THE ELEMENTS OF

SOLID GEOMETRY

CAMBRIDGE UNIVERSITY PRESS
London: FETTER LANE, E.C.
C. F. CLAY, Manager

Edinburgh: 100, PRINCES STREET
Berlin: A. ASHER AND CO.
Leipzig: F. A. BROCKHAUS
New York: G. P. PUTNAM'S SONS
Bombay and Calcutta: MACMILLAN AND CO., Ltd.

THE ELEMENTS OF SOLID GEOMETRY

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CAMBRIDGE:
AT THE UNIVERSITY PRESS
1909

First Edition 1905.
Reprinted 1909.

PREFACE.

IN English secondary schools, the study of Solid Geometry receives much less attention than in the corresponding institutions of Germany and the United States. As a rule, the course adopted covers little more than the first twenty-one propositions of Euclid's eleventh book. In consequence, however, of the recent changes in the teaching of plane geometry, it is probable that more time will in future be given to the subject; and the present volume, which embodies the notes that have been used in this school for the last twenty years, contains, it is hoped, the theorems of chief interest and importance.

I have to thank my colleague, Mr C. H. Richards, for his kindness in reading the manuscript and for useful criticisms. I should be grateful for notices of any errors that may be found in the text or exercises, and for suggestions for the improvement of the book, in case a second edition should be required.

CHARLES DAVISON.

KING EDWARD'S HIGH SCHOOL,
BIRMINGHAM,
September 1905.

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CHAPTER I.

LINES AND PLANES IN SPACE.

Planes and their Intersections, etc.

1. Def. 1. A **plane surface** or **plane** is a surface such that, if any two points be taken in it, the straight line joining them, or this straight line produced to any distance in either direction, lies entirely in that surface.

Def. 2. A plane is **determined** by certain conditions when no other plane can be made to satisfy those conditions.

Def. 3. A **straight line** is **perpendicular** to a **plane** when it is perpendicular to every line which can be drawn to meet it in that plane.

Def. 4. A **straight line** is **parallel** to a **plane** when, if it be produced to any distance in either direction, it does not meet the plane produced in every direction.

Def. 5. A **straight line** is **oblique** to a **plane** when it is neither parallel nor perpendicular to the plane.

Def. 6. Two **planes** are **parallel** to one another when, if they be produced to any distance in any direction, they do not meet.

EXERCISES.

1. AB and AC are two equal straight lines in a plane, and AD is perpendicular to the plane; if D be any point in AD, then DB is equal to DC.

2. If, from a point without a plane, equal straight lines be drawn to a plane, the extremities of these lines lie upon a circle.

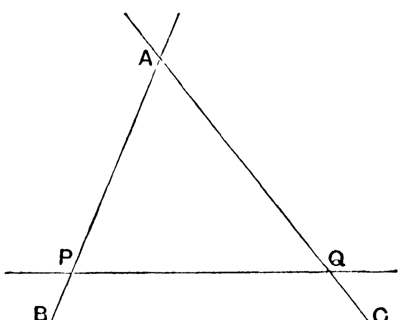
3. **If, from a point O without a plane, three equal straight lines OA, OB, OC be drawn to the plane, and also OD perpendicular to the plane, then D is the circumcentre of the triangle ABC.**

4. **Find the locus of a point in space equidistant from three given points.**

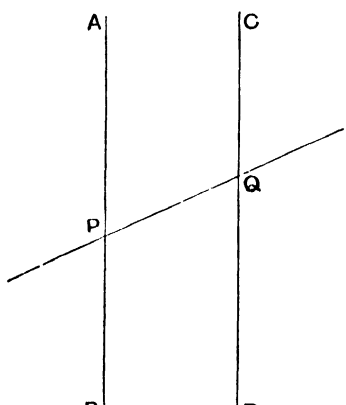
5. **Of two oblique lines drawn from a point to a plane and meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.**

6. A piece of wire is bent into three portions, each of the outer portions being twice the middle portion and at right angles to the plane containing the other two; prove that the distance between the extremities of the wire is three times the middle portion.

2. PROP. 1. **A plane is determined: (i) by two intersecting straight lines, (ii) by a straight line and a point without it, (iii) by three points not in the same straight line, and (iv) by two parallel straight lines.**



(i)—(iii)



(iv)

(i) Let AB, AC be two intersecting straight lines; to prove that one and only one plane can be drawn through AB, AC.

Let a plane pass through AB, and be turned about AB until it pass through C. In this position, let it be called the plane X.

Since the points A, C lie in the plane X,

∴ the straight line AC lies in the plane X.

∴ the plane X contains both the lines AB, AC.

Again, if P, Q be *any* two points in AB, AC, respectively, the straight line PQ, produced to any distance in either direction, lies in the plane X;

∴ the plane X is the only plane that contains the lines AB, AC;

i.e. two intersecting straight lines determine a plane.

(ii), (iii) The proofs are similar to the above.

(iv) Let AB, CD be two parallel straight lines; to prove that one, and only one, plane can be drawn through AB and CD.

Let a plane pass through AB, and be turned about AB until it pass through any point C in CD. In this position, let it be called the plane X.

If a straight line be drawn through C in the plane X parallel to AB, this line must coincide with CD; otherwise two straight lines would be drawn through C parallel to AB, which is impossible;

∴ the plane X contains both the lines AB and CD.

Again, if P, Q be *any* two points in AB, CD, respectively, the straight line PQ, produced to any distance in either direction, lies in the plane X;

∴ the plane X is the only plane that contains the lines AB, CD;

i.e. two parallel straight lines determine a plane.

EXERCISES.

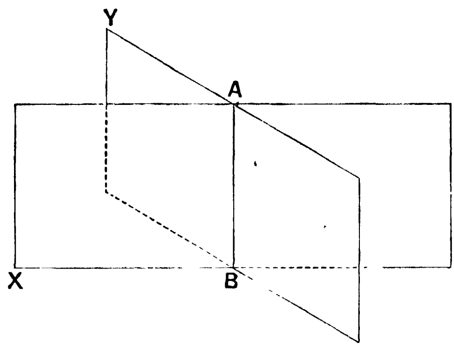
1. The sides of a parallelogram lie in one plane.

2. If three straight lines meet in one point, and all intersect a fourth straight line, which does not pass through that point, the three lines lie in one plane.

3. No two straight lines joining points in two lines which are not in the same plane, can be parallel.

4. If any two lines, drawn to intersect each of two given lines in space, meet one another, then the two given lines must either meet one another or be parallel.

√ 3. Prop. 2. **The common section of two planes is a straight line.**



Let X, Y be two intersecting planes; to prove that their common section is a straight line.

Let A, B be any two points on their common section,

∴ the straight line AB lies in both planes.

Again, since a straight line and a point determine a plane,

∴ no point outside AB can be common to both planes,

∴ the common section of the two planes is the straight line AB.

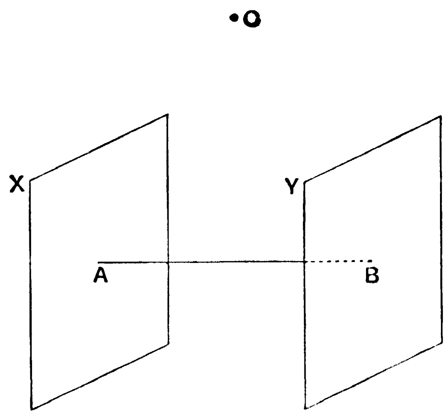
EXERCISES.

1. A number of unlighted candles stand upon a table, and a lighted candle shorter than any of the former, stands on the same table; prove that the shadows formed on the ceiling by the unlighted candles, if produced, all meet in one point.

2. Two planes, X, Y, intersect in the line AB; PQ is any line in the plane X not parallel to AB; prove that any plane through PQ intersects the plane Y in a straight line which meets AB in the same point as PQ.

3. **If three planes intersect one another, the lines of intersection either meet in a point or are parallel.**

4. PROP. 3. **Planes to which the same straight line is perpendicular are parallel.**



Let AB be perpendicular to each of the planes X, Y; to prove that the planes X and Y are parallel.

Let AB meet the planes X, Y, in A, B, respectively.

If the planes X, Y meet, let C be a point in their common section;

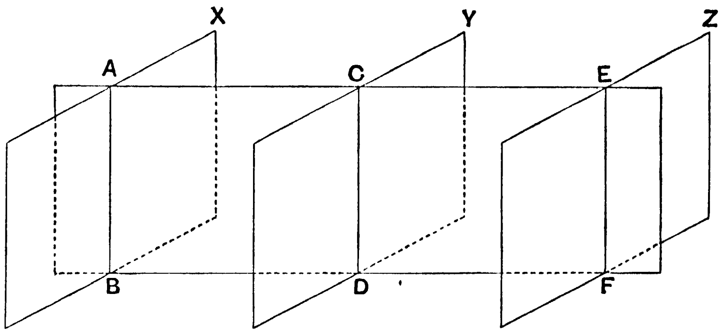
then AC lies in the plane X and BC in the plane Y, and therefore each of the angles BAC, ABC is a right angle, which is impossible;

∴ the planes X and Y are parallel.

COR. Through any point outside a given line, only one plane can be drawn perpendicular to that line.

For, if two planes could be drawn, they would be parallel to one another.

5. PROP. 4. **The common section of a series of two or more parallel planes with another plane is a series of two or more parallel straight lines.**



Let X, Y, Z, ... be a series of parallel planes, and let their common sections with the plane AF be the straight lines AB, CD, EF, ... ; to prove that the lines AB, CD, EF, ... are parallel.

For, if any two of the lines, say AB and CD, were to meet when produced in a point P, then the point P would be common to the two parallel planes X and Y, which is impossible;

∴ the common sections AB, CD, EF, ... are a series of parallel straight lines.

EXERCISES.

1. Parallel straight lines intercepted between parallel planes are equal.

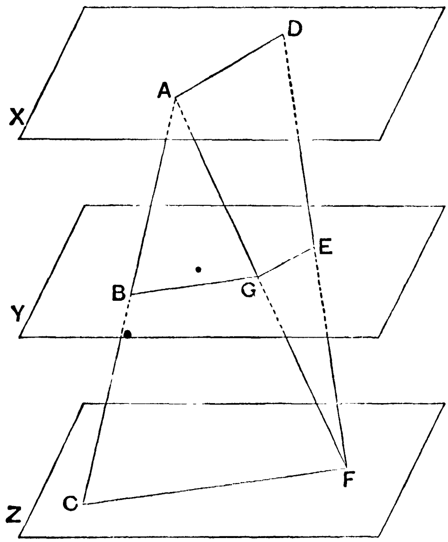
2. **Through the same point, there cannot pass two planes both parallel to the same plane.**

3. Three parallel straight lines are cut by parallel planes, and the points of intersection are joined; show that the triangles so formed are congruent.

4. **If a straight line be perpendicular to one of two parallel planes, it is also perpendicular to the other.**

5. **If two planes be parallel to a third plane, they are parallel to one another.**

6. Prop. 5. **If straight lines, not necessarily in the same plane, are cut by a series of parallel planes, their corresponding segments are proportional.**



Let AC, DF be cut by the parallel planes X, Y, Z in the points A, B, C and D, E, F; to prove that AB : BC = DE : EF.

Join AF, cutting the plane Y in G; join BG, CF, AD, GE.

Since the parallel planes Y, Z are cut by the plane ACF,

∴ the common sections BG, CF are parallel;

∴ AB : BC = AG : GF.

Since the parallel planes X, Y are cut by the plane FAD,

∴ the common sections AD, GE are parallel;

∴ AG : GF = DE : EF;

∴ AB : BC = DE : EF.

EXERCISES.

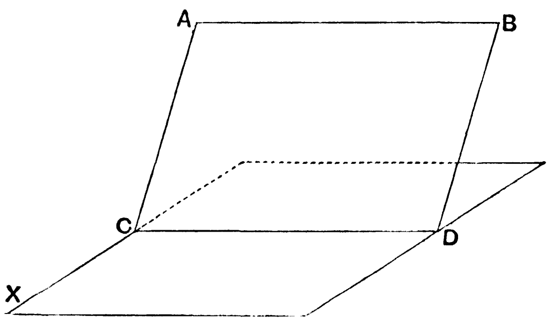
1. If the intercepts made on any straight line by three or more parallel planes be equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

2. If two straight lines, not in one plane, be cut by four parallel planes, the two segments intercepted by the first and second planes have the same ratio to one another as the two segments intercepted by the third and fourth planes.

3. If three straight lines, not in the same plane, connect two parallel planes, their mid-points lie on a plane parallel to the given planes.

4. O is a fixed point outside a fixed plane X; P is any point in X, and Q divides OP in a given ratio; find the locus of Q.

7. Prop. 6. **If a plane be drawn through one only of two parallel straight lines, it is parallel to the other line.**



Let AB, CD be two parallel straight lines, and X a plane passing through CD, but not through AB; to prove that the plane X is parallel to AB.

The common section of the plane AD and X is the straight line CD;

∴ if AB meet the plane X, it must meet it in a point of CD, which is impossible, since AB and CD are parallel.

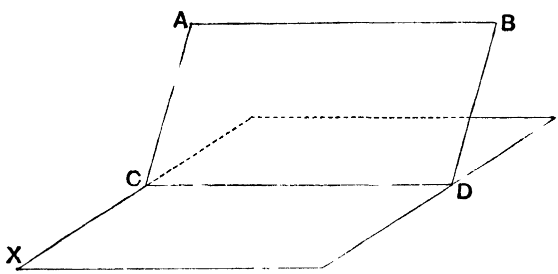
EXERCISES.

1. If three straight lines, not all in the same plane, be parallel, then any one of the lines is parallel to the plane containing the other two.

2. A straight line and a plane perpendicular to the same straight line are parallel.

3. Through a given point, a plane can be drawn parallel to each of two lines which are not in the same plane.

8. Prop. 7. **If a straight line be parallel to a plane, the common section of the plane with any plane passing through the given line is parallel to that line.**



Let AB be parallel to the plane X, and AD any plane through AB; to prove that CD, the common section of the two planes, is parallel to AB.

Every point in CD, or CD produced, lies in the plane X; hence, if AB meet CD, it must also meet the plane X, which is impossible;

$\therefore$ AB is parallel to CD.

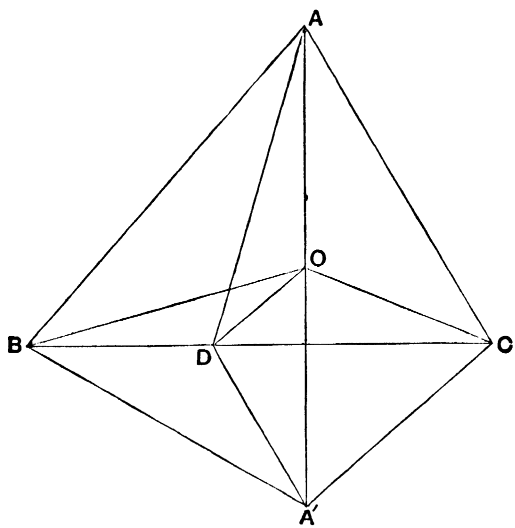
EXERCISES.

1. If a straight line AB be parallel to a plane, and parallel lines AC, BD be drawn to meet the plane in C, D, then AC is equal to BD.

2. If a straight line be parallel to a plane, a line drawn through any point in the plane parallel to the given line lies in the plane.

STRAIGHT LINES PERPENDICULAR TO PLANES.

√ 9. PROP. 8. **If a straight line be perpendicular to each of two other straight lines at their point of intersection, it is perpendicular to the plane containing them*.**



Let OA be perpendicular to each of the lines OB, OC at their point of intersection O; to prove that OA is perpendicular to the plane BOC.

In OB, OC, or OB, OC produced, take any points B, C, respectively; join BC, and in BC take any point D; then OD is *any* line in the plane BOC. Produce AO to A′, making OA′ equal to OA, and join B, D, C to A and A′.

* In order that a straight line may be perpendicular to a plane, it is *necessary* (def. 3) that it should be perpendicular to *every* line meeting it in that plane. The object of this proposition is to show that it is *sufficient* to prove it perpendicular to any *two* lines meeting it in that plane.

In the triangles AOB, A′OB,

$$AO = A'O, \text{ OB is common, } \angle AOB = \angle A'OB,$$

$$\therefore AB = A'B.$$

Similarly, $$AC = A'C.$$

In the triangles ABC, A′BC,

$$AB = A'B, \ AC = A'C, \text{ and BC is common,}$$

$$\therefore \angle ABC = \angle A'BC.$$

In the triangles ABD, A′BD,

$$AB = A'B, \text{ BD is common, and } \angle ABD = \angle A'BD,$$

$$\therefore AD = A'D.$$

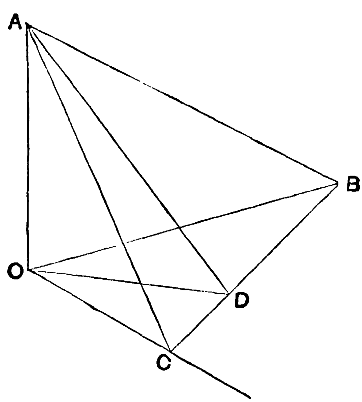
Lastly, in the triangles AOD, A′OD,

$$AO = A'O, \text{ OD is common, } AD = A'D,$$

$$\therefore \angle AOD = \angle A'OD;$$

∴ AO is perpendicular to *any* line OD in the plane BOC, and it is therefore perpendicular to the plane BOC.

Second Proof. Through B draw BC to *any* point C in OC or



CO produced, and draw OD perpendicular to BC; then, by varying the position of C, OD or DO produced may be given *any* position in the plane of OB, OC.

Since AOB, AOC, ODB are right angles,

$$\therefore AB^2 - AC^2 = OB^2 - OC^2 = DB^2 - DC^2;$$

$$\therefore AD \text{ is perpendicular to } BC.$$

Now,
$$AD^2 = AB^2 - DB^2$$
$$= OA^2 + OB^2 - OB^2 + OD^2$$
$$= OA^2 + OD^2;$$

$\therefore$ AO is perpendicular to OD, and is therefore perpendicular to the plane BOC.

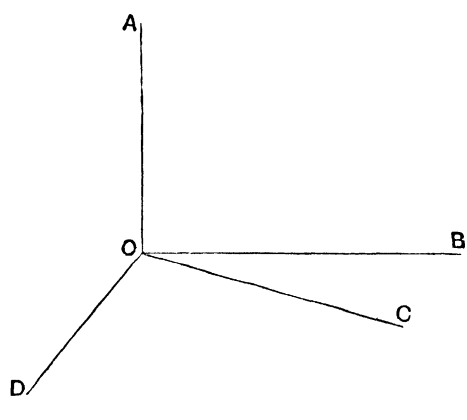
EXERCISES.

1. If a cube be defined as a solid figure bounded by six squares, show that each edge is perpendicular to two faces of the cube.

2. AB is a straight line; from any point A in it, AC and AD are drawn perpendicular to AB; and from any point B in it, BE and BF are drawn perpendicular to AB: prove that the plane CAD is parallel to the plane EBF.

3. Draw a plane through a given point perpendicular to a given straight line.

√**10.** Prop. 9. **All the perpendiculars to a straight line at a given point in it lie in a plane perpendicular to the line.**



Let OB, OC, OD, ... be all perpendicular to OA at the point O; to prove that OB, OC, OD, ... lie in a plane perpendicular to OA.

The plane which contains two of the lines, OB, OC, is perpendicular to OA, and therefore every line in this plane which meets OA is perpendicular to OA;

∴ the common section of the planes BOC, AOD is perpendicular to OA.

But OD is the only line in the plane AOD which is perpendicular to OA;

∴ OD lies in the plane BOC;

∴ all the lines OB, OC, OD, ... lie in a plane perpendicular to OA.

COR. 1. If the right angle AOB rotate about the arm OA, then OB traces out the plane through O perpendicular to OA.

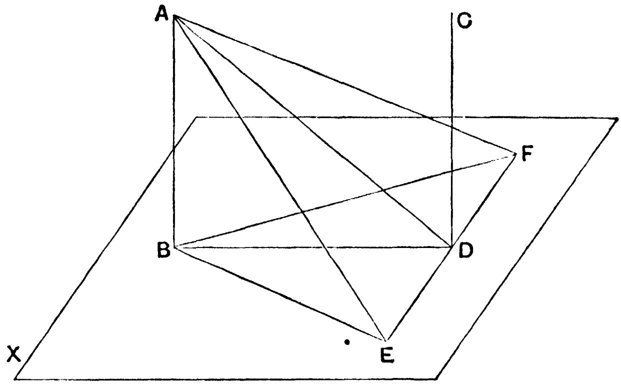
COR. 2. The locus of all points which are equidistant from two given points A, B, is the plane which bisects AB at right angles; for it is the plane traced out by the rotation of a line which bisects AB at right angles.

EXERCISES.

1. If two intersecting circles revolve about their line of centres, the points of intersection lie in a plane perpendicular to the line of centres.

2. Find the locus of a point, the difference between the squares of whose distances from two fixed points is constant.

11. PROP. 10. **If two lines be perpendicular to a plane, they are parallel to one another.**



Let AB, CD be both perpendicular to the plane X; to prove that AB is parallel to CD.

Join B and D, the feet of the perpendiculars; through D, *in the plane* X, draw a straight line EF perpendicular to BD, making DE equal to DF, and join BE, BF, AE, AD, AF.

In the triangles BDE, BDF,

BD is common, DE = DF, $\angle$ BDE = $\angle$ BDF,

$\therefore$ BE = BF.

In the triangles ABE, ABF,

AB is common, BE = BF, right $\angle$ ABE = right $\angle$ ABF,

$\therefore$ AE = AF.

In the triangles ADE, ADF,

AD is common, DE = DF, AE = AF,

$\therefore$ $\angle$ ADE = $\angle$ ADF,

$\therefore$ EF is perpendicular to AD, and also to BD and CD,

$\therefore$ CD lies in the same plane as AD and BD, i.e. in the same plane as AB and BD,

and since $\angle$s ABD, CDB are both right angles,

$\therefore$ AB is parallel to CD,

Cor. Only one perpendicular can be drawn from a given point to a given plane.

For straight lines which are perpendicular to the same plane are parallel to one another, and therefore cannot intersect in a point either inside or outside the plane.

EXERCISES.

1. Point out the mistake in the following reasoning: "Since AB and CD are both perpendicular to the plane X,

$\therefore$ $\angle$s ABD, CDB are right angles,

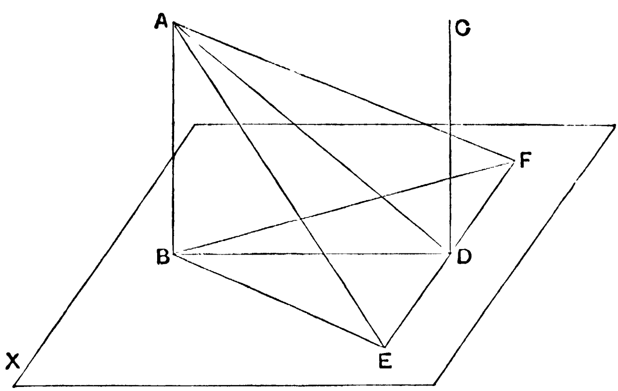
$\therefore$ $\angle$ ABD + $\angle$ CDB = two right angles,

$\therefore$ AB is parallel to CD."

2. Two parallel planes are equidistant from one another.

3. A and B are two fixed points in a plane X, and P a point in a plane Y such that the sum of the squares on AP, BP is constant; find the locus of the point P.

√12. Prop. 11. **If two straight lines be parallel, and one of them be perpendicular to a plane, the other is also perpendicular to the same plane.**



Let AB be parallel to CD, and AB be perpendicular to the plane X; to prove that CD is perpendicular to the plane X.

Let CD meet the plane X in D; through D, *in the plane* X, draw a straight line EDF perpendicular to BD, making DE equal to DF, and join BE, BF, AE, AD, AF.

In the triangles BDE, BDF,

$$\text{BD is common, DE} = \text{DF}, \angle \text{BDE} = \angle \text{BDF},$$

$$\therefore \text{BE} = \text{BF}.$$

In the triangles ABE, ABF,

$$\text{AB is common, BE} = \text{BF, right } \angle \text{ABE} = \text{right } \angle \text{ABF},$$

$$\therefore \text{AE} = \text{AF}.$$

In the triangles ADE, ADF,

$$\text{AD is common, DE} = \text{DF, AE} = \text{AF},$$

$$\therefore \angle \text{ADE} = \angle \text{ADF};$$

$\therefore$ ED is perpendicular to AD, and it is also perpendicular to BD;

$\therefore$ EF is perpendicular to the plane ABD.

