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METRIC DIFFERENTIAL GEOMETRY
OF CURVES AND SURFACES

METRIC DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

By

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PREFACE

This book is designed as a text for first-year graduate students of metric differential geometry, and it will also be found useful by those who wish to acquire some knowledge of this subject by independent reading. The treatment is elementary. Plane analytic geometry, three-dimensional analytic geometry, and calculus are prerequisite for understanding the developments of the text, but vectors are not used. The advantages of a treatment by means of vectors are well known, but it has been thought best, in order to make the discussion as elementary as possible, to refrain from employing them here.

Most of the material included is classic. It has been the common heritage of so many generations of geometers that no attempt has been made to give complete references to sources or to assign credit in every instance for the original discovery of results. An occasional reference has been given to more extensive treatises, particularly for the proofs of a few theorems which have been included here without proof. A bibliography of a few of the best-known treatises is appended at the end.

The reader who is acquainted with projective differential geometry will recognize that the definitions of those configurations which admit of projective definitions are stated in projective form. Moreover, whenever it is feasible to do so, definitions are stated in such a way as to be valid in hyperspace as well as in ordinary space. As a rule, no simplicity is lost in so doing, and anyone who goes on to study projective differential geometry and hyperspatial geometry will find the road made smoother by this method of treatment.

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CONTENTS

CHAPTER I

CURVES

1. Introduction. This book is primarily devoted to the analytic metric differential geometry of curves and surfaces in ordinary three-dimensional space. To say that a geometry is *analytic* means that it employs a coordinate system and thus avails itself of the powerful methods of algebra and analysis. Furthermore, *metric* geometry is characterized by saying that it is the study of properties of figures that are unchanged when the figures themselves are subjected to rigid motions, namely, translations and rotations. Examples of such properties are the distance between two points, the angle between two lines, and the area of a triangle. The fact that the ordinary *measures* of these and various other quantities are invariant under rigid motions has caused this geometry to be called *metric.*

The *differential,* or *infinitesimal,* geometry of a figure is concerned with properties of the figure that depend only on a neighborhood of one of its elements. For instance, the differential geometry of a plane curve, regarded as the locus of a point, is concerned with the properties of the curve in a neighborhood of one of its points, as distinguished from properties which depend on the entire curve. The familiar definition of the tangent at a point of a curve, which states that the tangent is the limit of the secant through this point and a neighboring point on the curve as the second point approaches the first along the curve, is perhaps the simplest geometrical definition having a characteristically differential nature; obviously, the tangent depends only on a neighborhood of its point of contact. On the other hand, the problem of determining the number of intersections of a straight line and a conic in the same plane demands for its solution a knowledge of the entire line and conic and is an instance of a problem not of an essentially differential character.

The definition of the tangent just referred to involves a limiting process. Such limiting processes occur frequently in differential geometry. For this reason the differential calculus is a very convenient tool for its study. Indeed, most books on the calculus freely employ the metric differential geometry of plane curves as an aid in interpreting the definition of the derivative and as a field of application of the principles of the calculus. So the reader is probably already familiar with the fact that, when the equation of a plane curve C is written with ordinary orthogonal cartesian coordinates x, y in the form

$$y = y(x)\,, \tag{1·1}$$

the equation of the tangent at a point $P(x, y)$ of C is

$$Y - y = y'(X - x) \qquad \left(y' = \frac{dy}{dx}\right), \tag{1·2}$$

in which X, Y are the coordinates of a variable point on the tangent. Moreover, *the equation of the normal* at the point P of the curve C is

$$y'(Y - y) + X - x = 0\,. \tag{1·3}$$

The element of arc length ds of C is given, except for sign, by the formula

$$ds^2 = dx^2 + dy^2\,, \tag{1·4}$$

and *the curvature* $1/\rho$ at the point P of the curve C is expressed in terms of derivatives of y by the formula

$$\frac{1}{\rho} = \frac{y''}{(1 + y'^2)^{3/2}}\,. \tag{1·5}$$

2. Definition and equations of a curve. In ordinary three-dimensional space let us establish a left-handed orthogonal cartesian coordinate system with the same unit of distance for all three axes. In this system any point P has coordinates x, y, z, as in Figure 1.

A curve may be described qualitatively as the locus of a point moving with one degree of freedom. A curve is also sometimes said to be the locus of a one-parameter family of points, or the locus of a single infinity of points. However valuable these descriptions may be for facilitating the visualization of a curve, none of them are sufficiently restrictive for our present purposes. So we proceed to define *a real proper analytic space curve.*

DEFINITION 1. *Let the coordinates x, y, z of a point P be given as single-valued real-valued analytic functions of a real independent variable t on an interval T of a t-axis, by equations of the form*

$$x = x(t)\,, \qquad y = y(t)\,, \qquad z = z(t)\,. \tag{2·1}$$

Further, suppose that the functions $x(t)$, $y(t)$, $z(t)$ are not all constant on T. Then the locus of the point P, as t varies on the interval T, is a real proper analytic curve C.

Some comments on the foregoing definition will perhaps clarify its meaning. Equations (2 1) are called *the parametric equations* of the curve C, the parameter being the variable t. We reserve the right to permit the parameter t to take on complex values if they are properly introduced in the sequel. Moreover, one or more of the coordinates

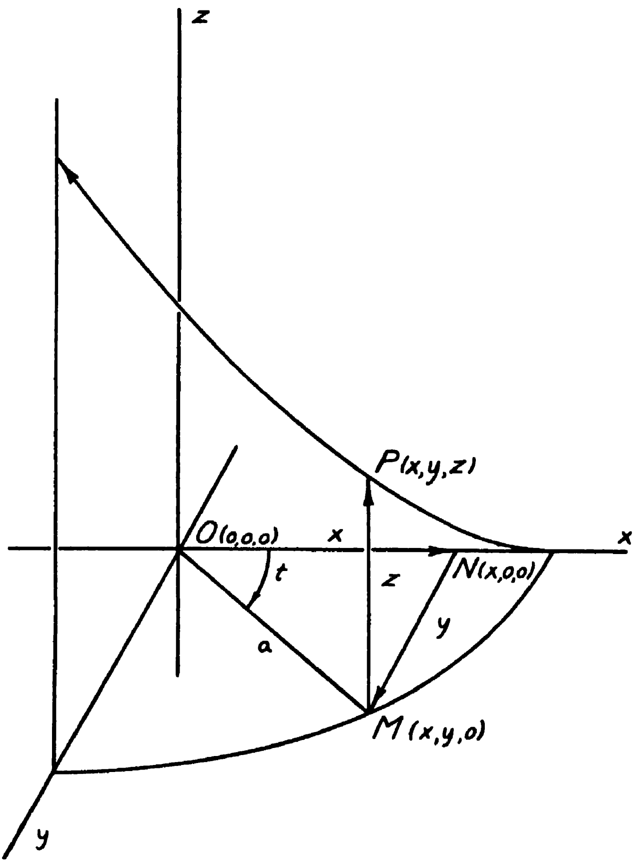

FIG. 1

x, y, z may, under suitable conditions, be allowed to be complex. The curve C would in this case be called *complex*, or perhaps, under suitable conditions, *imaginary*. To say that a curve is *proper* means that it does not reduce to a single fixed point, as it would do if the coordinates x, y, z were all constant. It is clear that at *an ordinary point* of a real proper analytic curve, i.e., a point where nothing exceptional occurs, the inequality

$$x'^2 + y'^2 + z'^2 > 0 \qquad \left(x' = \frac{dx}{dt}, \dots\right) \tag{2·2}$$

holds. Any point of such a curve where this inequality fails to hold is called *singular*, although the singularity may belong to the parametric representation being used for the curve defined as a point-locus, or may belong to the curve itself. Singular points will ordinarily be avoided hereinafter. A curve, or portion of a curve, which is free of singular points may be called *nonsingular*. Furthermore, we assume that the interval T is so small that values of the parameter t on the interval T and points (x, y, z) on the curve C are in one-to-one correspondence, so that the parameter t is *a coordinate* of the corresponding point (x, y, z) on the curve C.

To say that the functions x, y, z are *analytic* means, roughly, that they can be expanded into power series. More precisely, this statement means that, at each point t_0 within the interval T, each of these functions can be expanded into a Taylor's series of powers of the difference $t - t_0$, which converges when the absolute value $|t - t_0|$ is sufficiently small. It would be possible to study differential geometry under the hypothesis that the functions considered possess only a definite, and rather small, number of derivatives; but we assume analyticity in the interests of simplicity. So the word *function* will mean, for us, *analytic function;* and the word *curve* will mean a real proper nonsingular analytic curve unless the contrary is indicated.

Some examples of parametric equations of curves will now be adduced. First of all, the equations (2·1) may be linear, of the form

$$(2\cdot 3)\qquad x = a + lt\,, \qquad y = b + mt\,, \qquad z = c + nt\,,$$

in which a, b, c and l, m, n are constants. Then the curve C is *a straight line* through the fixed point (a, b, c) and with direction cosines *proportional to* l, m, n. If t is the algebraic distance from the fixed point (a, b, c) to the variable point (x, y, z) on the line, then l, m, n are the direction cosines of the line and satisfy the equation

$$(2\cdot 4)\qquad l^2 + m^2 + n^2 = 1\,.$$

As a second example, equations (2·1) may take the form

$$(2\cdot 5)\qquad x = t\,, \qquad y = t^2\,, \qquad z = t^3\,.$$

The curve C is then *a cubical parabola*. This is one form of *a twisted cubic*, which can be defined as the residual intersection of two quadric surfaces that intersect elsewhere in a straight line (see Exs. 2, 6, below). Finally, if equations (2·1) have the form

$$(2\cdot 6)\qquad x = a\cos t\,, \qquad y = a\sin t\,, \qquad z = bt \qquad (a > 0\,,\ b > 0)\,,$$

the curve C is *a left-handed circular helix*, or machine screw. This may be described as the locus of a point which revolves around the z-axis at a constant distance a from it and at the same time moves parallel to the z-axis at a rate proportional to the angle t of revolution (see Fig. 1). If we had supposed $b < 0$, then the helix would have been right-handed.

A curve can be represented analytically in other ways than by its parametric equations. For example, it is known that one equation in x, y, z represents a surface, and that two independent simultaneous equations in x, y, z, say

$$(2 \cdot 7) \qquad F(x, y, z) = 0\,, \qquad G(x, y, z) = 0\,,$$

represent the intersection of two surfaces, which is a curve. Equations (2·7) are called *implicit equations* of this curve. Sometimes it is convenient to represent a curve by implicit equations, when really the curve under consideration is only part of the intersection of the two surfaces represented by the individual equations (see Ex. 2).

If the implicit equations (2·7) be solved for two of the variables in terms of the third, say for y and z in terms of x, the result can be written in the form

$$(2 \cdot 8) \qquad y = y(x)\,, \qquad z = z(x)\,.$$

These equations represent the same curve as equations (2·7); and they, or the equations which similarly express any two of the coordinates of a variable point on the curve as functions of the third coordinate, are called *explicit equations* of the curve. Each of equations (2·8) separately represents a cylinder projecting the curve onto one of the coordinate planes. So equations (2·8) are a special form of equations (2·7) for which the two surfaces are projecting cylinders.

If the first of the parametric equations (2·1) of a curve C be solved for t as a function of x, and if the result is substituted in the remaining two of these equations, the explicit equations (2·8) of the curve C are obtained. From one point of view, the explicit equations (2·8) of a curve, when supplemented by the identity $x = x$, are parametric equations

$$(2 \cdot 9) \qquad x = x\,, \qquad y = y(x)\,, \qquad z = z(x)$$

of the curve, the parameter now being the coordinate x.

EXERCISES

1. Write explicit equations for the straight line (2 3), the cubical parabola (2 5), and the helix (2 6).

2. Show that the cubical parabola (2 5) is only part of the intersection of the two cylinders

$$y = x^2 , \qquad z = x^3 ,$$

and is also only a part of the intersection of the two quadric surfaces

$$z = xy , \qquad y^2 = zx .$$

3. Prove that the cubical parabola (2·5) intersects any plane

$$ax + by + cz + d = 0$$

in three points, and discuss special cases.

4. Prove that a necessary and sufficient condition that a curve (2 1) be a plane curve is

$$\text{(2 10)} \qquad \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0 \qquad \left(x' = \frac{dx}{dt}, \ldots\right).$$

5. Determine the function f so that the curve

$$x = a \cos t , \qquad y = a \sin t , \qquad z = f(t)$$

shall be a plane curve. What is then the form of the curve?

6. Compare the shapes of the four types of twisted cubics:

a) Cubical ellipse,

$$x = \frac{t^2}{t^2 + 1}, \qquad y = \frac{t}{t^2 + 1}, \qquad z = \frac{t^3}{t^2 + 1}.$$

b) Cubical hyperbola,

$$x = \frac{t^2}{t^2 - 1}, \qquad y = \frac{t}{t^2 - 1}, \qquad z = \frac{t^3}{t^2 - 1}.$$

c) Cubical parabola,

$$x = t^2 , \qquad y = t , \qquad z = t^3 .$$

d) Cubical hyperbolic parabola,

$$x = t^2 , \qquad y = t , \qquad z = \frac{t^3}{t - 1}.$$

3. Arc length. We shall now establish a formula for the square of the element of arc of a space curve analogous to the formula (1·4) for a plane curve. Referring to Figure 2, let us consider a point $P(x, y, z)$ on a curve C represented by equations (2 1) and consider also on C a neighboring point $Q(x + \Delta x, y + \Delta y, z + \Delta z)$. Draw the line PQ,

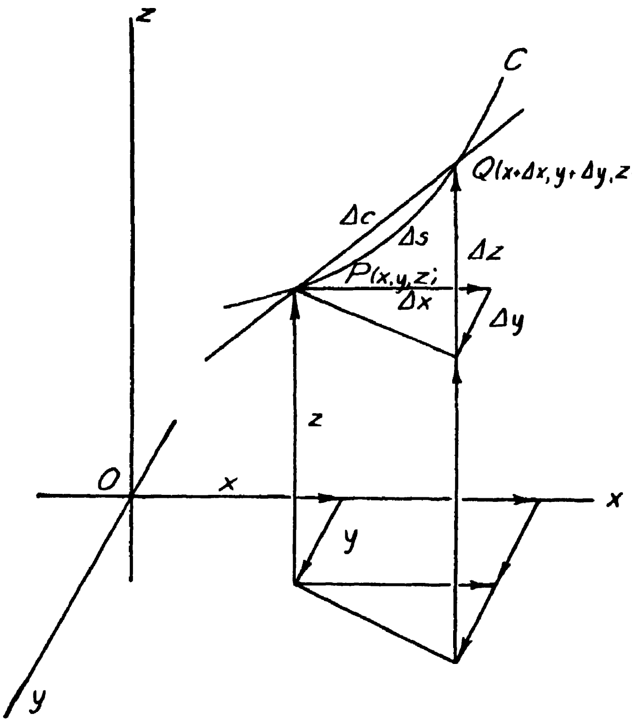

FIG. 2

and let the segment of this line between P and Q be denoted by Δc, while the arc of the curve C between P and Q is denoted by Δs. From elementary geometry it is known that

$$\Delta c^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 . \tag{3 1}$$

Let t be the value of the parameter corresponding to the point P, and let Δt be the increment of the parameter corresponding to displacement along the curve C from P to Q. Then elementary algebra gives

$$\left(\frac{\Delta c}{\Delta s}\right)^2 \left(\frac{\Delta s}{\Delta t}\right)^2 = \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2 . \tag{3 2}$$

Letting Δt approach zèro and assuming that the limit of the ratio of the chord Δc to the arc Δs is unity, i.e., that

$$(3\cdot 3) \qquad \lim_{\Delta s \to 0} \frac{\Delta c}{\Delta s} = 1 ,$$

we obtain

$$(3\cdot 4) \qquad \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 .$$

Multiplying through by dt^2, we arrive at the desired formula for *the square of the element of arc of a space curve, namely,*

$$(3\cdot 5) \qquad ds^2 = dx^2 + dy^2 + dz^2 .$$

A formula for calculating the length of a curve between two points on it can now be deduced. The inequality (2·2) assures us that $ds/dt \neq 0$. Let us solve equation (3·4) for ds, taking the positive square root, and then integrate with respect to t from an arbitrarily chosen fixed lower limit t_0 to a variable upper limit t. With the understanding that the arc length s is zero when $t = t_0$, we reach the required formula,

$$(3\cdot 6) \qquad s = \int_{t_0}^{t} (x'^2 + y'^2 + z'^2)^{1/2} dt \qquad \left(x' = \frac{dx}{dt}, \cdots\right).$$

It should be carefully noted in this connection that taking the positive square root is equivalent to making $ds/dt > 0$, so that s is an increasing function of t. Thus, a positive sense on the curve C has been chosen according to the following convention:

The positive sense on a curve defined by parametric equations is, by agreement, the sense in which the parameter increases.

This convention is not of a purely geometric nature, since the positive sense along a curve C, as agreed upon, is not determined when the curve is given but depends on the parameter used in the analytic representation of the curve. This parameter can be changed at will in the following way. Let us suppose that the parameter t is an arbitrary function of another parameter u, so that

$$(3\cdot 7) \qquad t = t(u) \qquad \left(\frac{dt}{du} \neq 0\right).$$

Then let us substitute this function in equations (2·1), obtaining

$$(3\cdot8)\qquad x = x(t(u))\,,\qquad y = y(t(u))\,,\qquad z = z(t(u))\,.$$

These are the parametric equations of the same curve C referred to the new parameter u. In particular, the transformation of parameter $t = -u$ reverses the positive sense on the curve C.

It may be worth noting that at the close of Section 2 the parameter was, in effect, specialized to be the coordinate x. For some purposes this choice of parameter is very convenient. It will be more convenient for our purposes, however, to choose the arc length s along a curve C, measured from an arbitrarily chosen fixed point on C, as the parameter for C. This choice is effected in the following way. Since $ds/dt \neq 0$, it follows that the integral in equation (3·6) can be inverted, so that this equation can be solved for t as a function of s; let the result of this solution be

$$(3\cdot9)\qquad t = t(s)\,.$$

Substituting in equations (2 1) as we did before, we now obtain *the parametric equations of the curve C with the arc length s as parameter:*

$$x = x(t(s))\,,\qquad y = y(t(s))\,,\qquad z = z(t(s))\,.$$

From the foregoing discussion the following conclusion can be drawn.

THEOREM 1. *It is no restriction on a curve to suppose that the parameter used in writing the parametric equations of the curve is the arc length s measured from an arbitrarily chosen fixed point of the curve.*

It will be supposed hereinafter, unless the contrary is indicated, that the parameter is the arc length s, and then accents will indicate derivatives with respect to s. When the parameter is not necessarily the arc length, some letter other than s will be used to denote the parameter. Equation (3·5) can be re-written in the form

$$(3\cdot10)\qquad x'^2 + y'^2 + z'^2 = 1 \qquad \left(x' = \frac{dx}{ds}\,,\cdots\right)$$

from which one obtains, by differentiation,

$$(3\cdot11)\qquad x'x'' + y'y'' + z'z'' = 0\,.$$

These equations will be used frequently in succeeding sections.

EXERCISES

1. Show that the parametric equations of the helix (2 6), when the parameter is the arc length s, measured from the point of the helix on the x-axis, become

$$(3\ 12)\quad x = a \cos \frac{s}{(a^2 + b^2)^{1/2}}, \quad y = a \sin \frac{s}{(a^2 + b^2)^{1/2}}, \quad z = \frac{bs}{(a^2 + b^2)^{1/2}}.$$

2. The length of one turn of the helix is $2\pi(a^2 + b^2)^{1/2}$.

3. When the equations of a curve are written in the explicit form (2 8), the arc length may be calculated by the formula

$$(3\ 13)\qquad s = \int_{x_0}^{x} (1 + y'^2 + z'^2)^{1/2} dx \qquad \left(y' = \frac{dy}{dx}, \cdots\right).$$

4. Show how to calculate the arc length when the equations of a curve are written in the implicit form (2 7).

5. The length of the curve

$$2ay = x^2, \qquad 6a^2z = x^3$$

from the origin to any point (x, y, z) is $x + z$.

6. The length of the curve

$$y = 2(ax)^{1/2} - x, \qquad z = x - \frac{2}{3}\left(\frac{x^3}{a}\right)^{1/2}$$

from the origin to any point (x, y, z) is $x + y - z$.

7. Find the length of the curve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad x = \frac{a}{2}(e^{z/a} + e^{-z/a})$$

from the point $(a, b, 0)$ to any point (x, y, z).

8. Investigate the most general transformation of parameter preserving the condition

$$(3\cdot 14)\qquad x'^2 + y'^2 + z'^2 = 1 \qquad \left(x' = \frac{dx}{dt}, \cdots\right).$$

4. The local trihedron. The local trihedron at a point of a space curve is a trihedron having for edges three mutually perpendicular straight lines through the point, of which one is *the tangent* and the other two are normals, called, respectively, *the principal normal* and *the binormal.* In order to define the local trihedron more explicitly, we proceed to make use of the hypothesis of analyticity formulated in Definition 1 of Section 2. Except when the contrary is indicated, the points of curves under consideration will always be *ordinary* points, i.e., nonsingular points at which nothing special or exceptional occurs.

Let us consider an analytic curve C whose parametric equations, with the arc length s measured from a fixed point P_0 on C as parameter, are

$$(4\cdot 1) \qquad x = x(s)\,, \qquad y = y(s)\,, \qquad z = z(s)\,.$$

As in Figure 3, let $P(x, y, z)$ be a point on C, corresponding to a certain value s of the parameter, and let $P_1(x_1, y_1, z_1)$ be a neighboring point

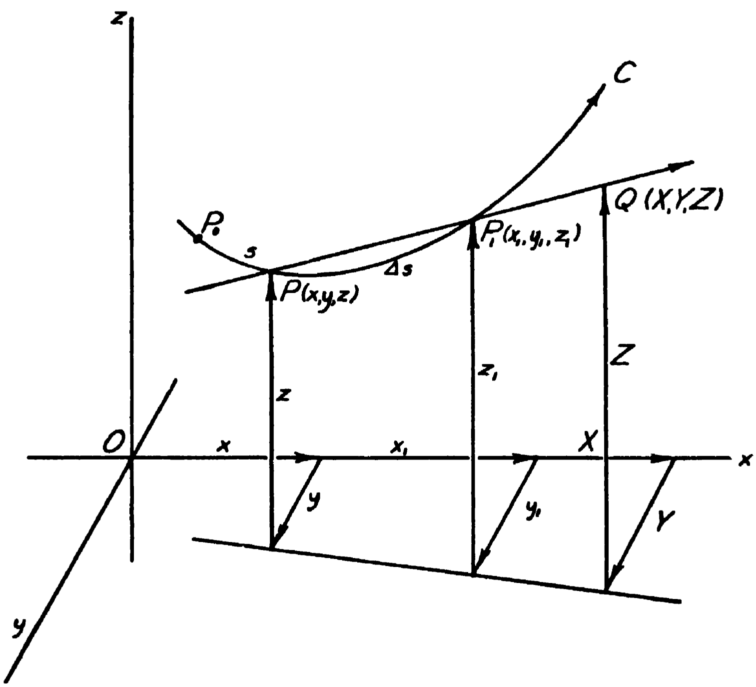

FIG. 3

on C, corresponding to a value $s + \Delta s$ of the parameter. Since the curve C is analytic, each of the functions x_1, y_1, z_1 can be expanded into a power series in Δs, so that we have

$$(4\cdot 2) \qquad \begin{cases} x_1 = x + x'\Delta s + \frac{1}{2}x''\Delta s^2 + \cdots, \\ y_1 = y + y'\Delta s + \frac{1}{2}y''\Delta s^2 + \cdots, \\ z_1 = z + z'\Delta s + \frac{1}{2}z''\Delta s^2 + \cdots \end{cases} \qquad \left(x' = \frac{dx}{ds}\,, \cdots\right).$$

Here each of the functions x_1, y_1, z_1 is to be evaluated at $s + \Delta s$, while each of $x, y, z, x', y', z', x'', \ldots$ is to be evaluated at s; moreover, Δs is supposed to be so small that all the series (4·2) converge. These series will be fundamental in much of what follows.

The tangent line, or briefly *the tangent*, of a space curve may be defined with the same language as the tangent of a plane curve.

DEFINITION 1. *The tangent line at a point P of a curve C is the limit of the secant line through P and a neighboring point P_1 on C, as P_1 approaches P along C.*

In order to find the equations of the tangent line, let us consider a variable point $Q(X, Y, Z)$ on the secant line PP_1, as in Figure 3. The so-called *two-point formula* for the equations of a straight line gives immediately the equations of the secant line, namely,

$$\frac{X - x}{x_1 - x} = \frac{Y - y}{y_1 - y} = \frac{Z - z}{z_1 - z}. \tag{4·3}$$

Substituting for x_1, y_1, z_1 the respective power series (4·2), multiplying through by Δs, and then letting Δs approach zero, we obtain *the equations of the tangent line at the point P of the curve C*,

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'} \qquad \left(x' = \frac{dx}{ds}, \cdots\right). \tag{4·4}$$

Here X, Y, Z are the coordinates of a variable point on the tangent, and x, y, z, x', y', z' are to be evaluated at the value of s corresponding to *the contact point P*.

Let us denote *the direction cosines of the tangent* by α, β, γ. Then, except possibly for sign, these are respectively equal to x', y', z'. To remove the ambiguity of sign, we choose the positive sense on the tangent according to the following convention:

The positive sense on the tangent is, by agreement, such that

$$\alpha = x', \qquad \beta = y', \qquad \gamma = z'. \tag{4·5}$$

If t is the algebraic distance along the tangent from the contact point $P(x, y, z)$ to the variable point (X, Y, Z), then each of the ratios in equations (4·4) is equal to t. Consequently, the equations of the tangent can be written in the parametric form,

$$X = x + \alpha t, \qquad Y = y + \beta t, \qquad Z = z + \gamma t. \tag{4·6}$$

We next define one of the faces of the local trihedron, called *the normal plane.*

DEFINITION 2. *The normal plane at a point P of a curve C is the plane through P perpendicular to the tangent line of C at P.*

The equation of the normal plane can be written by observing that the equation of any plane through the point (x, y, z) has the form

$$a(X - x) + b(Y - y) + c(Z - z) = 0, \tag{4·7}$$

in which a, b, c are proportional to the direction cosines of any line perpendicular to the plane. Since the tangent line is perpendicular to the normal plane, it follows that *the equation of the normal plane is*

$$\alpha(X - x) + \beta(Y - y) + \gamma(Z - z) = 0. \tag{4·8}$$

Another face of the local trihedron, called *the osculating plane,* is, in some sense, an analogue of the tangent line. A tangent line of a curve is sometimes described by saying that it is *a two-point line* of the curve, or also by saying that it intersects the curve in *two consecutive points* at its point of contact. Now, just as a line is determined by two points, so a plane is determined by three points. Consequently, in studying a space curve C, it is suggested to consider *a three-point plane,* or a plane that intersects C in *three consecutive points* at a point P. Such a plane is called *the osculating plane* of C at P, and P is referred to as its *point of osculation.*

DEFINITION 3. *The osculating plane at a point P of a curve C is the limit of the plane through P and two neighboring points P_1, P_2 on C and not collinear with P, as each of P_1, P_2 independently approaches P along C.*

In order to write the equation of the osculating plane, let us recall that the equation of the plane determined by the three points P, P_1, P_2 can be written in the form

$$\begin{vmatrix} X & Y & Z & 1 \\ x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix} = 0, \tag{4·9}$$

a variable point on the plane having coordinates X, Y, Z, the point P having coordinates x, y, z, and the point P_i having coordinates x_i, y_i, z_i $(i = 1, 2)$. Let us substitute for x_1, y_1, z_1 the power series

(4 2) with Δs_1 in place of Δs, and similarly substitute for x_2, y_2, z_2 the series (4·2) with Δs_2 in place of Δs. Then, taking suitable linear combinations of rows, and making use of certain elementary properties of determinants, we reduce the equation (4 9) whose left member is a determinant of the fourth order to the following equation, whose left member contains as a factor a determinant of only the third order:

$$(4\cdot 10) \quad \begin{vmatrix} X - x & Y - y & Z - z \\ x' + \cdots & y' + \cdots & z' + \cdots \\ x'' + \cdots & y'' + \cdots & z'' + \cdots \end{vmatrix} \tfrac{1}{2}\Delta s_1 \Delta s_2 (\Delta s_2 - \Delta s_1) = 0\,,$$

the dots indicating terms which vanish when Δs_1 and Δs_2 approach zero. Dividing this equation by the factor outside of the determinant, and letting Δs_1 and Δs_2 approach zero, we obtain *the equation of the osculating plane*,

$$(4\cdot 11) \quad \begin{vmatrix} X - x & Y - y & Z - z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0\,.$$

On interchanging rows and columns and writing only a typical row within parentheses, this equation becomes

$$(4\cdot 12) \quad (X - x, x', x'') = 0\,.$$

Another way of arriving at the equation of the osculating plane is instructive. Let us return to the equation (4 7) of any plane through the point $P(x, y, z)$, and substitute therein in place of X, Y, Z the power series (4·2) for x_1, y_1, z_1, writing Δs_1 in place of Δs to correspond to the point P_1. The result is a necessary and sufficient condition that the plane (4·7) contain P_1, or that it contain the secant line PP_1. Dividing through by Δs_1, and then letting Δs_1 approach zero, we obtain the following condition on a, b, c:

$$ax' + by' + cz' = 0\,.$$

This condition is necessary and sufficient that any normal to the plane (4·7) be perpendicular to the tangent line at the point P of the curve C, or, in other words, that the plane contain the tangent line of C at P. A plane satisfying this condition may be called *a two-point plane* of the curve C at the point P. Let us again substitute in equation (4·7) in place of X, Y, Z the power series (4·2) with subscripts 2, where we previously had subscripts 1, to correspond to the point P_2. Making use of the condition just found for a two-point plane, let us divide through by Δs_2^2 and then let Δs_2 approach zero; the result is the following condition on a, b, c:

$$ax'' + by'' + cz'' = 0 .$$

Elimination of a, b, c from equation (4·7) and the two equations just found yields again the equation (4·11) of the osculating plane.

It may be observed that the equation (4·7) and the two conditions just found on a, b, c leading to the equation of the osculating plane could have been obtained by complying with the following rule. *Write the most general equation of a plane in variable coordinates X, Y, Z, and demand that this equation be satisfied by the power series* (4·2) *identically in Δs as far as the terms in Δs^2.* This process will be employed later on in analogous situations.

All the lines of the one-parameter family of straight lines passing through a point P of a curve C and perpendicular to the tangent line of C at P are naturally called *normal lines*, or simply *normals*, of C at P. These lines lie, of course, in the normal plane of C at P and form a flat pencil with center at P. Among these normals there are two which are distinguished from the rest. They are used as the two remaining edges of the local trihedron and are named and defined in the following way.

DEFINITION 4. *The principal normal at a point P of a curve C is the normal line that lies in the osculating plane of C at P.*

DEFINITION 5. *The binormal at a point P of a curve C is the normal line that is perpendicular to the osculating plane of C at P.*

In arriving at *the equations of the binormal* it will be convenient to make use of an easily verified algebraic identity known as *Lagrange's identity*, namely,

$$(4\cdot 13) \quad \begin{cases} (a^2 + b^2 + c^2)(l^2 + m^2 + n^2) - (al + bm + cn)^2 \\ \qquad = (am - bl)^2 + (bn - cm)^2 + (cl - an)^2 , \end{cases}$$

in which a, b, c, l, m, n are any six numbers. If a, b, c are the direction cosines of a line, and if l, m, n are the direction cosines of another line, while θ is the angle between the two lines, then

$$a^2 + b^2 + c^2 = 1 , \qquad l^2 + m^2 + n^2 = 1 , \qquad \cos\theta = al + bm + cn ,$$
$$\sin^2\theta = (am - bl)^2 + (bn - cm)^2 + (cl - an)^2 ,$$

and so in this case Lagrange's identity reduces to the trigonometric identity

$$1 - \cos^2 \theta = \sin^2 \theta \, .$$

Denoting *the direction cosines of the binormal* by λ, μ, ν we observe that λ, μ, ν are respectively proportional to the coefficients of X, Y, Z in the equation (4·11) of the osculating plane, so that we may write

$$(4\cdot 14) \qquad \lambda = \rho(y'z'' - y''z') \, , \qquad \mu = \rho(z'x'' - z''x') \, , \qquad \nu = \rho(x'y'' - x''y') \, ,$$

where ρ is a proportionality factor to be determined. Squaring and adding equations (4·14), making use of Lagrange's identity in x', y', z' and x'', y'', z'', and keeping in mind equations (3 10) and (3·11), we obtain

$$(4\cdot 15) \qquad \frac{1}{\rho^2} = x''^2 + y''^2 + z''^2 \, .$$

This equation determines ρ except for sign. This ambiguity of sign is removed by the following convention with regard to the positive sense on the binormal, which enables us always to take the positive square root when solving for ρ:

The positive sense on the binormal is, by agreement, such that ρ is positive.

The equations of the binormal can immediately be written and are

$$(4\cdot 16) \qquad \frac{X - x}{\lambda} = \frac{Y - y}{\mu} = \frac{Z - z}{\nu} \, .$$

The equations of the principal normal can be written in the form

$$(4\cdot 17) \qquad \frac{X - x}{l} = \frac{Y - y}{m} = \frac{Z - z}{n} \, ,$$

in which the direction cosines l, m, n will be shown to have the values given by

$$(4\cdot 18) \qquad l = \rho x'' \, , \qquad m = \rho y'' \, , \qquad n = \rho z'' \, .$$

To make the demonstration, let us observe that, since the principal normal is perpendicular both to the binormal and to the tangent, we have

$$\begin{aligned} l(y'z'' - y''z') + m(z'x'' - z''x') + n(x'y'' - x''y') &= 0 \, , \\ lx' + my' + nz' &= 0 \, . \end{aligned}$$

Solving these equations for the ratios of l, m, n, and keeping always in mind the fundamental equations (3·10) and (3·11), we obtain

$$l = kx'', \qquad m = ky'', \qquad n = kz'', \tag{4·19}$$

where k is a proportionality factor to be determined. Squaring and adding these equations, we find by (4·15) that $k^2 = \rho^2$. The ambiguity in the sign of k is removed, and the proof completed, by making the following convention with regard to the positive sense on the principal normal:

The positive sense on the principal normal is, by agreement, such that $k = \rho$.

The three edges of the local trihedron, namely, the tangent, principal normal, and binormal lie by pairs in its three faces, namely, the osculating plane, the normal plane, and the rectifying plane, the last-named plane being defined as follows:

DEFINITION 6. *The rectifying plane at a point P of a curve C is the plane through P perpendicular to the principal normal of C at P.*

It follows at once that *the equation of the rectifying plane is*

$$l(X - x) + m(Y - y) + n(Z - z) = 0. \tag{4·20}$$

Incidentally, it may be observed that the equation (4·11) of the osculating plane can also be written in the form

$$\lambda(X - x) + \mu(Y - y) + \nu(Z - z) = 0. \tag{4·21}$$

EXERCISES

1. The tangent line at a point of a straight line is the line itself. The osculating plane at a point of a plane curve not a straight line is the plane of the curve, and the osculating plane at a point of a straight line is indeterminate.

2. The circular helix (2·6) crosses the generators of the cylinder $x^2 + y^2 = a^2$ at a constant angle, namely,

$$\cos^{-1} \frac{b}{(a^2 + b^2)^{1/2}}.$$

3. The principal normal at any point of the circular helix (2·6) intersects orthogonally the z-axis, i.e., the axis of the cylinder $x^2 + y^2 = a^2$ on which the helix lies.

4. The equations (4·4) of the tangent and (4·11) of the osculating plane are unchanged in form if the parameter is changed from the arc length s to any parameter t. In this case the formula (4·15) for ρ becomes

$$\frac{1}{\rho^2} = \frac{\Sigma x'^2 \Sigma x''^2 - (\Sigma x' x'')^2}{(\Sigma x'^2)^3}, \tag{4·22}$$

the summation being for cyclical permutation of x, y, z. Study the effect of this change of parameter on the remaining formulas of this section.

5. Through any point Q in space there pass three osculating planes of the cubical parabola (2 5). The three points of osculation of these planes lie in a plane which passes through Q.

6. The matrix

$$\begin{pmatrix} \alpha & \beta & \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{pmatrix}$$

is orthogonal. The determinant of this matrix has the value $+1$. The sum of the squares of the elements in any row (or column) is unity. The sum of the products of the elements in any row (or column) by the corresponding elements of a different row (or column) is zero. Each element of the matrix is equal to its own cofactor, so that

$$(4\ 23)\quad \begin{cases} \alpha = m\nu - n\mu\,, & \beta = n\lambda - l\nu\,, & \gamma = l\mu - m\lambda\,, \\ l = \mu\gamma - \nu\beta\,, & m = \nu\alpha - \lambda\gamma\,, & n = \lambda\beta - \mu\alpha\,, \\ \lambda = \beta n - \gamma m\,, & \mu = \gamma l - \alpha n\,, & \nu = \alpha m - \beta l\,. \end{cases}$$

7. If all the osculating planes of a curve pass through a fixed point, the curve is a plane curve.

8. Prove that all the normal planes of the curve

$$x = a\sin^2 t\,, \qquad y = a\sin t\cos t\,, \qquad z = a\cos t$$

pass through the origin.

9. Show that, of all lines through a point P of a curve C, the tangent line is *nearest* the curve, by proving that the distance from a point P_1 on C to the tangent is an infinitesimal of higher order, compared with the arc PP_1 as principal infinitesimal, than the distance from P_1 to any other line through P.

10. Show that, of all planes through the tangent line at a point of a curve, the osculating plane is *nearest* the curve, in the sense of Exercise 9.

5. Curvature and torsion. At each point of a plane curve there is a curvature, which may be calculated by the formula (1·5). But at each point of a space curve there are two curvatures, called, respectively, the *first* and *second* curvatures, or, better, the *curvature* and the *torsion,* which it is the purpose of this section to discuss.

Intuitively, *the curvature* at a point of a space curve, like that of a plane curve, is a measure of the rate at which the tangent at the point rotates when the point of contact moves along the curve. More precisely, the curvature at a point of a curve is the rate of change in the direction of the tangent line, per unit arc length of the curve. In still more exact language, the curvature may be defined as follows:

DEFINITION 1. *The curvature at a point P of a curve C is the limit of the ratio of the angle between the tangent at P and the tangent at a neighboring point P_1 of C, to the arc PP_1, as the point P_1 approaches P along C.*

If the angle between the two tangents at P and P_1 is denoted by $\Delta\theta$, and if the arc PP_1 is denoted by Δs, it follows immediately that the curvature of C at P is

$$\lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s}.$$

For the purpose of finding an expression for the curvature at a point of a curve in terms of derivatives of the coordinates of the point, it is advantageous to make use of a *spherical indicatrix,* in the following way. Let us construct a sphere with center at the origin and with unit radius. Let us then consider a point P on a curve C and on C a neighborhood of P such that no two of the tangents of C at points in this neighborhood are parallel to each other. Finally, let us draw radii of the sphere parallel to all these tangents. The locus of the extremities of these radii is a curve Γ on the sphere called *the spherical indicatrix of the tangents of the curve C.* The points of the indicatrix Γ are in one-to-one correspondence with the points of the curve C. Let Q and Q_1 be the points of Γ corresponding, respectively, to the points P and P_1 of C. Let the arc of the indicatrix between Q and Q_1 be denoted by $\Delta\sigma$, and let the arc of the great circle through Q, Q_1 be denoted by Δc. Assuming that

$$\lim_{\Delta\sigma \to 0} \frac{\Delta c}{\Delta\sigma} = 1,$$

we find

$$(5\cdot 1) \qquad \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \lim \frac{\Delta c}{\Delta s} = \lim \frac{\Delta c}{\Delta\sigma}\frac{\Delta\sigma}{\Delta s} = \lim \frac{\Delta\sigma}{\Delta s} = \frac{d\sigma}{ds},$$

since $\Delta\theta = \Delta c$, in the sense that the number of radians in $\Delta\theta$ is equal to the number of linear units (radii of the sphere) in Δc. But since the coordinates of a point at unit distance from the origin are equal to the direction cosines of the line from the origin to the point, it follows that the coordinates of the point Q are α, β, γ. The formula $(3\cdot 4)$ for the derivative of the arc of a curve can now be applied to the indicatrix Γ, along which the parameter is the arc length s on C. In this way, and by means of equations (4 5), (4 15), one obtains

$$(5\cdot2)\qquad \left(\frac{d\sigma}{ds}\right)^2 = \alpha'^2 + \beta'^2 + \gamma'^2 = x''^2 + y''^2 + z''^2 = \frac{1}{\rho^2}.$$

It follows that $d\sigma/ds = 1/\rho$ if the convention is made that the arc σ on the indicatrix shall be an increasing function of the arc s on the curve C. The result may be stated in the form of a theorem.

THEOREM 1. *The curvature at a point of a curve is* $1/\rho$, *the function* ρ *being positive and computed by equation* (4 15).

The torsion at a point of a curve is, intuitively, the rate at which the binormal at the point rotates when the point moves along the curve. The following definition of the torsion is obtained from the definition of the curvature by substituting the *binormal* for the *tangent.*

DEFINITION 2. *The torsion at a point* P *of a curve* C *is the limit of the ratio of the angle between the binormal at* P *and the binormal at a neighboring point* P_1 *of* C, *to the arc* PP_1, *as the point* P_1 *approaches* P *along* C.

If the angle between the two binormals at P and P_1 is denoted by $\Delta\phi$, and if the arc PP_1 is denoted by Δs as usual, then the torsion of C at P is

$$\lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s}.$$

The spherical indicatrix of the binormals of the curve C is constructed in the same way as the indicatrix of the tangents, the binormals being used in place of the tangents. If σ now denotes arc length on the indicatrix of the binormals, and if *the torsion is denoted by* $1/\tau$, it is easy to show that

$$(5\cdot3)\qquad \frac{1}{\tau^2} = \left(\frac{d\sigma}{ds}\right)^2 = \lambda'^2 + \mu'^2 + \nu'^2.$$

Therefore the torsion is given, except for sign, by the formula

$$(5\cdot4)\qquad \frac{1}{\tau} = \pm(\lambda'^2 + \mu'^2 + \nu'^2)^{1/2}.$$

The ambiguity of sign will be removed in the next section.

EXERCISES

1. The curvature at a point of a plane curve of the family defined by the equations

$$z = 0, \qquad M(x, y)dx + N(x, y)dy = 0$$

is given by the formula

$$\frac{1}{\rho} = \frac{MN(M_y + N_x) - N^2M_x - M^2N_y}{(M^2 + N^2)^{3/2}},$$

where the subscripts indicate partial derivatives.

2. The curvature of the circular helix (2 6) is given by the formula

$$(5\ 5) \qquad \frac{1}{\rho} = \frac{a}{a^2 + b^2}$$

and is therefore constant.

3. If the curvature of a real curve is zero at every one of its points, the curve is a straight line, and conversely.

4. Let a circle* of radius a be drawn on a sheet of paper which is then rolled into a circular cylinder of radius b. If the cylinder is placed with its axis on the z-axis and with the bottom of the rolled circle on the x-axis, the parametric equations of the curve are

$$x = b \cos\left(\frac{a}{b} \sin \frac{s}{a}\right), \qquad y = b \sin\left(\frac{a}{b} \sin \frac{s}{a}\right), \qquad z = a\left(1 - \cos \frac{s}{a}\right),$$

where s is the arc of the curve measured from the bottom. The curvature of the curve is given by the formula

$$(5\ 6) \qquad \frac{1}{\rho^2} = \frac{1}{a^2} + \frac{1}{b^2} \cos^4 \frac{s}{a}.$$

5. Consider a point P of a curve C and a point P_1 near P on C, in the positive sense from P. On the positive end of the tangent of C at P lay off a segment PM equal in length to the arc Δs of C between P and P_1. Denote the length of the segment P_1M by d. Then prove that the curvature $1/\rho$ of C at P is given by

$$(5\ 7) \qquad \frac{1}{\rho} = \lim_{\Delta s \to 0} \frac{2d}{\Delta s^2}.$$

* C. Smith, *Solid Geometry* (London: MacMillan & Co., 1910), p. 212, Ex. 20.

6. If in the formula (1 5) for the curvature of a plane curve the independent variable is changed from x to s and the convention is observed that the positive square root is to be taken in the denominator, the curvature is found to be given by

$$(5\cdot 8)\qquad \frac{1}{\rho} = x'y'' - x''y' \qquad \left(x' = \frac{dx}{ds}, \cdots\right),$$

and so may be either positive or negative in sign. But if $z = 0$ in (4 15) and if the convention is observed that the positive square root is to be taken, the curvature of a plane curve is found to be given by

$$(5\cdot 9)\qquad \frac{1}{\rho} = (x''^2 + y''^2)^{1/2},$$

and so is always positive. Reconcile these results with regard to both the magnitude and the sign of the curvature.

6. Frenet formulas. The Frenet formulas express the first derivatives, with respect to the arc length, of the nine direction cosines of the three edges of the local trihedron linearly in terms of these cosines themselves, the coefficients being, except possibly for sign, the curvature and torsion. When written in our customary notation *the Frenet formulas are*

$$(6\cdot 1)\qquad \begin{cases} \alpha' = \dfrac{l}{\rho}, & \beta' = \dfrac{m}{\rho}, & \gamma' = \dfrac{n}{\rho}, \\ l' = -\dfrac{\alpha}{\rho} - \dfrac{\lambda}{\tau}, & m' = -\dfrac{\beta}{\rho} - \dfrac{\mu}{\tau}, & n' = -\dfrac{\gamma}{\rho} - \dfrac{\nu}{\tau}, \\ \lambda' = \dfrac{l}{\tau}, & \mu' = \dfrac{m}{\tau}, & \nu' = \dfrac{n}{\tau}, \end{cases}$$

the accent indicating differentiation with respect to the arc length s.

The formulas (6·1) can be established in the following way. The three formulas in the first row result at once upon differentiating equations (4 5) and eliminating x'', y'', z'' by means of (4·18). In order to arrive at the three formulas in the third row, let us begin by differentiating the equations

$$\lambda^2 + \mu^2 + \nu^2 = 1, \qquad \alpha\lambda + \beta\mu + \gamma\nu = 0.$$

The derived equations can be reduced, by means of the first three of the Frenet formulas (already established) and the orthogonality of the principal normal and binormal, to

$$\lambda\lambda' + \mu\mu' + \nu\nu' = 0, \qquad \alpha\lambda' + \beta\mu' + \gamma\nu' = 0.$$

Solving these equations for the ratios of λ', μ', ν', and employing the equations in the second line of (4·23), we get

$$\lambda' = kl\,, \qquad \mu' = km\,, \qquad \nu' = kn\,,$$

where k is a proportionality factor to be determined. Squaring and adding these equations, and making use of equation (5·4), we find $1/\tau^2 = k^2$. Here is an opportunity to *keep the promise made at the end of Section* 5, *to remove the ambiguity in the sign of the torsion*, by the following convention with regard to it:

The sign of the torsion $1/\tau$ *is, by agreement, such that* $1/\tau = k$.

The Frenet formulas in the second line of (6·1) can be established by first differentiating the equations in the second line of (4·23), and then using the Frenet formulas already established, as well as the remaining equations (4·23).

In spite of the fact that the formula (5·4) for the torsion apparently contains an irrationality, it is possible to show, in the following way, that *the torsion is a rational function of the derivatives of the coordinates* x, y, z. Differentiation of equations (4·14), followed by some simple reductions, results in

$$(6\cdot2)\qquad \begin{cases} y'z''' - y'''z' = \dfrac{1}{\rho\tau}\,l - \dfrac{\rho'}{\rho^2}\,\lambda\,, \\[2ex] z'x''' - z'''x' = \dfrac{1}{\rho\tau}\,m - \dfrac{\rho'}{\rho^2}\,\mu\,, \\[2ex] x'y''' - x'''y' = \dfrac{1}{\rho\tau}\,n - \dfrac{\rho'}{\rho^2}\,\nu\,. \end{cases}$$

By means of these equations and (4·15) it is easy to verify the following formula,

$$(6\cdot3)\qquad \frac{1}{\tau} = -\frac{1}{x''^2 + y''^2 + z''^2}\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

after first expanding the determinant on the elements of the second row. This formula exhibits the torsion $1/\tau$ as a rational function of the first, second, and third derivatives of x, y, z.

Let us consider the system of differential equations

$$(6\cdot4)\qquad u' = \frac{v}{\rho}\,, \qquad v' = -\frac{u}{\rho} - \frac{w}{\tau}\,, \qquad w' = \frac{v}{\tau} \qquad (\rho > 0)\,,$$

in three unknown functions u, v, w, accents indicating differentiation with respect to an independent variable s, and ρ, τ being functions of s. The Frenet formulas (6·1) assert that, if the functions ρ, τ and the variable s have the significance usually attached to them in curve theory, then three triples of solutions of the differential equations (6·4) are

$$\alpha, l, \lambda\,; \qquad \beta, m, \mu\,; \qquad \gamma, n, \nu\,.$$

It would be possible to start out from the equations (6·4) as fundamental and base a theory of curves on them. First of all, by means of the theory of such differential equations, one would prove the following theorem.

THEOREM 1. *The differential equations* (6·4) *define a curve uniquely, except for its position in space; and for this curve* $1/\rho$ *and* $1/\tau$ *are, respectively, the curvature and the torsion, and* s *is the arc length.*

This method of developing curve theory is intimately related to another method, which takes the coefficients of equations (6·4) as fundamental and which is based on the following theorem.

THEOREM 2. *If two functions* f_1, f_2 *of a variable* s *are given, if* $f_1 > 0$, *and if* $1/\rho$, $1/\tau$ *are defined by placing*

$$(6\cdot 5) \qquad \frac{1}{\rho} = f_1(s)\,, \qquad \frac{1}{\tau} = f_2(s)\,,$$

then there exists a curve which is uniquely determined except for its position in space, and for which s *is the arc length and* $1/\rho$, $1/\tau$ *are, respectively, the curvature and the torsion.*

This theorem is scarcely more than a restatement of Theorem 1, and its proof may be made to depend on the integration of equations (6·4); but we shall not go into details* here. The equations (6·5) are of so much importance that they are given a name by the following definition.

DEFINITION 1. *The intrinsic equations of a curve are two equations expressing the curvature and torsion of the curve as functions of its arc length.*

The theory of curves can proceed from the intrinsic equations† as a starting-point, but this method of studying curves will not be developed to any great extent here. However, it seems proper to make a distinction between *intrinsic equations*, as just defined, and *natural equations*, defined in the following way.

* Eisenhart, *Differential Geometry*, pp. 22–28.

† Cesaro, *Geometria intrinseca*, Chap. IX

DEFINITION 2. *A natural equation of a curve is any equation connecting the curvature* $1/\rho$, *the torsion* $1/\tau$, *and the arc length* s *of the curve.*

Evidently a natural equation,

$$f\left(\frac{1}{\rho}, \frac{1}{\tau}, s\right) = 0,$$

of a curve imposes a condition on the curve, so that the curve has certain special properties; but there may be many curves having these properties. For example, $1/\tau = 0$ is a *natural* equation characterizing all plane curves not straight lines, and $1/\rho = 0$ is a natural equation characterizing all straight lines. An additional independent natural equation $g = 0$ of the curve specializes the curve still more. If the two natural equations $f = 0$, $g = 0$ be solved simultaneously for $1/\rho$, $1/\tau$ as functions of s, the intrinsic equations (6 5) result. Therefore, *two natural equations ordinarily determine a curve uniquely, except for its position in space.*

EXERCISES

1. The torsion of a plane curve not a straight line is zero. The torsion of a straight line is indeterminate.

2. If the parameter that varies along a curve is changed from the arc length s to any parameter t, the torsion of the curve is then given by the formula

$$\frac{1}{\tau} = -\frac{1}{\Sigma x'^2 \Sigma x''^2 - (\Sigma x' x'')^2}\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}, \tag{6 6}$$

the summation being for cyclical permutation of x, y, z.

3. The torsion of the left-handed circular helix (2 6) is given by the formula

$$\frac{1}{\tau} = -\frac{b}{a^2 + b^2}$$

and is therefore constant. Moreover, *the torsion of a left-handed helix is negative* when the coordinate system is left-handed. *The torsion of a right-handed helix is positive* when the coordinate system is left-handed. Discuss the situation if the coordinate system were right-handed.

4. In Definition 1 in Section 5 replace the tangent by the principal normal, and show that the resulting so-called *third curvature* is

$$\left(\frac{1}{\rho^2} + \frac{1}{\tau^2}\right)^{1/2}.$$

Carry out a similar investigation for any line through the point P, whose direction cosines are functions of the arc length s.

5. When a point P moves along a curve C, the local trihedron of C at P moves as a rigid body with a screw motion, which is compounded of a translation and a rotation. Show that the direction cosines of *the instantaneous axis of rotation** are proportional to

$$(6\ 7)\qquad \frac{\lambda}{\rho}-\frac{\alpha}{\tau}, \quad \frac{\mu}{\rho}-\frac{\beta}{\tau}, \quad \frac{\nu}{\rho}-\frac{\gamma}{\tau}.$$

This line lies in the rectifying plane. As in Exercise 4, show that the rate of change in the direction of the instantaneous axis, per unit arc length of the curve, is

$$\left(\tan^{-1}\frac{\tau}{\rho}\right)'.$$

6. The parametric equations of any plane curve can be written in the form

$$(6\ 8)\qquad x = a + \int_{s_0}^{s} \cos\sigma ds, \quad y = b + \int_{s_0}^{s} \sin\sigma ds, \quad z = 0,$$

where

$$\sigma = \int_{s_0}^{s} \frac{ds}{\rho} = \tan^{-1}\frac{dy}{dx}$$

and s_0 is a fixed value of s.

