THE BOOK WAS DRENCHED

THE QUADRILATERAL

AN INVESTIGATION
OF
ITS CHIEF PROPERTIES
AND A SYNOPSIS
OF
OLD THEOREMS AND PROBLEMS

BY
RICHARD SMYTH, M.A. (T.C.D.)

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"I am a transmitter, not a maker"

CONFUCIUS.

FOREWORD

THIS monograph on Quadrilaterals claims little originality, except in its mode of presentation of the subject, and the fact that it is issued without a single diagram. This, it is hoped, will be regarded by its readers as a compliment to their intelligence.

Trigonometrical formulæ are freely utilized, and it is assumed that its readers are acquainted with the principles of Inversion, Polars, and Coaxal Circles.

Symbols and contractions, used for brevity, are mainly those commonly employed: Perpendicular $\perp$, Circle $\odot$, Parallel $\|$ and Parallelogram $\|$m; also such obvious contractions as max. = maximum, ext. = external, pt. = point, etc. The abbreviation Quad is used throughout for Quadrilateral.

In the absence of diagrams, uniformity of structure and lettering is essential. It should therefore be noted that the Quad mostly referred to is the plane figure bounded by 4 rt. lines of unequal length—$a > d > c > b$—and lettered from left to right: the side AB, (a), being horizontal.

The Complete Quad: the plane figure formed by the mutual intersections of 4 rt. lines, which has 6 vertices—2 ext. vertices being the junctions of opposite sides—and a 3rd diagonal (the linear link of these), is referred to frequently. The allied plane figure—the Tetragon—is dealt with very briefly.

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THE QUADRILATERAL

I

TRANSFORMATIONS

(1) Any convex Polygon of more than four sides is reducible to a Quad in which 2 sides and 2 angles of the Polygon are retained.

Take for example the Heptagon ABCDEFG. Draw from F a ‖ to GE, meeting AG (produced) in X. Join XE, XD. Draw from E a ‖ to DX, meeting AG in Y. Join YD, YC. Draw from D a ‖ to CY, meeting AG in Z. Then the Quad ABCZ = the Heptagon.

For, having the same base and altitude, △ EFG = EXG and △ DEX = DYX and △ CDY = CZY ∴ ABCDEFG = ABCDEX = ABCDY = ABCZ.

(2) A Quad is reducible to a Triangle which retains one side of the Quad and an adjacent angle.

Draw through the vertex C, of the Quad ABCD, a rt. line ‖ to BD, meeting AD (produced) in K. Join BK. Then, as △ BCD = BDK, Q, the area of the Quad, = △ ABK, which retains ∠A and side AB.

(3) A Quad is reducible to a Triangle the lengths of whose sides are those of the diagonals.

Let the Quad be ABCD on base AB. Draw (below the base) a rt. line, BE, ‖ to and = AC, and join ED. Then Q = △ DBE, whose sides, BE (= AC = m) and BD (= n), are inclined at the interdiagonal angle, θ; and whose area = $\frac{1}{2}mn.\sin\theta$.

The equality is made evident by drawing the ‖m BEST, whose vertices, S, T, are the junctions of EA, BC, with the rt. line drawn through D ‖ to BE. For, obviously, (ABCD) = $\frac{1}{2}$(BEST) = △ DBE.

(4) A Quad may be transformed into a Polygon of any given species.

Let the spec' s be that of the Regular Hexagon. Construct on base AB of the Quad a regular hexagon, H. Reduce H, Q, to the $\triangle$s S, T. Let the Mean Proportional to base and altitude of S be h, and of T be q. (Then $h^2 : q^2 = \text{H} : \text{Q}$.) On AB find the point R, making AR = 4th proportional to h, q, AB. Finally, erect on AR a regular hexagon, P. Area of this = Q. For, from the law of similar figures, $\text{H} : \text{P} = \text{AB}^2 : \text{AR}^2 = h^2 : q^2$.

Cor. If in the above (3) a rt. line EF be drawn $\parallel$ to and = BD, the $\parallel$m DBEF, whose sides are the diagonals, is a graphic representation of the Quad ABCD. Its area = 2Q; the vertex A lies within it, and the linear links of A with the vertices B, E, F, D, of the $\parallel$m are the sides a, b, c, d, inclined to each other as in the Quad. And by means of this transformation, the problem: Given m, n, θ, and 2 opposite angles of a Quad, construct it, is readily solved.

II

AREA ESTIMATIONS

(1) In terms of the Diagonals, m, n, and Interdiagonal Angle, θ (*vide* I, 3)

$$\text{Q} = \tfrac{1}{2}mn.\sin\theta.$$

(2) To express Q in terms of the Sides and Diagonals; let DH be the altitude of the Equivalent Triangle DBE, and BX, DY be $\perp$s from B, D, on AC(= m); and let the sides AB, BC, CD, DA be respectively a, b, c, d.

Since $a^2 - b^2 = (\text{AX}^2 - \text{CX}^2) = \text{AC}.(\text{AX} - \text{CX})$, and $d^2 - c^2 = \text{AC}.(\text{AY} - \text{CY})$, we infer that $a^2 - b^2 + c^2 - d^2 = 2\text{XY}.\text{AC} = 2m.\text{XY}$. But XY is $\parallel$ and = BH, and $\text{BH} = \text{BD}.\cos\theta = n.\cos\theta$. $\therefore$ $2mn.\cos\theta = a^2 - b + c^2 - d^2$. Let this be termed 2V. Then, as $mn.\cos\theta = \text{V}$, and $mn.\sin\theta = 2\text{Q}$, evidently (squaring and adding) $m^2n^2 = 4\text{Q}^2 + \text{V}^2$. Hence the required formula:

$$16\text{Q}^2 = (2mn)^2 - (a^2 - b^2 + c^2 - d^2)^2.$$

(3) The area may be expressed as a function of the Sides and the sum (2σ) of a pair of Opposite Angles (e.g. B and D).

Equating the values of m^2 in $\triangle$s ACB, ACD, we deduce that $ab.\cos B - cd.\cos D = \frac{1}{2}(a^2 + b^2) - \frac{1}{2}(c^2 + d^2)$. Let this be W, and let $ab.\sin B + cd.\sin D$, which is twice the Area, be termed U. Adding $U^2 + W^2 (= a^2b^2 + c^2d^2 - 2abcd.\cos 2\sigma)$ to $V^2 - W^2$, which $= (V + W)$. $(V - W) = (a^2 - d^2).(c^2 - b^2) = (ac + bd)^2 - 2abcd - (a^2b^2 + c^2d^2)$; $U^2 + V^2 = (ac + bd)^2 - 4abcd.\cos^2 \sigma$. But $(2Q)^2 + V^2 = (mn)^2$; and substituting this new value of $(mn)^2$ in the foregoing estimate of the Area, (2), it becomes:

$$16Q^2 = 4(ac + bd)^2 - (a^2 - b^2 + c^2 - d^2)^2 \\ - 16abcd.\cos^2 \sigma.$$

Hence, finally, factorizing and putting s for the semi-perimeter,

$$Q = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd.\cos^2 \sigma}.$$

Cor. When A, B, C, D, are concyclic, as $\cos \sigma = o$, $mn = ac + bd$ (Ptolemy). Also, if S be the area of a Cyclic Quad, $S = \sqrt{(s - a . s - b . s - c . s - d)}$; and the general formula becomes: $Q = \sqrt{(S^2 - abcd.\cos^2 \sigma)}$.

(2) If $mn = Z$, then U, V, Z, are proportional to the sides and hypotenuse of a rt. angled triangle: U/V being the tangent of its acute angle, θ.

(3) When area and sides of the Quad ABCD are given, $\angle\sigma$ is known. Also the angles A and B (and, hence, the Quad) may be constructed. For we are given (*vide supra*) $ab.\sin B + cd.\sin D$, and $ab.\cos B - cd.\cos D$. Let the value of the former be v, and of the latter w. Then, as we are implicitly given $v.\sin B + w.\cos B$, which $= ab - cd.\cos 2\sigma = k$, we can find sin B. (And from a similar equation we can find sin A.) Or, geometrically, B may be found by the following method:

On a rt. line cut off segments OX, OY, OZ, whose lengths are v, w, k units, respectively. Place OY at rt. angles to OX, and inflect in the ⊙ circumscribed about (XOY) the chord OZ. Then the angle ZOY = B.

For, if $\phi = \angle$ZOY, and x, y be the angles XYO, YXO, the angle subtended by OZ at X $= \phi + y$. Hence OZ$(= k) =$ XY. $\sin(\phi + y) =$ XY.$(\sin\phi.\cos y + \cos\phi.\sin y)$

$$= \text{XY}.(\sin\phi.\sin x + \cos\phi.\sin y).$$

But XY.$\sin x =$ OX, and XY.$\sin y =$ OY. $\therefore k = v.\sin\phi + w.\cos\phi$. Hence $\phi =$ B.

III

THE TETRAGON

THE figure ABCD, regarded as a system of points connected by right lines, is a Tetragon; and the linear links AC and BD are tetragon sides. The Complete Tetragon, having for vertices the 4 pts., has 3 Summits, or Diagonal Points, the junctions of opposite "sides": G $= mn$, E $= ac$, F $= bd$. And EFG is its Diagonal Triangle.

MEDIANS

(1) The medians—links of the centres of opposite sides—of a tetragon are concurrent; and the summation (Σ) of their squares $= \frac{1}{4}\Sigma$ squares of the sides.

Let the midpts. of the sides AB and CD, BC and AD, CA and BD, be H and J, K and L, M and N; and lengths of corresponding medians be u, v, w.

Drawing the figure, we see that v, w; w, u; and u, v, are the pairs of diagonals of the ||ms KMLN, NJMH and HLJK. These, bisecting each other, meet in a common pt., O. Again, if the pairs of sides be a, c; b, d; m, n; applying a known theorem: In a $\triangle$ whose sides are x, y, the base z, and its median r; $4(x^2 + y^2) = 2z^2 + 2(2r)^2$—to each of the $\triangle$s KJH, KMN and JMH, we deduce that $m^2 + n^2 = 2(u^2 + v^2)$, and that $a^2 + c^2 = 2(v^2 + w^2)$, and $b^2 + d^2 = 2(w^2 + u^2)$. $\therefore \Sigma a^2 = 4\Sigma u^2$. Also $a^2 + b^2 + c^2 + d^2 - m^2 - n^2 = 4w^2$.

(2) If $\widehat{ac}$ be the acute inclination of a to c, etc., prove that, in the complete figure, $a^2 - b^2 + c^2 - d^2 = \pm 2mn.\cos\widehat{mn} = 2(ac.\cos\widehat{ac} - bd.\cos\widehat{bd})$; and also that

$$(a^2 + b^2 + c^2) - 2(ab.\cos\widehat{ab} - bc.\cos\widehat{bc} - ac.\cos\widehat{ac}) = d^2.$$

To the $\triangle$s JOK, KOM, MOH apply the theorem : In a $\triangle$ with sides x and y and base and its median, z and r; $x^2 - y^2 = 2zr.\cos\widehat{zr}$. In the $\triangle$ JOK—whose base, JK, is $\frac{1}{2}n$ and its median, $\frac{1}{2}$HK, is $\frac{1}{4}m$—evidently

$2(u^2 - v^2) = 2mn.\cos\widehat{mn}$. And similarly we see that $2(v^2 - w^2) = 2ac.\cos\widehat{ac}$, and $2(u^2 - w^2) = 2bd.\cos\widehat{bd}$.

