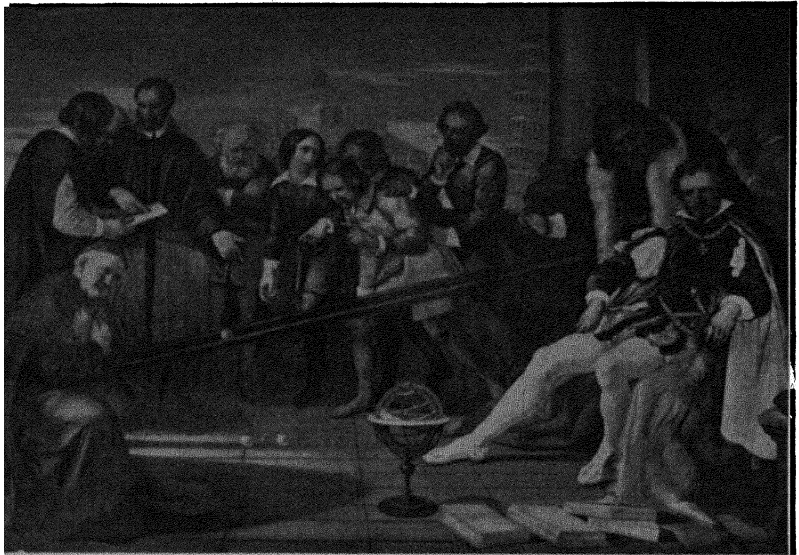


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A SECOND COURSE OF MECHANICS
AND PROPERTIES OF MATTER



Galileo's experiment to investigate the motion of a ball rolling down an inclined plane. The painting, by Giuseppe Bezzuoli, is one of a number of frescoes in the Tribuna di Galileo, a temple dedicated to Galileo in the Museum of Physics and Natural History, adjoining the Pitti Palace in Florence. Galileo, standing by the upper end of the incline, explains to Mazzoni the consequences to be deduced from the experiment. A kneeling peripatetic friar times the ball with his pulse. On the left, a group of professors search the pages of Aristotle to refute the argument of Galileo. In the middle background are the cathedral and leaning tower of Pisa; at the right are a tank and vessel for catching water, whose weight is a measure of the time taken by the ball's motion.

A SECOND COURSE OF MECHANICS AND PROPERTIES OF MATTER

BY

A. E. E. MCKENZIE, M.A.
Trinity, College, Cambridge

When Galileo rolled balls down an inclined
plane a new light burst upon all investigators
of Nature

IMMANUEL KANT



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ILLUSTRATIONS

GALILEO INVESTIGATES THE MOTION OF A BALL
ROLLING DOWN AN INCLINED PLANE
(From the fresco by GUISEPPE BEZZUOLI)

frontispiece

In addition to the text-figures, of which there are 199, the following portraits have been included:

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PREFACE

THE normal sixth form course in schools is followed herein, each topic being developed to about the standard required for University Scholarships.

The first three chapters cover the elements of statics, hydrostatics and dynamics. Mechanics is studied as part of their mathematical course by many science specialists in schools, and there is neither the need nor the time to cover all of the same ground again in physics periods. A descriptive summary of the principles, written from a physical standpoint, may therefore prove a useful revision course for such pupils, and at the same time provide an adequate introduction for biologists whose curriculum does not include mathematics.

A frequent overlap within physics itself, the kinetic theory of gases, has not been included in the book, as it can be treated more conveniently as part of the course in Heat.

One of the problems in writing a school text-book is to preserve a logical sequence and interconnection of ideas, and yet, at the same time, to include all those topics which are normally taught. For example, the theme of Chapter 1 is the equilibrium of forces, but in connection with the parallelogram law, mention must be made of relative velocity. A digression in the text has been avoided in this case by defining relative velocity in a worked answer to an example. [In the same chapter it is desirable to show that the three fundamental principles of statics, namely, the principles of moments, of the parallelogram of forces and of work, are mutually deducible. As proofs of these deductions in the text would have destroyed the balance of the chapter, they have been included with the examples to be worked by the student, hints being given where necessary.

When the writer of an elementary text-book reflects that none of the scientific ideas he presents are original, he can take comfort in the saying of Pascal: 'Qu'on ne dise pas que je n'ai rien dit de nouveau; la disposition des matières est nouvelle. . . . Et comme si les mêmes pensées ne formaient pas un autre corps de discours par une disposition différente, aussi bien que les mêmes mots forment d'autres pensées par leur différente disposition.'

I owe a great debt to Mr F. Record for reading the manuscript and making many valuable suggestions; also to Mr Kingsley Dove, to

Mr P. Haswell and to Mr V. M. Turnbull who shared the onerous task of checking the answers to the questions.

I must also express my thanks to those who provided illustrations; to Mr D. G. A. Dyson who took the photographs for Figs. 121, 179 and 192; to Dr F. P. Bowden for Figs. 197 and 198; to Kodak Ltd. Research Laboratories for Fig. 73; to Messrs Negretti and Zambra for Fig. 56; to Messrs W. & T. Avery Ltd. for Figs. 136 and 137; to Messrs Constable and Company for Fig. 151, taken from Dr Rosenhain's *Introduction to Physical Metallurgy*; to the Director of the Museum of Physics and Natural History, Florence, for the frontispiece; and to *Picture Post* Library for the portraits listed on p. vi.

I am very grateful to the different Examining Bodies for allowing me to reproduce questions. The following is the key to the acknowledgements made after each question: (O.S.) Oxford Scholarship, (C.S.) Cambridge Scholarship, (O. & C.) Oxford and Cambridge, (O.) Oxford Local, (C.) Cambridge Local, (L.) London, (N.) Northern Universities, (B.) Bristol, (D.) Durham, Higher School Certificates. I have stated elsewhere how much, in my opinion, the teaching of science in this country owes to the superb stock of questions accumulated by these Examining Bodies.

A. E. E. M.

CAMBRIDGE

April 1949

CHAPTER 1

STATICS

THE science of mechanics began with the generalisation into principles of practical knowledge of simple machines such as the lever, the pulley, the wheel and axle, the inclined plane and the screw. Records exist of the use of these machines by the ancient Egyptians and Assyrians; but apart from the two contributions of Archimedes (287–212 B.C.), which were the law of the lever and the principle of flotation, the science of mechanics did not develop until the sixteenth century.

This chapter reviews the elements of the branch of mechanics known as *statics*, which deals with the conditions for the equilibrium of forces. Force will be considered as a concept of which we have direct experience—a push, a pull, or an attraction such as weight.

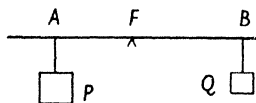


Fig. 1

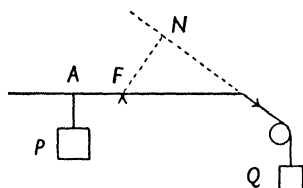


Fig. 2

Principle of Moments

It is obvious to anyone who uses a lever that the longer the arm between the pivot or fulcrum and the point at which a force is applied, the greater the turning effect. Archimedes discovered an exact, quantitative law applying to the equilibrium of two forces acting on a lever (Fig. 1):

$$P \times AF = Q \times FB.$$

Despite his ingenious efforts to deduce it geometrically the law is now recognised as an empirical fact, i.e. its basis is experiment.

Leonardo da Vinci (1452–1519) was one of the first to see clearly that, in the case of an oblique force, the effective lever arm is the perpendicular from the fulcrum to the line of action of the force. Thus, in Fig. 2,

$$P \times AF = Q \times FN.$$

The product of a force and the perpendicular distance from an axis on to the line of action of the force is known as the moment of the force about the axis.

A general condition for equilibrium may be stated in terms of moments in the following form:

Principle of moments: If a body is in equilibrium under the action of a number of forces, the algebraic sum of the moments of the forces about any axis is zero.

Moments may be taken about any axis because a body in equilibrium does not tend to rotate. But if moments are not taken about the fulcrum, the moment of the reaction of the support must be taken into account. In Fig. 3 this reaction is $P+Q$ (assuming the weight of the lever itself to be negligible), and its moment about F is zero.

Suppose moments are taken about one end, C , of the bar:

$$P \times CA - (P+Q) CF + Q \times CB = 0.$$

This reduces to the simple law of the lever,

$$P \times AF = Q \times FB,$$

confirming that the general principle holds.

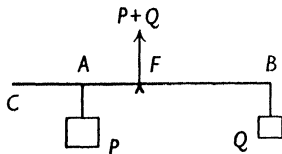


Fig. 3

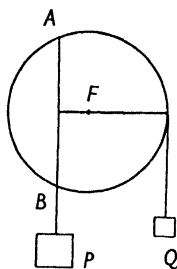


Fig. 4

The parallelogram law

A force has direction as well as magnitude, and hence is a *vector* quantity, as opposed to a *scalar* quantity, which has magnitude only. A force also has a definite line of action; its point of application, however, may be considered as any point on the line of action so long as the point is rigidly connected to the body. Thus, in Fig. 4, suppose that two weights P and Q , attached to a disc freely pivoted at its centre, counterbalance; P could be attached to the disc at any point on the vertical line AB without disturbing the equilibrium.

The vector quantities met with in mechanics are displacement, velocity, acceleration, force and momentum, the scalar quantities being mass, work and energy.

Two vector quantities cannot be added algebraically like scalar quantities; their resultant must be found geometrically. Suppose a point A

suffers simultaneously displacements AB and AD (Fig. 5); its resultant displacement is the diagonal AC of the parallelogram $ABCD$. This type of addition law applies to all the vector quantities enumerated above. Its statement for forces is as follows:

Parallelogram of forces. If two forces acting at a point are represented in magnitude and direction by the sides of a parallelogram drawn from that point, then their resultant is represented by the diagonal of the parallelogram drawn from that point.

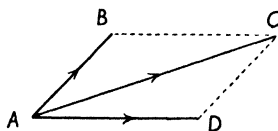


Fig. 5

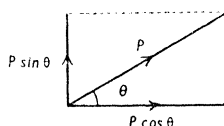
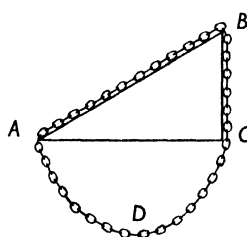
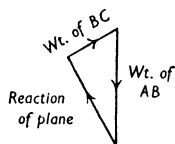


Fig. 6



(a)



(b)

Fig. 7

Non reacting

It might seem that this law is evident without recourse to experiment. But it rests on the mutual independence of forces, the effect of each force being the same as though the other were not acting, and this is an empirical fact.

Just as two forces may be compounded into a single resultant, so a single force may be resolved into two components. It is usually convenient to resolve a force into two components at right angles; thus the force P in Fig. 6 is equivalent to the two mutually perpendicular forces $P \cos \theta$ and $P \sin \theta$.

The principle of the parallelogram of forces was first explicitly stated by Sir Isaac Newton (1642–1727), but it was known and used earlier by Stevinus of Bruges (1548–1620). Stevinus derives his ideas from a consideration of an endless chain resting on a smooth inclined plane (Fig. 7(a)). The chain must be in equilibrium, otherwise perpetual motion, which was assumed by Stevinus to be impossible, would result. Considerations of symmetry show that, if the portion of the chain

ADC is removed, the portion ABC remains in equilibrium. The portion of the chain AB is supported by the weight of BC acting in the direction AB and by the reaction of the plane acting at right angles to AB . Thus Fig. 7 (b) represents the resolution of the weight of AB into two components at right angles.

The principle of the parallelogram of forces may be regarded as being just as empirical and fundamental as the principle of moments, and the two are mutually deducible (see Example 10, p. 21).

EXAMPLE

A ship A is travelling due north at 20 knots and a ship B due east at 15 knots. Find the relative velocity of A with respect to B .

If B is 10 sea miles due west of A at noon, what will be their shortest distance apart and when will this occur?

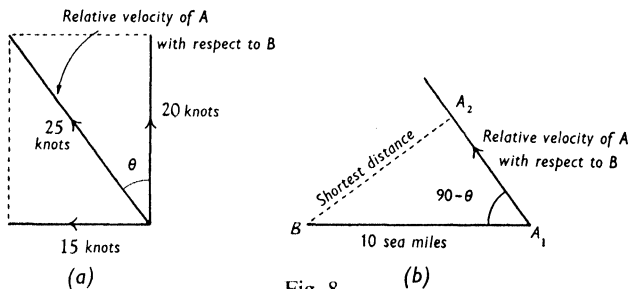


Fig. 8

The relative velocity of A with respect to B is the velocity A appears to have to an observer on B who regards himself as being at rest.

If both A and B are given additional velocities equal and opposite to that of B , then B is brought to rest, and A travels with the relative velocity required.

The relative velocity of A may be found by the parallelogram of velocities (Fig. 8 (a)).

$$\text{Diagonal} = \sqrt{(15^2 + 20^2)} = 25,$$

$$\tan \theta = 15/20,$$

$$\therefore \theta = 36^\circ 52'.$$

Thus the relative velocity of A with respect to B is 25 knots in a direction $36^\circ 52'$ west of north.

To find the shortest distance between B and A imagine B at rest and A moving with the relative velocity just determined (Fig. 8 (b)).

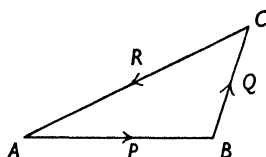
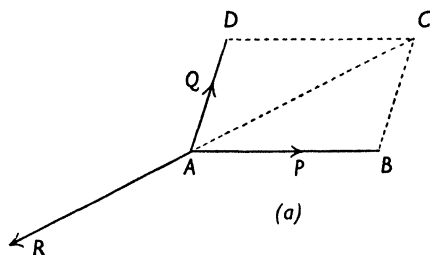
$$\begin{aligned} \text{Shortest distance} &= 10 \sin (90 - \theta) = 10 (20/25) \\ &= 8 \text{ sea miles.} \end{aligned}$$

$$\begin{aligned} \text{Time to reach shortest distance} &= \frac{A_1 A_2}{\text{relative velocity}} \\ &= \frac{10 \cos (90 - \theta)}{25} = \frac{6}{25} \text{ hr.} \\ &= 14.4 \text{ min.} \end{aligned}$$

Hence the shortest distance between the ships is 8 sea miles at 12.14 hr.

Three inclined forces in equilibrium

Three non-parallel, coplanar forces which are in equilibrium must be concurrent. For suppose that the line of action of the third force does not pass through the point of intersection of the other two forces. Then the third force has a moment about this point* whereas the other two forces have zero moments; hence the three forces could not be in equilibrium. The direction of the reaction of a hinge can sometimes be found because it must pass through the point of intersection of the other two forces acting on the hinged body.



(b) Triangle of forces

Fig. 9

The following theorem, known as the Triangle of Forces, is a corollary to the Parallelogram of Forces: *When three forces acting at a point can be represented in magnitude and direction by the sides of a triangle, taken in order, the forces are in equilibrium.*

Suppose that the three forces P , Q , R in Fig. 9 (a) can be represented in magnitude and direction (although not in position) by the sides AB , BC , CA respectively of the triangle ABC . AC represents the resultant of P and Q , and this is equal and opposite to R , which is represented by CA . Hence the three forces are in equilibrium.

The converse of the theorem of the triangle of forces is also true.

There is a further corollary to the Parallelogram of Forces:

Polygon of forces. If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon, taken in order, then the forces are in equilibrium.

* Strictly 'an axis through the point perpendicular to the plane of the forces'.

This can be proved by dividing the polygon $ABCDE$ into triangles (Fig. 10 (b)). In the triangle ABE the forces T , P and a force represented by BE are in equilibrium; similarly, S and forces represented by EB and BD are in equilibrium, as are Q , R and a force represented by DB . Since pairs of equal and opposite forces have been introduced along BE and BD , P , Q , R , S , T must be in equilibrium.

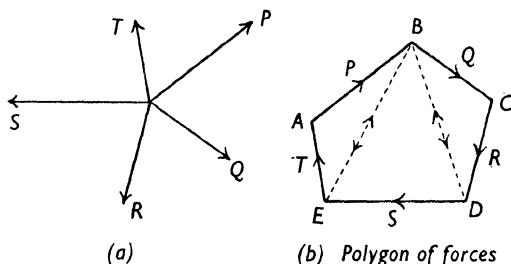


Fig. 10

Resultant of two parallel forces

In Fig. 3 the force $P+Q$ acting upwards at F counterbalances the two forces P and Q . It must therefore counterbalance the resultant of P and Q ; hence the resultant must be $P+Q$ acting downwards at F .

In the case of two unlike (i.e. opposed) parallel forces P and Q (Fig. 11), the resultant is clearly $P-Q$ acting at some point F , whose position may be found by the principle of moments. Since the moment of the resultant about F is zero, the sum of the moments of P and Q about F must be zero, i.e.

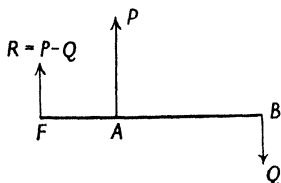


Fig. 11

$$P \times AF = Q \times FB.$$

The couple

A combination of two equal unlike parallel forces (Fig. 12), known as a *couple*, cannot be represented by a single force. A couple has no resultant and produces rotation, but not translation, of the body on which it acts. It can be balanced only by another couple.

The moment of a couple is the same about any axis, namely, Pd (Fig. 12). For example, the moment about an axis distant x from one of the forces is

$$P(x+d) - Px = Pd.$$

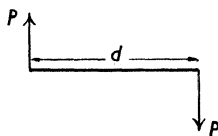


Fig. 12

General condition for equilibrium

Any system of coplanar forces, which is not in equilibrium, can be reduced either to a single force, or to a couple.

The following are sufficient conditions for the equilibrium of a system of coplanar forces:

(a) *Condition for translational equilibrium:* $\Sigma X=0$ and $\Sigma Y=0$, where X and Y represent components of the forces in two directions, usually taken at right angles.

(b) *Condition for rotational equilibrium:* the sum of the moments about any axis is zero.*

An example illustrating the application of these criteria is given on p. 12.

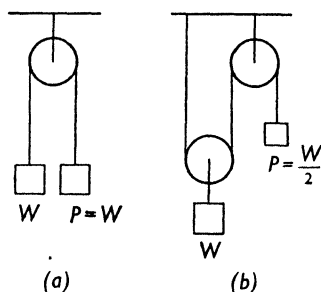


Fig. 13

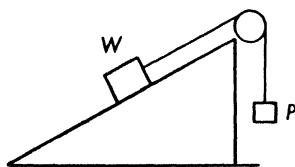


Fig. 14

Work

There is another method of considering the conditions for equilibrium. Stevinus realised that when a load W is raised by an effort P applied to a system of pulleys, the product of W and the distance moved by W is equal to the product of P and the distance moved by P . With a single pulley, $W=P$, and the distances moved are equal (Fig. 13 (a)); with the pulley system of Fig. 13 (b), the distance moved by P is double that moved by W and $P=\frac{1}{2}W$ (ignoring the weight of the lower pulley).

Galileo regarded the inclined plane in a similar way. Equilibrium is achieved (Fig. 14) when

$(P \times \text{vertical distance moved by } P) = (W \times \text{vertical distance moved by } W)$.
Although W moves on the incline it is the vertical distance through

* Other sufficient conditions for equilibrium are: (i) $\Sigma X=0$ and the sums of the moments about any two axes are zero; (ii) the sums of the moments about any three, non-collinear axes are zero.

which it moves that counts. Thus a new concept, that of work, was introduced.

Work is equal to the product of a force and the distance moved along the line of action of the force.

The principle of virtual work

In 1717 John Bernoulli enunciated the principle of virtual work which applies to all cases of equilibrium. Suppose a system of forces suffers a slight displacement consistent with the geometrical conditions. The product of each force and the displacement along its line of action is called the *virtual work*—virtual because the displacement is imagined. The system is in equilibrium if the total virtual work is zero.

EXAMPLE

An elastic ring of weight W rests on a smooth cone of semi-vertical angle θ as shown in Fig. 15. Find the tension in the ring.

The forces acting on the ring are (i) its weight W , which may be considered to act at its centre of gravity, the centre of the ring; (ii) its tension T , which acts along the circumference of the ring; (iii) the reaction R of the cone on the ring, which must be at right angles to the cone since the cone is smooth and hence can exert no frictional component along its surface.

Suppose that the centre of the ring is at a distance h below the vertex of the cone. Let $h \rightarrow h + \delta h$ so that the ring is stretched slightly. The radius of the ring increases by $\delta h \tan \theta$ and its circumference by $2\pi\delta h \tan \theta$.

Since the ring is in equilibrium the sum of the virtual work is zero:

$$\text{Virtual work done by weight} = W\delta h,$$

$$\text{Virtual work done by tension} = -T \cdot 2\pi\delta h \tan \theta.$$

The negative sign indicates that work must be done against T in stretching, whereas W does work in falling.

The virtual work done by R is zero, since the ring moves at right angles to the force.

$$\therefore W\delta h - T \cdot 2\pi\delta h \tan \theta = 0,$$

$$T = \frac{W}{2\pi} \cot \theta.$$

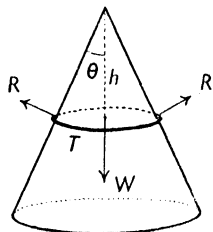


Fig. 15

Machines

A machine cannot deliver more work than is put into it. This is sometimes called the principle of work. Had this principle been firmly grasped in the past, much time and ingenuity devoted to attempting to devise perpetual-motion machines would have been saved.

In practice the useful work given out by a machine is less than the work put into it, some being expended in overcoming friction and some in moving the parts of the machine. Hence arises the concept of efficiency, defined as follows:

$$\text{Efficiency} = \frac{\text{Work got out}}{\text{Work put in}}.$$

The force applied to the machine is usually called the effort, P , and the force the machine overcomes is called the load W . Two further concepts refer to machines:

$$\text{Mechanical advantage} = \frac{\text{Load}}{\text{Effort}},$$

$$\text{Velocity ratio} = \frac{\text{Distance moved by effort}}{\text{Distance moved by load}}.$$

The reader should deduce that

$$\text{Efficiency} = \frac{\text{Mechanical Advantage}}{\text{Velocity Ratio}}.$$

EXAMPLE 1

Fig. 16 represents a Weston pulley tackle. A and B are two pulleys, rigidly attached so that they revolve together, and they are fitted with notches so that an endless chain, which passes round them and also round a pulley C , cannot slip. P is the effort and W the load. If there are 15 notches on A and 16 on B , find the velocity ratio. If an effort of 40 lb. is required to raise a load of 5 cwt., calculate the mechanical advantage and the efficiency.

After one complete revolution the effort has moved a distance equal to the circumference of B . The part of the chain passing round C has wound on to B but has wound off A . The load W has risen half the difference of the circumferences of B and A because the chain on both sides of C has been shortened.

$$\therefore \text{Velocity ratio} = \frac{16}{\frac{16-15}{2}} = 32.$$

$$\text{Mechanical advantage} = \frac{5 \times 112}{40} = 14.$$

$$\begin{aligned} \text{Efficiency} &= \frac{\text{Mechanical advantage}}{\text{Velocity ratio}} \\ &= \frac{14}{32} = 0.44. \end{aligned}$$

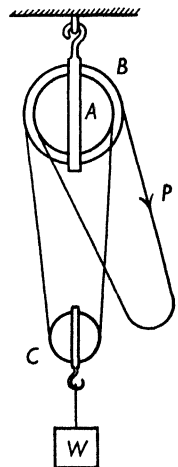


Fig. 16

As the efficiency is less than 50%, more than half of the effort is used to overcome friction and less than half the effort to support the load. Hence the friction alone is more than sufficient to support the load when the effort is removed, and the machine will not 'overhaul'.

EXAMPLE 2

The pitch of the screw in the press shown in Fig. 17 is $\frac{1}{2}$ inch and the total length of the handle is 18 inches. Two forces P , each equal to 20 lb. wt., are applied at right angles to the ends of the handle. If the efficiency is 35%, what is the load W that can be overcome?

When the screw turns through 1 complete revolution it moves down a distance equal to its pitch. In 1 revolution,

$$\text{Work done by } P = 2 \times 2\pi \times 9 \times 20 = 720\pi \text{ in. lb.}$$

$$\text{Work done against } W = \frac{1}{2}W \text{ in. lb.}$$

$$\therefore \text{Efficiency} = \frac{\frac{1}{2}W}{720\pi}$$

$$\text{also} = \frac{35}{100}.$$

$$\therefore W = 1580 \text{ lb. wt.}$$

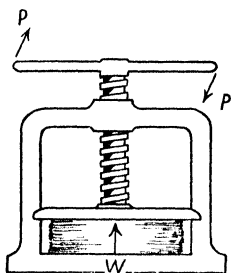


Fig. 17

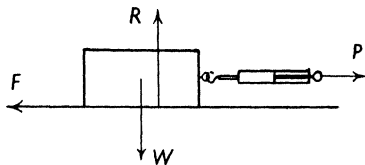


Fig. 18

Friction

Whenever a body slides, or tends to slide, on another an opposing force of friction is set up. It is common experience that the friction increases if the bodies are pressed together, as, for example, when the brake of a bicycle is applied. It is perhaps surprising to find that the force of friction is independent of the area of the brake block in contact with the rim, so long as the force or normal reaction between the brake block and the rim remains the same, and also that the friction is independent of the speed at which the bicycle is moving.

These facts are embodied in what are known as the laws of friction, which are only roughly true:

(1) Friction is independent of the area of the surfaces in contact.

(2) Friction is just sufficient to prevent sliding until the maximum or limiting value is reached, when movement is on the verge of commencing. Limiting friction F is directly proportional to the normal reaction R between the surfaces; the ratio F/R is called the *coefficient of friction* μ .

(3) Friction is independent of the relative velocities of the surfaces.

The laws can be demonstrated with the apparatus of Fig. 18, which represents a block of wood or metal to which a spring balance is attached, resting on a horizontal surface. If reasonably consistent results are to be obtained the surfaces must be clean and smooth. The reading of the spring balance increases until the body is about to slide, and when sliding commences the force of friction, called *kinetic* as opposed to *static* friction, is slightly less than the limiting friction. (The coefficients of static and kinetic friction are nearly equal and are often assumed to be the same.) The block can be turned on to its end, which has a smaller area than the bottom to show that friction is independent of the area of the surfaces in contact. The normal reaction R , which in Fig. 18 is equal to the weight of the block, W , can be varied by resting different weights on the top of the block, and in this way several values of F/R and hence of μ can be found. (Note that in Fig. 18 there are two couples which are in equilibrium; as P increases R moves farther away from W .)

The coefficient of friction varies widely for different materials. For ice sliding on ice it is between 0.2 and 0.03, while for copper sliding on copper it is between 0.8 and 1, and may even exceed 1. The nature and mechanism of friction will be discussed in Chapter 9.

The angle of friction

Fig. 19 represents a body which is on the point of sliding down an inclined plane. The three forces acting on the body, its weight W , the

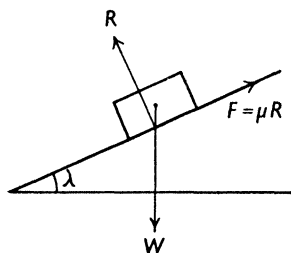


Fig. 19

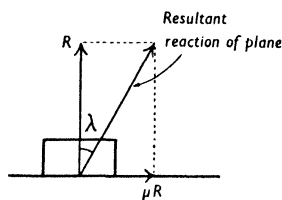


Fig. 20

normal reaction of the plane R , and the force of limiting friction μR , are in equilibrium. (Note that they are therefore concurrent.)

Resolving along the plane

$$W \sin \lambda = \mu R.$$

Resolving perpendicular to the plane

$$W \cos \lambda = R.$$

Dividing $\tan \lambda = \mu$.

Thus μ may be determined by finding the angle to which a surface must be tilted for a body just to commence to slide down. λ is called the *angle of friction*.

The resultant reaction between two horizontal surfaces when sliding is about to commence makes the angle λ with the normal (Fig. 20);

$$\tan \lambda = \frac{\mu R}{R} = \mu.$$

EXAMPLE

A uniform ladder, 13 ft. long and weighing 50 lb., rests with its upper end against a smooth wall and its lower end on the rough ground at a point 5 ft. from the wall. What must be the least coefficient of friction between the ground and the ladder to prevent slipping?

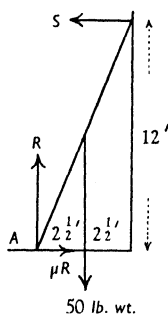


Fig. 21

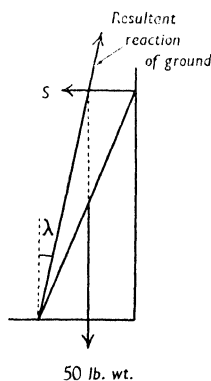


Fig. 22

The forces acting on the ladder are shown in Fig. 21. Since the wall is smooth, the reaction S is normal, and since the ladder is uniform its weight can be considered to act at its midpoint. If the ladder is on the point of slipping the forces exerted by the ground on it will be a normal reaction R and a force of limiting friction μR .

By the theorem of Pythagoras, the height above the ground of the upper end of the ladder is 12 ft.

Resolving vertically, $R = 50$.

Resolving horizontally, $\mu R = S$.

Taking moments about A , $12 S = 2\frac{1}{2} \times 50$.

$$\therefore S = 10\frac{5}{4} \text{ lb. wt.}$$

$$\therefore \mu = \frac{S}{R} = \frac{10\frac{5}{4}}{50} = 0.21.$$

The problem may be solved graphically as in Fig. 22. The resultant reaction of the ground must pass through the point of intersection of the lines of action S and of the 50 lb. wt., since three forces which are in equilibrium must be concurrent. The angle λ is measured and $\tan \lambda (= \mu)$ is found.

Centre of gravity

The weights of all the particles of which a body is composed form a system of parallel forces, and these are equivalent to a single resultant, the total weight of the body, acting at a point which is called the *centre of gravity* of the body.

The position of the centre of gravity does not change when a body is differently orientated in the earth's gravitational field—a fact which is not immediately obvious.

Consider the simple case of four particles, of weights w_1, w_2, w_3 and w_4 , situated in a straight horizontal line, and suppose that their centre

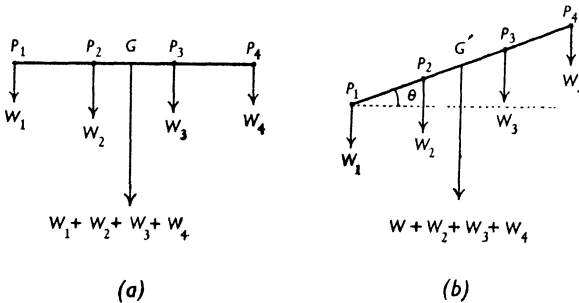


Fig. 23

of gravity is at G (Fig. 23 (a)). Let $P_2P_1 = x_2$, $P_3P_1 = x_3$, $P_4P_1 = x_4$ and $GP_1 = \bar{x}$. The moment of the resultant weight, acting at G , about any axis is equal to the sum of the moments of the weights of the separate particles.

Taking moments about an axis through P_1 perpendicular to the plane of the paper,

$$(w_1 + w_2 + w_3 + w_4) \bar{x} = w_2x_2 + w_3x_3 + w_4x_4.$$

$$\therefore \bar{x} = \frac{w_2x_2 + w_3x_3 + w_4x_4}{w_1 + w_2 + w_3 + w_4}.$$

Now suppose the line P_1P_4 to be turned through an angle θ (Fig. 23 (b)); let the new position of the centre of gravity be G' and $P_1G' = \bar{x}'$.

Taking moments about an axis through P_1 as before,

$$(w_1 + w_2 + w_3 + w_4) \bar{x}' \cos \theta = w_2x_2 \cos \theta + w_3x_3 \cos \theta + w_4x_4 \cos \theta$$

$$\therefore \bar{x}' = \frac{w_2x_2 + w_3x_3 + w_4x_4}{w_1 + w_2 + w_3 + w_4}.$$

Since $\bar{x}' = \bar{x}$, the position of G in the line P_1P_4 is unchanged by the alteration in the orientation of the system.

A similar argument can be applied to all the particles composing a solid body.

Centres of gravity in simple cases

From considerations of symmetry the centre of gravity of a rectangular or of a circular lamina must lie at the midpoint. (A lamina is a body of uniform thickness; here the thickness is assumed to be negligible to avoid the complication of a third dimension.)

The centre of gravity of a triangular lamina is at the intersection of the medians, which lies one-third of the way up each median. Suppose

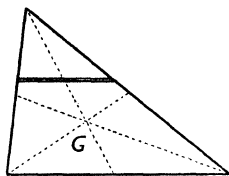


Fig. 24

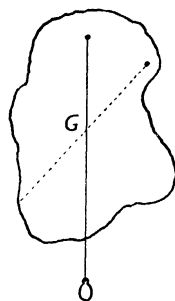


Fig. 25

the triangle to be divided into narrow strips, parallel to the base, as in Fig. 24. The centre of gravity of each strip is at its midpoint, and hence the centre of gravity of the triangle must be on the median, which is the straight line joining the midpoints. For similar reasons the centre of gravity of the triangle is on the other two medians, and it must therefore be at their point of intersection.

The centre of gravity G of a lamina, such as a piece of cardboard, may be found experimentally by suspending the lamina freely and, by means of a plumb line, marking on it the vertical through the point of suspension (Fig. 25). G must lie vertically below the point of suspension, otherwise the lamina would rotate. If at least two positions for the point of suspension are taken, G is determined because it is at the intersection of the verticals in the two positions.

To find the position of the centre of gravity G of a rod of non-uniform cross-section, such as a billiard cue, the rod may be balanced horizontally on a knife edge when G must lie vertically above the knife

edge (Fig. 26). The weight of the rod W may then be found by displacing the knife edge and balancing the system by means of a weight W' suitably placed (Fig. 27)

$$W \times FG = W' \times AF.$$

$$\therefore W = W' \frac{AF}{FG}.$$

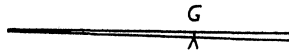


Fig. 26

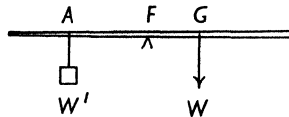


Fig. 27

Sometimes the centre of gravity of a body lies outside the body. For example, the centre of gravity of the L-shaped piece of wire in Fig. 28 is at G , where

$$DG \times \text{weight of } AB = EG \times \text{weight of } BC.$$

A knowledge of the position of the centre of gravity enables a prediction to be made of the angle to which a body can be tilted before it

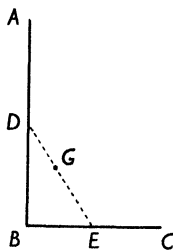


Fig. 28

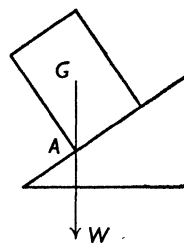


Fig. 29

topples over, assuming that it does not slide first. The body in Fig. 29 is on the verge of toppling because its centre of gravity G is vertically above A . If the plane on which it is resting is tilted farther, the line of action of W will lie outside the base and W will have an overturning effect about A . Buses are constructed with as low a centre of gravity as possible in order to minimise the danger of toppling over.

EXAMPLE

Find the position of the centre of gravity of the lamina $ABEFC$ in Fig. 30.

The centre of gravity of $ABCD$ is at its midpoint G_1 , and the centre of gravity of BEF is at G_2 , where $EG_2 = \frac{2}{3}a$. Suppose the centre of gravity of the whole lamina is at G , at distances $\bar{x}$, $\bar{y}$ from AD and DC respectively.

Let w be the weight per unit area of the lamina. The weight of $ABCD$ is $4abw$, and the weight of BEF is $\frac{1}{2}abw$.

Taking moments about AD ,

$$(4abw + \frac{1}{2}abw)\bar{x} = 4abw \cdot a + \frac{1}{2}abw(2a + \frac{1}{3}a).$$

$$\therefore \bar{x} = \frac{4a + \frac{7}{6}a}{4\frac{1}{2}} = \frac{3}{2}a.$$

Taking moments about DC ,

$$(4abw + \frac{1}{2}abw)\bar{y} = 4abw \cdot b + \frac{1}{2}abw(b + \frac{1}{3}b).$$

$$\therefore \bar{y} = \frac{4b + \frac{5}{6}b}{4\frac{1}{2}} = \frac{10}{9}b.$$

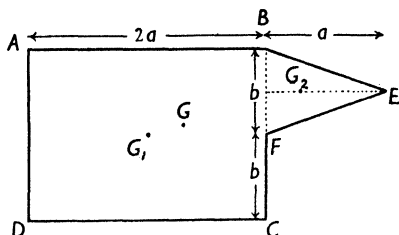


Fig. 30

Centre of mass

The term centre of mass (or centroid) is often used synonymously with centre of gravity and, under normal conditions, they are the same point. If a series of masses m_1 , m_2 , etc., are at distances x_1 , x_2 , etc., from a given axis, the distance of the centre of mass $\bar{x}$ from the axis is defined as

$$\bar{x} = \frac{\sum m_1 x_1}{\sum m_1}.$$

This definition of the position of the centre of mass is similar to that obtained for the centre of gravity of the four particles on p. 13.

If, however, a body is situated in a non-uniform gravitational field the weights of its particles are not in the ratio of their masses. Under such conditions a body may not possess a centre of gravity, since the weights of its particles may not be parallel forces and may not reduce to a single resultant force but to a couple which, moreover, will vary for different orientations of the body. The centre of mass, on the other hand, is independent of the gravitational field.

Stability

There are three types of equilibrium, represented in Fig. 31 and known by the self-descriptive names: *stable*, *neutral* and *unstable* equilibrium. On being given a slight displacement, the sphere tends to return to its original position in (a), to remain in the new position in (b) and to

become displaced further in (c); its centre of gravity rises in (a), remains at the same horizontal level in (b) and falls in (c).

The fact that a body tends to take the position in which its centre of gravity is at the lowest possible point enables the shape, known as a

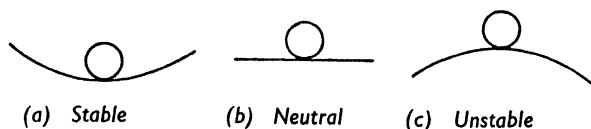


Fig. 31

catenary, of a hanging chain or flexible rope supported from its two ends, to be calculated mathematically. If the chain is drawn down as in Fig. 32 (b) its centre of gravity must rise, since, on release, it returns to its original shape.

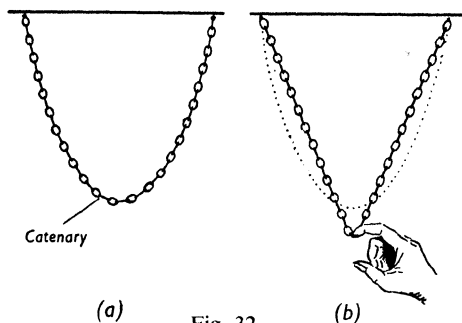


Fig. 32

The balance

An important practical example, involving a consideration of stability, is the common balance, whose essential feature is a symmetrical beam pivoted centrally. If the beam were pivoted at its centre of gravity it would be in neutral equilibrium and would remain at rest at any angle to the horizontal. The centre of gravity is arranged in practice to be below the point of support, so that it rises when the beam is deflected from the horizontal, and hence the equilibrium is stable. If the centre of gravity were above the point of support, the equilibrium would be unstable.

The sensitiveness of a balance is usually defined as the deflection, in scale divisions, when an excess weight of 1 mgm. is added to one scale pan. It will be seen that sensitiveness and stability are, to some extent, mutually exclusive.

Consider a beam AB (Fig. 33), of length $2l$ and weight W , pivoted at its midpoint F , and with its centre of gravity G at a distance d vertically below F . Suppose that the weights of the scale pans are w and that a small weight p placed in one of them causes a deflection θ . Taking moments about F ,

$$wl \cos \theta + Wd \sin \theta = (w+p)l \cos \theta.$$

$$\therefore \tan \theta = \frac{pl}{Wd}.$$

If θ is small, $\tan \theta = \theta$. $\therefore \theta/p = \frac{l}{Wd}.$

The sensitiveness is represented by θ/p ; for it to be high, l must be large, and W and d small. The beam is not made too long, however,

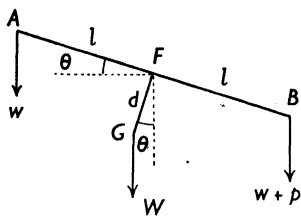


Fig. 33

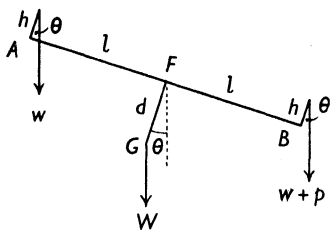


Fig. 34

because this means a long period of swing and slow weighing. The weight W is made as small as possible consistent with strength, by constructing the beam in the form of a lattice. The value of d can usually be varied by means of a small weight which slides on the pointer; by decreasing d the sensitiveness is increased but the equilibrium becomes less stable, the period of swing becomes longer and weighing more difficult.

The scale pans are suspended from knife edges on the beams. The ideal arrangement is that in Fig. 33, in which the three knife edges A , F , B are in a straight line, but in cheap balances this may not be exactly so. If the knife edges at the ends of the beam are above F the sensitiveness increases with the load; if below, the sensitiveness decreases with increasing load. Suppose the knife edges are at a distance h above the beam (Fig. 34). Taking moments about F ,

$$w(l \cos \theta - h \sin \theta) + Wd \sin \theta = (w+p)(l \cos \theta + h \sin \theta),$$

$$\frac{\tan \theta}{p} = \frac{l}{Wd - (2w+p)h}.$$

Thus as w increases, the sensitiveness increases.

Errors due to faults in a balance may be eliminated by double weighing. Suppose, for example, that a balance beam swings evenly about the horizontal position when the scale pans are unloaded, but that a body appears to have different weights, W_1 and W_2 , when weighed in the two pans. The error is probably due to a slight inequality in the lengths of the two arms of the beam. Suppose that these lengths are l_1 and l_2 and that the true weight of the body is W :

$$W_1 l_1 = W l_2,$$

$$W l_1 = W_2 l_2.$$

$$\therefore W = \sqrt{W_1 W_2}.$$

Another possible source of error is that the weights may not be accurate owing to wear or other causes. It is usual to assume that the largest weights in the box are accurate, and to balance say the 50 gm. weight against $20 + 20 + 5 + 2 + 2 + 1$ gm., 20 against $10 + 5 + 2 + 2 + 1$ gm., and so on, as a test of the accuracy of the weights.

SUMMARY

For the discussion of equilibrium there are three fundamental principles, mutually deducible, the principle of moments, the parallelogram of forces, and the principle of work. For the application of these principles it is convenient to devise the concept of centre of gravity, and therewith to derive the principle that if equilibrium is stable, a slight displacement causes a rise in the centre of gravity.

QUESTIONS

MOMENTS, GENERAL CONDITIONS FOR EQUILIBRIUM AND FRICTION

1. Define the *moment of a force* about a point.

Describe an experiment to verify the principle of moments.

A uniform circular table weighing 20 lb. is supported on three vertical legs uniformly spaced on its circumference of 44 in. Calculate the upward force required to tilt the table when applied (a) at one leg, (b) at a point on the circumference of the table midway between two adjacent legs.

(Take π as $\frac{22}{7}$.)

(N.)

2. Define a couple, and prove that it has the same moment about all points in its plane.

A uniform rectangular gate $ABCD$, of weight 200 lb., is hinged at A and D . AD is vertical and 4 ft. long, AB is 10 ft. If the reaction at A is horizontal, find its magnitude. State the horizontal and vertical components of the reaction at D , and hence show that two equal and opposite couples are acting on the gate.

3. Explain what is meant by the resultant of a number of coplanar parallel forces and show how it may be calculated.

A plank 10 ft. long and weighing 40 lb. rests symmetrically on two parallel rails, A and B , which are in the same horizontal plane and 6 ft. apart. A load of weight 100 lb. is moved along the plank to such a position that the total force on B is a maximum subject to the condition that the plank remains in equilibrium on the rails. What is then the force on B and where has the load been placed? (O.)

4. Define the *moment of a force*.

A reel of cotton consists of a cylinder of radius a with ends of radius b ($b > a$), and rests with both its ends on a horizontal table. The cotton thread leaves the reel on the lower side and is held under tension at right angles to the axis, and at an angle α to the table. Show that the reel will wind up or unwind according as α is less or greater than $\cos^{-1} a/b$. When the thread is inclined at this critical angle, find the tension necessary to *slide* the reel along the table, if its mass is M and the coefficient of friction between it and the table is μ . (O. & C.)

5. State the conditions of equilibrium of a body acted upon by a system of coplanar forces.

A sphere of weight W rests between two inclined planes inclined respectively at 30° and 60° to the horizontal. What is the ratio of the reactions of the two planes? (O. S.)

6. Summarise the various conditions which are being satisfied when a body remains in equilibrium under the action of three non-parallel forces.

A wireless aerial attached to the top of a mast 20 ft. high exerts a horizontal force upon it of 60 lb. wt. The mast is supported by a stay-wire running to the ground from a point 6 ft. below the top of the mast, and inclined at 60° to the horizontal. Assuming that the action of the ground on the mast can be regarded as a single force, draw a diagram of the forces acting on the mast, and determine by measurement or by calculation the force in the stay-wire. (C.)

7. Three forces in one plane act on a rigid body. What are the conditions for equilibrium?

The plane of a kite of mass 12 lb. is inclined to the horizon at 60° . The resultant thrust of the air on the kite acts at a point 10 in. above its centre of gravity, and the string is attached at a point 12 in. above the centre of gravity. Find the thrust of the air on the kite, and the tension in the string. (O. & C.)

8. State the principle of the *parallelogram of forces*, and describe an experiment to demonstrate its truth.

A uniform rod, 10 ft. long, weighing 10 lb. is fastened at one end to a vertical wall by a smooth hinge. A string 4 ft. long, fastened to a point 5 ft. vertically above the hinge, is attached to the rod at a point 3 ft. from the hinge. Give a diagram of the arrangement, drawn approximately to scale, and find (i) the tension in the string, (ii) the reaction of the hinge. (N.)

9. A uniform rod AB of mass 200 gm. is hinged to a wall at A , so that it can turn freely in a vertical plane, and is held horizontally by a string BC attached to a fixed point C vertically above A . The position of C is such that $AC = \frac{1}{2}BC$. A mass of 500 gm. hangs from the end B . Calculate the tension in the string, and the magnitude and direction of the reaction at the hinge. (C.)

10. Deduce the principle of the lever from the parallelogram of forces as follows. In Fig. 35 R is the resultant of forces P and Q obtained by the parallelogram of forces. Let P and Q act on a lever, of negligible weight, in a position such as that shown. If the lever is to be in equilibrium, the pivot F must lie on the line of action of R ; hence prove that

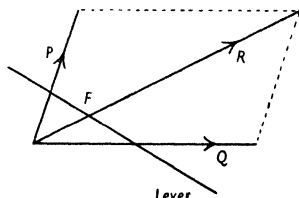


Fig. 35

$$P \times \text{perpendicular from } F \text{ on to } P = Q \times \text{perpendicular from } F \text{ on to } Q.$$

11. State the laws of friction for solids, and indicate how you would verify them by experiment.

A window curtain weighs 6 lb. and is supported by six equidistant *thin* curtain rings from a horizontal rod, so that the weight is equally distributed between the rings. If the coefficient of friction is 0.5 and the rings are 9 in. apart, find the work done against friction in drawing the curtain back to the position of the end ring.

(N.)

12. Explain the meaning of the term *coefficient of friction* and describe an experimental method of measuring it.

A rectangular block with a square base of side 10 cm. rests on a horizontal surface. If a horizontal force is applied near the bottom of one vertical face, the block slides. If the force is applied near the top, the block topples over. When the force is applied at 20 cm. from the bottom, the block sometimes slides and sometimes topples. Find the coefficient of friction between the block and the surface.

(C.)

13. A uniform ladder, 13 ft. long and weighing 60 lb., rests with its upper end against a smooth vertical wall and with its lower end 5 ft. from the wall on rough ground. By a graphical method find the magnitude and direction of the reaction at the foot of the ladder. Check your result by calculation. What is the least coefficient of friction between ground and ladder necessary to maintain equilibrium?

(C.)

14. A uniform rod rests in limiting equilibrium in contact with a horizontal floor and a vertical wall, the rod being in a vertical plane which is perpendicular to the wall. If the wall and the floor are equally rough, prove that the angle between the rod and the wall is twice the angle of friction.

(L.)

15. A man weighing 140 lb. climbs up a uniform ladder, 20 ft. long and 70 lb. in weight, which rests against a rough vertical wall at an angle of 45° . If the coefficient of friction at each end of the ladder is 0.5, how far will the man be able to climb up the ladder before it begins to slip?

(N.)

16. Distinguish between *static* and *sliding* (kinetic) friction and define the *coefficient of sliding friction*.

How would you investigate the laws of sliding friction between wood and iron?

An iron block, mass 10 lb., rests on a wooden plane inclined at 30° to the horizontal. It is found that the least force parallel to the plane which causes the block to slide *up* the plane is 10 lb. wt. Calculate the coefficient of sliding friction between wood and iron.

(N.)

17. A body of weight W rests upon a rough plane inclined to the horizontal at an angle θ , the coefficient of static friction between the body and the plane being μ . Find the least force that will just set the body in motion up the plane, and also the inclination (ϕ) of this force to the line of greatest slope of the plane. (O. & C.)

18. State the laws of friction and discuss how far these are obeyed in practice.

A weightless ladder of length l lies on a roof whose inclination to the vertical is θ° . The ladder is anchored at its lower end to a pivot fixed in the roof and otherwise only makes contact with the roof at its upper end where the coefficient of friction between it and the roof is μ . Find how far the ladder may be rotated about the pivot and still remain safe for a man to walk up. (O. S.)

PARALLELOGRAM OF VELOCITIES

19. Distinguish between scalar and vector quantities, and explain how vector quantities may be compounded.

A boatman, whose speed of rowing is u , crosses a river of width a to a point exactly opposite. Find the time of the journey if the speed of the stream is V (less than u). (C.)

20. A submarine moving on a northerly course at 10 knots is sighted at 10 a.m. from a ship steaming due east at 20 knots. The submarine is then 3 sea miles north-east of the ship. Find (a) the velocity of the submarine relative to the ship, (b) the nearest distance of approach and the time it occurs. (D.)

21. Two bodies A and B are moving in different directions. How may the velocity of B relative to A at a given instant be determined?

To a man in a car moving due eastward with a velocity of 20 m.p.h. the wind appears to be coming from a direction 60° east of south. When the car moves with the same speed due westward the wind appears to come from a direction 30° west of south. Find the velocity and true direction of the wind. (O. & C.)