7. For the plane curve of Exercise 6 verify the Frenet formulas

$$(6\ 9)\qquad \begin{cases} \alpha' = \dfrac{l}{\rho}, & \beta' = \dfrac{m}{\rho}, \\ l' = -\dfrac{\alpha}{\rho}, & m' = -\dfrac{\beta}{\rho}, \end{cases}$$

where

$$\alpha = \cos\sigma, \quad \beta = \sin\sigma, \quad l = -\sin\sigma, \quad m = \cos\sigma.$$

8. Use the result of Exercise 6 to verify the differential equations

$$(6\ 10)\qquad x'' = -\frac{1}{\rho}y', \quad y'' = \frac{1}{\rho}x',$$

* Ziwet and Field, *Introduction to Analytic Mechanics* (New York: Macmillan Co., 1912), Chap. IV.

and then prove that x and y both satisfy *the differential equation of plane curves*,

$$x''' + \frac{\rho'}{\rho} x'' + \frac{1}{\rho^2} x' = 0 \qquad \left(x' = \frac{dx}{ds}, \cdots\right). \tag{6 11}$$

9. Defining a *cylindrical helix* to be a curve which lies on a cylinder and crosses the generators of the cylinder at a constant angle, show that this curve is the same as a *gradient curve*, defined to be a curve whose tangents make a constant angle with a fixed direction. Prove that the spherical indicatrix of the tangents of such a curve is a circle, and that *a necessary and sufficient condition that a curve be a cylindrical helix is that the ratio of its curvature to its torsion be constant.*

10. If the principal normals of a curve are all parallel to a fixed plane, the curve is a cylindrical helix.

11. Find the parametric equations for the cylindrical helix whose intrinsic equations are $\rho = s$, $\tau = s$.

12 Calculate ρ and τ for the curve

$$x = a(t - \sin t), \qquad y = a(1 - \cos t), \qquad z = bt.$$

13. Discuss the curve for which

$$\rho = as, \qquad \tau = bs \qquad (a, b = \text{consts.})$$

7. The local coordinate system. Frequently in analytic geometry a special choice of the coordinate system simplifies the equations of the configuration which is being studied. A familiar illustration may be found in plane analytic geometry. There the general equation of an ellipse has six terms, whereas, when the origin of coordinates is at the center of the ellipse and the axes of coordinates coincide with the principal axes of the ellipse, the equation takes a simpler form, containing only three terms. This specially chosen coordinate system is said to be *covariantly connected with*, or *covariant to*, the ellipse, in the sense that the origin and coordinate axes are defined geometrically in terms of the ellipse in a way which is invariant under rigid motions in the plane. One of the first problems in any analytic geometric investigation of a configuration is to discover a covariant coordinate system, i.e., a coordinate system which is defined geometrically in terms of the configuration in a way which is invariant under the group of transformations defining the geometry in view, since the introduction of such a coordinate system usually simplifies the analysis.

We have not hitherto specialized our coordinate system beyond prescribing that it is a left-handed orthogonal cartesian system with the same unit of distance for all three axes. We now propose to choose a new coordinate system that will be covariantly connected with the point P and the curve C under consideration, so that we may expect to enjoy corresponding analytic simplifications. More specifically, let

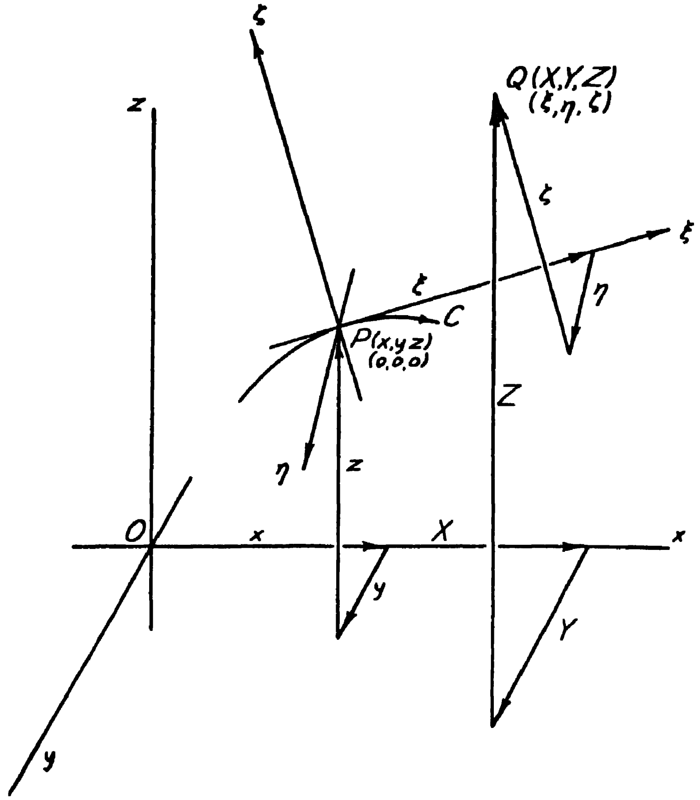

FIG. 4

us take the new x-axis along the tangent, the new y-axis along the principal normal, and the new z-axis along the binormal of C at P, positive senses on coinciding lines being the same, the unit of distance remaining the same as before, and C being supposed to be not a straight line. The new coordinate system will be called *the local coordinate system at the point P of the curve C*. From the equation (see Ex. 6 of Sec. 4)

$$\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} = 1 \,,$$

it follows that *the local coordinate system has the same orientation as the original coordinate system,* and therefore is left-handed. The truth of this assertion, and also that of the next statement below, will be assumed as well known, since transformation of coordinates is thoroughly discussed in any standard text on three-dimensional analytic geometry* (see Fig. 4).

The equations of transformation between the unspecialized (or old) coordinates X, Y, Z of a point Q and the local coordinates ξ, η, ζ of Q referred to the local coordinate system at a point P, with the old coordinates x, y, z, of a curve C are

$$(7\cdot 1)\qquad \begin{cases} X - x = \alpha\xi + l\eta + \lambda\zeta\,, \\ Y - y = \beta\xi + m\eta + \mu\zeta\,, \\ Z - z = \gamma\xi + n\eta + \nu\zeta\,. \end{cases}$$

Solution of these equations for ξ, η, ζ results in an equivalent set of equations,

$$(7\cdot 2)\qquad \begin{cases} \xi = \alpha(X - x) + \beta(Y - y) + \gamma(Z - z)\,, \\ \eta = l(X - x) + m(Y - y) + n(Z - z)\,, \\ \zeta = \lambda(X - x) + \mu(Y - y) + \nu(Z - z)\,. \end{cases}$$

Both sets of equations can be read off very conveniently from the accompanying table.

(7·3)

	$X - x$	$Y - y$	$Z - z$
ξ	α	β	γ
η	l	m	n
ζ	λ	μ	ν

The following theorem about transformation of direction cosines will be found useful in Chapter II.

THEOREM 1. *If A, B, C are the direction cosines of a line referred to the unspecialized coordinate system, and if a, b, c are the direction cosines*

* Snyder and Sisam, *Analytic Geometry of Space* (New York: Henry Holt & Co., 1914), p. 39.

of the same line referred to the local coordinate system at a point P of a curve C, then the equations of transformation between these cosines can be obtained from equations (7·1) above by writing A, B, C in place of $X - x$, $Y - y$, $Z - z$, and a, b, c in place of ξ, η, ζ.

To prove this theorem let P_1, P_2 be two points on the line under consideration, and denote the coordinates of these points in both systems by subscripts 1 and 2, respectively. Write equations (7·1) for the point P_1, and then for the point P_2, and subtract, to obtain

$$(7\cdot 4)\qquad \begin{cases} X_2 - X_1 = \alpha(\xi_2 - \xi_1) + l(\eta_2 - \eta_1) + \lambda(\zeta_2 - \zeta_1)\,, \\ Y_2 - Y_1 = \beta(\xi_2 - \xi_1) + m(\eta_2 - \eta_1) + \mu(\zeta_2 - \zeta_1)\,, \\ Z_2 - Z_1 = \gamma(\xi_2 - \xi_1) + n(\eta_2 - \eta_1) + \nu(\zeta_2 - \zeta_1)\,. \end{cases}$$

It is easy to verify, and besides is geometrically obvious, since the length of a line segment is preserved by rigid motion, that

$$(7\cdot 5)\ \begin{cases} [(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2]^{1/2} \\ \qquad = [(\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2 + (\zeta_2 - \zeta_1)^2]^{1/2} \end{cases}$$

Dividing the left member of each of equations (7·4) by the left member of (7·5), and similarly for the right members, and remembering the definition of the direction cosines of a line, we obtain

$$(7\cdot 6)\qquad \begin{cases} A = \alpha a + lb + \lambda c\,, \\ B = \beta a + mb + \mu c\,, \\ C = \gamma a + nb + \nu c\,, \end{cases}$$

thus completing the demonstration.

In the local coordinate system the power series expansions (4·2) for the parametric equations of a curve take an especially simple form. As many terms of the simplified expansions as desired can be calculated in the following way. In equations (4·2) replace x', y', z' by α, β, γ, respectively, according to (4·5). Then replace x'', y'', z'' by l/ρ, m/ρ, n/ρ, according to (4·18). To calculate similar expressions for x''', y''', z''', differentiate equations (4·18) to obtain

$$(7\cdot 7)\quad l' = \rho' x'' + \rho x'''\,,\qquad m' = \rho' y'' + \rho y'''\,,\qquad n' = \rho' z'' + \rho z'''$$

and then make use of the Frenet formulas (6 1); the required expressions are

$$(7\cdot 8)\qquad \begin{cases} x''' = -\dfrac{1}{\rho^2}\alpha + \left(\dfrac{1}{\rho}\right)' l - \dfrac{1}{\rho\tau}\lambda\,, \\ y''' = -\dfrac{1}{\rho^2}\beta + \left(\dfrac{1}{\rho}\right)' m - \dfrac{1}{\rho\tau}\mu\,, \\ z''' = -\dfrac{1}{\rho^2}\gamma + \left(\dfrac{1}{\rho}\right)' n - \dfrac{1}{\rho\tau}\nu\,. \end{cases}$$

Expressions for the fourth and higher derivatives of the coordinates can similarly be calculated, by repeated differentiation and use of the Frenet formulas. Substituting these derivatives in equations (4·2), and writing s in place of Δs, as we have a right to do if we suppose that the arc length s is measured from the point P under consideration, we obtain

$$(7\cdot 9)\qquad \begin{cases} x_1 = x + \xi\alpha + \eta l + \zeta\lambda\,, \\ y_1 = y + \xi\beta + \eta m + \zeta\mu\,, \\ z_1 = z + \xi\gamma + \eta n + \zeta\nu\,, \end{cases}$$

where ξ, η, ζ are the following power series,

$$(7\cdot 10)\begin{cases} \xi = s - \dfrac{1}{6\rho^2}s^3 + \dfrac{1}{8}\dfrac{\rho'}{\rho^3}s^4 + \cdots\,, \\ \eta = \dfrac{1}{2\rho}s^2 + \dfrac{1}{6}\left(\dfrac{1}{\rho}\right)' s^3 + \dfrac{1}{24}\left[\left(\dfrac{1}{\rho}\right)'' - \dfrac{1}{\rho^3} - \dfrac{1}{\rho\tau^2}\right] s^4 + \cdots\,, \\ \zeta = -\dfrac{1}{6\rho\tau}s^3 - \dfrac{1}{24}\left[\left(\dfrac{1}{\rho\tau}\right)' + \dfrac{1}{\tau}\left(\dfrac{1}{\rho}\right)'\right] s^4 + \cdots\,. \end{cases}$$

Comparison of equations (7 9) with (7·1) shows that ξ, η, ζ are the local coordinates of the point P_1 near the point P which is the origin of the local coordinates. So the following conclusion can be drawn.

THEOREM 2. *Equations* (7·10) *are the parametric equations of a curve C in the local coordinate system at an ordinary point P of C, the parameter being the arc length s measured from the point P.*

EXERCISES

1. The components ω_ξ, ω_η, ω_ζ of rotation of the local trihedron about the ξ-axis, η-axis, and ζ-axis, respectively (see Ex. 5 of Sec. 6), are given by

$$\omega_\xi = -\frac{1}{\tau}, \qquad \omega_\eta = 0, \qquad \omega_\zeta = \frac{1}{\rho}.$$

2. Prove that the equations of a circle in a general position can be written in the form

$$(7\cdot 11)\qquad \begin{cases} x = r(a_1 \cos t + a_2 \sin t) + h, \\ y = r(b_1 \cos t + b_2 \sin t) + k, \\ z = r(c_1 \cos t + c_2 \sin t) + l, \end{cases}$$

in which r is the radius, and (h, k, l) the center, while the remaining constants satisfy the conditions

$$a_1^2 + b_1^2 + c_1^2 = 1, \qquad a_2^2 + b_2^2 + c_2^2 = 1, \qquad a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

3. Differentiate equations (7·8), and use the results in proving that x, y, and z satisfy *the differential equation of space curves*,

$$(7\cdot 12)\qquad \begin{cases} x'''' + \left(2\dfrac{\rho'}{\rho} - \dfrac{\tau'}{\tau}\right)x''' + \left(\dfrac{\rho''}{\rho^2} + \dfrac{\rho'\tau'}{\rho\tau} + \dfrac{1}{\rho^2} + \dfrac{1}{\tau^2}\right)x'' \\ \qquad\qquad - \dfrac{1}{\rho^2}\left(\dfrac{\rho'}{\rho} - \dfrac{\tau'}{\tau}\right)x' = 0. \end{cases}$$

8. The osculating sphere and the osculating circle. Just as the tangent (or osculating) line at a point of a curve is determined by two consecutive points, and the osculating plane by three, so *the osculating sphere* is determined by four consecutive points.

DEFINITION 1. *The osculating sphere at a point P of a curve C, not a plane curve, is the limit of the sphere through P and three neighboring points P_1, P_2, P_3 on C, as each of P_1, P_2, P_3 independently approaches P along C.*

The local equation of the osculating sphere can be found by writing the most general equation of a sphere in local coordinates ξ, η, ζ and then demanding that this equation be satisfied by the power series (7·10) for ξ, η, ζ identically in s as far as the terms of the third degree. The resulting equation, after the squares are completed on η and ζ, is

$$(8\cdot 1)\qquad \xi^2 + (\eta - \rho)^2 + (\zeta + \tau\rho')^2 = \rho^2 + (\tau\rho')^2.$$

The center of this sphere evidently has the local coordinates $0, \rho, -\tau\rho'$, and the square of its radius R is given by the formula

$$(8\cdot 2) \qquad R^2 = \rho^2 + (\tau\rho')^2 .$$

Equations (7 1) yield the following formulas for *the unspecialized coordinates X_0, Y_0, Z_0 of the center P_0 of the osculating sphere,*

$$(8\ 3) \qquad \begin{cases} X_0 = x + \rho l - \tau\rho'\lambda , \qquad Y_0 = y + \rho m - \tau\rho'\mu , \\ Z_0 = z + \rho n - \tau\rho'\nu . \end{cases}$$

The osculating circle at a point P of a curve is naturally the circle that intersects the curve in three consecutive points at P.

DEFINITION 2. *The osculating circle at a point P of a curve C is the limit of the circle through P and two neighboring points P_1, P_2 on C and not collinear with P, as each of P_1, P_2 independently approaches P along C.*

The calculation of the local equations of the osculating circle will be much simplified by observing that this circle must lie in the osculating plane, $\zeta = 0$. Then, writing the most general equation of a circle in the osculating plane, in local coordinates ξ, η, and demanding that this equation be satisfied by the power series (7 10) for ξ, η identically in s as far as the terms of the second degree, we find *the local equations of the osculating circle,*

$$(8\ 4) \qquad \zeta = 0 , \qquad \xi^2 + (\eta - \rho)^2 = \rho^2 .$$

The next two theorems are immediate consequences of the foregoing discussion.

THEOREM 1. *At a point of a curve, the osculating plane intersects the osculating sphere in the osculating circle.*

THEOREM 2. *The radius ρ of the osculating circle at a point P of a curve C is the reciprocal of the curvature of C at P.*

The center $(0, \rho, 0)$ of the osculating circle is sometimes called *the center of curvature,* and the radius ρ of the osculating circle is sometimes called *the radius of curvature,* at a point of a curve. Associated with each point of a curve there is a noteworthy covariant line called *the polar line,* which is defined as follows:

DEFINITION 3. *The polar line associated with a point P of a curve C is the line through the center of the osculating circle perpendicular to the osculating plane of C at P.*

The local equations of the polar line are evidently

$$(8\cdot5) \qquad \xi = 0, \qquad \eta = \rho.$$

The unspecialized parametric equations of the same line are

$$(8\cdot6) \quad X = x + \rho l + t\lambda, \quad Y = y + \rho m + t\mu, \quad Z = z + \rho n + t\nu,$$

in which t is a parameter. Comparison of equations (8 3) and (8 6) yields the following theorem (see also Ex. 1).

THEOREM 3. *The center of the osculating sphere at a point P of a curve C lies on the polar line of C associated with P and is situated at the distance $|\tau\rho'|$ from the center of the osculating circle, on the positive side of the osculating plane if $\tau\rho'$ is negative, and on the negative side if $\tau\rho'$ is positive.*

Just as a curve which lies entirely in a plane is called *a plane curve*, so a curve drawn entirely on a sphere is called *a spherical curve*. Of course, a circle can be a spherical curve, since there is a one-parameter family of spheres that contain a given circle. Moreover, no other curve than a circle can be at the same time a spherical curve and a plane curve.

Spherical curves that are not plane curves will now be investigated. For the purpose of stating a theorem *let a function S be defined by placing*

$$(8\ 7) \qquad S = \frac{\rho}{\tau} + (\tau\rho')'.$$

THEOREM 4. *A necessary and sufficient condition that a curve C, not a plane curve, be a spherical curve is $S = 0$.*

To prove this let us differentiate equations (8 2) and (8 3) with respect to the arc length s of the curve C. The derived equations are reducible to

$$(8\cdot8) \quad X_0' = -S\lambda, \quad Y_0' = -S\mu, \quad Z_0' = -S\nu, \quad RR' = S\tau\rho'$$

A curve, not a plane curve, is evidently a spherical curve if, and only if, the osculating sphere at a point of the curve remains fixed when the point moves along the curve. In this case the center and radius are fixed, that is, $X_0' = 0$, $Y_0' = 0$, $Z_0' = 0$, $R' = 0$. Inspection of equations (8 8) now makes the truth of Theorem 4 evident. Moreover, the last of equations (8·8) contains the following theorem.

THEOREM 5. *A necessary and sufficient condition that the radius of the osculating sphere of a curve, not a plane curve, be constant is that the curvature of the curve be constant or else that the curve be spherical.*

If a curve C is not plane and is not spherical, then, as a point P moves along C, the center P_0 of the osculating sphere of C at P also moves along a curve C_0. The relations of these two curves are very interesting. For instance, the first three of equations (8 8) show that the tangent of C_0 at P_0 is parallel to the binormal of C at P. But the center P_0 of the osculating sphere lies on the polar line, by Theorem 3, and this line is parallel to the binormal. So the following conclusion can be drawn.

THEOREM 6. *The polar line corresponding to a point P of a nonplane nonspherical curve C is tangent to the locus C_0 of the centers of the osculating spheres of C, at the center P_0 of the osculating sphere of C at P.*

Moreover, squaring and adding the first three of equations (8 8), and denoting by s_0 arc length of the curve C_0, we find

$$ds_0^2 = S^2 ds^2 . \tag{8 9}$$

Making the convention that *the arc s_0 shall be an increasing function of the arc s*, we obtain

$$ds_0 = \pm S ds ,$$

the positive sign being taken when $S > 0$ and the negative when $S < 0$. Further results in this connection may be found in Exercise 6 below.

EXERCISES

1. The centers of the double infinity of two-point spheres lie in the normal plane, $\xi = 0$. The centers of the single infinity of three-point spheres lie on the polar line, $\xi = 0$, $\eta = \rho$.

2. Derive the formulas (8 3) for the nonspecialized coordinates of the center of the osculating sphere without using local coordinates.

3. At a point of a curve, the centers of all two-point circles that are located in the osculating plane lie on the principal normal; the centers of all two-point circles lie in the normal plane.

4. At a point of a curve, all three-point spheres contain the osculating circle.

5. Calculate the equations of the osculating circle without using local coordinates.

6. If $S > 0$, indicate by subscripts o quantities belonging to the locus C_0 of the centers of the osculating spheres of a curve C, and establish the following formulas,

$$
(8\ 10) \qquad \begin{cases} \alpha_0 = -\lambda\,, & \beta_0 = -\mu\,, & \gamma_0 = -\nu\,, \\ \rho_0 = \pm\tau S\,, & \tau_0 = \rho S\,, & \\ l_0 = \mp l\,, & m_0 = \mp m\,, & n_0 = \mp n\,, \\ \lambda_0 = \mp\alpha\,, & \mu_0 = \mp\beta\,, & \nu_0 = \mp\gamma\,, \end{cases}
$$

in which the upper signs are to be taken if $\tau > 0$, and the lower signs if $\tau < 0$. If $S < 0$, the signs of α_0, β_0, γ_0 and of l_0, m_0, n_0 are to be changed, and then the upper signs are to be taken if $\tau < 0$, and the lower signs if $\tau > 0$. Observe in both cases that τ_0 has the same sign as S, and that

$$
(\rho\rho_0)^2 = (\tau\tau_0)^2\,.
$$

7. The osculating plane at a point P_0 of the locus C_0 of the centers of the osculating spheres of a curve C is the normal plane at the corresponding point P of the curve C.

8. If a curve has constant curvature, the center of the osculating sphere coincides with the center of the osculating circle; the radius of the osculating sphere is constant; the curvature of the locus of the centers of the osculating spheres is constant and the same as the curvature of the first curve; the product of the torsions of the two curves is equal to the square of the common curvature; the two curves have the same principal normals; and each of the two curves is the locus of the centers of curvature of the other.

9. As a point P moves along a curve C, the tangent to the locus of the center of the osculating circle of C at P makes the same angle with the radius of the circle to P (principal normal of C) that the tangent of the locus of the center of the osculating sphere (polar line of C) makes with the radius of the sphere to P.

10. Calculate ρ and τ for the curve represented by the equations

$$
x = a \sin t \cos t\,, \qquad y = a \cos^2 t\,, \qquad z = a \sin t \qquad (a = \text{const.})\,,
$$

in which t is the parameter, and prove that this curve is spherical.

11. For the circular helix calculate the direction cosines of the tangent, principal normal, and binormal; write the equations of the osculating plane, normal plane, and rectifying plane and also of the osculating sphere and osculating circle; write the parametric equations of the locus of the center of the osculating sphere; calculate ρ and τ.

12. For the cubical parabola compute the same things as asked for in Exercise 11.

13. Show that local parametric equations of the osculating circle are

$$\xi = \rho \sin \frac{t}{\rho}, \qquad \eta = \rho\left(1 - \cos \frac{t}{\rho}\right), \qquad \zeta = 0,$$

where *t is arc length on the osculating circle.* Then show that unspecialized parametric equations of the osculating circle are

$$X = x + \rho\left[\alpha \sin \frac{t}{\rho} + l\left(1 - \cos \frac{t}{\rho}\right)\right],$$

$$Y = y + \rho\left[\beta \sin \frac{t}{\rho} + m\left(1 - \cos \frac{t}{\rho}\right)\right],$$

$$Z = z + \rho\left[\gamma \sin \frac{t}{\rho} + n\left(1 - \cos \frac{t}{\rho}\right)\right].$$

14. Reversing the positive sense on a curve reverses the positive sense on the tangent and binormal but leaves the positive sense on the principal normal always directed toward the center of curvature.

9. Shape of a curve. Much information about the shape of a curve C in the neighborhood of one of its points P could be acquired by studying the orthogonal projections of C onto the three faces of the local trihedron at P. The local equations of these projections would be obtained by eliminating s between equations (7 10) taken in pairs. However, the shape of the curve can be determined with sufficient accuracy for most purposes by studying only certain approximations to these projections, which can be investigated with very simple analysis, as follows:

In the equations (7 10) of a curve C, referred to the local trihedron at *an ordinary point* P of C, let all terms after the first in the right members be dropped. The resulting equations,

$$\text{(9 1)} \qquad \xi = s, \qquad \eta = \frac{1}{2\rho} s^2, \qquad \zeta = -\frac{1}{6\rho\tau} s^3,$$

represent a cubical parabola which has approximately the shape of the curve C in the neighborhood of the point P. Of course, the smaller the neighborhood of P is, the closer will be the approximation to C.

If s is eliminated from the first two of equations (9·1), and if the result is interpreted in the $\xi\eta$-plane, the equation of *an ordinary parabola,*

$$(9\cdot 2) \qquad 2\rho\eta = \xi^2,$$

is obtained. This parabola is approximately the orthogonal projection of the curve C onto the osculating plane, $\zeta = 0$, at the point P. In Figure 5(a) the projection of the curve C onto its osculating plane is seen from a point on the positive half of the binormal. The first of equations (9 1) shows how ξ changes when a variable point moves, in

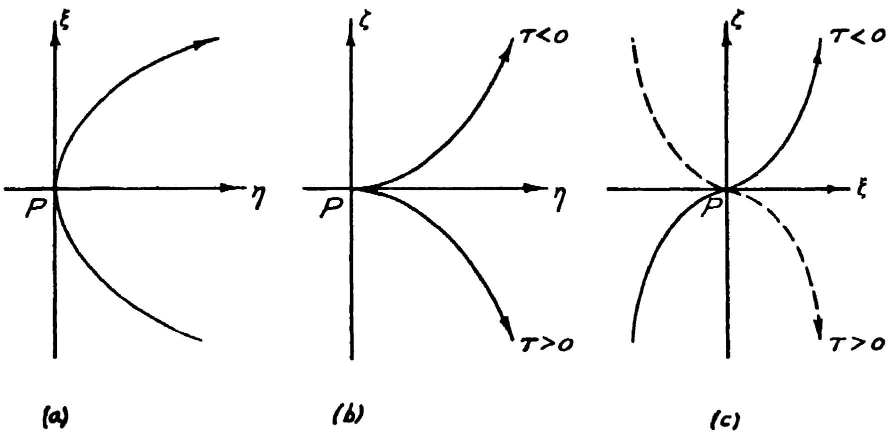

FIG. 5

the direction of increasing arc length s, over the portion of the curve C in a small neighborhood of the point P. The second of (9 1) similarly describes the behavior of η. Thus the following conclusion is justified.

THEOREM 1. *When a variable point P_1 runs along a curve in the positive direction through a fixed point P, the point P_1 passes through the normal plane at P from the negative to the positive side, remaining on the positive side of the rectifying plane.*

If s is eliminated from the last two of equations (9·1), and if the result is interpreted in the $\eta\zeta$-plane, the equation of *a semicubical parabola,*

$$(9\cdot 3) \qquad 2\rho\eta^3 = 9\tau^2\zeta^2,$$

results. This curve is approximately the orthogonal projection of the curve C onto the normal plane, $\xi = 0$, at P. In Figure 5(b) the projection of the curve C onto its normal plane is seen from a point on the negative half of the tangent. The third of equations (9·1) shows that ζ behaves in two different ways according as the torsion $1/\tau$ is positive or negative at the point P. These two cases are distinguished as follows:

THEOREM 2. *With the hypothesis of Theorem 1, the point P_1 passes through the osculating plane from the positive to the negative side if the torsion is positive, and from the negative to the positive side if the torsion is negative.*

Eliminating s between the first and third of equations (9·1), and interpreting the result in the $\zeta\xi$-plane, we get the equation of *a cubical parabola*,

$$6\rho\tau\zeta = -\xi^3, \tag{9·4}$$

which is approximately the orthogonal projection of the curve C onto the rectifying plane, $\eta = 0$, at P. In Figure 5(c) the projection of the curve onto the rectifying plane is seen from a point on the positive half of the principal normal. There are again two cases to be distinguished according as the torsion is positive or negative at P. When the torsion is positive, the projection lies in the second and fourth quadrants and is generated downward; when the torsion is negative, the projection lies in the first and third quadrants and is generated upward.

In the foregoing investigation it was supposed that the point P was *an ordinary point* on the curve C, i.e., a point at which the curve has no special property. If the curve C has a special property at a point P, which it does not have at every one of its points, then this point P may be called *an exceptional point* of the curve. There are three kinds of exceptional points which deserve mention here. First of all, when a curve C is represented by parametric equations of the form (2·1), if the curve does not reduce to a fixed point, then x', y', z' are not identically zero in the parameter t. However, it may happen that these derivatives all vanish for one or more isolated values of t, although they do not vanish identically. A point where this vanishing occurs is called, as we have already seen, *a singular point.* At a singular point the curve is, so to speak, temporarily like a fixed point. A variable point running along the curve stops when it comes to such a point and ordinarily reverses the direction of its motion. For this reason such a point is called *a stationary point* and is ordinarily *a cusp. At a stationary point the tangent line is indeterminate* when defined to be a line intersecting the curve in two coincident points at the point of tangency.

Secondly, it may happen that x'', y'', z'' vanish identically in t. If so, the curve C is a straight line and its curvature is identically zero. However, it may happen that the curvature vanishes for one or more isolated values of t, although it does not vanish identically. At a point where this occurs the curve is temporarily like a straight line. The tangent line stops its rotation instantaneously, and ordinarily reverses the direction of its rotation, when its contact point passes through such a point in running along the curve C. For this reason, at such an exceptional point the tangent is said to be *stationary*, and the point is ordinarily *an inflexion point. At an exceptional point of this kind the tangent line contains at least three consecutive points of the curve, and the osculating plane is indeterminate.*

Finally, if equation (2 10) is satisfied identically in t, the curve C is a plane curve. If, however, this equation is satisfied by isolated values of t but is not satisfied identically, the points corresponding to these isolated values of t are exceptional points at which the curve is temporarily like a plane curve. At such a point the osculating plane is said to be *stationary*, because, when its point of osculation runs along the curve and passes through such a point, the osculating plane ceases its rotation instantaneously and ordinarily reverses the direction of the rotation. *At an exceptional point of this kind the curve has at least four consecutive points in its osculating plane.*

It is evident that a point of a curve is either stationary or not; the tangent line is either stationary there or not; and the osculating plane is either stationary there or not. So, from this point of view, there are in all eight possibilities as to the shape of the curve in the neighborhood of the point. We have described the shape of the curve in the ordinary case, i.e., the case in which neither point nor line nor plane is stationary. The discussion of the shape of the curve in the other seven cases is left to the reader.

CHAPTER II

THE MOVING TRIHEDRON

1. Preliminary discussion. The method employed in this chapter for developing the theory of the moving trihedron is due to Professor G. A. Bliss, who presented it from time to time in his lectures on metric differential geometry.

In order to explain what is meant by *the moving trihedron* in curve theory, it is convenient to start with the usual fundamental left-handed orthogonal cartesian coordinate system with the same unit of distance for all three axes. This system will be referred to in the following discussion as *the fixed coordinate system*, and the same unit of distance will be used throughout. Let us consider a real proper nonsingular analytic curve C, not a straight line, whose parametric equations in the fixed coordinate system are

$$(1\ 1) \qquad x = x(s)\,, \qquad y = y(s)\,, \qquad z = z(s)\,,$$

the parameter s being the arc length measured from some fixed point to the ordinary point $P(x, y, z)$ on C.

Further, let us consider any point Q whose coordinates X, Y, Z in the fixed coordinate system are given as functions of s by equations of the form

$$(1\ 2) \qquad X = X(s)\,, \qquad Y = Y(s)\,, \qquad Z = Z(s)\,.$$

If the coordinates X, Y, Z of the point Q are constants, then Q is fixed, relative to the fixed coordinate system, when s varies. This case will be excluded hereinafter, unless the contrary is indicated. Then, as s varies, the point P moves along the curve C and the point Q generates a proper curve C_1, represented by the parametric equations (1 2). The points of the curves C and C_1 are in one-to-one correspondence, corresponding points P and Q being those that correspond to the same value of the parameter s.

At a point P of the curve C there is the local coordinate system introduced in Section 7 of Chapter I. In this coordinate system the coordinate axes lie along the three edges of the local trihedron of C at P, the ξ-axis lying along the tangent, the η-axis along the principal normal, and the ζ-axis along the binormal. The local coordinates ξ, η, ζ of the point Q that corresponds to the point P are related to the coordinates X, Y, Z of Q by the equations* (I 7·1), or (I·7·2), of transformation of coordinates. Therefore the coordinates ξ, η, ζ are themselves functions of s, and we have

$$(1 \cdot 3) \qquad \xi = \xi(s)\,, \qquad \eta = \eta(s)\,, \qquad \zeta = \zeta(s)\,.$$

When the point P moves along the curve C, the local trihedron of C at P also moves, of course, and hence is appropriately referred to as *the moving trihedron* at the point P of the curve C. Furthermore, the local coordinate system based on the moving trihedron may be designated as *the moving coordinate system* associated with, or at, the point P of the curve C. If the local coordinates ξ, η, ζ of the point Q are constants, then Q is fixed relative to the moving trihedron.

It is frequently of interest to study the properties of the curve C_1, and in particular to investigate its relations to the fundamental curve C. For this purpose it is usually advantageous to refer C_1 to the moving coordinate system associated with a variable point of the curve C. The theory that thus originates is capable of extensive applications in the study of pairs of curves with their points in one-to-one correspondence, and especially in the investigation of classes of curves C_1 which are defined in terms of a given curve C in such a way that the points of the curve C and of a curve C_1 are in one-to-one correspondence. In a situation of this kind, the definition of a curve C_1 is usually stated by imposing a relation between various parts of the local trihedrons of C_1 and C which holds at all pairs of corresponding points. It is not surprising, then, that the general theory of the moving trihedron, which will be developed in the next section, has as one of its essential features the calculation of the direction cosines of the edges of the local trihedron at any point Q of the curve C_1, referred to the local coordinate system at the corresponding point P of the curve C.

2. General theory. The general theory of the moving trihedron is based on the equations (I·7·1) of transformation of point coordinates, and the equations (I 7 6) of transformation of direction cosines. Equations (I·7·1) express analytically the relations between the coordinates X, Y, Z of a point Q, referred to the fixed coordinate system, and the coordinates ξ, η, ζ of the same point Q referred to the moving

* Roman numerals are employed in referring to equations in preceding chapters. For example, the equations (I·7·1) are equations (7·1) in Chapter I.

coordinate system at the corresponding point $P(x, y, z)$ of a curve C. Equations (I·7·6) similarly express analytically the relations between the direction cosines A, B, C of a line referred to the fixed coordinate system and the direction cosines a, b, c of the same line referred to the moving coordinate system.

First of all, some analytical consequences of equations (I 7 1) will be deduced, and then certain geometrical interpretations will be explained. If the equations (I 7·1) are differentiated with respect to the arc length s of the curve C, the derived equations can be reduced, by means of the Frenet formulas (I 6 1), to the form

$$(2\cdot 1) \qquad \begin{cases} X' = \alpha A_1 + lB_1 + \lambda C_1\,, \\ Y' = \beta A_1 + mB_1 + \mu C_1\,, \\ Z' = \gamma A_1 + nB_1 + \nu C_1\,, \end{cases}$$

in which the coefficients A_1, B_1, C_1 are defined by the formulas

$$(2\cdot 2) \quad A_1 = 1 - \frac{\eta}{\rho} + \xi'\,, \qquad B_1 = \frac{\xi}{\rho} + \frac{\zeta}{\tau} + \eta'\,, \qquad C_1 = -\frac{\eta}{\tau} + \zeta'\,,$$

the accent indicating differentiation with respect to s. Similarly, differentiation of equations (2 1) and reduction by the Frenet formulas lead to

$$(2\cdot 3) \qquad \begin{cases} X'' = \alpha A_2 + lB_2 + \lambda C_2\,, \\ Y'' = \beta A_2 + mB_2 + \mu C_2\,, \\ Z'' = \gamma A_2 + nB_2 + \nu C_2\,, \end{cases}$$

wherein the coefficients A_2, B_2, C_2 are defined by the formulas

$$(2\cdot 4) \qquad \begin{cases} A_2 = -\dfrac{B_1}{\rho} + A_1'\,, \qquad B_2 = \dfrac{A_1}{\rho} + \dfrac{C_1}{\tau} + B_1'\,, \\ C_2 = -\dfrac{B_1}{\tau} + C_1'\,. \end{cases}$$

Repetition of the process gives

$$(2\cdot 5) \qquad \begin{cases} X''' = \alpha A_3 + lB_3 + \lambda C_3\,, \\ Y''' = \beta A_3 + mB_3 + \mu C_3\,, \\ Z''' = \gamma A_3 + nB_3 + \nu C_3\,, \end{cases}$$

where A_3, B_3, C_3 are defined by

$$(2\ 6)\qquad \begin{cases} A_3 = -\dfrac{B_2}{\rho} + A_2', \qquad B_3 = \dfrac{A_2}{\rho} + \dfrac{C_2}{\tau} + B_2', \\ C_3 = -\dfrac{B_2}{\tau} + C_2'. \end{cases}$$

An easy induction would yield recursion formulas for the coefficients A_i, B_i, C_i obtained by further repetitions of this process, but it will not be necessary to calculate any more of these coefficients here.

Let us make the convention that the arc length s_1 of the curve C_1, measured from some fixed point thereon, shall be an increasing function of the arc length s of the curve C. Squaring and adding equations (2 1), and taking the positive square root of the sum in accordance with this convention, we obtain

$$(2\ 7)\qquad \frac{ds_1}{ds} = (X'^2 + Y'^2 + Z'^2)^{1/2} = (A_1^2 + B_1^2 + C_1^2)^{1/2}$$

The equality of the first and last members of these equations supplies, upon taking reciprocals, a formula for the derivative ds/ds_1 which is written as the first of equations (2 8) below, the summation therein being for cyclical permutation of A, B, C:

$$(2\ 8)\qquad \begin{cases} \dfrac{ds}{ds_1} = \dfrac{1}{(\Sigma A_1^2)^{1/2}}, \\ \dfrac{d^2s}{ds_1^2} = -\dfrac{\Sigma A_1 A_2}{(\Sigma A_1^2)^2}. \end{cases}$$

The second of these equations is obtained by differentiating the first by the ordinary rules of the calculus and using the definitions (2 4). Similar expressions for higher derivatives of s with respect to s_1 could be calculated by continuing this process, but they will not be needed hereinafter.

The first three derivatives of X with respect to s_1 may be calculated by means of the well-known formulas

$$(2\ 9)\qquad \begin{cases} \dfrac{dX}{ds_1} = X' \dfrac{ds}{ds_1}, \\ \dfrac{d^2X}{ds_1^2} = X'' \left(\dfrac{ds}{ds_1}\right)^2 + X' \dfrac{d^2s}{ds_1^2}, \\ \dfrac{d^3X}{ds_1^3} = X''' \left(\dfrac{ds}{ds_1}\right)^3 + 3X'' \dfrac{ds}{ds_1} \dfrac{d^2s}{ds_1^2} + X' \dfrac{d^3s}{ds_1^3}, \end{cases}$$

the derivatives of Y, Z being easily written by cyclical permutations of X, Y, Z. The equation in the second line of (2·9), and the similar equations for Y, Z, can be reduced, by means of (2·1), (2·3), and (2·8), to the form

$$(2\cdot10)\qquad \begin{cases} \dfrac{d^2X}{ds_1^2} = \alpha L + lM + \lambda N\,, \\[2ex] \dfrac{d^2Y}{ds_1^2} = \beta L + mM + \mu N\,, \\[2ex] \dfrac{d^2Z}{ds_1^2} = \gamma L + nM + \nu N\,, \end{cases}$$

in which the coefficients L, M, N are defined by the formulas

$$(2\cdot11)\qquad \begin{cases} L = \dfrac{1}{(\Sigma A_1^2)^2}(A_2\Sigma A_1^2 - A_1\Sigma A_1A_2)\,, \\[2ex] M = \dfrac{1}{(\Sigma A_1^2)^2}(B_2\Sigma A_1^2 - B_1\Sigma A_1A_2)\,, \\[2ex] N = \dfrac{1}{(\Sigma A_1^2)^2}(C_2\Sigma A_1^2 - C_1\Sigma A_1A_2)\,, \end{cases}$$

the summation being for cyclical permutations of A, B, C. Finally, direct calculation gives

$$(2\cdot12)\qquad \begin{cases} \dfrac{dY}{ds_1}\dfrac{d^2Z}{ds_1^2} - \dfrac{d^2Y}{ds_1^2}\dfrac{dZ}{ds_1} = \left(\dfrac{ds}{ds_1}\right)^3(\alpha P + lQ + \lambda R)\,, \\[2ex] \dfrac{dZ}{ds_1}\dfrac{d^2X}{ds_1^2} - \dfrac{d^2Z}{ds_1^2}\dfrac{dX}{ds_1} = \left(\dfrac{ds}{ds_1}\right)^3(\beta P + mQ + \mu R)\,, \\[2ex] \dfrac{dX}{ds_1}\dfrac{d^2Y}{ds_1^2} - \dfrac{d^2X}{ds_1^2}\dfrac{dY}{ds_1} = \left(\dfrac{ds}{ds_1}\right)^3(\gamma P + nQ + \nu R)\,, \end{cases}$$

where the coefficients P, Q, R are defined by placing

$$(2\cdot13)\qquad \begin{cases} P = B_1C_2 - B_2C_1\,, \qquad Q = C_1A_2 - C_2A_1\,, \\ \qquad R = A_1B_2 - A_2B_1\,. \end{cases}$$

Some of the foregoing formulas have geometrical interpretations. First of all, *the direction cosines of the tangent line* at a point Q of the curve C_1, referred to the fixed coordinate system, are

$$\frac{dX}{ds_1}\,,\qquad \frac{dY}{ds_1}\,,\qquad \frac{dZ}{ds_1}\,.$$

If each of equations (2 1) is multiplied through by ds/ds_1, the left members of the resulting equations are shown, by the equations in the first line of (2 9), to be exactly these direction cosines. The equations (I·7 6) of transformation of direction cosines can now be employed to complete the proof of the following theorem.

THEOREM 1. *The direction cosines of the tangent line at a point Q of the curve C_1, referred to the moving coordinate system associated with the corresponding point P of the curve C, are respectively proportional to the functions A_1, B_1, C_1 and are equal to*

$$A_1 \frac{ds}{ds_1}, \qquad B_1 \frac{ds}{ds_1}, \qquad C_1 \frac{ds}{ds_1}.$$

The curvature $1/\rho_1$ at the point Q of the curve C_1 is expressed in terms of derivatives of the coordinates of Q by the familiar formula

$$(2\cdot 14) \qquad \frac{1}{\rho_1^2} = \left(\frac{d^2X}{ds_1^2}\right)^2 + \left(\frac{d^2Y}{ds_1^2}\right)^2 + \left(\frac{d^2Z}{ds_1^2}\right)^2.$$

If the expressions found for the second derivatives in equations (2 10) are substituted in this formula, simple calculations suffice to demonstrate the following theorem.

THEOREM 2. *The curvature $1/\rho_1$ at a point Q of a curve C_1 is given by the formula*

$$(2\cdot 15) \qquad \frac{1}{\rho_1} = (L^2 + M^2 + N^2)^{1/2}.$$

The direction cosines of the principal normal at a point Q of the curve C_1, referred to the fixed coordinate system, are

$$\rho_1 \frac{d^2X}{ds_1^2}, \qquad \rho_1 \frac{d^2Y}{ds_1^2}, \qquad \rho_1 \frac{d^2Z}{ds_1^2}.$$

If each of equations (2·10) is multiplied through by ρ_1, and if proper account is taken again of the equations (I·7·6) for the transformation of direction cosines, the following theorem is easily proved.

THEOREM 3. *The direction cosines of the principal normal at a point Q of the curve C_1, referred to the moving coordinate system associated with the corresponding point P of the curve C, are respectively proportional to the functions L, M, N and are equal to*

$$\rho_1 L, \qquad \rho_1 M, \qquad \rho_1 N.$$

The first direction cosine of the binormal at a point Q of the curve C_1, referred to the fixed coordinate system, is

$$\rho_1 \left(\frac{dY}{ds_1} \frac{d^2Z}{ds_1^2} - \frac{d^2Y}{ds_1^2} \frac{dZ}{ds_1} \right) ;$$

and the other two direction cosines can be written by circular permutations of X, Y, Z. Equations (2·12) and the equations of transformation (I 7 6) can be used to prove the following theorem.

THEOREM 4. *The direction cosines of the binormal at a point Q of the curve C_1, referred to the moving coordinate system associated with the corresponding point P of the curve C, are respectively proportional to the functions P, Q, R and are equal to*

$$\rho_1 \left(\frac{ds}{ds_1} \right)^3 P , \qquad \rho_1 \left(\frac{ds}{ds_1} \right)^3 Q , \qquad \rho_1 \left(\frac{ds}{ds_1} \right)^3 R .$$

It may be remarked, in passing, that if $X^{(i)}$, $Y^{(i)}$, $Z^{(i)}$ are thought of as the components of a vector along the three fixed axes, then A_i, B_i, C_i ($i = 1, 2, \ldots$) are the components of the same vector along the moving axes, as is shown by equations (2 1), (2 3), (2 5), and so on.

The general theory developed up to this point in the present chapter will be applied in various special situations in the remaining sections of the chapter.

EXERCISES

1. Establish the formula

(2 16)
$$\frac{1}{\rho_1^2} = \frac{P^2 + Q^2 + R^2}{(A_1^2 + B_1^2 + C_1^2)^3} .$$

2. Prove that the torsion $1/\tau_1$ at a point Q of a curve C_1 is given by the formula

(2 17)
$$\frac{1}{\tau_1} = -\frac{1}{P^2 + Q^2 + R^2} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} .$$

3. Prove by (2 1) and (2 2) that the local coordinates ξ, η, ζ of a point Q which is fixed, relative to the fixed coordinate system, satisfy the differential equations

(2 18)
$$\xi' = -1 + \frac{\eta}{\rho}, \qquad \eta' = -\frac{\xi}{\rho} - \frac{\zeta}{\tau}, \qquad \zeta' = \frac{\eta}{\tau} .$$

4. Differentiate equations (I 7 6) and use the result to prove that the local direction cosines a, b, c of a line which is fixed, relative to the fixed coordinate system, satisfy the differential equations

$$a' = \frac{b}{\rho}, \quad b' = -\frac{a}{\rho} - \frac{c}{\tau}, \quad c' = \frac{b}{\tau}. \tag{2 19}$$

5. Find the differential equations which the local coordinates ξ, η, ζ of a point Q must satisfy if the curve C_1 generated by Q is a straight line.

6. Solve the problem of Exercise 5 if C_1 is restricted only to be a plane curve.

3. Involutes. As a first application of the method of the moving trihedron, we shall study the involutes of a given curve. These are defined in the following way.

DEFINITION 1. *An involute of a curve C is a curve that intersects the tangents of C at right angles.*

In other words, an involute of a curve is *an orthogonal trajectory* of the tangents of the curve. In order to find all the involutes of a given curve C, let us consider a point $P(x, y, z)$ on C, and a point Q on the tangent of C at P, which generates an involute C_1 of C. The local coordinates ξ, η, ζ of Q, and the direction numbers A_1, B_1, C_1 of the tangent to the involute C_1 at Q, satisfy the conditions

$$\eta = \zeta = 0, \quad A_1 = 0, \tag{3 1}$$

which are merely an analytic form of the definition of the involute C_1. Equations (3 1) and (2 2) imply

$$0 = 1 + \xi', \quad B_1 = \frac{\xi}{\rho}, \quad C_1 = 0. \tag{3 2}$$

Integration of the first of these equations leads to

$$\xi = c - s, \tag{3 3}$$

where c is an arbitrary constant. This equation validates the following statement.

THEOREM 1. *A given curve has a one-parameter family of involutes.*

The involutes of a curve cover the surface which is the locus of the tangents of the curve (later on called *the tangent developable* of the curve). In the moving coordinate system the coordinates ξ, η, ζ of the point Q generating a general involute C_1 are given by

$$\xi = c - s, \quad \eta = 0, \quad \zeta = 0. \tag{3 4}$$

The parametric equations of an involute in the fixed coordinate system are easily found, by the equations (I·7·1) of transformation, to be

$$(3\cdot 5)\qquad \begin{cases} X = x + \alpha(c - s)\,, \qquad Y = y + \beta(c - s)\,, \\ Z = z + \gamma(c - s)\,. \end{cases}$$

The last of equations (3·2) has the following interpretation.

THEOREM 2. *The tangent at a point of an involute of a curve C is perpendicular to the binormal at the corresponding point of C.*

The next theorem is a consequence of the conditions $A_1 = 0$, $C_1 = 0$.

THEOREM 3. *The tangent at a point of an involute of a curve C is perpendicular to the rectifying plane at the corresponding point of C.*

It is obvious that one could calculate for an involute C_1 all the functions introduced in the preceding section for a general curve C_1. On this basis an extensive theory of involutes could be constructed. However, the discussion will be restricted, for the present, to a consideration of two further properties of involutes, called, respectively, *the parallelism property* and *the string property*. The reason for the name *parallelism property* is made evident by the following theorem.

THEOREM 4. *The distance between two involutes of a curve C, measured along the tangents of C, is constant; and the tangents of the two involutes at points on each tangent of C are parallel.*

To prove this theorem, let us consider two involutes individualized by assigning two particular values c_1, c_2 to the constant c, and let the corresponding coordinates ξ be denoted by ξ_1, ξ_2. For these two involutes equation (3·3) gives

$$\xi_1 = c_1 - s\,, \qquad \xi_2 = c_2 - s\,.$$

If the value of s is the same in both equations, subtraction leads to

$$(3\cdot 6)\qquad \xi_2 - \xi_1 = c_2 - c_1\,.$$

This equation asserts the first part of the theorem. The second part is proved by observing that the tangents of the involutes are perpendicular to the same rectifying plane, by Theorem 3, and therefore are parallel lines.

The string property is incorporated in the following theorem, which is merely an interpretation of equation (3·3).

THEOREM 5. *If one end of a string of constant length c is fastened at* *the point where $s = 0$ on a curve C, and if the string is held taut and wound along C so as to remain tangent to C, the locus of the other end of the string is an involute of C.*

EXERCISES

1 Deduce equations (3 5) without using the theory of the moving trihedron.

2. Find the involutes of the circular helix (I 2 6), and prove that these involutes are all plane curves whose planes are perpendicular to the generators of the circular cylinder on which the helix lies. Prove that these involutes are also involutes of the circles cut on the cylinder by the planes in which they lie.

4. Evolutes. Another family of curves, associated with a given curve and amenable to investigation by the method of the moving trihedron, consists of the evolutes of the curve, which are defined as follows:

DEFINITION 1. *An evolute of a curve C is a curve whose tangents intersect C orthogonally.*

An immediate consequence of the definitions of involute and evolute may be stated as follows:

THEOREM 1. *When one curve is an involute of another curve, then the second curve is an evolute of the first.*

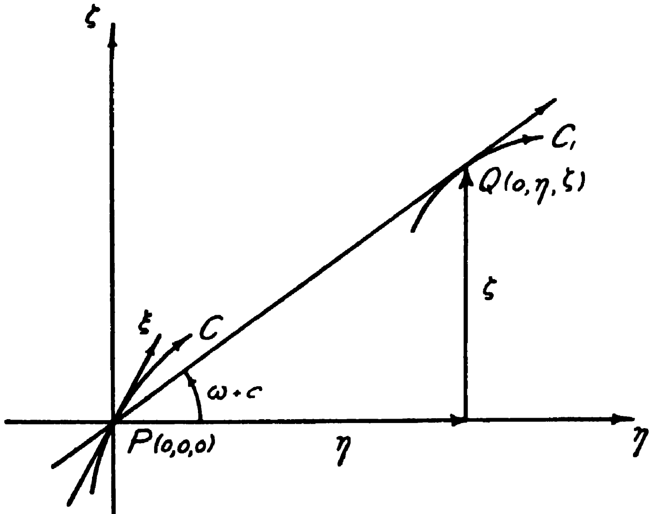

FIG. 6

In order to find all the evolutes of a given curve C, let us consider a point P on C, and a point Q which generates an evolute C_1 of C, as in Figure 6. It is understood that Q corresponds to P in the sense that the tangent to the evolute C_1 at Q passes through P perpendicular to the tangent of C at P. Therefore *the point Q lies in the normal plane of C at P.* Furthermore, the local coordinates ξ, η, ζ of the point Q and the direction numbers A_1, B_1, C_1 of the tangent to the evolute C_1 at Q, referred to the moving coordinate system of C at P, satisfy the necessary and sufficient conditions

$$(4\cdot1)\qquad \xi = 0\,, \qquad A_1 = 0\,, \qquad B_1 = k\eta\,, \qquad C_1 = k\zeta\,,$$

where k is a nonvanishing proportionality factor whose value is immaterial. The first two of these conditions and the definition of A_1 in (2·2) imply

$$(4\cdot2)\qquad \eta = \rho\,.$$

This result can be interpreted as follows:

THEOREM 2. *The point Q generating an evolute of a curve C, and corresponding to a point P of C, lies on the polar line associated with the point P of the curve C.*

The last two of the conditions (4·1) and the last two of the definitions (2·2), in the presence of (4·2), imply

$$(4\cdot3)\qquad \frac{\rho\zeta' - \zeta\rho'}{\rho^2 + \zeta^2} = \frac{1}{\tau}\,.$$

Let a function ω be defined by placing

$$(4\cdot4)\qquad \omega = \int_{s_0}^{s} \frac{ds}{\tau}\,,$$

where s_0 is an arbitrarily chosen fixed value of s. Integration of the differential equation (4·3) now gives

$$(4\cdot5)\qquad \tan^{-1}\frac{\zeta}{\rho} = \omega + c\,,$$

where c is an arbitrary constant. This equation confirms the following assertion.