Thus, since $(a^2 + c^2) - (b^2 + d^2) = 2(v^2 - u^2)$,

and m^2 and n^2 are, respectively, $a^2 + b^2 - 2ab.\cos\widehat{ab}$, and $b^2 + c^2 - 2bc.\cos\widehat{bc}$, the 2 identities are readily proved.

(3) The midpoints of the linear links AC, BD, EF, are collinear.

Let these midpts. be M, N, T; and cut off from EA and ED, the segments EP $= a$, and EH $= c$. Also from FA and FB, cut off F$p = d$ and F$h = b$. Since $\perp$s drawn from T on a and c are the halves of those drawn to them from F, we deduce that, if Apex-Base denote $\wedge$, Ta $-$ Tc $= \frac{1}{2}$(Fa $-$ Fc) $= \frac{1}{2}$Q $=$ Ma $+$ Mc $=$ Na $+$ Nc $= \triangle$ EPH added to either $\triangle$ TPH or MPH or NPH. The latter are $\therefore$ 3 equal $\triangle$s; and, as they are on the same side of a common base, MNT is a rt. line $\parallel$ to PH.

(4) In the Complete Tetragon the triangles whose vertical angles are E and F; the containing sides of the former being segments of EA and ED which respectively $= a$ and c, and of the latter parts of FA and FB which $= d$ and b, have equal and parallel bases.

As the $\triangle$s Fph and EPH may be interchanged in the above demonstration (3), ph, too, is $\parallel$ to MN: and we infer from identity (2) that $a + c^2 - 2ac.\cos$ E, or $(\mathrm{PH})^2$, $= b^2 + d^2 - 2bd.\cos$ F $= (ph)^2$.

It is also evident in the diagram of The Medians, where $\triangle$s KMN and JMN on common base, w, are similar to, and have sides that are halves of the sides of $\triangle$s EPH and Fph. Hence PH $= ph = 2w$.

Cor. $mn.\cos 0 = ac.\cos \mathrm{E} - bd.\cos \mathrm{F}$.

TETRAGONS and QUADS are RECIPROCAL FIGURES (*vide* IX)

IV

THE INSCRIBABLE (CYCLIC) QUAD

HERE, as $\cos\sigma = o$, $Q = S$. The angles of the Quad are known when the sides are given; for, since

$$(a^2 + b^2) - (c^2 + d^2) = 2(ab + cd).\cos B,$$

the values of $I \pm \cos B$ give

$$(ab + cd).\cos^2 \tfrac{1}{2}B = (s - c).(s - d)$$

and $(ab + cd).\sin^2 \frac{1}{2}B = (s - a).(s - b)$. Hence, also, Q, being $\frac{1}{2}(ab + cd).\sin B$, is the sq. root of $(s-a).(s-b).(s-c).(s-d)$.

(1) The radius, R, of the Circle, and the diagonals, m, n, of the Quad, may be expressed in terms of the sides.

Applying the formula for a △ of sides a, b, c, and area S, in its form $4S.R = abc$ (which may also be written $\sqrt{(bc)}.\sqrt{ca}.\sqrt{ab} = 4SR$) to each of the constituent △s whose sides are m, a, b; m, c, d, etc., we deduce the corresponding (alternatively written) formula for the Quad:

$$4QR = \text{sq. root of } (bc + ad).(ca + bd).(ab + cd).$$

For, if P_1, P_2, P_3, denote the couplets $(bc + ad)$, $(ca + bd)$, $(ab + cd)$; $4QR = mP_3 = nP_1 = \sqrt{(mnP_1P_3)}$. But $mn = P_2$ (PTOLEMY); hence the deduction.

Also it is evident that $m^2 = P_1P_2/P_3$ and $n^2 = P_2P_3/P_1$.

(2) The perpendiculars u, x, v, y, drawn from the midpts. U, X, V, Y of the sides, a, b, c and d, of a Cyclic Quad, to the opposite sides, are concurrent.

Let u meet v in Z. It may be proved that XZ is ⊥ to d (and for similar reasons, that YZ is ⊥ to b). For, as ⊥s to the sides at the midpts. meet in O, the circumcentre, UOVZ, is a ||m; so is UXVY. Let their diagonals (which bisect each other) meet in K. Then, as KO = KZ and KX = KY, XZYO is a ||m. ∴ XZ is || to OY. Hence, like OY, it is ⊥ to AD.

(3) The products of the perpendiculars from any pt., P, on the circumference of a circle to the pairs of opposite "sides" of an inscribed Tetragon are equal.

Let PS be a diameter, and let p, q; r, s; and t, w be the feet of ⊥s from P on the opposite sides AB, CD; CA, BD; BC, AD. Then, joining the vertices to P and to S, the similar △s PpB, PAS (e.g.) show that 2R.Pp =

PA.PB, and the similar △s PqC, PDS, that 2R.Pq = PC.PD. Hence PA.PB.PC.PD/4R^2—a constant = Pp.Pq = Pr.Ps = Pt.Pw.

(4) If a rt. line cut 3 of the sides of a variable Quad inscribed in a given ⊙ in fixed pts., it cuts the 4th side in a fixed pt.

ABCD is one position of the Quad. Let the transversal, T, of which the fixed pts. F, G, H, are on BC, CD, DA, meet AB produced in pt. X. Draw from A the chord AZ ∥ to T, and let CZ cut T at Y. Now C, D, A, Z being concyclic points and T being ∥ to AZ, angle GDH = GYC = Z ∴ the pts. C, D, H, Y are concyclic. Hence GD.GC = GH.GY. ∴ Y is a fixed pt. Also, since ∠D (or its supplement) = both ∠XBC and ∠XYC; XBYC is a cyclic Quad. Hence FX.FY = FB.FC and, Y being fixed, X is a fixed pt.

(5) Inscribe in a given circle a Quad whose sides shall pass each through a fixed pt.: the 4 pts. being within the circle.

Let L, M, N, S be the pts. and AB, BC, CD, DA the sides of the Quad on which they respectively lie. If o be centre and r the radius of the ⊙, let the link LM be produced to X so that LM.MX = $r^2 - o\text{M}^2$. Similarly XN is produced to Y, making XN.NY = $r^2 - o\text{N}^2$, and YS is produced to Z, making YS.SZ = $r^2 - o\text{S}^2$. Then the rt. line ZK, inclined to YZ at ∠YZK which = supplement of ∠LXY, passes through vertex A of Quad.

Construction. Complete the chord AS—meeting ⊙ again in D; and let the ⊙ again meet DN in C, and CM in B. Then pts. B, L, A are collinear; for (*vide infra*) angle CBL = 180°–D. ABCD is ∴ the required Quad.

(LBXC, XCYD and YDZA are obviously cyclic Quads. Let ∠CBL = B, and YXC = x, CDA = D, and YZA = z. Then, since B = LXY + x and z = D + x; B + D = z + LXY = 180°.)

(6) In any given ⊙, the Inscribed Quad of max. perimeter and area is the Square.

This follows from the fact that of all inscribed △s whose base is a fixed chord, the Isosceles has max. perimeter and area. For suppose all vertices but one to remain fixed, the perimeter and area of a variable △

on a fixed base becomes max. when the 2 sides meeting in this point are equal. The max. inscribed Quad $\therefore$ is that one whose sides subtend equal arcs.

V

THE CIRCUMSCRIBABLE QUAD

HERE the semiperimeter, s, is $a + c = b + d$. Hence

$$(s - a).(s - b).(s - c).(s - d) = abcd,$$

and $\therefore$ the general formula becomes: $Q = \sqrt{abcd}.\ \sin\sigma$. Let $k = \sqrt{(abcd)}$. Then, as $rs = Q$,

$$r = k.\sin\tfrac{1}{2}(A + C)/(a + c) = k.\sin\tfrac{1}{2}(B + D)/(b + d).$$

(1) $\sqrt{(ab)}.\sin\tfrac{1}{2}\widehat{ab} = \sqrt{(cd)}.\sin\tfrac{1}{2}\widehat{cd}$.

This important relation between adjacent sides and included angle is deduced from the identities: $(a - b)^2 = (c - d)^2$, and $a^2 + b^2 - 2ab.\cos B = c^2 + d^2 - 2cd.\cos D$. Hence $ab.\sin^2\tfrac{1}{2}B = cd.\sin^2\tfrac{1}{2}D$.

(2) If radial lines—passing through O, the In. centre—cut a, b, c, d, in G, R, S, T, and make with AO, BO, CO, DO, angles which, respectively, are $\tfrac{1}{2}C$, $\tfrac{1}{2}D$, $\tfrac{1}{2}A$, $\tfrac{1}{2}B$; these pts. divide each side in the ratio of adjoining sides.

Since $\angle AOG = \tfrac{1}{2}C$, the $\angle BOG = \tfrac{1}{2}D$; for sum of the $\angle$s of $\triangle AOB = 180° = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(C + D)$. Hence $GA : GO = \sin\tfrac{1}{2}C : \sin\tfrac{1}{2}A = \sin\tfrac{1}{2}\widehat{bc} : \sin\tfrac{1}{2}\widehat{ad} = \sqrt{(ad/bc)} = k/bc$. Also $GB : GO = \sin\tfrac{1}{2}\widehat{cd} : \sin\tfrac{1}{2}\widehat{ab} = k/cd$. $\therefore$ $GA : GB = d : b$. And $OG = OR = OS = OT = k/s$.

(3) Given the 4 sides of a Circumscribable Quad in magnitude, and one of them in position; find the Locus of the In. centre.

Let a be the fixed side. Find pt. G, dividing it in the ratio $d : b$. Then, as $GO = \sqrt{abcd}/s$, the Locus is a Circle. (Centre G, Radius GO.)

(4) The mid-diagonal line, X, of Circumscribable Quads passes through the In. centre, O; and O divides its segment between b and d in ratio $d : b$.

As the sum of the areas of $\triangle$s OAB, OCD $= \tfrac{1}{2}Q = (MAB) + (MCD) = (NAB) + (NCD)$, O is a pt. on X.

Again, if X cut b and d in S and H; the $\perp$s on X from B and D being equal, (ODH) : (OBS) = OH : OS = (OAH) : (OCS) ∴ also = (ODH) + (OAH) : (OBS) + (OCS) = (AOD) : (BOC) = AD : BC.

(5) In a given ⊙ the Circumscribed Quad of minimum perimeter and area is the Square.

Let AB be a side of any such Quad, touching the ⊙, centre, O, at P. And let X and Y denote the rt. lines OA and OB produced. When the intercept, AB, is bisected by P, the △ AOB has minimum area. (It is obvious that any other intercept through P is $>$AB, and that AOB is the min. intercepted △ whose base passes through P.) For, let another tangent, with contact pt. V, cut X, Y, in S, T; and let an intercept, st, be drawn through P and ∥ to ST. The △ TOS is $>$ △ tOs ∴ is $>$ △AOB. And, as (TOS) and (AOB) have the same altitude, AB is min. The Square, having each side bisected by its pt. of contact, is ∴ the Tangent-Quad of min. perimeter and area.