22. Explain what is meant by the *composition* and by the *resolution of velocities*, giving practical examples.

A wheel, of diameter 3 ft., is rolling along a horizontal road at 40 m.p.h. What are the directions and magnitudes of the velocities of the points on the edge of the wheel 27 in. above the level of the road? (O.)

23. Define *relative velocity*.

A foot passenger capable of moving with speed v wishes to cross the street in front of a motor car approaching with speed V on his side of the road. If the edge of the outer mudguard is a ft. from the kerb and the front of the mudguard is b ft. behind a point directly opposite the foot passenger at the instant when he moves, what is his chance of escape? (C. S.)

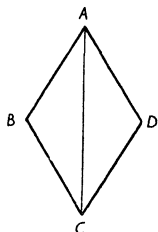


Fig. 36

VIRTUAL WORK

24. Deduce the Principle of the Lever from the Principle of Virtual Work.

25. A framework $ABCD$, consisting of four uniform, freely jointed rods, each of the same length and weight w , is hung from A , and the corners A , C are connected by a string as in Fig. 36. Find the tension in the string.

[Suppose the string to be stretched by a small amount and calculate the virtual works. At each joint there is a pair of equal and opposite forces whose virtual works cancel out.]

26. If the framework in question 25 is prevented from collapsing by a strut BD of length $1\frac{1}{2}$ times the length of each rod, instead of the thread AC , find the thrust in the strut.

27. Three equal smooth pencils, each of weight W , are tied together by a single loop of fine inextensible cotton so that each touches the other two. Prove that the bundle can rest in stable equilibrium on a smooth table only if the breaking tension of the cotton is not less than $W\frac{\sqrt{3}}{6}$. (O. S.)

MACHINES

28. Define the terms mechanical advantage and velocity ratio of a machine, and show in the ideal case of some simple machine that the mechanical advantage is equal to the velocity ratio.

A wheel and axle are of radius $1\frac{1}{2}$ ft. and 2 in. respectively, and the arrangement is used for lifting a load of 100 lb. wt. If the friction between the axle and its bearing is equivalent to a couple of 20 lb. wt. in., calculate (a) the velocity ratio, (b) the mechanical advantage, and (c) the efficiency of the machine. (O. & C.)

29. Discuss the principle of the conservation of energy with reference to any one simple machine, such as a screw-jack or pulley system.

The efficiency of a sheaved-pulley system of velocity ratio 4 is 60%. A boy weighing 112 lb. sits in a light cradle slung from the lower pulley block and is raised through a vertical distance of 2 ft. by a man pulling on the free end of the rope. Find the force the man exerts, and the work he does.

If, instead of being raised by someone else, the boy raises himself by pulling on the free end of the rope, find the force he must exert. (O.)

30. What is meant by the terms mechanical advantage, velocity ratio, efficiency? Show how they are related.

Illustrate your answer by reference to a differential pulley system consisting of two co-axial upper pulleys, a rising pulley and an endless chain. If the radii of the upper pulleys are 12 in. and 10 in. and the efficiency of the machine is 40%, calculate the effort necessary to raise a load of a ton weight. (N.)

31. A force of 14 lb. wt. is applied at right angles to the end of a spanner, of effective length 6 in., in order to tighten a nut on a screw with a pitch of $\frac{1}{16}$ in. If the efficiency is 30%, find the thrust of the nut in the direction of its axis.

32. The cranks of a bicycle are 7 in. long, the diameter of the wheels is 28 in., and the wheels over which the chain passes have 48 and 18 teeth respectively. If the bicycle weighs 32 lb., find the maximum gradient up which a 12-stone man can ride by putting all his weight on each pedal alternately. Allow for friction, using the fact that the rider freewheels without acceleration down a gradient of 1 in 200.

Since the rider allows his weight to provide the effort when ascending the hill, how does he perform the necessary work? Discuss whether or not his progress is jerky.

CENTRE OF GRAVITY AND STABILITY

33. Explain the term *centre of gravity*. Describe and explain how you would find experimentally the centre of gravity of a thin card of irregular shape.

A wooden stool 2 ft. 2 in. high consists of a square seat with a uniform vertical leg at each corner. The seat is 2 in. thick and of mass 2 lb. The total mass of the legs is $1\frac{1}{4}$ lb. Find the height of the centre of gravity. (N.)

34. A thin uniform circular disc of radius 1 ft. has a circular hole of radius 3 in. cut in it, the centre of the hole being 6 in. from the centre of the disc. Find the centre of gravity of the disc. (N.)

35. A uniform square plate of iron has sides 1 ft. in length. One corner is bent over so that the corner coincides with the centre of the square. Calculate the position of the centre of gravity of the plate. (C.)

36. Three rods of the same material and cross-section are joined together to form a triangle with sides 3, 4, 5 cm. respectively. Find the position of the centre of gravity.

37. Prove that the position of the centre of gravity of a uniform lamina in the shape of a semicircle of radius a is at a distance of $4a/3\pi$ from the base.

A rectangular strip of tin-plate $ABCD$ measures 4 cm. by 8 cm. A semicircular portion is stamped out of it by a circular punch, of radius 1 cm., the centre of the circle being at the midpoint of the long side AB . Show that the distance of the centre of gravity of the remainder from CD is 1.92 cm. approximately. (C.)

38. Prove that the centre of gravity of a uniform triangular lamina coincides with the centre of gravity of three equal weights at its vertices.

A regular pyramid, whose base is a regular hexagon, is made of uniform thin sheet metal. If a is the radius of the circumcircle of its base and $2a$ is the length of the slant edges, find the position of the centre of gravity of the total surface of the pyramid, and state its distance from the base. (L.)

39. Explain the meaning of the terms *stable* and *unstable* equilibrium of a rigid body.

A two-wheeled motor-car trailer is intended for use on rough mountain roads.

The wheels are 36 in. diameter, with a track of 26 in., the combined weight of wheel and axle being $\frac{1}{4}$ ton; a flat thin tray weighing 112 lb. is mounted horizontally 6 in. above the line of the axle.

Make a careful sketch of the trailer and calculate the angle of tilt at which it will topple over sideways on a sloping road. (N.)

40. A cylinder of radius 1 cm. and length 4 cm. is standing on end on an inclined plane, the angle of which is gradually increased. If the coefficient of friction between the cylinder and the plane is 0.3, find whether the cylinder will slide or topple first.

41. A right circular cone stands on its base on a plane board whose inclination to the horizontal can be varied. Find the least semi-vertical angle of the cone so that it slides before falling as the inclination of the board is increased. The coefficient of friction between base of cone and board is μ . The centre of mass of a cone is at a distance from the base equal to one-quarter of the height of the cone.

(O. & C.)

42. A square lamina, whose diagonal is of length l , rests in a vertical plane on two smooth pegs which are in the same horizontal line, distant a apart. Show that there are one or three positions of equilibrium according as l is greater than or less than $4a$. Examine their stability. (O. S.)

43. Find the position of the centre of a mass of a solid uniform hemisphere of radius a .

Such a hemisphere is placed with its curved surface on a smooth horizontal plane. Find its position of equilibrium and show that it is stable. (O. S.)

44. A uniform rod, of weight W and length L , is attached at one end to a smooth hinge and at the other to an elastic string of modulus $\frac{1}{2}W$. The unstretched length of the string is $\frac{1}{2}L$, and its free end is fixed vertically above the hinge at a height L . Show that the rod is in equilibrium when inclined at an angle of 60° to the vertical and examine whether the equilibrium is stable. (O. S.)

THE BALANCE

45. Describe a good type of physical beam balance and point out the conditions which such a balance should fulfil. On what does the sensitivity of the balance depend? How would you determine the sensitivity experimentally?

A body is weighed first in the left- and then in the right-hand pan of a balance, the respective weights being 9.842 gm. and 9.833 gm. Find the true weight of the body and the ratio of the lengths of the arms of the balance. (N.)

46. Describe a sensitive form of beam balance, giving diagrams and pointing out the purpose of its knife edges and of the method of hanging its pans. On what does the sensitiveness and the accuracy of such a balance depend?

With a balance of which the arms were 10 cm. long, it was found that 0.010 gm. extra load on one pan deflected the beam of mass 20 gm. through 1° and that this deflection was independent of the loads placed on the pans. What can you deduce from these measurements? (O.)

47. The arms of a balance are each 7 cm. long, the length of the pointer is 12 cm. and the mass of the beam is 50 gm. If the three knife edges are in a plane and the centre of gravity of the beam is 0.02 cm. below the central knife edge, how much will the end of the pointer be deflected when the difference in load in the pans is 1 milligram. (C.)

48. Two balances, made of the same material, are alike in all respects except that the linear dimensions of one are n times those of the other. Compare the angular deflections of the beams for a given difference in load. (C.)

CHAPTER 2

HYDROSTATICS

THE principles of hydrostatics, the study of fluids in equilibrium, can be deduced from the application of the laws of statics to the concept of a liquid.*

A liquid flows. It has no shape of its own, but takes that of the vessel which contains it. Thus the characteristic property of a liquid is that its parts slide over each other on the application of the slightest tangential force; a perfect liquid would offer no resistance to change of shape. In practice there is some friction between the parts, called viscosity, which is large in the case of a sticky liquid like treacle, but this affects motion only and so does not require consideration in hydrostatics.

The free surface of a liquid in equilibrium must be at right angles to the forces acting upon it; otherwise a flow of liquid would take place. When gravity is the only force acting, the surface is horizontal, but if a steady wind is blowing, the surface is inclined so as to be at right angles to the resultant of the force of the wind and of the force of gravity. The surface of a large expanse of smooth water is spherical, being everywhere at right angles to the direction of gravity.

A liquid exerts a force or thrust on the surface of a body with which it is in contact. The thrust exerted by the liquid is everywhere at right angles to the surface of the body; otherwise there would be a component of the thrust along the surface of the body, and the equal and opposite reaction by the surface of the body on the liquid would cause the liquid to flow.

The hydraulic press and pressure

A liquid has the property of transmitting the effects of a thrust exerted upon it to all its parts, whatever the shape of the vessel in which it is contained. In the hydraulic braking system of a car, for example, a thrust is transmitted from the foot-pedal to the brake drums via a liquid.

Fig. 37 represents the principle of the hydraulic press, whereby a small thrust F_1 can be made to produce a large thrust F_2 . The machine

* Gases will be considered later.

is capable of a very high mechanical advantage, and some hydraulic presses designed for forging steel are capable of delivering blows of 12,000 tons weight. Suppose the forces F_1 and F_2 applied to the two pistons in Fig. 37 are in equilibrium. If the system undergoes a small displacement, so that F_1 moves down a distance x_1 and F_2 moves up a corresponding distance x_2 , the total work done must be zero (Principle of Virtual Work). Hence, ignoring the effect of gravity,

$$F_1x_1 - F_2x_2 = 0.$$

But the volume of liquid leaving the narrow arm is equal to that entering the wider arm, i.e.

$$x_1a_1 = x_2a_2,$$

where a_1 and a_2 are the cross-sections of the arms.

$$\therefore \frac{F_1}{a_1} = \frac{F_2}{a_2}.$$

Thus although the thrust exerted by the liquid changes according to the areas of the arms, the quantity represented by Force/area is invariable. This quantity, *force per unit area*, is called *pressure*, and it is a concept more valuable and significant in the study of fluids than that of thrust.

The term thrust is usually employed to denote the total force on a surface, i.e. the product of pressure and area.

The *pressure at a point* is defined as the force per unit area on an extremely small area taken round the point, i.e.

$$\text{Pressure at a point} = \lim_{\text{Area} \rightarrow 0} \left(\frac{\text{Force}}{\text{Area}} \right).$$

The above method of deducing from the Principle of Virtual Work that pressure is transmitted undiminished through a liquid was used by Galileo and Pascal. But the principle can also be argued as follows. On subjection to pressure a liquid is very slightly compressed, thereby giving rise to forces of elasticity which counteract the pressure. If one part of a liquid were momentarily at a greater pressure than another it would be more compressed; it would therefore expand and compress the other part until the pressures were the same.

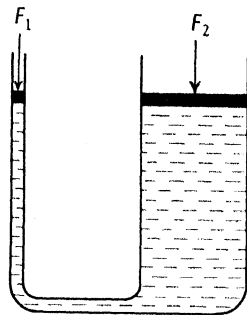


Fig. 37

Pressure in a liquid due to gravity

A liquid exerts a thrust by reason of its weight and the pressure in a liquid increases with the depth, as can be seen from the simple demonstration represented in Fig. 38.

To find the pressure p at depth h in a liquid of density ρ , consider a narrow metal cylinder, fitted with a frictionless piston, supported in the liquid as in Fig. 39 (a). Suppose the piston has an area a and is at a depth h below the surface of the liquid. If the upward thrust of the liquid on the piston is F , then a downward force of F must be exerted on the piston to keep it in position (ignoring the weight of the piston).

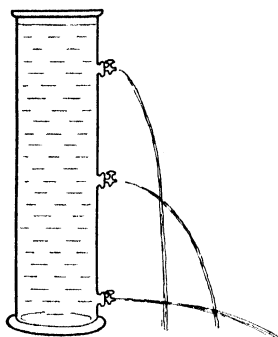


Fig. 38

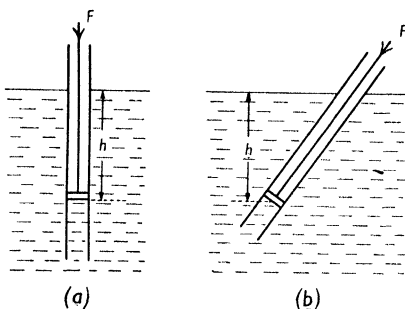


Fig. 39

Let the piston be forced down a very small distance x . A volume of liquid ax will be expelled from the cylinder and, in effect, is lifted a distance h because this volume of liquid causes a very slight rise in the surface of the liquid:

Work done on the piston = Gain in potential energy of the liquid, i.e.

$$Fx = axphg.$$

$$\therefore F/a = hpg.$$

Thus the pressure in the liquid at depth h is hpg and depends only on the depth and density of the liquid. It follows that the thrust on a dam is independent of the surface area of the reservoir.

Now suppose that the cylinder is tilted as in Fig. 39 (b), the vertical depth of the piston, which is supposed to be of very small area, still being h . Exactly the same argument applies as above, showing that the pressure exerted on the piston is hpg whatever the orientation of the piston.

EXAMPLE

Find the pressure at a depth of 50 cm. in water. The pressure of the atmosphere is 76 cm. of mercury and the density of mercury is 13.6 gm. per c.c.

Pressure due to water = 50×981 dynes per sq. cm.

Pressure due to atmosphere = $76 \times 13.6 \times 981$ dynes per sq. cm.

Pressure due to both = $981 (50 + 1033.6)$
 $= 1.06 \times 10^6$ dynes per sq. cm.

The hydrostatic paradox

Suppose the vessels in Fig. 40 contain water to the same depth h cm., and have bases of equal area a sq. cm.; the pressure of the water on the base of each is h gm. wt. per sq. cm.; the thrust of the base of each is

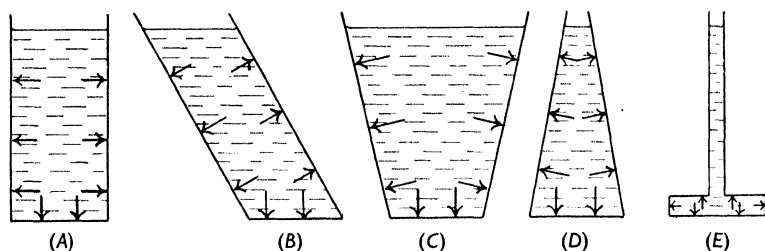


Fig. 40. The arrows show the directions of the thrusts exerted by the liquid on the walls of the vessels; the reactions of the walls on the liquids are in the opposite directions.

ha gm. wt., which is equal to the weight of water in vessel *A*. These vessels have shapes similar to those discussed by Pascal in his *Treatise on the Equilibrium of Liquids* published in 1663, and Pascal pointed out that if vessel *A* contains 100 lb. of water and vessel *E* only 1 oz. of water, the thrusts on the bases of *A* and *E* are both 100 lb. wt. This remarkable fact is known as the hydrostatic paradox. If the water in *E* were solidified and loosened from the sides of the vessel the thrust of the ice on the base of *E* would be only 1 oz. wt. But once the ice is melted the thrust increases to 100 lb. wt. The explanation is to be found in the upward pressure exerted by the water on that part of the vessel *E* facing the base, which reacts with an equal and opposite downthrust on the water. Ice, on the other hand, exerts no upthrust on the surface above it and hence receives no downthrust from this surface.

In vessel *B* the left-hand sloping side exerts an upthrust and supports some of the weight of the water, while the right-hand side exerts a downthrust on the water. In vessel *C* both sides help to support the

weight of the liquid, and in vessel *D* both sides exert a downthrust on the liquid.

Experimental proof that the bases of the vessels are subject to equal thrusts may be obtained by making each base detachable and supporting it from the arm of a balance (Fig. 41) to measure the thrust. The sides of the vessels must be supported firmly and independently of the balance.

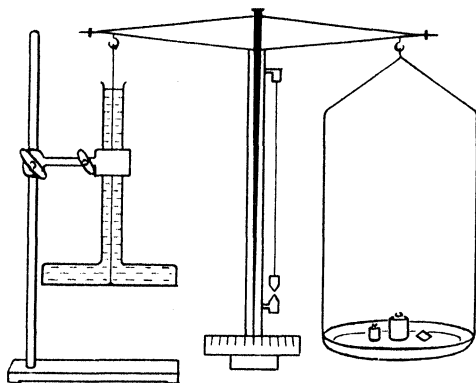


Fig. 41

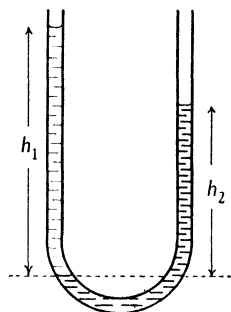


Fig. 42

Comparison of the densities of two liquids

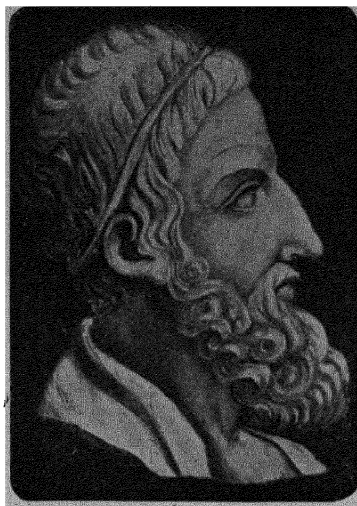
If two immiscible liquids of different densities, such as water and paraffin, are poured into the arms of a U tube, their surfaces will be at different levels. In Fig. 42 the liquid columns in the two arms of the U tube below the horizontal line counterbalance because they are the same liquid; hence a column of liquid of density ρ_1 and height h_1 balances a column of liquid of density ρ_2 and height h_2 . The pressures due to these two columns must be equal, i.e.

$$h_1 \rho_1 = h_2 \rho_2.$$

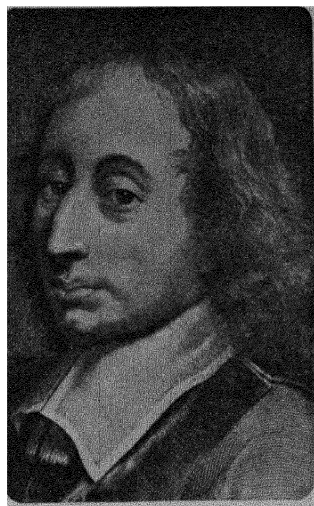
$$\therefore \frac{\rho_1}{\rho_2} = \frac{h_2}{h_1}.$$

Measurement of the heights of the two columns enables the densities of the liquids to be compared.

If the liquids mix, Hare's apparatus may be used (Fig. 43), the liquids being raised in the arms of the inverted U tube by suction at *A*. The pressures due to the two columns of height h_1 and h_2 are each equal to the pressure of the atmosphere minus the pressure of the air in the



ARCHIMEDES



BLAISE PASCAL

ARCHIMEDES (287–212 B.C.) was the greatest scientist of antiquity, being one of the few to practise the experimental method. Educated at Alexandria, not long founded by Alexander the Great, he was probably a pupil of Euclid. He discovered the law of the lever and developed the concept of centre of gravity. He discovered also the well-known principle in hydrostatics which bears his name and was the inventor of a revolving screw for raising water. His contributions to pure mathematics included a method of finding the circumference of a circle by means of inscribed and circumscribed polygons, leading to a value for the constant now known as π . Most of his life was spent in his native city of Syracuse. It was here that he met his death at the hands of a Roman soldier, when the city was taken during the Second Punic War, although the Consul Marcellus had given orders that his life should be spared.

BLAISE PASCAL (1623–1662) was one of the most remarkable of Frenchmen. Because of poor health, he was educated at home by his father, an eminent mathematician who refused to teach the boy Euclid lest he should neglect languages. But Blaise taught himself geometry, and by the age of seventeen had made original discoveries concerning conic sections. His main contributions to science were in hydrostatics. He performed numerous experiments to elucidate the principles of pressure in liquids and also devised the famous experiment of taking a barometer to the top of a mountain. His peculiar and, to many modern minds, his compelling fascination is the unique combination he presents of scientist and saint. On the night of 23rd November 1654 he underwent a mystical, religious experience and for the rest of his life carried a parchment, bearing his pact with God, sewn in the lining of his coat. His notes for a major work in defence of the Christian religion, the *Pensées*, are still widely read for the sake of their superb prose, their religious conviction and the profound knowledge they display of the human heart.

U tube above the liquids. Thus the equation $h_1\rho_1=h_2\rho_2$ holds as before.

Since the pressure due to a column of liquid depends only on the depth and not on the cross-section of the column, the arms of the U tube need not have equal or uniform cross-sections.

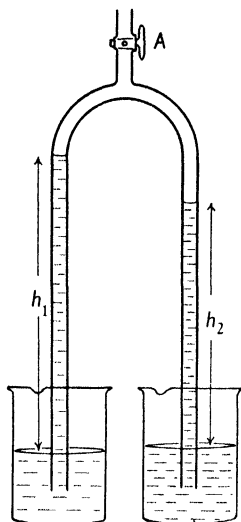


Fig. 43

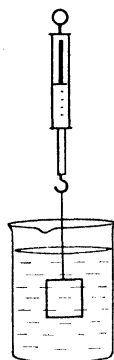


Fig. 44

The principle of Archimedes

When a body is totally or partially immersed in a fluid it experiences an upthrust equal to the weight of fluid displaced. This principle was discovered by Archimedes over two thousand years ago and applies to gases as well as to liquids.

To prove it theoretically, suppose the immersed body to be removed and the cavity to be filled with the fluid. The fluid in this cavity will be in equilibrium; it will have no tendency to move, since it is surrounded by identical fluid. Thus the rest of the fluid must exert on it an upthrust exactly equal to its weight, and this upthrust will be exerted on the solid when replaced.

The principle may be verified experimentally by finding the apparent loss in weight when a body of known volume is immersed in a liquid of known density. Thus in Fig. 44, if the body is a cube of side 2 cm., and hence of volume 8 c.c., its apparent loss in weight when immersed in water should be 8 gm.

If the truth of the principle is assumed, the experiment enables the density, or more strictly, the specific gravity of the solid to be determined.

Density is defined as $\frac{\text{Mass}}{\text{Volume}}$.

Specific gravity is defined as $\frac{\text{Mass of a substance}}{\text{Mass of an equal volume of water}}$.

EXAMPLE

A solid weighs 20.0 gm. in air, 15.0 gm. in water and 16.0 gm. in a liquid. Find the density of the solid and of the liquid.

By Archimedes' principle,

$$\begin{aligned}\text{Weight of water displaced by the solid} &= 20.0 - 15.0 \\ &= 5.0 \text{ gm.}\end{aligned}$$

$$\therefore \text{Specific gravity of solid} = \frac{20.0}{5.0} = 4.0.$$

$$\text{Weight of liquid displaced by the solid} = 20.0 - 16.0 = 4.0 \text{ gm.}$$

$$\therefore \text{Specific gravity of liquid} = \frac{4.0}{5.0} = 0.80.$$

EXAMPLE

A body, which floats in water, weighs 5.0 gm. in air. It is sunk in water by being attached to a sinker, and the two together weigh 10.0 gm. in water. The sinker alone weighs 25.0 gm. in water. Find the specific gravity of the body.

$$\begin{aligned}\text{Upthrust of the water on the body} &= (25.0 - 10.0) + 5.0 \\ &= 20.0 \text{ gm.}\end{aligned}$$

$$\therefore \text{Weight of water displaced by the body} = 20.0 \text{ gm.}$$

$$\therefore \text{Specific gravity of body} = \frac{5.0}{20.0} = 0.25.$$

Hydrometers

When a body floats it displaces a weight of fluid equal to its own weight. This is a clear consequence of the principle of Archimedes, but at the time of Galileo the Aristotelians were still maintaining that a body floats or sinks according to its shape, e.g. a leaf floats and a cube sinks. Galileo, in a simple but conclusive experiment, placed a conical lump of wax in water, in which it sank whatever its orientation, and he then caused it to float by dissolving salt in the water.

The principle of flotation finds application in the hydrometer, an instrument for measuring the specific gravity of a liquid. Fig. 45 (a) represents a constant weight hydrometer. The hydrometer is floated in the liquid whose specific gravity is required, and the reading on the scale inside the stem at the level of the surface of the liquid gives the

specific gravity direct. The hydrometer is designed to have a low centre of gravity so that it floats vertically and consists of a hollow glass vessel weighted at the bottom with mercury.

Fig. 45 (b) represents a constant immersion type known as Nicholson's hydrometer which can be used for the determination of the specific gravity of either a liquid or a solid. It consists of a hollow metal float attached at its upper end, by means of a slender neck carrying a fixed mark M , to a scale pan A , and at its lower end to a pan B .

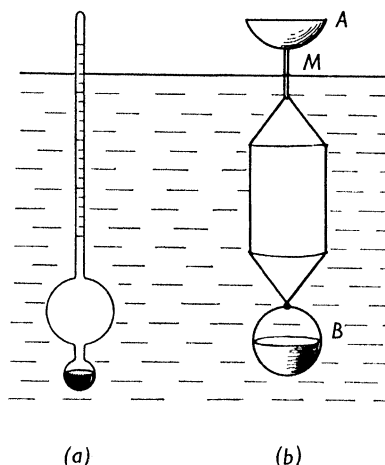


Fig. 45

To determine the specific gravity of a liquid the weight W of the hydrometer must first be found. The hydrometer is then floated in the liquid and immersed to the mark by placing a weight w_1 in the upper pan. This is repeated with the hydrometer floating in water; suppose the weight necessary to sink it to the mark in water is w_2 .

$$\text{Weight of liquid displaced} = W + w_1.$$

$$\text{Weight of water displaced} = W + w_2.$$

Since the volumes of liquid and water displaced are equal,

$$\text{Specific gravity of liquid} = \frac{W + w_1}{W + w_2}.$$

To determine the specific gravity of a solid the latter is placed first in the pan A , with the result that the weight which must be placed in A to sink the hydrometer in water to the mark is reduced from w_2 to w_3 .

It is then moved into pan *B*, being tied there if it tends to float, the necessary weight in *A* now being w_4 .

$$\text{Weight of solid in air} = w_2 - w_3.$$

$$\text{Upthrust on solid in water} = w_4 - w_3.$$

$$\therefore \text{Specific gravity of solid} = \frac{w_2 - w_3}{w_4 - w_3}.$$

EXAMPLE

An iceberg of density $0.916 \text{ gm. cm.}^{-3}$ floats in sea water of density $1.005 \text{ gm. cm.}^{-3}$. What fraction of its volume is submerged?

$$\text{Let total volume of iceberg} = V \text{ c.c.}$$

$$\therefore \text{Weight of iceberg} = 0.916 V \text{ gm.}$$

$$\text{Let volume of iceberg submerged} = v \text{ c.c.}$$

$$\therefore \text{Weight of sea water displaced} = 1.005 v \text{ gm.}$$

$$\therefore 0.916 V = 1.005 v.$$

$$\therefore \frac{v}{V} = \frac{0.916}{1.005} = 0.911.$$

Hence 0.911 of the volume of the iceberg is submerged.

Stability of flotation

An important practical problem of hydrostatics is the condition for the stable equilibrium of a ship. Fig. 46 represents a ship heeling over; *G* is the centre of gravity of the ship, through which its weight *W* acts, and *B* is the centre of gravity of the displaced water, called the *centre of buoyancy*, through which the upthrust acts. When the ship is on an even keel *B* and *G* are in the same vertical line *GM*, but when she heels over the shape of the displaced water changes, causing *B* to shift in position. The vertical line through *B* intersects the 'centre line' of the ship in *M* which is called the *metacentre*, *GM* being called the *metacentric height*.

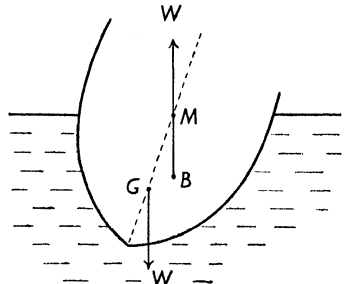


Fig. 46

It is clear that the two forces in Fig. 46 form a couple tending to right the ship. If *G* were above *M*, however, the couple would tend to make the ship heel over further. Equilibrium is stable if *G* is below *M* and unstable if *G* is above *M*. For maximum stability the centre of gravity of a ship should be as low as possible. It is unwise to stand up

in a rowing boat in a rough sea because the centre of gravity is thereby raised.

The metacentric height of a ship may be determined by finding the tilt θ of the mast when a heavy weight w is moved across the deck a distance d . The removal of w from a place on the deck is equivalent to the application of an upward force of w there and hence gives rise to a couple $wd \cos \theta$ (Fig. 47). The couple due to the ship's displacement, W , is $W \cdot GM \sin \theta$.

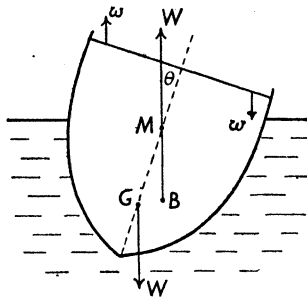


Fig. 47

$$W \cdot GM \sin \theta = wd \cos \theta,$$

$$GM = \frac{wd}{W} \cot \theta$$

$$= \frac{wd}{W\theta} \quad \text{if } \theta \text{ is small.}$$

The siphon

In Fig. 48 the liquid flows from vessel *A* to vessel *B* until the levels in the two vessels are the same; the arrangement is called a siphon. The difference in pressure causing the motion is $h\rho g$, where ρ is the density of the liquid. If the vessel *B* were removed, the liquid would, of course, still flow and continue to do so until the level in *A* was the same as the lower end of the right-hand arm of the tube, the pressure then being atmospheric at this level on both sides. The vertical height of the siphon tube must not exceed the height of the liquid which can be supported by the pressure of the atmosphere, because if it did so the liquid would run out of the tube on both sides and the siphon could not function.

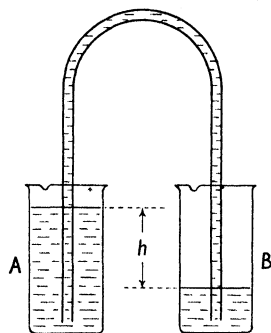


Fig. 48

Thrust on a plane surface immersed in a liquid

The thrust on a plane surface immersed in a liquid is equal to the product of the pressure at the centroid of the surface and of the area of the surface.

Suppose a surface of area S is immersed in a liquid of density ρ . Consider a narrow, horizontal strip of the surface, of area dS , at depth h (Fig. 49).

$$\text{Thrust on the strip} = g\rho h dS.$$

$$\text{Thrust on whole surface} = \int g\rho h dS = g\rho \int h dS.$$

$$\text{If depth of centroid} = \bar{h},$$

$$\bar{h}S = \int h dS \quad (\text{definition of centroid, p. 16}).$$

$$\therefore \text{Thrust on whole surface} = g\rho \bar{h}S$$

$$= \text{pressure at centroid} \times \text{area of surface}.$$

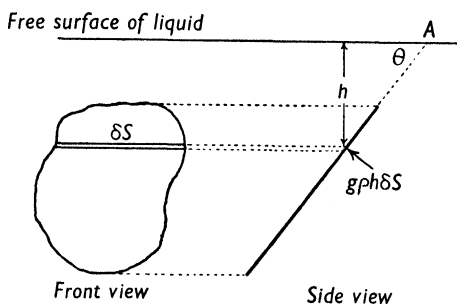


Fig. 49

Centre of pressure

The thrust on the immersed surface is everywhere at right angles to the surface and hence may be regarded as a set of parallel forces which are equivalent to a single resultant force. The point in the immersed surface at which this resultant force acts is called the *centre of pressure*. It is at a greater depth than the centroid because the pressures at points below the centroid are greater than the pressures above it. Thus in the case of a parallelogram with one side in the surface of a liquid the centre of pressure is two-thirds as far below the surface as the bottom side; in the case of a triangle with one side in the surface it is half-way down the median; and in the case of a triangle with a vertex in the surface and one side parallel to the surface it is three-quarters of the way down the median.

The position of the centre of pressure may be found by taking moments about the line in which the plane of the surface cuts the surface of the liquid, A in Fig. 49.

$$\text{Moment about } A \text{ of thrust on strip} = g\rho h dS \times h \operatorname{cosec} \theta.$$

(The thrust on one side only of the strip is being considered, but there is an equal and opposite thrust on the other side.)

Let depth of centre of pressure = H .

Moment of resultant thrust = sum of moments of thrusts on horizontal strips,

$$\text{i.e.} \quad H \operatorname{cosec} \theta \int g \rho h dS = \int g \rho h^2 \operatorname{cosec} \theta dS.$$

$$\therefore H = \frac{\int h^2 dS}{\int h dS}.$$

It can be seen that the position of the centre of pressure is independent of the density of the liquid and varies with the depth of immersion of the surface. The term $\operatorname{cosec} \theta$ disappears because H represents the vertical depth of the centre of pressure.

Allowance for the atmospheric pressure acting on the surface of the liquid can be made as follows. If the liquid is water and the atmospheric pressure is equivalent to a depth of 34 ft. of water, the effective surface of the liquid is taken as 34 ft. above the actual surface. In general, h must be increased by P/ρ , where P is the atmospheric pressure and ρ the density of the liquid.

EXAMPLE

Find the resultant thrust on a rectangular lamina of sides a and b immersed in a liquid of density ρ , the plane of the rectangle being vertical and one of the sides of length a being in the surface of the liquid (Fig. 50). Find also the position of the centre of pressure.

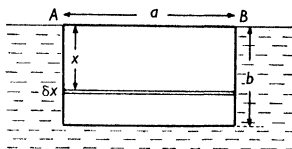


Fig. 50

Thrust on rectangle = (pressure at centre of gravity) $\times$ area

$$\begin{aligned} &= \frac{b}{2} \rho g ab \\ &= \frac{1}{2} ab^2 \rho g. \end{aligned}$$

Consider a horizontal strip at depth x and of width δx .

Thrust on strip = $x \rho g a \delta x$.

Moment of thrust about $AB = x^2 \rho g a \delta x$.

Let $\bar{x}$ = depth of centre of pressure. Taking moments about AB ,

$$\begin{aligned} \frac{1}{2} ab^2 \rho g \bar{x} &= \int_0^b x^2 \rho g a dx, \\ \frac{1}{2} ab^2 \rho g \bar{x} &= \frac{1}{3} b^3 \rho g a, \\ \bar{x} &= \frac{2}{3} b. \end{aligned}$$

The pressure of gases

The pressure of a gas differs fundamentally from that of a liquid in respect of its cause. A liquid exerts a pressure only under gravity or under an external force, applied, say, by a piston. A gas, on the other hand, exerts a pressure because of the mobility and kinetic energy of its molecules—what Boyle called its ‘spring’. It has a perpetual tendency to expand, and the bombardment of its molecules on the walls of a containing vessel exerts a pressure.

The pressure throughout a small volume of gas is uniform, but the effect of gravity on a large volume of gas, such as the atmosphere, is to cause a pressure gradient, the pressure in the lower parts being higher than in the upper parts. The attraction of the earth on the atmosphere also prevents it from expanding into space and escaping.

The compressibility of a gas, in contrast to that of a liquid, is considerable. Thus the density of the atmosphere at a height of 5 miles is only one-quarter of its value at sea level, whereas the density of sea water at a depth of 5 miles in the Pacific is not greatly different from its usual value.

For dealing with the pressure of gases, therefore, the relation between the pressure and volume of a gas, known as *Boyle’s law*, is necessary: *The pressure and volume of a fixed mass of gas are inversely proportional provided the temperature remains constant, i.e.*

$$p \propto \frac{1}{v}, \quad p = \text{pressure of gas}, \quad v = \text{volume of gas}.$$

$$\therefore pv = \text{constant}.$$

The variation of the pressure of the atmosphere with height

It is possible to derive a relation between the pressure of the atmosphere and height above sea level, assuming the atmosphere to be at constant temperature.

Let the pressure and density of the atmosphere at height h above sea level be p and ρ respectively. At height $h + \delta h$ let the pressure be $p + \delta p$.

The change of pressure δp is due to the weight per unit area of air between the heights h and $h + \delta h$ (shaded in Fig. 51), and since over this small difference in height the density of the air does not change appreciably,

$$\delta p = -g\rho\delta h.$$

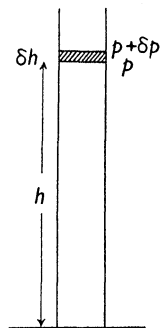


Fig. 51

The negative sign is necessary because δp is negative. It will be assumed that g does not change appreciably with change in h .

$$\rho = \frac{1}{v}, \quad \text{where } v = \text{volume of unit mass of gas,}$$

$$pv = \text{constant} \quad (\text{Boyle's law}).$$

$$\therefore \rho = kp, \quad \text{where } k \text{ is a constant.}^*$$

$$\therefore \delta p = -k g p \delta h.$$

$$\int_{p_0}^p \frac{dp}{p} = \int_0^h -k g dh, \quad \text{where } p_0 = \text{pressure at sea level.}$$

$$\therefore \log_e \frac{p}{p_0} = -kgh,$$

$$p = p_0 e^{-kgh}.$$

Thus the pressure of the atmosphere falls off exponentially with height (Fig. 52). The expression becomes more complicated when allowance is made for change of temperature with height and for the water-vapour content of the atmosphere.

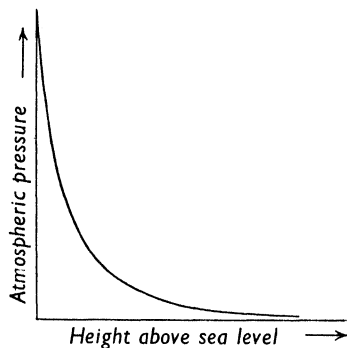


Fig. 52

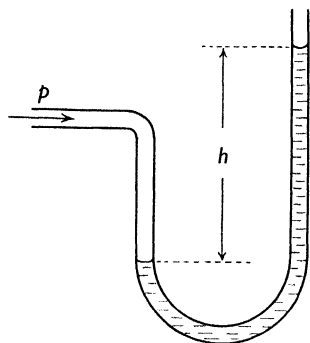


Fig. 53

The manometer and barometers

The pressure of a gas may be measured by balancing it against the pressure of a liquid. The manometer (Fig. 53) consists of a U tube containing a suitable liquid, such as a light oil or mercury, and the pressure p of the gas causes a difference in level h of the liquid arms:

$$p = B + h\rho g,$$

where B = atmospheric pressure, ρ = density of liquid.

* $k = \frac{1}{RT}$, where R is the gas constant for unit mass and T the absolute temperature.

The simplest and most accurate way of measuring the pressure of the atmosphere is by means of a mercury barometer, a simple form of which can be set up in a few minutes. A glass tube, about 80 cm. long and sealed at one end, is filled with mercury, care being taken to exclude air bubbles. The tube is then upturned with the thumb over the open end, and the thumb is removed under the surface of mercury in a trough. The mercury sinks so that its height in the tube above the

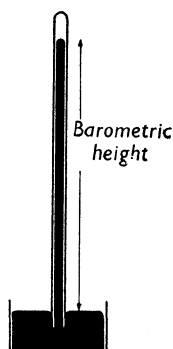


Fig. 54

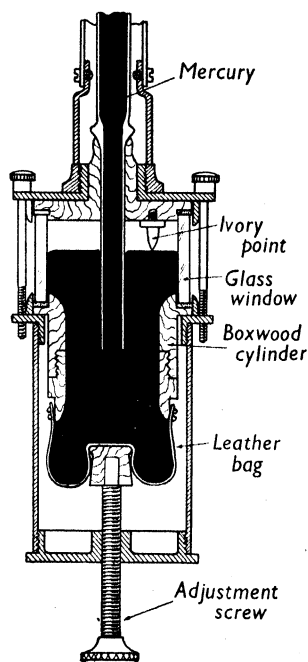


Fig. 55. Cistern of Fortin barometer.

level in the trough is about 76 cm., and this height is a measure of the atmospheric pressure (Fig. 54). When the atmospheric pressure changes, the level in the trough alters as well as the level in the tube. Hence the trough or cistern of the standard Fortin barometer, which is merely a mercury barometer with refinements, is made in the form of a leather bag which can be screwed up or down to keep the level in the trough at a fixed mark, the tip of an ivory pin (Fig. 55). In the standard Kew barometer the cistern is rigid and the scale is contracted to allow for movement of the mercury level in the cistern.

A portable instrument known as the aneroid barometer (Fig. 56) consists of a corrugated, evacuated box whose centre caves in, or is pulled out by a spring, when the atmospheric pressure increases or decreases respectively. The movement is magnified by a system of levers and causes a pointer to rotate. This instrument requires calibration against a mercury barometer.

EXAMPLE

The space above the mercury in a faulty barometer contains air, and when a correct barometer reads 76.0 cm. the faulty barometer reads 75.0 cm., there being then a space of 5.0 cm. above the mercury. What is the correct reading when the faulty barometer reads 75.5 cm.?

Let h cm. = correct reading when faulty barometer reads 75.5 cm.

	1st case	2nd case
Pressure of air in faulty barometer (cm.) =	76-75	$h-75.5$
Height of air in faulty barometer (cm.) =	5	4.5

By Boyle's law,

$$pv = \text{constant.}$$

$$\therefore (h - 75.5) 4.5 = 1 \times 5,$$

$$h = 76.6 \text{ cm.}$$

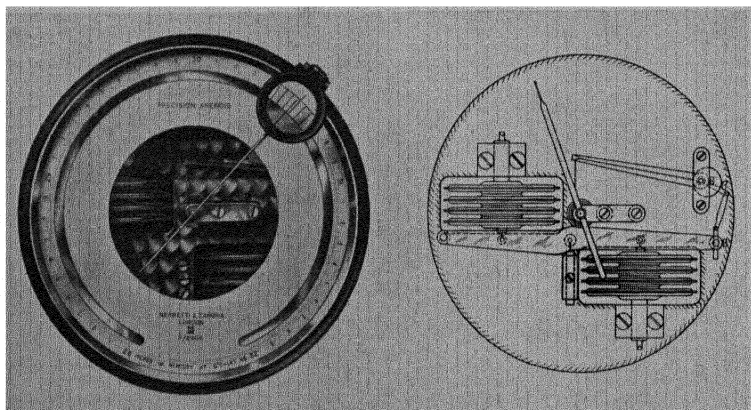


Fig. 56. A precision aneroid barometer, the movement being shown on the right. Two sets of four exhausted, diaphragm capsules, each set consisting of eight nickel-plated, hardened and tempered steel diaphragms, are fixed at one end to the frame of the instrument and their free ends are connected to a magnifying lever. Errors due to temperature changes are almost perfectly compensated by leaving a specific amount of dry air in the capsules. Messrs Negretti and Zambra claim, 'The inherent errors have been reduced to such small proportions that this aneroid barometer might well replace the mercurial barometer for scientific observations, since it requires no corrections for temperature or latitude.'

Air pumps

(1) *Geryk type.* Fig. 57 (a) represents the principle of the Geryk pump, commonly met with in the school laboratory, which is capable of reducing the pressure of the air in a vessel to about $\frac{1}{2}$ mm. of mercury. When the piston is raised air is forced out through the valve at the top and, when it is lowered below the inlet, air rushes from the vessel to be exhausted into the vacuum above the piston.

Fig. 57 (b) shows details of construction. The oil seal above the piston and at the bottom of the barrel, together with the accurate

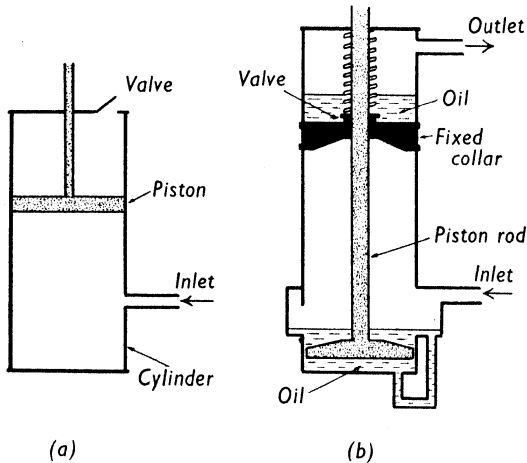


Fig. 57

shaping of the parts, eliminates dead space where trapped air could linger during operation. The annular space and side tube at the bottom of the cylinder enables oil which has collected below the piston to pass into the space above it.

(2) *Rotary type.* Fig. 58 represents a rotary oil-sealing type of pump, known as the Hyvac. The circular rotor rotates about an axis not

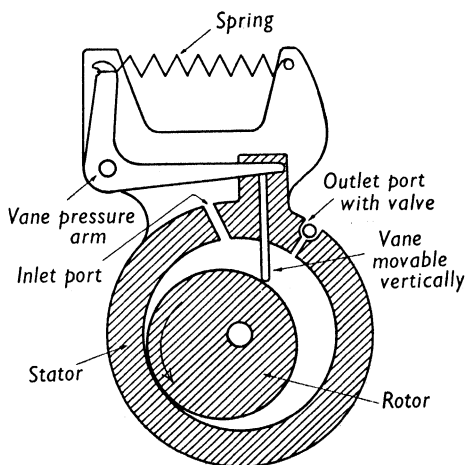


Fig. 58

through its own centre but through the centre of the stator, and consequently it always rubs against the stator. During rotation the air is swept along until it is expelled through the outlet port, which has a valve opening outwards. A spring-loaded, vertical, movable vane presses against the rotor, moving up and down as the latter rotates, and serves to separate the two sides of the pump. In the position shown in Fig. 58 air is coming in through the inlet port to the left of the vane from the vessel to be exhausted, and trapped air is being compressed to the right of the vane. Every time the point of contact between rotor and stator passes the inlet port a gulp of air is trapped, and during the ensuing revolution it is compressed and expelled.

The whole is contained in a box of oil and hence is oil-sealed; if there is any leakage along the rotor shaft or the vane, only oil enters the pump and is ejected through the valve. Often two pumps are arranged in series on a common horizontal shaft, and pressures down to about 10^{-3} mm. of mercury can be attained.

If a still higher vacuum is required a mercury diffusion pump, in which the vapour from boiling mercury sweeps out the air, can be used to complete the task begun by the rotary pump. In this way pressures down to 10^{-6} or even 10^{-7} mm. of mercury can be attained.

McLeod gauge

The minute pressures achieved by vacuum pumps are measured by admitting some of the gas into a gauge, compressing it a known amount until its pressure is directly measurable, and then applying Boyle's law.

Fig. 59 represents the McLeod gauge, the liquid it contains being mercury. The cistern *E* is first lowered until the mercury runs out of *A*, *B* and *D*, so that gas from the evacuated vessel, whose pressure p is required, can flow into the bulb *D* and the capillary tube above it. The cistern is raised, causing the gas in *D* to be compressed, until the mercury rises to a fixed mark *C*. The pressure of the compressed air is then $p+h$. Suppose the volume of the capillary tube above *C* is v

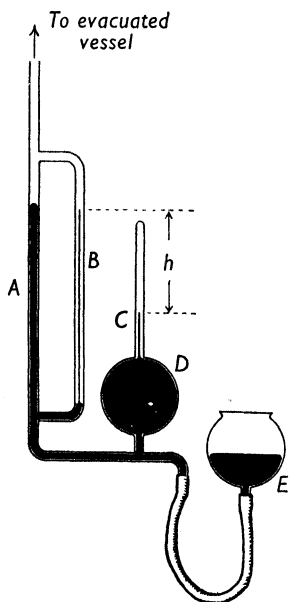


Fig. 59. McLeod gauge.

and the volume of D and of the whole capillary tube is V . Applying Boyle's law,

$$pV = (p + h)v.$$

The values of v and V are determined when the gauge is made and h is measured; hence p can be found. The capillary tube B has the same cross-section as the capillary tube above D , and its purpose is to eliminate surface-tension effects in the measurement of h .

SUMMARY

In hydrostatics no new fundamental principle, additional to those used in statics, is introduced, but a new concept, pressure, is invaluable. By applying statical principles to a liquid (which has the characteristic property that it flows when subjected to a tangential force) it can be shown that the pressure exerted on an immersed surface is at right angles to the surface, that an applied pressure is transmitted unchanged through a liquid, and that the pressure of a liquid under gravity is proportional to its depth and density but independent of its cross-section, the thrust exerted on a small immersed surface being independent of the orientation of the surface. The principle of Archimedes is also deducible by the same means. These principles were tested by experiment at every step during their discovery.

A gas, unlike a liquid, is very compressible, and its volume changes with pressure in accordance with Boyle's law. It exerts a pressure, by the bombardment of its molecules, in all directions. The effect of gravity on a large volume of gas is to cause a vertical pressure gradient.

QUESTIONS

1. Explain what is meant by the pressure at a point in a liquid, and describe an arrangement by which such a pressure can be measured.

Calculate the pressure at a depth of 5 ft. in treacle of specific gravity 1.3 if the pressure on its free surface is 15 lb. wt. per sq. in. and the liquid is at rest. Discuss whether the pressure would remain the same if the liquid were in motion.

(1 cu. in. of water weighs 0.57 oz.)

(O.)

2. Describe a hydraulic press and state the principles on which its action depends.

From the top of a barrel 2 ft. high, full of brine, projects a straight uniform tube inclined at an angle of 60° to the vertical, its bore having an area of 2 sq. in. The barrel will withstand a maximum pressure of 600 lb. wt. per sq. ft. What weight of brine poured into the tube will just cause the barrel to burst?

(1 cu. ft. of water weighs 62.5 lb. Specific gravity of brine = 1.20.)

(N.)