THEOREM 3. *A given curve has a one-parameter family of evolutes.*

The evolutes of a curve cover the surface which is the locus of the polar lines of the curve (later on called *the polar developable* of the curve). In the moving coordinate system the coordinates ξ, η, ζ of the point Q generating a general evolute C_1 are given by

$$(4\cdot6) \qquad \xi = 0, \qquad \eta = \rho, \qquad \zeta = \rho \tan(\omega + c).$$

Equations (I·7·1) now yield *the parametric equations of an evolute in the fixed coordinate system,*

$$(4\cdot7) \qquad \begin{cases} X = x + \rho[l + \lambda \tan(\omega + c)], \\ Y = y + \rho[m + \mu \tan(\omega + c)], \\ Z = z + \rho[n + \nu \tan(\omega + c)]. \end{cases}$$

As in the case of involutes, a very extensive theory could be developed by computing for an evolute C_1 the various functions introduced in Section 2 for a general curve C_1. However, the discussion will be confined, for the present, to establishing one more property of evolutes, called *the rotation property* and put in evidence by the following theorem.

THEOREM 4. *If each tangent of an evolute of a curve C is rotated through the same angle about the point P in which it meets C, remaining in the normal plane of C at P, then in their new positions these lines (originally tangents of an evolute) are still tangents of an evolute of C.*

The proof is very simple. Let two evolutes of a curve C be individualized by assigning two particular values c_1, c_2 to the constant c. The angle between the two tangents of these evolutes that pass through a point P of C is shown to be constant, as P moves along C, by the equation

$$(\omega + c_2) - (\omega + c_1) = c_2 - c_1 = \text{const.}$$

EXERCISES

1. Calculate the element of arc ds_1 of a general evolute of a curve, and by integration prove that

$$s_1 = \rho \sec(\omega + c) + h,$$

where h is an arbitrary constant.

2. A necessary and sufficient condition that a curve be a plane curve is $\omega = \text{const.}$

3. The one of the evolutes of a plane curve which is characterized by the condition $c = -\omega$ lies in the plane of the curve and is the locus of the centers of the osculating circles of the curve. Conversely, if the locus of the centers of the osculating circles of a curve C is an evolute of C, then C is a plane curve.

4. The evolutes of a plane curve C are the cylindrical helices on the cylinder which is the locus of the straight lines drawn perpendicular to the plane of C at the centers of the osculating circles of C.

5. Prove that the normal plane at a point of a curve is the rectifying plane at the corresponding point of an evolute C_1 of the curve, by showing that $P = 0$ for the evolute C_1.

5. Parallel curves. Parallel curves are defined as follows:

DEFINITION 1. *Two curves are parallel in case it is possible to set up a one-to-one correspondence between their points such that corresponding points are equally distant and the tangents at corresponding points are parallel.*

Two involutes of a curve C are parallel curves, for which corresponding points are on a tangent of C. A curve may be said to be parallel to itself, the distance between corresponding points being zero in this case.

All curves parallel to a given curve C, which is not a straight line and is not a plane curve, will now be determined. For this purpose let us consider a point P on the curve C and the corresponding point Q which generates a curve C_1 parallel to C. Let ξ, η, ζ denote, as usual, the local coordinates of Q, referred to the local coordinate system of C at P. The two hypotheses that the distance PQ is constant and that the tangent of C_1 at Q is parallel to the tangent of C at P, or, what is the same thing, is perpendicular to the principal normal and binormal of C at P, are expressed analytically by the equations

$$(5\cdot 1)\qquad \xi^2 + \eta^2 + \zeta^2 = c^2\,, \qquad B_1 = 0\,, \qquad C_1 = 0 \qquad (c = \text{const.})\,.$$

After the first of these equations is differentiated, and the formulas for B_1, C_1 are substituted in the other two from the definitions (2 2), we obtain

$$(5\cdot 2)\qquad \xi\xi' + \eta\eta' + \zeta\zeta' = 0\,, \qquad \eta' = -\frac{\xi}{\rho} - \frac{\zeta}{\tau}\,, \qquad \zeta' = \frac{\eta}{\tau}\,.$$

The result of eliminating η' and ζ' from these equations is

$$(5\cdot 3)\qquad \xi\left(\xi' - \frac{\eta}{\rho}\right) = 0\,.$$

There are three cases to be considered according as both factors vanish, or $\xi \neq 0$, or $\xi' - \eta/\rho \neq 0$.

In the first case the conditions $\xi = 0$, $\xi' - \eta/\rho = 0$ lead at once to $\xi = 0$, $\eta = 0$, $\zeta = 0$. Therefore the curve C_1 parallel to the curve C coincides with C in this case.

The second case is characterized by the conditions $\xi \neq 0$, $\xi' - \eta/\rho = 0$. Consequently, the solution of the problem is made to depend on the integration of the system of differential equations

$$(5\cdot 4) \qquad \xi' = \frac{\eta}{\rho}, \qquad \eta' = -\frac{\xi}{\rho} - \frac{\zeta}{\tau}, \qquad \zeta' = \frac{\eta}{\tau},$$

the only solutions of interest in this connection being those that satisfy the conditions

$$(5\cdot 5) \qquad \xi \neq 0, \qquad \xi^2 + \eta^2 + \zeta^2 = c^2 \quad (c = \text{const.} \neq 0).$$

The Frenet formulas (I 6 1) tell us that three sets of solutions of these equations are α, l, λ; β, m, μ; γ, n, ν. Therefore, the most general set of solutions ξ, η, ζ is given by

$$(5\cdot 6) \qquad \begin{cases} \xi = c_1\alpha + c_2\beta + c_3\gamma, \\ \eta = c_1 l + c_2 m + c_3 n, \\ \zeta = c_1\lambda + c_2\mu + c_3\nu, \end{cases}$$

where c_1, c_2, c_3 are arbitrary constants not all zero. These solutions satisfy the conditions (5·5). Therefore, equations (5 6) give the coordinates ξ, η, ζ, in the moving system, of the point Q generating a curve C_1 which is parallel to the curve C. *The parametric equations of the curves parallel to the curve C, in the fixed coordinate system,* are found by (I·7·1) to be, in this case,

$$(5\cdot 7) \qquad X = x + c_1, \qquad Y = y + c_2, \qquad Z = z + c_3.$$

Therefore *these curves parallel to the curve C are obtainable from C by a translation.*

The third case is characterized by the conditions $\xi = 0$, $\xi' - \eta/\rho \neq 0$. It is evident that $\eta \neq 0$, and so the solution of the problem is made to depend on the integration of the system of differential equations

$$(5\cdot 8) \qquad \eta' = -\frac{\zeta}{\tau}, \qquad \zeta' = \frac{\eta}{\tau},$$

the only solutions of interest in this connection being those that satisfy the conditions

$$(5\cdot 9) \qquad \xi = 0\,, \qquad \eta \neq 0\,, \qquad \eta^2 + \zeta^2 = c^2 \qquad (c = \text{const.} \neq 0)\,.$$

Elimination of ζ, and then of η, from (5 8) by means of the last of (5·9) results in

$$(5\ 10) \qquad \frac{d\eta}{(c^2 - \eta^2)^{1/2}} = -\frac{ds}{\tau}\,, \qquad \frac{d\zeta}{(c^2 - \zeta^2)^{1/2}} = \frac{ds}{\tau}\,.$$

Integrating these equations for η, ζ and re-writing the first of (5 9), we have

$$(5\ 11) \qquad \xi = 0\,, \qquad \eta = c\cos(\omega + h)\,, \qquad \zeta = c\sin(\omega + h)\,,$$

where ω is defined by equation (4·4) and h is an arbitrary constant. Equations (5·11) give the coordinates ξ, η, ζ, in the moving coordinate system, of the point Q generating a curve C_1 which is parallel to the curve C. *The parametric equations of the curves parallel to the curve C, in the fixed coordinate system,* are found by (I 7·1) to be, in this case,

$$(5\cdot 12) \qquad \begin{cases} X = x + c[l\ \cos(\omega + h) + \lambda \sin(\omega + h)]\,, \\ Y = y + c[m \cos(\omega + h) + \mu \sin(\omega + h)]\,, \\ Z = z + c[n\ \cos(\omega + h) + \nu \sin(\omega + h)]\,. \end{cases}$$

These are the curves (distinct from C since $c \neq 0$) which are parallel to C and are not obtainable from C by a translation. Another characterization of them will be based on the following theorem.

THEOREM 1. *The orthogonal trajectories of the normal planes of a curve C are parallel to C.*

For the purpose of the proof of this theorem, an orthogonal trajectory of the normal planes of a curve C is described analytically, in the notation of the theory of the moving trihedron, by the equations

$$(5\ 13) \qquad \xi = 0\,, \qquad B_1 = 0\,, \qquad C_1 = 0\,.$$

Consequently, the problem of determining all the orthogonal trajectories of the curve C is reduced by the last two of the formulas (2·2) to the analytic problem of solving the system of equations,

$$(5\cdot 14) \qquad \xi = 0\,, \qquad \eta' = -\frac{\zeta}{\tau}\,, \qquad \zeta' = \frac{\eta}{\tau}\,.$$

The result of eliminating τ from the last two of these equations is

$$\eta\eta' + \zeta\zeta' = 0,$$

and integration of this equation gives

$$\eta^2 + \zeta^2 = c^2 \qquad (c = \text{const.}) .$$

If $c = 0$, then $\xi = 0$, $\eta = 0$, $\zeta = 0$. Thus the curve C is exhibited as an orthogonal trajectory of its own normal planes. If $c \neq 0$, the problem of finding the orthogonal trajectories of the curve C is analytically the same as the problem solved in the third case above. So the theorem is proved. The promised characterization of the curves (5·12) is now at hand.

THEOREM 2. *The curves represented parametrically by equations* (5·12) *are the orthogonal trajectories of the normal planes of the curve* C *(exclusive of the curve* C *itself).*

The following theorem summarizes the results of the argument in this section.

THEOREM 3. *The curves parallel to a nonrectilinear nonplane curve* C *are the curves obtainable from* C *by a translation and the orthogonal trajectories of the normal planes of* C.

EXERCISES

1. Determine all curves parallel to a nonrectilinear plane curve, and show that the results are essentially those stated in Theorem 3.

2. Find the parametric equations, in the fixed coordinate system, of all curves parallel to the circle

$$(5\cdot15) \qquad x = r\cos\frac{s}{r}, \qquad y = r\sin\frac{s}{r}, \qquad z = k,$$

where r, k are constants and s is arc length.

3. The involutes of the evolutes of a curve C are the orthogonal trajectories of the normal planes of C, and hence are parallel to C.

4. Consider a point Q in the normal plane at a point P of a curve C. When P moves along C, demand that the locus C_1 of Q shall be such that its osculating plane at Q is the normal plane of C at P. Then prove that C_1 is the locus of the centers of the osculating spheres of C (sometimes called *the planar evolute* of C).

5. Prove that the coordinates ξ, η, ζ, in the moving coordinate system, of the point Q generating *an orthogonal trajectory* C_1 of the osculating planes of a curve C (sometimes called *a planar involute* of C) are

$$(5\cdot16)\qquad \begin{cases} \xi = -\sin\sigma\displaystyle\int_{\sigma_0}^{\sigma}\rho\sin\sigma d\sigma - \cos\sigma\int_{\sigma_0}^{\sigma}\rho\cos\sigma d\sigma\,, \\ \eta = -\cos\sigma\displaystyle\int_{\sigma_0}^{\sigma}\rho\sin\sigma d\sigma + \sin\sigma\int_{\sigma_0}^{\sigma}\rho\cos\sigma d\sigma\,, \\ \zeta = 0\,, \end{cases}$$

where σ is defined by

$$(5\cdot17)\qquad \sigma = \int_{s_0}^{s}\frac{ds}{\rho}\,,$$

the lower limits σ_0, s_0 being arbitrarily chosen fixed values of σ, s. Then prove that all these trajectories C have the same planar evolute, namely, the curve C itself.

6. Find all *the orthogonal trajectories of the rectifying planes of a curve C*. Prove that they include the involutes of C and that any two of them are parallel.

6. The transformation of Combescure. It may be recalled that in defining parallel curves two conditions were laid down, one referring to constant distance and one to parallel tangents. In this section the condition of constant distance is dropped, and a one-to-one correspondence with the condition of parallel tangents is studied briefly.

DEFINITION 1. *Two curves are said to correspond by a transformation of Combescure in case there is a one-to-one correspondence between their points such that their tangents at corresponding points are parallel.*

Since any curve parallel to a curve C corresponds to C by a transformation of Combescure, it is obvious that, when a curve C is given, there always exist other curves in the relation of a transformation of Combescure to C. A remark perhaps not so obvious is that the transformation of Combescure is actually more general than the transformation by parallelism, in the sense that there exist pairs of curves not parallel but related by a transformation of Combescure. The truth of this remark is made evident by comparing equations (5·7), or (5·12), which involve arbitrary constants at most, with the equations in the following theorem, which involve an arbitrary function.

THEOREM 1. *The parametric equations of the most general curve in the relation of a transformation of Combescure to a curve C are*

$$(6\cdot1)\quad X = \int_{s_0}^{s} \alpha f(s)ds\,, \qquad Y = \int_{s_0}^{s} \beta f(s)ds\,, \qquad Z = \int_{s_0}^{s} \gamma f(s)ds\,,$$

in which s is arc length on C, and α, β, γ *are the direction cosines of the tangent of C, while* $f(s)$ *is an arbitrary function of s.*

To prove this theorem, let us observe that the parametric equations of the most general curve C_1 whose points are in one-to-one correspondence with the points of a given curve C can be written in the form

$$(6\cdot2)\qquad X = X(s)\,, \qquad Y = Y(s)\,, \qquad Z = Z(s)\,,$$

the functions in the right members being arbitrary functions of the arc length s on C. If arc length on the curve C_1 is denoted by s_1, the direction cosines of the tangent at a point Q of the curve C_1 are found, by differentiating equations (6·2), to be given by

$$(6\cdot3)\qquad \frac{dX}{ds_1} = X'\frac{ds}{ds_1}\,, \qquad \frac{dY}{ds_1} = Y'\frac{ds}{ds_1}\,, \qquad \frac{dZ}{ds_1} = Z'\frac{ds}{ds_1}\,.$$

The derivative ds_1/ds can be found after squaring and adding these equations and taking the square root of the sum; the result is

$$(6\cdot4)\qquad \frac{ds_1}{ds} = \pm(X'^2 + Y'^2 + Z'^2)^{1/2}.$$

It is surely true that ds_1/ds is arbitrary, since the functions X, Y, Z are arbitrary. Let us write

$$(6\cdot5)\qquad \frac{ds_1}{ds} = f(s)\,,$$

where $f(s)$ is an arbitrary function of s. The tangent at a point Q of the curve C_1 is parallel to the tangent at the corresponding point P of the curve C if, and only if,

$$(6\cdot6)\qquad \frac{dX}{ds_1} = \alpha\,, \qquad \frac{dY}{ds_1} = \beta\,, \qquad \frac{dZ}{ds_1} = \gamma.$$

From equations (6 3), (6 5), and (6 6) one deduces the differential equations

$$(6\ 7)\qquad dX = \alpha f(s)ds\,, \qquad dY = \beta f(s)ds\,, \qquad dZ = \gamma f(s)ds\,,$$

integration of which reproduces equations (6·1) and so completes the proof.

EXERCISES

1. If two curves correspond by a transformation of Combescure, the principal normals at corresponding points of the curves are parallel.

2. If two curves correspond by a transformation of Combescure, the binormals at corresponding points of the curves are parallel.

3. If the points of two curves, neither of which is a plane curve, are in one-to-one correspondence so that the binormals at corresponding points of the curves are parallel, the correspondence is a transformation of Combescure.

4. Denoting by $1/\rho_1$ and $1/\tau_1$ the curvature and torsion of any curve (6·1) in the relation of a transformation of Combescure to a given curve C, prove that

$$f(s) = \frac{\rho_1}{\rho} = \frac{\tau_1}{\tau}\,,$$

and hence that the ratio of the curvature to the torsion of a curve is invariant under a transformation of Combescure.

7. Bertrand curves. This section is devoted to a brief discussion of a special class of curves called Bertrand curves, which are defined in the following way.

DEFINITION 1. *A Bertrand curve is a curve such that there exists another curve with the same principal normals.*

The existence of Bertrand curves is at once evident from the following theorem.

THEOREM 1. *Every plane curve is a Bertrand curve.*

To prove this theorem, consider the locus C_0 of the centers of the osculating circles of a plane curve C. The curve C has the tangents of C_0 for principal normals, and the same is true of any other orthogonal trajectory C_1 of the tangents of C_0. Hence, the curves C and C_1 have the same principal normals.

Another example of a Bertrand curve is any curve with constant curvature (see Chap. I, Sec. 8, Ex. 8). Two curves, C and C_1, with the same principal normals are called *associated Bertrand curves.* The method of the moving trihedron will now be used to prove the following theorem.

THEOREM 2. *A necessary and sufficient condition that a curve, not a plane curve and not a curve with constant curvature, be a Bertrand curve is that the curvature and torsion of the curve satisfy a linear equation of the form*

$$\frac{a}{\rho} + \frac{b}{\tau} = 1 \qquad (a, b = \text{consts.} \neq 0)\,. \tag{7·1}$$

To prove this theorem, consider a curve C, a point P on C, and a point Q with local coordinates ξ, η, ζ. Let Q now be on the principal normal of C at P, and, furthermore, let the principal normal at Q of the locus C_1 of the point Q coincide with the principal normal of C at P, or, what is the same thing, let the tangent and binormal of C_1 at Q be perpendicular to the principal normal of C. The analytic formulation of these necessary and sufficient conditions that C and C_1 be associated Bertrand curves is

$$\xi = 0\,, \qquad \zeta = 0\,, \qquad B_1 = 0\,, \qquad Q = 0\,, \tag{7·2}$$

the function Q being defined in (2·13). Equations (2·2) are equivalent in this case to

$$A_1 = 1 - \frac{\eta}{\rho}\,, \qquad 0 = \eta'\,, \qquad C_1 = -\frac{\eta}{\tau}\,. \tag{7·3}$$

The condition $B_1 = 0$ and the definitions of A_2, C_2 in (2·4) imply

$$A_2 = A_1'\,, \qquad C_2 = C_1'\,. \tag{7·4}$$

Moreover, the condition $Q = 0$ is equivalent, by the definition of the function Q, to

$$C_1A_2 - C_2A_1 = 0\,. \tag{7·5}$$

Since $\eta' = 0$, it immediately follows that

$$\eta = a \qquad (a = \text{const.} \neq 0)\,. \tag{7·6}$$

Since ρ is not constant, it follows from the first of equations (7·3) that $A_1 \neq 0$; and since the curve C is not plane, it follows from the third of equations (7·3) that $C_1 \neq 0$. Substituting from (7·4) into (7·5), we obtain

$$(7\cdot7) \qquad C_1A_1' - C_1'A_1 = 0 .$$

Integrating this equation, we find

$$(7\cdot8) \qquad A_1 = hC_1 \qquad (h = \text{const.} \neq 0) .$$

If the expressions for A_1 and C_1 in equations (7·3) are modified by writing a in place of η, and if these expressions are then substituted in equation (7·8), the latter equation becomes precisely equation (7·1) with $b = -ha$. Thus the condition (7·1) is proved to be *necessary*. *The sufficiency* of this condition is demonstrated by starting with a point Q having local coordinates $(0, a, 0)$. The definition of B_1 shows that $B_1 = 0$. Then the reasoning can be reversed from equation (7·8) to (7·5) to show that $Q = 0$.

It will be observed that *associated Bertrand curves intercept a constant distance on their common principal normals*. In the moving coordinate system associated with a Bertrand curve C, the coordinates ξ, η, ζ of the point Q generating a Bertrand curve C_1 associated with C are given by

$$(7\cdot9) \qquad \xi = 0 , \qquad \eta = a , \qquad \zeta = 0 \quad (a = \text{const.} \neq 0) .$$

The parametric equations of the associated Bertrand curve C_1, in the fixed coordinate system, are

$$(7\cdot10) \qquad X = x + la , \qquad Y = y + ma , \qquad Z = z + na .$$

EXERCISES

1. Prove geometrically, and also analytically, that equations (7·2) imply $L = 0$, $N = 0$.

2. If a curve C is such that there exists another curve with the same binormals, the curve C is a plane curve, and the two curves intercept a constant distance on their common binormals.

3. If A is the angle between the tangents at corresponding points of a Bertrand curve C and an associated Bertrand curve, then the linear relation (7·1) can be replaced by

$$\frac{a \sin A}{\rho} - \frac{a \cos A}{\tau} = \sin A \qquad (a = \text{const.} \neq 0) .$$

For a plane curve C we have $A = 0$, and for a curve C with constant curvature, $A = \pi/2$.

4. A circular helix is a Bertrand curve which has infinitely many circular helices for associated Bertrand curves.

5. The product of the torsions at corresponding points of two associated Bertrand curves is constant and positive.

6. The orthogonal trajectories of the principal normals of a circular helix are circular helices with the same principal normals.

7. Two orthogonal trajectories of the principal normals of any curve intercept a constant distance on the principal normals. The same is true for the binormals.

8. Prove that, if a curve C has the property that the locus of a point on a tangent at a constant distance from the contact point P is a straight line, as P varies along C, then C is a plane curve and is, in fact, *the tractrix*.

CHAPTER III

SURFACES

1. Definition of a surface. A surface can be described as a two-parameter family, or double infinity, of points. A surface can also be said to be the locus of a point moving with two degrees of freedom. A more precise definition of an analytic surface will be formulated presently.

One method of representing a surface analytically consists in first establishing the usual left-handed orthogonal cartesian coordinate system with the same unit of distance on all three axes and then imposing one condition on a variable point $P(x, y, z)$ by an equation of the form

$$F(x, y, z) = 0 . \tag{1·1}$$

Such an equation is called *the implicit equation* of the surface represented by it.

Certain very simple types of surfaces are already familiar. For example, if the equation (1·1) is linear in the variables x, y, z, the surface represented by it is a *plane*, which is the simplest surface of all. Perhaps the next simplest surface is the *sphere*. If the equation (1·1) is of the second degree, the surface represented by it is a *quadric surface*, of which the sphere is a special case. If equation (1·1) is homogeneous in x, y, z, it represents a *cone* whose vertex is at the origin. Finally, if one of the variables is missing from the implicit equation of a surface, the surface is a *cylinder* whose generators are parallel to the axis of the missing variable.

If the implicit equation (1·1) be solved for one of the variables as a function of the other two, say for z as a function of x, y, the resulting equation,

$$z = f(x, y) , \tag{1·2}$$

represents the same surface as before. Such an equation is called *the explicit equation* of the surface represented by it. The explicit equation (1·2) can be exhibited as a special case of the implicit equation (1·1) by transposing z to the right member and placing

$$F(x, y, z) = f(x, y) - z .$$

Although for some purposes the implicit and explicit equations of surfaces are very useful, the definition of a real proper analytic surface which will be used in this and succeeding chapters will be based on a parametric representation.

DEFINITION 1. *Let the coordinates x, y, z of a point P be given as single-valued real-valued analytic functions of two real independent variables u, v on a rectangle T in a uv-plane, by equations of the form*

$$(1\cdot3)\qquad x = x(u, v)\,, \qquad y = y(u, v)\,, \qquad z = z(u, v)\,.$$

Further, let the jacobians of x, y, z with respect to u, v be denoted by J_1, J_2, J_3, so that

$$(1\cdot4)\quad \begin{cases} J_1 = y_u z_v - y_v z_u\,, \qquad J_2 = z_u x_v - z_v x_u\,, \qquad J_3 = x_u y_v - x_v y_u \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left(x_u = \dfrac{\partial x}{\partial u}\,, \cdots\right), \end{cases}$$

and suppose that not all of J_1, J_2, J_3 vanish identically on the rectangle T. Then the locus of the point P, as u, v vary on T, is a real proper analytic surface S.

Equations (1·3) are called *parametric equations* of the surface S, the *parameters* being the variables u, v. We reserve the right to permit the parameters to take on complex values whenever they are properly introduced in the sequel. Moreover, one or more of the coordinates x, y, z may, under suitable conditions, be allowed to be complex.

To say that a surface is *proper* means that it does not reduce to a single fixed point and also that it does not reduce to a curve. Both of these degenerate cases are ruled out by the hypothesis that the jacobians J_i ($i = 1, 2, 3$) do not all vanish identically. For, if the locus S were to reduce to a fixed point P, the coordinates x, y, z of P would all be constant, and the jacobians J_i would all vanish identically. Furthermore, if the locus S were to reduce to a curve, this curve could be represented parametrically by equations of the form (I·2·1). If in these equations the parameter t is set equal to any function of u, v, the result is three equations, of the form (1·3), for which the jacobians J_i are easily proved, by actual calculation, to vanish identically. Conversely, *the identical vanishing of the jacobians J_i would imply that the locus of the point P was not a proper surface.* For, if the jacobians all vanish identically, then the functions x, y, z are three solutions of a linear homogeneous partial differential equation of the form

$$(1\cdot5)\qquad a\theta_u + b\theta_v = 0\,,$$

in which the coefficients a, b are functions of u, v. The theory of linear partial differential equations of the first order tells us how to integrate this equation. First form the associated ordinary differential equation

$$(1\cdot 6) \qquad bdu - adv = 0 .$$

This equation has an integral

$$(1\cdot 7) \qquad t(u, v) = \text{const.},$$

and the most general solution θ of equation $(1\cdot 5)$ is given by the formula

$$(1\cdot 8) \qquad \theta = F(t(u, v)) ,$$

the function F being arbitrary. Consequently the coordinates x, y, z are either all constant or are, at most, functions of a single parameter t, so that either P is a fixed point or else has for its locus a curve.

Even if the jacobians J_1, J_2, J_3 do not all vanish identically on the rectangle T, it may happen that they vanish simultaneously for one or more isolated pairs of values of u, v, or perhaps they vanish simultaneously along a curve $v = v(u)$ in T. Any point of a real proper analytic surface at which the jacobians J_1, J_2, J_3 vanish simultaneously is called *singular*, although the singularity may belong to the parametric representation being used for the surface defined as a point-locus, as in the case of the sphere discussed in the next section, or else the singularity may belong to the surface itself. Singular points will ordinarily be avoided hereinafter. A surface, or portion of a surface, which is free of singular points may be called *nonsingular*. In this book the word *surface* will ordinarily mean a real proper nonsingular analytic surface.

Elimination of u, v from the parametric equations $(1\cdot 3)$ of a surface S would lead to the implicit equation $(1\cdot 1)$ of S. Vice versa, if the implicit equation $(1\cdot 1)$ of a surface is given and a parametric representation of the surface is desired, let two of the variables, say x and y, be arbitrary functions of two parameters u, v, and then solve $(1\cdot 1)$ for z as a function of u, v. In particular, we might take $x = u$, $y = v$. Then solution of $(1\cdot 1)$ for z would lead to the explicit equation $(1\cdot 2)$ of the surface, except that u and v would occur in place of x and y, respectively. Indeed, the explicit equation $(1\cdot 2)$ of a surface, when supplemented by the identities $x = x$, $y = y$, becomes the parametric equations

$$(1\cdot 9) \qquad x = x\,, \qquad y = y\,, \qquad z = f(x, y)$$

of the same surface, the parameters now being the coordinates x, y.

EXERCISES

1. If the parametric equations (1 3) are linear in u, v, they represent a plane.

2. The equations

$$x = u + v\,, \qquad y = (u + v)^2\,, \qquad z = (u + v)^3$$

represent a cubical parabola, and not a proper surface.

2. Curves on surfaces. In the first part of this section the analytic representation of curves on surfaces will be explained. One-parameter families and *nets* of curves will be discussed, with particular reference to *parametric curves.* In the second part, some examples of the parametric equations of surfaces will be adduced for the purpose of illustration and for convenience of reference. *Parametric curves* on a surface are defined as follows:

DEFINITION 1. *The parametric curves on a surface S, which is represented by parametric equations of the form* (1 3), *are defined to be those curves on S along each of which one of the parameters varies while the other is constant.*

If the parameter v is held fixed while u varies, the locus of the variable point $P(x, y, z)$ is a curve on the surface S. This curve, which is sometimes denoted by C^u, is called a *u-curve* because its parameter is u. If v is given a different value and is again held fixed while u varies, the locus of the point P is another u-curve on the surface S. In this way, by placing v = const., a one-parameter family of u-curves is defined. These cover the surface S and are completely described by the differential equation $dv = 0$.

Similarly, interchanging the roles of the parameters u and v, we define a one-parameter family of *v-curves* on the surface S, along any one of which, denoted by C^v, the parameter v varies and u = const., so that $du = 0$. *The family of u-curves and the family of v-curves together constitute the parametric curves, which are completely represented analytically by the differential equation*

$$(2\cdot 1) \qquad du\,dv = 0\,.$$

Tangent lines of parametric curves are called *parametric tangents*, those of u-curves and of v-curves being named *u-tangents* and *v-tangents*, respectively. Sometimes the values of a pair u, v which locates a point P on a surface S are spoken of as *curvilinear coordinates* of P. Since on a surface S points P and pairs of values of u, v are in continuous one-to-one correspondence when the range T of the variables u, v is sufficiently small, the terminology is then justified. Parametric curves are also sometimes called *coordinate curves.*

Any curve C on a surface can be represented analytically in various ways. For example, in the parametric equations (1·3) of a surface S let the parameters u, v be functions of a third variable t, so that we have

$$u = u(t)\,, \qquad v = v(t)\,. \tag{2·2}$$

Then the locus of the point $P(x, y, z)$, as t varies, is a curve on S. Conversely, any curve C on S can be represented analytically by equations of the form (2·2), since along C each of u, v is a function of any parameter t that varies along C. Equations (2·2) are called *curvilinear parametric equations* of the curve represented by them.

Another way to represent a curve C on a surface is to impose a condition on the curvilinear coordinates u, v by making them satisfy an equation of the form

$$F(u, v) = 0\,. \tag{2·3}$$

If this equation be solved for one of the variables as a function of the other, say for v as a function of u, we obtain

$$v = v(u)\,. \tag{2·4}$$

If this function of u is substituted in place of v in equations (1·3), the resulting equations are parametric equations of the curve C, the parameter now being u. An equation (2·3) is referred to as *the curvilinear implicit equation* of the curve represented by it, while an equation (2·4) is designated as *the curvilinear explicit equation* of the curve which it represents. The equation $v =$ const. of a *u-curve* is a special case of the explicit equation (2·4), but the curves $u =$ const. are not represented by (2·4).

An equation of the form

$$F(u, v) = c \qquad (c = \text{const.}) \tag{2·5}$$

represents *a one-parameter family* of curves on the surface defined by the parametric equations (1 3). One of these curves passes through each point (u_0, v_0) of the surface, namely, that curve whose equation in curvilinear coordinates is

$$F(u, v) = F(u_0, v_0) .$$

The equation

$$F_u du + F_v dv = 0 , \tag{2·6}$$

obtained from (2 5) by differentiation, is called *the curvilinear differential equation* of the family of curves represented by (2 5). Conversely, *any equation of the form*

$$m du + n dv = 0 , \tag{2·7}$$

in which m, n *are functions of* u, v, *is the curvilinear differential equation of a one-parameter family of curves on the surface represented by equations* (1 3).

A net of curves on a surface is defined as follows:

DEFINITION 2. *A net of curves on a surface* S *is two one-parameter families of curves on* S *such that through each point* P *of* S *there passes just one curve of each family, the two tangents of the curves at* P *being distinct.*

It is convenient to speak of the two tangents at a point P of the two curves of a net that pass through P as simply *the tangents of the net* at P. *The curvilinear differential equation of a net* results when the two equations of the form (2 7) representing the two component families of the net are multiplied together. Since the two families of the net are distinct, corresponding coefficients in the equations of the families are not proportional; or, in other words, the ratio of the differentials dv and du computed from one equation is not identically equal to this ratio computed from the other equation. Moreover, it follows from the discussion in Section 4 below that, since at every point of the portion of the surface under consideration the tangents of the net are distinct, the two ratios cannot be equal at any point. Consequently, *the curvilinear differential equation of a net* can be written in the form

$$A du^2 + 2B du dv + C dv^2 = 0 \qquad (AC - B^2 \neq 0) , \tag{2·8}$$

in which the coefficients A, B, C are functions of u, v. Conversely, any equation of this form (with nonvanishing discriminant) represents a net of curves on the surface (1 3), since the left member of such an equation can be factored into two distinct linear factors which, when equated to zero, represent the two component families of the net, the two tangents of the curves of the two families being distinct at every point of the surface under consideration.

The equation (2 1) representing the parametric curves on a surface is a special case of equation (2 8). Indeed, the parametric curves on a surface, or else on a sufficiently restricted region of it, form a net, called *the parametric net*, and *equation* (2 1) *is the curvilinear differential equation of the parametric net.*

Some of the simplest examples of parametric equations of surfaces may aid in arriving at a better understanding of the general theory of surfaces. First of all, *the xy-plane* can be represented by the parametric equations

$$(2\ 9) \qquad x = u\,, \qquad y = v\,, \qquad z = 0\,.$$

In this case the jacobians J_i have the values given by

$$(2\ 10) \qquad J_1 = 0\,, \qquad J_2 = 0\,, \qquad J_3 = 1\,.$$

The u-curves are straight lines parallel to the x-axis, and the v-curves are straight lines parallel to the y-axis. So ordinary orthogonal cartesian coordinates in a plane are an instance of curvilinear coordinates on a surface, the parametric curves being straight and the surface being flat in this special case.

After the plane the next simplest surface is perhaps *the sphere*, which is usually defined to be the locus of a point moving at a constant distance from a fixed point. The following parametric equations of the sphere with center at the origin and with radius r can be read off directly from Figure 7:

$$(2\ 11) \quad x = r \sin u \cos v\,, \qquad y = r \sin u \sin v\,, \qquad z = r \cos u\,.$$

For this representation of the sphere the jacobians J_i have the values given by

$$(2\ 12) \qquad \begin{cases} J_1 = r^2 \sin^2 u \cos v\,, \qquad J_2 = r^2 \sin^2 u \sin v\,, \\ \qquad\qquad J_3 = r^2 \sin u \cos u\,. \end{cases}$$

In geographical language the u-curves are meridians of longitude and the v-curves are parallels of latitude, the curvilinear coordinates of a point being its colatitude u and its longitude v. It should be noted that, although the sphere itself, when defined as the locus of a point, is perfectly regular at every one of its points, nevertheless the jacobians J_i all vanish at the north and south poles, where $u = 0$ and $u = \pi$, respectively. The reason for this phenomenon is that there are infinitely many meridians of longitude through both poles, so that

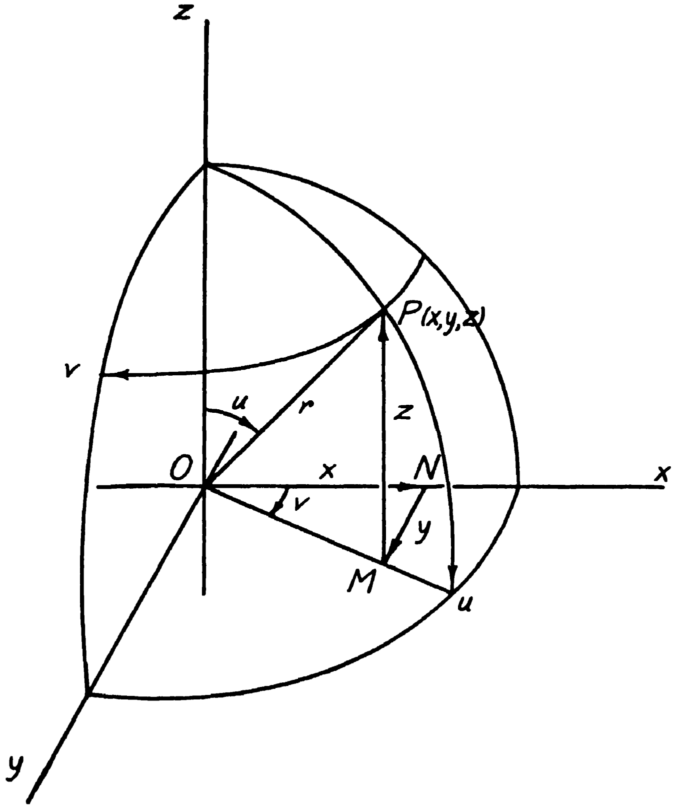

FIG. 7

at the poles the parametric curves fail to satisfy one of the conditions laid down in the definition of a net. The singularities at the poles therefore belong to the analytic representation used and not to the surface itself regarded as a point-locus.

A surface of revolution, or simply *a revolute*, is in some sense a generalization of the sphere. Simple parametric equations of a revolute are obtained in the following way. With reference to Figure 8, let us consider any curve C, in the zx-plane, which does not cross the z-axis and which has equations of the form

$$(2\cdot 13) \qquad y = 0, \qquad z = f(x).$$

When the curve C is revolved about the z-axis, the locus of C is a surface of revolution whose *axis of revolution* is the z-axis. Any point $P_0(x_0, 0, z_0)$ on C takes the position of a point $P(x, y, z)$ after C has

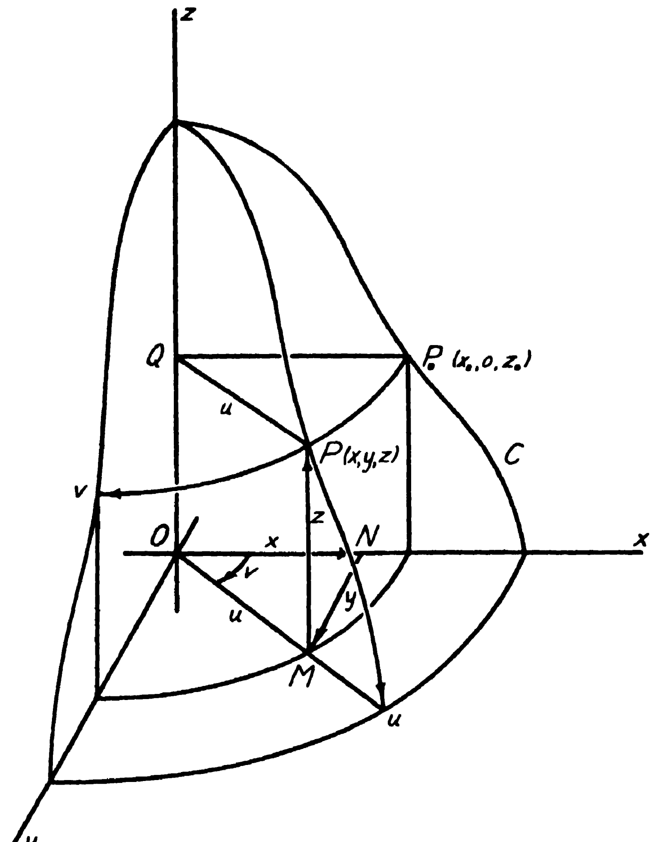

FIG. 8

been revolved through an angle v. It is obvious that $z = z_0$. Moreover, if a function u is defined by placing

$$(2\cdot 14) \qquad u = (x^2 + y^2)^{1/2}$$

and is restricted to be positive, one sees immediately that

$$x_0 = OM = u$$

and therefore that $z = f(u)$. It is now evident that *parametric equations of the surface of revolution are*

$$(2\cdot 15) \qquad x = u \cos v\,, \qquad y = u \sin v\,, \qquad z = f(u)\,.$$

For the revolute the jacobians J_i have the values given by

$$(2\cdot16)\quad \begin{cases} J_1 = -f'u\cos v\,, \qquad J_2 = -f'u\sin v\,, \qquad J_3 = u \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left(f' = \dfrac{df}{du}\right). \end{cases}$$

The u-curves and v-curves are still called "meridians of longitude" and "parallels of latitude," respectively. Comparison of Figures 7 and 8 shows that u in Figure 8 is $r \sin u$ in Figure 7. If in the last of the equations (2·15) of a revolute the function $f(u)$ is specialized by placing

$$f(u) = (r^2 - u^2)^{1/2}\,,$$

the revolute becomes a sphere. If u is now replaced by $r \sin u$, the resulting equation,

$$z = f(r\sin u) = r\cos u\,,$$

is the last of the equations (2·11) of the sphere.

EXERCISES

1. The surface represented by the equations

$$(2\cdot17)\qquad x = u\,, \qquad y = v\,, \qquad z = uv$$

is a *hyperbolic paraboloid*, whose explicit equation is

$$z = xy\,.$$

The u-curves are straight lines,

$$y = v\,, \qquad z = vx\,,$$

parallel to the zx-plane; the v-curves are straight lines,

$$x = u\,, \qquad z = uy\,,$$

parallel to the yz-plane. The jacobians J_i for this surface have the values given by

$$J_1 = -v\,, \qquad J_2 = -u\,, \qquad J_3 = 1\,.$$

2. Find the implicit equation and the jacobians J_i $(i = 1, 2, 3)$ for each of the following surfaces:

a) Ellipsoid,

$$(2\cdot18)\qquad x = a\sin u\cos v\,, \qquad y = b\sin u\sin v\,, \qquad z = c\cos u\,.$$

b) Hyperboloid of two sheets,

$$x = a \cosh u \cosh v\,, \qquad y = b \cosh u \sinh v\,, \qquad z = c \sinh u\,. \tag{2·19}$$

c) Hyperboloid of one sheet,

$$x = a \cosh u \cos v\,, \qquad y = b \cosh u \sin v\,, \qquad z = c \sinh u\,. \tag{2·20}$$

d) Elliptic paraboloid,

$$x = au \cos v\,, \qquad y = bu \sin v\,, \qquad z = u^2\,. \tag{2·21}$$

e) Hyperbolic paraboloid,

$$x = au \cosh v\,, \qquad y = bu \sinh v\,, \qquad z = u^2\,. \tag{2·22}$$

3. Discuss the following two surfaces of revolution:

a) The *catenoid* (surface of revolution of a catenary),

$$x = u \cos v\,, \qquad y = u \sin v\,, \qquad z = a \log \frac{u + (u^2 - a^2)^{1/2}}{b}\,. \tag{2·23}$$

b) The *anchor ring* (surface of revolution of a circle),

$$x = u \cos v\,, \qquad y = u \sin v\,, \qquad z = [b^2 - (u - a)^2]^{1/2} \quad (a > b)\,. \tag{2·24}$$

4. Defining a *conoid* to be the locus of a straight line which is perpendicular to and intersects a fixed straight line l, which is revolved about l, and which is, at the same time, translated along l, prove that the parametric equations of a conoid can be written in the form

$$x = u \cos v\,, \qquad y = u \sin v\,, \qquad z = g(v)\,. \tag{2·25}$$

Describe the parametric curves on this surface.

5. Discuss the following two conoids:

a) The *screw surface*,

$$x = u \cos v\,, \qquad y = u \sin v\,, \qquad z = av \qquad (a = \text{const.})\,. \tag{2·26}$$

b) The *cylindroid*,

$$x = u \cos v\,, \qquad y = u \sin v\,, \qquad z = \sin 2v\,. \tag{2·27}$$

6. Prove that the locus of the principal normals of a circular helix is a screw surface.

7. Find the parametric equations of the form (2·25) for the conoid whose implicit equation is

$$x^2z^2 + a^2y^2 = r^2x^2 ,$$

and determine the shape of the surface.

8. Discuss the *helicoid*,

$$(2\cdot 28) \qquad x = u \cos v , \qquad y = u \sin v , \qquad z = av + f(u) \qquad (a = \text{const.}) .$$

3. The first fundamental form. The word *form*, as used in algebra, means *homogeneous polynomial*. An n-ary p-adic form is a form of degree p in n variables. In particular, a *binary quadratic* form is a form of the second degree in two variables. *A differential form* is a form in the differentials of a set of variables, the coefficients of the form being functions of these variables. Binary quadratic differential forms appear frequently in the metric differential geometry of surfaces. For instance, the left member of equation (2·8) is form of this kind.

The first fundamental form in the metric theory of surfaces is a binary quadratic differential form which arises in the calculation of the element of arc of a curve on a surface. To discover it, let us consider a surface S represented analytically by the parametric equations (1·3), and on S consider a curve C whose curvilinear equation is known. Differentiating equations (1·3), we obtain

$$(3\cdot 1) \qquad \begin{cases} dx = x_u du + x_v dv , \qquad dy = y_u du + y_v dv , \\ \qquad\qquad dz = z_u du + z_v dv , \end{cases}$$

the differentials du, dv being supposed to be related by the equation resulting from differentiating the curvilinear equation of C. Let s denote arc length measured from some fixed point of the curve C. Then, squaring and adding equations (3·1), we find

$$(3\cdot 2) \qquad ds^2 = Edu^2 + 2Fdudv + Gdv^2 ,$$

where the coefficients E, F, G are defined by the formulas

$$(3\cdot 3) \qquad \begin{cases} E = x_u^2 + y_u^2 + z_u^2 , \qquad F = x_u x_v + y_u y_v + z_u z_v , \\ \qquad\qquad G = x_v^2 + y_v^2 + z_v^2 . \end{cases}$$

Thus the square of the element of arc of the curve C is calculated, and the way is prepared for the following definition.

Definition 1. *The first fundamental form is the form in the right member of equation* (3 2).

The coefficients E, F, G of the first fundamental form are sometimes called *the first fundamental coefficients* in the theory of surfaces. Some additional notations will be useful in the sequel. In the domain of real numbers the coefficients E, G are positive. Two functions A and C are now defined to be the positive square roots of E and G, respectively, so that

$$(3\cdot 4) \qquad A = E^{1/2}, \qquad C = G^{1/2}.$$

Furthermore, the first fundamental form is definitely positive in the domain of reals, since this form is equal to ds^2. Therefore, the discriminant $EG - F^2$ of this form is positive. In fact, Lagrange's identity (I·4·13) gives a relation,

$$(3\cdot 5) \qquad EG - F^2 = J_1^2 + J_2^2 + J_3^2,$$

between this discriminant and the jacobians J_i defined by (1·4). A function H is now defined to be the positive square root of this discriminant, so that

$$(3\cdot 6) \qquad H = (EG - F^2)^{1/2}.$$

The element of arc of a parametric curve can be expressed by a very simple formula. Let arc length of a u-curve be denoted by s^u and that of a v-curve by s^v. Since we have $dv = 0$ along a u-curve and $du = 0$ along a v-curve, the general formula (3 2) for the element of arc of any curve specializes, for the parametric curves, into

$$(3\cdot 7) \qquad ds^u = A\,du, \qquad ds^v = C\,dv,$$

where A, C are defined in (3·4), and the positive square roots are taken in order that the positive sense on a parametric curve may be the sense in which the parameter increases, in harmony with the convention made in Section 3 of Chapter I.

Then the length s of a curve C on a surface, measured from an arbitrarily chosen fixed point (u_0, v_0) to a variable point (u, v) on C, may be calculated by means of the integral

$$(3\ 8) \qquad s = \int_{(u_0,\, v_0)}^{(u,\, v)} (E\,du^2 + 2F\,du\,dv + G\,dv^2)^{1/2}.$$

In order to evaluate this integral along the curve C defined by a curvilinear implicit equation (2·3), one may differentiate this equation to get a relation between du and dv which is valid at points of C, and may then use (2·3) to express one variable as a function of the other, say v as a function of u. Thus the evaluation of the integral may be reduced to a quadrature. It is even easier to reduce the evaluation of the integral to a quadrature when the curve C is defined by its curvilinear parametric equations (2·2), by using t as independent variable.

EXERCISES

1. For the xy-plane represented by the parametric equations (2·9), verify the results,

$$(3\cdot 9) \qquad E = 1\,, \quad F = 0\,, \quad G = 1\,, \quad H = 1\,;$$

for the sphere (2·11),

$$(3\cdot 10) \qquad E = r^2\,, \quad F = 0\,, \quad G = r^2 \sin^2 u\,, \quad H = r^2 \sin u\,;$$

and for the revolute (2·15),

$$(3\cdot 11) \qquad E = 1 + f'^2\,, \quad F = 0\,, \quad G = u^2\,, \quad H = u(1 + f'^2)^{1/2}\,,$$

the positive square root being taken.

2. Calculate J_i, E, F, G, H for the surfaces whose parametric equations are written in the exercises of Section 2.

3. Use the integral (3·8) to verify that the total length of a meridian of the sphere (2·11) is $2\pi r$, that the length of the equator is $2\pi r$, and that the length of a parallel of latitude, $u = b$, is $2\pi r \sin b$.

4. When the equation of a surface is written in the explicit form (1·2), place

$$x = u\,, \quad y = v\,, \quad f_x = p\,, \quad f_y = q\,,$$

and verify the following results:

$$(3\cdot 12) \qquad \begin{cases} J_1 = -p\,, & J_2 = -q\,, \quad J_3 = 1\,, \\ E = 1 + p^2\,, & F = pq\,, \quad G = 1 + q^2\,, \quad H^2 = 1 + p^2 + q^2\,. \end{cases}$$

4. Tangent plane and normal line. Let us consider a surface S represented by the parametric equations (1·3), and on S a curve C whose curvilinear parametric equations are

$$(4\cdot 1) \qquad u = u(s)\,, \quad v = v(s)\,,$$

the parameter s being arc length measured from an arbitrarily chosen fixed point of C. The direction cosines α, β, γ of the tangent line at a point $P(x, y, z)$ of C are given by the formulas

$$(4\cdot 2)\qquad \begin{cases} \alpha = x' = x_u u' + x_v v' , \\ \beta = y' = y_u u' + y_v v' , \\ \gamma = z' = z_u u' + z_v v' \end{cases} \qquad \left(x' = \frac{dx}{ds}, \cdots\right).$$

Let functions belonging to the parametric curves be indicated by appropriate superscripts u or v. Then equations (4·2) and (3·7) lead to the following formulas for the direction cosines of the u-tangent and the v-tangent at a point of a surface:

$$(4\cdot 3)\qquad \begin{cases} \alpha^u = \dfrac{x_u}{A}, \quad \beta^u = \dfrac{y_u}{A}, \quad \gamma^u = \dfrac{z_u}{A}; \\ \alpha^v = \dfrac{x_v}{C}, \quad \beta^v = \dfrac{y_v}{C}, \quad \gamma^v = \dfrac{z_v}{C}. \end{cases}$$

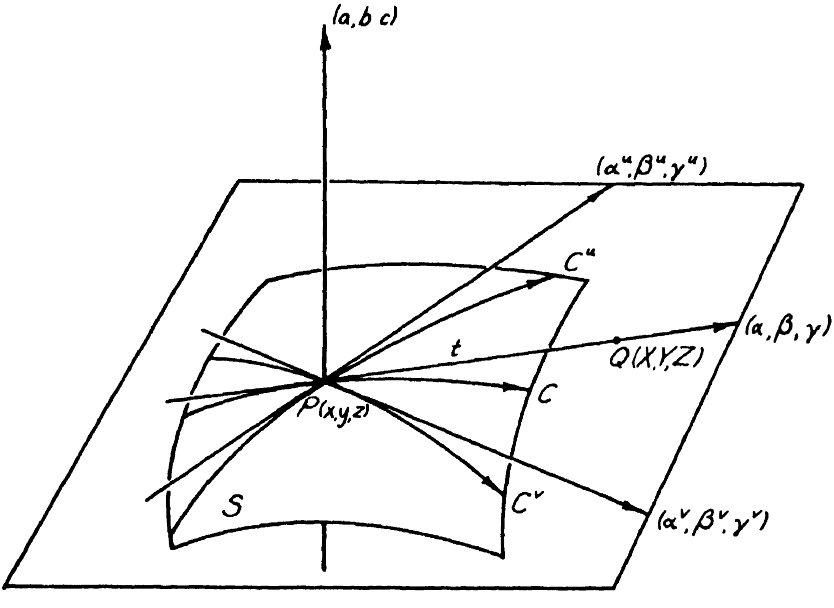

FIG. 9

A remarkable fact about the geometry of a surface, in the neighborhood of one of its points, will now be stated.

THEOREM 1. *The tangent lines at an ordinary point P on a surface S of all the curves that lie on S and pass through P lie in a plane.*

The proof of this theorem is illustrated by Figure 9. The parametric equations of the tangent line at the point P of the curve C on the surface S can be written in the form

$$(4\cdot4)\qquad \begin{cases} X = x + t(x_u u' + x_v v')\,, \\ Y = y + t(y_u u' + y_v v')\,, \\ Z = z + t(z_u u' + z_v v')\,, \end{cases}$$

in which t is the algebraic distance from the contact point $P(x, y, z)$ to a variable point $Q(X, Y, Z)$ on the tangent, and u', v' are calculated from the equations (4·1) of C. As the point Q moves along the tangent, the distance t varies. If the curve C is now allowed to vary in all possible ways, remaining on S and passing always through P, the ratio v'/u', or dv/du, varies. Under these circumstances equations (4·4) are the parametric equations of a plane, whose implicit equation is found, by eliminating the parameters tu' and tv' from equations (4·4), to be

$$(4\cdot5)\qquad \begin{vmatrix} X - x & Y - y & Z - z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = 0\,.$$

The tangent lines at the point P of all the curves that lie on the surface S and pass through P lie in this plane. The following definition of *the tangent plane* is based on the foregoing theorem.