VI

THE CYCLIC-CIRCUMSCRIBABLE QUAD

In this, combining the two varieties, $Q = \sqrt{(abcd)}$; $\sigma = 90°$ and $s = a + c = b + d$.

(1) If a Quad ABCD may be inscribed in one circle, Y, while circumscribed about another circle, X, its contact pts. with the latter—K, L, M, N—form a Quad whose diagonals meet at ∠90° on the line of centres (Oo) in a pt., I, which is the diagonal-junction of ABCD.

Since both Quads are Cyclic, the 4 diagonals meet in I, the Pole of the coincident 3rd diagonals (X.19). Let O be centre of Y and o the centre of X, and H the foot of $\perp$ from O—passing through o and I—to the 3rd diagonal. Now the ∠MIN is 90°, for the ∠s N, M of △ MIN are, respectively, the ∠s M, N of the △s CLM, AKN, which are complements of $\frac{1}{2}$C and $\frac{1}{2}$A ∴ ∠MIN = 180° − $\frac{1}{2}$(A + C) = 90°.

The midpts. of KLMN are concyclic; and the circle, Z, passing through them, is the Inverse of Y with respect to X.

D being Pole of MN, oD intersects the latter in its midpt., V. Let r be the radius of X, and R the radius of Y; then, the $\angle$ MIN being 90°, VI = VN $\therefore$ $o\mathrm{N}^2 = r^2 = o\mathrm{V}^2 + \mathrm{VI}^2$. Hence, if G be midpt. of oI, we deduce that $r^2 = 2\mathrm{G}o^2 + 2\mathrm{GV}^2$, and $\therefore$ that GV = a constant. And, as the above reasoning applies equally to any other side of KLMN, each of its sides is bisected by the circle, Z, whose centre is G and radius (ϱ) is GV. And this circle is the Inverse of Y; since $r^2 = o\mathrm{V}.o\mathrm{D}$.

The Three Circles, X, Y, Z, are Coaxal.

Let Oo meet the 3rd diagonal in H; and let IG = g and IH = h. Since $r^2 = o\mathrm{H}.o\mathrm{I} = 2g(2g + h)$, and, as r^2 also $= 2g^2 + 2\varrho^2$, then $\varrho^2 = g(g + h) = \mathrm{GI.GH}$. Hence, finally, as $\mathrm{OH.OI} = \mathrm{R}^2$ and $o\mathrm{H}.o\mathrm{I} = r^2$ and $\mathrm{GH.GI} = \varrho^2$, we conclude that HI—a segment of the line of centres—is diameter of a circle which cuts at 90° each of the ⊙s X, Y, Z. They are $\therefore$ coaxal circles (whose Limiting Points are I and H).

(2) When a Quad inscribed in one circle has for sides tangents to another circle, prove the Relation of the intercentre segment, w, to the radii:

$$^1/(\mathrm{R} + w)^2 + {}^1/(\mathrm{R} - w)^2 = {}^1/r^2$$

(which, if written $x^2 = y^2 + z^2$, resembles the Relation in $\triangle$s: $x = y + z$).

In Circle, centre, O, radius, R, inscribe a $\triangle$ ABC. Draw diameter, NS, ⊥ to the base, AC. Join D, on arc ASC, to A and C. Obviously BS and DN, the bisectors of $\angle$s B and D, meet in o, the In. centre; and, as $\frac{1}{2}$D is complement of $\frac{1}{2}$B, $(r/o\mathrm{B})^2 + (r/o\mathrm{D})^2 = 1$. Hence $^1/r^2$ = the sum of the reciprocals of the squares of oB and oD. But $o\mathrm{B}.o\mathrm{S} = o\mathrm{D}.o\mathrm{N} = \mathrm{R}^2 - w^2 = k$. Hence $o\mathrm{S}^2 + o\mathrm{N}^2$, or $2\mathrm{R}^2 + 2w^2 = k^2/r^2 = (\mathrm{R} + w)^2.(\mathrm{R} - w)^2/r^2$.

Another proof is derived from the identity (a) $r^2 = 2g^2 + 2\varrho^2$, where g is ½ of the linear link of o with I, the diagonal junction; and the fact that the ⊙ Z, of centre G and radius ϱ, is the Inverse of ⊙ Y, of radius R: r being the radius of Circle of Inversion. For, from the Law of Inverse Circles,

$$g/w = \varrho/\mathrm{R} = r^2/(\mathrm{R}^2 - w^2) = \text{a constant, K.}$$

Hence, from (a), $2\mathrm{K}^2(\mathrm{R}^2 + w^2) = r^2 = \mathrm{K}(\mathrm{R}^2 - w^2)$

$$\therefore (\mathrm{R} + w)^2 + (\mathrm{R} - w)^2 = (\mathrm{R}^2 - w^2)^2/r^2.$$

COR. The triangle formula ($x = y + z$) may be derived from the above figure. D coinciding with C, the Quad becomes $\triangle$ ABC. The rt. lines So and SC are equal, as $\angle o\mathrm{CS} = \mathrm{C}o\mathrm{S} = \frac{1}{2}(\mathrm{B} + \mathrm{C})$. Hence $o\mathrm{B}.o\mathrm{S}$, or $\mathrm{R}^2 - w^2 = o\mathrm{B}.\mathrm{CS} = 2\mathrm{R}r$ $\therefore$ $^1/(\mathrm{R} + w) + {}^1/(\mathrm{R} - w) = {}^1/r$.

(3) The interdiagonal angle of the Cyclic-circumscribable Quad, and the radii of inscribed and circumscribed circles, are functions of the sides.

$\mathrm{Q} = \mathrm{K} = \sqrt{(abcd)} = \frac{1}{2}mn.\sin\theta$; and $(4\mathrm{QR})^2 = \mathrm{P}_1\mathrm{P}_2\mathrm{P}_3$, $\therefore$
$\theta = \mathrm{Sin}^{-1}\ 2\mathrm{K}/(ac + bd)$
$r = \mathrm{K}/s = \sqrt{abcd/(a + c)(b + d)}$
$\mathrm{R} = \frac{1}{4}\sqrt{bc + ad)(ca + bd)(ab + cd)/abcd}$.

(4) When a circumscribed Quad has concyclic vertices, the In. circle pt. of contact divides each side in the ratio of the adjacent sides.

If G be the AB contact pt., $r = \mathrm{OG} = \mathrm{Q}/s$, and $\angle\mathrm{AOG} = 90° - \frac{1}{2}\mathrm{A} = \frac{1}{2}\mathrm{C}$. $\therefore$ GA : GB $= d : b$. (V, 2.)

VII

PENCILS AND RANGES

(*a*) INTRODUCTION

QUARTETS of concurrent rt. lines, each forming a Pencil of 4 Rays passing through a Focus, are defined and measured by the mutual relation of the segments cut by the Rays on any Transversal.

Let the transversal be a rt. line on which the rays (numbered left to right) 1, 2, 3, 4 cut out the Range ABCD. The anharmonic or Cross Ratio of AB/CB : AD/CD is constant, and specifies the Pencil. This A.R. may be termed the BC.AD : AB.CD ratio. Two others, viz. AB.CD : AC.BD and AC.BD : BC.AD, also are constant; and the three are shown by drawing through B, on Ray 2, a rt. line ‖ to Ray 4. Let this meet Ray 1 at M and Ray 3 at N. From similar $\triangle$s, if V be Focus, BN : VD = BC : CD and VD : MB = AD : AB; hence BC.AD : AB.CD = BN : MB. Now if $\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3$ denote, respectively, BC.AD; AC.BD; AB.CD; we have, since $\mathrm{P}_1 : \mathrm{P}_3$ = BN : MB, and, obviously,

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$P_2 = P_1 + P_3$; $MN : BN = P_2 : P_1$. And finally $P_1 : P_2 : P_3 = BN : NM : MB$ = a trio of constant ratios.

Let the transversal be a circle passing through the Focus, on which the Rays cut the Cyclic Range *abcd*. The A.R. of the latter is that of the Pencil; and this may be expressed in terms of the chords *ab*, *bc*, *cd*, *ad*. For, with V as centre and R as radius of inversion, we may invert the circle into a linear transversal, cut by the Pencil at A, B, C, D—the inverses of the pts. *a*, *b*, *c*, *d*. Then since, by similar $\triangle$s, $AB/ab = VA.VB/R^2$ and $CD/cd = VC.VD/R^2$, it is evident that if p_1, p_2, p_3 be related to *a*, *b*, *c*, *d*, as P_1, P_2, P_3 are to A, B, C, D, then $p_1 : p_2 : p_3 = P_1 : P_2 : P_3$.

COR. Equiangular Pencils are equicross, e.g. a Cyclic Range gives a constant Pencil for any position of its Focus, V, on the circle; so also are Pencils that have a common transversal. And if the intersections of three pairs of homologous Rays of equal Pencils be collinear, the 4th pair also intersect on the same rt. line. (Axiomatic.)

(2) The 6 cross ratios of the Range ABCD are P_2/P_3; P_3/P_1; P_1/P_2, and their reciprocals.

(3) The A.R. may be written (ABCD), (BADC), (CDAB), or (DCBA).

Harmonic System

When the constant cross ratio $AB/CB : AD/CD = -1$, the A.R. becomes an H.R. In the above, the line MN is bisected, and BN = BM — Criterion of an H. Pencil.

$$AB/BC = AD/CD,$$

$$\therefore (AB + BC)/(AB - BC) = (AD + CD)/(AD - CD),$$

or, *o* being midpt. of AC; $oB.oD = oA^2$. And if O be centre of BD; $OA.OC = OB^2$. Hence, when segments AD, BD and CD, of a Range, are in Harmonic Progression, the circle, X, on diameter AC (or BD) cuts orthogonally any circle, Y, of which BD (or AC) is chord (e.g. if *o* be centre of X; $oB.oD = oA^2 = r^2$ = Sq. of Tangent drawn from *o* to Y).

The converse of the foregoing, which is an obvious deduction, serves to prove an important theorem:—If from a point A, outside a circle, any secant be drawn, which cuts the Polar of A in C, and the circle in B and D, the Range ABCD is Harmonic. For, let the secant from A which passes through O, the centre of Y, cut Y in *b*, *d*, and the Polar in pt. c. Then—AcC being 90°—the circle, X, on diameter AC passes through c. But as $Abcd$ is an H.R. ($OA.Oc = Od^2$), any ⊙ of which Ac is chord (e.g. ⊙ AcC), cuts at 90° the ⊙ on bd ∴ arc BD cuts at 90° the ⊙ on AC; and ABCD is an Harmonic Range.

(b) Practical Applications

(1) In a Cyclic Quad, ABCD, the minimum chord drawn through G, the diagonal junction, is that whose segment intercepted by two opposite sides is bisected by G.

