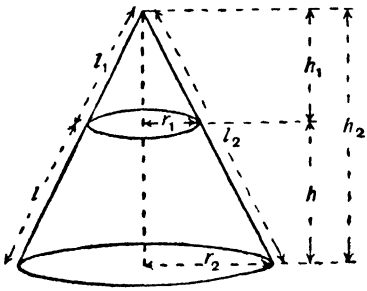


FIG. 123.

This result may be obtained also by using the formula $\frac{1}{3}\pi r^2 h$ for the volume of a cone, treating the frustum as the difference of the two cones shown in the figure.

The area of the curved surface of the frustum, which we shall denote by S, is the difference of the areas of the curved surfaces of the two cones shown in the figure.

Denoting the slant heights of the cones by l_1, l_2 and the slant height of the frustum by l, so that $l = l_2 - l_1$,

$$S = \pi r_2 l_2 - \pi r_1 l_1 = \pi r_2 l_2 \left(1 - \frac{r_1}{r_2}\frac{l_1}{l_2}\right).$$

But, from similar triangles, $\dfrac{r_1}{r_2} = \dfrac{l_1}{l_2}$

$$\therefore S = \pi r_2 l_2 \left(1 - \frac{l_1^2}{l_2^2}\right)$$

$$= \pi r_2 l_2 \left(\frac{l_2^2 - l_1^2}{l_2^2}\right) = \pi \frac{r_2}{l_2}(l_2 - l_1)(l_2 + l_1)$$

$$= \pi r_2 l\left(1 + \frac{l_1}{l_2}\right) = \pi r_2 l\left(1 + \frac{r_1}{r_2}\right)$$
$$= \pi(r_1 + r_2)l.$$

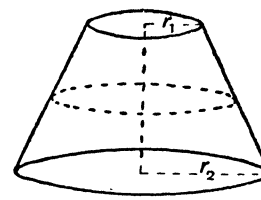


FIG. 124.

The radius of the mid-section of the frustum, i.e. the section by the plane mid-way between the ends and parallel to them (shown by the dotted circle in Fig. 124) is $\frac{1}{2}(r_1 + r_2)$. Its circumference $= 2\pi \cdot \frac{1}{2}(r_1 + r_2) = \pi(r_1 + r_2)$.

Hence we can express the surface area of a frustum of a cone in the form

$S =$ (circumference of mid-section) $\times$ (slant height).

Zone of a sphere

A *zone* of a sphere is a part cut off between two parallel planes.

Its plane faces are circles. If their radii are r_1, r_2 and the distance between the parallel planes is h (Fig. 125), it can be proved (see Part III) that

$$\text{volume of zone} = \tfrac{1}{6}\pi h\{3(r_1^2 + r_2^2) + h^2\}.$$

If the radius of the sphere is r, it can be proved (see Part III) that

$$\text{area of curved surface of zone} = 2\pi r h.$$

This is equal to the area cut off between the same planes on the cylinder which circumscribes the sphere and whose axis is perpendicular to the planes.

It should be noted that this area depends only on the distance between the planes and not on the positions of the planes.

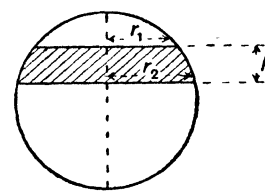


FIG. 125.

Segment (or cap) of a sphere

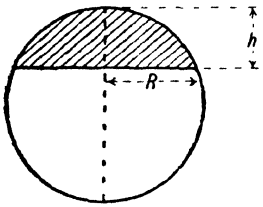


FIG. 126.

A *segment*, or *cap*, of a sphere is a part cut off by a plane (Fig. 126). It is a special case of a zone when one of the planes touches the sphere.

If the radius of the plane face is R and the thickness of the cap is h, the volume is obtained from the formula for the volume of a zone by putting $r_1 = 0$, $r_2 = R$.

$\therefore$ Volume of segment $= \frac{1}{6}\pi h(3R^2 + h^2)$.

Area of curved surface of segment $= 2\pi rh$.

Oblique frustum of a cylinder

If a cylinder is cut by a plane AB (Fig. 127) not parallel to the end faces, the volume $ABDC$ cut off is clearly equal to the volume of the cylinder $MNDC$, since vol. BPN = vol. APM.

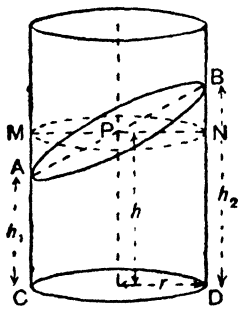


FIG. 127.

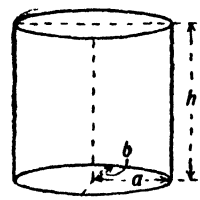


FIG. 128.

$\therefore$ Vol. of oblique frustum $ABDC = \pi r^2 h$.

If h_1, h_2 are the lengths of the shortest and longest generators of the frustum, $h = \frac{1}{2}(h_1 + h_2)$.

$\therefore$ Vol. of frustum $= \pi r^2\left(\frac{h_1 + h_2}{2}\right)$.

Also, area of curved surface of frustum = area of cylinder $MNDC = 2\pi rh = \pi r(h_1 + h_2)$.

Elliptic cylinder

If the cross-section of a cylinder is an ellipse whose principal axes are of lengths $2a$, $2b$,

volume of cylinder = (area of base) × height $= \pi ab.\ h$.

Exercise XXIII

1. An iron sphere of diameter 8 in. weighs 70 lb. What is the weight of one of diameter 6 in.?

2. A set of weights for scales are similar solids; the 4 oz. weight is $\frac{1}{2}$ in. thick. What are the thicknesses of the 1 oz., 8 oz. and 1 lb. weights?

3. The surface area of one sphere is eight times that of another; find the ratio of the radii and the ratio of their volumes.

4. Two equal spheres each of radius 5 cm. are melted down into a single sphere. What is its radius?

5. Two similar lead cylinders, one 3 in. long, the other 5 in. long, are melted down and made into a single similar cylinder. What is its length?

6. A pyramid 10 in. high is cut into two parts by a plane parallel to the base and 6 in. from the base. What is the ratio of the volumes of the two parts?

7. A cone of height 6 cm. is to be divided into two parts of equal weights by a plane parallel to the base. Find the distance of the plane from the base.

8. Find the volume of a right prism 1 ft. 6 in. long whose cross-section is an equilateral triangle of side 3 in.

9. A cubic foot of copper is drawn into a wire 800 yards long. What is the diameter of the wire?

10. A mound of earth is in the shape of a pyramid 2 ft. 6 in. high on a square base of side 3 ft. 3 in. Find the volume of earth in the mound.

11. Find the volume and surface-area of a regular tetrahedron of side 8 cm.

12. A sphere of 2 in. diameter is beaten out into a circular sheet 0·015 in. thick. Find the radius of the sheet.

13. A hollow cast-iron sphere weighs 240 lb. and its external diameter is 14 in. Find the thickness of the iron. [1 cu. in. of cast iron weighs 0·26 lb.]

14. Find the weight per foot length of lead pipe of $1\frac{1}{2}$ in. bore, $\frac{3}{16}$ in. thick. [1 cu. ft. of lead weighs 711 lb.]

15. A glass tube 15 cm. long, of external diameter 4 mm. weighs 4 gm. Find the inside diameter. [Sp. gr. of glass = 2·52.]

16. Find the volume of a spherical shell of internal radius r and thickness t, and show that if t is very small compared with r the volume is approximately $4\pi r^2 t$.

17. A reservoir containing a million gallons of water is emptied by a pipe of diameter 2 ft. 4 in. in 3 hours. Find the rate of flow in the pipe. [$6\frac{1}{4}$ gall. = 1 cu. ft.]

18. A pipe of diameter 3 ft. is to be replaced by two pipes, one having twice the diameter of the other, which shall together carry the same volume of water as the single pipe. Find their diameters.

19. Find the volume of the air-space under the hipped roof in Exercise XXII, Question 10 (p. 179).

[Hint.—Draw vertical planes through A and B so as divide the volume into a prism and two pyramids.]

20. A swimming bath is 110 ft. long and 25 ft. wide. The water is 3 ft. deep at one end and 6 ft. 6 in. at the other. Find the volume of the water.

21. Find the volume of a log of timber, 6 ft. long, which tapers from 8 in. square at one end to 3 in. square at the other.

22. A friction clutch is in the shape of a frustum of a cone of end diameters 5 in. and 7 in., the distance between the plane ends being 4 in. Find the area of the curved surface.

23. Fig. 129 shows a vertical cross-section of the head of a bolt, horizontal sections being circles. Find its volume.

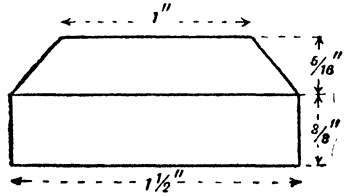


Fig. 129.

24. A hollow shaft 3 ft. 6 in. long is of uniform thickness throughout, and has a taper of 1 in 20. At the narrow end the external diameter is 4 in. and the internal diameter 3 in. Find the volume of metal in the shaft.

25. A glass ash-tray is in the form of a frustum of a cone with a spherical segment scooped out from the top, the dimensions being as shown in Fig. 130. Find the volume of glass.

26. The vertical cross-section of a casting is shown in 131, the bottom face being circular. Find the volume of metal in the casting.

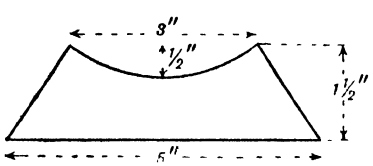


FIG. 130.

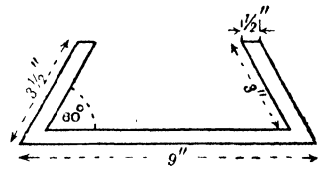


FIG. 131.

27. Find the *total* surface area of the frustum of a cylinder shown in Fig. 132.

28. Find the ratios of the surface areas of a cube, sphere and regular tetrahedron of equal volume.

29. A fine wire of diameter d is wound (close-coiled) on the outside of a cylinder of length l and diameter D. Find the length of the wire.

30. A right-angled triangle whose sides are 3 cm., 4 cm., 5 cm. revolves about its hypotenuse. Find the volume of the double cone generated.

31. Fig. 133 shows a section of a collar for a shaft. Find its volume.

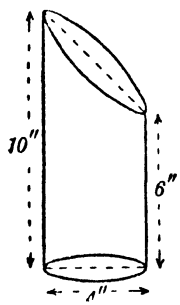


FIG. 132.

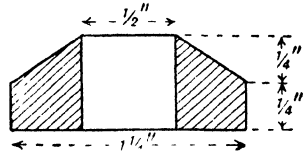


FIG. 133.

32. The largest possible cone of vertical angle 60° is cut out from a sphere. Find the ratio of its volume to that of the sphere.

33. An elbow-joint (Fig. 134) consists of two pieces of metal piping at right-angles. The external diameter of each pipe is 3 in. and the metal is $\frac{1}{8}$ in. thick. Find the volume of the metal.

34. Find the weight of the steel rivet shown in cross-section in Fig. 135, the total length of the rivet being $1\frac{5}{8}''$. [1 cu. in. of steel weighs 0·29 lb.]

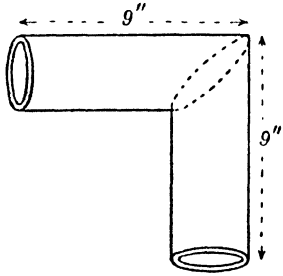


FIG. 134.

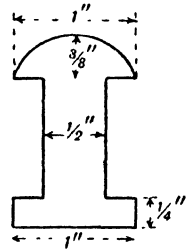


FIG. 135.

35. Prove that the volume of a segment of a sphere is $\frac{1}{3}\pi h^2(3r-h)$, where r is the radius of the sphere and h is the height of the segment.

36. If S is the area of the curved surface of a frustum of a cone, r_1 and r_2 the radii of its ends, l the slant height and θ the angle between the slant height and the base, prove by projection that $l\cos\theta = r_2 - r_1$ and $S\cos\theta = \pi(r_2^2 - r_1^2)$; deduce that $S = \pi(r_1 + r_2)l$.

37. Obtain the formula (p. 187) for the area of the curved surface of a frustum of a cone by regarding the surface as the limit of the sum of a number of trapezia.

TRIGONOMETRY

CHAPTER IX

RATIOS OF ACUTE ANGLES

The six trigonometric ratios

If the angles of a triangle are known, then the shape of the triangle is determined, though not its size. We can draw as many triangles as we please having their angles equal to the given angles ; they will all be " similar triangles " (p. 133).

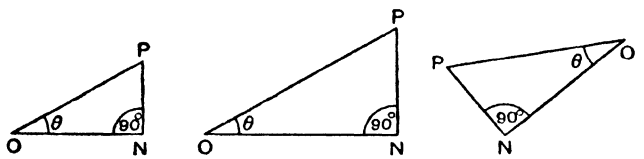


FIG. 136.

In Fig. 136 each of the triangles has $\widehat{ONP} = 90°$ and $\widehat{NOP} = \theta$. They are similar triangles and therefore their corresponding sides are proportional (p. 134) ; that is, for example, the ratio NP/OP is the same for all the triangles shown, though of course if θ had a different value this ratio would be different. The ratio NP/OP, which depends only on the value of θ, is defined as the *sine* of the angle θ, written as sin θ.

There are, altogether, six ratios between the three sides of the triangle ONP, viz. :

$$\frac{NP}{OP}, \frac{ON}{OP}, \frac{NP}{ON}, \frac{ON}{NP}, \frac{OP}{ON}, \frac{OP}{NP}.$$

Each of these ratios is independent of the size of the triangle and depends only on the value of the angle θ. They are spoken of as "the six trigonometric ratios of the angle θ," and are called the *sine*, *cosine*, *tangent*, *cotangent*, *secant* and *cosecant*, respectively, of the angle θ; they are written as sin θ, cos θ, tan θ, cot θ, sec θ, cosec θ, respectively.

The first three ratios have already been met with in Part I.

Definitions of the ratios

We can set out the definitions of the six ratios concisely in the following form.

Draw a right-angled triangle having one of its acute angles equal to the given angle θ. The side opposite to the right angle is called the hypotenuse. Now label the side which is opposite to the angle θ as "opposite side," and the remaining side as "adjacent side" (it is adjacent to the angle θ). Then, using the words "opposite" and "adjacent" as short for "opposite side" and "adjacent side,"

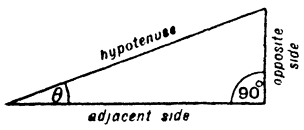


FIG. 137.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite}}.$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}.$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}, \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}.$$

Relations between the six ratios

It will be noticed from the definitions that three of these ratios are the reciprocals of the other three, viz.

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

[It will serve as a help in remembering these if we observe that in each of the three pairs of reciprocals there is one ratio with the prefix "co"; thus the reciprocal of sin is *co*sec, of sec is *co*s, of tan is *co*t.]

Also, referring to Fig. 136,

$$\tan\theta = \frac{NP}{ON} = \frac{\frac{NP}{OP}}{\frac{ON}{OP}} = \frac{\sin\theta}{\cos\theta}, \text{ and hence } \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Thus all the ratios can be expressed in terms of sin θ and cos θ.

We shall see later that, in the case of acute angles, any one of the ratios can be expressed in terms of any other, so that the values of all the ratios can be found if the value of *one* of them is known.

Tables of the ratios

The values of the ratios of angles from 0° to 90° are tabulated in books of trigonometric tables under the headings "natural sines," "natural secants," etc. In some books of tables only the values of sines, cosines and tangents are given; in that case the values of the other three ratios can be found from the relations above.

Where tables of cotangents, secants and cosecants are provided, however, the student is advised to make use of them; for not only does it save time and labour to read off, for example, sec 28° directly from the table of secants, instead of looking up cos 28° and then working out $\frac{1}{\cos 28^\circ}$, but there is less risk of mistake or inaccuracy since the numerical work in compiling the tables has been done by professional computers.

Example.—The top of a vertical pole is 9 ft. above the ground. Find, to the nearest inch, the length of the shadow cast on the ground when the elevation of the sun is 76°.

In Fig. 138, AB represents the pole, and BC the shadow on the ground. We are given that

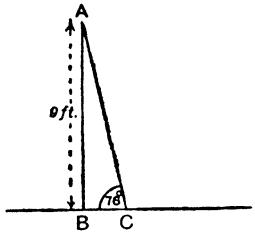


FIG. 138.

$AB = 9$ ft. and $\widehat{ACB} = 76°$.

In the right-angled triangle ABC,

$$\frac{BC}{AB} = \cot 76°$$

$$\therefore BC = AB \cot 76° = 9 \cot 76°$$

$$= 9 \times 0{\cdot}2493 \simeq 2{\cdot}244 \text{ ft.}$$

$$= 2 \text{ ft. } 3 \text{ in. (to the nearest inch).}$$

Hence the length of the shadow is 2 ft. 3 in. (to the nearest inch).

An alternative method is as follows:

$$\frac{AB}{BC} = \tan 76°$$

$$\therefore BC = \frac{AB}{\tan 76°} = \frac{9}{4{\cdot}0108} \simeq 2{\cdot}244 \text{ ft.} \simeq 2 \text{ ft. } 3 \text{ in.}$$

The former method is the easier, since it involves multiplying 9 by 0·2493 instead of dividing 9 by 4·0108.

As a general rule, it is best to use a ratio in which the unknown quantity (BC in the above example) occurs in the numerator.

Example.—A straight road up a hillside makes an angle of 10° with the horizontal. How far must a man walk along the road in order to rise a vertical height of 400 ft. ?

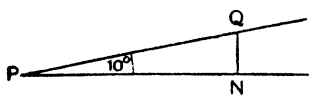


FIG. 139.

In Fig. 139, PQ represents the distance the man walks along the road. His vertical rise is NQ. $\therefore NQ = 400$ ft.

Here the unknown length, which we want to find, is PQ, and so we try to find a ratio in which PQ is the numerator and

in which the denominator is known. The only known length is NQ; so we take:

$$\frac{PQ}{NQ} = \text{cosec } 10°$$

$$\therefore PQ = NQ \text{ cosec } 10° = 400 \text{ cosec } 10°$$

$$= 400 \times 5{\cdot}7588 \simeq 2304 \text{ ft.}$$

Hence the man must walk a distance of 2304 ft. along the road.

If tables of cosecants are not available, we must work from a table of sines, thus:

$$PQ = 400 \text{ cosec } 10° = \frac{400}{\sin 10°} = \frac{400}{0{\cdot}1736} \simeq 2304 \text{ ft.}$$

Radian measure for angles

Angles are ordinarily measured in practical work (e.g. on protractors, galvanometer scales, etc.) in degrees and minutes, but there is another unit for angle which is extremely useful in theoretical work. It is the angle subtended at the centre of a circle by an arc equal in length to the radius, and it is called a *radian*. It is the angle $\widehat{AOB}$ in Fig. 140.

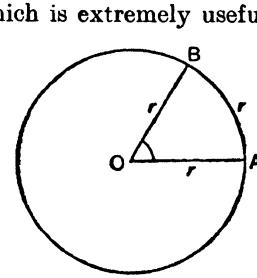


FIG. 140.

We can easily convert from degrees to radians or *vice versa*, just as we can convert from inches to centimetres or from grammes to pounds. For in a circle of radius r, half the circumference has a length πr and this arc subtends an angle 180° at the centre. Hence an arc of length r subtends an angle $\frac{180°}{\pi}$ at the centre. Thus, by our definition of a radian,

$$1 \text{ radian} = \frac{180°}{\pi} \simeq 57{\cdot}3° \simeq 57° \, 18'$$

(more accurately 1 radian $\simeq 57{\cdot}296° \simeq 57° \, 17' \, 45''$).

Hence also,

$$1^\circ = \frac{\pi}{180} \text{ radian} \simeq 0{\cdot}01745 \text{ radian.}$$

The rule for conversion is most easily remembered in the form :

$$\pi \text{ radians} = 180^\circ.$$

Length of arc of circle

If an arc of a circle subtends an angle θ radians (usually written as θ^c) at the centre, its length is θ times that of an arc which subtends 1 radian at the centre.

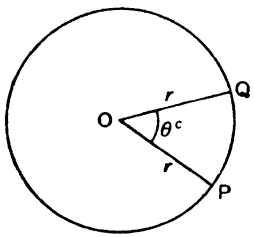


FIG. 141

Hence, if the radius of the circle is r,

$$\textit{length of arc} = r\theta.$$

If the angle is measured in *degrees*, say x°, then since $x^\circ = \frac{\pi x}{180}$ radians, the length of the arc is $r \cdot \frac{\pi x}{180}$.

Area of sector of circle

Referring to Fig. 141,

$$\frac{\text{area of sector } POQ}{\text{area of circle}} = \frac{\text{angle subtended by } PQ}{\text{angle subtended by whole circumference}}$$

$$= \frac{\theta}{2\pi}$$

$$\therefore \text{ area of sector } POQ = \pi r^2 \times \frac{\theta}{2\pi} = \tfrac{1}{2}r^2\theta.$$

Hence

$$\textit{area of sector of circle} = \tfrac{1}{2}r^2\theta.$$

Example.—An endless belt passes over two pulleys, whose radii are r_1 and r_2 and whose centres are distant d apart. Find the length of the belt, assuming it is taut throughout.

The belt may be open (Fig. 142 (*a*)) or crossed (Fig. 142 (*b*)). In the former case the two pulleys rotate in the same direction, in the latter case they rotate in opposite directions.

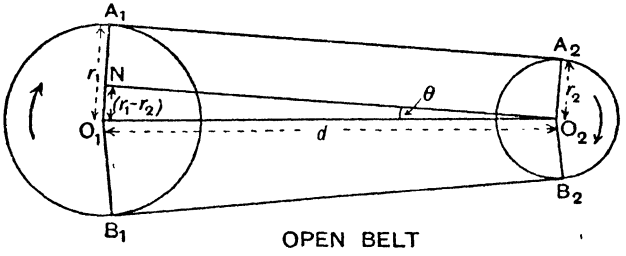


FIG. 142 (*a*).

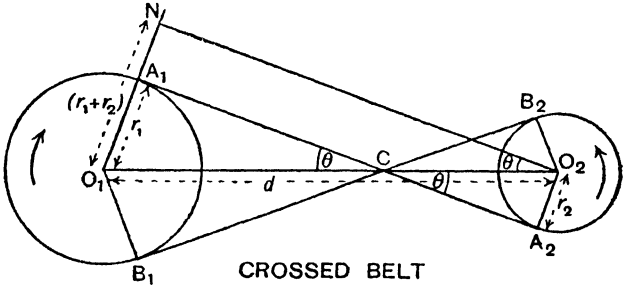


FIG. 142 (*b*).

O_1A_1, O_1B_1, O_2A_2 and O_2B_2 are the radii to the points of contact of the belt with the pulleys; O_1A_1 is parallel to O_2A_2, since each is perpendicular to the tangent A_1A_2. Also O_1B_1 is parallel to O_2B_2.

O_2N is drawn perpendicular to O_1A_1, or O_1A_1 produced.

For an open belt $O_1N = r_1 - r_2$; for a crossed belt $O_1N = r_1 + r_2$.

Let the angle between each of the straight parts of the belt and the line of centres be θ radians.

Let the total length of the belt be l.

I. For open belt

$O_2\widehat{O}_1N = \left(\frac{\pi}{2} - \theta\right)$ radians.

$\therefore$ Angle of lap on larger pulley $= 2\pi - 2\left(\frac{\pi}{2} - \theta\right) = (\pi + 2\theta)$ radians.

$\therefore$ Length of belt in contact with larger pulley $= r_1(\pi + 2\theta)$.

Also $O_1\widehat{O}_2A_2 = \left(\frac{\pi}{2} + \theta\right)$ radians.

$\therefore$ Angle of lap on smaller pulley $= 2\pi - 2\left(\frac{\pi}{2} + \theta\right) = (\pi - 2\theta)$ radians.

$\therefore$ Length of belt in contact with smaller pulley $= r_2(\pi - 2\theta)$.

Also $A_1A_2 = NO_2 = d \cos\theta$ and $B_1B_2 = A_1A_2 = d \cos\theta$.

$\therefore$ Total length of belt $= r_1(\pi + 2\theta) + r_2(\pi - 2\theta) + 2d \cos\theta$,

i.e. $l = \pi(r_1 + r_2) + 2\theta(r_1 - r_2) + 2d \cos\theta$.

The angle θ can be found from the relation $O_1N = O_1O_2 \sin\theta$,

i.e. $r_1 - r_2 = d \sin\theta$.

We can also write the expression for l in the form

$$l = \pi(r_1 + r_2) + 2\theta(r_1 - r_2) + 2(r_1 - r_2)\cot\theta,$$

on substituting for d in terms of r_1 and r_2.

$$\therefore\ l = \pi(r_1 + r_2) + 2(r_1 - r_2)(\theta + \cot\theta).$$

II. For crossed belt

Angle of lap on larger pulley $= 2\pi - 2\left(\frac{\pi}{2} - \theta\right) = (\pi + 2\theta)$ radians, as before.

$\therefore$ Length of belt in contact with larger pulley $= r_1(\pi + 2\theta)$.

Also $C\widehat{O}_2A_2 = \left(\frac{\pi}{2} - \theta\right)$ radians.

$\therefore$ Angle of lap on smaller pulley $= 2\pi - 2\left(\frac{\pi}{2} - \theta\right) = (\pi + 2\theta)$ radians.

$\therefore$ Length of belt in contact with smaller pulley $= r_2(\pi + 2\theta)$.
Also $A_1A_2 = NO_2 = d \cos \theta$ and $B_1B_2 = A_1A_2 = d \cos \theta$.

$$\therefore l = r_1(\pi + 2\theta) + r_2(\pi + 2\theta) + 2d \cos \theta$$
$$= \pi(r_1 + r_2) + 2\theta(r_1 + r_2) + 2d \cos \theta.$$

The angle θ is now given by the relation $r_1 + r_2 = d \sin \theta$.
The alternative form for l is :

$$l = \pi(r_1 + r_2) + 2(r_1 + r_2)(\theta + \cot \theta).$$

It will be noticed that, if the distance d is fixed, the length of a *crossed* belt depends only on $(r_1 + r_2)$, the *sum* of the radii of the pulleys, and not on r_1 and r_2 separately. This fact is made use of in fitting step or cone pulleys for varying the velocity ratio of two shafts.

Variation of the ratios as the angle increases from 0° to 90°

Since the hypotenuse is the longest side of a right-angled triangle, sin θ and cos θ are each less than 1, and their reciprocals, cosec θ and sec θ, are therefore each greater than 1 for all values of θ between 0° and 90°.

If OA and OB are two fixed radii of a circle, at right angles to each other (Fig. 143), and if OP is a radius making an angle θ with OA, then, as θ increases from 0° to 90°, OP turns from the position OA to the position OB.

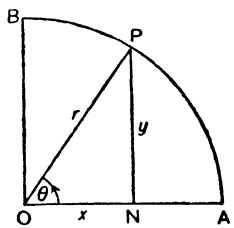


FIG. 143.

Suppose PN is perpendicular to OA, and that the radius of the circle is r units. Then if $ON = x$ units and $NP = y$ units,

$$\sin \theta = \frac{y}{r}, \cos \theta = \frac{x}{r}, \tan \theta = \frac{y}{x}, \cot \theta = \frac{x}{y}, \sec \theta = \frac{r}{x}, \text{cosec } \theta = \frac{r}{y}.$$

As θ increases from 0° to 90°, y increases from 0 to r and x decreases from r to 0. Hence sin θ increases from 0 to 1, and cos θ decreases from 1 to 0.

In tan θ, which is equal to y/x, the numerator increases and the denominator decreases, so that the fraction increases.

When $\theta = 0$, $\tan\theta = 0$; when θ is just less than 90°, y is nearly equal to r and x is nearly zero, so that y/x is then very large, and as θ approaches more and more closely to 90°, x becomes smaller and smaller and y/x, i.e. $\tan\theta$, increases without limit. We express this briefly by saying that $\tan\theta$ tends to infinity as θ tends to 90°, or, still more briefly, that $\tan 90° = \infty$ (read as $\tan 90° =$ infinity).

If the student looks at his tables he will find that $\tan 88° = 28{\cdot}64$, $\tan 89° = 57{\cdot}29$, $\tan 89°\ 30' = 114{\cdot}6$, $\tan 89°\ 48' = 286{\cdot}5$, $\tan 89°\ 54' = 573{\cdot}0$. Also, from seven-figure tables he will find that $\tan 89°\ 58' = 1718{\cdot}9$, $\tan 89°\ 59' = 3437{\cdot}7$.

The other three ratios are the reciprocals of the three already considered. Now when a quantity increases its reciprocal decreases, and *vice versa*, and hence we see that, as θ increases from 0° to 90°, $\cot\theta$ decreases from ∞ to 0, $\sec\theta$ increases from 1 to ∞, $\operatorname{cosec}\theta$ decreases from ∞ to 1.

The student can also verify these statements by considering how the ratios $\frac{x}{y}$, $\frac{r}{x}$ and $\frac{r}{y}$ vary as P goes from A to B.

Ratios of some important angles

The angles 30°, 45° and 60° are important because they occur so frequently in practical applications of trigonometry, and also because their ratios can be easily found without the use of tables.

If we draw a square $ABCD$ of side 1 unit and draw one diagonal AC (Fig. 144 (*a*)), then $C\widehat{A}B = A\widehat{C}B = 45°$, and, from Pythagoras' theorem, $AC = \sqrt{2}$ units. From the right-angled triangle ABC we can write down all the trigonometric ratios of 45°.

For example,

$$\sin 45° = \frac{BC}{AC} = \frac{1}{\sqrt{2}},\quad \tan 45° = \frac{BC}{AB} = 1,\quad \sec 45° = \frac{AC}{AB} = \sqrt{2}.$$

If we draw an equilateral triangle PQR (Fig. 144 (b)) and draw PM perpendicular to QR, then PM bisects QR and also bisects the angle QPR. Thus if we call the length of each side

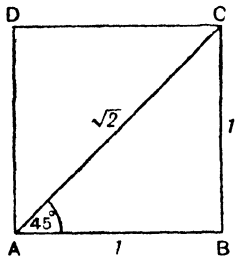


FIG. 144 (a).

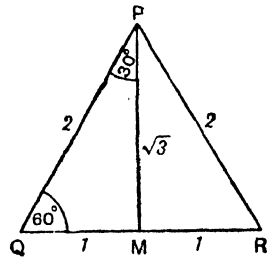


FIG. 144 (b).

of the triangle PQR 2 units, $QM = 1$ unit, and, from Pythagoras' theorem, $PM = \sqrt{(PQ^2 - QM^2)} = \sqrt{3}$ units. The angles of the triangle PQM are 60°, 90° and 30°, and from that triangle we can now write down all the trigonometric ratios of 60° and 30°.

For example, $\cos 60° = \dfrac{QM}{PQ} = \frac{1}{2}$, $\operatorname{cosec} 60° = \dfrac{PQ}{MP} = \dfrac{2}{\sqrt{3}}$,

$$\cot 30° = \frac{MP}{QM} = \sqrt{3}, \quad \sin 30° = \frac{QM}{PQ} = \frac{1}{2}.$$

The student should make a mental picture of the diagrams above, and should remember the values of $\sqrt{2}$ and $\sqrt{3}$, viz. $\sqrt{2} \simeq 1{\cdot}414$, $\sqrt{3} \simeq 1{\cdot}732$, so that he can convert the ratios to decimals if required.

Note on the use of tables

Since $\cos\theta$, $\cot\theta$ and $\operatorname{cosec}\theta$ all decrease as θ increases, the numbers in the difference columns in the tables of those ratios must be subtracted, not added.

Sin θ, tan θ and sec θ all increase with θ, and so the numbers in their difference columns must be added.

Example.—Find cot 42° 8′.

From tables, cot 42° 6′ = 1·1067

difference for 2′ = 0·0013

∴ cot 42° 8′ = 1·1067 − 0·0013 = 1·1054.

Example.—Find sec 66° 27′.

From tables, sec 66° 24′ = 2·4978,

difference for 3′ = 0·0050,

∴ sec 66° 27′ = 2·4978 + 0·0050 = 2·5028.

We are not likely to forget whether to add or subtract if we remember that 42° 8′ being greater than 42° 6′, its cotangent must be less than that of 42° 6′.

There is an easy rule for remembering whether to add or subtract differences. If we call the ratios which have the prefix *co-* (viz. cosine, cotangent, cosecant) the "co-ratios," the rule is: *subtract differences in the case of the co-ratios*, add in the case of the other ratios.

The same rule applies also to the tables of the logarithms of the trigonometric ratios (i.e. tables of log, sines, etc.).

The student will notice that in some of the tables there are certain ranges of values of the angle for which mean differences are not tabulated. For example, in four-figure tables of natural tangents no mean differences are tabulated when the angle is greater than about 77°. The reason is that the increase in the value of tan θ when θ increases from, say, 86° to 86° 6′ is 0·37 while the increase in tan θ when θ increases from 86° 48′ to 86° 54′ is 0·57 ; so that an increase of 1 minute in the angle between 86° and 86° 6′ corresponds to an increase of about 0·06 in the value of tan θ, which the same increase in the angle between 86° 48′ and 86° 54′ corresponds to an increase of about 0·10 in the value of tan θ. Thus no number in the difference column would be correct for the whole range of values in the line marked 86°.

In such cases the student can obtain a reasonably good value for the function by using proportional parts as shown in the following example.

Example.—Find tan 79° 34′.

From tables, tan 79° 30′ = 5·3955

tan 79° 36′ = 5·4486

∴ Difference for 6′ = 0·0531

∴ Difference for 4′ $\simeq \frac{4}{6} \times 0{\cdot}0531 = 0{\cdot}0354$

∴ tan 79° 34′ $\simeq$ 5·3955 + 0·0354 = 5·4309.

Exercise XXIV

1. Find, from tables,

(i) sin 55° 16′, (ii) tan 67° 19′, (iii) cot 48° 39′,
(iv) sec 12° 52′, (v) cos 81° 4′, (vi) cosec 33° 21′,
(vii) tan (0·571 radians), (viii) cosec (1·364 radians).

2. In Fig. 145 express as ratios, in terms of a, b, c,

(i) sin α,
(ii) cot β,
(iii) sec α,
(iv) tan α,
(v) cos β,
(vi) cosec α.

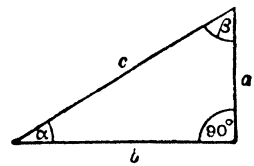
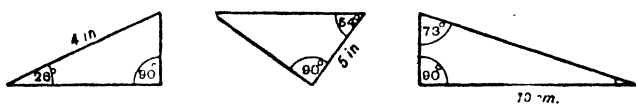


Fig. 145.

3. Find, without the use of tables,

sin 60°, cosec 30°, tan 60°, cosec 45°, sec 30°, cot 60°.

4. Find the lengths of the remaining sides of the triangles shown:



(i) (ii) (iii)

Fig. 146.

5. A man rows across a river in a direction making an angle of 55° with either bank. If the river is 30 yd. wide, how far does he have to row?

6. A pendulum 15 in. long swings, in a vertical plane, through an angle of 22° on either side of the vertical (Fig. 147). Find the vertical height through which its end rises.

7. In the preceding question, find the total horizontal distance through which the end moves in one complete oscillation.

8. A ladder, which leans against a vertical wall, makes an angle of 65° with the ground and reaches a window 12 ft. above the ground. Find the length of the ladder.

9. Find the area of the triangle in Fig. 148. Find also the distance of A from BC.

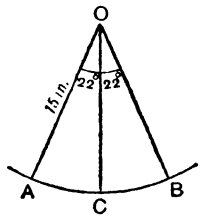


FIG. 147.

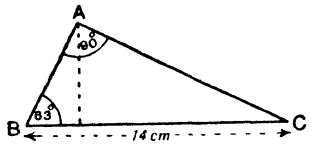


FIG. 148.

10. Find the angle θ (to the nearest minute) if:

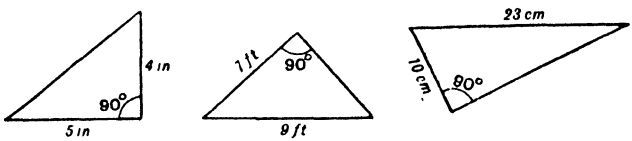
(i) $\sin\theta = 0{\cdot}55$, (ii) $\cot\theta = 0{\cdot}164$, (iii) $\tan\theta = 2{\cdot}26$.

11. Find the angle α (to the nearest minute) if:

(i) $\cos\alpha = 0{\cdot}973$, (ii) $\sec\alpha = 3{\cdot}453$, (iii) $\operatorname{cosec}\alpha = 1{\cdot}75$.

12. Draw an angle whose secant is 1·8. Measure the angle, and compare your answer with that given in the tables.

13. Find the remaining angles in the triangles shown:



(i) (ii) (iii)

FIG. 149.

14. Verify, from tables, that $\log\sec 58°\ 24' = -\log\cos 58°\ 24'$. Explain why the relation $\log\sec\theta = -\log\cos\theta$ is true for all values of θ.

15. A wireless mast 20 ft. high casts a shadow of length 14 ft. 6 in. on the ground. Find the angle of elevation of the sun.

16. A garage with a span roof has a rectangular floor of dimensions shown in Fig. 150, and the pitch of the roof is 28°. Find the area of the roof.

17. Water flows over a V-notch (Fig. 151) at a velocity of 40 ft. per sec. If the angle of the notch is 50° and the depth of the water is kept constant at 3 ft., find the volume which flows over in one minute.

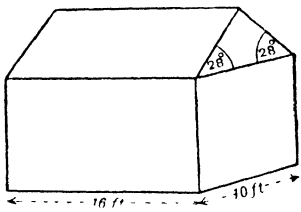


Fig. 150.

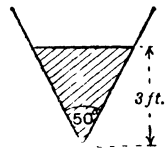


Fig. 151.

18. A telegraph pole of diameter 1 ft. 8 in. is strengthened by a wire cable, which passes once round the pole and is secured to the ground, as in Fig. 152. Find the length of cable required. Find also the distance of the foot of the cable from the centre of the pole.

19. Each leg of a step-ladder is 8 ft. long and it stands on level ground with its feet 5 ft. 6 in. apart. Find the angle which each leg makes with the ground, and the height of the top above the ground.

20. Find the lengths of the sides AB and BC of the metal plate in Fig. 153, and find the area of the plate.

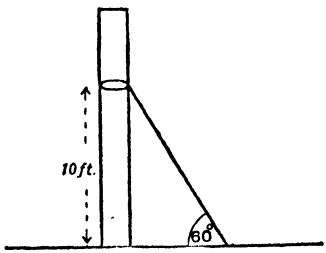


Fig. 152.

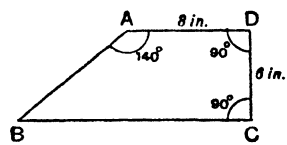


Fig. 153.

21. If a shell is fired with velocity v ft. per sec. from a gun at an elevation α, it reaches a height $\frac{1}{64}v^2 \sin^2 \alpha$ ft. and strikes the ground at a distance $\frac{1}{16}v^2 \sin \alpha \cos \alpha$ ft. from the gun, air resistance being neglected. If $v = 1200$ find (i) the distance at which it strikes the ground when $\alpha = 52°$, (ii) the angle of elevation necessary to reach a height of 9000 ft.

22. The two tangents from a point P to a circle of radius 5 cm. are inclined at an angle of 34° to each other. Find the length of either tangent, and the distance of P from the centre of the circle.

23. The power, p watts, in an alternating-current circuit is given by $p = vi$, where v is the voltage (in volts) and i is the current (in amperes). If $v = 200 \sin 50t$ and $i = 10 \sin (50t + \frac{\pi}{4})$, the angles being in radians, find the power when $t = 0{\cdot}01$.

24. A baulk of timber of the dimensions shown (Fig. 154) rests over a cylindrical drum of diameter 14 in. Find the heights of A and B above the ground.

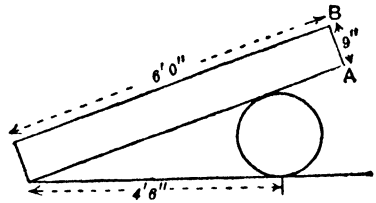


FIG. 154.

25. Draw the graph of $y = \sin x$ for values of x from 0° to 90°.

26. Draw the graph of $y = \cos x$ from $x = 0$ to $x = \frac{1}{2}\pi$ radians.

27. Draw the graph of $y = \tan x$ from $x = 0°$ to $x = 90°$.

28. A belt passes over two pulleys whose diameters are 2 ft. 2 in. and 1 ft., and whose centres are 4 ft. apart. Find the length of belt necessary (i) open, (ii) crossed.

Complementary angles

If the sum of two angles is a right-angle the angles are said to be "complementary"; each is called the "complement"

of the other. The complement of an angle θ is therefore the angle $90° - \theta$, or $\frac{\pi^c}{2} - \theta$. (Note. $\frac{\pi^c}{2}$ means $\frac{\pi}{2}$ radians.)

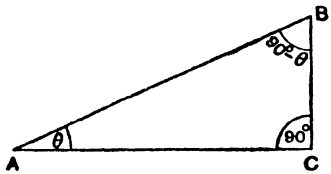


FIG. 155.

If we draw a right-angled triangle ABC with $B\hat{A}C = \theta$, then, since the three angles add up to 180°, $A\hat{B}C = 90° - \theta$. From the figure,

$$\sin(90° - \theta) = \frac{AC}{AB} = \cos\theta; \qquad \cos(90° - \theta) = \frac{BC}{AB} = \sin\theta;$$

$$\tan(90° - \theta) = \frac{AC}{BC} = \cot\theta; \qquad \cot(90° - \theta) = \frac{BC}{AC} = \tan\theta;$$

$$\sec(90° - \theta) = \frac{AB}{BC} = \operatorname{cosec}\theta; \quad \operatorname{cosec}(90° - \theta) = \frac{AB}{AC} = \sec\theta.$$

These are best remembered in words, thus:

cosine of any angle = sine of its complement;
cotangent of any angle = tangent of its complement;
cosecant of any angle = secant of its complement.

We can also interchange the words sine and cosine, tangent and cotangent, secant and cosecant in these relations, as is obvious since the complement of $90° - \theta$ is θ itself.

These relations show why we use the terms co-sine, co-tangent and co-secant, the prefix *co-* being the first part of the word complement.

By the aid of the three relations above we can use tables of sines, tangents and secants to find cosines, cotangents and cosecants. For example, cot 25° 12′ = tan (90° − 25° 12′) = tan 64° 48′ = 2·1251, from the table of tangents.

Identities

The meaning of the word *identity* in mathematics has been explained on p. 13.

The identity,

$$\sin^2\theta + \cos^2\theta \equiv 1$$

has been proved in Part I (p. 295).

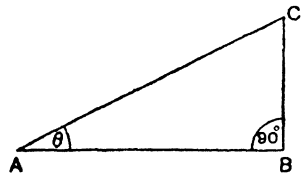


FIG. 156.

It follows at once from Pythagoras' theorem. For in Fig. 156 :

$$BC^2 + AB^2 = AC^2$$

$$\therefore \left(\frac{BC}{AC}\right)^2 + \left(\frac{AB}{AC}\right)^2 = 1$$

i.e. $\sin^2\theta + \cos^2\theta = 1.$

If we divide both sides of this identity by $\cos^2\theta$, we have

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

i.e. $$\tan^2\theta + 1 = \sec^2\theta.$$

Also, dividing both sides of the first identity by $\sin^2\theta$, we have

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

i.e. $$1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

As these three identities are often useful for simplifying trigonometric formulæ, we shall collect them together for reference :

$$\sin^2\theta + \cos^2\theta \equiv 1,$$

$$1 + \tan^2\theta \equiv \sec^2\theta,$$

$$1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$$

Example.—Prove that $\sin^2\theta + \sin^2\theta \tan^2\theta \equiv \tan^2\theta$.

L.H.S. (meaning " left-hand side " of the relation)

$$= \sin^2\theta\,(1 + \tan^2\theta)$$
$$= \sin^2\theta \,.\, \sec^2\theta$$
$$= \frac{\sin^2\theta}{\cos^2\theta}$$
$$= \tan^2\theta = \text{R.H.S.}$$

Example.—Simplify the expression $\dfrac{1+\cos x}{\sin x} + \dfrac{\sin x}{1+\cos x}$.

$$\text{Expression} = \frac{(1+\cos x)^2 + \sin^2 x}{\sin x\,(1+\cos x)}$$
$$= \frac{1 + 2\cos x + \cos^2 x + \sin^2 x}{\sin x\,(1+\cos x)}$$
$$= \frac{2 + 2\cos x}{\sin x(1+\cos x)}, \text{ since } \cos^2 x + \sin^2 x = 1,$$
$$= \frac{2(1+\cos x)}{\sin x(1+\cos x)}$$
$$= \frac{2}{\sin x}$$
$$= 2 \operatorname{cosec} x.$$

Given one ratio of an acute angle, to find the other ratios

Example.—If $\sin\alpha = 3/5$, find $\cot\alpha$.

Since $\sin^2\alpha + \cos^2\alpha = 1$

$$\therefore \cos^2\alpha = 1 - \sin^2\alpha = 1 - \tfrac{9}{25} = \tfrac{16}{25}$$
$$\therefore \cos\alpha = \tfrac{4}{5}$$
$$\therefore \cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{\cancel{5}} \times \frac{\cancel{5}}{3} = \frac{4}{3}.$$

The point to notice here is that we have not found the angle first, and have not used trigonometric tables at all.

[NOTE.—We took the positive square root for cos α above, viz. $\frac{4}{5}$, instead of $\pm\frac{4}{5}$, because we are assuming in this chapter that all the angles with which we deal are acute.]

A more direct method is as follows :

Draw a right-angled triangle ABC (Fig. 157) in which $BC = 3$ units, $AB = 5$ units. Then sin $\widehat{BAC} = \frac{3}{5}$ and therefore $\widehat{BAC} = \alpha$.

$$\therefore \cot \alpha = \frac{AC}{BC}.$$

But, from Pythagoras' theorem,

$$AC = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4 \text{ units};$$

$$\therefore \cot \alpha = \tfrac{4}{3}.$$

It will be seen that it is not necessary to draw the figure accurately ; we need only draw a right-angled triangle roughly,

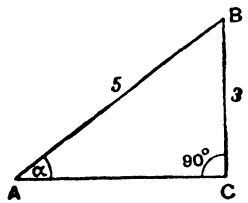


FIG. 157.

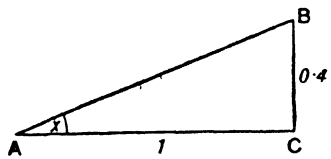


FIG. 158.

mark one of the angles α and call the lengths of the two sides, whose ratio is sin α, 3 and 5, or any two convenient numbers whose ratio is 3/5. We can then find the third side from Pythagoras' theorem, and so write down any of the ratios of the angle α.

Example.—If tan $x = 0{\cdot}4$, find sin x.

In $\triangle ABC$ (Fig. 158), let $\widehat{BAC} = x$.

$$\text{Then } \frac{BC}{AC} = \tan x = 0{\cdot}4.$$

Hence if $AC = 1$ unit, $BC = 0{\cdot}4$ units.

$$\therefore AB = \sqrt{1^2 + (0{\cdot}4)^2} = \sqrt{1 + 0{\cdot}16} = \sqrt{1{\cdot}16} \simeq 1{\cdot}077.$$

$$\therefore \sin x = \frac{BC}{AB} \simeq \frac{0{\cdot}4}{1{\cdot}077} \simeq 0{\cdot}371.$$

Exercise XXV

1. Verify from tables that $\cos 28° 36' = \sin (90° - 28° 36')$, and that $\tan 28° 36' = \cot (90° - 28° 36')$.

2. Verify the relations $\cos \theta = \sin (90° - \theta)$, etc., when $\theta = 60°$, without using tables.

3. If $\sin x = \cos 25°$, find x.

4. Show that $\sin (90° - \alpha) \cdot \cot (90° - \alpha) = \sin \alpha$.

5. Show that $\dfrac{\cos\left(\frac{\pi}{2} - \beta\right)}{\sec\left(\frac{\pi}{2} - \beta\right)} = \sin^2 \beta$.

6. If $\sin \theta = 5/13$, find $\tan \theta$.

7. If $\cos \phi = 1/2$, find $\operatorname{cosec} \phi$.

8. If $\tan \alpha = 3$, verify that $\dfrac{\sin \alpha - \cos \alpha}{\sec \alpha - \operatorname{cosec} \alpha} = 0{\cdot}3$.

9. Find, without tables, the value of $\tan \theta + \sec \theta$ when $\sin \theta = \frac{1}{2}$.

Simplify the following expressions:

10. $\dfrac{1}{\sin^2 \theta} - 1$.

11. $\sin^3 A + \sin A \cos^2 A$.

12. $\dfrac{\sec \alpha - \cos \alpha}{\sin \alpha}$.

13. $\dfrac{\cot \theta}{\sqrt{1 + \cot^2 \theta}}$.

14. $\dfrac{1}{1 + \sin x} + \dfrac{1}{1 - \sin x}$.

15. $\dfrac{\sin^2 A}{\tan A} - \dfrac{\cos^2 A}{\cot A}$.

Prove that:

16. $\sin^4 x - \cos^4 x \equiv 1 - 2 \cos^2 x$.

17. $\tan A + \cot A \equiv \sec A \operatorname{cosec} A$.

18. $\sec^2 \theta + \operatorname{cosec}^2 \theta \equiv \sec^2 \theta \operatorname{cosec}^2 \theta$.

19. $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B \equiv \sin^2 A - \sin^2 B$.

20. $\sqrt{\dfrac{1+\sin x}{1-\sin x}} \equiv \sec x + \tan x$. [Hint.—Multiply numerator and denominator by $\sqrt{1+\sin x}$.]

21. (Sec y − cos y) (cosec y − sin y) $\equiv$ sin y cos y.

Applications of trigonometry to solid figures

There are no new principles involved in applying trigonometry to solid figures. The student must be careful, however, to visualize the figure properly as a space figure, and must remember that right angles in space will not generally be represented by right angles in his diagram.

Example.—A room is 20 ft. long × 14 ft. wide × 10 ft. high, and a string is stretched from one corner of the ceiling to the opposite corner of the floor. Find the angle the string makes with the floor.

In Fig. 159, AE represents the string. AC is the projection of AE on the floor; hence $\widehat{CAE}$ is the required angle.

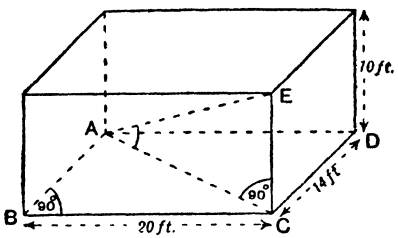


FIG. 159.

Since $\widehat{ABC}$ is a right angle,

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{14^2 + 20^2}$$
$$= \sqrt{196 + 400} = \sqrt{596} \simeq 24{\cdot}41 \text{ ft.}$$

Also $\widehat{ACE}$ is a right angle.

$$\therefore \cot \widehat{CAE} = \frac{AC}{CE} = \frac{24{\cdot}41}{10} = 2{\cdot}441$$

$$\therefore \widehat{CAE} = \underline{22^\circ\ 17'}.$$

[The student will notice that we found $\widehat{CAE}$ here from its cotangent instead of from its tangent. We did this merely because it is easier to divide 24·41 by 10 than to divide 10 by 24·41.]

Example.—A right pyramid has a square base, of side 4 cm., and its slant edges are each 5 cm. long. Find the height of the pyramid and the angle between any slant edge and the base.

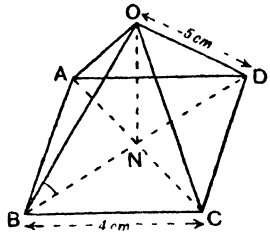


FIG. 160.

In Fig. 160, ON is perpendicular to the base $ABCD$. From symmetry N is the centre of the base.

Since $\widehat{BCD}$ is a right angle,

$$BD = \sqrt{BC^2 + CD^2} = \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2} \text{ in.}$$

$$\therefore BN = \tfrac{1}{2}BD = 2\sqrt{2} \text{ in.}$$

In the right-angled $\triangle OBN$ (in which $\widehat{N}$ is the right angle)

$$ON^2 = OB^2 - BN^2 = 5^2 - (2\sqrt{2})^2 = 25 - 8 = 17.$$

$$\therefore ON = \sqrt{17} \simeq 4{\cdot}123 \text{ cm.}$$

BN is the projection of the edge BO on the base. Hence angle between BO and base $= \widehat{OBN}$.

$$\text{Cos } \widehat{OBN} = \frac{BN}{OB} = \frac{2\sqrt{2}}{5} = \frac{2 \times 1{\cdot}414}{5} = \frac{2{\cdot}828}{5} \simeq 0{\cdot}566.$$

$$\therefore \widehat{OBN} \simeq 55° \, 32'.$$

Example.—A hillside is inclined at 15° to the horizontal, and a road is to be constructed on the hillside with a gradient of 1 (vertical rise) in 12 (along the road). What must be the angle between the road and the lines of steepest slope on the hillside ?

In Fig. 161, AC represents a portion of the road, AD and BC are the lines of steepest slope through A and C, and $ABFE$ is a horizontal plane. DE and CF are vertical lines.

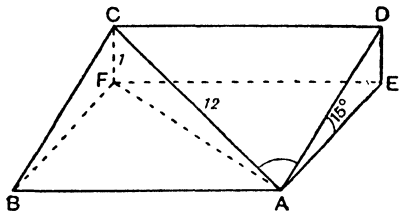


FIG. 161.

If $CF = 1$ unit, then $AC = 12$ units.

$\triangle CBF$ is a right-angled triangle and $\widehat{CBF} = 15°$.

$\therefore \dfrac{BC}{CF} = \text{cosec } 15°.$

$\therefore BC = 1 \text{ cosec } 15° = 3{\cdot}8637$ units.

The angle between the road and the lines of steepest slope $= \widehat{BCA}$ $(= \widehat{CAD})$.

$\triangle ABC$ is a right-angled triangle ($\widehat{ABC}$ being a right angle).

$\therefore \text{Cos } \widehat{BCA} = \dfrac{BC}{CA} = \dfrac{3{\cdot}8637}{12} \simeq 0{\cdot}322.$

$\therefore \widehat{BCA} \simeq 71° \ 13'.$

Exercise XXVI

1. The slant height of a right circular cone is 5 in., and the diameter of the base is 8 in. Find the vertical angle and the height of the cone.

2. Find the angle between two diagonals of a cube.

3. The roof of a lean-to shed slopes at 30° and the floor is square; find the angle of slope of a diagonal of the roof.

4. Fig. 162 represents a right triangular prism whose ends are equilateral triangles of side 6 in., and BCD is a section by a plane inclined at 40° to the ends. Find the lengths of AD and CD, and the area of the section BCD.

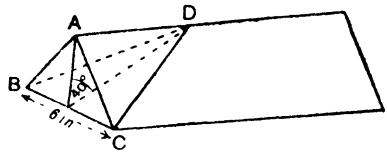


FIG. 162.

5. In Question 4, find the angle between CD and each of the faces of the prism through BC.

6. An electric light pendant in the shape of a hemispherical bowl of radius 8 in. is suspended from a point in the ceiling by three chains each 12 in. long, attached to points at equal distances apart on the rim. Find (i) the angle between each chain and the vertical, (ii) the angle between two chains.

7. The top and base of a reservoir embankment (Fig. 163) are squares of sides 180 yd. and 200 yd. respectively, and the vertical height is 15 ft. Find the slopes of the bank and of the slant edges.

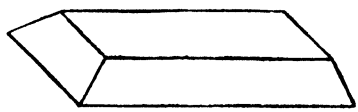


FIG. 163.

8. A hillside faces due South and has a slope of 30°. Find the gradient of a path on the hillside running in the direction N.W.

9. A vertical wall, facing East, is 16 ft. long and 5 ft. high. Find the area of its shadow on the ground when the sun is in the direction S.E. at an elevation of 60°.

Harder problems

In some problems the unknown quantity (length or angle) cannot be found directly. We have to denote it by a symbol, such as x or θ, and then write down what is stated in the

problem in terms of this unknown. We may then obtain an equation which we can solve for the unknown.

Example.—An aeroplane is observed at the same time by two anti-aircraft batteries, distant 1 mile apart, to be at elevations of 20° and 14°. Assuming that the aeroplane is travelling directly towards the two batteries, find its height and its horizontal distance from the nearer battery.

In Fig. 164, A represents the aeroplane, B and C the two batteries. AD is vertical.

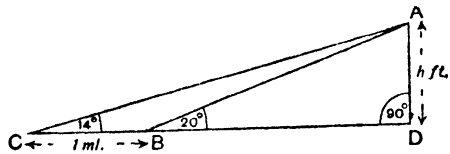


FIG. 164.

Using the foot as our unit of length, let $AD = h$. Then, from the right-angled triangle ACD,

$$\frac{CD}{h} = \cot 14° \qquad \therefore CD = h \cot 14°;$$

and, from the right-angled triangle ABD,

$$\frac{BD}{h} = \cot 20° \qquad \therefore BD = h \cot 20°.$$

$$\therefore CB = CD - BD = h(\cot 14° - \cot 20°).$$

But we are given that $CB = 1$ ml. $= 5280$ ft.

$$\therefore h(\cot 14° - \cot 20°) = 5280$$

i.e. $$h(4{\cdot}0108 - 2{\cdot}7475) = 5280$$

$$1{\cdot}2633h = 5280$$

$$h = \frac{5280}{1{\cdot}2633} \simeq 4180.$$

Hence the height of the aeroplane is 4180 ft. (approx.).

Its horizontal distance from B is BD

$BD = h \cot 20°$	*No.*	*Log*
$= 4180 \cot 20°$	4180	3·6212
$= 11480$ ft.	cot 20°	0·4389
$= 3827$ yd. $= 2$ ml. 307 yd.	11,480	4·0601

Example.—A man at the top of a mountain observes, by using a theodolite, that the angles of depression of two landmarks, one due South and the other due East, are 5° 18′ and 7° 33′ respectively. He finds from a map that the landmarks are both at 100 ft. above sea level and 10·7 miles apart. What is the height of the mountain ?

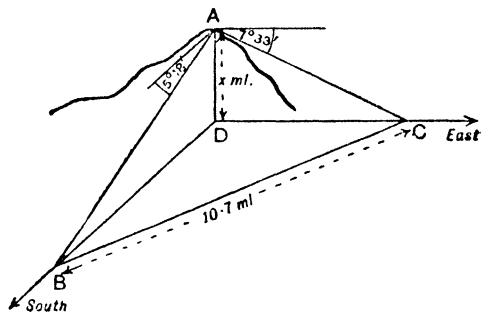


Fig. 165.

In Fig. 165, A is the man ; B, C are the two landmarks ; BDC is a horizontal plane and AD is vertical.

$\widehat{BAD} = 90° - 5° 18′ = 84° 42′$; $\widehat{CAD} = 90° - 7° 33′ = 82° 27′$.

Let $AD = x$ miles.

Then $\dfrac{BD}{x} = \tan 84° 42′$. $\quad \therefore BD = x \tan 84° 42′$ miles,

$\dfrac{CD}{x} = \tan 82° 27′$ $\quad \therefore CD = x \tan 82° 27′$ miles.