DEFINITION 1. *The tangent plane at a point P of a surface S is the plane that contains the tangent lines at P of all the curves that lie on S and pass through P.*

The equation (4·5) of the tangent plane can be written in the form

$$(4\cdot6)\qquad (X - x,\ x_u,\ x_v) = 0\,,$$

by interchanging rows and columns of the determinant and writing only a typical row within parentheses. The equation of the tangent plane can also be written in the form

$$(4\cdot7)\qquad J_1(X - x) + J_2(Y - y) + J_3(Z - z) = 0\,,$$

by expanding the determinant in (4 5) on the elements of the first row.

The point $P(x, y, z)$ is naturally called *the contact point* of the tangent plane of the surface S at P. The formulas (4·2) for the direction cosines of the tangent line at a point P of a curve on a surface show that, as long as the point P is fixed, the tangent line changes its orientation in the tangent plane if, and only if, the derivative dv/du changes. In fact, this derivative is a coordinate of the tangent line in the flat pencil composed of all the tangent lines in the tangent plane at the point P. It is convenient to give the name *direction* to this coordinate by a definition.

DEFINITION 2. *At a point of a curve on a surface the direction of the curve (or of its tangent) is the derivative dv/du calculated from the curvilinear equation of the curve.*

Evidently, the tangents of two curves at a point on a surface coincide if, and only if, the directions of the two curves at the point are equal. *The normal line*, or simply *the normal*, at a point of a surface is defined as follows:

DEFINITION 3. *The normal at a point P of a surface S is the line through P perpendicular to the tangent plane of S at P.*

The equation (4·7) of the tangent plane shows that, except possibly for sign, the direction cosines of the normal, which will be denoted by a, b, c, are proportional to the jacobians J_i. Hence, except possibly for sign, they are equal to these jacobians divided by H. The uncertainty with regard to their signs is removed by the following convention relative to the positive sense on the normal:

The positive sense on the normal is, by agreement, such that

$$(4\cdot 8) \qquad a = \frac{J_1}{H}, \qquad b = \frac{J_2}{H}, \qquad c = \frac{J_3}{H}.$$

The geometrical fact that the normal at a point of a surface is perpendicular to both parametric tangents at the point is affirmed analytically by the following easily proved identities:

$$(4\cdot 9) \qquad a\alpha^u + b\beta^u + c\gamma^u = 0, \qquad a\alpha^v + b\beta^v + c\gamma^v = 0.$$

With these should be associated the following:

$$(4\cdot 10) \qquad \begin{cases} a^2 + b^2 + c^2 = 1, \qquad aa_u + bb_u + cc_u = 0, \\ aa_v + bb_v + cc_v = 0. \end{cases}$$

Finally, direct calculation suffices to verify the identity

$$(4\cdot 11) \qquad \begin{vmatrix} \alpha^u & \beta^u & \gamma^u \\ \alpha^v & \beta^v & \gamma^v \\ a & b & c \end{vmatrix} = \frac{H}{AC}.$$

Since the right member is positive, so also is the determinant positive. Therefore *the trihedron whose edges are the u-tangent, the v-tangent, and the normal at a point of a surface is left-handed*, since this trihedron must have the same orientation as the fundamental coordinate trihedron.

EXERCISES

1. For the sphere (2 11) verify the results,

(4 12) $$a = \sin u \cos v\,, \qquad b = \sin u \sin v\,, \qquad c = \cos u\,;$$

and for the revolute (2 15),

(4 13) $$a = -\frac{f' \cos v}{(1 + f'^2)^{1/2}}\,, \qquad b = -\frac{f' \sin v}{(1 + f'^2)^{1/2}}\,, \qquad c = \frac{1}{(1 + f'^2)^{1/2}}\,.$$

2. Write the equations of the normal line at any point of the sphere, and prove that this line passes through the center of the sphere.

3. Write the equations of the normal line at any point of the revolute, and prove that this line intersects the axis of revolution at the point $(0, 0, f + u/f')$. The length of the segment of the normal between this point and the surface is

$$\frac{u}{f'}\,(1 + f'^2)^{1/2}\,.$$

4. The direction cosines of the normal at a point of a surface whose equation is written in the explicit form $z = f(x, y)$ are proportional to $f_x, f_y, -1$.

5. The direction cosines of the normal at a point of a surface whose equation is written in the implicit form $F(x, y, z) = 0$ are proportional to F_x, F_y, F_z. Interpret geometrically the equation

(4 14) $$F_x dx + F_y dy + F_z dz = 0\,,$$

which is obtained from the equation of the surface by differentiation.

6. When the equations of a curve are written in the implicit form

$$F(x, y, z) = 0\,, \qquad G(x, y, z) = 0\,,$$

the direction cosines of the tangent at a point of this curve are proportional to

$$F_y G_z - F_z G_y\,, \qquad F_z G_x - F_x G_z\,, \qquad F_x G_y - F_y G_x\,.$$

7. Find the equations of the tangent line at the point (1, 1, 1) of the curve

$$xyz = 1\,, \qquad y^2 = x\,.$$

8. Find the equations of the tangent line and normal plane at any point of the curve

$$x^2 + y^2 + z^2 = 4a^2, \qquad x^2 + z^2 = 2ax.$$

9. Find the equation of the normal plane at any point on the curve

$$ax^2 + by^2 + cz^2 = 1, \qquad x^2 + y^2 + z^2 = d^2.$$

5. Envelopes of surfaces. Just as in the plane a sufficiently regular one-parameter family of curves has an envelope which is a curve to which all the curves of the family are tangent, so in three-dimensional space a sufficiently regular one-parameter family of surfaces has an envelope which is a surface to which all the surfaces of the family are tangent. We shall discuss in this section the envelope of a one-parameter family of nonspecialized analytic surfaces, and in the next section shall restrict attention to the case of a one-parameter family of planes.

Let us consider a sufficiently regular analytic one-parameter family of surfaces represented by the equation

$$F(x, y, z, t) = 0, \tag{5·1}$$

in which t is the parameter. For an arbitrarily chosen and then fixed value t of the parameter, equation (5·1) represents a surface of the family. If a small increment Δt is given to t, we obtain the equation

$$F(x, y, z, t + \Delta t) = 0 \tag{5·2}$$

of a neighboring surface of the family. Any point on the curve of intersection of these two surfaces is also on the surface whose equation is

$$\frac{1}{\Delta t}[F(x, y, z, t + \Delta t) - F(x, y, z, t)] = 0.$$

If this curve of intersection approaches a limit when Δt approaches zero, this limiting curve is called *the characteristic curve*, or simply *the characteristic*, of the surface (5·1). In other words, *the characteristic of a surface is the curve in which a consecutive surface intersects it.* The implicit equations of the characteristic are

$$F(x, y, z, t) = 0, \qquad F_t(x, y, z, t) = 0, \tag{5·3}$$

in which the subscript denotes partial differentiation. The way is now prepared to define *the envelope.*

DEFINITION 1. *The envelope of a one-parameter family of surfaces is the locus of the characteristics of the surfaces.*

Elimination of t from equations (5·3) would yield the implicit equation of the envelope. If the elimination be performed by solving the second of (5·3) for t as a function of x, y, z and then substituting in the first, the equation of the envelope takes the form

$$F(x, y, z, t(x, y, z)) = 0 . \tag{5·4}$$

With this equation of the envelope it is easy to prove the following theorem.

THEOREM 1. *The envelope of a one-parameter family of surfaces is tangent to each surface of the family at each point of the characteristic of the surface.*

To prove this theorem, let us observe that the direction cosines of the normal at a point of the envelope (5·4) are proportional to

$$F_x + F_t t_x , \qquad F_y + F_t t_y , \qquad F_z + F_t t_z .$$

But at a point of the characteristic curve (5·3) we have $F_t = 0$. Therefore the direction cosines of the normal to the envelope are proportional to F_x, F_y, F_z, and so this line is also normal to the surface (5·1). Consequently, at each point of the characteristic (5·3), the surface (5·1) and the envelope (5·4) have the same tangent plane and hence are tangent to each other, as was to be shown.

The characteristics of all the surfaces of a one-parameter family lie on the envelope of the family and so form a one-parameter family of curves on a surface. If this family of curves is sufficiently regular, it has an envelope. To investigate this envelope, let us consider the characteristic (5·3) and a neighboring characteristic whose equations, obtained from (5·3) by giving to t an increment Δt, are

$$F(x, y, z, t + \Delta t) = 0 , \qquad F_t(x, y, z, t + \Delta t) = 0 . \tag{5·5}$$

Any point of intersection of these two characteristics has coordinates satisfying the equation

$$\frac{1}{\Delta t} [F_t(x, y, z, t + \Delta t) - F_t(x, y, z, t)] = 0 .$$

If this point of intersection approaches a limit when Δt approaches zero, the limiting point is called *a focal point* of the characteristic. In other words, *the focal points of a characteristic are the points in which a* *consecutive characteristic intersects it.* The coordinates of a focal point satisfy the equations

$$(5\cdot 6)\quad F(x, y, z, t) = 0\,, \qquad F_t(x, y, z, t) = 0\,, \qquad F_{tt}(x, y, z, t) = 0\,.$$

If these equations be solved for x, y, z as functions of t, the resulting equations,

$$(5\cdot 7)\qquad x = x(t)\,, \qquad y = y(t)\,, \qquad z = z(t)\,,$$

may be regarded as the parametric equations of a curve. This curve is the locus of the focal points of the characteristics of the surfaces of the family and is, by definition, the envelope of the characteristics of the surfaces of the family. This envelope has a special name, introduced in the following definition.

DEFINITION 2. *The edge of regression of the envelope of a one-parameter family of surfaces is defined to be the envelope of the characteristics of the surfaces.*

The following theorem is an analogue of Theorem 1.

THEOREM 2. *The edge of regression of the envelope of a one-parameter family of surfaces is tangent to each characteristic at each focal point of the characteristic.*

The proof of this theorem begins with the observation that the direction cosines of the tangent at a point of the edge of regression are proportional to x', y', z' calculated from equations (5 7), the accent denoting differentiation with respect to t. Since the functions of t in the right members of equations (5·7) satisfy the first two of equations (5·6) identically in t, differentiation of these equations gives

$$F_x x' + F_y y' + F_z z' + F_t = 0\,,$$

$$F_{tx} x' + F_{ty} y' + F_{tz} z' + F_{tt} = 0\,.$$

But at a focal point of a characteristic we have $F_t = 0$, $F_{tt} = 0$ by the last two of equations (5 6). Consequently, we find

$$x' = k(F_y F_{tz} - F_z F_{ty})\,,$$

$$y' = k(F_z F_{tx} - F_x F_{tz})\,,$$

$$z' = k(F_x F_{ty} - F_y F_{tx})\,,$$

where k is a nonvanishing proportionality factor whose value is immaterial. But the parentheses are proportional to the direction cosines of the tangent line of the characteristic (5·3) at the focal point. Therefore the edge of regression and the characteristic have the same tangent line at a focal point of the characteristic, and hence are tangent to each other there, as was to be shown.

As a working rule, *to find the envelope of a one-parameter family of surfaces* (5·1), *differentiate with respect to the parameter and eliminate the parameter. To find the edge of regression of the envelope, differentiate twice with respect to the parameter and solve for x, y, z as functions of the parameter.*

If, however, one happens to be considering a one-parameter family of surfaces associated with the points of a space curve, and if the equation of a surface of the family is written *in local coordinates*, it is convenient to know how to differentiate local coordinates in order to find the envelope by this method. It turns out that the conditions (II 2·18) on the local coordinates of a point which is fixed relative to the fixed system of coordinates are the differentiation formulas required, as will now be demonstrated. Let the equation of a surface of the family under consideration be

$$f(\xi, \eta, \zeta, s) = 0, \tag{5·8}$$

in which s is arc length along the curve and ξ, η, ζ are local coordinates. If the expressions given for ξ, η, ζ by the equations (I·7·2) of transformation of coordinates are substituted in equation (5·8) the result is the equation of the surface referred to the *fixed* coordinate system, namely,

$$f(\Sigma\alpha(X - x), \Sigma l(X - x), \Sigma\lambda(X - x), s) = 0, \tag{5·9}$$

the summation being for cyclical permutations. In order to find the characteristic of this surface, the procedure is to differentiate partially with respect to the parameter s, which appears in α, l, λ, x as well as explicitly. The result of the differentiation is the equation

$$f_\xi\xi' + f_\eta\eta' + f_\zeta\zeta' + f_s = 0, \tag{5·10}$$

where the accents indicate differentiation with respect to s. The derivatives ξ', η', ζ' can easily be calculated by differentiating equations (I·7·2) with respect to s while X, Y, Z are fixed. The expressions obtained as the result of this differentiation can be reduced by means of the Frenet formulas (I·6·1) and the equations (I 7 2) themselves to *the differentiation formulas for local point coordinates,*

$$(5\ 11)\quad \xi' = -1 + \frac{\eta}{\rho}, \quad \eta' = -\frac{\xi}{\rho} - \frac{\zeta}{\tau}, \quad \zeta' = \frac{\eta}{\tau} \qquad \left(\xi' = \frac{d\xi}{ds}, \cdots\right).$$

These formulas are observed to be the same as the conditions (II 2 18), and give the expressions to be used for ξ', η', ζ' in equation (5 10) to complete the solution of the envelope problem, as was to be shown. Some concrete applications of these formulas will be found in Exercise 2 below and in the next section.

EXERCISES

1. The equation of the envelope of the two-parameter family of surfaces represented by the equation

$$(5\ 12)\qquad F(x, y, z, u, v) = 0,$$

in which u, v are the parameters, is obtained by eliminating u and v from this equation and the derived equations

$$(5\ 13)\qquad F_u(x, y, z, u, v) = 0, \qquad F_v(x, y, z, u, v) = 0.$$

2 Use the differentiation formulas (5 11) to prove that the characteristic of the osculating sphere (I 8 1) at a point P of a nonspherical curve C, as P varies on C, is the osculating circle (I 8 4). Then prove that the edge of regression of the envelope of the osculating sphere of the curve C is the curve C itself.

3. The parametric equations of the envelope of the one-parameter family of plane curves

$$(5\ 14)\qquad x = x(u, a), \qquad y = y(u, a),$$

the parameter of the family being a, are found by solving the equation

$$(5\ 15)\qquad x_u y_a - x_a y_u = 0$$

for u as a function of a and substituting in (5 14).

4. The parametric equations of the envelope of the one-parameter family of surfaces

$$(5\ 16)\qquad x = x(u, v, a), \qquad y = y(u, v, a), \qquad z = z(u, v, a),$$

the parameter of the family being a, are found by solving the equation

$$(5\cdot17) \qquad \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_a & y_a & z_a \end{vmatrix} = 0$$

for v as a function of u, a and substituting in (5·16).

5. The parametric equations of the envelope of the two-parameter family of surfaces

$$(5\cdot18) \qquad x = x(u, v, a, b), \qquad y = y(u, v, a, b), \qquad z = z(u, v, a, b),$$

the parameters of the family being a, b, are found by solving the equations

$$(5\cdot19) \qquad \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_a & y_a & z_a \end{vmatrix} = 0, \qquad \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_b & y_b & z_b \end{vmatrix} = 0$$

for u, v as functions of a, b and substituting in (5·18).

6. Developable surfaces. This section is devoted to a special class of surfaces called *developable surfaces*, or simply *developables*. The reason for the name is that analytic developables are the only analytic surfaces that can be developed upon, or applied to, a plane so as to fit upon it exactly without stretching, tearing, or folding. This property, however, will not be discussed in detail until later (see Chap. 6, Sec. 4). The most general kind of developable surface is *the tangent developable of a curve*. The following definition of the tangent developable of a curve has the advantages of being usable in projective geometry as well as in metric geometry, and of being valid in the geometry of hyperspace as well as in that of ordinary space.

DEFINITION 1. *The tangent developable of a curve is the locus of the tangents of the curve.*

In order to obtain parametric equations of the tangent developable of a curve, let us consider a curve C whose parametric equations are supposed to be known, the parameter being the arc length s measured from a fixed point P_0 to the point $P(x, y, z)$ of C (see Fig. 10). The parametric equations of the tangent line of C at P are

$$(6\cdot1) \qquad X = x + \alpha t, \qquad Y = y + \beta t, \qquad Z = z + \gamma t,$$

in which t is the algebraic distance from P to a variable point $Q(X, Y, Z)$ on the tangent and α, β, γ are the direction cosines of the tangent. If s is held fixed while t varies, the point P is fixed and the point Q runs along the fixed tangent of C at P. But if s and t both vary, the point P varies along the curve C and the tangent line generates a developable surface, which is the locus of the point Q. Therefore *equations* (6·1) *are parametric equations of the tangent developable of the curve C, the parameters being s and t.*

The curve C is called *the cuspidal edge*, or *edge of regression*, of its tangent developable. Each tangent line of C is called *a generator* of the developable, and the point where a generator touches the cuspidal edge is called *the focal point* of the generator. The reason why some

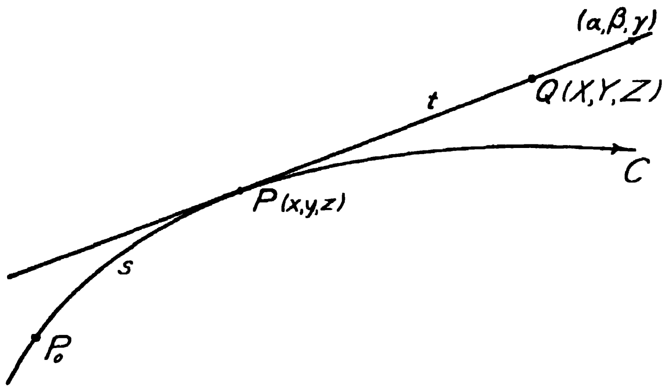

FIG 10

of the terminology of the theory of envelopes is used in this connection is contained in Theorem 2, and the reason for the name *cuspidal edge* is found in Exercise 7 below. The tangent developable of a curve is to be visualized as consisting of two sheets joined together so as to be tangent to each other along the curve itself as a sharp edge, namely, the edge of regression. One sheet is generated by the positive half-tangent, and the other by the negative half-tangent of the curve.

There are some special types of developables which should be mentioned. For example, if the edge of regression is a plane curve, the developable is all or part of the plane of the curve. So a *plane* is a very special type of developable. Moreover, a *cone* is a developable whose edge of regression has reduced to a fixed point. Equations (6 1) represent a cone if x, y, z are all constant and if the direction cosines α, β, γ of a generator are functions of any parameter u. The vertex of the cone is then the fixed point (x, y, z). *A flat pencil of lines* may be thought of as a degenerate cone, but leads to the plane as a developable, and need not be considered further here. A *cylinder* is a developable, since a cylinder may be thought of as a cone with its vertex at infinity. Equations (6 1) represent a cylinder if the direction cosines α, β, γ of a generator are all constant and do not belong to the tangent of the curve generated by the variable point (x, y, z). Finally, if a developable reduces to a curve at all, then it reduces to a *straight line*, which is the edge of regression of this improper developable. Only proper developables will ordinarily be considered hereinafter. An important property of a developable surface is enunciated in the following theorem.

THEOREM 1. *The tangent planes at all ordinary points on a generator of a developable coincide in one plane which, if the developable is not a cone or cylinder, is the osculating plane of the edge of regression at the focal point of the generator.*

The proof of this theorem may be made by calculating the equation of the tangent plane at a point of the developable (6 1). Using equation (4 5) with s, t in the place of u, v, we find that the equation of the tangent plane immediately reduces to the equation (I·4 11) of the osculating plane of the curve C and is independent of the parameter t. The special cases of the cone and cylinder offer no difficulties, the equation of the tangent plane at an ordinary point of either surface being independent of the parameter t.

Ordinarily a surface has a different tangent plane at every one of its points, and therefore has altogether a two-parameter family of tangent planes. However, Theorem 1 makes it clear that *a developable surface has only a single infinity of tangent planes at most.* It is natural to ask the converse question, whether every one-parameter family of planes can be regarded as the tangent planes of a developable surface. An affirmative answer is furnished by the following theorem.

THEOREM 2. *Every analytic one-parameter family of planes in three-dimensional space envelops a developable surface.*

To prove this theorem, let us consider a one-parameter family of planes represented by the equation

$$(6\cdot 2) \qquad a(t)x + b(t)y + c(t)z + d(t) = 0,$$

the coefficients a, b, c, d being analytic functions of the parameter t, and a, b, c being not all zero. Certain exceptional cases can be disposed of briefly. First, if all the planes of the family (6 2) pass through as many as three noncollinear points, then the planes of the family all coincide in one fixed plane, the ratios of the functions a, b, c, d being constants. In this case the envelope is the fixed plane, which is a kind of developable surface. Second, if the planes of the family (6·2) all pass through two, but no more than two, independent points, the envelope is the line joining these two points and is a degenerate developable; the planes of the family form an axial pencil with this line as axis. Similar remarks apply if the planes are all parallel to a fixed plane, the envelope being the line at infinity in this plane. In the third place, if the planes of the family (6·2) all pass through just one fixed point, the envelope is a cone with its vertex at this point, and if the planes of the family are all parallel to a fixed line, then the envelope is a cylinder with its generators parallel to this line. Having disposed thus of the special cases, let us suppose that the planes of the family (6·2) do not all pass through a fixed point and are not all parallel to a fixed line. Then equation (6·2) can be written in the form

$$a(t)x + b(t)y + tz + 1 = 0\,, \tag{6·3}$$

by suitably choosing new coefficients and a new parameter. The theory of envelopes, as developed in Section 5, can be applied to this family of planes. The result of differentiating equation (6·3) with respect to t is

$$a'x + b'y + z = 0 \qquad \left(a' = \frac{da}{dt}\,, \cdots\right), \tag{6·4}$$

and another differentiation produces

$$a''x + b''y = 0\,. \tag{6·5}$$

If the three equations (6·3), (6·4), and (6·5) be solved simultaneously for x, y, z, the result is *the parametric equations of the edge of regression of the envelope of the planes* (6·3),

$$x = \frac{b''}{D}\,, \qquad y = -\frac{a''}{D}\,, \qquad z = \frac{a''b' - a'b''}{D}\,, \tag{6·6}$$

where D is defined by placing

$$D = \begin{vmatrix} a & b & t \\ a' & b' & 1 \\ a'' & b'' & 0 \end{vmatrix} \tag{6·7}$$

and is not zero. The proof may be completed by first observing that the characteristic of the plane (6·3) is a straight line, and then appealing to Theorem 2 of the preceding section. Or, alternatively, if the equations of the tangent line at a point of the edge of regression (6·6) are calculated, it turns out that this line is precisely the characteristic whose equations are (6·3) and (6·4). Therefore the envelope of the planes (6·3) is the developable generated by the tangents of the curve (6·6). It may be remarked that, if the equation of the osculating plane at a point of the curve (6·6) is calculated, it reduces to the equation (6·3), as it should.

There are three interesting developables associated with a space curve C, which we shall now discuss briefly, using local coordinates and the differentiation formulas (5·11). The first of these developables is *the tangent developable* of the curve C, which, according to Definition 1, is the locus of the tangents of C. The tangent developable of the curve C could also be defined as the envelope of the osculating planes of C. For, the local equation of the osculating plane is $\zeta = 0$. Differentiation by (5·11) shows at once that *the characteristic of the osculating plane is the tangent line*, $\eta = 0$, $\zeta = 0$. Another differentiation shows that *the focal point of the tangent is the origin* (0, 0, 0). Therefore *the envelope of the osculating planes of a curve is the tangent developable of the curve, and the edge of regression of the envelope is the curve itself.*

The second remarkable developable associated with a curve C is called *the polar developable* of C and may be defined to be the envelope of the normal planes of C. The local equation of the normal plane is $\xi = 0$. One differentiation by (5·11) gives the local equations of the characteristic of the normal plane, namely, $\xi = 0$, $\eta = \rho$. Therefore *the characteristic of the normal plane is the polar line.* Another differentiation by (5·11) gives the focal point of the polar line, namely, the point $(0, \rho, -\tau\rho')$. This is the center of the osculating sphere. Therefore *the polar developable of a curve is the locus of the polar lines of the curve, and its edge of regression is the locus of the centers of the osculating spheres of the curve.* Evidently, the polar developable of a curve is the tangent developable of the locus of the centers of the osculating spheres of the curve.

Finally, *the rectifying developable* of a curve C is, by definition, the envelope of the rectifying planes of C. The local equation of the rectifying plane is $\eta = 0$. One differentiation by (5·11) gives the local equations of the characteristic of the rectifying plane, namely,

$$(6\cdot8) \qquad \eta = 0\,, \qquad \tau\xi + \rho\zeta = 0\,.$$

Therefore *the characteristic of the rectifying plane is the instantaneous axis* (I·6 7) *of the local trihedron.* Another differentiation by (5·11) gives *the focal point of the instantaneous axis,*

$$(6\cdot 9)\qquad \left(-\frac{r}{r'},\quad 0,\quad \frac{1}{r'}\right) \qquad \left(r = \frac{\rho}{\tau}\right),$$

provided that the curve C is not a cylindrical helix, i.e., provided that $r' \neq 0$. If the curve C is a cylindrical helix, the focal point of each characteristic is at infinity. In fact, by taking the z-axis parallel to the generators of the sustaining cylinder, which the helix crosses at a constant angle, the equations of the cylindrical helix become

$$x = x(s)\,,\qquad y = y(s)\,,\qquad z = cs + c_1 \quad (c,\, c_1 = \text{consts.})$$

the parameter s being arc length; and it is easy to prove that the rectifying developable of the helix is the sustaining cylinder itself, the equations of which are

$$x = x(s)\,,\qquad y = y(s)\,.$$

EXERCISES

1. Verify the following formulas for a developable, using s, t in place of u, v:

$$(6\;10)\quad \begin{cases} J_1 = -\dfrac{t\lambda}{\rho}, & J_2 = -\dfrac{t\mu}{\rho}, \quad J_3 = -\dfrac{t\nu}{\rho}, \\ E = 1 + \dfrac{t^2}{\rho^2}, & F = 1\,, \quad G = 1\,, \quad H = \pm\dfrac{t}{\rho}, \\ a = \mp\lambda\,, & b = \mp\mu\,, \quad c = \mp\nu\,, \end{cases}$$

the upper signs being used for the sheet of the developable generated by the positive half-tangent, and the lower signs for the other sheet. Why do the jacobians J_i vanish when $t = 0$?

2. Discuss the envelopes of the osculating planes, normal planes, and rectifying planes of a curve without the use of local coordinates.

3. The normal planes of a curve C, not a plane curve, pass through a fixed point if, and only if, C is a spherical curve.

4. The rectifying planes of a nonrectilinear curve pass through a fixed point if, and only if,

$$(6\;11)\qquad \frac{\rho}{\tau} = hs + k \qquad (h,\, k = \text{consts.})\,.$$

5. The implicit equation of the tangent developable of the cubical parabola (I·2·5) is

$$4(y - x^2)(zx - y^2) - (xy - z)^2 = 0 .$$

6. The normal plane at a point P of the edge of regression of a developable cuts the developable in a curve with a cusp at P, the cusp tangent being the principal normal of the edge of regression.

7. The osculating plane at a point P of the edge of regression of a developable cuts the developable in a generator counted twice and in a curve whose curvature at P is three-fourths the curvature of the edge of regression at P.

8. The rectifying plane at a point P of the edge of regression of a developable cuts the developable in a generator and in a curve with an inflexion at P, the inflexional tangent being the generator.

9. A necessary and sufficient condition that a surface $z = f(x, y)$ be developable is

$$f_{xx}f_{yy} - f_{xy}^2 = 0 . \tag{6 12}$$

10. A necessary and sufficient condition that a surface $F(x, y, z) = 0$ be developable is

$$\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_x \\ F_{yx} & F_{yy} & F_{yz} & F_y \\ F_{zx} & F_{zy} & F_{zz} & F_z \\ F_x & F_y & F_z & 0 \end{vmatrix} = 0 . \tag{6 13}$$

7. Ruled surfaces. Just as the locus of a point moving with one degree of freedom is a curve, so the locus of a straight line moving with one degree of freedom is a surface. Such a surface, however, is of a special type and is known as *a ruled surface.* The variable line is called *a generator,* or *ruling,* of the surface. A more precise definition of a ruled surface follows:

DEFINITION 1. *A ruled surface is a surface such that through each point of it passes at least one straight line lying entirely on it.*

Many examples of ruled surfaces are quite familiar. Perhaps the simplest of all ruled surfaces with real rulings are the plane and the cones and cylinders. The hyperbolic paraboloid and the hyperboloid of one sheet are ruled; in fact, each of these surfaces has on it two one-parameter families of rulings. The conoids, including the screw surface and the cylindroid, are ruled. Every developable surface is ruled, but not conversely. Some writers call a developable surface *a torse* and call a nondevelopable ruled surface *a scroll.*

Parametric equations of a ruled surface can be obtained in the following way. Let us consider a curve C whose parametric equations are known, the parameter being the arc length s measured from a fixed point P_0 to a point $P(x, y, z)$ of C (see Fig. 11). Let g be any line

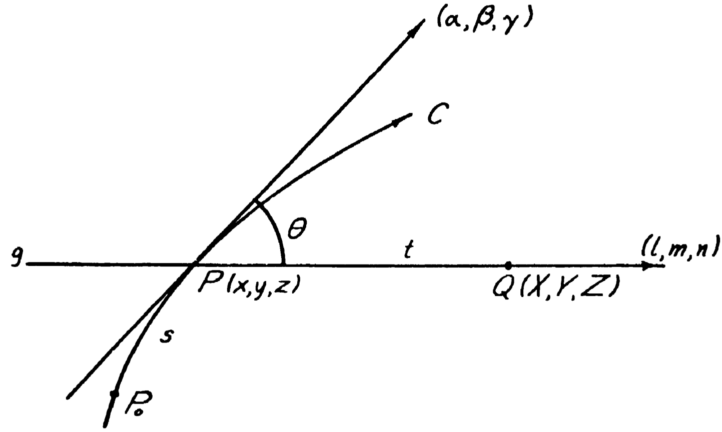

FIG. 11

through P, and let the direction cosines l, m, n of g be functions of s. Then parametric equations of the generator g are

$$(7\cdot 1) \qquad X = x + lt\,, \qquad Y = y + mt\,, \qquad Z = z + nt\,,$$

in which t is the algebraic distance from P to a variable point $Q(X, Y, Z)$ on g. If s is held fixed while t varies, the point P is fixed, and the point Q runs along the fixed generator g at P. But if s and t both vary, the point P varies along the curve C, and the line g generates a ruled surface, which is the locus of the point Q. Therefore, *equations* (7·1) *are parametric equations of a ruled surface, the parameters being s and t.*

The hypothesis that the direction cosines l, m, n are not all constant would prevent the ruled surface under consideration from being a cylinder, and in the following argument this hypothesis will be understood, unless the contrary is indicated. The curve C may be called *the director curve* associated with the equations (7·1) of a ruled surface. Any curve crossing all the generators of a ruled surface can be used as director curve.

Some useful notations and formulas of the theory of ruled surfaces will now be introduced. Let θ be the angle between the generator g and the tangent line at a point P of the director curve C. Further, let two functions p and q be defined by placing

$$(7\cdot 2) \quad p^2 = l'^2 + m'^2 + n'^2\,, \quad q = \alpha l' + \beta m' + \gamma n' \quad \left(l' = \frac{dl}{ds}, \cdots\right)$$

and agreeing that the positive square root will be taken for the function p, which is not zero since l, m, n are supposed to be not all constant. If s, t are used in place of u, v, direct calculation leads to the following special formulas in ruled surface theory for some of the familiar functions of general surface theory:

$$(7\cdot 3)\quad \begin{cases} J_1 = \beta n - \gamma m - (mn' - m'n)t\,, \\ J_2 = \gamma l - \alpha n - (nl' - n'l)t\,, \\ J_3 = \alpha m - \beta l - (lm' - l'm)t\,, \\ E = 1 + 2qt + p^2t^2\,, \qquad F = \cos\theta\,, \qquad G = 1\,, \\ H^2 = \sin^2\theta + 2qt + p^2t^2\,. \end{cases}$$

The positive sense on the director curve C is known, since the parametric equations of C are known. Consistency demands that the positive sense on all the s-curves on the ruled surface be that of increasing parameter s. The positive sense on the t-curves (generators) may, of course, be chosen at will but is determined when the parametric equations (7·1) are written. The positive sense on the normal is, as usual, such that the direction cosines a, b, c of the normal are given by the formulas (4·8). Then the tangent of an s-curve, the generator, and the normal at each point of a ruled surface form a left-handed trihedron.

In the special case in which the ruled surface (7·1) is a developable, the director curve may be its edge of regression. If the director curve C is so chosen, each generator g coincides with the tangent of C at the point where g meets C, and the following specializations occur:

$$\theta = 0\,, \qquad l = \alpha\,, \qquad m = \beta\,, \qquad n = \gamma\,, \qquad p = \frac{1}{\rho}\,, \qquad q = 0\,.$$

The next two theorems are concerned with the tangent planes at the points of a generator of a ruled surface.

THEOREM 1. *The tangent plane at a point of a ruled surface contains the generator through the point.*

THEOREM 2. *Each plane through a generator of a nondevelopable ruled surface is tangent to the surface at one, and only one, point of the generator, the point varying with the plane.*

The truth of the first theorem is geometrically evident when one recalls that the tangent plane at a point P of a ruled surface contains the tangent line at P of every curve on the surface through P, and that the generator through P is such a curve, which is its own tangent line. An analytic proof can also be constructed for the theorem. First write *the equation of the tangent plane at any point Q of a ruled surface,*

$$(7\cdot 4)\qquad \begin{vmatrix} X-x & Y-y & Z-z \\ \alpha + l't & \beta + m't & \gamma + n't \\ l & m & n \end{vmatrix} = 0 .$$

Then leave the value of s the same but replace t in equations (7·1) by t_1 to get the coordinates of any point Q_1 on the generator through Q. The proof is completed by showing that the coordinates of Q_1 satisfy equation (7·4) identically in t_1.

To prove Theorem 2, one observes that the equation of any plane whatever through a generator of a nondevelopable ruled surface can be written in the form

$$(7\cdot 5)\qquad a(X-x) + b(Y-y) + c(Z-z) = 0 ,$$

where a, b, c are restricted to be not all zero and to satisfy the condition

$$(7\cdot 6)\qquad al + bm + cn = 0 .$$

One then inquires whether it is possible to determine t so that equations (7·4) and (7·5) represent the same plane. It turns out that there is a unique value of t satisfying this requirement, since the two equations that express the equality of the ratios of the coefficients of $X - x$, $Y - y$, $Z - z$ in equations (7·4) and (7·5) are equivalent in virtue of the condition (7·6). This value of t, which we do not need to write, corresponds to the contact point of the given plane (7·5). The contact points are found to be different for different planes through the generator, in virtue of the hypothesis that the ruled surface is not developable.

Two skew lines in three-dimensional space have a common perpendicular, along which the shortest distance between these two lines is measured. This fact can be used to enrich the differential geometry of ruled surfaces in the following way. Let us consider, as in Figure 12, a generator g of a ruled surface R, with direction cosines l, m, n corresponding to a value s of the arc along the director curve C. Let us also consider a neighboring generator g_1 of R, with direction cosines l_1, m_1, n_1 corresponding to a value $s + \Delta s$ of the arc. Let λ, μ, ν be the direction cosines of the line perpendicular to both of g and g_1. Then we have

$$(7\cdot 7)\qquad \begin{cases} \lambda l + \mu m + \nu n = 0\,, \\ \lambda l_1 + \mu m_1 + \nu n_1 = 0\,, \end{cases}$$

and therefore

$$(7\cdot 8)\qquad \lambda = k(mn_1 - m_1 n)\,,\quad \mu = k(nl_1 - n_1 l)\,,\quad \nu = k(lm_1 - l_1 m)\,,$$

where k is a proportionality factor to be determined. Each of the cosines l_1, m_1, n_1 can be expanded into a power series in Δs,

$$(7\cdot 9)\qquad \begin{cases} l_1 = l + l'\Delta s + \cdots\,, \qquad m_1 = m + m'\Delta s + \cdots\,, \\ \qquad n_1 = n + n'\Delta s + \cdots\,; \end{cases}$$

and then equations (7·8) become

$$(7\cdot 10)\qquad \begin{cases} \lambda = k[(mn' - m'n)\Delta s + \cdots]\,, \\ \mu = k[(nl' - n'l)\Delta s + \cdots]\,, \\ \nu = k[(lm' - l'm)\Delta s + \cdots]\,. \end{cases}$$

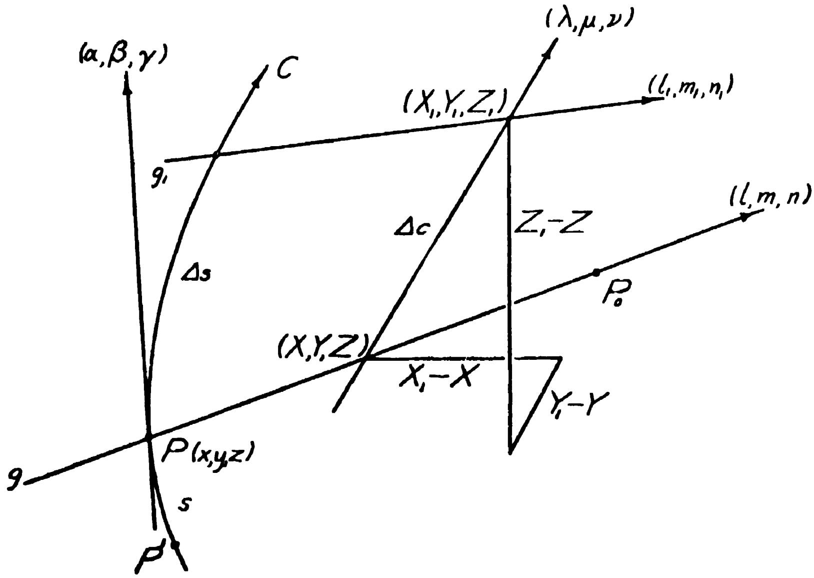

FIG. 12

Squaring and adding these equations, we find, by Lagrange's identity (I·4·13) and the definition of p in (7·2),

$$1 = k^2[p^2\Delta s^2 + \cdots]\,.$$

The ambiguity in the sign of k is removed by the following convention with regard to the positive sense on the common perpendicular:

The positive sense on the line perpendicular to the generators g and g_1 is, by agreement, such that

$$k = \frac{1}{p\Delta s + \cdots}. \tag{7.11}$$

If this value of k is substituted in equations (7.10), they become

$$\left\{\begin{aligned} &\lambda = \frac{1}{p}(mn' - m'n) + \cdots, \qquad \mu = \frac{1}{p}(nl' - n'l) + \cdots, \\ &\nu = \frac{1}{p}(lm' - l'm) + \cdots, \end{aligned}\right. \tag{7.12}$$

the omitted terms containing Δs as a factor. Letting Δs approach zero, we obtain the following result.

THEOREM 3. *The direction cosines of the line which is the limit of the common perpendicular of a generator g and a neighboring generator g_1 of a ruled surface R, as g_1 approaches g over R, are*

$$\frac{1}{p}(mn' - m'n), \qquad \frac{1}{p}(nl' - n'l), \qquad \frac{1}{p}(lm' - l'm). \tag{7.13}$$

This line may be called *the common perpendicular of the generator g and a consecutive generator.*

We shall now find a function which is sometimes spoken of as *the distance between consecutive generators.* For this purpose let us consider again two neighboring generators g and g_1. Let their common perpendicular meet g in the point (X, Y, Z) and g_1 in the point (X_1, Y_1, Z_1). The coordinate X_1 can be expanded into a power series in Δs and Δt,

$$\begin{aligned} X_1 &= X + X_s\Delta s + X_t\Delta t + \cdots \\ &= X + (a + l't)\Delta s + l\Delta t + \cdots, \end{aligned}$$

and there are similar expansions for Y_1, Z_1. Then the length Δc of the segment of the common perpendicular between g and g_1 is given by

$$\left\{\begin{aligned} \Delta c &= \lambda(X_1 - X) + \mu(Y_1 - Y) + \nu(Z_1 - Z) \\ &= \Sigma\left[\frac{1}{p}(mn' - m'n) + \cdots\right][(a + l't)\Delta s + l\Delta t + \cdots] \\ &= r\Delta s + \cdots, \end{aligned}\right. \tag{7.14}$$

the summation being for cyclical permutations, and the function r being defined by placing

$$(7\cdot 15)\qquad r = \frac{1}{p}\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}.$$

Evidently we have

$$(7\cdot 16)\qquad \lim_{\Delta s \to 0} \frac{\Delta c}{\Delta s} = r.$$

The function r is sometimes called *the distance between consecutive generators.* The following theorem is a justification for saying that "consecutive generators of a developable surface intersect."

THEOREM 4. *A ruled surface is developable if, and only if,* $r = 0$.

Evidently, Δc is ordinarily an infinitesimal of the same order as Δs, but the theorem asserts that for a developable surface Δc is an infinitesimal of higher order than Δs. The proof of the theorem can be made by considering the equation (7·4) of the tangent plane at a point Q of a ruled surface and the equation obtained therefrom by differentiation with respect to t. If $r \neq 0$, these two equations determine the generator through the contact point Q as the characteristic of the tangent plane when t varies. However, the characteristic is indeterminate and the tangent plane is fixed as Q varies along a generator if, and only if, $r = 0$. Therefore in case $r = 0$ the surface is a developable, as was to be shown.

The line perpendicular to a generator g of a ruled surface and to the consecutive generator intersects g in the point called, in the following definition, *the central point of g.*

DEFINITION 2. *The central point of a generator g of a ruled surface R is the limit of the point in which g is met by the common perpendicular of g and a neighboring generator g_1 of R, as g_1 approaches g over R.*

To find the coordinates of the central point P_0 of a generator g, we make the following calculation:

$$(7\cdot 17)\quad \begin{cases} 0 = \Sigma\lambda l_1 = \Sigma\lambda l = \Sigma\lambda(l_1 - l) = \dfrac{1}{\Delta c}\,\Sigma(X_1 - X)(l_1 - l) \\[2ex] \quad = \dfrac{1}{\Delta c}\,\Sigma[(\alpha + l't)\Delta s + l\Delta t + \cdots][l'\Delta s + \cdots] \\[2ex] \quad = \dfrac{1}{\Delta c}\,[(\Sigma\alpha l' + t\Sigma l'^2)\Delta s^2 + \Sigma l l'\Delta s\Delta t + \cdots] \\[2ex] \quad = \dfrac{\Delta s^2}{\Delta c}\,(q + p^2 t + \cdots). \end{cases}$$

Dividing through by $\Delta s^2/\Delta c$ and letting Δs approach zero, we find that *the value t_0 of t, corresponding to the central point P_0 of a generator g, is given by the formula*

$$(7\cdot 18) \qquad t_0 = -\frac{q}{p^2}.$$

If this expression for t_0 is substituted in place of t in equations (7 1), *the coordinates X_0, Y_0, Z_0 of the central point P_0 of a generator g* are found to be given by

$$(7\cdot 19) \quad X_0 = x - \frac{q}{p^2}\, l, \qquad Y_0 = y - \frac{q}{p^2}\, m, \qquad Z_0 = z - \frac{q}{p^2}\, n.$$

When the parameter s varies, equations (7 · 19) are the parametric equations of a curve called *the line of striction* in the following definition.

DEFINITION 3. *The line of striction on a ruled surface is the locus of the central points of the generators of the surface.*

The director curve C is the line of striction in case $q = 0$. The line of striction of a developable surface is the edge of regression of the surface.

This introduction to the theory of ruled surfaces will be concluded by investigating the behavior of the tangent plane at a point of a nondevelopable ruled surface as the point moves along a generator. *The central plane* of a generator is defined as follows:

DEFINITION 4. *The central plane of a generator of a ruled surface is the tangent plane of the surface at the central point of the generator.*

Let φ be the angle between the normal lines of a ruled surface at the central point P_0 of a generator g and at a point Q of g at an algebraic distance t from P_0. Then φ is equal to one of the angles between the central plane of g and the tangent plane of the surface at the point Q. We find, using the result of Exercise 4 below, that

$$(7\ 20) \qquad \begin{cases} \cos\varphi = \dfrac{1}{Hp}\,(J_1 l' + J_2 m' + J_3 n') \\[2ex] \qquad = \left(\dfrac{\sin^2\theta - \dfrac{q^2}{p^2}}{\sin^2\theta + 2qt + p^2t^2}\right)^{1/2}, \end{cases}$$

the details of the calculation being omitted. Elementary trigonometry gives

$$(7\cdot 21) \qquad \sin\varphi = \pm\frac{pt + \dfrac{q}{p}}{(\sin^2\theta + 2qt + p^2t^2)^{1/2}}.$$

From these equations and the identity in Exercise 3 below, we obtain

$$(7\cdot 22) \qquad \tan\varphi = \pm\frac{p}{r}(t - t_0),$$

where t_0 is given by the formula (7 18). The ambiguity of sign can be removed by the following convention with regard to the positive sense of rotation of the tangent plane:

The positive sense of rotation of the tangent plane is, by agreement, such that

$$(7\cdot 23) \qquad \tan\varphi = \frac{p}{r}(t - t_0).$$

Equation (7·23) implies the following theorem.

THEOREM 5. *The tangent plane at the point at* $-\infty$ *on a generator* g *of a nondevelopable ruled surface is perpendicular to the central plane of* g. *As the point of contact* Q *of the tangent plane runs along* g *to the central point* P_0 *of* g, *the tangent plane turns through an angle* $\pi/2$ *into coincidence with the central plane. As* Q *runs along* g *from* P_0 *to the point at* $+\infty$ *on* g, *the tangent plane turns through another angle* $\pi/2$, *in the same direction, and becomes perpendicular to the central plane of* g *again.*

Still further information about the behavior of the tangent plane of a ruled surface is contained in equation (7 23). The direction of rotation of the tangent plane, as its point of contact Q runs in the positive direction along the generator g, depends upon the sign of r since $p > 0$, or, what is equivalent, depends upon the sign of the determinant in the definition (7 15) of r. The following conclusion is immediately reached.

THEOREM 6. *With the hypotheses of Theorem* 5, *the rotation is in the positive direction if* $r > 0$, *and is in the negative direction if* $r < 0$.

Thus two types of ruled surfaces are distinguished. A ruled surface may be called *positively twisted* if $r > 0$, and *negatively twisted* if $r < 0$.

EXERCISES

1. The locus of the normal lines at the points of a generator of a nondevelopable ruled surface is a hyperbolic paraboloid. For a developable, the locus of the normal lines at the points of a generator is a plane.

2. Discuss *Cayley's cubic scroll*,

$$y - zx + x^3 = 0.$$

3. Prove the identity

$$p^2r^2 = p^2 \sin^2\theta - q^2 . \tag{7·24}$$

4. The direction cosines of the normal to a ruled surface at a point on the line of striction are

$$\frac{l'}{p}, \quad \frac{m'}{p}, \quad \frac{n'}{p} .$$

5. The line of striction on a screw surface (III 2 26) is the z-axis, i.e., *the axis* of the screw.

6. The line of striction of a hyperboloid of one sheet of revolution is the minimum circle on the surface.

7. Defining *the asymptotic tangent plane of a generator of a nondevelopable ruled surface* to be the tangent plane of the surface at the point at infinity on the generator, prove that the equation of the asymptotic tangent plane is

$$(mn' - m'n)(X - x) + (nl' - n'l)(Y - y) + (lm' - l'm)(Z - z) = 0 . \tag{7 25}$$

8. Defining *the asymptotic developable of a nondevelopable ruled surface* to be the envelope of the asymptotic tangent planes of the generators of the surface, investigate the asymptotic developable of the screw surface and of the hyperboloid of one sheet.

9. Defining *the parameter of distribution of a ruled surface* to be the function r/p, prove that the parameter of distribution is the cotangent of the angle between the central plane of a generator and the tangent plane of the surface at the point on the positive half of the generator at unit distance from the central point.

10. The product of the distances from the central point of a fixed generator of a nondevelopable ruled surface to any two points on the generator which are contact points of perpendicular tangent planes of the surface is constant and is equal to $-r^2/p^2$.

11. The line of striction of the ruled surface of binormals of a curve C is the curve C itself. At a point P of C, the central plane of the binormal is the rectifying plane of C, the normal to the surface is the principal normal of C; and the common perpendicular of consecutive binormals is the tangent of C. The parameter of distribution is $-\tau$. The ruled surface is positively twisted if, and only if, the torsion of the curve C is negative.

12. For the ruled surface of principal normals of a curve C, the distance from a point P of C to the central point of the principal normal of C at P is

$$\frac{\frac{1}{\rho}}{\frac{1}{\rho^2}+\frac{1}{\tau^2}}.$$

The common perpendicular of consecutive principal normals is parallel to the instantaneous axis of the local trihedron. The parameter of distribution is

$$-\frac{\frac{1}{\tau}}{\frac{1}{\rho^2}+\frac{1}{\tau^2}}.$$

The ruled surface is positively twisted if, and only if, the torsion of the curve C is negative.

13. At the central point of a generator of a nondevelopable ruled surface, the common perpendicular of consecutive generators, the generator, and the normal form a left-handed trihedron. The common perpendicular of consecutive generators is a tangent line of the surface and, with the generator, determines the central plane of the generator. For a developable the common perpendicular of consecutive generators is the binormal of the edge of regression and hence is perpendicular to the tangent plane.

14. If the distance between two consecutive generators of a ruled surface is of the second order with respect to the arc on the director curve, then the ruled surface is a developable and the distance is of the third order at least. If the distance is constantly of the fourth order, the distance is zero, so that the developable is a cone or a plane.

15. If the parameter that varies along a director curve C of a ruled surface is u, not necessarily arc length, and if l, m, n are merely proportional to the direction cosines of a generator of the surface, then equations (7·1) still represent the surface, although t is not necessarily the algebraic distance from the point $P(x, y, z)$ on C to the point (X, Y, Z) on the generator g through P. In this case equation (7·18), giving the value t_0 of t for the central point of the generator g, becomes

$$(7\cdot26)\qquad t_0\Sigma(mn' - nm')^2 + \Sigma(mz' - ny')(mn' - nm') = 0\,,$$

the accent denoting differentiation with respect to u. A necessary and sufficient condition that the ruled surface be developable is

$$(7\cdot27)\qquad \begin{vmatrix} x' & y' & z' \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0\,.$$

A necessary and sufficient condition that the ruled surface be a cylinder is

(7·28) $$\Sigma(mn' - nm')^2 = 0 .$$

If the ruled surface is a developable but not a cylinder, then the value t_0 of t for the focal point of the generator g can be written in any one of the forms

(7·29) $$t_0 = \frac{ny' - mz'}{mn' - nm'} = \frac{lz' - nx'}{nl' - ln'} = \frac{mx' - ly'}{lm' - ml'}$$

for which the denominator is not zero; this developable is a cone in case

(7·30) $$(x + t_0 l)' = (y + t_0 m)' = (z + t_0 n)' = 0 ;$$

the edge of regression of the developable is a plane curve in case

(7·31) $$\frac{(ny' - mz')'}{(mn' - nm')'} = \frac{(lz' - nx')'}{(nl' - ln')'} = \frac{(mx' - ly')'}{(lm' - ml')'} .$$

CHAPTER IV

CURVES ON SURFACES

1. Transformation of parameters. The primary purpose of this chapter is to study curves on surfaces, particularly certain systems of curves which appear naturally in investigating surfaces. Most of these systems are *nets* of curves having special properties. Some of these nets are *covariant* to their sustaining surfaces. For example, the *minimal* net, the *asymptotic* net, and the *lines of curvature* on a surface are covariant to it, i.e., are geometrically definable in terms of it in a way which is invariant under rigid motion in space.

The parametric net of curves on a surface is not necessarily covariant to the surface. In fact, it will be shown in this section that by a suitably chosen transformation of parameters the parametric net can be made to coincide with any prescribed net of curves on the surface. In particular, any covariant net can be made to be parametric. The geometry of a surface must evidently be independent of the analytical representation used for the surface, and hence must be independent of the choice of parametric net. Often the analysis employed in solving a particular problem can be simplified by choosing the parametric net suitably. If the parametric net is a covariant net, then properties of that net are really properties of the surface itself, and so the employment of a covariant parametric net is frequently advantageous.

Let us consider a surface S represented analytically by parametric equations of the form

$$(1\cdot1) \qquad x = x(u, v)\,, \qquad y = y(u, v)\,, \qquad z = z(u, v)\,.$$

Any net of curves on S can be defined by its curvilinear differential equation,

$$(1\cdot2) \qquad Adu^2 + 2Bdudv + Cdv^2 = 0 \qquad (AC - B^2 \neq 0)\,,$$

in which the coefficients A, B, C are functions of u, v. In particular, the curvilinear differential equation of the parametric net associated with the representation $(1\cdot1)$ is

$$(1\cdot3) \qquad dudv = 0\,.$$

It is possible to change the parameters without changing the parametric net. Indeed, it will now be shown that *the transformation of parameters*

$$(1\cdot 4) \qquad p = p(u)\,, \qquad q = q(v) \qquad (p_u q_v \neq 0)\,,$$

from u, v to p, q leaves the parametric net invariant. This transformation is effected by solving equations (1·4) for u and v as functions of p and q, respectively, and then substituting these functions in equations (1·1). Each u-curve becomes a p-curve, since if $v = \text{const.}$ then $q = \text{const.}$ Similarly, each v-curve becomes a q-curve. So the parameters that vary along the parametric curves have been changed, but the curves themselves have not been changed at all.