Let the chord, UV, cut AD and BC at Y and Z, respectively. Then, since A.UDCV ≕ B.UDCV, the Range UYGV ≕ UGZV, or its equivalent, VZGU. But, as the minimum chord drawn through a point within a circle is bisected by the point, UG = VG, ∴ GY = GZ.

(2) In the Complete Quad the apices of △ EFG are foci of Harmonic Pencils; and when A, B, C, D are concyclic, this △ is Self-conjugate.

For, drawing through G a rt. line ∥ to FB, which cuts EF in K, and the sides a, d, c of the Quad in X, Y, Z; obviously BC : CF = GX/GY; also = XZ/KZ and GZ/YZ; hence also = (XZ − GZ)/(KZ − YZ) = GX/KY ∴ GY=KY. And F.AGBE is ∴ an Harmonic Pencil: a rt. line drawn ∥ to Ray 3 from a pt. on Ray 1, having its segment (2.4.) intercepted by Rays 2 and 4, bisected at the pt. Similarly we prove that E.BGCF is an Harmonic Pencil. Hence if EG intersect BC in J, BJCF is an H. Range ∴ G.BECF is an H. Pencil.

When ABCD is a cyclic Quad; if FG cuts the sides a and c in X and Z, the Range EBXA is Harmonic—being formed on a by an H. Pencil F.EBGA. So also is ECZD. Hence the Polar of E passes through both X and Z: it is ∴ FG. And since these reasonings apply, *mutatis mutandis*, to the Pencil E.FCGB, and to the rt. line EG, the proposition is proved: FG, GE, EF are the Polars, respectively, of E, F, and G.

(3) In a Cyclic Quad each of the three diagonals is cut Harmonically by the other two.

Let DB and AC meet the 3rd diagonal respectively in H and K. That the sides, GK, GH, HK, of △ GHK, are Harmonic conjugates of AC, DB, EF, has been already proved. Independently: The △ EBF has a transversal, AC, cutting its sides in A, C, K; also 3 rt. lines, viz. FD, ED, BD—drawn from its vertices and concurrent in D, cut those sides in A, C, H.

Hence, by a well-known law, EA.BC.FK = AB.CF.KE,
and EA.BC.FH = − AB.CF.HE
∴ FK : KE = − FH : HE.

(4) In Cyclic Quads the diagonals are diameters of three coaxal circles.

Since FKEH, AGCK and DGBH are each an Harmonic Range, the circles of which EF, AC and DB are diameters are, respectively, cut at 90° by the circles of which KH, GK and GH are chords ∴ it is evident that the circle circumscribed about GHK cuts orthogonally the three diagonal circles; and, as their centres are collinear, the criterion for three coaxal circles is fulfilled.

Cor. The Radical Axis is the ⊥ drawn from centre of ⊙ GHK to MN, the mid-diagonal line.

VIII

HOMOGRAPHIC AND INVOLUTION SYSTEMS

(*a*) Introduction

Ranges or Pencils having equal cross-ratios are said to be Equicross, or Equi-anharmonic, or Homographic. Equicross ranges on the same rt. line (or, on 2 rt. lines) divide it (or them) homographically.

If X, Y, be 2 fixed pts. on 2 given rt. lines, x, y; and if on x we take a system of pts., A, B, C, R, and on y a corresponding system, L, M, N, S, such that the rectangles AX.LY = BX.MY = CX.NY = RX.SY = k^2; then ABCR and LMNS are Homographs.

For, superposing y on x, so that Y coincides with X, and erecting at X a ⊥ XK = k, the ⊙s ALK, BMK, CNK, RSK all have XK as tangent (since XA.XL, etc. = k^2 = XK^2) ∴ the Pencils K.ABCR and K.LMNS are equiangular, or equicross.

If y be superposed on x, so that Y does not coincide with X, the 2 systems of pts. A, B, C, R, etc. and L, M, N, S, etc. on x, divide it homographically. The pts. of one system which coincide with their corresponding pts. of the other are termed The Double Points.

We may find the Double Points of collinear Homographs when 3 pairs of homologous pts. of the system are given.

Let these pts. be A, L ; B, M ; C, N, and let P be one of the required D. Pts. By the given condition the A.R. of PABC = A.R. of PLMN. Hence PA.BC/PB.AC = PL.MN/PM.LN, and ∴ PA.PM/PL.PB = a constant. We are thus given the ratio of the squares of tangents from P to ⊙s of which AM and LB are diameters : P ∴ lies on a circle coaxal with them (X,16), and the pts. where this known circle cuts the rt. line ALBN are the Double Points. (For, if midpts. of AM, LB be u, w, the required ⊙ has centre v, such that vu/vw = given ratio = $T_1^2 : T_2^2$. Its radius = the tangent drawn from v to the circle whose diameter is the linear link of the Limiting Pts.)

Another method is given by Inversion, and the use of Pascal's Theorem; Let the homologous pts. be L, R ; and M, S ; and N, T; and let them appear on the rt. line in the order LMNTSR. Then invert the line into a circle, and let the respective inverses be A, E, C, F, B, D. The zigzag chords AB, BC, CD, etc., form the Hexagon ABCDEF ; which, being inscribed in a circle, has the junctions, X, Y, Z, of its "opposite sides"—AB with DE, etc.—collinear. The inverses of the pts. of intersection of XYZ with the circle are The Double Points.

(For, let one of these Double Pts. be P, and its inverse, p. The line points linked with their inverses form Rays of a Pencil, whose Focus is O, the centre of the Circle of Inversion ; and, as—with XYZ transversal—the Pencil D.PAEC = A.PDBF, then O.PAEC = O.PDBF ∴ pLMN = pRST.)

Involution. When 2 trios of pts. on a rt. line are so related that—o being a fixed pt. on the line—oA.oA′ = oB.oB′ = oC.oC′ = k^2 ; it is readily seen that ABCC′ and A′B′C′C are Equicross Ranges. (Erect ⊥ oK = k ; then K.ABCC′ and K.A′B′C′C are evidently equiangular, equicross Pencils with a common Ray.) The 2 sets of points thus related are Six Points in Involution.

(*b*) Practical Applications

(1) If 3 sides of a variable Quad, inscribed in a given circle, X, cut a given rt. line in fixed pts., the 4th side also cuts it in a fixed pt.

Let the transversal, T, meet the sides BC (produced), CD and DA, at F, G, and H, respectively; and meet the circle at R and S. Then, if T cut BA (produced) at Z, we can prove that Z is fixed. For, joining R and S to A and to C, FRGHSZ is a System of 6 Points in Involution: The Pencil C.GRFS = C.DRBS = A.DRBS = A.HRZS = A.ZSHR ∴ the Ranges GRFS and ZSHR —which may be written *acbc'* and *a'c'b'c*—are equicross. Hence Z is a fixed point. (It is the Inverse of G: the Circle of Inversion having its centre, *o*, on T, and its rad. = tangent, *t*, to X from *o*. For *o*G.*o*Z = *o*R.*o*S = t^2.)

(2) Insert between the internal diagonals of a complete Quad, ABCD, a rt. line which shall subtend given angles at E and F, the ext. vertices.

Let the ∠s, for example, be 30° and 45°. Choose on one of the diagonals, *m*, produced if necessary, a suitable trial point, X. This is joined to E and F, and ∠s XEY = 30° and XFZ = 45°, are formed: the pts. Y and Z being on diagonal *n*. These are nearly coincident pts. Now, as X, in successive trials, moves along the diagonal *m*, Y and Z will form 2 homographic divisions on *n*. The Double Points solve the problem.

(3) In a Quad—or a given circle—inscribe a Quad whose sides shall pass each through a fixed pt.

On AB, base of Quad—or arc of ⊙—select a pt., X, which nearly fulfils the condition; then, the 4 pts. being P, R, S, T, and XP meeting BC in *b*, *b*R meeting CD in *c*, and *c*S meeting DA in *d*, let *d*T meet AB in *x*. As X moves on AB to other trial positions, *x* moves correspondingly, and the pts. X and *x* will form 2 homographic divisions on the line AB—or on the curve. A Double Pt. gives the solution.

(4) The sides of a Tetragon cut in Involution any rt. line which intersects them.

Let the linear transversal meet CD (produced) at L, and AB (produced) at L′, and the other opposite "sides," AD, BC, at M, M′ and AC, BD, at N, N′. Join B and D to N, and let AC meet BD in G. It will be seen that each of the Ranges, LMNN′ and L′M′N′N = CANG. For, the Pencil D.CANG = D.LMNN′ = B.LMNN′ ; and B.CANG = B.M′L′NN′ = B.L′M′N′N.

Cor. The Centre of Involution is (obviously) the intersection of the transversal with the Radical Axis of the circles on diameters LL′, MM′ and NN′.

IX

RECIPROCATION

RECIPROCAL POLARS

The figure X, so related to the figure Z that its points and lines are respectively the Poles and Polars of lines and points in Z, with respect to a fixed circle (Auxiliary Circle), is termed the Reciprocal of Z.

The geometric properties of X and Z, being correlative, reciprocation duplicates theorems and problems: every Proposition leads to another, called its Reciprocal. To a number of collinear points corresponds a number of concurrent rt. lines ; to parallel lines, points collinear with K—the centre of Aux. ⊙ ; and to the ∠ between 2 lines, the ∠ subtended by 2 points at K.

Examples.—(1) Reciprocal of a Range (Pencil) = equicross Pencil (Range).

Since the Polars of 4 pts., A, B, C, D, on a rt. line L, are concurrent at *p*, the Pole of L, and the line K*p* is ⊥ to L, it is obvious that, if the Polar of A cut KA in *a*, and the Polar of B cut KB in *b*, etc., the Pencil K.ABCD and the Pencil *p.abcd* are equiangular and ∴ equicross.

(2) The sides of a circumscribable Quad are cut by any tangent to the inscribed circle in points whose A.R. is constant.

Reciprocating ; the variable tangent becomes a variable pt., T, viz., its pt. of contact ; and the Range pts. become the links of T with the 4 fixed pts., *a*, *b*, *c*, *d*, where the ○ touches sides of the Quad. Then, as the Pencil T.*abcd* is constant—being equiangular for all positions of its Focus, T, on the circumference—the Range formed by the Poles of its Rays is also constant.

(3) Any pt., P, linked with vertices of a complete Quad forms a Pencil in Involution.

Reciprocation transforms this to—The 6 "sides" of a Tetragon (the sides and diagonals of a Quad) are cut in Involution by any linear transversal—a theorem which has been already proved. (VIII, 4.)

(4) Reciprocate the theorem: The rectangles under the perpendiculars drawn from any pt. on the circumference of a circle to each pair of opposite sides of an inscribed Quad, are equal.

Through the variable pt., P, draw a tangent, T, and at the Quad vertices, A, B, C, D, draw tangents to form the circumscribed Quad EFGH. Let p, q, r, s denote $\perp$s from P on AB, BC, etc., and Ee, Ff, Gg, Hh, the $\perp$s from the vertices of EFGH on the tangent, T. E being Pole of AB, by Salmon's Theorem (if X, Y, be the Polars of pts. x, y, and o be the circle centre, $ox : oy = \perp$ from x to Y : $\perp$ from y to X : evident on drawing the figure) the ratio $Ee : p = Eo : Po$, and $Gg : r = Go : Po$, and $Ff : q = Fo : Po$; $Hh : s = Ho : Po$. Hence $Ee.Gg : Ff.Hh = oE.oG : oF.oH$. The Reciprocal Theorem ∴ is: The rectangles under perpendiculars to a variable tangent, drawn from each pair of opposite vertices of a circumscribed Quad, are in a constant ratio.