Since $\widehat{BDC}$ is a right angle, $BD^2 + CD^2 = BC^2$

$\therefore x^2(\tan^2 84° 42' + \tan^2 82° 27') = 10·7^2$

$x^2(116·2 + 56·94) = 10·7^2$

$173·14\, x^2 = 10·7^2$

$$x = \frac{10·7}{\sqrt{173·14}} = 0·8132$$

$\therefore AD = 0·8132$ ml. $= 4293$ ft.

Hence the height of the mountain $= 4293 + 100$

$= 4393$ ft.

	No.	*Log*
$\tan^2 84° 42'$	116·2	$2 \times 1·0326 = 2·0652$
		2·0652
$\tan^2 82° 27'$	56·94	$2 \times 0·8777 = 1·7554$
		1·7554
	$\frac{10·7}{\sqrt{173·14}}$	1·0294
		$\frac{1}{2}(2·2384) = 1·1192$
	x	$\bar{1}$ 9102

Exercise XXVII

1. A straight tunnel AB is bored horizontally through a mountain. The distance over the mountain is $6\frac{1}{4}$ miles, and the sides of the mountain slope at angles of 12° and 21°. Find the length of the tunnel. [Hint.—First find the height of C above AB.]

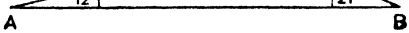


Fig. 166.

2. The angles of elevation of the top of a building from two windows of a house opposite are 44° 30′ and 37° 8′. If the windows are 15 ft. and 40 ft. above street level, what is the height of the building ?

3. A man in a boat observes that two lighthouses P and Q bear N. 20° W. and N. 32° E., and from his map he finds that Q is 10 miles due east of P. Find his distance from P.

4. An aeroplane is observed at the same instant from two places A and B, five miles apart, at elevations of 14° and 10°, being then vertically above some point between A and B. One minute later it is vertically above B. Find its height (in feet) and its speed (in ml. per hr.).

5. A surveyor, who wants to find the width of a river, observes the angle of elevation of the top of a tree on the bank directly opposite to be 8° 18′. He then walks downstream a distance of 50 yd. and finds the elevation to be 5° 30′. What is the width of the river ?

6. A wall 5 ft. high runs parallel with the side of a house, and a scaffold pole 17 ft. 6 in. long resting over the wall, with its foot on the ground and the top against the house, is inclined at 34° to the horizontal (the vertical plane of the pole being perpendicular to the wall). Find the distance between the wall and the house.

Miscellaneous Exercise XXVIII

1. A cotter 5 in. long is 1 in. wide at one end and $1\frac{1}{4}$ in. wide at the other end (Fig. 167). Find the angle of taper.

2. The lid of a desk, hinged along the top edge, is 18 in. from front edge to hinge, and slopes at 6° to the horizontal when closed. Through what vertical distance does the front edge rise when the lid is opened through 40° ?

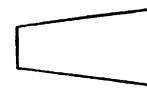


FIG. 167.

3. In a reciprocating engine (Fig. 168) the crank OA is 2 ft. long and the connecting rod AB is 10 ft. long. If $\widehat{AOB} = 72°$, find (i) the angle ϕ, (ii) the length OB, (iii) the distance of the crosshead B from top-dead-centre.

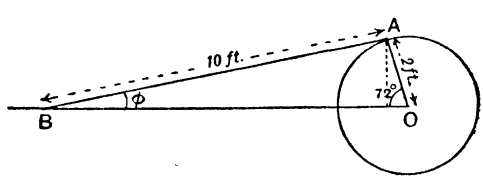


FIG. 168.

4. The pressure, p lb. per sq. in. between the faces of a cone clutch is given by the formula $p = \frac{6F}{\pi b D}$ cosec α, where b in. is the width of the face, α the cone angle, D in. the mean diameter and F lb. the force exerted by the lever. Find, by logarithms, the value of b required when $\alpha = 8° 15'$, $D = 14\frac{5}{8}$, $F = 600$, $p = 250$.

5. Fig. 169 shows the cross-section of a railway cutting which is 60 yd. long. Find the volume of earth removed.

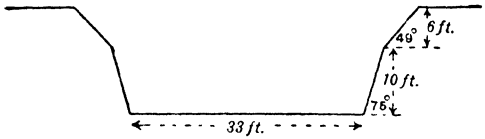


Fig. 169.

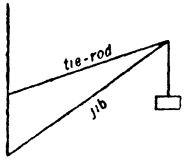


Fig. 170.

6. The jib of a crane (Fig. 170) is 14 ft. long and inclined at 54° to the vertical. The tie-rod is inclined at 72° to the vertical; find its length.

7. Vertical borings are made at points 300 yd. apart and coal is found at depths of 640 ft. and 725 ft. Find the dip of the coal seam to the nearest degree.

8. In the framework shown in Fig. 171, $AB = 16$ ft. Find the lengths of the sloping members and of CD.

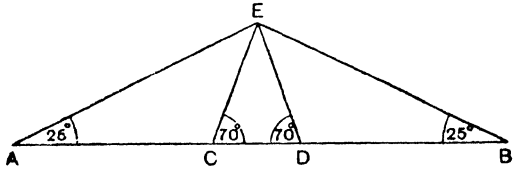


Fig. 171.

9. In the roof truss shown in Fig. 172, $AC = BC = 15$ ft., $AD = BE = 8$ ft. 6 in., $DE = 8$ ft., and the distance between the supports A and B is 24 ft. Find all the angles.

10. The jib of a navvy-crane (see Fig. 170) is 21 ft. long and its inclination to the horizontal can be varied between 30° and

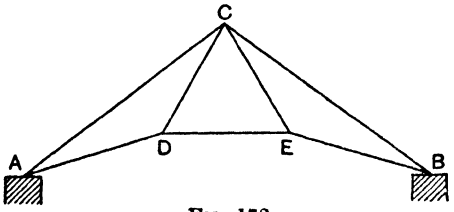


Fig. 172.

60°. The crane can turn completely round its vertical axis. Find the surface area of the ground which can be excavated. (Neglect the dimensions of the bucket.)

11. A surveyor's field-book reads as follows:

	Yards.	
	To *B*	
	435	
To *C* 85	385	
	185	150 to *E*.
To *D* 140	120	
	From *A*	

B is due north of *A*. Find the bearings of *D* and *E* from *C*. [Note.—If a line *PQ* points in the direction N. 60° E., *Q* is said to bear N. 60° E. from *P*.]

12. A tunnel *AB* is constructed through a mountain ridge (Fig. 173). *A*, *B*, *C* are 900, 1200, 1700 ft., respectively, above sea-level, and the sides *AC*, *BC* of the ridge slope at 27° and 34° to the horizontal. Find the gradient and the length of the tunnel.

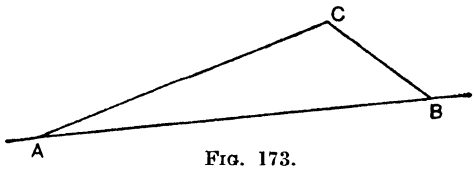


FIG. 173.

13. Is $\sin^2\theta = 1 - \cos\theta$ an identity or an equation? If it is an equation, solve it for θ.

14. The E.M.F. (e) generated in a coil of cross-sectional area A rotating with angular velocity ω about an axis perpendicular to a magnetic field of strength H is given by $e = 10^{-8}\omega HA \sin \omega t$. Calculate, by logarithms, the value of e when $t = 0{\cdot}006$, if $\omega = 50\pi$, $H = 30$ and $A = 140$.

15. A toggle joint (Fig. 174) consists of two rods *OA*, *AB*, each 15 in. long, hinged at *A*. The end *O* is fixed and *B* can

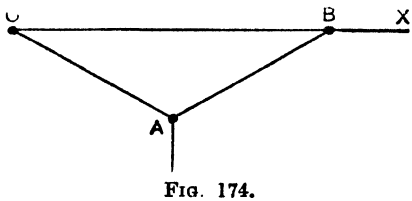


FIG. 174.

slide along the line OX. Initially $OB = 26$ in. Find the decrease in the angle OAB when B is pushed 6 in. towards O.

16. Two elevations of a hopper are shown in Fig. 175. Find the slope of the edge AB.

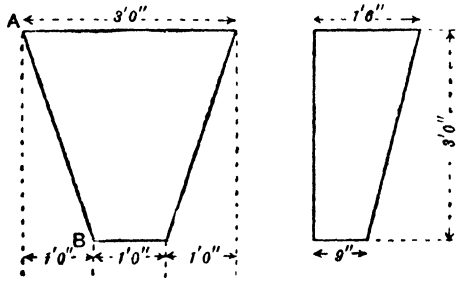


FIG. 175.

CHAPTER X

ANGLES OF ANY MAGNITUDE. PERIODIC FUNCTIONS

Angles of any magnitude

If a rod turns about one end, in one revolution it turns through 360°, in two revolutions through 720°, in one and a half revolutions through 540°, and so on. We have to distinguish between the two directions of rotation, since if it rotates through 60° in one direction its final position will be different from that which it would occupy if it rotated through 60° in the opposite direction. We therefore regard rotations in the *anti-clockwise* direction as positive, and rotations in the clockwise direction as negative. Thus a rotation of 220° means turning through 220° in the anti-clockwise direction (Fig. 176); a rotation of $-60°$ means turning through 60° in the clockwise direction (Fig. 177).

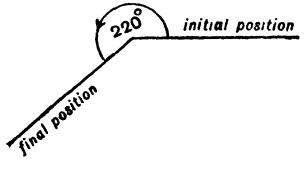


FIG. 176.

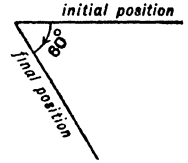


FIG. 177.

Definitions of the trigonometric ratios of angles of any magnitude

Let $X'OX$, $Y'OY$ be two lines at right angles, and suppose a line OP rotates about O from the direction OX through an angle θ into the position OP.

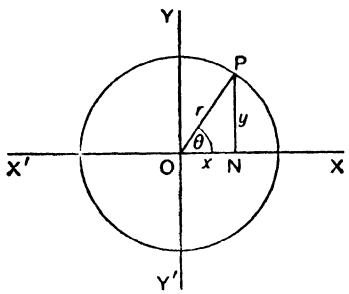


FIG. 178.

Let x and y be the co-ordinates of P referred to the axes OX and OY; i.e. x, y are the algebraic values of ON, NP, respectively, x being positive if P lies to the right of $Y'OY$ and y being positive if P lies above $X'OX$.

If $OP = r$ (always regarded as positive), we define $\sin\theta$, $\cos\theta$, etc., as follows:

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x},$$

and their reciprocals,

$$\text{cosec } \theta = \frac{r}{y}, \qquad \sec \theta = \frac{r}{x}, \qquad \cot \theta = \frac{x}{y}.$$

We take these as our definitions of the sine, cosine, etc., of any angle whatever its magnitude, whether positive or negative. When the angle lies between 0° and 90°, they agree with the definitions previously used (in Chap. IX).

The lines $X'OX$, $Y'OY$ divide the plane of the paper into four quadrants, which we number in the order in which they would be described in a positive rotation (Fig. 179).

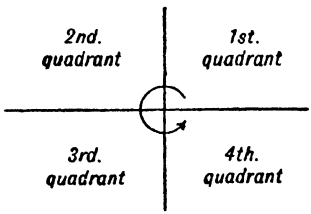


FIG. 179.

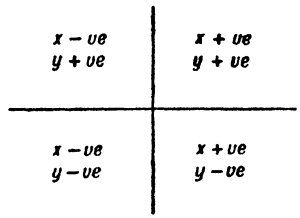


FIG. 180.

x is positive in the first and fourth quadrants, negative in the second and third; y is positive in the first and second quadrants, negative in the third and fourth.

Since r is always counted as positive, sin θ has the same sign as y, and cos θ has the same sign as x. The signs of sin θ and cos θ in the four quadrants are therefore as shown in Fig. 181.

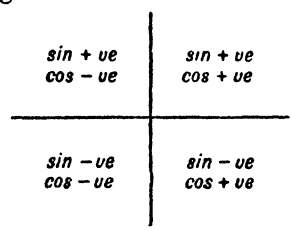


FIG. 181.

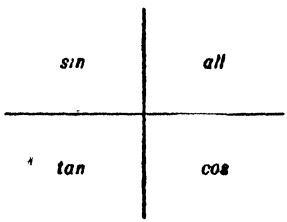


FIG. 182.

Sin θ is positive when OP lies in the first or second quadrant, negative when it lies in either of the other two quadrants. Cos θ is positive in the first and fourth quadrants and negative in the other two.

Since $\tan\theta = y/x$, $\tan\theta$ is positive when x and y have the same signs (i.e. both positive or both negative) which occurs when OP lies in the first or third quadrant. In the second and fourth quadrants x and y have opposite signs (one positive, the other negative) and therefore $\tan\theta$ is negative in those two quadrants.

Since the reciprocal of a positive number is positive, and the reciprocal of a negative number is negative, the ratios $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$ have the same signs as their reciprocals, viz. $\sin\theta$, $\cos\theta$ and $\tan\theta$.

The signs of the six ratios are most easily remembered by Fig. 182, where the word in any particular quadrant signifies that, in that quadrant, that ratio and its reciprocal are positive, and that all the other ratios are negative. For example, in the first quadrant all the ratios are positive; in the fourth quadrant $\cos\theta$ and its reciprocal $\sec\theta$ are positive, all the others are negative.

Fig. 182 may be "read" in an anti-clockwise (i.e. positive) direction, beginning with the first quadrant, as "*all, sin, tan, cos.*"

It will be noticed that each of the ratios is positive in two quadrants, and negative in two quadrants.

Example.—Find the value of $\cos 153°$.

Here OP lies in the second quadrant (Fig. 183), and x is negative,

i.e. $x = -ON$,

$$\therefore \cos 153° = \frac{x}{r} = -\frac{ON}{OP} = -\cos \widehat{NOP}$$

$$= -\cos(180° - 153°) = -\cos 27°$$

$$= -0{\cdot}8910.$$

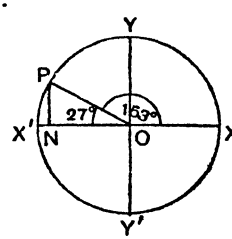


FIG. 183.

Example.—Find the value of tan 240°.

OP lies in the third quadrant (Fig. 184); x and y are both negative,

i.e. $x = -ON$, $y = -NP$

$$\therefore \tan 240° = \frac{y}{x} = \frac{-NP}{-ON} = +\frac{NP}{ON}$$

$$= \tan \widehat{NOP} = \tan 60° = \sqrt{3} = 1{\cdot}7321.$$

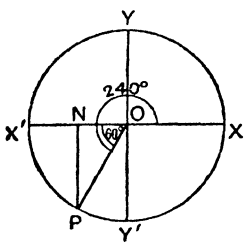


FIG. 184.

If the arm OP lies in the first quadrant we usually speak of the *angle* XOP as lying in the first quadrant, and so for the other quadrants.

Thus, we should say that the angle 153° lies in the second quadrant and that the angle 240° lies in the third quadrant.

Method for finding any trigonometric ratio of any angle

We notice from the two examples above that cos 153° is *numerically* equal to cos 27°, and tan 240° is numerically equal to tan 60°, and in the same way it will be seen that the sine, cosine, etc., of *any* angle $\widehat{XOP}$ are *numerically* equal to the sine, cosine, etc., of the acute angle that OP makes with the line $X'OX$. The sign to be attached to the ratio is determined by the signs of x and y, and may be obtained from the "*all, sin, tan, cos*" rule. Thus we have the following rule for finding the sine, cosine, etc., of any angle.

Rule.—

(1) Estimate, by a rough sketch, in which quadrant the angle lies, and determine the sign of the ratio from the " all, sin, tan, cos " rule.

(2) Find the acute angle between OP and $X'OX$ and write down the value of the corresponding ratio of that angle from tables.

After some practice the student will find that he need not draw even a rough sketch, but will be able to estimate the quadrant and the angle mentally.

Example.—Find sin 460°.

460° = 360° + 100° = one revolution + 100°.

The angle therefore lies in the second quadrant, and its sine is positive.

The acute angle required is 80° (Fig. 185).

∴ sin 460° = + sin 80° = 0·9848.

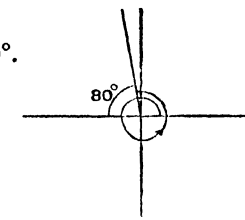


FIG. 185.

Example.—Find cot (− 54°).

− 54° lies in the fourth quadrant.

Cot θ is the reciprocal of tan θ and is therefore negative in the fourth quadrant.

∴ cot (− 54°) = − cot 54° = − 0·7265.

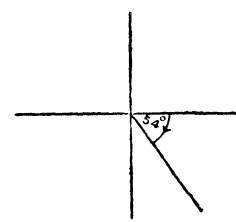


FIG. 186.

NOTE.—It will be observed that we can add, or subtract, any multiple of 360° (or 2π radians) without affecting the trigonometric ratios of the angle.

Thus in the example above, sin 460° = sin (460° − 360°) = sin 100°.

Also, addition or subtraction of any multiple of 180°

(or π radians) does not alter the *numerical values* of the trigonometric ratios, though it may change their sign. The sign can readily be found from the "*all, sin, tan, cos*" rule.

For example, $\cos 230° = -\cos 50°$; $\tan 230° = \tan 50°$.

Example.—The current, i ampères, in a circuit after t secs. is given by the formula $i = 5 \sin 100\pi t$, the angle being in radians. Find the current after 0·231 sec.

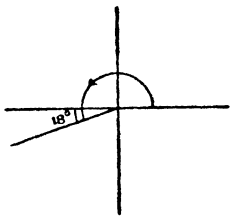


FIG. 187.

When $t = 0{\cdot}231$,

$i = 5 \sin 23{\cdot}1\pi = 5 \sin (22\pi + 1{\cdot}1\pi)$

$= 5 \sin 1{\cdot}1\pi$ (see note above)

$= 5 \sin (1{\cdot}1 \times 180)°$, since π radians $= 180°$,

$= 5 \sin 198°$.

198° is in the third quadrant, and its sine is negative.

$\therefore i = 5 \times (-\sin 18°) = -5 \sin 18°$

$= -5 \times 0{\cdot}3090 = -1{\cdot}545$ ampères.

NOTE.—When angles are expressed without units, the units are understood to be radians.

Exercise XXIX

State whether the following ratios are positive or negative:

1. sin 172°, sin 315°, $\cos \frac{5\pi}{4}$, tan 220°, cos (−31° 8′).

2. sin 600°, cot 195°, $\sin\left(-\frac{3\pi}{4}\right)$, sec 140°, $\tan \frac{7\pi}{8}$.

Express each of the following as the ratio of an acute angle, with the proper sign (+ or −):

3. cos 252°, sin 116°, sin (−10°), tan 187° 30′.

4. $\sin \frac{5\pi}{2}$, $\cos(-248°)$, $\tan 94°\ 15'$, $\cos \frac{13\pi}{4}$.

5. $\tan(-460°)$, $\sec 293°\ 42'$, $\sin 1{\cdot}82$, $\operatorname{cosec} 214°$.

Find the values of:

6. Sin 140°. 7. Sin 310° 36′. 8. Cos 212°.

9. $\cos \frac{7\pi}{4}$. 10. Tan 136° 15′. 11. Tan 195° 54′.

12. Sin 6. 13. $\cos \frac{5}{6}\pi$. 14. Sec 225°.

15. Cot 98° 30′. 16. Cosec 152°. 17. $\sin(-570°)$.

Verify the following relations for any acute angle θ (by drawing a rough sketch):

18. $\sin(180° - \theta) = \sin\theta$, $\cos(180° - \theta) = -\cos\theta$, $\tan(180° - \theta) = -\tan\theta$.

19. $\sin(180° + \theta) = -\sin\theta$, $\cos(180° + \theta) = -\cos\theta$, $\tan(180° + \theta) = \tan\theta$.

20. $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$.

21. By substituting $(-\theta)$ for θ in the relations $\sin(90° - \theta) = \cos\theta$, etc., on p. 209, and using the results of the previous question, prove that $\sin(90° + \theta) = \cos\theta$, $\cos(90° + \theta) = -\sin\theta$, $\tan(90° + \theta) = -\cot\theta$.

22. The voltage in a certain circuit at time t is $240 \sin 400t$. Find the voltage when $t = 1/80$.

23. The valve displacement, d ft., in a certain machine is given by the formula

$$d = 3{\cdot}5 + 4{\cdot}2 \sin\theta + 1{\cdot}75 \cos\theta,$$

where θ is the angle through which the crank has turned. Find the displacement when the crank has turned through 130°.

24. In which quadrant does the angle θ lie if:

(i) $\sin\theta$ is positive and $\cos\theta$ is negative?
(ii) $\sin\theta$ is negative and $\cos\theta$ is positive?
(iii) $\tan\theta$ is positive and $\cos\theta$ is negative?

25. Evaluate $\cos(3t + 0{\cdot}785)$ when $t = 0{\cdot}512$.

Graph of $\sin\theta$

By constructing a table of values of $\sin\theta$ for values of θ at intervals of, say, 15°, we can draw the graph of $\sin\theta$ for any range of values of θ that we please.

There is, however, a very simple and useful method of obtaining the graph.

If in Fig. 188 the rotating line OP is of length 1 unit, then $\sin\theta = \frac{NP}{OP} = NP$. Take a point O' on OA produced and let $O'X$ be taken as our axis for θ; mark off on $O'X$ a scale of

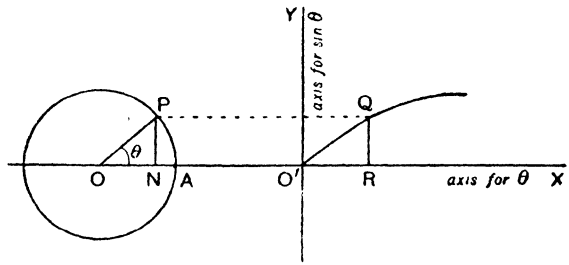


Fig. 188.

values for θ. If R is the point corresponding to the value $\theta = \widehat{AOP}$, the ordinate RQ on our graph has to be equal to $\sin\theta$, i.e. equal to NP. If, therefore, we draw PQ parallel to OA, and if this line cuts the ordinate through R in Q, then Q will be a point on the graph of $\sin\theta$.

By taking a number of different positions for OP and corresponding points R we get a number of points on the graph of $\sin\theta$. We then join these points by a smooth curve.

In Fig. 189 values of θ have been taken at intervals of 30° from $-30°$ to 480°. The lines OP_0, OP_1, OP_2, OP_3, . . .

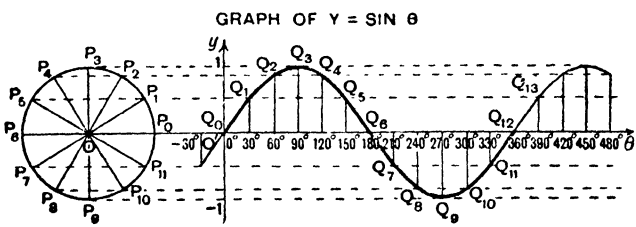


Fig. 189.

are the positions of the radius when $\theta = 0°, 30°, 60°, 90°, \ldots$, and $Q_0, Q_1, Q_2, Q_3, \ldots$ are the tops of the ordinates corresponding to those values.

By allowing the radius to rotate through an angle greater than 480° we could draw the graph for values of θ as large as we please, and by considering the radius rotating in the opposite direction we could obtain the graph for negative values of θ.

It is clear from the way in which the graph is drawn that the values of y are repeated during each successive revolution of the radius. For example, the value of y when $\theta = 420°$ is the same as when $\theta = 60°$, and the portion of the graph between $\theta = 330°$ and $\theta = 480°$ is identical with that between $\theta = -30°$ and $\theta = 120°$. The complete graph therefore consists of a succession of waves, repeated indefinitely in both directions. The thick line in the graph indicates one complete wave.

Graph of cos θ

If θ is any acute angle, $90° + \theta$ lies in the second quadrant.

Let $\widehat{AOP} = 90° + \theta$ in Fig. 190. Then $\widehat{BOP} = \theta$ and, since NP is parallel to OB, $\widehat{NPO} = \theta$.

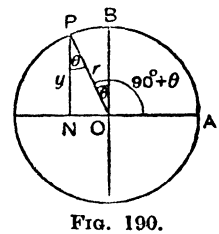


FIG. 190.

$$\sin(90° + \theta) = \frac{y}{r} = \frac{NP}{OP}, \text{ since } y \text{ is positive,}$$

$$= \cos \widehat{NPO} = \cos \theta.$$

(See also Exercise XXIX, Question 21.)

We have proved this relation only when θ is an acute angle, but it can be shown to be true for all values of the angle θ. (The student could verify this for himself by considering θ to lie in each of the four quadrants in turn.)

The ordinate of the cosine graph for any value of θ is therefor equal to the ordinate of the sine graph for the value $90° + \theta$. This means that the graph of $y = \cos \theta$ can be obtained from

the graph of $y = \sin\theta$ by merely moving the origin and the y-axis through 90° to the right; or, what is equivalent, by keeping the origin and y-axis fixed and moving the sine curve bodily through 90° to the left. We thus obtain the graph in Fig. 191.

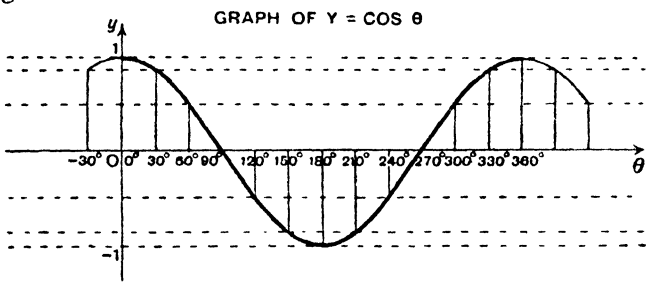


FIG. 191.

The cosine curve, being merely a displaced sine curve, also consists of a succession of waves, each of width 360° (or 2π radians). One complete wave, from $\theta = 0°$ to $\theta = 360°$, is indicated by the thick line in the graph.

Graphs of tan θ and cot θ

To draw the graph of tan θ we construct a table of values from a book of trigonometric tables and plot the points. So also for the graph of cot θ. The graphs are shown in Figs. 192 and 193.

It will be noticed that there are breaks in the curve $y = \tan\theta$ at $\theta = 90°$, 270°, etc., and at $\theta = -90°$, $-270°$, etc. We say that the curve is "discontinuous" at those points, in contrast to the curve of sin θ which is a continuous curve. The curve of cos θ is also continuous, but that of cot θ is discontinuous at $\theta = 0°$, 180°, 360°, etc., and at $\theta = -180°$, $-360°$, etc.

The curves $y = \sin\theta$ and $y = \cos\theta$ lie entirely between the two lines $y = -1$ and $y = 1$, but the curves $y = \tan\theta$ and $y = \cot\theta$ stretch indefinitely upwards and downwards.

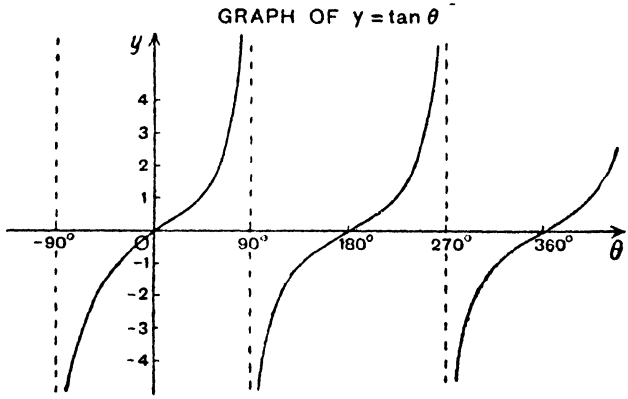


FIG. 192.

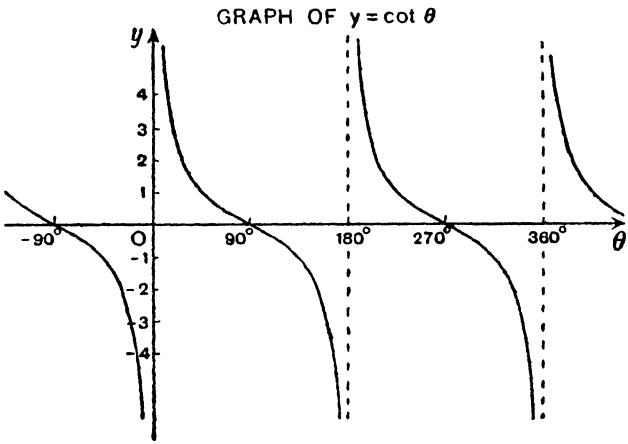


FIG. 193.

As θ approaches 90° from the left, i.e. by gradually increasing values, tan θ increases indefinitely; for tan 88° = 28·64, tan 89° = 57·29, tan 89° 30′ = 114·6, tan 89° 48′ = 286·5,

tan 89° 54′ = 573·0, and between 89° 54′ and 90° the value of tan θ increases without limit; we say briefly that tan θ approaches ∞. On the other hand, as θ approaches 90° from the right, i.e. by decreasing values, we find that tan 92° = −28·64 (being negative because 92° is in the second quadrant), tan 91° = −57·29, tan 90° 30′ = −114·6, tan 90° 12′ = −286·5, tan 90° 6′ = −573·0, and between 90° 6′ and 90° the value of tan θ becomes larger and larger numerically, but is still negative. We say briefly that tan θ approaches $-\infty$. Thus when θ is just less than 90°, tan θ is a very large positive number; when θ is just greater than 90°, tan θ is a very large negative number. We cannot give any definite value to tan 90°, since we might regard it equally well as being $+\infty$ or $-\infty$.

This fact need not worry the student, and it will cause no difficulty in practice, so long as he realizes how the graph behaves near the points where the breaks occur.

Graphs of sec θ and cosec θ

These graphs do not occur so frequently in practice as the graphs of the other ratios. The graph of cosec θ is shown in Fig. 194.

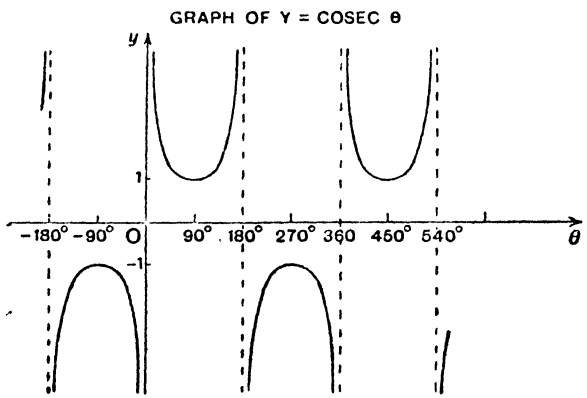


FIG. 194.

Since $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sin(90°+\theta)} = \operatorname{cosec}(90°+\theta)$, the graph of $\sec\theta$ is obtained by moving the curve $y = \operatorname{cosec}\theta$ bodily through 90° to the left.

Sec θ and cosec θ are both numerically greater than 1 (i.e. greater than $+1$ or less than -1) for all values of θ.

Periodic functions

We have seen that if we add 360°, or 720°, or any exact multiple of 360°, to an angle θ, the value of $\sin\theta$ is unaltered. The values of $\sin\theta$ repeat themselves at intervals of 360°.

A function which repeats itself at regular intervals is called a *periodic function*, and the interval between two successive repetitions is called the *period* of the function. Thus $\sin\theta$ is a periodic function of θ, its period being 360°, or 2π radians.

Cos θ also is a periodic function of θ with period 2π radians. Tan θ also repeats itself every 360°, but it repeats itself twice in an interval of 360°, as we can see from the graph between, say, $-90°$ and $+270°$. The *smallest* interval in which $\tan\theta$ repeats itself is 180°, and that is what is meant by the period. Thus $\tan\theta$ has a period of 180°, or π radians. So also has $\cot\theta$.

Sec θ and cosec θ are both periodic with period 2π radians.

Functions of the type $a \sin p\theta$ or $a \sin \omega t$ or $a \sin 2\pi ft$. Oscillations

The graph of $y = a \sin\theta$, where a is any fixed number, is obtained by drawing the graph of $y = \sin\theta$ and then multiplying each ordinate by the same number a. This is equivalent merely to altering the scale for y.

We could also obtain the graph of $y = a \sin\theta$ by the same construction as in Fig. 188, but using a crank OP of length a units instead of 1 unit.

Now let us see what the graph of $y = \sin 2\theta$ is like. This is the same as $y = \sin\theta$ except that 2θ takes the place of θ previously. Since the graph of $y = \sin\theta$ crosses the axis when $\theta = \ldots, -180°, 0°, 180°, 360°, \ldots$, the graph of $y = \sin 2\theta$

will cross the axis when 2θ has those values, i.e. when $\theta = \ldots, -90°, 0°, 90°, 180°, \ldots$.

The curve of sin θ reaches its maximum and minimum values when $\theta = \ldots, -90°, 90°, 270°, \ldots$, and therefore the curve of sin 2θ has its maximum and minimum values when $2\theta = \ldots, -90°, 90°, 270°, \ldots$, i.e. when $\theta = \ldots, -45°, 45°, 135°, \ldots$. Thus the graph of $y = \sin 2\theta$ is the same as that in Fig. 189, except that all the numbers on the θ-axis have now to be halved. This is equivalent to making the scale for θ twice as large.

The function sin θ repeats itself when θ increases by 360°, and so the function sin 2θ repeats itself when 2θ increases by 360°, i.e. when θ increases by 180°. Thus sin 2θ has a period of 180°, or π radians. There are *two* complete waves between $\theta = 0$ and $\theta = 360°$.

In the same way sin 3θ is periodic with a period for θ of $\frac{360°}{3}$, i.e. 120°, or $\frac{2\pi}{3}$ radians. There are three complete waves in the graph between $\theta = 0$ and $\theta = 360.°$

Generally, if p is any fixed number, the graph of $y = \sin p\theta$ is the same as the graph in Fig. 189, except that the numbers marked there on the θ-axis are now to be read as being the values of $p\theta$ instead of θ, i.e. the graph of $y = \sin p\theta$ is the same as the graph of $y = \sin \theta$, but with a different scale for θ. The function sin $p\theta$ has a period for θ of $\frac{360°}{p}$, or $\frac{2\pi}{p}$ radians.

The graph of $y = a \sin p\theta$ is obtained from that of $y = \sin p\theta$ by merely multiplying each ordinate by a.

The graph is shown in Fig. 195, angles being marked in radians. If p is a whole number there are p complete waves between 0 and 2π.

If a crank OP, of length a, rotates with angular velocity ω radians per sec. starting from the position OA (Fig. 196), then in t secs. it turns through an angle ωt radians. If Q is the projection of P on the diameter perpendicular to OA, then, as P goes round the circle, Q moves along that diameter from O to B then down to B', back to B, and so on, i.e. Q oscillates between

B and B'. The distance (y) of Q above O is equal to NP, which is $a \sin \omega t$. When P is in the third or fourth quadrant $\sin \omega t$ is negative and y is negative, indicating that Q is *below* O.

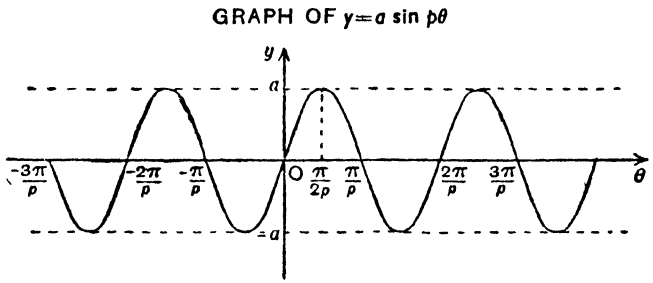


FIG. 195.

If we draw the graph of $y = a \sin \omega t$, marking values of t along the horizontal axis, the ordinate y will give the displacement of Q from O at the time given by the abscissa t.

The point Q executes one complete oscillation, i.e. one to-and-fro movement, as for example from B to B' and back to B, when the crank makes one complete revolution, i.e. when the angle ωt increases by 2π radians, i.e. when t increases by $\frac{2\pi}{\omega}$. The *period of the oscillation* is therefore $\frac{2\pi}{\omega}$ secs. The period is usually denoted by T. (Note that the period here is a time, not an angle.)

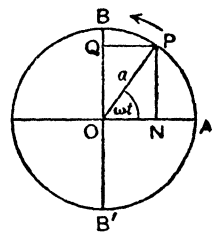


FIG. 196.

The number of oscillations per sec. is called the *frequency* of the oscillations and is usually denoted by f (or sometimes by n). It is clear that

$$\text{frequency} = \frac{1}{\text{period}},$$

i.e. $$f = \frac{1}{T} = \frac{\omega}{2\pi}, \text{ and hence } \omega = 2\pi f.$$

The maximum displacement of the moving point (Q) from its mean position (O) is called the *amplitude* of the oscillations. In Fig. 196 the amplitude is the length of OB, i.e. a.

Hence the mathematical equation representing oscillations of amplitude a and frequency f is $y = a \sin 2\pi ft$.

Functions of the type $a \sin (p\theta + \alpha)$ or $a \sin (\omega t + \alpha)$ or $a \sin (2\pi ft + \alpha)$

Suppose we want to draw the graph of $y = a \sin (\theta + \alpha)$, where α is any fixed angle. This is the same as $y = a \sin \theta$ except that $\theta + \alpha$ takes the place of θ. Now the curve $y = a \sin \theta$ crosses the axis when $\theta = \ldots, -\pi, 0, \pi, 2\pi, \ldots$, and hence the curve $y = a \sin (\theta + \alpha)$ crosses the axis when $\theta + \alpha$ has those values, i.e. when $\theta = \ldots, -\pi - \alpha, -\alpha, \pi - \alpha, 2\pi - \alpha, \ldots$ The points of crossing the axis are therefore all moved through an amount α to the left. The same is true also of any particular ordinate, so that the graph of $y = a \sin (\theta + \alpha)$ is obtained by moving the curve $y = a \sin \theta$ bodily to the left through an amount α.

If α is negative it means that the curve is moved to the right instead of to the left; for example, the curve $y = 2 \sin (\theta - 30°)$ is obtained by drawing the curve $y = 2 \sin \theta$ and moving it through 30° to the right.

Example.—Sketch the curve $y = 4 \sin (\theta + 60°)$.

We first draw the curve $y = 4 \sin \theta$ which is an ordinary sine curve of amplitude 4. The values of θ for this curve are marked above the axis. We now have to move the curve through 60° to the left, or, what is equivalent, we can move the y-axis through 60° to the right, so that all the numbers on the θ-axis read 60° less than originally. The new values of θ are marked below the axis. The curve is shown in Fig. 197.

To obtain the graph of $y = a \sin (p\theta + \alpha)$ we write this as $y = a \sin \left\{p\left(\theta + \frac{\alpha}{p}\right)\right\}$. This is the same as $y = a \sin p\theta$ except

that θ is replaced by $\theta + \frac{\alpha}{p}$. The effect of this is to move the curve bodily to the left through an amount $\frac{\alpha}{p}$; or, if we prefer, we can move the y-axis to the right through an amount α/p.

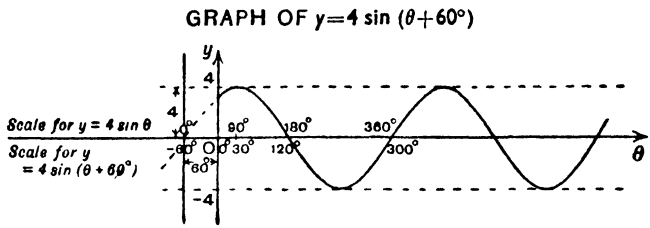


FIG. 197.

Example.—Sketch the curve $y = 2 \sin (3\theta - 150°)$, and find the smallest positive value of θ for which y is a maximum.

We first draw the curve $y = 2 \sin 3\theta$, which is a sine wave of amplitude 2 and period $\frac{360°}{3}$, i.e. 120°.

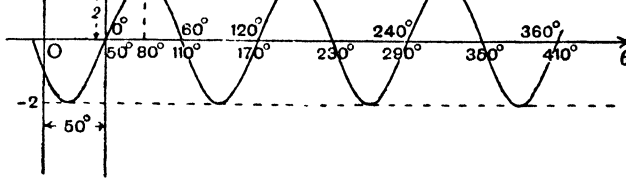


FIG. 198.

There are three waves between 0° and 360° and the curve crosses the axis at 0°, 120°, 240°, 360°, etc. The values of θ for this curve are marked above the axis.

We now write the required curve as $y = 2 \sin 3(\theta - 50°)$, so that we have to move our original curve through 50° to

the right, or, what is easier, we have to move our y-axis through 50° to the left. The values of θ for the new curve are marked below the axis. The smallest positive value of θ for which y is a maximum is half way between 50° and 110°, i.e. $\theta = 80°$.

The significance of the constant α can be illustrated by referring again to the rotating vector, or crank, on p. 238. If OP starts from a position OA' (Fig. 199) instead of from OA, and if $\widehat{AOA'} = \alpha$ (radians), then, after t secs., $\widehat{A'OP} = \omega t$ and hence $\widehat{AOP} = \omega t + \alpha$ (radians). The displacement of Q from O is therefore given by $y = a \sin(\omega t + \alpha)$.

The period of oscillation of Q, which is equal to the time of one revolution of the crank, is not affected by altering the starting position of the crank; the period is still $2\pi/\omega$ secs.

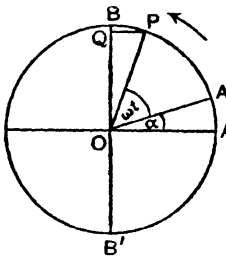


FIG. 199.

The angle AOP between OP and the line of reference OA, is called the *phase* at that instant, and α is called the *phase constant* or *phase displacement* (or sometimes the *epoch*). If we imagine two cranks rotating about O in Fig. 199 with the same angular velocity ω, but one starting from OA and the other from OA' at the same instant, the second crank will always be an angle α ahead of the first one, i.e. it "leads" the first one by an amount α. For this reason electrical engineers usually say that the oscillation $y = a \sin(\omega t + \alpha)$ has a *lead* of α. If α is negative it is called a *lag*.

The second crank is α/ω secs. ahead of the first in reaching any position; we might call α/ω the *time lead*.

If we have two oscillations of the same period but different amplitudes, such as

$$y = a \sin(\omega t + \alpha) \text{ and } y = b \sin(\omega t + \beta),$$

the phase of the first at time t is $\omega t + \alpha$, and the phase of the

second at that time is $\omega t + \beta$. The difference of these angles is $\alpha - \beta$; this is called the *phase difference* of the two oscillations. Electrical engineers would say that the first oscillation leads the second by an amount $\alpha - \beta$.

If $\alpha = \beta$ the two oscillations have the same phase at any time; they are said to be *in phase*; if α is not equal to β they are said to be out of phase.

It may be noted that, since $a \cos(p\theta + \beta) = a \sin(p\theta + \beta + \pi/2)$, the equation $y = a \cos(p\theta + \beta)$ represents an oscillation of the same amplitude (a) and period ($2\pi/p$) as $y = a \sin(p\theta + \beta)$ but leading it by $\pi/2$.

If we draw, with the same axes, the graphs of two oscillations which are in phase with each other, they will cross the time-axis at the same points.

Fig. 200 shows the graphs of two oscillations which are in phase, Fig. 201 the graphs of two oscillations which are out of phase with each other.

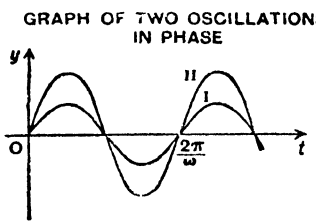


Fig. 200.

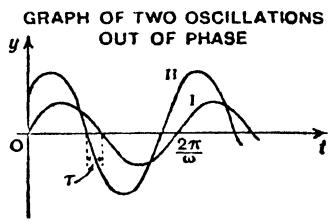


Fig. 201.

In each figure the amplitude of the curve II is twice that of I.

In Fig. 201, the oscillation II leads I by a time τ; for example, the maximum and minimum values of II occur a time τ before those of I. The angle of lead, or phase difference, is $\omega\tau$.

Summary.—The significance of the constants in the equation $y = a \sin(\omega t + \alpha)$ or $y = a \sin(2\pi f t + \alpha)$ is as follows:

a = amplitude; affects only the scale for y.

$$\left.\begin{aligned} \frac{2\pi}{\omega} &= \frac{1}{f} = \text{period} \\ f &= \frac{\omega}{2\pi} = \text{frequency} \end{aligned}\right\} \text{affects only the scale for } t.$$

α = phase displacement; affects only the position of the curve.

Mechanical and electrical oscillations

Oscillations of the type considered above are of frequent occurrence in mechanics and engineering. They are referred to as sine-wave oscillations, or *sinusoidal* oscillations or *simple harmonic* oscillations. The oscillation may be an actual vibration of a moving point or part of a machine, or it may be merely a convenient way of representing in a formula the value at any time of a quantity which varies between two extreme values $\pm a$ according to a sine law.

Thus in the case of a weight at the end of a spring, and in the motion of the bob of a pendulum (if the amplitude of the oscillations is small), the displacement is represented by an expression of the type above, and in alternating-current theory we meet with voltages given by expressions like $v = V \sin(2\pi ft - \phi)$. In the two former cases there is an actual oscillation—of the weight up and down in a vertical line or of the pendulum-bob in an arc of a circle—but in the third case there is no such motion, though the value of v does oscillate between $\pm V$.

In much the same way every sine-wave (or simple harmonic) oscillation can be pictured as produced by a rotating vector as on pp. 238, 242, though the crank may not be an actuality, but only a geometrical fiction which we use in order to give us a more concrete picture of the behaviour of the quantities with which we are dealing. The electrical engineer uses rotating vectors very frequently in this way.

Exercise XXX

1. Draw an accurate graph of $\sin\theta$ between $\theta = -180°$ and $\theta = +180°$, using a rotating vector as on p. 232. Find, from your graph, the value of $\sin 132°$. What angles are there between $-180°$ and $180°$ having their sine equal to 0·71 ?

State the periods of the following functions of θ (giving the answers in degrees and in radians).

2. (i) $\sin 4\theta$, (ii) $\cos 3\theta$, (iii) $\tan 5\theta$.

3. (i) $\sin\frac{\theta}{2}$, (ii) $\sin\left(\frac{3\theta}{2}+\frac{\pi}{6}\right)$, (iii) $\cot p\theta$.

State the frequencies of the following oscillations :

4. (i) $y = \sin 100\pi t$, (ii) $y = \sin 40t$, (iii) $y = \cos 2\pi t$.

Write down the amplitude, period and frequency of each of the following oscillations :

5. (i) $y = 5\sin(4t+1)$, (ii) $v = 230\sin(50\pi t - 0{\cdot}8)$.

6. (i) $s = 3{\cdot}6\sin\left(\frac{x}{2}+1{\cdot}4\right)$, (ii) $i = 10\sin(100\pi t + 6{\cdot}4)$,
(iii) $y = 12\cos(4t - 2{\cdot}8)$.

7. (i) $y = A\sin\frac{px}{q}$, (ii) $y = B\sin\left(\frac{px}{q}+r\right)$.

8. Draw the graph of $y = 0{\cdot}5\sin 2x$ between $x = 0$ and $x = 360°$. From your graph find the value of $\sin 250°$, and compare this with the value obtained by the use of tables.

9. Sketch on the same diagram (i.e. with the same axes and the same scales) the graphs of the functions $\sin x$, $\sin 2x$, $\sin\frac{x}{2}$ between $x = 0$ and $x = 2\pi$.

10. Draw a rough sketch of the curve $y = 2\tan 3x$ over two periods commencing at $x = 0$.

11. Sketch roughly the curves represented by the expressions in Questions 2–7, showing at least two complete waves in each case, and mark in your figure the amplitude, period and phase displacement, and the points where the curve crosses the axis.

12. If a simple pendulum of length l is pulled away from the vertical through a small angle θ_0 and then allowed to fall, the angle θ which it makes with the vertical at time t later is given

by the formula $\theta = \theta_0 \cos \sqrt{\frac{g}{l}}t$, where g is the "acceleration due to gravity" (i.e. 32·19 ft./sec.^{2}, or 981·2 cm./sec.^{2}). Find (i) the period of oscillation of a pendulum of length 1 ft., (ii) the length of a pendulum which beats seconds (i.e. which swings from left to right, or from right to left, in one sec.).

13. A crank OP, 3 ft. long, rotates at a uniform speed of 30 revs. per min., starting from a position OA. If Q is the projection of P on the line through O perpendicular to OA, find the equation for the distance of Q from O at a time t secs. later.

14. If the crank in Question 13 starts from some other position and passes through the position OA $\frac{1}{4}$ sec. after the start, find the equation for the distance of Q from O at a time t secs. after the start.

15. The voltage in a circuit t secs. after the current is switched on is $200 \sin (314t - 50)$. When is the voltage first zero, and when is it first a maximum?

16. What is the phase difference between the oscillations $y = 3 \sin \left(4t + \frac{\pi}{8}\right)$ and $y = 5 \sin \left(4t - \frac{\pi}{4}\right)$? Express it (i) as an angle, (ii) as a fraction of the period, (iii) as a time-lead or time-lag.

17. Draw an accurate graph of $y = 2 \sin (5x - 3)$ over one period starting from $x = 0$.

18. Write down the equation of the sine curve in Fig. 202.

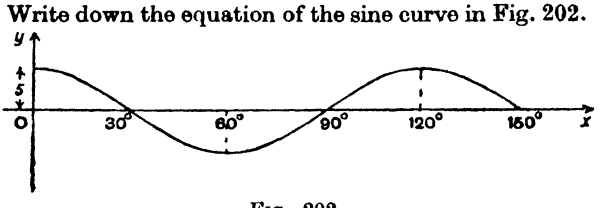


FIG. 202.

19. Write down the equation of the sine curve in Fig. 203.

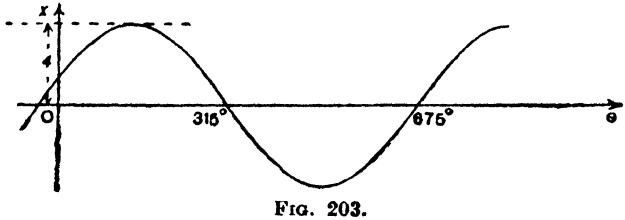


FIG. 203.

Some other periodic functions and their graphs.

The sine function is the simplest periodic function, but many other periodic functions occur in practice. For example, in electricity, although the sine wave is the ideal wave form for alternating currents, the voltage produced by a commercial generator is generally of a more complicated wave-type. We shall consider a few periodic functions which can be built up from pure sine functions.

1. Type $y = c + a \sin (p\theta + \alpha)$

This is a sine wave in which each ordinate is increased by the same amount c. Its graph is therefore obtained by drawing the curve $y = a \sin (p\theta + \alpha)$ and moving it a distance c up the y-axis, or, what is equivalent, lowering the y-axis through a distance c. The period is unaltered; it is still $\frac{2\pi}{p}$.

Example.—Fig. 204 shows the graph of $y = 1 + \sin 2\theta$. The value of y oscillates between 0 and 2 (instead of between -1 and $+1$).

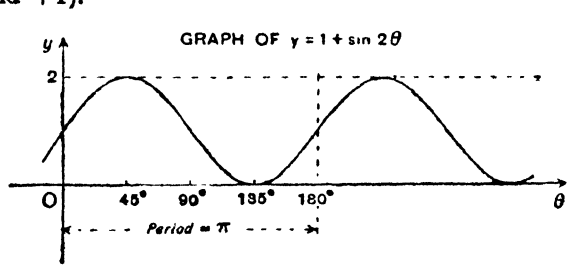


FIG. 204.

2. Type $y = a \sin \theta + b \cos \theta$

Example.—Draw the graph of $y = 3 \sin \theta + 2 \cos \theta$.

We draw the graph of $3 \sin \theta$ and in the same figure (i.e. using the same axes and the same scales) we draw the graph of $2 \cos \theta$. By means of a pair of dividers or compasses,

we now add the ordinates of the two curves, remembering that a negative ordinate has to be subtracted. The resulting curve, shown by a slightly thicker line, is the graph of $y = 3 \sin \theta + 2 \cos \theta$.

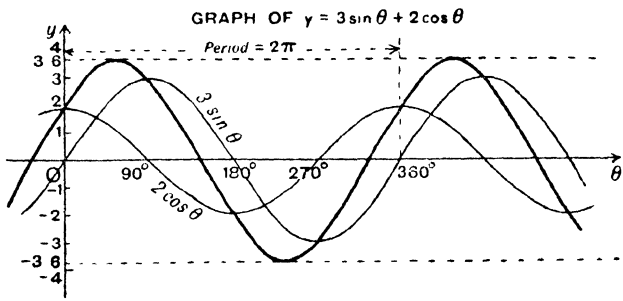


FIG. 205.

It will be noticed that the final curve is periodic, its period being 2π, which is the same as the period of each of the curves from which it is compounded. It can be shown that the final curve is itself a sine wave. In fact we shall show later (in Part III) that if any two sine waves of the same period are added together the resulting curve is itself a sine wave of the same period.

Note that $2 \cos \theta = 2 \sin (\theta + \pi/2)$, so that the component curves in this example are sine waves of the same period, but having a phase difference of $\pi/2$, i.e. a quarter of a period. The amplitude of the resultant curve is seen to be 3·6 (approx.). [Compare p. 292.]

3. Type $y = a \sin p\theta + b \sin q\theta$

Example.—Draw the graph of $y = 3 \sin \theta + 4 \sin 2\theta$.

Fig. 206 shows the result of adding the ordinates of the curves $y = 3 \sin \theta$ and $y = 4 \sin 2\theta$.

We are here adding two sine waves of different periods.

The resultant curve, shown by a slightly thicker line, is *not* a sine wave, though it *is* a periodic curve.

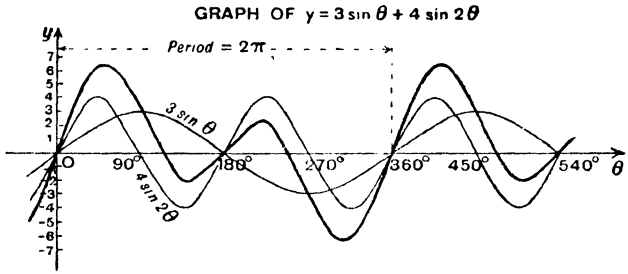


FIG. 206.

Example.—Draw the graph of $y = 4 \sin \theta + \sin 3\theta$.

Fig. 207 shows the result of adding the curves $y = 4 \sin \theta$ and $y = \sin 3\theta$.

Here again the resultant curve is *not* a sine wave, but it *is* periodic.

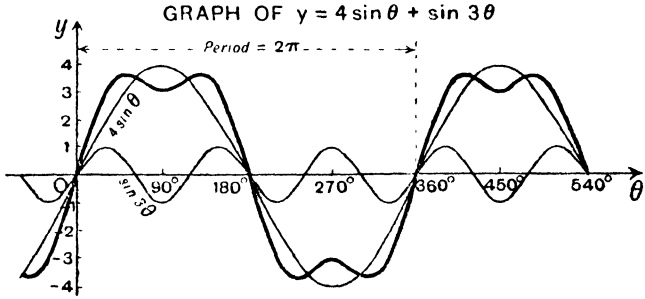


FIG. 207.

4. Type $y = a \sin (\theta + \alpha) + b \sin (2\theta + \beta) + c \sin (3\theta + \gamma) + \ldots$

This consists of adding a number of sine waves, the frequencies of the second, third, etc., terms on the right-hand side being twice, three times, etc., that of the first term. The component terms on the right-hand side are called, respectively, the first harmonic (or "fundamental"), second harmonic, third harmonic, etc. In the case of a musical note they

represent the fundamental tone and the harmonics or overtones. Compound waves of this type are of importance in many branches of physics and engineering.

The curve in Fig. 207 is more or less typical of waves of this type in which only odd harmonics occur (in the example shown only the first and third harmonics are present); such waves occur frequently in the study of alternating currents. In the curve in Fig. 206 there are only first and second harmonics; this type of wave occurs in considering valve motions.

Period and frequency of compound wave

When two sine waves of different periods are compounded (i.e. added together) we can find the period of the resultant wave (which is *not* a sine wave) without drawing a graph. The period of the resultant wave is in fact the least common multiple of the periods of the component waves, as the following example will show.

Example.—What is the period of the function $\sin 2x + \sin 4x$?

The period of $\sin 2x$ is $\frac{2\pi}{2}$, i.e. π radians, so that the function $\sin 2x$ returns to its original value when x increases by π or 2π or 3π, etc., radians.

The period of $\sin 4x$ is $\frac{2\pi}{4}$, i.e. $\frac{\pi}{2}$ radians, so that $\sin 4x$ returns to its original value when x increases by $\frac{\pi}{2}$ or π or $\frac{3\pi}{2}$ or 2π, etc., radians.

The smallest increase in x for which *both* functions, and therefore also their sum, return to their original values is π, which is the L.C.M. of π and $\frac{\pi}{2}$.

The period of the function $\sin 2x + \sin 4x$ is therefore π radians.

The same method applies if we compound more than two sine waves; we merely find the L.C.M. of the periods of all the components.

If we are given the frequencies instead of the periods we may first convert to periods, find their L.C.M. and then convert back to frequencies.

Example.—Two waves of frequencies 40 and 100 are compounded. What is the frequency of the compound wave?

The periods of the component waves are $\frac{1}{40}$ and $\frac{1}{100}$ sec. The L.C.M. of these is $\frac{1}{20}$ sec. This is the period of the compound wave, and its frequency is therefore 20.

In this example we notice that 20 is the H.C.F. of 40 and 100, and it is not difficult to show that the frequency of any compound wave is the H.C.F. of the component frequencies.

Exercise XXXI

1. A weight at the end of a spring is vibrating in a vertical line. The length, x in., of the spring at time t secs. is given by $x = 6 + 2 \cos 4\pi t$.

Draw a graph of x against t. What are the greatest and least lengths of the spring during the motion?

2. Draw the graph of $y = \sin x + \frac{1}{2} \sin 2x$ between $x = 0$ and $x = 4\pi$.

3. The current in an inductive circuit is given by:

$$i = 5 \sin 40t - 0{\cdot}2 \cos 40t.$$

Plot a graph of i against t over one period.

4. In a reciprocating engine the velocity (v) of the piston is given by $v = \omega r \left(\sin \theta + \frac{r}{2l} \sin 2\theta\right)$, where r is the length of the crank, l the length of the connecting-rod and ω the angular velocity of the crank.

If $r = 2$, $l = 10$ and $\omega = 6\pi$, draw a graph of v against θ, from $\theta = 0$ to $\theta = 4\pi$.

5. The voltage in a cable at time t secs. is equal to

$$200 \sin 100\pi t + 75 \sin 300\pi t.$$

Draw a graph to show how the voltage varies during the first $\frac{1}{15}$ sec.

6. Plot the curve $y = \sin(x - 40°) + 3 \sin(3x + 20°)$ from $x = 0°$ to $x = 360°$.

7. Plot the graph of $y = \sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta$ from $\theta = 0$ to $\theta = 2\pi$.

8. Plot the graph of d against θ in Question 23 of Exercise XXIX (p. 231).

Find, without drawing a graph, the periods of the following periodic functions of x:

9. $\text{Sin } 2x + 3 \sin 4x$.

10. $\text{Sin } 2x + 2 \sin 3x$.

11. $4 \sin \frac{1}{2}x - \sin \frac{1}{3}x$.

12. $\text{Sin } 3\pi x + \cos 5\pi x$.

13. $\text{Sin}\left(3x + \frac{\pi}{4}\right) + 2 \sin\left(4x - \frac{\pi}{8}\right)$.

14. $\text{Sin } x + \sin 2x + \sin 3x$.