But CD lies in the plane ABD, since CD is parallel to AB,

$\therefore$ ED is perpendicular to CD.

Again since AB is parallel to CD,

$$\angle \text{ABD} + \angle \text{CDB} = \text{two right angles.}$$

But $\angle$ ABD is a right angle,

$\therefore$ $\angle$ CDB is a right angle,

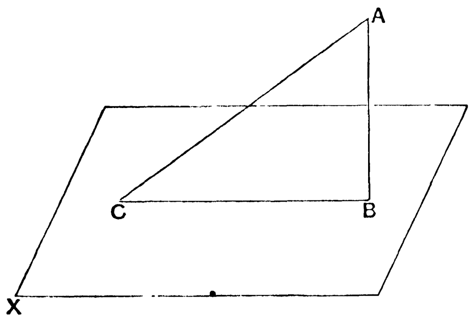
$\therefore$ CD is perpendicular to BD and EF, and therefore to the plane X.

EXERCISES.

1. AB and CD are two parallel straight lines; from a given point P, PM is drawn perpendicular to AB and MN is drawn perpendicular to CD; prove that PN is perpendicular to CD.

2. Two straight lines AC, BC intersect at right angles in C, and from A a straight line AD is drawn perpendicular to the plane in which they are; show that CD is perpendicular to BC.

√13. Prop. 12. **Of all straight lines that can be drawn from a given point to a plane, the perpendicular to the plane is the shortest.**

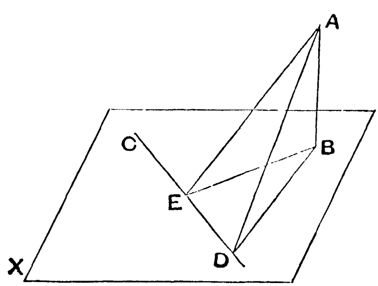


Let A be the given point, AB the perpendicular to the given plane X, and AC any other straight line drawn from the point to the plane; to prove that AB is less than AC.

Since AB is perpendicular to the plane X, and BC is a line in the plane,

∴ ∠ ABC is a right angle,
∴ ∠ ACB is less than ∠ ABC,
∴ AB is less than AC.

14. Prop. 13. **If two straight lines be drawn from a point outside a given plane, one perpendicular to the plane, and the other to a straight line in the plane, the line joining the feet of the perpendiculars is perpendicular to the line in the plane.**



Let AB be drawn from a point A outside the plane X, perpendicular to the plane, and AE perpendicular to the line CD in the plane; to prove that BE is perpendicular to CD.

Take any point D in CD, and join BD, AD.

Then, since ABD, AED, ABE are right angles,

$$\begin{aligned} BD^2 &= AD^2 - AB^2 \\ &= AE^2 + ED^2 - AB^2 \\ &= BE^2 + AB^2 + ED^2 - AB^2 \\ &= BE^2 + ED^2; \end{aligned}$$

$\therefore \angle$ BED is a right angle.

COR. If AB be drawn perpendicular to the plane, and BE perpendicular to CD, then it may be proved, in the same manner, that AE is perpendicular to CD.

EXERCISES.

1. Prove Prop. 13 by cutting off equal lengths EC, ED and joining CB, DB, CA, DA.

2. OA, OB, OC are three straight lines mutually at right angles; if AD be drawn perpendicular to BC, then OD is perpendicular to BC.

3. **Two planes, X, Y, intersect in the line AB; from any point O outside the planes, OM, ON are drawn perpendicular to them; prove that OM, ON lie in a plane perpendicular to AB.**

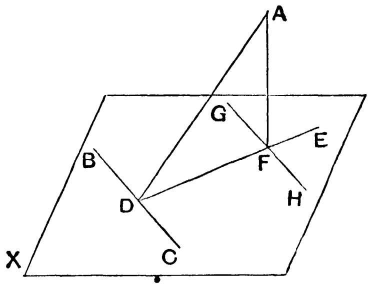
4. From a point P are drawn lines PA, PB perpendicular to two planes which intersect in the line XY; through A a straight line is drawn parallel to XY; show that this line is perpendicular to AB.

5. From a point E, EC, ED are drawn perpendicular to two planes CAB, DAB, which intersect in the line AB, and DF is drawn perpendicular to the plane CAB, meeting it in F; show that the line CF is perpendicular to AB.

6. Perpendiculars are drawn from a given point upon an infinite number of planes all passing through a given straight line; find the locus of the intersections of the perpendiculars with the planes.

7. Find the locus of points which are equidistant from two intersecting straight lines.

15. PROP. 14. **To draw a straight line perpendicular to a given plane from a given point, (i) outside the plane, (ii) in the plane.**



Let A be the given point; to draw a straight line from A perpendicular to the plane X.

(i) Let A be outside the plane X.

Draw any straight line BC in the plane; draw AD perpendicular to BC, DE in the plane X perpendicular to BC, and AF perpendicular to DE.

Through F draw GH parallel to BC, and therefore lying in the plane X.

Since BC is perpendicular to DA and DE,

∴ BC is perpendicular to the plane ADF.

But GH is parallel to BC,

∴ GH is perpendicular to the plane ADF and therefore to AF;

∴ AF is perpendicular to DE and GH, and therefore to the plane X.

(ii) Let A be in the given plane X.

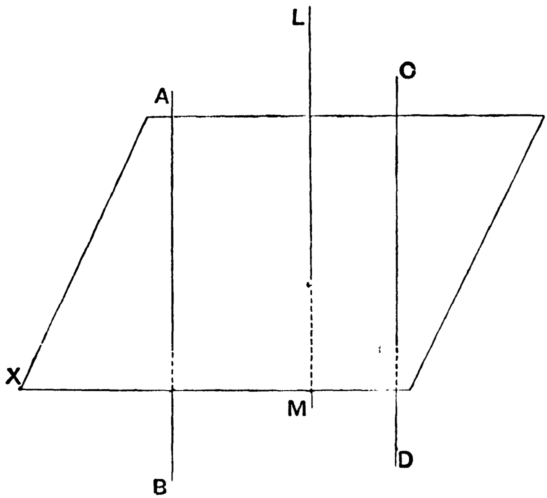
Take any point P outside the plane; draw PM perpendicular to the plane, and AL parallel to MP.

Since AL is parallel to MP,

∴ AL is perpendicular to the plane X.

Parallel Lines in Different Planes.

16. Prop. 15. **Straight lines which are parallel to the same straight line are parallel to one another.**



Let AB and CD be both parallel to LM; to prove that AB is parallel to CD.

Let X represent a plane perpendicular to LM.

Since AB and CD are both parallel to LM,

∴ AB and CD are both perpendicular to the plane X;

∴ AB is parallel to CD.

EXERCISES.

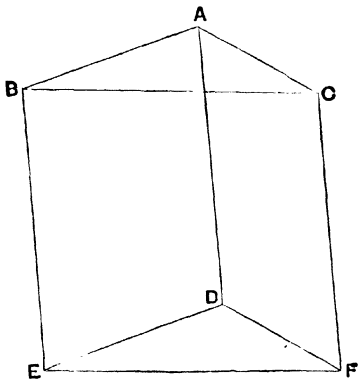
1. ABCD, ABFE are two parallelograms in different planes with a common side AB; prove that CDFE is a parallelogram.

2. If two straight lines be parallel to one another and if one of them be parallel to a plane, the other is also parallel to the plane.

3. If any two planes be cut by a third plane which is parallel to their line of intersection, their common sections with the third plane are parallel.

4. A line parallel to two intersecting planes is parallel to the intersection of these planes.

17. Prop. 16. **If two straight lines which meet one another be respectively parallel to two other straight lines which meet one another and lie in a different plane, the angle contained by the first pair of lines is equal to the angle contained by the second pair.**



Let AB, AC be respectively parallel to DE, DF, the two pairs of lines lying in different planes; to prove that $\angle$ BAC = $\angle$ EDF.

Make AB = DE, AC = DF, and join BC, EF, BE, AD, CF.

Since AB is equal and parallel to DE,

$\therefore$ BE is equal and parallel to AD.

Since AC is equal and parallel to DF,

$\therefore$ CF is equal and parallel to AD.

$\therefore$ BE is equal and parallel to CF,

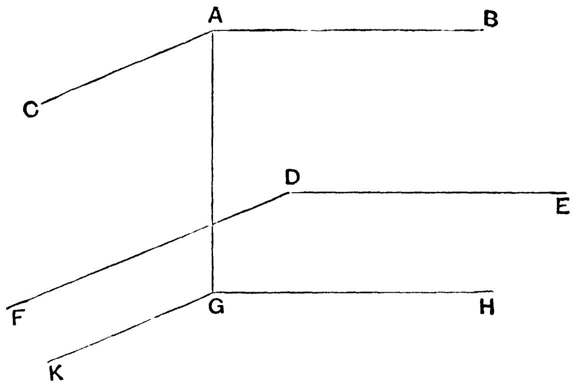
$\therefore$ BC = EF.

In the triangles ABC, DEF,

AB = DE, AC = DF, BC = EF,

$\therefore$ $\angle$ BAC = $\angle$ EDF.

18. Prop. 17. **If two intersecting straight lines in one plane be respectively parallel to two intersecting straight lines in another plane, the two planes are parallel.**



Let AB be parallel to DE, and AC to DF; to prove that the plane BAC is parallel to the plane EDF.

Draw AG perpendicular to the plane EDF, GH parallel to DE and therefore to AB, GK parallel to DF and therefore to AC.

Since AG is perpendicular to the plane EDF,

∴ AGH is a right angle, and therefore AG is also perpendicular to AB.

Similarly, AG is also perpendicular to AC;

AG is perpendicular to the plane BAC as well as to the plane EDF;

∴ plane BAC is parallel to the plane EDF.

EXERCISES.

1. If a series of points be taken so that the perpendiculars from them to a plane are of equal length, the points all lie in a plane parallel to the former.

2. Find the locus of points in space equidistant from two given parallel planes.

3. If each of two intersecting lines be parallel to a plane, the plane in which they lie is parallel to the plane.

4. If, from a point outside a plane, three straight lines be drawn to meet the plane, the plane which passes through their middle points is parallel to the given plane.

5. Three straight lines AO, BO, CO meet in a point O and do not lie in one plane; if AO, BO, CO be produced to A′, B′, C′, respectively, so that OA′=OA, OB′=OB, OC′=OC, then the planes ABC, A′B′C′ are parallel.

NON-PLANAR LINES.

19. DEF. 7. Straight lines are **non-planar** or **skew** when they do not lie in the same plane.

It is evident, from Prop. 1, that non-planar lines never intersect.

DEF. 8. The **angle between two non-planar lines** is the angle between one of the lines and a line drawn through any point in it parallel to the other (see Prop. 16).

DEF. 9. **Two non-planar lines are perpendicular** to one another when the angle between them is a right angle.

DEF. 10. A **skew quadrilateral** is one whose opposite sides are skew or non-planar.

EXERCISES.

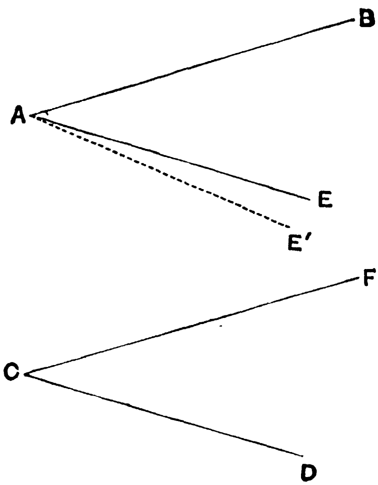
1. A sheet of paper ABCD, bounded by four straight lines, is folded about a line which meets AB in E and CD in F; state which pairs of lines are planar and which non-planar.

2. Two equal squares are in parallel planes, with the sides of the one respectively parallel to the sides of the other; prove that a diagonal of one square is parallel and perpendicular to the diagonals of the other.

3. The angle between one edge of a cube and a diagonal of the opposite face is half a right angle.

4. A line joining the middle points of two adjacent sides of a skew quadrilateral is equal and parallel to a line joining the middle points of the other two sides.

20. PROP. 18. **Through two non-planar lines, one and only one pair of parallel planes can be drawn.**



Let AB, CD be two non-planar lines; to prove that one and only one pair of parallel planes can be drawn through AB and CD.

Draw AE parallel to CD, and CF parallel to AB; then the plane BAE (containing AB) is parallel to the plane DCF (containing CD).

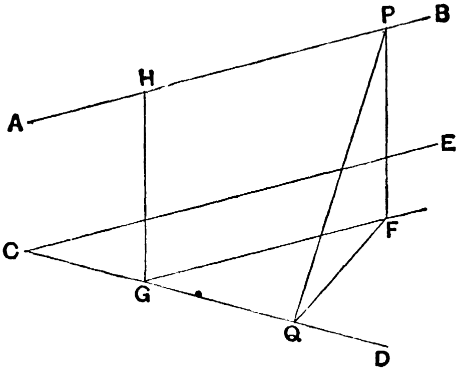
Now, any other plane through AB cuts the plane ACDE along a line AE′; and if this plane were parallel to the plane DCF, the common sections AE′, CD would be parallel, which is impossible, since AE is parallel to CD.

∴ one and only one pair of parallel planes can be drawn through the non-planar lines AB, CD.

EXERCISE.

AB, CD are two non-intersecting straight lines; show how to draw a plane through CD parallel to AB.

21. PROP. 19. **A straight line can be drawn perpendicular to both of two non-planar lines, and this line is the shortest distance between the two given lines.**



Let AB, CD be the two non-planar lines.

Draw CE parallel to AB; from any point P in AB, draw PF perpendicular to the plane DCE, FG parallel to CE (and therefore also to AB), meeting CD in G, and GH parallel to FP, meeting AB in H; to prove that GH is perpendicular to AB and CD and is the shortest distance between AB and CD.

Since HG is parallel to PF, and PF is perpendicular to the plane DCE,

∴ HG is perpendicular to the plane DCE and therefore to CD.

Again, HGFP is a parallelogram,

∴ GHP = ∠ PFG = a right angle,

∴ GH is perpendicular both to AB and CD.

Let Q be any point in CD, and join PQ, QF.

Since ∠ PFQ is a right angle,

∴ PQ > PF. But PF = HG;

∴ HG is less than any other straight line, such as PQ, joining AB and CD.

EXERCISES.

1. The shortest distance between two non-planar lines is perpendicular to the pair of parallel planes containing them.

2. If a straight line be parallel to a plane, the shortest distance from it to all lines in the plane which are not parallel to it is constant.

3. If AB be the shortest distance between two non-planar lines, and CD any other line joining them, the line joining the middle points of AB and CD is perpendicular to AB.

4. Find the locus of the middle points of all straight lines whose extremities lie on two non-planar lines.

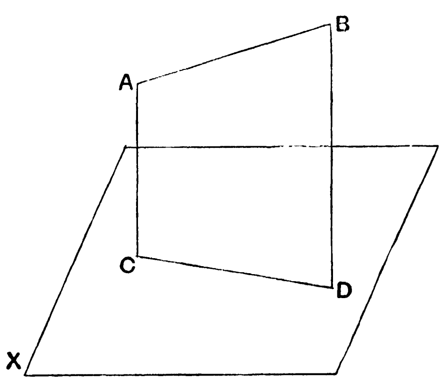
5. ABCD, CDEF are faces of a cube and G the middle point of the edge EF; draw the straight line which is perpendicular to AG and also to the edge BH.

PROJECTIONS OF LINES ON PLANES, ETC.

22. DEF. 11. The **projection of a point on a plane** is the foot of the perpendicular from the point to the plane.

DEF. 12. The **projection of a line on a plane** is the path traced out by the projection on the plane of a point which traces the given line from end to end.

√ **23.** PROP. 20. **The projection of a straight line on a plane is a straight line.**



Let AB be a straight line, and CD its projection on the plane X, C and D being the projections of A and B on the plane; to prove that CD is a straight line.

The plane which contains AC and AB contains all the lines drawn from points in AB parallel to AC, i.e. perpendicular to the plane X;

∴ the projection of AB on the plane X is the intersection of the plane ABDC with the plane X, and is therefore a straight line.

EXERCISES.

1. The projections of parallel straight lines on any plane are parallel.

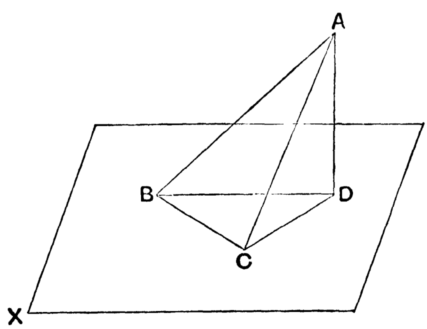
2. Two planes, X and Y, intersect in the straight line AB; from any point O in AB, OP, OQ are drawn in the plane X equally inclined to AB; prove that the angle between OP, OQ is less than the angle between their projections on the plane Y.

3. If two straight lines be at right angles, their projections on a plane parallel to one of them are at right angles.

4. OA and OB are two intersecting straight lines, OC is any straight line through their point of intersection, but not in their plane, which is such that the angles COA, COB are equal; show that the projection of OC on the plane AOB bisects the angle AOB.

24. Def. 13. The **inclination of a straight line to a plane** is the acute angle between the straight line and its projection on the plane.

25. Prop. 21. **The angle which a straight line makes with a plane is less than the angle which it makes with any straight line meeting it in that plane.**



Let AB be the straight line, and AD the perpendicular on the given plane X from a point A in AB, so that BD is the projection of AB on the plane X, and let BC be any other straight line in the plane; to prove that the angle ABD is less than the angle ABC.

From BC cut off BC equal to BD and join AC.

In the triangles ABD, ABC,

AB is common, $BC = BD$, and $AC > AD$ (since ADC is a right angle),

$$\therefore \angle ABC > \angle ABD.$$

EXERCISES.

1. If, from a point outside a plane, equal straight lines be drawn to a plane, they are equally inclined to the plane.

2. If two parallel straight lines meet a plane, they make equal angles with it.

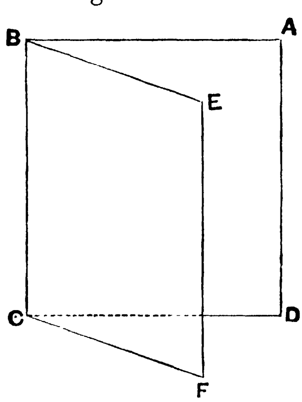
3. Any straight line is equally inclined to two parallel planes.

Dihedral Angles.

26. Def. 14. The inclination of one of two intersecting planes to the other is called a **dihedral angle.**

Def. 15. The two planes (AC, BF) are called the **faces** of the dihedral angle.

Def. 16. The line (BC) in which the faces intersect is called the **edge** of the dihedral angle.



A dihedral angle may be denoted by the letters at two points on its edge; or, if more than two planes intersect in that edge, by the letters at two points in the edge and at one point in each plane. Thus, the dihedral angle in the figure may be denoted by BC, or by ABCF.

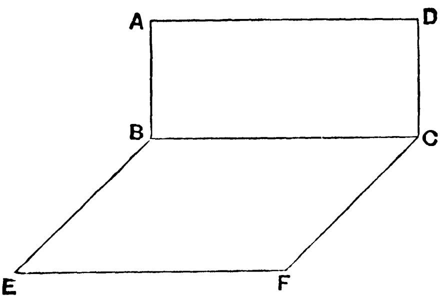
Let B, C be any two points in the edge, BA, CD lines perpendicular to BC in the plane AC, and BE, CF lines perpendicular to BC in the plane BF.

Since AB is parallel to CD, and BE to CF,

$$\therefore \angle ABE = \angle DCF.$$

Thus, if, from any point in the edge, two straight lines be drawn perpendicular to the edge, one in each plane, the magnitude of this angle is constant, and it may therefore be taken as a measure of the dihedral angle.

27. Def. 17. One **plane** is said to be **perpendicular to another plane** when the dihedral angle between the planes is a right angle.



Thus, if B be any point in the edge BC of a right dihedral angle, and if BA, BE, one in each plane, be perpendicular to BC, then AB is perpendicular to BC and BE, and therefore to the plane BF. Hence, an alternative definition to the above would be:

One **plane** is said to be **perpendicular to another plane** if any line drawn in one plane perpendicular to the common section of the two planes, be also perpendicular to the other plane.

EXERCISES.

1. **Vertically opposite dihedral angles are equal.**

2. A plane which bisects a dihedral angle also bisects the vertically opposite dihedral angle.

3. **The angle between the perpendiculars from any point to two intersecting planes is equal to the dihedral angle between the planes.**

4. If a plane cut two parallel planes, it makes with them on the same side interior dihedral angles together equal to two right angles.

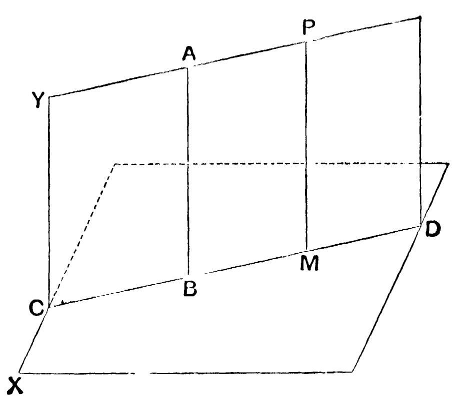
5. **If two planes be at right angles, a straight line drawn from any point in their common section perpendicular to one plane must lie in the other plane.**

6. If one plane be perpendicular to another, and a straight line be drawn from a point in the first perpendicular to the other, this straight line will meet the common section of the planes.

7. Two right-angled triangles BAC, ABD (A and B being respectively the right angles) are situated in planes perpendicular to one another; show that the line which joins the middle points of AB and CD is perpendicular to AB.

8. If a plane be drawn through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

28. PROP. 22. **If a straight line be perpendicular to a plane, every plane which passes through that line is perpendicular to the given plane.**



Let AB be perpendicular to the plane X; to prove that any plane Y drawn through AB is perpendicular to the plane X.

Let CD be the common section of the two planes. From *any* point M in CD draw MP in the plane Y perpendicular to CD.

Since ABM and PMD are both right angles,

∴ PM is parallel to AB,

∴ PM is perpendicular to the plane X,

∴ the plane Y is perpendicular to the plane X.

EXERCISES.

1. If the common section of two given planes be perpendicular to a third plane, each of the given planes is perpendicular to the third plane.

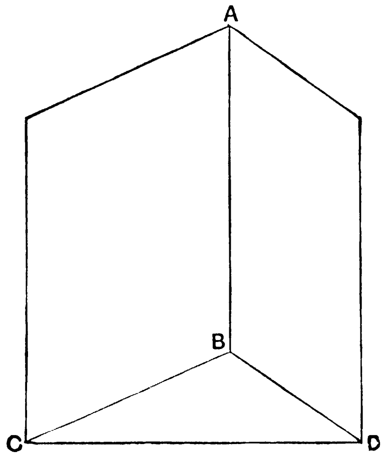
2. Draw a plane perpendicular to each of two intersecting planes.

3. Through any line in a plane, one and only one plane can pass perpendicular to the given plane.

4. Through any straight line oblique to a given plane, one and only one plane can pass perpendicular to the given plane.

5. If a straight line be parallel to a plane, any other plane perpendicular to the line is perpendicular to the plane.

√ **29.** Prop. 23. **If two planes be both perpendicular to a third plane, their common section is also perpendicular to that plane.**



Let the planes ABC, ABD be both perpendicular to the plane BCD; to prove that their common section AB is also perpendicular to the plane BCD.

If a perpendicular be drawn from B to the plane BCD, it must lie in each of the planes ABC, ABD, and therefore must be their common section.

∴ AB is perpendicular to the plane BCD.

EXERCISES.

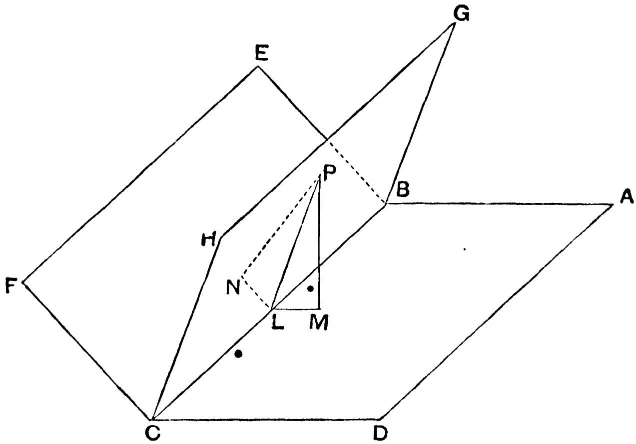
1. If two straight lines be parallel, the common section of any two planes passing through them is parallel to either.