3. Describe a method based on fluid pressure for determining the specific gravity of a liquid.

A mercury manometer is connected to the domestic water supply, and the difference between the two mercury levels is observed to be 50.0 cm. Calculate the pressure of the supply in lb. wt. per sq. in.

(The density of mercury = 13.60 gm. per c.c., 1 lb. wt. = 454 gm. wt., and 1 in. = 2.54 cm.) (N.)

4. State the principle of Archimedes and give a theoretical proof of it.

A glass of water has a lump of ice floating in it and is full to the brim. Show that no water will overflow when the ice melts. (O. & C.)

5. Explain why the buoyancy of water upon an immersed stone is practically the same, whatever the depth.

6. A body weighs 61 gm. in water and 59 gm. in a liquid of specific gravity 1.25. Calculate the weight and specific gravity of the body. (C.)

7. A piece of *lignum vitae* of specific gravity 1.35 is attached to a piece of oak of specific gravity 0.65. If the combined body just floats in water, what are the relative volumes of the pieces of wood? (O. S.)

8. A body floats in a liquid in a closed vessel. Discuss the effect of changing the pressure of the air in the vessel. (O. & C.)

9. State Archimedes' principle. Explain how very accurate weighing is affected by the buoyancy of air.

A litre of pure water has a mass of 1 kg. Calculate, to the nearest decigram, its apparent weight, when weighed on a balance in air of density 0.00128 gm. per c.c., using weights of density 8.9 gm. per c.c. (C.)

10. A substance of density 2.54 gm. per c.c. weighs 9.763 gm. in air at 17° C. and 754 mm. pressure. If the weights are brass (density 8.4 gm. per c.c.) calculate the weight *in vacuo* at N.T.P. Density of air at N.T.P. 0.00129 gm. per c.c. (B.)

11. Give a diagram to illustrate the shape of some form of common hydrometer which is used to measure the specific gravity of a liquid, and explain the reasons for the shape. Show the likely positions for the markings 1.30, 1.15 and 1.00.

If the weight of the hydrometer is 26 gm. and the distance between the 1.30 and 1.00 marks is 5 cm., what is the diameter of the cylindrical stem? (O.)

12. A hydrometer consists of a bulb attached to a cylindrical tube of radius 3 mm., the mass of the whole being 6 gm. What would be the distance between the graduations on the stem for water and a liquid of specific gravity 0.9? (O. & C.)

13. A Nicholson's hydrometer floating in water requires 7.15 gm. on the top of the pan to adjust the mark to the surface. When a piece of a mineral was on the upper pan 2.14 gm. additional weight was required, and when the mineral was on the lower pan 4.23 gm. was required on the upper pan to adjust the mark to the surface. Calculate the specific gravity of the mineral. (C.)

14. A straight plank of wood of uniform cross-section floats on the surface of water. One end is subsequently raised above the surface by means of a rope attached to the extremity of the plank. Why will the rope be vertical?

If the specific gravity of the wood is 0.75, find (a) what fraction of the length of the plank is immersed in the inclined position, (b) the tension of the rope in terms of the weight of the plank. (D.)

15. A cylinder of density σ , length l , and cross-section A floats with its axis vertical in a liquid of density ρ . The cylinder is pushed vertically down until it is just immersed. Find the work done when the liquid is contained in a very large vessel, and also when it is contained in a cylindrical vessel of cross-section B , comparable in magnitude to A . (O. & C.)

16. If a load of 20 tons, moved 50 ft. across the deck of a ship of 20,000 tons displacement, causes the ship to tilt through $\frac{1}{2}^\circ$, what is its metacentric height? (O.)

17. A thin-walled cylindrical vessel containing a non-viscous liquid has a small circular hole at the bottom. Show from considerations of energy or otherwise that the volume of liquid flowing through the hole per second is proportional to the square root of the height of liquid above it and to the square of the radius of the hole.

Describe briefly how you would verify this result experimentally. (O. & C.)

18. Find an expression for the work done in pumping v c.c. of liquid against a pressure p dynes per sq. cm.

A heart discharges 75 c.c. of blood at each beat against an average pressure of 10 cm. of mercury. Assuming that the pulse frequency is 72 per min., calculate the rate of working in watts.

(Density of mercury = $13.6 \text{ gm. cm.}^{-3}$, $g = 981 \text{ cm. sec.}^{-2}$) (C.)

19. Define pressure at a point in a fluid. Find the total thrust on the sides and vertical ends of a V-shaped trough 1 ft. deep, 2 ft. wide at the top, and 4 ft. long when filled with water, density 62.5 lb. per cu. ft. (O. & C.)

20. Prove that a small area A , immersed at a depth h below the surface of a liquid of density ρ , experiences a thrust $A\rho gh$.

A lock gate is 10 ft. wide and hinged along one vertical edge. Calculate the couple which would be required to start opening it when the height of water on one side is 5 ft. and on the other 4 ft.

(Density of water = 62.5 lb. per cu. ft.) (O. S.)

21. What do you understand by *the pressure at a point in a liquid* and by *the centre of pressure*?

One end of a tank full of water, density 62.5 lb. per cu. ft., is a vertical rectangle 5 ft. long and 3 ft. high. Find the thrust on it.

If it were hinged at the lower edge and kept in position by bolts at each end of the top edge, find the force on each bolt. (O.)

22. Describe how you would determine experimentally the centre of pressure of a plane surface held vertical under water. (O. S.)

23. Find the centre of pressure of a rectangular sheet a in. long and b in. wide, of uniform thickness, immersed in liquid of uniform density with one side of length b in. in the surface, the plane of the rectangle being inclined at an angle θ to the vertical.

If the rectangular sheet remain in the same position with respect to the vessel containing the liquid, and the depth of the liquid be increased by h in., find the new position of the centre of pressure. (L.)

24. A hollow cylinder closed at one end is sunk, closed end uppermost, into the sea. On pulling it up, it is found that the interior has been wetted for five-sixths of its length. Calculate the depth to which it had been sunk, assuming that the atmospheric pressure is 75 cm. of mercury, the density of mercury 13.6 gm. per c.c., and that of sea water 1.025 gm. per c.c. (O. & C.)

25. A diving bell of volume 50 cubic metres, full of air at atmospheric pressure and temperature, is lowered so that its mouth is at a depth of 15 metres below the surface of the sea. What volume of air, measured at atmospheric pressure and temperature, must be pumped into the bell to keep it full of air?

(Density of sea water = 1.02 gm. per c.c. Temperature of sea water = 5° C. Atmospheric pressure = 76 cm. of mercury. Atmospheric temperature = 20° C. Density of mercury = 13.6 gm. per c.c.) (C.)

26. State the law of floating bodies and describe an experiment to test its truth.

The gas-bag of an airship has a volume of a million cubic metres. What total weight can be lifted when it is filled with (a) hydrogen, (b) helium, the density of which is twice that of hydrogen?

(Density of hydrogen = 0.09 gm. per litre; density of air = 1.29 gm. per litre.) (N.)

27. State Archimedes' principle. Describe an experiment to show that the atmosphere exerts an upthrust on bodies immersed in it.

An 'inventor' reported that he had discovered a new gas which, when used for filling balloons, had a lifting power equal to twice that of hydrogen. Assuming that the density of air is 14.4 times that of hydrogen, show that the report must be erroneous. (C.)

28. State the conditions which must be fulfilled in order that a body may float wholly or partly submerged.

An inflated thin-walled rubber balloon containing a little mercury floats in a lake of fresh water with half its volume immersed. Show that there is a second possible position of equilibrium for the balloon, and show further that the equilibrium in the second case is unstable. Assume that the balloon remains spherical throughout. (C.)

29. Discuss the stability of an inverted test-tube containing a little air and just completely immersed in a liquid.

30. Give an account of the laws governing the equilibrium of floating bodies. How would you attempt to verify these laws experimentally?

A sealed spherical cellophane balloon has a diameter of 5 metres and weighs, with the apparatus it carries, 1 kg. It contains one-tenth of the volume of hydrogen required to fill it at atmospheric pressure. The balloon is allowed to ascend; if the cellophane does not expand and if the temperature of the atmosphere is assumed to

be constant at 0°C. at all heights, calculate at what height the envelope becomes full and the height to which the balloon rises. The pressure p at height h (km.) is related to that at the ground (p_0) by the relation $h = 20 \log_{10} (p_0/p)$.

(Density of air and hydrogen at 0°C. and atmospheric pressure are 1.29 and 0.09 gm. per litre respectively.) (O. & C.)

31. Explain the principle of the action of the siphon. Give an example of a siphon (a) in nature, (b) in the home, explaining the action in each case. (L.)

32. Dry air is trapped in the closed end of a vertical U tube by mercury which is poured in the open end. The length, l , of the column of air in the closed end is measured for different values of the pressure excess, h , of the mercury column.

The following table of results was obtained in an experiment:

l (in cm.)	30	28.2	26.7	25.3	24
h (in cm.)	5	10	15	20	25

Plot a suitable graph to show that Boyle's law is obeyed, and from the graph deduce the value of the atmospheric pressure during the experiment. (L.)

33. The space above the mercury in a faulty barometer contains air, and when a true barometer reads 30 in. the faulty barometer reads 29 in., there being then a space of 4 in. above the mercury. Find the height of the true barometer when this faulty barometer reads 28.75 in. (N.)

34. Describe two forms of barometer and discuss their relative merits for two practical purposes.

A mercury barometer is known to be defective and to contain a small quantity of air in the space above the mercury. When an accurate barometer reads 770 mm. the defective one reads 760 mm., and when the accurate one reads 750 mm. the defective one reads 742 mm. What is the true atmospheric pressure when the defective barometer reads 750 mm.? (C. S.)

35. A barometer in which the mercury column is 76 cm. high is placed in a lift. If the lift first accelerates upwards at $98 \text{ cm. per sec.}^2$, then runs at constant speed and finally slows down at a constant deceleration of $98 \text{ cm. per sec.}^2$, describe the variations in level of the mercury in the barometer.

(The effect of change of height in the pressure of the atmosphere can be ignored. Acceleration due to gravity = $981 \text{ cm. per sec.}^2$) (C. S.)

36. What is the relation between the pressure and the volume of a constant mass of gas at constant temperature?

Describe some form of air pump and explain its action.

The bell-jar and barrel of an exhaust pump of the piston type have volumes of 3 litres and 200 c.c. respectively. Find the ratio between the initial density of the air in the jar and the density after three strokes of the pump. (L.)

37. A piston pump of effective volume 200 c.c. is used to exhaust a vessel of volume 1 litre. How many complete strokes will be required to reduce the pressure of the air in the vessel to one-hundredth of its initial value? (Neglect the volume of the connecting tubes, etc., and assume that the temperature remains constant.) (O. & C.)

38. The pressure in a vacuum system can be measured by means of a McLeod gauge, which consists of a large bulb (which can be connected with the vacuum system) surmounted by a capillary tube into which the gas in the bulb can be compressed by means of mercury, and its pressure read. Make a sketch of such a gauge, pointing out the purpose of any features you incorporate. Calculate a suitable length and diameter of capillary if the bulb has a volume of 500 c.c. and you wish to measure pressures between $\frac{1}{10}$ mm. and $\frac{1}{100}$ mm. of mercury. (O. S.)

39. Describe briefly some form of pump for evacuating a large bell-jar to about 2 cm. of mercury.

A piston pump of dead space d and total volume v making n strokes per second is used to evacuate a large vessel of volume V ($V \gg v$) initially at atmospheric pressure P . Show that the limiting pressure reached is Pd/v , and obtain an expression for the rate of change of pressure in the vessel when the pressure has fallen to p . Hence find the time to reduce the pressure to twice the limiting pressure. (Assume Boyle's law and constant temperature throughout.) (O. S.)

40. Write an account of the production and measurement of low gaseous pressures. (B.)

CHAPTER 3

DYNAMICS

DYNAMICS is that branch of mechanics which deals with the relation between mechanical force and motion. It was created by Galileo (1564–1642), whose study of falling bodies and projectiles was stimulated by problems of artillery fire.

Falling bodies

Up to the time of Galileo the ideas of Aristotle concerning falling bodies were generally accepted. Aristotle maintained that every body has its proper place and when in its proper place it has no tendency to move, just as a fluid loses its apparent weight when immersed in itself. A body above its proper place tends to move down and a body below its proper place to move up; thus solid bodies tend to fall, and the hot gases from a fire to rise. Aristotle further laid down that the speed with which a solid body falls is proportional to its weight. Galileo's investigations originated in his successful attempt to disprove this, by dropping a 10 lb. and 1 lb. shot from the top of the Leaning Tower of Pisa and showing that they reached the ground simultaneously.*

Having demonstrated that all bodies fall with the same speed—discrepancies being accounted for by air resistance—Galileo began to speculate on the possibility of the motion being subject to a simple law. A falling body increases in speed as it falls. Galileo correctly rejected the idea that the velocity acquired might be proportional to the distance fallen; if such a law held a body would not fall spontaneously and would require to be given a start.

He then considered another simple possibility, that the velocity acquired might be proportional to the time of falling. Suppose a body falling from rest for a time t acquires a velocity v and traverses a distance s .

Assume $v \propto t$, i.e. $v = at$, where a is a constant.

* It has been suggested that this famous experiment is a myth, since Galileo never refers to it in his writings. It was described, however, by Viviani in his biography of Galileo, written only twelve years after the latter's death. A similar experiment was performed fifty years earlier by Simon Stevin, who allowed two balls of lead, one ten times the weight of the other, to fall 30 ft. on to a plank below, where 'they landed so evenly that there seemed to be only one thump'.

Then

$s = \text{average velocity} \times \text{time}$

$$= \frac{v}{2} t = \frac{1}{2} at^2.$$

Thus the hypothesis may be tested by ascertaining whether the distance traversed is proportional to the square of the time.

Galileo did this by 'diluting' gravity. He allowed a brass ball to run down a groove in an inclined plank, 12 yards long, the groove being lined with smooth parchment to diminish friction. By means of a water clock he timed the ball over different distances and verified that $s \propto t^2$. As might be expected from the primitive apparatus at his disposal Galileo's experiments were not very precise; but like those of J. J. Thomson in a different field, they were sufficiently precise to give confidence that the hypotheses they were testing were correct.

Instantaneous velocity and acceleration

Galileo's researches led him to invent two new concepts, those of *instantaneous velocity* and *instantaneous acceleration*. The average velocity of a moving body is the distance traversed divided by the time taken. If the time interval is made extremely small, this quotient becomes the instantaneous velocity; in calculus notation $v = \frac{ds}{dt}$. Again, the average acceleration of a moving body is the change in velocity divided by the time taken. If the time interval is made extremely small the quotient becomes the instantaneous acceleration, a ; $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Since the velocity of a falling body is proportional to the time of fall $v = kt$, so that $\frac{dv}{dt} = k$, where k is a constant, and hence the body must have a uniform acceleration. This acceleration is called the *acceleration due to gravity* and is denoted by g . Its value is approximately 32 ft. per sec. per sec., or 981 cm. per sec. per sec.

Uniformly accelerated linear motion

Consider a body moving in a straight line with a uniform acceleration a . Suppose at zero time its velocity is u , that after a time t its velocity is v and it has traversed a distance s . From the definition of a ,

$$v = u + at, \tag{i}$$

$s = \text{average velocity} \times \text{time}$

$$= \frac{u+v}{2} t.$$

Substituting for v from (i), $s = ut + \frac{1}{2}at^2$. (ii)

Eliminating t from equations (i) and (ii),

$$v^2 = u^2 + 2as. \quad (\text{iii})$$

These equations are useful and should be memorised.

Projectiles

The experiments with the inclined plane led to further discoveries. Galileo realised that the horizontal and vertical motions of the rolling ball could be regarded as independent, and considered separately. He

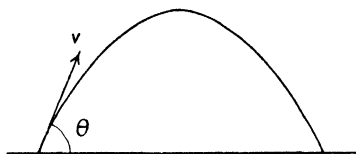


Fig. 60

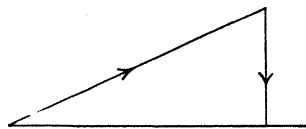


Fig. 61

applied the idea to a projectile and showed that its path was a parabola (Fig. 60). Before his time it had been suggested that a cannon ball proceeded in a straight line until it had spent its force and then fell vertically as in Fig. 61. He illustrated the problem by dropping objects from the masts of moving ships; these fall vertically in relation to the ship but in parabolas in relation to the sea.

Consider a projectile shot at an angle θ to the horizontal with a velocity v (Fig. 60). The vertical motion of the projectile is the same as that of a body thrown up vertically with a velocity $v \sin \theta$. The horizontal velocity of the projectile is $v \cos \theta$ and continues unchanged (ignoring air resistance), a remarkable but fundamental fact which will be justified later. The velocity has, of course, simply been resolved into two components at right angles.

It is now comparatively easy to find the time of flight, the maximum height attained by the projectile and the range.

Consider the vertical motion:

Let t = time to reach the highest point.

Initial velocity = $v \sin \theta$.

Final velocity = 0.

$$0 = v \sin \theta - gt \quad (\text{from equation (i) on p. 52}).$$

$$\therefore t = \frac{v \sin \theta}{g}.$$

The time taken by the projectile in falling is the same as that taken in rising.

$$\therefore \text{Time of flight} = 2t = \frac{2v \sin \theta}{g}.$$

Maximum height attained by projectile = $\frac{v^2 \sin^2 \theta}{2g}$ (from equation (iii) on p. 53).

Consider the horizontal motion:

$$\begin{aligned} \text{Horizontal range} &= v \cos \theta \times \text{time of flight} \\ &= \frac{2v^2 \sin \theta \cos \theta}{g} = \frac{v^2 \sin 2\theta}{g}. \end{aligned}$$

This is a maximum when $\sin 2\theta = 1$, i.e. when $\theta = 45^\circ$.

If the principle of treating vertical and horizontal motions independently is borne in mind, no attempt need be made to memorise these results.

Inertia

A further concept, even more valuable than those of acceleration and of the independence of vertical and horizontal motion, was derived from experiments with the inclined plane. Galileo allowed the ball which had descended one inclined plane to run up another and found that whatever their relative inclinations, the ball was always capable of rising to the same vertical height as that from which it had dropped—making a slight allowance for loss of motion due to friction. Thus if the first plane were steep and the second had a gentle inclination, the ball would run a much greater distance along the second than it had descended along the first. Galileo asked what would happen to the ball if the second plane were horizontal and all friction could be eliminated. He concluded that the ball would continue with uniform velocity with no tendency to stop.

He performed further experiments substituting a pendulum for the ball and the inclined plane, and by this expedient reduced friction to a negligible quantity. The pendulum bob

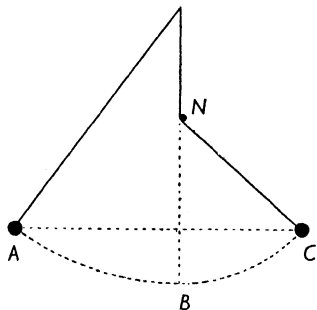


Fig. 62

of Fig. 62 swung down through the arc AB , and then, when the string hit the nail N , up through the steeper arc BC ; C and A were always the same vertical height above B .

Thus Galileo arrived at the idea that a body continues to move with uniform velocity if no force acts on it; the action of a force produces

not a velocity but an acceleration or change of velocity. Those who held that a constantly acting force was required to preserve the uniform motion of a body were involved in the awkward problem as to the nature of the force which sustains the motion of a stone which has just been thrown. Aristotle maintained that the air, compressed in front of the stone, rushed round to the back to prevent the formation of a vacuum and thus propelled the stone. Horizontal motion of the stone in a vacuum would therefore be impossible, but this did not trouble Aristotle as he denied the existence of a vacuum.

The property of a body which tends to resist change of motion is called *inertia* or *mass*. Its effects may be observed when passengers stand in moving buses and trams. When the bus starts the passengers tend to remain at rest, and to be left behind; when the bus stops they tend to continue. The effects are particularly pronounced if the bus stops or swerves suddenly.

Definition and units of mass

Mass is the numerical measure of inertia. Suppose two bodies B_1 and B_2 are held on a frictionless surface (e.g. perfectly smooth ice) against the ends of a compressed spring (Fig. 63). On being released the bodies move apart, being acted upon by the equal and opposite forces exerted by the spring. The more massive body B_1 acquires a smaller velocity than the less massive body B_2 . The ratio of the masses of the two bodies is defined as follows:

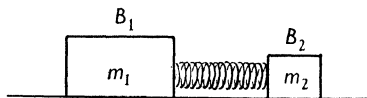


Fig. 63

$$\frac{\text{Mass of } B_1}{\text{Mass of } B_2} = \frac{\text{Velocity acquired by } B_2}{\text{Velocity acquired by } B_1}.$$

Using symbols,
$$\frac{m_1}{m_2} = \frac{v_2}{v_1}.$$

Suppose the spring is exerting a force on the two bodies for a time t . Then if a_1 and a_2 are the average accelerations of the bodies B_1 and B_2 respectively, $a_1 = \frac{v_1}{t}$ and $a_2 = \frac{v_2}{t}$; whence $\frac{v_2}{v_1} = \frac{a_2}{a_1}$.

Thus the definition of the ratio of the two masses can be written in the more convenient form

$$\frac{\text{Mass of } B_1}{\text{Mass of } B_2} = \frac{\text{Acceleration of } B_2}{\text{Acceleration of } B_1},$$

i.e.

$$\frac{m_1}{m_2} = \frac{a_2}{a_1}.$$

The ratio m_1/m_2 is found to be constant whatever the force exerted by the spring; when the force is greater, a_1 and a_2 increase in the same proportion. It is the same if found indirectly by the separate reaction of B_1 and B_2 with a third body B_3 . Hence m_1 and m_2 are invariable quantities.

For the measurement of mass some convenient body must be selected, and the masses of other bodies compared with its mass as a standard. The unit of mass on the metric system is the kilogramme, which is the mass of a lump of platinum preserved in the International Bureau of Weights and Measures at Sèvres. On the British system the unit is the pound, which is the mass of a lump of platinum kept in the Standards Office of the Board of Trade, London. Both standards are purely arbitrary, although Borda made the standard kilogramme to be as nearly as possible equal to the mass of 1000 c.c. of water at 4° C., the temperature of the maximum density of water. The box of 'weights' in the laboratory contains replicas of submultiples of the standard kilogramme.

Although he conceived the qualitative idea of inertia Galileo did not take this further step of devising the quantitative idea of mass. The conception of mass and its distinction from weight was one of Newton's imaginative achievements. The mass of a body is its resistance to motion and is a fundamental, invariable property. The weight of a body, which is equal to the attraction of the earth for the body, is a variable property depending on the distance of the body from the centre of the earth.

Definition and units of force

That which is capable of causing a change of motion of a body is called a force. The concept of force is implicit in the concept of inertia and the measure of a force is the change of motion it can produce. From the definition of mass above,

$$\frac{m_1}{m_2} = \frac{a_2}{a_1}.$$

$$\therefore m_1 a_1 = m_2 a_2.$$

The product of mass and acceleration is defined as the measure of the force, F, exerted on the body, i.e.

$$\mathbf{F} = m\mathbf{a}.$$

It may be asked how this definition of force corresponds with the measurement of force in statics by the extension produced in a spring

balance. A spring balance is calibrated by applying weights and the weight, W , of a body produces an acceleration g when the body falls freely, so that $W=mg$. Thus the spring balance may be regarded as calibrated in terms of mass times acceleration.

The units of force are defined as follows:

A force of 1 *dyne* gives to a mass of 1 gm. an acceleration of 1 cm. per sec. per sec.

A force of 1 *poundal* gives to a mass of 1 lb. an acceleration of 1 ft. per sec. per sec.

Thus in the equation $F=ma$, if m is expressed in grammes and a in cm. sec.⁻², then F is stated in dynes; if m is expressed in lb. and a in ft. sec.⁻², then F is stated in poundals.

Since $W=mg$, where W is the weight of a mass m , 1 lb. wt.* = 32 poundals, 1 gm. wt. = 981 dynes.

The units of work are defined as follows:

When a force of 1 dyne acts through 1 cm. 1 *erg* of work is done. 1 *joule* is 10⁷ ergs.

When a force of 1 poundal acts through 1 ft., 1 *ft. poundal* of work is done.

When a force of 1 lb. wt. acts through 1 ft., 1 *ft. lb.* of work is done.

Measurement of mass

The beam balance really compares weights, but its use for the comparison of masses is justified because, at a particular place, the weights of all bodies are proportional to their masses.

On the moon the weight of a body is only one-sixth of its weight on earth, and it would stretch a spring there only one-sixth as far as on the earth. Its mass, however, would be unchanged and a beam balance would give the same value for its mass in the two places because the weight of the standard would be reduced in the same proportion as that of the body.

It is not self-evident that the weights and masses of bodies made of different materials should be proportional to each other at a particular

* The lb. wt. is unsatisfactory for accurate scientific work because it varies with locality. The weight of an object if measured in this unit remains numerically the same and appears not to change with locality. The value of g at sea level in latitude 45° N. is 32.1740 ft. sec.⁻² Thus in the British Engineering System the pound-force is defined as 32.1740 poundals; this is clearly preferable to the variable unit, the pound-weight (lb. wt.). From the pound-force engineers derive a new unit of mass, the slug. A mass of 1 slug acquires an acceleration of 1 ft. sec.⁻² when acted upon by 1 pound-force. Thus the slug is equivalent to 32.1740 lb. The pound-force and the slug are not used in this book.

place. They must be proportional, however, because all bodies fall with the same acceleration at a particular place. A strict experimental test may be made with a pendulum which provides a continuous falling and rising. Newton compared the periodic times of pendulums having identical dimensions and hollow bobs, which he filled in turn with gold, silver, lead, glass, sand, common salt, wood, water and wheat; he found that they were exactly the same. He was particularly interested in the proportionality of mass and weight in connection with the attraction of the sun for the different planets.

Mass may also be measured by a method involving the interaction of two bodies, as in its definition. The apparatus, known as the ballistic balance, will be described later.

EXAMPLE

Two bodies, of mass m_1 and m_2 , where $m_1 > m_2$, hang from the ends of a string over a light frictionless pulley (Fig. 64). Find the acceleration of the system and the tension in the string.

Two forces act on each body, as shown in Fig. 64, where T is the tension of the string. Suppose a is the acceleration of the system and apply the law $F=ma$ to each body in turn:

$$m_1 g - T = m_1 a,$$

$$T - m_2 g = m_2 a.$$

Adding, $(m_1 - m_2) g = (m_1 + m_2) a.$

$$\therefore a = \frac{m_1 - m_2}{m_1 + m_2} g.$$

$$T = \frac{2m_1 m_2}{m_1 + m_2} g.$$

(If m_1 and m_2 are in lb. then T is in poundals.)

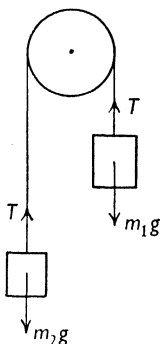


Fig. 64

Atwood's machine

The arrangement of Fig. 64 was adopted by Atwood at the end of the eighteenth century to demonstrate the laws of motion. It is a device by which the fall of bodies under gravity may be slowed down and is comparable with Galileo's inclined plane and rolling ball.

The equation obtained in the example above

$$(m_1 - m) g = (m_1 + m_2) a,$$

follows from a direct application of $F=ma$ to the system as a whole. Provided that the mass of the pulley is negligible, the total mass of the system is $(m_1 + m_2)$ and the resultant force on it is $(m_1 - m_2) g$.

By measuring the time t for the bodies to move a distance s from

rest the acceleration a may be found from the equation $s = \frac{1}{2}at^2$. The apparatus can thus be used to test the validity of the equation $F=ma$ and also to find a moderately accurate value for g .

(i) To show $a \propto F$:

Vary $(m_1 - m_2)$, keeping $(m_1 + m_2)$ constant, and measure a in each case.

(ii) To show $a \propto \frac{1}{m}$:

Vary $(m_1 + m_2)$ keeping $(m_1 - m_2)$ constant, and measure a in each case.

(iii) To find g :

$$g = \frac{m_1 + m_2}{m_1 - m_2} a.$$

The final velocity after falling a distance s can also be found by the apparatus in Fig. 65, which is a common form of Atwood's machine. The small body r , called a rider, rests on one of two equal masses p and cannot pass through the ring A , whereas p can. Thus the rider is left behind at A and the motion from A to B is unaccelerated, continuing with the final velocity v attained at A . Then $v = s'/t'$, where s' is the distance AB and t' is the time taken from A to B . Suppose p and r fell a distance s from rest in a time t before reaching A . The acceleration, a , in

this part of the motion is $\frac{v}{t} = \frac{s'}{tt'}$. Alternatively, assuming the acceleration to be constant, $a = \frac{2s}{t^2}$.

The friction of the bearings of the pulley may be compensated by adding a small extra weight such as a piece of wire on the descending side sufficient to prevent the motion from slowing down when there is no rider. The inertia of the pulley can also be allowed for (see question 29, p. 103).

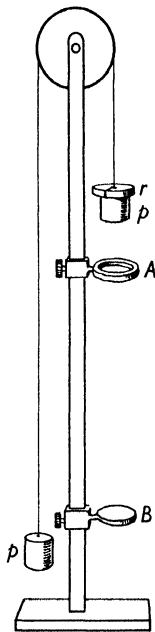


Fig. 65. Atwood's machine.

EXAMPLE

A body of mass m_2 , resting on a smooth plane inclined at θ to the horizontal, is attached by a string to a body of mass m_1 as shown in Fig. 66. Find the acceleration of the system and the tension in the string.

The forces acting on each body are shown in Fig. 66. T is the tension of the string and R the normal reaction of the plane. Suppose a is the acceleration and apply the law $F=ma$ to each body in turn.

For m_1 , $m_1g - T = m_1a$.

For m_2 , resolving along and perpendicular to the plane,

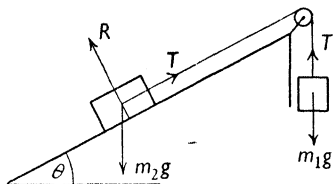


Fig. 66

$$T - m_2g \sin \theta = m_2a,$$

$$R - m_2g \cos \theta = 0.$$

From these three equations the three unknowns a , T and R may be found. Adding the first two equations,

$$a = \frac{m_1 - m_2 \sin \theta}{m_1 + m_2} g.$$

By substitution,

$$T = \frac{m_1 m_2 (1 + \sin \theta)}{m_1 + m_2} g.$$

Newton's laws of motion

The ideas underlying Newton's famous laws of motion have already been described. The following is a literal translation of the original Latin statement of the laws:

(1) Every body perseveres in its state of rest or of uniform motion in a straight line except in so far as it is compelled to change that state by impressed forces.

(2) Change of motion is proportional to the impressed moving force, and takes place in the direction of the straight line in which such force is impressed.

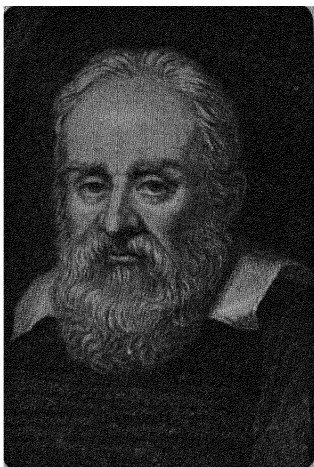
(3) Reaction is always equal and opposite to action; that is to say, the actions of two bodies upon each other are always equal and directly opposite.

The third law was assumed in the definition of mass and the second law in the definition of force. The first law is a special case of the second; it deals with zero acceleration and hence zero force.

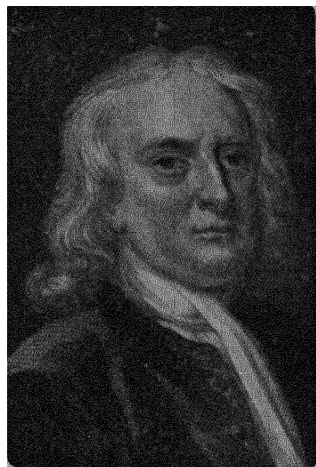
The modern term for 'change of motion' in the second law is 'change of momentum', *momentum* being defined as *the product of mass and velocity*. Force will be shown to be equal to rate of change of momentum (p. 63).

Action and reaction

The third law of motion introduces a principle unrealised by Galileo and is the most fundamental of the three. It will therefore be considered more fully.



GALILEO GALILEI



ISAAC NEWTON

GALILEO GALILEI (1564–1642) spent most of his life in a dangerous and pugnacious advocacy of the Copernican theory, which he conducted with zest and sarcasm at the expense of the Aristotelians and the Jesuits; but his discoveries in mechanics were more profound, if less spectacular, than those in astronomy. As a young man he was for three years a professor at the University of Pisa, and it was during this period that he performed his famous experiment at the Leaning Tower and began his creation of the subject dynamics. From Pisa he went to Padua, under the free Republic of Venice, and the next eighteen years were among the happiest and most fruitful of his life. In 1609 he constructed a telescope with which he discovered the lunar mountains, the moons of Jupiter, the rings of Saturn, the phases of Venus and the spots and rotation of the sun, discoveries which confirmed his belief in the Copernican theory. He left Padua and the protection of Venice to take a post in Florence. Here, for nearly thirty years, he was under the surveillance and power of the Inquisition. His belief that he could persuade the Church of the truth of the Copernican theory proved groundless and towards the close of his life he chose to recant rather than submit to torture and death.

ISAAC NEWTON (1642–1727), building on the work of Copernicus, Kepler and Galileo, established modern science and gave his name to an epoch. His formulation of the concept of mass and of the laws of motion, his discovery of the law of gravitation and invention of a new branch of mathematics, the calculus, enabled him to produce a mathematical theory of the universe which is still the core of physical science. In addition he revolutionised the study of light and made fundamental contributions in sound and on the flow and oscillation of liquids. His famous phrase ‘*Hypotheses non fingo*’ meaning that he did not put forward suppositions unless they had a basis in observation and experiment, crystallised the aim of the scientific method. His indifference to the publication of his discoveries is reminiscent of the attitude of Cézanne, who would leave his paintings lying by a bush. Some of Newton’s apparent indifference, however, sprang from his sensitiveness to criticism; he was involved in painful controversies during his lifetime with Hooke, with Leibnitz and with Flamsteed.

Forces always occur in pairs because, for a force to act, two bodies must always be involved. There is the body which exerts the force and the body which reacts with an equal and opposite force. Action and reaction are interchangeable and the distinction is purely arbitrary; the force on a rocket is regarded as the action, whereas the force on a gun is regarded as the reaction. Newton's third law is fundamental because it indicates that all problems in mechanics involve interaction.

When we walk we push backwards on the ground and the ground reacts by pushing forwards on us. All locomotion depends on the existence of reaction, as, for example, the reaction of the rails on the driving wheels of a locomotive, of the sea on the screw of a ship, and of the air on the propeller of an aeroplane.

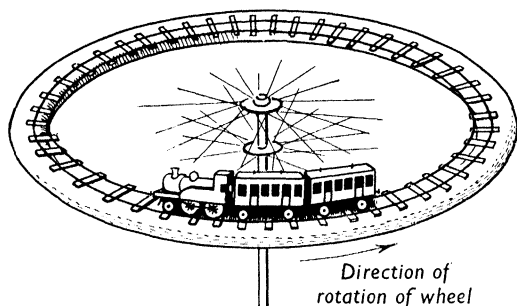


Fig. 67

While a railway train is being accelerated in a forward direction, the earth is being accelerated in the contrary direction. At any instant the backward momentum of the earth is equal to the forward momentum of the train. The mass of the earth is so large that its movement is inappreciable, but the effect can be demonstrated by running a toy train round a track fixed to the rim of a bicycle wheel (Fig. 67). When the train moves forward the wheel moves backward. If the train is stopped by a catch attached to the track, both train and wheel come to rest simultaneously, showing that their momenta are equal and opposite.

Newton considered the problem of a horse and cart. If the cart pulls back on the horse with the same force that the horse pulls forward on the cart, how is motion possible? The horizontal forces on the horse are the backward pull of the cart, P , and the forward thrust of the ground, F_1 (Fig. 68). The backward force exerted by the horse on the ground, also F_1 , may be ignored, because we are interested in the forward motion of the horse and not the resulting (very slight) backward motion of the earth. The horizontal forces on the cart are the forward

pull P and a backward force F_2 . The forward force F_2 exerted by the wheels on the earth may be ignored. It is well to notice, however, that there are two forces F_1 and two forces F_2 . In nearly all problems it is convenient to forget this, to consider only the forces exerted on the body in question and to ignore the equal and opposite forces exerted by the body on other bodies.

Forward motion of the horse and cart occurs when $F_1 > P > F_2$. If the ground were too slippery for the horse to get a grip and $F_2 > F_1$, the horse would not be able to cause the cart to move forward.

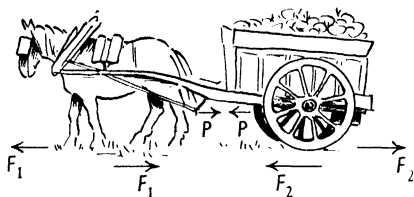


Fig. 68

Action and reaction are equal under all conditions. The backward pull of the cart on the horse is still equal to the forward pull of the horse on the cart when they are accelerating. If all frictional forces on the cart could be eliminated, the horse would still find that the cart pulled back on him while it was being accelerated. If the cart has a mass m and acceleration a , the horse must exert on it a force of $F=ma$; the cart exerts a backward, inertial force on the horse of ma .

The case is essentially the same as that of the two bodies B_1 and B_2 in Fig. 63, the cart corresponding to B_2 and the horse and the earth corresponding to B_1 . Thus the centre of gravity of the horse and the earth must be slightly accelerated in a direction opposite to that of the acceleration of the cart (which moves independently of the earth in the absence of friction).

Momentum and energy

Suppose a force F acts on a body of mass m and produces an acceleration a :

$$\mathbf{F} = m\mathbf{a} \quad (\text{p. 56}).$$

Suppose at zero time the velocity of the body is u and after a time t its velocity is v :

$$v = u + at \quad (\text{p. 52}).$$

$$\therefore a = \frac{v - u}{t}.$$

$$\therefore F = \frac{mv - mu}{t}.$$

Thus the force exerted on a body is equal to the rate of change of momentum.

The equation may be written

$$Ft = mv - mu.$$

Ft is called the *impulse* of the force; it can be seen from the equation to be equal to the change of momentum. If m is in lb., v in ft. per sec. and t in sec., then F is in poundals. Thus the units of momentum, mv , are the same as those of Ft , i.e. poundal-sec. Similarly, if m is in grammes and v in cm. per sec., the units of mv are dyne-sec.

Suppose the body traverses a distance s while its velocity increases from u to v :

$$v^2 = u^2 + 2as \quad (\text{p. 53}).$$

$$\therefore a = \frac{v^2 - u^2}{2s}.$$

$$\therefore Fs = \frac{1}{2}mv^2 - \frac{1}{2}mu^2.$$

Fs is the work done by the force; $\frac{1}{2}mv^2$ is called the *kinetic energy* of the body. The equation represents the statement that the *work done on the body is equal to the change in kinetic energy*.

Kinetic energy may be defined as energy due to motion; it is equal to the work that can be done by the body in being brought to rest.

If m is in lb. and v in ft. per sec., the units of $\frac{1}{2}mv^2$ are those of Fs , i.e. foot-poundals. Similarly, if m is in grammes and v in cm. per sec., $\frac{1}{2}mv^2$ is in ergs.

The momentum and energy equations have been derived from the equation $F = ma$. All three equations may be regarded as equally fundamental, and any one of them could be taken as a starting point from which the other two could be derived. Each equation is suited to solving particular types of problems.

Impulse and momentum are convenient concepts when considering forces which act for a short time, called impulsive forces, such as the blow of a cricket bat on a ball, the thrust on a wall of a jet of water, the reaction on a machine-gun when it projects a spray of bullets. The average force exerted by an impulse depends on the time for which it acts. A harder blow can be exerted on a steel chisel than on a chisel having a wooden handle because in the former case the blow acts for a shorter time. The springs of a motor-car or railway coach minimise percussive forces by making them act for an appreciable time.

The work-energy equation is useful for solving problems when the distance through which a force acts is known, such as in calculating

the braking force necessary to stop a car in a given distance. It cannot be used, however, when some unknown quantity of energy is lost in the form of heat, as on impact.

EXAMPLE

A hose pipe directs a jet of water, of area of cross-section 2 sq. in. and velocity 30 ft. per sec., at right angles to a wall. Calculate the force exerted on the wall assuming that the water is brought to a standstill by the impact.

(1 cu. ft. of water weighs 62·3 lb.)

Mass of water striking the wall per second

$$= 30 \times \frac{2}{144} \times 62\cdot3$$

$$= 25\cdot96 \text{ lb.}$$

Momentum of water striking the wall per second

$$= 25\cdot96 \times 30$$

$$= 779 \text{ poundal-sec.}$$

But Force = change of momentum per second

$$= 779 \text{ poundals}$$

$$= \frac{779}{16} = 48\cdot7 \text{ lb. wt.}$$

EXAMPLE

Assuming a coefficient of friction equal to 1 between the tyres and the ground, find the least distance in which a car travelling at 30 m.p.h. can pull up.

Let mass of car $= m \text{ lb.}$

Maximum braking force $= m \text{ lb. wt.} = 32 m \text{ poundals}$

Let distance in which car pulls up $= s \text{ ft.}$

Work done against braking force = Loss of kinetic energy of car

$$(30 \text{ m.p.h.} = 44 \text{ ft. per sec.}),$$

$$\text{i.e.} \quad 32ms = \frac{1}{2}m(44)^2.$$

$$\therefore s = 30\frac{1}{4} \text{ ft.}$$

The principle of the conservation of momentum

When two bodies collide, each exerts an equal and opposite force on the other (Newton's third law of motion). The forces act for the same time and hence the changes of momentum they produce are equal and opposite. Momentum, like velocity, is a vector quantity, so that momenta in opposite directions must be given positive and negative signs. Thus the total change of momentum is zero and the total momentum after collision is equal to the total momentum before. Suppose the bodies have masses m_1 and m_2 , velocities before collision u_1 and u_2 , velocities after collision v_1 and v_2 :

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2.$$

Velocities in the opposite direction to those marked in Fig. 69 must be counted as negative.

The principle applies not only to collision but to all cases of the interaction between two or more bodies. Thus the backward momentum of a gun is equal to the forward momentum of the shell. When a ball is bounced on the ground it suffers a considerable change of momentum, for not only does its downward momentum disappear but it receives an upward momentum. This change of momentum, however, is exactly counterbalanced by an opposite change of momentum of the earth.

The general principle may be stated as follows: *When bodies in a system interact there is no resultant change of momentum, provided that no external force acts on the system.*

The principle of the conservation of momentum, which is a corollary of the third law of motion, can be regarded as an extension of the

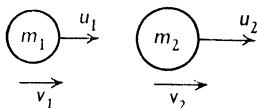


Fig. 69

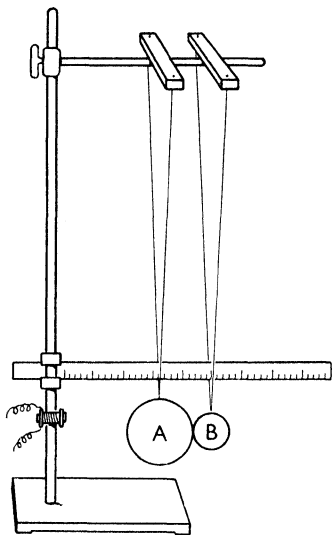


Fig. 70

first law of motion to a system of bodies. Mutual internal forces in a system cannot move the centre of gravity of the system as a whole. When a moving shell or bomb bursts in mid-air the pieces fly in all directions, but the centre of gravity of the system moves on as before.

The ballistic balance

The ballistic balance is a device for comparing masses. Alternatively, it can be used to demonstrate the principle of the conservation of momentum.

A convenient form is shown in Fig. 70. *A* and *B* are two steel balls supported by strings and just touching. *A* is drawn aside and held by an electromagnet. It is then released and collides with *B*. The horizontal displacements of *A* before and after collision, and of *B* after collision, are read on the horizontal scale. These are proportional to the velocities at collision as will now be shown.

Suppose a ball of mass m falls a vertical distance h (Fig. 71). The kinetic energy at the lowest point of its path is equal to the work done by gravity on the ball:

$$\frac{1}{2}mv^2 = mgh, \quad \text{where } v = \text{velocity at lowest point.}$$

$$\therefore v^2 = 2gh.$$

By Pythagoras,

$$l^2 = x^2 + (l-h)^2,$$

$$2lh = x^2 + h^2.$$

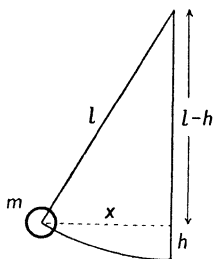


Fig. 71

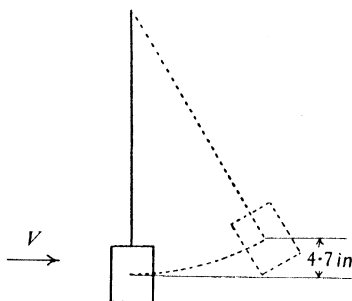


Fig. 72

If h is small compared with x , h^2 can be neglected.

$$\therefore 2lh = x^2,$$

$$h \propto x^2,$$

$$v^2 \propto h \propto x^2.$$

$$\therefore v \propto x.$$

If the masses of A and B are m_1 and m_2 , and their velocities before impact are u_1 and zero, and after impact v_1 and v_2 respectively, then

$$m_1v_1 + m_2v_2 = m_1u_1.$$

Substituting horizontal displacements for u_1 , v_1 and v_2 , the ratio of m_1 and m_2 can be found. Alternatively, if m_1 and m_2 are known, the accuracy of the equation can be demonstrated, thereby verifying the principle of the conservation of momentum.

EXAMPLE

A bullet of mass 1 oz. is fired into a box of sand of mass 20 lb. suspended so as to form a 'ballistic pendulum' (Fig. 72) and remains embedded in it. If the box swings up through a vertical height of 4.7 in., find the velocity of the bullet.

Let v ft. per sec. = velocity of box and bullet immediately after impact:

$$v = \sqrt{2gh} = \sqrt{2 \cdot 32 \cdot \frac{4.7}{12}}$$

$$= 5.0 \text{ ft. per sec.}$$

Let V ft. per sec. = velocity of bullet before impact.

Applying the principle of the conservation of momentum,

$$\frac{1}{16}V = (\frac{1}{16} + 20) 5.0,$$

$$V = 1600 \text{ ft. per sec.}$$

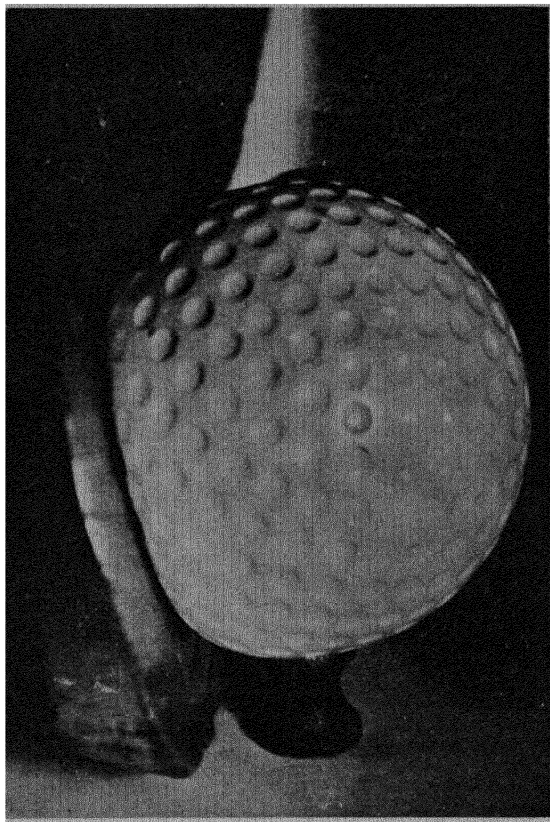


Fig. 73. Golf ball at the moment of impact.
A high speed flash study.

Impact

Collisions may be divided into three classes:

(i) inelastic collisions in which there is no rebound and bodies remain together after impact, e.g. the collision of two pieces of putty;

(ii) perfectly elastic collisions in which the relative velocity of separation is equal to the relative velocity of approach; these never occur in practice, glass and hardened steel being the nearest approach to perfectly elastic materials;

(iii) imperfectly elastic collisions in which the relative velocity of separation is less than the relative velocity of approach.

Newton introduced the concept *coefficient of restitution*, e , which is defined as *the ratio of the relative velocity of separation to the relative velocity of approach*. Thus suppose two bodies, masses m_1 and m_2 , have velocities u_1 and u_2 before impact and velocities v_1 and v_2 after impact. Then

$$v_2 - v_1 = -e(u_2 - u_1). \quad (1)$$

The negative sign is necessary if e is to be positive, since the relative velocity of two bodies is reversed on impact.

By the principle of the conservation of momentum,

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2. \quad (2)$$

Equations (1) and (2) state the two conditions which must be satisfied in solving a problem on direct collision.

When two bodies collide they are deformed (see Fig. 73) and then recover their shape. During restitution of shape a perfectly elastic body (for which $e=1$) would completely restore the energy expended in deforming it. In the case of an imperfectly elastic body some energy is lost in the form of heat.

EXAMPLE

Two spheres of masses 3 gm. and 5 gm. approach each other with speeds of 10 cm. per sec. and 2 cm. per sec. respectively and collide directly. If the coefficient of restitution is $\frac{3}{4}$, find the velocities after collision and the loss of kinetic energy.

Let v_1 and v_2 cm. per sec. be the velocities after collision (Fig. 74), taking the direction from left to right as positive.