The invariance of the parametric net under the transformation (1·4) can also be demonstrated in another way. Differentiation of (1·4) and subsequent multiplication give

$$(1\cdot 5) \qquad dpdq = p_u q_v du dv\,.$$

Therefore the equation (1·3) implies $dpdq = 0$, and hence the parametric net is invariant. Moreover, similar reasoning would show that *the transformation*

$$(1\cdot 6) \qquad p = p(v)\,, \qquad q = q(u) \qquad (p_v q_u \neq 0)$$

leaves the parametric net invariant, although it interchanges the parameters that vary on the two families of the net, so that u-curves become q-curves and v-curves become p-curves.

The converse question whether the transformations (1·4) and (1·6) are the only transformations that preserve the parametric net is answered in the affirmative by the following theorem.

THEOREM 1. *The transformations* (1·4) *and* (1·6) *are the only transformations of parameters that leave the parametric net invariant.*

The proof can be made by considering *the general transformation of parameters,*

$$(1\cdot 7) \qquad p = p(u, v)\,, \qquad q = q(u, v) \qquad (J = p_u q_v - p_v q_u \neq 0)\,.$$

The effect of this transformation on the parametric net can be calculated in the following way. Differentiation of equations (1·7) gives

$$(1\cdot 8) \qquad dp = p_u du + p_v dv\,, \qquad dq = q_u du + q_v dv\,,$$

and then multiplication results in the equation

$$(1\cdot 9)\qquad dpdq = p_uq_udu^2 + (p_uq_v + p_vq_u)dudv + p_vq_vdv^2\,.$$

The new parametric net is therefore the net whose equation in the old parameters u, v is written by setting the right member of equation (1·9) equal to zero. This net is the old parametric net in case

$$(1\cdot 10)\qquad p_uq_u = p_vq_v = 0\,, \qquad p_uq_v + p_vq_u \neq 0\,.$$

These conditions are satisfied if

$$p_v = q_u = 0\,, \qquad p_uq_v \neq 0\,,$$

and also if

$$p_u = q_v = 0\,, \qquad p_vq_u \neq 0\,,$$

and only if one or the other of these two sets of conditions is satisfied. Integration of these equations completes the proof.

It will now be shown that *the parametric net can be made to coincide with any prescribed net of curves on a surface.* Let a net (1·2) be given, and let it be required to choose a transformation of parameters (1·7) so that the new parametric net, $dpdq = 0$, shall coincide with the net (1·2). Comparison of the left member of (1·2) and the right member of equation (1·9) shows that it is sufficient to take for p and q in the transformation (1·7) two independent solutions of the two partial differential equations

$$(1\cdot 11)\qquad \frac{p_uq_u}{A} = \frac{p_uq_v + p_vq_u}{2B} = \frac{p_vq_v}{C}\,,$$

when $ABC \neq 0$. But a case in which a denominator vanishes offers no essential exception, and so the following theorem is proved.

THEOREM 2. *It is no restriction on a surface represented by parametric equations to suppose that any particular net of curves on the surface is the parametric net.*

The effect of the transformation (1·7) on some of the functions introduced in the preceding chapter will now be calculated. Differentiation of equations (1·1) gives

$$(1\cdot 12)\qquad x_p = x_uu_p + x_vv_p\,, \qquad x_q = x_uu_q + x_vv_q$$

and similar formulas for derivatives of y, z. The result of solving equations (1·8) for du, dv is

$$(1\cdot 13)\quad du = \frac{1}{J}(q_v dp - p_v dq)\,, \qquad dv = \frac{1}{J}(-q_u dp + p_u dq)\,,$$

where J is defined in equations (1·7). But if u, v are regarded as functions of p, q, direct differentiation yields

$$(1\cdot 14)\qquad du = u_p dp + u_q dq\,, \qquad dv = v_p dp + v_q dq\,.$$

Comparing (1·13) and (1·14), we obtain

$$(1\cdot 15)\quad u_p = \frac{q_v}{J}\,, \qquad u_q = -\frac{p_v}{J}\,, \qquad v_p = -\frac{q_u}{J}\,, \qquad v_q = \frac{p_u}{J}$$

and therefore have

$$(1\cdot 16)\qquad u_p v_q - u_q v_p = \frac{1}{J}\,.$$

Transformed functions being denoted by the usual letters with dashes above, the following formulas can be established by actual calculation, the details of which will not be reproduced here:

$$(1\cdot 17)\quad \begin{cases} \bar{J}_i = \dfrac{J_i}{J} & (i = 1, 2, 3)\,, \\ \bar{E} = Eu_p^2 + 2Fu_p v_p + Gv_p^2\,, \\ \bar{F} = Eu_p u_q + F(u_p v_q + u_q v_p) + Gv_p v_q\,, \\ \bar{G} = Eu_q^2 + 2Fu_q v_q + Gv_q^2\,, \\ \bar{H} = \pm\dfrac{H}{J}\,, \\ \bar{a} = \pm a\,, \quad \bar{b} = \pm b\,, \quad \bar{c} = \pm c\,, \end{cases}$$

the upper of the ambiguous signs being used when $J > 0$, and the lower when $J < 0$.

The invariance of the direction cosines a, b, c of the normal at a point of a surface under transformation of parameters is the analytic equivalent of the fact that the normal is covariant to the point and the surface, and does not depend, except for its sense, on the parametric curves. It may be observed that the jacobians J_t and the discriminant H are *relative invariants*, while the first fundamental coefficients E, F, G are not even relative invariants. There are many functions which are absolute invariants under transformation of parameters. Three examples of such functions are listed in the first three exercises below. The demonstrations by direct calculation may be somewhat laborious, but there are more elegant methods of confirming* the invariance.

EXERCISES

1. *The first differential parameter* $\Delta_1\varphi$ of a function $\varphi(u, v)$, defined by

$$\Delta_1\varphi = \frac{1}{H^2}(E\varphi_v^2 - 2F\varphi_v\varphi_u + G\varphi_u^2), \tag{1 18}$$

is an absolute invariant under the transformation (1 7) of parameters.

2. *The mixed differential parameter* $\Delta_1(\varphi, \psi)$ of two functions $\varphi(u, v)$ and $\psi(u, v)$, defined by

$$\Delta_1(\varphi, \psi) = \frac{1}{H^2}[E\varphi_v\psi_v - F(\varphi_v\psi_u + \varphi_u\psi_v) + G\varphi_u\psi_u], \tag{1 19}$$

is an absolute invariant under transformation of parameters.

3. The second differential parameter $\Delta_2\varphi$ of a function $\varphi(u, v)$, defined by

$$\Delta_2\varphi = \frac{1}{H}\left[\left(\frac{E\varphi_v - F\varphi_u}{H}\right)_v + \left(\frac{G\varphi_u - F\varphi_v}{H}\right)_u\right], \tag{1 20}$$

is an absolute invariant under transformation of parameters.

4. Verify the following formulas:

$$\begin{cases} \Delta_1 u = \dfrac{G}{H^2}, \qquad \Delta_1 v = \dfrac{E}{H^2}, \qquad \Delta_1(u, v) = -\dfrac{F}{H^2}, \\ \Delta_1 u\Delta_1 v - \Delta_1^2(u, v) = \dfrac{1}{H^2}, \\ E = \dfrac{\Delta_1 v}{\Delta_1 u\Delta_1 v - \Delta_1^2(u, v)}, \\ F = -\dfrac{\Delta_1(u, v)}{\Delta_1 u\Delta_1 v - \Delta_1^2(u, v)}, \\ G = \dfrac{\Delta_1 u}{\Delta_1 u\Delta_1 v - \Delta_1^2(u, v)}. \end{cases} \tag{1 21}$$

* Eisenhart, *Differential Geometry*, pp. 84–89.

5. If a transformation of parameters is made so that the two families whose equations in the old parameters u, v are

$$\varphi(u, v) = \text{const.}, \qquad \psi(u, v) = \text{const.}$$

become the new parametric curves, then the new fundamental coefficients $\overline{E}$, $\overline{F}$, $\overline{G}$ are given by the formulas

$$(1 \cdot 22) \qquad \begin{cases} \overline{E} = \dfrac{\Delta_1\psi}{\Delta_1\varphi\Delta_1\psi - \Delta_1^2(\varphi, \psi)}, \\ \overline{F} = -\dfrac{\Delta_1(\varphi, \psi)}{\Delta_1\varphi\Delta_1\psi - \Delta_1^2(\varphi, \psi)}, \\ \overline{G} = \dfrac{\Delta_1\varphi}{\Delta_1\varphi\Delta_1\psi - \Delta_1^2(\varphi, \psi)}. \end{cases}$$

6. The first fundamental form (III 3 2) is *absolutely invariant* under transformation of parameters.

7. The parametric curves on the revolute (III 2 15) are not changed by the transformation of parameters

$$(1 \cdot 23) \qquad p = \int_{u_0}^{u} \frac{1}{u}(1 + f'^2)^{1/2}du, \qquad q = v,$$

under which the first fundamental form for this surface becomes

$$(1 \cdot 24) \qquad u^2(dp^2 + dq^2).$$

2. Minimal curves. Hitherto all the variables and functions entering into the discussion have been supposed to be real, but in certain parts of the theory of curves and surfaces it is very convenient to permit the introduction of imaginary values. Without attempting to treat the subject of complex curves and surfaces exhaustively, we shall in this section consider briefly one type of imaginary curves called *minimal curves*, which are of great interest in some parts of the theory of surfaces. These considerations will lead to the introduction of some ideas from projective geometry at the close of the section.

Returning to the definition of a curve (Chap. I, Sec. 2, Def. 1), let us permit the parameter t to range over a suitable region of the complex plane, and also permit the coordinates x, y, z to take on complex values. The inequality (I 2·2) can now be dropped, and the definition of a minimal curve can be stated as follows:

DEFINITION 1. *A minimal curve is a nonsingular curve for which*

$$(2\cdot 1)\qquad x'^2 + y'^2 + z'^2 = 0 \qquad \left(x' = \frac{dx}{dt}, \cdots\right).$$

It is at once evident that *a nonsingular curve is a minimal curve if, and only if, $ds = 0$.* Certainly, minimal curves must be imaginary. That such curves exist is easily established. Indeed, the equations

$$(2\cdot 2)\qquad \begin{cases} x = \frac{1}{2}(1 - c^2)t + c_1, \qquad y = \dfrac{i}{2}(1 + c^2)t + c_2, \\ z = ct + c_3 \qquad\qquad (i^2 = -1), \end{cases}$$

in which c, c_1, c_2, c_3 are constants and t is a variable, represent a *minimal straight line,* as is easily verified. Moreover, the equations

$$(2\cdot 3)\qquad \begin{cases} x = \frac{1}{2}(1 - t^2)f'' + tf' - f, \\ y = i[\frac{1}{2}(1 + t^2)f'' - tf' + f], \\ z = tf'' - f' \qquad\qquad \left(f' = \dfrac{df}{dt}, \cdots\right), \end{cases}$$

in which f is any function of the variable t such that $f''' \neq 0$, represent a *minimal curve.* The problem of reducing the equations of any minimal straight line to the form (2 2), and of showing that the equations of any nonrectilinear minimal curve can be written in the form (2·3), need not be discussed* here.

Returning now to the definition of a surface (Chap. III, Sec. 1, Def. 1), let us permit the parameters u, v to range over a suitable region of the complex plane, and also permit the coordinates x, y, z to take on complex values. Since a necessary and sufficient condition that a nonsingular curve be a minimal curve is $ds = 0$, it follows that the minimal curves on a surface are those curves for which the first fundamental form vanishes. Thus we reach the following conclusion.

THEOREM 1. *The curvilinear differential equation of the minimal curves on a surface* (1·1) *is*

$$(2\cdot 4)\qquad E\,du^2 + 2F\,du\,dv + G\,dv^2 = 0.$$

Tangents of minimal curves may be called *minimal tangents* (see Ex. 3 below). If a surface is such that $H \neq 0$, the minimal curves

* *Ibid.*, pp. 47–49.

form a net called *the minimal net* on the surface. Inspection of equation (2·4) makes the truth of the following statement evident.

THEOREM 2. *Necessary and sufficient conditions that the minimal net on a surface be parametric are*

$$E = G = 0, \qquad F \neq 0. \tag{2·5}$$

By way of illustration let us suppose that the surface under consideration is a plane, and take it for the plane $z = 0$. In cartesian coordinates x, y the curvilinear differential equation of the minimal curves in the plane $z = 0$ is

$$dx^2 + dy^2 = 0. \tag{2·6}$$

This equation is equivalent to two linear differential equations, the integrals of which can be written in the form

$$x + iy = p, \qquad x - iy = q \qquad (i^2 = -1), \tag{2·7}$$

in which p, q are arbitrary constants. Thus the following theorem is proved.

THEOREM 3. *The minimal curves in a plane constitute two one-parameter families of straight lines.*

If equations (2·7) are solved for x and y, the resulting equations,

$$x = \tfrac{1}{2}(p + q), \qquad y = \frac{i}{2}(q - p), \tag{2·8}$$

are parametric equations of the xy-plane for which the minimal lines are parametric. In fact, direct calculation with p, q in place of u, v gives

$$E = G = 0, \qquad F = \tfrac{1}{2}.$$

For a second illustration let us suppose that the surface under consideration is the sphere (III 2·11). The curvilinear differential equation of the minimal curves on the sphere can be written in the form

$$\csc^2 u\,du^2 + dv^2 = 0. \tag{2·9}$$

This equation is equivalent to two linear equations, whose integrals are

$$p = e^{iv} \tan\frac{u}{2}, \qquad q = e^{-iv} \tan\frac{u}{2} \qquad (i^2 = -1), \tag{2·10}$$

where p, q are arbitrary constants. If these equations are solved for u, v as functions of p, q, and if these functions are substituted in equations (III·2·11), new parametric equations of the sphere are obtained, namely,

$$(2\cdot 11)\qquad x = \frac{r(p+q)}{1+pq}, \qquad y = \frac{ir(q-p)}{1+pq}, \qquad z = \frac{r(1-pq)}{1+pq}.$$

That the minimal net is the parametric net for this representation can be verified by calculating the first fundamental coefficients with p, q in place of u, v. The result is

$$E = G = 0, \qquad F = \frac{2r^2}{(1+pq)^2}.$$

The following theorem affords an insight into the nature of the minimal curves on the sphere.

THEOREM 4. *The minimal curves on a sphere are the two one-parameter families of (imaginary) rectilinear generators on the sphere.*

To demonstrate this theorem, it is sufficient to deduce from the implicit equation of the sphere,

$$x^2 + y^2 + z^2 = r^2,$$

the equations of the rectilinear generators, namely,

$$\frac{x+iy}{r+z} = \frac{r-z}{x-iy} = p, \qquad \frac{x-iy}{r+z} = \frac{r-z}{x+iy} = q,$$

in which p, q are parameters, and then to show that the result of solving these equations for x, y, z as functions of p, q is precisely equations (2·11).

Certain notions from analytic projective geometry are quite useful in interpreting some of the formulas of metric differential geometry. One of the most important projective invariants is *cross ratio*, which may be defined as follows:

DEFINITION 2. *The cross ratio r of four numbers a, b, c, d, in the order named, is defined by the formula*

$$(2\cdot 12)\qquad r = \frac{c-a}{c-b}\cdot\frac{d-b}{d-a}.$$

DEFINITION 3. *In case* $r = -1$, *the pairs* a, b *and* c, d *are said to separate each other harmonically.*

DEFINITION 4. *The cross ratio of four coplanar concurrent lines is defined to be the cross ratio of their coordinates in a suitable coordinate system in the flat pencil to which the lines belong.*

For example, the cross ratio of four lines in the xy-plane and through the origin is the cross ratio of their slopes. Again, the cross ratio of four lines in the tangent plane at a point of a surface and through the point of contact is the cross ratio of their directions dv/du. The notion of harmonic separation can be carried over to four lines of a flat pencil. The following theorem will now be proved.

THEOREM 5. *A necessary and sufficient condition that two intersecting lines be perpendicular to each other is that they separate harmonically the two minimal lines in their plane and through their point of intersection.*

For convenience let the two lines lie in the xy-plane and pass through the origin, and let their slopes be m, n. The slopes of the minimal lines through the origin in the xy-plane are i, $-i$. Direct calculation reduces the condition of harmonic separation,

$$\frac{m-i}{m+i}\cdot\frac{n+i}{n-i} = -1\,, \tag{2·13}$$

to the condition of orthogonality,

$$mn + 1 = 0\,. \tag{2·14}$$

Simple calculations, which need not be reproduced here, suffice to prove the following theorem.

THEOREM 6. *The two roots in* dv/du *of the quadratic equation*

$$A du^2 + 2B du dv + C dv^2 = 0 \qquad (AC - B^2 \neq 0) \tag{2·15}$$

separate harmonically the two roots of the equation

$$L du^2 + 2M du dv + N dv^2 = 0 \qquad (LN - M^2 \neq 0) \tag{2·16}$$

if, and only if,

$$AN - 2BM + CL = 0\,. \tag{2·17}$$

The function in the left member of equation (2·17) is given a name by the following definition.

DEFINITION 5. *The function*

$$AN - 2BM + CL \tag{2.18}$$

is called the harmonic invariant of the two binary quadratic forms in the left members of equations (2.15) *and* (2.16).

It is known, in fact, that the harmonic invariant is a relative invariant of the forms under the most general linear homogeneous transformation of the variables of the forms. An immediate corollary of Theorem 6 will now be stated.

THEOREM 7. *At a point of a surface the two tangents of one net separate harmonically the two tangents of another net if, and only if, the harmonic invariant of the two forms in the curvilinear differential equations of the nets vanishes.*

EXERCISES

1. The locus of all the minimal straight lines through a fixed point is a quadric cone with its vertex at the point (called *the isotropic cone* at the point). The equation of the isotropic cone at the origin is

$$x^2 + y^2 + z^2 = 0 .$$

2. The distance between any two distinct finite points on a minimal straight line, calculated by the usual distance formula, is zero. Similarly, the arc length between two points on a minimal curve is zero.

3. The tangent lines of a minimal curve are minimal straight lines.

4. The osculating planes of a minimal curve are *isotropic planes* (characterized by the condition $a^2 + b^2 + c^2 = 0$, the equation of a plane being $ax + by + cz + d = 0$).

5. Find the equations of transformation from the meridians and parallels to the minimal curves on the revolute (III 2.15).

6. The cross ratio of four numbers $a_1, \cdots, a_4$ is equal to the cross ratio of the four numbers $b_1, \cdots, b_4$ into which $a_1, \cdots, a_4$ are transformed by any *linear fractional transformation*,

$$b_i = \frac{Aa_i + B}{Ca_i + D} \qquad (i = 1, \cdots, 4\,;\, AD - BC \neq 0)\,. \tag{2.19}$$

7. The tangent plane at a point P of a sphere intersects the sphere in the two minimal straight lines that lie on the sphere and pass through P.

3. Angles between curves on a surface. When two curves on a surface intersect at a point P, it is customary to say that the angle between them is the angle between their tangents at P. However, there is ambiguity as to what is meant by the angle between the tangents, unless it is defined more precisely. When a surface is represented analytically by its parametric equations (1 1), the positive sense on each of the parametric tangents and on the normal at a point of the surface has already been defined. Moreover, when a curve on the surface is represented analytically by its curvilinear parametric equations, the positive sense on its tangent at any one of its points has been defined. For the purpose of formulating a precise definition of the angle between two tangents at a point of a surface, a positive sense of rotation in the tangent plane of the surface at the point is assigned by the following convention:

In the tangent plane at a point P of a surface represented by the parametric equations (1 1) *the positive sense of rotation about P is, by agreement, the sense in which the positive half of the u-tangent can be rotated through an angle less than π into the positive half of the v-tangent.*

The angle between the parametric tangents, or, more precisely, the angle from the u-tangent to the v-tangent, at a point of a surface is defined as follows:

DEFINITION 1. *The angle ω between the parametric tangents at a point of a surface represented by the parametric equations* (1 1) *is the smallest angle from the positive half of the u-tangent in the positive sense of rotation to the positive half of the v-tangent.*

The angle ω just defined satisfies the condition

$$0 < \omega < \pi .$$

The equations (III·4 3) for the direction cosines of the parametric tangents, and elementary trigonometry, can be used to establish the formulas

$$(3\ 1) \qquad \cos\omega = \frac{F}{AC}, \qquad \sin\omega = \frac{H}{AC},$$

in which H is defined by (III·3·6) and A, C by (III·3·4). The angle ω is acute or obtuse according as $F > 0$ or $F < 0$. If $F = 0$, then $\omega = \pi/2$, and the parametric curves intersect at right angles. The definition of *an orthogonal net* of curves will now be stated.

DEFINITION 2. *An orthogonal net of curves on a surface S is a net such that at each point of S the two curves of the net intersect at right angles.*

The following conclusion is immediate.

THEOREM 1. *A necessary and sufficient condition that the parametric net on a surface be orthogonal is $F = 0$.*

The foregoing considerations can be generalized from the parametric net to any net whatever on a surface. Let us consider, on a surface represented by the parametric equations $(1 \cdot 1)$, two curves C and C_1 intersecting at a point P. Let the curvilinear parametric equations of C and C_1 be, respectively,

$$(3 \cdot 2) \quad u = u(s) \,, \qquad v = v(s) \,; \qquad u = u_1(s_1) \,, \qquad v = v_1(s_1) \,,$$

the parameters s, s_1 being arc lengths and the subscript 1 referring always to the curve C_1. The angle between the tangents of C and C_1, or, more precisely, the angle from the tangent of C_1 to the tangent of C, at P is defined as follows:

DEFINITION 3. *The angle θ between the tangents of two curves C, C_1 intersecting at a point P on a surface is the smallest angle from the positive half of the tangent of C_1 in the positive sense of rotation to the positive half of the tangent of C.*

The angle θ just defined satisfies the condition

$$0 \leqq \theta < 2\pi \,.$$

The direction cosines of the tangents of the curves C, C_1 at the point P are given by

$$(3 \cdot 3) \qquad \alpha = x_u \frac{du}{ds} + x_v \frac{dv}{ds} \,, \qquad \alpha_1 = x_u \frac{du_1}{ds_1} + x_v \frac{dv_1}{ds_1} \,,$$

and symmetric formulas for β, β_1, γ, γ_1. Therefore $\cos\theta$ and, by elementary trigonometry, $\sin\theta$ can be calculated. The results are

$$(3 \cdot 4) \begin{cases} \cos\theta = \dfrac{1}{dsds_1}\,[Edudu_1 + F(dudv_1 + du_1dv) + Gdvdv_1] \,, \\[2ex] \sin\theta = -\dfrac{H}{dsds_1}\,(dudv_1 - du_1dv) \,, \end{cases}$$

provided that the sign of $\sin\theta$ is properly chosen. The reason for the choice made may not be evident a priori, but the sign chosen will be shown to be the correct one by specializing the curves C_1 and C to be the u-curve and the v-curve, respectively, at the point P.

If the curve C_1 is specialized to be the u-curve at the point P, we have

$$(3\cdot 5)\qquad dv_1 = 0\,, \qquad ds_1 = A du_1\,,$$

and therefore equations (3·4) reduce to

$$(3\cdot 6)\qquad \cos\theta = \frac{E}{A}\frac{du}{ds} + \frac{F}{A}\frac{dv}{ds}\,, \qquad \sin\theta = \frac{H}{A}\frac{dv}{ds}\,.$$

Incidentally, elementary trigonometry gives in this case

$$(3\cdot 7)\qquad \tan\theta = \frac{H dv}{E du + F dv}\,,$$

a formula that will be useful later on. If now the curve C is specialized to be the v-curve at P, the resulting simplifications,

$$du = 0\,, \qquad ds = C dv\,,$$

reduce equations (3·6) to equations (3·1), as they should do, since now $\theta = \omega$. The first of equations (3·4) implies the following theorem.

THEOREM 2. *A necessary and sufficient condition that two curves on a surface be perpendicular to each other at a point of intersection P is that the directions dv/du, dv_1/du_1 of the two curves at P satisfy the equation*

$$(3\cdot 8)\qquad E du du_1 + F(du dv_1 + du_1 dv) + G dv dv_1 = 0\,.$$

The bilinear form in the left member of this equation is known as *the polar form* of the first fundamental form with respect to the variables du_1, dv_1. Theorem 2 can be used to establish the following result.

THEOREM 3. *A necessary and sufficient condition that a net* (1·2) *be an orthogonal net is*

$$(3\cdot 9)\qquad AG - 2BF + CE = 0\,.$$

The demonstration may be made by dividing equation (3·8) by $du du_1$, by supposing that dv/du, dv_1/du_1 are the roots of equation (1·2), and by making use of the well-known formulas for the sum and product of the roots of a quadratic equation in terms of the coefficients of the equation.

Inspection of equation (3·9) and reference to Theorems 5 and 7 of the preceding section suffice to demonstrate the following theorem.

THEOREM 4. *A necessary and sufficient condition that a net of curves on a surface be an orthogonal net is that at each point of the surface the tangents of the net separate the minimal tangents of the surface harmonically.*

In order to discover a formula for *the element of area dA* of a surface, we may proceed intuitively as follows. As in Figure 13, let us think

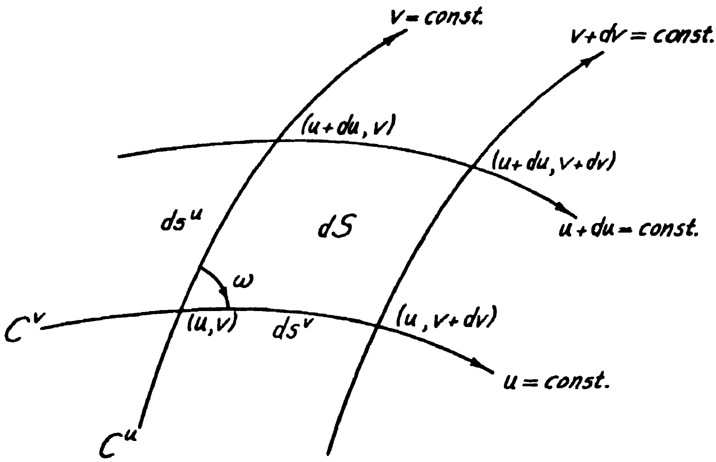

FIG. 13

of dA as a small parallelogram bounded by two u-curves corresponding to constant values of v and $v + dv$, and two v-curves corresponding to constant values of u and $u + du$. Two adjacent sides of the parallelogram are ds^u, ds^v, and the included angle is ω. Consequently, the area dA of the parallelogram is given by

$$(3\cdot 10) \qquad dA = \sin \omega ds^u ds^v = H du dv \, .$$

The area A of a region of a surface may be found by evaluating with suitably chosen limits the double integral in the formula

$$(3\cdot 11) \qquad A = \int\!\!\int H du dv \, .$$

For a different approach to the problem of calculating the area of a region of a surface, and an analytically rigorous deduction of the formula (3·11), the reader may consult Goursat-Hedrick, *Mathematical Analysis*, I, 272.

EXERCISES

1. Calculate the area of a sphere by means of the integral (3 11).

2. Prove that, on a surface (1 1), any one-parameter family of curves represented by the curvilinear differential equation

$$mdu + ndv = 0 ,$$

in which m, n are functions of u, v, has a family of orthogonal trajectories represented by the equation

$$(En - Fm)du + (Fn - Gm)dv = 0 . \tag{3 12}$$

3. On the screw surface (III 2 26) the curves represented by the curvilinear differential equation

$$du^2 - (u^2 + a^2)dv^2 = 0$$

form an orthogonal net.

4. Find the curvilinear differential equation of the curves crossing the generators of a developable surface at a constant angle, and check the result by showing that the equation is that of the involutes of the edge of regression of the developable when the angle is a right angle.

5. The curvilinear equation of the curves (called *loxodromes*) crossing the meridians of a revolute (III 2 15) at a constant angle a is

$$v \cot a = \int_{u_0}^{u} \frac{1}{u} (1 + f'^2)^{1/2} du + c \qquad (c = \text{const.}) . \tag{3 13}$$

6. The equation of the loxodromes on a sphere (III 2 11) is

$$v \cot a = \log \tan \frac{u}{2} + c \qquad (c = \text{const.}) \tag{3 14}$$

7. If a man (regarded as a point) on a spherical earth of radius r starts from the point $(u_0, 0)$ and crosses the meridians of longitude at the angle $\pi/4$, how far has he traveled when he has reached the latitude $u = u_0 + \pi/2$? How far has he traveled when he has crossed all the meridians once? How far when, and if, he has reached the South Pole?

8. A cone of revolution of acute semivertical angle α, about the z-axis and with its vertex at the origin, is represented by the parametric equations

$$x = u \cos v \sin \alpha , \qquad y = u \sin v \sin \alpha , \qquad z = u \cos \alpha . \tag{3 15}$$

The curvilinear equation of the curves on the upper nappe of this cone which cut the generators at a constant angle a is

$$u = he^{kv} \qquad (h = \text{const.};\ k = \sin a \cot a)\,. \tag{3·16}$$

Prove that the orthogonal projections of these curves onto the xy-plane are logarithmic spirals.

9. Prove that the curvilinear equation of the orthogonal trajectories of the generators of a ruled surface (III 7 1) is

$$t = -\int_{s_0}^{s} \cos\theta ds + c \qquad (c = \text{const.})\,. \tag{3·17}$$

Hence prove that any two orthogonal trajectories intercept a constant distance on all generators.

10. A necessary and sufficient condition that two families of curves,

$$\varphi(u, v) = \text{const.}\,, \qquad \psi(u, v) = \text{const.}\,, \tag{3·18}$$

form an orthogonal net on a surface is $\Delta_1(\varphi, \psi) = 0$ (see Sec. 1, Ex. 5).

11. If a family of curves, $\varphi(u, v) = \text{const.}$, has the property that $\Delta_2(\varphi) = 0$ (see Sec. 1, Ex. 3), then the orthogonal trajectories of these curves can be found by integrating the system of partial differential equations

$$\psi_u = \frac{1}{H}(F\varphi_u - E\varphi_v)\,, \qquad \psi_v = \frac{1}{H}(G\varphi_u - F\varphi_v)\,. \tag{3·19}$$

The family, $\psi(u, v) = \text{const.}$, of orthogonal trajectories also has the property that $\Delta_2\psi = 0$.

12. If the two families (3 18) are such that equations (3 19) are satisfied, then

$$\Delta_1(\varphi, \psi) = 0\,, \qquad \Delta_1\varphi = \Delta_1\psi\,. \tag{3·20}$$

If these families were taken for the parametric curves, it would be true that $\bar{F} = 0$, $\bar{E} = \bar{G}$.

13. If *an isothermally orthogonal net* is defined to be a net such that there exists a transformation of parameters which makes it parametric and also makes $F = 0$, $E = G$, prove that necessary and sufficient conditions that the parametric net be isothermally orthogonal are

$$F = 0\,, \qquad \left(\log\frac{E}{G}\right)_{uv} = 0\,. \tag{3·21}$$

(Parameters for which $F = 0$, $E = G$ may be called *isothermic parameters*.)

14. Conditions (3 20) are sufficient that the two families (3 18) form an isothermally orthogonal net. The conditions

$$\Delta_1(\varphi, \psi) = 0\,, \qquad \Delta_2(\varphi) = 0 \tag{3·22}$$

are also sufficient therefor.

15. Necessary and sufficient conditions that the two families (3·18) form an isothermally orthogonal net are

$$\Delta_1(\varphi, \psi) = 0\,, \qquad \frac{\Delta_2\varphi}{\Delta_1\varphi} = f(\varphi)\,, \tag{3 23}$$

where f is an arbitrary function.

16. If the parametric net, $dudv = 0$, on a surface is isothermally orthogonal, with $F = 0$, $E = G$, then equations (3·19) become

$$\psi_u = -\varphi_v\,, \qquad \psi_v = \varphi_u\,. \tag{3 24}$$

Hence show that other isothermally orthogonal nets on the surface can be obtained by taking for φ, ψ in equations (3 18) the functions φ, ψ defined by

$$\varphi + i\psi = f(u + iv) \qquad (i^2 = -1)\,, \tag{3 25}$$

where f is an arbitrary function.

17. If *a complex variable on a surface* is defined to be $\varphi + i\psi$, where the families

$$\varphi(u, v) = \text{const.}\,, \qquad \psi(u, v) = \text{const.}$$

constitute an isothermally orthogonal net, then the equations of the most general isothermally orthogonal net on the surface can be obtained by equating to arbitrary constants the real and imaginary components of an arbitrary function of the complex variable $\varphi + i\psi$ on the surface.

18. The meridians and parallels on a surface of revolution constitute an isothermally orthogonal net.

19. The rectilinear generators and the circular helices on a screw surface constitute an isothermally orthogonal net.

20. In the xy-plane the lines $x = \text{const.}$ and the lines $y = \text{const.}$ form an isothermally orthogonal net, as do also the lines $\theta = \text{const.}$ and the circles $r = \text{const.}$ of an ordinary polar coordinate system with pole at the origin and with the polar line along the x-axis.

21. A system of confocal ellipses and hyperbolas forms an isothermally orthogonal net in a plane.

22. If the minimal curves constitute the parametric net $dudv = 0$, on a surface, the transformation

$$u = p + iq\,, \qquad v = p - iq \tag{3·26}$$

makes the new parametric net, $dpdq = 0$ isothermally orthogonal.

4. Asymptotic curves: the second fundamental form. *The asymptotic curves*, or simply *the asymptotics*, on a surface have various characteristic properties, and hence may be defined in different ways. As defined below, they are covariant to the surface both in metric and in projective geometry. Their curvilinear differential equation leads to the definition of *the second fundamental form*, and in studying some of their properties the so-called *Weingarten differential equations* are introduced.

DEFINITION 1. *A curve C on a surface S is an asymptotic curve in case at each point of C the osculating plane of C coincides with the tangent plane of S.*

The tangents and directions of asymptotic curves are called *asymptotic tangents* and *asymptotic directions*, respectively. The reason for the name *asymptotic* will be found in Theorem 1 in Section 4 of Chapter V. Two theorems will now be demonstrated by synthetic arguments.

THEOREM 1. *Every straight line on a surface is an asymptotic curve.*

THEOREM 2. *On a plane every curve is an asymptotic curve.*

Theorem 1 may be demonstrated by first observing that the tangent plane of a surface S at a point P of a straight line on S contains the line, since it contains the tangent line at P of every curve on S through P, and the straight line is such a curve, which is its own tangent line at P. Moreover, the tangent plane also osculates the line at P, since every plane through a straight line is an osculating plane of the line. The second theorem is proved by remarking that the osculating plane at a point of a plane curve is the plane of the curve, and that the plane is its own tangent plane at every one of its points.

In deducing the curvilinear differential equation of the asymptotic curves on a surface, the first step is to write *the equation of the osculating plane at a point of any curve on a surface.* Let us consider a surface S with the parametric equations (1·1), and on S a curve C defined by the curvilinear parametric equations

$$u = u(t)\,, \qquad v = v(t)\,.$$

Let (X, Y, Z) be a variable point on the osculating plane at a point $P(x, y, z)$ of C. Then the equation of the osculating plane of C at P is

$$(4\cdot 1) \qquad (X - x, x', x'') = 0 ,$$

in which a determinant is indicated by writing only a typical row within parentheses, the other two rows being obtained by replacing X, x by Y, y and then by Z, z, and in which x', x'' are total derivatives given by

$$(4\cdot 2) \qquad \begin{cases} x' = x_u u' + x_v v' & \left(x' = \dfrac{dx}{dt}, \cdots\right), \\ x'' = x_{uu}u'^2 + 2x_{uv}u'v' + x_{vv}v'^2 + x_u u'' + x_v v'' . \end{cases}$$

At the point P the osculating plane of the curve C and the tangent plane of the surface S already have in common the tangent line of C. These planes will coincide in case the osculating plane contains one other line through P in the tangent plane. Ordinarily both the u-tangent and the v-tangent at P are distinct from the tangent of C, and certainly one or the other of them must be. Let us suppose that the u-tangent is distinct from the tangent of C. The parametric equations of the u-tangent can be written in the form

$$(4\cdot 3) \qquad X = x + x_u w , \qquad Y = y + y_u w , \qquad Z = z + z_u w ,$$

in which w is a parameter. If the expressions for x', x'', X in equations (4·2) and (4·3) are substituted in equation (4·1), the result can be reduced by elementary properties of determinants to the following *necessary and sufficient condition that the curve C be an asymptotic curve,*

$$(4\cdot 4) \qquad (x_u, x_v, x_{uu}u'^2 + 2x_{uv}u'v' + x_{vv}v'^2) = 0 .$$

If three functions L, M, N are defined by the formulas

$$(4\cdot 5) \qquad \begin{cases} L = \dfrac{1}{H}(x_{uu}, x_u, x_v) , \qquad M = \dfrac{1}{H}(x_{uv}, x_u, x_v) , \\ N = \dfrac{1}{H}(x_{vv}, x_u, x_v) , \end{cases}$$

then equation (4·4) can be reduced to *the curvilinear differential equation of the asymptotic curves on a surface,*

$$(4\cdot 6) \qquad L du^2 + 2M du dv + N dv^2 = 0 .$$

The way is now prepared to define *the second fundamental form.*

DEFINITION 2. *The second fundamental form is the form in the left member of equation* (4·6).

The coefficients L, M, N of the second fundamental form are sometimes called *the second fundamental coefficients* in the theory of surfaces. It will be recalled that the direction cosines a, b, c of the normal at a point of a surface are given by

$$(4\cdot 7) \qquad \begin{cases} a = \dfrac{1}{H}(y_u z_v - y_v z_u)\,, \qquad b = \dfrac{1}{H}(z_u x_v - z_v x_u)\,, \\ \qquad\qquad c = \dfrac{1}{H}(x_u y_v - x_v y_u)\,. \end{cases}$$

Therefore the definitions (4·5) are equivalent to

$$(4\cdot 8) \qquad L = \Sigma a x_{uu}\,, \qquad M = \Sigma a x_{uv}\,, \qquad N = \Sigma a x_{vv}\,,$$

the summation being for terms obtained by replacing a, x by b, y and then by c, z. Still other useful formulas can be deduced for L, M, N. By direct substitution from the formulas (4·7) for a, b, c, it is easy to verify the identities

$$(4\cdot 9) \qquad \Sigma a x_u = 0\,, \qquad \Sigma a x_v = 0\,,$$

which express analytically the orthogonality of the normal and the parametric tangents at a point of a surface. Differentiation of equation (4·9) leads to

$$(4\cdot 10) \qquad \begin{cases} \Sigma a x_{uu} + \Sigma a_u x_u = 0\,, \qquad \Sigma a x_{uv} + \Sigma a_v x_u = 0\,, \\ \Sigma a x_{uv} + \Sigma a_u x_v = 0\,, \qquad \Sigma a x_{vv} + \Sigma a_v x_v = 0\,. \end{cases}$$

Therefore the formulas (4·8) are equivalent to

$$(4\cdot 11) \qquad \begin{cases} L = -\Sigma a_u x_u\,, \qquad M = -\Sigma a_v x_u = -\Sigma a_u x_v\,, \\ \qquad\qquad N = -\Sigma a_v x_v\,. \end{cases}$$

It is not difficult to deduce the effect of a transformation of parameters (1·7) on the second fundamental coefficients L, M, N. One way to do this is based on the formulas (4·8) for L, M, N. The last three of the formulas (1·17) give the effect of the transformation (1·7) on the direction cosines a, b, c. From the calculus we have the formulas

$$x_{pp} = x_{uu}u_p^2 + 2x_{uv}u_pv_p + x_{vv}v_p^2 + x_uu_{pp} + x_vv_{pp}\,,$$

$$x_{pq} = x_{uu}u_pu_q + x_{uv}(u_pv_q + u_qv_p) + x_{vv}v_pv_q + x_uu_{pq} + x_vv_{pq}\,,$$

$$x_{qq} = x_{uu}u_q^2 + 2x_{uv}u_qv_q + x_{vv}v_q^2 + x_uu_{qq} + x_vv_{qq}$$

and similar formulas for y, z. Then forming the expressions analogous to (4·8) for $\bar{L}$, $\bar{M}$, $\bar{N}$, we find

$$\bar{L} = \pm(Lu_p^2 + 2Mu_pv_p + Nv_p^2)\,,$$

$$\bar{M} = \pm[Lu_pu_q + M(u_pv_q + u_qv_p) + Nv_pv_q]\,,$$

$$\bar{N} = \pm(Lu_q^2 + 2Mu_qv_q + Nv_q^2)\,,$$

the upper of the ambiguous signs being used when the jacobian J of the transformation (1·7) is positive, and the lower when J is negative.

According to Theorem 2 above, if the surface under consideration is a plane, then every curve on it is an asymptotic curve, i.e., the asymptotic curves are indeterminate, and therefore $L = 0$, $M = 0$, $N = 0$. Another way to reach the same conclusion is to start with parametric equations of the xy-plane in the form

$$x = u\,, \qquad y = v\,, \qquad z = 0\,,$$

and to show by actual calculation that for this representation we have $L = 0$, $M = 0$, $N = 0$, conditions that are invariant under any transformation of parameters, as well as under any rigid motion of the plane in space.

Conversely, if $L = 0$, $M = 0$, $N = 0$, the asymptotic curves are indeterminate and the question arises whether the surface under consideration is a plane. An affirmative answer is furnished by the following theorem.

Theorem 3. *Necessary and sufficient conditions that a surface be a plane are*

$$L = M = N = 0\,. \tag{4·12}$$

The necessity was proved above. In proving the sufficiency it will be convenient to use the so-called *Weingarten differential equations*, which will now be deduced. It is easy, by direct substitution, to verify the identities

$$(4 \cdot 13) \qquad \begin{cases} H(cy_u - bz_u) = Fx_u - Ex_v\,, \\ H(cy_v - bz_v) = Gx_u - Fx_v\,, \end{cases}$$

and four others arising from these by circular permutations. Moreover, differentiation of the relation

$$(4 \cdot 14) \qquad a^2 + b^2 + c^2 = 1$$

produces the equations

$$(4 \cdot 15) \qquad \Sigma a a_u = 0\,, \qquad \Sigma a a_v = 0\,.$$

Equations (4 11) and (4 15) can be grouped into two sets of three equations each, each set being written in a column as follows:

$$(4 \cdot 16) \qquad \begin{cases} \Sigma x_u a_u = -L\,, & \Sigma x_u a_v = -M\,, \\ \Sigma x_v a_u = -M\,, & \Sigma x_v a_v = -N\,, \\ \Sigma a a_u = 0\,; & \Sigma a a_v = 0\,. \end{cases}$$

The first set can be solved for a_u, b_u, c_u, since the determinant of the coefficients reduces to the function H, which is not zero. Similarly, the second set can be solved for a_v, b_v, c_v. The result of the solution is that not only a, x but also b, y and c, z satisfy the following equations, which are called *the Weingarten differential equations:*

$$(4 \;17) \qquad \begin{cases} a_u = \dfrac{1}{H^2}(FM - GL)x_u + \dfrac{1}{H^2}(FL - EM)x_v\,, \\ a_v = \dfrac{1}{H^2}(FN - GM)x_u + \dfrac{1}{H^2}(FM - EN)x_v\,. \end{cases}$$

It is evident from these equations that if $L = 0$, $M = 0$, $N = 0$ for a surface, then a, b, c are constant. By means of equations (4 9) with a, b, c constant it is easy to show that the tangent plane at a point of the surface is fixed when the point varies on the surface. Therefore the surface is a plane, as was to be proved.

The asymptotic curves on developable surfaces have a special property, which is incorporated in the next two theorems.

THEOREM 4. *The asymptotic curves on a developable surface not a plane form only a one-parameter family (counted twice) and coincide with the generators of the surface.*

THEOREM 5. *If the asymptotic curves on a surface form only a one-parameter family, the surface is a developable not a plane.*

The proof of Theorem 4 is accomplished by direct calculation. Using the equations (III 6 1) of the tangent developable of a curve and substituting in the definitions (4·5) with s, t in place of u, v, we find

$$L = \pm \frac{t}{\rho\tau}, \qquad M = 0, \qquad N = 0, \tag{4·18}$$

on the assumption that the developable under consideration is not a cone or cylinder. Since by hypothesis the developable is not a plane, it follows that $L \neq 0$, and so the differential equation of the asymptotics becomes $ds^2 = 0$. Consequently the asymptotics on the developable are the generators counted twice. Incidentally, it is easy to verify that $LN - M^2 = 0$ for the developable (III 6 1) and that this equation is invariant under the transformation of parameters (1·7). The arguments for the cone and for the cylinder present no essential differences from the foregoing, and the same conclusion can be obtained for them.

To prove Theorem 5 let us observe that the hypothesis of this theorem is equivalent to supposing that $LN - M^2 = 0$ while not all of L, M, N vanish identically. Hence the surface is not a plane. Let the family of asymptotics be taken for the v-curves; then $N = 0$ and therefore $M = 0$. The second of the Weingarten differential equations (4 17) tells us that $a_v = 0$, $b_v = 0$, $c_v = 0$. Therefore the tangent plane is fixed when v varies, so that the surface has only a one-parameter family of tangent planes and must be a developable, as was to be proved.

Let us now restrict attention to those surfaces for which $LN - M^2$ does not vanish identically. If a real surface (or portion of a surface) with real parameters is such that $LN - M^2 < 0$, the asymptotic curves are real; where $LN - M^2 > 0$, the asymptotics are imaginary. If $LN - M^2$ happens to vanish at a point on a surface while not vanishing identically, such a point is called *a parabolic point*. Ordinarily, the conditional equation $LN - M^2 = 0$ defines a curve which is the locus of parabolic points on a surface. This curve is called *the parabolic curve* on the surface and separates the region where the asymptotic curves are real from the region where they are imaginary.

Moreover, it may happen that L, M, N vanish simultaneously at one or more points on a surface but do not all vanish identically; such a point is called *a flat point*, or *a planar point*, on the surface.

The truth of the following statement is evident on inspection of the equation (4·6) of the asymptotic curves.

THEOREM 6. *Necessary and sufficient conditions that the asymptotic net on a nondevelopable surface be parametric are*

$$L = N = 0\,, \qquad M \neq 0\,. \tag{4·19}$$

The question may arise whether it is possible for the asymptotic net and the minimal net on a surface to coincide. Necessary conditions that they coincide on a surface not a plane are

$$L = hE\,, \qquad M = hF\,, \qquad N = hG \qquad (h \neq 0)\,,$$

where h is a proportionality factor, which will now be proved to be a constant. Let the net under consideration be taken for the parametric net. Then $E = 0$, $G = 0$, $L = 0$, $N = 0$, so that the Weingarten equations (4·17) become

$$a_u = -hx_u\,, \qquad a_v = -hx_v\,. \tag{4·20}$$

If the first of these equations is differentiated with respect to v and the second with respect to u, and the results compared, we obtain

$$h_v x_u = h_u x_v\,.$$

This equation is also satisfied by y and by z. If either of h_u and h_v were different from zero, the jacobians J_i defined by (III·1 4) would all vanish, and then the surface would not be a proper nonsingular surface. It follows that $h_u = 0$, $h_v = 0$; and therefore h = const. Integration of equations (4 20) and of the same equations with b, y and c, z leads to

$$a = -hx + c_1\,, \qquad b = -hy + c_2\,, \qquad c = -hz + c_3\,,$$

where c_1, c_2, c_3 are arbitrary constants. Squaring and adding these equations, we obtain

$$(hx - c_1)^2 + (hy - c_2)^2 + (hz - c_3)^2 = 1\,.$$

Therefore, if the minimal net coincides with the asymptotic net on a surface not a plane, the surface is a sphere. That the minimal net actually does coincide with the asymptotic net on a sphere is proved by calculating the six fundamental coefficients for the sphere (III·2·11). The results are

$$E = r^2\,, \qquad F = 0\,, \qquad G = r^2 \sin^2 u\,,$$
$$L = -r\,, \qquad M = 0\,, \qquad N = -r \sin^2 u\,,$$

and the factor h is $-1/r$. The conclusion will now be stated.

THEOREM 7. *The minimal net coincides with the asymptotic net on a surface not a plane if, and only if, the surface is a sphere.*

EXERCISES

1. The curvilinear differential equation of the asymptotic curves on a surface of revolution (III 2 15) is

$$f''du^2 + uf'dv^2 = 0\,.$$

2. The curvilinear differential equation of the asymptotic curves, distinct from the generators, on a nondevelopable ruled surface (III 7 1) has the form of an *equation of Riccati*,

$$\frac{dt}{ds} = f + 2gt + ht^2\,, \tag{4 21}$$

where f, g, h are functions of s. Calculate f, g, h for a general ruled surface, for the ruled surface of the binormals of a curve, and for the ruled surface of the principal normals of a curve.

3. On the anchor ring (III 2 24) find the parabolic curve, the region where the asymptotic curves are real, and the region where they are imaginary.

4. The curvilinear differential equation of the asymptotic curves on a surface $z = f(x, y)$ is

$$r dx^2 + 2s dx dy + t dy^2 = 0\,, \tag{4 22}$$

where r, s, t are defined by

$$r = f_{xx}\,, \qquad s = f_{xy}\,, \qquad t = f_{yy}\,. \tag{4 23}$$

5. The asymptotic curves on a surface $F(x, y, z) = 0$ are represented by the equation

$$F_{xx}dx^2 + F_{yy}dy^2 + F_{zz}dz^2 + 2F_{yz}dydz + 2F_{zx}dzdx + 2F_{xy}dxdy = 0\,. \tag{4 24}$$

6. Prove that the explicit equation of a surface can be written in the form

$$(4\cdot 25)\quad \begin{cases} z = \frac{1}{2}(a_0x^2 + 2a_1xy + a_2y^2) + \frac{1}{6}(b_0x^3 + 3b_1x^2y \\ \qquad\qquad + 3b_2xy^2 + b_3y^3) + \cdots, \end{cases}$$

by taking the origin on the surface and the z-axis along the normal at the origin. Use this equation to prove that the tangent plane at a point of a surface intersects the surface in a curve with a *node* (double point) at the point, the nodal tangents being the asymptotic tangents. Prove from the equation (4 25) that the derivatives y', y'' along an asymptotic curve at the origin satisfy the equations

$$(4\cdot 26)\quad \begin{cases} 0 = a_0 + 2a_1y' + a_2y'^2 & \left(y' = \dfrac{dy}{dx}, \cdots\right), \\ 0 = b_0 + 3b_1y' + 3b_2y'^2 + b_3y'^3 + 2(a_1 + a_2y')y'', \end{cases}$$

while the same derivatives along the corresponding branch of the plane curve of intersection of the surface and the tangent plane at the origin satisfy

$$(4\cdot 27)\quad \begin{cases} 0 = a_0 + 2a_1y' + a_2y'^2, \\ 0 = b_0 + 3b_1y' + 3b_2y'^2 + b_3y'^3 + 3(a_1 + a_2y')y''. \end{cases}$$

7. Prove that the curvature $1/\rho$ of any curve $y = y(x)$ on the surface (4 25) at the origin is given by

$$(4\cdot 28)\quad \frac{1}{\rho^2} = \frac{1}{(1 + y'^2)^3}\,[(1 + y'^2)(a_0 + 2a_1y' + a_2y'^2)^2 + y''^2].$$

For an asymptotic curve this formula reduces to

$$\frac{1}{\rho} = \frac{y''}{(1 + y'^2)^{3/2}},$$

if proper choice of sign is made. If the x-axis is chosen to be tangent to an asymptotic curve, show that the curvature and torsion of this curve at the origin have the values

$$(4\cdot 29)\quad \frac{1}{\rho} = -\frac{b_0}{2a_1}, \qquad \frac{1}{\tau} = -a_1,$$

while the curvature of the corresponding branch of the curve of intersection of the surface and the tangent plane at the origin is given by

$$(4\cdot 30)\quad \frac{1}{\rho} = -\frac{b_0}{3a_1}.$$

Compare the two curvatures in (4·29) and (4·30).

8. Prove that the second fundamental form of a surface is $-\Sigma dx da$, just as the first fundamental form is Σdx^2.

9. The asymptotic curves on a quadric surface are the rectilinear generators of the surface.

10. Prove that the asymptotic curves on the screw surface (III 2 26) are the parametric curves and form an orthogonal net.

11. Prove that the asymptotic curves on the catenoid (III 2 24) form an orthogonal net.

12. The locus of the tangent lines of the curved asymptotics (4 21) at the points of a fixed generator of a ruled surface is ordinarily a hyperboloid of one sheet.

13. Calculate L, M, N for the surfaces whose parametric equations are given in the exercises of Section 2 in Chapter III.

14. The locus of an inflexion point on a meridian of a surface of revolution is the parabolic curve, or part of it, on the surface.