2. If two planes be at right angles, and a line A in one of them be perpendicular to a line B in the other, one at least of the lines A, B is perpendicular to the common section of the planes.

3. AB is perpendicular to a plane X, AL, BM are perpendicular to a second plane Y; prove that LM is perpendicular to the common section of the planes X, Y.

4. **Three straight lines mutually at right angles meet at P; any plane is drawn cutting them at A, B, C and PO is perpendicular to the plane ABC; prove that O is the orthocentre of the triangle ABC.**

30. PROP. 24. **Every point in the plane which bisects a dihedral angle is equidistant from the faces of the angle.**



Let P be any point in the plane CG which bisects the dihedral angle ABCF; and let PM, PN be perpendicular to the planes AC, BF, respectively; to prove that PM is equal to PN.

Draw PL perpendicular to BC, so that LM, LN are both perpendicular to BC.

In the triangles PLM, PLN,

$\angle$ PLM = $\angle$ PLN (since the plane CG bisects the dihedral angle ABCF),

right $\angle$ PML = right $\angle$ PNL, and PL is common,

$\therefore$ PM = PN.

EXERCISES.

1. Find the locus of a point which is equidistant from two given planes.

2. If two planes intersect, the planes which bisect the adjacent dihedral angles are perpendicular to one another.

3. Find the locus of a point which is equidistant from three given planes which are not all parallel.

Polyhedral Angles.

31. Def. 18. A **solid angle** is formed by three or more planes which meet at a point and do not intersect in one line.

Def. 19. The plane angles which form a solid angle are called its **faces.**

Def. 20. The lines in which the faces intersect are called its **edges.**

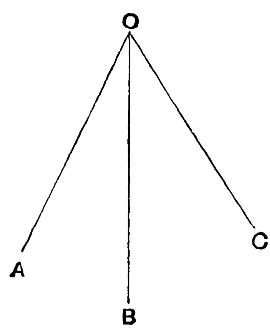
Def. 21. The point in which the edges meet is called the **vertex** of the solid angle.

Def. 22. A **trihedral angle** is a solid angle formed by three plane angles.

Def. 23. A **polyhedral angle** is a solid angle formed by more than three plane angles.

Def. 24. A polyhedral angle is said to be **convex** when every section made by a plane which cuts all its edges is a convex polygon; or, when the angle lies entirely on one side of each face.

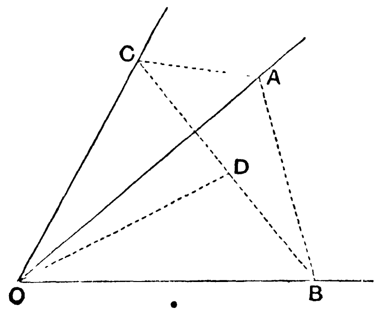
A polyhedral angle is denoted by its vertex, as O, or by its vertex and the edges taken in order, as O-ABC.



EXERCISE.

If the planes forming a solid angle be cut by parallel planes, the common sections with the former planes form similar figures.

√**32.** PROP. 25. **Of the plane angles which contain a trihedral angle, any two are together greater than the third.**



Let the solid angle at O be contained by the three plane angles BOC, COA, AOB; to prove that any two of these angles are together greater than the third.

Of the three angles (supposed not all equal, in which case the proof is obvious), let BOC be that which is not less than either of the others and greater than the angle BOA.

Draw OD in the plane BOC, making the angle BOD equal to the angle BOA, and make OD equal to OA. Through D draw any line in the plane BOC, meeting OB in B and OC in C.

In the triangles AOB, DOB,

$$OA = OD,\ OB \text{ is common, and } \angle AOB = \angle DOB,$$

$$\therefore AB = DB.$$

Now, $AB + AC > BC$, i.e. $> BD + DC$,

$$\therefore AC > DC.$$

In the triangles AOC, DOC,

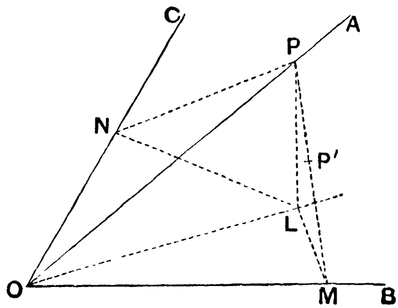
$$OA = OD,\ OC \text{ is common, and } AC > DC,$$

$$\therefore \angle AOC > \angle DOC,$$

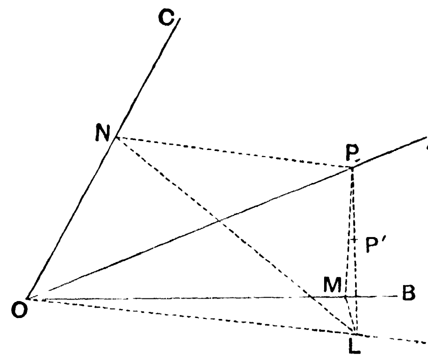
$$\therefore \angle AOB + \angle AOC > \angle BOC.$$

Second Proof. In OA, any one of the edges of the trihedral angle, take any point P; draw PL perpendicular to the

(i)



(ii)



plane BOC, PM perpendicular to OB and PN to OC; then LM is perpendicular to OB and LN to OC.

Since PLM is a right angle, PM > LM, and there is a point P′ in MP, such that P′M = LM. Now in the triangles P′MO, LMO,

$$P'M = LM,\ OM \text{ is common, and } \angle P'MO = \angle LMO,$$

$$\therefore \angle P'OM = \angle LOM,$$

$$\therefore \angle POM > \angle LOM.$$

(i) If L lie within the angle BOC,

then $\angle$ POM > $\angle$ LOM, and similarly $\angle$ PON > $\angle$ LON,

$\therefore$ $\angle$ POM + $\angle$ PON > $\angle$ LOM + $\angle$ LON,

i.e. $\angle$ AOB + $\angle$ COA > $\angle$ BOC.

(ii) If L lie on one of the bounding lines OB, OC, or outside the angle BOC, then one of the angles COA, AOB, is greater than $\angle$ BOC, i.e.

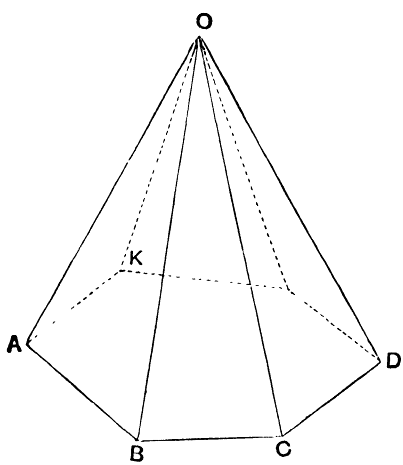
$\angle$ COA + $\angle$ AOB > $\angle$ BOC.

EXERCISES.

1. Any one of the plane angles forming a polyhedral angle is less than the sum of the other plane angles.

2. The sum of the angles of a skew quadrilateral is less than four right angles.

33. PROP. 26. **The sum of the plane angles of any convex polyhedral angle is less than four right angles.**



Let the solid angle at O be contained by the plane angles AOB, BOC, COD, ... KOA; to prove that the sum of these angles is less than four right angles.

Let the plane angles AOB, BOC, COD, ... KOA be cut by a plane, the sections forming the convex polygon ABCD ... K.

The sum of the angles of the triangles AOB, BOC, COD, ... KOA

= twice as many right angles as there are triangles,

= twice as many right angles as the polygon ABCD ... K has sides,

= sum of the interior angles of the polygon ABCD ... K + four right angles.

But $$\angle OAK + \angle OAB > \angle KAB,$$
$$\angle OBA + \angle OBC > \angle ABC,$$

and so on ;

∴ sum of base angles of triangles AOB, BOC, COD, ... KOA > sum of interior angles of polygon ABCD ... K ;

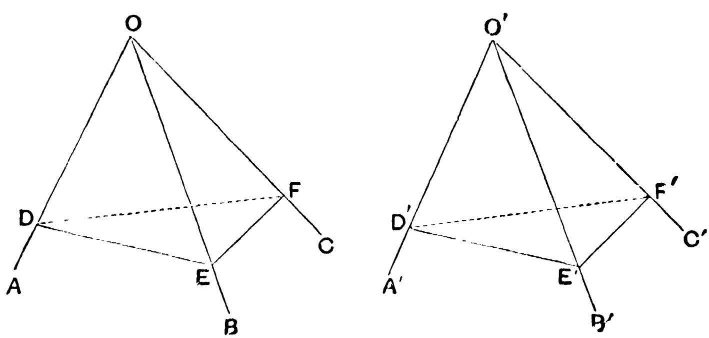
∴ sum of plane angles at O is less than four right angles.

EXERCISES.

1. How many solid angles can be formed by plane angles each of which is equal to : (*a*) 60°, (*b*) 90°, (*c*) 108°, (*d*) 50°, (*e*) 30°, (*f*) 70°, (*g*) 40°, (*h*) 45° ?

2. How many solid angles can be formed by plane angles of 60° and 90° ?

34. PROP. 27. **If each of two trihedral angles be contained by plane angles which are respectively equal to one another, their corresponding dihedral angles are equal.**



Let the two trihedral angles O-ABC, O′-A′B′C′, have the angles BOC, COA, AOB respectively equal to the angles B′O′C′, C′O′A′, A′O′B′; to prove that the dihedral angle OA is equal to the dihedral angle O′A′, etc.

Along OA, O′A′, mark off equal lengths OD, O′D′ and in the planes OAB, OAC, respectively, draw DE, DF perpendicular to OA, and in the planes O′A′B′, O′A′C′, respectively, draw D′E′, D′F′ perpendicular to O′A′. Join EF, E′F′.

In the triangles ODE, O′D′E′,

$$\angle DOE = \angle D'O'E', \text{ right } \angle ODE = \text{right } \angle O'D'E',$$

$$\text{and } OD = O'D',$$

$$\therefore\ DE = D'E' \text{ and } OE = O'E'.$$

Similarly, $DF = D'F'$ and $OF = O'F'$.

Again, in the triangles OEF, O′E′F′,

$$OE = O'E',\ OF = O'F', \text{ and } \angle EOF = \angle E'O'F',$$

$$\therefore\ EF = E'F'.$$

Lastly, in the triangles DEF, D′E′F′,

$$DE = D'E',\ DF = D'F',\ EF = E'F',$$

$$\therefore\ \angle EDF = \angle E'D'F',$$

dihedral $\angle OA =$ dihedral $\angle O'A'$; etc.

EXERCISES.

1. If two of the dihedral angles of a trihedral angle be equal, the opposite plane angles are equal.

2. If two plane angles of a trihedral angle be equal, the opposite dihedral angles are equal.

3. In a trihedral angle, the greater plane angle is subtended by the greater dihedral angle, and conversely.

4. If the plane angles of two trihedral angles be equal, each to each, the trihedral angles may be either superposed or placed so that the edges of the one are the prolongations of the edges of the other.

5. **If each of two trihedral angles have the dihedral angles between their faces respectively equal, their corresponding plane angles are equal.**

6. **Two trihedral angles are equal when a plane angle and the adjacent dihedral angles of one are respectively equal to a plane angle and the adjacent dihedral angles of the other and are similarly placed.**

7. **Two trihedral angles are equal when two plane angles and the included dihedral angle of the one are respectively equal to two plane angles and the included dihedral angle of the other and are similarly placed.**

8. **Two trihedral angles are equal when the three plane angles of the one are respectively equal to the three plane angles of the other and are similarly placed.**

9. In any trihedral angle, the three planes passing through the bisectors of the plane angles and perpendicular to these planes respectively intersect in the same line.

10. In any trihedral angle, the three planes bisecting the three dihedral angles intersect in the same line.

11. In any trihedral angle, the three planes through the edges and the respective bisectors of the opposite plane angles intersect in the same line.

12. In any trihedral angle, the three planes through the edges perpendicular to the opposite faces respectively intersect in the same line.

MISCELLANEOUS EXERCISES.

1. A straight line cannot be perpendicular to each of two intersecting planes.

2. If equal straight lines OA, OB, OC be drawn to a plane from a point O outside it, and if D be any point inside the triangle ABC, prove that OD is less than either OA, OB, OC.

3. Prove that the middle points of the sides of a skew quadrilateral are the angular points of a parallelogram, and that the plane of this parallelogram is parallel to the diagonals of the quadrilateral.

4. If three straight lines, which do not all lie in one plane, be cut in the same ratio by three planes, two of which are parallel; show that the third will be parallel to the other two, if its intersections with the three straight lines be not collinear.

5. Give a construction for drawing a line which shall be equally inclined to three given non-planar lines meeting in a point.

6. Find the locus of a point which is equidistant from three non-planar lines which meet in a point.

7. If a straight line be equally inclined to three straight lines which meet it and lie in one plane, it is at right angles to that plane.

8. Into how many regions is space divided by: (i) one plane, (ii) two intersecting planes, (iii) two parallel planes, (iv) three planes not intersecting in one straight line, (v) four planes no three of which intersect in one straight line or pass through one point?

9. From the extremities of two parallel straight lines AB, CD, parallel straight lines A*a*, B*b*, C*c*, D*d* are drawn, meeting a plane in *a*, *b*, *c*, *d*; prove that AB is to CD as *ab* is to *cd*, taking the case in which A, B, C, D are on the same side of the plane.

10. Three equal straight lines PA, PB, PC are drawn forming the edges of a solid angle with each of its plane angles equal to two-thirds of a right angle; if N be the foot of the perpendicular from P on the plane ABC, show that the square on PN is double of the square on AN.

11. If three planes have a common line of intersection, any two parallel lines are cut in the same proportion by the three planes.

12. Three straight lines not in the same plane, but parallel to and equidistant from one another, are intersected by a plane, and the points of intersection joined; when will the triangle so formed be equilateral and when isosceles?

13. OA, OB, OC are three equal lines mutually perpendicular to one another; if ON be perpendicular to the plane ABC, $3 \cdot ON^2 = OA^2$.

14. Two planes meet at any angle; find the shortest distance on their surfaces from a point on one to a point on the other.

15. Find a point in a given plane, such that the sum of its distances from two given points outside the plane and on the same side of it may be a minimum.

16. **If OA, OB, OC be mutually at right angles to one another, the triangle ABC is acute-angled.**

17. If two straight lines in one plane be equally inclined to another plane, they are equally inclined to the common section of the two planes.

18. Two perpendicular planes intersect in the line AB; from a point C in this line, on opposite sides of AB, are drawn CE and CF in one of the planes so that the angles ACE, ACF are equal; prove that CE and CF make equal angles with any line through C in the other plane.

19. ABCD is the face of a cube, AE a diagonal of the cube, and AF an edge parallel to the edge CE; show that AE is perpendicular to the plane BDF.

20. Through the angular points of a parallelogram ABCD, parallel straight lines AP, BQ, CR, DS are drawn, which are cut by any plane in the points P, Q, R, S, respectively; prove that PQRS is also a parallelogram, and that the sum of AP, CR is equal to the sum of BQ, DS.

21. Through a given point, draw a straight line to intersect two non-planar lines; and show that only one straight line can be so drawn.

22. Draw a straight line perpendicular to a given plane and so as to intersect two given non-planar lines.

23. **If the triangles ABC, A′B′C′ in different planes be such that AB and A′B′ meet, as also BC and B′C′, and CA and C′A′; the lines AA′, BB′, CC′ meet in one point or are parallel.**

24. **If two triangles ABC and A′B′C′ in different planes be so situated that the lines AA′, BB′, CC′ meet in a point, then the pairs of corresponding sides AB, A′B′; AC, A′C′; BC, B′C′ intersect in collinear points.**

25. Show how to cut a tetrahedral angle so that the section may be a parallelogram.

26. AB, BC, CD are the sides of a cube, of which AD is a diagonal; prove that the dihedral angle between the planes of ABD, ACD is equal to two-thirds of a right angle.

27. If the sides of a skew quadrilateral ABCD be cut by a plane in the four points E, F, G, H; prove that EA . FB . GC . HD = EB . FC . GD . HA.

28. a, b, c are three non-intersecting straight lines, of which a and b are at right angles to c; x, y, z are respectively the shortest distances between b and c, c and a, a and b; prove that z is equal to the shortest distance between x and y.

29. Through a given point, draw a straight line which shall make equal angles with each of three non-planar lines.

30. A, B, are fixed points in one of two non-planar lines AB, CD; find a point P in CD such that the sum of the squares on AP, BP may be a minimum.

CHAPTER II.

THE POLYHEDRON.

35. DEF. 25. A **polyhedron** is a solid bounded on all sides by planes. The bounding planes are called its **faces**, the lines in which the faces intersect its **edges**, and the points in which the edges meet its **vertices**.

DEF. 26. A **regular polyhedron** is one which has all its faces regular polygons.

DEF. 27. A **tetrahedron** is a polyhedron with four faces, a **hexahedron** with six, an **octahedron** with eight, a **dodecahedron** with twelve, and an **icosahedron** with twenty faces.

DEF. 28. A **diagonal** of a polyhedron is a straight line joining any two vertices not lying in the same face.

36. PROP. 28. **If V be the number of vertices, E the number of edges, and F the number of faces, in any polyhedron, then** $E + 2 = F + V$ (Euler's Theorem)*.

Imagine the polyhedron to be re-constructed, starting with a single face, and replacing one by one all the faces of the polyhedron in such a manner that, when any face is added, the edges which newly come in contact are all consecutive edges.

When any face is added, let r be the number of edges of this face which come in contact with those of the polyhedron previously constructed; then, since these r edges are all consecutive, the number of vertices which come in contact is $r + 1$. In other words, the addition of a new face implies the addition of one more new edge than new vertices.

In the first face laid down, the number of edges is the same as the number of vertices, and the addition of the last face adds

* Before reading this theorem, exercises 1 and 2 (p. 48) may be worked.

neither new edges nor new vertices; so that the number of edges in the completed polyhedron exceeds the number of vertices by $F - 2$, i.e.

$$E - V = F - 2$$

or

$$E + 2 = F + V.$$

37. PROP. 29. **There are five, and only five, regular polyhedra.**

The angle of an equilateral triangle contains 60°, of a square 90°, of a regular pentagon 108°, of a regular hexagon 120°, of a regular heptagon $128\frac{4}{7}$°, etc.

A solid angle can therefore be formed by the junction of 3, 4 or 5 angles of an equilateral triangle (the sum of such angles being 180°, 240° or 300°), but not by six or more such angles (the sum of such angles being 360° or more); also by the junction of 3 angles of a square (the sum of such angles being 270°), but not by 4 or more; also by the junction of 3 angles of a regular pentagon (the sum of such angles being 324°), but not by 4 or more; but not by 3 angles of a regular hexagon (the sum of such angles being 360°), or by 3 angles of a regular polygon with a higher number of sides.

Thus, there are five, and only five, regular polyhedra.

38. PROP. 30. **To find the number of vertices, edges and faces in the different regular polyhedra.**

Consider, first, the regular polyhedron, each of whose solid angles is formed by the junction of three equilateral triangles.

Let V be the number of vertices, E the number of edges, and F the number of faces, in this polyhedron.

At every vertex, three edges meet, and each edge joins two vertices,

$$\therefore\ 2E = 3V, \text{ or } E = \frac{3V}{2}.$$

At every vertex, three faces meet, and each face adjoins three vertices,

$$\therefore\ 3F = 3V, \text{ or } F = V.$$

But, by Euler's theorem,

$$E + 2 = F + V,$$

$$\therefore \frac{3V}{2} + 2 = V + V,$$

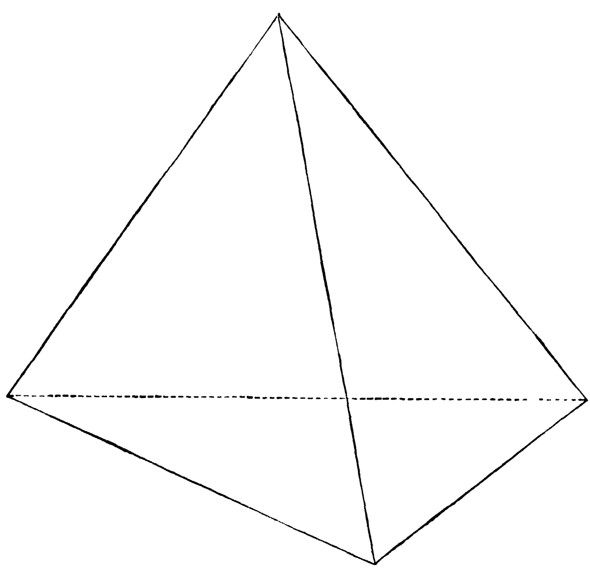
or $$V = 4;$$

$\therefore$ there are 4 vertices, 6 edges and 4 faces.

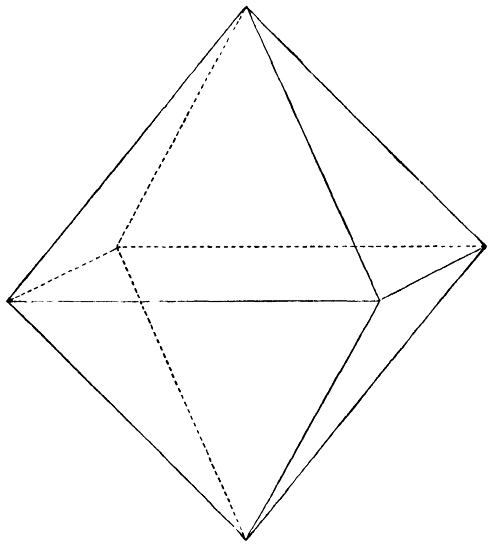
Similarly, it may be shown that, if each solid angle of a regular polyhedron is formed by the junction of:

4 equilateral triangles,	$V = 6$,	$E = 12$,	$F = 8$,
5 equilateral triangles,	$V = 12$,	$E = 30$,	$F = 20$,
3 squares,	$V = 8$,	$E = 12$,	$F = 6$,
3 regular pentagons,	$V = 20$,	$E = 30$,	$F = 12$.

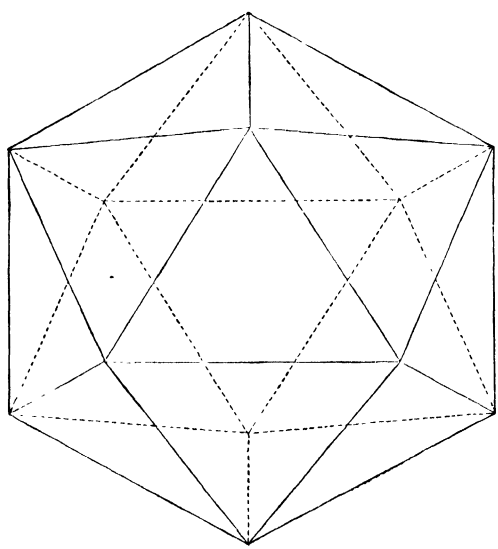
The five regular polyhedra are therefore the tetrahedron, the octahedron, the icosahedron, the cube or hexahedron, and the dodecahedron. They are represented in the accompanying figures.



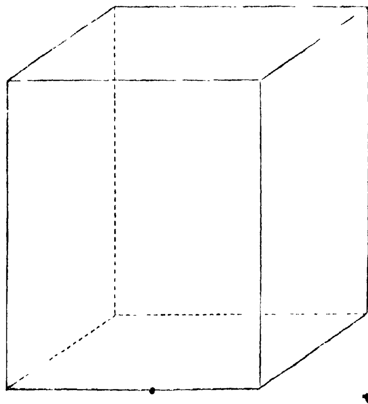
Regular Tetrahedron.



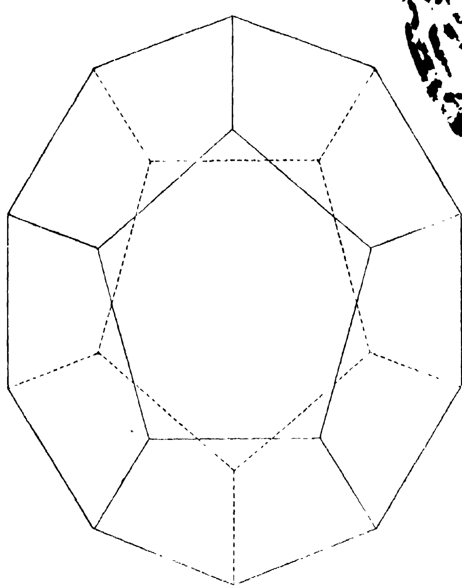
Regular Octahedron.



Regular Icosahedron.



Cube.



Regular Dodecahedron.

EXERCISES.