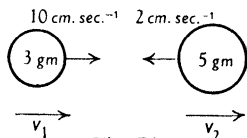


Fig. 74

From the definition of coefficient of restitution,

$$v_2 - v_1 = -\frac{3}{4}(-2 - 10). \quad (1)$$

By the principle of the conservation of momentum,

$$\begin{aligned} 3v_1 + 5v_2 &= 3 \times 10 - 5 \times 2 \\ &= 20. \end{aligned} \quad (2)$$

Multiply (1) by 3 and add to (2):

$$\begin{aligned} 8v_2 &= 47, \\ v_2 &= 5\frac{7}{8} \text{ cm. sec.}^{-1} \\ \therefore v_1 &= -3\frac{1}{8} \text{ cm. sec.}^{-1} \end{aligned}$$

The negative sign indicates that v_1 is in a direction opposite to that indicated in Fig. 74.

$$\text{K.E. before impact} = \frac{1}{2} \cdot 3 \cdot 10^2 + \frac{1}{2} \cdot 5 \cdot 2^2 = 160 \text{ ergs.}$$

$$\text{K.E. after impact} = \frac{1}{2} \cdot 3 \left(3\frac{1}{8}\right)^2 + \frac{1}{2} \cdot 5 \cdot \left(5\frac{7}{8}\right)^2 = 101 \text{ ergs.}$$

$$\therefore \text{Loss of kinetic energy} = 160 - 101 = 59 \text{ ergs.}$$

Determination of the coefficient of restitution

The coefficient of restitution may be determined by allowing a sphere to bounce on a flat surface. Suppose the sphere falls a vertical height h and bounces to a vertical height h' :

$$\frac{\text{Velocity of separation}}{\text{Velocity of approach}} = \frac{\sqrt{2gh'}}{\sqrt{2gh}}.$$

$$\therefore e = \sqrt{\frac{h'}{h}}.$$

The coefficient of restitution may also be found with the apparatus of Fig. 70. Using the nomenclature on p. 67,

$$e = \frac{v_2 - v_1}{u}.$$

The same value for e is obtained whatever the comparative masses of the balls A and B .

The principle of the conservation of mechanical energy

When bodies collide there is conservation of momentum, mv , but not of kinetic energy, $\frac{1}{2}mv^2$. This is possible because momentum is a vector quantity whereas kinetic energy is scalar. Thus the total momentum of two bodies of equal masses, approaching each other with equal velocities, is zero; but their total kinetic energy is twice the kinetic energy of either.

The kinetic energy lost on impact is converted into heat and perhaps also into sound; if all forms of energy are taken into account no energy is lost. In mechanics, however, we are not concerned

with heat produced and we are particularly interested in the conditions necessary for the conservation of mechanical energy, i.e. the conditions under which the equation, $Fs = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$, holds good.

When a body moves from one position to another in a field of force, if the work done by the body against the field is independent of the path taken, the field of force is called a *conservative field*. In such a field mechanical energy is conserved.

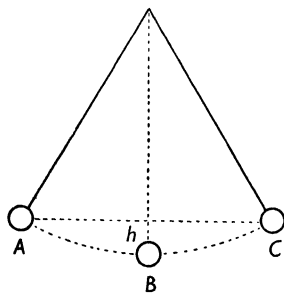


Fig. 75

Gravity is a typical conservative field. During its motion along the path AB the pendulum bob in Fig. 75 acquires the same kinetic energy as if it had fallen vertically a distance h .

During the travel from B to C the kinetic energy is all expended in doing work against the field, and the work done in moving from B to C is called the *potential energy*. Thus if m is the mass of the pendulum bob, and v is its velocity at B ,

$$\frac{1}{2}mv^2 = mgh.$$

Kinetic energy Potential energy

At any point in its path the sum of its kinetic and potential energies is constant. This is the form in which the principle of the conservation of mechanical energy is usually applied. The concept of potential energy applies only to conservative fields, e.g. gravitational, electric and magnetic fields. *The potential energy of a body at a point is the work done in taking the body from some chosen zero level to the point.*

When forces due to friction or air resistance exist, a system ceases to be conservative; the work done against these forces is not independent of the path taken and does not change in sign when the direction of motion is reversed. However, if the forces and the path are known it is sometimes possible to calculate the work done, as in the example on p. 72.

Power

Power is rate of doing work. The units employed are defined as follows:

1 *horse power* is 550 ft. lb. per sec.

1 *watt* is 1 joule per sec.

1 *kilowatt* is 1000 watts.

The power developed by an engine is a variable quantity and depends on the rate of fuel consumption. In the case of a steam locomotive there are, in fact, three limitations to the power that can be developed. First, the couple transmitted to the driving wheels is limited by the maximum steam pressure in the boiler and by mechanical dimensions such as the area of the pistons. Secondly, the draw-bar pull is limited by the force which can be exerted between the driving wheels and the rails; this depends on the coefficient of friction and the weight on the driving wheels. Thirdly, the speed is limited by the rate of generation of steam in the boiler; as the speed increases the pressure of the steam falls until the force it exerts is unable to produce further acceleration.

EXAMPLE

A motor coach weighing 6 tons travels up an incline of 1 in 20 at 30 m.p.h. If the resisting forces (due to friction and air resistance) are 15 lb. wt. per ton, calculate the horse-power developed by the engine of the coach.

Work done per second against resisting forces $= 6 \times 15 \times 44$ ft. lb.

Work done per second against gravity $= 6 \times 2240 \times \frac{1}{20}$ ft. lb.

Total work done per second $= 3960 + 29,568$
 $= 33,528$ ft. lb.

Power developed $= \frac{33,528}{550}$
 $= 61$ H.P.

SUMMARY

This chapter has dealt with the relation between force and motion. Mass and force were first defined, on the basis of Newton's laws of motion, and from these definitions were derived three important relations, which may be expressed mathematically as follows:

$$F = ma,$$

$$Ft = mv - mu,$$

$$Fs = \frac{1}{2}mv^2 - \frac{1}{2}mu^2.$$

In this way new concepts, impulse (Ft), momentum (mv), work (Fs), and kinetic energy ($\frac{1}{2}mv^2$), were introduced.

From the second of the equations, together with Newton's third law of motion, follows the principle of the conservation of momentum. The third equation is a limited statement of the principle of the conservation of energy, as it applies to conservative systems only; a full consideration of conservation of energy must take account of the dissipation of energy in the form of heat, which does not occur in a conservative system.

QUESTIONS

[Take $g = 32$ ft. sec.⁻² or 981 cm. sec.⁻² unless otherwise stated.]

KINEMATICS

1. A stone is thrown vertically upward with a velocity of 96 ft. sec.⁻¹. Find the interval of time between the two instants when it is 80 ft. from the ground.

(O. & C.)

2. A particle falls from rest freely under gravity. In the last second of its motion it falls $\frac{9}{16}$ of the total fall. How far did it fall?

(O. & C.)

3. Fig. 76 represents apparatus for determining the acceleration due to gravity, g , by direct fall. A smoked glass plate is held vertically between wooden guides by a piece of thread and a tuning fork is clamped in such a position that a bristle attached to one of its prongs just touches the plate. The tuning fork is set vibrating and the plate is allowed to fall by burning the thread, with the result that a wavy trace is made on the plate.

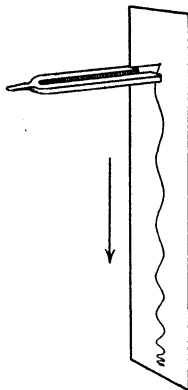


Fig. 76

Show that the difference in length of any two successive sets of the same number of waves is constant.

Using a tuning fork of frequency 256 c.p.s. the difference in length of two successive sets of 10 waves is 1.50 cm. What value does this give for g ?

4. Describe in detail the falling-plate experiment by which the frequency of a tuning fork may be determined. In a trace obtained in this experiment a group of 9 complete waves measures 6.72 cm. and starts 1.69 cm. from the beginning of the trace. What is the frequency of the fork? (L.)

5. A coach is slipped from a train travelling uniformly at 60 miles per hour when the train is 1 mile from a station. If the coach is uniformly retarded so as to stop at the station, where is the train at the moment the coach stops at the station? (C. S.)

6. A stone is dropped from a tower 100 metres high situated on the equator. Find how far, and in what direction, from the point vertically beneath the point of release (as measured by a plumb-line) is the place where the stone strikes the ground. (N.B. The earth is rotating about its axis.) (O. & C.)

7. Deduce the laws of uniformly accelerated motion.

A motor cycle and a car start from rest at the same place at the same time and travel in the same direction. The cycle accelerates uniformly at 4 ft. per sec. per sec. up to 40 m.p.h., and the car at 2 ft. per sec. per sec. up to 60 m.p.h. Calculate the time and distance at which car overtakes cycle. (O. & C.)

8. Describe a method by which you could verify experimentally that freely falling bodies near the earth move with constant acceleration.

A lift moves with uniform velocity through a height of 600 ft. in 30 sec. At what height must the lift be when a stone is dropped from it, if the stone reaches the ground at the same time as the lift reaches the end of its upward journey? With what velocity will the stone strike the ground? (O. & C.)

9. During each third of the interval of time taken by a car to travel a journey from rest to rest, the car's acceleration is constant. By means of a speed-time graph, prove that the average speed for the journey is two-thirds of the car's speed at the middle of the interval. Prove also that, if one n th of the journey is travelled in the first half of the interval, the accelerations during the 1st, 2nd and 3rd thirds of the interval are in the ratios

$$(8-n):8(n-2):(8-7n).$$

(O.)

10. (a) Galileo's rejection of the idea that the velocity acquired by a falling body is proportional to the distance fallen was based on the following fallacious argument. Suppose a body, falling from rest, reaches a final velocity v after falling a distance s . Taking its average velocity as $\frac{1}{2}v$ the time taken in falling is $s/\frac{1}{2}v = 2s/v$. Now suppose the body falls a distance $2s$ from rest and acquires a velocity $2v$. Taking its average velocity as v , the time taken in falling is $2s/v$, which is the same time as that taken to fall the distance s . Hence it must traverse the second half of its path instantaneously.

Point out the flaw in this argument.

(b) Investigate the motion of a body which moves so that its velocity is proportional to the distance traversed and obtain a relation between the distance traversed, s , and the time, t . What do you deduce about (i) its acceleration, (ii) its initial velocity?

11. Water issues in a circular jet from a small hole in the bottom of a container. Show that a value for the acceleration due to gravity may be calculated from measurements of the diameters at various depths. Indicate how the effects of surface tension may be taken into account. (O. S.)

PROJECTILES

12. Show how, from a knowledge of the velocity of projection of a body under gravity, its subsequent position at any instant may be found.

A tennis ball is returned from a point close to the ground on the back line and just clears the net, hitting the ground on the back line at the other side of the net. If the height of the net is 3 ft. 3 in., and the distance from the net to either back line is 39 ft., find the velocity with which the ball left the racquet. (C.)

13. Show that, under certain ideal conditions, the path of a particle projected near the surface of the earth is a parabola. Find the range of a particle projected from the surface of the earth with velocity v at an angle α to the horizontal. What factors affect the result in practice?

An aeroplane flies at a height h at a constant speed u in a straight horizontal line, so as to pass vertically over a certain gun. At the instant when the aeroplane is vertically over it, the gun fires a shell which hits the aeroplane. Neglecting the effects of the atmosphere, deduce the minimum muzzle velocity of the shell and the correct angle of elevation at this velocity. (O. & C.)

14. A shell is fired from a gun with a muzzle velocity of 480 ft. per sec. at a target $3600\sqrt{3}$ ft. distant at the same height as the gun. Neglecting air resistance, find the angle of elevation. What are the limits within which a wireless pole 224 ft. high can be placed between the gun and the target without being hit? (O. & C.)

15. Particles are simultaneously projected in all directions with speed u ft. per sec. from a stationary balloon. Prove that t seconds later all the particles, whose motion has not been interrupted, will be situated on a sphere. Find the centre and radius of the sphere. (O.)

16. Show that the path of a projectile is a parabola if air resistance is neglected, and find the direction which will give the maximum horizontal range for a given velocity of projection.

Calculate the muzzle velocity and the height reached by a shell which would have a maximum horizontal range of 60 miles if air resistance could be neglected. (O.)

17. What is the greatest range of a projectile fired at initial velocity 500 ft. per sec. up a slope inclined at 20° to the horizon? (O.)

18. An observer standing close beside an anti-aircraft gun notices that a shell explodes 5 sec. after it has been fired. The sound of the explosion reaches him 9 sec. later. If the angle of elevation of the gun is 45° , calculate to within the nearest 100 ft. the height at which the shell explodes.

(The velocity of sound in air is 1110 ft. per sec., $g = 32$ ft. sec. $^{-2}$) (C. S.)

NEWTON'S LAWS OF MOTION

19. (a) A man on a weighing machine (1) stands still, (2) moves down into the knees bend position, (3) moves up again. Compare and explain the readings of the weighing machine in the three cases.

(b) Discuss how, in the light of Newton's third law of motion, one team can pull another team over in a tug-of-war. Show that when the rope accelerates one team must pull harder on it than the other, but that when one team pulls the other over with a steady velocity, the two pulls must be equal.

(c) A man throws a ball against a wall and catches it after the rebound. Consider the conservation of momentum, taking the earth into account.

(d) In Fig. 77 the beam is balanced while the mass $M+m$ is prevented from falling by being tied to the pulley with an extra thread. What happens to the beam when this thread is cut and $M+m$ begins to descend and M ascend?

(e) Explain why the load on the front wheels increases when a motor-car accelerates.

(f) Two perfectly elastic bodies, of equal mass and moving in the same straight line, collide. Show that, to satisfy the two conditions governing collision, they must exchange velocities.

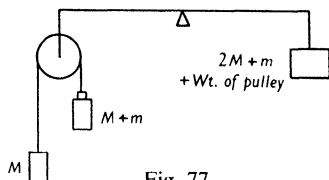


Fig. 77

20. What is the distinction between the *mass* of a body and its *weight*? What evidence is there that the weights of bodies are proportional to their masses? (O.)

21. A car, travelling at 40 m.p.h., can brake to a standstill in 30 yards. If the driver weighs 200 lb., what retarding force is necessary to keep him in his seat? (O. & C.)

22. A body slides down a plane with uniform velocity when the latter makes an angle of 30° with the horizontal. What will be the acceleration of the body down the plane when the inclination is increased to 45° ? (B.)

23. If a particle has an initial velocity u and travels a distance s under a uniform acceleration f , show that the final velocity v is given by $v^2 = u^2 + 2fs$.

A string which passes over a light frictionless pulley supports at its ends masses of 10 gm. and 12 gm. Find what will be the velocity of the system when the string has moved 22 cm. (O. & C.)

24. A trolley of mass 1912 gm., at rest on a horizontal table, is pulled by a horizontal string, which, after passing over a light pulley at the edge of the table, supports a 50 gm. wt. Assuming that friction may be neglected, calculate the acceleration of the trolley and the distance it moves in 2 sec. (C.)

25. Describe an experiment by which the acceleration due to gravity may be determined by direct measurement of the time of fall of a body.

A passenger in a lift noted that at one stage of his journey a spring balance carrying a mass of 10 kg. registered 12 kg. What was then the acceleration of the lift?

What would the spring balance read if the lift were falling (a) freely, (b) at constant speed? (N.)

26. From Newton's laws of motion derive a relation between force, mass and acceleration.

A mine cage weighing 1 ton is being lowered down a shaft. The tension in the cable is 5 cwt. What is the force exerted on the floor of the cage by a miner weighing 200 lb. standing on it?

The miner throws a lump of coal up 6 ft. vertically and catches it as it descends again. How long elapses between his throwing and catching it? (O. S.)

27. An observer in a closed railway carriage is equipped with a delicate spring balance and a plumb-line. While the carriage is at rest he determines the weight of a piece of metal and the position of the freely hanging plumb-line. What information could he derive from simultaneous observations with these two pieces of equipment when the carriage is in motion?

What would be observed inside the carriage if it was (a) going up an incline of 1 in 10 with uniform speed, (b) moving up the same incline with an acceleration of 3.2 ft. sec.⁻²? (C. S.)

28. Two weights, of 500 gm. each, hang freely on the ends of a weightless cord which runs over a frictionless pulley. If they are initially at rest, calculate how long it is after a 1 gm. weight has been added to one of them that this latter has dropped 1 metre, and find its velocity at this point. What is the tension in the string?

If, instead, the weights were to run on frictionless planes at 45° to the vertical, what then would be the time, the velocity, and the tension? (O. S.)

29. A body *A* rests on a horizontal table, the coefficients of static and kinetic friction between the body and the table being μ_1 and μ_2 respectively ($\mu_1 > \mu_2$). A light string, attached to *A*, passes over the edge of the table and supports a body *B*, hanging freely. If the system is in limiting equilibrium, and is then set in motion, show that its acceleration is $\frac{(\mu_1 - \mu_2)g}{\mu_1 + 1}$. If *B* reaches the ground after *A* has travelled a distance *s*, so that the string becomes slack, how much farther will *A* be able to travel along the table before it comes to rest? (O. & C.)

30. Explain how Newton's first and second laws of motion respectively define *force* and the way in which it is to be measured. Show that the third law leads to the principle of conservation of momentum, illustrating your answer by two practical examples.

A chain hanging vertically and weighing 3 lb. per ft. is being wound on a drum of 3 ft. diameter, the free end of the chain being coiled on the floor. The drum

rotates uniformly with a speed of 90 r.p.m., its axis being horizontal and 20 ft. above the floor. Draw a diagram to scale showing how the tension in the chain, at the point where it first makes contact with the drum, varies with the length of the unwound chain as the last 40 ft. are being wound up. Use the diagram to deduce the total energy required for this operation. Show on the diagram how much of this is potential energy, and explain what form the remainder of the energy takes.

($g = 32.2$ ft. per sec. per sec.) (N.)

MOMENTUM

31. State the principle of *conservation of linear momentum*; describe briefly, and explain, an experiment to verify it.

A shell, mass 1 cwt., is fired from a gun of mass 5 tons with a muzzle velocity of 2000 ft. per sec. The gun recoils 2 ft. Determine (i) the initial velocity of recoil, (ii) the average resisting force. (C.)

32. A fire hose, emitting 120 gallons of water per minute, will, when directed vertically, throw a jet to a height of 100 ft. What force will the jet exert if it is directed horizontally against a vertical wall? It may be assumed that after striking the wall the water falls vertically down to the ground.

(A gallon of water weighs 10 lb.) (C.)

33. A seagull weighs 2 lb. and has an effective wing area of 1 sq. ft. If the density of air is 0.06 lb. ft.⁻³, calculate what vertical air speed is required for the bird to hover. (C. S.)

34. Show how Newton's second law of motion leads to a definition of force.

A horizontal jet of water delivering 500 kg. per sec. at a speed of 4 m. per sec. strikes the blades of a water wheel. The blades are moving with half the speed of the water and the water drops off with no horizontal velocity. Find the force exerted on the wheel. (C.)

35. A uniform fine chain, of mass m gm. per cm. length, is suspended so that its lower end just touches a horizontal table, and is then allowed to fall. Show that, at the instant when a length x cm. has reached the table, the force exerted on the table is $3mgx$ dynes. (O.)

36. A target weighing 299 lb. is suspended by two long parallel cords so that it can swing upwards parallel to itself, the cords keeping in the same vertical plane. A cannon shell of mass 1 lb. is fired into the target in a horizontal direction through its centre of gravity and becomes embedded in the target which swings upwards, rising through a distance of 1 ft. Calculate the velocity of the cannon shell. (N.)

37. A hammer of 60 lb. wt. falls 9 ft. from rest on to a light peg which it drives into the ground in $\frac{1}{12}$ sec. from the moment of impact. Calculate the average resistance of the ground in lb. wt. (N.)

38. A pile driver weighing 400 lb. falls through 10 ft. and drives a pile weighing 1000 lb. through a distance of 3 in. Find the average resistance to the pile in lb. wt., and the energy dissipated in heat per stroke. (O. S.)

39. State Newton's second and third laws of motion, and from them deduce the principle of the conservation of momentum.

An aircraft has 8 machine-guns directed straight ahead. Each gun fires bullets of mass 1 oz. at the rate of 1200 per min. with muzzle velocity (measured when the gun is stationary) of 2400 ft. per sec. By how much (in ft. per sec.) is the speed of the aircraft reduced in 5 sec. when all the guns are firing, if the total mass of the machine and pilot is 6000 lb.? Find also the backward force (in lb. wt.) exerted on the aircraft by the firing. (N.)

40. Describe the ballistic pendulum and explain how it can be used to determine the velocity of a rifle bullet. If the mass of the pendulum is M , its length l , the mass of the bullet m , and if the lower end of the pendulum moves through a horizontal distance d , find expressions for the velocity of the bullet and the fractional loss of kinetic energy at impact. Assume that the pendulum can be regarded as 'simple', and that d is small compared with l . (O. & C.)

41. State the law of conservation of momentum, and explain how it is concerned in the jet propulsion of aircraft. Discuss whether jet propulsion is feasible beyond the limits of the atmosphere.

Describe how you would verify the law of conservation of momentum experimentally in the laboratory. (O.)

42. State the principle of the conservation of momentum.

Two masses M and M' are connected by a string passing over a pulley. The heavier mass M rests on a platform and the lighter mass M' is raised H cm. and then allowed to drop. With what velocity will the heavier mass leave the platform; with what velocity will it return to the platform; and how long will it be in the air?

(C. S.)

IMPACT

43. Two smooth balls A and B , whose weights are 1 lb. and 2 lb., moving in opposite directions with velocities of 40 ft. per sec. and 30 ft. per sec. respectively, strike each other directly. Find the velocity of each after impact if the coefficient of restitution is 0.75 and state the direction in which each will move. (D.)

44. Define 'coefficient of restitution'. Describe an experiment to determine its value in the case of a steel sphere dropped on to a rigid horizontal steel plate.

A helium atom moving in a straight line with velocity V makes direct impact with a hydrogen atom at rest. Assuming that the two atoms behave like perfectly elastic spheres and that the mass of the helium atom is four times that of the hydrogen atom, calculate (a) the velocity of the hydrogen atom after collision, (b) the percentage energy lost by the helium atom in the collision. (N.)

45. A stone of mass M rests on a horizontal sheet of ice. A block of ice of mass m is propelled along the sheet and strikes the stone with velocity u and then sticks to the stone. If the coefficient of sliding friction between the stone and the ice is n and the friction between ice and ice can be neglected, how far will the combined mass of ice and stone travel before coming to rest? (C. S.)

46. A rubber ball of mass 4 oz. falls from a height of 10 ft. on to a concrete floor. It rebounds to a height of 4 ft. What is the coefficient of restitution? What change of momentum occurs at the second rebound? To what height will it rise on the third rebound? (C.)

47. State the laws of collision.

Two spherical balls whose masses are in the ratio 3:1 are suspended side by side so that their centres are in a horizontal line and the balls hang just touching. The smaller ball is drawn aside and let go. It strikes the larger ball with a velocity of 20 cm. per sec. and the collision is perfectly elastic. Find the velocities of the two balls (a) after the first collision, (b) after the second collision. (C.)

48. A ball is dropped on to a horizontal plane from a height h ft. above it. Find the time that elapses before the ball comes to rest in terms of h and e , the coefficient of restitution. (O.)

49. A mass of 400 gm. hangs from a light spring of such stiffness that an additional load of 100 gm. would cause an extension of 1.5 cm. A mass of 100 gm. falls from a height of 50 cm. on to the 400 gm. mass, and adheres to it. How far will the combined masses descend after the impact? (C.)

50. Two perfectly elastic balls moving along the same straight line are in collision; prove that the energy transferred from one to the other is proportional to the product of their relative velocities and the sum of their momenta, and is, when these quantities are given, greatest when the masses of the balls are equal. (C. S.)

51. A particle of mass 1 gm. travelling with a velocity of 1000 cm. per sec. collides with a second particle of mass 4 gm. which is free to move. After collision the first particle is found to have been deflected through an angle of 60° and to be moving with a velocity of 500 cm. per sec. Calculate the velocity and direction of the second particle, and the energy lost in the collision. (O. & C.)

52. A stream of particles, each of mass m gm. moving with velocity u cm. per sec., is impinging normally on a plate which is being moved towards the particles at a constant velocity U cm. per sec. The coefficient of restitution for the collision is e . Find the change in kinetic energy of each particle, and also the force employed to maintain the motion of the plate if n particles impinge on it per second. (N.)

ENERGY

53. Define *potential energy*, *kinetic energy*. Show that the kinetic energy of a particle of mass m moving with a velocity v is $\frac{1}{2}mv^2$.

Show also that, in the case of a particle falling freely under gravity, the sum of its kinetic and potential energies is constant. (O. & C.)

54. Explain the doctrine of the conservation of energy, illustrating your explanation by the case of a simple pendulum swinging in the air.

A small pendulum bob, supported by a thread 50 cm. long, vibrates through an angle of 30° at the point of suspension. Calculate the velocity of the pendulum bob at its lowest point. (N.)

55. A 20 lb. weight falls vertically from rest through a distance of 400 ft., and its velocity is then found to be 155 ft. per sec. Calculate the average resistance of the air to the motion of the body. (D.)

56. In passing through a wood block 8 cm. thick the velocity of a bullet is reduced from 700 metres per sec. to 300 metres per sec. If the mass of the bullet is 10 gm., find the average resistance to penetration. (C.)

57. If the coefficient of friction between the tyres of a car and the road is 0.6, find the least distance in which the car can be brought to rest from a speed of 15 miles an hour (a) on the level, (b) when descending a hill of gradient 1 in 6.

(O. & C.)

58. A bobsleigh is travelling at 30 m.p.h. when it starts ascending an incline of 1 in 100. If it comes to rest after travelling 500 ft. up the slope, calculate the proportion of the energy lost in friction and deduce the coefficient of friction between the runners and the snow.

(O. & C.)

59. What is the principle of the conservation of energy?

Give three examples which illustrate the law. Discuss whether it is an independent principle or a deduction from Newton's laws of motion.

(O. S.)

POWER

60. Explain the meaning of (a) work, (b) power, (c) efficiency of a machine.

A friction band passes round the flywheel of an engine; one end of the band supports an 84 lb. weight and the other is attached to a spring balance. The diameter of the flywheel is 2 ft. It is found that, when the flywheel is making 1320 r.p.m., the spring balance reads 24 lb. What horse-power is the engine developing? If the total power supplied by the combustion of the fuel is 3×10^6 ft. lb. per min., what is the overall efficiency of the engine?

(1 h.p. = 550 ft. lb. per sec.)

(O. & C.)

61. A motor-car weighing 2000 lb. climbs at 15 m.p.h. a hill with a gradient of 1 in 8. If the frictional force is 50 lb. wt., calculate the horse-power of the car.

(1 h.p. = 550 ft. lb. wt. per sec.)

(N.)

62. A motor-car weighing 3000 lb. and driven by an engine of 24 h.p. is capable of maintaining a maximum speed of 75 m.p.h. on a level road. What is the effective force resisting the motion of the car? Assuming this retarding force to remain constant, what would be the maximum speed of the car up a gradient of 1 in 10? (C.)

63. A rope is coiled a few times round a cylindrical post and it is found that a small pull (T_1) applied at one end is able to sustain a much larger pull (T_2) at the other end. Show that $T_2/T_1 = e^{\mu\theta}$, where μ is the coefficient of friction between rope and post and θ the angle of lap in radians of the rope round the post. [Hint. Use the method on p. 91.]

Suggest an experimental method for verifying the formula and show how you would plot your results and use the graph to determine the value of μ .

A band brake makes contact over half the circumference of a pulley of 6 inches diameter which revolves at the rate of 200 rev. per minute. The tensions on the two sides are 300 lb. wt. and 100 lb. wt. respectively. Find (a) the coefficient of friction between the band and pulley, (b) the horse-power delivered to the pulley. (N.)

CHAPTER 4

ROTATION

Uniform motion in a circle

When a body is moving in a circle with constant speed its direction, and hence its velocity, is continuously changing; it is therefore continuously accelerated. Because the speed in the direction of motion is unchanged, the acceleration must be at right angles to the path of the body, i.e. towards the centre of the circle. Thus to whirl a stone, attached to the end of a string, in a circle, a pull must be exerted along the string.

Consider a particle moving in a circle of radius r with constant speed v (Fig. 78). Suppose in a time t the particle moves from a position

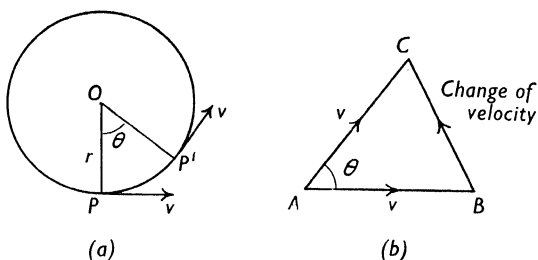


Fig. 78

P to P' and the radius OP turns through an angle θ . The velocities of the particle at P and at P' may be represented by AB and AC respectively in Fig. 78(b); AB and AC are in directions perpendicular to OP and OP' respectively and hence $\angle CAB = \theta$. By the parallelogram of velocities, the two velocities represented by AB and BC are together equivalent to the velocity AC . Thus BC represents the change in velocity when the particle moves from P to P' . The instantaneous acceleration of the particle is the rate of change of velocity, i.e. the instantaneous value of BC/t .

$BC = v\theta$ because θ is very small.

$$\begin{aligned} \therefore \text{Acceleration of particle} &= \frac{BC}{t} = \frac{v\theta}{t} \\ &= \frac{v}{t} \frac{\text{arc } PP'}{r} \quad \left(\text{since } \theta = \frac{\text{arc } PP'}{r} \right) \\ &= \frac{v^2}{r} \quad \left(\text{since } v = \frac{\text{arc } PP'}{t} \right). \end{aligned}$$

If the mass of the particle is A , the force exerted on the particle is mv^2/r , the product of mass and acceleration.

It is often convenient to express this in terms of the angular velocity, ω , which is the angle in radians through which OP rotates in 1 sec.:

$$\omega = \frac{\theta}{t} = \frac{\frac{\text{arc } PP'}{r}}{t} = \frac{v}{r}.$$

Thus the force which must be exerted on the particle towards the centre of the circle is mv^2/r or $m\omega^2r$.

If the particle is not moving with a uniform speed there must, in addition to the force of $m\omega^2r$ towards the centre of the circle, be also a force along the tangent equal to the product of m and the acceleration along the tangent, i.e.

$$mr \frac{d\omega}{dt} \text{ or } mr \frac{d^2\theta}{dt^2}.$$

Centripetal and centrifugal force

It has already been explained that a horse pulling a cart of mass m with an acceleration a feels a backward pull of ma , quite apart from any frictional forces. There is, in fact, an inertial force of ma equal and opposite to the applied force of ma . Similarly in circular motion, there is an inertial force of $m\omega^2r$ away from the centre of the circle, equal and opposite to the applied force of $m\omega^2r$. The applied force towards the centre of the circle is called the *centripetal force* (*peto*, I seek); the inertial force away from the centre of the circle is called the *centrifugal force* (*fugo*, I flee).

Consider a stone being whirled at the end of a string; the string exerts a centripetal force on the stone and the stone exerts an equal centrifugal force on the string. In the case of an elastic string the existence of the centrifugal force is revealed by the stretching which occurs. If the speed of rotation is increased these forces must increase and eventually the string may break, in which case the stone will fly off at a tangent.

Balancing of machines

The rotating parts of a machine must, during manufacture, be given rigorous balancing tests because, if the centre of gravity does not coincide with the axis of rotation, the resulting centrifugal force may cause violent vibrations. Suppose the centre of gravity of a flywheel, of mass 5 tons, is $\frac{1}{16}$ in. from the centre of the shaft and the speed is 1000 r.p.m.

Centrifugal force $= m\omega^2 r$

$$= 5 \left(\frac{1000 \cdot 2\pi}{60} \right)^2 \times \frac{1}{120} \times \frac{1}{32} \text{ tons wt.}$$

$$= 14.3 \text{ tons wt.}$$

A flywheel can be balanced by removing metal from the heaviest side. But in such a case as the driving wheel of a locomotive (Fig. 79), where the effects of cranks and connecting rods have to be counteracted, balance weights must be employed.

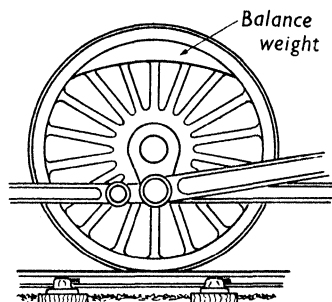


Fig. 79. Locomotive driving wheel.

Looping the loop

The ball in Fig. 80 will stay in contact with the track while describing the vertical circle if, at the highest point of the circle, the centrifugal force is not less than the force of gravity. Subject to the same condition,

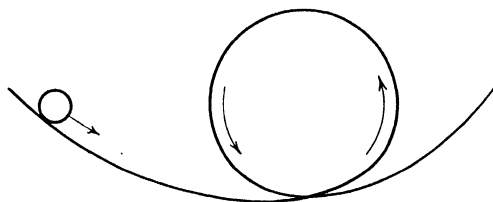


Fig. 80

a bucket of water can be whirled in a vertical circle without spilling, and a pilot, who is not strapped into his aeroplane, can loop the loop without falling out.

The governor

Fig. 81 represents the governor, an invention of James Watt for making an engine run at constant speed. Two balls are carried by arms which are freely pivoted to a collar *A*, fixed to a vertical shaft *S* which is geared to the engine. The arms are connected by links to *B*, which is a sleeve fitting loosely on the shaft *S* and which

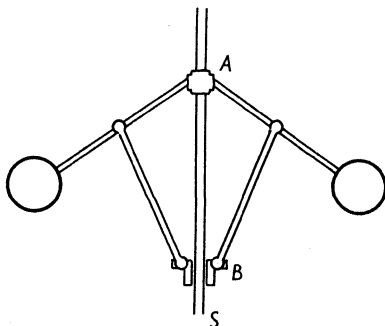


Fig. 81

is connected by levers, not shown in Fig. 81, to the throttle valve of the engine.

The balls rotate with the shaft S , and when the speed of the engine increases they rise, causing B also to rise. This has the effect of reducing the steam supply and slowing down the engine. When the speed decreases the balls and B fall, and more steam is admitted, thereby speeding up the engine again. The way the balls rise with increasing speed can be seen on a large scale in the behaviour of the suspended cars of a roundabout at a fair.

A small governor is fitted to the motor of a gramophone; movement of B (Fig. 81) alters the pressure of a small felt pad which acts as a brake on the turn-table.

The principle of the governor can be investigated by considering the simple arrangement in Fig. 82, known as a *conical pendulum*. A

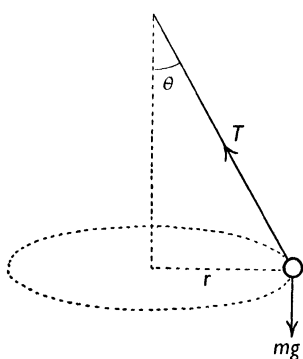


Fig. 82

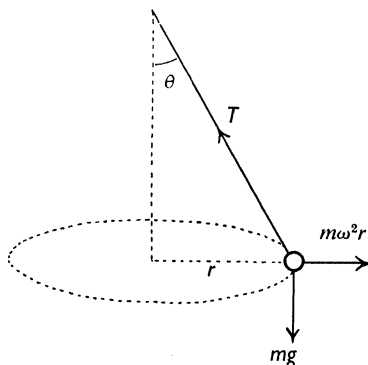


Fig. 83

bob of mass m , attached to a light string, is projected in a horizontal circle of radius r with angular velocity ω . Suppose the string makes an angle θ with the vertical and its tension is T . The horizontal component of T provides the necessary centripetal force and the vertical component of T supports the weight of the bob.

$$T \sin \theta = m\omega^2 r,$$

$$T \cos \theta = mg.$$

Dividing

$$\tan \theta = \frac{\omega^2 r}{g}.$$

The problem can be reduced to a statical one by inserting the centrifugal force (Fig. 83) and regarding the three forces as being in equi-

librium. This is an application of d'Alembert's principle, which asserts that any system can be treated as being in equilibrium if all the inertial forces (each the product of mass and acceleration) are considered with the applied forces.

Banking

A railway track is banked on a curve in order that the necessary centripetal force on a train may be provided by the horizontal component

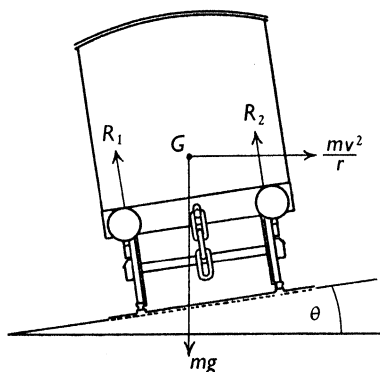


Fig. 84

of the normal reaction of the rails on the wheels (Fig. 84). If the track were horizontal the centripetal force could be provided only by the force between the flanges of the wheels and the rails, and the resulting grinding action would cause unnecessary wear. A track can be correctly banked only for a particular speed. At speeds greater than this there is a force between the flanges of the wheels and the outer rail; at speeds less than this there is a force between the flanges of the wheels and the inner rail.

Suppose for a speed v the correct angle of banking for a curve, of radius r , is θ . Let R_1 and R_2 be the normal reactions of the rails for a train of mass m . Consider the centrifugal force mv^2/r to be acting at the centre of mass G and, by d'Alembert's principle, treat the system as being in equilibrium.

Resolving horizontally,

$$(R_1 + R_2) \sin \theta = \frac{mv^2}{r}.$$

Resolving vertically, $(R_1 + R_2) \cos \theta = mg$.

Dividing $\tan \theta = \frac{v^2}{rg}$.

Thus for a speed of 60 m.p.h. a track of radius $\frac{1}{4}$ mile must be banked at an angle

$$\tan^{-1} \frac{88^2}{440 \times 3 \times 32} = 10\frac{1}{2}^\circ.$$

A motor-car turning a corner is often assisted by the camber on the road, but the centripetal force is largely provided by the friction between the tyres and the road. A racing track, however, is banked and shaped like a saucer. The faster a car travels the more it moves to the outside of the track where the gradient of the banking is steeper.

An aeroplane must bank in order to turn at all, and the centripetal force is provided by the component of the lift, L (Fig. 85).

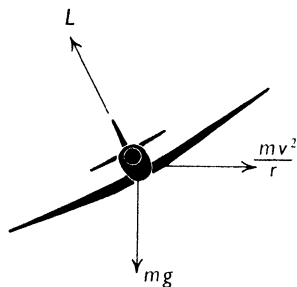


Fig. 85

Upsetting and skidding

(1) *A cyclist.* A cyclist must lean inwards when travelling round a corner or he will be upset. Suppose the cyclist and his machine have a total mass m and that they are turning (Fig. 86) to the left in a circle of radius r at a speed v . R and F represent the normal reaction and the force of friction, respectively, exerted by the ground on the tyres. The centrifugal force mv^2/r has been inserted and hence the system may be regarded as in equilibrium.

Taking moments about A ,

$$mg \ AG \sin \theta = \frac{mv^2}{r} \ AG \cos \theta.$$

$$\therefore \tan \theta = \frac{v^2}{rg}.$$

Thus the cyclist must lean at this angle θ to remain in equilibrium. If he were vertical his weight mg would pass through A and there would be no moment to counterbalance that of mv^2/r about A . The forces in Fig. 86 really represent two couples since $R=mg$ and $F=mv^2/r$; when the cyclist is vertical one of these couples disappears and hence the other is operative.

The maximum value of F is μmg , where μ is the coefficient of friction. The cyclist will skid if

$$\frac{mv^2}{r} > \mu mg,$$

i.e. if $v^2 > \mu rg$.

Thus the cyclist is likely to skid (a) if he is going too fast (when v is large), (b) if the road is slippery (when μ is small), and (c) if he turns too sharply (when r is small).

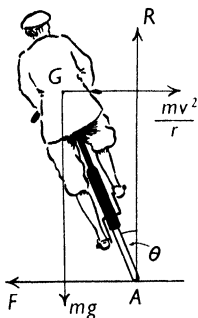


Fig. 86

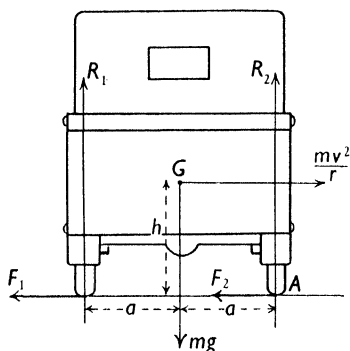


Fig. 87

(2) *A vehicle.* Fig. 87 represents a vehicle, of mass m , turning to the left in a circle of radius r , at speed v . R_1 and R_2 are the normal reactions, and F_1 and F_2 the forces of friction, exerted by the ground on the tyres. Suppose the vertical height of the centre of gravity G is h , and the distance between the wheels is $2a$. The centrifugal force is inserted in Fig. 87 and the system regarded as in equilibrium.

When the vehicle is about to be upset it will travel on the wheels on one side only, i.e. $R_1 = 0$. It will upset if the moment of the centrifugal force about A is greater than the moment of the weight, i.e.

$$\text{if } \frac{mv^2}{r} h > mga,$$

$$\text{if } v^2 > \frac{arg}{h}.$$

Thus a vehicle is least likely to upset if the distance between the wheels ($2a$) is large and if its centre of gravity is low (h is small).

The maximum value of $F_1 + F_2$ is μmg , where μ is the coefficient of friction. The vehicle will skid if

$$\frac{mv^2}{r} > \mu mg,$$

i.e. if $v^2 > \mu rg$.

A reckless driver going too fast can minimise the chance of a skid by 'cutting' a corner as in Fig. 88; the path ADC is a curve of greater radius r than the path ABC

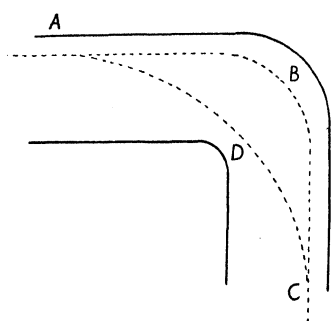


Fig. 88

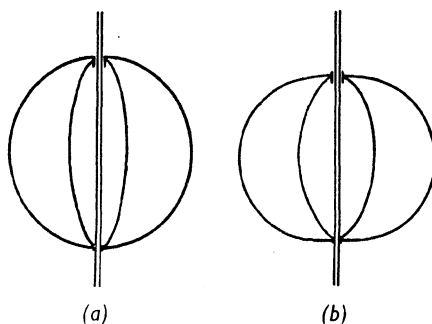


Fig. 89

The earth's rotation

Fig. 89 represents an experiment to illustrate the flattening of the earth at the poles and the bulging at the equator due to the effect of centrifugal force. Two flexible, circular metal hoops, arranged at right angles to each other, are fixed at their lowest points to a vertical spindle while their highest points are free to slide up and down the spindle. On being rotated the shape changes from (a) to (b) in Fig. 89. The angular velocity ω of each part of the hoops is constant, but the distance r from the axis of rotation varies from zero at the poles to a maximum at the equator. Hence the centrifugal force $m\omega^2r$ is zero at the poles and greatest at the equator. This has caused, in the case of the earth which behaves as a plastic body, a difference of 27 miles in the equatorial and polar diameters. The effect was predicted by Newton and verified by a French expedition to Lapland, led by the pompous Maupertuis who, said Voltaire, behaved as though he had flattened the poles himself. The effect is more pronounced in the case of some other planets which spin faster than the earth.

The earth's rotation causes the weight of a body to decrease as it is taken from the poles to the equator, and this is additional to the effect due to the earth's shape. The accelerations due to gravity at the equator, latitude 45° , and the poles are 978.0, 980.6 and 983.2 cm. sec.⁻², respectively. It should therefore be easiest to set up an athletic record, such as the high jump, at the equator.

The earth's rotation also causes a plumb-line to hang slightly out of the vertical everywhere except at the poles and on the equator.

Suppose that the attraction of the earth on a mass m , at a place P , in latitude λ , is mg_0 (Fig. 90). The centrifugal force is $m\omega^2 r$ as shown, where ω is the angular velocity of the earth. The observed weight of the body is mg , which is the resultant of the forces mg_0 and $m\omega^2 r$, and whose direction is inclined at an angle θ to OP . Resolving along, and perpendicular to, OP ,

$$mg \cos \theta = mg_0 - m\omega^2 r \cos \lambda,$$

$$mg \sin \theta = m\omega^2 r \sin \lambda.$$

From these two equations knowing g_0 , ω , r and λ the values of g and θ may be obtained. The maximum value of $g_0 - g$ occurs at the equator and is $3.5 \text{ cm. sec.}^{-2}$; θ is greatest in latitude 45° where its value is $6'$.

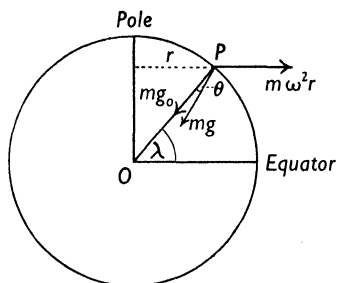


Fig. 90

Centrifuges

Honey is extracted from the honey-comb by rotating the latter at high speed. The force tending to keep the honey in the comb is less than the centrifugal force; hence, as happens to a whirling stone when the string breaks, the honey flies off tangentially. Similarly, the centrifugal drying machine operates by whirling damp clothes and fabrics in a perforated metal cylinder; as the cylinder rotates water is flung out through the perforations.

Centrifuges are designed to separate, by means of rapid rotation, substances of different density suspended in a liquid. Particles of greater density than the liquid move away from the axis of rotation and those of smaller density towards the axis. Thus, when a vessel containing milk is rotated, the cream collects at the centre and can be removed easily. The effect is very similar to the floating or sinking of bodies of different densities in a stationary liquid. The centrifugal force $m\omega^2 r$, on the liquid increases with r , and hence the pressure in the liquid increases from the axis outwards. If a suspended body has a density less than that of the liquid its outward centrifugal force is less than the thrust inwards of the liquid upon it. This inward thrust is similar to the Archimedes upthrust and is due to the difference in pressure on the two sides of the immersed body. Similarly, the centrifugal force on a body of greater density than the liquid is greater than the inward thrust of the liquid.

To investigate how the pressure in a liquid varies with the distance from the axis of rotation, suppose a liquid of density ρ is contained in

a closed, vertical, cylindrical vessel, which it just fills, and the cylinder is rotating with angular velocity ω about its axis. Consider a ring of liquid of radius x , thickness δx and vertical height δh , with its centre O on the axis of rotation (Fig. 91). Let the pressures in the liquid at distances x and $x + \delta x$ from the axis of rotation be p and $p + \delta p$.

Resultant inward thrust on ring $= \delta p \cdot 2\pi x \delta h$.

Centrifugal force on ring $= m\omega^2 r$

$$= 2\pi x \delta x \delta h \rho \omega^2 x.$$

$$\therefore \delta p \cdot 2\pi x \delta h = 2\pi x \delta x \delta h \rho \omega^2 x.$$

$$\therefore \delta p = \rho \omega^2 x \delta x$$

Integrating, $\int dp = \int \rho \omega^2 x dx,$

$$p = \frac{1}{2} \rho \omega^2 x^2 + C.$$

C , the constant of integration, is the value of p when $x=0$, i.e. the pressure at O , which is zero at the surface of the liquid and increases with depth as in a liquid at rest.

EXAMPLE

Show that the free surface of a liquid rotating with a uniform angular velocity ω is a paraboloid of revolution.

Consider a particle of liquid in the surface, mass m , whose co-ordinates with respect to the axes OX and OY are (x, y) (Fig. 92). The resultant R of the weight of

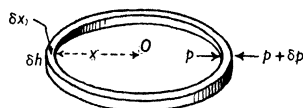


Fig. 91

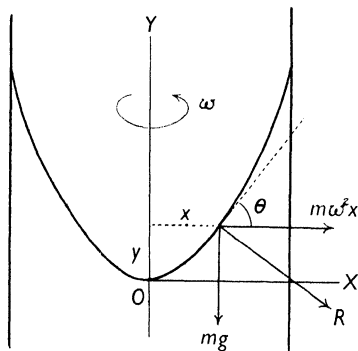


Fig. 92

the particle mg , and the centrifugal force, $m\omega^2 x$, must be at right angles to the surface (see p. 26). R is, of course, counterbalanced by the thrust of the rest of the liquid on m .

$$\therefore \tan \theta = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g}.$$

But $\tan \theta = dy/dx$, the slope at (x, y) .

$$\therefore \frac{dy}{dx} = \frac{\omega^2 x}{g}.$$

Integrating,

$$y = \frac{\omega^2 x^2}{2g} + C.$$

When $x=0$, $y=0$; hence $C=0$.

$$\therefore y = \frac{\omega^2 x^2}{2g}.$$

This is the equation of a parabola.

Tension in a revolving string

If a belt or chain, which is rotating at high speed over a pulley, is pushed off the pulley it will run about like a rigid hoop and jump over obstacles. A cowboy's rotating lariat behaves similarly. The centrifugal force causes the tension in the rope to be extremely high.

Suppose a rope of mass m per unit length is rotating in the form of a circle of radius r , with angular velocity ω , and that its tension is T . Consider an arc AB subtending an angle 2θ at the centre of the circle O (Fig. 93). The rest of the rope exerts tangential forces T , T on AB at A and B . The resultant of these two forces is $2T \sin \theta$ in the direction PO . The mass of AB is $m \cdot AB$ and the centrifugal force on AB is $m \cdot AB \cdot \omega^2 r$ along OP .

$$\begin{aligned} \therefore 2T \sin \theta &= m \cdot AB \cdot \omega^2 r \\ &= m \cdot 2r\theta \cdot \omega^2 r. \quad (AB = r \cdot 2\theta) \end{aligned}$$

If AB is very small, $\sin \theta = \theta$,

$$\therefore T = mr^2 \omega^2.$$

Thus a rotating body is in a state of elastic stress. Racing cars sometimes cast their tyres, which stretch due to the tension set up in them. The speed beyond which flywheels can safely be rotated is limited by their tensile strength; they tend to burst at too high speeds.

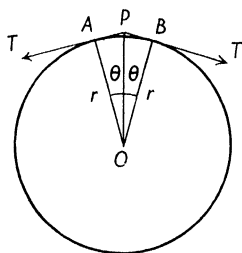


Fig. 93

Kinetic energy of rotation and moment of inertia

A rotating flywheel may be regarded as a large number of particles revolving in circles of varying radii. Its kinetic energy is equal to the sum of the kinetic energies of all the particles of which it is composed.

Suppose the flywheel has an angular velocity ω . Then a particle of mass m at distance r from the centre (Fig. 94) will have a velocity of $r\omega$.

$$\therefore \text{Kinetic energy of particle} = \frac{1}{2}m(r\omega)^2.$$

$$\begin{aligned} \therefore \text{Kinetic energy of whole flywheel} &= \Sigma \frac{1}{2}mr^2\omega^2 \\ &= \frac{1}{2}\omega^2 \Sigma mr^2 \text{ (since } \omega \text{ is the same for all particles).} \end{aligned}$$

The quantity Σmr^2 is called the *moment of inertia* of the flywheel and is denoted by I . Thus the kinetic energy of the flywheel is $\frac{1}{2}I\omega^2$. I represents rotational inertia and corresponds to mass in linear motion. The effect of mass in rotation clearly depends on its distribution; thus a flywheel with light spokes having nearly all its mass concentrated in its rim has a much greater moment of inertia than a flywheel of equal mass in the form of a uniform disc. The former would be more difficult to start and to stop because it would acquire a greater kinetic energy for a given angular velocity.