5. The fundamental differential equations. The fundamental differential equations in the metric differential theory of surfaces consist of five linear homogeneous partial differential equations which express the three second partial derivatives of the coordinates of a variable point on the surface, and the two first partial derivatives of the direction cosines of the normal of the surface at the point, linearly in terms of these cosines themselves and the two first partial derivatives of the coordinates of the point. Two of these equations are *the Weingarten differential equations* (4 17). The other three, called *the Gauss differential equations,* will now be deduced. If the equations (III 3 3) defining the first fundamental coefficients E, F, G be differentiated with respect to u and v, the derived equations can be written in the form

$$(5\cdot 1)\begin{cases} E_u = 2\Sigma x_u x_{uu}\,, & F_u = \Sigma x_v x_{uu} + \tfrac{1}{2}E_v\,, & G_u = 2\Sigma x_v x_{uv}\,, \\ E_v = 2\Sigma x_u x_{uv}\,, & F_v = \Sigma x_u x_{vv} + \tfrac{1}{2}G_u\,, & G_v = 2\Sigma x_v x_{vv}\,. \end{cases}$$

These equations and equations (4·8) can be grouped into three sets of three equations each, each set being written in a column as follows:

$$(5\cdot 2)\begin{cases} \Sigma x_u x_{uu} = \tfrac{1}{2}E_u\,, & \Sigma x_u x_{uv} = \tfrac{1}{2}E_v\,, & \Sigma x_u x_{vv} = F_v - \tfrac{1}{2}G_u\,, \\ \Sigma x_v x_{uu} = F_u - \tfrac{1}{2}E_v\,, & \Sigma x_v x_{uv} = \tfrac{1}{2}G_u\,, & \Sigma x_v x_{vv} = \tfrac{1}{2}G_v\,, \\ \Sigma a x_{uu} = L\,; & \Sigma a x_{uv} = M\,; & \Sigma a x_{vv} = N\,. \end{cases}$$

The first set can be solved for x_{uu}, y_{uu}, z_{uu}, since the determinant of the coefficients reduces to the function H, which is not zero. Similarly, the second set can be solved for x_{uv}, y_{uv}, z_{uv}, and the third set can be solved for x_{vv}, y_{vv}, z_{vv}. The result of the solution is that not only x, a but also y, b and z, c satisfy *the Gauss differential equations*, namely, the first three of the following equations, the last two being *the Weingarten differential equations*, which are re-written for convenience of reference,

$$(5\cdot3)\qquad \begin{cases} x_{uu} = \Gamma^1_{11}x_u + \Gamma^2_{11}x_v + La\,, \\ x_{uv} = \Gamma^1_{12}x_u + \Gamma^2_{12}x_v + Ma\,, \\ x_{vv} = \Gamma^1_{22}x_u + \Gamma^2_{22}x_v + Na\,, \\ a_u = \dfrac{1}{H^2}(FM - GL)x_u + \dfrac{1}{H^2}(FL - EM)x_v\,, \\ a_v = \dfrac{1}{H^2}(FN - GM)x_u + \dfrac{1}{H^2}(FM - EN)x_v\,, \end{cases}$$

where the coefficients Γ^i_{jk} $(i, j, k = 1, 2)$ are defined by the formulas

$$(5\cdot4)\qquad \begin{cases} \Gamma^1_{11} = \dfrac{1}{2H^2}(GE_u + FE_v - 2FF_u)\,, \\ \Gamma^1_{12} = \dfrac{1}{2H^2}(GE_v - FG_u)\,, \\ \Gamma^1_{22} = \dfrac{1}{2H^2}(-FG_v - GG_u + 2GF_v), \\ \Gamma^2_{11} = \dfrac{1}{2H^2}(-FE_u - EE_v + 2EF_u)\,, \\ \Gamma^2_{12} = \dfrac{1}{2H^2}(EG_u - FE_v)\,, \\ \Gamma^2_{22} = \dfrac{1}{2H^2}(EG_v + FG_u - 2FF_v)\,. \end{cases}$$

The coefficients Γ^i_{jk}, with the understanding that $\Gamma^i_{jk} = \Gamma^i_{kj}$, are sometimes called *the Christoffel three-index symbols of the second kind for the first fundamental form*, although the notation is not that of Christoffel. From the point of view of the theory of quadratic differential forms the functions Γ^i_{jk} are regarded* as the components of an invariant called *the fundamental affine connection* of the first fundamental form.

The thirteen coefficients of the fundamental differential equations (5·3) are not arbitrary functions of u, v but must satisfy certain conditions called *integrability conditions.* These are themselves partial differential equations and may be calculated in the following way. Since all the functions involved in equations (5·3) are analytic, the order of differentiation of x, and also of a, is surely immaterial. But two of the third derivatives of x, namely x_{uuv}, x_{uvv}, and one of the second derivatives of a, namely a_{uv}, can be calculated from equations (5·3) in two ways. These three pairs of derivatives must be equal, so that

$$(5\cdot 5)\qquad (x_{uu})_v = (x_{uv})_u\,, \qquad (x_{uv})_v = (x_{vv})_u\,, \qquad (a_u)_v = (a_v)_u\,.$$

After the right members of equations (5·3) are substituted in equations (5·5) and the differentiations are performed, the second derivatives of x and the first derivatives of a can be eliminated from the result by means of the equations (5·3). Thus, three equations of the form

$$(5\cdot 6)\qquad a_i x_u + b_i x_v + c_i a = 0 \qquad (i = 1, 2, 3)$$

are obtained. Since these equations are satisfied not only by x, a but also by y, b and z, c, and since the determinant (x_u, x_v, a) reduces to H and therefore is not zero, it follows that

$$(5\cdot 7)\qquad a_i = b_i = c_i = 0 \qquad (i = 1, 2, 3)\,.$$

These equations are the integrability conditions. They express relations between the six fundamental coefficients and their derivatives of order not higher than the second. Although it might appear that there were nine integrability conditions, it turns out that they are not all independent. In fact, actual calculation,† the details of which need not be reproduced here, shows that only three of the conditions (5·7) are independent. They are *the condition of Gauss,*

$$(5\cdot 8)\qquad \frac{LN - M^2}{H} = \left(\frac{H\Gamma^2_{11}}{E}\right)_v - \left(\frac{H\Gamma^2_{12}}{E}\right)_u \qquad (E \neq 0)\,,$$

* O. Veblen, *Invariants of Quadratic Differential Forms* (Cambridge Tract No. 24) (Cambridge: University Press, 1927), p. 35.

† Bianchi, *Lezioni di geometria differenziale* (3d ed., 1922), I, 171.

and the *two conditions of Codazzi*,

$$(5\cdot 9)\qquad \begin{cases} L_v - \Gamma^1_{12}L - \Gamma^2_{12}M = M_u - \Gamma^1_{11}M - \Gamma^2_{11}N\,, \\ N_u - \Gamma^2_{12}N - \Gamma^1_{12}M = M_v - \Gamma^2_{22}M - \Gamma^1_{22}L\,. \end{cases}$$

When the integrability conditions of a system of linear partial differential equations are satisfied, the system is said to be *completely integrable*. We have seen that, when the parametric equations (1·1) of a surface are given, the first fundamental coefficients E, F, G can be calculated by the formulas (III·3 3) and H by (III 3 6). The direction cosines a, b, c of the normal are given by (4·7), the second fundamental coefficients L, M, N by (4 8), and the components Γ^i_{jk} of the fundamental affine connection by (5·4). Furthermore, x, a and also y, b and z, c satisfy the fundamental differential equations (5 3) of Gauss and Weingarten, and consequently the six fundamental coefficients necessarily satisfy three integrability conditions, namely, the condition (5·8) of Gauss and the two conditions (5 9) of Codazzi. The system (5·3) as we have computed it, starting with the parametric equations (1·1), is completely integrable, since, as we have seen, the three integrability conditions of this system are satisfied.

The method that we have used in developing the theory of surfaces is not the only method that could be used. For example, it would be possible to start with a system of linear homogeneous partial differential equations of the form (5 3) with coefficients which are analytic functions of u, v subject only to satisfying the integrability conditions (5·7). It would be possible to prove, then, that such a completely integrable system of equations possesses three pairs of solutions x, a; y, b; z, c, such that the locus of the point $P(x, y, z)$ is a surface S and such that a, b, c are the direction cosines of the normal of S at P. The equations would determine the surface S uniquely except for its position in space.

A closely related method of defining a surface consists in starting with the six fundamental coefficients subject to the conditions of Gauss and Codazzi, and proving an existence theorem with regard to the surface. In fact, although the proof need not be reconstructed* here, the fundamental theorem for this method of studying surfaces may be stated as follows:

THEOREM 1. *When six analytic functions E, F, G, L, M, N of two independent variables u, v are given, subject to the inequality $EG - F^2 > 0$, and satisfying the condition of Gauss and the two conditions of*

* *Ibid.*, I, 177. Eisenhart, *Differential Geometry*, p. 157.

Codazzi, there exists a surface, uniquely determined except for its position in space, for which E, F, G and L, M, N are, respectively, the first and second fundamental coefficients.

EXERCISES

1. Necessary and sufficient conditions that the asymptotic curves on a nondevelopable surface be parametric is that the coordinates x, y, z of a variable point on the surface satisfy a completely integrable system of partial differential equations of the form

$$(5\ 10)\qquad \begin{cases} x_{uu} = a_1 x_u + b_1 x_v\,, \\ x_{vv} = a_2 x_u + b_2 x_v\,, \end{cases}$$

and do not satisfy any equation of the form

$$(5\ 11)\qquad x_{uv} = a_3 x_u + b_3 x_v\,.$$

2. Write out equations (5 4), (5 8), and (5·9) in the following special cases:

1. $F = 0$.
2. $M = 0$.
3. $F = 0,\ M = 0$.
4. $F = 0,\ E = G$.
5. $E = 1,\ F = 0$.
6. $L = 0,\ N = 0$.

3. Write out equations (5·4) for the developable, cone, cylinder, plane, and sphere, and verify equations (5 8) and (5·9) for these surfaces.

6. Conjugate nets. Conjugate nets of curves on a surface constitute a special class of nets which can be defined in various ways. The definition stated below is equally valid in both projective and metric geometry and has the further advantage of being available for the geometry of hyperspace. A conjugate net is not necessarily covariant to its sustaining surface, although there exist conjugate nets which are covariant to a given surface, one of which will be studied in the next section.

DEFINITION 1. *A net of curves on a surface is a conjugate net in case the tangents of the curves of one family of the net at the points of each fixed curve of the other family form a developable surface.*

Although the two families of the net do not play symmetrical roles in the definition as stated, it will turn out that the two families of a conjugate net are interchangeable. The two tangents of the two curves of a conjugate net at a point of a surface are called *conjugate tangents,* and the directions of the two curves at the point are called *conjugate directions.* The reason for the name *conjugate* will be found in Theorem 2 in Section 4 of Chapter V.

A condition satisfied by conjugate directions at a point of a surface will now be deduced. For this purpose let us consider two distinct one-parameter families of curves,

$$dv - f(u, v)du = 0, \qquad dv_1 - g(u, v)du_1 = 0, \tag{6·1}$$

on a surface S represented by equations (1 1). The subscripts 1 in the equation of the second family are used merely to distinguish notationally between the direction of this family and that of the first

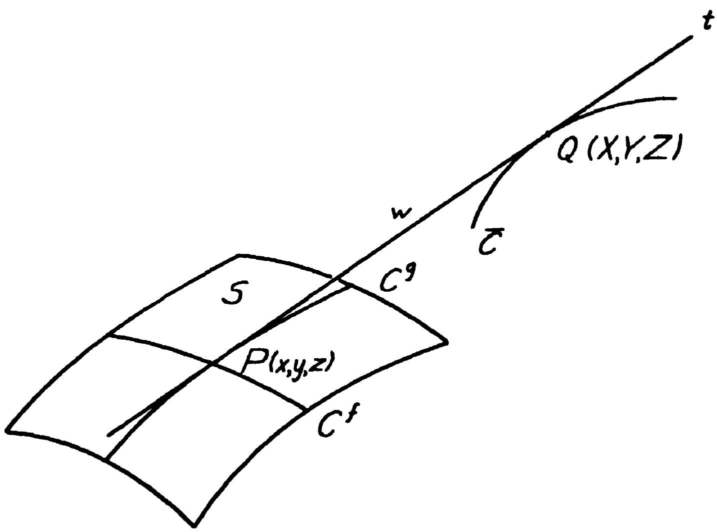

Fig. 14

family at a point of the surface. Exclusion of the v-curves by supposing that the equations (6·1) are solved for dv and dv_1 will not impair the generality of the final result. Referring to Figure 14, let us consider on the surface S the two curves C^f, C^g of the two families (6·1) through a point $P(x, y, z)$. The tangent line t of C^g at P has the parametric equations

$$\begin{cases} X = x + w(x_u + x_v g), \qquad Y = y + w(y_u + y_v g), \\ \qquad Z = z + w(z_u + z_v g), \end{cases} \tag{6·2}$$

in which w is a parameter which is *proportional to* the distance from the point P to a variable point $Q(X, Y, Z)$ on t. If the tangent t generates a developable surface as the point P varies along the curve C^f, and if Q is the corresponding focal point of t, then the locus of Q is a curve $\overline{C}$ to which the line t is tangent, or else in exceptional cases the point Q may be a fixed finite point or a fixed point at infinity. Let us postpone consideration of these exceptional cases. Then the total derivatives of X, Y, Z with respect to the independent variable u along the curve $\overline{C}$, the parameter w being now a function of u, v, must be respectively proportional to the total derivatives of x, y, z with respect to u along the curve C^g. The latter derivatives are given by

$$(6\cdot3)\qquad \frac{dx}{du} = x_u + x_v g\,, \qquad \frac{dy}{du} = y_u + y_v g\,, \qquad \frac{dz}{du} = z_u + z_v g\,.$$

Total differentiation of equations (6·2) along the curve $\overline{C}$ and use of the Gauss differential equations (5·3) lead to

$$(6\cdot4)\qquad \frac{dX}{du} = X_u + X_v f = l x_u + m x_v + n a$$

and similar formulas with Y, y, b and Z, z, c, in which the coefficients l, m, n are defined by the formulas

$$(6\cdot5)\qquad \begin{cases} l = 1 + w_u + w_v f + w[\Gamma^1_{11} + (f+g)\Gamma^1_{12} + fg\Gamma^1_{22}]\,, \\ m = f + g(w_u + w_v f) \\ \qquad\qquad + w[g_u + g_v f + \Gamma^2_{11} + (f+g)\Gamma^2_{12} + fg\Gamma^2_{22}]\,, \\ n = w[L + (f+g)M + fgN]\,. \end{cases}$$

The condition of proportionality of the derivatives is equivalent to the two conditions

$$(6\cdot6)\qquad m = lg\,, \qquad n = 0\,.$$

If the point Q were to coincide with the point P, then we should have $w = 0$, and therefore $f = g$, so that the two families (6·1) would not be distinct as supposed. Consequently we have $w \neq 0$, and so the condition $n = 0$ becomes

$$(6\cdot7)\qquad L + (f+g)M + fgN = 0\,,$$

or, after elimination of f, g by means of equations (6·1),

$$(6\cdot8)\qquad L\,du\,du_1 + M(du\,dv_1 + du_1\,dv) + N\,dv\,dv_1 = 0\,.$$

In the exceptional case in which the point Q is a fixed finite point we have, in place of the conditions (6·6), the conditions

$$l = m = n = 0,$$

since the derivatives of the coordinates X, Y, Z must be zero, and so equation (6·8) is obtained again. In the exceptional case in which the point Q is a fixed point at infinity, the direction cosines of the tangent t must be constants. Consequently we have

$$x_u + x_v g = c(E + 2Fg + Gg^2)^{1/2} \qquad (c = \text{const.})$$

with similar equations for y, z, and so after differentiation we again reach the same equation (6·8). Conversely, equation (6·8) is easily shown to be sufficient that the two directions dv/du, dv_1/du_1 be conjugate. The following theorem may now be stated.

THEOREM 1. *A necessary and sufficient condition that the directions dv/du and dv_1/du_1 of two curves intersecting at a point of a surface be conjugate directions is that they satisfy equation* (6·8).

The bilinear form in the left member of equation (6·8) is known as *the polar form* of the second fundamental form with respect to the variables du_1, dv_1. The symmetry of this equation in dv/du and dv_1/du_1 shows that the two families of a conjugate net are interchangeable. Theorem 1 can be used to prove the following theorem.

THEOREM 2. *A necessary and sufficient condition that a net* (1·2) *be a conjugate net is*

$$AN - 2BM + CL = 0. \qquad (6\cdot9)$$

The reasoning in the proof is similar to that used in demonstrating Theorem 3 of Section 3. If equation (6·8) is divided by $du\,du_1$, if dv/du, dv_1/du_1 are supposed to be the roots of equation (1·2), and if the elementary symmetric functions of the roots of a quadratic equation are properly employed, the proof is immediate. The next theorem is a corollary of Theorem 2.

THEOREM 3. *A necessary and sufficient condition that the parametric net on a surface be a conjugate net is $M = 0$.*

The theory of conjugate nets is related to the asymptotic curves and the second fundamental form in the same way as the theory of orthogonal nets is related to the minimal curves and the first fundamental form. Theorem 2 above and Theorem 7 of Section 2 imply the following theorem.

THEOREM 4. *A net of curves on a surface in ordinary space is a conjugate net if, and only if, the tangents of the net at each point of the surface separate harmonically the asymptotic tangents at the point.*

The first of the conditions (6·6) is equivalent to

$$(6\cdot 10)\quad \begin{cases} g - f = w\{g_u + g_v f + \Gamma_{11}^2 + (f+g)\Gamma_{12}^2 + fg\Gamma_{22}^2 \\ \qquad\qquad - g[\Gamma_{11}^1 + (f+g)\Gamma_{12}^1 + fg\Gamma_{22}^1]\}\,, \end{cases}$$

and determines that value of w which, when substituted in equations (6·2), yields the coordinates of the focal point Q of the line t. This focal point is called *the Laplace transformed point*, or *the ray point*, of the curve C^f at the point P.

If in the condition (6·8) for the conjugacy of two directions, hitherto supposed distinct, the two directions be allowed to coincide, so that the subscripts 1 may be erased, the asymptotic directions appear as self-conjugate directions. Each family of the asymptotic net on a surface may therefore be regarded as the limit of a conjugate net the two families of which have approached coincidence. In this limiting situation we have $f = g$ and $w = 0$, so that the focal point Q coincides with the point P.

The tangent plane at a point P of a surface S envelops a developable surface when P varies along a curve C^f on S. It will now be shown that this developable is precisely the developable generated by the tangent t which is conjugate to the tangent of C^f at P, by proving the following theorem.

THEOREM 5. *As the point of contact P of a tangent plane of a surface S varies along a curve C^f on S, the characteristic of the tangent plane is the tangent which is conjugate to the tangent of C^f at P.*

The theory of envelopes can be applied to the one-parameter family of tangent planes,

$$(6\cdot 11)\qquad \Sigma a(X - x) = 0\,,$$

whose parameter is the independent variable u along the curve C^f. Differentiation of equation (6·11) along C^f gives

$$(6\cdot 12)\qquad \Sigma(a_u + a_v f)(X - x) = 0\,,$$

in virtue of the identities (4·9). Equations (6·11) and (6·12) together represent the characteristic of the tangent plane (6·11), which therefore passes through the contact point $P(x, y, z)$. If a curve C^g passes through the point P tangent to this characteristic, then the coordinates X, Y, Z of a point on the tangent of C^v at P are given by equations of the form

$$(6\cdot 13) \qquad X = x + w(x_u + x_v g) ,$$

where w is a parameter, and these coordinates must satisfy equations (6·11) and (6·12). Substitution in (6·11) furnishes no new information, but from (6·12) we get

$$(6\cdot 14) \qquad \Sigma(a_u + a_v f)(x_u + x_v g) = 0 .$$

Multiplying out and making use of equations (4·11), we arrive at equation (6·7), thus completing the proof.

EXERCISES

1. Every net of curves in a plane is a conjugate net.

2. On a developable surface not a plane the generators and any other one-parameter family of curves form a conjugate net.

3. The meridians and parallels on a surface of revolution form a conjugate net.

4. The parametric curves on the conoid (III 2·25) form a conjugate net.

5. The parametric curves on the helicoid (III 2·28) form a conjugate net.

6. The curvilinear differential equation of the family of curves conjugate to the family represented by

$$mdu + ndv = 0 ,$$

where m, n are functions of u, v, is

$$(6\cdot 15) \qquad (Ln - Mm)du + (Mn - Nm)dv = 0 .$$

7. Under a transformation of parameters (1·7) the second fundamental coefficients L, M, N are cogredient to the first fundamental coefficients E, F, G, except possibly for sign. Account for the ambiguity of sign.

8. Build differential parameters with respect to the second fundamental form analogous to those of Exercises 1, 2, and 3 of Section 1 for the first fundamental form. Pursue the analogy between conjugacy and orthogonality as studied in Exercises 10–14 of Section 3, defining, in particular, *isothermal*

conjugacy and proving that necessary and sufficient conditions that the parametric net on a surface be isothermally conjugate are

$$(6\ 16)\qquad M = 0\,, \qquad \left(\log \frac{L}{N}\right)_{uv} = 0\,.$$

9. The meridians and parallels on a surface of revolution form an *isothermally conjugate* net.

10. A necessary and sufficient condition that the parametric net on a surface be a conjugate net is that the coordinates x, y, z of a variable point on the surface satisfy a *partial differential equation of Laplace* of the form

$$(6\ 17)\qquad x_{uv} = ax_u + bx_v\,,$$

where a, b are functions of u, v.

11. On any surface the family of plane sections made by planes of an axial pencil is conjugate to the family of curves of contact of cones circumscribing the surface and having their vertices on the axis of the pencil.

12. Defining *a surface of translation* to be the locus of a curve which is translated so that its various points trace congruent curves, show that the equations of a surface of translation can be written in the form

$$(6\ 18)\qquad x = U_1 + V_1\,, \qquad y = U_2 + V_2\,, \qquad z = U_3 + V_3\,,$$

where U_1, U_2, U_3 are functions of u alone and V_1, V_2, V_3 are functions of v alone. A surface of translation can be generated as a surface of translation in two ways. The parametric curves for the representation (6 18) form a conjugate net. A developable of tangents of one family at points of a curve of the other family is a cylinder. An elliptic paraboloid and a hyperbolic paraboloid are surfaces of translation.

13. As the point of contact P of a tangent plane of a surface S, not a developable, varies along an asymptotic curve C on S, the characteristic of the tangent plane is the tangent of C at P.

14. Deduce the condition (6 8) for conjugate directions from equation (III 7 27) by placing

$$x' = x_u + x_v f\,, \qquad l = x_u + x_v g\,,$$

$$l' = x_{uu} + x_{uv}(f + g) + x_{vv} fg + x_v(g_u + g_v f)\,.$$

Determine the focal point of the tangent l by means of equation (III 7 26), and discuss special cases.

7. The lines of curvature. The lines of curvature on a surface constitute a net of curves which is covariant to the surface. This net is both conjugate and orthogonal and will presently be shown to be the only orthogonal conjugate net on the surface. However, this property will not be used as a definition. Instead we prefer to define a line of curvature as follows:

DEFINITION 1. *A curve C on a surface S is a line of curvature in case the ruled surface of normals of S at points of C is a developable surface.*

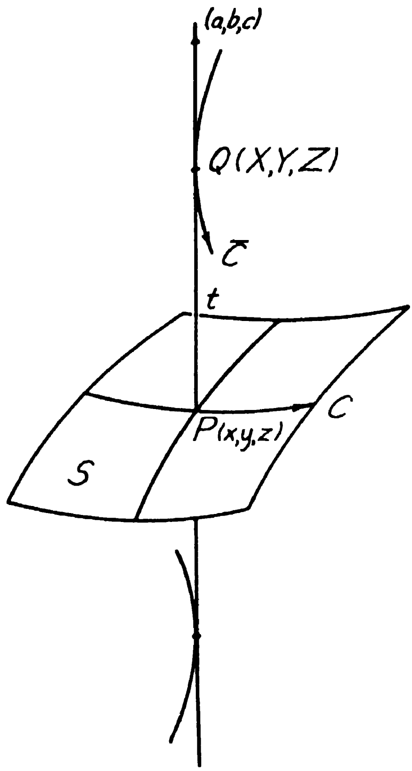

FIG. 15

The reason for the name *line of curvature* will be found in Theorem 1 in Section 2 of Chapter V. The curvilinear differential equation of the lines of curvature will now be deduced. Referring to Figure 15, let us consider a surface S represented by (1·1), a point $P(x, y, z)$ on S, and the normal of S at P. Let us also consider a curve C on S through P, whose curvilinear parametric equations are

$$(7\cdot 1) \qquad u = u(s), \qquad v = v(s),$$

in which s is arc length on C. The parametric equations of the normal of S at P are

$$(7\cdot 2) \qquad X = x + at\,, \qquad Y = y + bt\,, \qquad Z = z + ct\,,$$

in which the parameter t is the algebraic distance from the point P to a variable point $Q(X, Y, Z)$ on the normal. If the normal at the point P generates a developable surface as P varies along the curve C, and if Q is the corresponding focal point of the normal, then the locus of Q is ordinarily a curve $\overline{C}$ to which the normal is tangent. Therefore the total derivatives of X, Y, Z with respect to the arc length s of C, which is a parameter along $\overline{C}$, must be respectively proportional to the direction cosines a, b, c of the normal, t now being a function of u, v. Total differentiation of (7·2) and use of the Weingarten differential equations (4·17) lead to

$$(7\cdot 3) \qquad \begin{cases} X' = X_u u' + X_v v' \\ \quad\; = l x_u + m x_v + t' a \end{cases} \qquad \left(X' = \frac{dX}{ds}\,, \cdots\right)$$

and similar formulas for Y', Z', in which the coefficients l, m are defined by

$$(7\cdot 4) \qquad \begin{cases} l = u' + t\left[\dfrac{u'}{H^2}(FM - GL) + \dfrac{v'}{H^2}(FN - GM)\right], \\ m = v' + t\left[\dfrac{u'}{H^2}(FL - EM) + \dfrac{v'}{H^2}(FM - EN)\right]. \end{cases}$$

The condition of proportionality of X', Y', Z' and a, b, c is equivalent to the two conditions

$$l = 0\,, \qquad m = 0\,.$$

In the exceptional case in which the point Q remains fixed as the point P varies along the curve C the same two conditions, and also $t' = 0$, are obtained; in the exceptional case of parallel normals along the curve C, the expressions in brackets in equations (7·4) vanish; but the conclusions presently to be reached are valid even in these cases. After placing $l = 0$, $m = 0$ in equations (7·4), elimination of t leads to *the differential equation of the lines of curvature,*

$$(7\cdot 5) \qquad (EM - FL)du^2 + (EN - GL)du\,dv + (FN - GM)dv^2 = 0\,.$$

The form of equation (7·5) shows that the lines of curvature on a surface S ordinarily form a net, two lines of curvature passing through each point P of S. The normal of S at P therefore ordinarily has two focal points. To determine them, we eliminate the ratio v'/u' from equations (7·4) after placing $l = 0$, $m = 0$. The result is

$$(7\cdot 6)\quad (LN - M^2)t^2 - (EN - 2FM + GL)t + EG - F^2 = 0\,.$$

Solution of this equation gives the distances from the point P to the focal points of the normal corresponding to the two lines of curvature through the point P.

The quadratic form in the left member of the equation (7·5) of the lines of curvature on a surface is known as *the jacobian* of the two fundamental forms of the surface. It vanishes identically if $L = 0$, $M = 0$, $N = 0$; in this case the surface is a plane. It also vanishes identically if L, M, N are proportional to E, F, G but do not themselves all vanish; in this case the surface is a sphere. Examination of the three coefficients in the jacobian shows that they vanish only in these two cases. Thus the following theorem is proved.

THEOREM 1. *The lines of curvature are indeterminate on the plane and the sphere, and on no other surface.*

When the lines of curvature are under consideration, it is ordinarily understood that the sustaining surface is not a plane or a sphere. Points at which the coefficients of the jacobian vanish simultaneously while not vanishing identically are called *umbilical points* and will be avoided hereinafter. The discriminant of the jacobian is shown to be negative, when all the functions considered are real, by an easily verified identity, namely,

$$(7\cdot 7)\left\{\begin{aligned}&(EM - FL)(FN - GM) - \tfrac{1}{4}(EN - GL)^2\\ &\quad = -\frac{H^2}{E^2}(EM - FL)^2 - \left[\tfrac{1}{2}(EN - GL) - \frac{F}{E}(EM - FL)\right]^2.\end{aligned}\right.$$

An immediate conclusion may be drawn.

THEOREM 2. *The lines of curvature on a real surface are real.*

It is easy to verify that the harmonic invariant of the jacobian and the first fundamental form vanishes, and also that the harmonic invariant of the jacobian and the second fundamental form vanishes. Moreover, the jacobian can, without difficulty, be shown to be the only form, except possibly for a factor, whose harmonic invariant with the two fundamental forms does vanish. In this manner one proves the following theorem.

THEOREM 3. *The lines of curvature on a surface not a plane or a sphere constitute the only orthogonal conjugate net on the surface.*

Inspection of equation (7·5) shows that the lines of curvature are parametric in case

$$EM - LF = 0, \qquad GM - NF = 0, \qquad EN - GL \neq 0.$$

These two equations, in the presence of the inequality, are equivalent to $F = 0$, $M = 0$. Thus we reach the following conclusion, which can also be inferred from Theorem 3 of this section, Theorem 3 of Section 6, and Theorem 1 of Section 3.

THEOREM 4. *The lines of curvature on a surface not a plane or a sphere are parametric if, and only if,* $F = 0$, $M = 0$.

EXERCISES

1. The lines of curvature on a surface of revolution are the meridians and parallels. Describe the developables generated by the normals of the surface along these curves.

2. The lines of curvature on a developable consist of the generators and their orthogonal trajectories.

3. Calculate the differential equation of the lines of curvature for each of the surfaces whose equations are given in the exercises of Section 2 in Chapter III.

4. The equation of the lines of curvature on a surface $z = f(x, y)$ is

$$(7\cdot 8)\quad \begin{cases} [(1 + p^2)s - pqr]dx^2 + [(1 + p^2)t - (1 + q^2)r]dxdy \\ \qquad\qquad - [(1 + q^2)s - pqt]dy^2 = 0, \end{cases}$$

where we have placed

$$p = f_x, \qquad q = f_y, \qquad r = f_{xx}, \qquad s = f_{xy}, \qquad t = f_{yy}.$$

5. Prove by means of Theorem 4 in Section 7 of Chapter III that the differential equation of the lines of curvature on a surface $F(x, y, z) = 0$ is

$$(7\cdot 9)\qquad \begin{vmatrix} dx & dy & dz \\ F_x & F_y & F_z \\ dF_x & dF_y & dF_z \end{vmatrix} = 0.$$

6. Defining *an isothermic surface* to be a surface on which the lines of curvature form an isothermally orthogonal net, prove that a surface of revolution is isothermic.

7. At a point of a surface where the asymptotic tangents are real, the tangents of the lines of curvature bisect the angles between the asymptotic tagents.

8. For the hyperbolic paraboloid

$$x = \frac{a}{2}(u + v), \qquad y = \frac{b}{2}(u - v), \qquad z = \tfrac{1}{2}uv,$$

find the explicit equation of the surface, the curvilinear equation of the asymptotic curves, and the curvilinear equation of the lines of curvature. Show that this surface is isothermic.

9. Deduce the equation (7 5) of the lines of curvature from equation (III 7 27) by placing

$$l = a, \qquad m = b, \qquad n = c.$$

Determine the focal points of the normal by means of equation (III 7 26), and discuss special cases.

8. Geodesics. The *geodesic curves*, or simply *geodesics*, on a surface constitute a covariant two-parameter family of curves and may be defined in the following way.

DEFINITION 1. *A curve C on a surface S is a geodesic in case at each point on C the osculating plane of C contains the normal line of S.*

If a surface has a straight line on it, the normal of the surface at any point of the line determines with the line a plane which is an osculating plane of the line. This reasoning justifies the following assertion.

THEOREM 1. *Every straight line on a surface is a geodesic.*

The curvilinear differential equation of the geodesics on a surface (1·1) will now be deduced. Substituting the expressions for X, Y, Z from the equations (7 2) of the normal line into the equation (4·1) of the osculating plane at a point of a curve on the surface, we obtain

$$(8\cdot 1) \qquad (a, x', x'') = 0,$$

where the parentheses are used to indicate a determinant. Substituting the expressions for x', x'' from equations (4·2) into this equation, and making use of the Gauss differential equations (5 3) to eliminate the second derivatives of x, we find

$$(8\cdot 2)\left\{\begin{aligned}&(a,\ x_u u' + x_v v',\ [\Gamma^1_{11}x_u + \Gamma^2_{11}x_v]u'^2 + 2[\Gamma^1_{12}x_u + \Gamma^2_{12}x_v]u'v' \\ &\qquad + [\Gamma^1_{22}x_u + \Gamma^2_{22}x_v]v'^2 + x_u u'' + x_v v'') = 0.\end{aligned}\right.$$

By use of elementary properties of determinants the left member of this equation can be written as the product of a certain factor times the determinant

$$(a, x_u, x_v) ,$$

which is equal to H and therefore is not zero. Consequently the other factor must vanish. The result of setting this factor equal to zero is *the curvilinear differential equation of the geodesics,*

$$(8\cdot 3)\quad \begin{cases} u'v'' - u''v' = -\Gamma_{11}^{2}u'^{3} - (2\Gamma_{12}^{2} - \Gamma_{11}^{1})u'^{2}v' \\ \qquad\qquad + (2\Gamma_{12}^{1} - \Gamma_{22}^{2})u'v'^{2} + \Gamma_{22}^{1}v'^{3} , \end{cases}$$

in which the accent indicates differentiation with respect to any parameter along a geodesic.

Other convenient forms for the equation of the geodesics can be obtained by specializing the parameter in equation (8·3). If the parameter along a geodesic is supposed to be u, then $u' = 1$, $u'' = 0$, and so equation (8·3) reduces to

$$(8\cdot 4)\quad v'' = -\Gamma_{11}^{2} - (2\Gamma_{12}^{2} - \Gamma_{11}^{1})v' + (2\Gamma_{12}^{1} - \Gamma_{22}^{2})v'^{2} + \Gamma_{22}^{1}v'^{3} .$$

Again, if the parameter in equation (8·3) is arc length along a geodesic not a minimal curve, one has simultaneously with equation (8·3) also

$$(8\cdot 5)\qquad Eu'^{2} + 2Fu'v' + Gv'^{2} = 1 .$$

Differentiation of this equation results in

$$(8\cdot 6)\quad \begin{cases} 2(Eu' + Fv')u'' + 2(Fu' + Gv')v'' + E_u u'^{3} \\ \qquad + (E_v + 2F_u)u'^{2}v' + (G_u + 2F_v)u'v'^{2} + G_v v'^{3} = 0 . \end{cases}$$

Solution of equations (8·3) and (8·6) for u'', v'' gives *the curvilinear differential equations of the geodesics with arc length along a geodesic as parameter.*

$$(8\cdot 7)\qquad \begin{cases} u'' + \Gamma_{11}^{1}u'^{2} + 2\Gamma_{12}^{1}u'v' + \Gamma_{22}^{1}v'^{2} = 0 , \\ v'' + \Gamma_{11}^{2}u'^{2} + 2\Gamma_{12}^{2}u'v' + \Gamma_{22}^{2}v'^{2} = 0 . \end{cases}$$

Since the differential equation (8·4), for instance, is of the second order, it follows that *the geodesics on a surface constitute a two-parameter family of curves.* There is a unique geodesic through a given point in a given direction on a surface. There is also a unique geodesic through two given points not too far apart on a surface.

If the curves v = const. are geodesics, then $\Gamma^2_{11} = 0$. If, further, the curves u = const. are the orthogonal trajectories of these geodesics, then also $F = 0$. The formula in (5·4) for the symbol Γ^2_{11} shows, in this case, that $E_v = 0$; consequently E is a function of u alone. If a new parameter $\bar{u}$ is introduced by the transformation

$$(8\cdot 8) \qquad \bar{u} = \int_{u_0}^{u} E^{1/2}\, du\,,$$

the parameter v being unchanged, then the transformed coefficient $\bar{E}$ is shown to be unity by the second of equations (1·17). Thus the following statement is validated.

THEOREM 2. *If the u-curves are geodesics on a surface, and if the v-curves are their orthogonal trajectories, then by a suitable choice of parameter u along the u-curves, the first fundamental form of the surface can be reduced to*

$$(8\cdot 9) \qquad du^2 + Gdv^2\,.$$

The significance of this choice of the parameter u is made clear by remarking that

$$(8\cdot 10) \qquad ds^u = du\,.$$

Consequently, *the parameter u in the form* (8·9) *differs, at most, by a constant from the arc length measured from an arbitrarily chosen fixed point on a geodesic of the family.* Let us calculate the arc length s^u along a geodesic between two fixed orthogonal trajectories whose equations are $u = u_0$ and $u = u_1$. The result is (see Fig. 16)

$$(8\cdot 11) \qquad s^u = u_1 - u_0\,.$$

Since the arc length s^u is independent of v, it is the same for all the geodesics of the family now used for the u-curves. Hence the following conclusion can be drawn.

THEOREM 3. *The arc length measured along a geodesic belonging to a one-parameter family of geodesics, between two orthogonal trajectories of the family, is the same for all geodesics of the family.*

It is natural then to make the following definition of geodesic parallels.

DEFINITION 2. *The orthogonal trajectories of a one-parameter family of geodesics are geodesic parallels.*

When a curve C is given, there is a unique one-parameter family of geodesics crossing C at right angles. If a constant length is laid off from C along these geodesics, the locus of the end-points of these geodesic arcs is a curve geodesically parallel to C. Consequently, *a family of geodesic parallels is determined when one of them is given.*

One of the best-known properties of geodesics is their minimizing property, incorporated in the following theorem.

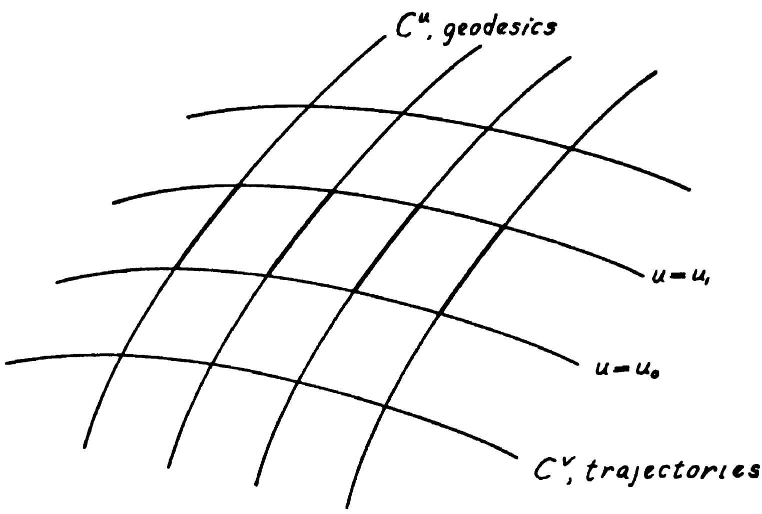

FIG. 16

THEOREM 4. *If the distance between two points not too far apart on a surface is measured along curves joining them and lying on the surface, the shortest distance between the points is measured along the geodesic joining them.*

The method of proof* may be suggested. The integral (III·3·8) for the length of a curve between two points P_0, P_1 on a surface can be written in the form

$$(8\cdot 12) \qquad s = \int_{u_0}^{u_1} f(u, v, v')du \qquad \left(v' = \frac{dv}{du}\right),$$

where the function f is given by the formula

$$(8\cdot 13) \qquad f(u, v, v') = (E + 2Fv' + Gv'^2)^{1/2}.$$

* Bolza, *Vorlesungen uber Variationsrechnung* (Leipzig: Teubner, 1909), pp. 209 and 228

Euler's equation for the extremals of the integral (8·12) is

$$f_{v'v'}v'' = f_v - f_{uv'} - f_{vv'}v' . \tag{8·14}$$

When this equation is calculated with the particular formula (8 13) for the function f, the extremals are found to be the geodesics (8·4), and every sufficiently short arc of such an extremal minimizes the integral (8·12).

EXERCISES

1. The geodesics on a plane are straight lines, and on a sphere are great circles.

2. Any curve is a geodesic on its rectifying developable.

3. A cylindrical helix is a geodesic on the cylinder whose generators it crosses at a constant angle.

4. The meridians on a surface of revolution are geodesics.

5. Every geodesic on a developable surface has the surface for rectifying developable.

6. The curvilinear differential equation of the geodesics on the sphere (III 2·11) is

$$v'' = -2 \cot u \, v' - \sin u \cos u \, v'^3 .$$

Show that this is the equation of all great circles on the sphere by differentiating the equation

$$A \sin u \cos v + B \sin u \sin v + C \cos u = 0$$

of these circles twice with respect to u and eliminating the arbitrary constants A, B, C.

7. Minimal curves on a surface are geodesics.

8. Calculate the differential equation of the geodesics on a developable and on a revolute.

9. Defining *a geodesic circle* to be the locus of a point on a surface whose distance from a fixed point P of the surface, measured along the one-parameter family of geodesics through P, is constant, show that any ordinary circle is a geodesic circle on a sphere. The parallels on a revolute are geodesic circles.

10. All concentric geodesic circles on a surface are geodesic parallels.

CHAPTER V

CURVATURE

1. Radius of normal curvature. The differential, or infinitesimal, geometry of a surface is the theory of those properties of the surface that depend only on a neighborhood of a general one of its points. In the last two chapters considerable information about such properties has already been brought to light. By way of illustration it is sufficient to mention the tangent plane, the normal line, the minimal tangents, the asymptotic tangents, and the tangents of the lines of curvature at a point of a surface.

The primary purpose of this chapter is to study still more intensively the differential geometry of a surface. Particular attention will be paid to the curvatures at a point of a surface and to the curvature and torsion of curves lying on the surface and passing through the point.

We begin by investigating the curvature of all curves that lie on a surface and pass through a point P with the same tangent line at P. For this purpose, let us consider a surface S whose parametric equations are

$$(1\cdot 1) \qquad x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v),$$

and on S a curve C, not a minimal curve, whose curvilinear parametric equations are

$$(1\cdot 2) \qquad u = u(s), \qquad v = v(s),$$

the parameter s being arc length on C. At a point $P(x, y, z)$ on C the direction cosines l, m, n of the principal normal of C are given by

$$(1\cdot 3) \quad l = \rho x'' = \rho(x_{uu}u'^2 + 2x_{uv}u'v' + x_{vv}v'^2 + x_u u'' + x_v v'')$$

and two similar formulas, in which the accent denotes differentiation with respect to s, and ρ is the radius of curvature of C at P. The normal plane of the curve C certainly contains the normal line of the surface S at the point P, whose direction cosines a, b, c are given by the formulas (IV·4·7). A positive sense of rotation in the normal plane of the curve C is assigned by the following convention.

In the normal plane at a point P of a curve C on a surface the positive sense of rotation about P is, by agreement, the sense in which the positive half of the principal normal can be rotated through the angle $\pi/2$ into the positive half of the binormal.

The angle between the principal normal and the surface normal is defined as follows and is illustrated by Figure 17, in which the abbreviation *P.N.* is used for the principal normal, *S.N.* for the surface normal, and *Bin.* for the binormal, and the positive sense on the tangent is *from* the reader.

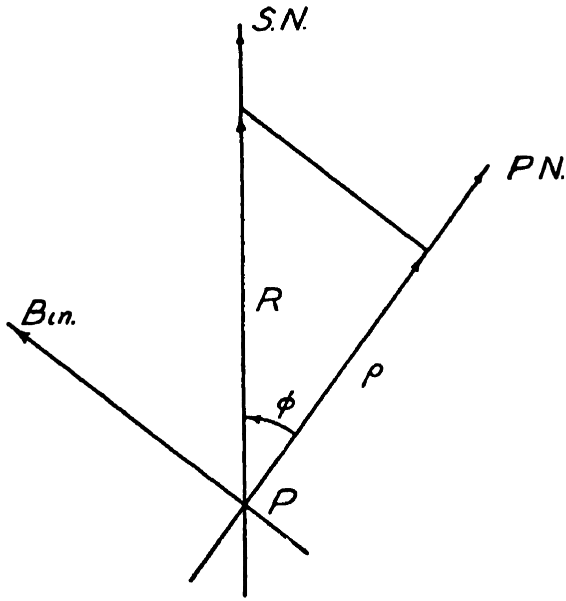

FIG. 17

DEFINITION 1. *The angle φ between the principal normal, at a point P of a curve C on a surface S, and the normal of S at P is the smallest angle from the positive half of the principal normal in the positive sense of rotation to the positive half of the surface normal.*

According to this definition, the angle φ satisfies the condition

$$0 \lesseqgtr \varphi < 2\pi .$$

Making use of equations (1·3), (IV·4 8), and (IV 4 9), we find

$$(1\cdot 4) \qquad \cos\varphi = \Sigma al = \rho\Sigma ax'' = \rho(Lu'^2 + 2Mu'v' + Nv'^2) ;$$

and then, using the formula (III·3·2) for ds^2, we obtain

$$(1\ 5) \qquad \frac{\cos\varphi}{\rho} = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} .$$

This is a very noteworthy result with many implications, some of which will be investigated immediately.

The right member of equation (1·5) is obviously the ratio of the second fundamental form to the first fundamental form of a surface S. It depends on the six fundamental coefficients and the ratio dv/du. The formulas for the six coefficients are calculated from the equations (1·1) of the surface S in terms of the variables u, v. Their values at the particular point P under consideration are found by substituting in them the curvilinear coordinates of P. The direction dv/du is calculated from the curvilinear equation of a curve C at P. But all other curves on S through P tangent to the tangent line of C at P have the same direction dv/du. Therefore the right member of (1·5) is the same for all these curves, of which there are infinitely many, and the left member of (1·5) must be the same for all of them. Thus the following theorem is proved.

THEOREM 1. *Every curve on a given surface S, through a given point P on S, and tangent to a given line at P has at P the same value of the ratio* $\cos \varphi/\rho$, *where φ is the angle between the principal normal of the curve and the normal of the surface and ρ is the radius of curvature of the curve at P.*

Let us suppose that the right member of equation (1·5) is not zero. With this hypothesis we are excluding the possibility that the surface S under consideration might be a *plane*, for which L, M, N vanish identically, and are excluding *flat points* on a surface not a plane, for which L, M, N vanish simultaneously while not vanishing identically. Moreover, we are excluding the possibility that the curve C might be an *asymptotic curve* on S, along which the second fundamental form vanishes, and are excluding *curves on S which are tangent to an asymptotic curve* at the point P under consideration.

The right member of equation (1·5), being supposed not to vanish, is either positive or negative when all the numbers under consideration are real. If the second form is positive for a particular curve C at a point P on a surface S, then this form and $\cos \varphi$ are positive for every curve on S through P tangent to C, and for all these curves the angle φ must satisfy one or the other of the two conditions

$$0 \leqq \varphi < \frac{\pi}{2}, \qquad \frac{3\pi}{2} < \varphi < 2\pi .$$

If the second form is negative for a particular curve C on S at P, then this form and $\cos \varphi$ are negative for every curve on S through P tangent to C, and for all these curves the angle φ must satisfy the condition

$$\frac{\pi}{2} < \varphi < \frac{3\pi}{2}.$$

All the planes containing a line t which lies in the tangent plane at a point P of a surface S and passes through P form an axial pencil and cut S in a one-parameter family of plane curves. All these curves pass through P tangent to the line t at P. The osculating plane at P of any particular curve C of this family is, of course, the plane that contains C. There are infinitely many curves on the surface S, through the point P, tangent to the line t at P, and having for osculating plane at P the plane of the particular curve C under consideration. All these curves have at the point P not only the same tangent line and osculating plane but also the same osculating circle, the same center of curvature, the same radius of curvature ρ, the same principal normal (both in position and in sense), and the same angle φ. The plane curve C may be regarded as a typical and comparatively easily visualizable curve of this class of curves having the plane of C for osculating plane.

Among all the plane curves of section of the surface S made by the planes of the axial pencil with the line t as axis, there is one of especial interest, namely, the curve cut on the surface S by that plane through t which also contains the normal of S at the point P. This curve is called *a curve of normal section* according to the following definition.

DEFINITION 2. *The curve of normal section C_n for a direction dv/du at a point P of a surface S is the curve of intersection of S and the plane that contains the normal of S and cuts the tangent plane of S in the tangent line in the direction dv/du at P.*

If the second fundamental form is positive in the direction of a curve of normal section C_n at a point P of a surface S, then $\varphi_n = 0$ and hence $\cos \varphi_n = 1$, the subscript n indicating quantities belonging to C_n. In this case the curvature $1/\rho_n$ of the curve C_n at P is precisely equal to the right member of equation (1·5), and the center of curvature of C_n lies on the positive half of the normal of S at P. If, on the contrary, the second fundamental form is negative in the direction of the curve C_n, then $\varphi_n = \pi$ and hence $\cos \varphi_n = -1$. In this case the curvature $1/\rho_n$ of C_n at P is equal to the negative of the right member of (1·5), and the center of curvature of C_n lies on the negative half of the normal of S at P. A *radius of curvature of a curve* is always positive. But it is very convenient to introduce a *radius of* *normal curvature of a surface,* which may be either positive or negative in accordance with the following definition.

DEFINITION 3. *The radius of normal curvature R for a direction dv/du at a point P of a surface S is the radius of curvature ρ_n of the associated curve of normal section C_n if the second fundamental form is positive, and is $-\rho_n$ if the second form is negative.*

It follows directly from the definition that the formula

$$(1\cdot 6)\qquad \frac{1}{R} = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

is valid in all cases. The reciprocal $1/R$ of the radius of normal curvature is called the *normal curvature* for the direction dv/du at the point P of the surface S. The osculating circle at P of the associated curve of normal section C_n is called the *circle of normal curvature,* and its center is called the *center of normal curvature,* for the direction dv/du at the point P of the surface S. The center of normal curvature lies on the positive half of the normal at the distance R from P when $R > 0$, and lies on the negative half of the normal at the distance $-R$ from P when $R < 0$.

Comparison of equations (1·5) and (1·6) shows that

$$(1\cdot 7)\qquad R\cos\varphi = \rho\,.$$

This equation expresses a very simple relation between the radius of normal curvature R, for a given direction at a point P of a surface S, and the radius of curvature ρ of any other plane curve of section of S in the same direction at P. This relation is illustrated by Figure 17 and holds also for the radius of curvature ρ of any other curve on S through P in the given direction and having for osculating plane at P the plane producing the section. This relation is stated by the so-called *Theorem of Meusnier* as follows:

THEOREM 2. *If the center of normal curvature, for a given direction at a point P of a surface S, is projected onto the osculating plane of any other curve C on S through P in the same direction, the projection is the center of curvature of the curve C.*

The next theorem is equivalent to the Theorem of Meusnier.

THEOREM 3. *The osculating circles of all the curves on a surface S, through a point P and tangent to the same line t at P, are the circles which the osculating planes of the curves cut on the sphere of radius $|R|$ with its center at the center of normal curvature of S at P for the direction of t.*

EXERCISES

1. Prove by the formula (1 5) that every straight line on a surface is an asymptotic curve.

2. Prove by (1 5) that if a curve, not an asymptotic curve, on a surface has an inflexion at a point P, then the curve is tangent to an asymptotic curve at P.

3. Prove by (1 5) that $\varphi = \pi/2$ or else $\varphi = 3\pi/2$ for a nonrectilinear asymptotic curve which is free of inflexion points.

4. The radius of normal curvature R is numerically greater than the radius of curvature ρ of any other plane curve of section of a surface in the same direction at the same point.

5. For the direction of an asymptotic tangent at a point of a surface the normal curvature $1/R$ is zero.

2. Principal normal curvatures. The normal curvature $1/R$ at a fixed point P of a given surface S is a function of the direction at P for which the normal curvature is defined. When the plane making the associated normal section turns around the normal line of S at P, the normal curvature ordinarily varies and has a maximum and a minimum value. These extreme values are called *the principal normal curvatures* at the point P of the surface S and will now be studied.

To investigate the normal curvature $1/R$ at a point P of a surface S as a function of its direction, it is convenient to re-write equation (1·6) in the form

$$(2\cdot 1)\qquad \frac{1}{R} = \frac{L + 2Mh + Nh^2}{E + 2Fh + Gh^2} \qquad \left(h = \frac{dv}{du}\,,\ du \neq 0\right),$$

and then to investigate $1/R$ as a function of h. The assumption $du \neq 0$ will not impair the generality of the final results. Two special cases should be disposed of at once. If the surface S is a *plane*, then L, M, N vanish identically, and the normal curvature is zero for every direction at the point P, as it is also at a *flat point* on any surface, where L, M, N happen to vanish simultaneously without vanishing identically. If the surface S is a *sphere*, then L, M, N are respectively proportional to E, F, G, and the normal curvature is constant for every direction at the point P, as it is also at an *umbilical point* on any surface, where the proportionality may happen to hold without its holding at all points on the surface. These two cases will be excluded from the discussion hereinafter unless the contrary is indicated.