1. If a tetrahedron be re-constructed, face by face, state the number of edges and of vertices in the partially constructed figure for each value of F from 1 to 4.

2. If a cube be re-constructed, face by face, state the number of edges and of vertices in the partially constructed figure for each value of F from 1 to 6.

3. There is no polyhedron of less than four faces.

4. The number of vertices in a polyhedron cannot be less than four, but may be four or any number greater than four.

5. No polyhedron can have less than six edges, and, with the exception of six edges, none can have less than eight edges.

6. **If the faces of a polyhedron be all triangular, the number of faces is $2V-4$, where V is the number of vertices.**

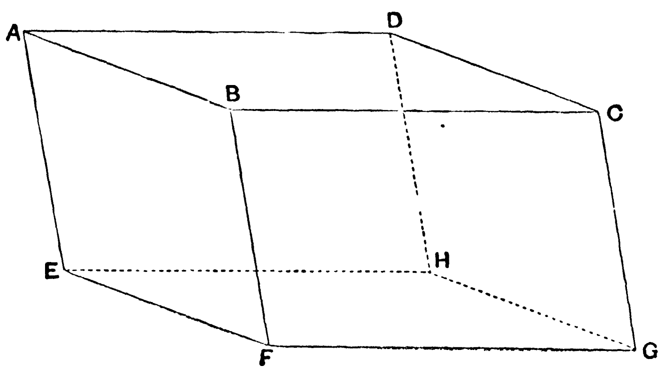
THE PARALLELEPIPED.

39. DEF. 29. A **parallelepiped** is a polyhedron bounded by three pairs of parallel planes.

DEF. 30. A **cube** is a parallelepiped whose faces are all squares.

DEF. 31. A **rectangular parallelepiped** or **cuboid** is a parallelepiped whose faces are all rectangles.

40. PROP. 31. **The opposite faces of a parallelepiped are equal parallelograms.**



Let ABCG be the parallelepiped; to prove that the opposite faces AF, DG are equal parallelograms, etc.

Since the parallel faces AF, DG are cut by the plane AC,

$\therefore$ the common sections AB, DC are parallel.

Since the parallel faces AH, BG are cut by the plane AC,

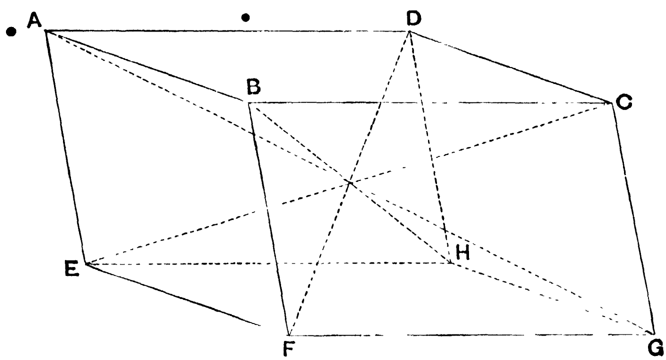
$\therefore$ the common sections AD, BC are parallel.

$\therefore$ ABCD is a parallelogram; and, similarly, the other faces may be shown to be parallelograms.

Again, AB is equal and parallel to DC, and AE to DH,

$\therefore \angle$ BAE $= \angle$ CDH, and therefore AF, DG are equal parallelograms.

41. Prop. 32. **The diagonals of a parallelepiped are concurrent and bisect one another.**



Let ABCG be the parallelepiped; to prove that the diagonals AG, BH, CE, DF are concurrent and bisect one another.

Since AB is equal and parallel to DC and therefore to HG,

$\therefore$ ABGH is a parallelogram,

$\therefore$ its diagonals AG, BH bisect one another.

Since AE is equal and parallel to DH and therefore to CG,

$\therefore$ ACGE is a parallelogram,

$\therefore$ its diagonals AG, CE bisect one another.

Since AD is equal and parallel to BC and therefore to FG,

∴ AFGD is a parallelogram,

∴ its diagonals AG, DF bisect one another.

Lastly, since BC is equal and parallel to FG and therefore to EH,

∴ BCHE is a parallelogram,

∴ its diagonals BH, CE bisect one another.

∴ the four diagonals AG, BH, CE, DF bisect one another and are therefore concurrent.

EXERCISES.

1. The opposite faces of a cube or cuboid are parallel to one another.

2. The diagonals of a cuboid are equal.

3. **The square on the diagonal of a cuboid is equal to the sum of the squares on the three edges meeting in one extremity of the diagonal.**

4. **The square on the diagonal of a cube is equal to three times the square on one of its edges.**

5. In an oblique parallelepiped, the sum of the squares on the four diagonals is equal to the sum of the squares on the twelve edges

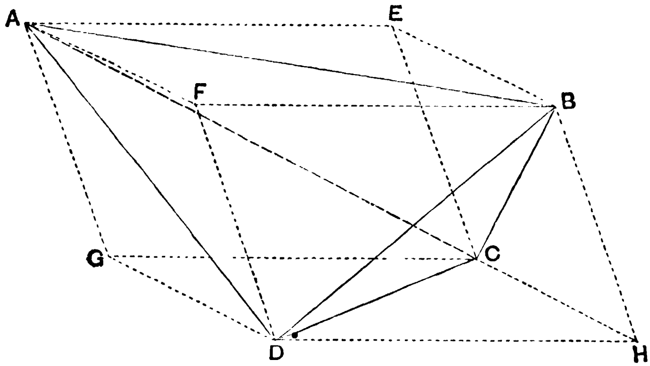
6. If the edges AB, CD of a parallelepiped ABCG be perpendicular to the edges AH, DE respectively, then four faces are rectangles.

7. If the diagonals of a parallelepiped be equal, the parallelepiped is a cuboid.

THE TETRAHEDRON.

42. A tetrahedron contains six edges, and each edge meets four others; two edges which do not meet are called **opposite** edges; there are thus three pairs of opposite edges in a tetrahedron.

Through two non-planar lines, such as a pair of opposite edges of a tetrahedron, one and only one pair of parallel planes can be drawn. The three pairs of parallel planes that are drawn through the three pairs of opposite edges of a tetrahedron thus form a parallelepiped, which is called the **circumscribing parallelepiped of** the tetrahedron.



Thus, if AEBH be a parallelepiped, the inscribed tetrahedron is obtained by drawing the diagonals AB, CD, AD, BC, AC, BD of the faces of the parallelepiped; so that ABCD is the inscribed tetrahedron.

Since EF is equal and parallel to CD, EH to AD, and FH to AC, we see that:

(i) if two opposite edges of a tetrahedron be equal, the corresponding faces of the circumscribing parallelepiped are rectangles;

(ii) if two opposite edges of a tetrahedron be at right angles, the corresponding faces of the parallelepiped are rhombuses.

43. *Example.* If two pairs of opposite edges of a tetrahedron be mutually at right angles, the third pair are also at right angles.

Let ABCD be the tetrahedron, and let AB be perpendicular to CD, and AD to BC; to prove that BD is perpendicular to AC.

Let AEBH be the circumscribing parallelepiped.

Since AB is perpendicular to CD,

$\therefore$ AEBF is a rhombus, and therefore AE = AF.

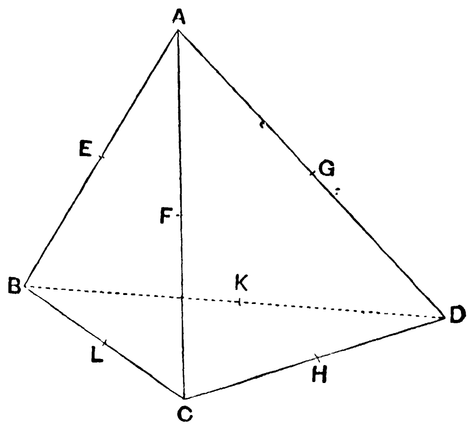
Since AD is perpendicular to BC,

∴ AFDG is a rhombus, and therefore AF = AG ;

∴ AE = AG, and therefore AECG is a rhombus,

∴ AC is perpendicular to BD.

44. Prop. 33. **The straight lines joining the middle points of opposite edges of a tetrahedron are concurrent and bisect one another.**



Let E, H be the middle points of the opposite edges AB, CD of the tetrahedron ABCD ; F, K of AC, BD ; and G, L of AD, BC ; to prove that EH, FK, GL are concurrent and bisect one another

Since AE = EB and AG = GD,

∴ EG is parallel to BD.

Since CL = LB and CH = HD,

∴ LH is parallel to BD,

∴ EG is parallel to LH.

Similarly, EL is parallel to AC and therefore to GH.

∴ EGHL is a parallelogram,

∴ its diagonals EH, GL bisect one another.

Similarly, FGKL is a parallelogram, and its diagonals FK, GL bisect one another;

∴ EH, FK, GL bisect one another and are therefore concurrent.

COR. The planes which pass through each edge of a tetrahedron and the middle point of the opposite edge are concurrent.

EXERCISES.

1. **In a regular tetrahedron, three times the square on an altitude is equal to twice the square on an edge.**

2. The altitude of a regular tetrahedron is three times the perpendicular from its foot on any of the other faces.

3. The middle points of the edges of a regular tetrahedron are the vertices of a regular octahedron.

4. The circumscribing parallelepiped of a regular tetrahedron is a cube.

5. The shortest distance between two opposite edges of a regular tetrahedron is one-half of the diagonal of the square on an edge.

6. **In any tetrahedron, the straight lines joining each vertex to the centroid of the opposite face meet in a point and divide one another in the ratio of 3 to 1.**

7. The sum of the squares on the edges of any tetrahedron is four times the sum of the squares on the straight lines joining the middle points of opposite edges.

8. The sum of the squares on any two opposite edges of a tetrahedron together with four times the square on the line which joins the middle points of those edges is equal to the sum of the squares on the other four edges of the tetrahedron.

9. If each edge of a tetrahedron be perpendicular to the opposite edge, the lines joining the middle points of pairs of opposite edges are equal in length.

10. One edge AD of the tetrahedron ABCD is perpendicular to the edge BC; prove that the sum of the squares on AB, CD is equal to the sum of the squares on AC, BD.

11. If each edge of a tetrahedron be equal to the opposite edge, the straight lines which join the middle points of opposite edges are mutually perpendicular.

12. If each edge of a tetrahedron be equal to the opposite edge, the straight line joining the middle points of any two opposite edges is perpendicular to each of those edges.

13. If each edge of a tetrahedron be equal to the opposite edge, the squares on the lines joining the middle points of opposite edges are equal to $\frac{1}{2}(b^2+c^2-a^2)$, $\frac{1}{2}(c^2+a^2-b^2)$, $\frac{1}{2}(a^2+b^2-c^2)$, where a, b, c are the lengths of the edges.

14. If the edges **AB**, **CD** of a tetrahedron be perpendicular to one another, the perpendiculars from **C** and **D** on **AB** meet **AB** in the same point.

15. Points **P**, **Q**, **R**, **S** are taken in the edges **AB**, **AC**, **DB**, **DC**, respectively, of a tetrahedron **ABCD**, such that **AP** : **PB** = **DR** : **RB** and **AQ** : **QC** = **DS** : **SC**; prove that the straight lines **PQ**, **RS** either meet or are parallel.

Prisms and Pyramids.

45. Def. 32. A **prism** is a solid figure bounded by plane faces, two of which are congruent polygons with their corresponding edges parallel, and the others pass through every pair of corresponding edges.

Def. 33. The two congruent polygons are called the **bases** of the prism; the other faces are called the **lateral faces.** The intersections of the lateral faces are called the **lateral edges.**

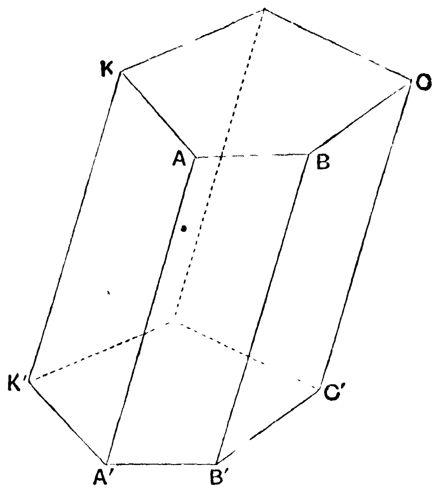
Def. 34. The **altitude** of a prism is the perpendicular distance between the planes of its bases.

Def. 35. The prism is called a **triangular, quadrilateral,** etc., prism according as each base is a triangle, quadrilateral, etc. It is evident that a parallelepiped is a prism whose bases are parallelograms.

Def. 36. A **regular hexagonal,** etc., prism is one in which the bases are regular hexagons, etc.

DEF. 37. A **truncated prism** is the portion of a prism included between the base and a plane section inclined to the base.

√**46.** PROP. 34. **The lateral faces of a prism are parallelograms.**



Let AB … KK′ be the prism, of which AB … K, A′B′ … K′ are the bases; to prove that the lateral faces ABB′A′, etc., are parallelograms.

Since AB is equal and parallel to A′B′,

AA′ is equal and parallel to BB′,

ABB′A′ is a parallelogram.

COR. The lateral edges of a prism are equal and parallel.

47. DEF. 38. If the lateral edges of a prism be perpendicular to the bases, the prism is called a **right** prism, if otherwise an **oblique** prism.

DEF. 39. A **right section** of a prism is a section made by a plane perpendicular to all the lateral edges of the prism.

48. DEF. 40. A **pyramid** is a polyhedron, one of whose faces is a polygon, and the others are triangles having a common vertex.

DEF. 41. The polygonal face is called the **base** of the pyramid; the other faces are called the **lateral faces**; the intersections of the lateral faces are called the **lateral edges.**

DEF. 42. The **altitude** of a pyramid is the perpendicular distance of the vertex from the base.

DEF. 43. The pyramid is called a **triangular, quadrilateral,** etc, pyramid, according as the base is a triangle, quadrilateral, etc. It is evident that a tetrahedron is a triangular pyramid.

DEF. 44. A **regular hexagonal**, etc., pyramid is one in which the base is a regular hexagon, etc.

DEF. 45. A **truncated pyramid** is the portion of a pyramid included between the base and a plane section inclined to the base.

DEF. 46. A **frustum** of a pyramid is the portion of a pyramid included between the base and a plane section parallel to the base. It is evident that every lateral face of a frustum of a pyramid is a trapezium.

DEF. 47. The **slant height** of a regular pyramid is the altitude of any lateral face. The **slant height** of a frustum of a regular pyramid is the altitude of any lateral face.

NUMERICAL EXERCISES.

1. Find the lateral area of the surface of an equilateral triangular prism, the edge of the base being 4 ins. and the height of the prism 6 ins.

2. Find the total area of the surface of a regular hexagonal prism, the edge of the base being 2 ins. long and the height of the prism 8 ins.

3. Find the area of the surface of a regular tetrahedron, the edge of which is 10 ins. long.

4. Find the lateral surface of a regular pyramid on a square base, the edge of the base being 6 ins. long, and the height of the pyramid 4 ins.

5. Find the area of the surface of a regular octahedron, the edge of which is 6 ins. long.

6. Find the total area of the surface of a regular hexagonal pyramid, the edge of the base being 3 ins. long and the height of the pyramid 5 ins.

7. Find the lateral area of a frustum of a regular pentagonal pyramid, the edges of the bases being respectively 5 and 3 ins. in length, and the slant height of the frustum 12 ins.

8. Find the total area of a frustum of a pyramid, the bases of which are squares whose sides are respectively 8 and 5 ins. and the height of the frustum 2 ins.

[*Answers.* **1.** 72 sq. ins. **2.** 116·78 sq. ins. **3.** 173·21 sq. ins. **4.** 60 sq. ins. **5.** 124·71 sq. ins. **6.** 74·10 sq. ins. **7.** 240 sq. ins. **8.** 154 sq. ins.]

EXERCISES.

1. If a plane cutting a prism be perpendicular to one of the lateral edges, it is perpendicular to them all.

2. The lateral edges of a regular pyramid are equal, and the lateral faces are equal isosceles triangles.

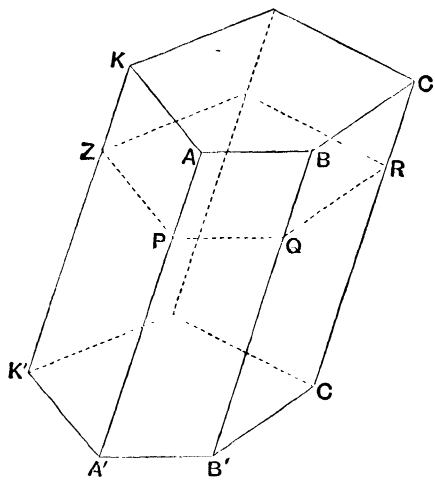
3. **Every lateral face of a frustum of a pyramid is a trapezium.**

4. **The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one-half its slant height.**

5. **The lateral area of a frustum of a regular pyramid is equal to one-half the sum of the perimeters of its bases multiplied by its slant height.**

PLANE SECTIONS OF POLYHEDRA.

49. PROP. 35. **The common section of a prism and a plane parallel to the bases is a polygon which is congruent with the bases.**



Let ABC ... KK′ be the prism, and PQR ... Z the section of the prism by a plane parallel to either base; to prove that the figure PQR ... Z is congruent with the figure ABC ... K or A′B′C′ ... K′.

Since the parallel planes ABC and PQR are cut by the plane ABB′A′,

∴ AB is parallel to PQ. But AP is parallel to BQ,

∴ ABQP is a parallelogram, and therefore PQ = AB.

Similarly, QR = BC, RS = CD, etc.

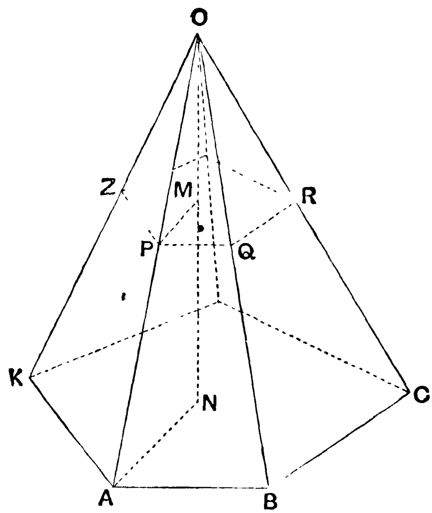
Again, since PQ is parallel to AB and QR to BC,

$$\therefore \angle PQR = \angle ABC.$$

Similarly, ∠ QRS = ∠ BCD, etc.

∴ the figure PQR ... Z is congruent with the figure ABC ... K.

√50. Prop. 36. **The common section of a pyramid and a plane parallel to the base is: (i) similar to the base, and (ii) such that its area is to the area of the base as the square of the distance of the plane from the vertex to the square of the distance of the base from the vertex.**



Let OAB .. K be the pyramid, PQ ... Z its section by a plane parallel to the base AB ... K, and OMN a perpendicular to the base, cutting the section PQ ... Z in M; to prove (i) that the figures PQ ... Z, AB ... K are similar, and (ii) that

$$\text{fig. PQ} \ldots \text{Z} : \text{fig. AB} \ldots \text{K} = \text{OM}^2 : \text{ON}^2.$$

(i) Since the parallel planes PQR, ABC are cut by the plane OAB,

∴ PQ is parallel to AB.

Similarly, QR is parallel to BC, RS to CD, etc.

$$\therefore \angle \text{Q} = \angle \text{B}, \ \angle \text{R} = \angle \text{C}, \text{ etc.}$$

∴ the figures are equiangular.

Again, since PQ is parallel to AB, the triangles OPQ, OAB are similar,

$\therefore$ PQ : AB = OQ : OB = QR : BC, for a similar reason.

$\therefore$ the sides about the equal angles Q, B are proportional, and so on.

$\therefore$ the figures PQ ... Z, AB ... K are similar.

(ii) Join PM, AN. Then, since the parallel planes PQR, ABC are cut by the plane OAN,

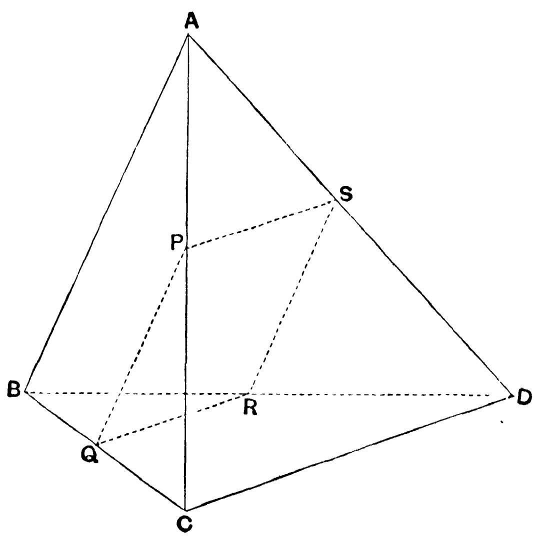
$\therefore$ PM is parallel to AN and therefore the triangles OPM, OAN are similar, so that OP : OA = OM : ON.

Now, fig. PQ ... Z : fig. AB .... K = $PQ^2 : AB^2$,

$= OP^2 : OA^2$,

$= OM^2 : ON^2$.

51. Prop. 37. **The common section of a tetrahedron and a plane parallel to a pair of opposite edges is a parallelogram.**



Let ABCD be the tetrahedron, and PQRS its section by a plane parallel to the opposite edges AB, CD; to prove that PQRS is a parallelogram.

Since AB is parallel to the plane PR, and since ABC, ABD are planes passing through AB, their common sections, PQ, RS, with the plane PR are parallel to AB, and therefore to one another.

Similarly, PS, QR are parallel to CD and therefore to one another.

$\therefore$ PQRS is a parallelogram.

EXERCISES

1. Every plane section of a parallelepiped which cuts two opposite faces is a parallelogram.

2. Any section of a cuboid made by a plane passing through one of its edges is a rectangle.

3. A cuboid is bisected by all planes drawn through the straight line joining the centres of a pair of opposite faces.

4. Show that, in general, a plane which cuts the six faces of a cube cuts them in a hexagon whose opposite sides are parallel, and show how to make the section a regular hexagon.

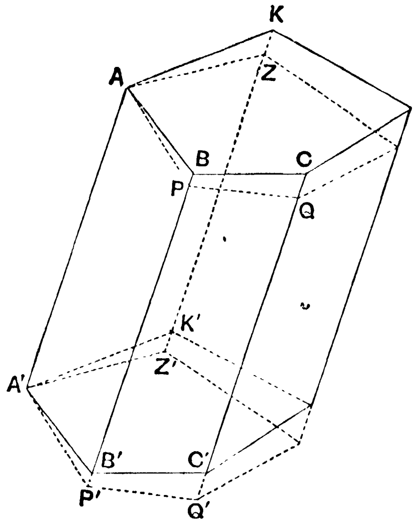
5. If the common section of a tetrahedron and a plane be a parallelogram, the plane is parallel to a pair of opposite edges.

6. If two opposite edges of a tetrahedron be equal, the section of the tetrahedron by a plane parallel to those edges is a parallelogram of constant perimeter.

7. Under what conditions is it possible to draw a plane so as to cut the faces of a tetrahedron in four lines forming a square?

VOLUMES OF SIMPLE POLYHEDRA.

52. PROP. 38. **The volume of an oblique prism is equal to the product of its right section and a lateral edge.**



Let AA′B′K′ be the oblique prism, and AA′P′Z′ a prism having a right section APQ ... Z of the given prism for its base, and the lateral edge AA′ of the prism for its altitude; to prove that the volume of the prism AA′B′ ... K′ is equal to the volume of the prism AA′P′Z′.

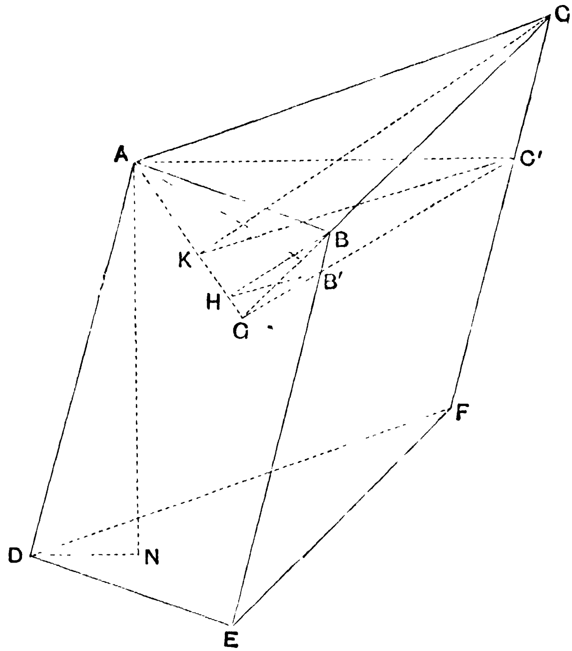
Since AB, A′B′ are parallel, and AP, A′P′ are parallel,

∴ BP = B′P′ and ∠ ABP = ∠ A′B′P′; also CQ = C′Q′ and ∠ BCQ = ∠ B′C′Q′; and so on;

∴ the truncated prism ACQ may be made to coincide with the truncated prism A′C′Q′;

∴ volume of prism AA′B′K′ is equal to that of the prism AA′P′Z′.