(5) If the vertices of a Quad, circumscribed to a circle, Z, be joined to any pt., V, external to Z, the 4 links form with the tangents from V to Z, a Pencil in Involution.

Reciprocal: The intersections of any linear transversal with a circle and with the sides of an inscribed Quad, are 6 pts. in Involution. (*vide* VIII, 1.)

(6) The 2 Quads, one of which is inscribed in a circle, the other circumscribed to it by drawing tangents at the vertices of the former, have collinear 3rd diagonals, whose ends form an Harmonic Range. (*vide* X, 19.)

Reciprocal: The 2 pairs of internal diagonals of such Quads are concurrent, and form an Harmonic Pencil.

Their Reciprocal forms, elsewhere proved, implicitly prove these 2 theorems.

(7) About a given Circle circumscribe a Quad, each vertex of which shall lie on one of 4 given rt. lines.

Let a, b, c, d be the pts. of contact of the sides, and let P, R, S, T be the Poles of the 4 rt. lines. We have then the problem: To construct an Inscribed Quad, *abcd*, whose sides pass through, respectively, the given pts. P, R, S, and T. (*vide* IV, 5.)

* * * * *

The Reciprocal Polar of a Circle (X) *is a Conic* (Z): an *Ellipse, Parabola,* or *Hyperbola,* according as K (Aux. circle centre) is within, on, or outside X.

To every pt. (junction of 2 coincident tangents) of Z, corresponds a tangent (linear link of 2 coincident pts.) of X, ∴ Z is Locus of the Pole of any tangent to X. Let PT be tangent to X at a pt. P, and KT a ⊥ to it—K being centre, and k the radius of Aux. ⊙—and let O be centre, and R, the radius of X. Find pts. M on KT and N on KO, such that KO.KN = KT.KM = k^2. Let NL be drawn ⊥ to, and ML ∥ to, KO. The Locus of M, the Pole of PT, is required. It is given by the figure. For (KMLN) and (KOPT) are similar, ∴ KM/ML = KO/OP = e. The locus ∴ is a Conic whose Focus is K, whose eccentricity (e) = KO/R, and whose Directrix is the Reciprocal Polar of O. And it is obvious that $e = 1$, or is $\gtrless 1$, if KO = R, or is $\gtrless$ R.

X

MISCELLANEA

MISCELLANEOUS PROPOSITIONS

(1) The Law of Collinearity of pts. on the sides of a Quad is, as in the Triangle: Equality of the continued products of alternate Segments.

If ⊥s drawn from the vertices A, B, C, D to a transversal, T, be p, q, r, s, and the segments of the sides cut by T be a_1, a_2, b_1, b_2, etc.; then, as $a_1 : a_2 = p : q$ and $b_1 : b_2 = q : r$, etc., $a_1.b_1.c_1.d_1 = a_2.b_2.c_2.d_2$.

Cor. The Law for any Polygon of N sides, N_1, N_2 being segments of the Nth side, is—

$$a_1.b_1.c_1 \ldots N_1. : a_2.b_2.c_2 \ldots N_2 = \pm 1.$$

(Negative when N is odd.)

(2) Circumscribe about, and inscribe in a Quad ABCD, a Quad whose species is given.

Let the species be that of the figure GHST, and let the diagonal GS of the latter make angles u, v, with GT, ST, respectively. On the sides AB, CD, of the Quad, describe circle-segments containing angles which are the supplements of the $\angle$s G and S; then draw in these the chords BL, inclined at $\angle u$ to AB, and CK, inclined at $\angle v$ to CD; and complete the circles ABL and CDK. The pts. where the rt. line LX again intersects them are, obviously, 2 opposite vertices of the required Quad.

To construct the inscribed Quad we circumscribe, as above, about GHST a Quad, *abcd*, similar to ABCD, and divide the sides AB, BC, CD, DA by pts. g, h, s, t in the same ratios as ab, bc, cd, da are divided by G, H, S, T. The Quad *ghst*, being similar to GHST, is that required.

(3) Prove, by the aid of the Calculus, that the Quad formed by 4 rt. lines has max. area when its vertices are concyclic.

The consecutive sides being a, b, c, d; let angle $\widehat{ab} = \theta$ and $\widehat{cd} = \phi$. Area $= u = \frac{1}{2}(ab. \sin \theta + cd. \sin \phi)$. This being maximum, $du/d\theta = 0$. Hence

$$ab \cos \theta + cd \cos \phi . d\phi/d\theta = 0.$$

Now, as $c^2 + d^2 - 2cd \cos \phi = a^2 + b^2 - 2ab \cos \theta$, we have $cd.d\phi/d\theta = ab. \sin \theta/\sin \phi$. Hence the equation for $du/d\theta$ reduces to: $\sin \theta. \cos \phi + \cos \theta. \sin \phi = 0$. $\theta + \phi \therefore$ is 180°; the criterion of a Cyclic Quadrilateral.

(4) All of the Quads formed from quartets of 5 intersecting rt. lines have their mid-diagonal lines concurrent.

Draw the Quad ABCD so that E, the ac junction, may be on the left. Below, let a linear transversal, T, intersect the produced sides a and c (beyond E) at H and G, and d, b, at S, R. Let (4), (3), (2) and (1) mark, below T, the ends of BA, CD, DA and CB: (5) being T.

Omitting in turn any 3 successive lines (5), (4), (3), the Quads then formed, viz., ABCD, CDSR and ABRS, have for 3rd diagonals FE, FG, FH, and their internal diagonals are DB, DR, AR. Let midpts. of those be L, M, N, and of the latter be l, m, n. Now LM obviously bisects FD; so does lm. Hence LM, lm meet at x, the midpt. of FD. Similarly, it is evident that MN, mn meet at y, the midpt. of FR, and that the link, xy, is ∥ to DR. The position of ln is obscure; but if AB, to which LN is ∥, meet DR at V, a new Quad appears: FBVD, whose 3rd diagonal is AR and its internal diagonals are DB and FV. Hence ln (linking midpts. of diagonals DB, AR) meets LN in z, midpt. of FV; and as V is on DR, and x, y are midpts. of FD, FR, the rt. line xy passes through z.

We have ∴ two △s—LMN and lmn—of which pairs of corresponding sides, MN, mn, etc., meet at pts. x, y, z, on a rt. line; they are ∴ in perspective, and the rt. lines Ll, Mm, Nn, linking their vertices, are concurrent.

And as each trio in the above includes 2 Quads of preceding trio, the pt. of concurrence is the same for all.

(5) All the Triangles formed from trios of 4 pts. (the 4 component triangles of a Tetragon) have concurrent Nine Pt. Circles.

Let U, V, and G, H, be the midpts. of the sides AB, CD, and BC, AD, of a tetragon; and M, N, those of AC, BD. The 9 pt. ⊙s of △s BDC and BDA are circumcircles of △s NVG and NUH, respectively. Let them intersect in Z. The ⊙s MUG and MVH pass through Z. For the angles which NG, NU, subtend at Z are those which they subtend, respectively, at V, H; and, NVGB and NHUB being ∥ms, these are the ∠s that BD makes with BC and BA. Their sum = B = ∠UMG. Hence UG subtends equal angles at Z and M. Similarly, we prove that ∠VZH = ∠VMH = D. (The sides of the ∥m HUGV subtend at Z the angles A, B, C, D.)

(6) The mid-diagonal line cuts the opposite sides of the Quad proportionately, and its segment intercepted by a pair of opposite sides is cut proportionately by the internal diagonals.

If ⊥s are drawn from the vertices to X, the mid-diagonal line; the former is readily proved. The latter—X cutting the sides AB, BC, CD, DA in P, R, S, T—is proved by regarding BC as a transversal to the △ DNS, and AD as a transversal to △ BNP (etc.): the products of alternate segments being equated.

(7) ABCD is a variable Quad. If the vertices A, B, C, D, move on a circle, Y, while the ext. vertices, E and F, move on another fixed circle, Z; the midpt. of the 3rd diagonal, T, moves on a fixed circle, X.

Let O, *o*, be centres and R, *r* the radii of Y, Z respectively. The ⊙ on diameter EF cuts Y orthogonally, for E lies on the Polar of F (*vide* VII, 2), and EF, being a chord of Z, is ⊥ to *o*T. $\therefore oT^2 = r^2 - (\frac{1}{2}EF)^2$, and $OT^2 = R^2 + (\frac{1}{2}EF)^2$. Hence $OT^2 + oT^2 = R^2 + r^2 =$ a constant. T $\therefore$ lies on a circle, X, whose centre is the midpt. of O*o*, and the square of whose diameter is $2R^2 + 2r^2 - oO^2$.

(8) Bisect a Square by placing within it a Greek Cross, whose alternate ext. (projecting) vertices shall have contact with it.

Let the Cross, a 12-sided figure formed by 5 equal squares, be denoted by the ext. vertices ABCDEFGH; and let *abcd* be the Square. Also let $ab = y$ and $AB = x =$ I″. Then, by the given condition $y^2 = 10 \therefore y =$ the diagonal, AE, of the rectangle ABEF whose sides are I″ and 3″. We therefore infer that the contact pts. are the midpts. of the Square and that the angle $\widehat{xy}$, or BA*b*, $=$ BEA $= \tan^{-1}1/3 = \theta$. Hence the construction: From midpts. of the Square sides draw, in anti-clockwise direction, the rt. lines making $\angle\theta$ with the sides: *m*, with *ab*; *n*, with *bc*; *m′*, with *cd*; and *n′*, with *da*. Then draw ||s to *m*, through the midpts. of *bc* and *da*; and ||s to *n*, through the midpts. of *ab* and *cd*. The intersections of the 2 pairs of || lines with *m* and *m′* and with *n* and *n′* show the Greek Cross in the Square, having its alternate ext. vertices resting on the sides: A on *ab*, C on *bc*, E on *cd*, and G on *da*.

A mechanical proof is given by H. E. Dudeney (to whom I am indebted for this and the following problem): With a pair of scissors cut out the Cross and remove it; then the remnants, cut away, can be patched together to fit into the Cross. These—4 equal pentagons—are also seen to compose the Cross, by drawing 2 rt. lines in it, viz. AE and CG.

(9) Divide a Square into 4 equal Hexagons.

This is solved by drawing 2 zigzag lines—which produce the ancient *Swastika* symbol—across the square, linking pairs of opposite sides. Let *a*, *b*, *c*, *d* be its vertices, and *m*, *n*, *u*, *v* be midpts. of *ab*, *bc*, *cd*, *da*. Join

a to u, and draw through n, and v, rt. lines ∥ to au. Then, from o, the centre (diagonal junction) of the square, draw a ⊥ to these, cutting them in N and V. Similarly, ∥s are drawn to bv through m and u; the ⊥s on which, from o, cut them in the pts. M and U. The Swastika component lines are mMUu and nNVv.