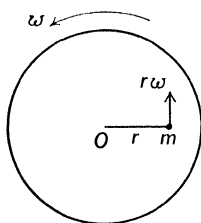


Fig. 94

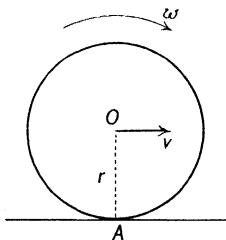


Fig. 95

Because of their ability to store kinetic energy flywheels are incorporated in engines to keep them running steadily. A steam roller needs a large flywheel because it runs at a slow speed; a motor-car engine is fitted with a smaller flywheel because it runs at a higher speed.

It is sometimes convenient to write $I = Mk^2$, where M is the total mass of a rotating body and k is called the *radius of gyration*; k is clearly the distance from the axis of rotation at which the whole mass of the body would need to be placed in order to have the same moment of inertia as the body.

A body rolling along the ground

A body rolling along the ground has kinetic energy of both translation and rotation.

Consider a wheel, of mass m , of radius r , and of moment of inertia I , rolling with angular velocity ω (Fig. 95) and moving with velocity v .

$$\text{Kinetic energy of translation} = \frac{1}{2}mv^2.$$

$$\text{Kinetic energy of rotation} = \frac{1}{2}I\omega^2.$$

$$\therefore \text{Total kinetic energy of wheel} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

It can be shown as follows that $v = r\omega$. When the wheel rolls so as to turn through 1 revolution, its centre O moves forward a distance $2\pi r$; suppose this occurs in a time t .

$$\text{Velocity of centre, } v = \frac{2\pi r}{t} = r \frac{2\pi}{t} = r\omega.$$

It is of interest to note that the velocity of any point on the wheel may be found by compounding the translational velocity v of the wheel as a whole with the velocity due to the angular rotation, using the parallelogram of velocities. The point A has a velocity $v - r\omega = 0$; it is momentarily at rest and is called the instantaneous centre of rotation. The lower half of the wheel is rotating in the opposite direction to the motion of the wheel as a whole, while the upper half is rotating in the same direction. In photographs of swiftly moving vehicles sometimes the lower parts of the wheels are distinct and the upper parts blurred; the upper parts move appreciably while the exposure is being made, whereas the lower parts do not.

Experimental determination of the moment of inertia of a flywheel

(1) *Fixed flywheel.* The moment of inertia I of a flywheel fixed in rigid supports may be determined by allowing a body of mass m , attached to one end of a string wrapped round the axle of the flywheel (Fig. 96), to fall a known distance h . Suppose the body acquires a velocity v and the flywheel an angular velocity ω .

Loss of potential energy of falling body = Kinetic energy of translation of body + Kinetic energy of rotation of flywheel, i.e.

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

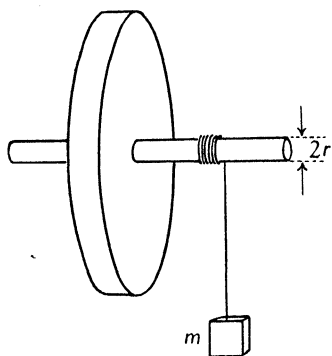


Fig. 96

If r is the radius of the axle of the flywheel, $v = r\omega$. There are two ways of finding v or ω . The time t taken by the body to fall from rest a distance h may be measured. Then the average velocity of fall = h/t ; the final velocity, $v = 2h/t$. A second and more accurate method is to allow the string to become detached from the axle after the body has

fallen the distance h and to count the number of revolutions n made by the flywheel in coming to rest, noting the time t' occupied in the process. Then the average angular velocity of the flywheel is $\frac{2\pi n}{t'}$; the angular velocity when the string became detached, ω is $\frac{4\pi n}{t'}$.

Knowing m , h , v and ω the value of I may be calculated.

(2) *Flywheel mounted on an axle which rolls down inclined rails.* Suppose a flywheel of mass M and moment of inertia I acquires a linear velocity v and angular velocity ω in rolling down inclined rails, thereby descending a vertical distance h (Fig. 97).

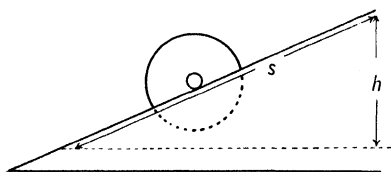


Fig. 97

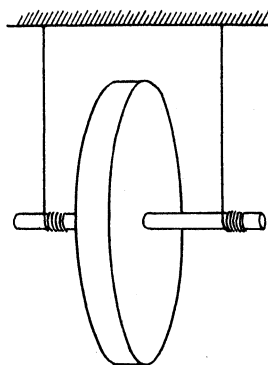


Fig. 98

Loss of potential energy of flywheel = Kinetic energy of translation of flywheel + Kinetic energy of rotation of flywheel,

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2.$$

If r is the radius of the axle, $v = r\omega$. The value of v may be found by measuring the time t required by the flywheel to roll a distance s down the incline from rest; $v = 2s/t$. Knowing m , h , v and r , I may be calculated.

(3) *Flywheel mounted on an axle suspended by strings.* Another method of finding the moment of inertia of a flywheel is to support it by two vertical strings wrapped round the axle and allow it to fall vertically as in Fig. 98. Using the same nomenclature as in method (2) above

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2.$$

If t is the time taken by the flywheel to fall from rest a vertical distance h , $v = 2h/t$. Also $\omega = v/r$. Hence I may be calculated.

The theorem of parallel axes

The moment of inertia of a body about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity plus Mh^2 , where M is the mass of the body and h is the distance of the axis from the centre of gravity.

Consider the lamina in Fig. 99; let AB represent the axis of rotation and CGD the parallel axis through the centre of gravity G . Suppose a particle of the lamina, mass m , is distant x from CD and hence distant $x+h$ from AB . Let I represent the moment of inertia of the body about AB and I_G the moment of inertia about the axis through G :

$$\begin{aligned} I &= \sum m(x+h)^2 \\ &= \sum mx^2 + \sum 2mxh + \sum mh^2 \\ &= I_G + 2h \sum mx + h^2 \sum m. \end{aligned}$$

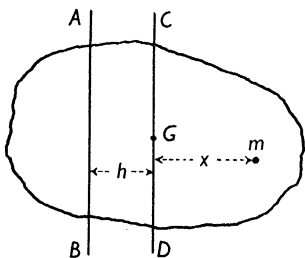


Fig. 99

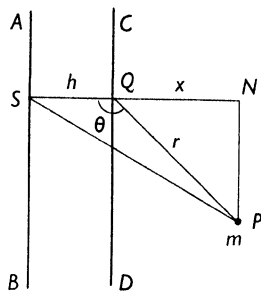


Fig. 100

But $\sum mx = 0$ because it represents the total moment about an axis through the centre of gravity. Also $\sum m = M$. Thus

$$I = I_G + Mh^2.$$

The theorem can be shown to hold not only for a lamina but for any body as follows.

Consider a particle, mass m , which is at a point P outside the plane containing the axes AB and CD (Fig. 100). PQ and PS are the perpendiculars to CD and AB respectively. PN is perpendicular to SQ produced.

$$I = \sum m \cdot PS^2.$$

But $PS^2 = r^2 + h^2 - 2rh \cos \theta$ (using the letters in Fig. 100).

Also $x = r \cos (180 - \theta) = -r \cos \theta$.

$$\begin{aligned} \therefore I &= \sum m(r^2 + h^2 + 2hx) \\ &= I_G + Mh^2, \text{ since } \sum mx = 0 \text{ as before.} \end{aligned}$$

The theorem of perpendicular axes

The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia about any two perpendicular axes in the plane that intersect on the first axis.

Fig. 101 represents a lamina in the plane of the paper. OX and OY represent two axes mutually at right angles and in the plane of the lamina. OZ is an axis perpendicular to the lamina. Consider a particle of the lamina, mass m , distant x , y and r from the three axes OY , OX and OZ respectively:

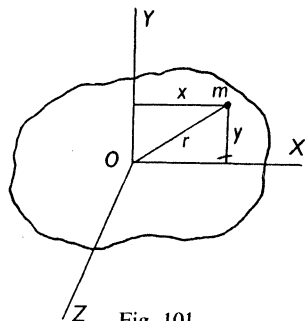


Fig. 101

Moment of inertia of lamina
about $OX = I_x = \sum my^2$.

Moment of inertia of lamina
about $OY = I_y = \sum mx^2$.

Moment of inertia of lamina
about $OZ = I_z = \sum mr^2$.

But
$$r^2 = y^2 + x^2,$$

$$\therefore I_z = I_x + I_y.$$

Calculation of the moment of inertia from the mass and dimensions of a body

The moment of inertia of a regular body can be calculated by integration. Consider two simple examples.

(1) *Rectangular lamina.* Fig. 102 (a) represents a rectangular lamina, length l , breadth b , mass ρ per unit area. To find its moment of inertia about an axis YY' through its centre of gravity G in the plane of the lamina and perpendicular to its length, consider a narrow strip of the lamina parallel to and distant x from YY' and of width δx :

Moment of inertia of strip about $YY' = b\delta x\rho \cdot x^2$

$$\begin{aligned} \text{Moment of inertia of whole lamina about } YY' &= \int_{-l/2}^{+l/2} b\rho x^2 dx \\ &= \frac{1}{2} \left[\frac{1}{3} b\rho x^3 \right]_{-l/2}^{+l/2} = \frac{b\rho l^3}{12} \\ &= M \frac{l^2}{12}, \end{aligned}$$

where M is the mass of the lamina.

Similarly, the moment of inertia of the lamina about $XX' = M \frac{b^2}{12}$.

Hence by the theorem of perpendicular axes (Fig. 102 (b)):

$$\text{Moment of inertia of the lamina about } ZZ' = M \frac{l^2 + b^2}{12}.$$

By the theorem of parallel axes (Fig. 102 (c)),

$$\begin{aligned} \text{Moment of inertia of the lamina about axis } AB &= M \frac{l^2}{12} + M \left(\frac{l}{2} \right)^2 \\ &= M \frac{l^2}{3}. \end{aligned}$$

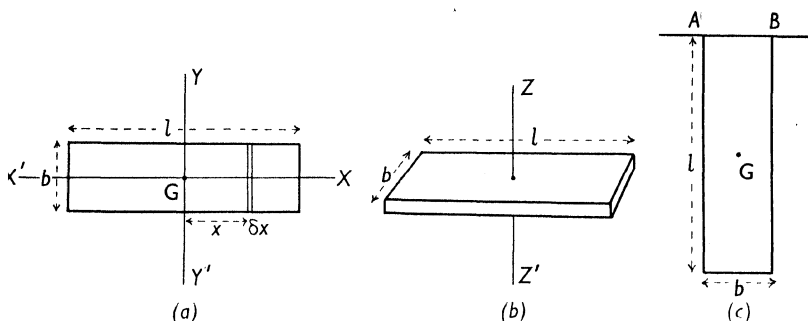


Fig. 102

(2) *Disc.* The moment of inertia of a disc about an axis through its centre and perpendicular to its plane may be found by considering a ring, of radius x and width δx (Fig. 103). Suppose the radius of the disc is a , its mass per unit area is ρ and its total mass is M .

Moment of inertia of ring about axis
through $O = 2\pi x \delta x \rho x^2$

Moment of inertia of disc about axis

$$\begin{aligned} \text{through } O &= \int_0^a 2\pi \rho x^3 dx \\ &= \frac{\pi \rho a^4}{2} = M \frac{a^2}{2}. \end{aligned}$$

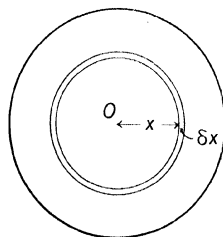


Fig. 103

By the theorem of perpendicular axes, the moment of inertia of the disc about a diameter is $M \frac{a^2}{4}$ (the moment of inertia about two perpendicular diameters is $M \frac{a^2}{4}$ in each case).

Comparison of linear and angular motion

The tendency of a force to produce angular acceleration is called the moment of the force or *torque*. The equations for angular motion are analogous to those for linear motion with torque G substituted for force F , angular displacement θ for linear displacement s , angular

velocity ω or $\dot{\theta}$ for linear velocity v , angular acceleration $\ddot{\theta}$ for linear acceleration a , and moment of inertia I for mass m .

Angular motion

$$G = I\ddot{\theta}$$

$$Gt = I\omega$$

$$G\theta = \frac{1}{2}I\omega^2$$

Linear motion

$$F = ma$$

$$Ft = mv$$

$$Fs = \frac{1}{2}mv^2$$

The quantity $I\omega$ is called the *angular momentum* or moment of momentum.

Newton's laws applied to angular motion could be stated as follows:

(1) A rotating body will continue to turn about a fixed axis with undiminished angular speed unless acted upon by an external torque.

An example of this law is the constant rotation of the earth on its axis. The attraction of the sun acts on the earth's centre and hence does not affect its rotation.

(2) The rate of change of angular momentum of a body is proportional to the applied torque.

(3) When one body exerts a torque upon another body the latter exerts an equal and opposite torque on the first body about the same axis of rotation.

Conservation of angular momentum

The principle of the conservation of angular momentum is analogous to that for linear momentum. *The angular momentum of any system on which no external torque acts is constant.*

If a small body, such as a 'conker' on the end of a string, is whirled in a circle and the string is allowed to wrap itself round the arm of the whirler, the angular velocity will increase as the string shortens. The moment of inertia, I , of the body decreases because its distance from the axis of rotation decreases, and hence the angular velocity ω must increase so that $I\omega$ may remain constant.

Another demonstration of the principle can be made by a person holding a heavy weight in each hand and sitting in a chair which rotates freely about a vertical axis. When the arms are extended sideways the moment of inertia I is increased because the weights are further from the axis of rotation. The speed of rotation ω therefore decreases so that $I\omega$ may remain constant; the speed of rotation increases again if the weights are drawn inwards to the body.

A ballet dancer pirouetting on her toes, or a skater whirling on the ice, can slow down her rate of rotation by extending both arms and a leg; by drawing them in again she increases her speed of rotation. An

acrobat turning a somersault instinctively curls himself up in the air, so decreasing his moment of inertia and increasing his speed of rotation; he straightens himself out again to slow down just before his feet reach the ground.

The earth moves in an ellipse with the sun at one focus. At one end of the ellipse it is nearer to the sun and hence travels more quickly than at the other end.

Laplace's theory of the origin of the solar system was accepted for a century, but the application of the principle of the conservation of angular momentum shows it to be untenable. Laplace assumed that the solar system originated from a huge, diffuse, slowly rotating nebula which gradually cooled and condensed. As it contracted it rotated faster and faster until the centrifugal force became equal to the force of gravity. Rings of matter then split off and condensed to form the planets. The fact that all the planets rotate in the same direction, in almost circular orbits in the same plane, supports the theory. If, however, all the mass and angular momentum of the planets could be collected into the sun its speed of rotation would be such that the centrifugal force at its equator would be only 5% of gravity. Hence the planets could not have broken away from the sun as Laplace imagined.

The motion of a rigid body

If a force acts through the centre of mass of a rigid body it produces translational motion only; if it acts elsewhere it produces both translational and rotational motion. This can be demonstrated by flicking with the finger a pencil resting on a smooth table; unless it is flicked at its centre of gravity the pencil rotates as it moves away.

The translational motion of the centre of mass G is the same as if all the mass were concentrated at G and all the forces were moved parallel to themselves to act at G . The rotational motion about G may be obtained by equating the rate of change of angular momentum about G to the sum of the moments about G of all the forces acting on the body.

SUMMARY

A particle of mass m , revolving in a circle of radius r with uniform angular velocity ω , must be subject to a centripetal force $m\omega^2r$ and must exert a centrifugal force $m\omega^2r$. In problems such as those on the banking of tracks the centrifugal force may be considered as acting at the centre of gravity of the moving body, and the problem may then be regarded as one of statical equilibrium.

When considering the rotation of a rigid body the distribution of its mass must be taken into account, and this is done by means of a new concept, moment of inertia, $I = \sum mr^2$. Moment of inertia in rotary motion corresponds to mass in linear motion; for example, kinetic energy of rotation is $\frac{1}{2}I\omega^2$, as compared with kinetic energy of linear motion $\frac{1}{2}mv^2$. A typical problem dealt with in this chapter is the motion of a flywheel rolling down an inclined plane. The loss of potential energy is equal to the sum of the kinetic energies of translation and rotation; if the flywheel has descended a vertical distance h from rest,

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2.$$

QUESTIONS

[Take $g = 32$ ft. sec.⁻² or 981 cm. sec.⁻²]

1. Deduce an expression for the force required to cause a particle to move in a circle with uniform speed.

A mass of 2 lb. on a smooth horizontal table is describing a circle of radius 3 ft. uniformly at 30 r.p.m. It is attached to the centre of the circle by a string. What is the tension in the string? (O.)

2. A road over a bridge has the form of a vertical circular arc of radius 60 ft. What is the greatest speed in m.p.h. at which a car can cross the bridge without leaving the ground at the crest of the road? (D.)

3. A shilling is placed on a gramophone record rotating at the rate of 78 r.p.m., and it is found that the shilling remains on the record if its centre is within 6 cm. of the centre of the record, whereas outside this distance it slides off. Explain this, and deduce the coefficient of friction between the shilling and the record. (O. S.)

4. In a 'loop-the-loop' railway, the cars, after descending a steep incline, run round the inside of a vertical circular track, 20 ft. in diameter, making a complete turn over. Assuming there is no friction, find the minimum height above the top of the circular track from which the cars must start. (N.)

5. A flywheel weighs 10 tons and its centre of gravity is $\frac{1}{8}$ in. from the centre of the shaft. Find the pull on the shaft when the wheel is making 240 r.p.m. If the shaft is 6 in. in diameter and the coefficient of friction between the bearing surfaces is 0.01, how many horse-power will be wasted due to the wheel being out of balance? (D.)

6. Fig. 104 shows a portion of the governor of a steam engine. When the system is rotating once in a second the dimensions are as shown. If the masses A and B are each 4 lb., find (i) the centripetal force on each, and (ii) the vertical force on the sliding collar C . (N.)

7. A particle is placed at the highest point of a smooth sphere of radius r . It is then given an infinitesimal displacement. At what point will it leave the sphere? (C.)

8. Show that, if the earth is assumed to be a sphere of uniform density, the value of the gravitational acceleration will vary from the equator to the poles owing to the rotation.

Assuming the radius of the earth to be 6.4×10^8 cm. and g at the pole to be 983 cm. per sec. per sec., find approximately the magnitude and direction of g in latitude 60° . (C.)

9. Calculate the angle at which a curve of radius 352 ft. should be banked so as to avoid side-slip when a motor-car is travelling round it at a speed of 30 m.p.h. (N.)

10. A skater is moving on one foot in a circle of radius 20 ft. at 10 m.p.h. At what angle with the vertical will the line passing through his centre of gravity and the edge of his skate be inclined? (D.)

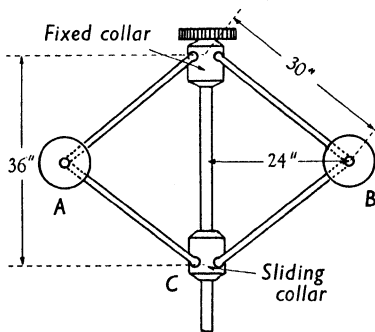


Fig. 104

11. Why does a child's hoop begin to follow a curved path when it becomes inclined to the vertical?

12. Explain why a lorry going round a corner at too great a speed is apt to overturn.

A symmetrically loaded lorry weighs 5 tons, and the height of its centre of gravity is 5 ft. above the ground in a vertical plane midway between the wheels. The breadth of the wheel base may be taken to be 6 ft. 3 in. If there is no side-slip, what is the maximum speed at which the lorry can take a curve of mean radius 6 yards without beginning to overturn?

Determine the minimum tangential force and the minimum coefficient of friction to prevent side-slip. (O. & C.)

13. Explain why a motor-cycle combination (side-car on left) is liable to overturn when taking a left-hand corner at speed.

Assuming that the centre of mass of the combination is 2 ft. from the ground and 1 ft. to the left of the motor-cycle, calculate the maximum speed of the combination in a circle of radius 50 ft. Assume that the road surface is horizontal and that there is no skidding. (O. S.)

14. Discuss the possibility of a motor-cyclist riding round the inside surface of a vertical cylinder. (C. S.)

15. Two particles resting on a rough horizontal disc are connected by a piece of string which is drawn tight and lies along a radius. The disc is now set spinning and is speeded up until the particles begin to slip. Show that, if the coefficient of friction between each particle and the disc is the same, the speed at which they slip is the same as that at which a similar particle would slip if placed at their centre of gravity. (C. S.)

16. One end of a light flexible string of length 100 cm. is attached to a fixed point and supports at the other end a small bob of mass 100 gm. The whole forms a conical pendulum, the bob describing a circle in a horizontal plane. Show that

$$n = \frac{30}{\pi} \sqrt{\left(\frac{g}{100 \cos \theta} \right)},$$

where n is the number of revolutions per minute and θ is the angle the string makes with the vertical.

If the string breaks under a tension of 300 gm. wt., find the maximum angle the string can make with the vertical. (O. & C.)

17. A closed cylindrical can of radius a and height h is just filled with a fluid of density ρ . It is then rotated with angular velocity ω about its own axis, held vertical.

Prove that the total thrust on the top of the can will be $\frac{\pi \rho \omega^2 a^4}{4}$ and find the total thrust on the bottom. (O. S.)

18. An india-rubber band has a mass of 4 gm. per metre when stretched on the circumference of a wheel of 10 cm. radius, the stretching force being 20,000 dynes. Find how many revolutions per second the wheel must make so that the band may not press upon the wheel. (N.)

19. Deduce the laws of conservation of momentum and angular momentum from Newton's laws of motion.

A particle of unit mass is projected so as to move on the smooth inner surface of an inverted cone of semi-vertical angle α , whose axis is vertical. When it is at a distance s_0 from the vertex its angular velocity about the axis is β and its kinetic energy is γ . Show that its distance from the vertex will always lie between the values of s defined by the equation

$$2\{\gamma + g \cos \alpha (s_0 - s)\} = \frac{\beta^2 s_0^4}{s^2} \sin^2 \alpha.$$

(O. S.)

MOMENT OF INERTIA

20. Explain what is meant by *moment of inertia* and *angular momentum*.

A thin hollow cylinder, open at both ends and weighing 96 lb., (a) slides with a speed of 10 ft. per sec., without rotating, (b) rolls with the same speed, without slipping. Compare the kinetic energies of the cylinder in the two cases. (N.)

21. Find the acceleration with which a hoop of radius 2 ft. rolls down a plane inclined at 30° to the horizontal. (O.)

22. If the diameter of a penny is 3 cm. and its mass is 9 gm., calculate its total energy when it rolls along a table with a velocity of 6 cm. per sec. (N.)

23. A peg-top of moment of inertia 100 gm. cm.² is started by winding 30 cm. of string round the peg of the top and drawing it away with a constant force of 100 gm. weight. What is the angular velocity of the top? (C.)

24. Two gear wheels, of equal thickness, of the same material and having radii in the ratio 2:1, are mounted on parallel frictionless spindles, but are separated so as not to mesh with one another. The larger wheel is set spinning at a speed of 10 rev. per sec., and the wheels are then brought into mesh. What is the resulting speed of each wheel? (C. S.)

25. A cord is wound round the horizontal axle (whose radius is 1.2 cm.) of a flywheel, and a weight of 2 kg. is attached to the free end of the cord. Starting from rest, the weight is released from the axle after falling through 100 cm. After the weight is released the flywheel is observed to make 14 turns in 5.8 sec. Neglecting friction, calculate (a) the kinetic energy of the weight at the moment of release, (b) the moment of inertia of the flywheel. (C.)

26. A flywheel, which can turn about a horizontal axis, is set in motion by a 500 gm. weight hanging from a thin string that passes round the axle. After the wheel has made 5 revolutions, the string is detached from the axle and the weight drops off. The wheel then makes 7 revolutions before being brought to rest by friction. The radius of the axle is 2.0 cm., and at the instant when the weight drops off the angular velocity of the wheel is 10 radians per sec. Assuming that the work done against friction in each revolution is always the same, calculate the moment of inertia of the flywheel about its axis of rotation. (O. & C.)

27. Explain the meaning and usefulness of the quantity 'moment of inertia', and deduce expressions for (a) the angular acceleration of a body of moment of inertia I about a given axis in terms of a couple G applied to the body about that axis, (b) the kinetic energy of rotation when the body is rotating with angular velocity ω about that axis.

A steel cylinder of radius 3 cm. and mass 2000 gm. is supported horizontally in frictionless bearings on a light shaft passing through its axis of symmetry. A long light cord with one end fixed to the cylinder is wrapped round it a large number of times, and a weight of 100 gm. fastened to its free end. Find (a) the linear acceleration of the weight, (b) the tension in the cord, (c) the angular acceleration of the cylinder. (The moment of inertia of a cylinder of mass M and radius a about its axis is $\frac{1}{2}Ma^2$.) (O. & C.)

28. A pair of parallel rails is supported in a horizontal position and the axle of a wheel rests on the rails. A thread is wrapped round the axle and a weight hung on the end of the thread. As the weight falls the wheel moves along the rails. How would you determine the moment of inertia of the wheel with this arrangement? (D.)

29. Masses of 95 gm. and 105 gm., hanging freely, are connected by a light string which passes over a pulley of mass 20 gm. When released the system moves with an acceleration of 46.7 cm. per sec.² Calculate a value of g if the mass of the pulley is (a) neglected, (b) taken into account. Regard the pulley as a simple disc of moment of inertia $\frac{1}{2}Mr^2$, and assume that no kinetic energy is lost in friction. (N.)

30. Define the terms 'moment of force' and 'moment of inertia'.

A thin uniform rod of mass m and length l , pivoted freely at its base, is allowed to fall from a vertical position. Calculate its angular velocity and acceleration when it has turned through an angle θ . Prove any formula you use.

(It may be assumed that the moment of inertia of the rod about an axis through its end and perpendicular to its length is $\frac{1}{3}ml^2$.) (O. & C.)

31. Calculate the moment of inertia of a solid body about any axis, given the moment of inertia about a parallel axis through the centre of gravity.

Hence show how the moment of inertia, about any axis, of a solid body made up of regular shapes may be calculated. Illustrate your answer by calculating the moment of inertia of a square plate of side $2a$, that has a circular hole of diameter a cut out of its centre, about an axis perpendicular to the plate and through one corner.

How would you check your calculation experimentally? (C.)

CHAPTER 5

SIMPLE HARMONIC MOTION

THE motion of a body which continually retraces its path at equal intervals of time is called periodic motion. There is a special type of periodic motion, common in nature, in which *the acceleration of the body is always directed towards, and is proportional to its distance from, a fixed point*; this is known as *simple harmonic motion* (S.H.M.). Since force is equal to the product of mass and acceleration, the *force on a body executing S.H.M. must also be directed towards, and be proportional to the distance from, the fixed point*.

All motion due to elastic displacement is simple harmonic because the restoring force is proportional to the displacement, and the fact that the vibrations of musical instruments are fundamentally of this type is responsible for the name harmonic. Further examples are the vibrations of a mass attached to the end of a spiral spring and of a pendulum executing small oscillations.

Investigation of S.H.M.

Suppose a body of mass m , attached to the end of a spiral spring (Fig. 105) is pulled down a distance a below its equilibrium position and then released. It will start to oscillate up and down.

Consider the body at time t after its release, when its centre of gravity is at N , distant x below the equilibrium position O . ON is called the *displacement* of the body. If μ is the force required to stretch the spring unit distance, the restoring force on the body at N is μx (the force being proportional to the distance).

Mass $\times$ acceleration = Force,

$$\text{i.e.} \quad m \cdot \frac{d^2x}{dt^2} = -\mu x. \quad (1)$$

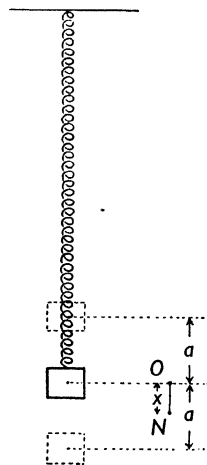


Fig. 105

(The negative sign indicates that, whereas x is measured in the direction ON , the restoring force is in the opposite direction NO .)

Let

$$\frac{\mu}{m} = \omega^2.$$

Then

$$\frac{d^2x}{dt^2} = -\omega^2 x. \quad (2)$$

This substitution simplifies the expressions which follow; ω is a constant since μ and m are constants. The solution to the differential equation (2) is*

$$x = a \cos \omega t, \quad (3)$$

where a is a constant, as can be verified by differentiating equation (3) twice:

$$\frac{dx}{dt} = -\omega a \sin \omega t, \quad (4)$$

$$\frac{d^2x}{dt^2} = -\omega^2 a \cos \omega t$$

$$= -\omega^2 x \quad (\text{which is equation (2)}).$$

In equation (3) a is the maximum value of x , since the maximum value of a cosine is 1; a is called the *amplitude* of the motion (Fig. 105). Equation (3) enables the displacement x of the body at any time t to be calculated.

Equation (4) enables the velocity dx/dt of the body at any time t to be calculated.

Periodic time

In moving from its lowest point to its highest point and back the body is said to execute one *vibration* or one *cycle*. The time required to execute one vibration is called the *periodic time* T . The number of vibrations per second is called the *frequency* f .

The periodic time is four times the time required for the body to move from its lowest point to its undisturbed position. Zero time was taken when the body was at its lowest point; i.e. when $x=a$, $t=0$. Suppose when $x=0$ for the first time, $t=t'$. Substituting in

$$x = a \cos \omega t, \quad (\text{equation (3)})$$

we have

$$0 = a \cos \omega t'.$$

$$\therefore \omega t' = \frac{\pi}{2}, \quad \left(\text{since } \cos \frac{\pi}{2} = 0 \right)$$

$$t' = \frac{\pi}{2\omega}.$$

$$\therefore T = 4t' = \frac{2\pi}{\omega},$$

* See also p. 109.

$$\text{i.e.} \quad T = 2\pi \sqrt{\frac{m}{\mu}}, \quad \left(\omega^2 = \frac{\mu}{m} \right)$$

$$\text{i.e.} \quad \text{Periodic time} = 2\pi \sqrt{\frac{\text{Mass}}{\text{Force per unit displacement}}}$$

This useful equation, which applies to all linear S.H.M., should be memorised. If m is in grammes, μ must be in dynes per cm.; if m is in lb., μ must be in poundals per ft.

T is independent of the amplitude a and the motion is said to be *isochronous*. This important property makes the pendulum well suited to control clocks because its rate of swing does not change if its amplitude fluctuates slightly.

In considering the vibrations in Fig. 105 the mass of the spring has been ignored. The spring has one end at rest and its effective mass is one-third of its total mass. The effective mass of the spring should be added to the mass of the body m .

$$\text{Thus } T = 2\pi \sqrt{\frac{m + \frac{1}{3}s}{\mu}}, \text{ where } s \text{ is the mass of the spring.}$$

EXAMPLE 1

A weighted test-tube, of total mass 20 gm. and area of cross-section 2 sq. cm., floats vertically in water. The tube is depressed a small amount and then released. Show that its motion is S.H.M. and find the time of oscillation.

If the test-tube is pushed down a distance x cm., by Archimedes' principle,

$$\text{Restoring force} = 2x \text{ gm. wt.}$$

Since the restoring force is proportional to the displacement, x , the motion is S.H.M.:

$$\begin{aligned} \text{Restoring force per unit displacement} &= 2 \text{ gm. wt. per cm.} \\ &= 2 \times 981 \text{ dynes per cm.} \end{aligned}$$

$$\begin{aligned} T &= 2\pi \sqrt{\frac{m}{\mu}} \\ &= 2\pi \sqrt{\frac{20}{2 \times 981}} \\ &= 0.63 \text{ sec.} \end{aligned}$$

EXAMPLE 2

A body moving with S.H.M. has a velocity of 4 ft. per sec. at a distance of 3 ft. from the mean position and 3 ft. per sec. at a distance of 4 ft. from the mean position. What is its velocity at the mean position? What are the amplitude and periodic time of the motion?

From equations (3) and (4) on p. 106,

$$\begin{aligned} \frac{dx}{dt} &= -\omega a \sin \omega t = -\omega a \sqrt{1 - \cos^2 \omega t} \\ &= -\omega a \sqrt{1 - \frac{x^2}{a^2}} = -\omega \sqrt{a^2 - x^2}. \end{aligned}$$

Substitute in the equation $\frac{dx}{dt} = -\omega \sqrt{(a^2 - x^2)}$;

$$\text{When } x=3, \frac{dx}{dt} = -4. \quad \therefore 4 = \omega \sqrt{(a^2 - 9)}.$$

$$\text{When } x=4, \frac{dx}{dt} = -3. \quad \therefore 3 = \omega \sqrt{(a^2 - 16)}.$$

Solving these two equations, $a=5$ and $\omega=1$.

$$\text{At mean position } x=0, \quad \therefore \frac{dx}{dt} = \pm \omega a = \pm 5.$$

Hence velocity in mean position = 5 ft. per sec.

$$\text{Amplitude} = a = 5 \text{ ft.}$$

$$\text{Periodic time} = \frac{2\pi}{\omega} = 2\pi \text{ sec.}$$

Energy in S.H.M.

As the body in Fig. 105 vibrates up and down, the form of its energy continually changes. At its highest and at its lowest points its kinetic energy is zero but its potential energy (due to the spring) is a maximum; at O its kinetic energy is a maximum and its potential energy is zero.

The potential energy at the lowest (or highest) point is equal to the work done against the restoring force in moving the body to this point from O .

$$\begin{aligned} \therefore \text{Potential energy at lowest point} &= \int_0^a \mu x dx \\ &= \frac{1}{2} \mu a^2. \end{aligned}$$

At any position of the body the sum of its kinetic and potential energies must be $\frac{1}{2} \mu a^2$.

A check on this can be made by finding the kinetic energy of the body when it is at O .

$$\text{Velocity of body} = -\omega a \sin \omega t \quad (\text{equation (4)}).$$

But when body is at O , time $t' = \frac{\pi}{2\omega}$ (see above).

$$\begin{aligned} \therefore \text{Velocity at } O &= -\omega a \sin \left(\omega \frac{\pi}{2\omega} \right) \\ &= -\omega a. \end{aligned} \quad \left(\sin \frac{\pi}{2} = 1 \right)$$

$$\begin{aligned} \therefore \text{Kinetic energy at } O &= \frac{1}{2} m v^2 = \frac{1}{2} m \omega^2 a^2 \\ &= \frac{1}{2} \mu a^2. \end{aligned} \quad \left(\omega^2 = \frac{\mu}{m} \right)$$

Phase

The general solution of the equation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

is

$$x = a \cos(\omega t + \epsilon),$$

where a and ϵ are constants. The solution $x = a \cos \omega t$ holds only if $t=0$ when $x=a$. It was arranged that this should be so by taking the time from the instant when the body in Fig. 105 was released from its lowest point.

If, however, zero time is taken when the body is at O , $t=0$ when $x=0$. Substituting in the above equation,

$$0 = a \cos \epsilon,$$

$$\therefore \epsilon = \frac{\pi}{2}. \quad \left(\cos \frac{\pi}{2} = 0 \right)$$

Hence

$$x = a \cos \left(\omega t + \frac{\pi}{2} \right),$$

or

$$x = -a \sin \omega t.$$

The value of ϵ depends on the position of the body in its path at the instant which is taken as zero time. The two motions represented by

$$x_1 = a \cos \omega t \quad \text{and} \quad x_2 = a \cos(\omega t + \epsilon)$$

are said to differ in *phase* by ϵ . The motions have the same frequency $\omega/2\pi$ and the same amplitude a but they are out of step.

Geometrical representation of S.H.M.

It can be shown that the projection of uniform circular motion on a diameter is S.H.M.

Consider a point P (Fig. 106) moving with uniform angular velocity ω in a circle of radius a , and let N be the projection of P on a diameter XOX' . As P revolves in the circle, N moves backwards and forwards along the diameter. Suppose, at a time t , $\hat{POX} = \theta$ and $ON = x$; then

$$x = a \cos \theta.$$

The velocity of P is $a\omega$ along the tangent at P and the velocity of N , dx/dt , is the resolved part along OX of the velocity of P .

$$\therefore \frac{dx}{dt} = -a\omega \sin \theta.$$

(The negative sign indicates that the direction of dx/dt is opposite to that of x increasing.)

The acceleration of P is $a\omega^2$ along the radius PO and the acceleration of N , d^2x/dt^2 , is the resolved part along OX of the acceleration of P .

$$\begin{aligned}\therefore \frac{d^2x}{dt^2} &= -a\omega^2 \cos \theta \\ &= -\omega^2 x.\end{aligned}$$

Since ω is constant the acceleration is proportional to the displacement and hence the motion of N is S.H.M.

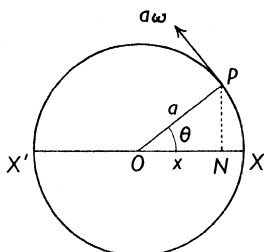


Fig. 106

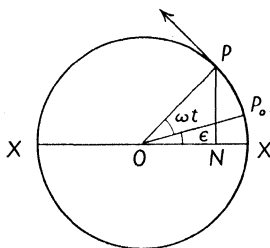


Fig. 107

The time T required by N to execute one complete vibration (i.e. to travel backwards and forwards along the diameter) is equal to the time required by P to describe one revolution. Thus

$$T = \frac{2\pi}{\omega}.$$

Since T is independent of the radius of the circle the periodic time of S.H.M. is independent of the amplitude.

Suppose at time $t=0$, P is at X ; then $\theta = \omega t$ and

$$x = a \cos \omega t.$$

If, however, at time $t=0$, P is at P_0 (Fig. 107), where $\angle XOP_0 = \epsilon$, then

$$x = a \cos (\omega t + \epsilon).$$

The angle ϵ represents the difference of phase between P_0 and X .

The simple pendulum

Angular motion as well as linear motion may be simple harmonic; an example is the *simple pendulum*, consisting of a small massive bob at the end of a (theoretically) weightless string.

Suppose that the bob has a mass m and that the string, of length l , makes a small angle θ with the vertical (Fig. 108). The weight of the

bob can be resolved into a component $mg \sin \theta$ perpendicular to the string and a component $mg \cos \theta$ along the string; the former component tends to restore the pendulum to its vertical position while the latter component has no effect on the motion.

$$\begin{aligned}\text{Restoring force} &= mg \sin \theta \\ &= mg \theta. \quad (\text{if } \theta \text{ is small, } \sin \theta = \theta)\end{aligned}$$

Displacement along the arc $= l\theta$.

$$\therefore \text{Restoring force} = \frac{mg}{l} \times \text{displacement along the arc.}$$

Since the restoring force is proportional to the displacement, the motion of the bob along the arc must be simple harmonic.

$$\begin{aligned}\therefore T &= 2\pi \sqrt{\frac{\text{Mass}}{\text{Force per unit displacement}}} \\ &= 2\pi \sqrt{\frac{m}{\frac{mg}{l}}} \\ &= 2\pi \sqrt{\frac{l}{g}}.\end{aligned}$$

This equation applies only when θ is small and does not hold for large amplitudes.

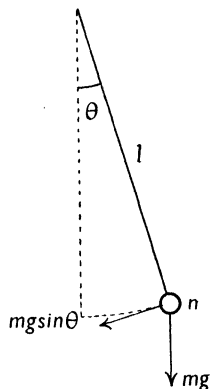


Fig. 108

General case of angular S.H.M.

The following alternative treatment of the simple pendulum is a convenient introduction to the general case of angular S.H.M.

The moment of inertia I of the bob about the point of support of the string is ml^2 .

$$\text{Restoring couple} = mg \theta l.$$

But in angular motion,

$$\text{Couple} = I \frac{d^2\theta}{dt^2}, \quad (\text{p. 98})$$

$$\text{i.e.} \quad -mg \theta l = ml^2 \frac{d^2\theta}{dt^2}.$$

(The negative sign indicates that the couple acts in a direction opposite to θ increasing.)

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{g}{l} \theta.$$

This has the same form as the equation for linear simple harmonic

motion on p. 106, and hence the periodic time is given by $T=2\pi/\omega$, where $\omega^2=g/l$.

$$\therefore T=2\pi\sqrt{\frac{l}{g}}.$$

The result can be generalised as follows. When a body vibrates about an axis under the action of a couple which is proportional to the angular displacement from the undisturbed position, the body executes an angular S.H.M., for which

$$T=2\pi\sqrt{\frac{I}{c}},$$

where T =periodic time,

I =moment of inertia of the body about the axis of vibration,

c =the couple for unit angular displacement (e.g. in dyne-cm. per radian).

The compound or rigid pendulum

Fig. 109 represents a rigid body of mass m , whose centre of gravity is at G and which is free to vibrate under gravity about an axis through O_1 (called the centre of suspension) perpendicular to the paper. A body which vibrates in this way is called a *compound* or a *rigid pendulum*.

Restoring couple about

$$\begin{aligned} O_1 &= mgh_1 \sin \theta \\ &= mgh_1 \theta, \text{ if } \theta \text{ is small.} \end{aligned}$$

Since the restoring couple is proportional to the angular displacement the vibrations are simple harmonic.

The restoring couple per unit angular displacement is mgh_1 .

$$\therefore T=2\pi\sqrt{\frac{I_1}{mgh_1}},$$

where T is the periodic time and I_1 is the moment of inertia of the body about O_1 .

If k is the radius of gyration of the body in Fig. 109 about G , the moment of inertia about G is mk^2 .

$$\therefore I_1 = m(k^2 + h_1^2). \quad (\text{theorem of parallel axes, p. 94})$$

$$\therefore T=2\pi\sqrt{\frac{k^2 + h_1^2}{h_1 g}}.$$

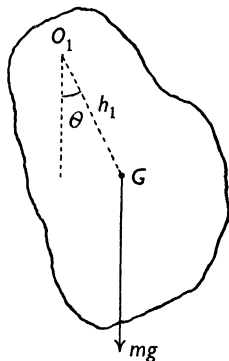


Fig. 109

Minimum periodic time

Suppose that, for some other centre of suspension O_2 , such that $O_2G = h_2$ (Fig. 110), the periodic time is the same as about O_1 . Then

$$\frac{k^2 + h_2^2}{h_2} = \frac{k^2 + h_1^2}{h_1},$$

i.e. $k^2(h_1 - h_2) = h_1h_2(h_1 - h_2),$

so that either $h_2 = h_1$ or $k^2 = h_1h_2$.

For the former possibility O_1 and O_2 are symmetrically placed with respect to the centre of gravity; for the latter, the arrangement is asymmetrical except when $h_1 = k = h_2$.

The periodic time is a minimum in the symmetrical position. T is a minimum when $\frac{k^2 + h_1}{h_1g}$ is a minimum.

But
$$\frac{k^2 + h_1^2}{h_1} = \frac{(k - h_1)^2 + 2kh_1}{h_1} = \frac{(k - h_1)^2}{h_1} + 2k.$$

This has the minimum value of $2k$ when $h_1 = k$.

$$\therefore \text{Minimum period, } T = 2\pi \sqrt{\frac{2k}{g}}.$$

Simple equivalent pendulum

The distance O_1O_2 , $h_1 + h_2$, is equal to the length of the simple pendulum which has the same period as the compound pendulum about O_1 or O_2 , known as the *simple equivalent pendulum*. This can be proved as follows:

$$k^2 = h_1h_2, \quad \therefore \frac{k^2}{h_1} = h_2.$$

$$\therefore \frac{k^2 + h_1^2}{h_1} = \frac{k^2}{h_1} + h_1 = h_2 + h_1.$$

$$\therefore T = 2\pi \sqrt{\frac{h_1 + h_2}{g}}.$$

Taking O_1 as the centre of suspension, O_2 is called the *centre of oscillation* and vice versa; if the whole mass of the body could be concentrated at O_2 , the period of vibration about O_1 would be unaltered.

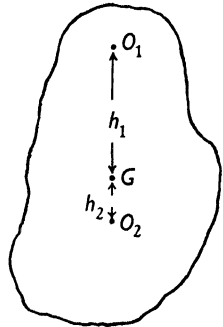


Fig. 110

Experimental investigation of the compound pendulum

The way in which the periodic time of a compound pendulum, in the form of a rectangular bar, varies with the distance of the centre of suspension from the centre of gravity may be investigated by the arrangement of Fig. 111. A metal bar is drilled with holes, sufficiently large to enable a knife edge to be inserted through them, and the periodic time about different axes is measured. The position of the centre of gravity of the bar is determined by balancing on a knife edge.



Fig. 111

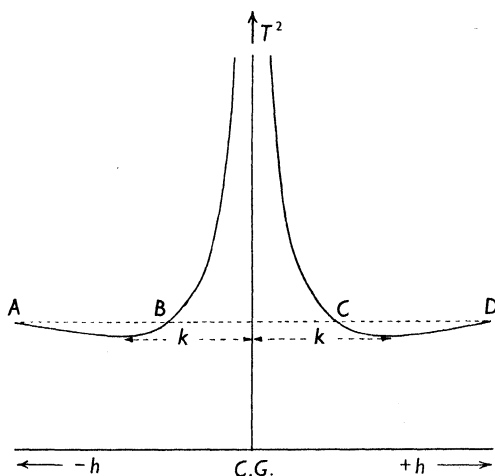


Fig. 112

The graph of the periodic time T and the distance h of the axis of suspension from the centre of gravity is similar to that in Fig. 112. The minima are at positions $h=k$, where k is the radius of gyration of the bar. A line such as $ABCD$ cuts the graph in four points which represent four different points on the bar about which the period of vibration is the same. The distances AC and BD are equal to the length of the simple equivalent pendulum.

Centre of percussion

Suppose that the centre of suspension O_1 of a compound pendulum is free to move, say on the ring of a retort stand (Fig. 113), and the position of the centre of oscillation O_2 is determined by finding the length of the simple pendulum which has the same period. If the compound pendulum is struck a sharp blow with a hammer at O_2 there is

no jar on the suspension. If, however, the blow is struck above or below O_2 the suspension is jarred and moves bodily. Thus O_2 is called the *centre of percussion*.

We have seen that the period of the pendulum about O_1 is the same as if the whole mass of the pendulum were concentrated at O_2 . A blow F at X (Fig. 114) is equivalent to a blow F at O_2 plus a couple $F(O_2X)$. The blow at O_2 sets the pendulum vibrating in the normal manner and the couple jars the system.

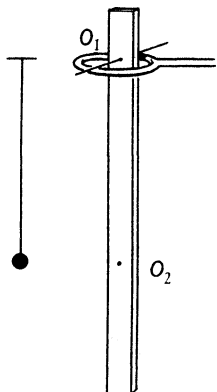


Fig. 113

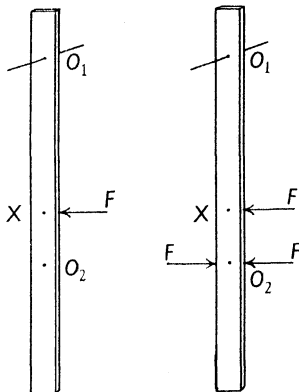


Fig. 114

A cricket ball should be struck by the bat at its centre of percussion with respect to the hands of the batsman; a good hammer is constructed so that its centre of percussion is in line with the driving surface.

Determination of g

The most accurate method of determining the acceleration due to gravity is by means of a pendulum. The period of a simple pendulum is given by the equation $T = 2\pi \sqrt{\frac{l}{g}}$, and hence a determination of g involves the measurement of a time T and of a length l .

The earliest determinations were made with simple pendulums; for example, in 1669 Picard used a copper ball, 1 in. in diameter, suspended by an aloe fibre (which is unaffected by moisture), and in 1792 Borda and Cassini used a platinum ball, $1\frac{1}{2}$ in. in diameter, hung from a fine iron wire about $12\frac{3}{4}$ ft. long. A simple pendulum has considerable disadvantages, however. Its swings die away too quickly for a long time interval to be measured unless a heavy bob and a strong wire are used, in which case it ceases to be an ideal simple pendulum; nor can the

length l , the distance between the point of support and the centre of gravity of the bob, be measured very accurately.

Suggestions were put forward, therefore, for making use of the interchangeability of the centres of suspension and of oscillation of a compound pendulum.

Kater's reversible pendulum.

In 1817 Captain Kater made his celebrated determination of g in London, using what is known as Kater's reversible pendulum. The pendulum consisted of a brass bar to which two steel knife edges A and B (Fig. 115) were attached at such a distance apart (about 39.4 in.) that the period was very slightly greater than that of the pendulum of a standard clock, beating seconds (i.e. $T=2$ sec.). C and D were two adjustable weights, and the larger weight C was moved until the times of swing about the two knife edges were very nearly equal, whereupon C was fixed in position; the smaller weight D was then adjusted, by means of a screw, until the times of swing were as nearly equal as it was possible to make them. The distance between the two knife edges, equal to the length of the simple equivalent pendulum, was measured to $\frac{1}{10,000}$ in. by placing the pendulum horizontally under two microscopes arranged so as to be vertically above the knife edges, and then replacing the pendulum with a standard scale.

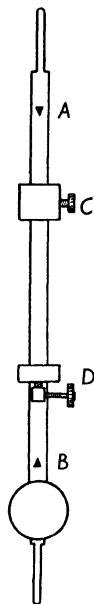


Fig. 115

The period was measured by the method of coincidences. The pendulum was set up in front of a standard clock (Fig. 116), and a small disc of white paper of the same width as the end of the pendulum was attached to the bob of the clock pendulum. The swings of both pendulums were observed through a telescope 9 ft. away, and when the two pendulums were in phase, so that they passed through their vertical positions at exactly the same instant, the white disc was completely obscured by the end of the pendulum in front of it. Successive coincidences occurred every 530 sec., during which time the reversible pendulum made 528 swings (i.e. half oscillations).

$$\therefore \text{Half period of reversible pendulum} = \frac{530}{528} = 1.003787 \text{ sec.}$$

If an error of 1 swing were made in observing the coincidences, and the observations should have been 529 swings in 531 seconds,

$$\text{Half-period of convertible pendulum} = \frac{531}{529} = 1.0037807 \text{ sec.}$$

This represents an error of only 1 part in 100,000.

The standard clock was checked every 24 hr. by stellar measurements while the experiments were in progress, so that the time of swing of the pendulum was really compared with the rate of rotation of the earth.