Simple harmonic oscillations of the following frequencies are compounded. What is the resultant frequency in each case?

15. 50, 200. **16.** 30, 45. **17.** 225, 250.

18. f, nf, where n is a whole number. Express your conclusion in this case in words.

CHAPTER XI

TRIGONOMETRIC EQUATIONS

Graphical solution of equations

We have had examples in algebra of solving equations by means of graphs. We can also solve equations in trigonometry by graphical methods.

Example.—Solve the equation $x = 2 \sin x$ for x.

[NOTE.—As explained on p. 230, $\sin x$ is understood to mean $\sin$ (x radians).]

If we draw the graphs of $y = x$ and $y = 2 \sin x$ on the same axes and with the same scales, then at the points where the curves intersect, the ordinates for the two curves will be equal, i.e. $x = 2 \sin x$. Hence the roots of the equation are the values of x at the points where the curves intersect.

GRAPHICAL SOLUTION OF EQUATION $x = 2 \sin x$

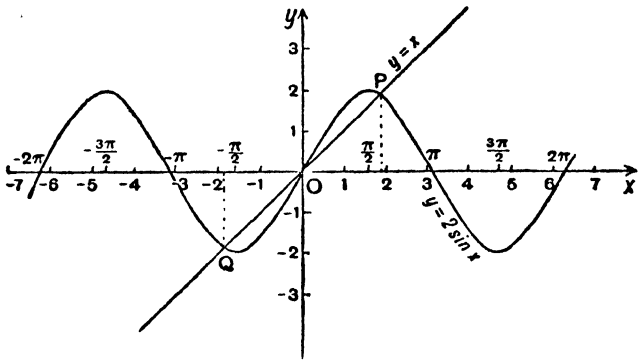


FIG. 208.

The graphs of $y = x$ and $y = 2 \sin x$ are shown in Fig. 208. The x-axis is scaled (underneath the axis) at unit intervals, but the most convenient values to take for graphing the sine curve are $\pi/2$, π, $3\pi/2$, etc.; these values are marked above the axis. The curves intersect at three points, viz. O, P and Q. It is evident that they will not intersect again however far we continue the graphs to the right or left. The abscissæ of the points O, P and Q are 0, 1·9 and $-1·9$, approx. The equation therefore has three roots, viz. $x = 0$ and $x = \pm 1·9$ (approx.).

We could get a closer approximation to the roots by drawing the parts of the graph near P and Q on a larger scale. We need only enlarge one portion, say that near P, since the root at Q is clearly the negative of that at P. Fig. 209 shows the portion between $x = 1·8$ and $x = 2·0$ on a larger scale.

Table of values for 2 sin (x radians) between $x = 1·8$ and $x = 2$.

.. ..	1·8	1·85	1·9	1·95	2·0
Angle in degrees	103·14°	106·00°	108·87°	111·73°	114·60°
x ..	0·9738	0·9613	0·9462	0·9289	0·9092
$\sin x$ ..	1·9476	1·9226	1·8924	1·8578	1·8184

[NOTE.—1 radian $\simeq 57·3° = 57° 18'$.]

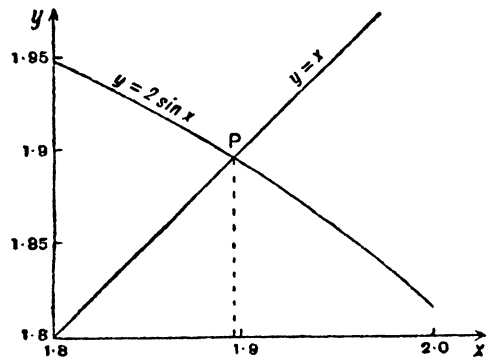


FIG. 209.

From Fig. 209 a more accurate value of the x at P is 1·896. The roots of the equation are therefore 0 and $\pm 1{\cdot}896$ (approx.).

[To test in the equation :

$$\begin{aligned}\sin(1{\cdot}896 \text{ radians}) &= \sin 108^\circ\ 38' \\ &= \sin(180^\circ - 108^\circ\ 38') \\ &= \sin 71^\circ\ 22' = 0{\cdot}9476.\end{aligned}$$

$\therefore$ When $x = 1{\cdot}896$, $2 \sin x = 2 \times 0{\cdot}9476 = 1{\cdot}8952 = 1{\cdot}895$ to three places of decimals.]

It is often advisable to rewrite the equation in a different form so as to make the graphs which have to be drawn as simple as possible.

For example, if we wish to solve the equation $x^2 \tan x = 1$, instead of finding the intersections of the graphs of $y = x^2 \tan x$ and $y = 1$, we might write the equation in the form $x^2 = \cot x$ (by dividing both sides of the equation by $\tan x$), and find the intersections of the graphs of $y = x^2$ and $y = \cot x$, which are curves of well-known shape. Alternatively we could put the equation into the form $\tan x = \dfrac{1}{x^2}$, and find the intersections of the graphs of $y = \tan x$ and $y = \dfrac{1}{x^2}$.

Angles having a given sine, cosine or tangent, etc.

To find the angles whose sines are equal to $\frac{1}{2}$ is equivalent to solving the equation $\sin\theta = \frac{1}{2}$ for θ. We could do this graphically by drawing the graph of $y = \sin\theta$ and finding where it cuts the line $y = \frac{1}{2}$.

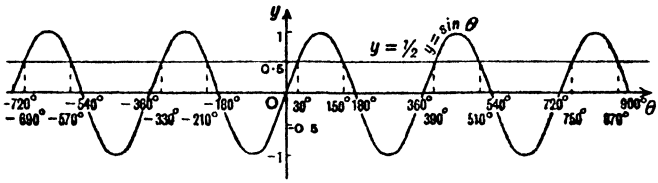


FIG. 210.

From Fig. 210 it is seen that the line $y = \frac{1}{2}$ cuts the curve $y = \sin\theta$ in an infinite number of points. The values of θ at those points are . . ., $-690°$, $-570°$, $-330°$, $-210°$, $30°$, $150°$, $390°$, $510°$, $750°$, $870°$, . . .

Similarly, if we take any other value between -1 and $+1$, call it k, we see that the line $y = k$ cuts the curve $y = \sin\theta$ in an infinite number of points; that is, the equation $\sin\theta = k$ has an infinite number of roots.

Since the curve $y = \sin\theta$ repeats itself every $360°$ we need only to find the roots between $\theta = 0°$ and $\theta = 360°$ or the roots which lie within any other period, say between $-180°$ and $+180°$. If we then add or subtract multiples of $360°$, we shall obtain all the other roots.

The same is true if we want to find all the angles having a given cosine, or tangent, or cotangent, secant or cosecant.

We can find the roots between $0°$ and $360°$ without drawing a graph, as the following examples will show.

Example.—Find all the values of θ for which $\sin\theta = \frac{1}{2}$.

First find the angles between $0°$ and $360°$. Since $\sin\theta$ is positive, θ must lie in the first or second quadrant.

The angle in the first quadrant whose sine is $\frac{1}{2}$ is 30°. The angle in the second quadrant is 180° − 30°, i.e. 150° (Fig. 211).

All the other values are found by adding or subtracting multiples of 360°. Thus we get

. . ., −690°, −330°, 30°, 390°, 750°, . . .

and . . ., −570°, −210°, 150°, 510°, 870°, . . .

the dots indicating that we can continue indefinitely in each direction.

All these angles are included in the general formula

$$\theta = 30° \pm n \,.\, 360° \text{ or } 150° \pm n \,.\, 360°,$$

where n denotes *any* integer.

Example.—Find θ if $\cos\theta = -0{\cdot}6202$.

First to find the angles between 0° and 360° which satisfy this equation. Since $\cos\theta$ is negative, θ must lie in the second or third quadrant.

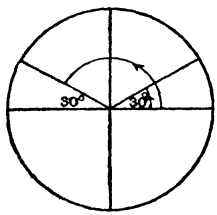


Fig. 211.

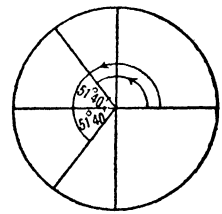


Fig. 212.

The acute angle whose cosine is 0·6202 is (from tables) 51° 40′. The required angles (shown in Fig. 212) are therefore 180° − 51° 40′, i.e. 128° 20′, and 180° + 51° 40′, i.e. 231° 40′. All the other angles are obtained by adding or subtracting multiples of 360° ; they are

. . ., −591° 40′, −231° 40′, 128° 20′, 488° 20′, 848° 20′, . . .

and

. . ., −488° 20′, −128° 20′, 231° 40′, 591° 40′, 951° 40′, . . .

or, more concisely,

$$128° \; 20' \pm n \,.\, 360° \text{ or } 231° \; 40' \pm n \,.\, 360°,$$

where n is any integer.

It has already been observed (p. 227) that each of the six trigonometric ratios is positive in two quadrants and negative in the other two. Thus, whichever of the six ratios is given, there will be two angles between 0° and 360° which have the given ratio.

Thus we have the following *rule* :

To find all the angles having a given sine, cosine, tangent, etc., first write down the two angles between 0° *and* 360° *having the given ratio, and then add or subtract multiples of* 360°.

In the case of the tangent we can write down the angles still more easily. If we refer to the graph of $\tan\theta$ on p. 235, we see that all the angles having a given tangent differ by multiples of 180° ; so that we need only to find *one* angle having the given tangent and then add or subtract multiples of 180°.

For example, all the angles whose tangents are equal to 1 are included in the formula $45° \pm n \,.\, 180°$, where n is any integer.

Similarly, the general solution of the equation $\tan\theta = \tan\alpha$ is $\theta = \alpha \pm n \,.\, 180°$.

The notation $\sin^{-1} x$ is often used to denote an angle whose sine is equal to x, with similar meanings for $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, etc. Thus in the last example we have found all the values of $\cos^{-1}(-0{\cdot}6202)$. The student should be careful to remember that $\sin^{-1} x$ does *not* mean $\sin x$ raised to the power -1.* To avoid ambiguity it is usual to write $\sin x$ to the power -1 as $(\sin x)^{-1}$.

Example.—Find all the angles θ between 0° and 360° which satisfy the equation $5 \sin(2\theta - 45°) = 1{\cdot}47$.

$$\sin(2\theta - 45°) = \frac{1{\cdot}47}{5} = 0{\cdot}294$$

$$\therefore\ 2\theta - 45° = \sin^{-1} 0{\cdot}294.$$

* In Continental and American books the notation arc sin x, arc cos x, etc., is used in place of $\sin^{-1} x$, $\cos^{-1} x$, etc.

The two values of $\sin^{-1} 0{\cdot}294$ between 0° and 360° are 17° 6′ and 162° 54′.

$\therefore 2\theta - 45° = 17° 6' \pm n \cdot 360°$

or $162° 54' \pm n \cdot 360°$, where n is any integer.

$\therefore 2\theta = 62° 6' \pm n \cdot 360°$

or $207° 54' \pm n \cdot 360°$

$\therefore \theta = 31° 3' \pm n \cdot 180°$

or $103° 57' \pm n \cdot 180°$, where n is any integer.

i.e. $\theta =$ . . ., $-148° 57'$, 31° 3′, 211° 3′, 391° 3′, . . .

or . . ., $-76° 3'$, 103° 57′, 283° 57′, 463° 57′, . . .

We have to pick out the values of θ between 0° and 360°; they are

31° 3′, 211° 3′ and 103° 57′, 283° 57′;

or, arranging in order,

31° 3′, 103° 57′, 211° 3′ and 283° 57′.

To find an angle when its sine and cosine are given

If an angle is known to be acute we can find the angle (from tables) if we are given *either* its sine *or* its cosine (or its tangent or cotangent, etc.).

If an angle is known to lie between 0° and 360° and the sine is given, there are two possible values for the angle: e.g. if the sine is $\frac{1}{2}$, the angle may be 30° or 150°. Similarly, if the cosine is given there are two possible values. If, however, we are given both the sine *and* the cosine then the angle is determined.

For example, suppose we are given that $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2}$ and that θ lies between 0° and 360°. Then, since $\sin\theta = \frac{1}{2}$, θ must be either 30° or 150°; also, since $\cos\theta = -\frac{\sqrt{3}}{2}$, θ must be either 150° or 210°. The only possible value for θ, therefore, is 150°.

The quickest way to find the angle is to use the "*all, sin, tan, cos* rule" (p. 227). Thus, in the above example, since $\sin\theta$ is positive, θ lies in the 1st or 2nd quadrant; since $\cos\theta$

is negative, θ lies in 2nd or 3rd quadrant. The only possible quadrant therefore is the 2nd. The value of the angle can now be found either from its sine *or* from its cosine.

Example.—Find θ if $\sin\theta = -\frac{4}{5}$ and $\cos\theta = \frac{3}{5}$, assuming that θ lies between 0° and 360°.

Since $\sin\theta$ is $-ve$, θ lies in the 3rd or 4th quadrant.

Since $\cos\theta$ is $+ve$, θ lies in the 1st or 4th quadrant.

Hence θ must lie in the 4th quadrant.

From a table of sines we find that the acute angle whose sine is $\frac{4}{5}$, i.e. 0·8, is 53° 8′.

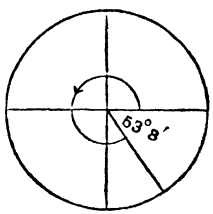


FIG. 213.

Hence $\theta = 360° - 53°\ 8'$
$= 306°\ 52'$.

Exercise XXXII

[Exercises 1–7 are to be solved *graphically*.]

1. Solve the equation $\cos$ (x radians) $= 1 - \frac{x}{2}$.

2. Find the smallest positive root of the equation $x \tan x = 1$, the angle being in radians.

3. Find the roots of the equation $\sin(x + 25)° = \cos 2x°$ which lie between 0 and 180.

4. When a crossed belt of length l in. passes over two pulley wheels of radii r_1 and r_2 in., the distance d in. between their centres is given by $d = (r_1 + r_2)\ \text{cosec}\ \theta$, where θ is determined from the equation

$$l = 2(r_1 + r_2)\left(\frac{\pi}{2} + \theta + \cot\theta\right)$$ (see p. 201).

If $l = 80$, $r_1 = 6$ and $r_2 = 4$, find θ and hence d.

5. The force on a piston is proportional to $\cos\theta + \frac{r}{l}\cos 2\theta$, where r and l are the lengths of the crank and connecting-rod. If $l = 4r$, find the smallest positive value of θ for which the force on the piston is zero.

6. Fig. 214 shows a cross-section of a gutter full of water. The area of the cross-section of the water is $\frac{1}{2}r^2(\theta - \sin\theta)$ sq. in. If $r = 3$ and the area is $4\frac{1}{2}$ sq. in., find the value of θ, and express the angle in degrees.

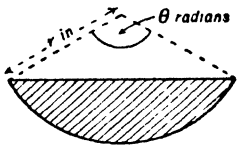


FIG. 214.

7. In a certain type of ammeter the current, i amps., when the deflection of the needle is $\theta°$ is given by

$$i = 0{\cdot}735\sqrt{\frac{\theta}{\sin(\theta + 30)°}}.$$

Find the deflection when a current of 3 amps. passes through the instrument.

8. In which quadrant does an angle lie if its sine and cosine are (i) both positive, (ii) both negative ?

Find the two values between 0° and 360° of the following :

9. $\text{Sin}^{-1}\ 0{\cdot}33$.

10. $\text{Cos}^{-1}\ 0{\cdot}472$.

11. $\text{Sin}^{-1}\ (-0{\cdot}92)$.

12. $\text{Tan}^{-1}\ 4{\cdot}23$.

13. $\text{Sec}^{-1}\ 1{\cdot}40$.

14. $\text{Cos}^{-1}\ (-0{\cdot}845)$.

Find the values between $-180°$ and 180° of the following :

15. $\text{Cot}^{-1}\ 0{\cdot}708$.

16. $\text{Sin}^{-1}\ (-0{\cdot}265)$.

Find, in radians, the angles between 0 and 2π which satisfy the following equations :

17. $\text{Sin}\ \theta = \frac{1}{4}$.

18. $\text{Tan}\ x = -1$.

19. $\text{Cos}\ \alpha = 0{\cdot}2$.

20. $\text{Cosec}\ \theta = 2$.

Find all the values of θ between 0° and 360° which satisfy the following equations :

21. $2\cos 4\theta = 0{\cdot}75$.

22. $3\sin\frac{\theta}{2} = 2{\cdot}58$.

23. $\text{Sin}\ 2\theta = -0{\cdot}5$.

24. $5\tan 3\theta = 9{\cdot}325$.

25. The angular displacement ϕ degrees of a pendulum at time t sec. is given by $\phi = 15\sin 3\pi t$. Find the times during the first second at which the displacement is $+10°$.

26. The voltage in a circuit t sec. after the current is switched on is $200\sin(314t - 50)$ volts. Find, by calculation, (i) when the voltage is first zero, and (ii) when it first reaches its maximum value.

27. The displacement s in. of a slide-piece in a certain mechanism at time t sec. is given by $s = 6\sin(3t + 0{\cdot}8)$. Find the

smallest positive value of t for which the displacement is (i) $+3$ in., (ii) -3 in.

Find the angles θ, between 0° and 360°, which satisfy the following:

28. Sin $\theta = 0{\cdot}81$ and cos $\theta = -0{\cdot}5864$.

29. Sin $\theta = -\frac{5}{13}$ and cos $\theta = \frac{12}{13}$.

30. Sin $\theta = -\frac{8}{17}$ and cos $\theta = -\frac{15}{17}$.

Trigonometric equations

Example.—Find all the angles between 0° and 360° which satisfy the equation

$$\tan^2\theta - 4\tan\theta + 1 = 0.$$

This is a quadratic equation in tan θ; solving the quadratic in the ordinary way we have:

$$\tan\theta = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$= 2 \pm 1{\cdot}7321$$

$$= 3{\cdot}7321 \text{ or } 0{\cdot}2679.$$

When $\tan\theta = 3{\cdot}7321$, $\theta = 75° \pm n \,.\, 180°$.

When $\tan\theta = 0{\cdot}2679$, $\theta = 15° \pm n \,.\, 180°$.

$\therefore$ The possible values of θ between 0° and 360° are 15°, 75°, 195° and 255°.

If two or more ratios occur in the equation we must transform the equation into a form in which only one ratio occurs.

Example.—Solve the equation

$$\sin^2\theta = \cos\theta.$$

Since $\sin^2\theta = 1 - \cos^2\theta$, the equation can be written in the form

$$1 - \cos^2\theta = \cos\theta,$$

which contains only $\cos\theta$.

$$\therefore \cos^2\theta + \cos\theta - 1 = 0.$$

This is a quadratic in $\cos\theta$; solving the quadratic we have

$$\cos\theta = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

$$= \frac{-1 \pm 2{\cdot}23607}{2}$$

$$= \frac{1{\cdot}23607}{2} \text{ or } -\frac{3{\cdot}23607}{2}$$

$$= 0{\cdot}6180 \text{ or } -1{\cdot}6180.$$

The latter value is impossible, since $\cos\theta$ cannot be less than -1 for any real angle θ.

$$\therefore \cos\theta = 0{\cdot}6180$$

$$\therefore \theta = 51^\circ\ 50' \pm n \,.\, 360^\circ$$

$$\text{or } 308^\circ\ 10' \pm n \,.\, 360^\circ.$$

Example.—Find the angles x between 0° and 360° which satisfy the equation

$$\sin 2x = \tfrac{1}{2}\cos 2x.$$

Dividing both sides of the equation by $\cos 2x$, we have

$$\tan 2x = \tfrac{1}{2},$$

which contains only one ratio, viz. $\tan 2x$.

$$\therefore 2x = 26^\circ\ 34' \pm n \,.\, 180^\circ, \text{ where } n \text{ is any integer.}$$

$$\therefore x = 13^\circ\ 17' \pm n \,.\, 90^\circ, \text{ where } n \text{ is any integer.}$$

The values of x between 0° and 360° are

$$13^\circ\ 17',\ 103^\circ\ 17',\ 193^\circ\ 17',\ 283^\circ\ 17'.$$

Exercise XXXIII

Find the angles between 0° and 360° which satisfy the following equations :

1. $4 \sin^2 \theta = 1$.
2. $\text{Tan } x = 5 \cot x$. [Hint.—Write $\cot x$ as $\frac{1}{\tan x}$.]
3. $\text{Sin}^2 \theta - 2 \cos^2 \theta + 1 = 0$.
4. $\text{Cos}^2 x = 3 (1 + \sin x)$.
5. $\text{Tan } x + 2 \cot x - 3 = 0$ [see hint to Question 2].
6. $\text{Sec}^2 \theta = \frac{1}{2} \tan \theta + 1$.
7. $\text{Cos}^2 \frac{x}{2} = 0{\cdot}4$.
8. $\text{Sin } 3x = \cos 3x$.
9. $\text{Sin } x + \tan x = 0$.
10. The angular velocity of a connecting-rod in an engine is

$$\frac{\omega \cos \theta}{\sqrt{\left(\frac{l^2}{r^2} - \sin^2 \theta\right)}}.$$

If $r = 1{\cdot}4$ and $l = 7$, find the values of θ for which the angular velocity is $\frac{1}{8}\omega$.

11. Barr's formula for the efficiency of a worm-gearing is

$$e = \frac{\tan \alpha \ (1 - \mu \tan \alpha)}{\mu + \tan \alpha}.$$

If $e = 0{\cdot}93$ and $\mu = 0{\cdot}02$ find the angle α. (Only the smallest positive value is required here.)

12. Fig. 215 shows a cam which rotates about O and imparts a vertical motion to the follower F, the follower moving up and down in a vertical line through O. Show that, when the point of contact P lies on the arc AB, the height (h) of P above O is given by

$$h = d \cos \theta + \sqrt{b^2 - d^2 \sin^2 \theta}.$$

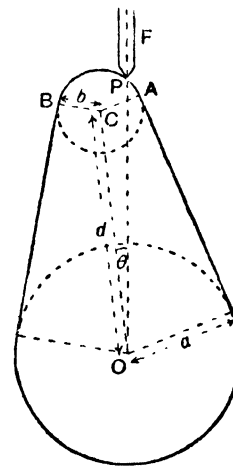


FIG. 215.

If $b = \frac{1}{2}$ in. and $d = 2\frac{3}{4}$ in., find the value of θ when $h = 3{\cdot}1$ in.

Some useful identities

The relation $\sin^2\theta+\cos^2\theta\equiv 1$ was proved earlier for any *acute* angle θ, but it is easily proved to be true for any angle θ whatever its magnitude, positive or negative.

For in Fig. 178, $ON^2+NP^2=OP^2$ in whatever quadrant OP lies.

$$\therefore \left(\frac{ON}{OP}\right)^2+\left(\frac{NP}{OP}\right)^2=1.$$

But $\frac{ON}{OP}=\pm\cos\theta$ and $\frac{NP}{OP}=\pm\sin\theta$, the signs to be taken depending on the quadrant in which OP lies. The $-$ signs disappear when we square them and thus in any case we have

$$\cos^2\theta+\sin^2\theta\equiv 1.$$

So also the relations

$$1+\tan^2\theta\equiv\sec^2\theta \text{ and } 1+\cot^2\theta\equiv\operatorname{cosec}^2\theta$$

are true for all values of θ.

It can be proved also that the relations:

$$\sin(90^\circ-\theta)=\cos\theta, \qquad \cos(90^\circ-\theta)=\sin\theta,$$
$$\tan(90^\circ-\theta)=\cot\theta, \qquad \cot(90^\circ-\theta)=\tan\theta,$$
$$\sec(90^\circ-\theta)=\operatorname{cosec}\theta, \qquad \operatorname{cosec}(90^\circ-\theta)=\sec\theta,$$

are true, whatever the angle θ, by drawing figures in which the angle lies in different quadrants.

Example.—Verify the relation $\sin(90^\circ-\theta)=\cos\theta$ when $\theta=120^\circ$.

$$\sin(90^\circ-120^\circ)=\sin(-30^\circ)$$
$$=-\sin 30^\circ \text{ (Fig. 216)}$$
$$=-\tfrac{1}{2}$$
$$\cos 120^\circ=-\cos 60^\circ=-\tfrac{1}{2}.$$

Hence the relation is true when $\theta=120^\circ$.

If θ is an acute angle, $180^\circ-\theta$ is an angle in the second quadrant and, from the rule on p. 229, we have from Fig. 217 that

$$\sin(180^\circ-\theta)=\sin\theta, \quad \cos(180^\circ-\theta)=-\cos\theta,$$
$$\tan(180^\circ-\theta)=-\tan\theta.$$

Also if θ is acute, $(-\theta)$ is in the fourth quadrant and therefore, from Fig. 217,

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta, \quad \tan(-\theta) = -\tan\theta.$$

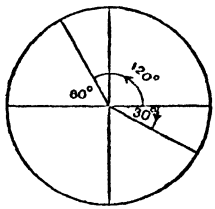


FIG. 216.

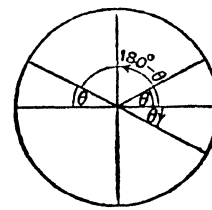


FIG. 217.

These six relations can also be proved to be true whatever the magnitude of the angle θ, positive or negative.

Example.—Verify that $\cos(180° - \theta) = -\cos\theta$ when $\theta = -110°$.

$$\cos\{180° - (-110°)\} = \cos(180° + 110°) = \cos 290°$$
$$= \cos 70°, \text{ from Fig. 218.}$$
$$-\{\cos(-110°)\} = -(-\cos 70°), \text{ from Fig. 219}$$
$$= \cos 70°.$$

Hence the relation is true when $\theta = -110°$.

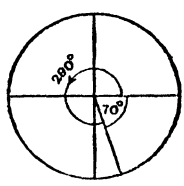


FIG. 218.

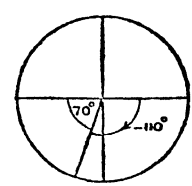


FIG. 219.

The following identical relations can also be proved to be true for all angles θ:

$$\sin(90° + \theta) = \cos\theta, \quad \cos(90° + \theta) = -\sin\theta,$$
$$\tan(90° + \theta) = -\cot\theta;$$

$$\sin(180° + \theta) = -\sin\theta, \quad \cos(180° + \theta) = -\cos\theta,$$
$$\tan(180° + \theta) = \tan\theta.$$

We shall prove only the first; the proofs of the remainder can be left as an exercise for the student.

$\text{Sin}(90° + \theta)$
$= \sin\{90° - (-\theta)\} = \cos(-\theta)$ from the relations on p. 264
$= \cos\theta.$

The relations between the ratios of an angle θ and the ratios of the angles $(-\theta)$, $180° - \theta$ and $180° + \theta$ are illustrated on the graphs of the ratios.

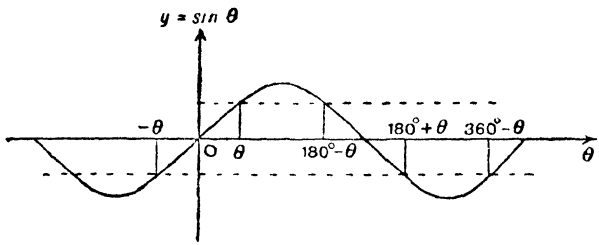


FIG. 220.

For example, the graph of $y = \sin\theta$ in Fig. 220 shows that

$$\sin(180° - \theta) = \sin\theta,$$

and that $$\sin(180° + \theta) = \sin(-\theta) = -\sin\theta.$$

[In Fig. 220, we have marked θ as an angle between 0° and 90°; the student is advised to take other values for θ on the graph and to verify these relations for the values he chooses.]

The student is not expected to remember all these relations. In practice, if we want any one of them we draw a rough figure in which θ is *acute*, and read off the relation from the figure. We then know that it is true for all values of θ.

This same remark applies to expressions of the form

given stated scale. To distinguish between the two opposite directions (or " senses ") on a line we often put an arrowhead on the line.

We usually speak of " the vector $\overrightarrow{AB}$ " when we mean the vector represented by a line AB (in the sense *from A to B*) in a vector diagram. A and B are called the *initial* and *final* points of the vector respectively. If the sense of the vector were from B to A, so that the arrowhead would be reversed in Fig. 221, we should write it as " the vector $\overrightarrow{BA}$."

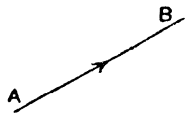


FIG. 221.

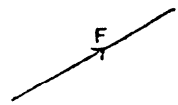


FIG. 222.

We sometimes denote a vector by a single letter with an arrow over it; thus, if the length of the vector in Fig. 222 be F, we should speak of " the vector $\vec{F}$." The arrow over the F is necessary to distinguish between a vector and a number or length. Many books use a heavier type instead of an arrow, and write the vector as $\mathbf{F}$; electrical engineers use a dot and write it as $\dot{F}$.

Another useful way of writing a vector is to take some standard direction, such as OX in Figs. 223 and 224, from which to measure angles. Then if the angle between OX and the direction of $\vec{F}$ is α, we write the vector $\vec{F}$ as F_α; electrical engineers often write it as $F \angle \alpha$.

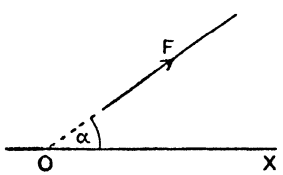


FIG. 223.

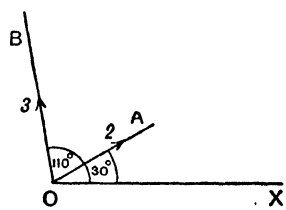


FIG. 224.

For example, in Fig. 224, $\overrightarrow{OA} = 2_{30°}$, $\overrightarrow{OB} = 3_{110°}$, referred to OX as direction of reference.

[NOTE.—It should be carefully observed that when we speak of the direction OX we mean the direction *from O to X*. The easiest way to make quite sure that we label the angle correctly in the diagram is to take a pen and lay it along OX with the nib pointing towards X and then turn the pen about O (in an anti-clockwise direction) until it falls along the vector with the nib pointing in the direction of the arrow. The angle through which the pen has turned is the angle α.

For example, vector $\overrightarrow{EO}$ in Fig. 225 is $2_{240°}$, *not* $2_{60°}$.]

The *resultant* of two vectors is defined by the "parallelogram law"; that is, the resultant of the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Fig. 226 is the vector $\overrightarrow{OC}$, where OC is the diagonal through O of the parallelogram formed on the sides OA and OB.

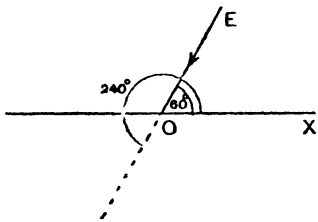


FIG. 225.

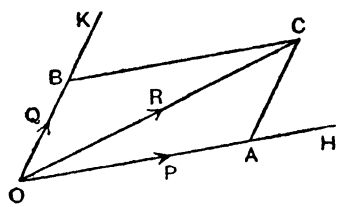


FIG. 226.

This is the law by which, in mechanics, we compound two forces acting at a point or two velocities.

The vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the *components* of the vector $\overrightarrow{OC}$ in the directions OH and OK. The components of a vector in any two directions can be found by reversing the construction for the resultant, that is by drawing lines through C parallel to the given directions cutting them in A and B.

It is frequently necessary to find the components of a vector in two directions which are at right angles, such as OX, OY in Fig. 227. In that case the magnitudes of the components are called the *resolved parts* or *resolutes* of the vector

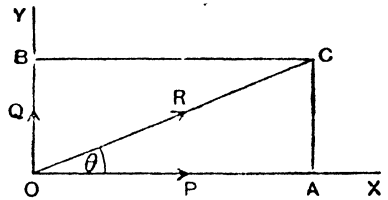


FIG. 227.

in those directions. If the resolved parts along OX, OY are P and Q and the magnitude of the resultant is R, and if the direction of the resultant makes an acute angle θ with OX, then, from Fig. 227,

$$P = R\cos\theta, \quad Q = R\sin\theta;$$

Also $$R = \sqrt{P^2 + Q^2}, \quad \tan\theta = \frac{Q}{P}.$$

Resultant of any number of vectors

In Fig. 226, since $OACB$ is a parallelogram, AC is equal and parallel to OB. We can therefore find the resultant $\vec{R}$ (i.e. $\overrightarrow{OC}$) by drawing OA and AC, of lengths P and Q, parallel to the vectors $\vec{P}$ and $\vec{Q}$. It is not necessary to draw the whole parallelogram. The construction is shown in Fig. 228.

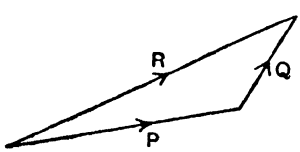


FIG. 228.

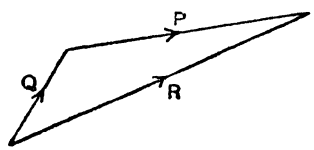


FIG. 229.

We might describe the construction briefly thus: Place the vectors $\vec{P}$, $\vec{Q}$ end to end, so that the initial point of one vector coincides with the final point of the other; the resultant is then represented by the line joining the free ends, in the direction *from* the first point *to* the last.

The order in which we take the vectors makes no difference, as is shown by Fig. 229, which is equivalent to drawing the triangle OBC of Fig. 226.

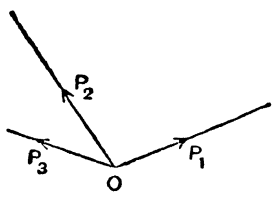


FIG. 230 (*a*).

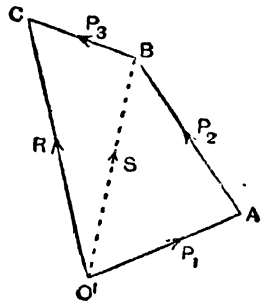


FIG. 230 (*b*).

Now suppose we have three vectors, $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$ (Fig. 230 (*a*)). $\vec{P}_1$ and $\vec{P}_2$ have a resultant $\vec{S}$ (Fig. 230 (*b*)). Compounding $\vec{S}$ with $\vec{P}_3$, they have a resultant $\vec{R}$. Thus $\vec{R}$ is the result of compounding the three vectors $\vec{P}_1$, $\vec{P}_2$ and $\vec{P}_3$. The order in which we take the three vectors makes no difference to the final result. The vector $\vec{R}$ is called the resultant of the three given vectors.

If $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$ represent three forces acting at O, they are equivalent to a single force $\vec{R}$ acting at O.

We can find the resultant of any number of vectors by extending the construction above. The rule is: Place the vectors end to end, so that the initial point of each vector

coincides with the final point of the previous one, that is, so that the direction of the arrows as we go round the figure $O'ABC$ is continuous. The resultant is represented by the line joining the free ends in the direction from the first point to the last.

$O'A$, AB, BC (in Fig. 230 (*b*)) may be regarded as the links of a broken chain, and CO' as the final link which completes the chain. This link *reversed* is the resultant. Hence the above rule is called the *chain rule*.

Example.—Find, graphically, the resultant of the vectors $2{\cdot}5_{30°}$, $4_{70°}$, $2_{125°}$, $6_{212°}$.

Take OX as direction of reference. Taking a scale of 1 unit = 1 cm, draw OA of length 2·5 cm. making an angle 30° with OX (Fig. 231). From A draw AB of length 4 cm., making an angle 70° with OX. Similarly, draw BC and CO to represent $2_{125°}$ and $6_{212°}$.

Join OD. Then $\overrightarrow{OD}$ is the resultant.

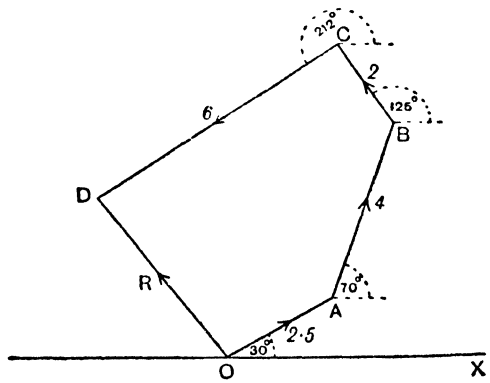


FIG. 231.

By measurement it is found that $R = 4{\cdot}3$ cm. which represents 4·3 units (on our scale) and $\widehat{XOD} = 128°$.

Hence the resultant is the vector $4{\cdot}3_{128°}$.

[In Fig. 231 the scale has been reduced for convenience of printing. The student should use as large a scale as possible, of course.]

Example.—A body is pulled simultaneously by forces of 20 lb. wt. due N., 25 lb. wt. due E. and 16 lb. wt. in a direction S.W. Find the direction in which it moves.

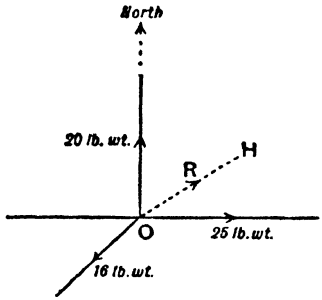


FIG. 232 (*a*).

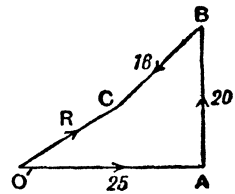


FIG. 232 (*b*).

The forces are shown in Fig. 232 (*a*). Their resultant is found by the chain rule in Fig. 232 (*b*); it is represented by $\overrightarrow{O'C}$.

The body moves in the direction of the resultant force, *i.e.* in the direction *OH* (which is drawn parallel to *O'C*). By measurement we find that *OH* makes an angle $56\frac{1}{2}°$ with the northerly direction, i.e. the body moves in the direction N. $56\frac{1}{2}°$ E.

Resolved part of a vector in any direction

In Fig. 227 *OA* is the resolved part of $\overrightarrow{OC}$ in the direction *OX*; it is the projection of *OC* on that direction.

If $\overrightarrow{OC}$ represents a force, $\overrightarrow{OA}$ may be regarded as the *effective part* of the force in the direction *OX*, since the component $\overrightarrow{OB}$ has no tendency to move the body in the direction *OX*.

Now suppose the angle θ between OX and OC is an obtuse angle, as in Fig. 233.

The force $\overrightarrow{OC}$ is equivalent to the two forces $\overrightarrow{OA}$ and $\overrightarrow{OB}$. The effective part so far as motion along OX is concerned is $\overrightarrow{OA}$, which is a force of magnitude P in the direction OX' opposite to OX. The effective force in the direction OX is therefore of magnitude $-P = -R\cos(180° - \theta)$ $= -R \times (-\cos\theta) = R\cos\theta$.

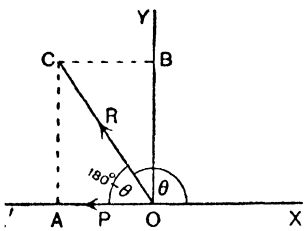


Fig. 233.

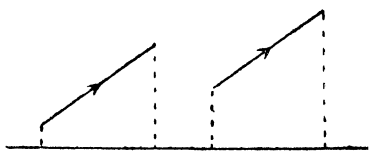


Fig. 234.

[Check.—Since θ is obtuse, $\cos\theta$ is negative, and hence the resolved part in the direction OX is negative, which is correct.]

In the same way, if we take any other value of θ, say in the third or fourth quadrants, we find that in every case the resolved part of $\overrightarrow{R}$ along OX is $R\cos\theta$.

It may also be seen that the resolved part along OY (perpendicular to OX) is $R\sin\theta$ in every case. Thus : *the resolved parts of a vector R_θ along OX, OY are always $R\cos\theta$ and $R\sin\theta$.*

It is clear that the resolved parts in any given direction of two equal and parallel vectors are themselves equal (Fig. 234).

To solve the equations $r\cos\theta = a$, $r\sin\theta = b$

Equations of this type in which a and b are given, and we have to solve for r and θ, occur very frequently.

Since $r\cos\theta$ and $r\sin\theta$ are the resolved parts along OX

and OY of the vector r_θ, our problem is simply to find the magnitude (r) and direction (θ) of the vector whose resolved parts along OX and OY are a and b. This can be done rapidly from a rough sketch.

Example.—If $r\cos\theta = 3$ and $r\sin\theta = 4$, find r and θ.

The resolved parts of r_θ along OX, OY are 3 and 4 respectively. Hence the vector is as shown in Fig. 235.

From Pythagoras' theorem,

$$r^2 = 3^2 + 4^2 = 9 + 16 = 25,$$

$$\therefore r = 5.$$

(We take the positive square root for r obviously.)

Also, from the figure, $\theta = \tan^{-1}\frac{4}{3} = \tan^{-1} 1{\cdot}3333 = 53°\ 8'$.

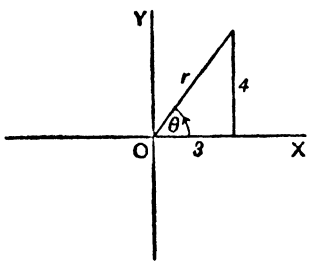


Fig. 235.

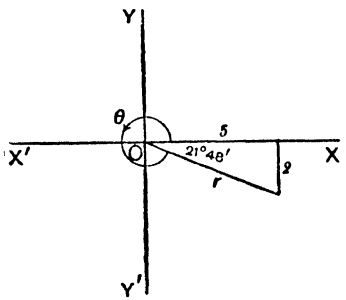


Fig. 236.

We might, of course, take θ to be $53°\ 8' \pm$ any multiple of $360°$, but in every case in practice only the simplest value of θ is required.

Example.—Solve the equations $r\cos\theta = 5$, $r\sin\theta = -2$.

The resolved part along OY is negative, viz. -2, in this case; the vector r_θ is therefore as shown in Fig. 236,

$$r^2 = 5^2 + 2^2 = 25 + 4 = 29$$

$$\therefore r = \sqrt{29} = 5{\cdot}385.$$

From the figure, the angle θ is in the fourth quadrant.
The acute angle between OX and the vector

$$= \tan^{-1}\tfrac{2}{5} = \tan^{-1} 0{\cdot}4 = 21^\circ\ 48',$$

$$\therefore\ \theta = 360^\circ - 21^\circ\ 48' = 338^\circ\ 12'.$$

We sometimes prefer to take θ as being $-21^\circ\ 48'$. Both values for θ satisfy the equations, with the value of r as found.

Resolved part of the resultant of any number of vectors

We shall now show that :

The resolved part in any direction of the resultant of any number of vectors = the sum of the resolved parts, in that same direction, of the component vectors.

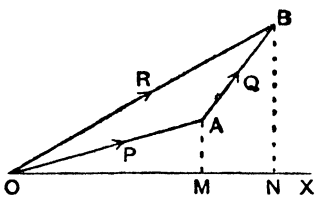


Fig. 237 (*a*).

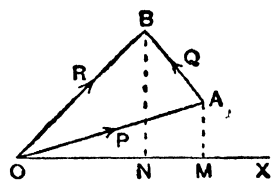


Fig. 237 (*b*).

$\vec{P}, \vec{Q}$ are any two vectors, and $\vec{R}$ their resultant. OX is any direction ; M, N are the projections of A, B on OX.

In Fig. 237 (*a*) the resolved parts of $\vec{P}$ and $\vec{Q}$ along OX are OM, MN. Their sum $= OM + MN = ON =$ the resolved part of $\vec{R}$ along OX.

In Fig. 237 (*b*) the resolved part of $\vec{P}$ is OM, and the resolved part of $\vec{Q}$ along OX is $-NM$ (being negative since the angle between OX and $\vec{Q}$ is obtuse). Their sum $= OM - NM = ON$ = the resolved part of $\vec{R}$ along OX.

Our statement is, in fact, true whatever the directions of $\vec{P}$ and $\vec{Q}$.

It is easily seen that the statement is equally true however many vectors there are.

Analytical method for finding the resultant of any number of vectors

The above gives us an easy method of finding the resultant of any number of vectors *by calculation*. Take any two convenient directions OX, OY at right angles. Resolve each force into its two components along OX and OY. Let the algebraic sum of the components along OX, OY be X and Y respectively. Then X and Y are the components along OX and OY of the resultant, which is therefore found by compounding them in the usual way. If the resultant is of magnitude R and makes an angle θ with OX,

$$R\cos\theta = X, \qquad R\sin\theta = Y.$$

From these equations we can find R and θ.

We shall take the two examples which have been worked graphically on pp. 273–4.

Example.—Find, by calculation, the resultant of the vectors $2{\cdot}5_{30°}$, $4_{70°}$, $2_{125°}$, $6_{212°}$.

Take the line of reference as OX and the line perpendicular to it (the line 90°) as OY.

Vector	*Component along OX*	*Component along OY*
$2{\cdot}5_{30°}$	$2{\cdot}5\cos 30° = 2{\cdot}5\times 0{\cdot}8660 = 2{\cdot}1650$	$2{\cdot}5\sin 30° = 2{\cdot}5\times 0{\cdot}5 = 1{\cdot}25$
$4_{70°}$	$4\cos 70° = 4\times 0{\cdot}3420 = 1{\cdot}3680$	$4\sin 70° = 4\times 0{\cdot}9397 = 3{\cdot}7588$
$2_{125°}$	$2\cos 125° = 2\times(-0{\cdot}5736) = -1{\cdot}1472$	$2\sin 125° = 2\times(0{\cdot}8192) = 1{\cdot}6384$
$6_{212°}$	$6\cos 212° = 6\times(-0{\cdot}8480) = -5{\cdot}0880$	$6\sin 212° = 6\times(-0{\cdot}5299) = -3{\cdot}1794$
	$X = -2{\cdot}7022$	$Y = -3{\cdot}4678$

The resultant therefore has component $-2{\cdot}7022$ along OX and $3{\cdot}4678$ along OY.

Drawing a rough sketch (Fig. 238) we see that:

$$R = \sqrt{(2{\cdot}7022)^2 + (3{\cdot}4678)^2}$$
$$= \sqrt{7{\cdot}302 + 12{\cdot}03} = \sqrt{19{\cdot}33} \simeq 4{\cdot}397 \simeq 4{\cdot}40.$$

Also the acute angle between R and OX'

$$= \tan^{-1} \frac{3{\cdot}4678}{2{\cdot}7022} = 52^\circ\ 5'$$

Hence $\theta = 180^\circ - 52^\circ\ 5' = 127^\circ\ 55'.$

Thus $\vec{R} = 4{\cdot}40_{127^\circ\ 55'}.$

(This is of course more accurate than the value found graphically on p. 273.)

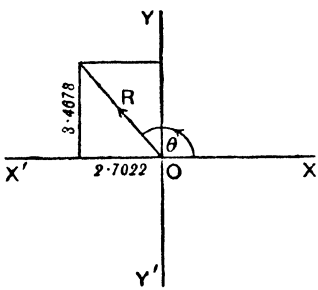


FIG. 238.

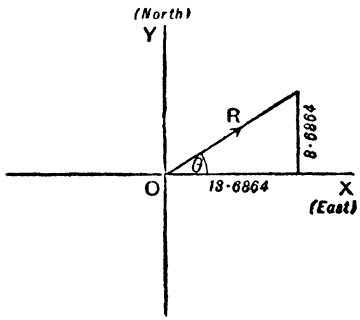


FIG. 239.

Example.—A body is pulled simultaneously by forces of 20 lb. wt. due N., 25 lb. wt. due E. and 16 lb. wt. in a direction S.W. Find the direction in which it moves.

Take OX, OY in directions E. and N. respectively.

Vector	*Component along OX*		*Component along OY*	
20 lb. wt. due N. $=20_{90^\circ}$	$20 \cos 90^\circ$	$= 0$	$20 \sin 90^\circ$	$= 20$
25 lb. wt. due E. $=25_{0^\circ}$	$25 \cos 0^\circ$	$= 25$	$25 \sin 0^\circ$	$= 0$
16 lb. wt. S.W. $=16_{225^\circ}$	$16 \cos 225^\circ$ $= 16 \times (-0{\cdot}7071) = -11{\cdot}3136$		$16 \sin 225^\circ$ $= 16 \times (-0{\cdot}7071) = -11{\cdot}3136$	
		$X = 13{\cdot}6864$		$Y = 8{\cdot}6864$

The resultant has components 13·6864 lb. wt. due E. and 8·6864 lb. wt. due N. (shown in Fig. 239).

θ is in the first quadrant and is equal to

$$\tan^{-1}\frac{8\cdot6864}{13\cdot6864}=32^\circ\ 24'.$$

The body moves in the direction of the resultant force, i.e. in the direction N. 57° 36′ E.

Example.—Show that the resultant of two equal forces, each of magnitude P, inclined at an angle α to each other, is a force of magnitude $2P\cos\frac{\alpha}{2}$ in the direction bisecting the angle between them.

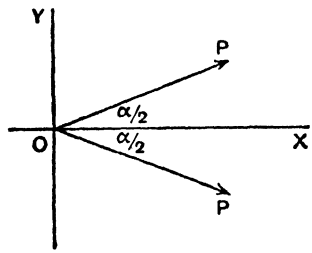


FIG. 240.

Choose the line bisecting the angle between the forces as OX.

The force $P_{\alpha/2}$ has components $P\cos\frac{\alpha}{2}$ along OX and $P\sin\frac{\alpha}{2}$ along OY.

The force $P_{-\alpha/2}$ has components $P\cos\frac{\alpha}{2}$ along OX and $-P\sin\frac{\alpha}{2}$ along OY.

Hence the resultant has components $2P\cos\frac{\alpha}{2}$ along OX and 0 along OY; i.e. the resultant is a force of magnitude $2P\cos\frac{\alpha}{2}$ along the bisector of the angle.

Exercise XXXV

[Additional easier exercises on vectors will be found in Part I (pp. 266–269).]

Find, graphically and by calculation, the magnitude and direction of the resultant of each of the following pairs of vectors, expressing its direction by the angle which it makes with the first vector in each case:

1. Forces of 3 lb. wt. and 10 lb. wt. inclined at 60° to each other.

2. Velocities of 40 ft. per sec. and 15 ft. per sec. inclined at 150°.

3. Forces of 5 tons wt. and 8 tons wt. at right angles.

Find, graphically and by calculation, the magnitude and direction of the resultant of the following pairs of vectors:

4. Forces of 10 tons wt. due N. and 7 tons wt. in the direction S.E.

5. Velocities of 25 m.p.h. N. 30° E. and 18 m.p.h. N. 55° W.

6. Express the vectors shown in Fig. 241 in the form P_α, taking OX as the direction of reference, and find their resolved parts along OX and OY:

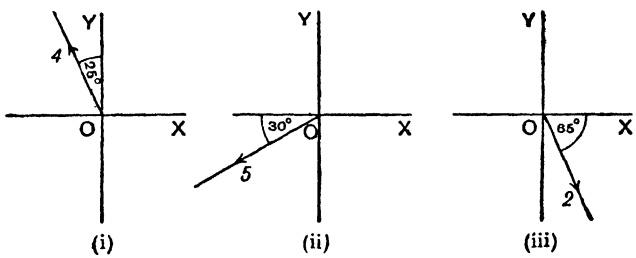


FIG. 241.

7. Find the resultant of the vectors $6_{20°}$ and $9_{230°}$.

8. A man's normal walking pace is 4 m.p.h. If he walks at his normal pace across the deck of a ship which is moving forward at 20 knots, find his actual velocity. [1 knot = a speed of 1 nautical ml. per hr. = 6080 ft. per hr. $\simeq$ 1·15 m.p.h.]

9. An aeroplane is climbing at an angle of 18° to the horizontal. If its horizontal speed is observed to be 135 m.p.h., what is its actual speed? How long does it take to climb 1000 ft. (vertically)?

10. The vector $5_{65°}$ is the resultant of two vectors, one of which is $6_{30°}$; find the other vector.

11. A man can row a boat at 4 m.p.h. in still water. If he wishes to row directly across a river flowing at 3 m.p.h., in what direction must he keep the boat headed?

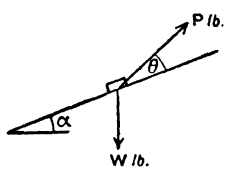


FIG. 242.

12. A body of weight W lb. is pulled up a smooth plane, inclined at an angle α to the horizontal, by a force P lb. wt. inclined at an angle θ to the plane. What is the resolved part of the resultant force on the body up the plane?

13. A force of 10 tons wt. acts in a direction N. 70° E. Find, graphically, its components in the directions E. and N.E.

14. Find, graphically and by calculation, the resultant of two velocities of magnitudes 20 ft. per sec. and 32 ft. per sec. in perpendicular directions.

15. A man walks 3 ml. due N., then 2 ml. S. 36° E. and finally 5 ml. due E. Find the distance and bearing of his final position from the starting-point.

16. Find, *graphically*, the resultant of the vectors $10_{60°}$, $4_{110°}$, $8_{165°}$.

17. Take the vectors in Question 16 in a different order and verify that the resultant is the same as that previously obtained.

18. Show graphically that the following three forces are in equilibrium: 7·07 tons N. 60° E., 27·32 tons N. 45° W., 26·38 tons S. 30° E.

[A number of forces are in equilibrium if their resultant is zero.]

19. Three forces of magnitudes 4·5 lb., 6 lb. and 9 lb. are in equilibrium. Find the angles between their directions.

Find r and θ from the following equations (taking r to be positive):

20. $r\cos\theta = 5$, $r\sin\theta = 12$. **21.** $r\cos\theta = 4$, $r\sin\theta = -3$.

22. $r \cos \theta = 2{\cdot}5$, $r \sin \theta = 1{\cdot}4$. 23. $r \cos \theta = -1{\cdot}6$, $r \sin \theta = 2{\cdot}7$.

24. Find, *by calculation*, the resultant of the vectors in Question 16.

25. A truck is pulled along a light railway by two men hauling on ropes inclined to the rails at 20° on either side. If the force along the rails necessary to move the truck is 230 lb. wt., find the pull which each man must exert.

26. Fig. 243 shows a crane supporting a load of 3 tons. If the stresses in the jib and tie are P tons and Q tons respectively, find the values of P and Q *graphically*. [The three forces shown must be in equilibrium.]

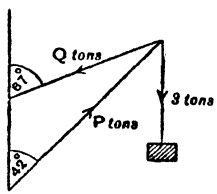


FIG. 243.

27. Find P and Q in Question 26 *by calculation*.

28. If P_α is any vector, show that the resultant of the vectors P_α, $P_{\alpha+\frac{2\pi}{3}}$, $P_{\alpha+\frac{4\pi}{3}}$ is zero.

29. From the result of Question 28 deduce, by resolving along OX and OY (where OX is the direction of reference) that:

$$\cos \alpha + \cos\left(\alpha + \frac{2\pi}{3}\right) + \cos\left(\alpha + \frac{4\pi}{3}\right) \equiv 0$$

and

$$\sin \alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) \equiv 0.$$

[The latter identity occurs in electrical engineering in the theory of a three-phase generator.]

30. Find, *by calculation*, the resultant of the following vectors:

$$4_{15°},\ 2{\cdot}5_{80°},\ 5_{136°},\ 3{\cdot}4_{(-70)°}.$$

31. Find, *by calculation*, the magnitude and direction of the single force which is equivalent to the following forces acting at a point: 10 lb. wt. due E., 6 lb. wt. N. 24° W., 12·4 lb. wt. S. 35° W., 8·2 lb. wt. S. 57° E.

*

CHAPTER XIII

TRIGONOMETRIC RATIOS OF THE SUM AND DIFFERENCE OF TWO ANGLES

RATIOS OF SMALL ANGLES

Sine and cosine of the sum of two angles

Let $\vec{P}$, $\vec{Q}$ be any two vectors at right angles and $\vec{R}$ their resultant. Take any line OX of reference (Fig. 244) and let the angle between OX and $\vec{P}$ be α; then $\vec{P} = P_\alpha$ and $\vec{Q} = Q_{90°+\alpha}$. Let the angle between $\vec{P}$ and $\vec{R}$ be β; then $\vec{R} = R_{\alpha+\beta}$.

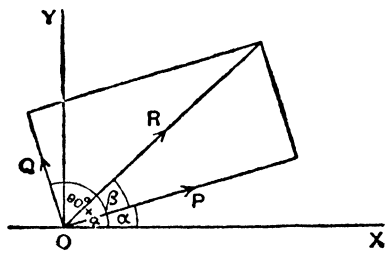


FIG. 244.

Now the resolved part of $\vec{R}$ along OX = sum of resolved parts of $\vec{P}$ and $\vec{Q}$ along OX.

$$\therefore R\cos(\alpha+\beta) = P\cos\alpha + Q\cos(90°+\alpha)$$
$$= P\cos\alpha - Q\sin\alpha,$$

since $\cos(90°+\alpha) = -\sin\alpha$ (p. 265).

But $P = R\cos\beta$ and $Q = R\sin\beta$.

$$\therefore R\cos(\alpha+\beta) = R\cos\beta\cos\alpha - R\sin\beta\sin\alpha$$
$$\therefore \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

If we take resolved parts along OY, instead of along OX, we have,

$$\begin{aligned} R \sin(\alpha+\beta) &= P \sin\alpha + Q \sin(90^\circ+\alpha) \\ &= P \sin\alpha + Q \cos\alpha \\ &= R\cos\beta \sin\alpha + R \sin\beta \cos\alpha \\ \therefore \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha \sin\beta. \end{aligned}$$

These two formulæ are extremely important for all applications of trigonometry, and the student must commit them to memory. They are usually quoted in the forms:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

These formulæ are true whatever the angles A and B, and whether the angles are positive or negative. The formulæ are therefore still true if we write $-B$ in place of B.

$$\begin{aligned} \therefore \sin(A-B) &= \sin A \cos(-B) + \cos A \sin(-B) \\ &= \sin A \cos B + \cos A \,.\, (-\sin B) \\ &= \sin A \cos B - \cos A \sin B. \end{aligned}$$

$$\begin{aligned} \text{and } \cos(A-B) &- \cos A \cos(-B) - \sin A \sin(-B) \\ &= \cos A \cos B - \sin A \,.\, (-\sin B) \\ &= \cos A \cos B + \sin A \sin B. \end{aligned}$$

Collecting these four formulæ we have:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B;$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

where upper signs are to be taken together and lower signs together.

The student should notice the following points which will help him to remember these formulæ:

(i) the $\sin(A \pm B)$ formulæ contain "mixed products," viz. $\sin A \cos B$ and $\cos A \sin B$;

the $\cos(A \pm B)$ formulæ contain products of like ratios, viz. $\cos A \cos B$ and $\sin A \sin B$;

(ii) in the $\sin(A \pm B)$ formulæ the signs ($\pm$) on the two sides of the identity are the same; in the $\cos(A \pm B)$ formulæ the signs are opposite.

Example.—Find the values of $\sin 75°$ and $\cos 75°$ without using tables.

$$\begin{aligned}\sin 75° &= \sin(45° + 30°)\\ &= \sin 45° \cos 30° + \cos 45° \sin 30°\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{(\sqrt{3}+1)\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}\\ &= \frac{2{\cdot}449 + 1{\cdot}414}{4} = \frac{3{\cdot}863}{4} \simeq 0{\cdot}966.\end{aligned}$$

$$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45° \cos 30° - \sin 45° \sin 30°\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{(\sqrt{3}-1)\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}\\ &= \frac{2{\cdot}449 - 1{\cdot}414}{4} = \frac{1{\cdot}035}{4} \simeq 0{\cdot}259.\end{aligned}$$

Example.—Prove that $\sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) = 0$ for all values of α.

$$\begin{aligned}&\sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right)\\ &= \sin\alpha + \left(\sin\alpha \cos\frac{2\pi}{3} + \cos\alpha \sin\frac{2\pi}{3}\right)\\ &\qquad + \left(\sin\alpha \cos\frac{4\pi}{3} + \cos\alpha \sin\frac{4\pi}{3}\right)\end{aligned}$$

$$= \sin\alpha + (\sin\alpha \cos 120° + \cos\alpha \sin 120°) + (\sin\alpha \cos 240° + \cos\alpha \sin 240°)$$

$$= \sin\alpha + \sin\alpha \times (-\tfrac{1}{2}) + \cos\alpha \times \frac{\sqrt{3}}{2} + \sin\alpha \times (-\tfrac{1}{2}) + \cos\alpha \times \left(-\frac{\sqrt{3}}{2}\right)$$

$$= \sin\alpha - \tfrac{1}{2}\sin\alpha + \frac{\sqrt{3}}{2}\cos\alpha - \tfrac{1}{2}\sin\alpha - \frac{\sqrt{3}}{2}\cos\alpha$$

$= 0$, for all values of α.