If the normal curvature $1/R$, regarded as a function of its direction h, has a maximum or a minimum, then the corresponding value of h must satisfy the equation

$$(2\cdot2)\qquad \frac{d}{dh}\left(\frac{1}{R}\right) = 0\,.$$

Performing the indicated differentiation shows that the critical values of h must be the roots of the equation

$$(2\cdot3)\left\{\begin{aligned}&(E + 2Fh + Gh^2)(M + Nh)\\ &\qquad\qquad - (L + 2Mh + Nh^2)(F + Gh) = 0\,,\end{aligned}\right.$$

which simplifies into

$$(2\cdot4)\qquad EM - FL + (EN - GL)h + (FN - GM)h^2 = 0\,.$$

When h is replaced by dv/du, this equation reduces to the differential equation (IV·7·5) of the lines of curvature whose discriminant, as written in Section 7 of Chapter IV, was shown by the identity (IV·7·7) to be negative, all the functions under consideration being real. Therefore the two roots of equation (2·4) are real and distinct. That one of these roots actually makes the normal curvature $1/R$ a maximum and the other root makes it a minimum can be shown by examining the sign of the second derivative,

$$\frac{d^2}{dh^2}\left(\frac{1}{R}\right)\cdot$$

Upon actual calculation it turns out that the sign of this derivative is the same as the sign of the linear function

$$(2\cdot5)\qquad EN - GL + 2(FN - GM)h\,,$$

provided that h is a root of equation (2·4). The demonstration may be completed by solving (2·4) by the quadratic formula and substituting the roots, one at a time, in the linear function (2·5). One of these roots does give this function a negative value and therefore makes $1/R$ a maximum; the other root gives it a positive value and hence makes $1/R$ a minimum. The outstanding results of the foregoing argument will now be summarized.

THEOREM 1. *At an ordinary point P on a surface not a plane or a* *sphere there is just one direction for which the normal curvature is a maximum and just one for which it is a minimum. These two directions are the directions of the lines of curvature at P.*

This theorem gives the reason for the name *lines of curvature.* The maximum and minimum normal curvatures at a point of a surface are called *principal normal curvatures.* The associated centers and circles of normal curvature are called, respectively, *principal centers of normal curvature* and *principal circles of normal curvature.* The associated radii of normal curvature, i.e., the radii of normal curvature for the directions of the lines of curvature, are also a maximum and a minimum and are called *principal radii of normal curvature* in the following definition.

DEFINITION 1. *The principal radii R_1, R_2 of normal curvature at an ordinary point P of a surface S not a plane or a sphere are the maximum and minimum radii of normal curvature of S at P, without notational discrimination as to which is maximum and which is minimum.*

The principal radii R_1, R_2 can be proved to be the roots of the equation

$$(2\cdot 6)\quad (LN - M^2)R^2 - (EN - 2FM + GL)R + EG - F^2 = 0\,,$$

by eliminating h directly from equations (2 1) and (2·4). The same conclusion can be reached by using equation (2·3) instead of (2·4) and employing the following rather indirect method of elimination. Equations (2·1) and (2·3), taken together, are equivalent to

$$\frac{1}{R} = \frac{M + Nh}{F + Gh} = \frac{L + Mh}{E + Fh}$$

and hence to

$$(2\cdot 7)\qquad \begin{cases} E + Fh - R(L + Mh) = 0\,, \\ F + Gh - R(M + Nh) = 0\,. \end{cases}$$

Elimination of h from these equations is easily effected and produces equation (2·6).

Equations (2·7) have the virtue of associating a unique principal radius R_1 or R_2 with a given direction h of a line of curvature, and vice versa. Since equation (2·6) is equivalent to the equation (IV·7·6) determining the focal points of the normal, the following inference can be made.

THEOREM 2. *At a point P of a surface S the principal radii of normal* *curvature are the algebraic distances from P to the focal points of the normal regarded as a generator of the two developable surfaces of normals intersecting S in the lines of curvature through P.*

The focal points of the normal are therefore the principal centers of curvature at a point P of a surface S. When the point P varies over the surface S, the locus of each of the focal points of the normal is a surface called an *evolute surface*, or also a *surface of centers*, of S. The normals of S are tangent to both of these surfaces, and the two points of contact of a normal may be called *corresponding* points. Each evolute surface is the locus of the edges of regression of the developable surfaces of normals of S which intersect S in one family of the lines of curvature.

EXERCISES

1. The principal radii of normal curvature of a surface $z = f(x, y)$ are the roots of the equation

$$(2.8) \qquad (rt - s^2)R^2 - [(1 + q^2)r - 2pqs + (1 + p^2)t]HR + H^4 = 0,$$

where

$$(2.9) \qquad \begin{cases} p = f_x, \quad q = f_y, \quad r = f_{xx}, \quad s = f_{xy}, \quad t = f_{yy}, \\ H^2 = 1 + p^2 + q^2. \end{cases}$$

2. The principal radii of normal curvature of a surface $F(x, y, z) = 0$ are the roots of the equation

$$(2.10) \qquad \begin{vmatrix} F_{xx} - \frac{H}{R} & F_{xy} & F_{xz} & F_x \\ F_{xy} & F_{yy} - \frac{H}{R} & F_{yz} & F_y \\ F_{xz} & F_{yz} & F_{zz} - \frac{H}{R} & F_z \\ F_x & F_y & F_z & 0 \end{vmatrix} = 0,$$

where

$$(2.11) \qquad H^2 = F_x^2 + F_y^2 + F_z^2.$$

3. Find the principal radii of normal curvature of the surface

$$2z = 6x^2 - 5xy - 6y^2$$

at the origin.

4. Prove that for the surface of revolution of a parabola about its directrix one of the principal radii of normal curvature is always double the other.

5. When the lines of curvature are parametric on a surface, the principal normal curvatures $1/R_1$, $1/R_2$ at a point of the surface are given by the formulas

$$\frac{1}{R_1} = \frac{L}{E}, \qquad \frac{1}{R_2} = \frac{N}{G}, \tag{2 12}$$

the first corresponding to the u-curve and the second to the v-curve.

6. The principal normal curvatures at a point of a surface of revolution are given by the formulas

$$\frac{1}{R_1} = \frac{f''}{(1+f'^2)^{3/2}}, \qquad \frac{1}{R_2} = \frac{f'}{u(1+f'^2)^{1/2}}. \tag{2 13}$$

The radius R_1 is equal in length to the radius of curvature of a meridian, while the radius R_2 is equal in length to the segment of the normal between the surface and the axis of revolution.

7. When the lines of curvature are parametric on a surface, the Weingarten differential equations (IV 4 17) become the so-called *equations of Rodrigues*,

$$a_u = -\frac{1}{R_1} x_u, \qquad a_v = -\frac{1}{R_2} x_v, \tag{2 14}$$

and the conditions of Codazzi (IV 5 9) become

$$\begin{cases} 2\left(\dfrac{1}{R_1}\right)_v = \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)(\log E)_v, \\ 2\left(\dfrac{1}{R_2}\right)_u = \left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)(\log G)_u. \end{cases} \tag{2 15}$$

8. The principal normal curvatures at a point of a developable surface (III 6 1) are given by the formulas

$$\frac{1}{R_1} = \pm\frac{\rho}{t\tau}, \qquad \frac{1}{R_2} = 0, \tag{2 16}$$

the positive sign being used on the sheet of the developable generated by the positive half of the tangent of the edge of regression, and the negative on the other sheet.

9. On each evolute surface of a surface S the curves corresponding to the lines of curvature on S form a conjugate net.

10. On each evolute surface of a surface S the edges of regression (of developables composed of normals of S) of which it is the locus are geodesic curves.

3. Gaussian curvature and mean curvature. In studying the shape of a surface in the neighborhood of one of its points, frequent use is made of two curvatures called, respectively, *the Gaussian curvature* and *the mean curvature*, which are defined as follows:

DEFINITION 1. *The Gaussian curvature K at a point P of a surface S is the product of the principal normal curvatures of S at P, and the mean curvature k of S at P is half the sum of the principal normal curvatures.*

It should be remarked that some geometers define the mean curvature to be exactly the sum of the principal normal curvatures. From Definition 1 and equation (2 6) the following formulas for K, k are obtained:

$$(3\ 1)\qquad \begin{cases} K = \dfrac{1}{R_1R_2} = \dfrac{1}{H^2}(LN - M^2)\,, \\[2ex] k = \dfrac{1}{2}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right) = \dfrac{1}{2H^2}(EN - 2FM + GL)\,. \end{cases}$$

At first glance one would think that both curvatures K, k depend on all six fundamental coefficients. But *the equation of Gauss* (IV 5 8) shows that the Gaussian curvature K is expressible in terms of the first fundamental coefficients E, F, G and their first and second partial derivatives. The precise formula thus expressing K is that in the following theorem.

THEOREM 1. *The Gaussian curvature K at a point of a surface is expressed in terms of the coefficients E, F, G and their first and second derivatives by the formula*

$$(3\ 2)\qquad K = \frac{1}{H}\left[\left(\frac{H\Gamma_{11}^{2}}{E}\right)_v - \left(\frac{H\Gamma_{12}^{2}}{E}\right)_u\right] \qquad (E \neq 0)\,,$$

in which H is defined by (III 3 6) *and* Γ_{11}^2, Γ_{12}^2 *by* (IV 5 4).

The Gaussian curvature K vanishes at every point of a plane, since $L = 0$, $M = 0$, $N = 0$ for a plane. Moreover, K vanishes at every point of a developable surface, since $LN - M^2 = 0$ for a developable. Conversely, if K vanishes at every point of a surface, the surface is a developable, by Theorems 3 and 5 in Section 4 of Chapter IV. Thus the following theorem is proved.

THEOREM 2. *The Gaussian curvature is zero at every point of a surface if, and only if, the surface is a developable.*

Points on a surface at which the Gaussian curvature vanishes without vanishing at every point of the surface are *parabolic points*. Let us now consider a surface S which is not a developable and examine the form of S in the neighborhood of a point P which is not a parabolic point. If $K > 0$ at the point P, then the principal radii R_1, R_2 of normal curvature of the surface S at the point P have the same sign. Since all the other radii of normal curvature of S at P lie between R_1 and R_2, these also have the same sign. Therefore all the centers of normal curvature are on one side of the tangent plane of S at P, and so the surface in the neighborhood of the point P lies entirely on one side of its tangent plane at P. In this case P is called an *elliptic point*. *In a region on a surface in which all the points are elliptic, the asymptotic curves are imaginary.*

If $K < 0$ at a point P of a surface S, then the principal radii R_1, R_2 of S at P have opposite signs. Therefore some of the centers of normal curvature of S at P lie on one side of the tangent plane, and some on the other. It follows that in the neighborhood of P part of the surface S is on one side of its tangent plane at P and part on the other. In this case P is called a *hyperbolic point*. *In a region on a surface in which all the points are hyperbolic, the asymptotic curves are real.*

In consequence of the continuity of a polynomial the value of the second fundamental form changes sign only by passing through zero. Hence *the asymptotic directions at a hyperbolic point separate the directions for which the normal curvature $1/R$ is positive from the directions for which $1/R$ is negative.*

Just as developable surfaces can be characterized by the condition $K = 0$, so an important class of surfaces called *minimal surfaces* can be defined by the condition $k = 0$.

DEFINITION 2. *A minimal surface is a surface at every point of which the mean curvature is zero.*

A minimal curve on a real surface being imaginary, one might suspect that a minimal surface must be imaginary; but such is not the case. The relation between minimal curves and minimal surfaces is stated in Exercise 7, below. The reason why minimal surfaces were given this name is that they came into prominence in connection with a problem of minimizing the area of a surface, called *the problem of Plateau*, which has received the attention of many distinguished mathematicians.* This problem of the calculus of variations can be formulated sufficiently precisely for our purposes as follows:

* For a historical sketch see Darboux, *Leçons* (3d ed.), I, 319.

Given a simply closed curve C in space and a connected surface S bounded by C, to determine S so that the inclosed area of S shall be a minimum.

Just as it was found that if a curve minimizes the arc length integral (III 3 8) then the curve is a geodesic (IV 8 4), so it can be shown* that if a surface minimizes the area integral (IV 3·11) then it must be such that its mean curvature vanishes, i.e., must be a minimal surface. It should be noted that the vanishing of the mean curvature is only a *necessary* condition for a minimum area; there is no guaranty without further evidence that if $k = 0$ then the area is actually a minimum.

Inspection of the formulas (3 1) for k makes the truth of the following statement evident.

THEOREM 3. *A minimal surface is characterized by any one of the three equivalent conditions*

$$(3\ 3)\qquad k = 0\,, \qquad \frac{1}{R_1} + \frac{1}{R_2} = 0\,, \qquad EN - 2FM + GL = 0\,.$$

A plane should be classed among the minimal surfaces. For a minimal surface not a plane the second of the conditions (3 3) gives $R_1 + R_2 = 0$, and so implies the following two theorems.

THEOREM 4. *At each point P of a minimal surface S not a plane the centers of principal normal curvature lie on opposite sides of S at equal distances from P.*

THEOREM 5. *The Gaussian curvature is negative at every point of a minimal surface not a plane.*

The third of the conditions (3 3) implies the following theorem.

THEOREM 6. *On a minimal surface not a plane the asymptotic curves form an orthogonal net, and the minimal curves form a conjugate net.*

Developable surfaces, for which $K = 0$, and minimal surfaces, for which $k = 0$, are two special types of surfaces of a much larger class, called *Weingarten surfaces*, which are defined as follows:

DEFINITION 3. *A Weingarten surface is a surface for which there exists a relation between the two principal normal curvatures, of the form*

$$(3\ 4)\qquad f\left(\frac{1}{R_1},\ \frac{1}{R_2}\right) = 0\,,$$

where f is an arbitrary function.

* Bolza, *Vorlesungen uber Variationsrechnung* (Leipzig and Berlin, 1910), p. 667.

Other surfaces of Weingarten are surfaces of constant Gaussian curvature, surfaces of constant mean curvature, revolutes, and helicoids.

EXERCISES

1. The curvatures K, k for a surface $z = f(x, y)$ are given by the formulas

$$(3\ 5)\qquad K = \frac{rt - s^2}{(1 + p^2 + q^2)^2}, \qquad k = \frac{(1 + q^2)r - 2pqs + (1 + p^2)t}{2(1 + p^2 + q^2)^{3/2}}.$$

2. The Gaussian curvature K is positive at every point of an ellipsoid and at every point of a hyperboloid of two sheets; is negative at every point of a hyperboloid of one sheet and at every point of a hyperbolic paraboloid; but is positive at some points, zero at some points, and negative at some points on an anchor ring.

3. At a point on a nondevelopable ruled surface (III 7 1) the Gaussian curvature K is given by the formula

$$(3\ 6)\qquad K = -\frac{q^2 - p^2 \sin^2\theta}{(\sin^2\theta + 2qt + p^2t^2)^2}$$

and is therefore negative at every finite point of the surface (see Chap III, Sec. 7, Ex. 3). Prove that K has the same value for two points on a generator g equally distant from the central point of g and that the value of K at the central point is algebraically less than the value of K at any other point of g, the minimum value of K being

$$(3\ 7)\qquad -\frac{p^4}{q^2 - p^2 \sin^2\theta}.$$

The limit of the Gaussian curvature K at a point P of a generator g, as P approaches infinity along g, is zero.

4. Calculate the Gaussian curvature K at a point of the ruled surface of binormals of a curve (see Chap. III, Sec. 7, Ex. 11), and show that the minimum value of K for all points on a generator is $-1/\tau^2$.

5. Calculate K for the ruled surface of principal normals of a curve (see Chap. III, Sec. 7, Ex. 12), and show that the minimum value of K for all points on a generator is

$$(3\ 8)\qquad -\frac{\left(\dfrac{1}{\rho^2} + \dfrac{1}{\tau^2}\right)^2}{\dfrac{1}{\tau^2}}.$$

6. The Gaussian curvature of a helicoid is constant along a helix.

7. If the minimal curves on a real minimal surface are parametric, then

$$E = 0, \quad F \neq 0, \quad G = 0, \quad M = 0,$$

and consequently

$$\Sigma x_u x_{uv} = 0, \quad \Sigma x_v x_{uv} = 0, \quad \Sigma a x_{uv} = 0,$$

whence

$$x_{uv} = 0, \quad y_{uv} = 0, \quad z_{uv} = 0,$$

so that

$$(3\ 9) \qquad x = U_1 + V_1, \quad y = U_2 + V_2, \quad z = U_3 + V_3.$$

Therefore a real minimal surface is a surface of translation of a minimal curve along a conjugate imaginary minimal curve (see Chap. IV, Sec. 6, Ex. 12).

8. If a surface of revolution not a plane is a minimal surface, then the surface is a catenoid.

9. A screw surface is a minimal surface (see Chap. IV, Sec. 4, Ex. 10).

10. A necessary and sufficient condition that the asymptotic curves on the two evolute surfaces of a surface S correspond is that S be a Weingarten surface.

4. Equation of Euler: indicatrix of Dupin. *The equation of Euler* is a modified form of the equation (1 6) for the normal curvature $1/R$ defined for a direction at a point of a surface. The *indicatrix of Dupin* is a certain conic in the tangent plane at a point of a surface, which serves to depict the shape of the surface in the neighborhood of the point.

In order to deduce *the equation of Euler*, let us consider a surface S and suppose that the lines of curvature on S are parametric, so that $F = 0$, $M = 0$. Then equation (1 6) becomes

$$(4\ 1) \qquad \frac{1}{R} = \frac{L du^2 + N dv^2}{E du^2 + G dv^2}.$$

On the tangent line in which a plane of normal section intersects the tangent plane at a point P of S, the positive sense is that which agrees with the positive sense on the curve of normal section. Let θ be the smallest angle from the positive half of the u-tangent to the positive half of this line, in the positive sense of rotation in the tangent plane as prescribed in Section 3 of Chapter IV. Then formulas (IV·3·6) give

$$(4\cdot 2) \qquad \cos\theta = A\,\frac{du}{ds}, \qquad \sin\theta = C\,\frac{dv}{ds},$$

where A, C are defined in (III·3·4). By means of these formulas and equation (III·3·2), equation (4·1) becomes

$$(4\cdot 3) \qquad \frac{1}{R} = \frac{L}{E}\cos^2\theta + \frac{N}{G}\sin^2\theta .$$

Finally, equations (2·12) can be used to reduce equation (4·3) to

$$(4\cdot 4) \qquad \frac{1}{R} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2} .$$

DEFINITION 1. *Equation* (4·4) *is called the equation of Euler.*

The equation of Euler can be used to deduce the equation of *the indicatrix of Dupin* at a point P of a surface S not a plane. There are three cases to be considered, according as the Gaussian curvature K is positive, zero, or negative, i.e., according as the point P is elliptic, parabolic, or hyperbolic.

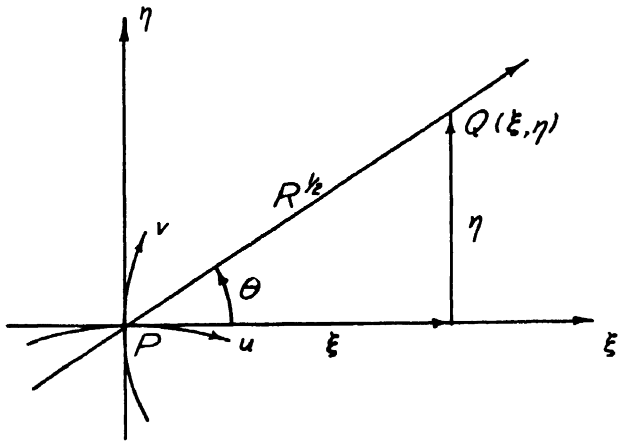

FIG. 18

In the first case ($K > 0$) the principal radii R_1, R_2 are both positive or both negative. If they are both negative, they can be made to become positive by reversing the positive sense on the normal of the surface S at the point P, and this change can be effected by merely changing the sign of the parameter u. Consequently, R_1, R_2 may be supposed to be both positive, and then R is positive for all directions at P. In the tangent plane of S at P let us choose P as origin of a cartesian coordinate system with the ξ-axis along the u-tangent and with the η-axis along the v-tangent, positive senses agreeing on coincident lines. Then on the positive half of the tangent of the curve of normal section making the angle θ with the ξ-axis (see Fig. 18) lay off a segment of length $R^{1/2}$ to locate a point Q. The coordinates ξ, η of Q are given by the formulas

$$(4 \cdot 5) \qquad \xi = R^{1/2} \cos \theta\,, \qquad \eta = R^{1/2} \sin \theta\,.$$

Solving for $\cos \theta$, $\sin \theta$ and substituting in the equation of Euler (4·4), we obtain the equation of the locus of the point Q, namely,

$$(4 \cdot 6) \qquad \frac{\xi^2}{R_1} + \frac{\eta^2}{R_2} = 1\,.$$

The locus is an ellipse.

DEFINITION 2. *The ellipse represented by equation* (4·6), *at a point P where $K > 0$ on a surface S, is called the indicatrix of Dupin of S at P.*

A reason for the name *elliptic point* is now obvious. Passing to the second case ($K = 0$), we observe that in this case one of the principal normal curvatures is zero. Let us suppose that $1/R_1 = 0$. Then $1/R_2 \neq 0$ for a surface not a plane; and R_2, if not already positive, can be made to become positive by reversing the positive sense on the normal. The limiting form of the ellipse (4·6) in this case is two parallel straight lines,

$$(4 \cdot 7) \qquad \eta = \pm R_2^{1/2}\,.$$

DEFINITION 3. *The two parallel straight lines represented by equations* (4·7), *at a point P where $K = 0$ on a surface S not a plane, is called the indicatrix of Dupin of S at P.*

A reason for the name *parabolic point* is now at hand, although at a parabolic point the indicatrix is the degenerate parabola consisting of two parallel straight lines and is not a nondegenerate parabola. In the third case ($K < 0$) the two principal radii R_1, R_2 differ in sign, and R_1, if not already positive, can be made to become positive by changing the positive sense on the normal. On the positive half of the tangent of the curve of normal section making the angle θ with the ξ-axis let us lay off a segment of length $R^{1/2}$ to locate a point Q when $R > 0$, and a segment of length $(-R)^{1/2}$ to locate Q when $R < 0$.

The coordinates ξ, η of Q in the two cases are given by the respective pairs of equations

$$(4\cdot 8)\qquad \begin{cases} \xi = R^{1/2}\cos\theta, & \eta = R^{1/2}\sin\theta\,; \\ \xi = (-R)^{1/2}\cos\theta, & \eta = (-R)^{1/2}\sin\theta\,. \end{cases}$$

Solving the pair of equations in each line for $\cos\theta$, $\sin\theta$ and substituting in the equation of Euler (4·4), we obtain the equations of the locus of Q in the two cases, namely,

$$(4\cdot 9)\qquad \frac{\xi^2}{R_1} - \frac{\eta^2}{(-R_2)} = \pm 1\,.$$

The locus is two conjugate hyperbolas.

DEFINITION 4. *The two conjugate hyperbolas represented by equations* (4·9), *at a point P where $K < 0$ on a surface S, are called the indicatrix of Dupin of S at P.*

A reason for the name *hyperbolic point* is now obvious. Moreover, reasons for the names *asymptotic tangents* and *conjugate tangents* are contained in the next two theorems.

THEOREM 1. *At a hyperbolic point on a surface the asymptotic tangents of the surface are the asymptotes of the indicatrix of Dupin.*

THEOREM 2. *At an elliptic or hyperbolic point of a surface conjugate tangents lie along conjugate diameters of the indicatrix of Dupin.*

The proof of Theorem 1 is simple. The equation of Euler (4·4) shows that the slopes of the asymptotic tangents, i.e., the values of $\tan\theta$ when $1/R = 0$, are given by

$$(4\cdot 10)\qquad \tan^2\theta = -\frac{R_2}{R_1}\,.$$

But the equation of the asymptotes of the hyperbolas (4·9), namely,

$$(4\cdot 11)\qquad \eta^2 = -\frac{R_2}{R_1}\,\xi^2\,,$$

shows that the slopes of the asymptotes are precisely the slopes of the asymptotic tangents, whence the theorem follows. Incidentally, equation (4·10) shows that the tangents of the lines of curvature bisect the angles between the asymptotic tangents.

For the proof of the second theorem, it may be observed that the condition (IV·6·8) for the conjugacy of two tangents in the directions

dv/du, dv_1/du_1 can be advantageously modified when $F = 0$, $M = 0$. In fact, by means of formulas (2·12) and (4·2), and formulas (4·2) with du_1, dv_1, θ_1 in place of du, dv, θ, respectively, the condition for conjugacy can be reduced to

$$\tan\theta \tan\theta_1 = -\frac{R_2}{R_1}. \tag{4·12}$$

This is precisely the condition satisfied by the slopes of conjugate diameters either of the ellipse (4·6) or of the hyperbolas (4·9), whence the theorem follows.

EXERCISES

1. At a point of a surface the sum of the normal curvatures for any two orthogonal directions is constant.

2. At a point of a surface the sum of the radii of normal curvature for any two conjugate directions is constant.

3. Prove that the mean curvature k at a point of a surface is the average of all the normal curvatures $1/R$ with respect to the inclination angle θ, in other words, verify the formula

$$k = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{R}\,d\theta. \tag{4·13}$$

4. Prove by means of (4·12) that the angle $\theta_1 - \theta$ between two conjugate tangents at a point of a surface is given by the formula

$$\tan(\theta_1 - \theta) = \frac{R_2\cot\theta + R_1\tan\theta}{R_2 - R_1}. \tag{4·14}$$

5. Prove by (4·14) that at an elliptic point of a surface there is a real pair of conjugate tangents such that the angle between them is a minimum. The slopes of these conjugate tangents are given by the formula

$$\tan^2\theta = \frac{R_2}{R_1}. \tag{4·15}$$

Hence the tangents of the lines of curvature bisect the angles between these tangents. Calculate the minimum angle between conjugate tangents.

6. Calculate the angle between the asymptotic tangents at a hyperbolic point of a surface.

7. Show that equations (4·2) are equivalent to

$$\cos\theta = \frac{ds^u}{ds}, \qquad \sin\theta = \frac{ds^v}{ds}.$$

Hence show that

$$ds^2 = ds^{u2} + ds^{v2},$$

the lines of curvature being parametric.

5. A local coordinate system. When the lines of curvature on a surface are parametric so that $F = 0$, $M = 0$, a very convenient *local coordinate system* at a point of the surface is that for which the ξ-axis is along the u-tangent, the η-axis is along the v-tangent, and the ζ-axis is along the normal of the surface at the point, positive senses agreeing on coincident lines.

The *equations of transformation*, between the general coordinates X, Y, Z of any point Q and the local coordinates ξ, η, ζ of Q referred to the local coordinate system at a point $P(x, y, z)$ of a surface S, can be read off from the accompanying table, which is analogous to the table (I·7·3) and in which A, C are defined by placing

$$A = E^{1/2}, \qquad C = G^{1/2}, \tag{5 1}$$

the positive square roots being taken.

(5 2)

	$X-x$	$Y-y$	$Z-z$
ξ	x_u/A	y_u/A	z_u/A
η	x_v/C	y_v/C	z_v/C
ζ	a	b	c

Power-series expansions for the local coordinates ξ, η, ζ of a point Q on the surface S and near the point P can be found in the following way. Each of the general coordinates X, Y, Z of Q can be expanded into a Taylor's series of the form

$$X = x + x_u\Delta u + x_v\Delta v + \tfrac{1}{2}(x_{uu}\Delta u^2 + 2x_{uv}\Delta u\Delta v + x_{vv}\Delta v^2) + \cdots,$$

in which the increments Δu, Δv correspond to displacement on the surface S from the point P to the point Q. By means of the fundamental differential equations (IV·5 3), the equations obtained therefrom by differentiation, and the conditions of Gauss and Codazzi, the second and higher derivatives of x can be expressed uniquely as linear combinations of x_u, x_v, a. Therefore $X - x$ can be expressed in the form

$$(5 \cdot 3) \qquad X - x = \xi \frac{x_u}{A} + \eta \frac{x_v}{C} + \zeta a ,$$

in which the coefficients ξ, η, ζ are the local coordinates of the point Q on the surface S and are analytically represented by the series

$$(5 \cdot 4) \begin{cases} \xi = A[\Delta u + \frac{1}{2}(\Gamma^1_{11}\Delta u^2 + 2\Gamma^1_{12}\Delta u\Delta v + \Gamma^1_{22}\Delta v^2) + \cdots] , \\ \eta = C[\Delta v + \frac{1}{2}(\Gamma^2_{11}\Delta u^2 + 2\Gamma^2_{12}\Delta u\Delta v + \Gamma^2_{22}\Delta v^2) + \cdots] , \\ \zeta = \frac{1}{2}(L\Delta u^2 + N\Delta v^2) + \frac{1}{6}[(L_u + \Gamma^1_{11}L)\Delta u^3 + 3\Gamma^1_{12}L\Delta u^2\Delta v \\ \qquad + 3\Gamma^2_{12}N\Delta u\Delta v^2 + (N_v + \Gamma^2_{22}N)\Delta v^3] + \cdots . \end{cases}$$

There are, of course, formulas similar to (5 3) for $Y - y$ and $Z - z$, but the series (5·4) for ξ, η, ζ are the same for each of the differences $X - x$, $Y - y$, $Z - z$. The result may now be stated.

Theorem 1. *The equations* (5·4) *are the parametric equations of a surface S referred to the local coordinate system at a point of S, the parameters being Δu, Δv.*

An expanded form of the explicit equation of the surface S can be deduced by eliminating Δu, Δv from equations (5 4). The elimination can be accomplished by supposing that ζ is a power series in ξ, η with undetermined coefficients and then demanding that the series (5 4) satisfy this equation identically in Δu, Δv as far as terms of any desired degree. The result to terms of the third degree can be reduced, by the aid of equations (2·12) and (2·15), to

$$(5 \cdot 5) \quad \zeta = \frac{1}{2}\left(\frac{\xi^2}{R_1} + \frac{\eta^2}{R_2}\right) + \frac{1}{6}(b_0\xi^3 + 3b_1\xi^2\eta + 3b_2\xi\eta^2 + b_3\eta^3) + \cdots ,$$

where the coefficients $b_0, \ldots, b_3$ are defined by the formulas

$$(5 \cdot 6) \begin{cases} b_0 = \dfrac{1}{A}\left(\dfrac{1}{R_1}\right)_u , & b_1 = \dfrac{1}{C}\left(\dfrac{1}{R_1}\right)_v , \\ b_3 = \dfrac{1}{C}\left(\dfrac{1}{R_2}\right)_v , & b_2 = \dfrac{1}{A}\left(\dfrac{1}{R_2}\right)_u . \end{cases}$$

By way of illustrating the use of equation (5 5), let us drop all terms after those of the second degree, obtaining thus an approximation to the equation of the surface. If the point P of the surface S that is now the origin is an elliptic point, and if R_1, R_2 are both positive, the equations of the curve in which a plane parallel to the tangent plane, on the positive side of the tangent plane and at an infinitesimal distance e from it, cuts the surface S are found to be approximately

$$(5\cdot 7) \qquad \zeta = e\,, \qquad \frac{\xi^2}{2eR_1} + \frac{\eta^2}{2eR_2} = 1\,.$$

The section is therefore approximately an ellipse which is similar to the indicatrix of Dupin (4·6). If the point P is a hyperbolic point, the section of the surface S by a plane parallel, and very near, to the tangent plane of S at P is approximately a hyperbola, which is similar to one or the other of the two conjugate hyperbolas composing the indicatrix of Dupin (4·9), according as the section is on one side or the other of the tangent plane.

EXERCISES

1. In the local coordinate system at a point P of a surface S the equation of any sphere tangent to S at P has the form

$$(5\cdot 8) \qquad \xi^2 + \eta^2 + \zeta^2 - 2n\zeta = 0\,,$$

where n is a parameter. Such a sphere intersects S in a curve with a double point at P whose double-point tangents are represented by the equations

$$(5\cdot 9) \qquad \zeta = 0\,, \qquad \left(1 - \frac{n}{R_1}\right)\xi^2 + \left(1 - \frac{n}{R_2}\right)\eta^2 = 0\,.$$

If these tangents coincide at all, they coincide with the tangent of one or the other of the lines of curvature at P, and in each case the center of the sphere is at the corresponding principal center of normal curvature.

2. The locus of the circles of normal curvature at a point of a surface is the algebraic surface of the fourth order whose equation is

$$(5\cdot 10) \qquad (\xi^2 + \eta^2 + \zeta^2)\left(\frac{\xi^2}{R_1} + \frac{\eta^2}{R_2}\right) = 2\zeta(\xi^2 + \eta^2)\,.$$

3. The locus of the centers of curvature, at a point P of a surface S, of all curves lying on S and passing through P is the algebraic surface of the fourth order whose equation is

$$(5\cdot 11) \qquad (\xi^2 + \eta^2 + \zeta^2)\left(\frac{\xi^2}{R_2} + \frac{\eta^2}{R_1}\right) = \zeta(\xi^2 + \eta^2)\,.$$

4. If the x-axis and y-axis for the expansion (IV 4 25) are the tangents of the lines of curvature at the origin, which is now supposed not to be an umbilical point, then $a_1 = 0$, $a_0 \neq a_2$. Use equation (2 8) to obtain the expansions

$$(5\ 12)\qquad \begin{cases} \dfrac{1}{R_1} = a_0 + b_0 x + b_1 y + \cdots, \\ \dfrac{1}{R_2} = a_2 + b_2 x + b_3 y + \cdots. \end{cases}$$

Hence show that the expansion (IV 4 25) becomes

$$(5\ 13)\qquad \begin{cases} z = \dfrac{1}{2}\left(\dfrac{x^2}{R_1} + \dfrac{y^2}{R_2}\right) + \dfrac{1}{6}\left[\left(\dfrac{1}{R_1}\right)_x x^3 + 3\left(\dfrac{1}{R_1}\right)_y x^2 y \right. \\ \qquad\qquad \left. + 3\left(\dfrac{1}{R_2}\right)_x x y^2 + \left(\dfrac{1}{R_2}\right)_y y^3\right] + \cdots. \end{cases}$$

5. A tangent line at an ordinary point P of a surface S is an asymptotic tangent if, and only if, it intersects S in three coincident points at P.

6. Every plane, except the tangent plane, through an asymptotic tangent at an ordinary point P of a surface S intersects S in a curve with an inflexion at P.

6. Geodesic curvature. *The geodesic curvature* at a point of a curve on a surface is, except perhaps for sign, the curvature of the orthogonal projection of the curve onto the tangent plane of the surface at the point. Before defining the sign of the geodesic curvature, it is convenient to introduce *the tangential normal* at a point of a curve on a surface.

DEFINITION 1. *The tangential normal at a point P of a curve C on a surface S is the normal line of C that lies in the tangent plane of S at P.*

Since the positive sense on the tangent at a point of a curve and the positive sense on the normal at a point of a surface have already been agreed upon, a positive sense on the tangential normal may be determined by the following convention.

At a point P of a curve C on a surface S the positive sense on the tangential normal is, by agreement, such that the tangent of C, the tangential normal, and the normal of S form a left-handed trihedron.

The direction cosines of the tangential normal at a point $P(x, y, z)$ of a curve C on a surface S are easily shown to be

$$(6 \cdot 1)\qquad bz' - cy', \qquad cx' - az', \qquad ay' - bx',$$

where a, b, c are the direction cosines of the normal of S at P, and the accent denotes differentiation with respect to arc length on C. The angle between the principal normal and the tangential normal is defined as follows and is illustrated by Figure 19, in which the abbreviation $T.N.$ is used for the tangential normal, and the positive sense on the tangent is *from* the reader.

DEFINITION 2. *The angle ψ between the principal normal and the tangential normal at a point P of a curve C on a surface is the smallest angle from the positive half of the principal normal in the positive sense of rotation to the positive half of the tangential normal.*

According to this definition, the angle ψ satisfies the condition

$$0 \leqq \psi < 2\pi .$$

Moreover, the angle ψ is connected with the angle φ (see Sec. 1, Def. 1) by one or the other of the two relations

$$\psi = \varphi - \frac{\pi}{2}, \qquad \psi = \varphi + \frac{3\pi}{2}, \tag{6.2}$$

the latter holding only when $0 \leqq \varphi < \pi/2$. In both cases elementary trigonometry gives

$$\cos \psi = \sin \varphi . \tag{6.3}$$

Preparatory to formulating a precise definition of the geodesic curvature at a point P of a curve C on a surface S, let us project the curve C orthogonally into a curve C_1 on the tangent plane at the point P of the surface S, by means of a cylinder S_1 whose generators are parallel to the normal of S at P. The projected curve C_1 and the original curve C have the same tangent line at P, and the principal normal of C_1 coincides with the tangential normal of C at P except that their positive senses may or may not agree, i.e., the center of curvature of C_1 may lie on the positive half of the tangential normal or else on the negative half. Moreover, the tangential normal of the curve C is the normal line of the cylinder S_1 at the point P, and the curve C_1 is the curve of normal section of the cylinder S_1 in the direction of C at P.

Equation (1.7) is equivalent to

$$\frac{\cos \varphi}{\rho} = \frac{1}{R}, \tag{6.4}$$

where φ is the angle between the principal normal of C and the normal of S, while ρ is the radius of curvature of C and R is the radius of normal curvature of S for the direction of C at P. The equation analogous to (6·4), when the curve C is thought of as a curve on the cylinder S_1, is

$$\frac{\cos \psi}{\rho} = \frac{1}{r}, \tag{6·5}$$

where r is the radius of normal curvature of the cylinder S_1 for the direction of the curve C at the point P. *Geodesic curvature* can now be defined.

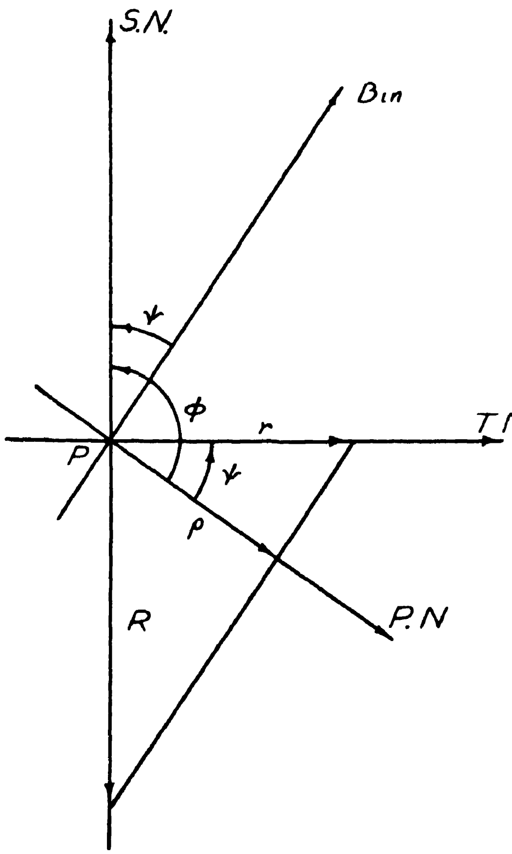

FIG. 19

DEFINITION 3. *The geodesic curvature at a point P of a curve C on a surface S is the normal curvature $1/r$, at P and for the direction of C, of the cylinder projecting C orthogonally onto the tangent plane of S at P.*

An equivalent characterization of the geodesic curvature is contained in the following theorem.

THEOREM 1. *The geodesic curvature $1/r$ at a point P of a curve C on a surface S is the curvature of the orthogonal projection C_1 of C onto the tangent plane of S at P if the center of curvature of C_1 lies on the positive half of the tangential normal of C at P, and is the negative of the curvature of C_1 if the center of curvature of C_1 lies on the negative half of the tangential normal.*

It is natural to call r the *radius of geodesic curvature* of the curve C at the point P. The center of curvature of the curve C_1 is called *the center of geodesic curvature* of C at P. Equations (6·3) and (6·5) imply

$$\frac{\sin \varphi}{\rho} = \frac{1}{r}. \tag{6·6}$$

Squaring and adding equations (6·4) and (6·6), we obtain

$$\frac{1}{\rho^2} = \frac{1}{R^2} + \frac{1}{r^2}. \tag{6·7}$$

The geometrical relation among the radii ρ, R, r is described by the following theorem and is illustrated by Figure 19.

THEOREM 2. *If at the center of curvature corresponding to a point P of a curve C on a surface S a line is drawn in the normal plane and perpendicular to the principal normal of C at P, this line meets the normal of S in the center of normal curvature of S for the direction of C at P and meets the tangential normal in the corresponding center of geodesic curvature of C.*

A formula for the geodesic curvature at a point of a curve on a surface will now be calculated. Equation (6·5), the formulas (6·1) for the direction cosines of the tangential normal, and the well-known formulas for the direction cosines of the principal normal at a point of a curve give us

$$\frac{1}{r} = \Sigma x''(bz' - cy'), \tag{6·8}$$

the summation being for cyclical permutations of x, y, z and of a, b, c. Evaluating the total first derivatives y', z' and rearranging, we obtain

$$\frac{1}{r} = \Sigma x''[(bz_u - cy_u)u' + (bz_v - cy_v)v'], \tag{6·9}$$

and then, by the aid of the identities (IV·4·13),

$$\frac{1}{r} = \frac{1}{H} \Sigma x''[(Ex_v - Fx_u)u' + (Fx_v - Gx_u)v']. \tag{6·10}$$

Evaluating the total second derivative x'' and making use of the Gauss differential equations (IV 5 3) to eliminate x_{uu}, x_{uv}, x_{vv}, we arrive at the desired *formula for the geodesic curvature* $1/r$ *at a point of a curve on a surface,*

$$(6\ 11)\quad \begin{cases} \dfrac{1}{r} = H[u'v'' - u''v' + \Gamma^2_{11}u'^3 + (2\Gamma^2_{12} - \Gamma^1_{11})u'^2v' \\ \qquad\qquad - (2\Gamma^1_{12} - \Gamma^2_{22})u'v'^2 - \Gamma^1_{22}v'^3], \end{cases}$$

in which *the accent denotes differentiation with respect to the arc length of the curve,* measured from some arbitrarily chosen fixed point thereon, and the symbols Γ^i_{jk} are defined by the formulas (IV·5 4). A glance at the equation (IV 8 3) of the geodesic curves on a surface now suffices to substantiate the following statement.

THEOREM 3. *If the geodesic curvature is zero at every point of a curve on a surface, the curve is a geodesic curve.*

Let the radii of geodesic curvature of the u-curve and the v-curve at a point of a surface be denoted by r_1, r_2, respectively. Then the formula (6 11) and the expressions (III 3 7) for the elements of arc of the parametric curves yield the formulas

$$(6\ 12)\qquad \frac{1}{r_1} = \frac{H\Gamma^2_{11}}{E^{3/2}}, \qquad \frac{1}{r_2} = -\frac{H\Gamma^1_{22}}{G^{3/2}}.$$

If, further, the parametric curves form an orthogonal net, these formulas specialize into

$$(6\ 13)\qquad \frac{1}{r_1} = -\frac{E_v}{2EG^{1/2}}, \qquad \frac{1}{r_2} = +\frac{G_u}{2GE^{1/2}}.$$

Let us consider on a surface S a closed curve C which does not cross itself. Such a curve is called *simply closed.* If the curve C can be continuously shrunk over its interior to a point, the interior is called *a simply connected region.* For example, a circle on a sphere is simply closed and divides the sphere into two simply connected regions. *Total curvature* of a simply connected region will now be defined.

DEFINITION 4. *The total curvature of a simply connected region is the integral*

$$(6\ 14)\qquad \iint KH\,du\,dv$$

extended over the region.

A very powerful formula called *the Gauss-Bonnet integral formula* will now be introduced without proof:*

$$\int \frac{ds}{r} + \int\int KH du dv = \int d\theta . \tag{6·15}$$

Here $1/r$ is the geodesic curvature of a simply closed curve C bounding a simply connected region of a surface; s is arc length along C; and θ is the angle between the tangent at a variable point P of C and the u-tangent at P, given by equations (IV 3 6). The simple integrals are to be extended around the boundary curve C, and the double integral is to be evaluated over the interior region. Some applications of this formula will be found in Exercises 8, 9, and 10 below; and it will now be applied to prove the following theorem concerning a *geodesic triangle*, i.e., a triangle bounded by three geodesic arcs on a surface.

THEOREM 4. *The total curvature of a simply connected region bounded by a geodesic triangle on a surface is*

$$A + B + C - \pi ,$$

where A, B, C are the interior angles of the triangle.

In Figure 20 let the simple integrals be extended around the boundary of the triangle with its interior on the left. Since the boundary

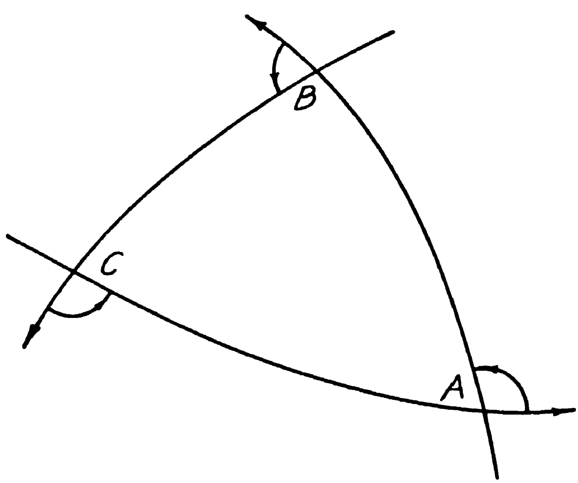

FIG. 20

is composed of geodesic arcs, the geodesic curvature $1/r$ vanishes, so that the first integral in (6·15) is zero. The total angle turned through

* For the proof see Darboux, *op. cit.*, III, 125.

by the tangent line is 2π diminished by the supplements of the angles A, B, C, the sum of which is $3\pi - A - B - C$. Therefore the total angle turned through is $A + B + C - \pi$, and this is the total curvature, as was to be proved.

On a surface with constant Gaussian curvature the total curvature of a simply connected region becomes the product of the Gaussian curvature and the area of the region. In particular, Theorem 4 gives, for a simply connected region bounded by a geodesic triangle on a surface of constant Gaussian curvature,

$$K\Delta = A + B + C - \pi\,, \tag{6·16}$$

where Δ is the area of the triangle. If the surface is a developable, then $K = 0$, and hence

$$A + B + C = \pi\,. \tag{6·17}$$

If the developable is a plane, the geodesics are straight lines and equation (6·17) expresses a well-known theorem.

EXERCISES

1. For a surface of revolution (III 2·15) the radius r_2 of geodesic curvature at a point P of a parallel is given by the formula

$$r_2 = u(1 + f'^2)^{1/2}\,.$$

Show that the radius r_2 is the segment of the tangent to the meridian at the point P, measured from P to the point where the tangent intersects the axis of revolution.

2. The accompanying table gives the cosines of the angles between the tangent, principal normal, and binormal at a point of a curve on a surface, referred to its lines of curvature, and the edges of the local trihedron associated with the point of the surface.

	u-tangent	v-tangent	normal
tangent	$\cos\theta$	$\sin\theta$	0
P.N.	$-\sin\theta\sin\varphi$	$\cos\theta\sin\varphi$	$\cos\varphi$
Bin.	$\sin\theta\cos\varphi$	$-\cos\theta\cos\varphi$	$\sin\varphi$

3. In the local coordinate system at a point of a surface the coordinates of the ray-points of the lines of curvature C^u and C^v are, respectively, 0, r_1, 0 and $-r_2$, 0, 0.

4. The local equation of the osculating plane at a point P of a curve C on a surface referred to its lines of curvature is

$$\xi \sin\theta - \eta\cos\theta + \zeta\tan\varphi = 0\,,$$

and the local equations of the tangent line of C at P are

$$\zeta = 0\,,\qquad \eta = \xi\tan\theta\,,$$

where

$$\tan\theta = \frac{C}{A}\frac{dv}{du}\,,\qquad \tan\varphi = \frac{R}{r}\,.$$

5. The local equations of the osculating planes of the lines of curvature C^u, C^v at a point of a surface are, respectively,

$$r_1\eta - R_1\zeta = 0\,,\qquad r_2\xi + R_2\zeta = 0\,. \tag{6 18}$$

6. The geodesic curvature at a point of a curve, $\varphi(u, v) = \text{const.}$, is given in terms of differential parameters by the formula

$$\frac{1}{r} = -\frac{\Delta_2\varphi}{(\Delta_1\varphi)^{1/2}} - \Delta_1\left(\varphi, \frac{1}{(\Delta_1\varphi)^{1/2}}\right)\,, \tag{6 19}$$

which reduces to the *Formula of Bonnet*,

$$\left\{\begin{aligned} \frac{1}{r} = \frac{1}{H}\Bigg\{&\left[\frac{F\varphi_v - G\varphi_u}{(E\varphi_v^2 - 2F\varphi_v\varphi_u + G\varphi_u^2)^{1/2}}\right]_u \\ &+ \left[\frac{F\varphi_u - E\varphi_v}{(E\varphi_v^2 - 2F\varphi_v\varphi_u + G\varphi_u^2)^{1/2}}\right]_v\Bigg\}\,. \end{aligned}\right. \tag{6 20}$$

7. If the parameters on a surface are *geodesic polar coordinates*, i.e., the distance r, measured from a fixed point P_0 along a geodesic through P_0 to a variable point, and the angle θ which this geodesic makes with a fixed geodesic at P_0, then the first fundamental form of the surface can be reduced to

$$dr^2 + (r^2 - \tfrac{1}{3}K_0r^4 + \cdots)d\theta^2\,, \tag{6 21}$$

where K_0 is the Gaussian curvature of the surface at the point P_0.

8. The total curvature of any closed surface of the same connectivity as the sphere, i.e., a closed surface which can be divided into two simply connected regions by one simply closed curve, is 4π.

9. The total curvature of the anchor ring, which can be made into a simply connected region by two simply closed curves, is zero.

10. The total curvature of a closed surface which can be made into a simply connected region by $2p$ simply closed curves is $4\pi(1 - p)$. For the sphere $p = 0$, and for the anchor ring $p = 1$. (The number p is called the *genus* of the surface, and $2p + 1$ is called its *connectivity*.)

11. Establish the formula

$$\frac{1}{r} = \theta' + \frac{E^{1/2}}{r_1} u' + \left(\frac{G^{1/2}}{r_2} - \omega_v\right) v' \qquad \left(\theta' = \frac{d\theta}{ds}, \cdots\right),$$

where r is given by (6 11); r_1, r_2 by (6 12); ω by (IV 3 1); θ by (IV 3 6); and s is arc length. When the lines of curvature are parametric, this formula becomes

$$\text{(6 22)} \qquad \frac{1}{r} = \theta' + \frac{\cos\theta}{r_1} + \frac{\sin\theta}{r_2}.$$

12. Formulas for the differentiation of the local coordinates defined in Section 5 are the following, in which A, C are defined by (5 1):

$$\text{(6 23)} \qquad \begin{cases} \xi_u = A\left(-1 + \dfrac{\eta}{r_1} + \dfrac{\zeta}{R_1}\right), & \xi_v = C\dfrac{\eta}{r_2}, \\ \eta_u = -A\dfrac{\xi}{r_1}, & \eta_v = C\left(-1 - \dfrac{\xi}{r_2} + \dfrac{\zeta}{R_2}\right), \\ \zeta_u = -A\dfrac{\xi}{R_1}, & \zeta_v = -C\dfrac{\eta}{R_2}. \end{cases}$$

13. When the lines of curvature are parametric, the condition of Gauss can be written in the form

$$\text{(6 24)} \qquad \frac{1}{R_1R_2} = \frac{1}{G^{1/2}}\left(\frac{1}{r_1}\right)_v - \frac{1}{E^{1/2}}\left(\frac{1}{r_2}\right)_u - \frac{1}{r_1^2} - \frac{1}{r_2^2},$$

and the conditions of Codazzi become

$$\text{(6 25)} \qquad \begin{cases} \dfrac{1}{G^{1/2}}\left(\dfrac{1}{R_1}\right)_v = -\dfrac{1}{r_1}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right), \\ \dfrac{1}{E^{1/2}}\left(\dfrac{1}{R_2}\right)_u = +\dfrac{1}{r_2}\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right). \end{cases}$$

7. Geodesic torsion. In this section a formula for the torsion at a point of a curve on a surface will be computed. The *geodesic torsion* will be defined, and a formula obtained for it. These results will then be applied in various special situations.

An indirect method will now be used to compute a formula for the torsion at a point P of a curve C on a surface S. Let the direction cosines of the binormal of C at P be denoted, as usual, by λ, μ, ν, and let the angle between the binormal of C and the normal line of S be defined as follows:

DEFINITION 1. *The angle ψ_1 between the binormal, at a point P of a curve C on a surface S, and the normal of S at P is the smallest angle from the positive half of the binormal in the positive sense of rotation to the positive half of the surface normal.*

Comparison of the angle ψ_1 just defined with the angle ψ between the principal normal and the tangential normal (see Sec. 6, Def. 2, and Fig. 19) shows that $\psi_1 = \psi$. Therefore the formula

$$\cos \psi_1 = \Sigma \lambda a \tag{7·1}$$

becomes, by the aid of (6·3),

$$\sin \varphi = \Sigma \lambda a \, . \tag{7·2}$$

Differentiation with respect to the arc length s on the curve C and the subsequent use of a Frenet formula give

$$\cos \varphi \varphi' = \Sigma \frac{l}{\tau} a + \Sigma \lambda a' \qquad \left(\varphi' = \frac{d\varphi}{ds}, \cdots\right) . \tag{7·3}$$

By means of the formula

$$\cos \varphi = \Sigma l a \, , \tag{7·4}$$

equation (7·3) can be reduced to

$$\left(\varphi' - \frac{1}{\tau}\right) \cos \varphi = \Sigma \lambda a' \, . \tag{7·5}$$

The right member of this equation can be transformed by the formulas (I 4·14) and (I 4·18) for λ, l to produce the equality

$$\Sigma \lambda a' = \Sigma l(b'z' - c'y') \, . \tag{7·6}$$

When the total derivatives in the right member of this equation are evaluated and the Weingarten differential equations (IV·4 17) are used, and also the formula (7·4), one obtains

$$(7\cdot7)\qquad \Sigma l(b'z' - c'y') = -\frac{\cos\varphi}{T},$$

where $1/T$ is defined by the formula

$$(7\cdot8)\quad \frac{1}{T} = -\frac{(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2}{H(Edu^2 + 2Fdudv + Gdv^2)}.$$

From the foregoing calculations we obtain

$$(7\cdot9)\qquad \left(\varphi' - \frac{1}{\tau} + \frac{1}{T}\right)\cos\varphi = 0.$$

If $\cos\varphi \neq 0$, the desired *formula for the torsion* $1/\tau$ *at a point of a curve on a surface* is established, namely,

$$(7\cdot10)\qquad \frac{1}{\tau} = \frac{1}{T} + \varphi'.$$

It is true that $\cos\varphi = 0$ for an asymptotic curve, but the formula (7·10) can be proved (see Ex. 1, below) to be valid also for asymptotic curves. The *geodesic torsion* at a point of a curve on a surface will now be defined.