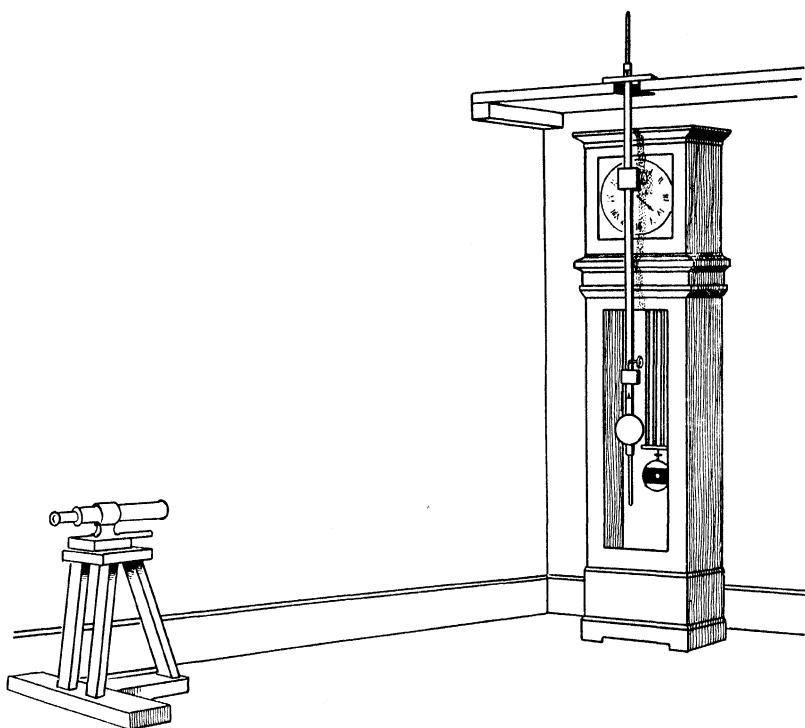


Fig. 116

An allowance was made for the effect of the buoyancy of the air on the pendulum.

The final result, corrected for the height above sea level of Portland Place where the observations were made, was expressed as 39.13829 in. for the length of the seconds pendulum at sea level in the latitude of London.

Bessel's theory of the reversible pendulum

Bessel showed that it is quite simple to correct for slight differences between the two periodic times of a reversible pendulum. Suppose the two periods are T_1 and T_2 , which are very nearly equal, and the dis-

tances of the two knife edges from the centre of gravity of the pendulum are h_1 and h_2 :

$$T_1 = 2\pi \sqrt{\frac{k^2 + h_1^2}{h_1 g}},$$

$$T_2 = 2\pi \sqrt{\frac{k^2 + h_2^2}{h_2 g}}.$$

Squaring, rearranging and subtracting,

$$\begin{aligned} \frac{T_1^2}{4\pi^2} h_1 g - \frac{T_2^2}{4\pi^2} h_2 g &= h_1^2 - h_2^2. \\ \therefore \frac{4\pi^2}{g} &= \frac{T_1^2 h_1 - T_2^2 h_2}{h_1^2 - h_2^2} \\ &= \frac{1}{2} \left(\frac{T_1^2 + T_2^2}{h_1 + h_2} + \frac{T_1^2 - T_2^2}{h_1 - h_2} \right). \end{aligned}$$

The second term is small since T_1 is very nearly equal to T_2 , so that the value of $h_1 - h_2$ need not be known to a high order of accuracy. A sufficiently accurate estimate of its value can be obtained by balancing the pendulum horizontally on a knife edge to find the position of its centre of gravity.

Bessel showed that for very accurate determinations corrections for the following must be made.

(1) *Air correction.* (a) The buoyancy of the air, tending to decrease the restoring force; (b) the increase in inertia due to the air carried with the pendulum; (c) the viscosity of the air acting as a drag on the pendulum. All these three effects tend to increase the period.

(2) *Knife edges.* If the knife edges are not perfectly sharp they roll on the supports thereby slightly increasing the effective length of the pendulum.

(3) *Yielding of the support.* The support may have a natural period of vibration which is superimposed on that of the pendulum.

(4) *Finite arc of swing.* An appreciable amplitude of swing of α radians causes the period to increase in the ratio $1 + \frac{\alpha^2}{16} : 1$.

Comparative determinations of g

As with many physical quantities, an absolute determination of g is difficult, but comparative measurements can be made with ease. Gravity determinations throughout the world are referred to the value of g at a particular pier in the laboratory of the Geodetic Institute, Potsdam, where an elaborate absolute determination was made between

the years 1894 and 1906. Several reversible pendulums similar to Kater's were used, all the corrections mentioned above being applied, and the final probable error was 3 parts in 1,000,000. To compare the value at any place with that of Potsdam, it is necessary to find the period of a suitable pendulum, such as that in Fig. 117, at the place in question, transport the pendulum to Potsdam and find its period there.

Then $\frac{g_1}{g_2} = \frac{T_2^2}{T_1^2}$, so long as the temperature, air pressure, amplitude of swing, and the rigidity of the pier are the same in both places.

The variation of g with latitude, due to the change of distance from the centre of the earth, is employed to find the figure, or shape, of the earth. In 1880 von Sterneck introduced gravity pendulums which beat $\frac{1}{2}$ sec. and which were only one-quarter as long as those formerly employed (Fig. 117). This resulted in an increase in accuracy and decrease in cost, with the result that gravity stations throughout the world increased from about a hundred to several thousand in the following ten years.

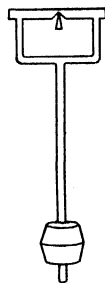


Fig. 117

A gravity pendulum, vibrating in a closed chamber in which the air pressure is less than $\frac{1}{10}$ atm., and starting with a total arc of vibration of $1\frac{1}{2}^\circ$, will swing for 12 hr. and still have an arc of $25'$. By timing over such a long interval an accuracy of 1 part in 1,000,000 is attained. Since it is a hopeless task to attempt to count the oscillations over a period of 12 hr., the method of coincidences is employed. The interval between two coincidences, known as the coincidence interval, is measured for 3 or 4 coincidences when the pendulum starts oscillating and then for 3 or 4 more after about 12 hr. The number of coincidences, and hence the total number of oscillations, in the exact period of about 12 hr. can thus be obtained. The coincidences are obtained between the gravity pendulum and the pendulum of an electrically driven clock which is checked by wireless time signals sent out from the master clock at Greenwich.

The value of g changes with altitude as well as with latitude, and the variation is about 1 part in 10,000 for every 1000 ft. of altitude. Owing to local differences in the densities of the rocks there are also variations of 1 or 2 parts in 10,000, the investigation of which is one of the methods of prospecting for oil and minerals. In view of the smallness of these local anomalies great accuracy in the measurement of the relative value of g is essential.

EXAMPLE

Show that the motion of a steel ball, rolling to and fro in a concave mirror, is simple harmonic and find its period.

Let r and R be the radii of the ball and the mirror respectively. The ball moves to and fro in a circular arc like the bob of a simple pendulum of length $R-r$, but in addition it rolls. Its moment of inertia about its centre is $\frac{2}{5}mr^2$, where m is its mass.

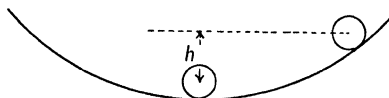


Fig. 118

Suppose in moving from one extreme position to its undisturbed position it falls through a vertical height h (Fig. 118) and acquires a velocity v :

$$\begin{aligned} mgh &= \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{r^2} & (\text{p. 94}) \\ &= \frac{1}{2}mv^2 + \frac{1}{2} \cdot \frac{2}{5}mr^2 \frac{v^2}{r^2} \\ &= \frac{1}{2} \cdot \frac{7}{5}mv^2. \\ \therefore v^2 &= \frac{5}{7} \cdot 2gh. \end{aligned}$$

But for a simple pendulum, $v^2 = 2gh$. Hence the velocity of the ball rolling in the concave mirror is at all points $\sqrt{\frac{5}{7}}$ of the velocity of the bob of a simple pendulum of length $R-r$. The motion is therefore simple harmonic and the period T is given by

$$T = 2\pi \sqrt{\frac{7(R-r)}{5g}}.$$

EXAMPLE

The motion of a particle is compounded of a simple harmonic motion along the x -axis and a simple harmonic motion, with the same centre and of the same period, along the y -axis, the amplitude of the latter being twice that of the former and its phase one-quarter of a period different. Show that the path of the particle is an ellipse and that the maximum velocity is twice the minimum velocity. (C.)

Let the two S.H.M.'s be represented by the equations

$$x = a \sin \omega t,$$

$$y = 2a \sin \left(\omega t + \frac{\pi}{2} \right).$$

Squaring,

$$x^2 = a^2 \sin^2 \omega t,$$

$$\frac{y^2}{4} = a^2 \cos^2 \omega t.$$

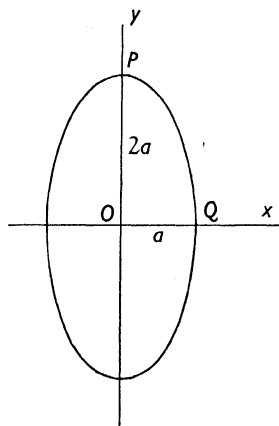


Fig. 119

Adding,
$$x^2 + \frac{y^2}{4} = a^2,$$

which is the equation of an ellipse.

$$\frac{dx}{dt} = a\omega \cos \omega t.$$

$\therefore$ Maximum velocity along x -axis is $a\omega$

$$\frac{dy}{dt} = -2a\omega \sin \omega t.$$

$\therefore$ Maximum velocity along y -axis is $2a\omega$.

Maximum resultant velocity occurs at Q (Fig. 119) and is $2a\omega$.

Minimum resultant velocity occurs at P and is $a\omega$.

SUMMARY

The periodic motion in a straight line of a body on which the force is always directed towards, and proportional to the distance from, a fixed point, is called simple harmonic motion: $m \frac{d^2x}{dt^2} = -\mu x$. The equation of the path, in its simplest form, is $x = a \cos \omega t$ (where $\omega^2 = \mu/m$), and the periodic time is given by $T = 2\pi \sqrt{\frac{m}{\mu}}$.

The projection of uniform circular motion on a diameter can be shown to be S.H.M.

Angular motion also may be simple harmonic; for example, the motion of a simple or of a compound pendulum, and the theory is analogous to that for linear S.H.M. The acceleration due to gravity may be determined most accurately by means of a reversible (compound) pendulum.

QUESTIONS

[Take $g = 32 \text{ ft. sec.}^{-2}$ or $981 \text{ cm. sec.}^{-2}$]

1. What is simple harmonic motion and why is its study of practical importance? Calculate the periodic time for small vertical oscillations of a mass of 200 gm. hung on the end of a light spiral spring, given that the spring will be extended 5 cm. if a load of 60 gm. is hung on it. (O.)

2. What is meant by *simple harmonic motion*? Explain the conditions under which it occurs. What experiment would you make to test whether this condition is satisfied in the case of (a) the oscillation of a mass at the end of a spiral spring, (b) the torsional oscillation of a mass at the end of a wire which is fixed at the top?

How may the acceleration due to gravity be found experimentally, using a spiral spring? (O.)

3. Investigate the vertical vibrations of a mass attached to the end of a spiral spring.

If a mass of 500 gm. is attached to a light unstretched spiral spring and released, find (a) the maximum velocity it will require, (b) how far it will drop. The spring is extended by 1 cm. for an additional load of 100 gm. (C. S.)

4. In the HCl molecule the force required to alter the distance between the atoms from its equilibrium value is 5.4×10^5 dynes per cm. What is the fundamental frequency of the vibration of the molecule, assuming the vibration to be simple harmonic, and the mass of the Cl atom to be infinite compared to that of the H atom which is 1.66×10^{-24} gm.? (C. S.)

5. A uniform U-tube contains a column of liquid of height 25 cm. in each limb. The liquid in one limb is depressed and then released; find the period of oscillation. (L.)

6. A test-tube of weight 6 gm. and external diameter 2 cm. is floated vertically in water by placing 10 gm. of mercury at the bottom of the tube. The tube is depressed a small amount and then released. Find the time of oscillation. (O. & C.)

7. A deep vessel is filled, half with water and half with benzene (sp.gr. 0.88), which is immiscible with water. A thin cylinder, of length 12 cm. and sp.gr. 0.92, is placed in the vessel and constrained to move with its axis vertical. At what position does the cylinder come to rest? If the cylinder be displaced vertically from this position and released, what will be the period of the subsequent oscillations? The effects of viscosity may be neglected. (C. S.)

8. A light elastic string of unstretched length l is laid on a smooth horizontal table and stretched to a length $l+a$ between fixed points A and B (Fig. 120). A mass m is attached to its midpoint, drawn slightly aside and released. Show that the motion of m is simple harmonic and that the periodic time T is given by

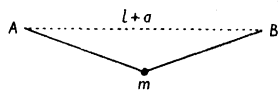


Fig. 120

$$T = 2\pi \sqrt{\frac{m(l+a)}{4\mu a}},$$

where μ is the force required to produce unit extension of the string.

9. A uniform sphere of radius r rolls without slipping down a plane inclined to the horizontal at an angle θ . Find the acceleration of its centre of gravity. (Moment of inertia of a sphere about a diameter $= \frac{2}{5}Mr^2$.)

A concave mirror of radius R is fixed rigidly with its axis vertical and the sphere is placed on it. The sphere is displaced slightly from its equilibrium position and is then allowed to roll from rest down a line of greatest slope. Show that it will execute simple harmonic motion, and find the period of the motion. (C. S.)

10. A small body vibrates with S.H.M. along a straight line with frequency 4 per sec. and maximum displacement 2 cm. What is the maximum velocity of its motion? (O.)

11. A particle vibrates with S.H.M. along a straight line. Its greatest acceleration is $5\pi^2$ cm. sec.⁻² and when its distance from the equilibrium position is 4 cm. the velocity of the particle is 3π cm. sec.⁻¹ Find the amplitude and period of oscillation of the particle. (O. & C.)

12. Define simple harmonic motion, and show that the total energy (kinetic + potential) of a particle executing simple harmonic motion is proportional (a) to the square of the amplitude, (b) to the square of the frequency.

The total energy of a particle executing a simple harmonic motion of period 2π sec. is 10,240 ergs. $\pi/4$ sec. after the particle passes the midpoint of the swing its displacement is $8\sqrt{2}$ cm. Calculate the amplitude of the motion and the mass of the particle. (O. & C.)

13. On a given day the depth at high water over a harbour bar is 32 ft., and at low water $6\frac{1}{4}$ hr. earlier it is 21 ft. If high water is due at 3.20 p.m., what is the earliest time at which a ship drawing 28 ft. 6 in. can cross the bar, assuming the rise and fall of the tide to be simple harmonic? (C. S.)

14. A mass of 100 gm. is suspended from a spring, and it is found that the addition of 20 gm. causes the spring to extend by 5 cm. The mass of 100 gm. is set into oscillation with an amplitude of 3 cm., and when it is at the extreme upper limit of its oscillation, a mass of 10 gm. which has fallen through 20 cm. strikes it and sticks to it. Calculate the time that elapses before the combined masses pass for the first time through the new centre of oscillation. Neglect the mass of the spring. (C. S.)

15. Show that the motion of the piston of a steam engine is approximately simple harmonic if the connecting rod is long compared with the crank.

16. The shadow of a small marble fastened near the edge of a horizontal gramophone turn-table is projected by a parallel beam of light on to a vertical screen on which the light falls normally. Show that if the turn-table is rotating with uniform angular velocity, the motion of the shadow can be represented by an equation of the form $x = A \sin(pt - \delta)$, and explain the significance of the terms A , p and δ .

PENDULUMS

17. Deduce a formula for the period of vibration of a simple pendulum.

A simple pendulum is mounted in a mine lift which can be hoisted in such a way as to reach a high speed quickly.

Trace the effect of the motion of the lift on the swinging pendulum.

Neglect any change of g with the lift position. (N.)

18. Describe how you would measure accurately the length (l) and time (t) of one vibration of a simple pendulum.

A pendulum whose point of support is inaccessible can be shortened by measured lengths. In an experiment the following observations were made:

Pendulum shortened by (cm.)	0	20	40	60	80	100
Time for 40 swings (sec.)	113.5	107.6	101.5	94.7	88.1	80.5

Plot a graph to show the relation between the length shortened and the square of the time of one vibration. Choose the scale so that the graph may be prolonged to enable the original length of the pendulum to be determined. Hence using the

formula $t = 2\pi \sqrt{\frac{l}{g}}$, calculate the value of g . (C.)

19. A clock which normally ticks every quarter second is mounted on the centre of a gramophone turn-table. Show that, if the pendulum is perfectly rigid, the clock will lose $14\frac{1}{2}$ min. per hour when the turn-table rotates at 78 r.p.m. (O. S.)

20. Explain and describe the method of coincidences for comparing the time of vibration of a pendulum with that of a standard pendulum. Why are the results obtained by this method more accurate when the times of vibrations of the pendulums are nearly equal than when they are far apart? (C. S.)

21. Find an expression for the periodic time of a rigid pendulum and explain how a very accurate value of the acceleration due to gravity can be found by using a reversible rigid pendulum. (O.)

22. A uniform thin rod performs small oscillations in a vertical plane about a smooth horizontal axis which passes through the rod at a distance of 10 cm. from its centre of gravity. The time of oscillation is 1.09 sec. Derive an expression for the time of oscillation and calculate the length of the rod. About what other axes will the rod oscillate with this period? (N.)

23. What is meant by the *radius of gyration* of a rotating body? How would you find its value experimentally for a uniform rod oscillating about one end?

A uniform rod 4 ft. long is held at one end in the hand. At what distance from the hand must a blow be delivered at right angles to the rod so that no shock may be felt by the person holding the rod? (D.)

24. A knitting pin, soldered on the circumference of a circular metal disc of uniform thickness so that it is normal to the plane of the disc, rests on two supports, and the disc is set swinging in its own plane. The diameter of the disc is 32 cm. Calculate its time of oscillation. The thickness of the pin may be taken as negligible. (D.)

25. A hoop, of radius 2 ft., is hung on the point of a nail projecting vertically from a horizontal rod and allowed to vibrate. Calculate its periodic time when it is (a) vibrating in its own plane, (b) vibrating about an axis through the point of the nail and practically tangential to the hoop. (D.)

26. The beam of a balance weighs 150 gm. and its moment of inertia is 5000 gm. cm.² Each arm of the balance is 10 cm. long. When set swinging the beam makes one complete oscillation in 6 sec. How far is the centre of gravity of the beam below its point of support, and through what angle would the beam be deflected by a weight of 1 milligram placed in one of the scale pans? (C.)

27. A uniform thin rod AB of mass m and length l can rotate in a vertical plane about A as a pendulum. A particle of mass $2m$ is also fixed on the rod at a distance x from A . Find x so that the periodic time of a small swing may be a minimum. (L.)

COMBINATION OF S.H.M.'s

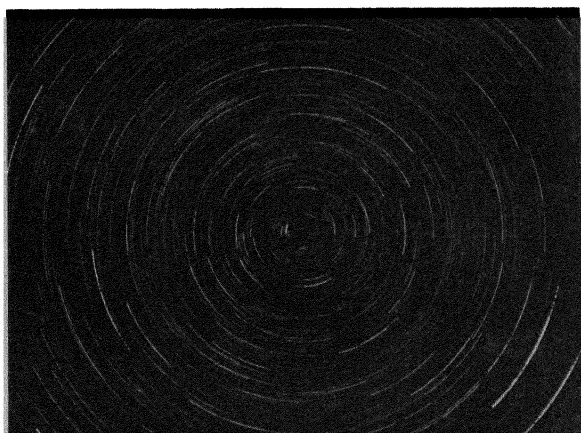
28. Show that uniform circular motion can be regarded as the resultant of two perpendicular S.H.M.'s of the same period and amplitude. What will be the resultant of two perpendicular S.H.M.'s of the same period but unequal amplitudes?

29. Find graphically the resultant of two S.H.M.'s at right angles to one another (a) when the amplitudes and periods are equal and one vibration differs in phase by $\frac{1}{2}\pi$ from the other, (b) when the amplitudes are equal, the period of one is twice as great as that of the other and the slower vibration is $\frac{1}{2}\pi$ ahead of the other. (B.)

CHAPTER 6

GRAVITATION

THE principles of dynamics and the theory of gravitation were developed by Newton mainly to account for the motions of the heavenly bodies. The astronomical discoveries which preceded his work require therefore some consideration.



D. G. A. Dyson

Fig. 121. Trails of the circumpolar stars. The dotted nature of some of the trails is caused by clouds obscuring the stars for short periods during the exposure.

Successive glances at the heavens at intervals of a few hours on some clear bright night show that the stars appear to be revolving. They revolve about a point called the North Celestial Pole, which is on the prolongation of the earth's axis and is near to the Pole Star. The camera makes the motion obvious (see Fig. 121).

If the stars were visible in the daytime, the sun would be seen to slip slowly backwards among them, about twice its breadth in one day. The path of the sun moving among the stars is marked by the constellations of the Zodiac, the Ram, the Bull, the Twins, the Crab, the Lion, the Virgin, the Scales, the Scorpion, the Archer, the Goat, the Water Carrier and the Fish, which are the monthly milestones in its yearly backward slide. These constellations, which are quite arbitrary

groupings of stars, were mapped out for convenience in identification by Babylonian astronomers about 2700 B.C.

The stars revolve together without changing their relative positions, but there are certain heavenly bodies, called the planets (Greek, wanderers) which move among the stars, sometimes forward and sometimes backward (see Fig. 122). Five of the planets, Mercury, Venus, Mars, Jupiter and Saturn, can be seen with the naked eye and were known to the ancients; three others, Uranus, Neptune and Pluto, were not discovered until after the invention of the telescope.

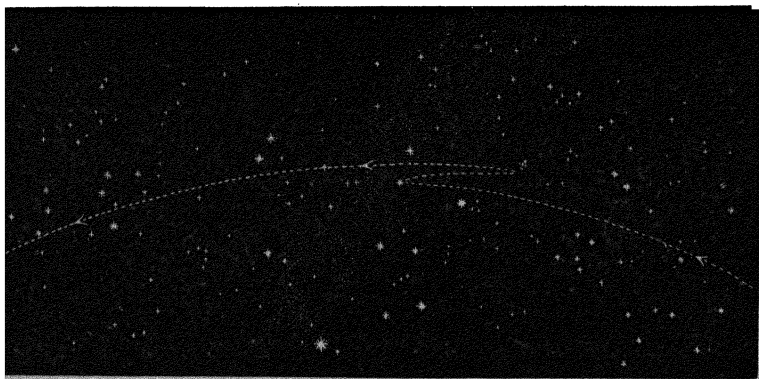


Fig. 122. Path of Mars through the stars.

The Greek astronomer Hipparchus compiled a carefully made catalogue of the nightly positions of 1000 stars and of the planets. From this catalogue Ptolemy, who worked in Alexandria about A.D. 100, devised a theory of the motions of the heavens which was accepted for 1400 years; his book, the *Almagest*, acquired an authority almost equal to that of the bible. Ptolemy suggested that the whole of the heavens, carrying the stars, revolved round a stationary Earth. To explain the forward and retrograde motion of the planets among the stars he postulated that the planets revolve in circles whose centres revolve in larger circles round the earth (see Fig. 123). The smaller circles are termed epicycles and the larger circles deferents. By a suitable choice of radii and velocities the theory accounted quite accurately for the observations.

The Ptolemaic system was first challenged in 1543 in a book by a Polish monk, Nicolaus Copernicus, entitled *Concerning the Revolutions of the Heavenly Bodies*. Copernicus suggested that just as a man aboard

ship moving on smooth water might imagine that the lights of the coast were moving past him while he himself was stationary, so the same illusion might be experienced by a watcher of the heavens. Instead of the whole universe revolving round the earth, a simpler and equally satisfactory hypothesis was the rotation of the earth in the opposite direction. Copernicus took an important step further. He revived the theory of the Greek astronomer, Aristarchus, that the earth is not the centre of the universe but revolves round the sun, and that the rest of the planets also revolve round the sun. He spent many years making observations of Mars, the most easily observed planet, and was much struck by its variations in brightness. Night by night it

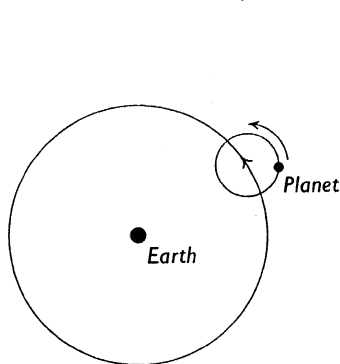


Fig. 123

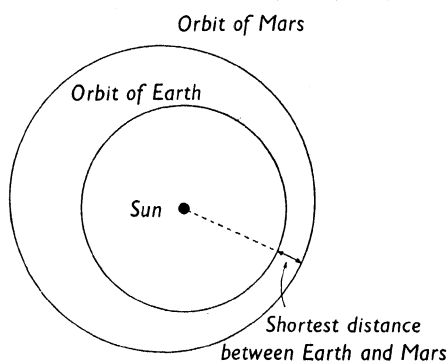


Fig. 124

moves forwards among the stars, then stops and begins to move back. When it is on the turn it appears largest and brightest; Copernicus explained this by assuming that the Earth and Mars are then nearest to each other during their revolutions round the sun (see Fig. 124). He retained circular orbits and epicycles, but his theory was a much neater geometrical solution than that of Ptolemy, since it involved only 34 epicycles instead of 80.

He hesitated to publish his book until he was dying. Nevertheless, the Pope, to whom the book was dedicated, expressed himself pleased with it, as he did not take it seriously. Martin Luther, on the other hand, was contemptuous; did not the bible say that Joshua commanded the sun to stand still and not the earth?

Those who felt the force of the theory objected that if the earth rotated, bodies would be hurled from its surface. If a stone is thrown up it does not fall to the west of the thrower, but vertically. Moreover, and this is a more serious objection, no relative motion or parallax of the stars had been noticed such as is observed among objects at different

distances when one is moving in a ship. It is now known that this parallax does exist; it was first measured by Bessel in 1838, but it is extremely small because the stars are at such enormous distances from the earth.

The full implications of the Copernican theory were not realised until nearly a century later, when Galileo was made to recant his belief in it, and Giordano Bruno was burnt at the stake for refusing to do so.

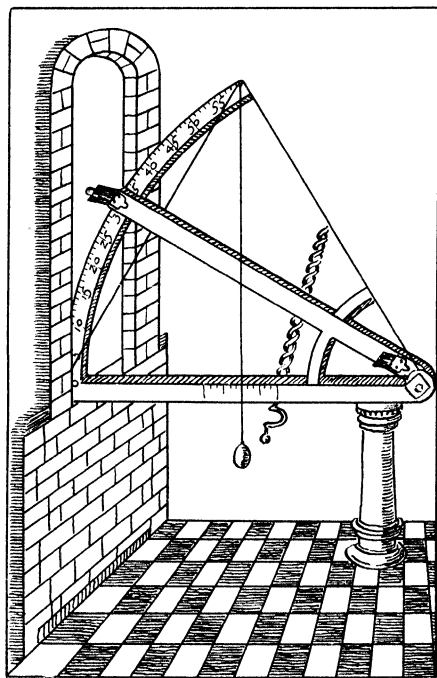


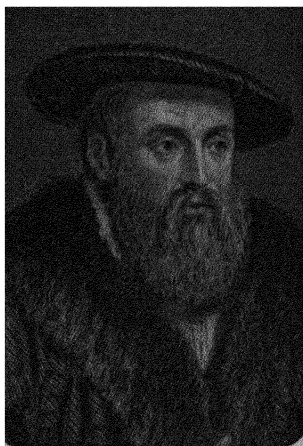
Fig. 125. Tycho Brahé's 'Quadrant' (from an old print in the British Museum). The movable arm is adjusted by means of the screw until the two sights are in line with the star under observation. The altitude of the star is read off the graduated scale.

There now appeared, in the history of astronomy, two great figures, diverse in temperament and intellect, but each the complement of the other. These were Tycho Brahé (1546–1601), an imperious nobleman and a brilliant astronomical observer, and Johannes Kepler (1571–1630), impecunious and of humble birth, endowed with a strong speculative and mathematical ability.

For twenty-one years, at his observatory in Denmark, Tycho Brahé measured the times of transit of the stars and planets (i.e. when they



TYCHO BRAHÉ



JOHANNES KEPLER

TYCHO BRAHÉ (1546–1601), unsurpassed as a practical astronomer, achieved the greatest accuracy possible in observations without telescopes. His fine observatory of Uraniborg, built for him by Frederick II, King of Denmark, was equipped with instruments of his own construction. Here, for twenty-one years, he made nightly observations of the positions of the planets and of the stars, introducing corrections for errors in the divisions of his quadrant and for the refraction of the air. He was of fiery temper and, as a young man, lost part of his nose in a duel; this he replaced by an artificial nose made of an alloy of silver. On the death of Frederick he was driven from Uraniborg by his enemies and the observatory fell into ruin. He sought refuge in Prague under the patronage of Rudolph II, Emperor of Bohemia, and it was here that he was joined by Kepler as an assistant. After the short space of a year in Prague he died, leaving his observations to Kepler.

JOHANNES KEPLER (1571–1630) succeeded Tycho Brahé at Prague under the title of ‘Imperial Mathematician’ at an excellent salary which was seldom paid. He records that he was forced to supplement his income by practising astrology, ‘the foolish and disreputable daughter of astronomy, without which the wise old mother would starve.’ His superb and quite astonishing achievement, after many years of effort, was the discovery from Tycho’s observations of the laws concerning the orbits of the planets. His profound belief in the mathematical nature of the universe is well illustrated in another of his discoveries which, he said, gave him the most intense pleasure and which is now known to be valueless. He believed that the sizes of the orbits of the six planets known in his day had been so adjusted by the Creator that they fitted exactly round the sphere and round the vertices of the only five regular solids which possess equal edges and angles, those with 4, 6, 7, 12 and 20 faces, the six solids fitting exactly inside each other like the familiar child’s toy. He was also the founder of geometrical optics.

cross the meridian at the place of observation) and their highest altitudes (i.e. angles of elevation above the horizon). His wooden quadrant with its brass scale and rotating arm carrying sights rather like a rifle (see Fig. 125), was of great size for the sake of accuracy; it had a radius of 10 ft. and was capable of measuring to one minute of arc compared with the normal error of Hipparchus of half a degree. Tycho's observations were used by navigators for hundreds of years, much as the *Nautical Almanack* is used to-day, but, even more important, they passed at his death into the hands of his assistant Kepler.

Kepler accepted the Copernican theory which Tycho rejected, and was saved from persecution by living in a Protestant country. He had a profound, mystical belief in the mathematical nature of the universe, like the followers of Pythagoras centuries earlier, and he spent a lifetime juggling with Tycho's figures. Tycho's best observations had been of the planet Mars, and Kepler tried nineteen possible orbits for this planet. At length, when the last of his three famous laws had dawned upon him, he was filled with ecstasy: 'I will indulge in my sacred fury; I will triumph over mankind by the honest confession that I have stolen the golden vases of the Egyptians to build up a tabernacle for my God.'

Kepler's laws

(1) *The planets move round the sun in ellipses having the sun at one focus.*

(2) *Each planet revolves in such a way that an imaginary line joining it to the sun sweeps out equal areas in equal times (Fig. 126).*

(3) *The squares of the times of revolution of the planets bear a constant ratio to the cubes of their mean distances from the sun.*

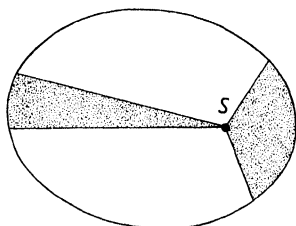


Fig. 126

The table below illustrates the third law. The mean distances of the planets from the sun, d , are measured in terms of the earth's distance which is taken as 1, and the time of revolution T is measured in years.

Planet	T	d	T^2/d^3
Mercury	0.24	0.39	0.97
Venus	0.62	0.72	1.03
Earth	1.00	1.00	1.00
Mars	1.88	1.52	1.01
Jupiter	11.9	5.20	1.00
Saturn	29.5	9.54	1.00
Uranus	84.0	19.2	1.00
Neptune	165	30.1	1.00
Pluto	248	39.5	0.99

Kepler speculated that the forces which kept the planets in their orbits might be magnetic. The theory which held the day, however, before Newton appeared upon the scene, was the vortex theory of Descartes. It was supposed that the whole of space was filled with a fluid, and that whirls or vortices in this fluid, centred in the sun, swept the planets round in their orbits. One fundamental misconception embodied in the theory was that the planets required a steady application of force to keep them moving, whereas the only force required is one between each planet and the sun to change its direction of motion.

Newton's Principia

The time was now at hand for England to make her contribution to this record of astronomical discovery, and it was to be the crown of the whole. About 1645 a group of men interested in natural philosophy began to meet together, partly as a result of the influence of the scientific philosophy of Lord Bacon and partly as a distraction from the distressing state of the country in which the Civil War was then raging. From these small beginnings grew the Royal Society, receiving its Royal Charter in 1662 from Charles II, who, said Pepys, 'mightily laughed at them for spending time only in weighing air'. One of the tasks of Robert Hooke, the Curator of the Royal Society, was to provide three or four experimental demonstrations for each weekly meeting. Men like Isaac Newton, Robert Boyle, Christopher Wren, Edmund Halley and Samuel Pepys were Fellows.

The Fellows of the Royal Society were well acquainted with the work of Kepler, and about 1685 several of them, including Hooke, Wren and Halley, were convinced that a planet would move in an ellipse if it were attracted to the sun with a force which varied according to the inverse square of its distance away. But they could not prove it. Halley said that he would ask Newton if he could provide a proof. Newton at this time was Lucasian Professor of Mathematics at Cambridge, and seldom attended meetings of the Society, which were held mainly in London and Oxford.

Halley journeyed to Cambridge and was astonished to find that Newton had worked out a proof years before but had lost his papers. Halley soon realised that this proof, which was beyond the powers of any other members of the Royal Society, was but a small item in what Newton had already achieved. Newton was still upset by the controversy which resulted from the paper he had submitted to the Royal Society 13 years earlier on his experiments on the dispersion of light,

and was resolved to publish no more of his discoveries. 'He was of the most fearful, cautious and suspicious temper that ever I knew', said Whiston. Furthermore, having solved all the problems which arose out of his theory of gravitation he had lost interest in the subject—as he wrote in a letter to Hooke in 1679, 'my affection to philosophy being worn out'.

Halley set about the task of cajoling Newton to write what Lagrange later called 'the greatest production of the human mind', the *Principia* or *Mathematical Principles of Natural Knowledge*. Halley was not a rich man, but he agreed to defray the cost of publication himself. While the work was in preparation Hooke, also a man of genius and of a temperament even more difficult and explosive than that of Newton himself, made his usual claim that some of his ideas had been stolen. He wanted an acknowledgement of his priority in the discovery of the inverse square law. Newton, stung to the quick, at first refused to publish the third book of the *Principia*, its outstanding part, and wrote sarcastically to Halley, 'Now is not this very fine? Mathematicians that find out, settle and do all the business, must content themselves with being nothing but dry calculators and drudges; and another that does nothing but pretend and grasp at all things must carry away all the invention.' He was mollified, however, when the Royal Society took his part against Hooke.

The *Principia* was published in 1687. It was written in eighteen months, not from accumulated notes but re-created step by step, an almost superhuman feat. The mathematics was cast in geometrical form because of Newton's admiration for the geometers of the ancient world. It also gave him satisfaction to make the proofs difficult 'to avoid being baited by little smatterers in mathematics'. The result was unfortunate, as it impeded the future progress of English mathematics. 'Nobody since Newton has been able to use geometrical methods to the same extent for the like purposes', wrote Dr Whewell, 'and as we read the *Principia* we feel as when we are in an ancient armoury where the weapons are of gigantic size; and as we look at them we marvel at what manner of man he was who could use as a weapon what we can scarcely lift as a burden.'

The preface and the first book of the *Principia* deals with general definitions, states the laws of motion and formulates clearly, for the first time, the law of the composition of forces, proves Kepler's laws by assuming an inverse square law and deals with the laws of impact. It is, in fact, the first modern text-book of mechanics, nearly all original.

In the second book Newton is concerned to show that 'the hypothesis of vortices is utterly irreconcilable with astronomical phenomena and rather serves to perplex than explain the heavenly motions'. He deals therefore with motion in fluids and viscosity, 'which I conceive may be of use in the building of ships'. An account is given too of pendulums and wave motion, the argument being everywhere supported by a wealth of experiment, an unsurpassed combination of mathematical and experimental ingenuity.

Having laid down the principles of mechanics in the first two books, Newton goes on to make a spectacular application of them in the third book. He finds the masses of the planets in terms of the earth's mass, makes a celebrated guess that the specific gravity of the earth is between 5 and 6 (the modern value is 5.5), calculates the masses of the sun and the planets which have satellites, works out the flattening of the earth at the poles, shows that the pull on a spheroid does not act as its centre, as it does in the case of a sphere, and hence calculates the conical motion of the earth's axis known as the precession of the equinoxes (which Airy regarded as his most astonishing achievement). He also works out the main irregularities of the moon's motion due to the pull of the sun, gives the fundamental theory of the tides, and shows how to predict the return of comets by calculating their orbits.

Truly the lines of Lucretius at the base of his statue in Trinity are apt: 'Qui genus humanum ingenio superavit' ('who excelled the human race in power of thought').

Newton the man (1642-1727)

There was no indication among Newton's forebears of genius to come, nor did he show any outstanding signs of it in his boyhood. He was fond of constructing models, and it is known that he made a water clock and a model windmill.

In 1661 he went to Trinity College, Cambridge, where Isaac Barrow, the Lucasian Professor of Mathematics, recognised and encouraged his great powers and later resigned his chair in Newton's favour when Newton was only 27 years of age.

In 1665 the university was closed because of the plague and Newton went home to Lincolnshire. Here, during two years of complete quiet and leisure, his three greatest discoveries germinated, the theory of gravitation, the invention of the calculus, and the theory of the nature of white light. He was then 23 and 'in those days', he said, 'I was in

the prime of my age for invention and minded mathematics and philosophy more than at any time since'.

These discoveries were fully developed in the *Principia* and in his *Opticks*, the latter work not being published until he was 61 and Hooke was dead.

In 1692, at the age of 50, he suffered from a nervous breakdown. His friends, realising the unhealthiness of his life of solitary introspection in Cambridge, procured for him the post of Master of the Mint. He moved to London, became the most famous man of his day, was knighted by Queen Anne, and lived to a ripe old age. He was elected President of the Royal Society in 1703 and re-elected each year until his death in 1727. His authority in scientific affairs was absolute. He made no further scientific discoveries during this period, although his powers were in no way impaired.

When he left Cambridge for London in 1696 he packed up a box of manuscripts containing over a million words in his handwriting. These papers, on his death, passed to his niece and from her to the Portsmouth family; they are known as the Portsmouth Collection. They reveal his intense interest in theology, history and alchemy. Indeed, to the greatest of all mathematical physicists, mathematics and physics seem to have been but a side-line. The object of science, for Newton as for all the natural philosophers of the seventeenth century, was to demonstrate the divine will in nature and not to increase the material welfare of mankind, despite the picture drawn by Lord Bacon of the benefits of science in his fabulous island New Atlantis.

The box contained many pamphlets arguing against the orthodox belief in the Trinity. Newton abandoned this belief as a young man, but kept his disbelief a lifelong secret to avoid controversy and persecution. He was thus unable to take Holy Orders and hence to become Master of Trinity College.

The box contained also the records of his vast and secret researches in alchemy. So far from being the first completely modern man of the Age of Enlightenment, as Voltaire would have us believe, he was, says Lord Keynes, the last of the magicians. Having solved the riddle of the heavens by pure thought he conceived that the riddle of the whole universe could be grasped in a similar way if only he could find the clues. He looked for these clues in the measurements of Solomon's temple, in the Book of Revelation and in the cryptic verses and traditions of the alchemists handed down from Babylonian times, to the repetition of whose real and imaginary experiments he devoted much effort. 'At spring and the fall of the leaf', wrote his amanuensis,

Humphrey Newton, 'he used to employ about six weeks in his laboratory, the fire scarcely going out either day or night, he sitting up one night, and I another, till he had finished his chemical experiments, in the performances of which he was the most accurate, strict, exact. What his aim might be I was not able to penetrate into, but his pains, his diligence at these set times made me think he aimed at something beyond the reach of human art and industry'.

After 1696 the alchemical researches were not pursued further: 'And he looked very seldom, I expect, into the chest where, when he had left Cambridge, he had packed the evidences of what had occupied and so absorbed his intense and flaming spirit in his rooms and his garden and elaboratory between the Great Gate and the Chapel. . . . After his death Bishop Horsley was asked to inspect the box with a view to publication. He saw the contents with horror and slammed the lid. A hundred years later Sir David Brewster looked into the box. He covered up the traces with carefully selected extracts and some straight fibbing.'*

The contents of the box were put up for auction in 1936. Lord Keynes was able to purchase and reassemble about half of the papers, but the rest have been dispersed and have found their way to the United States of America.

The scientific revolution

Some historians now regard as the beginning of the modern world, not the Renaissance but the scientific revolution which culminated in the *Principia*. The physical synthesis of the seventeenth century, based on the concept of inertia, whereby nature came to be regarded as an automatic mechanism rather than the haunt of spirits or of occult forces, was the decisive break with medieval tradition and the fundamental new idea in the civilisation of the West.

The law of gravitation

Newton's law of gravitation states: *Every particle of matter attracts every other particle of matter with a force which varies directly as the product of the masses of the particles and inversely as the square of the distance between them, i.e.*

$$F = G \frac{m_1 m_2}{r^2},$$

* Lord Keynes, in *Newton Tercentenary Celebrations*, Cambridge University Press 1947.

where F is the force exerted on two masses m_1 and m_2 at a distance r apart and G is a universal constant known as the *constant of gravitation*. The force of attraction between two bodies is found by experiment to be independent of the material of the bodies or of their physical states, such as temperature.

Newton proved the law of gravitation from Kepler's laws (p. 130). From Kepler's first law it can be shown that the force between a planet and the sun must vary inversely as the square of their distance apart. From the second law it follows that the angular momentum of a planet is constant, and hence that the only force acting upon it is one towards the sun. From the third law it can then be deduced that the force between each planet and the sun is proportional to the product of their masses.

The mathematics of these deductions is too difficult to be given here. However, some insight may be gained from the following simple deduction of Kepler's third law for the special case of a circular orbit assuming the truth of the law of gravitation. Consider a body of mass m moving in a circle of radius r under the gravitational attraction of a body of mass M at the centre of the circle (Fig. 127), and suppose that the period of revolution is T .

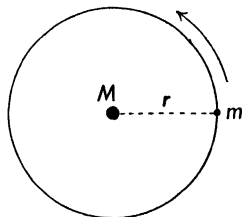


Fig. 127

Gravitational attraction = Centrifugal force

$$\begin{aligned} \text{i.e.} \quad G \frac{Mm}{r^2} &= m \left(\frac{2\pi}{T} \right)^2 r. \quad \left(\frac{2\pi}{T} = \text{angular velocity} \right) \\ \therefore \frac{T^2}{r^3} &= \frac{4\pi^2}{GM}. \end{aligned}$$

The value of T^2/r^3 is constant for all values of m and r since $4\pi^2/GM$ is a constant.

The mass of the earth

A sphere of matter attracts other bodies as though its whole mass were concentrated at the centre of the sphere.

A body of mass m on the earth's surface is attracted by the earth with a force equal to its weight mg . If M is the mass of the earth and R its radius,

$$mg = G \frac{mM}{R^2}.$$

$$\therefore g = G \frac{M}{R^2}.$$

The value of G can be obtained from experiment as will be described later, and hence, since the radius of the earth R is known, the mass of the earth M can be calculated. Taking

$$G = 6.67 \times 10^{-8} \text{ dyne cm.}^2 \text{ gm.}^{-2},$$

$$g = 981 \text{ cm. sec.}^{-2},$$

$$R = 6.37 \times 10^8 \text{ cm.,}$$

$$\therefore M = \frac{981(6.37 \times 10^8)^2}{6.67 \times 10^{-8}} \text{ gm.}$$

$$= 5.96 \times 10^{27} \text{ gm.}$$

$$= 5.87 \times 10^{21} \text{ tons.}$$

The density of the earth can be calculated by dividing its mass by its volume and the value is $5.52 \text{ gm. cm.}^{-3}$

The motion of the moon

Newton first tested his theory of gravitation by the motion of the moon. Suppose the orbit of the moon is a circle of radius R_1 (actually it is an ellipse which is nearly a circle), and that its period of revolution is T . The angular velocity of the moon is $2\pi/T$, and hence its acceleration towards the centre of its orbit is $(2\pi/T)^2 R_1$. But the acceleration of a body at the surface of the earth is g . By the law of gravitation,

$$\frac{\text{Acceleration at moon}}{\text{Acceleration at earth's surface}} = \frac{(\text{Radius of earth})^2}{(\text{Radius of moon's orbit})^2}$$

$$\text{i.e.} \quad \frac{\left(\frac{2\pi}{T}\right)^2 R_1}{g} = \frac{R^2}{R_1^2},$$

where R is the radius of the earth. The theory may be checked by calculating each side of this equation from the following data

$$T = 27.32 \text{ days,}$$

$$R = 238,800 \text{ miles (mean value, centre of earth to centre of moon),}$$

$$g = 32.17 \text{ ft. sec.}^{-2} \text{ (at sea level latitude } 45^\circ),$$

$$R = 3959 \text{ miles (mean value),}$$

$$\frac{\left(\frac{2\pi}{T}\right)^2 R_1}{g} = \frac{\left(\frac{2\pi}{27.32 \cdot 24.60 \cdot 60}\right)^2 \cdot 238,800 \cdot 1760.3}{32.17} = 2.777 \times 10^{-4},$$

$$\frac{R^2}{R_1^2} = \frac{3959^2}{238,800^2} = 2.748 \times 10^{-4}.$$

The two sides of the equation agree to 1 part in 100.

The masses of the sun and the planets

Let E , S , M be the masses of the earth, sun and moon respectively. Assume that the orbits of the earth round the sun and of the moon round the earth are circles, that the radii of these orbits are R_1 and R_2 and the periods of revolution are T_1 and T_2 respectively.

Gravitational attraction = Centrifugal force

$$G \frac{ES}{R_1^2} = E \left(\frac{2\pi}{T_1} \right)^2 R_1,$$

$$G \frac{ME}{R_2^2} = M \left(\frac{2\pi}{T_2} \right)^2 R_2.$$

Dividing,

$$\frac{S}{E} = \frac{R_1^3 T_2^2}{R_2^3 T_1^2}.$$

Hence the mass of the sun in terms of the mass of the earth may be determined provided that R_1 , R_2 , T_1 and T_2 are known. Similarly, the mass of any planet which has a satellite can be calculated in terms of the mass of the sun.

The relative masses of the sun and the planets are shown in the following table:

Name	Mass Earth=1	Mean density	
		Earth=1	Water=1
Sun	329,390	0.25	1.39
Mercury	0.04	0.70	3.8
Venus	0.81	0.94	5.20
Earth	1.000	1.00	5.527
Mars	0.106	0.71	3.90
Jupiter	314.50	0.25	1.36
Saturn	94.07	0.12	0.63
Uranus	14.40	0.24	1.34
Neptune	16.72	0.23	1.28

The discovery of Neptune

A triumph of the theory of gravitation which produced an impression among the non-scientific public, perhaps as profound as any other discovery in science, was the prediction of the existence of the planet Neptune from calculations based on the disturbance of the orbit of Uranus. In 1845 Adams, a young English mathematician, conjectured that these disturbances were due to the effect of an unknown planet whose position he calculated. Possibly from scepticism, Airy, the Astronomer Royal, made no attempt to look for the planet, while Challis, the director of the Cambridge Observatory, who actually observed an unknown star with the appearance of a disc, did not trouble to verify that it was the planet predicted by Adams. The

following year a French mathematician Leverrier made the same calculation and communicated his results to the Berlin observatory. The planet was at once found in the exact place predicted.

History was repeated in 1930 when the planet Pluto was discovered by American astronomers from perturbations of the orbit of Neptune.

The tides

The tides are caused by the attraction of the moon and of the sun.

Fig. 128 shows the effect of the moon on the oceans. There is a high tide at the point *A* nearest to the moon and also at the point *B* farthest from her. The mass of the earth may be considered as concentrated

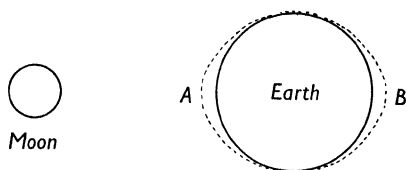


Fig. 128

at its centre and its attraction by the moon is counterbalanced by centrifugal force. At *A* the moon's attraction on the oceans is greater than the centrifugal force, while at *B* it is less; hence there is a resultant force on the oceans, at *A* towards the moon, and at *B* away from her.

As the earth rotates the positions of high tide move over its surface because they remain at the points nearest to, and farthest from, the moon. Thus two high tides and two low tides occur every 24 hr. 50 min., the interval which elapses between two successive lunar transits of the meridian (the moon is revolving round the earth as well as the earth rotating, which accounts for the interval not being 24 hr.).

Three-fifths of tidal action is caused by the moon and two-fifths by the sun. At full and at new moon the earth, sun and moon are collinear so that the solar and lunar effects reinforce each other; the tides are then at their highest and are called spring tides. When the sun and the moon are pulling at right angles the tides are at their lowest and are known as neap tides.

The earth's gravitational field

The concept of a gravitational field is quite as legitimate, if not quite so useful, as are the concepts of magnetic and electric fields. The strength or intensity of a gravitational field is defined as the force exerted on unit mass, and hence is equal to *g*.

Since an inverse square law holds in each case, many of the propositions and proofs in electrostatics apply also to gravitation. The charge on the surface of a sphere behaves at all external points as though it were concentrated at the centre of the sphere, and the electric field inside a hollow charged sphere is zero. Similarly, the mass of a spherical shell behaves as though it were concentrated at the centre of the shell and the gravitational field inside a hollow shell is zero. The latter proposition merely implies that a particle at any point inside a spherical shell is attracted equally in all directions.

There is also an exact parallel between electrostatic and gravitational potential. The difference in gravitational potential between two points is defined as the work done in taking unit mass from one point to the other; and zero potential is defined as that at an infinite distance from all attracting matter. The electrostatic potential due to a point charge q , at distance r in *vacuo* is q/r ; similarly, the gravitational potential due to a particle of mass m at distance r is $G \frac{m}{r}$.

The fact that electrostatic potential depends on the nature of the dielectric has led some investigators to look for a similar effect in gravitation. The gravitational attraction between two spheres was measured when they were separated by air, and also when slabs of different materials were interposed between them, but no effect was observed.

The velocity of escape

The moon has no atmosphere because its gravitational field is not sufficiently strong to prevent gases from escaping. For a similar reason the light gases, hydrogen and helium, whose thermal velocities are much higher than those of oxygen and nitrogen, are very rare in the earth's atmosphere.