[For another method of proving this identity, see Exercise XXXV, Question 29.]

Example.—Expand $5 \sin(200t - 1{\cdot}6)$, the angle being in radians.

$$5\sin(200t - 1{\cdot}6) = 5(\sin 200t \,.\, \cos 1{\cdot}6 - \cos 200t \,.\, \sin 1{\cdot}6)$$

$$1{\cdot}6 \text{ radians} = \frac{1{\cdot}6 \times 180°}{\pi} = \frac{288°}{\pi} \simeq 91{\cdot}66° = 91°\ 40' \text{ to the nearest minute.}$$

$$\therefore 5\sin(200t - 1{\cdot}6) = 5(\sin 200t \cos 91°\ 40' - \cos 200t \sin 91°\ 40')$$
$$= 5\{\sin 200t \times (-0{\cdot}0291) - \cos 200t \times 0{\cdot}9996\}$$
$$= -0{\cdot}1455 \sin 200t - 4{\cdot}998 \cos 200t.$$

Example.—Show that $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.

$$\sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}$$
$$= \sin x \,.\, \frac{1}{\sqrt{2}} + \cos x \,.\, \frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(\sin x + \cos x).$$

Multiply both sides by $\sqrt{2}$

$$\therefore \sin x + \cos x = \sqrt{2} \sin \left(x + \frac{\pi}{4}\right).$$

This result is often useful and is worth remembering.

Formulæ for sin $2A$ and cos $2A$

$$\text{Sin } (A+B) = \sin A \cos B + \cos A \sin B$$

$$\therefore \sin 2A = \sin (A+A) = \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A$$

$$\therefore \sin 2A = 2 \sin A \cos A.$$

$$\text{Cos } (A+B) = \cos A \cos B - \sin A \sin B$$

$$\therefore \cos 2A = \cos (A+A) = \cos A \cos A - \sin A \sin A$$
$$= \cos^2 A - \sin^2 A.$$

If in this result we write $\sin^2 A = 1 - \cos^2 A$ or $\cos^2 A = 1 - \sin^2 A$ we get two other forms for cos $2A$; viz.:

$$\cos 2A = \cos^2 A - (1 - \cos^2 A) = 2 \cos^2 A - 1$$

$$\text{and } \cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2 \sin^2 A.$$

Thus

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2 \cos^2 A - 1$$
$$= 1 - 2 \sin^2 A.$$

The last two formulæ express cos $2A$ in terms of cos A alone or sin A alone, and are very useful on that account. From them we get:

$$2 \cos^2 A = 1 + \cos 2A \quad \text{and} \quad 2 \sin^2 A = 1 - \cos 2A$$

$$\therefore \cos^2 A = \tfrac{1}{2}(1 + \cos 2A) \quad \text{and} \quad \sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$$

All the formulæ printed in thick type above are important, and the student must either memorize them or be able to obtain them rapidly as we have done above.

Example.—Find the values of sin $22\frac{1}{2}°$ and cos $22\frac{1}{2}°$ without using tables, and compare them with the values given in the tables.

In the formula $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$ put $A = 22\frac{1}{2}°$.

$$\therefore \sin^2 22\tfrac{1}{2}° = \tfrac{1}{2}(1 - \cos 45°) = \tfrac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) = \tfrac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$= \tfrac{1}{2}(1 - 0{\cdot}7071) = \tfrac{1}{2}(0{\cdot}2929) = 0{\cdot}14645$$

$$\therefore \sin 22\tfrac{1}{2}° = \pm\sqrt{0{\cdot}14645} = \pm 0{\cdot}3827.$$

Since $22\frac{1}{2}°$ lies in the first quadrant its sine is positive

$$\therefore \sin 22\tfrac{1}{2}° = 0{\cdot}3827.$$

From the formula for $\cos^2 A$ in terms of cos $2A$,

$$\cos^2 22\tfrac{1}{2}° = \tfrac{1}{2}(1 + \cos 45°) = \tfrac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \tfrac{1}{2}(1 + 0{\cdot}7071)$$

$$= \tfrac{1}{2}(1{\cdot}7071) = 0{\cdot}85355$$

$$\therefore \cos 22\tfrac{1}{2}° = \pm\sqrt{0{\cdot}85355} = \pm 0{\cdot}9239.$$

Since $22\frac{1}{2}°$ lies in the first quadrant its cosine is positive

$$\therefore \cos 22\tfrac{1}{2}° = 0{\cdot}9239.$$

On looking up sin $22\frac{1}{2}°$ and cos $22\frac{1}{2}°$ in tables we find exactly the values obtained here, so that the values we have obtained are correct to four decimal places.

Example.—Find expressions for sin 3θ and cos 3θ in terms of sin θ and cos θ.

Sin $3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= (2 \sin\theta \cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta$, using the formulæ for sin 2θ and cos 2θ;

$= 2 \sin\theta \cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2 \sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2 \sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$= 3 \sin\theta - 4\sin^3\theta.$

$$\begin{aligned}\cos 3\theta &= \cos(2\theta+\theta)\\ &= \cos 2\theta \,.\, \cos\theta - \sin 2\theta \sin\theta\\ &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta\\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

The student will see that by continuing in this way we could express the sine and cosine of 4θ, 5θ, etc., in terms of $\sin\theta$ and $\cos\theta$.

Exercise XXXVI

1. Verify the formulæ for $\sin(A+B)$ and $\sin(A-B)$ when $A=70°$, $B=40°$.

2. Verify the formulæ for $\cos(A+B)$ and $\cos(A-B)$ when $A=45°$, $B=225°$.

Use the formulæ for $\sin(A\pm B)$ and $\cos(A\pm B)$ to verify the following identities:

3. $\sin(90°+\theta)\equiv\cos\theta$.

4. $\cos(90°+\theta)\equiv-\sin\theta$.

5. $\sin(180°+\theta)\equiv-\sin\theta$.

6. $\cos(180°+\theta)\equiv-\cos\theta$.

7. $\cos(270°+\theta)\equiv\sin\theta$.

8. $\sin(180°-\theta)\equiv\sin\theta$.

9. If $i=5\sin\left(300t+\frac{\pi}{6}\right)$, express i in the form

$a\sin 300t + b\cos 300t$, finding the values of a and b.

Find the value of i when $t=0{\cdot}02$ from both forms and verify that the two values are equal.

10. Express $20\sin(100\pi t - 0{\cdot}65)$ in the form

$a\sin 100\pi t - b\cos 100\pi t$, finding the values of a and b.

11. Prove that $\cos\left(x-\frac{\pi}{3}\right)+\cos\left(x+\frac{\pi}{3}\right)\equiv\cos x$.

12. If $\sin\alpha=\frac{3}{5}$, $\sin\beta=\frac{12}{13}$, and α, β are acute angles, find the value of $\sin(\alpha+\beta)$, without using trigonometric tables.

13. If θ is an acute angle and $\sin\theta=0{\cdot}8$, find the values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$, without using trigonometric tables.

Express each of the following as a single trigonometric ratio:

14. $\sin 20°\cos 50° + \cos 20°\sin 50°$.

15. Cos 85° cos 25° + sin 85° sin 25°.

16. Cos $\frac{\pi}{3}$ cos $\frac{\pi}{6}$ − sin $\frac{\pi}{3}$ sin $\frac{\pi}{6}$.

17. Sin 10° cos 40° − cos 10° sin 40°.

18. Sin 5θ cos 2θ − cos 5θ sin 2θ.

19. Cos $\frac{\alpha}{2}$ cos $\frac{3\alpha}{2}$ − sin $\frac{\alpha}{2}$ sin $\frac{3\alpha}{2}$.

20. $1 - 2\sin^2 15°$.

21. Sin $\frac{x}{2}$ cos $\frac{x}{2}$.

22. The reading of a wattmeter is

$$W = VI\{\cos(30° - \phi) + \cos(30° + \phi)\}\text{watts};$$

prove that $W = \sqrt{3}VI\cos\phi$.

23. Prove that $\sin x - \cos x \equiv \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$.

Find the values of the following ratios, without using trigonometric tables:

24. Sin 105°. **25.** Cos 15°. **26.** Sin $67\frac{1}{2}°$.

27. Verify that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ when $\theta = \frac{\pi}{6}$ radians.

28. By writing 4θ as $2 \times 2\theta$, prove that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

29. The force P required to pull a weight W up a rough plane, of inclination α and coefficient of friction μ, is given by $P = W(\sin\alpha + \mu\cos\alpha)$. If $\mu = \tan\lambda$, prove that $P = W\frac{\sin(\alpha + \lambda)}{\cos\lambda}$.

30. If the force in Question 29 is inclined at an angle θ to the plane, $P = W(\sin\alpha + \mu\cos\alpha)/(\cos\theta + \mu\sin\theta)$. Prove that $P = W\sin(\alpha + \lambda)/\cos(\theta - \lambda)$.

Prove the following identities:

31. Sin $(A + B)$. sin $(A - B) \equiv \sin^2 A - \sin^2 B$.

32. $(\sin\theta + \cos\theta)^2 \equiv 1 + \sin 2\theta$.

33. $\cos^4 x - \sin^4 x \equiv \cos 2x$.

34. $\cos 2\alpha - \cos 2\beta \equiv 2(\cos^2\alpha - \cos^2\beta) = 2(\sin^2\beta - \sin^2\alpha)$.

35. $\frac{1 - \cos\theta}{1 + \cos\theta} \equiv \tan^2\frac{\theta}{2}$. **36.** Cosec x + cot x = cot $\frac{x}{2}$.

37. The efficiency e of a certain screw-gearing is given by $e = \dfrac{1-\mu\tan\alpha}{1+\mu\cot\alpha}$. If $\mu = \tan\lambda$, prove that $e = \tan\alpha \cdot \cot(\alpha+\lambda)$.

Find the angles x between 0° and 360° which satisfy the following equations:

38. $\cos 2x = \sin x$. **39.** $\sin 2x = \sin x$.

40. $\sin 3x = \frac{1}{2}\sin x$. **41.** $3\sin(x+60°) = \cos x$.

42. Use the result of Question 23 to find the angles between 0° and 360° which satisfy the equation $\sin x - \cos x = 1$.

43. The distance x of a piston from one end of its stroke is given by:

$$x = r(1-\cos\theta) + \frac{r^2}{2l}(1-\cos 2\theta).$$

If $r = 6$ in., $l = 3$ ft., find the values of θ (between 0° and 360°) for which $x = 10$ in.

44. Solve the equation $\sin(\theta-40°) = 2\cos(\theta+25°)$, giving the values of θ between 0° and 360°.

To express $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta+\alpha)$

In the examples on p. 287 we have seen that

$$-0{\cdot}1455\sin 200t - 4{\cdot}998\cos 200t = 5\sin(200t - 1{\cdot}6)$$

and

$$\sin x + \cos x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right).$$

We can similarly transform any expression of the type $a\sin\theta + b\cos\theta$ into the form $r\sin(\theta+\alpha)$, where r and α have suitable values. For

$$\begin{aligned} r\sin(\theta+\alpha) &= r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \\ &= r\cos\alpha \cdot \sin\theta + r\sin\alpha \cdot \cos\theta. \end{aligned}$$

This is the same as $a\sin\theta + b\cos\theta$, if

$$r\cos\alpha = a \quad \text{and} \quad r\sin\alpha = b.$$

From these two equations we can find r and α, when a and b are given, as explained on p. 275.

Example.—Express $3\sin\theta + 2\cos\theta$ in the form $r\sin(\theta+\alpha)$.

$$\begin{aligned} r\sin(\theta+\alpha) &= r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \\ &= r\cos\alpha \cdot \sin\theta + r\sin\alpha \cdot \cos\theta. \end{aligned}$$

This is identical with $3 \sin\theta + 2\cos\theta$, if

$$r\cos\alpha = 3, \qquad r\sin\alpha = 2.$$

From Fig. 245, $r = \sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13} \simeq 3{\cdot}606.$

$$\alpha = \tan^{-1}\tfrac{2}{3} \text{ (acute angle)}$$
$$= \tan^{-1} 0{\cdot}6667 \simeq 33^\circ\ 42'$$

$$\therefore\ 3\sin\theta + 2\cos\theta \equiv 3{\cdot}606\sin(\theta + 33^\circ\ 42').$$

This shows that the result of adding the two " sine waves " $3\sin\theta$ and $2\cos\theta$ (the latter being a sine wave whose phase differs by a quarter of a period from that of the former) is a sine wave of amplitude 3·606, which leads the first of the two component waves by 33° 42′.

[Compare p. 247, where the curves have actually been added graphically. The resultant curve, shown in Fig. 205, has an amplitude of 3·6 (approx.) and it crosses the axis of θ from below to above when $\theta = -33^\circ$ (approx.). Thus its equation is $y = 3{\cdot}6\sin(\theta + 33^\circ)$, approximately.]

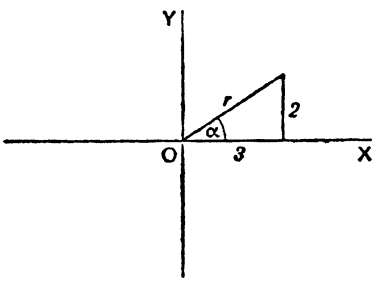


FIG. 245.

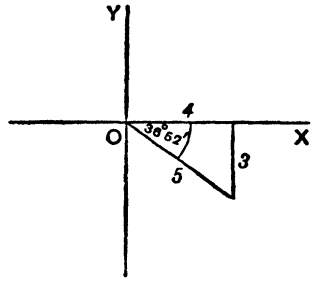


FIG. 246.

Example.—Convert the expression $4\sin\omega t - 3\cos\omega t$ into the form $r\sin(\omega t + \alpha)$.

$$r\sin(\omega t + \alpha) = r(\sin\omega t\cos\alpha + \cos\omega t\sin\alpha)$$
$$= r\cos\alpha\,.\,\sin\omega t + r\sin\alpha\,.\,\cos\omega t.$$

This is identical with $4\sin\omega t - 3\cos\omega t$ if

$$r\cos\alpha = 4, \qquad r\sin\alpha = -3.$$

From Fig. 246, $r = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$, and α is in the fourth quadrant.

The acute angle shown $= \tan^{-1} \frac{3}{4}$

$= \tan^{-1} 0{\cdot}75 = 36° 52'.$

Hence $\alpha = -36° 52'.$

[We could equally well write $\alpha = 360° - 36° 52' = 323° 8'$, but it is usual to take the *numerically smallest* value for α (i.e. the value between $-180°$ and $+180°$) in examples of this type.]

$$\therefore 4 \sin \omega t - 3 \cos \omega t = 5 \sin (\omega t - 36° 52').$$

In the general case, $a \sin \theta + b \cos \theta = r \sin (\theta + \alpha)$ where

$$r \cos \alpha = a \text{ and } r \sin \alpha = b.$$

We cannot draw the figure unless we know whether a and b are positive or negative ; but, squaring and adding, we have

$$r^2 (\cos^2 \alpha + \sin^2 \alpha) = a^2 + b^2$$

$$\therefore r^2 = a^2 + b^2$$

$\therefore r = \sqrt{a^2 + b^2}$ (taking the positive value for r) ;

dividing, we have $\dfrac{r \sin \alpha}{r \cos \alpha} = \dfrac{b}{a}$.

$$\therefore \tan \alpha = \frac{b}{a}, \qquad \therefore \alpha = \tan^{-1} \frac{b}{a}.$$

The particular value of $\tan^{-1} \dfrac{b}{a}$ to be taken has to be decided by the signs of a and b which fix the quadrant in which α lies. Thus we have the general formula :

$$a \sin \theta + b \cos \theta \equiv \sqrt{a^2 + b^2} \sin \left(\theta + \tan^{-1} \frac{b}{a} \right),$$

where the value of $\tan^{-1} \dfrac{b}{a}$ is determined by the quadrant in which α lies.

This shows that $y = a \sin \theta + b \cos \theta$ is a sine wave of amplitude $\sqrt{a^2 + b^2}$, having a lead of $\tan^{-1} \dfrac{b}{a}$ compared with

$y = a \sin \theta$. If $\tan^{-1} \frac{b}{a}$ is in the third or fourth quadrant it has a lag instead of a lead.

Example.—Solve the equation $15 \sin \theta + 8 \cos \theta = 10$.

$15 \sin \theta + 8 \cos \theta = r \sin (\theta + \alpha)$ where $r \cos \alpha = 15$, $r \sin \alpha = 8$.

Hence, from Fig. 247, $r = \sqrt{225 + 64} = \sqrt{289} = 17$

$$\alpha = \tan^{-1} \tfrac{8}{15} \text{ (in first quadrant)}$$
$$= \tan^{-1} 0{\cdot}5333$$
$$= 28° 4'$$

$\therefore 15 \sin \theta + 8 \cos \theta = 17 \sin (\theta + 28° 4')$.

The equation can therefore be written in the form

$17 \sin (\theta + 28° 4') = 10$.

$\therefore \sin (\theta + 28° 4') = \frac{10}{17}$

$= 0{\cdot}5882$

$\therefore \theta + 28° 4' = 36° 2' \pm n \,.\, 360°$,

or $143° 58' \pm n \,.\, 360°$,

where n is any integer,

$\therefore \theta = 7° 58' \pm n \,.\, 360°$

or $115° 54' \pm n \,.\, 360°$.

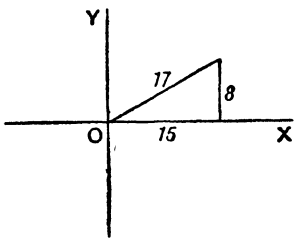


FIG. 247.

Example.—The displacement, x in., of a slide valve is given by $x = 1{\cdot}6 + 0{\cdot}3 \sin 20\pi t - 1{\cdot}1 \cos 20\pi t$, where t is the time in sec. and the angles are in radians. Find the greatest and least values of x and the times at which they occur.

$$0{\cdot}3 \sin 20\pi t - 1{\cdot}1 \cos 20\pi t = r \sin (20\pi t + \alpha),$$

where $r = \sqrt{(0{\cdot}3)^2 + (1{\cdot}1)^2} = \sqrt{0{\cdot}09 + 1{\cdot}21}$

$= \sqrt{1{\cdot}30} = 1{\cdot}14$

and $\alpha = -\left(\text{the acute angle whose tangent is } \frac{1{\cdot}1}{0{\cdot}3}\right)$

$= -74° 45'$.

Since the angle $20\pi t$ is in radians we require α in radians.

$$\alpha = -1{\cdot}3046 \text{ (radians)}.$$

$$\therefore\ 0{\cdot}3 \sin 20\pi t - 1{\cdot}1 \cos 20\pi t = 1{\cdot}14 \sin (20\pi t - 1{\cdot}3046).$$

$$\therefore\ x = 1{\cdot}6 + 1{\cdot}14 \sin (20\pi t - 1{\cdot}3046).$$

As t varies, $\sin (20\pi t - 1{\cdot}3046)$ oscillates between -1 and $+1$, and therefore x oscillates between $1{\cdot}6 - 1{\cdot}14$ and $1{\cdot}6 + 1{\cdot}14$, i.e. between 0·46 and 2·74. These are the least and greatest values of x.

The greatest value of x occurs when $\sin (20\pi t - 1{\cdot}3046) = +1$,

i.e. when $20\pi t - 1{\cdot}3046 = \frac{\pi}{2} \pm n \,.\, 2\pi$, where n is any integer,

i.e. when $20\pi t = 2{\cdot}8754 \pm n \,.\, 2\pi$,

i.e. when $t = \frac{2{\cdot}8754}{20\pi} \pm \frac{1}{10}n$

$$= 0{\cdot}0458 \pm 0{\cdot}1n.$$

Since t denotes time, we take only positive values for t.

Hence x attains its greatest value, viz., 2·74, when

$$t = 0{\cdot}0458,\ 0{\cdot}1458,\ 0{\cdot}2458, \text{ etc.}$$

Similarly, x attains its least value, viz. 0·46, when $\sin (20\pi t - 1{\cdot}3046) = -1$,

i.e. when $20\pi t - 1{\cdot}3046 = -\frac{\pi}{2} \pm n2\pi$,

i.e. when $20\pi t = -0{\cdot}2662 \pm n2\pi$

i.e. when $t = -\frac{0{\cdot}2662}{20\pi} \pm \frac{1}{10}n$

$$= -0{\cdot}0042 \pm 0{\cdot}1n,$$

i.e. when $t = 0{\cdot}0958,\ 0{\cdot}1958,\ 0{\cdot}2958,$ etc.

As we should expect, since the graph of x against t is a sine wave, these values are mid-way between the values of t at which x has its greatest value.

Exercise XXXVII

Convert the following expressions into the form $r \sin(\theta + \alpha)$:

1. $7 \sin\theta + 4 \cos\theta$.
2. $1{\cdot}5 \sin\theta + 2{\cdot}8 \cos\theta$.
3. $5 \sin\theta - 12 \cos\theta$.
4. $100 \sin\theta - 82 \cos\theta$.
5. $4{\cdot}62 \cos\theta + 1{\cdot}93 \sin\theta$.

6. Express $11 \sin 300t - 25 \cos 300t$ in the form $r \sin(300t - \alpha)$, giving α in radians.

7. What are the greatest and least values of $a \sin\theta + b \cos\theta$?

8. What are the greatest and least values of $3 \sin\theta - 4 \cos\theta$, and at what values of θ (between 0° and 360°) do they occur?

9. Express $1{\cdot}6 \sin 2\pi ft + 0{\cdot}3 \cos 2\pi ft$ in the form $r \sin(2\pi ft + \theta)$, giving θ in radians.

10. Convert $2 \cos\left(\theta - \frac{\pi}{6}\right)$ to the form $r \sin(\theta + \alpha)$.

11. Express $15 \sin 60\pi t + 8 \cos 60\pi t$ in the form $r \sin(60\pi t + \alpha)$, giving the value of α in radians.

Make a sketch of the graph of the function against t for one complete cycle from $t = 0$, indicating the period and the amplitude and the value of the function at the beginning and end of the interval.

Find the angles θ between 0° and 360° which satisfy the following equations:

12. $4{\cdot}2 \sin\theta + 5{\cdot}5 \cos\theta = 2{\cdot}7$.
13. $9 \sin\theta - 16 \cos\theta = -12$.
14. $\text{Sin } \frac{1}{2}\theta + \cos\frac{1}{2}\theta = \frac{1}{2}$.
15. $\text{Sec } \theta = 1 + 2 \tan\theta$.

16. The turning moment T tons ft. on the crank shaft of an engine is given by $T = 6 + 2{\cdot}5 \sin 2\theta - 3{\cdot}8 \cos 2\theta$. Find the values of θ between 0° and 180° for which $T = 3$.

17. What is the greatest value of T in Question 16, and for what values of θ does it occur?

18. Express $11{\cdot}1 \sin 3\omega t - 8{\cdot}5 \cos 3\omega t$ in the form $r \sin(3\omega t + \alpha)$, all angles being in radians; thence solve the equation $11{\cdot}1 \sin 3\omega t - 8{\cdot}5 \cos 3\omega t = 10$ for t, when $\omega = 60$.

19. For what values of θ between 0° and 360° is $3 \sin\theta + 4 \cos\theta$ positive?

20. The force P required to pull a weight W up a rough plane of inclination α and coefficient of friction μ is given by

$$P = W(\sin\alpha + \mu \cos\alpha).$$

Find the inclination of the steepest plane up which a man exerting a force of 120 lb. wt. can pull a weight of 200 lb., the coefficient of friction being 0·4.

21. The potential difference, v volts, which must be applied to send a current I sin ωt ampères through a circuit of resistance R ohms and inductance L henrys is given by

$$v = I\,(R \sin \omega t + L\omega \cos \omega t).$$

Express v in the form $v = IZ \sin(\omega t + \phi)$.

[Z is called the *impedance* (or "apparent resistance") of the circuit.]

Find the value of Z and ϕ if $R = 8$, $L = 0{\cdot}03$, $\omega = 100\pi$.

22. What are the greatest and least values of each of the following:

(i) $A + B \sin(\omega t + \alpha)$; (ii) $a \sin\theta + b \cos\theta + c$;

(iii) $A + B \sin(\omega t + \alpha) + C \cos(\omega t + \alpha)$?

Approximate values of sin θ, cos θ and tan θ when θ is small

In this paragraph the angles are understood to be measured in *radians*; that is, θ is a number and sin θ means sin (θ radians), cos θ means cos (θ radians), and so on.

The value of the ratio $\dfrac{\sin\theta}{\theta}$ when θ is small is important for later work and so we shall draw up a table to see how this ratio behaves as θ decreases towards zero.

θ	..	1	0·5	0·2	0·1
sin θ	..	0·84147	0·47943	0·19867	0·099831
$\dfrac{\sin\theta}{\theta}$	..	0·84147	0·95886	0·99335	0·99831

θ	..	0·08	0·06	0·04	0·02
sin θ	..	0·0799138	0·0599645	0·0399913	0·0199972
$\dfrac{\sin\theta}{\theta}$	..	0·99892	0·99941	0·99978	0·99986

It will be noticed that as we made θ smaller and smaller we used first five-figure and then seven-figure tables, the reason being that the numerator and denominator in the fraction $\dfrac{\sin\theta}{\theta}$ were so nearly equal that greater accuracy became necessary as we proceeded.

It is seen from the table above that as θ decreases, the ratio $\frac{\sin\theta}{\theta}$ becomes more and more nearly equal to 1. The accuracy of our tables prevents us from finding the ratio when θ is less than 0·02 with a satisfactory degree of precision, but it seems reasonable to conclude that as θ approaches indefinitely closely * to 0, the ratio $\frac{\sin\theta}{\theta}$ approaches indefinitely closely to 1.

We can, in fact, *prove* by geometry that this is so. The proof is given in Part III.

We usually write the above result in the form

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 ;$$

the left-hand side is read as " the limit, as θ approaches 0, of $\frac{\sin\theta}{\theta}$."

When θ is very small, $\sin\theta$ is therefore very nearly equal to θ, the radian measure of the angle.

This is very useful in finding the sines of small angles and is often used in calculations arising in surveying.

Example.—Find the value of sin 1° 42′ 35″.

$$1^\circ = \frac{\pi}{180} \text{ radians.}$$

$$\therefore\ 1^\circ\ 42'\ 35'' = \left(1 + \frac{42}{60} + \frac{35}{3600}\right)^\circ = \frac{6155^\circ}{3600}$$

$$= \frac{6155}{3600} \times \frac{\pi}{180} \text{ radians} = \frac{6155 \times 3{\cdot}14159}{3600 \times 180} \text{ radians}$$

$$= 0{\cdot}02984 \text{ radians.}$$

$$\therefore \sin 1^\circ\ 42'\ 35'' \simeq 0{\cdot}02984.$$

* $\frac{\sin\theta}{\theta}$ cannot be evaluated when θ is *exactly* equal to 0, since $\frac{\sin 0}{0} = \frac{0}{0}$, which is meaningless.

This approximation is actually correct to five decimal places; by taking a more accurate value for π we could, if required, evaluate sin 1° 42′ 35″ still more accurately.

If we tabulate the values of the ratio $\frac{\tan\theta}{\theta}$ for values of θ decreasing to 0 we obtain the following:

θ	1	0·5	0·2	0·1
$\tan\theta$ ..	1·55741	0·54631	0·20260	0·1003323
$\frac{\tan\theta}{\theta}$ ..	1·55741	1·09262	1·01300	1·00332

θ ..	0·08	0·06	0·04	0·02
$\tan\theta$ ..	0·0801702	0·0600726	0·0400233	0·0200012
$\frac{\tan\theta}{\theta}$ ..	1·00213	1·00121	1·00058	1·00006

It is natural to conclude that as θ approaches indefinitely closely * to 0, the ratio $\frac{\tan\theta}{\theta}$ approaches indefinitely closely to 1.

This also can be *proved* to be true (see Part III).

Thus $$\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1,$$

the angle being measured in radians.

Hence when θ is very small, $\tan\theta$ is approximately equal to θ.

The student will notice from the tabulated values above that when θ is very small, $\sin\theta$ is *slightly less* than θ, while $\tan\theta$ is *slightly greater* than θ.

When θ is very small, $\cos\theta \simeq 1$. We can obtain a better approximation however by using the approximation for $\sin\theta$.

For $\cos\theta=\cos\left(2\times\frac{\theta}{2}\right)=1-2\sin^2\frac{\theta}{2}$.

* $\frac{\tan\theta}{\theta}$ cannot be evaluated when θ is *exactly* equal to 0, since $\frac{\tan 0}{0}=\frac{0}{0}$, which is meaningless.

If θ is very small, so also is $\frac{\theta}{2}$, and therefore $\sin\frac{\theta}{2} \simeq \frac{\theta}{2}$.

$$\therefore \cos\theta \simeq 1-2\left(\frac{\theta}{2}\right)^2 = 1-2\left(\frac{\theta^2}{4}\right) = 1-\frac{\theta^2}{2}.$$

Hence we have the following approximations when θ is very small:

$$\sin\theta \simeq \theta, \quad \cos\theta \simeq 1-\frac{\theta^2}{2}, \quad \tan\theta \simeq \theta.$$

It is possible to obtain better approximations for $\sin\theta$ and $\tan\theta$. It can be proved (by methods which are too difficult to explain here) that $\sin\theta \simeq \theta - \frac{\theta^3}{6}$ and $\tan\theta \simeq \theta + \frac{\theta^3}{3}$.

Example.—Find the angle subtended by a circular target 5 ft. in diameter at a distance of half a mile.

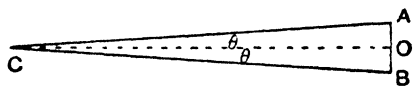


FIG. 248.

In Fig. 248, AB is a diameter of the target and O its centre.
$CO = \frac{1}{2}$ ml. $= 2640$ ft.

If $\widehat{ACO} = \theta$,

$$\tan\theta = \frac{AO}{CO} = \frac{2\frac{1}{2}}{2640} = \frac{5}{5280}.$$

Since θ is very small, $\tan\theta \simeq \theta$,

$$\therefore \theta \simeq \frac{5}{5280} \text{ radians}$$
$$= \frac{5}{5280} \times \frac{180^c}{\pi} = \frac{15^\circ}{88\pi}$$
$$= 0{\cdot}05426^\circ$$
$$= 3{\cdot}2556'$$
$$= 3'\ 15{\cdot}3''.$$

$$\therefore \text{Required angle} = \widehat{ACB} = 2\theta = 6'\ 30{\cdot}6''$$
$$\simeq 6'\ 31''.$$

Exercise XXXVIII

1. Show that the approximation $\sin\theta \simeq \theta$ is correct to 3 significant figures if the angle is less than 4° 20′, and correct to 4 significant figures if it is less than 1° 57′.

2. If x is small show that $\sin x' \simeq 0{\cdot}00029x$, and evaluate sin 24′.

3. Find the value of cos 89° 40′ without tables.

4. If the distance of the Sun from the Earth is 92,000,000 miles and it subtends an angle of 32′ at the Earth, find the diameter of the Sun.

5. The angle of depression of a ship at sea when observed from the top of a cliff 240 ft. high is 41′. Find the distance of the ship from the foot of the cliff. (Neglect the curvature of the Earth.)

6. If an arc AB of a circle subtends an angle of 20° at the centre, find from tables the ratio length of chord AB : length of arc AB.

7. If an arc AB of a circle of radius 1 in. subtends an angle of 10° at the centre, find the difference of the lengths arc AB − chord AB. [Take $\pi = 3{\cdot}14159$, $\sin 5° = 0{\cdot}087156$.]

8. Show that, if α is small, $\sin(\theta+\alpha) \simeq \sin\theta + \alpha\cos\theta$, the angles being in radians.

Deduce that $\sin\left(\frac{\pi}{4}+\alpha\right) \simeq \frac{1}{\sqrt{2}}(1+\alpha)$, and hence find the value of sin 46° to three significant figures without tables. [Take $\pi = 3{\cdot}1416$, $\sqrt{2} = 1{\cdot}4142$.]

9. Show that a better approximation to $\sin\left(\frac{\pi}{4}+\alpha\right)$ is $\frac{1}{\sqrt{2}}\left(1+\alpha-\frac{\alpha^2}{2}\right)$, and find the value of sin 46° to five significant figures by using this approximation. [Take $\pi = 3{\cdot}14159$, $\sqrt{2} = 1{\cdot}41421$.]

Miscellaneous Exercise XXXIX

(Harder examples)

Prove the following identities :

1. $\cot\theta - \tan\theta \equiv 2\cot 2\theta$.

2. $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} \equiv \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$.

3. $\text{Sin}^2\,\alpha+\sin^2\left(\alpha+\frac{2\pi}{3}\right)+\sin^2\left(\alpha+\frac{4\pi}{3}\right)\equiv\frac{3}{2}$.

4. If $\cos 2\theta=0{\cdot}4$ and θ is an acute angle, find $\sin\theta$, $\cos\theta$ and $\tan\theta$, without the aid of tables.

5. When a rod rests with its ends on two smooth planes of inclinations α and β, its inclination θ to the horizontal is given by the equation

$$\sin(\alpha+\beta)\,.\,\cos\theta=2\sin\alpha\,.\,\cos(\beta-\theta).$$

Prove that $$\tan\theta=\frac{\sin(\beta-\alpha)}{2\sin\alpha\sin\beta}.$$

6. The acceleration of a piston is $r\omega^2(\cos\theta\pm\frac{r}{l}\cos 2\theta)$, the + and − signs corresponding to out-stroke and in-stroke respectively. If $\frac{r}{l}=\frac{1}{5}$, find the values of the crank angle θ for which the acceleration of the piston is zero.

7. The equation $\frac{\sin(\beta+\theta)}{\sin(\beta-\theta)}=\frac{\cos(30°-\phi)}{\cos(30°+\phi)}$ occurs in the design of a power-factor meter for alternating currents. Show that $\tan\theta=\frac{1}{\sqrt{3}}\tan\beta\tan\phi$.

8. Express $3\cos\omega t-4\cos(\omega t-60°)$ in the form

$$a\sin\omega t+b\cos\omega t,$$

finding the values of a and b, and thence convert the expression into the form $c\sin(\omega t+\alpha)$.

9. Find numbers a and b which make

$a\sin(\theta-30°)+b\sin(\theta+60°)$ identically equal to $2\sin\theta$.

10. If $\sin\theta+\cos\phi=p$ and $\cos\theta-\sin\phi=q$, show that $\sin(\theta-\phi)=\frac{1}{2}(p^2+q^2)-1$.

11. Find all the values of x between 0 and 10 which satisfy the equation $6\sin\frac{\pi x}{4}+8\cos\frac{\pi x}{4}=5$.

12. Prove that:

$$\lim_{h\to 0}\left\{\frac{\sin(x+h)-\sin(x-h)}{2h}\right\}=\cos x,$$

the angles being measured in radians.

CHAPTER XIV

TRIANGLE FORMULÆ

SOLUTION OF TRIANGLES

Relations between the sides and angles of a triangle

The three sides and the three angles of a triangle are often called its six " parts " or " elements."

In order to construct a triangle we do not need to know all the six parts; for example, it will be sufficient if we know the lengths of the three sides. The three angles are therefore determined by the lengths of the sides, and hence it must be possible to find a formula expressing the angles of any triangle in terms of the sides. There are also other relations, the most important of which are given below.

If ABC is any triangle, the lengths of the sides BC, CA, AB are usually denoted by a, b, c, respectively. [NOTE.—a is the side opposite the angle A, b opposite B and c opposite C.]

The Sine Rule

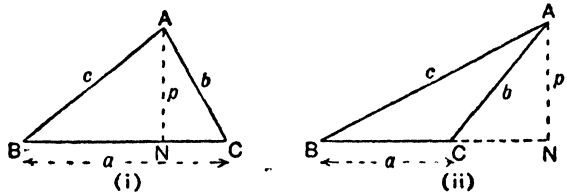


FIG. 249.

In Fig. (i) the triangle is acute-angled, in Fig. (ii) it is obtuse-angled. In each case AN is perpendicular to BC (or BC produced).

In Fig. (i):

$$\frac{p}{c} = \sin B$$

In Fig. (ii):

$$\frac{p}{c} = \sin B$$

$\therefore p = c \sin B$ $\qquad$ $\therefore p = c \sin B$

$\frac{p}{b} = \sin C$ $\qquad$ $\frac{p}{b} = \sin (180° - C) = \sin C$

$\therefore p = b \sin C$ $\qquad$ $\therefore p = b \sin C.$

Hence in each case,

$$b \sin C = c \sin B$$

$$\therefore \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, by drawing the perpendicular from C to AB, it can be shown that $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

We may express the *sine rule* in words thus :

The sides of a triangle are proportional to the sines of the opposite angles.

Example.—If, in a triangle ABC, $BC = 4{\cdot}3$ in., $\angle A = 65°$, $\angle B = 36°$, find the length of the side AC.

Here we are given that $a = 4{\cdot}3$ in. and we want to find b.

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\therefore b = \frac{a \sin B}{\sin A}$$

$$= \frac{4{\cdot}3 \sin 36°}{\sin 65°}$$

$$= 2{\cdot}789$$

$$\therefore AC \simeq 2{\cdot}79 \text{ in.}$$

No.	*Log.*
4·3	0·6335
sin 36°	$\bar{1}$·7692
	0·4027
sin 65°	$\bar{1}$·9573
2·789	0·4454

Example.—P, Q are two ends of a surveyor's base-line, 1200 yd. long, and a landmark R is observed from P and Q.

The angles QPR and PQR are found to be 28° and 105° 20′ respectively. What is the distance of R from P, and what is its shortest distance from the line PQ ?

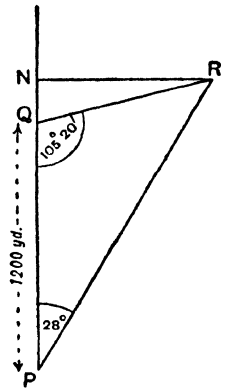


Fig. 250.

In the triangle PQR,

$$\begin{aligned} P\hat{R}Q &= 180° - (28° + 105° 20') \\ &= 180° - 133° 20' \\ &= 46° 40' \end{aligned}$$

$$\text{But } \frac{PR}{\sin P\hat{Q}R} = \frac{PQ}{\sin P\hat{R}Q}$$

$$\therefore \frac{PR}{\sin 105° 20'} = \frac{1200}{\sin 46° 40'}$$

$$\therefore PR = \frac{1200 \sin 105° 20'}{\sin 46° 40'}$$

$$= \frac{1200 \sin 74° 40'}{\sin 46° 40'}$$

$$\simeq 1591 \text{ yd.}$$

	No.	*Log.*
	1200	3·0792
	sin 74° 40′	$\bar{1}$·9842
		3·0634
−	sin 46° 40′	$\bar{1}$·8618
	1591	3·2016

Draw RN perpendicular to PQ produced.

In $\triangle PRN$,

	No.	*Log.*
$\frac{RN}{PR} = \sin 28°$	1591	3·2016
$\therefore RN = PR \sin 28° = 1591 \sin 28°$	$\sin 28°$	$\bar{1}$·6716
$\simeq 746{\cdot}7$ yd.	746·7	2·8732

The Cosine Rule

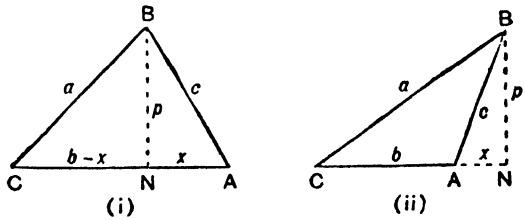


FIG. 251.

In Fig. (i), the angle A is acute; in Fig. (ii) the angle A is obtuse.

Draw BN perpendicular to CA, or CA produced.

Let $AN = x$. Then,

in Fig. (i) $CN = b - x$.

From Pythagoras' theorem,

$$a^2 = (b-x)^2 + p^2$$
$$c^2 = x^2 + p^2$$
$$\therefore a^2 - c^2 = (b-x)^2 - x^2$$
$$= b^2 - 2bx$$

But $\frac{x}{c} = \cos A$

$$\therefore x = c \cos A$$
$$\therefore a^2 - c^2 = b^2 - 2bc \cos A$$

in Fig. (ii) $CN = b + x$.

From Pythagoras' theorem,

$$a^2 = (b+x)^2 + p^2$$
$$c^2 = x^2 + p^2$$
$$\therefore a^2 - c^2 = (b+x)^2 - x^2$$
$$= b^2 + 2bx.$$

But $x = c \cos (180° - A)$

$$\therefore x = -c \cos A$$
$$\therefore a^2 - c^2 = b^2 - 2bc \cos A.$$

Hence in each case,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

In a similar manner we can prove that,

$$b^2 = c^2 + a^2 - 2ca \cos B$$

and

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

From the formula $a^2 = b^2 + c^2 - 2bc \cos A$, we have,

$$2bc \cos A = b^2 + c^2 - a^2$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Similarly, $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ and $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$.

[Note that the angle A is contained between the sides b and c, the angle B between the sides c and a, the angle C between the sides a and b ; thus the denominator in the formula for cos A is twice the product of the sides containing the angle.]

The cosine formula enables us to find the angles of a triangle when its sides are given.

Example.—Find the largest angle of the triangle whose sides are 8 cm., 11 cm., 10 cm.

The largest angle is the one opposite the longest side, that is the angle opposite the side of length 11 cm.

Let $a = 8$, $b = 11$, $c = 10$.

The required angle is B.

$$\text{Cos } B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$= \frac{10^2 + 8^2 - 11^2}{2 \times 10 \times 8} = \frac{100 + 64 - 121}{160} = \frac{43}{160}$$

$$= 0{\cdot}2688$$

$$\therefore B = 74^\circ\ 24'.$$

Example.—In the crane represented in Fig. 252, the upright AB is 10 ft., the tie-rod is 22 ft. long, and the angle BAC is 115°. Find the length of the jib BC.

Here $c = 10$, $b = 22$, $\widehat{A} = 115°$.

$\therefore a^2 = b^2 + c^2 - 2bc \cos A$

$= 484 + 100 - 440 \cos 115°$

$= 584 + 440 \cos 65°$, since

$\cos 115° = -\cos 65°$.

$= 584 + (440 \times 0{\cdot}4226)$

$= 584 + 185{\cdot}944$

$= 769{\cdot}944$

$\therefore a \simeq 27{\cdot}75$

$\therefore BC \simeq 27{\cdot}75$ ft.

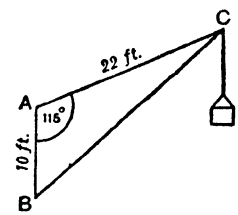


FIG. 252.

Example.—The crank of a steam-engine is 1 ft. long and the connecting-rod is 4 ft. 6 in. long. If the crank makes an angle of 55° with the line of stroke of the piston, find (i) the angle between the connecting-rod and the line of stroke, (ii) the distance of the crosshead from the end of its stroke.

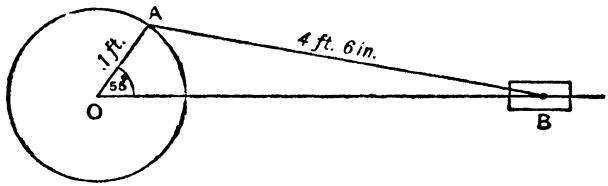


FIG. 253.

We can find $A\widehat{B}O$ from the sine rule:

$$\frac{\sin A\widehat{B}O}{AO} = \frac{\sin A\widehat{O}B}{AB}$$

$$\therefore \frac{\sin A\widehat{B}O}{1} = \frac{\sin 55°}{4{\cdot}5}$$

$$\therefore \sin A\widehat{B}O = \frac{0{\cdot}8192}{4{\cdot}5}$$

$$= 0{\cdot}1820$$

$\therefore A\widehat{B}O = 10° 29'$, since, from the figure, it must be acute.

Hence $O\widehat{A}B = 180° - (A\widehat{O}B + A\widehat{B}O) = 180° - 65° 29' = 114° 31'$.

We can now find OB either from the cosine rule or from the sine rule.

Using the cosine rule,

$$\begin{aligned} OB^2 &= OA^2 + AB^2 - 2OA \,.\, AB \cos \widehat{OAB} \\ &= 1 + (4{\cdot}5)^2 - 9 \cos 114° 31' \\ &= 1 + 20{\cdot}25 + 9 \cos 65° 29' \\ &= 1 + 20{\cdot}25 + (9 \times 0{\cdot}4150) \\ &= 1 + 20{\cdot}25 + 3{\cdot}7350 \\ &= 24{\cdot}985 \end{aligned}$$

$$\therefore OB \simeq 5{\cdot}00 \text{ ft.}$$

When B is at the end of its stroke, $OB = OA + AB = 5$ ft. 6 in.

Hence in the given position, B is 6 in. from the end of its stroke.

Resultant of two vectors

If AB, AC are two vectors of magnitudes P, Q, inclined to each other at an angle α, their resultant AD is obtained by the parallelogram law. Suppose its magnitude is R and that its direction makes an angle θ with that of AB.

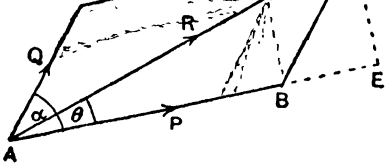


FIG. 254.

In $\triangle ABD$, $AD^2 = AB^2 + BD^2 - 2AB \,.\, BD \cos \widehat{ABD}$.

But $BD = AC = Q$ and $\widehat{ABD} = 180° - \widehat{BAC} = 180° - \alpha$.

$$\begin{aligned} \therefore R^2 &= P^2 + Q^2 - 2PQ \cos (180° - \alpha) \\ &= P^2 + Q^2 + 2PQ \cos \alpha \end{aligned}$$

$$\therefore R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}.$$

Draw the perpendicular, DE, from D to AB produced.

$$DE = BD \sin \widehat{EBD} = Q \sin \alpha$$

$$BE = BD \cos \widehat{EBD} = Q \cos \alpha$$

$$\therefore AE = AB + BE = P + Q \cos \alpha$$

$$\therefore \tan \theta = \frac{DE}{AE} = \frac{Q \sin \alpha}{P + Q \cos \alpha}.$$

Fig. 254 has been drawn for the case in which α is an acute angle. The student should verify that the formulæ for R and $\tan \theta$ are still true if α is an obtuse angle.

Area of a triangle

The area of a triangle is often denoted by $\triangle$ (Greek capital "delta").

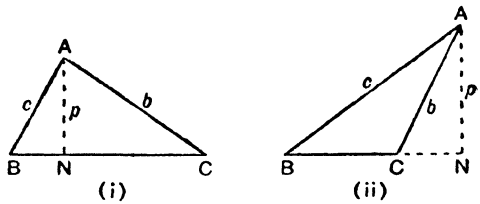


FIG. 255.

In each of Figs. (i) and (ii), $\triangle = \frac{1}{2}BC \cdot AN = \frac{1}{2}a \cdot p$.

In Fig. (i), $\frac{p}{b} = \sin C$.

In Fig. (ii), $\frac{p}{b} = \sin \widehat{ACN} = \sin (180° - C) = \sin C$.

Hence in each case, $p = b \sin C$.

$$\therefore \triangle = \tfrac{1}{2}ab \sin C$$

Similarly it can be proved that $\triangle = \frac{1}{2}bc \sin A$ and $\triangle = \frac{1}{2}ca \sin B$.

Thus, the area of a triangle = ½ product of any two sides × sine of angle between them.

Area of a regular polygon

A regular polygon is one whose sides are all equal and whose angles are all equal.

A circle can be drawn to circumscribe a regular polygon, i.e. to pass through all its vertices. [Fig. 256 shows a regular

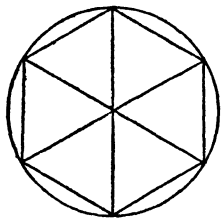


FIG. 256.

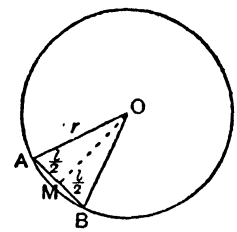


FIG. 257.

hexagon and its circumscribing circle.] By joining each of the vertices to the centre of the circle, the polygon is divided up into a number of equal triangles, and the area of the polygon is the sum of the areas of those triangles.

Suppose the polygon has n sides, each of length l. Let O be the centre of its circumscribing circle and OAB (Fig. 257) one of the n triangles into which the polygon is divided.

Let M be the mid-point of AB; then OM is perpendicular to AB.

$$\widehat{AOB} = \frac{2\pi}{n} \text{ radians} = \frac{360^\circ}{n}.$$

$$\therefore \widehat{AOM} = \frac{\pi}{n} \text{ radians} = \frac{180^\circ}{n}.$$

$$AM = \frac{l}{2}. \quad \therefore OM = \frac{l}{2} \cot \frac{\pi}{n}.$$

$$\therefore \text{Area } \triangle AOB = \tfrac{1}{2} AB \times OM = \frac{l}{2} \times \frac{l}{2} \cot \frac{\pi}{n} = \tfrac{1}{4} l^2 \cot \frac{\pi}{n}.$$

$$\therefore \text{Area of polygon} = n \times \text{area } \triangle AOB = \underline{\tfrac{1}{4} n l^2 \cot \frac{\pi}{n}}.$$

If the radius of the circumscribing circle is r, we can express the area as follows :

$$\text{Area of } \triangle AOB = \tfrac{1}{2}OA \,.\, OB \sin \widehat{AOB} = \tfrac{1}{2}r^2 \sin \frac{2\pi}{n}.$$

$$\therefore \text{ Area of polygon} = n \times \text{area } \triangle AOB = \tfrac{1}{2}nr^2 \sin \frac{2\pi}{n}.$$

Area of a segment of a circle

By a *segment* of a circle is meant the part cut off by a chord. Any chord such as AB in Fig. 258 divides a circle into two segments, a " major segment " ADB and a " minor segment " ACB. In Fig. 258 the minor segment cut off by the chord AB is shown shaded.

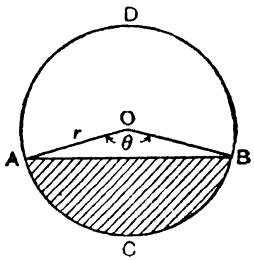


FIG. 258.

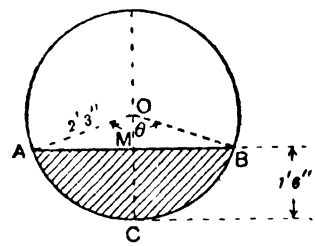


FIG. 259.

Let the arc of the segment subtend an angle θ radians at the centre of the circle ; let the radius of the circle be r.

Area of segment ACB

$$= \text{area of sector } OACB - \text{area of } \triangle OAB$$
$$= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}OA \,.\, OB \sin\theta$$
$$= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2 \sin\theta$$
$$= \tfrac{1}{2}r^2(\theta - \sin\theta).$$

Exercise. —Verify that this formula is true also for a major segment, where θ is the angle (in radians) subtended by the arc of the major segment. [In Fig. 258 the angle subtended by the major segment ADB is the reflex angle AOB.]

Example.—A cylindrical boiler 10 ft. long and 4 ft. 6 in. in diameter, with its axis horizontal, contains water to a depth of 1 ft. 6 in. Find the volume of the water.

A cross-section of the boiler is shown in Fig. 259.

$$OM = OC - MC$$
$$= 2' 3'' - 1' 6'' = 9''.$$

If $\widehat{AOB} = \theta$, $\widehat{AOM} = \frac{\theta}{2}$.

$$\therefore \cos\frac{\theta}{2} = \frac{OM}{OA} = \frac{\frac{3}{4}}{2\frac{1}{4}} = \frac{3}{9} = \frac{1}{3} = 0{\cdot}3333$$

$$\therefore \frac{\theta}{2} = 70° 32' = 1{\cdot}2310 \text{ radians}$$

$$\therefore \theta = 2{\cdot}462 \text{ radians}$$

$\therefore$ Area of sector $OACB = \frac{1}{2}r^2\theta = \frac{1}{2} \times (2{\cdot}25)^2 \times 2{\cdot}462$
$= 6{\cdot}233$ sq. ft.

Area of $\triangle OAB$
$= \frac{1}{2} . OA . OB \sin\theta$,
$= \frac{1}{2} \times (2{\cdot}25)^2 \times \sin 141° 4'$
$= \frac{1}{2} \times (2{\cdot}25)^2 \times \sin 38° 56'$
$= 1{\cdot}591$ sq. ft.

	No.	*Log.*
	2·25	0·3522
	$(2{\cdot}25)^2$	0·7044
	sin 38° 56′	1·7982
		0·5026
	2	0·3010
	1·591	0·2016

$\therefore$ Area of cross-section of water (shaded area)
$= 6{\cdot}233 - 1{\cdot}591 = 4{\cdot}642$ sq. ft.

$\therefore$ Volume of water $= 10 \times 4{\cdot}642$
$= 46{\cdot}42$ cu. ft.

Formula for the area of a triangle in terms of the sides

We have seen that $\triangle = \frac{1}{2}ab \sin C$.

But $\sin^2 C = 1 - \cos^2 C$
$= (1 + \cos C)(1 - \cos C)$

$$= \left(1 + \frac{a^2+b^2-c^2}{2ab}\right)\left(1 - \frac{a^2+b^2-c^2}{2ab}\right)$$

$$= \left(\frac{a^2+b^2+2ab-c^2}{2ab}\right)\left(\frac{c^2-a^2-b^2+2ab}{2ab}\right)$$

$$= \frac{\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}}{4a^2b^2}$$

$$= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2b^2}$$

It is usual to write $a+b+c \equiv 2s$, so that s is the semi-perimeter of the triangle.

Then $a+b-c = 2s-2c = 2(s-c)$, and similarly

$$a-b+c = 2(s-b) \quad \text{and} \quad b+c-a = 2(s-a).$$

Hence $$\sin^2 C = \frac{16s(s-a)(s-b)(s-c)}{4a^2b^2}$$

$$\therefore \sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)},$$

and, substituting in the formula $\triangle = \frac{1}{2}ab \sin C$, we have

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}$$

Example.—The field $ABCD$ represented in Fig. 260 has the following dimensions: $AB = 325$ yd., $BC = 100$ yd., $CD = 240$ yd., $DA = 210$ yd.; $\angle A = 70°$. Find its area.

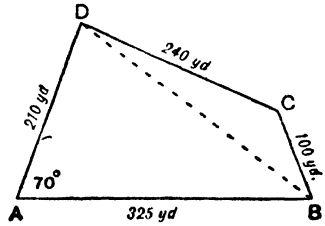


FIG. 260.

Divide the area up into two triangles by joining BD.

Area $\triangle ABD$
$= \frac{1}{2} \times 210 \times 325 \times \sin 70°$
$= 105 \times 325 \times \sin 70°$
$\simeq 32{,}070$ sq. yd.

No.	Log.
105	2·0212
325	2·5119
sin 70°	$\bar{1}$·9730
32070	4·5061

To find the area of $\triangle BCD$ we need to know either the angle BCD or the side BD.

Applying the cosine formula to $\triangle ABD$,

$BD^2 = 210^2 + 325^2 - 2 \times 210 \times 325 \cos 70°$
$= 44100 + 105625 - 46690$
$= 103035$
$\therefore BD \simeq 321$ yd.

420	2·6232
325	2·5119
cos 70°	$\bar{1}$·5341
46690	4·6692

Denoting the sides of $\triangle BCD$ by a, b, c,
$a = 240$, $b = 321$, $c = 100$.
$\therefore s = \frac{1}{2}(a+b+c) = (240 + 321 + 100)$
$= \frac{1}{2} \times 661 = 330{\cdot}5$.
$\therefore s - a = 90{\cdot}5$, $s - b = 9{\cdot}5$, $s - c = 230{\cdot}5$
$\therefore$ Area $\triangle BCD$
$= \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{330{\cdot}5 \times 90{\cdot}5 \times 9{\cdot}5 \times 230{\cdot}5}$
$\simeq 8{,}091$ sq. yd.

330·5	2·5191
90·5	1·9566
9·5	0·9777
230·5	2·3626
	7·8160
8091	3·9080

$\therefore$ Area of field
$\simeq 32{,}070 + 8{,}091$
$= 40{,}161$ sq. yd. $\simeq 8{\cdot}3$ acres.

Area of a quadrilateral

Since the lengths of the sides of a quadrilateral do not fix its shape it is not possible to express the area of a quadrilateral in terms of its sides.

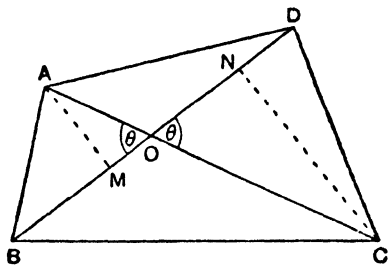


FIG. 261.

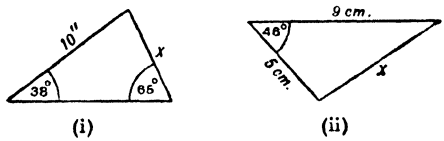
The following formula is however sometimes useful.
In Fig. 261, AM and CN are drawn perpendicular to BD.
Let the angle between the diagonals be θ.
Area of quadrilateral $ABCD$

$$= \text{area } \triangle ABD + \text{area } \triangle BCD$$
$$= \tfrac{1}{2}BD \,.\, AM + \tfrac{1}{2}BD \,.\, CN$$
$$= \tfrac{1}{2}BD \,.\, AO \sin\theta + \tfrac{1}{2}BD \,.\, CO \sin\theta$$
$$= \tfrac{1}{2}BD(AO + CO) \sin\theta$$
$$= \tfrac{1}{2}BD \,.\, AC \sin\theta.$$

That is, in words, area of a quadrilateral $= \frac{1}{2} \times$ product of diagonals $\times$ sine of angle between them.

Exercise XL

1. Find the side x in the triangles shown :—



(i) (ii)

FIG. 262.

2. Two angles of a triangle are 46° and 73°, and the side opposite the smaller angle measures 8 cm. What is the length of the side opposite the larger angle?

3. In a $\triangle PQR$, $\angle P = 34° 30'$, $\angle Q = 68°$ and $PQ = 5$ in. Find the length of QR.

4. The sides of a triangle measure 13 ft., 26 ft., 19 ft. Find the smallest angle of the triangle.

5. Two sides of a triangle measure 18 in. and 23 in. and contain an angle 62° 10′ between them. What is the length of the third side of the triangle ?

6. The sides of a triangle are in the ratio 3 : 5 : 7. Find the greatest angle.

7. Two angles of a triangle are 39° and 114°, and its shortest side is 7·5 cm. Find its longest side.

8. Find the area of the triangle in Question 4.

9. Find the area of the triangle in Question 3.

10. In a $\triangle DEF$, $DE = 8{\cdot}2$ in., $\angle D = 61°$, $\angle E = 43°$. Find the length of DF and the area of the triangle.

11 Two landmarks A, B are on opposite sides of a mountain, but both are visible from a point C. It is found that $AC = 2$ ml. 320 yd., $CB = 3$ ml. 850 yd. and $\angle ACB = 57° 40'$. What is the distance from A to B as the crow flies ?