(Or, briefly, draw rt. lines through m, n, u, v, inclined to the sides of Square at $\angle \tan^{-1} 1/2$. Then draw ⊥s to them from o.)

Cor. If the Square be transformed into a Greek Cross—ABpCDrEFsGHt—whose int. vertices are p, r, s and t, a mechanical proof may be given. Join midpt. of Bp with that of Fs, and midpt. of Dr with that of Ht. The 4 component Hexagons are then seen. They can be patched into the Square.

(10) In a Cyclic Quad, ABCD, if midpts. of the 3 diagonals (m, n, t) be M, N, T, and w = MN; $(m^2 - n^2)/2mn = w/t$.

Let K, the circumcircle, have centre O, radius R, and let R_1, R_2, R_3, be the semidiagonals: radii of the circles, X, Y, Z, whose diameters are m, n, t, respectively. Z cuts K orthogonally (for, if G be the mn junction, EFG is a Self-Conjugate △, ∴ if ⊙ on EF cuts OF in f, the Polar of F is Ef, and OF.Of = R^2). Hence $OT^2 = R^2 + R_3^2$. Again, since the ⊙s X, Y, Z are coaxal, if V be a pt. of intersection of X and Y, TV = R_3. Draw GH ⊥ to OT. Then, G being the Pole of t, GH is the Polar of T, and ∴ OT.OH = R^2. Now the ⊙ on OG passes through M, N, H ∴ TM.TN = TO.TH = OT^2 − OT.OH = $OT^2 - R^2 = R_3^2 = TV^2$. Hence TV is tangent to the ⊙ around △ MNV. The △s MTV, NTV are ∴ equiangular, and the ratio MV : NV = TM : TV = TV : TN = $R_1 : R_2$. Obviously, then, $R_1/R_2 - R_2/R_1 = 2w/t$ $(= \overline{TM} - \overline{TN} \div TV)$; or, $m^2 - n^2 = 2mnw/t$.

(11) In a given Circle inscribe a Quad whose diagonals are 3 given rt. lines.

As MN $= w = t.(m^2 - n^2)/2mn$; and the squares on OM, ON are respectively $R^2 - (\frac{1}{2}m)^2$ and $R^2 - (\frac{1}{2}n)^2$, we know the 3 sides of △ MON whose vertex, O, is a fixed pt. Constructing this △, and drawing the known chords AC and BD, ⊥ to OM and ON, respectively, the Quad is evident.

(12) A rt. line, L, revolves clockwise about a fixed pt., P, which lies outside and to the rt. of a Quad, and to lt. of a fixed Circle, X. Prove that, if L in one of its positions cut the sides AB, BC, CD, DA of the Quad in the pts. a, b, c, d, and if Pz be a segment of L such that

$$1/Pz = 1/Pa + 1/Pb + 1/Pc + 1/Pd\,;$$

the Locus of z is a rt. line.

Let P be below AB, and L so cut the sides that P, a, b, c, d, may be consecutive pts. Now 3 Rays of a Pencil, Dd, Dc, and the link DP, are seen. Draw the Harmonic Conjugate of DP, and let it cut L at the pt. e. Then Pced being an H. Range, Pc, Pe, Pd are in H. Progression $\therefore$ $1/Pc + 1/Pd = 2/Pe$.

Now let a rt. line, U, bisecting DP and drawn || to De, cut L in f (the midpt. of Pe) and the Side BC in M. (The sum $1/Pc + 1/Pd$ becomes $1/Pf$.) Then find pt. g on L, making M.Pfgb an Harmonic Pencil; and let the rt. line, V, bisecting MP and drawn || to Mg, cut L in h. (As $1/Pb + 1/Pf = 2/Pg = 1/Ph$; the latter is the sum $1/Pb + 1/Pc + 1/Pd$). And if V cuts the Side AB in N; proceeding as before, we find the pt. s on L, making the Pencil N.Phsa harmonic. Finally, the rt. line, W, drawn || to Ns from the midpt. of NP, is the required Locus.

For, if W cut L in z; $1/Pz = 2/Ps = 1/Pa + 1/Ph = 1/Pa + 1/Pb + 1/Pc + 1/Pd$.

(13) Let L, as it revolves round P, now intersect X, and become the Secant L_1, which intercepts a chord DC; then Secant L_2, intercepting diameter EF; and finally the Secant L_3, intercepting the chord AB:—the Quad CDEF may be constructed when the Projection (cd) of CD on L_2 is known. (*vide Harmonic System*, page 18.)

Let the Polar of P cut EF in G, and CD in H. Then, PD, PH, PC, being in H. Progression, so are their proportionals: Pd, PG, Pc. And, as PdGc is an H. Range; if o be midpt. of PG, $oc.od$ ($= o\mathrm{G}^2$) is known; also $oc - od$ is given $\therefore$ oc and od are known lengths, and $\perp$s erected on EF at c and d cut the ⊙ in C and D.

(14) When, in the above, the segment of L_2 intercepted by the tangents drawn to X at A and B is known, the Quad ABFE may be constructed.

Let mn be the intercept, and T the pt. where the tangents meet. The Polar of P passes through T, and, as T.PAGB is an H. Pencil, PmGn is an H. Range. Hence —o being the midpt. of PG—$om.on$ is known. We are given $on - om$ ∴ m and n are fixed pts. Tangents drawn from them to X touch the ⊙ at A and B.

(15) If the length (z) of the Diameter Secant, L_2, be known; also the angles x and y, which it makes with L_1 and L_3, the area of the Quad ABCD is known.

The centre of X being O, let θ and ϕ be the ∠s OAB and OCD—which are known, since OP ($= z - R$) is given. Now Q is the sum of 4 isosceles △s, having O for common vertex and radii for sides. Area is $\frac{1}{2}R^2$ multiplied by sum of the sines of ∠s AOB, BOC, COD and DOA. A summation which is readily reduced to

$$Q = R^2. \sin(\theta + \phi)\,[\cos(\theta - \phi) + \cos(x + y)].$$

(16) If a variable Quad be inscribed in one circle of a coaxal system and 3 of its sides touch, in every position, fixed ⊙s of the system; the 4th side also, in every position of Quad, touches a coaxal circle.

Let AB and ab be 2 positions of one of the sides—chords of ⊙, X—intersecting in G, and touching ⊙, Y, at H and K respectively. The △s GAa and GBb are similar, and GH = GK ∴ Aa : Bb = AG : bG = aG : BG, hence also = AG + aG : BG + bG = AH + aK : BH + bK. Now, by a well-known theorem, AH, aK, and BH, bK, being tangents from points on a circle, X, drawn to a circle, Y, of a coaxal system; the ratio of the squares of these tangents is that of the ⊥s from A, a, and B, b, on Radical Axis. Hence, from the above identities—if M and m be such ⊥s drawn from A and a, and N and n, those from B and b — Aa : Bb = $\sqrt{M} + \sqrt{m} : \sqrt{N} + \sqrt{n}$.

Similarly drawing BC, bc, tangents to another ○ of the system, it is evident that, if S, s, and P, p, be the ⊥s on the Radical Axis from C, c, and D, d (CD and cd being 2 positions of 3rd side of the Quad, which touch another coaxal ⊙) Bb : Cc = $\sqrt{N} + \sqrt{n} : \sqrt{S} + \sqrt{s}$, and that C$c$: Dd = $\sqrt{S} + \sqrt{s} : \sqrt{P} + \sqrt{p}$. Hence (continued products), we deduce

$$Dd : Aa = \sqrt{P} + \sqrt{p} : \sqrt{M} + \sqrt{m}.$$

DA and da are ∴ tangents to a ⊙ of the system.

(The theorem cited is readily proved. Let AU be ⊥ on line of centres, and let the latter meet the Radical Axis in V. If x, y, be centres of X, Y, and R, r, their radii, $Ax = R$; and $AH^2 = Ay^2 - r^2 = Ay^2 - Ax^2 + (Ax^2 - r^2) = Uy^2 - Ux^2 + Vx^2 - Vy^2$. Hence $AH^2 = UV.2xy = 2xy.M$. Similarly we prove $aK^2 = 2xy.m$. ∴ Tangent AH : Tangent $aK = \sqrt{M} : \sqrt{m}$. And the theorem in its general form is an obvious deduction, viz.: If X, Y, Z be coaxal circles, the tangents drawn from any pt., F, in X to Y and Z, have a constant ratio; and, conversely, if tangents from F to Y and Z have a constant ratio, the locus of F is a circle coaxal with Y and Z.)

(17) If a variable Quad, ABCD, inscribed in a circle, X, move so that its diagonals, AC, BD, are tangents to another circle, Y; its opposite sides, AB, CD, are tangents to a 3rd circle, coaxal with X and Y.

In any chosen position, ABCD; if H, K, be the contact pts. of AC, BD, with circle Y, and U, V, be the intersections of HK with AB and CD, respectively; the △s UAH and VDK are, obviously, equiangular. So also are △s UBK and VCH. Hence AU : AH = DV : DK, and BU : BK = CV : CH. We ∴ conclude that a coaxal circle touches AB at U, and CD at V. For, if Z be this ⊙, the ⊙ X, on which A, B, C, D lie, is coaxal with Y and Z (*vide supra*).

(18) Given the 4 sides and the area, a Quad may be constructed.

We may regard one side as fixed and then find one of the remaining vertices. This method, outlined in Casey's *Geometry*, depends on the fact that when a △ of given species has one vertex fixed and another moves on a given ⊙, the locus of the 3rd vertex is a given ⊙. Let ABCD be the Quad. If (below AB) a rt. line, BE, be drawn ∥ to and = AC, the △ DBE, whose sides are diagonals inclined at $\angle\theta$, represents the area, Q; and if DH be its altitude, we are given BE.DH. We are also given BE.BH, which $= V = \frac{1}{2}(a^2 - b^2 + c^2 - d^2)$ ∴ DH/BH, or $\tan\theta$: i.e. the species of △ DBH, is known. Now AB (a) being fixed, E lies on a given circle, Y, whose radius $AE = b$; and, as BE.BH is known, H lies on a given circle, X, the Inverse of Y.

The vertex D of △ DBH lies ∴ on a known ⊙, whose centre is not A. Where this ⊙ cuts the circle, centre A, radius d is vertex D. The sequel is obvious.

Expanding this analysis, a definite construction may be given. Let Y cut BE again in F; and from H draw a ‖ to AF, meeting AB in K. Then K is the centre of X, and KH is its radius, r. Obviously $KH/AE = r/b = BK/a = BH/BF = BE.BH/BE.BF = V/(a^2 - b^2)$. Hence BK and r are given; and if on BK a rt.-angled △ be erected (∠K being 90°) similar to △ BHD, its vertex, O, is centre of required circle. The radius, OD, is known ($= \varrho$); for BKO, BHD being similar, △s BOD, BKH, are also similar (having = ∠s at B and the sides about these ∠s proportional) ∴ $\varrho/r = OD/KH = DB/BH = \sec\theta$.