The velocity with which a rifle bullet must be fired vertically upwards so that it never comes down again (ignoring air resistance), called the *velocity of escape*, can be calculated as follows.

Let m be the mass of the bullet and M the mass of the earth.

Force on bullet when it is at distance x from the centre of the earth

$$= G \frac{mM}{x^2}.$$

$\therefore$ Work done against the gravitational field when bullet moves upwards a distance δx

$$= G \frac{mM}{x^2} \delta x.$$

∴ Total work done while bullet escapes

$$= \int_R^\infty G \frac{mM}{x^2} dx, \quad \text{where } R = \text{radius of earth,}$$

$$= G \frac{mM}{R}.$$

(This is, of course, the mass of the bullet times the gravitational potential at the earth's surface.)

Let v be the velocity of escape, i.e. the initial velocity of the bullet.
Initial kinetic energy of bullet = work done in escaping.

$$\frac{1}{2}mv^2 = G \frac{mM}{R},$$

$$v^2 = \frac{2GM}{R}.$$

By substituting values of G , M , and R in this expression it can be shown that $v = 36,700$ ft. per sec.

Determination of G

The methods which have been employed to determine the gravitational constant G fall into two classes: (i) large-scale methods involving the attraction of a mountain or the change in weight of a body when taken down a mine shaft, (ii) laboratory methods. The latter are much the more accurate, but the former have considerable historical interest.

Bouguer's mountain experiment

The first attempt to measure G was made in 1740, by the Frenchman Bouguer, in Peru. He selected Chimborazo, a mountain 20,000 ft. high in the Andes, and attempted to measure how much it deflected a plumb-line from the vertical. In a sense, he was really determining the mass of the earth, since he was comparing the attraction of the mountain, whose mass he estimated from its dimensions and density, with the attraction of the earth. But if the mass of the earth is determined, G can be calculated and vice versa (see p. 136).

Bouguer's first station was on the slopes of Chimborazo just above the snow-line, and his second station was several miles away, at about the same altitude and on the same parallel of latitude, sufficiently far away from the mountain for its attraction to be negligible (Fig. 129). He determined the position of the true vertical by observing through a telescope a star which was directly overhead; at the second station the

plumb-line hung parallel to the telescope, and at the first it was deflected from it by an angle θ , which had a value of about 8 sec. If M and m are the masses of the mountain and of the plumb-bob respectively, d is the distance of the plumb-bob at the first station from the centre of gravity of the mountain, G is the gravitational constant and g is the acceleration due to gravity at that altitude and latitude, then the forces acting on the plumb-bob are as shown in Fig. 129,

$$\tan \theta = \frac{\frac{GMm}{d^2}}{mg} = \frac{GM}{d^2 g}.$$

As θ , M , d and g are known, G can be calculated.

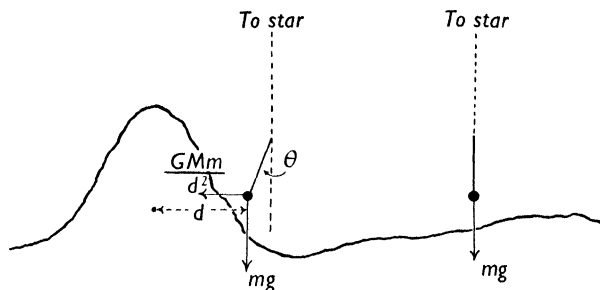


Fig. 129

Bouguer made his observations exposed to intense cold and a ferocious wind, and was unable to do more than demonstrate that the method was practicable.

Maskelyne's mountain experiment

In 1774 Maskelyne, the Astronomer Royal, repeated Bouguer's experiment on Shiehallion, a mountain in Scotland. A more elaborate survey than had been possible in the Andes was made to find the mass and position of the centre of gravity of the mountain. Observations were made at two stations on the north and south slopes, so that the angle between the two positions of the plumb-line gave twice the deflection caused by the mountain after due allowance had been made for the angle between the verticals at the two places (the directions in which the plumb-lines would have hung if the mountain had not been attracting them) due to the difference in latitude.

Maskelyne's result was of the right order but it was not very accurate. It is now known that there is a fundamental defect in the mountain method. Sir George Everest and Archdeacon Pratt showed that a

plumb-bob in the Ganges plain was deflected much less by the Himalayas than would be expected from their size and density. Pratt explained this by what is called isostasy. Mountains float in hydrostatic equilibrium on rocks below the earth's crust rather like an iceberg floating in the sea. They have a subterranean root of light rock floating in denser rocks and, as a result, it has even been known for a plumb-line to be deflected in what is apparently the wrong direction. Nowadays the method is used to investigate the composition of the earth's crust, the value of G being assumed.

Airy's mine experiment

In 1854 Airy determined G by measuring the acceleration due to gravity at the top and at the bottom of the shaft of Harton coal-mine, Sunderland, by the usual method of timing the swing of a pendulum.

The acceleration due to gravity at the top of the shaft, g_1 , is given by

$$g_1 = G \frac{M}{R^2},$$

where M is the mass of the earth and R is its radius (see p. 136).

Suppose h is the depth of the shaft and M' is the mass of the spherical shell, thickness h , shown in Fig. 130. The gravitational field due to the shell, inside the shell, is zero (see p. 140). Hence the gravitational field at the bottom of the shaft, g_2 , is due solely to a sphere of mass $M - M'$ and radius $R - h$,

$$g_2 = G \frac{M - M'}{(R - h)^2}.$$

Dividing,

$$\frac{g_2}{g_1} = \frac{M - M'}{M} \frac{R^2}{(R - h)^2}.$$

M' can be calculated, knowing R , h and the mean density of the shell. Thus M , and hence G , can be determined.

The Cavendish experiment

The first determination of G by a laboratory method was made by Cavendish in 1798 with apparatus designed by Michell, who died before he could use it. Two small lead spheres, each of mass m , were supported at the ends of a long wooden rod which was suspended by a fine torsion

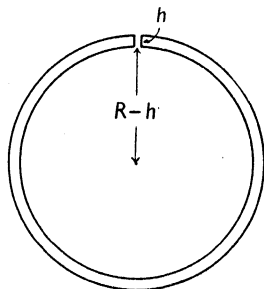


Fig. 130

wire made of copper (Fig. 131 (a)). Two large lead spheres, each of mass M , were placed near to the small spheres, as in Fig. 131 (b), and by their attraction caused the rod to deflect until the couple set up by the twisting of the torsion wire counteracted their effect. The force of attraction between each small and large sphere was, of course, minute, but the controlling couple was also made very small by using a torsion wire which was long and thin. Suppose that, when equilibrium was attained, the distance between the centres of each pair of large and small spheres was d and the deflection of the torsion rod was θ ; suppose

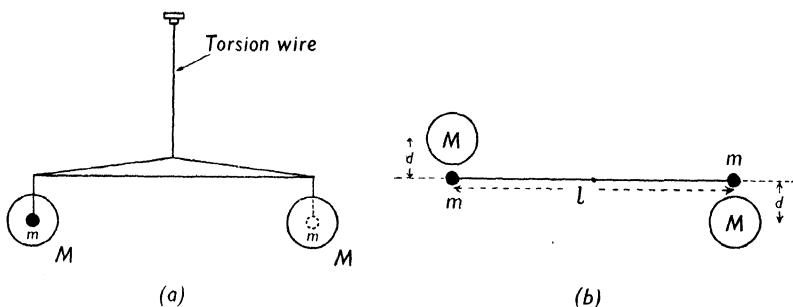


Fig. 131

l was the length of the torsion rod and c the restoring couple in the wire per unit angle of twist,

$$G \frac{mM}{d^2} l = c\theta.$$

The angle actually measured was 2θ , because the large spheres were moved so as to be on the opposite sides of the small spheres, thereby causing an equal deflection in the opposite direction.

The value of c was obtained by measuring the period of vibration T of the torsion rod and its attached spheres; $T = 2\pi \sqrt{\frac{I}{c}}$, I being the moment of inertia of the vibrating system, which could be calculated from the masses and dimensions of its parts.

The apparatus was made on a big scale; the torsion rod was 6 ft. long, the torsion wire over 3 ft. long, and the large spheres weighed over 3 cwt. each.

Cavendish installed it in an outhouse in his garden on Clapham Common. One of the chief experimental difficulties was the vibration set up by convection currents; even while the measurements were being taken the small spheres were never still, and this despite the fact that Cavendish remained outside the outhouse making his observations with

a telescope through a hole in the wall. To measure the deflection 2θ a fixed ivory scale was situated just beyond the arc of swing of the torsion rod, and a vernier scale was carried on each end of the rod. It was unnecessary to enter the outhouse to move the massive 3 cwt. spheres; they were hung from a strong cross-bar which could be swivelled round by pulling on a rope hanging outside the outhouse.

The accuracy of Cavendish's value for G was remarkable (see table on p. 148) considering the date at which it was obtained, although it is amusing to recall that he made a slip in calculating the mean of his twenty-nine results.

Boys's modification of the Cavendish experiment

In 1895 Boys repeated the Cavendish experiment. By reducing the size of the apparatus he was able to reduce convection effects very considerably and hence to increase the accuracy. The experiment was performed in a

vault under the Clarendon Laboratory, Oxford, and a complicated system of curtains and screens was devised to prevent air currents.

The torsion rod was a strip of mirror 0.9 in. long and was suspended from a fine quartz fibre (Fig. 132). Two small gold spheres hung from the torsion rod at different levels and were attracted by larger lead spheres. Had the four spheres all been at the same level each gold ball would have been attracted appreciably by both lead spheres owing to the shortness of the torsion rod. The quartz fibre, torsion rod and gold spheres were enclosed in a glass tube of diameter $1\frac{1}{2}$ in. which could be exhausted of air—again a device for preventing draughts. The deflection was measured by observing through a telescope the reflection of a scale in the mirror which formed the torsion rod. Deflections in opposite directions were obtained by moving the lead spheres to the opposite sides of the gold spheres, as in the Cavendish experiment.

The change in the sensitivity of the apparatus, as a result of the

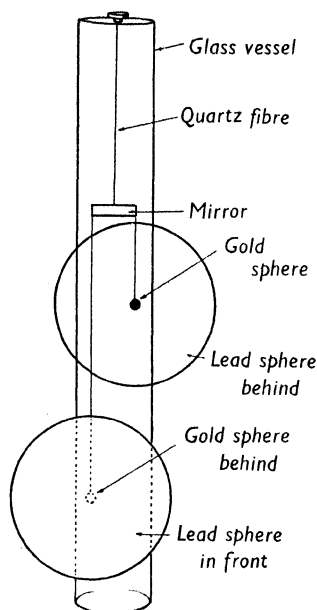


Fig. 132

change in dimensions, is worth careful consideration. It was shown, in the account of the Cavendish experiment, that

$$c\theta = G \frac{Mm}{d^2} l$$

and

$$T = 2\pi \sqrt{\frac{I}{c}}.$$

Taking $I = 2m\left(\frac{l}{2}\right)^2$ (ignoring the mass of the torsion rod),

$$c = \frac{4\pi^2 I}{T^2} = \frac{2\pi^2 ml^2}{T^2}.$$

$$\begin{aligned} \therefore \theta &= G \frac{Mm}{d^2} l \times \frac{T^2}{2\pi^2 ml^2} \\ &= G \frac{T^2}{2\pi^2} \frac{M}{d^2} \frac{1}{l}. \end{aligned}$$

This is the relation between G and the quantities which are actually measured.

The maximum desirable value of T , employed by both Boys and Cavendish, is of the order of 5 min.; if the period is longer the swings become so sluggish as to reduce, rather than increase, the accuracy of measurement.

In the equation $\theta = G \frac{T^2}{2\pi^2} \frac{M}{d^2} \frac{1}{l}$, consider the effect on θ of reducing the dimensions of the apparatus by say $\frac{1}{2}$, subject to the condition that T must not change. M is reduced by $(\frac{1}{2})^3$ since it is proportional to the (radius)³, while d^2 is reduced by $(\frac{1}{2})^2$ and l by $\frac{1}{2}$. Thus the value of θ is unaltered. Boys, however, did not alter all the dimensions of the apparatus in the same ratio:

	Cavendish	Boys
Diameter of small spheres	2 in. (lead)	$\frac{1}{4}$ in. (gold)
Diameter of large spheres	12 in.	$4\frac{1}{2}$ in.
Length of torsion rod	6 ft.	0.9 in.

The mass of the small spheres, m , does not appear in the final expression for the result, but it effects T . The advantage of gold over lead is due to its higher specific gravity, 19.3 as against 11.3. Thus gold spheres are smaller than lead spheres of the same mass and enable d , the distance between the centres of the large and small spheres, to be reduced.

The mass of the large spheres in the Boys experiment was about 18 lb. as compared with 350 lb. in the Cavendish experiment. But the

loss of sensitivity was not so serious as might appear because the reduction in the radii of the spheres enabled d to be reduced. The radii of the spheres were reduced by about one-third and hence the term M/d^2 in the final result was reduced by $(\frac{1}{3})^3 \div (\frac{1}{3})^2 = \frac{1}{3}$, taking the distance d as being approximately the same as the radius of a large sphere.

The most striking change of dimensions was in the length of the torsion rod. This had a greater effect on the moment of inertia of the suspended system, which is proportional to l^2 , than on the deflecting gravitational couple, which is proportional to l , and hence enabled the controlling couple due to torsion to be reduced far more than the deflecting couple was reduced. In fact $\theta \propto 1/l$, as was shown in the above expression for θ in terms of the measured quantities. Thus the fundamental reason for the superiority of Boys's experiment lay in the remarkable properties of quartz, great tensile strength combined with a small rigidity.* Boys was able to use extremely fine quartz fibres of diameter about 0.0125 mm., which were strong enough to support the torsion rod and gold spheres, and which offered a very small resistance to torsion. A further advantage of quartz is that it is perfectly elastic and returns at once to its zero position when the twisting couple is removed.

Heyl's experiment

The torsion-balance method of measuring G was taken to the limit of its accuracy by Boys and has now been superseded by a vibration method used first by Braun and later in 1927 by Heyl, whose value is regarded as the most accurate so far obtained.

The principle of the method is shown in Fig. 133. The period of vibration of a light rod, suspended from a torsion wire and carrying at its ends two small spheres each of mass m , is measured when the rod lies (a) perpendicular to, (b) along, the line joining two large masses M and M . The controlling couple in the position of Fig. 133 (a) is due to the torsion of the wire only, whereas in Fig. 133 (b) it is due to the torsion of the wire and also to the gravitational attraction between the masses m and M .

Suppose I is the moment of inertia of the vibrating system and c is the restoring couple per unit angle, due to torsion; the time of swing, T_1 , in Fig. 133 (a) is given by

$$T_1 = 2\pi \sqrt{\frac{I}{c}}.$$

* The general meanings of the terms 'tensile strength' and 'rigidity' are obvious; for their definitions see pp. 155 and 161.

In the equilibrium position of Fig. 133 (b) there is a gravitational force on each of the small masses m of $G \frac{Mm}{d^2}$, and this force changes in magnitude and direction as the rod vibrates. If, as an approximation, it is assumed that the force does not change appreciably in magnitude and direction, each mass m may be likened to the bob of a pendulum and the restoring couple per unit angular displacement (for the two masses) is $2G \frac{Mm l}{d^2}$ (see p. 111). Thus T_2 , the time of swing, is given by

$$T_2 = 2\pi \sqrt{\frac{I}{c + G \frac{Mm}{d^2} l}}$$

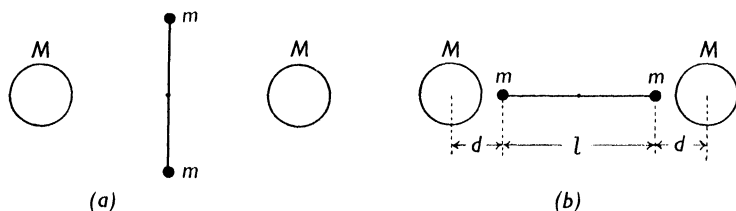


Fig. 133

This expression is, of course, only a rough approximation, but it indicates how c can be eliminated from the equations for T_1 and T_2 and G calculated.

Heyl used small spheres of gold, platinum and optical glass, each of mass about 50 gm., hanging from the ends of a light aluminium rod suspended from a very fine torsion wire of tungsten. He preferred tungsten to quartz because the latter is liable to break, unexpectedly and unaccountably. The large masses were cylinders of forged and machined steel, of mass about 66 kg. The experiment was performed in the constant-temperature room of the American Bureau of Standards, 35 ft. below ground.

Summary of values obtained for G

Date	Experimenter	Type of experiment	G (c.g.s. units)
1775	Maskelyne	Mountain	7.4×10^{-8}
1798	Cavendish	Torsion balance	6.754
1854	Airy	Mine	5.7
1895	Boys	Torsion balance	6.658
1927	Heyl	Vibration	6.670 *

* Birge estimates the probable error as 0.005, thus giving as the best value for $(6.670 \pm 0.005) \times 10^{-8}$ dyne cm.² gm.⁻²

Relativity and gravitation

The reader is probably aware that Einstein, starting with simple and innocuous-looking premises such as that all motion is relative, has made strange and revolutionary deductions about space and time. Newton postulated an absolute space and an absolute time which are the same for all observers, but Einstein has shown that space, time and mass are different for different observers who are moving with respect to each other. The magnitudes of the effects, it is true, are negligible unless the relative velocity of the observers is of the order of the velocity of light. But what becomes of the law of gravitation, $F = G \frac{m_1 m_2}{r^2}$, if no precise value can be assigned to m_1 , m_2 and r ?

It is well known to every aviator that it is impossible to devise an instrument which will show the direction of the vertical in an aeroplane. Suppose, for example, a plumb-line is used. Whenever the aeroplane accelerates, the plumb-line is drawn from the vertical as though it were being attracted in another gravitational field besides that of the earth. Einstein's general theory of relativity is based on his principle of equivalence; 'A gravitational field of force at any point of space is in every way equivalent to an artificial field of force resulting from acceleration, so that no experiment can possibly distinguish between them.'

To an aviator in the cockpit of a stationary or uniformly moving aeroplane a beam of light appears to pass in a straight line across the cockpit, but if his aeroplane is accelerating the light appears to bend (or would do so if the effect were not so minute). By the principle of equivalence, bending of light should occur also in a gravitational field, and Einstein predicted a slight bending of light when it passed very close to a large mass such as the sun. This was confirmed in 1919 by observations, during an eclipse of the sun, of light passing very close to the sun from a convenient star.

If light follows a curved path in space it is legitimate to regard space itself as curved instead of using the concept of a gravitational field to account for the path of the light. According to Newton's laws of motion, bodies depart from their motion in a straight line when they are acted upon by gravitational forces; according to Einstein's principle of least distance a body tends to move along the shortest path, called a geodesic, in the four-dimensional continuum, composed of space and time, and there is no need to postulate gravitational forces. The planets follow their elliptic orbits, not because they are being attracted to the

sun, but because the sun, by virtue of its great mass, has caused space in its vicinity to be curved.

Thus Einstein has replaced the theory of gravitation by a new geometry of space. His mathematical theory proved successful in explaining the precession of the orbit of Mercury, where Newton's theory of gravitation failed completely. The orbit of Mercury slowly precesses (Fig. 134), with a period of 3,000,000 years, and this could be explained, on the theory of gravitation, only by the influence of another planet, for which a name, Vulcan, was chosen, but which has never been found. Einstein showed that the orbits of all the planets precess but the effect is inappreciable, except in the case of Mercury which is nearest to the sun, and hence in the most 'curved' part of the continuum.

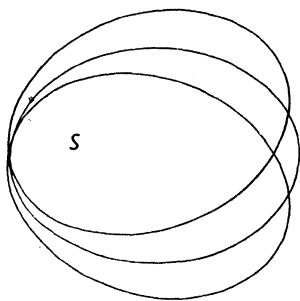


Fig. 134. Ellipse precession
(one focus fixed)

The theory of gravitation is still used for all practical purposes, including the calculations for the *Nautical Almanac*.

SUMMARY

The law of gravitation may be stated mathematically as follows:

$$F = G \frac{m_1 m_2}{r^2}.$$

It was deduced from Kepler's laws. By equating gravitational attraction to centrifugal force it may be tested by the motion of the moon, and the masses of the sun and of the planets may be found in terms of the mass of the earth.

The mass of the earth may be calculated, if G is determined experimentally, from the equation

$$g = G \frac{M}{R^2},$$

where M is the mass of the earth and R is its radius.

The most accurate determinations of G were those of Boys and Heyl.

QUESTIONS

[Take $g=981$ cm. sec.⁻² unless otherwise stated.]

1. Explain what is meant by the gravitation constant and describe a method of measuring it.

Two spherical balls of masses 10 kg. and 50 kg. respectively, when placed with their centres 30 cm. apart, attract each other with a force equal to the weight of $\frac{1}{24}$ milligram. If the radius of the earth, assumed spherical, is 6×10^8 cm., find the mean density of the earth. (C. S.)

2. Assuming that the earth describes a circular orbit of radius 1.5×10^8 km. about the sun in 365 days, calculate the mass of the sun.

($G=6.66 \times 10^{-8}$ c.g.s. unit.) (C. S.)

3. Give a brief survey of the evidence that led Newton to formulate his law of gravitation.

A small satellite revolves round a planet of mean density 10 gm. per c.c., the radius of the satellite's orbit being *very little* greater than the radius of the planet. If the constant of gravitation is 6.66×10^{-8} c.g.s. unit, calculate the time of revolution of the satellite. (N.)

4. The distance of Jupiter from the sun is $5.2 \times$ the distance of the earth from the sun. Assuming that both orbits are circular, find the time Jupiter takes to complete its orbit if the earth takes 1 year. (B.)

5. A given pendulum is moved from sea-level to a height of 4 miles. What is the percentage change in the period of oscillation? The radius of the earth is 4000 miles. (O.)

6. Compare the value of the acceleration due to gravity on the surface of Mars with its value on the earth's surface. (The radius of Mars is $0.53 \times$ the radius of the earth, and its mean density is $0.71 \times$ the mean density of the Earth.)

What assumption have you made in applying the law of gravitation in this problem? (N.)

7. Assuming that the diameter of the sun is 109 times that of the earth, and that the mass of the sun is 330,000 times that of the earth, compare the values of solar gravity on the sun's surface and terrestrial gravity on the earth's surface. (O. & C.)

8. Sketch a form of apparatus which has been used to determine the constant G in Newton's law of gravitation, and indicate the quantities which must be measured so that the value of the constant may be calculated.

Calculate the value of G , assuming that the earth is a homogeneous sphere of radius 4000 miles and density 342 lb. per cu. ft. and that the acceleration due to gravity at its surface is 32 ft. per sec. per sec. State the units in which your result is expressed. (N.)

9. Assuming that the earth is at rest and the moon describes a circular orbit round it, find the radius of the orbit given that the moon takes 28 days (2.4×10^6 sec.) to complete the orbit, that the radius of the earth is 6400 km. and that the acceleration of gravity at the earth's surface is 1000 cm. per sec. per sec. (C. S.)

10. How has the mean density of the earth been determined?

A spherical planet of uniform density makes one revolution about its own axis per day. What is the minimum density it must have to prevent the loss of loose particles from the equator?

($G = 6.66 \times 10^{-8}$ c.g.s. unit.) (O. S.)

11. Describe Boys's method of measuring the gravitation constant G , and show how the result is deduced from the observations.

Two pieces of apparatus for this experiment are similar in all respects, but the linear dimensions of the torsion bar, the attracting and attracted masses and their distance apart are n times as large in one as in the other. The lengths of the two suspensions are adjusted so that the times of vibration of the torsion bars with their attached masses are the same. Show that the deflection of the torsion bar will be the same in each case. (C. S.)

12. A vertical tungsten fibre supports a horizontal rod of length 4 cm. The rod carries a 50 gm. sphere immediately below one end and an identical sphere some 20 cm. below the other end. The period of oscillation of this system is 200 sec. A 5 kg. sphere can be placed close to the upper mass or removed completely.

Estimate the deflection of the suspended system resulting when the apparatus is used to determine G , and describe with experimental detail how the determination could be made.

(Assume $G = \frac{2}{3} \times 10^{-8}$ c.g.s. unit, radius of 5 kg. sphere = 5 cm., radius of 50 gm. sphere = 1 cm., mass of the rod = 9 gm.) (B.)

13. Prove that a uniform spherical shell of matter exerts no gravitational attraction in its interior. Give an expression for the gravitational attraction due to the shell at an external point.

Assuming that the earth is built up of concentric shells, each of uniform density, show that it is possible to determine the gravitational constant (G) and the value of the earth's mean density (Δ) by pendulum experiments carried out at the top and bottom of a deep-mine shaft.

A mass of 5 kg. is weighed on a balance at the top of a tower 20 metres high. The mass is then suspended from the pan of the balance by a fine wire 20 metres long and is reweighed. Find the change in weight in milligrams. Assume that the radius of the earth is 6330 km. (N.)

14. What is meant by the gravitational potential? How does it vary with the distance from the centre of the earth? What initial velocity would be required to project a body beyond the attractive force of the earth? (Radius of earth is 6.4×10^8 cm.) (C. S.)

15. Prove that the least velocity with which a particle must be projected from the surface of a planet of radius R and density ρ in order that it may escape completely is

$$R \sqrt{\left(\frac{8\pi}{3} G \rho\right)},$$

where G is the gravitational constant. Calculate this velocity in the case of the moon from the following data:

Mean density of earth = 5.52 gm. per c.c.

Mean density of moon = 3.36 gm. per c.c.

Mean radius of earth = 6380 km.

Mean radius of moon = 1740 km.

Acceleration of gravity at earth's surface = 980 cm. per sec. per sec. (O. S.)

16. Suppose that the law expressing the gravitational attraction between two particles of mass M_1 and M_2 distance R apart had been $F = K \frac{M_1^2 M_2^2}{R}$, discuss what seem to you the most interesting consequences of such a law. (C. S.)

17. Show that the gravitational force due to a sphere of matter of uniform density is, at all points outside it, the same as that due to a particle of mass equal to that of the sphere situated at its centre. Show further that a hollow shell of uniform density bounded by two concentric spheres exerts no gravitational attraction at any point within it. Hence show that if a smooth straight tunnel be constructed between any two points on the surface of the earth (assumed to be a sphere of uniform density) a particle released at one end will execute simple harmonic oscillations in the tunnel with a period independent of the length of the tunnel. Calculate the time taken by the particle in travelling from one end of the tunnel to the other.

(Mean density of earth = $5.52 \text{ gm. cm.}^{-3}$ Gravitation constant

$$G = 6.66 \times 10^{-8} \text{ cm.}^3 \text{ gm.}^{-1} \text{ sec.}^{-2}, \pi = 3.1416.)$$

(C. S.)

18. What is meant by the dimensions of a physical quantity? What are the dimensions of the constant of gravitation G ? A sphere of incompressible liquid is distorted from the spherical form and released. Deduce by the method of dimensions an expression for the period of its subsequent oscillations, assuming that the only forces which need to be considered arise from its own gravitational attraction.

(C. S.)

CHAPTER 7

ELASTICITY

THE behaviour of a piece of metal when subjected to a stretching force is of great interest to engineers. Fig. 135 represents the load-extension curve for mild steel, a material much used for constructional purposes, which may be in the form of a short bar, called a test-piece, or of a wire. The portion OA is a straight line, showing the extension to be proportional to the load; over this range the metal returns to its original length when the load is removed and the deformation is said

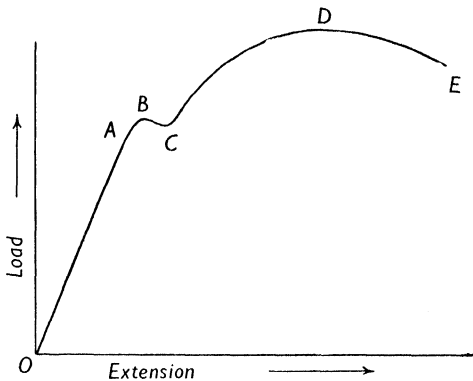


Fig. 135

to be *perfectly elastic*. Beyond A the metal acquires a permanent set, i.e. it does not return to its original length when the load is removed and the deformation is said to be *plastic*. A is known as the *elastic limit*;^{*} the extension there is of the order of $\frac{1}{1000}$ th of the original length.

B is called the *yield point*. When using the testing machine in Figs. 136 and 137, in which the load is applied by a hydraulic ram and determined by measuring the pressure of the liquid in the straining cylinder, there is actually a well-marked drop in the load at the yield point,

^{*} The point at which the graph ceases to be a straight line, known as the limit of proportionality, occurs shortly before the elastic limit, but the two points are so close together that they are usually regarded as being identical.

owing to the rapidity with which the material expands over the range *BC*. The test-piece undergoes a visible change. If it is rough the scale flakes off, and if it has a polished machined surface the latter becomes dull and frosted. Some kind of internal rearrangement seems to be taking place inside the metal and, after its momentary weakness over the range *BC*, it hardens and stands up to an increased load over the range *CD*, although there is no proportionality between load and extension.

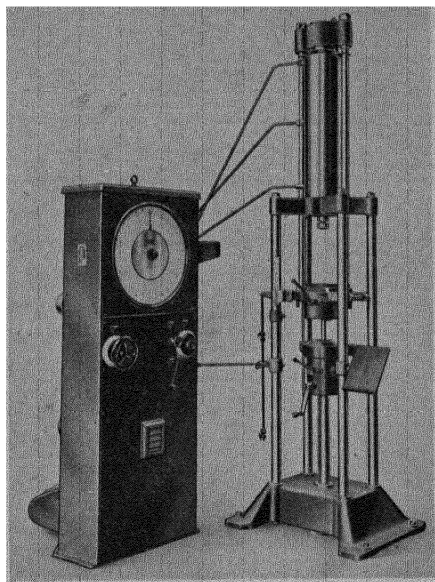


Fig. 136. Hydraulic, self-indicating, Universal Testing Machine. The load is obtained by measuring the pressure of the fluid in the straining cylinder, top right, and is indicated by the pointer on the dial.

As the metal is stretched its area of cross-section decreases and this change is constant throughout its length until *D* is reached. It then develops a 'neck' or 'waist' in one or more places, where it stretches rapidly because of the weakness caused by the continually diminishing cross-section, and fracture, represented by *E* on the graph, occurs at one of these necks. The load at *D*, the highest that can be sustained, divided by the original area of cross-section, is called the *tensile strength* or *breaking stress*.

Brass and the non-ferrous metals give a slightly different curve, with no kink *BC* in the curve, and hence no well-defined yield point.

Young's modulus

While the interest of the engineer is primarily focused on the yield point and tensile strength of materials (which form part of the study of the Strength of Materials), the interest of the physicist tends to be confined to the range *OA* in Fig. 135, that is, to the study of elasticity.

The concepts of elasticity, to which most of this chapter will be devoted, are of great importance because they are essential to the theory of all mechanical vibrations, including sound waves.

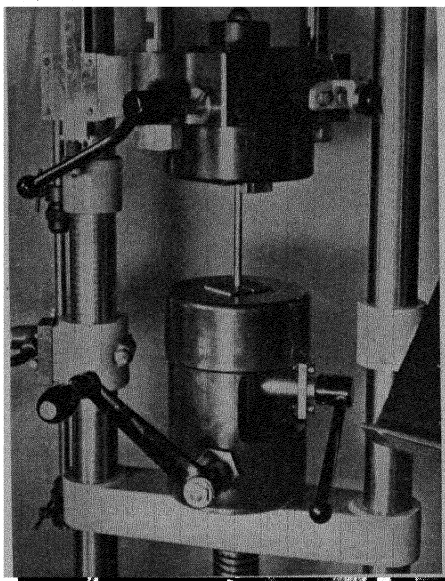


Fig. 137. Close-up view of test-piece being strained.

The fact that when a wire is stretched *the extension is proportional to the load* until the elastic limit is reached was discovered by Hooke and is known as *Hooke's law*.

The extension depends, not only upon the load, but also upon the original length and cross-section of the wire and upon the material of which it is composed. The elastic properties of different materials can be compared by means of a concept known as Young's modulus, *E*.

$$\text{Young's modulus} = \frac{\text{Stretching force per unit area of cross-section}}{\text{Increase in length per unit length}},$$

i.e.

$$E = \frac{F/A}{l/L}$$

where F =stretching force, A =area of cross-section, l =increase in length, L =original length.

Force per unit area, F/A , is known as *stress* and change in length per unit length, l/L , as *strain*, so that Young's modulus is Stress/Strain. Stress is usually measured in dynes per sq. cm., and since strain is a pure ratio, Young's modulus is also measured in dynes per sq. cm.

EXAMPLE

A steel wire of length 5 metres and area of cross-section 2 sq. mm. stretches 3 mm. when loaded with 26 kg. Calculate (a) Young's modulus for steel, (b) the energy stored in the wire.

$$\begin{aligned}\text{Young's modulus} &= \frac{F/A}{l/L} = \frac{\frac{26 \times 1000 \times 981}{0.02}}{\frac{0.3}{500}} \\ &= 21 \times 10^{11} \text{ dynes per sq. cm.}\end{aligned}$$

$$\begin{aligned}\text{Energy stored in wire} &= \text{work done in stretching the wire} \\ &= \text{average force} \times \text{extension} \\ &= \frac{1}{2} \text{ final load} \times \text{extension} \\ &= \frac{1}{2} \times 26 \times 1000 \times 981 \times 0.3 \\ &= 3.83 \times 10^6 \text{ ergs.}\end{aligned}$$

Determination of Young's modulus

Young's modulus may be determined by measuring the extension of a wire of the material when subjected to a suitable load. Since a metal wire passes its elastic limit when stretched much beyond $\frac{1}{1000}$ th of its original length, the load cannot be made very large, and for a measurable extension the length of the wire should be several metres; if the wire is 4 metres long the maximum permissible extension is about 4 mm. Two possible sources of error, a yielding of the support when the wire is loaded, and a change in length in the wire due to a change in temperature, can be eliminated by the arrangement in Fig. 138. A wire of the same material as that under test is hung from the same support and kept taut by a suitable weight. This wire carries a scale marked in mm.; to the wire under test is clamped a vernier scale which moves along the edge of the other scale.

An alternative and more accurate arrangement, due to Dr Searle, is shown in Fig. 139. Two wires of the same material suspended from the same support are clamped to a frame carrying a sensitive spirit level, one end of which can be raised or lowered by a micrometer screw. The wires are first made taut and kinks removed by attaching suitable loads to them, and the bubble in the level is adjusted so as to be central. One of the wires is then given a suitable additional load and the

screw is turned until the level is restored. The movement of the screw, which can be measured to $\frac{1}{1000}$ mm., is equal to the extension of the

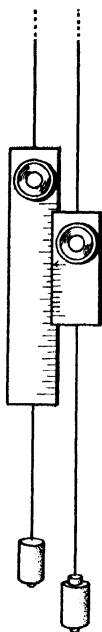


Fig. 138

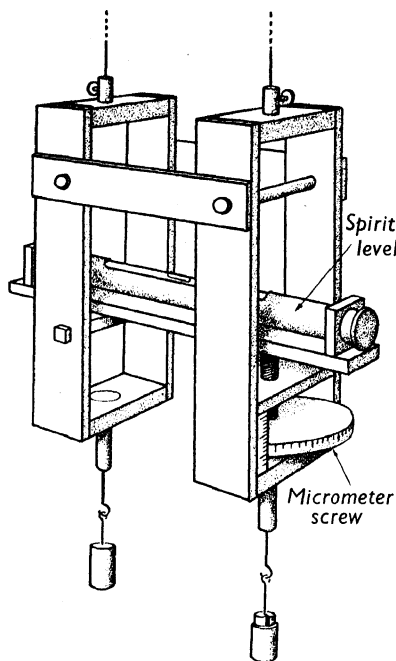


Fig. 139

wire. Two series of pairs of values of load and extension, with increasing and decreasing loads, are usually taken and a load-extension graph plotted. The gradient of the graph gives a mean value of load/extension.

The diameter of the wire under test is measured in two directions at right angles at various points along its length by means of a micrometer screw gauge, and the length of the wire is measured by a metre rule. Young's modulus is calculated as in the above example.

To find Young's modulus, when using the testing machine in Figs. 136 and 137, an instrument called an extensometer is attached to the

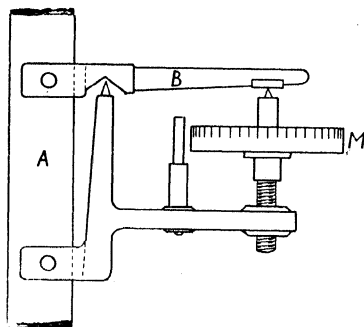


Fig. 140. Cambridge extensometer.

136 and 137, an instrument called an extensometer is attached to the

test-piece in order to measure its extension. In Fig. 140, *A* is the test-piece, *M* a micrometer and *B* is a vibrating reed. The dimensions of the instrument are such that the part of *B* which touches the micrometer screw moves down a distance equal to 5 times the extension of the test-piece. *M* is turned until it just touches *B* as *B* vibrates, both before and after the stress is applied.

Poisson's ratio

When a wire is stretched its cross-section decreases, but not sufficiently to prevent its volume from increasing slightly. The ratio of the lateral contraction per unit width to the longitudinal extension per unit length is found to be constant and is known as *Poisson's ratio*, usually denoted by σ . This may be expressed as follows:

$$\text{Poisson's ratio} = \frac{\text{Transverse strain}}{\text{Longitudinal strain}},$$

$$\text{i.e.} \quad \sigma = \frac{\delta r/r}{\delta L/L}$$

where

r = radius of wire,

L = length of wire.

For steel the value of σ is about 0.3. It can be shown by simple arithmetic that the volume of a wire or bar would remain unchanged when stretched by a small amount if σ were 0.5.

The three types of strain

Elasticity is that property of matter whereby a body regains its original shape after being deformed. So far, only stretching has been considered, but a body may also be deformed by compressing, shearing, bending, twisting and so on.

Any deformation of a body can be resolved, however, into three elementary types of strain, which can be considered independently even though they occur together, so long as the elastic limit is not exceeded.

These strains are (1) extension or contraction per unit length as already considered; (2) change of volume per unit volume without change of shape such as would occur if the body were immersed in a liquid subjected to a high pressure; (3) change of shape without change of volume due to a shearing stress (Fig. 141).

In Fig. 141 F is called the shearing force and θ the angle of shear. The sheared body may be thought of as a pack of cards resting on a

table. The couple shown could be applied by pressing the hand down flat on top of the cards and pushing horizontally. Only one couple is shown in Fig. 141, but there must be an equal and opposite couple to prevent rotation, and this is provided by the downward force on the cards due to their weight plus any necessary applied vertical force, and by the upward reaction of the table. If the angle of shear is appreciable, successive cards slide over one another, the elastic limit is exceeded

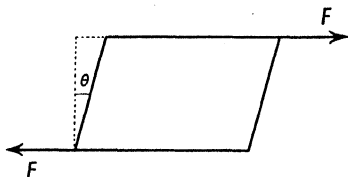


Fig. 141

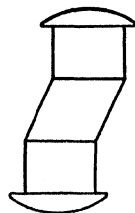


Fig. 142. Sheared rivet.

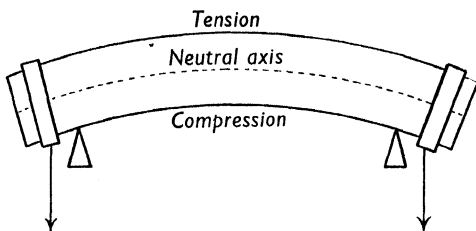


Fig. 143

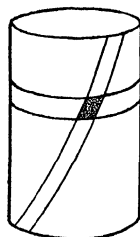


Fig. 144

and there is a permanent set. When a rivet is permanently sheared as in Fig. 142, successive planes of the metal may be regarded as having slid over one another in the same kind of way.

The bending of a rod involves stretching and compression and hence depends on Young's modulus, as can be seen from Fig. 143. The twisting of a wire is an example of shear; a small square on the surface becomes a rhombus when the wire is twisted (Fig. 144). The extension of a spiral spring also involves shear because the wire forming the spring does not increase in length, but is twisted, when the spring is stretched.

The three moduli of elasticity

Hooke's law may be generalised as follows to cover all three types of deformation, and it holds so long as the deformation is small: *stress is proportional to strain*. This fundamental law is the basis of the theory

of elasticity, and enables three moduli of elasticity to be formulated, each being defined as stress/strain.

Stress is always force per unit area; in stretching or in contracting, and in compression or in dilatation, it is normal to the surface, whereas in shearing it is tangential. Strain in the three cases is defined as (1) change in length per unit length, (2) change in volume per unit volume, and (3) angle of shear (θ in Fig. 141).

The three moduli are called Young's modulus E , the bulk modulus (or volume elasticity) K , and the modulus of rigidity (or shear or torsion modulus) N .

$$E = \frac{\text{Stretching force per unit area of cross-section}}{\text{Increase in length per unit length}},$$

$$K = \frac{\text{Compressive (or tensile) force per unit area}}{\text{Decrease (or increase) in volume per unit volume}},$$

$$N = \frac{\text{Tangential force per unit area}}{\text{Angle of shear}}.$$

Since strain is always dimensionless, each modulus is measured in the same units as stress, e.g. dynes per sq. cm. or tons per sq. in.

The bulk modulus is sometimes called the incompressibility; its reciprocal $1/K$ is called the compressibility and represents strain per unit stress, in contrast to a modulus of elasticity which is stress per unit strain.

The bulk modulus

The only modulus of elasticity which applies to a fluid is the bulk modulus, since a fluid flows if an attempt is made to subject it to a pressure in one direction and not in all other directions at the same time, or if it is subjected to a shearing stress.

Suppose a change of pressure from p to $p + \delta p$ causes a change of volume from v to $v - \delta v$; then, from the definition of the bulk modulus K ,

$$K = \frac{\delta p}{-\frac{\delta v}{v}} = -v \frac{dp}{dv}.$$

While the fluid is being compressed heat is developed. This heat may be removed as fast as it is generated, so that the temperature remains constant, and the change is then said to be isothermal; if no heat is removed, with the result that the temperature rises, the change is said to be adiabatic.

In the case of a gas, if the change is isothermal, the product of pressure and volume is constant (Boyle's law), i.e.

$$pv = \text{constant.}$$

Differentiating, $p dv + v dp = 0.$

$$\therefore -v \frac{dp}{dv} = p.$$

Thus the isothermal bulk modulus of a gas is equal to its pressure.

If the change is adiabatic

$$pv^\gamma = \text{constant,}$$

where γ is a constant, its value for air being 1.41. Differentiating,

$$p\gamma v^{\gamma-1} dv + v^\gamma dp = 0,$$

$$-v \frac{dp}{dv} = \gamma p.$$

Thus the adiabatic bulk modulus of a gas is γp , and it is this modulus which applies to sound waves since the compressions and rarefactions, being very rapid, take place adiabatically, the temperature of the gas in compression rising and in rarefaction falling.

The bulk modulus of a gas, whether isothermal or adiabatic, is obviously not a constant quantity; moreover, its determination requires only the measurement of the pressure of the gas. The bulk modulus of a solid or a liquid, on the other hand, is not a variable quantity, and its accurate determination is extremely difficult. The bulk modulus of a solid can, however, be calculated from the other two moduli (see p. 168).

Torsion

The modulus of rigidity is most conveniently determined from the torsion of a wire or cylindrical rod of the material.

Consider a thin cylindrical tube of radius r , thickness δr and length l (Fig. 145), the lower end of which is fixed and the upper end twisted through an angle ϕ . The material is sheared through an angle θ , and if θ is small, as will be the case if r is small compared with l ,

$$AA' = l\theta.$$

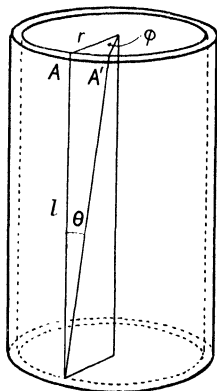


Fig. 145

But

$$AA' = r\phi.$$

$$\therefore \theta = \frac{r\phi}{l}.$$

If N is the modulus of rigidity of the material and F is the tangential force on the upper end of the tube,

$$\text{Stress} = \frac{F}{\text{Area of cross-section of tube}} = \frac{F}{2\pi r \delta r}$$

and

$$N = \frac{\text{Stress}}{\theta} = \frac{\frac{F}{2\pi r \delta r}}{\frac{r\phi}{l}}.$$

$$\therefore F = \frac{N\phi 2\pi r^3 \delta r}{l}.$$

$$\therefore \text{Twisting couple} = Fr = \frac{N\phi 2\pi r^3 \delta r}{l}.$$

If G is the couple which must be exerted on a solid cylinder of the material of radius a and length l to produce the same angle of twist ϕ ,

$$G = \int_0^a \frac{N\phi 2\pi r^3 \delta r}{l} = \frac{\pi N a^4 \phi}{2l}.$$

This expression for a solid cylinder has been obtained by a process of integration because the shear is a maximum in the surface of the cylinder and decreases from the surface to the axis, where it is zero. Thus the central fibres are the last to part when a wire is twisted until it breaks in two.

EXAMPLE

A steel shaft of diameter 3 in. and length 10 ft. transmits a torque of 2000 lb. ft. What is the twist in the shaft if the modulus of rigidity for steel is 11×10^6 lb. in.⁻²?

$$G = \frac{\pi N a^4 \phi}{2l}.$$

$$\begin{aligned} \therefore \phi &= \frac{2000 \cdot 12 \cdot 2 \cdot 10 \cdot 12}{\pi \cdot 11 \cdot 10^6 \cdot 1 \cdot 5^4} \\ &= 0.032 \text{ radians} \\ &= 1.9^\circ. \end{aligned}$$

Experimental determination of N

The modulus of rigidity of a material in the form of a wire may be found by applying a couple as in Fig. 146 and measuring the angle of

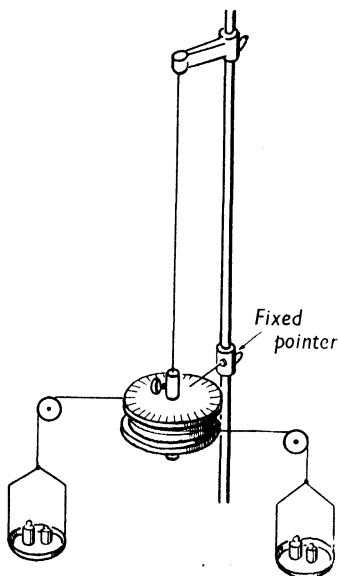


Fig. 146

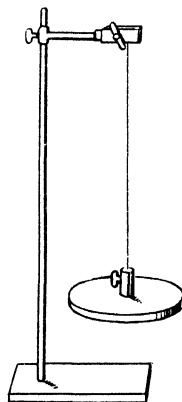


Fig. 147

twist, ϕ radians. If W is the total weight of each scale pan and contents and R is the radius of the pulley clamped to the wire,

$$G = WR = \frac{\pi N a^4 \phi}{2l},$$

using the same nomenclature as above.

$$\therefore N = \frac{2lWR}{\pi a^4 \phi}.$$

The radius of the wire, a , must be measured with especial care, since its fourth power enters into the result, and hence an error of 1% in its determination causes an error of 4% in the resulting value of N .

As an alternative to this statical method a dynamical method may be used. A suitable body of known moment of inertia I , such as a uniform disc, is clamped to the lower end of the wire (Fig. 147), and the periodic time T of torsional oscillations is measured. The oscillations are simple

harmonic because the restoring couple G is proportional to the angular displacement ϕ , viz.

$$G = \frac{\pi N a^4}{2l} \phi. \quad (\text{see above})$$

But

$$T = 2\pi \sqrt{\frac{I}{G/\phi}}. \quad (\text{p. 112})$$

$$\therefore T^2 = 4\pi^2 \frac{I}{\frac{\pi N a^4}{2l}},$$

$$N = \frac{8\pi I}{a^4 T^2}.$$

These torsional oscillations are a rare instance of a motion which continues to be simple harmonic even when the amplitude is large. The length of the wire is, as a rule, large compared with the radius, and hence the angle of shear, which is $r\phi/l$, remains smaller than the elastic limit even when ϕ is comparatively large.

Shear equivalent to an extension together with a perpendicular contraction

Shear may be regarded as the relative sliding of parallel planes of the material (p. 160), but alternatively it may be looked upon as a simultaneous extension and contraction in two mutually perpendicular directions. In Fig. 148 is shown a section $ABCD$ of a cube sheared through an angle θ into the position $A'B'CD$. The shear is greatly exaggerated since the proof which follows is true only for an infinitesimal shear and approximately true for a small shear. Thus in the proof θ must be taken as very small. The diagonals of the rhombus $A'B'CD$ are mutually at right angles and $B'D$ has extended, whereas $A'C$ has contracted.

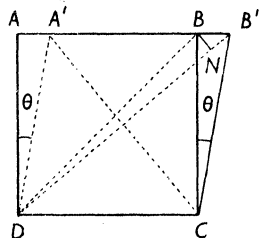


Fig. 148

$$\text{Strain along } DB = \frac{DB' - DB}{DB} = \frac{NB'}{BD},$$

where BN is drawn perpendicular to DB' . Since $\hat{BB'N}$ is very nearly 45° ,

$$NB' = BB' \div \sqrt{2}. \quad \text{Also } BD = BC \times \sqrt{2}.$$

$$\therefore \frac{NB'}{BD} = \frac{BB' \div \sqrt{2}}{BC \times \sqrt{2}} = \frac{1}{2} \frac{BB'}{BC} = \frac{1}{2} \tan \theta = \frac{1}{2} \theta.$$

Thus the strain along DB is $\frac{1}{2}\theta$.

Similarly, the strain along AC (a contraction) is $\frac{1}{2}\theta$. Thus a shear θ of a cube is equivalent to strains $\frac{1}{2}\theta$, one an extension and one a contraction, along the diagonals.

Relation between the elastic constants

Two of the four elastic constants, E , σ , K and N , are sufficient to determine completely the elastic properties of an *isotropic* material, i.e. a material which has the same elastic properties in all directions.