12. A trench is 6 ft. 6 in. deep and 3 ft. 6 in. wide at the bottom, and the sides slope at 80° to the horizontal. Find the length of the shortest plank which will reach from the bottom of one side to the top of the opposite side.

13. The dimensions of a derrick-crane are as shown in Fig. 263. Find the height of A above BC.

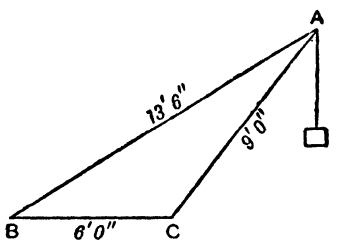


FIG. 263.

14. A motor-car door is prevented from swinging open to its fullest extent by a strap 10 in. long. The ends of the strap are fixed to points 6 in. and 4½ in. from the line of hinges. Find the greatest angle through which the door can open.

15. A wireless station P is 70 miles due North of another station Q. A ship finds that it is S. 28° 12′ W. from P and N. 67° 35′ W. from Q. Find its distance from P.

16. The cantilever structure in Fig. 264 has the dimensions shown. Find the lengths of the members BC, AC, CD.

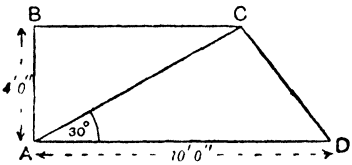


FIG. 264.

17. Two forces of 4 lb. wt. and 6 lb. wt. act in directions inclined at 50° to each other. What is the magnitude of the resultant force ?

18. An aeroplane, whose maximum speed in the absence of any wind is 240 m.p.h., flies at full throttle heading in a direction N.E. The pilot finds that his actual direction is N. 29° E. and his actual speed 219 m.p.h. Find the velocity and direction of the wind.

19. Find the magnitude and direction of the resultant of two forces of 2 tons wt. and 5 tons wt. in directions inclined at 110° to each other.

20. Two sides of an acute-angled triangle measure 8·2 in. and 9·9 in., and its area is 31·5 sq. in. Find the length of the third side. [Hint.—First find the angle between the given sides.]

21. Find the area of a regular octagon of side 4 cm.

22. A regular 7-sided polygon is inscribed in a circle of radius 3 in. Find its perimeter and its area.

23. A hexagonal nut fits into a $\frac{3}{4}$-in. spanner and screws on to a $\frac{1}{2}$-in. bolt. Find the area of its section.

24. A regular pentagon has an area of 25 sq. cm. Find the length of its side.

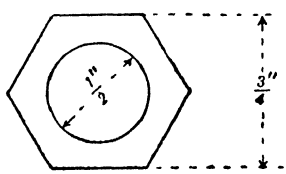


FIG. 265.

25. A gutter whose cross-section is an arc of a circle is 2 in. deep and 6 in. wide across the top (Fig. 266). Find the area of its cross-section.

26. A straight tunnel for an underground railway is to be 600 yd. long and its cross-section is to have the dimensions shown in Fig. 267. Find the volume of earth to be excavated.

FIG. 266.

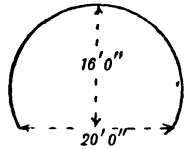


FIG. 267.

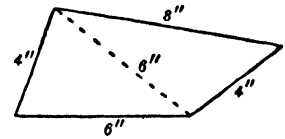


FIG. 268.

27. Find the area of the metal plate in Fig. 268, from the dimensions given, using the formula on p. 315.

28. Draw Fig. 268 to scale. Measure the other diagonal and the angle between the diagonals, and calculate its area from the formula on p. 317. Compare this with the value obtained in Question 27.

Solution of triangles

A triangle can be constructed if certain of its parts are given, such as its three sides, or two sides and the included angle, or one side and two angles (see Part I, Ch. XI). When three such parts are given the remaining three parts can be found either graphically, by drawing the triangle, or more accurately by calculation, using the sine and cosine rules. We usually speak of this as " solving " the triangle.

The cases in which it is possible to solve a triangle are set out below.

I. Given the three sides

The angles can be calculated from the cosine rule.

It is only necessary to calculate two of the angles in this way, since the third angle can then be found from the fact that the sum of the three angles is equal to 180°. The student

may, however, prefer to calculate all three angles and to use the fact about the angle-sum being equal to 180° as a check.

Example.—Solve the following triangle: $a = 6{\cdot}3$ in., $b = 2{\cdot}9$ in., $c = 8{\cdot}2$ in.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{8{\cdot}41 + 67{\cdot}24 - 39{\cdot}69}{2 \times 2{\cdot}9 \times 8{\cdot}2}$$

$$= \frac{35{\cdot}96}{47{\cdot}56} = 0{\cdot}7561$$

$$\therefore \underline{A = 40° \ 53'}$$

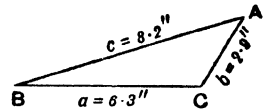


FIG. 269.

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{67{\cdot}24 + 39{\cdot}69 - 8{\cdot}41}{2 \times 8{\cdot}2 \times 6{\cdot}3}$$

$$= \frac{98{\cdot}52}{103{\cdot}32} = 0{\cdot}9535$$

$$\therefore \underline{B = 17° \ 32'}$$

$$C = 180° - (A + B) = 180° - 58° \ 25'$$

$$= \underline{121° \ 35'}.$$

[*or*

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{39{\cdot}69 + 8{\cdot}41 - 67{\cdot}24}{2 \times 6{\cdot}3 \times 2{\cdot}9}$$

$$= -\frac{19{\cdot}14}{36{\cdot}54} = -0{\cdot}5237.$$

Since cos C is negative, the angle C is obtuse.

$$\therefore C = 180° - 58° \ 25' = 121° \ 35'.$$

Check: $A + B + C = 40° \ 53' + 17° \ 32' + 121° \ 35' = 180° \ 0'$.]

As an alternative method, having found one angle by the cosine rule, we may then find the second angle by using the sine rule, which is more suited for logarithmic calculations.

Care must be exercised however in finding angles from the sine rule, since an angle and its supplement have the same sine; so that, if we are given the sine of an angle, there are

two angles, each less than 180°, having that sine, one acute and the other obtuse. We have then to decide in some way which is the correct angle to take. We can avoid any ambiguity by calculating the two smaller angles of the triangle, as we know that they must both be acute, since a triangle cannot have more than one obtuse angle (if any).

Thus, in the previous example, the smallest angle, being opposite the smallest side, is B.

From the cosine formula, we find (as above) that $B = 17° 32'$. The next angle in order of magnitude is A (opposite the second shortest side).

$$\frac{\sin A}{\sin B} = \frac{a}{b}$$

$$\therefore \sin A = \frac{a \sin B}{b}$$

$$= \frac{6{\cdot}3 \sin 17° 32'}{2{\cdot}9}$$

$\therefore A = 40° 52'$, since A must be acute.

	No.	*Log.*
	6·3	0·7993
	sin 17° 32′	$\bar{1}$·4789
		0·2782
	2·9	0·4624
	sin A	$\bar{1}$·8158

Then $C = 180° - (A + B) = 180° - 58° 24' = 121° 36'$.

[The student should note that, having found log sin A, which is $\bar{1}$·8158, we do not need to find sin A; we simply read off the angle A directly from a table of log sines.]

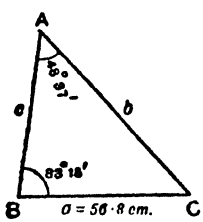


Fig. 270.

II. Given one side and two angles

The third angle is found from the fact that the sum of the angles is equal to 180°. The two unknown sides are then calculated from the sine rule.

Example.—Solve the following triangle: $a = 56{\cdot}8$ cm., $A = 48° 37'$, $B = 83° 15'$.

$C = 180° - (A + B) = 180° - 131° 52' = 48° 8'.$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\therefore b = \frac{a \sin B}{\sin A}$$

$$= \frac{56{\cdot}8 \sin 83° 15'}{\sin 48° 37'}$$

$$= 75{\cdot}16 \text{ cm.}$$

No.	*Log.*
56·8	1·7543
sin 83° 15′	$\bar{1}$·9969
	1·7512
sin 48° 37′	$\bar{1}$·8752
75·16	1·8760

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\therefore c = \frac{a \sin C}{\sin A}$$

$$= \frac{56{\cdot}8 \sin 48° 8'}{\sin 48° 37'}$$

$$= 56{\cdot}37 \text{ cm.}$$

56·8	1·7543
sin 48° 8′	$\bar{1}$·8720
	1·6263
sin 48° 37′	$\bar{1}$·8752
56·37	1·7511

III. Given two sides and the included angle

The third side is found from the cosine rule, and one of the other angles is then found either from the cosine rule or from the sine rule. Here again, in using the sine rule, in order to avoid ambiguity we find one of the two smaller angles, which we know to be acute. The third angle is the supplement of the sum of the other two.

Example.—Solve the triangle for which $b = 119{\cdot}2$ ft., $c = 35{\cdot}6$ ft., $A = 26° 16'$.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$= 14{,}209 + 1267 - 7610$$

$$= 7866$$

$$a \simeq 88{\cdot}7 \text{ ft.}$$

No.	*Log.*
119·2	2·0763
35·6	1·5514
cos 26° 16′	$\bar{1}$·9526
3805	3·5803

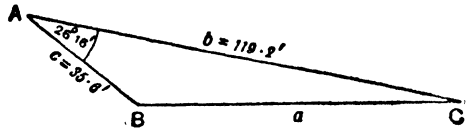


FIG. 271.

The two smaller sides are c and a, and hence the two smaller angles are C and A. Since A is given we shall calculate C.

$$\frac{\sin C}{\sin A} = \frac{c}{a}$$

$$\therefore \sin C = \frac{c \sin A}{a}$$

$$= \frac{35{\cdot}6 \sin 26° 16'}{88{\cdot}7}$$

$$\therefore C = 10° 14', \text{ since } C \text{ must be acute.}$$

$$\therefore B = 180° - (A + C)$$
$$= 180° - 36° 30' = 143° 30'.$$

	No.	*Log.*
	35·6	1·5514
	sin 26° 16′	$\bar{1}$·6460
		1·1974
	88·7	1·9479
	sin C	$\bar{1}$·2495

IV. Given two sides and a non-included angle

In this case there may be two possible solutions, i.e. there may be two triangles satisfying the given data. It is usually easy to decide whether there are two solutions or only one by drawing a rough figure. The following examples show how to proceed.

Example.—Solve the triangle given that $a = 7{\cdot}2$ in., $b = 11{\cdot}3$ in., $A = 34°$.

If we try to draw the figure we find that there are two possible triangles, viz. ACB_1 and ACB_2 in Fig. 272.

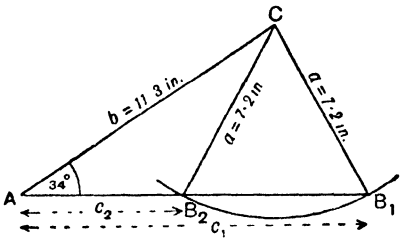


FIG. 272.

$$\frac{\sin B}{\sin A} = \frac{b}{a}$$

$$\therefore \sin B = \frac{b \sin A}{a}$$

$$= \frac{11{\cdot}3 \sin 34°}{7{\cdot}2}$$

$$\therefore B = 61° 23'$$

	No.	*Log.*
	11·3	1·0531
	sin 34°	$\bar{1}$·7476
		0·8007
	7·2	0·8573
	sin B	$\bar{1}$·9434

or $B = 180° - 61° 23' = 118° 37'$.

In Fig. 272, $\angle CB_1A = 61° 23'$, $\angle CB_2A = 118° 37'$.

(i) If $B = 61° 23'$,

$$C = 180° - (A + B)$$

$$= 180° - 95° 23'$$

$$= 84° 37'.$$

Also $$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\therefore c = \frac{a \sin C}{\sin A}$$

$$= \frac{7{\cdot}2 \sin 84° 37'}{\sin 34°}$$

$$\simeq 12{\cdot}8$$

	No.	*Log.*
	7·2	0·8573
	sin 84° 37′	$\bar{1}$·9981
		0·8554
	sin 34°	$\bar{1}$·7476
	12·81	1·1078

(ii) If $B = 118° 37'$,

$$C = 180° - (A + B)$$
$$= 180° - 152° 37'$$
$$= 27° 23'.$$

Also $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$

$$\therefore c = \frac{a \sin C}{\sin A}$$
$$= \frac{7{\cdot}2 \sin 27° 23'}{\sin 34°}$$
$$\simeq 5{\cdot}92$$

No.	*Log.*
7·2	0·8573
sin 27° 23′	$\bar{1}$·6627
	0·5200
sin 34°	$\bar{1}$·7476
5·021	0·7724

There are therefore two possible solutions, viz. :

$$B = 61° 23', \quad C = 84° 37', \quad c = 12{\cdot}8 \text{ in.},$$

or $$B = 118° 37', \quad C = 27° 23', \quad c = 5{\cdot}92 \text{ in.}$$

Example.—Solve the triangle given that $b = 21$ ft., $c = 16$ ft., $B = 50°$.

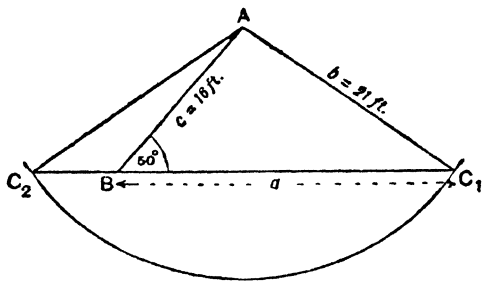


Fig. 273.

Drawing the figure in this case, we find that there is only one possible triangle, viz. the triangle ABC_1, in Fig. 273.

$$\frac{\sin C}{\sin B} = \frac{c}{b}$$

$$\therefore \sin C = \frac{c \sin B}{b}$$

$$= \frac{16 \sin 50°}{21}$$

$$\therefore C = 35° 42'$$

	No.	*Log.*
	16	1·2041
	sin 50°	$\bar{1}$·8843
		1·0884
	21	1·3222
	sin C	$\bar{1}$·7662

or $C = 180° - 35° 42' = 144° 18'$

If $C = 35° 42'$, $A = 180° - (B + C) = 180° - 85° 42' = 94° 18'$.

If $C = 144° 18'$, $A = 180° - (B + C) = 180° - 194° 18' = -14° 18'$,

which is impossible. Thus the latter value of C is impossible, and there is only one solution.

The fact that there is only one solution will always reveal itself in this way, even if we do not draw the figure.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\therefore a = \frac{b \sin A}{\sin B}$$

$$= \frac{21 \sin 94° 18'}{\sin 50°}$$

$$= \frac{21 \sin 85° 42'}{\sin 50°}$$

$$\simeq 27·3$$

	No.	*Log.*
	21	1·3222
	sin 85° 42'	$\bar{1}$·9988
		1·3210
	sin 50°	$\bar{1}$·8843
	27·33	1·4367

There is therefore only one solution, viz.

$$C = 35° 42', A = 94° 18', a = 27·3 \text{ ft.}$$

NOTE.—In both the examples above, the *given* angle is acute. If the *given* angle is *obtuse*, then the angle found from

the sine formula must be acute, since a triangle cannot have two obtuse angles. In that case therefore there is only one solution.

If the *given* angle is *acute*, a comparison of Figs. 272 and 273 shows that there are two solutions or one solution according as the side opposite the given angle is less or greater than the other given side. There is one exception to this rule, viz. when, in Fig. 272, B_1 and B_2 coincide; in that case there is only one triangle, which is right-angled at B.

Alternative method of solution, by dividing into right-angled triangles

It might be pointed out that while the methods for solving triangles given above are the most direct, any triangle can be solved without the use of the sine or cosine rule, by dividing it into two right-angled triangles as shown in the examples below.

Example.—Solve the triangle for which

$$b = 119{\cdot}2 \text{ ft.}, \; c = 35{\cdot}6 \text{ ft.}, \; A = 26° \; 16'.$$

[Compare p. 323.]

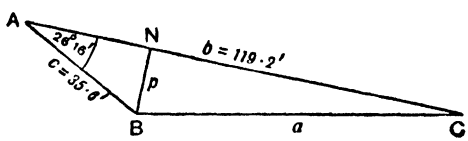


Fig. 274.

Draw BN perpendicular to AC.

Then $\triangle$'s ABN, CBN are both right-angled triangles.

Let $BN = p$.

$\frac{p}{35{\cdot}6} = \sin 26° 16'$

$\therefore p = 35{\cdot}6 \sin 26° 16'$

$= 15{\cdot}75$

$\frac{AN}{p} = \cot 26° 16'$

$\therefore AN = p \cot 26° 16'$

$= 15{\cdot}75 \cot 26° 16'$

$= 31{\cdot}93$

$\therefore CN = 119{\cdot}2 - 31{\cdot}93 = 87{\cdot}27$

Hence

$\tan C = \frac{p}{CN} = \frac{15{\cdot}75}{87{\cdot}27}$

$\therefore C = 10° 14'.$

$\therefore B = 180° - (A + C)$

$= 180° - 36° 30' = \underline{143° 30'}.$

No.	*Log.*
35·6	1·5514
sin 26° 16′	$\bar{1}$·6460
$p = 15{\cdot}75$	1·1974
15·75	1·1974
cot 26° 16′	0·3067
$AN = 31{\cdot}93$	1·5041
15·75	1·1974
87·27	1·9408
tan C	$\bar{1}$·2566
15·75	1·1974
sin 10° 14′	$\bar{1}$·2496
$a = 88{\cdot}67$	1·9478

Also $\frac{a}{p} = \operatorname{cosec} C$

$\therefore a = p \operatorname{cosec} C = \frac{15{\cdot}75}{\sin 10° 14'} = 88{\cdot}67 \simeq \underline{88{\cdot}7 \text{ ft.}}$

Example.—Solve the following triangle: $a = 56{\cdot}8$ cm., $A = 48° 37'$, $B = 83° 15'$.

[Compare p. 322.]

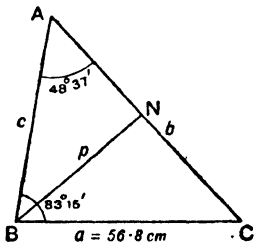


FIG. 275.

Draw BN perpendicular to AC; let $BN = p$.

$C = 180° - (A + B)$

$= 180° - 131° 52' = \underline{48° 8'}$

	No.	*Log.*
	56·8	1·7543
$\frac{p}{56·8} = \sin C$	sin 48° 8′	$\bar{1}$·8720
$\therefore p = 56·8 \sin C = 56·8 \sin 48° 8'$	$p = 42·30$	1·6263
$= 42·30$		
	56·8	1·7543
$\frac{CN}{a} = \cos C$	cos 48° 8′	$\bar{1}$·8244
$\therefore CN = a \cos C = 56·8 \cos 48° 8'$	37·90	1·5787
$= 37·90$		
	42·30	1·6263
Also $\frac{AN}{p} = \cot 48° 37'$	tan 41° 23′	$\bar{1}$·9450
$\therefore AN = 42·30 \cot 48° 37'$	37·27	1·5713
$= 42\ 30 \tan 41° 23'$		
$= 37·27$	42·30	1·6263
$\therefore b = AC = AN + CN$	sin 48° 37′	$\bar{1}$·8752
$= 37·27 + 37·90$	56·37	1·7511
$= \underline{75·17 \text{ cm.}}$		

Finally $\frac{c}{p} = \operatorname{cosec} 48° 37'$

$\therefore c = 42·30 \operatorname{cosec} 48° 37'$

$= \frac{42·30}{\sin 48° 37'} = \underline{56·37 \text{ cm.}}$

Exercise XLI

Solve the following triangles by calculation. Also draw the triangles accurately to scale and measure the remaining sides and angles, and compare the results with those obtained by calculation.

1. $a = 6·7$ cm., $b = 6$ cm., $c = 4$ cm.
2. $a = 19$ yd., $b = 48$ yd., $c = 36$ yd.

3. $b = 3$ in., $A = 32°$, $C = 74°$.

4. $c = 28$ ft., $B = 45°$, $C = 106°$.

5. $a = 2{\cdot}3$ cm., $b = 3{\cdot}6$ cm., $C = 60°$.

6. $a = 101$ ml., $c = 78$ ml., $B = 11°\ 25'$.

7. $b = 3{\cdot}5$ in., $c = 4$ in., $B = 52°$.

8. $a = 6$ cm., $b = 5$ cm., $A = 77°\ 42'$.

9. $a = 1$ ft., $c = 2$ ft., $C = 135°$.

10. Find, by calculation, the remaining sides and angles in the triangles in Fig. 276.

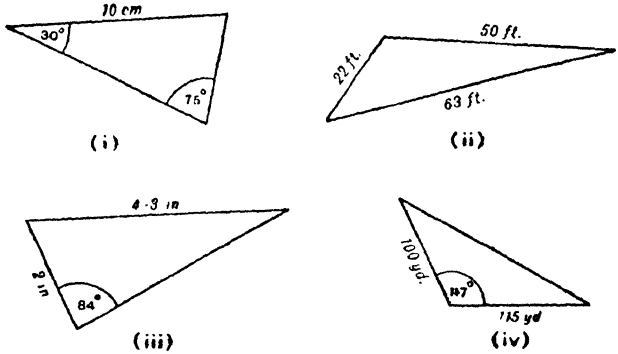


FIG. 276.

11. If $A = 62°$, $B = 37°\ 30'$ and $b = 40$ ft., find c.

12. Find the area of the triangle DEF in which $DE = 1{\cdot}6$ in., $\widehat{EDF} = 35°$ and $\widehat{DFE} = 124°$.

13. In a triangle PQR, $PQ = 15$ cm., $PR = 17$ cm. and $\angle P = 43°\ 30'$; find the angle Q.

14. ABC is a triangle of area 18 sq. in., and $AB = 5$ in., $BC = 9$ in. Find the angle ACB.

15. DEF is a triangle of area 41 sq. yd., and $\angle D = 17°$, $DE = 15$ yd. Find the lengths of the other sides.

16. A weight is suspended from a ceiling by two chains, of lengths 2 ft. and 4 ft., attached to points in the ceiling 5 ft. apart. Find the distance of the weight below the ceiling.

17. A mechanism, shown in Fig. 277, consists of two pistons which slide in cylinders whose axes are inclined to each other at 120°, and a connecting-rod 3 ft. long. Find OB when OA = 2ft.

If the least and greatest distances of A from O are 2 ft. and 2 ft. 6 in. find the travel of the piston B.

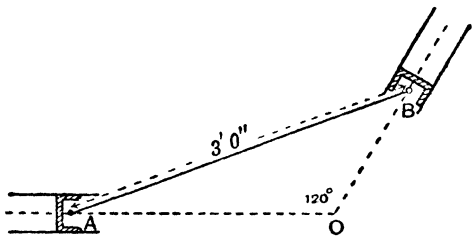


FIG. 277.

18. The mid-points of the legs of a step-ladder are connected by a rope 3 ft. 3 in. long. When the ladder is fully opened out the legs make angles 54° and 73° with the horizontal. Find the length of each leg of the ladder.

Harder problems

Example.—A surveyor, using a theodolite mounted at the top of a stake AC which is 4 ft. 9 in. high, observes the angles of elevation of two marks D and E on a vertical pole at B to be 12° 40′ and 9° 15′ (Fig. 278). The marks are 2 ft. 6 in. apart, and the lower one is 3 ft. 0 in. above the ground. Find the height of B above A.

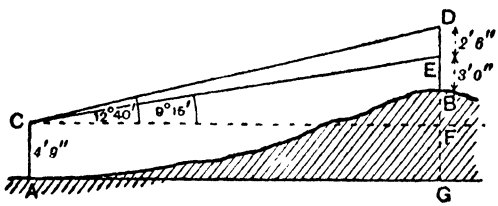


FIG. 278.

Draw CF and AG horizontal, cutting the vertical through B in F and G. Then $FG = CA = 4$ ft. 9 in.

Required height
$= BG = BF + 4$ ft. 9 in. $= (EF - 3$ ft. 0 in.$) + 4$ ft. 9 in.
$= EF + 1$ ft. 9 in.

Now $EF = EC \sin 9° 15'$, and we can find EC by applying the sine rule to $\triangle CDE$.

For $\widehat{CDE} = 90° - 12° 40' = 77° 20'$,

and $\widehat{DCE} = 12° 40' - 9° 15' = 3° 25'$.

$$\therefore \frac{EC}{\sin 77° 20'} = \frac{DE}{\sin 3° 25'} = \frac{2{\cdot}5}{\sin 3° 25'}$$

$$\therefore EC = \frac{2{\cdot}5 \sin 77° 20'}{\sin 3° 25'}$$

$$\therefore EF = EC \sin 9° 15'$$

$$= \frac{2{\cdot}5 \sin 77° 20' \,.\, \sin 9° 15'}{\sin 3° 25'}$$

No.	*Log.*
2·5	0·3979
sin 77° 20′	$\bar{1}$·9893
sin 9° 15′	$\bar{1}$·2061
	$\bar{1}$·5933
sin 3° 25′	$\bar{2}$·7752
6·579	0·8181

$\simeq 6{\cdot}58$ ft.

$= 6$ ft. 7 in.

$\therefore$ Height of B above A
$= 6$ ft. 7 in. $+ 1$ ft. 9 in.
$= 8$ ft. 4 in.

Example.—The roof truss represented in Fig. 279 has the dimensions shown. Find the length of AE.

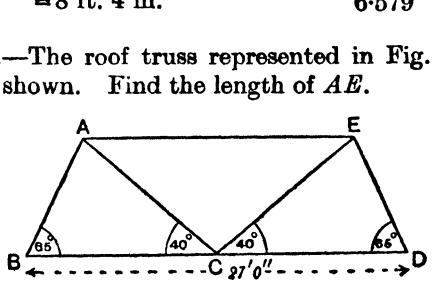


FIG. 279.

From symmetry C is the mid-point of BD.

$$\therefore BC = 13 \text{ ft. } 6 \text{ in.}$$

$$B\hat{A}C = 180° - (65° + 40°)$$

$$= 75°$$

$$\therefore \text{In } \triangle ABC, \frac{AC}{\sin 65°} = \frac{BC}{\sin 75°} = \frac{13{\cdot}5}{\sin 75°}$$

$$\therefore AC = \frac{13{\cdot}5 \sin 65°}{\sin 75°}$$

Since AE is parallel to BC, $E\hat{A}C = A\hat{C}B = 40°$.

$\therefore$ In the isosceles $\triangle ACE$,

	No.	*Log.*
$AE = 2AC \cos E\hat{A}C$	27	1·4314
	sin 65°	$\bar{1}$·9573
$= 2 \times \frac{13{\cdot}5 \sin 65°}{\sin 75°} \times \cos 40°$	cos 40°	$\bar{1}$·8843
		1·2730
$= \frac{27 \sin 65° \cos 40°}{\sin 75°}$	sin 75°	$\bar{1}$·9849
$= 19{\cdot}41$ ft.	19·41	1·2881

NOTE.—In this example the known length, BC, and the required length, AE, are sides of different triangles, so that we cannot apply the sine rule directly. We therefore use the common side of those two triangles, viz. AC, as a "link" to pass from one triangle to the other.

Example.—An aeroplane is sighted simultaneously from two stations A and B, B being one mile north-east of A. To the observer at A the aeroplane appears due north at an elevation of 34°; to the observer at B it appears in a direction N. 72° W. Find the height of the aeroplane.

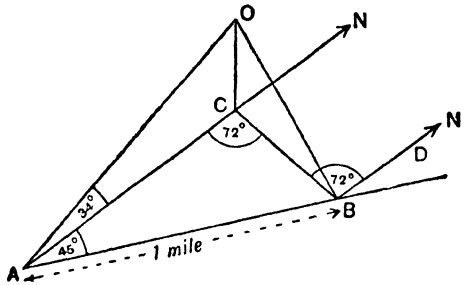


FIG. 280.

In Fig. 280 O represents the aeroplane and C is the point on the ground vertically below O.

$$\widehat{ACB} = \text{alternate angle } \widehat{CBD} = 72°.$$

$$\therefore \widehat{ABC} = 180° - (45° + 72°) = 63°.$$

$$\therefore \text{In } \triangle ABC, \frac{AC}{\sin 63°} = \frac{AB}{\sin 72°} = \frac{1}{\sin 72°}$$

$$\therefore AC = \frac{\sin 63°}{\sin 72°}.$$

In $\triangle ACO$, the angle at C is a right angle,

$$\therefore \frac{OC}{AC} = \tan 34°$$

$$\therefore OC = AC \tan 34°$$

$$= \frac{\sin 63° \tan 34°}{\sin 72°}$$

$$= 0{\cdot}632 \text{ mile}$$

$$= 0{\cdot}632 \times 5280 \text{ ft.}$$

$$= 3337 \text{ ft.}$$

	No.	*Log.*
	$\sin 63°$	$\bar{1}{\cdot}9499$
	$\tan 34°$	$\bar{1}{\cdot}8290$
		$\bar{1}{\cdot}7789$
	$\sin 72°$	$\bar{1}{\cdot}9782$
	$0{\cdot}6320$	$\bar{1}{\cdot}8007$

Exercise XLII

1. A ship sights two lighthouses both due North. After steaming 5 miles due East their bearings are N. 50° W. and N. 35° W. Find the distance between the lighthouses.

2. A rod 6 in. long rests at an inclination of 18° to the horizontal across a groove formed by two planes each inclined at 54° to the horizontal (Fig. 281). Find the height of the lower end of the rod above the bottom of the groove.

3. A tower stands on a hillside which may be regarded as a plane of inclination 15° to the horizontal. The elevation of the top of the tower from one point on the hillside is 20°, and from a point 100 yd. farther up the hill towards the tower the elevation is 28°. Find the height of the tower.

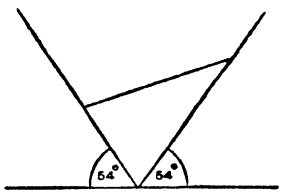


FIG. 281.

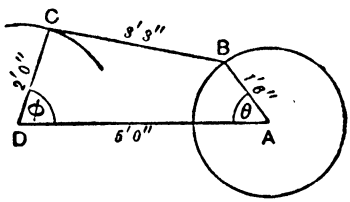


FIG. 282.

4. The dimensions of a four-bar linkage are shown in Fig. 282. Find the angle ϕ when $\theta = 45°$.

[Hint.—Join BD and find the angles BDA, BDC separately.]

5. The elevation of the top of a wireless mast from one window of a house is α. From a window a distance h higher up, in the same vertical line, the elevation is β. Show that the height of the top of the mast above the level of the first window is $h \dfrac{\sin \alpha \cos \beta}{\sin (\alpha - \beta)}$.

Calculate this height when $h = 20$ ft., $\alpha = 30°$, $\beta = 22°$.

6. Find the lengths of the three unmarked members of the framework in Fig. 283.

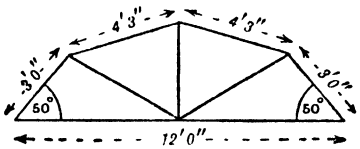


FIG. 283.

7. To find the distance of a landmark O from a point A from which it is invisible, a surveyor takes readings from two points B and C from which it is visible. His measurements are recorded in Fig. 284. Find the distance of O from A.

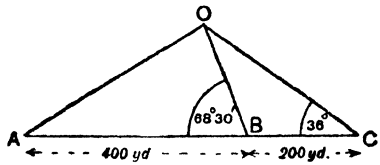


FIG. 284

8. A man at A wishes to find the distance between two points P and Q (Fig. 285) on the opposite side of a river which he cannot cross. He walks a measured distance to a point B and observes the angles BAP, BAQ, ABP, ABQ. Show how he can calculate the distance PQ from his measurements.

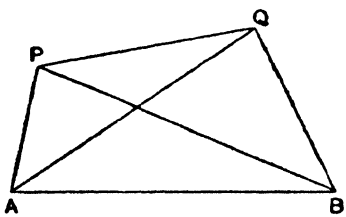


FIG. 285.

9. A steamer is observed from the top of a cliff 530 ft. high to bear S. 23° W. at an angle of depression 4° 30′. Four minutes later it is observed in a direction S. 41° E. and the angle of depression is then 6° 12′. Find the course and speed of the ship (assumed constant).

10. A man making a geographical survey wishes to find the height of a point B above his station at A, but the ground between A and B is impassable. He therefore chooses a second station C and makes the following measurements: elevations of B and C from A, distance AC, bearings of B from A and C. Show how he can calculate the required height from these data.

11. Use the sine rule to prove the theorem on p. 128, viz. that the bisectors of an angle of a triangle divide the opposite side in

the ratio of the sides containing the angle. [Hint.—Use AX, or AY (Fig. 52), as a link between the triangles containing the required sides.]

Miscellaneous Exercise XLIII

1. A vertical mast 50 ft. high stands on a hillside, of inclination 12°, facing North. Find, to the nearest inch, the length of its shadow when the Sun is due South at an elevation 31°.

2. A ship steaming due East is seen 6 miles away in the direction N. 50° W. Ten minutes later it is observed in the direction N. 30° W. Find its speed.

3. The floor of a hall is octagonal in shape and its area is 3000 sq. ft. Find the length of each side.

4. A man wishing to go from A to B can either cycle along a straight path which runs direct from A to B or go by car along a main road which runs in two straight parts AC, CB (Fig. 286). If he can cycle at 8 m.p.h. and his average speed by car is 25 m.p.h., which is the quicker way ?

5. P, Q are two landmarks on a straight road, P being $\frac{1}{4}$ mile due West of Q. A house bears S. 67° E. from P, and S. 42° W. from Q; find, to the nearest yard, its distance from the road.

6. Find, to the nearest inch, the lengths of the members AD and BD in the framed structure shown in Fig. 287.

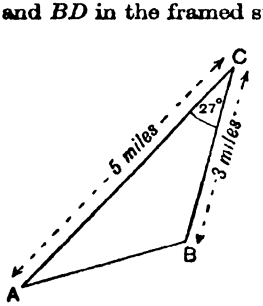


Fig. 286.

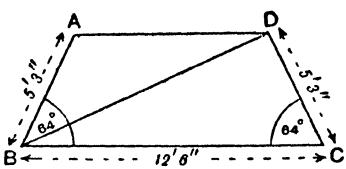


Fig. 287.

7. A cylindrical boiler, 12 ft. 0 in. long and 5 ft. 0 in. in diameter, has two furnace tubes, each 1 ft. 3 in. in diameter, running through it (Fig. 288). Find the volume of water (in gallons) required to fill the boiler to a depth of 4 ft. 0 in. [1 cu. ft. $\simeq$ 6·23 gallons.]

8. Two ships leave the same port at the same time. One steams at 12 knots in the direction N. 60° E., the other at 15 knots in the direction S. 75° E. How far apart are they 3 hours later, and what is then the bearing of the second ship from the first? [1 knot = a speed of 1 nautical mile per hour; 1 nautical mile $\doteq 1{\cdot}15$ statute miles.]

9. A circular railway track is to be constructed so as to pass through three points A, B, C. If $BC = 64$ yd., $CA = 35$ yd., $AB = 82$ yd., find the radius of the circle.

[Hint.—If O is the centre of the circle circumscribing a triangle ABC, and ON is perpendicular to BC (Fig. 289),

$$\widehat{BON} = \tfrac{1}{2}\,\widehat{BOC} = \tfrac{1}{2}(2\widehat{BAC}) = \widehat{BAC}.$$

$$\therefore \frac{BN}{BO} = \sin \widehat{BON} = \sin A.$$

$$\therefore \text{Radius} = BO = \frac{BN}{\sin A} = \frac{a}{2 \sin A}.]$$

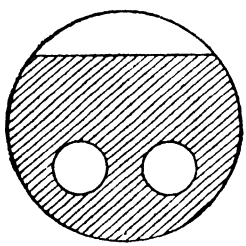


FIG. 288.

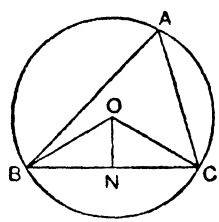


FIG. 289.

10. Two anti-aircraft batteries, P and Q, are six miles apart, Q being due North of P. An aeroplane, flying horizonally, is observed from P at an elevation of 40° due North, and at the same instant from Q at an elevation of 27° South. One minute later it is observed from Q at an elevation of 70° due South. Find the speed of the aeroplane.

11. Solve the triangle ABC given that $B = 30°$, $a = 4{\cdot}4$ in. $b + c = 6{\cdot}5$ in.

[Hint.—If BA is produced to D so that $AD = AC$ (Fig. 290), then $BD = b + c$. The angle D can now be calculated, and $\widehat{A} = 2\widehat{D}$.]

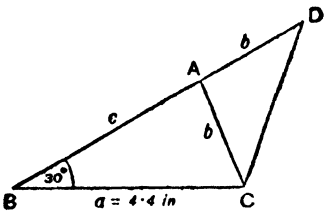


FIG. 290.

12. In the "quick-return motion" shown in Fig. 291, PQ is a rod 1 ft. 6 in. long whose end P describes a circle of radius 3 in., while its end Q moves along a straight guide distant 10 in. from the centre of the circle. Find x when $\theta = 130°$.

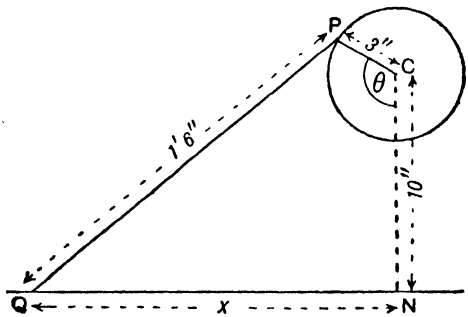


FIG. 291.

CALCULUS

CHAPTER XV

DIFFERENTIATION

One of the most powerful methods in modern mathematics is that of the calculus, the ideas of which were conceived by Archimedes in the third century B.C. By dividing a segment of a parabola into thin strips and adding together their areas, he found an approximation to the area of the segment. He then obtained closer and closer approximations by taking more and thinner strips. By this method of exhaustion he found the area of the segment exactly.

About 1586 Stevinus of Bruges used a method similar to that of Archimedes to find the thrust of a liquid on a surface, and a little later a Jesuit priest, Cavalieri, extended the method to find the volumes of solids. The methods used by these mathematicians to find the whole area, or volume, by dividing it up into small parts is now called " integration," i.e. finding the whole.

In the beginning of the seventeenth century the French mathematician Fermat considered the ratio of infinitesimally small increments and so laid the foundation on which Newton (1642–1727) and Leibniz (1646–1716) later built the theory of " differentiation " or finding rates of change from the ratios of small differences. It was due to the genius of Newton and Leibniz that a great advance was made. Newton conceived the idea of continuous change, and rate of change at an instant or flux, and he described his new subject as " fluxions."

He found that his knowledge of rates of change could be applied to calculate areas and volumes, that is to perform integrations, much more easily than by the method of exhaustion described above. Leibniz discovered the method of differentiation about the same time and we are specially indebted to him for his notation, which is essentially that now in general use.

In the last two centuries calculus has been developed to such an extent that it is now used to deal with problems in every branch of technical science.

Tank filling at a constant rate

Suppose water starts to flow into an empty tank at a constant rate of 2 cu. ft. per min.; then after t min. the volume of water in the tank is $2t$ cu. ft. If we call this volume V cu. ft., $V = 2t$. The graph of this equation is a straight line through the origin of gradient 2. Thus, the rate at which the water is flowing in, or in other words, the rate of increase of V, is equal to the gradient of the graph of V against t. In the same way, if the tank had 3 cu. ft. in it when the water started to

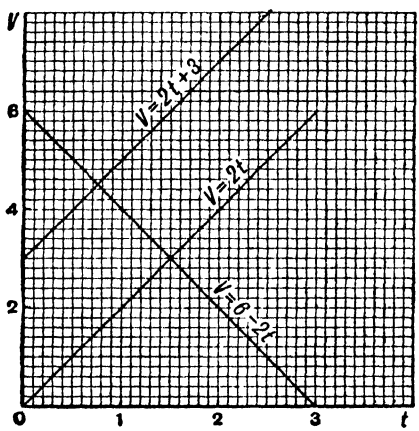


FIG. 292.

flow in, the volume after t min. would be given by $V = 2t + 3$. This equation also has a straight line graph of gradient 2. Thus, we see that, so long as the water flows in at a constant rate, the graph of V against t is a straight line, and the rate of flow, or the rate of increase of V is the gradient of the line. The graphs of $V = 2t$ and $V = 2t + 3$ are shown in Fig. 292.

Tank emptying at a constant rate

If a tank has 6 cu. ft. of water in it and water flows out at 2 cu. ft. per min., the volume V cu. ft. after t sec. is given by $V = 6 - 2t$. Thus the graph of V is a straight line, but one with a negative gradient -2. Hence, if we regard a rate of decrease of V of 2 cu. ft. per min. as the same as a rate of increase of V of -2 cu. ft. per min., it is still true that the rate of increase of V is the gradient of the graph of V against t. The graph of $V = 6 - 2t$ is shown in Fig. 292.

We conclude that, if the relationship between V and t is $V = at + b$, V is increasing at a constant rate a.

Tank filling and emptying at a variable rate

The following table gives the volume of water, V cu. ft., in a tank, which was filled at a variable rate, t min. after the water began to flow in. After a time a plug was pulled out and the tank began to empty.

t min. ..	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
V cu. ft.	2	5·50	8·90	11·15	12·05	11·00	8·60	5·45	1·20

Fig. 293 shows the graph of V against t. From $t = 1$ to $t = 1\frac{1}{2}$, V increased by $(11{\cdot}15 - 8{\cdot}90) = 2{\cdot}25$, so that, if the water had flowed in at a constant rate during this time, it would have flowed in at $\dfrac{2{\cdot}25}{\frac{1}{2}} = 4{\cdot}5$ cu. ft. per min. If the water had flowed in at a constant rate the graph of V would have been the chord AB instead of the curve from A to B, and the constant rate of flow, 4·5 cu. ft. per min., is the gradient of the chord AB. *This constant rate of flow is called the average*

(or mean) rate of increase of V in the time from $t=1$ *to* $t=1\frac{1}{2}$. Similarly, we find that, if we take an interval of time from $t=1$ to $t=1\frac{1}{8}$, V increases by $(9{\cdot}60-8{\cdot}90)=0{\cdot}70$, and for this increase the water would have had to flow at a constant rate of $\frac{0{\cdot}70}{\frac{1}{8}}=5{\cdot}60$ cu. ft. per min., or the gradient of the chord AC. Again, for a still smaller interval from $t=1$ to $t=1{\cdot}05$, the

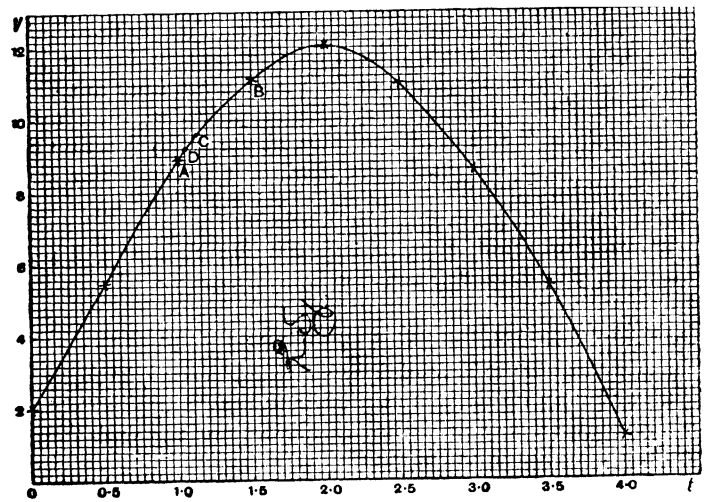


FIG. 293.

water would have had to flow at a constant rate of $\frac{9{\cdot}20-8{\cdot}90}{0{\cdot}05}$ $=6{\cdot}00$ cu. ft. per min., or the gradient of the chord AD. Now as we take shorter and shorter intervals of time we get chords which are closer and closer to the tangent at A and the corresponding constant rates of increase of V approach the gradient of the tangent at A. For this reason we call the gradient of the tangent at A *the rate of increase of V at the instant* $t=1$. Notice that it is a rate which can never

be measured exactly by using a stop-watch; it can only be approximated to by taking a very short interval of time, such as $\frac{1}{20}$th sec.

Therefore *the rate of increase of V at any value of* t

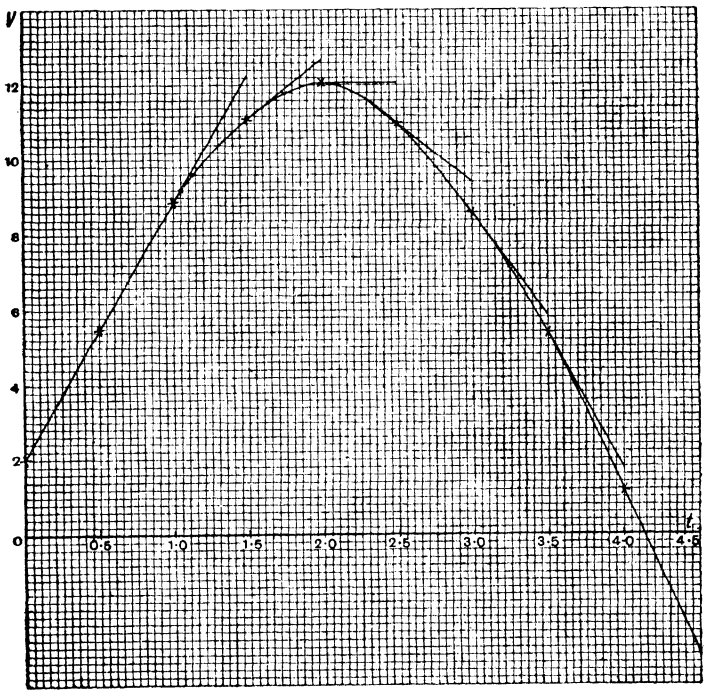


FIG. 294.

is the gradient of the tangent to the graph of V against t *at the point given by that value of* t. The rate of increase of V at the instant when the time is t may be denoted by $\dot{V}$; this notation was introduced by Newton.

In Fig. 294 the graph of Fig. 293 is repeated and the tangents at $t=0$, $\frac{1}{2}$, 1, etc., are drawn. The increase of V along each

tangent during $\frac{1}{2}$ min. is read from the graph and entered in the second line of the table below. The third line of the table shows the values of $\dot{V}$, which are the rates of increase of V, and are obtained by dividing the increments of V in the second line by $\frac{1}{2}$.

t min. ..	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
Increase of V along tangent in $\frac{1}{2}$ min. ..	3·5	3·5	3·5	1·55	0	−1·6	−2·7	−3·6	−4·8
$\dot{V}\left(\frac{\text{cu. ft.}}{\text{min.}}\right)$ ..	7·0	7·0	7·0	3·1	0	−3·2	−5·4	−7·2	−9·6

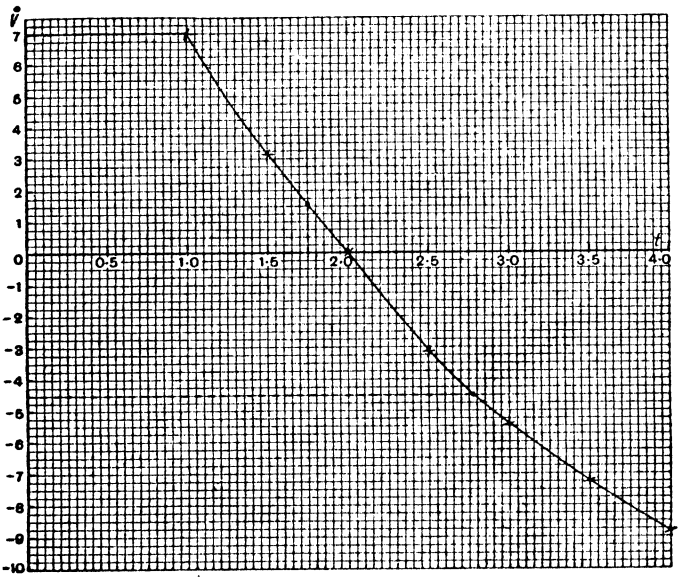


FIG. 295.

After $t=2$, V decreases and so its rate of increase and also the gradient of the tangent are negative.

Fig. 295 shows the graph of $\dot{V}$ plotted from this table. From this graph we can read the rate of increase of V at any

instant, and find at what instant it is increasing or decreasing at a given rate. For example, at $t = 1\frac{3}{4}$, $\dot{V} = 1{\cdot}5$, so the water is flowing in at 1·5 cu. ft. per min. Again, the water is flowing out at 4·5 cu. ft. per min. when $\dot{V} = -4{\cdot}5$, which is at $t = 2{\cdot}8$.

The graph of $\dot{V}$ is sometimes called the *derived curve* of the graph of V.

Difference ratio

Let the volume in the tank be V cu. ft. after t sec., and V′ cu. ft. after t' sec. In Fig. 296 P and Q are the points on the graph given by t and t', and hence $PK = MN = t' - t$

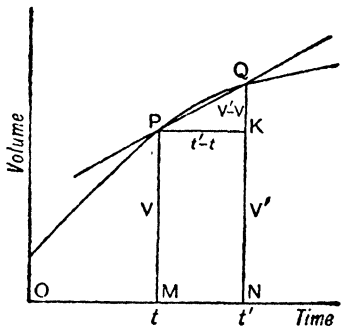


Fig. 296.

and $KQ = NQ - MP = V' - V$. As the time increases by $t' - t$, or PK, the volume increases by $V' - V$, or KQ.

$\therefore$ Average rate of increase of the volume in the time $t' - t$

$$= \frac{V' - V}{t' - t} = \frac{KQ}{PK} = \text{gradient of the chord PQ.}$$

$\dfrac{V' - V}{t' - t}$ may be called a difference ratio because it is the ratio of the differences of the volumes to the difference of the times.

The increase in the time from t to t' is written δt, where

the Greek letter δ is used as shorthand for the words "the increase of." Thus,

$$t' - t = \delta t = \text{the increase of } t.$$

Notice that δt does not mean $\delta \times t$; the δ cannot be separated from the letter that follows it, just as in a trigonometrical ratio like sin 20° the word *sin* cannot be separated from the 20°, being meaningless by itself.

In just the same way

$$V' - V = \delta V = \text{the increase of } V.$$

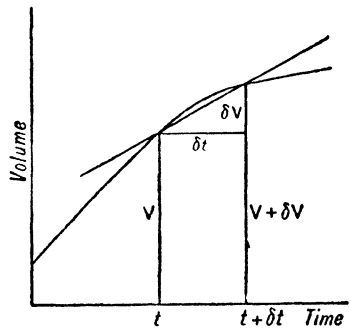


FIG. 297.

Fig. 297 shows the same ourve as Fig. 296, but t' and V' are replaced by $t + \delta t$ and $V + \delta V$ respectively.

$$\therefore \text{ the difference ratio } \frac{V' - V}{t' - t} = \frac{\delta V}{\delta t}.$$

Now if we take the time t' closer and closer to the time t, that is take δt smaller and smaller, the difference ratio (as we have already seen on p. 344) approaches closer and closer to $\dot{V}$, the rate of increase of V at the instant when the time is t, or the gradient of the tangent at P.

Leibniz suggested that, because the rate of increase of V at the instant t is obtained from the difference ratio $\frac{\delta V}{\delta t}$ by taking

δt smaller and smaller, it should be denoted by a similar notation, and so he wrote $\frac{dV}{dt}$ for $\dot{V}$. Hence,

$$\frac{dV}{dt} = \dot{V} = \text{the rate of increase of V at the instant } t.$$

The notation $\frac{dV}{dt}$ is used more frequently than the dot notation of Newton.

Because $\frac{dV}{dt}$ is obtained from a difference ratio, when we find $\frac{dV}{dt}$ from V we are said to "differentiate" V, and, for reasons which we shall see in the next volume, $\frac{dV}{dt}$ is called "the differential coefficient of V with respect to t" or "the derivative of V with respect to t."

Velocity and acceleration

If a body moves a distance s in a straight line in a time t, its velocity at the instant of time given by t is the rate of increase of s, or in symbols, $v = \dot{s}$ or $\frac{ds}{dt}$.

The graph obtained by plotting s against t is called the space-time or distance-time graph, and since $v = \frac{ds}{dt}$ the velocity at time t is the gradient of the tangent of the space-time graph at the time t.

The acceleration of the body is the rate of increase of its velocity, namely $\dot{v}$ or $\frac{dv}{dt}$. This is the gradient of the tangent to the graph of v against t, which is called the velocity-time graph.

Example.—If a body starts at a velocity of u ft./sec. and its velocity increases at a uniform acceleration of f ft./sec.^{2}, find the velocity after t sec. What type of curve is the velocity-time graph ?

Since the velocity increases f ft./sec. every second, it increases ft. ft./sec. in t sec.

$$\therefore v = u + ft.$$

The velocity-time graph is the graph of this equation, which is a straight line of gradient f.

Example.—The lift s in. of a cam follower after t sec. during one revolution of the cam is given by the following table.

t ..	0	0·04	0·08	0·12	0·16	0·20	0·24	0·28	0·32
s ..	0	0·072	0·372	0·912	1·488	0·912	0·372	0·072	0

Draw a space time graph for the motion of the follower, and from it draw graphs showing the velocity in ft./sec. and the acceleration in ft./sec.^{2}.

Fig. 298 (a) shows the space-time graph. From this graph by measuring gradients the following table is constructed.

t	0	0·04	0·08	0·12	0·16	0·20	0·24	0·28	0·32
v (in./sec.)	0	3·9	11·1	21·0	0	−21·0	−11·1	−3·9	0
v (ft./sec.)	0	0·325	0·925	1·75	0	−1·75	−0·925	−0·325	0

From this table the velocity time graph in Fig. 298 (b) is drawn. The acceleration time graph in Fig. 298 (c) is constructed in a similar way. Fig. 298 (b) and (c) show that the follower starts with an acceleration of 7·5 ft./sec.^{2}, moves with increasing acceleration until $t = 0{\cdot}1$; its acceleration then diminishes but its velocity still increases reaching its greatest value 1·75 ft./sec. at $t = 0{\cdot}12$. The curve in Fig. 298 (b) is the derived curve of the curve in Fig. 298 (a), and the curve in Fig. 298 (c) the derived curve of the curve in Fig. 298 (b).

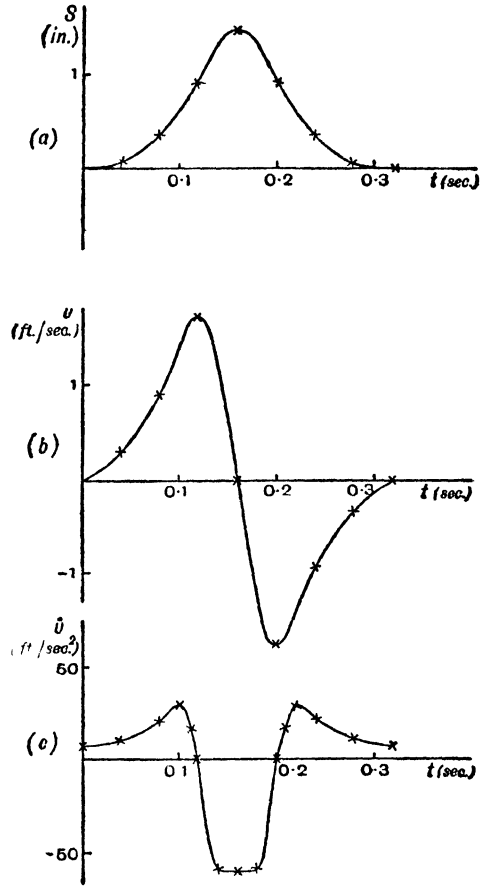


FIG. 298.

Exercise XLIV

1. A racing motor car attains its full speed from rest in 48 sec. and then covers a measured mile in 12 sec. Find in miles per hour (*a*) its speed over the measured mile, (*b*) its average speed in the first 48 sec., (*c*) its average speed in the first 60 sec.

2. When some water is being boiled its temperature rises from 15° C. to 100° C. in 10 min. ; find the average rate of increase of temperature in this time.

3. If a tank contains 40 gallons of water and the water runs out at 4 gallons per minute, find the number of gallons (n) left in the tank after t sec. Draw the graph of n against t. Express the rate of decrease of n as a negative rate of increase.

4. The following table gives the space s ft. passed over by a projectile in the bore of a gun in t sec.

s	..	0	1·2	3·6	7·0	11·3	25
t	..	0	0·002	0·004	0·006	0·008	0·01

Draw a graph of s against t, and from it deduce the graph of $\dot{s}$ against t. State the muzzle velocity of the gun.

5. The following table gives the height in inches of two boys of the same age at various ages.

Age	1	2	3	4	5	6	7	8	9	10	11	12
Height of A ..	31·1	34·8	37·7	40·0	42·3	44·9	47·0	49·5	51·8	54·2	56·5	58·2
Height of B ..	30·9	33·0	35·0	37·5	40·0	44·0	49·0	52·7	55·3	56·7	58	59

Using the same axes draw graphs to show the heights of both boys. Show that the height of one boy increases very nearly at a uniform rate and find this rate. Estimate approximately (*a*) during what time the other boy is growing faster than this boy, (*b*) when he is growing fastest, (*c*) his fastest rate of growth.

6. The relation between the distance and time for an electric tramcar starting from rest is given by :

Time in sec. ..	0	10	20	30	40	50	60	70
Distance in ft.	0	41	170	410	680	905	1070	1195

Draw a distance-time graph, and from it deduce the velocity-time and acceleration-time graphs (first and second derived curves). Estimate the velocity and acceleration after 45 sec.

7. If a man's height is x in. after t years and increases δx in. in a further time δt year, at what average rate has his height increased in the time δt ? How would you express the rate of increase of his height when he is exactly t years old ?

8. The velocity of a train in the first 90 sec. of its motion is given by :

v ft./sec. ..	0	6	12·0	27·7	40·5	47·2	49·5	44·2	35·2	30	27·7
t sec. ..	0	5	10	20	30	40	50	60	70	80	90

Plot a graph of v against t, and deduce from it the graph of $\frac{dv}{dt}$, which is the acceleration-time graph. At what time is the acceleration zero, and at what time has the train (a) an acceleration 1·1 ft./sec.^{2}, (b) a retardation 0·4 ft./sec.^{2} ?

9. When a condenser is being charged the charge on a plate, Q coulombs, increases with the time t sec. according to the following table :

t ..	0	0·01	0·02	0·03	0·04	0·05	0·06
Q ..	0	0·04	0·0625	0·0775	0·0865	0·0920	0·0962

Draw a graph of Q against t, and assuming that the charging current i ampères is given by $i = \frac{dQ}{dt}$, draw a graph of i against t. What is the greatest value of i ?

10. T° C. is the temperature of a body which has been cooling for t sec. In what direction does the graph of T against t slope ? What is the sign of $\frac{d^2T}{dt}$?

11. The lift x in. of a cam follower after t sec. is given by the following table :

240t ..	0 to 8	9	10	11	11·2	11·6	12·0	12·4	12·8	13	14	15	16 to 24
x ..	0	0·1	0·25	0·66	0·76	0·95	1·0	0·95	0·76	0·66	0·25	0·1	0

Draw a space-time graph for the motion of the follower, and from it draw velocity-time and acceleration-time graphs. At what values of t is the velocity numerically greatest ? Describe the way in which the acceleration varies as t increases from 0 to 1/10.