53. PROP. 39. **The volume of a triangular prism is equal to the product of its base and altitude.**



Let ABCF be an oblique triangular prism, ABC, DEF its bases, and AN its altitude; to prove that the volume of the prism = base ABC × altitude AN.

Let AB′C′ be a right section of the prism through A, and therefore perpendicular to BB′, CC′, and let the planes ABC, AB′C′ intersect in the line AG. Draw BH, CK perpendicular to AG, so that B′H, C′K are also perpendicular to AG.

Now,

$$\triangle ACG : \triangle AC'G = \tfrac{1}{2}AG \,.\, CK : \tfrac{1}{2}AG \,.\, C\,K$$

$$= CK : C'K;$$

and $\triangle ABG : \triangle AB'G = BH : B'H$

$= CK : C'K$ (by similar triangles BHB', CKC').

$$\therefore \triangle ACG - \triangle ABG : \triangle AC'G - \triangle AB'G = CK : C'K,$$

i.e. $$\triangle ABC : \triangle AB'C' = CK : C'K.$$

But AN is perpendicular to the plane ABC, and AD is perpendicular to the plane $AB'C'$,

$$\therefore \angle DAN = \angle \text{ between planes } ABC,\ AB'C' = \angle CKC',$$

and right $\angle AND$ = right $\angle KC'C$,

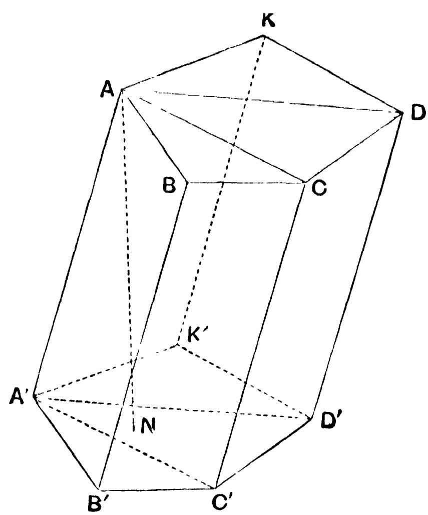
$\therefore$ triangles AND, $KC'C$ are similar,

$$\therefore AD : AN = KC : KC',$$

$$\therefore \triangle ABC : \triangle AB'C' = AD : AN,$$

$$\therefore \text{volume of prism} = \triangle AB'C' \times AD = \triangle ABC \times AN.$$

54. PROP. 40. **The volume of any prism is equal to the product of its base and altitude.**



Let ABC ... KK′ be any prism, ABC ... K, A′B′C′ ... K′ its bases, and AN its altitude; to prove that the volume of the prism

= base ABC ... K × altitude AN.

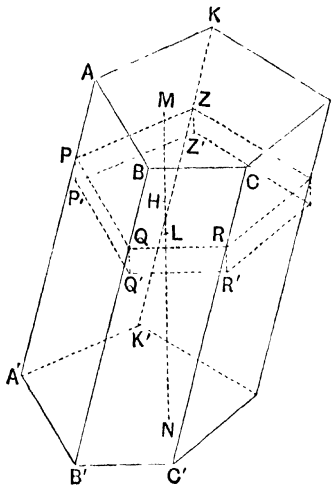
The prism may be divided into triangular prisms of the same altitude AN by planes passing through the edge AA′ and the diagonally opposite edges CC′, DD′, etc.

The volume of each triangular prism = its base × altitude AN,

∴ sum of volumes of triangular prisms = sum of their bases × altitude AN,

∴ volume of given prism = base ABC ... K × altitude AN.

Second Method. Let MN be an altitude of the prism ABC...KK′.



Divide the altitude into any number of equal parts, and let HL be one of them. Let PQR ... Z be the section of the prism by a plane through H parallel to the bases, and let planes through the sides of this polygon and parallel to the altitude of the prism form with the parallel planes through H, L the right prism PQR ... ZZ′.

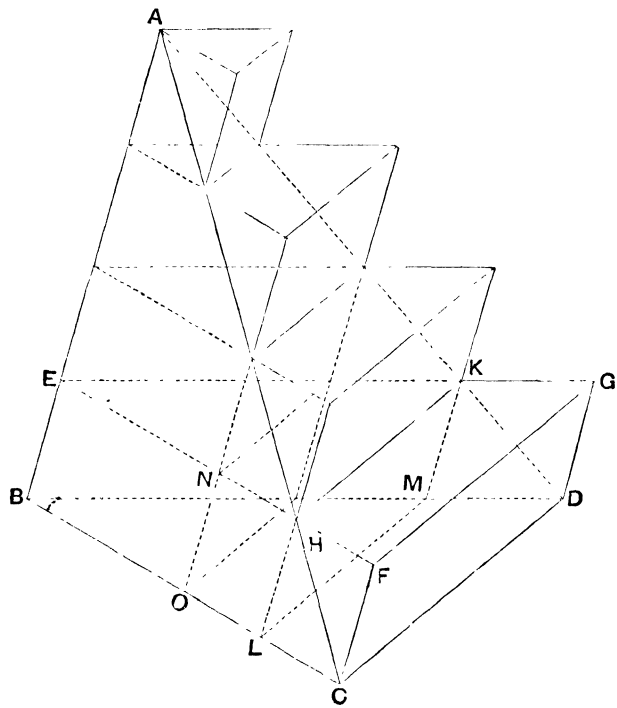
Since the area of the section PQR ... Z is equal to that of the base ABC ... K,

$\therefore$ the volume of the slice $=$ area of base $\times$ HL,

$\therefore$ the volume of the prism $=$ the sum of the volumes of all the slices

$=$ area of base $\times$ altitude.

55. PROP. 41. **The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude when the number of such prisms is indefinitely increased.**



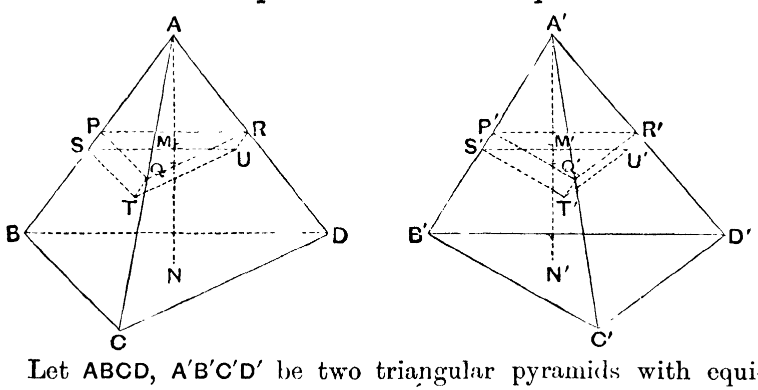
Let ABCD be a triangular pyramid. Divide one edge, AB, into any number of equal parts, and, through all the points of division and A, let planes be drawn parallel to the base BCD, intersecting the face ACD in a series of parallel lines HK, etc. Through these lines, let planes be drawn parallel to the edge AB. Then the two series of parallel planes with the two faces ABC, ABD form the two series of inscribed and circumscribed prisms of equal altitude, EBLM being the lowest prism of the inscribed series, and EBCD that of the circumscribed series. It is evident that the volume of the pyramid is greater than the sum of the volumes of the inscribed prisms, and less than the sum of the volumes of the circumscribed prisms.

Now the difference between the volumes of the lowest prisms of each series is the prism HLCD; the difference between the volumes of the succeeding prisms of the two series is the prism NOLM; and so on;

∴ the difference between the sum of the volumes of the two series of prisms is equal to the lowest circumscribed prism EBCD.

But, when the number of prisms is made indefinitely great, the volume of the prism EBCD can be made less than any quantity we please; i.e. the limits of the sum of the volumes of the two series of prisms are equal to one another, and therefore also equal to the intermediate volume of the pyramid.

56. Prop. 42. **Two triangular pyramids with equivalent bases and equal altitudes are equal in volume.**

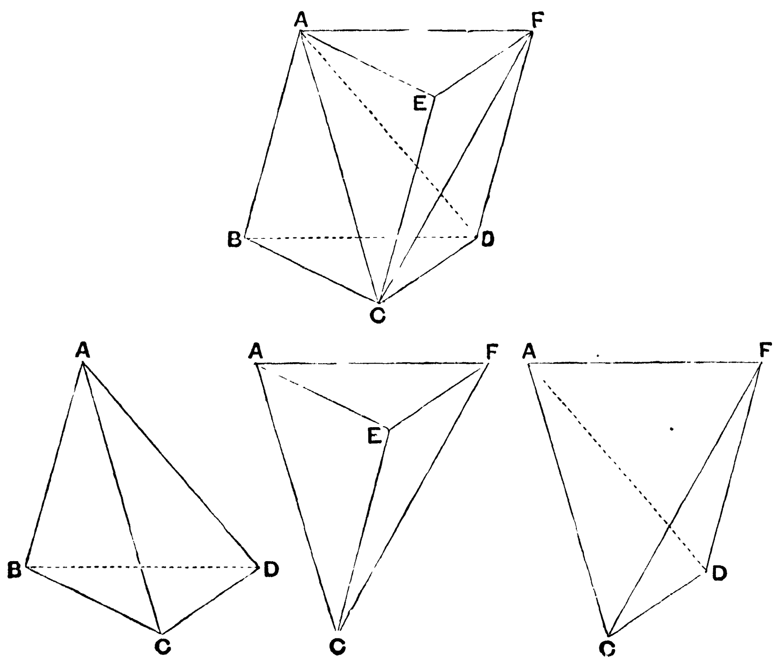


Let ABCD, A′B′C′D′ be two triangular pyramids with equivalent bases BCD, B′C′D′, and equal altitudes AN, A′N′; to prove that the pyramids ABCD, A′B′C′D′ are equal in volume.

From the altitudes AN, A′N′, cut off equal parts AM, A′M′; and let planes through M, M′ parallel to the bases BCD, B′C′D′ intersect the pyramid in the triangles PQR, P′Q′R′. Then the triangles PQR, P′Q′R′ are equal in area, and therefore equi-altitudinal prisms having the triangles PQR, P′Q′R′ as bases and lateral faces parallel to AB, A′B′ respectively, are equal in volume.

Now, if the altitudes AN, A′N′ be each divided into a number of parts each equal to the altitude of such prisms, the sum of all such prisms inscribed in the pyramid ABCD is equal to the sum of all those inscribed in the pyramid A′B′C′D′, and therefore, ultimately, the number of inscribed prisms in each being indefinitely increased, the volumes of the two pyramids are equal.

57. Prop. 43. **The volume of a triangular pyramid is one-third of the product of its base and altitude.**



Let ABCD be a triangular pyramid; to prove that its volume is equal to one-third of the product of its base and altitude.

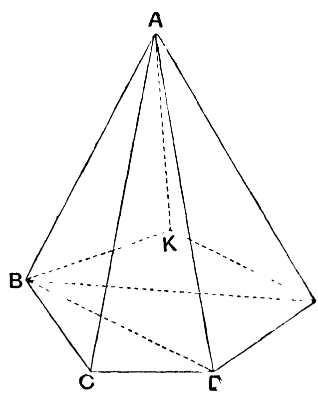
Through the sides of the base BCD let planes parallel to AB form, with the base BCD and a plane through A parallel to BCD, the triangular prism ABCDF.

Now, the planes ACD and ACF divide this prism into three triangular pyramids ABCD, CAEF and ACDF (shown separately below); of which ABCD and CAEF are on equal bases BCD, AEF and of equal altitude, and are therefore equal in volume; and the pyramids CAEF and ACDF are on equal bases CEF, CDF and of the same altitude (having the same vertex A);

hence, the triangular prism BCDF is divided into three equal triangular pyramids, one of which is the given pyramid ABCD;

∴ the volume of a triangular pyramid is equal to one-third of the product of its base and altitude.

√58. Prop. 44. **The volume of any pyramid is equal to one-third of the product of its base and altitude.**



Let ABCD ... K be any pyramid on the polygonal base BCD ... K; to prove that the volume of the pyramid is equal to one-third of the product of its base and altitude.

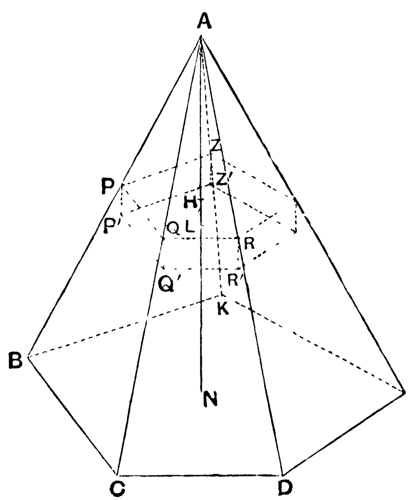
The pyramid may be divided into triangular pyramids of the same altitude by planes passing through AB and the diagonally opposite edges AD, etc.

The volume of each triangular pyramid is equal to one-third of the product of its base and altitude;

∴ as all such pyramids have the same altitude, the sum of the volumes of all the triangular pyramids so formed is equal to one-third of the product of the sum of the bases and the common altitude;

∴ the volume of the given pyramid is equal to one-third of the product of its base BCD ... K and altitude.

Second Method. Let AN be the altitude of the pyramid ABC ... K; and let A be the area of the base and H the length of the altitude. Divide the altitude into n equal parts each of length h, so that $H = nh$. Let HL be one of these parts; and let AH contain r of them, so that $AH = rh$. Let PQR ... Z be the section of the pyramid by a plane through H parallel to the base, and let planes through the sides of this polygon and parallel to the altitude of the pyramid form with the parallel planes through H, L, the right prism PQR ... ZZ′.



Since area of section PQR ... Z : area of base = $AH^2 : AN^2$;

$$\therefore \text{ area of section PQR ... Z} = \frac{r^2}{n^2} \cdot A,$$

$$\therefore \text{ volume of slice PQR ... ZZ}' = \frac{r^2}{n^2} \cdot Ah.$$

$\therefore$ volume of pyramid

$= $ sum of volumes of such slices from $r = 0$ to $r = n - 1$

$$= \frac{Ah}{n^2}\{1^2 + 2^2 + 3^2 + \dots + (n-1)^2\}$$

$$= \frac{Ah}{n^2} \cdot \frac{(n-1)\,n\,.\,(2n-1)}{6}$$

$$= \frac{Ah}{6}\left(2n - 3 + \frac{1}{n}\right)$$

$= \frac{1}{3}Ahn$, when n is indefinitely increased, and therefore h indefinitely diminished

$= \frac{1}{3}AH.$

NUMERICAL EXERCISES.

1. Find the number of c. ft. of earth removed per yard length from a cutting in level ground, 12 ft. in depth, the breadth of the base of the cutting being 15 ft., and the slopes of both sides 45°.

2. Find the volume of a regular hexagonal prism, the side of the base being 4 ins. long, and the height of the prism 15 ins.

3. A piece of wood is in the form of a regular pyramid on a square base; the side of the base is 6 ins. and the height of the pyramid 8 ins.; find the total area of the surface, and the volume, of the wood.

4. The length of each edge of a regular tetrahedron is 4 ins.; find the area of the surface, and the volume, of the tetrahedron.

5. A frustum of a pyramid is contained between two parallel planes distant 9 ft apart, and the areas of the parallel faces are 5 and 20 sq. ft.; find the height of the pyramid and the volume of the frustum.

6. A tub has a square bottom whose side is 8 ins. and a square top whose side is 14 ins.; if the depth be 6 ins., find the volume of the tub.

7. The top of a bath is a rectangle 7 ft. long and 3 ft. 6 ins. wide; the bottom is a rectangle 6 ft. long and 2 ft. 9 ins. wide; the depth is 2 ft.; prove that it will hold about 254 gallons, one gallon containing 277·274 c. ins.

8. A railway embankment is carried across a valley at right angles to its axis; all the surfaces of the embankment are planes, the top being a rectangle 80 yds. long and 25 ft. wide, the bottom 64 yds. long and 15 yds. wide; if the height of the embankment be 16 ft., find its volume.

[*Answers.* **1.** 972. **2.** 623·54 c. ins. **3.** 138·53 sq. ins., 96 c. ins. **4.** 27·71 sq. ins., 7·54 c. ins. **5.** 18 ft., 105 c. ft. **6.** 744 c. ins. **8.** 4527 c. yds., 11 c. ft.]

EXERCISES.

1. The plane which passes through two diagonally opposite edges of a parallelepiped divides it into two prisms of equal volume.

2. If a regular tetrahedron and a regular octahedron have edges of the same length, the volume of the octahedron is four times that of the tetrahedron.

3. If the area of one face of a regular tetrahedron be A, find the volume of the tetrahedron.

4. The altitude of a regular tetrahedron is equal to the sum of the four perpendiculars drawn from any point within it upon the four faces.

5. **If H be the height of a frustum of a pyramid, B, B′ the areas of its bases, the volume of the frustum is** $\frac{1}{3}H(B+B'+\sqrt{BB'})$.

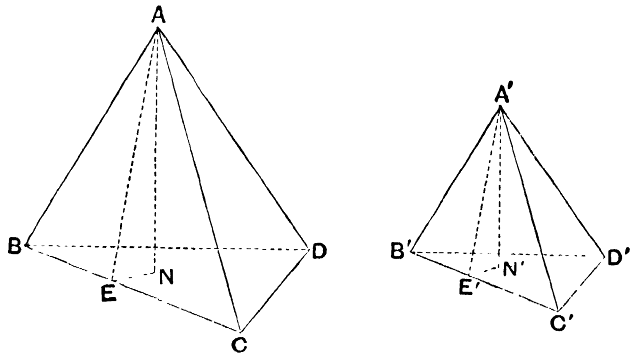
6. The top of a tub is a rectangle whose sides are a, b, and the bottom is a rectangle whose sides are parallel to those of the other and of lengths a', b'; if the depth be h, prove that the volume is $\frac{1}{6}h(2ab+2a'b'+ab'+a'b)$.

7. **If a right-angled triangular prism be cut by a plane, the volume of the truncated part is equal to a prism of the same base and of height equal to one-third the sum of the three edges.**

SIMILAR POLYHEDRA.

59. DEF. 48. Two polyhedra are said to be **similar** when they have the same number of faces, similar and similarly placed, and have their corresponding polyhedral angles equal.

60. PROP. 45. **The volumes of two similar tetrahedra are to one another as the cubes of corresponding edges.**



Let ABCD, A′B′C′D′ be similar tetrahedra, AN, A′N′ their altitudes, and BC, B′C′ a pair of corresponding edges, to prove that the volumes of the tetrahedra are to one another as BC^3 to $B'C'^3$.

Draw AE perpendicular to BC, and A′E′ to B′C′, then NE is perpendicular to BC and N′E′ to B′C′;

and, since the trihedral angles B, B′ are equal, $\angle AEN = \angle A'E'N'$.

Now, the triangles AEN, A′E′N′ are similar, as also are the triangles ABE, A′B′E′, and the triangles ABC, A′B′C′;

$$\therefore AN : A'N' = AE : A'E'$$
$$= AB : A'B'$$
$$= BC : B'C';$$

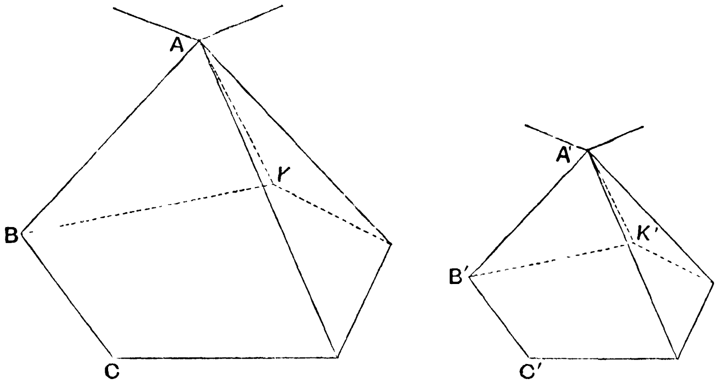
also $$\triangle BCD : \triangle B'C'D' = BC^2 : B'C'^2;$$

$\therefore$ volume of tetrahedron ABCD : volume of tetrahedron A′B′C′D′

$$= BC^3 : B'C'^3.$$

COR. The volumes of two similar pyramids are to one another as the cubes of corresponding edges.

61. Prop. 46. **The volumes of similar polyhedra are to one another as the cubes of the lengths of corresponding edges.**



Let A, A′ be corresponding vertices of two similar polyhedra, BC, B′C′ corresponding edges, and BC ... K, B′C′ ... K′ corresponding faces; to prove that the volumes of the polyhedra are to one another as BC^3 to $B'C'^3$.

The polyhedra may be divided into a number of similar pyramids by planes through the corresponding vertices A, A′ and corresponding edges of the polyhedra*; and the volumes of corresponding pyramids are to one another as the cubes of corresponding edges, which are to one another as BC^3 to $B'C'^3$; and therefore the sum of the volumes of the pyramids composing one polyhedron is to that of the pyramids composing the other as BC^3 to $B'C'^3$.

MISCELLANEOUS EXERCISES.

1. Any three faces of a tetrahedron are together greater than the fourth face.

2. A tetrahedron stands on an equilateral base and the angles at the vertex are right angles; show that the sum of the perpendiculars on the faces from any point on the base is constant.

* The proof of this is left as an exercise to the student.

3. **If V be the number of vertices of a convex polyhedron, the sum of the angles of all the faces is equal to 4V − 8 right angles.**

4. **The volumes of two tetrahedra which have a common solid angle are to one another as the products of the three edges terminating in the common vertex.**

5. If OA, OB, OC be the edges of a parallelepiped, and OD a diagonal, then OD is divided by the plane ABC in a point of trisection.

6. An open box in the shape of a cube lies on a horizontal plane with the diagonals of its base pointing to the four cardinal points; show that, at noon, when the length of the shadow of a vertical post is $1/2\sqrt{2}$ of the height of the post, the portions of the interior surface in sunlight and shade are as 37 : 43.

7. Find the size of a cube which will completely stop up a tube of uniform bore, the section of which is a regular hexagon of given magnitude.

8. OA, OB, OC are three edges of a cuboid of lengths a, b, c, respectively; prove that the area of the triangle ABC is $\frac{1}{2}\sqrt{(b^2c^2+c^2a^2+a^2b^2)}$.

9. The shortest distance between the diagonal of a cube and an edge which it does not meet is half the diagonal of one of the faces.

10. If the centroids of the faces of a tetrahedron be joined, a tetrahedron is formed similar to the given tetrahedron and containing one-twentyseventh of its volume.

11. A tetrahedron has an equilateral base, and the other three faces are isosceles triangles; prove that the opposite edges are at right angles.

12. The shortest distance between two opposite edges of a regular tetrahedron is inclined at half a right angle to each of the four edges which it does not meet.

13. If, from the orthocentre of a face of a tetrahedron, a line be drawn perpendicular to that face, it will intersect the perpendiculars to the other three faces drawn from the opposite vertices.

14. The six planes that bisect the dihedral angles of a tetrahedron are concurrent.

15. The six planes that bisect the edges of a tetrahedron at right angles are concurrent.

16. If a plane bisect the internal dihedral angle between two faces of a tetrahedron, it divides the opposite edge into two segments proportional to the areas of those faces.

17. If the three plane angles at the vertex of a tetrahedron be bisected and the points in which the bisecting lines meet the sides of the base be joined with the opposite angles of the base, the three lines so drawn are concurrent.

18. In any tetrahedron, the line joining the middle points of one pair of opposite edges is perpendicular to the shortest line between either of the other pairs of opposite edges.

19. If the three pairs of opposite edges of a tetrahedron be equal, the faces are acute-angled triangles.

20. **The volume of a tetrahedron is one-third that of its circumscribing parallelepiped.**

21. All tetrahedra are of equal volume in which each of a pair of opposite edges remains of the same length and in the same straight line.

22. Of all plane sections of a tetrahedron which are parallel to two opposite edges, the section of greatest area bisects all the other edges.

23. If a plane section of a tetrahedron be a rhombus, the side of the rhombus is half the harmonic mean between a pair of opposite edges.

24. If one edge AB of a tetrahedron be perpendicular to the opposite edge, the perpendiculars drawn from A and B to the opposite faces are concurrent.