Definition 2. *The geodesic torsion at a point P of a curve C on a surface S is the torsion at P of the geodesic curve on S that passes through P tangent to C.*

From the fact that either $\varphi = 0$ or $\varphi = \pi$ for a geodesic curve, it follows that $\varphi' = 0$ for a geodesic, and the following theorem can be stated.

Theorem 1. *The geodesic torsion at a point of a curve on a surface is the function* $1/T$ *defined by the formula* (7·8).

The next theorem is a corollary of Theorem 1 and is confirmed by reference to the equation (IV 7 5) of the lines of curvature.

Theorem 2. *If the geodesic torsion is zero at every point of a curve on a surface, the curve is a line of curvature, and conversely.*

Equation (7·10) also entails as a further consequence the following theorem.

THEOREM 3. *A necessary and sufficient condition that the torsion and the geodesic torsion be equal at every point of a curve C on a surface S is that the angle φ between the principal normal of C and the normal of S be constant along C.*

Geodesics and asymptotics are examples of curves along which $\varphi = \text{const.}$ The formula (7·8) shows that the geodesic torsion $1/T$ is the same at a point P of a surface S for all curves on S through P tangent to the same line at P. In particular, the geodesic torsion is zero for all curves tangent to a line of curvature at a point of a surface.

The geodesic torsions $1/T_1$, $1/T_2$ of the u-curve and the v-curve, respectively, at a point of a surface are shown by (7·8) to be given by the formulas

$$(7\cdot 11)\qquad \frac{1}{T_1} = -\frac{1}{HE}(EM - FL)\,,\qquad \frac{1}{T_2} = -\frac{1}{HG}(FN - GM)\,.$$

If, further, the parametric net is orthogonal, these formulas specialize into

$$(7\cdot 12)\qquad \frac{1}{T_1} = -\frac{M}{H}\,,\qquad \frac{1}{T_2} = +\frac{M}{H}\,.$$

A conclusion can now be drawn.

THEOREM 4. *The geodesic torsions of two orthogonal curves at a point of a surface differ only in sign.*

Finally, a proof of the so-called *Theorem of Joachimsthal* will be sketched. This theorem can be stated as follows:

THEOREM 5. *If two surfaces intersect at a constant angle along a curve, the curve of intersection is a line of curvature on both surfaces or neither. Conversely, if a curve of intersection of two surfaces is a line of curvature on both surfaces, they intersect at a constant angle.*

For the proof of the direct part of the theorem, consider two surfaces S_1, S_2 intersecting in a curve and let φ_1, φ_2 and T_1, T_2 be associated with the two surfaces S_1, S_2, respectively. Substitution in equation (7·10) and subsequent subtraction lead to

$$0 = \frac{1}{T_2} - \frac{1}{T_1} + (\varphi_2 - \varphi_1)'\,.$$

But $\varphi_2 - \varphi_1 = \text{const.}$ by hypothesis. Therefore $T_2 = T_1$, and consequently T_1, T_2 vanish together or else together fail to vanish, so that the desired conclusion is reached. For the proof of the converse part of the theorem, observe that the hypothesis is equivalent to the equations

$$\frac{1}{T_1} = 0, \qquad \frac{1}{T_2} = 0.$$

Consequently, equation (7 10) gives

$$(\varphi_2 - \varphi_1)' = 0.$$

Therefore $\varphi_2 - \varphi_1 = \text{const.}$, and the surfaces intersect at a constant angle.

EXERCISES

1. Taking the asymptotic curves as parametric on a nondevelopable surface S, so that $L = 0$, $N = 0$, $M \neq 0$, show by direct calculation that the torsions $1/\tau_1$, $1/\tau_2$ of the u-curve and the v-curve, respectively, at a point P of S are given by the formulas

(7 13) $$\frac{1}{\tau_1} = -\frac{M}{H}, \qquad \frac{1}{\tau_2} = +\frac{M}{H}.$$

Prove that the formula (7 10) gives these results in this case.

2. By means of (7 13) show that the torsions of the two asymptotic curves at a point P of a surface S differ only in sign, and that the product of these torsions is the Gaussian curvature of S at P.

3. If a line of curvature is a geodesic, then the line of curvature is a plane curve. But a plane line of curvature is not necessarily a geodesic curve.

4. If an asymptotic curve is a geodesic, then the asymptotic curve is a straight line.

5. Defining the *total geodesic curvature* at a point P of a curve C on a surface S to be the rate of change in the direction of the tangential normal of C at P per unit arc length of C, prove that the total geodesic curvature is given by the formula

(7 14) $$\left(\frac{1}{r^2} + \frac{1}{T^2}\right)^{1/2},$$

where $1/r$ and $1/T$ are, respectively, the geodesic curvature and the geodesic torsion of C at P.

6. Prove that the geodesic torsion $1/T$ at a point P of a curve C on a surface is given by the formula

$$\frac{1}{T} = \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \sin 2\theta \,, \tag{7 15}$$

where θ is the angle that the tangent of C at P makes with the tangent of the line of curvature associated with the principal radius R_1.

7. When a surface S is cut by a plane, or a sphere, at a constant angle, the curve of intersection is a line of curvature on S.

CHAPTER VI

TRANSFORMATIONS OF SURFACES

1. The general analytic transformation. A *transformation* between two surfaces is a one-to-one correspondence between their points. When the points of two surfaces are in one-to-one correspondence, each surface may be said to be transformed into the other, or to be mapped or represented upon the other. Each surface may also be said to be a transform of the other, or to be a map or representation of the other. The transformations considered in this chapter are not the most general imaginable but are supposed to be sufficiently regular so that they can be represented by analytic functions. After a few properties of the general analytic transformation are studied in this section, more special transformations will be discussed in the remaining sections of this chapter.

Let us consider two surfaces S, S_1 whose parametric equations are, respectively,

$$(1\ 1)\qquad \begin{cases} x = x(u, v)\,, & y = y(u, v)\,, & z = z(u, v)\,; \\ x_1 = x_1(p, q)\,, & y_1 = y_1(p, q)\,, & z_1 = z_1(p, q)\,. \end{cases}$$

Moreover, let us confine attention to a region of S on which points $P(x, y, z)$ and pairs of parameter values (u, v) are in one-to-one correspondence, and similarly consider only a region of S_1 in which points $P_1(x_1, y_1, z_1)$ and pairs (p, q) are in one-to-one correspondence. Then a one-to-one correspondence between points P and P_1 implies a one-to-one correspondence between pairs of parameter values (u, v) and (p, q), and vice versa. When such a correspondence exists, each of p, q is a *function* of u, v, and vice versa, according to the definition of a function as a correspondence. This reasoning leads to the following conclusion.

THEOREM 1. *A transformation between a surface referred to parameters u, v and a second surface referred to parameters p, q is represented analytically by equations of the form*

$$(1\ 2)\qquad p = p(u, v)\,, \qquad q = q(u, v) \qquad (J = p_u q_v - p_v q_u \neq 0)\,.$$

Any pair of equations (1·2) in which the functions are single-valued and analytic represents a transformation between the two surfaces S, S_1 with the equations (1·1). *Corresponding curves* on S and S_1 are curves generated by corresponding points. *Corresponding tangents* are the tangent lines of corresponding curves at corresponding points. With the usual definition of a *projectivity* between two flat pencils of lines as a one-to-one correspondence between their lines such that the cross ratio of any four lines of one pencil is equal to the cross ratio of the four corresponding lines in the other pencil, the following theorem can be proved.

THEOREM 2. *When the points P, P_1 of two surfaces S, S_1 are in one-to-one correspondence, the induced transformation relating the tangent lines of curves at a point P on S to the tangent lines of the corresponding curves at the corresponding point P_1 on S_1 is a projectivity.*

The proof begins by differentiating equations (1·2) to obtain

$$(1\cdot 3) \qquad dp = p_u du + p_v dv\,, \qquad dq = q_u du + q_v dv\,.$$

Division expresses the direction dq/dp of a curve at a point P_1 on the surface S_1 as a linear fractional function of the direction dv/du of the corresponding curve at the corresponding point P on S:

$$(1\cdot 4) \qquad \frac{dq}{dp} = \frac{q_u + q_v \dfrac{dv}{du}}{p_u + p_v \dfrac{dv}{du}}\,.$$

Since the cross ratio of four tangents at P is the cross ratio of their directions dv/du, and similarly at P_1, and since cross ratio is invariant under any linear fractional transformation (see Chap. IV, Sec. 2, Ex. 6), it follows that corresponding tangents are related by a projectivity, as was to be proved.

It is possible to simplify the equations (1·2) of a general analytic transformation between two surfaces. Let the functions $p(u, v)$, $q(u, v)$ be substituted for p, q in the equations of the surface S_1; a transformation of parameters is thereby effected on S_1, from p, q to u, v, and the result is that now both surfaces S and S_1 are referred to the same parameters u, v, so that corresponding points have the same curvilinear coordinates. Moreover, the surfaces have not been specialized, nor the generality of the correspondence impaired. Thus the following conclusion is reached.

THEOREM 3. *It is no restriction on a transformation between two surfaces, nor on the surfaces themselves, to suppose that the parameters on one of the surfaces have been chosen so that corresponding points have the same curvilinear coordinates.*

Obviously, the parameters on the other surface still remain arbitrary. When corresponding points have the same curvilinear coordinates, any equation connecting the curvilinear coordinates is the equation of a curve on one of the surfaces and at the same time is the equation of the corresponding curve on the other surface. Moreover, *when corresponding points have the same curvilinear coordinates, corresponding directions at corresponding points are equal.* In fact, the effect of the transformation of parameters which has just been carried out on the surface S_1 may be expressed by saying that it has simplified the equations (1 2) of the transformation between the two surfaces so that these equations have become

$$p = u\,, \qquad q = v\,.$$

Equation (1 4) therefore reduces to

$$\frac{dq}{dp} = \frac{dv}{du}\,,$$

and the statement is proved. A theorem concerning *corresponding orthogonal nets* will now be proved.

THEOREM 4. *If a transformation between two surfaces S, S_1 is such that the minimal curves on S and S_1 do not correspond, there exist an orthogonal net on S and an orthogonal net on S_1 which correspond to each other, and there is only one such pair of corresponding orthogonal nets.*

When corresponding points on the two surfaces S, S_1 have the same curvilinear coordinates, the hypothesis of the theorem is that the first fundamental coefficients E, F, G of S are not respectively proportional to the coefficients E_1, F_1, G_1 of S_1. The only binary quadratic differential form whose harmonic invariant with the first fundamental form of each of the surfaces S, S_1 vanishes is, except possibly for a factor, the jacobian of the two fundamental forms. When the jacobian is set equal to zero, the resulting curvilinear differential equation,

$$(1\cdot5)\quad (EF_1 - E_1F)du^2 + (EG_1 - E_1G)dudv + (FG_1 - F_1G)dv^2 = 0\,,$$

represents an orthogonal net on the surface S and also represents an orthogonal net on the surface S_1. These orthogonal nets correspond and constitute the only pair of corresponding orthogonal nets on the two surfaces.

The next theorem is an analogue of the preceding one and relates to *corresponding conjugate nets.*

THEOREM 5. *If a transformation between two nondevelopable surfaces S, S_1 is such that the asymptotic curves on S and S_1 do not correspond, there exist a conjugate net on S and a conjugate net on S_1 which correspond to each other, and there is only one such pair of corresponding conjugate nets.*

The proof runs parallel to the proof of Theorem 4 but involves the second fundamental coefficients L, M, N of the surface S and L_1, M_1, N_1 of S_1, in place of the first fundamental coefficients. When corresponding points have the same curvilinear coordinates, the curvilinear differential equation of the unique pair of corresponding conjugate nets is

$$(1\cdot 6)\quad \begin{cases} (LM_1 - L_1M)du^2 + (LN_1 - L_1N)du\,dv \\ \qquad\qquad + (MN_1 - M_1N)dv^2 = 0\,. \end{cases}$$

EXERCISES

1. The orthogonal nets (1·5) that correspond on two surfaces when the minimal curves do not correspond are real when the surfaces and parameters are real

2. Discuss the realness of the conjugate nets (1·6) that correspond on two two nondevelopable surfaces when the asymptotic curves do not correspond.

3. If just one family of asymptotic curves on a nondevelopable surface corresponds to just one family of asymptotic curves on another, then these families, each counted twice, may be regarded as constituting a pair of corresponding degenerate conjugate nets, and there is no other pair of corresponding conjugate nets.

4. If the asymptotic curves correspond on two nondevelopable surfaces, then every conjugate net on either surface corresponds to a conjugate net on the other. A similar statement concerning orthogonal nets is true if the minimal curves correspond.

5. If a surface $z = f(x, y)$ is projected orthogonally onto the xy-plane, the differential equations of the two families of the orthogonal net on the surface that corresponds to an orthogonal net on the plane can be written in the form

$$(1\cdot 7)\qquad p\,dx + q\,dy = 0\,,\qquad q\,dx - p\,dy = 0\qquad (p = f_x,\quad q = f_y)\,.$$

The first family consists of the horizontal contour lines cut by the planes $z =$ const. on the surface. The second family consists of the lines of steepest slope on the surface.

6. Consider a one-to-one correspondence between the points P, P_1 of two surfaces S, S_1 and discuss the induced transformation relating the osculating planes of curves at a point P on S to the osculating planes of the corresponding curves at the corresponding point P_1 on S_1.

2. Conformal representation. If two surfaces S and S_1 differ only in their positions in space, so that S could be made to coincide with S_1 by a rigid motion, and if corresponding points on S and S_1 are those that the rigid motion would bring into coincidence, then the surface S_1 is certainly a transform of S. A rigid motion preserves all the properties of a surface that are considered in metric geometry, since metric geometry is, by definition, the study of invariants under rigid motion. No other transformation than a rigid motion can preserve all the metric properties of a surface. Consequently many questions arise, among which are the following. Given a particular transformation of surfaces, what properties does it preserve? What is the most general transformation that preserves a given property? Given two surfaces, is it possible to transform one into the other by a certain type of transformation, i e., by a transformation preserving a certain property? Given a surface, what are all the surfaces into which it can be transformed by a certain type of transformation?

One of the most important classes of transformations of surfaces consists of the so-called *conformal representations*. These are the transformations that preserve angles and are defined precisely as follows:

DEFINITION 1. *A transformation between two surfaces S, S_1 is a conformal representation in case the angle between any two curves intersecting at a point on S is equal to the angle between the corresponding curves at the corresponding point on S_1.*

On the basis of this definition it is easy to prove the following theorem.

THEOREM 1. *When the points of two surfaces S, S_1 are in one-to-one correspondence and corresponding points have the same curvilinear coordinates u, v, a necessary and sufficient condition that the correspondence be a conformal representation is that the coefficients E_1, F_1, G_1 of S_1 be respectively proportional to the coefficients E, F, G of S*

The proof makes use of formulas (IV 3·4) for the cosine and sine of the angle θ between two curves C, C_1 intersecting at a point on the surface S. In order to obtain the same formulas for the cosine and sine of the angle θ_1 between the corresponding curves at the corresponding point on the surface S_1, it is sufficient to replace therein E, F, G by E_1, F_1, G_1, respectively. If we have

$$(2\cdot 1) \qquad E_1 = h^2E\,, \qquad F_1 = h^2F\,, \qquad G_1 = h^2G \qquad (h \neq 0)\,,$$

where h^2 is an arbitrary function of u, v and the positive square root is to be taken for h, it is easy to verify that

$$(2\cdot 2) \qquad \cos\theta_1 = \cos\theta\,, \qquad \sin\theta_1 = \sin\theta\,.$$

Then $\theta_1 = \theta$, and the sufficiency of the conditions (2·1) is proved. To prove the necessity, let us observe that, if all angles are preserved, then surely right angles are preserved. Therefore the conditions of orthogonality,

$$(2\cdot 3) \qquad \begin{cases} Edudu_1 + F(dudv_1 + du_1dv) + Gdvdv_1 = 0\,, \\ E_1dudu_1 + F_1(dudv_1 + du_1dv) + G_1dvdv_1 = 0\,, \end{cases}$$

must be equivalent for all directions dv/du, dv_1/du_1. Consequently conditions of the form (2·1) must be satisfied.

The formula (III 3 8) for the length s of an arc of a curve on a surface shows that, when a surface S is represented conformly upon another surface S_1 and corresponding points have the same curvilinear coordinates u, v, then the length s_1 of the corresponding arc of the corresponding curve on S_1 is related to the length s by the equation

$$(2\cdot 4) \qquad s_1 = hs\,,$$

where h is a function of u, v which is the same function for all pairs of corresponding arcs. Moreover, the formula (IV 3 11) for the area A of a region of S shows that the area A_1 of the corresponding region of S_1 is related to the area A by the equation

$$(2\cdot 5) \qquad A_1 = h^2A\,.$$

Ordinarily, conformal representation does not preserve *size*, either of length or of area. But it may be said to preserve *shape*, since it preserves angles. It is sometimes said that *infinitesimal triangles* are similar under conformal representation. A corollary of Theorem 1 may be stated as follows:

THEOREM 2. *A transformation between two surfaces is conformal if, and only if, the minimal curves on the two surfaces correspond.*

The proof is simple. If the points of two surfaces S, S_1 are in one-to-one correspondence and corresponding points have the same curvilinear coordinates, then the minimal curves on S and S_1 correspond if, and only if, the equations of the minimal curves, namely,

$$Edu^2 + 2Fdudv + Gdv^2 = 0,$$

$$E_1du^2 + 2F_1dudv + G_1dv^2 = 0,$$

are equivalent. If equations of the form (2 1) are satisfied, certainly the equations of the minimal curves are equivalent, and conversely.

The next theorem explains, at least in part, the reason why conformal mapping is of such great interest.

THEOREM 3. *Any surface can be represented conformly upon any other surface.*

The proof depends on the solution of two partial differential equations. Consider the two surfaces S, S_1 represented by the equations (1 1), and let it be required to find a transformation of parameters (1 2) on S_1 which will validate equations of the form (2 1). Referring to equations (IV 1 17) to recall the effect of a transformation of parameters on E, F, G, we see that our problem reduces to finding two functions $u(p, q)$ and $v(p, q)$ satisfying two partial differential equations

$$(2\ 6)\quad \begin{cases} \dfrac{1}{E}[Eu_p^2 + 2Fu_pv_p + Gv_p^2] = \dfrac{1}{F}[Eu_pu_q + F(u_pv_q + u_qv_p) + Gv_pv_q] \\ \qquad\qquad = \dfrac{1}{G}[Eu_q^2 + 2Fu_qv_q + Gv_q^2]. \end{cases}$$

These equations are known to admit infinitely many pairs of solutions for u, v as functions of p, q. Hence any surface can be represented conformly on any other surface, and indeed the representation is possible in infinitely many ways.

EXERCISES

1. The equations (IV 3 19) are invariant under conformal representation for which corresponding points have the same curvilinear coordinates.

2. Isothermally orthogonal nets are preserved by conformal representation.

3. If the parametric net, $dudv = 0$, on a surface S is isothermally orthogonal, and if the parametric net, $dpdq = 0$, on a surface S_1 is also isothermally orthogonal, then a conformal transformation between S and S_1 is defined by

$$p + iq = f(u + iv) \qquad (i^2 = -1)\,, \tag{2 7}$$

where f is an arbitrary function.

4. The most general conformal representation of a surface S upon a surface S_1 is obtained by finding a complex variable $\varphi + i\psi$ on S (see Chap. IV, Sec. 3, Ex. 17) and a complex variable $\varphi_1 + i\psi_1$ on S_1 and by placing

$$\varphi_1 + i\psi_1 = f(\varphi + i\psi)\,, \tag{2 8}$$

where f is an arbitrary function.

5. The most general conformal representation of a surface S upon a plane referred to ordinary cartesian coordinates x_1, y_1 is obtained by finding a complex variable $\varphi + i\psi$ on S and placing

$$x_1 + iy_1 = f(\varphi + i\psi)\,, \tag{2 9}$$

where f is an arbitrary function.

6. The most general conformal representation of a plane, referred to cartesian coordinates x, y, upon a plane referred to cartesian coordinates x_1, y_1 is obtained by placing

$$x_1 + iy_1 = f(x + iy)\,, \tag{2 10}$$

where f is an arbitrary function.

7. If the parametric net, $dudv = 0$, on a surface is isothermally orthogonal, the equations

$$x = u\,, \qquad y = v$$

define a conformal representation of the surface upon the xy-plane.

8. The equations

$$x = q\,, \qquad y = p\,, \tag{2 11}$$

where p, q are defined by placing

$$p = \int_{u_0}^{u} \frac{1}{u}(1 + f'^2)^{1/2}\,du\,, \qquad q = v\,, \tag{2 12}$$

define a conformal map of the revolute (III 2 15) upon the xy-plane. The meridians, v = const., map into straight lines parallel to the y-axis; and the parallels, u = const., map into straight lines parallel to the x-axis. The loxodromes (IV 3 13) map into straight lines,

$$y - x \cot \alpha + c = 0 . \tag{2 13}$$

9. The sum of the angles in a triangle whose sides are three loxodromic arcs on a surface of revolution is two right angles.

3. Conformal maps of the sphere upon the plane. Among all the conformal representations of a sphere upon a plane, there are two of special interest, namely, *Mercator's chart* and *stereographic projection.* These will be considered briefly in this section. *Mercator's chart* is defined as follows:

DEFINITION 1. *Mercator's chart of the sphere* (III 2 11) *upon the xy-plane is the representation*

$$x = q , \qquad y = p , \tag{3 1}$$

where the parameters p, q on the sphere are defined by placing

$$p = \log \cot \frac{u}{2} , \qquad q = v . \tag{3 2}$$

This representation can be shown to be conformal by observing that the first fundamental form

$$r^2 du^2 + r^2 \sin^2 u dv^2 \tag{3 3}$$

of the sphere (III 2 11) becomes, under the transformation of parameters (3 2),

$$r^2 \sin^2 u(dp^2 + dq^2) , \tag{3·4}$$

while the first fundamental form for the xy-plane is

$$dx^2 + dy^2 . \tag{3 5}$$

So corresponding points now have equal curvilinear coordinates, and the first fundamental coefficients of the two surfaces are respectively proportional.

Some further properties of Mercator's chart are immediately evident. The meridians of longitude, v = const., become straight lines parallel to the y-axis. The parallels of latitude, u = const., become straight lines parallel to the x-axis. The loxodromes (IV 3·14) become the straight lines represented by

$$(3\cdot6) \qquad y + x \cot a - c = 0 \qquad (a, c = \text{consts.}) .$$

Except for nonessential modifications, Mercator's chart can be shown (although the proof* will not be reproduced here) to be the only conformal map of the sphere upon the plane in which the loxodromes correspond to straight lines. These properties explain why Mercator's chart is so extensively used in making maps of the earth's surface, particularly for mariners.

Stereographic projection of a sphere upon a plane is defined as follows:

DEFINITION 2. *Stereographic projection of a sphere upon a plane is the correspondence established by selecting one of the extremities of the*

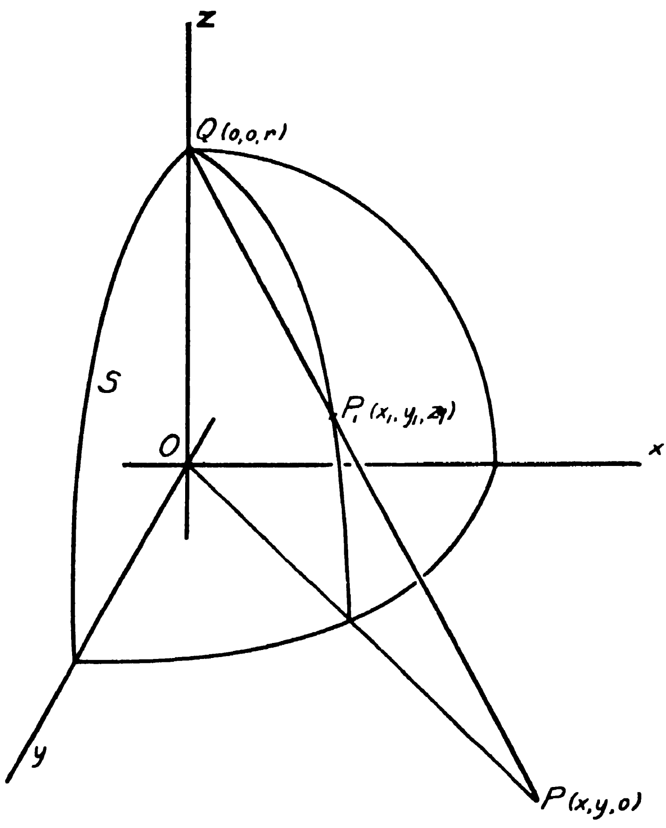

FIG. 21

* Scheffers, *Geometrie* (3d ed.), II, 102.

diameter perpendicular to the plane as a center of perspectivity, then drawing lines from this point to the points of the sphere, and finally marking the intersection points of these lines and the plane.

In order to write the equations of a stereographic projection, let us consider a sphere S with radius r and center at the origin, as in Figure 21. From the point $Q(0, 0, r)$ draw a line to any point $P_1(x_1, y_1, z_1)$ on the sphere and produce it to meet the plane $z = 0$ in the point $P(x, y, 0)$. The collinearity of the points Q, P_1, P is expressed by the equations

$$(3 \cdot 7) \qquad x_1 = kx\,, \qquad y_1 = ky\,, \qquad z_1 = (1 - k)r\,,$$

where k is a proportionality factor to be determined. Since the point P_1 is on the sphere S, we have

$$(3 \cdot 8) \qquad x_1^2 + y_1^2 + z_1^2 = r^2\,.$$

The value of k is found by substituting the expressions given by equations (3·7) for x_1, y_1, z_1 in equation (3·8) and solving the resulting equation for k. When this value of k is used in (3·7), the result is *the equations of the stereographic projection of the sphere S onto the xy-plane,*

$$(3 \cdot 9) \qquad \begin{cases} x_1 = \dfrac{2r^2x}{x^2 + y^2 + r^2}\,, \qquad y_1 = \dfrac{2r^2y}{x^2 + y^2 + r^2}\,, \\ z_1 = \dfrac{r(x^2 + y^2 - r^2)}{x^2 + y^2 + r^2}\,. \end{cases}$$

The cartesian coordinates x, y in the xy-plane can be regarded as curvilinear coordinates on the sphere S. Direct calculation of the first fundamental coefficients E_1, F_1, G_1 of the sphere leads to

$$(3 \cdot 10) \qquad E_1 = G_1 = \frac{4r^4}{(x^2 + y^2 + r^2)^2}\,, \qquad F_1 = 0\,.$$

Since these coefficients are proportional to the coefficients 1, 0, 1 for the plane, the stereographic projection is conformal. It is now easy to prove the following theorem.

THEOREM 1. *Circles through the center Q of perspectivity on the sphere correspond to straight lines in the xy-plane, and all other circles on the sphere correspond to circles in the xy-plane.*

The demonstration is made by showing that a linear equation of the form

$$(3\cdot 11) \qquad Ax_1 + By_1 + C(z_1 - r) = 0$$

transforms by (3 9) into a linear equation in x, y but that any linear equation in x_1, y_1 not of this form transforms by (3 9) into the equation of a circle in the xy-plane.

It could further be shown,* although the proof will not be reproduced here, that *stereographic projection is the most general conformal representation of the sphere upon the plane possessing the property that circles on the sphere correspond to circles in the plane.*

* *Ibid.*, p. 96.

EXERCISES

1. If in Exercise 8 of Section 2 the revolute is specialized to be a sphere by placing

$$(3\cdot 12) \qquad f(u) = (r^2 - u^2)^{1/2}$$

and if $u_0 = r$, compare the map given by (2 11) for the sphere with Mercator's chart (3·1).

2. Writing the equations of the minimal lines in the xy-plane in the form (IV 2 7), eliminate x, y from equations (3 9) to find the equations of the sphere referred to the parameters p, q of the plane. Prove that the transformation (from the parameters p, q to new parameters p_1, q_1)

$$(3\ 13) \qquad p = \frac{r}{q_1}, \qquad q = \frac{r}{p_1}$$

reduces these equations to equations essentially the same as the equations (IV 2 11) of a sphere referred to its minimal lines.

4. Applicability. When a surface is thought of as a thin, flexible, inextensible sheet and is then bent, or deformed, continuously without tearing or folding, it assumes infinitely many shapes between the initial and final ones. There are certain properties of the surface which are unchanged by this process and which the final form of the surface has in common with the original. When the surface is not stretched or contracted at all, it is clear that the length of a curve on the surface is not changed during the bending process. This property is taken as the base of the following definition of *applicable surfaces.*

DEFINITION 1. *A surface S is applicable to another surface S_1 in case there exists a one-to-one correspondence between the points of S and those of S_1 such that the length of any curve between any two points on S is equal to the length of the corresponding curve between the corresponding points on S_1.*

When a surface (or portion of a surface) S is applicable to another surface (or portion of a surface) S_1, then S_1 is applicable to S, and the transformation between the two surfaces is called *an isometric map.*

It is to be expected that *not every surface is applicable to a given surface.* For example, it is intuitively evident that a hemisphere is not applicable to a plane, since an attempt to flatten out the skin of half an orange will result in tearing or otherwise mutilating it (see Th. 4, below). So a criterion for applicability is desirable. This is furnished by the following theorem.

THEOREM 1. *When the points of two surfaces S, S_1 are in one-to-one correspondence and corresponding points have the same curvilinear coordinates u, v, a necessary and sufficient condition that the correspondence be an isometric map is that the coefficients E, F, G of S be respectively equal to the coefficients E_1, F_1, G_1 of S_1.*

The proof makes use of the formula (III 3 8) for the arc length s of a curve between two points on a surface S. The length s_1 of the corresponding curve between the corresponding points on the surface S_1 is obtained by attaching subscripts 1 to E, F, G. If $s_1 = s$ for every pair of corresponding curves, one of which joins two points on one surface and the other the corresponding two points on the other surface, the conditions

$$(4\cdot 1) \qquad E_1 = E\,, \qquad F_1 = F\,, \qquad G_1 = G$$

are found to be necessary. They are obviously also sufficient.

Since equations (4 1) are a special form of equations (2 1), it follows that *every isometric map is a conformal representation.* In fact, any metric invariant in surface theory which has been expressed by a formula involving only the coefficients E, F, G and their partial derivatives with respect to u, v is an invariant under *bending*, or isometric mapping. No further proof need be given for the following theorem.

THEOREM 2. *Angles, areas, geodesic curvature, Gaussian curvature, geodesics, and minimal curves are invariant under bending.*

When two surfaces are given by their parametric equations referred to *the same* parameters, a transformation between the surfaces can be established by making points with the same curvilinear coordinates correspond. Then if the coefficients E, F, G of one surface are respectively equal to the coefficients E_1, F_1, G_1 of the other at all pairs of corresponding points, the correspondence is an isometric map; in other words, the surfaces are applicable. Moreover, if two surfaces are given by their parametric equations referred to *different* parameters, it may be possible to make a transformation of parameters on one (or both) of the surfaces so that the parameters on the two surfaces become equal and so that the coefficients E, F, G of the one surface are then respectively equal to the coefficients E_1, F_1, G_1 of the other at points with the same curvilinear coordinates. If such is the case, then the two surfaces are applicable. An illustration of the method just suggested for proving two surfaces applicable is found in the demonstration of the following theorem.

THEOREM 3. *Every developable surface is applicable to the plane.*

The proof will be constructed for a developable surface(III 6 1) not a cone or cylinder; these special cases would offer no difficulties. The first fundamental coefficients E, F, G of the developable are given by the formulas

$$(4\cdot2) \qquad E = 1 + \frac{t^2}{\rho^2}, \qquad F = 1, \qquad G = 1,$$

where $1/\rho$ is the curvature of the edge of regression C of the developable. Let the plane under consideration be the plane $z = 0$. In cartesian coordinates x, y the first fundamental coefficients for this plane are given by

$$E = 1, \qquad F = 0, \qquad G = 1,$$

but other curvilinear coordinates will now be introduced. Let σ be defined by placing

$$(4\cdot3) \qquad \sigma = \int_0^s \frac{ds}{\rho},$$

and consider the plane curve C_1 whose equations are

$$(4\cdot4) \qquad x = \int_0^s \cos \sigma ds, \qquad y = \int_0^s \sin \sigma ds, \qquad z = 0,$$

in which the parameter s is arc length on the edge of regression C and where the functions σ, ρ belong to C. Easy calculation shows that s is also arc length on C_1, that $1/\rho$ is also curvature of C_1, and that σ also belongs to C_1. Let a point P with curvilinear coordinates s, t on the developable correspond to a point Q in the plane defined as follows: at the point R on C_1 with the same value of s as that of P draw the tangent of C_1 and let Q be the point on this tangent at the algebraic distance t from R. Then the curvilinear coordinates of the point Q are the same s, t as those of P. In fact, the parametric equations of the plane referred to s, t are

$$(4\ 5)\qquad \begin{cases} x = \displaystyle\int_0^s \cos\sigma ds + t\cos\sigma\,, \\ y = \displaystyle\int_0^s \sin\sigma ds + t\sin\sigma\,, \\ z = 0\,. \end{cases}$$

Calculation of E, F, G for the plane referred to the parameters s, t now gives precisely the formulas (4 2), and so the theorem is proved.

The converse of Theorem 3 is easy to prove. If a surface is applicable to the xy-plane, then, if necessary, a transformation of parameters on the surface can be made so that points on the surface corresponding to points (x, y) in the plane will also have curvilinear coordinates x, y. Then since $E = 1$, $G = 1$, and $F = 0$ for the xy-plane, it follows that $E = 1$, $G = 1$, and $F = 0$ for the surface. The condition (IV 5 8) of Gauss now gives $LN - M^2 = 0$ for the surface, and therefore the surface is developable. The conclusion may be stated as follows:

THEOREM 4. *The only surfaces applicable to the plane are developable surfaces.*

Application to the plane is often spoken of as *development upon* the plane. The reason for the name *developable surface* is now evident.

THEOREM 5. *A surface is a developable surface if, and only if, it is developable upon, or applicable to, a plane.*

EXERCISES

1. When a developable surface is developed upon a plane, the geodesics on the surface become straight lines in the plane.

2. When the rectifying developable of a curve is developed upon a plane, the curve itself becomes a straight line.

3. When the polar developable of a curve is developed upon a plane, the curve itself becomes a fixed point.

4. Every screw surface is applicable to some catenoid.

5. Every helicoid is applicable to some surface of revolution.

6. Determine the function $g(v)$ so that the conoid

$$x = u \cos v, \qquad y = u \sin v, \qquad z = g(v)$$

shall be applicable to some surface of revolution.

7. The revolute whose meridians are logarithmic curves,

$$x = u \cos v, \qquad y = u \sin v, \qquad z = \log u,$$

and the screw surface whose parametric equations are

$$x = u \cos v, \qquad y = u \sin v, \qquad z = v$$

have the same Gaussian curvature K at each pair of corresponding points (u, v), the formula for K being

$$K = -\frac{1}{(1 + u^2)^2};$$

but the two surfaces are not applicable.

8. If the developable circumscribing a surface along a curve C is developed upon a plane, the geodesic curvature of C is the ordinary curvature of the curve into which C is transformed on the plane.

5. Equiareal maps. Along with conformal representation, which preserves angles, and isometric mapping, which preserves lengths, should be associated *equiareal mapping*, which preserves areas.

DEFINITION 1. *A surface S is said to be mapped equiareally upon a surface S_1 in case there is a one-to-one correspondence between the points of S and S_1 such that the area of any suitably restricted region of S is equal to the area of the corresponding region of S_1.*

An analytic criterion for an equiareal map can be deduced as follows. Let us consider two surfaces S, S_1 represented analytically by equations (1·1). The areas A, A_1 of corresponding suitably restricted regions of S, S_1 are given by

$$(5\cdot 1) \qquad A = \iint H\,du\,dv, \qquad A_1 = \iint H_1\,dp\,dq.$$

A one-to-one correspondence between the points of S, S_1 is established by the formulas (1·2). If the parameters on the surface S_1 are transformed by the equations (1·2) from p, q to u, v so that corresponding points may have the same curvilinear coordinates u, v, the formula for the area A_1 becomes*

$$A_1 = \int\int H_1 |J| du dv , \tag{5·2}$$

where J, the jacobian of the transformation, is given by

$$J = p_u q_v - p_v q_u \tag{5·3}$$

and where the vertical bars denote absolute value. The characteristic condition for an equiareal map, namely, $A_1 = A$, is therefore equivalent to

$$H_1 J = \pm H . \tag{5·4}$$

Thus the following conclusion is reached.

THEOREM 1. *The transformation*

$$p = p(u, v) , \qquad q = q(u, v) \qquad (J = p_u q_v - p_v q_u \neq 0) \tag{5·5}$$

between a surface S referred to parameters u, v and a surface S_1 referred to parameters p, q is an equiareal map of S upon S_1 if, and only if, the functions p, q are solutions of one or the other of the two partial differential equations

$$p_u q_v - p_v q_u = \pm \frac{H}{H_1} , \tag{5·6}$$

where H is defined by (III 3 6) *for S and H_1 is the same function for S_1.*

It is evident that any surface can be mapped equiareally upon any other, since two functions p, q can be made to satisfy one condition of the form (5·6). If the surface S_1 is the xy-plane, then $H_1 = 1$ and a corollary of Theorem 1 can be stated.

THEOREM 2. *The transformation*

$$x = x(u, v) , \qquad y = y(u, v) , \tag{5·7}$$

between a surface S referred to parameters u, v and the xy-plane, is an equiareal map of S upon the xy-plane in case the functions x, y are solutions of one or the other of the two partial differential equations

$$x_u y_v - x_v y_u = \pm H . \tag{5·8}$$

* Goursat-Hedrick, *Mathematical Analysis* (Boston: Ginn & Co., 1904), I, 266.

By way of illustration let it be required to map the revolute (III·2 15) upon the xy-plane. For the revolute the function H is given by the formula

(5 9) $$H = u(1 + f'^2)^{1/2},$$

and so the problem is to solve the equations

(5·10) $$x_u y_v - x_v y_u = \pm u(1 + f'^2)^{1/2}.$$

A particular pair of solutions x, y of the first of these equations is given by

(5 11) $$x = v, \qquad y = -\int_{u_0}^{u} u(1 + f'^2)^{1/2} du.$$

In the equiareal map defined by these equations, the meridians become straight lines parallel to the y-axis, and the parallels become straight lines parallel to the x-axis.

An equiareal map of the sphere (III 2 11) upon the xy-plane can be easily constructed. For the sphere we have

(5·12) $$H = r^2 \sin u,$$

and so the problem is to solve the equations

(5 13) $$x_u y_v - x_v y_u = \pm r^2 \sin u.$$

A particular pair of solutions x, y of the first of these equations is given by

(5·14) $$x = rv, \qquad y = r \cos u.$$

In the equiareal map thus defined, the meridians and parallels become straight lines parallel to the y-axis and x-axis, respectively. This map is called *Lambert's projection* and can easily be shown to be the map obtained by circumscribing a cylinder about the sphere along the equator; making two points on the sphere and cylinder, respectively, correspond if they lie on a straight line intersecting at right angles the diameter of the sphere which is parallel to the generators of the cylinder; and afterward developing the cylinder upon the plane.

EXERCISES

1. If the points of two surfaces S, S_1 are in one-to-one correspondence and corresponding points have the same curvilinear coordinates, a necessary and sufficient condition that the correspondence be equiareal is $H_1 = \pm H$.

2. If a transformation between two surfaces is both conformal and equiareal, the surfaces are applicable.

3. Describe the map, of the sphere upon the plane, defined by

$$x = rv \sin u\,, \qquad y = ru\,. \tag{5·15}$$

4. Describe the map, of the sphere upon the plane, defined by

$$x = 2rv(1 - \cos u)^{1/2}\,, \qquad y = r[1 - (1 - \cos u)^{1/2}]\,. \tag{5·16}$$

6. Spherical indicatrix. *The spherical indicatrix* of a surface is a map or representation of the surface on a sphere with unit radius and with its center at the origin. To define this map, let us consider such a region S of the surface as has no two normals parallel, and draw radii of the sphere parallel to the normals at all the points of S. The locus of the extremities of these radii is a region S_1 of the sphere which is called *the spherical indicatrix* of the region S. To any point $P(x, y, z)$ of the region S corresponds a point $Q(a, b, c)$ of the region S_1, and conversely to any point Q of S_1 corresponds a point P of S.

Since the coordinates a, b, c of a point Q are the direction cosines of the normal of the surface S at the corresponding point P, equations (IV 4·7) for a, b, c may be regarded as the parametric equations of the indicatrix. Corresponding points P, Q evidently have the same curvilinear coordinates u, v. The implicit equation of the indicatrix is simply

$$a^2 + b^2 + c^2 = 1\,. \tag{6·1}$$

If a point P traces a curve C on a surface S, the corresponding point Q traces a corresponding curve C_1 on the spherical indicatrix S_1 of S. If s_1 denotes arc length on C_1, then

$$ds_1^2 = \Sigma(a_u du + a_v dv)^2\,, \tag{6·2}$$

the summation being for cyclical permutations of a, b, c. By means of the Weingarten differential equations (IV 4·17) this equation can be reduced to

$$ds_1^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2\,, \tag{6·3}$$

where the first fundamental coefficients e, f, g of the spherical indicatrix are given by the formulas

$$(6\cdot4)\qquad \begin{cases} e = \Sigma a_u^2 = -KE + 2kL\,, \\ f = \Sigma a_u a_v = -KF + 2kM\,, \\ g = \Sigma a_v^2 = -KG + 2kN\,, \end{cases}$$

in which K, k are, respectively, the Gaussian and mean curvatures of the surface S. If a function h, analogous to H, is defined by placing

$$(6\cdot5)\qquad h = (eg - f^2)^{1/2}$$

and by agreeing to take the positive square root, direct calculation gives

$$(6\cdot6)\qquad h = \pm KH\,,$$

the positive sign being used when $K > 0$ and the negative sign when $K < 0$.

The element of area dA on the surface S is given by the formula (IV·3 10). The element of area dA_1 on the spherical indicatrix S_1 is given by the analogous formula

$$(6\cdot7)\qquad dA_1 = h\,du\,dv\,.$$

By the aid of equation (6 6) it is easy to show that

$$(6\cdot8)\qquad \frac{dA_1}{dA} = \pm K\,.$$

This result may be stated in the form of a theorem.

THEOREM 1. *If ΔA is a small portion of a surface S including a point P, and if ΔA_1 is the corresponding portion of the spherical indicatrix of S, then*

$$(6\cdot9)\qquad \lim \frac{\Delta A_1}{\Delta A} = \pm K\,,$$

where the limit is taken as ΔA shrinks to the point P and where K is the Gaussian curvature of S at P.

This theorem gives a geometrical interpretation of the Gaussian curvature K which is quite analogous to the usual definition of the curvature $1/\rho$ of a plane curve C. If one considers an arc Δs, including a point P on C, and constructs a *circular indicatrix* C_1 *of the normals of* C, the corresponding arc Δs_1 of C_1 is such that

$$\lim \frac{\Delta s_1}{\Delta s} = \pm \frac{1}{\rho},$$

the limit being taken as Δs shrinks to the point P, and $1/\rho$ being the curvature of C at P.

EXERCISES

1. The spherical representation of a surface S is conformal if, and only if, S is a sphere or else is a minimal surface.

2. A necessary and sufficient condition that the tangent at each point of a curve C on a surface be parallel to the tangent at the corresponding point of the spherical image of C is that C be a line of curvature.

3. A necessary and sufficient condition that the tangent at each point of a curve C on a surface be perpendicular to the tangent at the corresponding point of the spherical image of C is that C be an asymptotic curve.

4. The total curvature of a simply connected region of a surface is, except possibly for sign, the area of the spherical indicatrix of the region.

5. The net of curves on a surface S which correspond to the minimal lines on the spherical indicatrix of S is represented by the differential equation

$$(6.10) \qquad e\,du^2 + 2f\,du\,dv + g\,dv^2 = 0$$

and is the net each family of which is conjugate to a family of minimal curves on S. At each point of S the tangents of the lines of curvature bisect the angles between the tangents of this net.

6. The angle between two conjugate tangents at a point P of a surface S is supplementary to the angle between the corresponding tangents of the spherical indicatrix, or is equal to it, according as the Gaussian curvature of S at P is positive or negative.

7. Parallel surfaces. If the normal lines of a surface are also the normals of a second surface, it is evident that the tangent planes of the two surfaces at points on the same normal are parallel; hence the following definition of *parallel surfaces.*

DEFINITION 1. *Two surfaces are called parallel in case they have the same normal lines.*

The following theorem will now be proved.

THEOREM 1. *The parametric equations of a surface S_1 parallel to a surface S can be written in the form*

$$(7\cdot 1) \qquad x_1 = x + at , \qquad y_1 = y + bt , \qquad z_1 = z + ct ,$$

in which t is the constant algebraic distance from a point $P(x, y, z)$ of S to the corresponding point $P_1(x_1, y_1, z_1)$ of S_1 and a, b, c are the direction cosines of the normal of S at P.

First of all, it is evident that the equations of any transversal surface S_1 of the normals of a surface S can be written in the form (7 1) when t is, in general, a function of u, v. The burden of the demonstration consists in showing that S_1 is parallel to S if, and only if, $t =$ const. Differentiation of equations (7 1), followed by use of the Weingarten differential equations (IV 4 17), leads to

$$(7\cdot 2) \begin{cases} x_{1u} = \left[1 + \dfrac{t}{H^2}(FM - GL)\right]x_u + \dfrac{t}{H^2}(FL - EM)x_v + t_u a , \\ x_{1v} = \dfrac{t}{H^2}(FN - GM)x_u + \left[1 + \dfrac{t}{H^2}(FM - EN)\right]x_v + t_v a , \end{cases}$$

and similar formulas for the derivatives of y_1, z_1. The normal at each point P of the surface S is also the normal at the corresponding point P_1 of the transversal surface S_1 if, and only if, in addition to the conditions

$$(7\cdot 3) \qquad \Sigma a x_u = 0 , \qquad \Sigma a x_v = 0 ,$$

the further conditions

$$(7\cdot 4) \qquad \Sigma a x_{1u} = 0 , \qquad \Sigma a x_{1v} = 0$$

are also satisfied. From equations (7 2), (7 3), and (7 4) one obtains, as a consequence,

$$(7\cdot 5) \qquad t_u = 0 , \qquad t_v = 0 ;$$

and therefore $t =$ const. Conversely, if $t =$ const., equations (7 2) and (7 3) imply that the conditions (7 4) must be satisfied. Thus the proof is completed. A corollary of Theorem 1 may be stated as follows:

Theorem 2. *If segments of constant length are marked off from a surface S on the normals on one side of S, the locus of the extremities of these segments is a surface parallel to S.*

It is quite possible that a surface S_1 represented by equations (7·1) may not be a proper surface. For example, if the surface S is a sphere and if the distance t is the radius of the sphere measured toward the inside of the sphere, the surface S_1 degenerates and becomes merely the center of the sphere. Again, if the surface S is a right circular cylinder, the distance t can be chosen so that S_1 is merely the axis of the cylinder. In what follows, the surface S_1 will be supposed to be a proper surface.

Calculation of the first fundamental coefficients E_1, F_1, G_1 for a surface S_1 parallel to a surface S yields the formulas

$$(7\cdot 6)\qquad \begin{cases} E_1 = (1 - t^2K)E - 2t(1 - tk)L\,, \\ F_1 = (1 - t^2K)F - 2t(1 - tk)M\,, \\ G_1 = (1 - t^2K)G - 2t(1 - tk)N\,, \end{cases}$$

in which K, k are, respectively, the Gaussian and mean curvatures of S. From these formulas one obtains

$$(7\cdot 7)\qquad H_1 = H(1 - 2tk + t^2K)\,,$$

if t is sufficiently small so that the expression in the parenthesis is positive. Calculation of the second fundamental coefficients for S_1 gives the formulas

$$(7\cdot 8)\qquad \begin{cases} L_1 = tKE + (1 - 2tk)L\,, \\ M_1 = tKF + (1 - 2tk)M\,, \\ N_1 = tKG + (1 - 2tk)N\,. \end{cases}$$

Finally, the Gaussian and mean curvatures K_1, k_1 at a point of the surface S_1 are given by the formulas

$$(7\cdot 9)\qquad K_1 = \frac{K}{1 - 2tk + t^2K}\,, \qquad k_1 = \frac{k - tK}{1 - 2tk + t^2K}\,.$$

If $F = M = 0$, then $F_1 = M_1 = 0$. Moreover, if $F = M = 0$, differentiation of equations (7·5) and use of the equations (V·2·14) of Rodrigues lead to the formulas

$$(7\cdot 10)\qquad x_{1u} = \left(1 - \frac{t}{R_1}\right)x_u\,, \qquad x_{1v} = \left(1 - \frac{t}{R_2}\right)x_v$$

and to symmetric formulas for y_1, z_1. The result can be stated as follows:

THEOREM 3. *The lines of curvature correspond on two parallel surfaces, and the tangents to corresponding lines of curvature at corresponding points are parallel.*

The formulas (7 9) can be used to verify the following statement.

THEOREM 4. *If a nonminimal surface has constant mean curvature k, and if the parallel surface S_1 for which $t = 1/(2k)$ is proper, then the Gaussian curvature of S_1 is constant and given by $K_1 = 4k^2$; if the parallel surface S_2 for which $t = 1/k$ is proper, the mean curvature of S_2 is constant and given by $k_2 = -k$.*

The formula for k_1 in the second of the formulas (7 9) can be used to prove the next theorem.

THEOREM 5 *If a nondevelopable surface has constant Gaussian curvature K and if the two parallel surfaces S_1, S_1' for which $t = \pm K^{-1/2}$ are proper, then the mean curvatures of S_1, S_1' are constant and given by*

$$k_1 = \mp\tfrac{1}{2}K^{1/2}.$$

EXERCISES

1. The spherical indicatrices of two parallel surfaces are identical

2. The function

$$\frac{k^2 - K}{K^2} \tag{7 11}$$

is an absolute invariant under the transformation from a surface to a parallel surface.

3. Show that the formulas (7 7) and (7 9) can be written in the form

$$\left\{\begin{aligned} &H_1 = H\left(1 - \frac{t}{R_1}\right)\left(1 - \frac{t}{R_2}\right), \\ &K_1 = \frac{1}{(R_1 - t)(R_2 - t)}, \qquad k_1 = \frac{1}{2}\left(\frac{1}{R_1 - t} + \frac{1}{R_2 - t}\right). \end{aligned}\right. \tag{7 12}$$

If R_1', R_2' are the principal radii of normal curvature of the surface S_1, show that

$$R_1' = R_1 - t, \qquad R_2' = R_2 - t. \tag{7 13}$$

Parallel surfaces have the same principal centers of normal curvature at corresponding points.

4. Discuss the effect of taking t large enough that the expression in parenthesis in the formula (7.7) for H_1 is no longer positive.

5. Establish the formula

$$(7.14) \qquad ds_1^2 = \left| \left(\frac{t}{R} - 1\right)^2 + t^2\left(\frac{1}{R_1} - \frac{1}{R}\right)\left(\frac{1}{R} - \frac{1}{R_2}\right) \right| ds^2 .$$

6. The ratios r_1/R_1, r_2/R_2 are absolute invariants under the transformation from a surface to a parallel surface. Interpret these invariants geometrically

BIBLIOGRAPHY

BIANCHI, L. *Lezioni di geometria differenziale*. 3d ed. Pisa: Spoerri, 1922. Deutsche Übersetzung von M. Lukat. Leipzig: Teubner, 1899.

BLASHKE, W. *Vorlesungen uber Differentialgeometrie*, Vol. I. 3d ed. Berlin: Springer, 1930.

CESARO, E. *Lezioni di geometria intrinsica*. Napoli: Tipografia della R. Accademia delle Scienze, 1896. *Vorlesungen uber naturliche Geometrie*. Deutsche Aufgabe von G. Kowalewski. Leipzig: Teubner, 1926.

DARBOUX, G. *Leçons sur la théorie générale des surfaces*. 2d ed. Paris: Gauthier-Villars, 1914.

EISENHART, L. P. *Differential Geometry*. Boston: Ginn, 1909.

SCHEFFERS, G. *Anwendung der differential und integral Rechnung auf Geometrie*. 3d ed. Berlin: De Gruyter, 1923.

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