12. The following table gives the length of the day at intervals of 28 days from Jan. 1. The unit for the time t is 28 days, and l is the length of the day.

t ..	0	1	2	3	4	5	6
l ..	7 h. 51 m.	8 h. 54 m.	10 h. 35 m.	12 h. 26 m.	14 h. 14 m.	15 h. 46 m.	16 h. 23 m.

t ..	7	8	9	10	11	12
l ..	16 h. 7 m.	14 h. 35 m.	12 h. 57 m.	11 hr. 11 m.	9 h. 26 m.	8 h. 6 m.

Draw a graph of l against t, and from it draw a graph of $\frac{dl}{dt}$ against t. At what time during the year is the length of day (a) increasing most quickly, (b) decreasing most quickly ?

13. If a body is thrown up vertically at 80 ft./sec. its height h ft. after t sec. is given by $h = 80t - 16t^2$. Draw the graph of h against t from $t = 0$ to 5. Find the gradients of the tangents at

$t = 0, \frac{1}{2}, 1$, etc., and hence draw a graph of $\frac{dh}{dt}$ against t. At what time is $\frac{dh}{dt}$ equal to (*a*) 2 (*b*) -2? What is the meaning of $\frac{dh}{dt}$ being negative?

Rate of increase with respect to other quantities than time.

Suppose that y is a function of x, that is, y is a quantity whose value depends on the value of x. Let y increase by δy when x increases by δx. Then the ratio of the increase in

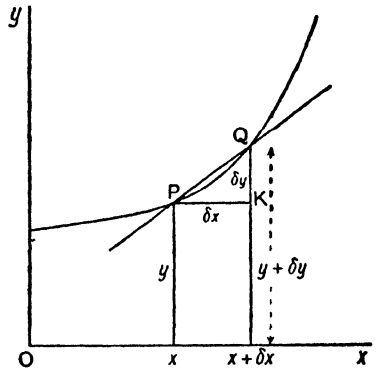


FIG. 299.

y to the increase in x is $\frac{\delta y}{\delta x}$. This ratio is similar to the difference ratio $\frac{\delta V}{\delta t}$ on p. 348. V and t in Fig. 297 are replaced by y and x in Fig. 299. $\frac{\delta y}{\delta x}$ is the gradient of the chord PQ in Fig. 299, just as $\frac{\delta V}{\delta t}$ is the gradient of PQ in Fig. 297.

Hence as δx is made smaller and smaller, $\frac{\delta y}{\delta x}$ approaches a value equal to the gradient of the tangent at P, and this is

denoted by $\frac{dy}{dx}$. It is "*the rate of increase of y with respect to x*," just as $\frac{dV}{dt}$ is the rate of increase of V with respect to the time t. It is also called "*the differential coefficient of* y *with respect to* x" or "*the derivative of* y *with respect to* x."

Notice that the words "rate of increase" are used although the variable x is not time. The words "with respect to x" show that the rate is not a time rate.

The following are examples of such rates of increase:

If a quantity of heat, Q calories, is required to raise the temperature of one gram of a liquid from some given temperature to T° centigrade, the rate of increase of Q with respect to T is the specific heat s of the liquid at the temperature T°, or in symbols $s = \frac{dQ}{dT}$.

If a beam rests on supports at the same level the shearing force S at a distance x from one end is the rate of increase of the bending moment M with respect to x, or $S = \frac{dM}{dx}$.

Approximate method of finding a rate of increase from a table of values

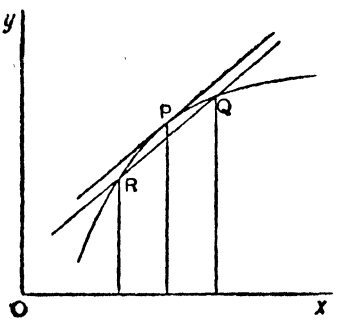


FIG. 300.

If R and Q (Fig. 300) are points near to a point P on a graph, such that the ordinate of P is midway between the ordinates at R and Q, then, provided the curve does not bend sharply between R and Q, the tangent at P is very nearly parallel to RQ. In other words the rate of increase of y at P is nearly the average rate of increase from R to Q.

This method of approximating to the rate of increase of y with respect to x often enables us to find it directly from a table of values of y and x, without drawing a graph, provided that the table gives values at sufficiently small intervals.

Example.—The following table, taken from steam tables, shows the volume V cu. ft. of 1 lb. of saturated steam at a pressure of 100 lb. per sq. in. for values of the temperature T° C. from T = 190 to 260. From the table find the value of $\frac{dV}{dT}$, the rate of increase of V with respect to T, at T = 200, and also tabulate the values of $\frac{dV}{dT}$ at T = 195, 205, 215, etc. [In this case the coefficient of expansion of steam at constant pressure is given by $\frac{dV}{dT}$].

T ..	..	190	200	210	220	230
V ..	..	4·7690	4·8901	5·0101	5·1289	5·2467

The value of $\frac{dV}{dT}$ at T = 200 is nearly equal to the mean rate of increase of V from T = 190 to T = 210.

$$\therefore \text{ at } T = 200, \frac{dV}{dT} \simeq \frac{5{\cdot}0101 - 4{\cdot}7690}{20}$$

$$\simeq \frac{0{\cdot}2411}{20}$$

$$= 0{\cdot}01205.$$

Thus the coefficient of expansion of the steam at 200° C. is 0·02105 cu. ft. per degree approximately.

The work of calculating $\frac{dV}{dT}$ at T = 195, 205, etc., which are midway between the values of T given in the table is best set out as below :

T.	*V.*	*Increase of V in 10°.*	$\frac{dV}{dT}$.
190	4·7690		
195		0·1211	0·01211
200	4·8901		
205		0·1200	0·01200
210	5·0101		
215		0·1188	0·01188
220	5·1289		
225		0·1178	0·01178
230	5·2467		

In this table the number 0·1211 in the third column is the increase in V, namely (4·8901 − 4·7690), in the 10° from T = 190 to T = 200. The number 0·01211 in the last column is obtained by dividing 0·1211 by 10, the number of degrees. This gives the average rate of increase from T = 190 to 200, which we take to be the value of $\frac{dV}{dT}$ at T = 195.

Exercise XLV

1. Draw a graph of y from the following table, and hence draw a graph of $\frac{dy}{dx}$ against x.

x ..	1	1·5	2	3	4	5	6	7	8
y ..	10	7·5	6·5	6	6·25	6·8	7·5	8·43	9·25

Find approximately the values of x at which (*a*) $\frac{dy}{dx} = -\frac{1}{2}$, (*b*) $\frac{dy}{dx} = 0{\cdot}5$.

2. Draw a graph of $y = x^2$ from $x = 0$ to 4. Measure the gradients of the tangents at $x = 0, \frac{1}{2}, 1, \ldots$ and hence plot the values of $\frac{dy}{dx}$ against x. Verify that the plotted points lie on a

straight line through (0, 0). Draw this straight line and find the expression of which it is the graph.

3. The following table gives the distance y in. moved by a piston of a motor car engine while the crank rotates through θ degrees. Draw a graph of y against θ and find from it the values of $\frac{dy}{d\theta}$ at $\theta = 30$, 90 and 120 respectively.

θ ..	0	15	30	45	60	75	90	105	120	135	150	165	180
y ..	0·77	0·83	1·02	1·30	1·66	2·07	2·46	2·83	3·16	3·43	3·62	3·74	3·77

4. The weight W lb. of a cubic foot of distilled water varies with the temperature T° C. according to the following table :

T	0	4	10	20	40	60	80
W	62·34	62·35	62·33	62·23	61·87	61·31	60·89

Draw a graph of W against T and hence draw a graph of $\frac{dW}{dT}$ against T.

5. Plot a graph of the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, from $r = 0$ to 2, taking values of r at intervals of $\frac{1}{4}$. From the graph find the values of $\frac{dV}{dr}$ at $r = 1$ and $r = 1{\cdot}5$.

6. From a table of logarithms to base 10, $\log 2{\cdot}1 = 0{\cdot}3222$, $\log 1{\cdot}9 = 0{\cdot}2788$. Find from these the approximate value of the rate of increase of $\log_{10} x$ with respect to x when $x = 2$.

7. An alternating current i ampères is given by $i = 10 \sin 200t$, where the angle is in radians. Calculate the values of i at $t = 0{\cdot}0048$ and $t = 0{\cdot}0052$, and deduce the approximate value of $\frac{di}{dt}$ at $t = 0{\cdot}005$.

8. From a table of tangents find the values of tan 44° and tan 46°, and deduce the approximate value of $\frac{d(\tan x^\circ)}{dx}$ at $x = 45$.

9. Use the formula $a^2 - b^2 = (a+b)(a-b)$ to factorize $2{\cdot}01^2 - 1{\cdot}99^2$ and deduce the approximate value of $\frac{d(x^2)}{dx}$ at $x = 2$. In the same way find the values of $\frac{d(x^2)}{dx}$ at $x = 3$, 4, 5 and 6, and compare the values obtained with the values of $2x$.

10. From a table of sines of angles in radians find the value of $\frac{d(\sin x \text{ radians})}{dx}$ at $x = 0$, 0·2, 0·4, 0·6, 0·8, 1·0, 1·2, 1·4, and verify

that the values obtained are very nearly equal to the corresponding values of $\cos x$.

11. From a table of the values of $\log_e x$ find the values of $\frac{d(\log_e x)}{dx}$ at $x = 1, 2, 3, 4$ and 5 and verify that the values obtained are nearly equal to the corresponding values of $\frac{1}{x}$.

To find the rate of increase of y with respect to x when y is a given function of x

We have seen on p. 343 that if $V = at + b$, the rate of increase of V with respect to t is a, or $\frac{dV}{dt} = a$. In the same way, if $y = ax + b$, $\frac{dy}{dx} = a$. We will now consider how to find the rates of increase of some other simple functions.

Rate of increase of y when $y = x^2$

Fig. 301 shows the graph of $y = x^2$. At P, $x = 2$, and at Q, $x = 2 + h$, where h is any number.

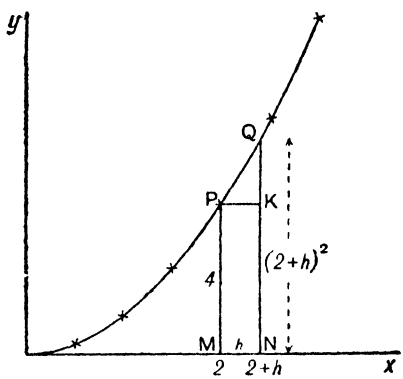


FIG. 301.

$\therefore$ MP $= 2^2$ and NQ $= (2+h)^2$

$\therefore$ increase in y = KQ $= (2+h)^2 - 2^2 = 4 + 4h + h^2 - 4 = 4h + h^2$,

and the increase in x is h,

$$\therefore \frac{\text{increase in } y}{\text{increase in } x} = \frac{\text{KQ}}{\text{PK}} = \frac{4h+h^2}{h}.$$

Now whatever value h has, except zero,*

$$\frac{4h+h^2}{h} = 4 + h.$$

Hence by taking h smaller and smaller the ratio of the increments becomes nearer and nearer to 4. Now the value of $\frac{dy}{dx}$ at $x=2$ is the number which the difference ratio approaches as Q approaches P, that is, as h approaches 0. Therefore, at $x=2$, $\frac{dy}{dx}=4$. This is the exact value of the rate of increase at $x=2$, not an approximate one, as in the case of a rate obtained graphically.

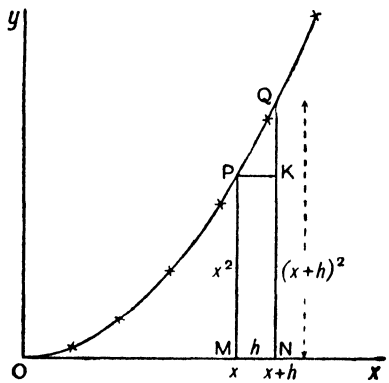


FIG. 302.

* If we put $h=0$ in $\frac{4h+h^2}{h}$ we get $\frac{0}{0}$, which has no meaning.

Now put x instead of 2, and consequently $(x+h)$ instead of $(2+h)$ as in Fig. 302. Then

$$\text{increase in } y = (x+h)^2 - x^2 = x^2 + 2hx + h^2 - x^2 = 2hx + h^2.$$

$$\therefore \frac{\text{increase in } y}{\text{increase in } x} = \frac{2hx + h^2}{h} = 2x + h,$$

provided h is not zero.

Now, in the same way as when x was 2, the difference ratio can be made as near to $2x$ as we like by making h small enough, and so the rate of increase of y at the value x is $2x$.

$$\therefore \text{ if } y = x^2, \frac{dy}{dx} = 2x \text{ for every value of } x.$$

In obtaining $\frac{dy}{dx}$ we have used h for the increment in x, instead of δx, because the work is easier to follow. We now write out the same equations, using δx instead of h.

$$\delta y = (x+\delta x)^2 - x^2 = x^2 + 2x \,.\, \delta x + (\delta x)^2 - x^2 = 2x \,.\, \delta x + (\delta x)^2.$$

and

$$\frac{\delta y}{\delta x} = \frac{2x \,.\, \delta x + (\delta x)^2}{\delta x} = 2x + \delta x.$$

As δx is made smaller and smaller $\frac{\delta y}{\delta x}$ becomes closer and closer to $2x$,

$$\therefore \frac{dy}{dx} = 2x.$$

If $y = ax^2$, where a is a fixed number,

$$\delta y = a(x+\delta x)^2 - ax^2 = a\{(x+\delta x)^2 - x^2\}.$$

We have seen above that

$$(x+\delta x)^2 - x^2 = 2x \,.\, \delta x + (\delta x)^2$$

$$\therefore \frac{\delta y}{\delta x} = \frac{a\{2x \,.\, \delta x + (\delta x)^2\}}{\delta x} = a\{2x + \delta x\}.$$

Making δx smaller and smaller the difference ratio approaches $a \times 2x$,

$$\therefore \frac{dy}{dx} = a \times 2x = 2ax.$$

Notice here that the factor a can be written outside at each step, so that the rate of increase of ax^2 with respect to x is a times the rate of increase of x^2. Just as ax^2 is a times x^2, so ax^2 increases a times as fast as x^2.

Example.—The area A of a circle of radius r is given by $A = \pi r^2$. Find $\frac{dA}{dr}$.

In the formula $A = \pi r^2$, A, r and π take the place of y, x and a in $y = ax^2$,

$$\therefore \frac{dA}{dr} = 2\pi r.$$

If the radius of a circle increases from r to $(r + \delta r)$ as in Fig. 303 the area increases by a circular strip of width δr, and the area of this strip is the difference between the areas of the two circles which is $\pi(r + \delta r)^2 - \pi r^2$.

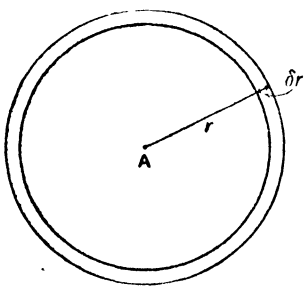


FIG. 303.

$$\therefore \delta A = \pi\{r^2 + 2r \,.\, \delta r + (\delta r)^2\} - \pi r^2$$
$$= \pi r^2 + 2\pi r \,.\, \delta r + \pi(\delta r)^2 - \pi r^2$$
$$\therefore \delta A = 2\pi r \,.\, \delta r + \pi(\delta r)^2$$
$$\therefore \frac{\delta A}{\delta r} = 2\pi r + \pi \delta r.$$

This approaches $2\pi r$ as δr approaches 0, and hence $\frac{dA}{dr} = 2\pi r$.

If δr is small in the equation $\delta A = 2\pi r \,.\, \delta r + \pi(\delta r)^2$, the last term is small compared with $2\pi r \,.\, \delta r$, so that $\delta A \simeq 2\pi r \,.\, \delta r$. The geometrical significance of this is that $2\pi r \,.\, \delta r$ is the area

the strip would have if it were a *straight* strip of width δr and length $2\pi r$., which is the circumference of the inner circle.

The differential coefficient of $\frac{1}{x}$

Fig. 304 shows the graph of $y = \frac{1}{x}$. P and Q are the points whose abscissæ are x and $x+h$. From P to Q

$$\text{increase in } y = \frac{1}{x+h} - \frac{1}{x}$$

$$= \frac{x-(x+h)}{(x+h)x}$$

$$= \frac{-h}{(x+h)x}$$

$$\therefore \frac{\text{increase in } y}{\text{increase in } x} = \frac{\frac{-h}{(x+h)x}}{h}$$

$$= -\frac{1}{(x+h)x}.$$

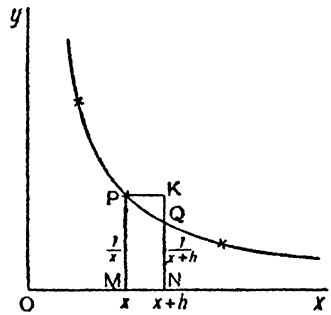


FIG. 304.

This difference ratio approaches the value $-\frac{1}{x^2}$ as h approaches 0, and the value which it approaches is the value of $\frac{dy}{dx}$ at P. Hence, $\frac{dy}{dx} = -\frac{1}{x^2}$ at P.

$$\therefore \text{ when } y = \frac{1}{x}, \quad \frac{dy}{dx} = -\frac{1}{x^2}.$$

Note that actually y decreases from P to Q. This is shown by a negative increase.

The differential coefficient of x^n

The rates of increase of ax^2 and $\frac{1}{x}$ found above are special cases of the general formula :

$$\frac{d(ax^n)}{dx} = nax^{n-1}.$$

This formula is true for all numerical values of a and n; a proof will be given in Part III.

By putting $n=2$ in the formula we get:

$$\frac{d(ax^2)}{dx}=2ax^{2-1}=2ax.$$

In the same way, since $\frac{1}{x}=x^{-1}$,

$$\frac{d\left(\frac{1}{x}\right)}{dx}=\frac{d(x^{-1})}{dx}=-x^{-1-1}=-x^{-2}=-\frac{1}{x^2}.$$

These two results have been proved above.

Examples.—Using the formula for $\frac{d(ax^n)}{dx}$ find the values of

$$\frac{d(4x^{10})}{dx},\quad \frac{d\left(\frac{2}{x^3}\right)}{dx},\quad \frac{d(m^{5\cdot 8})}{dm},\quad \frac{d(4\sqrt{t})}{dt}.$$

$$\frac{d(4x^{10})}{dx}=10.4x^{10-1}=40x^9.$$

$$\frac{d\left(\frac{2}{x^3}\right)}{dx}=\frac{d(2x^{-3})}{dx}=-3.2x^{-3-1}=-6x^{-4}=-\frac{6}{x^4}.$$

$$\frac{d(m^{5\cdot 8})}{dm}=5\cdot 8m^{5\cdot 8-1}=5\cdot 8m^{4\cdot 8}.$$

$$\frac{d(4\sqrt{t})}{dt}=\frac{d(4t^{\frac{1}{2}})}{dt}=\frac{4}{2}t^{\frac{1}{2}-1}=2t^{-\frac{1}{2}}=\frac{2}{\sqrt{t}}.$$

Example.—When a certain gas expands adiabatically the pressure and volume are related by the equation $pv^{1\cdot 4}=C$, where C is a constant. Find $\frac{dp}{dv}$ and show that it equals $\frac{-1\cdot 4p}{v}$.

Dividing the given equation by $v^{1\cdot4}$, $p = Cv^{-1\cdot4}$,

$$\therefore \frac{dp}{dv} = -1\cdot4Cv^{-1\cdot4-1} = -1\cdot4Cv^{-2\cdot4} = \frac{-1\cdot4C}{v^{2\cdot4}}.$$

But $C = pv^{1\cdot4}$,

$$\therefore \frac{dp}{dv} = \frac{-1\cdot4pv^{1\cdot4}}{v^{2\cdot4}} = \frac{-1\cdot4p}{v^{2\cdot4-1\cdot4}} = \frac{-1\cdot4p}{v}.$$

Example.—The candle-power C of a certain electric lamp is related to the voltage V by the equation $C = 2\cdot5 \times 10^{-7}V^4$. Find the differential coefficient of C with respect to V and calculate its value when (*a*) $V = 200$, (*b*) $C = 50$.

Since $C = 2\cdot5 \times 10^{-7}V^4$

$$\therefore \frac{dC}{dV} = 2\cdot5 \times 10^{-7} \cdot 4V^3 = 10^{-6} \cdot V^3,$$

or the candle-power is increasing at $10^{-6}V^3$ per volt.

When $V = 200$, $\frac{dC}{dV} = 10^{-6} \times 200^3 = 8$.

When $C = 50$, $2\cdot5 \times 10^{-7}V^4 = 50$

$\therefore V^4 = 2 \times 10^8$

$\therefore V = \sqrt[4]{2} \times 10^2$

$\doteq 1\cdot189 \times 10^2 = 118\cdot9$.

Hence $\frac{dC}{dV} = 10^{-6}\ V^3 = \frac{10^{-6}V^4}{V} = \frac{200}{V} \doteq 1\cdot682$.

Rate of increase of the sum of two expressions

If $y = 2x^2 + 5x^3 - 3x^4$, when x increases by any amount the increase in y

= increase in $2x^2$ + increase in $5x^3$ − increase in $3x^4$.

$$\therefore \frac{\text{increase in } y}{\text{increase in } x} = \frac{\text{increase in } 2x^2}{\text{increase in } x} + \frac{\text{increase in } 5x^3}{\text{increase in } x} - \frac{\text{increase in } 3x^4}{\text{increase in } x}.$$

This is true however small the increase in x may be, and so it must be true when each difference ratio is replaced by the rate of increase at x.

$$\therefore \frac{dy}{dx} = \frac{d(2x^2)}{dx} + \frac{d(5x^3)}{dx} - \frac{d(3x^4)}{dx}$$

$$= 2.2x + 5.3x^2 - 3.4x^3$$

$$= 4x + 15x^2 - 12x^3.$$

We conclude that *the rate of increase of the sum of two or more expressions is the sum of their rates of increase taken separately.*

Example.—A body moves s ft. in t sec. where $s = 12t - t^3$. Find its velocity at $t = 0, 1, 2, 3$.

$$\text{Velocity} = \dot{s} = \frac{d(12t)}{dt} - \frac{d(t^3)}{dt} = 12 - 3t^2.$$

Therefore the values of $\dot{s}$ are given by

t	0	1	2	3
$\dot{s}$	12	9	0	−15

The last value shows that at $t = 3$ the body is moving back towards its starting point.

Exercise XLVI

1. If a body fall s ft. in t sec. from rest, $s = 16t^2$.

   (*a*) Find the average rate at which it falls in the first 3 sec. and also the average rate at which it falls in the next second.

   (*b*) Find the average velocity from $t = 3$ to $t = 3{\cdot}1$, and from $t = 3$ to $t = 3{\cdot}01$.

   (*c*) Show that the average velocity from $t = 3$ to $t = 3 + h$ is $(96 + 16h)$ ft./sec., and from this formula find the average velocity from $t = 3$ to $t = 3{\cdot}0001$ and from $t = 3$ to $3{\cdot}000001$.

2. Show that, using the same formula $s = 16t^2$ as in Question 1, the average velocity while the time increases from t to $(t+h)$ is $(32t + 16h)$. Find the average velocities for $h = 0{\cdot}01$ and $h = 0{\cdot}0001$. What is the velocity after t sec.?

3. If P and Q are the points with abscissæ x and $(x+h)$ on the graph of $y = 6x^2$, find the gradient of PQ and the gradient of the tangent at P.

4. If $y = 6x^2$, what is the value of δy when x increases by δx? Find the value of $\frac{\delta y}{\delta x}$ and deduce from it the value of $\frac{dy}{dx}$.

5. If $y = \frac{1}{x}$ find the value of δy when x increases by δx. Hence find the value of $\frac{dy}{dx}$.

6. If $y = x^3$ find the increase in y when x increases by h, and hence find the ratio of the increase in y to the increase in x in its simplest form. Deduce the value of $\frac{dy}{dx}$.

7. If $y = ax + b$ show that $\frac{\delta y}{\delta x}$ has the same value for all values of δx. What is the meaning of this? What is the value of $\frac{dy}{dx}$?

8. Find the rate of increase of $\frac{1}{x^2}$ with respect to x by the same method as in question 5.

In the following questions use the formula $\frac{d(x^n)}{dx} = nx^{n-1}$.

Find the values of $\frac{dy}{dx}$ when y is:

9. $3x^3$. **10.** $4 - 2x^2$. **11.** $90x^3$. **12.** $2x^4 - \frac{1}{3}x^6$.

13. $4x^3 - 7x^2 + 21x - 50$. **14.** $0{\cdot}02x - 2{\cdot}54x^2 - 0{\cdot}72x^3$.

15. $\frac{1}{x^3}$. **16.** $\frac{1}{8x^4}$. **17.** $3x^{1{\cdot}6}$. **18.** $\frac{1}{4x^2} - \frac{1}{2x}$.

Find the values of the following differential coefficients:

19. $\frac{d(5t^2)}{dt}$. **20.** $\frac{d(\frac{1}{5}n^3)}{dn}$. **21.** $\frac{d(0{\cdot}1r^2)}{dr}$.

22. $\frac{d(2000x^4)}{dx}$. 23. $\frac{d(7l^7)}{dl}$. 24. $\frac{d(4p^3+5p^4)}{dp}$.

25. $\frac{d\left(\frac{1}{n}x^n\right)}{dx}$. 26. $\frac{d(ut+\frac{1}{2}qt^2)}{dt}$. 27. $\frac{d\left(\frac{100}{v}\right)}{dv}$.

28. $\frac{d\left(\frac{1}{r^k}\right)}{dr}$. 29. $\frac{d(2\times 10^{-6}\times V^{4\cdot 8})}{dV}$. 30. $\frac{d(4\sqrt{z})}{dz}$.

31. Find the gradient of the tangent to the curve $y=x^3-x$ at $x=\frac{1}{2}$ and at $x=1\frac{1}{2}$. For what value of x is the gradient of the tangent zero? Find the value of y at this point.

32. If a body is thrown upwards at 80 ft./sec. under gravity its height h ft. after t sec. is given by $h=80t-16t^2$. Find its velocity at $t=2$ and $t=3$. When is its velocity zero and how high is it then?

33. Find the gradients of the tangents to $y=x^2-3x+2$ at the points where the curve crosses the axis of x.

34. Show that the rate of increase of the volume of a sphere with respect to the radius equals the area of the surface of the sphere.

35. Find the co-ordinates of the points on the graph of $y=x^3-3x^2+6$ at which the tangents are horizontal.

Maxima and minima

To find for what values of x the value of y is a maximum or a minimum we have to find the values of x at which the tangent to the graph is horizontal (as at A and B in Fig. 303); these are the values of x which make $\frac{dy}{dx}=0$. If the curve is below the tangent on both sides as at A, the value of y is a maximum at A, whereas, if the curve is above the tangent as at B, the value of y is a minimum.

Now just to the left of A the value of $\frac{dy}{dx}$ is positive, and just to the right of A it is negative.

Therefore y has a maximum value at A if $\frac{dy}{dx}=0$ at A, and if $\frac{dy}{dx}$ is positive just to the left of A, and negative just to the right of A.

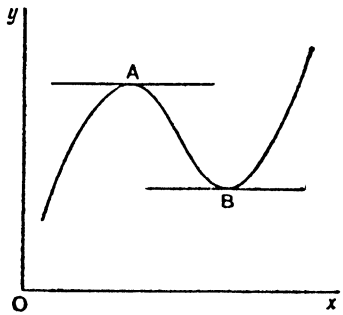


FIG. 305.

In the same way it can be seen that y has a minimum value at B if $\frac{dy}{dx}=0$ at B, and if $\frac{dy}{dx}$ is negative just to the left of B, and positive just to the right of B.

Example.—Find the greatest rectangular area that can be enclosed by 400 yds. of fencing.

If one side of the rectangle is x yds. the length of a perpendicular side is $(200-x)$ yds. Hence the area A sq. yds. is given by :

$$A = x(200-x) = 200x - x^2$$

$$\frac{dA}{dx} = 200 - 2x$$

$$\therefore \frac{dA}{dx} = 0 \text{ when } 200 - 2x = 0,$$

from which $x = 100$.

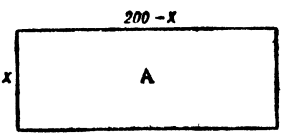


FIG. 306.

When $x = 100$, $A = 100 \times 100 = 10{,}000$.

If x is just less than 100, say 99·9, $\frac{dA}{dx}$ is positive, and if x is just greater than 100, $\frac{dA}{dx}$ is negative. Therefore, 10,000 is the maximum value of A.

Fig. 307 shows the graph of A against x.

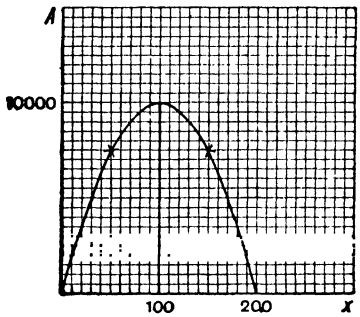


FIG. 307.

Exercise XLVII

Find any maximum and minimum values of y, and sketch a rough graph of y when :

1. $y = 2x - x^2$.
2. $y = x^3 - 3x^2$.
3. $y = 1 - x - x^2$.
4. $y = t^2 - 6t$.
5. $y = 1 - 2v + v^2$.
6. $y = r^3 - 3r$.

7. The bending moment of a certain beam at a distance x from one end is given by $M = \frac{1}{2}wx(l - x)$. Find its maximum value.

8. A closed tank is to be made with a square base to hold 40 gallons. If the edge of the tank is x ft., show that its surface area S ft.^{2} is given by $S = 2x^2 + \frac{25 \cdot 6}{x}$. Hence find its most economical shape. [1 gallon of water occupies 0·16 cu. ft.]

9. The horse-power transmitted by a belt moving at v ft./sec. is proportional to $fv - \frac{wv^3}{g}$, where f lb. wt./ft.^{2} is the maximum allowable stress in the belt, and w lb./ft.^{3} is the weight per unit

volume of the belt. If $f = 23{,}800$ and $w = 62$, find the value of v for which the horse-power is a maximum.

10. The power W watts given to an external circuit by a battery of internal resistance r and electromotive force E volts when the current is i ampères, is given by $W = Ei - ri^2$. If $E = 12$, $r = 16$, find for what current W is a maximum.

11. If $y = 4x - x^2$, find the least and greatest values of y between $x = 1$ and $x = 2{\cdot}5$.

12. If a beam AB of weight W lb. wt. and length l ft. is clamped horizontally at B and supported at A at the same level, its bending moment M lb. wt. ft. at x ft. from A is given by $M = \frac{W}{l}(\frac{3}{8}lx - \frac{1}{2}x^2)$. Show that M has a maximum value at $x = \frac{3}{8}l$, and calculate this maximum. Sketch a graph of M and use it to show that M is *numerically* greatest at $x = l$.

CHAPTER XVI

INTEGRATION

To find the distance moved by a body from its velocity-time graph

If the velocity v is uniform, the distance it moves in a time t is vt.

The graph of the velocity against the time is the straight line DC in Fig. 308. In this figure OB represents t, OD represents v and hence the distance moved vt is represented by $OD \times OB$, which is the area of the rectangle OBCD. For convenience we shall say that the distance moved is equal to the area OBCD.

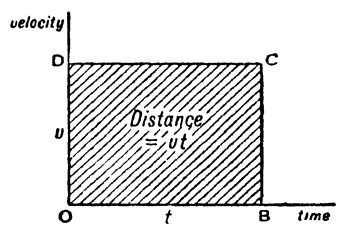


FIG. 308.

Variable velocity

When a body is moving with variable velocity its velocity-time graph is a curve, such as that in Fig. 309. Suppose we require to find the distance it moves in the time AB from $t=a$ to $t=b$. Divide the area ABCD into a number of strips by ordinates at equal intervals of t. During one of these intervals, such as MN, the velocity is nearly uniform and its value is nearly equal to PM throughout the interval. Hence

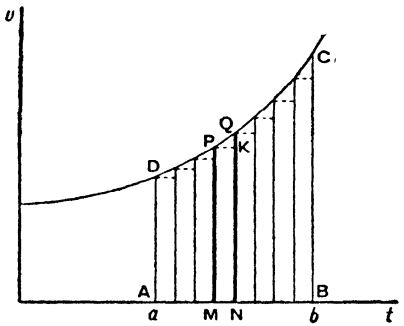


FIG. 309.

the distance the body moves in the time MN is nearly equal to the area PMNK. Similarly, the distance moved in each interval is nearly equal to the area under the dotted horizontal line, and therefore, adding up the distances moved in the intervals, the whole distance moved in the time AB is nearly equal to the total area under the dotted lines. By dividing AB into smaller and smaller intervals of time the total area under the dotted lines approximates more and more closely to the distance moved and also to the area ABCD bounded by the t axis, the curve DC and the ordinates AD and BC, which is called the area under the graph from $t=a$ to $t=b$. For this reason we conclude that the distance travelled in the time from $t=a$ to $t=b$ is *exactly* equal to the area ABCD under the graph of the velocity v from $t=a$ to $t=b$.

The following example shows how the distance can be calculated.

Example.—During the minute after the instant at which an accelerating car has attained a velocity of 12 m.p.h. its velocity is given by :

Time (sec.) ..	0	10	20	30	40	50	60
Velocity (m.p.h.)	12	34	49	59	65½	69	70

Find the distance it moves in (*a*) the whole minute, (*b*) the last 20 sec.

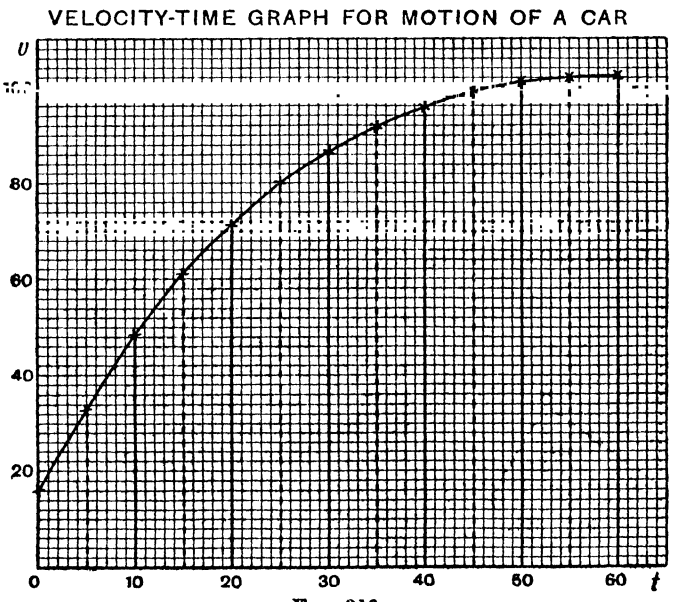


FIG. 310.

The work is simplified by first finding the velocity in ft. per sec.

t (sec.) ..	0	10	20	30	40	50	60
v (ft./sec.) ..	17·6	49·9	71·9	86·5	96·1	101·2	102·7

From this table a graph is drawn in Fig. 310. The area under it is divided into six strips by ordinates at intervals of 10 sec. The ordinates at the middle of each interval are shown by dotted lines. The area of each strip is nearly equal to mid-ordinate × base.

$\therefore$ Distance travelled in 1 min.

$$= \text{area under graph from } t=0 \text{ to } t=60$$
$$\simeq 10\{\text{sum of mid-ordinates}\}$$
$$\simeq 10\{33{\cdot}4+61+80+91{\cdot}6+98{\cdot}4+102{\cdot}2\}$$
$$\simeq 10\times 466{\cdot}6 \text{ ft.}$$
$$\simeq 4670 \text{ ft.}$$

Also, distance moved in last 20 sec.

$$\simeq 10\{98{\cdot}4+102{\cdot}2\}$$
$$\simeq 2010 \text{ ft.}$$

Example.—A mine cage drops for a time at uniform acceleration and then slows up with uniform retardation. If it comes to rest after it has dropped 800 ft. in 25 sec. find its maximum speed.

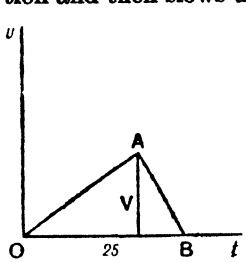


FIG. 311.

Since the cage starts from rest at uniform acceleration the velocity-time graph is a straight line OA through the origin until the cage stops accelerating. Then, while the cage is slowing up the velocity-time graph is another straight line AB because the retardation is constant.

The maximum velocity V is the ordinate of A, and the total distance travelled is the area under the graph, i.e. the area of the triangle OAB. But the area of the triangle is $\frac{1}{2}V \times 25$.

$$\therefore \tfrac{1}{2}V \times 25 = 800$$
$$\therefore V = 64 \text{ ft./sec.}$$

Exercise XLVIII

1. The relationship between the time and velocity of a body is given by :

t sec. ..	0	1	2	3	4	5	6
v ft./sec. ..	0	0·5	3	7·5	14	22·5	35·5

Draw a graph of v against t, and deduce the distance passed over in 6 sec.

2. When a train is travelling at 40 m.p.h. steam is shut off and the brakes applied. The following table gives its speed after t sec.

t sec. ..	0	12	24	36	48	60
v m.p.h. ..	40	30	23	20·4	14	12

How far does the train travel in this time?

3. A flywheel slowing down has speeds given by the following table :

t min.	0	1	2	3	4	5
N. revs. per min.	320	200	116	60	23	0

Draw a graph of N against t, and from it find the total number of turns the flywheel makes in coming to rest.

4. If the velocity-time graph for the motion of a train between two stations is OABC shown in Fig. 312, find the distance between the stations. Find also the acceleration in ft./sec.^{2} while the train is getting up speed.

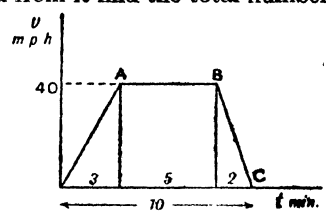


FIG. 312.

Space-time graph from the velocity-time graph

The following table gives the velocity, v miles per hour, of an electric train as it goes from one station to another at t hr. after it has left the first station. The first line gives the values of $60t$, which is the time in minutes.

$60t$ ..	0	1	2	3	4	5	6	7—13	14	15	16
v ..	0		39	47	52	54	55·7	56	50	43	0

From 7th to 13th minute the velocity is constant, so that this part of the graph of v is a straight line. Fig. 313 shows the graph of v. The area under it is divided into strips by ordinates at intervals of one minute.

VELOCITY-TIME GRAPH FOR AN ELECTRIC TRAIN

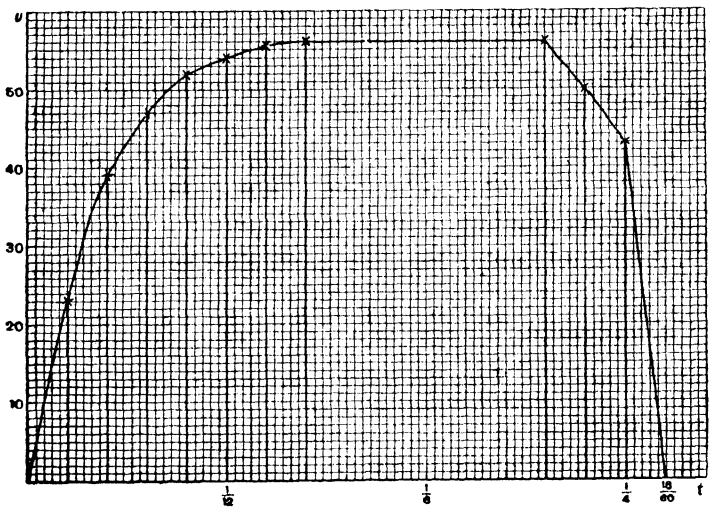


FIG. 313.

The distance in miles travelled in the first minute is the area of the first strip, i.e. it is represented by this area.

The following table gives the mid-ordinates of the strips in miles per hour, the distance travelled in each minute (except from $t=\frac{7}{60}$ to $t=\frac{13}{60}$ where the whole distance travelled in this time is given), and in the last column the distance travelled in the first t hr. In this column :

the distance from $t=0$ to $t=\frac{2}{60}=0{\cdot}217+0{\cdot}533=0{\cdot}750$.

Also : $1{\cdot}473=$ the distance from $t=0$ to $t=\frac{3}{60}$
$=$(distance from $t=0$ to $t=\frac{2}{60}$) + (distance from $t=\frac{2}{60}$ to $t=\frac{3}{60}$)
$=0{\cdot}750+0{\cdot}723=1{\cdot}473$.

t hr.	*Mid-ordinate in m.p.h.*	*Distance travelled (ml.).*	*Distance in t hr.*
0			
	13·0	0·217	
$\frac{1}{60}$			0·217
	32·0	0·533	
$\frac{2}{60}$			0·750
	43·4	0·723	
$\frac{3}{60}$			1·473
	49·4	0·823	
$\frac{4}{60}$			2·296
	53·0	0·883	
$\frac{5}{60}$			3·179
	55·0	0·917	
$\frac{6}{60}$			4·096
	55·8	0·930	
$\frac{7}{60}$			5·026
	56·0	5·600	
$\frac{13}{60}$			10·626
	55·0	0·883	
$\frac{14}{60}$			11·509
	46·2	0·770	
$\frac{15}{60}$			12·279
	21·4	0·357	
$\frac{16}{60}$			12·636

From this table a graph of the distance travelled is plotted against t; it is the thick curve in Fig. 314. This graph can be used to find how long the train takes for any given distance; e.g. when the distance is 9 miles, $t = (11{\cdot}25/60)$, so the train takes (11·25/60) hr., i.e. 11¼ min. for the first 9 ml.

If the distance from the first station is s ml., then this is the graph of s against t, but instead of s being measured from the station it might be measured from a station two miles farther back, then the graph of s is the dotted curve in Fig. 314, which is 2 units above the other curve.

Wherever s is measured from, $v = \frac{ds}{dt}$ and hence the area under the graph of $\frac{ds}{dt}$ for any interval of time, say from $t = a$ to $t = b$, is equal to the increase in the value of s from $t = a$ to $t = b$.

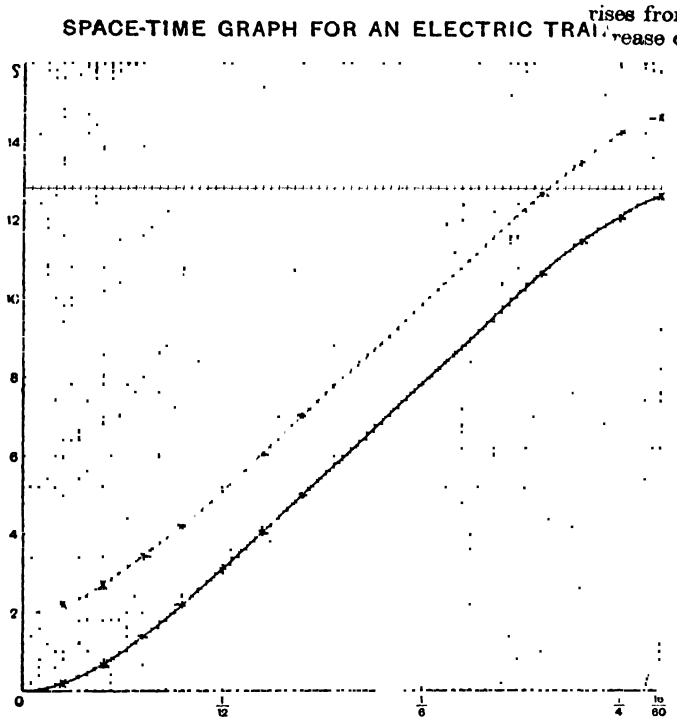


FIG. 314.

Integration

By dividing up the area under the velocity-time graph into a large number of very small parts we have seen that the distance travelled is exactly equal to the area under the velocity-time graph. This process of calculating the value of a quantity by dividing it up into a larger and larger number of smaller and smaller parts is called integration. Just as the word "integer" means "whole number," so "integration" means finding the whole from the sum of the parts. This method was first used by Archimedes, who lived in Syracuse from 287 B.C.

is nearly ..., to find the volumes of a sphere and a cone, and to under ...ther similar problems.

The relationship between differentiation and integration

We have seen that the area under the graph of $\frac{ds}{dt}$ for any interval of time, say from $t=a$ to $t=b$, is the distance travelled by the body in that time or the increase in the value of s in that time.

Thus the graph of $\frac{ds}{dt}$ is found from the graph of s by measuring the gradients of tangents, or differentiation, and the graph of s can be found from the graph of $\frac{ds}{dt}$ (provided we know the value of s at $t=0$) by adding up areas, or integration. The graph of s is called an integral curve or sum curve of the graph of t.

Integration with respect to any variable

If we write z in place of s and x in place of t the differential coefficient $\frac{ds}{dt}$ becomes $\frac{dz}{dx}$, and hence the area under the graph of $\frac{dz}{dx}$ from $x=a$ to $x=b$ equals the increase of z from $x=a$ to $x=b$. We will consider below the special case of $z=x^3$.

The area under the graph of $3x^2$ from $x=a$ to $x=b$

If $z=x^3$, $\frac{dz}{dx}=3x^2$. Fig. 315 shows the graph of x^3 above the graph of $3x^2$. The area under the graph of $3x^2$ from $x=a$ to $x=b$ is divided into a number of strips by ordinates at equal intervals of x. Ordinates are also drawn to the graph of x^3 at the same intervals. Consider the area of one of the

strips PMNQ under the graph $3x^2$. Since $\frac{d(x^3)}{dx}$ = base of ordinate MP is equal to the gradient of the tangent at p on the graph of x^3, which is approximately $\frac{kq}{pk}$.

$$\therefore \text{MP} \simeq \frac{kq}{pk} = \frac{kq}{\text{MN}}.$$

$$\therefore \text{the area of PMNK} = \text{MP} \times \text{MN}$$

$$\simeq \frac{kq}{\text{MN}} \times \text{MN}$$

$$\simeq kq.$$

Thus each of the steps similar to kq along the graph of x^3

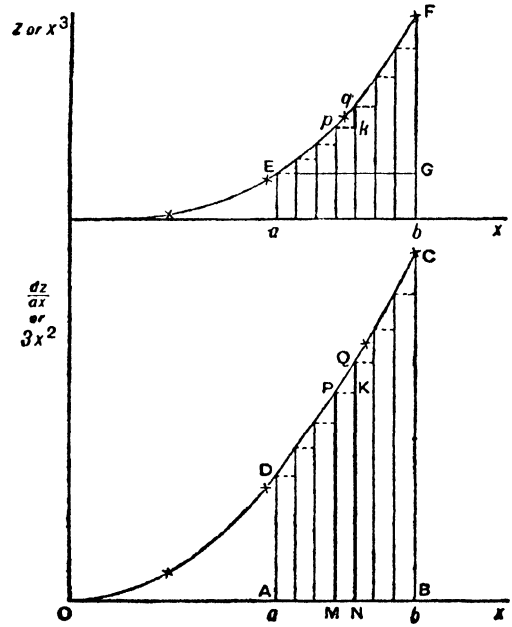


FIG. 315.

is nearly equal to the area of the corresponding rectangle under the graph of $3x^2$.

Hence, by addition, GF, the total increase in x^3 from E to F, is nearly equal to the sum of the areas of the rectangles below the dotted lines in the lower graph.

If the intervals are made smaller and smaller this approximation becomes more and more accurate. Hence we conclude that the area ABCD under the graph of $3x^2$ from $x=a$ to $x=b$ is exactly equal to GF, the increase of x^3 as x increases from a to b.

But at E, $x^3=a^3$, and at F, $x^3=b^3$.

$\therefore$ GF $=b^3-a^3$.

$\therefore$ area under the graph of $3x^2$ from $x=a$ to $x=b$ is (b^3-a^3).

$\left[x^3\right]_a^b$ is written as short for the words "the increase of x^3 from $x=a$ to $x=b$." Using this notation, the area under the graph of $3x^2$ from $x=1$ to $x=2$

$$=\left[x^3\right]_1^2=2^3-1^3=7.$$

Example.—Find the area under the graph of x^3 from $x=2$ to $x=4$.

The differential coefficient of x^4 is $4x^3$.

$\therefore$ if $z=\frac{1}{4}x^4$, $\dfrac{dz}{dx}=\frac{1}{4}\cdot 4x^3=x^3$.

$\therefore$ area under the graph of x^3 from $x=2$ to $x=4$.

$$=\left[\tfrac{1}{4}x^4\right]_2^4=\tfrac{1}{4}4^4-\tfrac{1}{4}\cdot 2^4=64-4=60.$$

Every expression such as $\frac{1}{4}x^4+7$, $\frac{1}{4}x^4-0{\cdot}3$, $120+\frac{1}{4}x^4$ obtained by adding a constant to $\frac{1}{4}x^4$ has the same differential coefficient as $\frac{1}{4}x^4$, namely x^3. Every such expression is included in $\frac{1}{4}x^4+c$ where c is any constant whatever. Thus we could write :

Area under the graph of x^3 from $x=2$ to $x=4$

$$=\left[\tfrac{1}{4}x^4+c\right]_2^4=[\tfrac{1}{4}4^4+c]-[\tfrac{1}{4}2^4+c]=64+c-4-c=60.$$

Hence the addition of c makes no difference to the calculation of the area under a curve between two ordinates. It is equivalent to the addition of 2 to s to get the upper graph in Fig. 314.

Exercise XLIX

1. The speed v ft./sec. of a motor car after t sec. from rest is given by :

t ..	0	5	10	15	20	25	30
v ..	0	12·6	19·9	26·7	33·7	44·4	50·4

Draw a velocity-time graph and from it find by the mid-ordinate rule the distances described in successive intervals of five seconds. Hence make a table of the distance s ft. described in t sec., and draw a graph of s against t. From it estimate the time taken for the first 200 yds.

2. When a condenser is being charged the current flowing into the plate is given by :

t sec. ..	0	0·1	0·2	0·3	0·4	0·5	0·6
i mA ..	200	122	74	44	28	16	10

Assuming that the current in milli-amperes is the rate of increase of the charge Q milli-coulombs on the plate, and that $Q=0$ at $t=0$, draw a graph to show the values of Q from $t=0$ to $t=0{\cdot}6$.

3. When a voltage V is applied to an inductance of $\frac{1}{2}$ henry the voltage is the rate of increase of $\frac{1}{2}i$, where i is the current in ampères, or $V=\frac{1}{2}\frac{di}{dt}$. The following table gives the values of the voltage at intervals of 0·01 sec. Given that $i=0$ at $t=0$, draw a graph of i against t.

t ..	0	0·01	0·02	0·03	0·04	0·05
V ..	200	198	162	118	62	0

From the graph find the value of t when $i=8{\cdot}5$.

4. Using the table in Question 1, p. 375, draw a graph of the distance described against the time for values of t from 0 to 6.

5. From the table in Question 2, p. 375, draw a graph of the distance s ft. described in t sec. from $t=0$ to $t=60$.

6. During one revolution of a cam the follower is required to have an acceleration in ft./sec.^{2} given by the following table, the

first value being when the follower is at rest in its lowest position. Draw the acceleration-time graph and deduce from it graphs to show the velocity in ft./sec. and the lift of the follower in inches at any time.

t	..	0	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0·09
v	..	0	35	120	200	200	200	200	120	35	0

t	..	0·10	0·11	0·12	0·13	0·14	0·15	0·16	0·17	0·18
v	..	−35	−120	−200	−200	−200	−200	−120	−35	0

7. Draw a curve from the following table and deduce from it an integral curve, showing the integral of y with respect to x.

x	..	0	1	2	3	4	5
y	..	6·00	7·33	9·00	10·80	12·67	14·57

8. If the values of $\frac{dz}{dx}$ are given by the following table, and $z=2$ at $x=0$, draw a graph of z. Find the value of x which makes $z=5$.

x	..	0	2	4	6	8
$\frac{dz}{dx}$	..	0·71	1	1·25	1·42	1·58

9. If $\frac{dz}{dx}=2x$, what is the value of z? Find the area under the graph of $2x$ from $x=a$ to b. Draw the graph and find the area by using the formula for the area of a trapezium.

10. If $\frac{ds}{dt}=\frac{1}{2}t^2$, what is the value of s? Find the value of s at $t=2$ if $s=1$ at $t=0$.

11. A body describes s ft. in t sec. If its velocity after t sec. is $32t$, find how far it goes from $t=1$ to $t=4$.

Write down the functions whose rates of increase with respect to x are:

12. $6x$.

13. $6x+5$.

14. $9x^3$.

15. $9x^3+6x+5$.

16. $10x^5$.

17. $\frac{1}{x^2}$.

18. $1{\cdot}8x^{0{\cdot}8}$.

19. $6x+\frac{1}{x^2}$.

The notation used in integration

Let us now consider how the area under $y = 3x^2$ from $x = 1$ to $x = 2$ is obtained by adding small strips (this is the area found on p. 381). Suppose the area is divided up into ten strips by ordinates at 1·1, 1·2, 1·3, etc., as in Fig. 316. The sum of the areas of these strips is nearly :

$$(3 \times 1^2)0{\cdot}1 + (3 \times 1{\cdot}1^2)0{\cdot}1 + (3 \times 1{\cdot}2^2)0{\cdot}1 + \ldots + (3 \times 1{\cdot}9^2)0{\cdot}1.$$

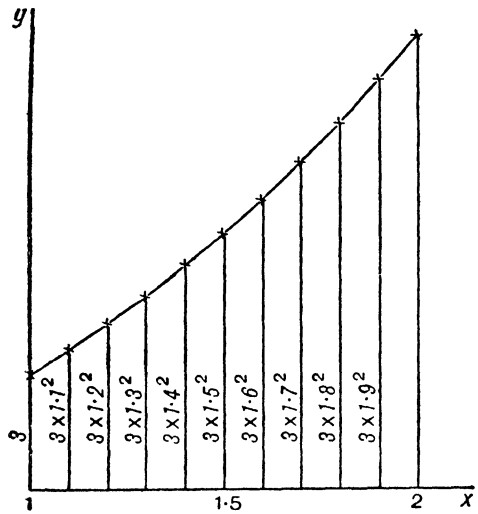


Fig. 316.

If we denote the small division 0·1 by the symbol δx then this sum is the sum of all the values of $3x^2 \,.\, \delta x$ obtained by putting $x = 1,\ 1{\cdot}1,\ 1{\cdot}2,\ \ldots\ 1{\cdot}9$. We use the shorthand $\sum_{x=1}^{x=2} 3x^2 \,.\, \delta x$ for this sum ; Σ being the capital sigma of the Greek alphabet, and standing here for the words " the sum of all the values of."

Since $\sum_{x=1}^{x=2} 3x^2 \,.\, \delta x$ is only an approximation for this area, a new notation is needed for the exact area. The notation

that everyone uses to-day is the notation invented by Leibniz, $\int_1^2 3x^2dx$. In this notation $\int$ is an elongated S, standing for "sum," just as Σ does, but with this difference, that $\int_1^2 3x^2dx$ implies that in finding the sum the elements $3x^2 . \delta x$ have been made smaller and smaller. It is read, "the integral of $3x^2dx$ from $x=1$ to $x=2$," and $x=1$ and $x=2$ are called the lower and upper limits of integration. We have seen that the area under $y=3x^2$ from $x=1$ to 2 equals the increase in the value of x^3+c from $x=1$ to 2, because

$$\frac{d(x^3+c)}{dx}=3x^2.$$

Hence $$\int_1^2 3x^2dx=\Big[x^3+c\Big]_1^2=(2^3+c)-(1^3+c)=7.$$

For this reason we write :

$$\int 3x^2dx=x^3+c.$$

$\int 3x^2dx$ is called the *indefinite integral* of $3x^2dx$ because its value (x^3+c) depends on x. It does not give the definite area until we have substituted numerical values of x. On the other hand $\int_1^2 3x^2dx$ is called a *definite integral* because it has a definite value, and equals the area under $y=3x^2$ from $x=1$ to $x=2$.

In the same way, since

$$\frac{d(x^4+c)}{dx}=4x^3,$$

$$\int 4x^3dx=x^4+c,$$

and since

$$\frac{d(x^{n+1}+c)}{dx}=(n+1)x^n,$$

$$\int (n+1)x^ndx=x^{n+1}+c.$$

This last result suggests differentiating $\frac{ax^{n+1}}{n+1}$ in order to find the integral of $ax^n dx$.

$$\frac{d\left(\frac{ax^{n+1}}{n+1}+c\right)}{dx}=\frac{(n+1)ax^n}{n+1}=ax^n.$$

$$\therefore \int ax^n dx=\frac{ax^{n+1}}{n+1}+c.$$

This is the formula for integrating any power of x except x^{-1}. [The integral of x^{-1} will be found in Part III.] Using suitable values of a and n we find,

$$\int 2x^4 dx=\frac{2x^{4+1}}{4+1}+c=\frac{2x^5}{5}+c.$$

$$\int \tfrac{1}{3}t^7 dt=\tfrac{1}{3}\cdot\frac{t^{7+1}}{7+1}+c=\frac{t^8}{24}+c.$$

If l is a constant, $\int\left(\frac{z}{l}\right)^3 dz=\int\frac{1}{l^3}\cdot z^3 dz=\frac{1}{l^3}\cdot\frac{z^4}{4}+c=\frac{z^4}{4l^3}+c.$

$$\int\frac{1}{x^2}dx=\int x^{-2}dx=\frac{x^{-2+1}}{-2+1}+c=-x^{-1}+c=-\frac{1}{x}+c.$$

$$\int r^{2\cdot 7}dr=\frac{r^{2\cdot 7+1}}{2\cdot 7+1}+c=\frac{r^{3\cdot 7}}{3\cdot 7}+c.$$

Example.—Find the area under $y=\frac{1}{10}x^3$ from $x=2$ to $x=3$.

$$\text{Area}=\int_2^3 \tfrac{1}{10}x^3 dx=\left[\tfrac{1}{10}\frac{x^4}{4}+c\right]_2^3$$

$$=\left(\tfrac{1}{10}\frac{3^4}{4}+c\right)-\left(\tfrac{1}{10}\cdot\frac{2^4}{4}+c\right)$$

$$=\frac{81-16}{40}=1\tfrac{3}{8}.$$

Notice again that in finding any definite integral it is not necessary to include the constant c. However, in all other applications of integration except those involving the evaluation of definite integrals the constant should be included. The problem that follows illustrates this.

Example.—If a body has an acceleration $\frac{1}{2}t^2$ after t sec., find its velocity after t sec. given that its velocity at $t=0$ is 2 ft./sec.

If the velocity is v ft./sec.,

$$\frac{dv}{dt} = \tfrac{1}{2}t^2.$$

$$\therefore v = \int \tfrac{1}{2}t^2 dt$$

$$= \tfrac{1}{2} \cdot \frac{t^3}{3} + c = \tfrac{1}{6}t^3 + c.$$

But $v = 2$ at $t = 0$,

$$\therefore 2 = 0 + c.$$

$$\therefore c = 2.$$

$$\therefore v = \tfrac{1}{6}t^3 + 2.$$

Exercise L

Using the formula $\int ax^n dx = \dfrac{ax^{n+1}}{n+1} + c$, find:

1. $\int x^7 dx$. **2.** $\int 10x^4 dx$. **3.** $\int \tfrac{1}{2}v^4 dv$. **4.** $\int 0{\cdot}1t^9 dt$.

5. $\int \sqrt{r}\, dr$. **6.** $\int \dfrac{dx}{2x^2}$. **7.** $\int \dfrac{dm}{m^4}$. **8.** $\int 12p^{2{\cdot}6} dp$.

Evaluate the following integrals:

9. $\int_1^3 2x dx$. **10.** $\int_0^2 6x^2 dx$. **11.** $\int_{\frac{1}{2}}^{1\frac{1}{2}} x^3 dx$. **12.** $\int_0^6 4\pi r^2 dr$.