25. In a tetrahedron ABCD, each edge is perpendicular to the opposite edge; prove that the orthogonal projections on the face BCD of the middle points of the edges AB, AC, AD lie on the nine-points circle of the triangle BCD.

26. If each pair of opposite edges of a tetrahedron be at right angles, the six middle points of opposite edges, and the six points in which the shortest distances between opposite edges meet those edges, lie on the same sphere.

27. Prove that a tetrahedron can be formed of any four congruent acute-angled triangles.

28. If the angles at each vertex of a tetrahedron be together equal to two right angles, then each edge is equal to the opposite edge.

29. If four points be so situated that the distance between each pair is equal to the distance between the other pair, the angles subtended at any one of these points by each pair of others are together equal to two right angles.

30. If the four faces of a tetrahedron be equal in area, they are congruent triangles.

31. Show that a cube may be cut by a plane so that the section shall be a square greater in area than the face of the cube in the proportion of 9 to 8.

32. P, Q, R, S are points on the edges AB, BC, CD, DA respectively of a tetrahedron; AQ, CP intersect in X; AR, CS in Y; show that, if BY and DX intersect, then P, Q, R, S lie in one plane.

33. If ABCD, A′B′C′D′ be two tetrahedra such that the lines AA′, BB′, CC′, DD′ are concurrent, the lines of intersection of corresponding faces are coplanar.

CHAPTER III.

CYLINDER AND CONE.

THE CYLINDER.

62. DEF. 49. A **right circular cylinder** is the solid generated by the revolution of a rectangle about one of its sides.

The side about which the rectangle revolves is called the **axis** of the cylinder; the opposite side traces out the **curved** or **lateral surface**; while the two ends of the rectangle trace out the **bases** of the cylinder.

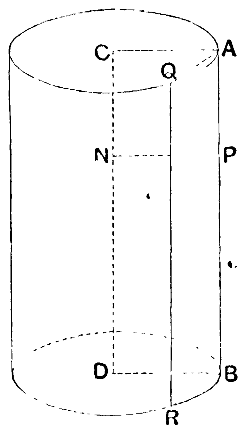
It is evident that the bases are circles whose planes are perpendicular to the axis, and therefore parallel to one another.

Since the revolving side of the rectangle is in all positions parallel to the axis, a right circular cylinder might also be defined as the surface traced by a straight line which is always parallel to a fixed straight line and passes through the circumference of a circle whose centre is in the fixed straight line and plane perpendicular to that line. And, more generally, a cylindrical surface is the surface generated by a straight line which is always parallel to a straight line fixed in space and passes through a given curve.

DEF. 50. An **oblique circular cylinder** is a cylinder whose bases are parallel circles, to which the generating lines are oblique.

DEF. 51. **A truncated cylinder** is the portion of a cylinder included between a base and any plane intersecting the cylinder.

√ **63.** PROP. 47. **The common section of a cylinder and a plane is a circle if the plane be perpendicular to the axis, and a rectangle if the plane be parallel to the axis.**



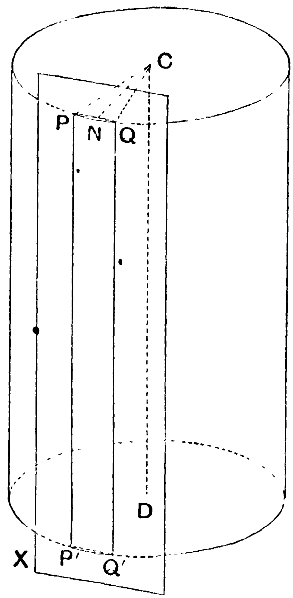
Let the cylinder be traced out by the revolution of the rectangle ABDC about the side CD; let P be any point in the opposite side AB.

(i) Draw PN perpendicular to CD. Then the line PN traces out a plane perpendicular to the axis CD, and the point P traces out a circle in this plane.

Hence, the section of a cylinder by a plane perpendicular to the axis is a circle.

(ii) Let AB, QR be two positions of the revolving side AB of the generating rectangle ABDC. Then AB and QR are equal and parallel to DC, and are therefore perpendicular to the base of the cylinder. Hence, the plane containing AB and QR intersects the lateral surface of the cylinder in the parallel lines AB, QR, and the bases in the parallel lines AQ, BR, which are perpendicular to AB; and therefore the section of the cylinder by a plane parallel to the axis is a rectangle.

64. Let X represent a plane parallel to the axis of the

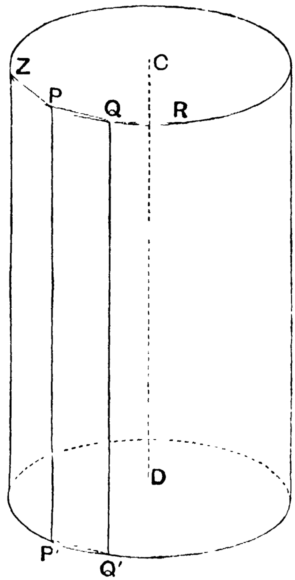


cylinder and intersecting the cylinder in the rectangle PQQ′P′. From C, the centre of one base, draw CN perpendicular to the end PQ of the rectangle. Then $PQ^2 = 4PN^2 = 4(CP^2 - CN^2)$.

Now, if the plane X move parallel to itself in the direction CN, CN increases, and consequently PQ diminishes; and, ultimately, when CN is equal to the radius of the base, PQ is equal to zero.

DEF. 52. When the section of a cylinder by a plane parallel to its axis becomes a rectangle of zero width, the plane is said to **touch** the cylinder, and the plane is called a **tangent-plane** to the cylinder

65. Prop. 48. **To find (i) the lateral surface, (ii) the volume, of a right circular cylinder.**



Let $PQR \ldots Z$ represent a regular polygon inscribed in one of the bases of the cylinder; the planes through each side of the polygon parallel to the axis intersect the cylinder in a series of rectangles which form the faces of a right prism inscribed in the cylinder. Let PQ be one of the sides of the polygon, and $PQQ'P'$ the corresponding face of the prism.

(i) The lateral surface of the prism is the sum of all the faces such as $PQQ'P'$.

Now, the area of the face $PQQ'P'$ $PP' \times PQ$;

$\therefore$ the lateral surface of the prism $= PP' \times$ perimeter of polygon $PQR \ldots Z$.

But, when the number of sides of this polygon is indefinitely increased, the perimeter of the polygon becomes the circumference of the base, and the lateral surface of the prism becomes the lateral surface of the cylinder;

$\therefore$ lateral surface of cylinder = circumference of base × altitude of cylinder.

(ii) The volume of the prism = area of polygon PQR ... Z × PP′, and therefore in the limit, when the number of sides in the polygon is indefinitely increased,

volume of cylinder = area of base × altitude of cylinder.

66. If r be the radius of the base and h the altitude of the cylinder, we have:

$$\begin{aligned} \text{lateral surface of cylinder} &= 2\pi rh, \\ \text{total} \quad ,, \qquad ,, \quad &= 2\pi rh + 2\pi r^2 \\ &= 2\pi r(h + r), \\ \text{volume of cylinder} &= \pi r^2 h. \end{aligned}$$

NUMERICAL EXERCISES.

1. Find the total surface and volume of a cylinder in which the radius of the base is 4 ins. and the altitude 10 ins. ($\pi = 3{\cdot}1416$.)

2. A square whose side is of length 1 ft. revolves about one side; find to the nearest sq. in. and c. in. respectively the area of the surface and the volume of the figure traced out. ($\pi = 3\frac{1}{7}$.)

3. A tape 22 yards long and $\frac{1}{100}$ of an inch thick, is wrapped round an axis $\frac{1}{2}$ inch in diameter; find the diameter of the roll. ($\pi = 3\frac{1}{7}$.)

4. Find the volume of a cylindrical tube, 2 ft. long, the bore being 4 ins., and the thickness of the tube one inch. ($\pi = 3{\cdot}1416$.)

5. A sheet of paper 11 ins. long and 6 ins. wide, is curved so as to form the lateral surface of a right circular cylinder 11 ins. high; find its volume. ($\pi = 3\frac{1}{7}$.)

Answers. **1.** 351·86 sq. ins., 502·66 c. ins. **2.** 1810 sq. ins., 5431 c. ins. **3.** 3·21 ins. **4.** 376·99 c. ins. **5.** $31\frac{1}{2}$ c. ins.

EXERCISES.

1. Find the volume of a right circular cylinder by summation, as in the second method of Art. 54.

2. Find the volume of an oblique circular cylinder by summation.

3. A rectangular sheet of paper a ins. long and b ins. wide is curved so as to form the lateral surface of a right circular cylinder a ins. high; find the volume of the cylinder.

4. The tangent-planes to a cylinder intersect it in straight lines parallel to the axis of the cylinder.

THE CONE.

67. DEF. 53. A **right circular cone** is the solid generated by the revolution of a right-angled triangle about one of the sides containing the right angle.

The side about which the triangle revolves is called the **axis** of the cone; the hypotenuse traces out the **curved or lateral surface** of the cone; while the other side traces out the **base of** the cone.

It is evident that the base is a circle whose plane is perpendicular to the axis of the cone. The extremity of the axis through which the generating hypotenuse passes is called the **vertex** of the cone; the other extremity is the centre of the base.

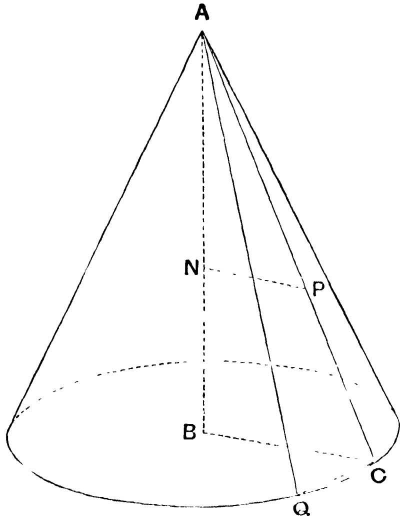
Since the revolving side of the right-angled triangle always passes through a fixed point, a right circular cone might also be defined as the surface traced out by a straight line which passes through a fixed point and also through the circumference of a circle so situated that the line joining its centre to the fixed point is perpendicular to the plane of the circle. And, generally, a conical surface is the surface generated by a straight line which passes through a fixed point and also through any given curve.

DEF. 54. An **oblique circular cone** is a cone whose base is a circle, such that the line joining its centre to the vertex of the cone is oblique to the plane of the circle.

DEF. 55. A **truncated cone** is the portion of the cone included between the base and any plane intersecting the cone.

DEF. 56. A **frustum** of a cone is the portion of the cone included between the base and a plane parallel to the base and intersecting the cone.

√68. Prop. 49. **The common section of a cone and a plane is a circle if the plane be perpendicular to the axis, and a triangle if the plane pass through the vertex.**



Let the cone be traced out by the revolution of the right-angled triangle ABC about the side AB; let P be any point on the hypotenuse AC.

(i) Draw PN perpendicular to AB. Then the line PN traces out a plane perpendicular to the axis AB, and the point P traces out a circle in this plane.

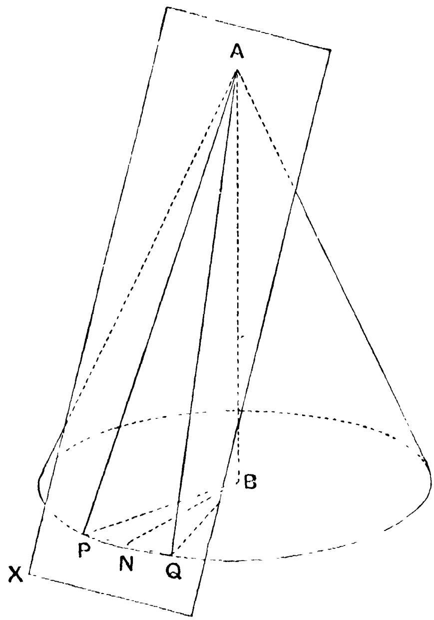
Hence, the section of a cone by a plane perpendicular to the axis is a circle.

(ii) Let AC, AQ be two positions of the revolving hypotenuse AC.

Then the plane containing AC, AQ intersects the base of the cone in the straight line CQ.

Hence, the section of the cone by a plane passing through the vertex A is the isosceles triangle ACQ.

69. Let X represent a plane passing through the vertex A

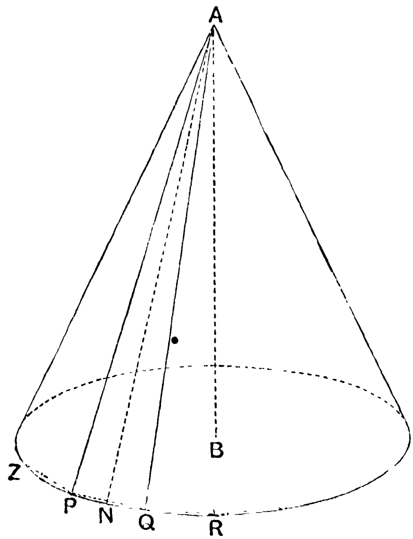


of a cone and intersecting the cone in the isosceles triangle APQ. From B, the centre of the base, draw BN perpendicular to PQ. Then $PQ^2 = 4PN^2 = 4(BP^2 - BN^2)$.

Now, if the plane X turn about the vertex A so that N, the middle point of PQ, moves in the direction BN, BN increases, and consequently PQ diminishes; and, ultimately, when BN is equal to the radius of the base, PQ is equal to zero.

DEF. 57. When the section of a cone by a plane passing through its vertex becomes a triangle of zero base, the plane is said to **touch** the cone, and the plane is called a **tangent-plane** to the cone.

70. PROP. 50. **To find (i) the lateral surface, (ii) the volume, of a right circular cone.**



Let PQR ... Z represent a regular polygon inscribed in the base of the cone; the planes passing through each side of the polygon and the vertex intersect the cone in a series of isosceles triangles which form the faces of a right pyramid inscribed in the cone. Let PQ be one of the sides of the polygon, and APQ the corresponding face of the pyramid.

(i) The lateral surface of the pyramid is the sum of all the faces such as APQ. Now, if AN be perpendicular to PQ, the area of the face APQ $= \frac{1}{2}$AN . PQ;

∴ the lateral surface of the pyramid

$= \frac{1}{2}$AN × perimeter of polygon PQR ... Z.

But, when the number of sides of this polygon is indefinitely increased, the perimeter of the polygon becomes the circumference of the base, the altitude AN of the triangle APQ becomes equal to the slant side of the cone, and the lateral surface of the pyramid becomes the lateral surface of the cone.

$\therefore$ the lateral surface of the cone $= \frac{1}{2}$ circumference of base $\times$ slant side of cone.

(ii) The volume of the pyramid

$$= \tfrac{1}{3} \text{ area of polygon } \mathsf{PQR} \ldots \mathsf{Z} \times \mathsf{AB},$$

and therefore in the limit, when the number of sides in the polygon is indefinitely increased,

$$\text{volume of cone} = \tfrac{1}{3} \text{ area of base} \times \text{altitude of cone.}$$

71. If r be the radius of the base, and h the altitude of the cone, we have therefore

$$\begin{aligned} \text{lateral surface of cone} &= \pi r \sqrt{r^2 + h^2}, \\ \text{total} \quad " \qquad " \quad &= \pi r \sqrt{r^2 + h^2} + \pi r^2 \\ &= \pi r \left(\sqrt{r^2 + h^2} + r\right), \\ \text{volume of cone} &= \tfrac{1}{3}\pi r^2 h. \end{aligned}$$

72. PROP. 51. **To find the lateral surface of a frustum of a given cone.**

Let PQR ... Z represent a regular polygon inscribed in the base of the cone; then, since the ends of the frustum are parallel, the planes through each side of the polygon and the vertex intersect the frustum in a series of trapezia, which form the faces of the frustum of a right pyramid inscribed in the frustum of the cone. Let PQ be one side of the polygon and PQQ′P′ the corresponding face of the frustum of the pyramid.

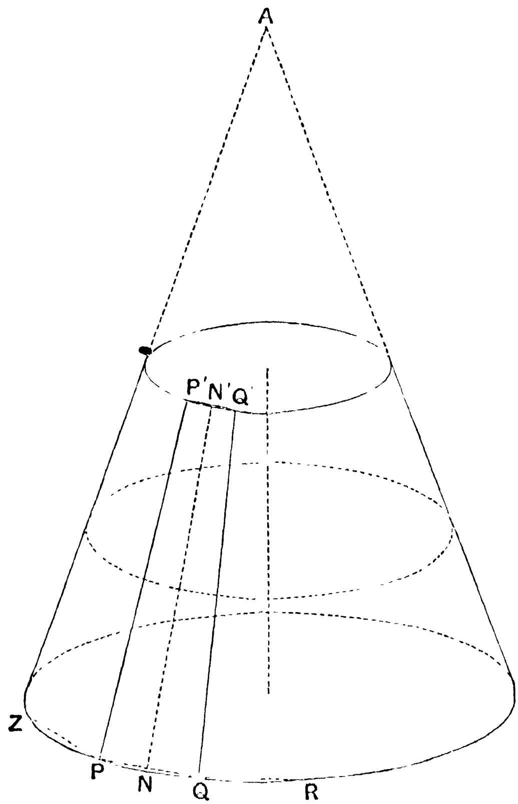
The lateral surface of the frustum of the pyramid is the sum of all the faces such as PQQ′P′.

Now, if N be the middle point of PQ, and if AN cut P′Q′ in N′, the area of this face $= \frac{1}{2}$NN′ . (PQ + P′Q′);

$\therefore$ lateral surface of the frustum of the pyramid

$= \frac{1}{2}$NN $\times$ sum of perimeters of polygons PQR ... Z, P′Q′R′ ... Z .

But, when the number of sides of the polygons is indefinitely increased, the perimeter of each polygon becomes the circumference of the corresponding base of the frustum, the altitude NN′ becomes the slant side of the frustum of the cone, and the lateral surface of the frustum of the pyramid becomes the lateral surface of the frustum of the cone; moreover, half the sum



of the circumferences of the bases of the frustum is equal to the circumference of the section of the frustum parallel to, and midway between, the two bases;

∴ the lateral surface of the frustum of the cone

= circumference of section of frustum parallel to, and midway between, the two bases × slant side of frustum.

NUMERICAL EXERCISES.

1. Find the total surface and volume of a right circular cone whose altitude is 8 ins. and radius of base 6 ins. ($\pi = 3{\cdot}1416$.)

2. An isosceles triangle, whose equal sides are each 5 ins. long and base 8 ins., revolves about the base; find the surface of the solid generated. ($\pi = 3{\cdot}1416$.)

3. A frustum of a cone is 10 ins. high, and the radii of its bases are respectively 7 and 5 ins.; find the altitude of the complete cone.

4. An equilateral triangle, whose side is 10 ins., revolves about one side; find to the nearest cubic inch the volume of the solid generated. ($\pi = 3\frac{1}{7}$.)

5. The radii of two circular sections of a right circular cone are 10 and 7 ins., the distance between the planes of the sections being 3 ins.; find the area of the surface included between the two sections. ($\pi = 3{\cdot}1416$.)

6. The radii of the bases of a frustum of a right circular cone are 12 and 8 ins., and the area of the lateral surface is $20\pi\sqrt{241}$ sq. ins.; find the altitude of the frustum.

7. The trunk of a tree is in the form of a frustum of a right circular cone, 30 ft. high, the radii of the ends being 2 ft. and $1\frac{1}{2}$ ft.; find the volume of the trunk. ($\pi = 3\frac{1}{7}$.)

8. The slant side of a cone is 25 ft. long, and the area of its lateral surface is 550 sq. ft.; find its volume. ($\pi = 3\frac{1}{7}$.)

9. Wine is poured into a conical wine-glass till half the inner surface of the glass is below the surface of the wine; show that the glass is a little more than 35 per cent. full.

10. If a piece of thin paper in the shape of a semicircle of radius $\sqrt{3}$ ft., just covers the lateral surface of a right circular cone, find the vertical angle of the cone and its volume. ($\pi = 3\frac{1}{7}$.)

11. A circular sector of radius 10 ins. and angle 120° is made into a conical bag by bringing its straight edges together; prove that the volume of the bag is about 109·7 c. ins. ($\pi = 3{\cdot}1416$.)

12. A cone 12 ins. high is to be divided into three portions of equal volume by planes parallel to the base: find the altitudes of the several portions to the hundredth part of an inch

Answers. **1.** 301·59 sq. in., 301·59 c. ins. **2.** 94·25 sq. ins. **3.** 35 ins. **4.** 786 c. ins. **5.** 226·58 sq. ins. **6.** 15 ins. **7.** $290\frac{5}{7}$ c. ft. **8.** 1232 c. ft. **10.** 60°, $1\frac{5}{28}$ c. ft. **12.** 8·32, 2·16, 1·52 ins.

EXERCISES.

1. Find the volume of a right circular cone by summation, as in the second method of Art. 59.

2. Find the volume of an oblique circular cone by summation.

3. Given the height of a right circular cone and the radius of its base, show how to construct a circle whose area shall be equal to the total surface of the cone.

4. A parallelogram and a triangle, upon the same base and between the same parallels, revolve round the base; prove that the solid generated by the triangle is one-third of that generated by the parallelogram.

5. A triangle makes a complete revolution round one of its sides; find in terms of the sides the surface and volume of the solid generated.

6. An equilateral triangle revolves about a line through an angular point parallel to the opposite side; find the volume of the solid generated.

7. A regular hexagon ABCDEF is described about a circle, and is revolved (i) about the diagonal CF, and (ii) about a diameter through two points of contact; find the volumes of the solids generated.

8. Show how to bisect the lateral surface of a right circular cone by a plane parallel to the base.

CHAPTER IV.

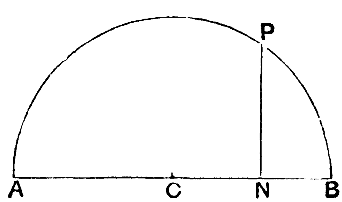
THE SPHERE.

73. Def. 58. A **sphere** is a surface generated by the revolution of a semicircle about its diameter. The centre of the semicircle is the **centre** of the sphere, and any line through the centre and terminated by the sphere is a **diameter** of the sphere.

Since any point on the arc of a semicircle is at the same distance from the centre, the following may be given as an alternative definition : A sphere is a surface every point of which is at the same distance from a point within it called the centre.

Sections of Spheres.

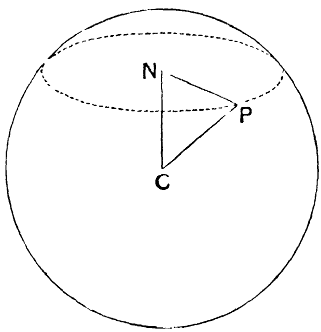
74. Prop. 52. **The common section of a plane and a sphere is a circle.**



Let P be any point on the semicircle APB, whose diameter is AB and centre C.

Draw PN perpendicular to AB. Let the figure revolve about the diameter AB; then the semicircle APB traces out the surface of a sphere, the line NP a plane perpendicular to AB, and the point P a circle whose centre is N, and this circle is the common section of the plane and sphere.

Second Proof. Let P be any point on the common section of the plane and sphere.



From the centre C of the sphere, draw CN perpendicular to the plane, and join NP, CP.

Since CNP is a right angle,

$$\therefore\ PN^2 = CP^2 - CN^2;$$

but CP, CN are both of constant length;

∴ PN is of constant length,

∴ the common section of the plane and sphere is a circle whose centre is N and radius is $\sqrt{CP^2 - CN^2}$.

Cor. 1. If the plane pass through the centre of the sphere, the radius of the circular section is equal to the radius of the sphere.

Cor 2. If the plane do not pass through the centre of the sphere, the radius of the circular section is less than that of the sphere.

COR. 3. The perpendicular from the centre of a sphere to any intersecting plane meets the plane in the centre of the circular section by the plane.

COR. 4. All planes which are equidistant from the centre of the sphere form equal circular sections, and *vice versâ.*

75. DEF. 59. A **great circle** is the section of a sphere made by a plane which passes through the centre.

DEF. 60. A **small circle** is the section of a sphere made by a plane which does not pass through the centre.

It follows that the radius of a great circle is equal to that of the sphere, while the radius of a small circle is less than that of the sphere.

DEF. 61. The **axis** of any circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle.

DEF. 62. The **poles** of any circle of a sphere are the extremities of its axis.

DEF. 63. Either end of any diameter of a sphere is said to be **antipodal** to the other.

NUMERICAL EXERCISES.

1. The radius of a sphere is 8 ins.; find the circumference of a small circle the plane of which bisects a radius at right angles.