CONS. Fix the side $AB = a$. Cut off the segment $BK = ra/b = Va/(a^2 - b^2)$. Erect KO ⊥ to AB and $= BK \tan\theta$. With centre O and radius $r \sec\theta = Vb \sec\theta/(a^2 - b^2)$, describe a ⊙. Then, D is intersection of this ⊙ with the ⊙, centre A, radius d.

(19) The diagonals of any Quad circumscribed about a Circle (or an Ellipse) form a Self-Conjugate Triangle.

Drawing the Quad ABCD in complete form (ext. vertex E being the DC, AB junction) let U, and V, and W, denote the junctions AC, EF and EF, DB and DB, AC. In the Circle (as also in the Ellipse), any chord is divided harmonically by a pt. P on it, and the Polar of P; also tangents meet in the Pole of their chord of contact. Let a, b, c and d—a being on AB—be the pts. of contact of the sides with the circle (or conic); then $abcd$ is an inscribed Quad. Let its ab, cd and bc, ad and ac, bd junctions be e and f and g, respectively. Then by the harmonic properties of the inscribed Quad, as already proved (*vide* VII), efg is a Self-Conjugate △ ∴ fg = Polar of e, and eg = Polar of f. Now, as E is Pole of ac and F is the Pole of bd, their junction, g, is the Pole of EF. But Pole of ef is g ∴ fFeE is a rt. line: *The 2 Quads have collinear 3rd Diagonals.* Also, A and C are the respective Poles of ad and bc (which meet in f). Hence f = Pole of AC; so is e the Pole of BD ∴ W, the AC, BD junction = Pole of $ef = g$: *The int. diagonal junctions of the 2 Quads are coincident.* And as ∴ ACe

and BDf are rt. lines, U coincides with e, and V with f.
Q.E.D.

Cor. The concurrent internal diagonals form an H. Pencil.

(20) The Locus of a pt. which linked with 4 fixed pts. forms a Pencil whose A.R. is constant, is a Conic. (For, if A, B, C, D, be vertices of a Quad inscribed in an Ellipse, and P a variable pt. on the curve, the Pencil P.ABCD is constant.)

Let S be the Focus nearest the Directrix, X, and let the Pencil—P being any chosen pt.—intersect X in the pts. a, b, c, d. Then a being on the AP ray, aS bisects the exterior angle formed by SA and SP (for if m, p, be the feet of $\perp$s from A and P, on X, $SA/SP = Am/Pp = Aa/Pa$, and similarly $SB/SP = Bb/Pb$). Hence $\angle aSb = \frac{1}{2}$ (supplt. of ASP $-$ supplt. of BSP) $= \frac{1}{2}$ ASB, which is constant. Similarly $\angle bSc = \frac{1}{2}$BSC, etc. ∴ A.R. of S.$abcd$ (=P.ABCD) is constant.

XI

GYMNASIUM

(Arena of Mental Exercise)

(1) The Quad of given area has min. perimeter—and the Quad of given perimeter has max. area—when it is equilateral.

(2) Of all the Parallelograms that can be formed with diagonals of given lengths, the Rhombus has the max. area.

(3) Divide a Parallelogram into 3 equal parts by a rt. line drawn parallel to one of the diagonals.

(4) By a rt. line drawn from vertex, C, to base, AB, of the Quad ABCD, cut off $\frac{1}{3}$th its area.

(5) In a given Rectangle, when possible, inscribe a Square (without using the circum. square).

(6) Construct a Parallelogram being given its area, one of its angles and the difference between its diagonals.

(7) In a given circle inscribe a Trapezium whose area and the sum of whose parallel sides are given.

(8) Describe a Rectangle of given area whose sides pass, each, through a fixed pt.

(9) Construct with 4 given rt. lines a Cyclic Quad.

(10) In a given Quad inscribe a Parallelogram whose angles are known.

(11) Construct a Quad, being given (*a*) the 4 sides and a pair of opposite angles ; (*b*) the 4 sides and $\angle$ of intersection of 2 opposite sides.

(12) Express the area of a Tetragon in terms of its sides a, a' ; b, b' ; c, c', by a formula derived from a study of the Medians.

(13) Construct a Quad equal in area to a Regular Octagon of 1″ side.

(14) Find the point within a Quad whose linear links with the midpts. of the sides divide the Quad into 4 equal parts.

(15) Express the 3rd diagonal of a Cyclic Quad in terms of the angles and sides.

(16) The sides of a Cyclic Circumscribed Quad are 9″, 5″, 3″, 7″. Find its angles.

(17) When the sides of one Quad are, respectively, parallel to those of another, the rt. lines linking corresponding vertices are concurrent.

(18) A Quad is simultaneously inscribed in a circle, X, and circumscribed about a circle, Y. If the radius of X be 7″ and that of Y be 4·8″, find the distance between the centres of X and Y.

(19) Given the 4 sides of a Cyclic Quad, one of whose diagonals bisects the other ; find the length of the bisector.

(20) The Bi-rectangular Quad, OAZB, has its acute angle at O. (1) Being given $\angle$O and the length of the 3rd diagonal, find locus of Z (the opposite vertex). (2) Express area (OAZB) in terms of sides OA, OB, and the angle O. Also, (3) Prove that, θ being the interdiagonal $\angle$, $\sin\theta$ is a function of OA, OB, and angle O.

(21) A point, P, within a Quad, is linked with the vertices by the rt. lines g, h, s, t, and G, H, S, T, are the feet of the $\perp$s from P on the sides ; find P (*a*) when the sum of the squares of g, h, s, t is minimum, and (*b*) when the area (GHST) is a minimum.

(22) Prove that in any Quad whose external vertices are E, F, and the internal diagonals m, n, that, Q being the area of the Quad :

$$4Q^2 = (mn + ac.\cos E - bd.\cos F).(mn - ac.\cos E + bd.\cos F).$$

(23) ABCD is a Cyclic Quad. If the Reciprocal Polar of $\triangle$ ABC be $\triangle$ xyz, and the vertex D of the Quad be the centre of Aux. $\odot$; the points x, y, z, D are concyclic.

(24) From a point, o, outside a square, a rt. line is drawn

cutting the sides in a, b, c, d. If $1/oa + 1/ob + 1/oc + 1/od = 1/ox$; find pt. x.

(25) If ABCD be a Quad inscribed in a Circle (or Ellipse), the intersections of AC, BD, and of BC, AD, and of tangents to the curve at C, and D, are collinear.

(26) The intersections of any chord of an Ellipse with the curve and with the sides of an inscribed Quad, are 6 Points in Involution.

(27) If focal chords AC, BD, of an Ellipse are diagonals of an inscribed Quad, and P be any point on the curve, the Pencil P.ABCD is Harmonic.

(28) Given the sides of a Cyclic Tetragon, find the area of the Rectangle formed by joining the In. centres of the Component-triangles.

(29) If, in every position of a variable Quad inscribed in an Ellipse, the diagonals subtend a rt. angle at a fixed pt. on the curve; the locus of G, the junction of the diagonals, is a rt. line.

(30) The Envelope of the sides of a variable Quad whose vertices move on a given circle, and whose diagonals meet at rt. angles in a fixed point, is a Conic. (Reciprocate.)

XII

ADDENDA

THE COMPLETE-QUAD COMPONENT TRIANGLES

(1) THE 4 △s formed by the mutual intersections of the sides of a Quad have concyclic Circumcentres and collinear Orthocentres.

Denote the circumcircles of these △s, ABF, BCE, CDF, DAE, by the numbers 1, 2, 3, 4. They contain a, b, c, d, as respective chords; and they intersect in a common pt., P. (For, if PC be common chord of ⊙, BCE and CDF; PCDF being a Cyclic Quad, $\angle$AFP $= \angle$ PCE $= \angle$ PBE ∴ ⊙ ABF—and, for similar reasons, ⊙ ADE—passes through P.) Let their respective centres be K_1, K_2, K_3, K_4; and their common chords be g = PD = (34), h = PA = (41), s = PB = (12), and t = PC = (23). Then, if G, H, S, T, denote the ⊥s drawn to these chords at their midpts., it is readily seen that the junction $\overline{HS} = K_1$ and $\overline{ST} = K_2$, and $\overline{TG} = K_3$, and $\overline{GH} = K_4$; also that the rt. line K_1K_3 subtends equal angles at K_2, and K_4, viz. $\angle\ \widehat{gh} = \widehat{st} = E$. The centres are ∴ concyclic.

That the Orthocentres are collinear, follows from the fact that, by a known theorem, the rt. line linking P with the Orthocentre of any inscribed △ is bisected by Simson's Line, Z. Hence, as Z is common to the 4 △s, the Orthocentres lie on rt. line ∥ to Z.

The theorem referred to is thus proved: AFB is an inscribed △. Let P be on the arc FB and k be the Orthocentre. Join Bk and produce the link to meet AF in v and the circle in K. From P ⊥s, Pp, Pq, are drawn to AF, BF; and the link pq (= Z, Simson's Line) cuts PK in s and AF in w.

As Kv, kv, subtending the same angle (90°—A), at F, are equal, the △s Kvw, kvw, are equal and congruent. And since PFpq is a semicircle and Pp, BK are parallel, spP, spw are isosceles △s, and wk is ∥ to pq. Hence pq bisects Pw and is ∥ to wk; it ∴ bisects Pk. (Also, the angle at which Z cuts FA is the complement of the angle PAB.)

(2) The Mid-diagonal Line, X, is perpendicular to Z, Simson's Line; and the sides a, b, c, d, and the radii of the corresponding circumcircles, are inversely proportional to the sines of the angles at which X meets them.

Since X is ∥ to the base of a △ whose sides, collinear with BC and AD, respectively, are b and d (III, 4), it cuts these sides of the Quad at ∠s whose Sines are in the ratio of $d : b$. It is required to prove that this is the ratio of the Cosines of the ∠s at which Z cut them. Drawing the figure, it is evident that the ⊥s from P on the sides of the Quad meet Z at ∠s that are the complements of the ∠s which Z makes with these sides. Hence, if the ⊥ on AB be Pa, and on BC be Pb, etc., and R_1, R_2, R_3, R_4, be radii of the ⊙s 1, 2, 3, 4, $\cos Zb : \cos Zd = Pd : pb = PD : PC$. The latter are chords of different ⊙s, 4 and 2, which subtend = arcs (∠E) ∴

$$\cos Zb : \cos Zd = R_4 : R_2 = d : b = \sin Xb : \sin Xd.$$

That X is $\perp$ to Z may be proved geometrically as follows :—

Let X cut FA, FB, in A′, B′, and let x be the angle FA′B′. We can show that z, the $\angle$ at which Z cuts FA, is $90° - x$. For, obviously, the $\triangle$s PAD, PBC, are equiangular ; hence PA : PB = AD : BC = FA′ : FB′, and (P being on the ⊙ FAB) $\angle$APB = $\angle$AFB. Hence, the containing sides being proportional, the $\triangle$s FA′B′, PAB, are similar $\therefore$ x = PAB = $90° - z$ (*vide supra*).

(3) If t be the 3rd diagonal and w be MN (the linear link of midpts. of the internal diagonals) ; then the area of the Quad ABCD = $wt \sin Xt$.