13. $\int_{\frac{1}{2}}^1 t^4 dt$. **14.** $\int_2^3 \dfrac{dx}{x^2}$. **15.** $\int_1^2 \dfrac{20 dv}{t^{1{\cdot}2}}$. **16.** $\int_2^4 \dfrac{dt}{3\sqrt{t}}$.

17. $\int_{-a}^{+a} x^6 dx.$ 18. $\int_{v_1}^{v_2} \frac{dv}{v^n}.$ 19. $\int_0^{+1} k^{2\cdot 6} dk.$

Find the values of:

20. $\int (1-x^2)dx.$ 21. $\int (2v-4v^3)dv.$ 22. $\int (x^3-x)dx.$

Find the following areas:

23. Under the curve $y=4x^2$ from $x=0$ to 3.

24. Under the curve $y=x^n$ from $x=a$ to b [n not equal to -1].

25. Under the curve $y=2\sqrt{x}$ from $x=1$ to 4.

26. Find by integration the area under the line $y=x$ from $x=a$ to b. Verify your result by using the formula for the area of a trapezium.

27. If the velocity of a body is $32t$ ft./sec. after t sec., express the distance travelled from $t=1$ to 2 as an integral and evaluate the integral.

28. If the acceleration of a body is $4t$ ft./sec.^{2} after t sec., find its velocity v ft./sec. after t sec., given that $v=6$ at $t=0$. How fast is it travelling after 5 sec.?

29. Draw the graph of $y=x^2$ from 0 to 2. Divide the area which is bounded by the curve, the axis of x and the ordinate at $x=2$, into ten strips by ordinates at $x=0{\cdot}2$, $0{\cdot}4$, etc. Hence draw an integral curve. Verify that the gradients of the integral curve at $x=\frac{1}{2}$ and $x=1\frac{1}{2}$ are the values of y at these values of x. Find, by using the formula for $\int x^2 dx$, the equation of the integral curve.

30. Find the co-ordinates of the points A and B in which the graph of $y=5x-x^2-6$ cuts the axis of x. Hence find the area bounded by AB and the part of the graph which is above AB.

Applications of integration.

The volume of a solid

If the cross-sections of a solid by parallel planes have the same area A and the length of the solid is x, the volume of the

solid is Ax, which is the area of the rectangle under the graph of A against x (Fig. 317).

If the cross-section is variable we suppose the solid divided up into a number of thin slices by parallel planes at equal intervals δx. Let A be the area of a section at a distance x from one end as in Fig. 318. The graph of A against x is shown in Fig. 319. Throughout a slice the cross-section has nearly the same value and hence the volume of the cross-section between $x = \text{OM}$ and $x = \text{ON}$.

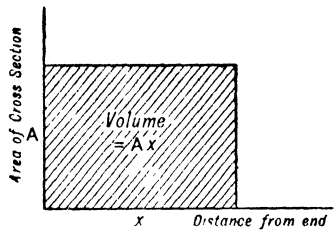


FIG. 317.

$$\doteq \text{MP} \times \text{ON} = \text{area of PMNK}.$$

Hence the total volume of the solid between the sections at $x = a$ and $x = b$ is approximately the sum of the areas under the dotted line in Fig. 319. By taking the thickness of the

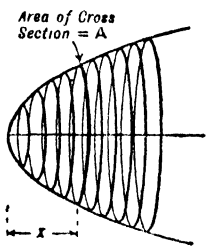


FIG. 318.

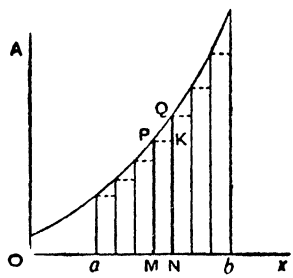


FIG. 319.

slices smaller and smaller we see that the total volume from $x = a$ to $x = b$ must be exactly equal to the area under the graph of A from $x = a$ to $x = b$. In symbols,

Volume from $x = a$ to $x = b$ is equal to $\int_{x=a}^{x=b} A\,dx$.

As in the case of velocity and time the volume obtained is

only numerically equal to the area, or is represented by the area, i.e. its units are not those of an area.

Example.—To find the volume of a right circular cone of height h and base radius r.

Fig. 320 shows the cone with its axis OB horizontal. A section perpendicular to the axis by a plane at a distance x from O is also shown. Let this section have radius y.

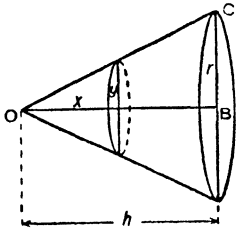


FIG. 320.

From Fig. 320 we see by similar triangles that

$$\frac{y}{x} = \frac{r}{h}$$

$$\therefore y = \frac{rx}{h}.$$

Hence the area of the section of radius y is $\pi y^2 = \dfrac{\pi r^2 x^2}{h^2}$.

Therefore the volume of the cone equals the area under the graph of $\dfrac{\pi r^2 x^2}{h^2}$ against x from $x = 0$ to $x = h$, which is the integral of $\dfrac{\pi r^2 x^2}{h^2}$ between the limits 0 and h.

$$\therefore \text{Volume of cone} = \int_0^h \frac{\pi r^2 x^2}{h^2}\, dx$$

$$= \left[\frac{\pi r^2}{h^2} \cdot \frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi r^2}{h^2} \cdot \frac{h^3}{3} - 0$$

$$= \tfrac{1}{3}\pi r^2 h.$$

Work done by a variable force

If a constant force F lb. wt. moves a body x ft. it does work Fx ft. lb. wt. In this case the graph of the force F against

the distance x is a straight line, as in Fig. 321, and the work done Fx is the area shaded.

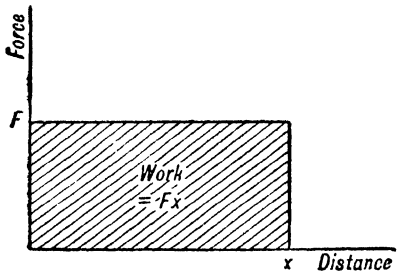


FIG. 321.

Now, if F is variable, the graph of F against x is a curve, as shown in Fig. 322. But in any short distance MN the force F is very nearly constant so that the work done is nearly equal to PM × MN or the area of the rectangle PMNK. Hence, by the same arguments as in finding the volume on p. 389,

the work done by F as the body moves from $x=a$ to $x=b$,

= the area under the graph of F from $x=a$ to $x=b$

$$= \int_a^b F\,dx.$$

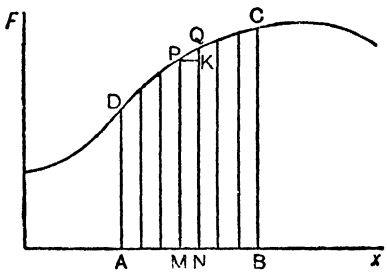


FIG. 322.

Work done by an expanding gas

Let the gas be contained in a cylinder of cross-section S ft.^{2} Let its pressure be p $\frac{\text{lb. wt.}}{\text{ft.}^2}$ and its volume v ft.^{3} Suppose that as the gas expands it pushes a piston along the cylinder. Then the force on the piston is pS lb. wt. Now, if the volume of the gas increases by a small amount δv ft.^{3}, the piston moves $\frac{\delta v}{\text{S}}$ ft. In this movement the force on the piston is nearly constant and hence it does work which is nearly

$$p\text{S lb. wt.} \times \frac{\delta v}{\text{S}} \text{ ft.} = p\delta v \text{ ft. lb. wt.}$$

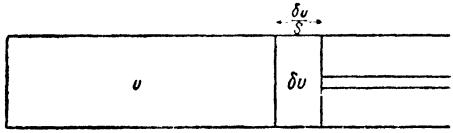


FIG. 323.

But $p\delta v$ is nearly the area of a strip under the p, v graph from v to $v+\delta v$, and hence, by the same arguments as before, the work done by a gas as it expands from a volume v_1

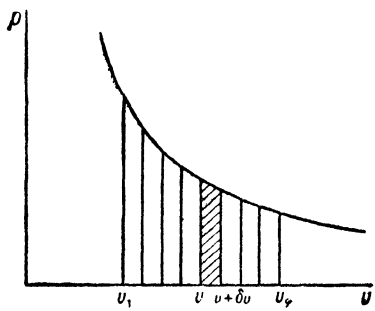


FIG. 324.

to a volume v_2 equals the area under the p, v graph from $v = v_1$ to $v = v_2$, or $\int_{v_1}^{v_2} p\,dv$.

Approximate evaluation of integrals

We have seen in Part I that the area under a curve can be found approximately by either of the following rules :

(*a*) The mid-ordinate rule.

Divide the area into a number of strips by ordinates at equal intervals and measure the ordinates at the middle of each interval. Then, if the strips are of width h :

Area under the curve $\simeq h \times$ sum of mid-ordinates.

(*b*) Simpson's rule.

Divide the area into $2n$ strips by ordinates having lengths $y_1, y_2, \ldots y_{2n+1}$ at equal intervals h. Then area under the curve

$$\simeq \tfrac{1}{3}h[y_1 + y_{2n+1} + 4(y_2 + y_4 + \ldots + y_{2n}) + 2(y_3 + y_5 + \ldots + y_{2n-1})]$$

$= \tfrac{1}{3}h[$sum of end ordinates $+ 4($sum of even ordinates$) + 2($sum of odd ordinates$)]$.

We have already used the mid-ordinate rule on p. 374 in finding the area under a velocity-time graph. Either of these rules can be used to approximate to the value of any quantity such as a volume or work which can be represented by the area under a curve. In other words these rules enable us to integrate approximately.

Example.—The force F lb. wt. on a body when it has moved x ft. is given by the following table. Find the work done by the force when the body moves 12 ft.

F ..	14	15·5	17·3	20·0	23·5	27·5	38·5
x ..	0	2	4	6	8	10	12

Using Simpson's rule :

$$\text{Work done} = \text{area under F, } x \text{ graph from } x=0 \text{ to } 10$$
$$\simeq \tfrac{2}{3}\{14+38{\cdot}5+4(15{\cdot}5+20{\cdot}0+27{\cdot}5) + 2(17{\cdot}3+23{\cdot}5)\}$$
$$= \tfrac{2}{3}\{52{\cdot}5+4(63{\cdot}0)+2(40{\cdot}8)\}$$
$$\simeq 257 \text{ ft. lb. wt.}$$

Average or mean ordinate of a graph

Suppose the area under a graph of y against x from $x=a$ to b is divided into n strips by ordinates at equal intervals, and that the lengths of the mid-ordinates of the intervals are $z_1, z_2, z_3 \ldots z_n$.

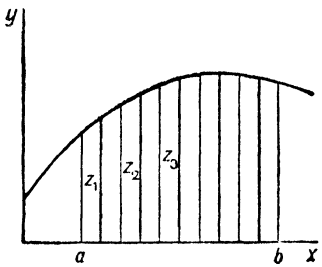


FIG. 325.

Then by the mid-ordinate rule :

$$\text{Area under curve} \simeq \text{width of strip} \times \text{sum of mid-ordinates}$$
$$\simeq \frac{b-a}{n} \times (z_1+z_2+z_3+ \ldots +z_n)$$
$$\simeq (b-a) \times \left(\frac{z_1+z_2+z_3+ \ldots +z_n}{n}\right)$$
$$\simeq (b-a) \times \text{average of mid-ordinates.}$$

$$\therefore \text{Average of mid-ordinates} \simeq \frac{\text{area under curve}}{\text{base of the area}}.$$

This becomes more and more accurate the larger the number of ordinates we take. For this reason we call the ratio of the area to the base the average or mean ordinate of the graph from $x=a$ to b.

For example, the average force acting on the body considered on p. 393,

$$=\frac{257 \text{ ft. lb. wt.}}{12 \text{ ft.}} \simeq 21{\cdot}4 \text{ lb. wt.}$$

Also the average velocity of the electric train considered on p. 375

$$=\frac{12{\cdot}64 \text{ miles}}{16 \text{ min.}}=\frac{12{\cdot}64 \times 60}{16} \text{ m.p.h.} = 47{\cdot}4 \text{ m.p.h.}$$

Since the area under the graph of y against x from $x=a$ to b is $\int_a^b y\,dx$ we have the following result:

Mean value of y from $x=a$ to $x=b$ is equal to $\dfrac{\int_a^b y\,dx}{b-a}$.

Exercise LI

1. A field ABCD has straight sides along AB, BC and AD, and the angles at A and B are right angles. The off-sets of the curved side CD from AB are given by:

Distance from A	0	30	60	90	120	150	180 yds.
Off-set	70	110	130	170	160	130	120 yds.

Find the area of the field in acres to the nearest $\frac{1}{10}$th acre.

2. The area A sq. ft. of the cross-section of a reservoir by a horizontal plane at y ft. from the bottom is given by:

y	0	5	10	15	20	25	30
A	10,500	12,820	15,000	17,040	18,940	20,000	21,400

Find the volume of water of reservoir in gallons when the water is (*a*) 20 ft. deep, (*b*) 30 ft. deep.

3. The following table gives the area A sq. ft. of the cross-section of a log at x ft. from one end. Find the volume of the log.

x ..	0	3	6	9	12	15	18
A ..	1	1·1	1·2	1·32	1·6	2·3	3·6

4. During the expansion stroke of an engine the thrust of the piston is given by :

Clearance of piston x in.—
0·77 0·83 1·02 1·30 1·66 2·07 2·46 2·83 3·16 3·43 3·62 3·74 3·77

Thrust T lb. wt.—
500 455 340 242 172 126 99 82 70 62 58 55 55

Find the work done by the force T during the stroke, and find the average thrust on the piston during the stroke.

5. The pressure and volume of a gas are given by :

p lb./ft.^{2} ..	100	44	28·6	21	16·6	13·8
v ft.^{3}	50	100	150	200	250	300

Find the work done by the gas in expanding from 50 to 300 cu. ft. Find also the mean pressure during the expansion.

6. The area A sq. ft. of the vertical cross-section of a railway cutting at x ft. from a certain point on the railway is given by :

A ..	180	190	210	235	265	297	332	375	425	460	475
x ..	0	20	40	60	80	100	120	140	160	180	200

Find the volume of earth to be excavated from $x=0$ to $x=200$, and deduce the mean area of a vertical section.

7. The acceleration of a body moving in a straight line is $\frac{dv}{dt}$ where v is the velocity after a time t. What does the area under the acceleration-time graph from $t=a$ to $t=b$ represent ?

The acceleration of a body is given by the following table :

Acceleration $\frac{dv}{dt}$ (ft./sec.^{2})	0·3	0·38	0·52	0·60	0·90	1·12	1·40
Time (sec.)	0	4	8	12	16	20	24

If the body is moving at 10 ft./sec. at $t=0$, find its velocity at $t=24$.

8. Use the table in Question 2 to construct a graph of the volume of water in the reservoir against the depth of the water.

9. Use the table in Question 7 to construct a graph of the velocity of the body against the time.

10. Show that the area of the cross-section of the cone, formed by revolving the line $y = kx$ from $x = 0$ to h about Ox, by a plane perpendicular to Ox at a distance x from the origin is $\pi k^2 x^2$. Deduce that the volume of the cone is $\int_0^h \pi k^2 x^2 dx$. Evaluate this and hence find a formula for the volume of a cone of height h and base radius r.

11. The force required to extend a spring x in. is sx lb. wt. where s is a constant, the stiffness of the spring. Show that the work done in increasing the extension from a in. to b in. is $\frac{1}{2}s(b^2 - a^2)$ in. lb. wt.

12. When a certain gas is expanding adiabatically the pressure p lb. wt./ft.^{2} and volume v ft.^{3} are related by $pv^{1\cdot3} = 200$. Find p in terms of v and hence find the work done by the gas in expanding from $v = \frac{1}{2}$ to $v = 1$, and the mean value of p during the expansion.

13. A chain of length 10 ft., which weighs $\frac{1}{2}$ lb. wt. per foot, is coiled on the floor. One end is lifted vertically. What force is required to do this when x ft. of the chain are off the floor? Show that the work done in lifting the whole chain just clear of the floor is $\int_0^{10} \frac{1}{2}x dx$ ft. lb. wt. Hence find the work done.

14. If the relation between the pressure and volume of a gas is $pv^n = c$ where c is a constant and n is not equal to 1, find the work done as the gas expands from a volume v_1 to a volume v_2, and show that it equals $\frac{p_1v_1 - p_2v_2}{n-1}$, where p_1, p_2 are the pressures when the volume is v_1 and v_2 respectively.

15. Find the mean value of the ordinate of $y = x^2$ from $x = 0$ to a.

16. If O, A, B are the points (0, 0) (2, 4), (3, 0) respectively, find the mean ordinate of the graph OAB.

17. Show by graphs that the areas under $y = \sin^2 x$ and $y = \cos^2 x$ from $x = 0$ to π are equal. Using the identity $\sin^2 x + \cos^2 x = 1$, find the value of each area and deduce that the mean value of $\sin^2 x$ from $x = 0$ to $x = \pi$ is $\frac{1}{2}$.

18. Find by integration the volume of a cone having a height of 12 in. and a base radius of 4 in. Do not quote the formula.

19. If a pyramid has a height h and a base of area A, show that the area of a cross-section parallel to the base at distance x from the vertex is $\frac{Ax^2}{h^2}$. Hence prove that its volume is $\frac{1}{3}Ah$.

20. Show that the volume of a frustum of a cone having end radii 2 in. and 3 in. respectively and length 8 in. is $\int_{16}^{24} \frac{\pi x^2}{64} dx$, and hence evaluate it.

ANSWERS

EXERCISE I (Page 4)

1. (*a*) $3\sqrt{3}=5{\cdot}20$; (*b*) $3\sqrt[3]{2}=3{\cdot}78$; (*c*) $4\sqrt{5}=8{\cdot}95$; (*d*) $7\sqrt{3}=12{\cdot}1$; (*e*) $12\sqrt{2}=17{\cdot}0$.

2. (*a*) $\frac{\sqrt{2}}{2}=0{\cdot}707$; (*b*) $\frac{\sqrt{5}}{10}=0{\cdot}224$; (*c*) $2\sqrt{3}=3{\cdot}46$; (*d*) $2\sqrt{5}=4{\cdot}47$.

3. (*a*) $7^{\frac{1}{2}}$; (*b*) $3^{\frac{1}{3}}$; (*c*) $\frac{1}{2^{\frac{1}{4}}}$; (*d*) $(\frac{2}{3})^{\frac{1}{2}}$; (*e*) $5^{\frac{7}{2}}$.

4. (*a*) $t^{\frac{1}{2}}$; (*b*) $x^{\frac{3}{2}}$; (*c*) $y^{\frac{5}{2}}$; (*d*) $2m^{\frac{1}{2}}$; (*e*) $x^{\frac{2}{3}}y^{\frac{2}{3}}$.

5. (*a*) $32a$; (*b*) 27 ; (*c*) y^4/x^3.

6. (*a*) $1/8n^4$; (*b*) $3r^2/20$; (*c*) w/W.

7. The latter. **8.** The former.

9. (*a*) $x^{\frac{7}{2}}$; (*b*) $x^{-\frac{2}{3}}$; (*c*) $x^{-\frac{1}{2}}$; (*d*) $x^{1{\cdot}8}$.

10. (*a*) 2 ; (*b*) $y^{\frac{1}{2}}$; (*c*) $3a^{\frac{2}{3}}b^{\frac{1}{3}}$; (*d*) $a^{\frac{2}{2}}$.

11. (*a*) $16/81$; (*b*) $2r^2$; (*c*) $432x^{\frac{2}{1}\frac{5}{2}}$; (*d*) p^3/q.

12. (*a*) $p^{\frac{3}{3}}$; (*b*) 50 ; (*c*) $a^{\frac{7}{4}}b^{-\frac{1}{8}\frac{5}{8}}$.

13. (*a*) $1/v^2$; (*b*) $m^{\frac{1}{2}}/n^{\frac{3}{2}}$; (*c*) $\frac{1}{3}q^{\frac{1}{2}}r^{\frac{3n}{2}}$.

14. (*a*) $1/\sqrt{y}$; (*b*) q^3/p^3 ; (*c*) $\sqrt[3]{y/x}$; (*d*) m/lt^2.

15. (*a*) l ; (*b*) $\sqrt[4]{x}/\sqrt{y}$; (*c*) $\sqrt{3/pq}$.

16. $y=2x^{\frac{1}{4}}$. **18.** $r=\sqrt{\mathrm{V}/\pi h}$.

EXERCISE II (Page 9)

1. (*a*) $2x^2-8x$; (*b*) $2\mathrm{M}^2+3\mathrm{MN}$; (*c*) $2\omega t+\frac{4\pi}{3}$; (*d*) a^3-a^2h.

2. (*a*) $3x^2+5x$; (*b*) $3\theta+7\alpha$; (*c*) $6y^2+2ay+2a^2$; (*d*) $2r_1l_1$.

3. (*a*) $x^2+a^2+b^2$; (*b*) $4p^2$; (*c*) $-m^2-2mn-5n^2$.

4. (*a*) $2m+2n$; (*b*) $-3x+5y$; (*c*) $4ab$; (*d*) $nt-90$; (*e*) a^2-b^2-ca.

5. (*a*) $7m+3n$; (*b*) $2p-6q$; (*c*) $x^2+4ax+3a^2$; (*d*) $6{\cdot}8t^2-10{\cdot}6t$.

6. (*a*) $2a^3-5a^2h-ah^2$; (*b*) $-4+7\cos\theta$; (*c*) $2-7\cos\mathrm{A}$; (*d*) $\sin x-\cos x$.

7. (*a*) $t^6+2t^7-6t^8$; (*b*) $4a+2b-7c$; (*c*) $-6x$; (*d*) $3\cos x$.

8. (*a*) $6x^2-x-12$; (*b*) $l^3-2l^2m+lm^2-2m^3$; (*c*) $8-14x+3x^2$; (*d*) $2p^3-p^2q-pq^2$.

9. (*a*) $4x^3-14x^2+20x-16$; (*b*) $x^4+x^3-6x^2-17x-21$; (*c*) $l^2m+lp+lm^2p+mp^2$; (*d*) $1+e^3+e^4$.

10. (*a*) x^2+5x+4 ; (*b*) $a^2-ab-6b^2$; (*c*) $p^2-3pq-4q^2$; (*d*) $2y^2+3y-35$; (*e*) $6x^3+7x^2-x-2$; (*f*) $l^3+l^2m-2lm^2$.

11. (*a*) $7+2x+x^2$; (*b*) $4y^2+12y+9$; (*c*) $p^2-4pq+4q^2$; (*d*) $4a^2r^2-20ars+25s^2$.

12. (*a*) m^2-n^2 ; (*b*) $\frac{r^2}{s^2}-1$; (*c*) $a^2-b^2-2bc-c^2$.

13. (*a*) 28561; (*b*) 17956; (*c*) 9604; (*d*) 2209.
14. (*a*) $\frac{7}{6a}$; (*b*) $\frac{3x}{14}$; (*c*) $\frac{a^2+b^2}{a^2b}$; (*d*) $\frac{1-2y}{y}$.
15. (*a*) $\sqrt{2}-1=0{\cdot}414$; (*b*) $\frac{5}{2}(4+\sqrt{6})=16{\cdot}12$; (*c*) $\frac{1}{3}(5+\sqrt{7})=2{\cdot}55$.
16. 0·02502. **17.** 0·02944.
18. (*a*) $x^2+2xy+y^2$ or $(x+y)^2$; (*b*) $4ab$.
19. (*a*) $-2xy-y^2$ or $-y(2x+y)$; (*b*) $1-\sin^2\theta$ or $\cos^2\theta$.
20. (*a*) $8+2\sin^2 x$; (*b*) $\frac{1}{2}wx^2-\frac{1}{2}wxl$ or $\frac{1}{2}wx(x-l)$.
21. (*a*) $3+3x-x^2$; (*b*) x^3-5. **22.** $8y^3+36y^2+54y+27$.
23. $a^3-3a^2b+3ab^2-b^3$, k^3-3k^2+3k-1.

EXERCISE III (Page 16)

1. (*a*) 3; (*b*) $4\frac{1}{2}$; (*c*) $1\frac{2}{3}$; (*d*) $1\frac{1}{2}$; (*e*) $\frac{2}{3}$; (*f*) 2.
2. (*a*) $p=3(q+r)$; (*b*) $q=\frac{1}{2}p$; (*c*) $R=5r$;
(*d*) $h=\frac{S}{2\pi r}-\frac{r}{2}$; (*e*) $E=\frac{4Wl^3}{bd^3y}$; (*f*) $v=\frac{uf}{u-f}$.
3. (*a*) 1/9; (*b*) 2; (*c*) 8/9; (*d*) −0·171.
4. (*a*) $p=\frac{2}{3}D^2$; (*b*) $p=\frac{100A}{100+rt}$; (*c*) $g=\frac{4\pi^2 l}{T^2}$;
(*d*) $h=\frac{r}{3}+\frac{V}{\pi r^2}$; (*e*) $C=\frac{1}{4\pi^2 lf^2}$; (*f*) $r=\frac{h^2+y^2}{2h}$.
5. $2s-2a$. **6.** (*a*) $k=\frac{m-x}{1+mx}$; (*b*) $m=\frac{k+x}{1-kx}$; (*c*) $x=\frac{m-k}{1+km}$.
7. $t=\frac{n_1d_1-n_2d_2}{n_2-n_1}$ **8.** $\theta_m=\frac{x\theta_1}{\theta_1-y}$. **10.** (*c*).
11. $\frac{1}{2\pi\sqrt{LC}}$. **12.** $\frac{35344\ EI}{N^2L^4}$. **13.** $(Tc^2-T^2)/2Tc$.
14. $\frac{R_1^2R_4^2-R_2^2R_3^2}{R_2^2L_3^2-R_1^2L_4^2}$. **15.** $6\frac{1}{8}$, $5\frac{3}{8}$ in. **16.** 22·6 in.
17. 5 min. **18.** 21, 44. **19.** 600.
20. 12 ohm. **21.** 157·5 lb. **23.** −40.
24. 144; $\frac{12n}{7}$. **25.** 32·5 knots; $\frac{20n+3000}{n-100}$ knots.
26. 2·3 mm. **27.** 23·5 miles per gallon.
28. £7 10*s*. 0*d*.; £5 10*s*. 0*d*.; 160 units.

EXERCISE IV (Page 22)

1. (*a*) ft./sec., velocity; (*b*) ft., distance; (*c*) ft./sec., velocity; (*d*) ft., distance.
2. (*a*) amps; (*b*) watts; (*c*) watts.
3. (*b*), (*b*). **4.** ft. lb. wt.
5. 1 lb. wt.$=32{\cdot}19$ lb. mass$\times\frac{\text{ft.}}{\text{sec.}^2}$.
6. $\frac{1\text{ lb. mass. ft.}}{\text{sec.}^2}$, 1 dyne$=\frac{1\text{ gram. cm.}}{\text{sec.}^2}$, 1 kilogram wt.$=981\times10^3$ dynes.
7. $4{\cdot}45\times10^5$ dynes. **8.** lb. wt./in.^{2}.

EXERCISE V (Page 25)

1. $6i$ volts. 2. $\frac{1}{2}x^2$. 3. $16t^2$ ft.
4. $\pi(2r+1)$ in.^{2}. 5. $(n+430)$ pence. 6. $A\frac{3}{2}$.
7. $\frac{1}{2(y+3)}$. 9. $\frac{1}{4}u^2-u$. 10. $A=\pi r^2$; $r=\sqrt{\frac{A}{\pi}}$.
11. $t=\frac{pv}{R}-273$. 12. $\sqrt{a^2+b^2}$. 13. (*a*) $\frac{1}{4}\pi h^3$; (*b*) $2\pi r^2$.
14. $\frac{ab}{\sqrt{a^2+b^2}}$. 15. $\frac{\sqrt{R^2-Z^2}}{ZRC}$.
16. (*a*) $\frac{4y+5}{y+3}$; (*b*) $\frac{5-3x}{4-x}$. 17. $\frac{v(u-r)}{u(v-r)}$.

EXERCISE VI (Page 29)

1. $x=\frac{1}{3}$, $y=\frac{5}{3}$.
2. $x=14\frac{3}{19}\simeq 14{\cdot}16$, $y=-4\frac{17}{19}\simeq -4{\cdot}89$.
3. $x=\frac{1}{50}=0{\cdot}02$, $y=1\frac{3}{50}=1{\cdot}06$.
4. $x=\frac{1}{3}$, $y=\frac{1}{7}$. 5. $x=15{\cdot}9$, $y=18{\cdot}5$.
6. $x=1{\cdot}58$, $y=1{\cdot}45$. 7. $x=0{\cdot}467$, $y=-0{\cdot}795$.
8. $x=1{\cdot}09$, $y=1{\cdot}30$. 9. $p=9$, $q=4$.
10. $a=0{\cdot}0442$, $b=1{\cdot}047$. 11. $m=3$, $c=-8$
12. $x=\frac{5}{4}$, $y=\frac{1}{8}$. 13. $x=\frac{rE}{R(r+s)+rs}$, $y=\frac{sE}{R(r+s)+rs}$.
14. $P=23{\cdot}2$, $N=28{\cdot}9$.
15. $T_1=50+25\sqrt{3}\simeq 93{\cdot}3$, $T_2=50\sqrt{3}-25\simeq 61{\cdot}6$.
16. $k=\frac{e(D-d)}{DdR}$. 19. $f=\frac{1}{2\pi\sqrt{LC}}$.
21. $a=(BC-1)/A$, $k=A(C-1)/(BC-1)$, $k'=A(B-1)/(BC-1)$.
23. $V^2=(a_1^2-a_2^2)/(T_1^2-T_2^2)$, $d^2=(a_1^2T_2^2-a_2^2T_1^2)/(T_1^2-T_2^2)$.
24. $h=\frac{\sqrt{3}}{2}d$.

EXERCISE VII (Page 38)

1. Min. $y=-13{\cdot}75$; $2{\cdot}16$, $-0{\cdot}66$.
2. Max. $R=5$, Min. $R=-3$; $-1{\cdot}64$, $0{\cdot}17$, $1{\cdot}81$.
3. $1{\cdot}36$, $-0{\cdot}473$. 4. Min. $t=6{\cdot}92$ at $m=1{\cdot}52$.
5. Max. $z=97{\cdot}4$ at $y=-3{\cdot}65$, min. $z=-97{\cdot}4$ at $y=3{\cdot}65$.
6. $1{\cdot}56$, $0{\cdot}403$. 7. Max. $p=5{\cdot}33$.
8. $-1{\cdot}7$, $0{\cdot}3$, $2{\cdot}2$, $2{\cdot}16$. 11. Max. $y=2{\cdot}24$; $74°$.
20. -3, 5. 21. $-\frac{1}{2}$, 4. 22. $\frac{4}{3}$, $\frac{7}{3}$.
23. 2, $-1\frac{2}{5}$. 24. $\frac{5}{9}$, $-\frac{160}{9}$. 25. $-0{\cdot}8$, 200.
26. $3y-4x=1$. 27. $5y+8x=68$. 28. $x+y=3$.
31. $23{\cdot}1$. 32. $x=-3$, $y=8$; $x=\frac{5}{3}$, $y=\frac{16}{9}$.
33. $40{\cdot}6$, $69{\cdot}0$. 34. 1430. 35. $4{\cdot}025$.

EXERCISE VIII (Page 42)

1. (*a*) $a(a+b)$; (*b*) $x(x-y)$; (*c*) $(a+h)(a-h)$.
2. (*a*) $q(2p-3q)$; (*b*) $(p+q)(p-q)$; (*c*) $(2l+m)(2l-m)$.
3. (*a*) $(x+3)(x-3)$; (*b*) $(11+r)(11-r)$; (*c*) 240×2.
4. (*a*) $(3c+4r)(3c-4r)$; (*b*) $8(k+5)(k-5)$; (*c*) $(\frac{1}{2}a+r)(\frac{1}{2}a-r)$.
5. (*a*) $(pq+ab)(pq-ab)$; (*b*) $p^2(q+b)(q-b)$;
(*c*) $5(2pq+3ab)(2pq-3ab)$.
6. (*a*) $3(x-1)(x+1)$; (*b*) $2x(x-l)$.
7. (*a*) $(l+2)(l-2)(l^2+4)$; (*b*) $l^2(l+4)(l-4)$; (*c*) $3r(2a+r)$.
8. (*a*) $(2x+3y)(2x-3y)(4x^2+9y^2)$; (*b*) $4(a+b)c$.
9. (*a*) $\left(\frac{t}{T}+1\right)\left(\frac{t}{T}-1\right)$; (*b*) $(r^n+1)(r^n-1)$; (*c*) $\left(k+\frac{3}{m}\right)\left(k-\frac{3}{m}\right)$.
10. (*a*) 255; (*b*) 256; (*c*) 1·1808.
11. (*a*) 1200; (*b*) 1725; (*c*) 60736.
12. 7·11. **13.** $5{\cdot}04\times 10^{12}$.
15. (*a*) $(a-2)(x-1)$; (*b*) $(l+m)(l+n)$.
16. (*a*) $(k-l)(a-b)$; (*b*) $(t-3s)(t+1)$.
17. (*a*) $(a+b+c)(a+b-c)$; (*b*) $(a-b+c)(a-b-c)$.
18. (*a*) $(a+b-c)(a-b+c)$; (*b*) $(2k+l+3m)(2k+l-3m)$.
19. (*a*) $(x+1)(x+4)$; (*b*) $(x-1)(x-4)$; (*c*) $(x+1)(x-4)$.
20. (*a*) $(x+5)(x+18)$; (*b*) $(x-5)(x-18)$; (*c*) $(x-5)(x+18)$.
21. (*a*) $(p+7)(p+9)$; (*b*) $(p+7)(p-9)$; (*c*) $(p-7)(p+9)$.
22. (*a*) $(y+3)(y+5)$; (*b*) $(y-3)(y+5)$; (*c*) $(y+3)(y-5)$.
23. (*a*) $(x+4)(x-1)$; (*b*) $(x+7)(x-4)$; (*c*) $(x-7)(x+4)$.
24. (*a*) $(l-2)(l+3)$; (*b*) $(l+2)(l-3)$; (*c*) $(l-1)(l-6)$.
25. (*a*) $(m-7)(m+2)$; (*b*) $(m+7)(m-2)$; (*c*) $(m-1)(m\times 14)$.
26. (*a*) [illegible]; (*b*) $(p-q)(p-2q)$; (*c*) $(p-2q)(p+q)$.
27. (*a*) [illegible]; (*b*) $(l-2m)(l+m)$; (*c*) $(l-6m)(l+5m)$.
28. (*a*) $(pq-2)(pq-3)$; (*b*) $(p-2q)(p-3q)$; (*c*) $(pq-3)(pq+2)$.
29. $xy\,(y-x)$. **30.** $-2x\,(a-x)$.
31. $(l-x)(l+x)(x+n)$. **32.** $5t\,(2r-t+2)$.
33. (*a*) $(1-\cos\theta)(1+\cos\theta)$; (*b*) (cos A−sin A)(cos A+sin A);
(*c*) (4 cos A+3 sin A)(2 cos A+sin A).
34. (*a*) $(2\tan\theta-1)(2\tan\theta+1)$; (*b*) $(1+\sin\theta)(1+2\cos\theta)$;
(*c*) $(\cos x-\sin x)(\cos x+\sin x)$, after putting $\sin^2 x+\cos^2 x=1$.

EXERCISE IX (Page 47)

1. $(2x+1)(x+1)$. **2.** $(3t+4)(t+1)$. **3.** $(3t+2)(t+2)$
4. $(4y+1)(y-2)$. **5.** $(2m+n)(m-2n)$. **6.** $(4x+1)(x-4)$.
7. $(3z-2)(2z-1)$. **8.** $(3l+1)(l-2)$. **9.** $(3+5k)(1+k)$.
10. $x-7$. **11.** $y+1$. **12.** $a+b$.
13. $4p+q$. **14.** $x-1$; -2. **15.** $3x^2+x+4$, 5.
16. $4y+1$, $10y$. **17.** $\frac{1}{3}t^2+\frac{1}{9}t-\frac{17}{27}$, $4\frac{10}{17}$. **20.** $(y-1)(y^2+y+1)$.
21. $(x+a)(x^2+ax+a^2)$. **22.** $(3+p)(9-3p+p^2)$.
23. $20(l+2)(l^2-2l+4)$. **24.** x^2-lx-l^2; $x(x-l)(x^2-lx-l^2)$.
25. 9·05 cu. in.; 0·0002 in.

EXERCISE X (Page 52)

1. b^2/x. **2.** $4n^2/3lm$. **3.** $\frac{2}{5}r^2$.
4. $24x/r$. **5.** $(x-y)/(x+z)$. **6.** $\frac{1}{4}(p+2)$.
7. a/c. **8.** $(x+3)/(x+1)$. **9.** $(r-s)/(r+s)$.
10. $1/(p+2q)$. **11.** $(z+1)/(z+4)$. **12.** $x/(x+5)$.
13. y^2-y-6. **14.** $c^2/(a+2c)$. **15.** $(x+3)/(x+1)$.
16. r^2-rx. **17.** $pq/(p+q)$. **18.** $-6/ab$.
19. $(rs-1)/(rs-s^2)$. **20.** $(m+n)/(m-n+2)$.
21. $1/(2x-1)$. **22.** $y/(y-1)$.
23. $(m-n)/(2m+3n)$. **24.** $(2l-1)/(l-2)$.

25. $\dfrac{1}{(x+3)(x+4)}$. **26.** $\dfrac{3+t}{t(3-t)}$. **27.** $\dfrac{4c-5a}{6abc}$.

28. $\dfrac{m+4}{m^2+m}$. **29.** $\dfrac{1}{l-l^2}$. **30.** $\dfrac{16y}{1-9y^2}$.

31. $\dfrac{3a^2}{a^2-b^2}$. **32.** $\dfrac{1}{y+2}$. **33.** $\dfrac{4rs}{(r^2-s^2)^2}$.

34. $-\dfrac{2}{x}$. **35.** $\dfrac{1}{2z^2-z-1}$. **36.** $\dfrac{6pq}{p^2-4q^2}$.

37. $\dfrac{4}{(x-1)(x-2)(x+3)}$. **38.** $-\dfrac{2}{x(x-1)(2x-1)}$.

39. $\dfrac{2}{(t-s)(3t+1)}$. **40.** $-\dfrac{xy}{(x-y)(x^2+y^2)}$.

41. (*a*) x; (*b*) does not simplify.
42. (*a*) does not simplify; (*b*) 2.
43. (*a*) does not simplify; (*b*) $m+n$.
44. (*a*) $2/mn$; (*b*) does not simplify.
45. (*a*) $(k-l)/(k-l)$; (*b*) does not simplify.
50. 1. **51.** $-\frac{5}{3}$. **52.** 1. **53.** 2.
54. $\frac{1}{6}Wab(a+2b)$. **56.** *No.* **57.** $\frac{1}{2}(a^2+b^2)$.
58. $\dfrac{rr_1+rr_2}{r+r_1+r_2}$, $\dfrac{rr_1+rr_2+r_1r_2}{r_1+r_2}$.

EXERCISE XI (Page 67)

1. $\pm\sqrt{\frac{3}{2}} \simeq \pm 1{\cdot}225$. **2.** $\pm\frac{1}{5}$. **3.** $\pm\sqrt{20} \simeq \pm 4{\cdot}472$.
4. $\frac{1}{3}$ or $\frac{5}{3}$. **5.** 1 or 2. **6.** -1 or 5.
7. -1 or $\frac{5}{2}$. **8.** $-\frac{2}{3}$ or $\frac{1}{4}$. **9.** $-\frac{3}{2}$ or $-\frac{1}{2}$.
10. $-\frac{5}{7}$ or 1. **11.** -2 or 3. **12.** -11 or 2.
13. -2 or $-\frac{11}{2}$. **14.** -4 or -1. **15.** -1 or 5.
16. -4 or -2. **17.** -3 or 8. **18.** -8 or -1.
19. 9. **20.** $\frac{1}{9}$.
21. to **30.** Answers same as 5–14. **31.** $-2{\cdot}56$ or $1{\cdot}56$.
32. $-2{\cdot}281$ or $-0{\cdot}219$. **33.** $-2{\cdot}52$ or $1{\cdot}19$.
34. $-0{\cdot}640$ or $0{\cdot}39$. **35.** $0{\cdot}13$ or $3{\cdot}87$.
36. $-0{\cdot}395$ or $4{\cdot}27$. **37.** $-0{\cdot}24$ or $2{\cdot}95$.
38. $-0{\cdot}0865$ or $2{\cdot}47$. **40.** $-\frac{1}{4}l \pm \sqrt{\frac{1}{4}l^2-k^2}$.

42. 0·551 or 5·449. **43.** −2·680 or 0·186.
44. −1·14 or 6·14. **45.** −22·25 or 2·25.
46. $\frac{-a\pm\sqrt{a^2+4bc}}{2c}$. **47.** $\frac{1}{2}(c\pm\sqrt{c^2-4fc})$.
48. $r=\frac{W}{2\pi slt}-\frac{1}{2}t$, $t=-r\pm\sqrt{\frac{W}{\pi sl}+r^2}$.

EXERCISE XII (Page 70)

1. $\frac{1}{2}$. **2.** 1·55.
3. (*a*) after 2 or $\frac{17}{4}$ sec.; (*b*) after $\frac{1}{4}$ or 5 sec.
4. 9 ft. 6 in., 2 ft. 6 in. **5.** 3·804.
6. 50, 2·45; $\frac{1}{2}\{\sqrt{(a+b)^2+0{\cdot}04ab}-(a+b)\}$.
7. 0·44; $\{\sqrt{v^4+4g^2l^2}-v^2\}/2g$.
8. 5·98 in. **9.** (*a*) 11·86; (*b*) $\frac{1}{2}$.
10. $a=\frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right)l\simeq 0{\cdot}21\ l$.
11. $L=\frac{1}{4\pi^2n^2C}\pm\frac{\sqrt{Z^2-R^2}}{2\pi n}$; 0·1434, 0·0592, 0·4160.
12. 51·06, $v=\left\{t+\sqrt{t^2-2d\left(\frac{1}{f}+\frac{1}{f'}\right)}\right\}\Big/\left(\frac{1}{f}+\frac{1}{f'}\right)$.
13. $l(\sqrt{n^2+n}-n)$ where $n=w_b/w_t$.

EXERCISE XIII (Page 77)

1. (*a*) −0·472, 8·472; (*b*) 4, 4.
2. 9. **3.** −2, −1. **4.** 2; −4, −3.
5. $l\sqrt{1\pm\sqrt{8/15}}\simeq 1{\cdot}315\ l$ or $0{\cdot}519\ l$.
6. −3·64, −0·64, 4·28. **7.** 0·782.
8. 6·53. **9.** 7 ft. 10 in.

EXERCISE XIV (Page 85)

1. 0·1512. **2.** 6·767. **3.** 1552. **4.** 0·06908.
5. 0·3375. **6.** 0·01394. **7.** 14·37. **8.** 1·868.
9. 0·9024. **10.** 0·2460. **11.** 5780. **12.** 0·3267.
13. 0·004914. **14.** 1·722. **15.** 0·04455. **16.** 2·467.
17. 0·9942. **18.** 0·8373. **19.** 128·7. **20.** 0·3675.
21. 1·63, −0·63.
22. 2, 5, 0·30, 0·78, 0·95; $\bar{1}$·30, 2·78, $\bar{2}$·95.
27. 0·76, 2·99. **28.** 2·92, 4·30.
29. 1·79, −1·79, 2·48. **30.** 5·21.

EXERCISE XV (Page 92)

1. $3{\cdot}775\times10^{-8}$.
2. 7907.
3. 30·96.
4. 400.
5. $2{\cdot}156\times10^{9} gm/cm^{2}$.
6. 2355.
7. 0·99.
8. 75·8, 37.
9. 0·001296.
10. 1745.
11. 17·2.
13. 19350 ; 8·2.
15. 174·5, 95·13 ft./min.
16. 1821.
17. $l=13.71(\mathrm{EI}/w\mathrm{N}^{2})^{\frac{1}{4}}$.
18. $t=(c/v)^{8}$.
19. $v_1=v_2(\mathrm{T}_2/\mathrm{T}_1)^{\frac{1}{\gamma-1}}$.
20. $\mathrm{A}=0{\cdot}312\ \mathrm{I}^{1\cdot22}$.
21. $\mathrm{R}=\left(1-\frac{\mathrm{E}}{100}\right)^{-4}$.
22. $n=\frac{\log \mathrm{C}-\log p}{\log v}$.
23. $\mathrm{L}=0{\cdot}33\ \mathrm{Q}^{0\cdot98}\ \mathrm{H}^{-1.44}$.
24. $r=100\left\{1-\left(\frac{\mathrm{A}}{\mathrm{P}}\right)^{\frac{1}{n}}\right\}$.
25. $\sqrt{\frac{2\pi^2 r^3 \mathrm{P}}{p}}$; $\sqrt[3]{\frac{\rho l^2}{2\pi^2 \mathrm{P}}}$; 0·0097, 0·0154, 0·0284, 0·0450.
26. $p=\frac{b^{16}}{a^{15}}$; $v=\left(\frac{a}{b}\right)^{15}$; 58·48.

EXERCISE XVI (Page 98)

1. 2.
2. 1·673.
3. −3·322.
4. −3.
6. 1·3389, 7·519.
7. 0·1786.
8. −1·18.
9. 0·0795.

EXERCISE XVII (Page 105)

1. $l_1 : l+l_1$.
2. 11 : 2.
3. $\pm\sqrt{5} : 1$.
4. 3 : 1.
5. $\left(1+\sin\frac{\pi}{n}\right)\Big/\left(1-\sin\frac{\pi}{n}\right)$.
6. $b/(a+b)$.
8. $m_1 : m_2=(g+f)\ ;\ (g-f)$.
10. (i) 13 ; (ii) $-\frac{8}{3}$; (iii) 9.
11. (i) 14/23 ; (ii) 27/23.
12. $\mathrm{R}=\frac{71\mathrm{V}^2}{90}$, 1818, 5385, 9896, 100·7.
13. $\mathrm{V}=2{\cdot}654\sqrt{x}$, 3·75 cu. ft./min. ; 5·75 ; 2·505.
14. $\frac{1}{8}$.

15. $1/\sqrt{n}$. **16.** $i=\frac{25}{R}$.

17. $v_m=r\sqrt{\frac{l_m}{l}}$.

18. $z=kxy$ where k is constant; $k=\frac{1}{2}$; 0·4, $2\frac{2}{5}$.

19. $x=kft^2$ where k is constant, $k=\frac{1}{8}$; 37·5, 20; $t=\sqrt{\frac{8x}{f}}$.

20. $V=\frac{2}{3}\frac{5}{4}r^2h$, 18 : 1.

21. $p=\frac{kT}{v}$ where k is constant; $2{\cdot}31\times10^3$; 21·644, 4·62.

22. $R=\frac{kl}{d^2}$; $k=5{\cdot}625\times10^{-6}$; (*a*) 2·11; (*b*) 6044 cm.

23. $I=khr^4$. **24.** $N=klr^4\theta$.

25. $M=kWl^2$. **26.** $E=kf^2/s$.

27. $L=kn^2A/l$. **28.** $T=k\sqrt{l}/d^2$.

29. $C=5{\cdot}89\times10^{-13}\times\frac{(n-1)A}{d}$, 1472 $\mu\mu$F.

30. $F=\frac{3{\cdot}82\times10^{-7}Ax}{l}$ where all lengths are in inches, 75 lb. wt.

32. $R=kl^2/w$, 117 ohms. **33.** $f\propto\sqrt{d}$.

EXERCISE XVIII (Page 117)

1. $a=\frac{7}{6}$, $b=-\frac{2}{3}\frac{5}{3}$; $78\frac{1}{2}$. **2.** $\frac{1}{u}=0{\cdot}061-\frac{1{\cdot}24}{v}$; 16·4 cm.

3. $u=\frac{2}{3}\frac{5}{3}$, $f=\frac{2}{1}\frac{5}{8}$. **4.** $3{\cdot}92\times10^{-7}x^{3\cdot23}$, 109.

5. $\frac{1}{2}W^{\frac{1}{2}}$, 9 lb. **6.** $a=1{\cdot}608$, $b=-0{\cdot}0442$.

7. $R=100+0{\cdot}4t$, $R=100{\cdot}8\{1+0{\cdot}00397(t-2)\}$.

8. $p=24{\cdot}83+0{\cdot}088t$. **9.** $W=1{\cdot}07l+1{\cdot}48$.

10. $B=28{\cdot}1d^2+0{\cdot}6$. **11.** $e=\frac{90}{l}+19{\cdot}1$.

12. $W=\frac{415}{x}-3{\cdot}7$. **13.** $l=h+\frac{225}{h}$.

14. $H=0{\cdot}633\sqrt{d}+\frac{0{\cdot}625}{\sqrt{d}}$. **15.** $f=\frac{3{\cdot}75\times10^8}{7{\cdot}5\times10^3+x^2}$.

16. $pv^{1\cdot27}=101$. **17.** $t=7{\cdot}95h^{-\frac{1}{2}}$.

18. $I=8{\cdot}9\times10^{-7}V^4$.

EXERCISE XIX (Page 129)

3. 1·95 in. from A, 0·65 in. from B; 3·9 in. from A, 1·3 in. from B.

4. $4\frac{1}{2}$. **5.** 2·14. **10.** 6·4 ft.

EXERCISE XX (Page 144)

1. (i) Not similar. Sides not proportional; (ii) Similar; (iii) Similar; (iv) Not similar. Angles not equal; (v) Similar; (vi) Similar.
2. 7·5 cm., 5 cm., 6·25 cm., 8·75 cm.
3. BC = 9 ft., CD = 20 ft. 3 in., DE = 6 ft. 9 in., EF = FA = 15 ft. 9 in.; area = $430\frac{5}{16}$ sq. ft.
4. 15·3 in., 3·5 in.
5. $\frac{5}{6}$ mile.
6. 76 ft.
7. 4 ft. 8 in. (to the nearest inch).
8. 250,000 miles (approx.).
9. 5 ft. 4 in.
10. 70 ft.
11. 1 : 2.
12. $(2\frac{1}{2}, 5)$.
13. R(3, $5\frac{1}{2}$); S($1\frac{1}{2}$, 6).
14. (8, 4·8), (4·8, 8), (1·6, 8), (−1·6, 3·2).
22. 7 : 5; $1\frac{1}{2}$ in. from A and B.
24. 9 in., 24 in.
25. 10 in., 16 in.
26. 6 miles to 1 inch.
27. 91·08 sq. in.
28. 1·304 sq. in.; 20·86 sq. in.
30. $at/(a-b)$.
32. 3 cm.
36. 9 in.

EXERCISE XXI (Page 163)

1. 54·55 sq. ft.
2. 9·2 in. or 5 ft. 2·8 in.
3. 19 ft. 7 in.
4. 122·5 sq. in.
5. 10 ft. 11·6 in.
6. 1,067 ft.
7. 6·63 cm.
8. 7·26 cm.
9. 0·19 in.
10. $3\frac{1}{8}$ in.
11. 0·27 in., 1·23 in.
12. 5 ft. $2\frac{1}{2}$ in.
15. 1·97 in.
16. 3·8 in.
19. 11·91 cm., 10 67 cm.
20. (i) $\sqrt{c^2-(a-b)^2}$, (ii) $\sqrt{c^2-(a+b)^2}$.

EXERCISE XXII (Page 178)

1. 10·5 in.
2. (i) $\sqrt{3a}$; (ii) 35° 16′; (iii) 54° 44′.
3. (i) 53° 8′; (ii) 21 ft.; (iii) 36·37 ft.
4. (i) 54° 44′; (ii) 70° 32′.
6. 54° 44′.
7. 10° 20′.
8. 20° 42′, 37° 46′.
9. (i) 62° 4′; (ii) 69° 27′.
10. (i) 379·3 sq. ft.; (ii) 8·96 ft.; (iii) 33° 56′; (iv) 35° 32′, 63° 26′.
11. 35° 16′.
12. 1 in 10·64.
13. 14° 26′.
14. 8 sq. in.
15. 138 sq. in.
16. 7·3 sq. ft. (approx.).
17. 7·746 in.
18. 6,082 ft.
19. (i) 3,558 ml.; (ii) 2,662 ml.
20. 9,920 ml.

EXERCISE XXIII (Page 189)

1. 29·53 lb.
2. 0·315 in., 0·630 in., 0·794 in.
3. 2·83; 22·63.
4. 6·30 cm.
5. 5·34 in.
6. 117 : 8.
7. 1·24 in.
8. 70·15 cu. in.

9. 0·276 in. **10.** 8·8 cu. ft.
11. 60·3 cu. cm.; 110·9 sq. cm. **12.** 9·43 in.
13. 2·03 in. **14.** 4·91 lb. **15.** 1·59 mm.
17. 208 ft. per min. **18.** 2·68 ft., 1·34 ft. **19.** 641¼ cu. ft.
20. 13,060 cu. ft. (approx.). **21.** 1·35 cu. ft.
22. 77·7 sq. in. **23.** 1·05 cu. in. **24.** 369 cu. in.
25. 17·4 cu. in. **26.** 52·6 cu. in. **27.** 130·8 sq. in.
28. 1 : 0·806 : 1·201. **29.** $\pi lD/d$. **30.** 30·16 cu. cm.
31. 0·368 cu. in. **32.** 9/32. **33.** 16·93 cu. in.
34. 2·63 oz.

EXERCISE XXIV (Page 205)

1. (i) 0·8218; (ii) 2·3926; (iii) 0·8800; (iv) 1·0258; (v) 0·1552, (vi) 1·8190; (vii) 0·6424; (viii) 1·0218.
2. (i) a/c; (ii) a/b; (iii) c/b; (iv) a/b; (v) a/c; (vi) c/a.
3. $\frac{\sqrt{3}}{2}$, 2, $\sqrt{3}$, $\sqrt{2}$, $\frac{2}{\sqrt{3}}$, $\frac{1}{\sqrt{3}}$.
4. (i) 1·754 in., 3·595 in.; (ii) 6·882 in., 8·506 in.; (iii) 3·057 cm., 10·457 cm.
5. 36·6 yd. **6.** 1·09 in.
7. 22·48 in. **8.** 13 ft. 3 in.
9. 39·6 sq. cm.; 5·66 cm.
10. (i) 33° 22′; (ii) 80° 41′; (iii) 66° 8′.
11. (i) 13° 21′; (ii) 73° 10′; (iii) 34° 51′.
12. 56° 15′.
13. (i) 38° 40′, 51° 20′; (ii) 38° 56′, 51° 4′; (iii) 25° 46′, 64° 14′.
15. 54° 3′ **16.** 181·2 sq. ft.
17. 10,072 cu. ft.
18. 16 ft. 9 in.; 6 ft. 7 in. (to the nearest inch).
19. 69° 54′, 7 ft. 6 in. **20.** 9·334 in., 15·151 in.; 69·453 sq. in.
21. (i) 8·27 ml.; (ii) 39° 14′. **22.** 16·35 cm.; 17·10 cm.
23. 920 watts. **24.** 18·35 in., 27·05 in.
28. (i) 13 ft. 0¾ in.; (ii) 13 ft. 7¼ in.

EXERCISE XXV (Page 213)

3. 65°. **6.** 5/12. **7.** $2\sqrt{3}/3$. **9.** $\sqrt{2}$. **10.** $\cot^2\theta$.
11. sin A. **12.** tan a. **13.** cos θ **14.** 2 sec.^2x. **15.** 0.

EXERCISE XXVI (Page 216)

1. 106° 16′; 3 in. **2.** 70° 32′.
3. 22° 12′. **4.** 4·36 in., 7·42 in., 20·34 sq. in.
5. 36°, 44° 28′. **6.** (i) 41° 49′; (ii) 70° 32′.
7. 26° 34′, 19° 28′. **8.** 1 (vertical) in 2·65 (along the path).
9. 32·66 sq. ft.

EXERCISE XXVII (Page 220)

1. 6·01 ml. **2.** 124 ft.
3. 10·76 ml. **4.** 2,730 ft. (approx.), 176 m.p.h.
5. 132 ft. **6.** 7 ft. 1 in.

EXERCISE XXVIII (Page 221)

1. 2° 52′. **2.** 11·95 in.
3. (i) 10° 58′; (ii) 10·44 ft.; (iii) 1·56 ft.
4. 2·184. **5.** 4,122 cu. yd.
6. 11·91 ft. **7.** 5° 24′.
8. $AE=BE=8{\cdot}83$ ft., $CE=DE=3{\cdot}97$ ft., $CD=2{\cdot}72$ ft.
9. $C\hat{A}D=C\hat{B}E=17°\ 7'$, $A\hat{C}D=B\hat{C}E=20°$, $A\hat{D}C=B\hat{E}C=142°\ 53'$
$C\hat{D}E=C\hat{E}D=56°\ 52'$, $D\hat{C}E=66°\ 16'$.
10. 693 sq. ft. **11.** S 11° 43′ W., S. 49° 36′ E.
12. 1 (vertical) in 7·77 (along the tunnel); 2,330 ft.
13. An equation; 0°, 90°. **14.** $5{\cdot}34\times10^{-3}$.
15. 36° 30′. **16.** 67° 23′.

EXERCISE XXIX (Page 230)

1. +, −, −, +, +. **2.** −, +, −, −, −.
3. $-\cos 72°$, $+\sin 64°$, $-\sin 10°$, $+\tan 7°\ 30'$.
4. $+\sin \pi/2$, $-\cos 68°$, $-\tan 85°\ 45'$, $-\cos \pi/4$.
5. $+\tan 80°$, $+\sec 66°\ 18'$, $+\sin(\pi-1{\cdot}82)=+\sin 1{\cdot}3216$, $-\operatorname{cosec} 34°$.
6. 0·6428. **7.** −0·7593. **8.** −0·8480. **9.** 0·7071.
10. −0·9573. **11.** 0·2849. **12.** −0·2790. **13.** −0·3827.
14. −1·4142. **15.** −0·1495. **16.** 2·1301. **17.** 1/2.
22. −230·1. **23.** 5·5923.
24. (i) Second; (ii) fourth; (iii) third. **25.** −0·6818.

EXERCISE XXX (Page 245)

1. 0·7431; 45° 14′, 134° 56′.
2. (i) 90°, $\frac{1}{2}\pi^c$; (ii) 120°, $\frac{2}{3}\pi^c$; (iii) 36°, $\frac{1}{5}\pi^c$.
3. (i) 720°, $4\pi^c$; (ii) 240°, $\frac{4}{3}\pi^c$; (iii) $\frac{180°}{p}$, $\frac{\pi^c}{p}$.
4. (i) 50; (ii) $\frac{20}{\pi}\simeq 6{\cdot}4$; (iii) 1.
5. (i) 5, $\frac{1}{2}\pi$, $\frac{2}{\pi}$; (ii) 230, $\frac{1}{2\cdot5}$, 25.
6. (i) 3·6, 4π, $\frac{1}{4\pi}$; (ii) 10, $\frac{1}{1\cdot0}$, 50; (iii) 12, $\frac{1}{2}\pi$, $\frac{2}{\pi}$.
7. (i) A, $\frac{2\pi q}{p}$, $\frac{p}{2\pi q}$; (ii) B, $\frac{2\pi q}{p}$, $\frac{p}{2\pi q}$.
12. (i) 0·111 sec.; (ii) 3·26 ft. (or 99·4 cm.).
13. $y=3\sin \pi t$.