2. Find the area of the circular section of a sphere of 13 ins. radius made by a plane whose distance from the centre is 5 ins.

3. In a sphere of 29 ins. radius, the diameter of a circular section is 42 ins.; find the distance of the intersecting plane from the centre of the sphere.

4. The radii of two circular sections of a sphere are 6 and 4 ins. respectively, and the distance between their planes is 2 ins.; find the radius of the sphere.

5. The radii of two parallel circular sections of a sphere are 8 and 5 ins., and the radius of the sphere is 10 ins.: find the distance between their planes.

6. Within a hollow sphere of 1 ft. radius is placed a right prism, the ends of which are equilateral triangles; the side of one of these being 1 ft. in length, and the surface of the sphere being in contact with all the six angular points of the prism, find in c. ins. the volume of the latter.

Answers. **1.** 43·53 ins. **2.** 452·39 sq. ins. **3.** 20 ins. **4.** 7·21 ins. **5.** 2·66 or 14·66 ins. **6.** 1222 c. ins.

EXERCISES.

1. Of two small circles of a sphere, the greater is nearer the centre than the less.

2. Draw a plane through a given point within a sphere so that the area of the circular section may be a minimum.

3. The centres of parallel circular sections of a sphere lie on a diameter of the sphere.

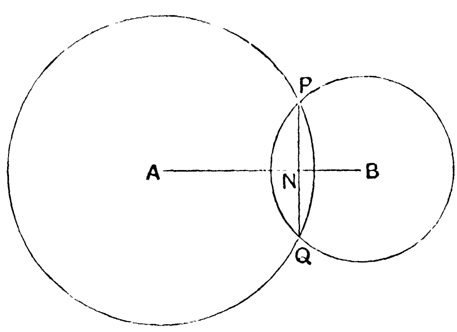
4. Through a fixed point, any three planes are drawn at right angles to one another so that each intersects a fixed sphere of radius a, prove that the sum of the areas of the three circles of intersection is constant, being equal to $\pi(3a^2 - c^2)$, where c is the distance of the fixed point from the centre of the sphere.

5. The points of intersection of two great circles are the poles of the great circle joining their poles.

6. If the pole of one great circle lie on another great circle, then the pole of the latter lies on the former great circle.

7. Two small circles which have the same poles intercept equal arcs of the great circles which pass through the poles.

76. PROP. 53. **The common section of two spheres is a circle.**



Let two circles, whose centres are A and B, intersect in P, Q; then AB bisects PQ at right angles in N.

Let the figure revolve about the line of centres AB; then each circle traces out a sphere, the line PQ a plane perpendicular to AB, and the point P a circle whose centre is N; and this circle is the common section of the two spheres.

COR. The line of centres passes through the centre of the circular section.

NUMERICAL EXERCISES.

1. The distance between the centres of two spheres whose radii are 15 and 36 ins. respectively, is 39 ins.; find the diameter of their circle of intersection.

2. Two spheres, of 10 and 8 ins. radii, intersect one another in a circle whose radius is 2 ins.; find the distance between their centres.

Answers. **1.** 27·69 ins. **2.** 17·54 ins.

EXERCISES.

1. Prove that a sphere can be inscribed in a cylinder, and will touch it along the circumference of a circle.

2. If a sphere be placed inside a hollow cone, it will touch the cone along the circumference of a circle.

3. Find the locus of a point, the sum of the squares of whose distances from the two fixed points is constant.

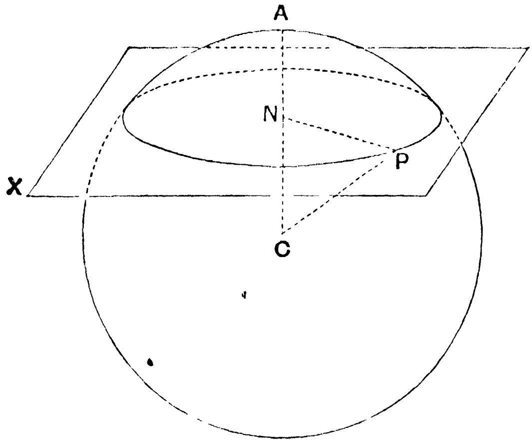
4. A fixed point O outside a given plane is joined to any point P in the plane; on OP a point Q is taken so that the rectangle OP . OQ is constant; show that Q lies on a fixed sphere.

5. The radii of two spheres are a and b, and the distance between their centres is c $(c < a + b)$; prove that the radius r of the circle of intersection is given by

$$2cr = \sqrt{[(a+b+c)(b+c-a)(c+a-b)(a+b-c)]}.$$

Tangent-Planes and Tangent-Lines.

77. Let X represent a plane cutting a sphere with centre C in a circle with centre N and radius NP; so that $NP^2 = CP^2 - CN^2$.



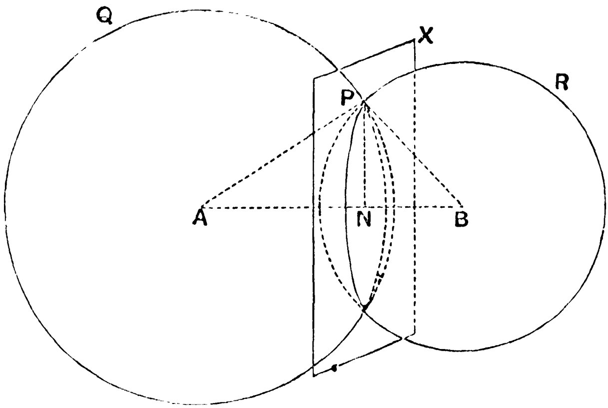
Now, if the plane X move parallel to itself in the direction CA, CN increases, and consequently NP diminishes; and, ultimately, when CN is equal to the radius, NP is equal to zero.

Def. 64. When the section of a sphere by a plane is a point-circle (i.e. a circle of radius zero), the plane is said to **touch** the sphere, and the plane is called a **tangent-plane** to the sphere.

√ **78.** Prop. 54. **The tangent-plane at any point of a sphere is perpendicular to the radius drawn to the point of contact.**

The perpendicular from the centre of a sphere on any intersecting plane meets it in the centre of the circular section made by the plane; and, if the plane move parallel to itself until, in the limit, the section becomes a point-circle, the centre of the circle ultimately coincides with the point of contact, and therefore the radius to the point of contact is perpendicular to the tangent-plane.

79. Let X represent the plane of the circular section of



two intersecting spheres Q and R, with centres A and B; let P be any point on the section and N its centre; so that

$$NP^2 = AP^2 - AN^2.$$

Now, if the sphere R move so that its centre moves in the direction AB, AN increases, and consequently NP decreases, and, ultimately, when AN is equal to the radius of the sphere Q, NP is equal to zero.

DEF. 65. When the common section of two spheres is a point-circle, each sphere is said to **touch** the other. If each sphere lie outside the other, the spheres touch one another **externally**; if one lie inside the other, they touch one another **internally.**

√ **80.** PROP. 55. **If two spheres touch one another, the line of centres passes through the point of contact.**

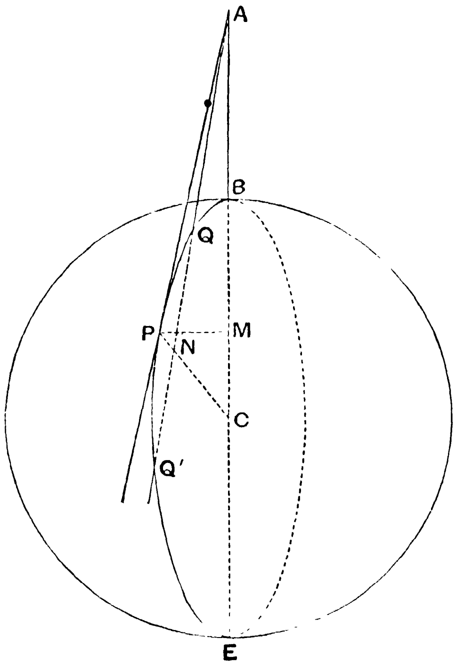
The line of centres passes through the centre of the common circular section; and, if one sphere move in either direction, until, in the limit, the section becomes a point-circle, the centre of the circle ultimately coincides with the point of contact; and therefore the line of centres passes through the point of contact.

COR. 1. If two spheres touch one another externally, the distance between their centres is equal to the sum of the radii.

COR. 2. If two spheres touch one another internally, the distance between their centres is equal to the difference of the radii.

COR. 3. If two spheres touch one another, they have a common tangent-plane at the point of contact.

81. Let a plane passing through the centre C of a sphere



intersect it in the circle BPE, and let any line in the plane of this circle meet the circle, and therefore also the sphere, in the points Q, Q′, and the diameter BCE in A. Draw CN perpendicular to QQ′ and therefore bisecting it and meeting the circle BPE in P. Then $QN^2 = CQ^2 - CN^2$.

Now, if the line AQQ′ move parallel to itself in the plane of the circle and in the direction CP, CN increases, and consequently QN diminishes; and, ultimately, when CN is equal to the radius of the sphere, QN is zero.

DEF. 66. When the two points in which a straight line intersects a sphere coincide, the straight line is said to **touch** the sphere, and the line is called a **tangent-line** to the sphere.

√ **82.** PROP. 56. **A tangent-line at any point of a sphere is perpendicular to the radius drawn to the point of contact.**

The perpendicular from the centre of a sphere on any intersecting line meets it at the middle point of the corresponding chord; and, if the line move parallel to itself until, in the limit, the two points of intersection coincide, the middle point of the chord ultimately coincides with the point of contact; and therefore the radius to the point of contact is perpendicular to the tangent-line.

COR. 1. A tangent-line to a sphere is also a tangent to the circle in which the plane containing the line and the centre of the sphere intersects the sphere.

COR. 2. An infinite number of tangent-lines may be drawn at any point of a sphere, and these all lie in the tangent-plane at the point.

COR. 3. All tangent-lines drawn to a sphere from an external point are equal in length.

For (see figure of Art. 81), $AP^2 = AC^2 - CP^2$, which is constant.

COR. 4. The points of contact of all tangent-lines drawn to a sphere from an external point lie on a circle whose plane is perpendicular to the diameter through the external point.

For, if PM be drawn perpendicular to ABE,

$$AM^2 - CM^2 = AP^2 - CP^2,$$

which is constant, so that M is a fixed point.

EXERCISES.

1. Draw a tangent plane to a sphere so as to pass through any given line.

2. **If chords of a sphere be drawn so as to pass through a fixed point, the rectangles contained by the segments of the chords are equal.**

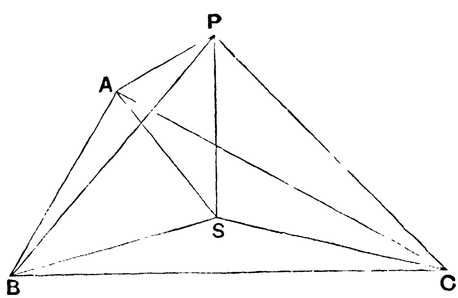
3. **If two spheres intersect one another, the tangent-lines drawn from any point in the plane of their common section to the spheres are equal.**

4. If three spheres intersect one another, their planes of intersection intersect in a line perpendicular to the plane through the centres of the spheres; and tangent-lines drawn from any point in this line to the three spheres are equal.

5. If four spheres intersect, two and two, the planes of their common sections are concurrent.

CIRCUMSCRIBED AND INSCRIBED SPHERES OF A TETRAHEDRON.

83. PROP. 57. **If a straight line be drawn through the circumcentre of a triangle perpendicular to its plane, every point in this line is equidistant from the angular points of the triangle.**



Let S be the circumcentre of the triangle ABC, SP perpendicular to the plane ABC, and P any point in SP; to prove that PA = PB = PC.

Since $$AS^2 = BS^2 = CS^2,$$

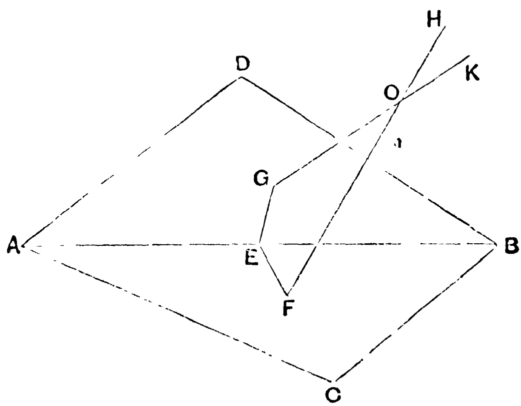
$$\therefore AS^2 + SP^2 = BS^2 + SP^2 = CS^2 + SP^2,$$

$\therefore AP^2 = BP^2 = CP^2$, since SP is perpendicular to SA, SB, SC,

$$\therefore AP = BP = CP.$$

COR. Hence, all spheres which pass through the three points A, B, C have their centres in SP.

√ **84.** PROP. 58. **Through four non-planar points, one and only one sphere can be drawn.**



Let A, B, C, D be four non-planar points; to prove that one and only one sphere can be drawn through A, B, C, D.

Let E be the middle point of AB, F, G the circumcentres of the triangles ABC, ABD, respectively, so that EF and EG are perpendicular to AB. Draw FH, GK perpendicular to the planes ABC, ABD, respectively.

Then, a straight line drawn from E to any point in FH is perpendicular to AB, i.e. FH lies in the plane through E perpendicular to AB. Similarly, GK lies in the same plane;

$\therefore$ FH, GK intersect, in O say.

Now, every point in FH, and in no other line, is equidistant from A, B and C; and every point in GK, and in no other line, is equidistant from A, B and D;

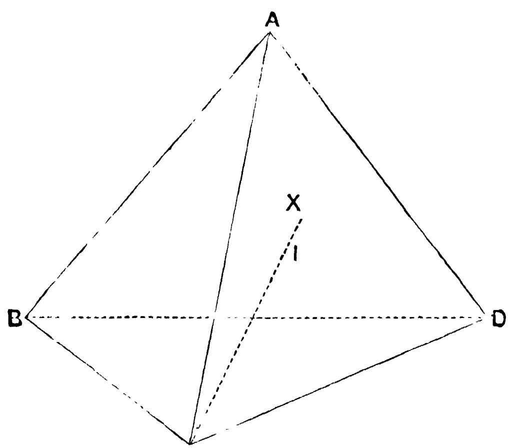
∴ O is the only point equidistant from A, B, C, D;

∴ one and only one sphere can be drawn through four non-planar points.

85. The preceding theorem might have been enunciated thus: One and only one sphere can pass through the angular points of a tetrahedron.

DEF. 67. The sphere which passes through the angular points of a tetrahedron is called the **circumscribed sphere** of the tetrahedron, its centre is the **circumcentre,** and its radius the **circumradius,** of the tetrahedron.

86. PROP. 59. **One and only one sphere can be drawn to touch the four faces of a tetrahedron.**



Let ABCD be a tetrahedron; to prove that one, and only one, sphere can be drawn to touch the four faces.

Every point in the plane which bisects the dihedral angle ABCD, and no other point, is equidistant from the faces ABC, DBC; also every point in the plane which bisects the dihedral angle ACDB, and no other point, is equidistant from the faces ACD, BCD;

∴ every point on the common section CX of these two planes, and no other point, is equidistant from the three planes ABC, ACD, BCD.

Again, every point in the plane which bisects the dihedral angle ABDC, and no other point, is equidistant from the planes ABD, CBD;

∴ the point I, in which this bisecting plane meets the common section CX of the other bisecting planes, and no other point, is equidistant from the four planes ABC, ACD, ADB, BCD;

∴ one and only one sphere can be drawn to touch these four planes.

87. DEF. 68. The sphere which touches the four faces of a tetrahedron is called the **inscribed sphere** of the tetrahedron, its centre is the **incentre**, and its radius the **inradius**, of the tetrahedron.

NUMERICAL EXERCISES.

1. Find the radii of the inscribed and circumscribed spheres of a cube whose edge is 6 ins. long.

2. Find the radii of the inscribed and circumscribed spheres of a regular tetrahedron whose edge is 9 inches long.

3. Find the radius of the sphere which touches one face of a regular tetrahedron and the other three faces produced, the edge of the tetrahedron being 2 ft. long.

4. Find the radii of the inscribed and circumscribed spheres of a regular octahedron whose edge is 9 ins. long.

Answers. **1.** 3 ins., 5·20 ins. **2.** 1·84, 5·51 ins. **3.** 9·80 ins. **4.** 3·67, 6·36 ins.

EXERCISES.

1. How may four points in space be arranged so that (i) no sphere, and (ii) any number of spheres, can be drawn through them?

2. Find the radii of the inscribed and circumscribed spheres of a cube.

3. Find the radii of the inscribed and circumscribed spheres of a regular tetrahedron.

4. Find the radius of the sphere which touches one face of a regular tetrahedron, and the other three faces produced.

5. Find the radii of the inscribed and circumscribed spheres of a regular octahedron.

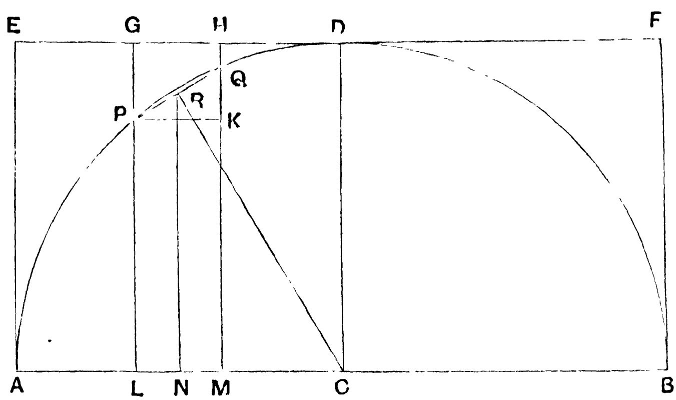
6. Four planes intersect so as to form a tetrahedron; how many spheres may as a rule touch them all?

AREA AND VOLUME OF A SPHERE.

88. DEF. 69. A **zone** is the portion of the surface of a sphere included between two parallel planes.

If one of the planes be a tangent-plane to the sphere, the zone is called a **zone of one base.**

√ **89.** PROP. 60. **To find the area of the surface of a sphere.**



Let ADB be a semicircle of radius r, AB a diameter, C its centre, CD a radius perpendicular to AB, and AE, BF, EDF tangents to the semicircle at A, B, D, forming with the diameter the rectangle AEFB.

Let P, Q be two neighbouring points on the semicircle, R the middle point of the chord PQ. Draw PL, QM, RN perpendicular to AB, and produce LP, MQ to meet EF in G, H. Draw PK perpendicular to QM, and join CR. Then, if the figure revolve about AB, the semicircle ADB traces out a sphere, the rectangle AEFB its circumscribing cylinder, and PQ a frustum of a cone whose axis is AB.

In the triangles QKP, CNR,

right $\angle$ QKP = right $\angle$ CNR, and $\angle$ QPK = complement of $\angle$ PRN
$= \angle$ CRN,

$\therefore$ the triangles QKP, CNR are similar,

$\therefore$ PQ : PK = CR : RN,

$\therefore$ PQ . RN = PK . CR = GH . GL,

$\therefore$ PQ . 2π . RN = GH . 2π . GL,

$\therefore$ surface of frustum traced by PQ

= lateral surface of cylinder traced by GH;

$\therefore$ when PQ is very small,

element of surface traced by arc PQ

= element of lateral surface of cylinder traced by GH.

Summing up all such elements formed by small arcs of the semicircle from A to B, we have

surface of sphere = lateral surface of circumscribing cylinder

$= 2r \cdot 2\pi r$

$= 4\pi r^2.$

COR. If PQ be a finite arc, we have, in the same way: the area of any zone of the sphere is equal to the area of the corresponding zone of the lateral surface of the circumscribing cylinder.

NUMERICAL EXERCISES.

1. Find the area of the surface of a sphere whose radius is 6 ins.

2. The area of the surface of a sphere is 732 sq. ins.; find the radius of the sphere.

3. The areas of three circles of a sphere of radius one foot are respectively, one-quarter, one-eighth and one-sixteenth, of the area of the surface of the sphere; find the distances of the intersecting planes from the centre of the sphere.

4. Find what fraction of the surface of a sphere 1 yard in diameter is visible from a point 20 feet from its centre.

5. Find the area of a polar ice-cap in square miles which extends $5°$ from the pole in all directions, taking the earth's radius to be 4000 miles, $\cos 5°$ to be $\cdot 996195$ and $\pi = 3\frac{1}{7}$.

6. If the sky be covered with a uniform canopy of cloud at a height of 2 miles; find in square miles the area of the cloud-surface visible to an observer at the sea-level. ($\pi = 3\frac{1}{7}$.)

Answers. **1.** 452·39 sq. ins. **2.** 7·63 ins. **3.** 0, 8·49, 10·39 ins. **4.** $\frac{37}{80}$. **5.** 382674 sq. miles. **6.** 50311 sq. miles.

EXERCISES.

1. Show how to divide the surface of a sphere into any given number of equal parts by drawing parallel planes.

2. Equal circular sections of a sphere cut off zones of equal areas, and conversely.

3. How much of the earth's surface may be seen by a person raised n radii above it?

4. Two right cones stand on a common circular base; show how to determine the radius of a sphere whose surface is equal to the sum of the surfaces of the cones.

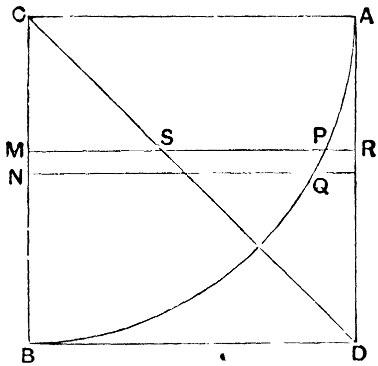
5. If the area of the spherical cap cut off by a small circle of a sphere be equal to the lateral area of the cone which stands on the circle and has its vertex at the centre, show that the height of the cap is equal to two-thirds of the radius of the sphere.

6. The area of the surface enclosed by the circle drawn with a pair of compasses is the same whether the circle be described on a plane or a sphere.

7. Two fixed concentric spheres cut off a zone of constant area from the surface of any sphere passing through their common centre.

8. There are two spheres of radii a and b respectively, and the distance between their centres is $c\,(c < a+b)$; find the area of that part of the surface of the first sphere which is included within the second.

90. PROP. 61. **To find the volume of a sphere.**



Let ABC be a quadrant of a circle of radius r, C its centre, CA, CB the bounding radii, and AD, BD the tangents at A, B, forming with the radii CA, CB, the square ACBD.

Let P, Q be two neighbouring points on the quadrant, PM, QN perpendiculars to CB; let MP cut CD in S and, when produced, meet AD in R.

Let the figure revolve about the radius CB; then the arc AB traces out a hemisphere, the square ACBD its circumscribing cylinder, and the triangle CBD a cone.

Since $$\angle MSC = \angle SCA = \angle SCM,$$

$$\therefore CM = MS,$$

$$\therefore MR^2 = CP^2 = CM^2 + MP^2 = MS^2 + MP^2,$$

$$\therefore \pi . MR^2 . MN = \pi . MS^2 . MN + \pi . MP^2 . MN,$$

∴ elemental slice of cylinder = elemental slice of cone + elemental slice of hemisphere,

∴ volume of cylinder = volume of cone + volume of hemisphere,

$$\therefore \text{volume of hemisphere} = \pi r^2 . r - \tfrac{1}{3}\pi r^2 . r = \tfrac{2}{3}\pi r^3,$$

$$\therefore \text{volume of sphere} = \tfrac{4}{3}\pi r^3.$$

Second Method. Let the radius CB be divided into n equal parts, each of length h, so that $CB = nh$; let $CM = mh$.

Since $PM^2 = CP^2 - CM^2$,

$$\therefore \pi . PM^2 . MN = \pi . CP^2 . MN - \pi . CM^2 . MN,$$

$\therefore$ volume of elemental slice of hemisphere resulting from the revolution of PMNQ

$$= \pi . CA^2 . MN - \pi . CM^2 . MN$$

$$= \pi r^2 . h - \pi . m^2 h^2 . h,$$

$\therefore$ volume of hemisphere

$$= \pi r^2 . r - \pi h^3 \left[0^2 + 1^2 + 2^2 + \ldots + (n-1)^2\right]$$

$$= \pi r^3 - \pi h^3 . \frac{(n-1) n (2n-1)}{6}$$

$$= \pi r^3 - \tfrac{1}{6}\pi h^3 (2n^3 - 3n^2 + n)$$

$$= \pi r^3 - \tfrac{1}{3}\pi r^3, \text{ since } nh = r, \text{ and } n^2h^3 \text{ and } nh^3 \text{ are very small when } h \text{ is indefinitely diminished,}$$

$$= \tfrac{2}{3}\pi r^3,$$

$$\therefore \text{volume of sphere} = \tfrac{4}{3}\pi r^3.$$

NUMERICAL EXERCISES.