The external vertices being E = ac, and F = bd, we can readily prove that $\triangle$ FMN, whose base is w, and altitude $\frac{1}{2}t. \sin Xt$, is equal to $\frac{1}{4}$th of Q ; which obviously proves the proposition.

For o being the midpt. of CD, let oM = x and oN = y, and let Apex Base denote $\triangle$; then, of the 3 $\triangle$s which compose Fw, one, Fx = Dx; another, Fy = Cy $\therefore$ $\triangle$ FMN = Quad MNCD =

$$\tfrac{1}{2} \triangle \text{ BCD} + \tfrac{1}{2} \triangle \text{ BMD} = \tfrac{1}{2}(\text{BCDM}) =$$
$$\tfrac{1}{2}(\triangle \text{ BCM} + \triangle \text{ CDM}) = \tfrac{1}{4} (\text{ABCD}).$$

Cor. The Quad is quadrisected by linking M, ~~and~~ N, with the vertices C, D, Also $\triangle$ EMN = $\triangle$ FMN. (For vertex E and side BC may be used instead of F and CD in the foregoing $\therefore$ $\triangle$ Ew = $\triangle$ Fw ; and as their altitudes are $\therefore$ equal, MN produced bisects EF : proving that midpts. of the 3 diagonals are collinear.)

(4) Denoting by ϕ the inclination of 3rd diagonal to mid-diagonal line : $8Q^2 = 2t.^2(2w. \sin \phi)^2 =$

$$(a^2 + b^2 + c^2 + d^2 - 2ac. \cos E - 2bd. \cos F).t^2 \sin^2 \phi.$$

As proved in the study of the Medians, $2w$ = PH = ph; and the squares of PH and ph are, respectively,

$$a^2 + c^2 - 2ac. \cos E \text{ and } b^2 + d^2 - 2bd. \cos F.$$

Also, as proved above, $2Q = 2wt, \sin \phi$.

(5) If in the Component $\triangle$ AFB any 2 pts. be taken in the sides, C and U in BF, and D and V in AF, the diagonal junctions in the Quads ABCD, ABUV and CDVU are collinear.

Let the diagonal junctions, AC, BD, etc. (ranged from below upwards), be X, Y, Z, and let the junction AC, BV be R, and AU, CV be S.

Join YC, also YX, YZ. Then, obviously, the Pencil

$$Y.VZSC = U.VZSC = U.VDAF = B.VDAF = B.RXAC.$$

$$\text{Hence } Y.VZSC = Y.RXAC.$$

And, of the Rays of these equal Pencils, one is common, 2 pairs (YV, YR and YS, YA) are collinear ∴ the 4th pair YZ, YX must form one rt. line.

COMPONENT TRIANGLES OF A CYCLIC TETRAGON

THE triangles formed from trios of concyclic points are those whose Orthocentres re-form the Tetragon; and whose Nine-Point Circles have concyclic centres; and whose In-centres form a Rectangle.

The Tetragon reappears when the Orthocentres, g, h, s, t, of the △s ABD, ABC, CDB, CDA, are joined. In the △ ABC, if U, W be midpts. of AB, BC, O being circumcentre, OU, OW are ⊥s to these sides; and, h being the Orthocentre, it is evident that

△s OUW, hCA are similar, having ∥ sides. Hence, as CA = 2UW, hC = 2OU; and if X be midpt. of Ch, the figure COUX is a ∥m. Radius of circumcircle OC (= R) is ∴ equal to UX, the diameter (= 2ϱ) of the Nine-pt. ⊙. And Oh meets UX in its midpt., n. (For the △s OUn, Chn, are equal and congruent.) Hence n is the centre of the Nine-pt. Circle of △ABC.

As, in △ ADB, Dg also = 2OU, we infer that CDgh is a ∥m. And if OV be ⊥ from O on CD, since, for similar reasons, 2OV = both Bs and At, ABst is a ∥m ∴ ($ghst$) = (CDAB).

(2) That m, n, x, z, the centres of the Nine-pt. Circles of △s ABD, ABC, CDB, CDA, are concyclic, is an inference from the foregoing: for, these pts., being the midpts. of Og, Oh, Os, Ot, form a tetragon whose sides mn, nx, xz, zm, are respectively ∥ to (and = half) the sides of ($ghst$). The latter being similar to CDAB, $mnxz$ is a cyclic tetragon. And its area is, obviously, $\frac{1}{4}$(ABCD).

COR. The pt. Z, common to these Nine-pt. Circles (*vide*

X, 5), is, in Cyclic Tetragons, the centre of the circumscribed ⊙ of *mnxz*, and the diagonal junction in the ||ms *gh*CD, *st*AB. It is also the pt. of concurrence of the perpendiculars drawn from the midpt. of each side of ABCD to the opposite side. For, if the diagonals *g*C, *h*D meet in P, since *g*C is bisected in P and *g*O is bisected in *m*, P*m* is || to OC, and=its half ($=\varrho$); and, as this reasoning applies equally to *n*, *x*, and *z*, P must coincide with Z. Again, as $mZ = nZ = xZ = zZ = \varrho$, it is obvious that Z is the centre of the circle (*mnxz*).

Also, as the rt. line joining the midpts. of AB, BC subtends at Z the angle which it subtends at the midpt. of AC, viz. ∠B, and $B = 180° - D$, the ⊥s from U, W, to CD, DA, meet in Z.

(3) The In. centres, G, H, S, T, of the △s ABD, ABC, CDB, CDA, are pts. of intersection of pairs of circles, whose centres *k*, *u*, *v*, *w*, are midpts. of the (lower) arcs AB, BC, CD, DA; and whose radii are *k*A, *u*B, *v*C, *w*D. (For, if C be apex of any inscribed △ on base AB, and H, its In. centre, the △ *k*AH has equal ∠s at A, H, $= \frac{1}{2}A + \frac{1}{2}C$ ∴ $kH = kA = kB$.)

ABGH, BCHS, CDST, DATG, are Cyclic Quads. Hence the angle at any vertex (G) of the Quad. GHST = ½ sum of the ∠s of △ whose In. centre it is: $G = \angle GHA + GTA + TAH = GBA + GDA + \frac{1}{2}A = 90°$ ∴ GHST is a Rectangle.

THE SQUARE ROAD TO Π

To find the exact value of π, and thus to Square the Circle, was, for medieval geometers, a fascinating problem. In their approximations many methods were used. The following approach through the square (Casey's Method) may fitly conclude this study of the Quadrilateral.

A square, ABCD, is circumscribed about a circle, and its contact points—E on AB, F on BC, etc.—are joined to form an inscribed square EFGH.

The diagonal DB cuts the circle in M, N, and the chord EF in L, bisecting it at rt. angles.

Tangents to the ⊙ at M, N, intercepted by sides of the circumsquare, are sides of the circumoctagon—whose contact points are vertices of the inscribed octagon.

We then proceed to prove the important theorem : If P_1, Q_1 be the respective areas of the inscribed and circumscribed Squares ; and if P_2, Q_2 be areas of the corresponding Octagons :

P_1, P_2, Q_1 are in Geom. progression, and
Q_1, Q_2, P_2 are in Harmonic progression.

Let O be the centre, and R, the radius of the circle, and let the tangent at M meet AD in K. Join H to M, N, F, and O to K.

Since EF is the Polar of B, OL : ON = ON : OB. The $\triangle$s FOL, FON, FOB, are $\therefore$ in Geom. progression. Their areas are, respectively, $\frac{1}{8}P_1$, $\frac{1}{8}P_2$, $\frac{1}{8}Q_1$, $\therefore$ P_2 is the Geom. Mean of P_1, Q_1.

We can prove that Q_2 is the H. Mean of Q_1, P_2 : Denoting by x, y, z, the figures OHD, OHKM, OHM, which are, respectively, $\frac{1}{8}Q_1$, $\frac{1}{8}Q_2$, $\frac{1}{8}P_2$; evidently $(x - y) = \triangle$DKM, and $(y - z) = \triangle$HKM. And, since KO is ‖ to HN, the ratio of these $\triangle$s, viz., DK : KH = DO : R = $\triangle$ OHD : $\triangle$ OHM = $x : z$. Hence x, y, z (and $\therefore$ Q_1, Q_2, P_2) are in Harmonic progression.

Substituting Polygons of n and $2n$ sides for Square and Octagon in the above demonstration, we have proved (implicitly) that :

If p, q denote the *reciprocals* of the areas of 2 polygons of the same number of sides inscribed in, and circumscribed about a circle, and p_2, q_2, the corresponding values for polygons of twice the number of sides :

p_2 = Geom. Mean of p and q.
q_2 = Arith. Mean of p_2 and q.

Thus, from p, q we find p_2, their GM ; then the AM of this and q gives q_2.

In like manner from p_2 q_2, we can find p_3, q_3, related to p_2 and q_2, as these are to p and q. Repeating this process ~~n times~~, when n is a large number we find the p_n, q_n values nearly equal. And, as the circle area is intermediate between their reciprocals—2 polygons of 2^n sides—we can find π.

For example, in a circle of unity radius beginning with the inscribed and circum. square, these estimations are : $p = \cdot 5$, $q = \cdot 25$. Whence $p_2 = \cdot 35355$, $q_2 = \cdot 30177$, $p_3 = \cdot 32648$, $q_3 = \cdot 31413$. Finally, n being [illegible], $p_n =$

$q_n = \cdot 31831$. Hence π, the reciprocal of this, is 3·1416 : a fairly close approximation. (It is Vieta's 355/113.)

A more expeditious and accurate method is given by the Calculus : The identities $\tan^{-1} x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5$ [the integral of $(1 + x^2)^{-1}$], and $\pi/4 = 4 \tan^{-1} 1/5 - \tan^{-1} 1/70 + \tan^{-1} 1/99$, enable us to estimate π, to as close an approximation as we desire ; and some investigators have thus calculated its value to 700 places of decimals !

* * * * *

In conclusion, some devices for drawing a rt. line equal to the circumference of a circle may be recalled :

(1) The fact that $\sqrt{2} (= 1 \cdot 415)$ is the side of a square inscribed in a circle of 1″ radius, and that the value of π may be written $3 + \frac{1}{10} \sqrt{2}$, suggested the following method :

Inscribe a Square in a Circle of 1″ radius. A rt. line may be drawn whose length equals 3 Diameters + $\frac{1}{5}$th of length of a Side of the Square. This (being 2π) is the required Circumference.

(2) Another method was that of doubling the Hypotenuse of a rt.-angled Triangle whose sides are 1″ and 3″:

A method suggested by the value $(2\sqrt{10})$ given to 2π by the Indian Geometers, who, inscribing in a circle of unit *diameter* a series of polygons of 12, 24, 48 and 96 sides ; and calculating their perimeters to be the Sq. Roots of 9·65, 9·81, 9·86 and 9·89, concluded that $\sqrt{10}$ is probably the perimeter of an inscribed polygon of a thousand sides.

"*Haec studia adolescentiam alunt, senectutem oblectant, secundas res ornant, adversis perfugium ac solatium praebent, delectant domi, non impediunt foris, pernoctant nobiscum, perigrinantur, rusticantur.*"

CICERO.

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