14. $y=3\sin\pi(t-1/4)$.

15. After $\dfrac{50-15\pi}{314}\simeq 0{\cdot}00916$ sec.; after $\dfrac{50-15{\cdot}5\pi}{314}\simeq 0{\cdot}00416$ sec.

16. (i) $\frac{3}{8}\pi$ radians; (ii) $\frac{1}{4}$ of a period; (iii) the first oscillation leads the second by $3\pi/32$.

18. $y=5\sin(3x+90°)$, *i.e.* $y=5\cos 3x$.

19. $x=4\sin(\frac{1}{2}\theta+22\frac{1}{2}°)$.

EXERCISE XXXI (Page 251)

1. 8 in., 4 in. **9.** π. **10.** 2π. **11.** 12π.
12. 2. **13.** 2π. **14.** 2π. **15.** 50.
16. 15. **17.** 25.
18. *f*. If one of the frequencies is an integral multiple of the other, the resultant frequency is the smaller of the two.

EXERCISE XXXII (Page 259)

1. 0, 1·11, 3·70. **2.** 0·86. **3.** 21·7, 141·7.
4. 0·4724 (radians), 21·98. **5.** 77°.
6. 1·935; 110° 51′. **7.** 10·9°. **8.** (i) First; (ii) third.
9. 19° 16′, 160° 44′. **10.** 61° 50′, 298° 10′. **11.** 246° 56′, 293° 4′.
12. 76° 42′, 256° 42′. **13.** 44° 25′, 315° 35′. **14.** 147° 40′, 212° 20′.
15. 54° 42′, −125° 18′. **16.** −15° 22′, −164° 38′.
17. $\theta=0{\cdot}2528$ or $2{\cdot}8888$. **18.** $x=\frac{1}{3}\pi$ or $\frac{7}{4}\pi$.
19. $\alpha=1{\cdot}3695$ or $4{\cdot}9137$. **20.** $\theta=\frac{1}{3}\pi$ or $\frac{5}{6}\pi$.
21. 17°, 73°, 107°, 163°, 197°, 253°, 287°, 343°.
22. 118° 38′, 241° 22′. **23.** 105°, 165°, 285°, 345°.
24. 20° 36′, 80° 36′, 140° 36′, 200° 36′, 260° 36′, 320° 36′.
25. 0·077, 0·256, 0·744, 0·923 sec.
26. (i) After $\dfrac{50-15\pi}{314}\simeq 0{\cdot}00916$ sec.; (ii) after $\dfrac{50-15{\cdot}5\pi}{314}\simeq 0{\cdot}00416$ sec.
27. (i) 0·606; (ii) 0·955.
28. 125° 54′. **29.** 337° 23′. **30.** 208° 4′.

EXERCISE XXXIII (Page 263)

1. 30°, 150°, 210°, 330°. **2.** 65° 54′, 114° 6′, 245° 54′, 294° 6′.
3. 35° 16′, 144° 44′, 215° 16′, 324° 44′. **4.** 270°.
5. 45°, 63° 26′, 225°, 243° 26′. **6.** 0°, 26° 34′, 180°, 206° 34′.
7. 101° 32′, 258°, 28′. **8.** 15°, 75°, 135°, 195°, 255°, 315°.
9. 0°, 180°, 360°. **10.** 51° 53′, 128° 7′, 231° 53′, 308° 7′.
11. 16° 9′. **12.** 7° 1′.

EXERCISE XXXIV (Page 267)

5. $-\sin\theta$. **6.** $-\cos\theta$. **7.** $\sin\theta$.
8. $\tan\theta$. **9.** $-\cos\theta$. **10.** $-\cot\theta$.
11. 1. **12.** $\sin\theta$. **13.** $\tan^3 x$.
19. $x^2+y^2=2$. **20.** $26x^2-14xy+5y^2=81$. **21.** $4x^2-y^2=4$.

EXERCISE XXXV (Page 281)

1. 11·79 lb. wt. at 47° 16′ with the force of 3 lb. wt.
2. 28·03 ft. per sec. at 15° 31′ with the velocity of 40 ft. per sec.
3. 9·434 tons wt. at 58° with the force of 5 tons wt.
4. 7·072 tons wt., N. 44° 25′ E.
5. 32·05 m.p.h., N. 4° 2′ W.
6. (i) $4_{115°}$; −1·6904, 3·6252; (ii) $5_{210°}$; −4·330, −2·5; (iii) $2_{295°}$; 0·8152, −1·8126.
7. $4{\cdot}845_{268°\ 16'}$.
8. 23·3 m.p.h. at 9° 52′ with direction of ship's motion.
9. 141·9 m.p.h.; 15½ sec.
10. $3{\cdot}44_{151°\ 35'}$.
11. 41° 24′ with the bank.
12. $P\cos\theta - W\sin\alpha$.
13. 5·98 tons wt. E., 4·84 tons wt. N.E.
14. 37·74 ft. per sec. at 58° with the velocity of 20 ft. per sec.
15. 6·33 ml., N. 77° 23′ E.
16. $15{\cdot}06_{105°\ 47'}$.
19. 62° 43′ between the first and second force, 153° 37′ between the second and third, 143° 40′ between the third and first.
20. $r=13$, $\theta=67°\ 23'$.
21. $r=5$, $\theta=-36°\ 52'$.
22. $r=2{\cdot}865$, $\theta=29°\ 15'$.
23. $r=3{\cdot}138$, $\theta=120°\ 39'$.
24. $15{\cdot}06_{105°\ 47'}$.
25. 122·4 lb. wt.
26. P=6·534, Q=4·750.
27. See answers to Question 26.
30. $4{\cdot}21_{61°\ 44'}$.
31. 11·72 lb. wt. in direction S. 38° 42′ E.

EXERCISE XXXVI (Page 290)

9. $4{\cdot}33\sin 300t + 2{\cdot}5\cos 300t$; 1·191.
10. $15{\cdot}92\sin 100\pi t - 12{\cdot}11\cos 100\pi t$.
12. 63/65.
13. 24/25, −7/25, −24/7.
14. sin 70°.
15. cos 60°.
16. $\cos\frac{1}{7}\pi$.
17. sin (−30°), *i.e.* −sin 30°.
18. $\sin 3\theta$.
19. $\cos 2\alpha$.
20. cos 30°.
21. $\frac{1}{2}\sin x$.
24. $\dfrac{1+\sqrt{3}}{2\sqrt{2}} \simeq 0{\cdot}9659$.
25. $\dfrac{1+\sqrt{3}}{2\sqrt{2}} \simeq 0{\cdot}9659$.
26. $\frac{1}{2}\sqrt{2+\sqrt{2}} \simeq 0{\cdot}9239$.
38. 30°, 150°, 270°.
39. 0°, 60°, 180°, 300°.
40. 0°, 52° 15′, 127° 45′, 180°, 232° 15′, 307° 45′.
41. 133° 11′, 313° 11′.
42. 90°, 180°.
43. 123° 24′, 236° 36′.
44. 56° 43′, 236° 43′.

EXERCISE XXXVII (Page 297)

1. $8{\cdot}062\sin(\theta+29°\ 45')$.
2. $3{\cdot}176\sin(\theta+61°\ 40')$.
3. $13\sin(\theta+292°\ 37')$ *or* $13\sin(\theta-67°\ 23')$.
4. $129{\cdot}3\sin(\theta+320°\ 39')$ *or* $129{\cdot}3\sin(\theta-39°\ 21')$.

5. $5{\cdot}007 \sin(\theta + 67° 19')$. **6.** $27{\cdot}31 \sin(300t - 1{\cdot}1563)$.
7. $\pm\sqrt{a^2+b^2}$. **8.** ± 5; 143° 8′, 323° 8′.
9. $1{\cdot}628 \sin(2\pi ft + 0{\cdot}1853)$. **10.** $2 \sin(\theta + \frac{1}{3}\pi)$.
11. $17 \sin(60\pi t + 0{\cdot}4899)$; $\frac{1}{30}$; 17, 8.
12. 104° 24′, 330° 20′. **13.** 19° 50′, 160° 10′.
14. 249° 38′. **15.** 0°, 126° 52′, 360°.
16. 7° 42′, 138° 58′. **17.** 10·549; 73° 20′.
18. $13{\cdot}98 \sin(3wt - 0{\cdot}6536)$; $0{\cdot}0081 \pm n \times 0{\cdot}0349$ *or* $0{\cdot}0167 \pm n \times 0{\cdot}0349$.
19. 0° to 126° 52′, 306° 52′ to 360°. **20.** 12° 3′.
21. $Z = \sqrt{R^2 + L^2\omega^2}$, $\phi = \tan^{-1}(L\omega/R)$; $Z = 12{\cdot}36$, $\phi = 0{\cdot}867$.
22. (i) $A+B$, $A-B$; (ii) $c+\sqrt{a^2+b^2}$, $c-\sqrt{a^2+b^2}$; (iii) $A+\sqrt{B^2+C^2}$, $A-\sqrt{B^2+C^2}$.

EXERCISE XXXVIII (Page 302)

4. 850,000 miles (approx.). **5.** 3·8 miles (approx.).
6. 0·9949. **7.** 0·00022 in.
8. 0·719. **9.** 0·71934.

EXERCISE XXXIX (Page 302)

4. 0·5477, 0·8367, 0·6546. **6.** ±79° 16′, ±100° 44′.
8. $a = -2\sqrt{3}$, $b = 1$; $3{\cdot}606 \sin(\omega t + 163° 54')$.
9. $a = \sqrt{3}$, $b = 1$. **11.** 2·153, 7·486.

EXERCISE XL (Page 317)

1. (i) 6·792 in.; (ii) 6·594 cm. **2.** 10·64 cm.
3. 2·901 in. **4.** 28° 32′. **5.** 21·60 in.
6. 120°. **7.** 15·10 cm. **8.** 118 sq. ft.
9. 6·724 sq. in. **10.** 5·764 in.; 20·66 sq. in.
11. 2 ml. 1,690 yd. **12.** 8 ft. 0 in. **13.** 7·17 ft.
14. 144° 7′. **15.** 65·03 ml.
16. $BC = 6{\cdot}93$ ft., $AC = 8$ ft., $CD = 5{\cdot}04$ ft. **17.** 9·102 lb. wt.
18. 67·2 m.p.h. from direction S. 71° 3′ E.
19. 4·71 tons wt. at an angle 86° 28′ with the force of 2 tons wt.
20. 7·926 in. **21.** 78·25 sq. cm.
22. 18·23 in.; 24·63 sq. in. **23.** 0·291 sq. in.
24. 3·812 cm. **25.** 8·67 sq. in.
26. 19,950 cu. yd. (approx.). **27.** 22·93 sq. in.
28. See answer to Question 27.

EXERCISE XLI (Page 330)

1. $A = 81° 29'$, $B = 62° 19'$, $C = 36° 12'$.
2. $A = 20° 24'$, $B = 118° 14'$, $C = 41° 22'$.
3. $B = 74°$, $a = 1{\cdot}654$ in., $c = 3$ in.
4. $a = 14{\cdot}13$ ft., $b = 20{\cdot}61$ ft., $A = 29°$.

5. $c=3{\cdot}758$ cm., $A=39°\ 6'$, $B=80°\ 54'$.
6. $b=29$ ml. (approx.), $A=136°\ 24'$, $C=32°\ 11'$.
7. $a=3{\cdot}985$ in., $A=63°\ 46'$, $C=64°\ 14'$, *or* $a=0{\cdot}941$ in., $A=12°\ 14'$, $C=115°\ 46'$.
8. $c=4{\cdot}550$ cm., $B=54°\ 30'$, $C=47°\ 48'$.
9. $b=1{\cdot}164$ ft., $A=20°\ 42'$, $B=24°\ 18'$.
10. (i) 10 cm., 5·177 cm., 75° ; (ii) 18° 11′, 45° 13′, 116° 36′ ; (iii) 4·022 in., 27° 33′, 68° 27′ ; (iv) 183·5 yd., 29° 3′, 33° 57′.
11. 64·82 ft. **12.** 0·317 sq. in.
13. 77° 10′ **14.** 33° 41′ *or* 18° 26′.
15. $DF=18{\cdot}70$ yd., $EF=6{\cdot}18$ yd.
16. 1·52 ft. **17.** 1·45 ft. ; 0·62 ft. **18.** 6·59 ft., 7·78 ft.

EXERCISE XLII (Page 336)

1. 2·945 ml. **2.** 3 in.
3. 43·8 ft. **4.** 66° 56′.
5. 66·62 ft. **6.** 4·68 ft., 3·52 ft., 4·68 ft.
7. 379 yd. **9.** 15·7 knots, N. 66° 38′ E.

EXERCISE XLIII (Page 338)

1. 8 ft. 5 in. **2.** 12·37 knots.
3. 24·93 ft. **4.** By car.
5. 135 yd. **6.** $AD=7'\ 10''$, $BD=11'\ 3''$.
7. 1,075 gall. (approx.). **8.** 32·1 nautical miles ; S. 22° 36′ E.
9. 43·2 yd. **10.** 183 m.p.h.
11. $A=78°\ 36'$, $C=71°\ 24'$, $b=2{\cdot}244$ in., $c=4{\cdot}256$ in.
12. 15·78 in.

EXERCISE XLIV (Page 351)

1. (*a*) 300 m.p.h. ; (*b*) 150 m.p.h. ; (*c*) 180 m.p.h.
2. 8·5° C. per min. **3.** $n=40-\frac{t}{15}$; $-\frac{1}{15}$ gal. per min.
4. Approx. 10,000 ft./sec.
5. 2·225 in. per year : (*a*) 4½ to 8½ yr. ; (*b*) 6 to 7 yr. ; (*c*) 5 in. per yr.
6. 22 ft./sec., $-0{\cdot}6$ ft./sec.^{2}. **7.** $\frac{\delta x}{\delta t}$ in. per yr. ; $\frac{dx}{dt}$ in. per yr.
8. After 50 sec. : (*a*) after about 30 sec. ; (*b*) after about 65 sec.
9. 5. **10.** Negative.
11. 0·045, 0·055. **12.** (*a*) $t=2$ to 3 ; (*b*) $t=9$ to 11.
13. 1¼ sec., 3¼ sec. ; body is falling.

EXERCISE XLV (Page 357)

1. (*a*) 1·6 ; (*b*) 4·6. **3.** 0·016, 0·026, 0·02.
5. 4π, 9π. **6.** 0·22.
7. 8·192, 8·624, 1080. **8.** 0·9657, 1·0355, 0·0349.

EXERCISE XLVI (Page 366)

1. (*a*) 48 ft./sec., 112 ft./sec. ; (*b*) 97·6 ft./sec., 96·16 ft./sec. ; (*c*) 96·0016 ft./sec. ; (*d*) 96·000016 ft./sec.
2. $(32t+0\cdot16)$ ft./sec., $32t+0\cdot0016$ ft /sec., $32t$ ft./sec.
3. $12x+6h$, $12x$.
4. $12x\delta x+6\delta x^2$, $12x+6\delta x$, $12x$.
5. $-\frac{\delta x}{x(x+\delta x)}$, $-\frac{1}{x^2}$.
6. $3x^2h+3xh^2+h^3$, $3x^2+3xh+h^2$, $3x^2$.
7. $\frac{d\eta}{dx}=a$.
8. $\frac{2}{x^3}$.
9. $9x^2$.
10. $-4x$.
11. $720x^7$.
12. $8x^3-2x^5$.
13. $12x^2-14x+21$.
14. $0\cdot02-5\cdot08x-2.16x^2$.
15. $-3/x^4$.
16. $-1/2x^5$.
17. $4\cdot8^{0\cdot6}$.
18. $-\frac{1}{2x^3}+\frac{1}{2x^2}$.
19. $10t$.
20. $\frac{3}{5}x^2$.
21. $0\cdot2r$.
22. $8000x^3$.
23. $49t^6$.
24. $8p+20p^3$.
25. x^{n-1}.
26. $u+gt$.
27. $-100/v^2$.
28. $-k/rk^{+1}$.
29. $9\cdot6\times10^{-6}V^{3\cdot8}$.
30. $2/\sqrt{z}$.
31. $-\frac{1}{2}$, $1\frac{1}{2}$, $\frac{1}{2}$, $-\frac{1}{4}$.
32. 16 ft./sec. upwards, 16 ft./sec. downwards, after $2\frac{1}{2}$ sec., 100 ft.
33. -1, 1.
35. (0, 6), (2, 2).

EXERCISE XLVII (Page 370)

1. Max. 1.
2. Max. 0 ; min. -4.
3. Max. $1\frac{1}{4}$.
4. Min. -9.
5. Min. 0.
6. Max. 2 ; min. -2.
7. $wl^2/8$.
8. A cube of side $\sqrt[3]{6\cdot4}$ ft. $\simeq1\cdot86$ ft.
9. 64.
10. $\frac{3}{8}$ amps.
11. Least 3 ; greatest 4.
12. $9Wl^2/128$.

EXERCISE XLVIII (Page 375)

1. 64 ft.
2. 660 yd.
3. 550 rev.
4. 5 ml., $\frac{44}{135}$ ft./sec.^{2}.

EXERCISE XLIX (Page 382)

1. 26 sec.
3. 0·023.
8. 3·2.
9. x^2+c, b^2-a^2.
10. $\frac{1}{6}t^3+c$, $2\frac{1}{3}$.
11. 240 ft.
12. $3x^2+c$.
13. $3x^2+5x+c$.
14. $3x^3+c$.
15. $3x^3+3x^2+5x+c$.
16. $\frac{5}{6}x^6+c$.
17. $-\frac{1}{x}+c$.
18. $x^{1\cdot8}+c$.
19. $3x^2-\frac{1}{x}+c$

EXERCISE L (Page 387)

1. $\frac{1}{8}x^8+c$. **2.** $2x^5+c$. **3.** $\frac{1}{10}v^5+c$.
4. $0{\cdot}01t^{10}+c$. **5.** $\frac{2}{3}\sqrt{r^3}+c$. **6.** $-1/2x+c$.
7. $-1/3m^3+c$. **8.** $\frac{1}{3}^0p^{3\cdot6}+c$. **9.** 8.
10. 16. **11.** $1\frac{1}{4}$. **12.** $\frac{4}{3}\pi a^3$.
13. $\frac{31}{1600}$. **14.** $\frac{1}{8}$. **15.** $100\left(1-\frac{1}{\sqrt[5]{2}}\right)$.
16. $\frac{2}{3}(2-\sqrt{2})$. **17.** $\frac{2a^7}{7}$. **18.** $\frac{1}{n-1}\left(\frac{1}{v_1^{n-1}}-\frac{1}{v_2^{n-1}}\right)$
19. $\frac{5}{18}$. **20.** $x-\frac{1}{3}x^3+c$. **21.** v^2-v^4+c.
22. $\frac{1}{4}x^4-\frac{1}{2}x^2+c$. **23.** 36. **24.** $\frac{1}{n+1}(b^{n+1}-a^{n+1})$.
25. $\frac{28}{3}$. **26.** $\frac{1}{2}(b^2-a^2)$. **27.** 48 ft.
28. $(2t^2+6)$ ft./sec., 56 ft./sec.
29. $y=\frac{1}{3}x^3+c$, where c is the value of y at $x=o$.
30. $x=2$ and $x=3$, $\frac{1}{6}$.

EXERCISE LI (Page 395)

1. 5 acres.
2. (*a*) 1,863,000 gal. ; (*b*) 3,117,000 gal.
3. 29·1 cu. ft. **4.** 460 in. lb. wt., 153 lb. wt.
5. 7960 ft. lb. ; 31·8 lb. wt./ft.^{2}.
6. 62310 cu. ft., 316 sq. ft.
7. Increase in velocity from $t=a$ to $t=b$; 27·3 ft./sec.
12. 154 ft. lb. wt., 308 lb. wt./ft.^{2}.
13. $\frac{1}{2}x$ lb. wt., 25 ft. lb. wt. **15.** $\frac{1}{3}a^2$.
16. 2. **17.** $\frac{1}{2}\pi$.
18. 64π cu. in. **20.** $\frac{152\pi}{3}$ cu. in.

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— [illegible]
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LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	[illegible]		0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	[illegible]	[illegible]	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	[illegible]	[illegible]	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	[illegible]	[illegible]	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	[illegible]	[illegible]	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	[illegible]	[illegible]	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	[illegible]		7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	[illegible]	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	[illegible]	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTI-LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	[illegible]	[illegible]	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	[illegible]	[illegible]	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTI-LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	[illegible]	[illegible]	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	[illegible]	[illegible]	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	[illegible]	[illegible]	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	[illegible]	[illegible]	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

NATURAL SINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Differences 1′	2′	3′	4′	5′
°															
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Differences 1′	2′	3′	4′	5′
°															
45	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	1
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993					
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998					
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000					

NATURAL COSINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Subtract Differences 1′	2′	3′	4′	5′
°															
0	1·000	1·000	1·000	1·000	1·000	1·000	0̄999	0̄999	0̄999	0̄999					
1	·9998	9998	9998	9997	9997	9997	9996	9996	9995	9995					
2	·9994	9993	9993	9992	9991	9990	9990	9989	9988	9987					
3	·9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	·9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	·9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	1
6	·9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	·9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	·9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	·9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	·9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	4	4
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	·9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	·8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	·8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	·8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	·8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	·8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	·8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	·8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	·8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	·8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	·8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	·7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	·7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	·7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	·7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	·7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

NATURAL COSINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Subtract Differences 1′	2′	3′	4′	5′
°															
45	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	·6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4225	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	13
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	13
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	11	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	14
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15

NATURAL TANGENTS

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Differences 1′	2′	3′	4′	5′
°															
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36′	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

NATURAL TANGENTS

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Differences 1′	2′	3′	4′	5′
°															
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	26	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	$\bar{0}057$	$\bar{0}145$	$\bar{0}233$	$\bar{0}323$	$\bar{0}413$	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	91
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2·9042	9208	9375	9544	9714	9887	$\bar{0}061$	$\bar{0}237$	$\bar{0}415$	$\bar{0}595$	29	58	87	116	145
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972					
77	4·3315	3662	4015	4373	4737	5107	5483	5864	6252	6646					
78	4·7046	7453	7867	8288	8716	9152	9594	$\bar{0}045$	$\bar{0}504$	$\bar{0}970$					
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80	5·6713	7297	7894	8502	9124	9758	$\bar{0}405$	$\bar{1}066$	$\bar{1}742$	$\bar{2}432$					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	$\bar{0}264$					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	$\bar{0}285$					
83	8·1443	2636	3863	5126	6427	7769	9152	$\bar{0}579$	$\bar{2}052$	$\bar{3}572$					
84	9·514	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					

LOG. SINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Differences 1′	2′	3′	4′	5′
°															
0	−∞	$\bar{3}\cdot242$	$\bar{3}\cdot543$	$\bar{3}\cdot719$	$\bar{3}\cdot844$	$\bar{3}\cdot941$	$\bar{2}\cdot020$	$\bar{2}\cdot087$	$\bar{2}\cdot145$	$\bar{2}\cdot196$					
1	$\bar{2}\cdot2419$	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	$\bar{2}\cdot5428$	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	$\bar{2}\cdot7188$	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4	$\bar{2}\cdot8436$	8543	8647	8749	8849	8946	9042	9135	9226	9315					
5	$\bar{2}\cdot9403$	9489	9573	9655	9736	9816	9894	9970	$\bar{0}$046	$\bar{0}$120	13	26	39	52	66
6	$\bar{1}\cdot0192$	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	$\bar{1}\cdot0859$	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	$\bar{1}\cdot1436$	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	$\bar{1}\cdot1943$	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	$\bar{1}\cdot2397$	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	$\bar{1}\cdot2806$	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	$\bar{1}\cdot3179$	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	$\bar{1}\cdot3521$	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	$\bar{1}\cdot3837$	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	$\bar{1}\cdot4130$	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	$\bar{1}\cdot4403$	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	$\bar{1}\cdot4659$	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	$\bar{1}\cdot4900$	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	$\bar{1}\cdot5126$	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	$\bar{1}\cdot5341$	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	$\bar{1}\cdot5543$	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	$\bar{1}\cdot5736$	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	$\bar{1}\cdot5919$	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	$\bar{1}\cdot6093$	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	$\bar{1}\cdot6259$	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	$\bar{1}\cdot6418$	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	$\bar{1}\cdot6570$	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	$\bar{1}\cdot6716$	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	$\bar{1}\cdot6856$	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	$\bar{1}\cdot6990$	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	$\bar{1}\cdot7118$	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	$\bar{1}\cdot7242$	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	$\bar{1}\cdot7361$	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	$\bar{1}\cdot7476$	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	$\bar{1}\cdot7586$	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	$\bar{1}\cdot7692$	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	$\bar{1}\cdot7795$	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	$\bar{1}\cdot7893$	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	$\bar{1}\cdot7989$	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	$\bar{1}\cdot8081$	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	$\bar{1}\cdot8169$	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	$\bar{1}\cdot8255$	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	$\bar{1}\cdot8338$	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	$\bar{1}\cdot8418$	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

LOG. SINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Differences 1′	2′	3′	4′	5′
°															
45	1·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	1·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	1·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	1·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	1·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	1·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	1·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	1·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	1·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	1·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	1·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	1·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	1·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	1·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	1·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	1·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	1·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	1·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	1·9499	9503	9506	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	1·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	1·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	1·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	1·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	1·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	1·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	1·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	1·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	1·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	1·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	1	2
74	1·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	1·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	1·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	1·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	1·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	1·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80	1·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	1·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	1·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	0	1	1
83	1·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975					
84	1·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983					
85	1·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989					
86	1·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994					
87	1·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997					
88	1·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999					
89	1·9999	9999	0000	0000	0000	0000	0000	0000	0000	0000					

LOG. COSINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Subtract Differences 1′	2′	3′	4′	5′
°															
0	0·0000	0000	0000	0000	0000	0000	0000	0000	0000	9999					
1	1·9999	9999	9999	9999	9999	9999	9998	9998	9998	9998					
2	1·9997	9997	9997	9996	9996	9996	9996	9995	9995	9994					
3	1·9994	9994	9993	9993	9992	9992	9991	9991	9990	9990					
4	1·9989	9989	9988	9988	9987	9987	9986	9985	9985	9984					
5	1·9983	9983	9982	9981	9981	9980	9979	9978	9978	9977					
6	1·9976	9975	9975	9974	9973	9972	9971	9970	9969	9968					
7	1·9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	0	1	1
8	1·9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9	1·9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10	1·9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11	1·9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12	1·9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13	1·9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14	1·9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	1·9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	1·9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	1	2
17	1·9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	1·9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	1·9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20	1·9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	1·9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	1·9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	1·9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	1·9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	1·9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	1·9537	9533	9529	9525	9522	9518	9514	9510	9506	9503	1	1	2	3	3
27	1·9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	1·9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	1·9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30	1·9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	1·9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32	1·9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	1·9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	1·9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	1·9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	1·9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	1·9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	1·8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	1·8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40	1·8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	1·8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42	1·8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	1·8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44	1·8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

LOG. COSINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Subtract Differences 1′	2′	3′	4′	5′
°															
45	1·8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46	1·8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47	1·8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48	1·8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49	1·8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50	1·8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51	1·7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52	1·7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53	1·7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54	1·7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55	1·7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56	1·7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57	1·7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58	1·7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59	1·7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60	1·6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61	1·6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62	1·6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63	1·6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64	1·6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65	1·6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66	1·6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67	1·5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68	1·5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69	1·5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70	1·5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71	1·5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72	1·4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73	1·4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74	1·4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75	1·4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76	1·3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77	1·3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78	1·3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79	1·2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80	1·2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81	1·1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82	1·1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83	1·0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84	1·0192	0120	0046	9970	9894	9816	9736	9655	9573	9489	13	26	39	52	66
85	2·9403	9315	9226	9135	9042	8946	8849	8749	8647	8543					
86	2·8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87	2·7188	7041	6889	6731	6567	6397	6220	6035	5842	5640					
88	2·5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89	2·242	2·196	2·145	2·087	2·020	3·941	3·844	3·719	3·543	3·242					

LOG. TANGENTS

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Differences 1′	2′	3′	4′	5′
°															
0	−∞	$\bar{3}$·242	$\bar{3}$·543	$\bar{3}$·719	$\bar{3}$·844	$\bar{3}$·941	$\bar{2}$·020	$\bar{2}$·087	$\bar{2}$·145	$\bar{2}$·196					
1	$\bar{2}$·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	$\bar{2}$·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	$\bar{2}$·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4	$\bar{2}$·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331					
5	$\bar{2}$·9420	9506	9591	9674	9756	9836	9915	9992	$\bar{0}$·068	$\bar{0}$·143	13	26	40	53	66
6	$\bar{1}$·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	$\bar{1}$·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	$\bar{1}$·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	$\bar{1}$·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	$\bar{1}$·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	$\bar{1}$·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	$\bar{1}$·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	$\bar{1}$·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	$\bar{1}$·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	$\bar{1}$·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	$\bar{1}$·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	$\bar{1}$·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	$\bar{1}$·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	$\bar{1}$·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	$\bar{1}$·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	$\bar{1}$·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	$\bar{1}$·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	$\bar{1}$·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	$\bar{1}$·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	$\bar{1}$·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	$\bar{1}$·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	$\bar{1}$·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	$\bar{1}$·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	$\bar{1}$·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	$\bar{1}$·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	$\bar{1}$·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	$\bar{1}$·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	$\bar{1}$·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	$\bar{1}$·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	$\bar{1}$·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	$\bar{1}$·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	$\bar{1}$·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	$\bar{1}$·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	$\bar{1}$·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	$\bar{1}$·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	$\bar{1}$·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	$\bar{1}$·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	$\bar{1}$·9697	9712	9727	9742	9757	9772	9788	9803	9818	9833	3	5	8	10	13
44	$\bar{1}$·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

LOG. TANGENTS

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Differences 1′	2′	3′	4′	5′
°															
45	·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50	·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55	·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60	·2380	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	7	10	13	16
65	·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69	·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70	·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80	·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81	·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83	·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84	·9784	9857	9932	0008	0085	0164	0244	0326	0409	0494	13	26	40	53	66
85	1·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446					
86	1·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663					
87	1·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357					
88	1·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	1·7581	8038	8550	9130	9800	0591	1561	2810	4571	7581					

SQUARES

	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
10	1000	1020	1040	1061	1082	1103	1124	1145	1166	1188	2	4	6	8	11	13	15	17	19
11	1210	1232	1254	1277	1300	1323	1346	1369	1392	1416	2	5	7	9	12	14	16	18	21
12	1440	1464	1488	1513	1538	1563	1588	1613	1638	1664	3	5	8	10	13	15	18	20	23
13	1690	1716	1742	1769	1796	1823	1850	1877	1904	1932	3	5	8	11	14	16	19	22	24
14	1960	1988	2016	2045	2074	2103	2132	2161	2190	2220	3	6	9	12	15	17	20	23	26
15	2250	2280	2310	2341	2372	2403	2434	2465	2496	2528	3	6	9	12	16	19	22	25	28
16	2560	2592	2624	2657	2690	2723	2756	2789	2822	2856	3	7	10	13	17	20	23	26	30
17	2890	2924	2958	2993	3028	3063	3098	3133	3168	3204	4	7	11	14	18	21	25	28	32
18	3240	3276	3312	3349	3386	3423	3460	3497	3534	3572	4	7	11	15	19	22	26	30	33
19	3610	3648	3686	3725	3764	3803	3842	3881	3920	3960	4	8	12	16	20	23	27	31	35
20	4000	4040	4080	4121	4162	4203	4244	4285	4326	4368	4	8	12	16	21	25	29	33	37
21	[illegible]	[illegible]	4494	4537	4580	4623	4666	4709	4752	4796	4	9	13	17	22	26	30	34	39
22	[illegible]	[illegible]	4928	4973	5018	5063	5108	5153	5198	5244	5	9	14	18	23	27	32	36	41
23	5290	5336	5382	5429	5476	5523	5570	5617	5664	5712	5	9	14	19	24	28	33	38	42
24	5760	5808	5856	5905	5954	6003	6052	6101	6150	6200	5	10	15	20	25	29	34	39	44
25	6250	6300	6350	6401	6452	6503	6554	6605	6656	6708	5	10	15	20	26	31	36	41	46
26	6760	6812	6864	6917	6970	7023	7076	7129	7182	7236	5	11	16	21	27	32	37	42	48
27	7290	7344	7398	7453	7508	7563	7618	7673	7728	7784	6	11	17	22	28	33	39	44	50
28	7840	7896	7952	8009	8066	8123	8180	8237	8294	8352	6	11	17	23	29	34	40	46	51
29	8410	8468	8526	8585	8644	8703	8762	8821	8880	8940	6	12	18	24	30	35	41	47	53
30	9000	9060	9120	9181	9242	9303	9364	9425	9486	9548	6	12	18	24	31	37	43	49	55
31	9610	9672	9734	9797	9860	9923	9986				6	13	19	25	32	38	44	50	57
31								1005	1011	1018	1	1	2	3	3	4	4	5	6
32	1024	1030	1037	1043	1050	1056	1063	1069	1076	1082	1	1	2	3	3	4	5	5	6
33	1089	1096	1102	1109	1116	1122	1129	1136	1142	1149	1	1	2	3	3	4	5	5	6
34	1156	1163	1170	1176	1183	1190	1197	1204	1211	1218	1	1	2	3	3	4	5	6	6
35	1225	1232	1239	1246	1253	1260	1267	1274	1282	1289	1	1	2	3	4	4	5	6	6
36	1296	1303	1310	1318	1325	1332	1340	1347	1354	1362	1	1	2	3	4	4	5	6	7
37	1369	1376	1384	1391	1399	1406	1414	1421	1429	1436	1	2	2	3	4	5	5	6	7
38	1444	1452	1459	1467	1475	1482	1490	1498	1505	1513	1	2	2	3	4	5	5	6	7
39	1521	1529	1537	1544	1552	1560	1568	1576	1584	1592	1	2	2	3	4	5	6	6	7
40	1600	1608	1616	1624	1632	1640	1648	1656	1665	1673	1	2	2	3	4	5	6	6	7
41	1681	1689	1697	1706	1714	1722	1731	1739	1747	1756	1	2	2	3	4	5	6	7	7
42	1764	1772	1781	1789	1798	1806	1815	1823	1832	1840	1	2	3	3	4	5	6	7	8
43	1849	1858	1866	1875	1884	1892	1901	1910	1918	1927	1	2	3	4	4	5	6	7	8
44	1936	1945	1954	1962	1971	1980	1989	1998	2007	2016	1	2	3	4	5	5	6	7	8
45	2025	2034	2043	2052	2061	2070	2079	2088	2098	2107	1	2	3	4	5	5	6	7	8
46	2116	2125	2134	2144	2153	2162	2172	2181	2190	2200	1	2	3	4	5	6	7	7	8
47	2209	2218	2228	2237	2247	2256	2266	2275	2285	2294	1	2	3	4	5	6	7	8	9
48	2304	2314	2323	2333	2343	2352	2362	2372	2381	2391	1	2	3	4	5	6	7	8	9
49	2401	2411	2421	2430	2440	2450	2460	2470	2480	2490	1	2	3	4	5	6	7	8	9
50	2500	2510	2520	2530	2540	2550	2560	2570	2581	2591	1	2	3	4	5	6	7	8	9
51	2601	2611	2621	2632	2642	2652	2663	2673	2683	2694	1	2	3	4	5	6	7	8	9
52	2704	2714	2725	2735	2746	2756	2767	2777	2788	2798	1	2	3	4	5	6	7	8	9
53	2809	2820	2830	2841	2852	2862	2873	2884	2894	2905	1	2	3	4	5	6	7	9	10
54	2916	2927	2938	2948	2959	2970	2981	2992	3003	3014	1	2	3	4	6	7	8	9	10

SQUARES

	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
55	3025	3036	3047	3058	3069	3080	3091	3102	3114	3125	1	2	3	4	6	7	8	9	10
56	3136	3147	3158	3170	3181	3192	3204	3215	3226	3238	1	2	3	5	6	7	8	9	10
57	3249	3260	3272	3283	3295	3306	3318	3329	3341	3352	1	2	3	5	6	7	8	9	10
58	3364	3376	3387	3399	3411	3422	3434	3446	3457	3469	1	2	4	5	6	7	8	9	11
59	3481	3493	3505	3516	3528	3540	3552	3564	3576	3588	1	2	4	5	6	7	8	10	11
60	3600	3612	3624	3636	3648	3660	3672	3684	3697	3709	1	2	4	5	6	7	8	10	11
61	3721	3733	3745	3758	3770	3782	3795	3807	3819	3832	1	2	4	5	6	7	9	10	11
62	3844	3856	3869	3881	3894	3906	3919	3931	3944	3956	1	3	4	5	6	8	9	10	11
63	3969	3982	3994	4007	4020	4032	4045	4058	4070	4083	1	3	4	5	6	8	9	10	11
64	4096	4109	4122	4134	4147	4160	4173	4186	4199	4212	1	3	4	5	6	8	9	10	12
65	4225	4238	4251	4264	4277	4290	4303	4316	4330	4343	1	3	4	5	7	8	9	10	12
66	4356	4369	4382	4396	4409	4422	4436	4449	4462	4476	1	3	4	5	7	8	9	11	12
67	4489	4502	4516	4529	4543	4556	4570	4583	4597	4610	1	3	4	5	7	8	9	11	12
68	4624	4638	4651	4665	4679	4692	4706	4720	4733	4747	1	3	4	5	7	8	10	11	12
69	4761	4775	4789	4802	4816	4830	4844	4858	4872	4886	1	3	4	6	7	8	10	11	13
70	4900	4914	4928	4942	4956	4970	4984	4998	5013	5027	1	3	4	6	7	8	10	11	13
71	5041	5055	5069	5084	5098	5112	5127	5141	5155	5170	1	3	4	6	7	9	10	11	13
72	5184	5198	5213	5227	5242	5256	5271	5285	5300	5314	1	3	4	6	7	9	10	12	13
73	5329	5344	5358	5373	5388	5402	5417	5432	5446	5461	1	3	4	6	7	9	10	12	13
74	5476	5491	5506	5520	5535	5550	5565	5580	5595	5610	1	3	4	6	7	9	10	12	13
75	5625	5640	5655	5670	5685	5700	5715	5730	5746	5761	2	3	5	6	8	9	11	12	14
76	5776	5791	5806	5822	5837	5852	5868	5883	5898	5914	2	3	5	6	8	9	11	12	14
77	5929	5944	5960	5975	5991	6006	6022	6037	6053	6068	2	3	5	6	8	9	11	12	14
78	6084	6100	6115	6131	6147	6162	6178	6194	6209	6225	2	3	5	6	8	9	11	13	14
79	6241	6257	6273	6288	6304	6320	6336	6352	6368	6384	2	3	5	6	8	10	11	13	14
80	6400	6416	6432	6448	6464	6480	6496	6512	6529	6545	2	3	5	6	8	10	11	13	14
81	6561	6577	6593	6610	6626	6642	6659	6675	6691	6708	2	3	5	7	8	10	11	13	15
82	6724	6740	6757	6773	6790	6806	6823	6839	6856	6872	2	3	5	7	8	10	12	13	15
83	6889	6906	6922	6939	6956	6972	6989	7006	7022	7039	2	3	5	7	8	10	12	13	15
84	7056	7073	7090	7106	7123	7140	7157	7174	7191	7208	2	3	5	7	8	10	12	14	15
85	7225	7242	7259	7276	7293	7310	7327	7344	7362	7379	2	3	5	7	9	10	12	14	15
86	7396	7413	7430	7448	7465	7482	7500	7517	7534	7552	2	3	5	7	9	10	12	14	16
87	7569	7586	7604	7621	7639	7656	7674	7691	7709	7726	2	4	5	7	9	11	12	14	16
88	7744	7762	7779	7797	7815	7832	7850	7868	7885	7903	2	4	5	7	9	11	12	14	16
89	7921	7939	7957	7974	7992	8010	8028	8046	8064	8082	2	4	5	7	9	11	13	14	16
90	8100	8118	8136	8154	8172	8190	8208	8226	8245	8263	2	4	5	7	9	11	13	14	16
91	8281	8299	8317	8336	8354	8372	8391	8409	8427	8446	2	4	5	7	9	11	13	15	16
92	8464	8482	8501	8519	8538	8556	8575	8593	8612	8630	2	4	6	7	9	11	13	15	17
93	[illegible]	[illegible]	[illegible]	8686	8705	8724	8742	8761	8780	8798	8817	2	4	6	7	9	11	13	15	17
94	[illegible]	[illegible]	[illegible]	8874	8892	8911	8930	8949	8968	8987	9006	2	4	6	8	9	11	13	15	17
95	9025	9044	9063	9082	9101	9120	9139	9158	9178	9197	2	4	6	8	10	11	13	15	17
96	[illegible]	[illegible]	[illegible]	9254	9274	9293	9312	9332	9351	9370	9390	2	4	6	8	10	12	14	15	17
97	[illegible]	[illegible]	[illegible]	9448	9467	9487	9506	9526	9545	9565	9584	2	4	6	8	10	12	14	16	18
98	9604	9624	9643	9663	9683	9702	9722	9742	9761	9781	2	4	6	8	10	12	14	16	18
99	9801	9821	9841	9860	9880	9900	9920	9940	9960	9980	2	4	6	8	10	12	14	16	18

RECIPROCALS

	0	1	2	3	4	5	6	7	8	9	Subtract Differences								
											1	2	3	4	5	6	7	8	9
1·0	1·0000	9901	9804	9709	9615	9524	9434	9346	9259	9174	9	18	27	36	45	55	64	73	82
1·1	·9091	9009	8929	8850	8772	8696	8621	8547	8475	8403	8	15	23	30	38	45	53	61	68
1·2	·8333	8264	8197	8130	8065	8000	7937	7874	7813	7752	6	13	19	26	32	38	45	51	58
1·3	·7692	7634	7576	7519	7463	7407	7353	7299	7246	7194	5	11	16	22	27	33	38	44	49
1·4	·7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	5	10	14	19	24	29	33	38	43
1·5	·6667	6623	6579	6536	6494	6452	6410	6369	6329	6289	4	8	13	17	21	25	29	33	38
1·6	·6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	4	7	11	15	18	22	26	29	33
1·7	·5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	6	10	13	16	20	23	26	29
1·8	·5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	3	6	9	12	15	18	20	23	26
1·9	·5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	3	5	8	11	13	16	18	21	24
2·0	·5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	2	5	7	10	12	14	17	19	21
2·1	·4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	2	4	7	9	11	13	15	17	20
2·2	·4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	2	4	6	8	10	12	14	16	18
2·3	·4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	2	4	5	7	9	11	13	14	16
2·4	·4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	2	3	5	7	8	10	12	13	15
2·5	·4000	3984	3968	3953	3937	3922	3906	3891	3876	3861	2	3	5	6	8	9	11	12	14
2·6	·3846	3831	3817	3802	3788	3774	3759	3745	3731	3717	1	3	4	6	7	9	10	11	13
2·7	·3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	1	3	4	5	7	8	9	11	12
2·8	·3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	1	2	4	5	6	7	9	10	11
2·9	·3448	3436	3425	3413	3401	3390	3378	3367	3356	3344	1	2	3	5	6	7	8	9	10
3·0	·3333	3322	3311	3300	3289	3279	3268	3257	3247	3236	1	2	3	4	5	6	7	9	10
3·1	·3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	1	2	3	4	5	6	7	8	9
3·2	·3125	3115	3106	3096	3086	3077	3067	3058	3049	3040	1	2	3	4	5	6	7	8	9
3·3	·3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	1	2	3	4	4	5	6	7	8
3·4	·2941	2933	2924	2915	2907	2899	2890	2882	2874	2865	1	2	3	3	4	5	6	7	8
3·5	·2857	2849	2841	2833	2825	2817	2809	2801	2793	2786	1	2	2	3	4	5	6	6	7
3·6	·2778	2770	2762	2755	2747	2740	2732	2725	2717	2710	1	2	2	3	4	5	5	6	7
3·7	·2703	2695	2688	2681	2674	2667	2660	2653	2646	2639	1	1	2	3	4	4	5	6	6
3·8	·2632	2625	2618	2611	2604	2597	2591	2584	2577	2571	1	1	2	3	3	4	5	5	6
3·9	·2564	2558	2551	2545	2538	2532	2525	2519	2513	2506	1	1	2	3	3	4	4	5	6
4·0	·2500	2494	2488	2481	2475	2469	2463	2457	2451	2445	1	1	2	2	3	4	4	5	5
4·1	·2439	2433	2427	2421	2415	2410	2404	2398	2392	2387	1	1	2	2	3	3	4	5	5
4·2	·2381	2375	2370	2364	2358	2353	2347	2342	2336	2331	1	1	2	2	3	3	4	4	5
4·3	·2326	2320	2315	2309	2304	2299	2294	2288	2283	2278	1	1	2	2	3	3	4	4	5
4·4	·2273	2268	2262	2257	2252	2247	2242	2237	2232	2227	1	1	2	2	3	3	4	4	5
4·5	·2222	2217	2212	2208	2203	2198	2193	2188	2183	2179	0	1	1	2	2	3	3	4	4
4·6	·2174	2169	2165	2160	2155	2151	2146	2141	2137	2132	0	1	1	2	2	3	3	4	4
4·7	·2128	2123	2119	2114	2110	2105	2101	2096	2092	2088	0	1	1	2	2	3	3	4	4
4·8	·2083	2079	2075	2070	2066	2062	2058	2053	2049	2045	0	1	1	2	2	3	3	3	4
4·9	·2041	2037	2033	2028	2024	2020	2016	2012	2008	2004	0	1	1	2	2	2	3	3	4
5·0	·2000	1996	1992	1988	1984	1980	1976	1972	1969	1965	0	1	1	2	2	2	3	3	4
5·1	·1961	1957	1953	1949	1946	1942	1938	1934	1931	1927	0	1	1	2	2	2	3	3	3
5·2	·1923	1919	1916	1912	1908	1905	1901	1898	1894	1890	0	1	1	1	2	2	3	3	3
5·3	·1887	1883	1880	1876	1873	1869	1866	1862	1859	1855	0	1	1	1	2	2	2	3	3
5·4	·1852	1848	1845	1842	1838	1835	1832	1828	1825	1821	0	1	1	1	2	2	2	3	3

RECIPROCALS

	0	1	2	3	4	5	6	7	8	9	Subtract Differences 1	2	3	4	5	6	7	8	9
5·5	·1818	1815	1812	1808	1805	1802	1799	1795	1792	1789	0	1	1	1	2	2	2	3	3
5·6	·1786	1783	1779	1776	1773	1770	1767	1764	1761	1757	0	1	1	1	2	2	2	3	3
5·7	·1754	1751	1748	1745	1742	1739	1736	1733	1730	1727	0	1	1	1	2	2	2	2	3
5·8	·1724	1721	1718	1715	1712	1709	1706	1704	1701	1698	0	1	1	1	1	2	2	2	3
5·9	·1695	1692	1689	1686	1684	1681	1678	1675	1672	1669	0	1	1	1	1	2	2	2	3
6·0	·1667	1664	1661	1658	1656	1653	1650	1647	1645	1642	0	1	1	1	1	2	2	2	2
6·1	1639	1637	1634	1631	1629	1626	1623	1621	1618	1616	0	1	1	1	1	2	2	2	2
6·2	·1613	1610	1608	1605	1603	1600	1597	1595	1592	1590	0	1	1	1	1	1	2	2	2
6·3	·1587	1585	1582	1580	1577	1575	1572	1570	1567	1565	0	0	1	1	1	1	2	2	2
6·4	·1563	1560	1558	1555	1553	1550	1548	1546	1543	1541	0	0	1	1	1	1	2	2	2
6·5	1538	1536	1534	1531	1529	1527	1524	1522	1520	1517	0	0	1	1	1	1	2	2	2
6·6	·1515	1513	1511	1508	1506	1504	1502	1499	1497	1495	0	0	1	1	1	1	2	2	2
6·7	·1493	1490	1488	1486	1484	1481	1479	1477	1475	1473	0	0	1	1	1	1	2	2	2
6·8	1471	1468	1466	1464	1462	1460	1458	1456	1453	1451	0	0	1	1	1	1	2	2	2
6·9	1449	1447	1445	1443	1441	1439	1437	1435	1433	1431	0	0	1	1	1	1	1	2	2
7·0	·1429	1427	1425	1422	1420	1418	1416	1414	1412	1410	0	0	1	1	1	1	1	2	2
7·1	1408	1406	1404	1403	1401	1399	1397	1395	1393	1391	0	0	1	1	1	1	1	2	2
7·2	1389	1387	1385	1383	1381	1379	1377	1376	1374	1372	0	0	1	1	1	1	1	2	2
7·3	1370	1368	1366	1364	1362	1361	1359	1357	1355	1353	0	0	1	1	1	1	1	2	2
7·4	1351	1350	1348	1346	1344	1342	1340	1339	1337	1335	0	0	1	1	1	1	1	2	2
7·5	1333	1332	1330	1328	1326	1325	1323	1321	1319	1318	0	0	1	1	1	1	1	1	2
7·6	·1316	1314	1312	1311	1309	1307	1305	1304	1302	1300	0	0	1	1	1	1	1	1	2
7·7	·1299	1297	1295	1293	1292	1290	1288	1287	1285	1284	0	0	0	1	1	1	1	1	1
7·8	·1282	1280	1279	1277	1276	1274	1272	1271	1269	1267	0	0	0	1	1	1	1	1	1
7·9	·1266	1264	1263	1261	1259	1258	1256	1255	1253	1252	0	0	0	1	1	1	1	1	1
8·0	·1250	1248	1247	1245	1244	1242	1241	1239	1238	1236	0	0	0	1	1	1	1	1	1
8·1	·1235	1233	1232	1230	1229	1227	1225	1224	1222	1221	0	0	0	1	1	1	1	1	1
8·2	·1220	1218	1217	1215	1214	1212	1211	1209	1208	1206	0	0	0	1	1	1	1	1	1
8·3	·1205	1203	1202	1200	1199	1198	1196	1195	1193	1192	0	0	0	1	1	1	1	1	1
8·4	·1190	1189	1188	1186	1185	1183	1182	1181	1179	1178	0	0	0	1	1	1	1	1	1
8·5	·1176	1175	1174	1172	1171	1170	1168	1167	1166	1164	0	0	0	1	1	1	1	1	1
8·6	·1163	1161	1160	1159	1157	1156	1155	1153	1152	1151	0	0	0	1	1	1	1	1	1
8·7	·1149	1148	1147	1145	1144	1143	1142	1140	1139	1138	0	0	0	1	1	1	1	1	1
8·8	·1136	1135	1134	1133	1131	1130	1129	1127	1126	1125	0	0	0	1	1	1	1	1	1
8·9	·1124	1122	1121	1120	1119	1117	1116	1115	1114	1112	0	0	0	1	1	1	1	1	1
9·0	·1111	1110	1109	1107	1106	1105	1104	1103	1101	1100	0	0	0	1	1	1	1	1	1
9·1	·1099	1098	1096	1095	1094	1093	1092	1091	1089	1088	0	0	0	1	1	1	1	1	1
9·2	·1087	1086	1085	1083	1082	1081	1080	1079	1078	1076	0	0	0	0	1	1	1	1	1
9·3	·1075	1074	1073	1072	1071	1070	1068	1067	1066	1065	0	0	0	0	1	1	1	1	1
9·4	·1064	1063	1062	1060	1059	1058	1057	1056	1055	1054	0	0	0	0	1	1	1	1	1
9·5	·1053	1052	1050	1049	1048	1047	1046	1045	1044	1043	0	0	0	0	1	1	1	1	1
9·6	·1042	1041	1040	1038	1037	1036	1035	1034	1033	1032	0	0	0	0	1	1	1	1	1
9·7	·1031	1030	1029	1028	1027	1026	1025	1024	1022	1021	0	0	0	0	1	1	1	1	1
9·8	·1020	1019	1018	1017	1016	1015	1014	1013	1012	1011	0	0	0	0	1	1	1	1	1
9·9	·1010	1009	1008	1007	1006	1005	1004	1003	1002	1001	0	0	0	0	0	1	1	1	1

DEGREES TO RADIANS

Differences—	1′	2′	3′	4′	5′
	3	6	9	12	15

	Radians	6′	12′	18′	24′	30′	36′	42′	48′	54′
°										
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332
2	·0349	0367	0384	0401	0419	0436	0454	0471	0489	0506
3	·0524	0541	0559	0576	0593	0611	0628	0646	0663	0681
4	·0698	0716	0733	0750	0768	0785	0803	0820	0838	0855
5	·0873	0890	0908	0925	0942	0960	0977	0995	1012	1030
6	·1047	1065	1082	1100	1117	1134	1152	1169	1187	1204
7	·1222	1239	1257	1274	1292	1309	1326	1344	1361	1379
8	·1396	1414	1431	1449	1466	1484	1501	1518	1536	1553
9	·1571	1588	1606	1623	1641	1658	1676	1693	1710	1728
10	·1745	1763	1780	1798	1815	1833	1850	1868	1885	1902
11	·1920	1937	1955	1972	1990	2007	2025	2042	2059	2077
12	·2094	2112	2129	2147	2164	2182	2199	2217	2234	2251
13	·2269	2286	2304	2321	2339	2356	2374	2391	2409	2426
14	·2443	2461	2478	2496	2513	2531	2548	2566	2583	2601
15	·2618	2635	2653	2670	2688	2705	2723	2740	2758	2775
16	·2793	2810	2827	2845	2862	2880	2897	2915	2932	2950
17	·2967	2985	3002	3019	3037	3054	3072	3089	3107	3124
18	·3142	3159	3176	3194	3211	3229	3246	3264	3281	3299
19	·3316	3334	3351	3368	3386	3403	3421	3438	3456	3473
20	·3491	3508	3526	3543	3560	3578	3595	3613	3630	3648
21	·3665	3683	3700	3718	3735	3752	3770	3787	3805	3822
22	·3840	3857	3875	3892	3910	3927	3944	3962	3979	3997
23	·4014	4032	4049	4067	4084	4102	4119	4136	4154	4171
24	·4189	4206	4224	4241	4259	4276	4294	4311	4328	4346
25	·4363	4381	4398	4416	4433	4451	4468	4485	4503	4520
26	·4538	4555	4573	4590	4608	4625	4643	4660	4677	4695
27	·4712	4730	4747	4765	4782	4800	4817	4835	4852	4869
28	·4887	4904	4921	4939	4957	4974	4992	5009	5027	5044
29	·5061	5079	5096	5114	5131	5149	5166	5184	5201	5219
30	·5236	5253	5271	5288	5306	5323	5341	5358	5376	5393
31	·5411	5428	5445	5463	5480	5498	5515	5533	5550	5568
32	·5585	5603	5620	5637	5655	5672	5690	5707	5725	5742
33	·5760	5777	5794	5812	5829	5847	5864	5882	5899	5917
24	·5934	5952	5969	5986	6004	6021	6039	6056	6074	6091
35	·6109	6126	6144	6161	6178	6196	6213	6231	6248	6266
36	·6283	6301	6318	6336	6353	6370	6388	6405	6423	6440
37	·6458	6475	6493	6510	6528	6545	6562	6580	6597	6615
38	·6632	6650	6667	6685	6702	6720	6737	6754	6772	6789
39	·6807	6824	6842	6859	6877	6894	6912	6929	6946	6964
40	·6981	6999	7016	7034	7051	7069	7086	7103	7121	7138
41	·7156	7173	7191	7208	7226	7243	7261	7278	7295	7313
42	·7330	7348	7365	7383	7400	7418	7435	7453	7470	7487
43	·7505	7522	7540	7557	7575	7592	7610	7627	7645	7662
44	·7679	7697	7714	7732	7749	7767	7784	7802	7819	7837

DEGREES TO RADIANS

Differences—	1′	2′	3′	4′	5′
	3	6	9	12	15

°	Radians	6′	12′	18′	24′	30′	36′	42′	48′	54′
45	·7854	7871	7889	7906	7924	7941	7959	7976	7994	8011
46	·8029	8046	8063	8081	8098	8116	8133	8151	8168	8186
47	·8203	8221	8238	8255	8273	8290	8308	8325	8343	8360
48	·8378	8395	8412	8430	8447	8465	8482	8500	8517	8535
49	·8552	8570	8587	8604	8622	8639	8657	3674	8692	8709
50	·8727	8744	8762	8779	8796	8814	8831	8849	8866	8884
51	·8901	8919	8936	8954	8971	8988	9006	9023	9041	9058
52	·9076	9093	9111	9128	9146	9163	9180	9198	9215	9233
53	·9250	9268	9285	9303	9320	9338	9355	9372	9390	9407
54	·9425	9442	9460	9477	9495	9512	9529	9547	9564	9582
55	·9599	9617	9634	9652	9669	9687	9704	9721	9739	9756
56	·9774	9791	9809	9826	9844	9861	9879	9896	9913	9931
57	·9948	9966	9983	0001	0018	0036	0053	0071	0088	0105
58	1·0123	0140	0158	0175	0193	0210	0228	0245	0263	0280
59	1·0297	0315	0332	0350	0367	0385	0402	0420	0437	0455
60	1·0472	0489	0507	0524	0542	0559	0577	0594	0612	0629
61	1·0647	0664	0681	0699	0716	0734	0751	0769	0786	0804
62	1·0821	0838	0856	0873	0891	0908	0926	0943	0961	0978
63	1·0996	1013	1030	1048	1065	1083	1100	1118	1135	1153
64	1·1170	1188	1205	1222	1240	1257	1275	1292	1310	1327
65	1·1345	1362	1380	1397	1414	1432	1449	1467	1484	1502
66	1·1519	1537	1554	1572	1589	1606	1624	1641	1659	1676
67	1·1694	1711	1729	1746	1764	1781	1798	1816	1833	1851
68	1·1868	1886	1903	1921	1938	1956	1973	1990	2008	2025
69	1·2043	2060	2078	2095	2113	2130	2147	2165	2182	2200
70	1·2217	2235	2252	2270	2287	2305	2322	2339	2357	2374
71	1·2392	2409	2427	2444	2462	2479	2497	2514	2531	2549
72	1·2566	2584	2601	2619	2636	2654	2671	2689	2706	2723
73	1·2741	2758	2776	2793	2811	2828	2846	2863	2881	2898
74	1·2915	2933	2950	2968	2985	3003	3020	3038	3055	3073
75	1·3090	3107	3125	3142	3160	3177	3195	3212	3230	3247
76	1·3265	3282	3299	3317	3334	3352	3369	3387	3404	3422
77	1·3439	3456	3474	3491	3509	3526	3544	3561	3579	3596
78	1·3614	3631	3648	3666	3683	3701	3718	3736	3753	3771
79	1·3788	3806	3823	3840	3858	3875	3893	3910	3928	3945
80	1·3963	3980	3998	4015	4032	4050	4067	4085	4102	4120
81	1·4137	4155	4172	4190	4207	4224	4242	4259	4277	4294
82	1·4312	4329	4347	4364	4382	4399	4416	4434	4451	4469
83	1·4486	4504	4521	4539	4556	4573	4591	4608	4626	4643
84	1·4661	4678	4696	4713	4731	4748	4765	4783	4800	4818
85	1·4835	4853	4870	4888	4905	4923	4940	4957	4975	4992
86	1·5010	5027	5045	5062	5080	5097	5115	5132	5149	5167
87	1·5184	5202	5219	5237	5254	5272	5289	5307	5324	5341
88	1·5359	5376	5394	5411	5429	5446	5464	5481	5499	5516
89	1·5533	5551	5568	5586	5603	5621	5638	5656	5673	5691