1. Find the volume of a sphere whose radius is (i) 5 ins., (ii) 10 ins. ($\pi = 3{\cdot}1416$.)

2. Find the radius of a sphere whose volume is 22 c. ft. 792 c. ins. ($\pi = 3\frac{1}{7}$.)

3. A right circular cone whose altitude is 9 ins. and radius of base $3\sqrt{3}$ ins., has a sphere inscribed in it; prove that the volume of the part of the cone which is outside the sphere is 45π c. ins.

4. A leaden sphere, one inch in diameter, is beaten out into a circular sheet of uniform thickness of ·01 inch; find the radius of the sheet.

5. The radius of a hemisphere is 7 ins.; find the altitude of a right circular cone on the same base, having (i) the same curved surface, (ii) the same volume, as the hemisphere.

6. Find, to the nearest cubic inch, the volume of a spherical shell, whose internal radius is 9 ins. and thickness 1 in. ($\pi = 3\frac{1}{7}$.)

7. A spherical shell, 1 ft. in external diameter and 2 ins. thick, is made of metal having a mass of 504 lbs. per c. ft.; find, to the nearest pound, the mass of metal in the shell. ($\pi = 3{\cdot}1416$.)

Answers. **1.** 523·60 c. ins., 4188·80 c. ins. **2.** 21 ins. **4.** 4·08 ins. **5.** 12·12, 14 ins. **6.** 1136 c. ins. **7.** 186 lbs.

EXERCISES.

1. The volume of a sphere is two-thirds of the volume of its circumscribing cylinder.

2. If a sphere and a cube have equal surfaces, compare their volumes.

3. On the base of a hemisphere of radius r is constructed a cone whose volume is equal to that of the hemisphere; find the altitude of the cone.

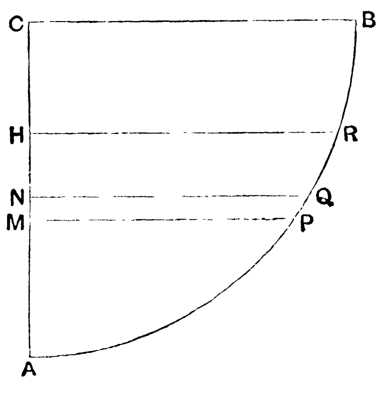
4. If the altitude of a right circular cone be equal to half the radius of its base, find the radius of the sphere whose volume is equal to that of the cone.

5. A cone is inscribed in a sphere, and its volume is one-quarter of that of the sphere; show that this can be done in two ways.

91. Def. 70. A **segment** of a sphere is the portion of it contained between two parallel planes.

If one of the planes be a tangent-plane to the sphere, the segment is called a **segment of one base.**

√**92.** Prop. 62. **To find the volume of a spherical segment of one base.**



Let r be the radius of the quadrant, and let $\mathsf{AH} = a$. Divide HA into n parts each equal to h, so that $a = nh$. Let MN be one of these parts, and let $\mathsf{AM} = mh$.

Since $$\mathsf{PM}^2 = \mathsf{CP}^2 - \mathsf{CM}^2 = \mathsf{CA}^2 - \mathsf{CM}^2,$$

$$\therefore \pi . \mathsf{PM}^2 . \mathsf{MN} = \pi . \mathsf{CA}^2 . \mathsf{MN} - \pi . \mathsf{CM}^2 . \mathsf{MN},$$

$\therefore$ volume of elemental slice of spherical segment resulting from the revolution of PMNQ

$$\begin{aligned}&= \pi . \mathsf{CA}^2 . \mathsf{MN} - \pi . \mathsf{CM}^2 . \mathsf{MN}\\ &= \pi r^2 . h - \pi (r - mh)^2 . h\\ &= 2\pi r h^2 m - \pi h^3 m^2;\end{aligned}$$

$\therefore$ volume of spherical segment

$$\begin{aligned}&= 2\pi r h^2 [0 + 1 + 2 + \ldots + (n-1)] - \pi h^3 [0^2 + 1^2 + 2^2 + \ldots + (n-1)^2]\\ &= 2\pi r h^3 \frac{(n-1)n}{2} - \pi h^3 . \frac{(n-1)n(2n-1)}{6}\\ &= \pi r (nh)^2 - \tfrac{1}{3}\pi (nh)^3, \text{ omitting very small quantities}\\ &= \pi r a^2 - \tfrac{1}{3}\pi a^3\\ &= \pi a^2 (r - \tfrac{1}{3}a).\end{aligned}$$

Cor. The volume of the spherical segment contained between planes at distances b and a from A is

$$\begin{aligned}&= \pi b^2 (r - \tfrac{1}{3}b) - \pi a^2 (r - \tfrac{1}{3}a)\\ &= \pi (b - a) [r (b + a) - \tfrac{1}{3}(b^2 + ab + a^2)]\end{aligned}$$

NUMERICAL EXERCISES.

1. The radius of a sphere is 15 ins.; find the volume of a spherical segment of one base whose height is 6 ins. ($\pi = 3{\cdot}1416$.)

2. The height of a segment of a sphere of 10 ins. radius is 3 ins.; find the ratio of the volume of the segment to that of the sphere.

3. A spherical ball, whose diameter is 1 ft., floats in water with its highest point 2 ins. above the surface; find the ratio of the volume immersed to the total volume of the ball.

4. If the curved surface of a segment of a sphere of radius 5 ins. is one-tenth of the surface of the sphere; find the volume of the segment. ($\pi = 3\frac{1}{7}$.)

Answers. **1.** 1470·27 c. ins. **2.** 243 : 4000. **3.** 25 : 27. **4.** $14\frac{2}{3}$ c. ins.

EXERCISES.

1. A sphere is cut by a plane, which divides the perpendicular diameter in parts which are as $n : 1$; show that the ratio of the volumes of the two segments of the sphere is $n^3 + 3n^2 : 1 + 3n$.

2. A spherical shell, whose outer and inner radii are a, b, respectively, is cut by a plane whose distance from the centre is c $(c < b)$; prove that the ratio of the volumes into which the shell is divided by the plane is

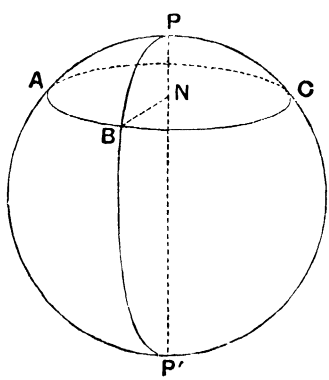
$$2(a^2 + ab + b^2) - 3c(a + b) : 2(a^2 + ab + b^2) + 3c(a + b).$$

3. A segment of a circle revolves about any diameter not intersecting it; prove that the volume generated by it is one-sixth of that of a circular cylinder whose radius is equal to the chord and whose altitude is equal to the projection of the chord on the diameter.

SPHERICAL TRIANGLES, ETC.

93. DEF. 71. The **distance between two points on the surface of a sphere** is the minor arc of the great circle passing through them.

✓94. PROP. 63. **All points on the circumference of a circle of a sphere are equally distant from its poles.**



Let A, B be any two points on the circumference of the circle ABC, P, P′ the poles of the circle; to prove that the points A, B are equally distant from P or P′.

Let PP′ meet the plane ABC in N; then N is the centre of the circle ABC;

$$\therefore AN^2 + NP^2 = BN^2 + NP^2,$$

$$\therefore AP^2 = BP^2,$$

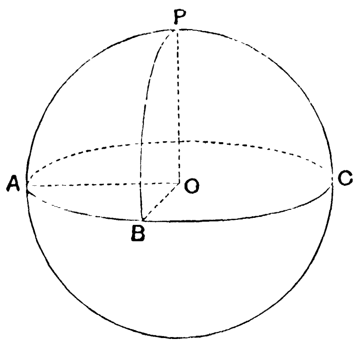
$$\therefore \text{chord } AP = \text{chord } BP,$$

$$\therefore \text{arc } AP = \text{arc } BP, \text{ and also arc } AP' = \text{arc } BP'.$$

COR. All points on the circumference of a great circle are at the distance of a quadrant from either pole.

95. DEF. 72. The **polar distance** of a circle of a sphere is the distance of any point on the circumference from the nearer pole, if a small circle, or from either pole, if a great circle.

√ **96.** PROP. 64. **If a point on the surface of a sphere lie at the distance of a quadrant from each of two other points (not the ends of a diameter), that point is the pole of the great circle passing through the two points.**



Let the point P on the surface of a sphere be at the distance of a quadrant from the two points A, B, which are not the ends of a diameter; to prove that P is the pole of the great circle ABC.

Since the arcs PA, PB are quadrants,

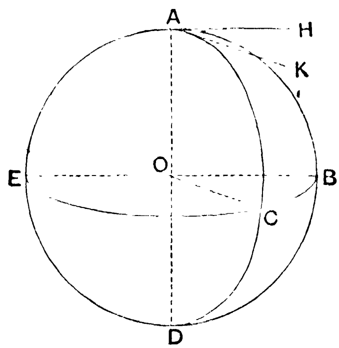
∴ PO is perpendicular to both OA, OB;

∴ PO is perpendicular to the plane ABC, or P is the pole of the great circle ABC.

97. DEF. 73. The **angle between two intersecting curves** is the angle between the tangents to the curves at the point of intersection.

DEF. 74. A **spherical angle** is the angle between two intersecting arcs of great circles.

√**98.** PROP. 65. **A spherical angle: (i) is equal to the dihedral angle between the planes of the great circles forming the angle; and (ii) is measured by an arc of a great circle of which the vertex of the angle is the pole, included between the planes of the great circles.**



Let ABD, ACD be the arcs of great circles forming the spherical angle A; AH, AK tangents to these circles at A; BCE the great circle of which A is the pole; to prove that the spherical angle A (i) is the dihedral angle between the planes of the great circles, and (ii) is measured by the arc BC of the circle BCE intercepted by the planes of the great circles.

(i) Since AH, AK are in the planes ABD, ACD, respectively, and are both perpendicular to the common section AD of the planes;

$\therefore$ the spherical angle A is the dihedral angle HAK between the planes.

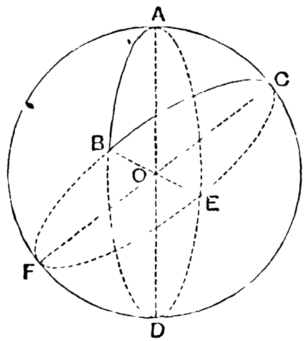
(ii) Since OB, OC are also in the planes ABD, ACD, respectively, and are both perpendicular to AD,

∴ angle BOC is also the dihedral angle between the planes;

∴ spherical angle A is equal to the angle BOC and is therefore measured by the arc BC.

99. Def. 75. A **spherical triangle** is the portion of the surface of a sphere bounded by three arcs of great circles.

The bounding arcs are called the **sides** of the spherical triangle. The **angles** of the spherical triangle are the spherical angles between adjacent sides.



For instance, if BCE, CAF, ABD be three great circles, ABC is one of the spherical triangles formed by their intersection; BC, CA, AB are its sides, and, being proportional to the angles BOC, COA, AOB, subtended by them at the centre of the sphere, are measured by these angles.

Thus, the parts of a spherical triangle are measured by the angles of the trihedral angle OABC which it subtends at the centre O of the sphere—the sides of the spherical triangle by the face angles of the trihedral angle, and the angles of the spherical triangle by the dihedral angles of the trihedral angle.

Def. 76. A spherical triangle is **equilateral** when its sides are all equal to one another, and **isosceles** when two of its sides are equal.

100. PROP. 66. **Any two sides of a spherical triangle are together greater than the third.**

Let BC, CA, AB (see figure of last article) be the sides of the spherical triangle ABC; to prove that arcs CA, AB are together greater than arc BC, etc.

Since OABC is a trihedral angle,

$$\therefore \angle COA + \angle AOB > \angle BOC,$$
$$\therefore \text{arc } CA + \text{arc } AB > \text{arc } BC.$$

101. DEF. 77. A **spherical polygon** is the portion of the surface of a sphere bounded by four or more arcs of great circles.

DEF. 78. A spherical polygon is **convex** when the corresponding polyhedral angle is convex.

102. PROP. 67. **The sum of the sides of a convex spherical polygon is less than four right angles.**

The sum of the face angles of the corresponding polyhedral angle is less than four right angles;

∴ the sum of the corresponding sides is less than four right angles.

EXERCISES.

1. If three sides of a spherical triangle be quadrants, the three angles are right angles.

2. Any side of a spherical polygon is less than the sum of the remaining sides.

3. **The angles at the base of an isosceles spherical triangle are equal.**

4. The greater angle of every spherical triangle has the greater side opposite to it.

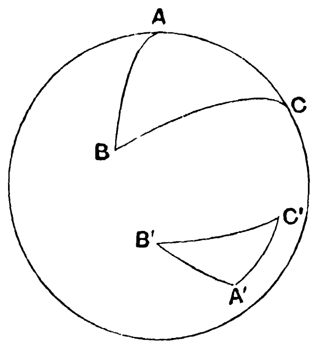
5. A cyclic spherical quadrilateral has the sum of one pair of opposite angles equal to the sum of the other pair.

6. **If the vertical angle of a spherical triangle be equal to the sum of the two angles at the base, the circumcentre of the triangle is at the middle point of the base, and the triangle formed by the chords of the sides of the triangle is right-angled.**

7. If OA, OB, OC are the edges of any convex trihedral angle, and OP is a line within it; prove that the angles AOB, AOC are together greater than the angles POB, POC.

103. DEF. 79. The **polar triangle** of a spherical triangle is that which has the poles of the sides of the given triangle for its vertices.

104. PROP. 68. **If one spherical triangle be the polar triangle of another, the second spherical triangle is also the polar triangle of the first.**



Let A′B′C′ be the polar triangle of the spherical triangle ABC; to prove that ABC is the polar triangle of A′B′C′.

Since B′ is the pole of AC,

$\therefore$ the arc AB′ is a quadrant.

Since C′ is the pole of AB,

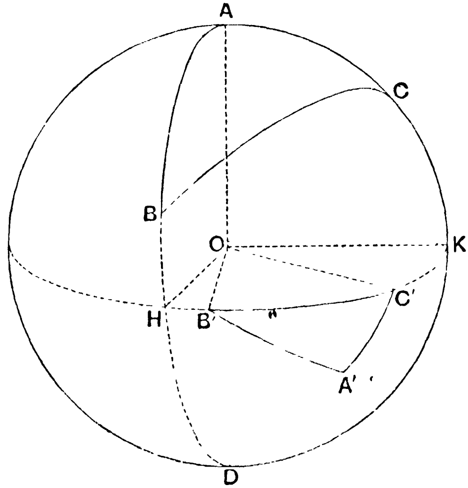
$\therefore$ the arc AC′ is a quadrant,

$\therefore$ A is the pole of B′C′.

Similarly, B is the pole of C′A′, and C of A′B′;

$\therefore$ ABC is the polar triangle of A′B′C′.

105. Prop. 69. **In two polar triangles, each angle of the one is the supplement of the opposite side of the other.**



Let ABC, A'B'C' be two polar triangles; to prove that A + a' = two right angles.

Let the great circles ABD, ACD cut B'C' in H, K, then

$$\angle \text{HOC}' + \angle \text{KOB}' = \text{two right angles},$$

$$\therefore \angle \text{HOK} + \angle \text{B}'\text{OC}' = \text{two right angles},$$

$$\therefore \text{A} + a' = \text{two right angles, etc.}$$

106. Prop. 70. **The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.**

Since A + a' = B + b' = C + c' = two right angles,

$$\therefore \text{A} + \text{B} + \text{C} = \text{six right angles} - (a' + b' + c').$$

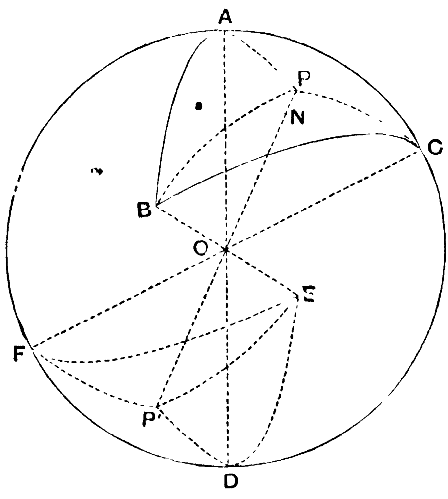
But $a' + b' + c'$ is less than four right angles and greater than zero;

$\therefore$ A + B + C is greater than two right angles and less than six right angles.

107. Def. 80. The excess of the sum of the angles of a spherical triangle over two right angles is called the **spherical excess** of the triangle.

Def. 81. One spherical triangle is said to be **symmetric** with another when the vertices of one triangle are antipodal to the vertices of the other.

108. Prop. 71. **A spherical triangle is equivalent to its symmetric spherical triangle.**



Let AD, BE, CF be diameters of the sphere, so that ABC, DEF are symmetric spherical triangles; to prove that the triangles ABC, DEF are equivalent.

Draw ON perpendicular to the plane ABC, and produce it to meet the sphere in P.

Then, $$OA^2 = OB^2 = OC^2,$$

$$\therefore AN^2 = BN^2 = CN^2,$$

$$\therefore AP^2 = BP^2 = CP^2,$$

$\therefore$ BPC, CPA, APB are isosceles spherical triangles.

Similarly, if PO produced meet the sphere in P′, it is perpendicular to the plane DEF, and EP′F, FP′D, DP′E are isosceles spherical triangles. Moreover, BPC, EP′F are symmetric spherical triangles, etc.

Now, if the spherical triangle BPC be applied to the triangle EP′F, so that P falls on P′, arc PB along arc P′E, then arc PC falls along the arc P′F, since ∠ BPC = ∠ EP′F; also B falls on E and C on F, since arc PB = arc P′E and arc PC = arc P′F;

∴ the triangle PBC coincides with the triangle P′EF.

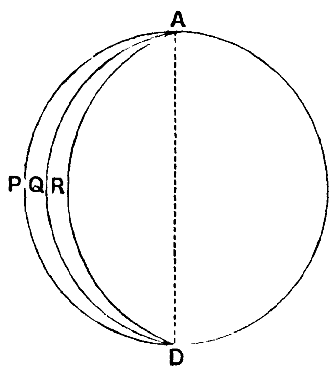
Similarly, the triangle PCA coincides with the triangle P′FD, and the triangle PAB with the triangle P′DE;

∴ the spherical triangle ABC is equivalent to its symmetric spherical triangle DEF.

109. Def. 82. The portion of the surface of a sphere included between two great circles is called a **lune.**

Def. 83. The **angle of a lune** is the angle between the great circles which bound it.

110. Prop. 72. **The area of a lune is to the area of the surface of the sphere as the angle of the lune is to four right angles.**



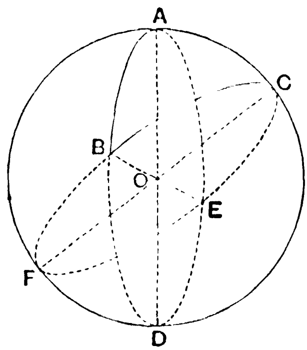
Let the lunes APDQ, AQDR,... have equal angles, then the lune APDQ may be made to coincide with the lune AQDR, etc.

Hence, if the angles of two lunes be commensurable, the areas of the lunes are to one another as their angles, and therefore the area of a lune is to the area of the surface of the sphere as the angle of the lune is to four right angles.

111. If the angle of a lune of a sphere of radius r contain A radians, the area of the lune

$$= \frac{A}{2\pi} \times 4\pi r^2$$
$$= 2Ar^2.$$

112. Prop. 73. **The area of a spherical triangle is measured by the spherical excess of the triangle.**



Let ABC be the spherical triangle, DEF its symmetric spherical triangle.

$$\text{Area of lune } ABDC = 2Ar^2,$$
$$\text{,, ,, } BAEC = 2Br^2,$$
$$\text{,, ,, } CAFB = 2Cr^2.$$

$$\text{But the lune } BAEC = \triangle ABC + \triangle AEC$$
$$= \triangle ABC + \triangle DBF.$$

$\therefore$ sum of areas of lunes ABDC, BAEC, CAFB

$$= \text{area of hemisphere} + 2\triangle ABC,$$

$$\therefore \text{ twice area of triangle } ABC = 2Ar^2 + 2Br^2 + 2Cr^2 - 2\pi r^2,$$

$$\therefore \text{ area of } \triangle ABC = (A + B + C - \pi)\, r^2.$$

EXERCISES.

1. What are the conditions that a spherical triangle may be self-polar?

2. If ABC be a spherical triangle, and A′B′C′ its polar triangle, and if the sides B′A′, B′C′ (produced, if necessary) cut the side AC at D and E respectively, prove that $AD = CE$.

3. **If two angles of a spherical triangle be equal, the triangle is isosceles.**

4. The greater side of every spherical triangle has the greater angle opposite to it.

5. Find the number of square miles in a triangle on the earth's surface of which the spherical excess is $1''$, taking the radius of the earth as 3963 miles. ($\pi = 3\frac{1}{7}$.) [76·2 sq. miles.

6. The exterior angle of a spherical triangle is less than the sum of the two interior opposite angles.

7. **The sum of the exterior angles of a convex spherical polygon is less than four right angles.**

8. The sum of the interior angles of a spherical pentagon is less than ten, and greater than six, right angles.

9. The area of a convex spherical polygon is $(2\pi - \sigma)\, r^2$, where σ is the sum of the exterior angles and r is the radius of the sphere.

10. What relation connects the area of a spherical triangle with the sum of the sides of its polar triangle?

MISCELLANEOUS EXERCISES.

1. The sum of the squares on three concurrent chords of a sphere at right angles to one another is constant.

2. A sphere is inscribed in a cube, and a plane passing through the other extremities of three concurrent edges cuts the sphere in a circle about which a square is described; show that the area of the square is two-thirds of the area of a face of the cube.

3. Spheres drawn through a fixed point to cut a fixed sphere at right angles have their centres on a fixed plane.

4. A cube and a regular octahedron have a common circumscribing sphere; show that their surfaces are in the same ratio as their volumes.

5. From the circumcentre of the triangle ABC, a perpendicular to its plane is drawn of length equal to the side of the square inscribed in that circle; show that the radius of the sphere which passes through A, B, C and the extremity of the perpendicular is three-quarters of the perpendicular.

6. **The inverse of a sphere with respect to a point on its surface is a plane perpendicular to the diameter through the point of inversion, and with respect to any other point is another sphere.**

7. Circles are drawn on a sphere so as to cut a given small circle at right angles; prove that their planes pass through a fixed point.

8. A sphere cuts two fixed planes at right angles to one another in circles the sum of whose areas is less by a given amount than half the area of the surface of the sphere; prove that the centre of the sphere lies on a fixed circular cylinder.

9. Show that the cylinder inscribed in a sphere which has its lateral surface a maximum has its altitude equal to its diameter.

10. If the base of a right circular cone be a circle of a sphere, the cone will intersect the sphere in another circle.

11. A cylindrical hole is bored through the centre of a sphere; prove that the volume remaining is equal to that of a sphere whose diameter is the length of the hole.

12. Prove that all spheres cutting two given non-intersecting spheres at right angles have their centres on a fixed plane and pass through two fixed points.

13. Prove that a sphere can be described through the middle points of the six edges of a tetrahedron, having its centre at the centroid of the tetrahedron.

14. If the opposite edges of a tetrahedron be equal, the inscribed sphere touches the faces at their circumcentre.

15. If a sphere touch the six edges of a tetrahedron, the sum of one pair of opposite edges is equal to the sum of either of the other pairs of opposite edges.

16. If the four sides of a skew quadrilateral touch a sphere, the points of contact are concyclic.

17. If a sphere touch the six edges of a tetrahedron, the three lines joining the points of contact of opposite edges are concurrent.

18. A given tetrahedron moves so that three of its faces touch respectively three concentric spheres; prove that the fourth face also touches a concentric sphere.

19. If a sphere be drawn touching three given spheres, the plane through the points of contact will pass through one of four fixed straight lines.

20. If O be any point within a tetrahedron ABCD, the sum of the three angles BOC, COD, DOB is greater than the sum of the angles BAC, CAD, DAB.

21. Within the area of a given triangle is described a triangle the sides of which are parallel to those of the given one; prove that the sum of the angles subtended by the sides of the interior triangle at any point not in the plane of the triangles, is less than the sum of the angles subtended at the same point by the sides of the exterior triangle.

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