Just as in thermal expansion the coefficient of cubical expansion is three times the coefficient of linear expansion, so we might expect the

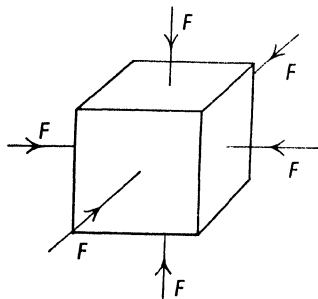


Fig. 149

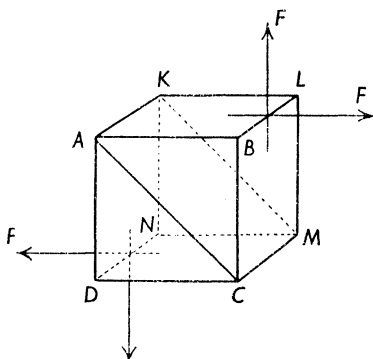


Fig. 150

strain for a given compressive stress, as in Fig. 149, to be three times the strain for a linear stress of the same magnitude, and hence that $K = \frac{1}{3}E$. This would be true if $\sigma = 0$, but, owing to the lateral extension which results from each longitudinal contraction, the relation is

$K = \frac{E}{3(1-2\sigma)}$, as will be shown presently.

Similarly, if $\sigma = 0$, $N = \frac{1}{2}E$, since, as has been proved above, a shear θ is equivalent to strains $\frac{1}{2}\theta$ along the diagonals. But σ is never zero and it will be shown that $N = \frac{E}{2(1+\sigma)}$.

In Fig. 149 suppose the cube originally had unit side and consider the effect of one pair of parallel, opposite forces F . They produce a contraction F/E along the direction in which they are acting (from the definition of E since both length and cross section are unity) and an extension $\sigma F/E$ in the other two directions at right angles (from the

definition of σ). The same applies to each of the three pairs of forces F . Thus the length of each side of the cube becomes $1 - \frac{F}{E} + 2\sigma \frac{F}{E}$.

$$\begin{aligned}\text{Change of volume} &= 1 - \left\{ 1 - \frac{F}{E}(1 - 2\sigma) \right\}^3 \\ &= 3 \frac{F}{E}(1 - 2\sigma),\end{aligned}$$

neglecting higher powers of F/E since E is very large.

$$\begin{aligned}\therefore K &= \frac{F}{3 \frac{F}{E}(1 - 2\sigma)} \quad \text{from the definition of } K \\ &= \frac{E}{3(1 - 2\sigma)}.\end{aligned}$$

Now consider the cube in Fig. 150 which is being sheared by four forces, each of F dynes. If the side of the cube is 1 cm. the tangential stresses on the four faces on which these forces act are F dynes per sq. cm. Since the strain is uniform throughout the cube the stresses on all planes in the cube parallel to $ABLK$ are F dynes per sq. cm., and the stresses on all planes parallel to $BLMC$ are likewise F dynes per sq. cm.

The force perpendicular to the diagonal plane $AKMC$, causing an extension of the cube in a direction parallel to BD , is the resultant of the forces on $AKLB$ and on $BLMC$, i.e. $2F \cos 45^\circ = \frac{2F}{\sqrt{2}}$. The area of the diagonal plane $AKMC$ is $\sqrt{2}$ sq. cm., and hence the stress across it is $\frac{2F}{\sqrt{2} \times \sqrt{2}} = F$ dynes per sq. cm. The stress across the other diagonal plane, $BDNL$, is the same, F dynes per sq. cm., but in this case it causes a contraction in a direction parallel to the diagonal AC .

The strain along BD , due to the stress F in this direction, is $\frac{F}{E}$ (from the definition of Young's modulus E). But there is an additional strain along BD of $\sigma \frac{F}{E}$, which is also an extension, due to the stress F parallel to AC , causing a contraction along AC .

$$\begin{aligned}\therefore \text{Total strain along } BD &= \frac{F}{E}(1 + \sigma) \\ &= \frac{\theta}{2},\end{aligned} \quad (\text{see p. 165})$$

where θ is the angle of shear.

But $\theta = \frac{F}{N}$. (definition of N)

$$\therefore \frac{F}{E}(1+\sigma) = \frac{F}{2N}.$$

$$\therefore N = \frac{E}{2(1+\sigma)}.$$

Eliminating σ from the equations for K and N above,

$$K = \frac{NE}{9N - 3E}.$$

Much the simplest way of finding K for a solid is to measure E and N and to substitute their values in this equation.

Material	Elastic moduli (dynes per sq. cm.)			Poisson's ratio
	E	K	N	σ
Steel	$19.5-20.6 \times 10^{11}$	18.1×10^{11}	$7.9-8.9 \times 10^{11}$	0.25-0.33
Copper	12.4-12.9	14.3	3.9-4.0	0.26
Aluminium	7.05	7.46	2.67	0.34
Brass	9.7-10.2	10.7	3.5	0.34-0.40
Constantan	16.3	15.5	6.11	0.32
German silver	11.6	—	4.5	0.37
Phosphor bronze	12.0	—	4.36	0.38
Quartz fibre	5.18	1.4	3.0	0.10
Rubber	0.05	—	0.00016	0.48

The structure of metals

The theory of elasticity rests on the fundamental assumption that materials are isotropic, i.e. that they have the same elastic properties in all directions. This implies an internal homogeneity and lack of structure which most materials do not possess. Metals, for example, when viewed under the microscope, are seen to be aggregates of crystals (Fig. 151). The crystal grains, however, are usually very small and their axes are arranged at random, so that the metal is statistically homogeneous and approximately isotropic. Nevertheless, the study of the internal structure of a metal is of the greatest importance for the understanding of plastic deformation.

To prepare a specimen for examination under a microscope, the metal is cut with a saw and the surface scratches are removed by polishing. The surface is then etched with a suitable solution, such as a dilute acid, to dissolve the 'flowed' material which covers the surface even after polishing with the finest powder, and to reveal the structure by eating into certain constituents of the surface. Thus the boundaries

of the crystal grains, consisting of amorphous or non-crystalline metal, are attacked more readily by the etching solution than the crystals themselves, and minute channels are formed. These channels do not reflect light into the microscope and hence appear black, showing clearly the shapes of the crystals.

Fig. 151 (b) shows the same surface as in Fig. 151 (a) after the metal has been strained beyond the elastic limit; a number of dark lines have appeared. These dark lines are called slip bands, and they were explained by Ewing and Rosenhain in 1899 as due to slipping along the cleavage planes of the crystals (Fig. 152). Until the elastic limit is

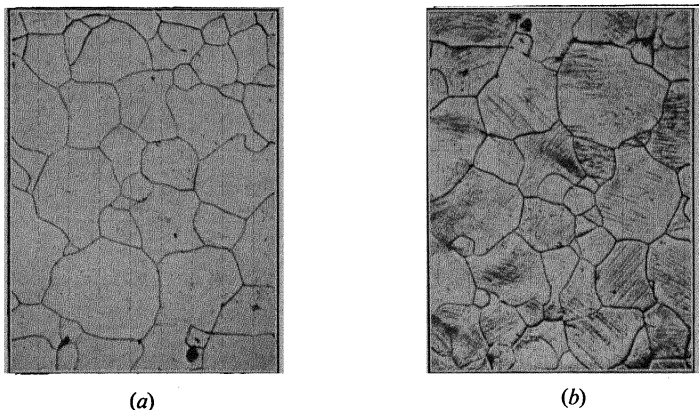


Fig. 151. Crystals in iron (a) before straining, (b) after considerable plastic strain. The additional black lines in (b), called 'slip bands', are due to the sliding of crystal planes over each other (see Fig. 152).

reached the crystals are temporarily distorted while the load is applied, but, beyond the elastic limit, slipping inside the crystals always occurs. There is no slipping of one crystal past another, and the boundaries seem to be stronger than the crystals themselves.

In 1914 methods of preparing large single metal crystals, in the form of thin rods, were discovered. The idea of a crystal with a cylindrical boundary may seem odd, but the essential condition is that the crystal axes should all be parallel throughout the rod. The remarkable thing about these single metal crystals is that they are as weak as plasticine and can be stretched at first by the fingers. Slip takes place in one direction only but not equally in all planes parallel to this direction, so that a series of layers or steps is developed. If a suitable steady load is applied the slip occurs in a series of jumps over a period of time and

seems to take place in quite arbitrary positions. A cylindrical rod consisting of a single crystal of cadmium can be extended to five times its original length and becomes a flat ribbon because the slip occurs in one particular direction.

Single crystals show a remarkable hardening as a result of deformation. A single crystal of silver has been known to increase by as much as 92 times in strength when stretched to just over twice its length. The phenomenon can be illustrated by a simple experiment. Hold a copper wire horizontally and run a bunsen flame along it. The heat causes crystallisation in the wire, not, it is true, into a single crystal,

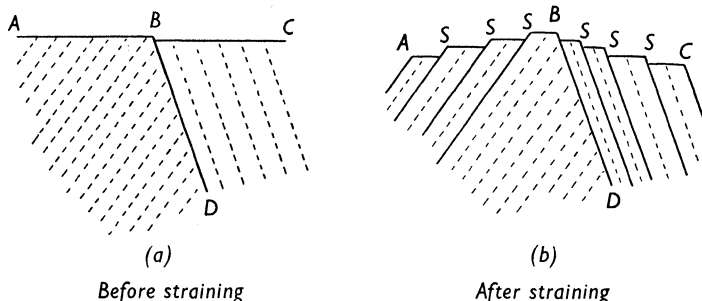


Fig. 152. In (a) *ABC* represents the smooth, polished surface of two adjacent crystals, the difference in level being due to etching. *BD* represents the boundary between the crystals and the dotted lines represent the potential planes of slip. In (b) slip has taken place in some of these planes, causing slip bands marked *S*.

but nevertheless the wire becomes so soft as to be bent easily by a very small weight placed on the end. On being bent backwards and forwards a few times the wire fully recovers its original stiffness.

Ordinary polycrystalline metals can similarly be strengthened by 'cold working' or rough treatment at a low temperature. When a wire is stretched beyond the elastic limit and unloaded it will be found on reloading that its elastic limit and yield point have been permanently raised (Fig. 153). But the elasticity or stress per unit strain is unchanged. The hardening produced by cold working can be removed by suitable heat treatment, known as annealing. The heating and consequent softening of the copper wire described above is an example of the annealing process; its subsequent hardening by being bent backwards and forwards is an example of cold working.

This suggests the clue to the strength of metals. Cold working breaks up the crystal grains into smaller units, whereas annealing causes recrystallisation and the formation of larger grains, facts which can be

further verified by examination with X-rays. In a single crystal slipping begins at a weak spot and then proceeds throughout the whole crystal as a small cut will develop into a tear right across a fabric. In a polycrystalline material the slip is confined to one crystal grain and stops at the boundary just as a tear in a fabric stops at a seam.

To produce great strength why should not the crystals of a metal be broken up completely? Paradoxically, in the words of Sir Lawrence Bragg, in order to be strong a metal must be weak. It is because slip can occur in it that a metal is not brittle like glass or like concrete, and can be drawn into wire, forged, rolled and generally deformed without breaking. Its strength and hardness can be varied enormously, however, by heat treatment and by altering its composition through the addition of carbon or other metals.

A single crystal is much weaker than would be expected from theoretical considerations, suggesting that the crystal lattice is not perfect but contains misfits. Whether these weak spots are inevitable is not known, but further study may throw light on the problem.

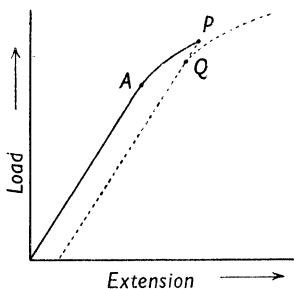


Fig. 153. A is the elastic limit. If the load is increased to P and then removed, on reloading the new curve is shown dotted and the new elastic limit is Q , which represents a higher load than A .

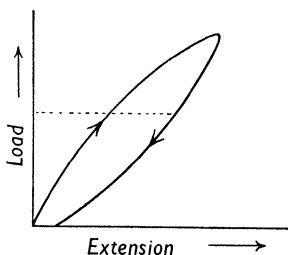


Fig. 154

Elastic hysteresis

In some substances, notably glass, the strain produced by a load takes time (perhaps hours or even days) to reach its final value and, when the load is removed, it takes time to disappear. The effect on the normally obtained load-extension curve is shown in Fig. 154; the extension produced by a load, represented by the dotted line, is less than might be expected during loading, and greater during unloading. The phenomenon is called *elastic hysteresis*, and, during the cycles of loading and unloading, energy is dissipated in the material in the form of heat.

Since the work done by the load during loading and the work done on the load during unloading are proportional to the areas under the two curves in Fig. 154, the energy dissipated is equal to the difference in these areas, i.e. the area inside the loop.

There is very little elastic hysteresis in the cases of the metals or quartz.

A possible explanation of elastic hysteresis in the case of a polycrystalline material is that certain atoms at the boundaries of the crystals have different positions of stable equilibrium and may, when the body is subjected to stress, transfer themselves from one crystal to another. The fact that elastic after-effect, or the slow recovery from deformation, may be shortened by heating supports the theory, because an increase in thermal agitation would assist atoms to return to their original positions.

Elastic fatigue

Rapid, repeated application and removal of a load weakens an elastic body and the body will break under an alternating stress of a magnitude far below the normal, steady breaking stress. The importance of this phenomenon, known as *elastic fatigue*, in engineering practice is apparent when, for example, the piston and connecting rods of a locomotive, which suffer tension and compression during each revolution of the crankshaft, are considered.

The phenomenon is explained by crystal slip. Under alternating stress, wear and tear occurs at the slip, and in due course a crack appears which weakens the surrounding crystals and causes fracture.

The elasticity of rubber

A crystalline body can be extended by only a fraction of 1% of its length before reaching its elastic limit, whereas rubber can be stretched by 800% of its length. Rubber's enormous range of elasticity should not be confused with a high modulus of elasticity. Young's modulus for rubber is, in fact, only about $\frac{1}{50}$ th of that of steel; it requires a much smaller force to stretch it than steel.

The rubber molecule is built up of units of isoprene, C_5H_8 , and contains on an average about 4000 of these units. Its 20,000 carbon atoms are spread out in a chain making it enormously long and thin; its actual length is about $\frac{1}{400}$ mm. Rubber in bulk has been compared to a mass of very long, intertwined, wriggling snakes. When it is subjected to a stretching force, the molecules tend to uncoil, and when the force is removed, they coil up again. This accounts for its great extensibility.

SUMMARY

The fundamental law of elasticity is that stress is proportional to strain. Three moduli of elasticity, each defined as stress/strain, cover all elastic deformations. Relations between the moduli can be deduced for isotropic materials.

Plastic deformation, in the case of metals, may be interpreted in terms of crystal slip.

QUESTIONS

[Take $g = 981 \text{ cm. sec.}^{-2}$]

1. Define *stress*, *strain*, *Young's modulus*. Describe a method of measuring Young's modulus for a material in the form of a wire, and indicate the percentage accuracy you would expect to obtain in each of the measurements you make.

Find the work done in producing an extension of 2 mm. in a steel wire of original length 2 metres and diameter 0.8 mm.

(Young's modulus for steel = 2×10^{12} dynes per sq. cm.) (C.)

2. State Hooke's law and describe, with the help of a rough graph, the behaviour of a copper wire which hangs vertically and is loaded with a gradually increasing load until it finally breaks. Describe the effect of gradually reducing the load to zero (a) before, (b) after, the elastic limit has been reached.

A uniform steel wire of density 7.8 gm. per c.c. weighs 16 gm. and is 250 cm. long. It lengthens by 1.2 mm. when stretched by a force of 8 kg. weight. Calculate (a) the value of Young's modulus for the steel, (b) the energy stored in the wire. (N.)

3. Define *Young's modulus of elasticity*.

Make a sketch of a simple form of extensometer and describe how it is used.

A wire 580 cm. long and of mean radius 0.059 cm. is loaded with a number of 2 kg. weights, and the following extensions are observed:

Load (kg.)	2	4	6	8	10	12
Extension (cm.)	0.085	0.165	0.235	0.303	0.370	0.441

Determine Young's modulus from these data. (L.)

4. Calculate the force required to prevent a steel wire of 1 sq. mm. cross-section from contracting when it cools from 60°C. to 15°C. , if Young's modulus for steel is 20×10^{11} dynes per sq. cm. and its coefficient of linear expansion is 11×10^{-6} per $^\circ \text{C.}$ Express your result in kilograms weight. (N.)

5. The rubber cord of a catapult is pulled back until its original length has been doubled. Assuming that the cross-section of the cord is 2 mm. square, and that Young's modulus for rubber is 10^8 c.g.s., calculate the tension in the cord. If the two arms of the catapult are 6 cm. apart, and the unstretched length of the cord is 8 cm., what is the stretching force? (O. & C.)

6. A mass m is hung at the end of an unstretched spring of negligible inertia and released from rest. Show that the maximum tension in the spring is $2mg$. (Assume that Hooke's law is obeyed.) (C.)

7. When a rubber cord is stretched the change in volume is very small compared with the change in shape. What will be the numerical value of Poisson's ratio for rubber, i.e. the ratio of the fractional decrease in diameter of the stretched cord to its fractional increase in length? (L.)

8. A uniform, horizontal, rigid bar of mass 100 kg. is supported horizontally by three equal, vertical wires *A*, *B* and *C*, each of initial length 1 metre and cross-section 1 sq. mm. *B* is a copper wire passing through the centre of the bar; *A* and *C* are steel and are arranged symmetrically one on each side of *B*. If Young's moduli for copper and steel are 1.5×10^{12} dynes per sq. cm. and 2×10^{12} dynes per sq. cm. respectively, find the tension in each wire and the extension. (O. & C.)

9. Define *stress*, *strain*, *elastic limit*, *Young's modulus*.

A thin uniform wire of radius *r* is clamped horizontally between rigid supports *A* and *B*, so that it is just taut and the length of wire between *A* and *B* is $2l$. A mass *m* is hung from the midpoint *C*, and displaces *C* through a vertical distance *y*, small compared with *l*. Find expressions for the stress and strain in the wire, and for Young's modulus, in terms of *m*, *l*, *r* and *y*, neglecting in each case squares and higher powers of *y/l* in comparison with unity. (O. & C.)

10. A steel cylindrical shell 3 ft. long, $\frac{1}{4}$ in. thick and 18 in. internal diameter can just be fitted over a solid steel core when the shell is heated through 231° C. The shell and core are allowed to cool to their original (room) temperature. Calculate the tension in the shell in tons weight. Neglect any change in diameter of the core. Find also the thrust exerted on the core per square inch of its surface.

(Young's modulus for steel = 1260 tons weight per sq. in.; the coefficient of linear expansion of steel = $0.000012 \text{ deg.}^{-1} \text{ C.}$) (N.)

11. What conditions must a body satisfy in order to be perfectly elastic? How far do actual bodies satisfy the conditions? A bar of steel is heated to a temperature of 500° C. Its ends are then clamped at two points 1 metre apart and the bar is allowed to cool. Find at what temperature it will break assuming the following values.

(Coefficient of expansion of steel 1.1×10^{-6} per $^{\circ}\text{C.}$; Young's modulus for steel 2×10^{12} dyne per sq. cm. Breaking stress for steel 8×10^9 dyne per sq. cm.) (C. S.)

12. Calculate the bursting pressure in atmospheres of a spherical leather football 25 cm. in diameter with a wall thickness of 2 mm.

(Tensile strength of leather is 0.3×10^9 dynes per sq. cm., 1 atm. is 10^6 dynes per sq. cm.) (C. S.)

13. Give a brief account of the effect of mechanical forces on gases, liquids and solids. State what quantities are necessary to describe the deformation of a solid under an applied force, and describe how *one* of them may be determined. (O. S.)

14. Assuming that the bulk modulus of water is 2.2×10^4 kg. per sq. cm., calculate the approximate depth below the surface of an ocean of pure water at which the density is 1% greater than at the surface. Compare the law of increase of pressure with the depth upon which you base your conclusions with the law of decrease of pressure with increasing height in an atmosphere supposed to be at the same temperature throughout. (C. S.)

15. Find the relation between the elasticity and density of a perfect gas (*a*) under isothermal conditions, (*b*) under adiabatic conditions. (C. S.)

16. What is meant by the modulus of rigidity of a material?

A wire of radius of cross-section a cm., and length l cm., made of material of modulus of rigidity n dyne-cm.⁻², is fixed at one end, and a couple of C dyne-cm. rotates the other end through θ radians. Verify, by the method of dimensions or otherwise, that C is proportional to $na^4\theta/l$.

In a particular torsion experiment, C and l are each measured to within 1%, and θ and a are each measured to within 2%. The value obtained for n is 7.5×10^{11} dyne-cm.⁻² Find the limits within which the true value of n should lie. (O.)

17. What is the work done in twisting through 90° a wire of length 50 cm. and diameter 0.09 cm. if its coefficient of rigidity is 8×10^{11} dynes per sq. cm.? (O.)

18. Explain the term *modulus of rigidity*. Describe briefly how this modulus can be determined experimentally for a copper wire.

A ship's propeller shaft, of 10 metres length and 15 cm. radius, twists through an angle of 1° when the shaft rotates at 60 r.p.m. If the modulus of rigidity (n) is 9×10^{11} dynes per sq. cm., calculate the power transmitted by the shaft.

(Couple per radian of twist = $\frac{\pi n r^4}{2l}$, where r is the radius and l is the length of the cylinder under twist.) (B.)

19. A torsion pendulum consisting of a brass disc suspended by a fine brass wire performs small torsional oscillations of period $T = 2\pi \sqrt{\frac{2Il}{\pi n a^4}}$, where I is the moment of inertia of the disc about its axis of oscillation, l and a the length and radius of the wire respectively, and n the rigidity modulus of brass. Find the percentage change in the period due to a change of temperature of 5°C . The coefficient of linear expansion of brass is 0.000018 and the temperature coefficient of the modulus of rigidity is -4.6×10^{-4} , both per $^\circ\text{C}$. (N.)

CHAPTER 8

SURFACE TENSION

THE surface of a liquid has a tendency to shrink and to acquire a minimum area, as can be seen whenever water drips from a tap; the water gathers itself into spherical drops, the sphere being the form which has a minimum surface area for a given volume.

This characteristic of a liquid surface is due to what is called *surface tension*, and the surface may be imagined to be in tension like an elastic membrane. If a rubber membrane, attached to a ring, is allowed to

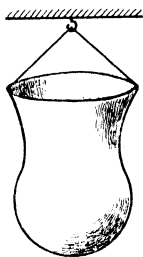


Fig. 155

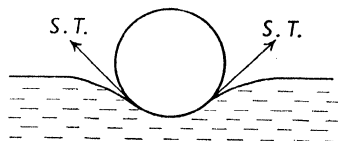


Fig. 156. A needle floating on the surface of water is supported partly by forces due to surface tension, *S. T.*, and partly by buoyancy equal to the weight of water displaced.

fill with water, it assumes the same shape as a drop of water about to detach itself from a tap (Fig. 155). A steel needle, if placed gently on water so that it does not break the surface, will float (Fig. 156). Insects such as the water spider can run about on the surface of a pond without getting wet. The common gnat and the blood-sucking mosquitoes, responsible for the transmission of malaria and yellow fever, lay their eggs in stagnant water, and the larvae which hatch out make use of surface tension to hang from the surface when they come up to breathe. They do this by opening flaps at the end of a breathing tube through the tail (Fig. 157). If, however, the surface of the water is covered with a thin layer of oil, which reduces the surface tension, the larvae are unable to support themselves from the surface and perish. By this simple expedient tracts of country hitherto dangerous or even deadly to man, such as the Panama Canal Zone and the Roman Campagna, have been made habitable.

The analogy between a liquid surface and an elastic membrane is not really a close one, as will be seen later when the cause of surface tension

is considered. Moreover, the force exerted by a liquid surface is independent of the area of the surface and does not, like the force exerted by an elastic membrane, become greater when the surface is stretched. Although it is true that a soap bubble can be inflated like a rubber balloon, the pressure inside the balloon increases as it is blown up, whereas that inside the soap bubble decreases.

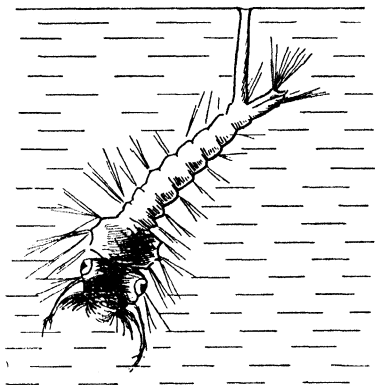


Fig. 157

Definition of surface tension

The experiment indicated in Fig. 158 suggests a straightforward way of defining and measuring surface tension. EF is a wire free to move on a rectangular wire frame $ABCD$. A soap film is spread over $ABCD$, and then the part $EBCF$ of the film is punctured with a hot pin. The wire EF at once moves sharply to the left as the film $AEFD$ shrinks. The force exerted on unit length of EF by the film surface may be taken as a measure of the surface tension. A soap film has two surfaces, one on each side of the frame $AEFD$, and hence the surface tension is half the actual force per unit length on EF . Applying the idea generally, it may be assumed that across any line in a liquid surface there is a force perpendicular to the line and tangential to the surface; the *force per unit length* is defined as the surface tension. The units usually employed are dynes per cm.

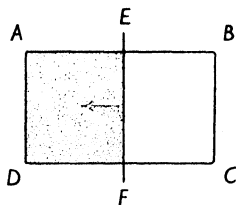


Fig. 158

Determination of surface tension from the force exerted by a film

The surface tension of a soap solution may be determined by measuring the force exerted by a film on a rectangular frame of fine wire. The frame is dipped into the soap solution and a film is formed on it; it is suspended from the arm of a balance and weights are placed in the pan attached to the other arm of the balance until the beam is horizontal (Fig. 159). The film is punctured and, as the downward force exerted on the frame by the film has now ceased, weights m gm. must be removed from the balance pan to restore the beam to its horizontal position.

Let surface tension of soap solution $= \gamma$ dynes per cm.

Length of frame $= l$ cm.

Downward force exerted by film on frame $= 2l\gamma$ dynes.

(there are surfaces on each side)

$$\therefore 2l\gamma = mg,$$

$$\gamma = \frac{mg}{2l} \text{ dynes per cm.}$$

In the case of a liquid like water a thin rectangular glass plate is used instead of a wire frame.* The glass plate is weighed and then adjusted so that its lower edge is in the surface of the liquid when the balance beam is horizontal. The weight, m gm., additional to the weight of the plate, necessary to pull the plate out of the surface of the liquid, is found. If the length of the glass plate is l cm. and its thickness is t cm.,

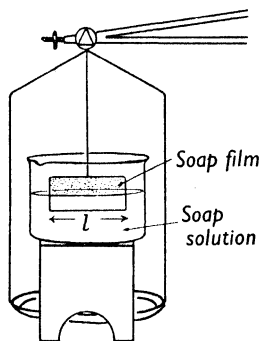


Fig. 159

Force to pull plate out of the surface $= \gamma \times \text{perimeter of plate,}$

i.e.

$$\gamma(2l + 2t) = mg.$$

$$\therefore \gamma = \frac{mg}{2(l+t)}.$$

* A vertical film of water cannot be formed on a wire frame because the tension at the top of the film must be greater than at the bottom to support the weight of the film, and the surface tension of a pure liquid cannot vary. The surface tension of a soap solution, on the other hand, varies with the concentration of the soap, being smaller the greater the concentration. The surface of a soap film has a greater concentration of soap than the interior, and, if the film is in danger of breaking, fresh liquid from the interior goes to the threatened place and increases the surface tension, thereby strengthening the weak spot. This variation in the surface tension of a solution is, however, very small, and can be ignored in nearly all problems.

If the lower edge of the glass plate is immersed below the surface of the liquid there will be an upthrust on it equal to the weight of liquid displaced; it is placed in the surface to avoid this complication. The glass plate and the beaker must be thoroughly cleaned, first with caustic soda and then, after rinsing, with dilute nitric acid, before the experiment, and not touched with the fingers subsequently, since a trace of grease lowers the surface tension of water considerably.

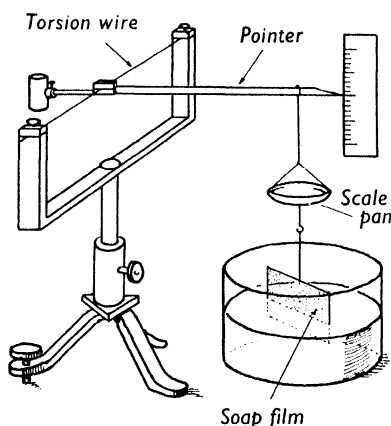


Fig. 160

Instead of a balance, the apparatus in Fig. 160, due to Dr G. F. C. Searle, may be used to find the force exerted on the frame or glass plate. The frame is suspended from a light pointer attached to a horizontal torsion wire stretched between rigid supports. After the film is ruptured weights m gm. are placed in the scale pan so that the pointer returns to the same position on the scale as it occupied before. Then

$$2l\gamma = mg.$$

Excess pressure inside a spherical bubble or drop

If a soap bubble is blown, say on the bowl of a clay pipe, and then the stem of the pipe is left open to the atmosphere, the bubble will gradually decrease in size showing that the tension of the film is squeezing out the air. The excess pressure, above that of the atmosphere, inside a spherical soap bubble can be calculated as follows.

Suppose the bubble has radius r , the excess pressure inside it is p and its surface tension is γ . Imagine that through its centre is a plane, weightless partition (which will not disturb its equilibrium) dividing it

into two hemispheres. Imagine also that the atmospheric pressure is zero, so that the actual pressure inside the bubble is p . (The excess pressure inside the bubble is due to the tension of the film and will remain unaltered whatever the value of the atmospheric pressure.)

Consider the equilibrium of one of the hemispheres fitted with the partition. Two sets of forces which counterbalance will be exerted upon it by the other hemisphere: (i) the thrust of the air on the partition equal to $\pi r^2 p$, tending to push the hemispheres apart, (ii) the pull of

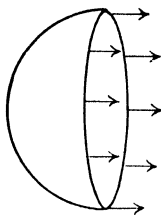


Fig. 161

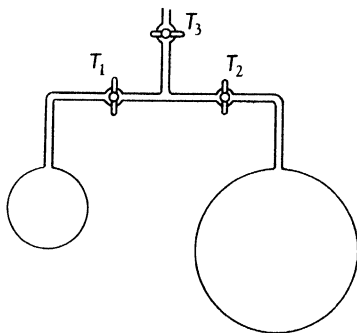


Fig. 162

the film indicated by the arrows in Fig. 161 and equal to $2 \times 2\pi r\gamma$ (there are inside and outside surfaces) tending to keep the hemispheres together.

$$\therefore \pi r^2 p = 4\pi r\gamma,$$

$$p = \frac{4\gamma}{r}.$$

Thus the pressure inside a spherical soap bubble becomes less as the radius increases. If the two bubbles in Fig. 162 are put in communication by opening the taps T_1 and T_2 (T_3 being closed), the smaller bubble will shrink and the larger bubble expand.

A spherical drop of liquid has only a single surface and hence the excess pressure inside it is $2\gamma/r$. To prove this from first principles, as above, there is no need to use the device of a weightless partition since the drop is not hollow.

The pressure on the concave side of any spherical liquid surface is always greater than that on the convex side by an amount $2\gamma/r$.

Excess pressure inside a cylindrical bubble

Suppose a cylindrical soap bubble is formed between the rims of two equal funnels which are dipped into a soap solution, placed together and then drawn apart, the pressure of the air inside the bubble being

adjusted to make the bubble cylindrical before the stems of the funnels are closed to the atmosphere.

An expression for the excess pressure p inside the bubble can be obtained by imagining the bubble to be divided by a plane weightless partition through its axis and then by considering the equilibrium of one-half of the bubble (Fig. 163). As in the case of the spherical bubble it will be assumed that the atmospheric pressure is zero, and hence that the actual pressure inside the bubble is p .

Suppose that the radius of the bubble is r , its height h and its surface tension γ . The forces exerted on one-half of the bubble fitted with the partition are as follows: (i) the pressure of the air on the partition, equal to the product of p and the area of the partition, $2rhp$, tending to force the two halves of the bubble apart; (ii) the pull of the film on the two vertical sides of length h , indicated by the arrows in Fig. 163 and equal to $2 \times 2h\gamma$ (there are inside and outside surfaces), tending to keep the two halves of the bubble together; (iii) the upward and downward pulls on the film by the rims of the funnels which are equal and opposite and hence are in equilibrium.

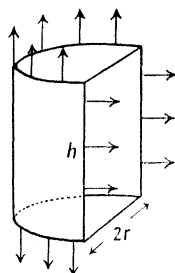


Fig. 163

$$\therefore 2hrp = 4h\gamma,$$

$$p = \frac{2\gamma}{r}.$$

The excess pressure inside a cylinder of liquid is given by $p = \gamma/r$, since in this case there is only one surface.

General expression connecting surface tension and curvature of a surface

Suppose that the pressure inside a cylindrical soap bubble is allowed to become equal to the atmospheric pressure outside. The bubble takes the shape shown in Fig. 164 and is said to have zero curvature, since at any point there are two curvatures in perpendicular planes which are equal and oppositely directed. Thus at the point A in Fig. 164 the film has a curvature in the plane of the paper and also at right angles to the paper, the radii both being R but in opposite senses. A surface having oppositely directed curvatures, like a horse's saddle, is called an anticlastic surface; the form of the bubble with zero curvature (Fig. 164) is called a catenoid.

The two radii marked at A in Fig. 164 are called the principal radii of curvature at the point. The general formula connecting the difference

in pressure, p , across the surface whose principal radii of curvature are R_1 and R_2 and whose surface tension is γ is

$$p = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

A proof of this is given in Appendix I, p. 206.

The following are special cases:

(i) Spherical drop: $R_1 = R_2 = r$. $\therefore p = \frac{2\gamma}{r}$.

In the case of a spherical soap bubble, which has two surfaces, $p = \frac{4\gamma}{r}$.

(ii) Cylindrical drop: $R_1 = r$, $R_2 = \infty$. $\therefore p = \frac{\gamma}{r}$.

In the case of a cylindrical soap bubble $p = \frac{2\gamma}{r}$.

(iii) Plane surface: $R_1 = R_2 = \infty$. $\therefore p = 0$.

(iv) Catenoid: $R_1 = -R_2$. $\therefore p = 0$.

EXAMPLE

Two flat plates are squeezed close together with a little water between them (Fig. 165). If the distance apart of the plates is d cm., the area of the plates in contact with the water is A sq. cm. and the surface tension of the water is γ dynes per cm., calculate the force normal to the plates required to pull them apart.

The pressure inside the liquid will be less than the atmospheric pressure by an amount p , where

$$p = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

$R_1 = \frac{1}{2}d$ since the water wets the plates and the shape of the section of the water surface shown in Fig. 165 is semicircular.

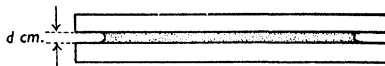


Fig. 165

R_2 is the radius of the disc of water between the plates (area A sq. cm.); since it is large compared with d , $1/R_2$ can be neglected.

$$\therefore p = \frac{2\gamma}{d}.$$

$$\therefore \text{Force required to pull the plates apart} = \frac{2\gamma A}{d}.$$

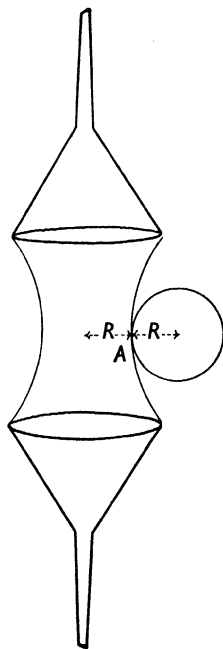


Fig. 164

Stability of cylindrical films

A cylindrical film, whose length is greater than its circumference, is unstable. This can be demonstrated with the apparatus of Fig. 166. Cylindrical soap bubbles are formed between circular supports *A* and

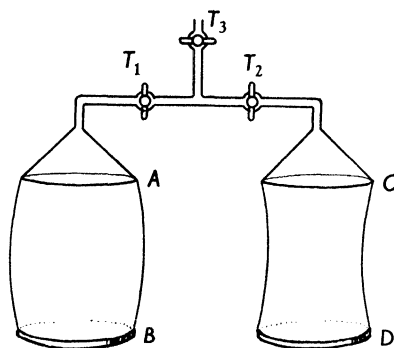


Fig. 166

B, *C* and *D*. By adjusting the air pressure inside each, one of the bubbles is made to bulge slightly and the other one to cave in slightly. When the two bubbles are put into communication by opening the taps T_1 and T_2 , one of the following will happen: (i) if the height of each

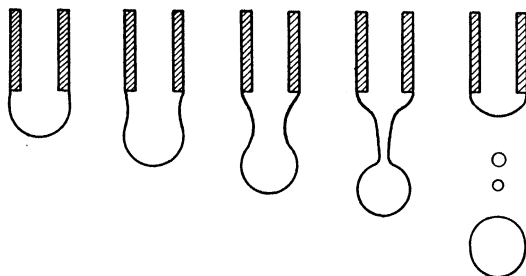


Fig. 167. Falling drop.

bubble is less than half its circumference, *AB* will shrink a little, and *CD* will swell out, both tending to become cylindrical; (ii) if the height of each bubble is more than half its circumference, *AB* will bulge further and *CD* will shrink until it collapses.

The two bubbles may be imagined as placed end to end to form a single bubble with a length equal to the sum of their lengths. Such a bubble will be unstable if its length exceeds its circumference. A very slight displacement, such as a bulge near one end, will become greater,

one-half of the bubble swelling out and the other shrinking until the bubble has divided and formed two spherical bubbles, one large and one small.

This explains why a liquid jet breaks up into drops and what happens in the fall of a growing drop from a tap (Fig. 167). The cylindrical film formed round a thin glass fibre, which has been dipped into water, breaks up into beads, and the same phenomenon can often be seen on a spider's web on a damp autumn morning. The process of breaking up into drops is retarded in the case of very viscous liquids so that it is possible to obtain long fine threads of treacle, glass or quartz.

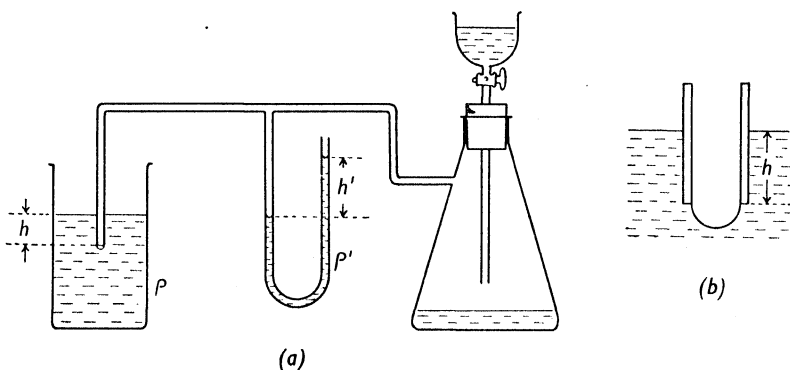


Fig. 168

Jaeger's method of measuring surface tension

The surface tension of soap solution may be determined by measuring the excess pressure inside a spherical soap bubble of known radius. Jaeger made the method a general one by blowing a bubble of air (or other suitable gas) from a tube in the interior of a liquid (Fig. 168 (a)). The bubble will have the shape of a segment of a sphere if the radius of the tube is small, and as its volume increases its radius decreases until it is hemispherical, as in Fig. 168 (b), when its radius is equal to that of the tube and the pressure inside it is a maximum. After passing the hemispherical form the bubble becomes unstable because its radius increases, and hence the equilibrium pressure inside it becomes less as more air goes in. Let r be the radius of the tube, h the depth of its dip (the technical term for the end of the tube) below the surface of the liquid, ρ the density of the liquid and γ its surface tension, and B the atmospheric pressure on the surface of the liquid.

$$\text{Maximum pressure inside the bubble} = \frac{2\gamma}{r} + B + g\rho h.$$

The maximum pressure is measured by means of a manometer containing a liquid of low density such as xylol. Suppose the difference in level in the arms of the manometer is h' and the density of the liquid used in the manometer is ρ' (Fig. 168 (a)):

$$\frac{2\gamma}{r} + B + g\rho h = B + g\rho'h'.$$

$$\therefore \frac{2\gamma}{r} = g(\rho'h' - \rho h).$$

The dropping funnel on the right of Fig. 168 (a) is a convenient method of blowing the bubbles under the liquid in the vessel on the left. As liquid enters the flask from the funnel it decreases the volume of the air in the flask and hence gradually increases the pressure in the bubble.

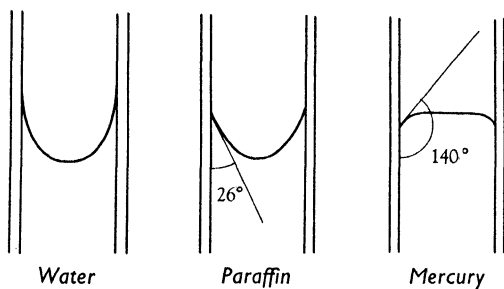


Fig. 169

The meniscus and angle of contact

The surface of a liquid is usually curved where it meets a solid surface. Fig. 169 shows the surface, called a *meniscus*, of each of three liquids in narrow glass tubes.

The shape of the water meniscus is due to the fact that the *adhesion* between water and glass, i.e. the attraction of the glass molecules for the water molecules, is greater than the *cohesion* of water, i.e. the attraction of the water molecules for each other; and the same is true of paraffin. The meniscus of mercury curves the opposite way because the cohesion of mercury is greater than the adhesion between mercury and glass.

The angle at which the liquid surface meets the solid surface is called the *angle of contact*. For water in contact with clean glass it is 0° , for paraffin 26° and for mercury 140° .

Most organic liquids, like water, have zero angle of contact with a clean glass surface. Contamination of the surfaces has a considerable

effect on the angle of contact, and the value for water may be as high as 15° on greasy glass. The angle of contact for water on different metals varies between 3° and 11° ; for water on paraffin wax it is 105° .

A simple and accurate method of finding the angle of contact is shown in Fig. 170. The glass plate is tilted until the surface of the liquid is horizontal where it meets the glass on one side of the plate; θ is the angle of contact. The surface can be 'scraped' clean of contamination before taking a measurement by drawing across it the edge of a sheet of blotting paper or a 'barrier' consisting of a strip of plate glass.

It has been found that the angle of contact differs when the liquid is advancing over a dry surface from the value when the liquid is receding

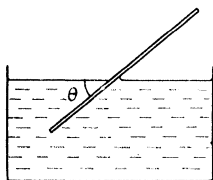


Fig. 170

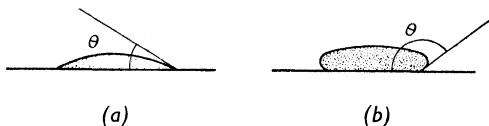


Fig. 171

over a wet one. This is particularly marked on a contaminated surface and may be observed in rain drops running down a window-pane. Every now and again the drops stick, and it can be seen that the lower edge of a drop makes a much larger angle with the pane than the upper edge. In the experiment of Fig. 170 readings are taken about a minute after the dry plate has been slightly lowered into the liquid (advancing angle), and at about the same interval after the plate has been slightly raised out of the liquid (receding angle).

When the angle of contact is zero, or nearly so, the liquid is said to *wet* the surface completely. If the angle of contact is appreciable but less than 90° the liquid is said to *wet* the surface, and if the angle is greater than 90° the liquid is said not to *wet* the surface. Thus in Fig. 171 (a) represents a drop on a surface which it wets and (b) a drop on a surface which it does not wet, θ being the angle of contact in each case.

Measurement of surface tension with a capillary tube

Most liquids tend to rise in a narrow capillary tube as ink rises through the pores of blotting paper or oil through the wick of a lamp (Fig. 172). The phenomenon, known as *capillarity*, may be regarded as due to the

difference of pressure set up as a result of the curvature of the meniscus. If the tube is narrow the meniscus has a spherical form and, if the angle of contact is zero, the radius of the meniscus is equal to that of the tube, r . The pressure just above the concave side of the meniscus must exceed that just below the convex side by $2\gamma/r$, where γ is the surface tension of the liquid (see p. 180). The capillary tube acts as a manometer and the difference of pressure it records is $g\rho h$, where ρ is the density of the liquid and h is the height the liquid rises in the

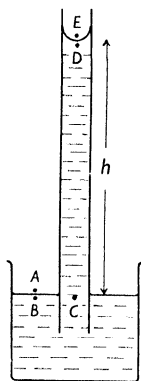


Fig. 172

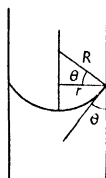


Fig. 173

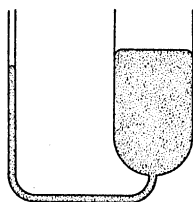


Fig. 174

capillary tube to compensate for the deficiency of pressure below the meniscus*.

$$\therefore \frac{2\gamma}{r} = g\rho h,$$

$$\gamma = \frac{rg\rho h}{2}.$$

The phenomenon provides an accurate method of measuring the surface tension of a liquid. The radius r may be found with a travelling microscope after breaking the tube at the point to which the meniscus

* In Fig. 172,

Pressure at A = Pressure at B (since the liquid surface is flat here)

= Pressure at C .

= Pressure at D + $g\rho h$.

Pressure at E = Pressure at D + $\frac{2\gamma}{r}$.

Assuming the pressures at A and E to be the same (i.e. ignoring the change in atmospheric pressure with height h),

$$\frac{2\gamma}{r} = g\rho h.$$

rose, or by introducing a thread of mercury into the tube and measuring its length and weight. The latter method is less satisfactory than the former if the cross-section of the tube is not accurately uniform, because it gives an average value of r over a length of the tube instead of the value at the meniscus.

If the angle of contact of the liquid is not zero, but has a value θ , the meniscus is still of spherical form, if the tube is sufficiently narrow, but it is less than a hemisphere, and its radius R is greater than that of the tube r . It can be seen from Fig. 173 that

$$R = \frac{r}{\cos \theta}.$$

Since the difference in pressure across the meniscus is $2\gamma/R$, the equation for the surface tension becomes

$$\gamma = \frac{r g \rho h}{2 \cos \theta}.$$

Hence θ must be known before γ can be calculated. The method is commonly used to determine θ , using a value of γ obtained from a method not involving the angle of contact.

In the case of a liquid like mercury, which does not wet the glass, the pressure set up by the convex meniscus forces the liquid down the capillary tube, producing a depression rather than an elevation of the liquid in the tube. The depression can best be demonstrated and measured by means of a U tube having as one arm a narrow capillary tube and as the other a much wider tube (Fig. 174).

Ferguson's method

There are several variants of the capillary tube method of measuring surface tension. One is to force the meniscus down the capillary tube to the level of the liquid outside by increasing the air pressure over the meniscus. This increase in air pressure, p , can be measured by a manometer employing, preferably, a lighter liquid than that whose surface tension is being measured. The surface tension can then be obtained from the relation $p = 2\gamma/r$.

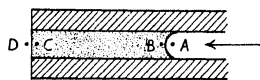


Fig. 175

Another variant, due to Ferguson, is accurate and has the advantage that only a very small quantity of liquid is required. The liquid is drawn into the end of a horizontal thick-walled capillary tube, and air pressure is applied in the direction of the arrow in Fig. 175 until the meniscus at the end of the tube is accurately plane. The pressures at

D , C and B are atmospheric, while that at A exceeds atmospheric by $2\gamma/r$, where r is the radius of the meniscus, which is the same as that of the tube if the angle of contact is zero and r does not exceed about

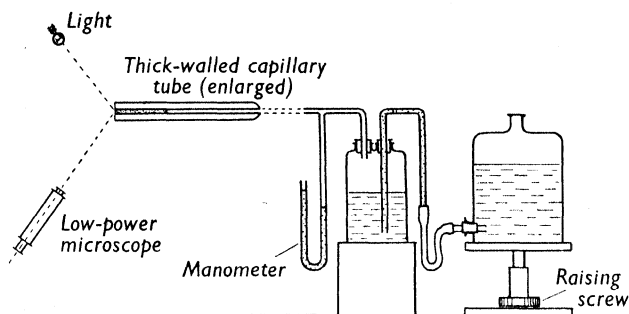


Fig. 176. Ferguson's apparatus for measuring surface tension.

1 mm. The complete apparatus is shown in Fig. 176. When the bottle on the right is raised (by a screw) water passes from it to the next bottle, thereby increasing the air pressure which is measured by the manometer. The stage at which the meniscus at the end of the capillary tube is plane can be judged precisely by illuminating it with a lamp and observing it through a low-power microscope; when the meniscus is plane it appears to be uniformly illuminated.

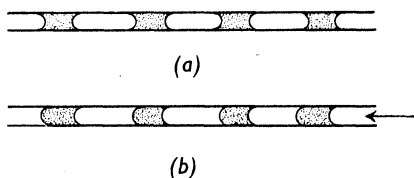


Fig. 177

Ferguson's method is reminiscent of what are known as Jamin's tubes. A large difference of pressure can be maintained between the ends of a capillary tube containing a number of short threads of liquid without blowing out the liquid. Fig. 177 (a) shows the threads before, and (b) after, pressure is applied in the direction of the arrow; the curvature of the left-hand meniscus of each thread changes, with the result that there is a drop of pressure across each meniscus as we proceed from right to left.

If the capillary tube in Fig. 172 is pushed so far down into the liquid that a height of less than h is left protruding above the surface, the

liquid will rise to the top of the tube, and then its curvature will become reduced sufficiently for equilibrium to be established.

The curvature of the mercury meniscus in a thermometer or barometer may differ when the mercury is rising from that when it is falling. The adjustment of the curvature of the meniscus to neutralise changes of pressure is responsible for the jerky motion of the mercury in these instruments.

Attractions between floating bodies

Light bodies floating in liquids, such as bubbles on the surface of a cup of tea, tend to cluster together. This is an instance of capillarity. If the liquid wets two bodies it rises between them by capillary attrac-

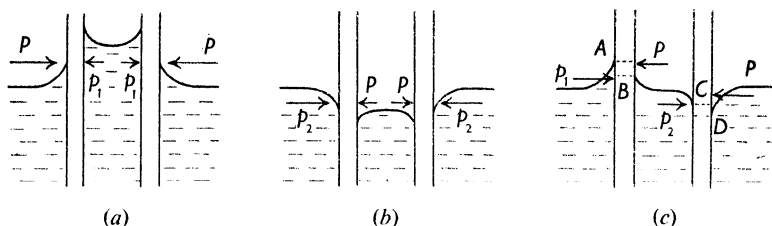


Fig. 178. (a) Bath bodies wetted, $P > p_1$; (b) bath bodies unwetted, $p_2 > P$;
(c) one body wetted, the other unwetted, $p_2 > P > p_1$.

tion when they come sufficiently close together. In Fig. 178 (a) the atmospheric pressure P is greater than the pressure p_1 of the liquid because of the pressure deficiency under the curved meniscus between the bodies; hence the bodies are forced together. Two bodies which are both unwetted by the liquid are forced together also; in Fig. 178 (b) $p_2 > P$.

A body which is wetted by the liquid and one which is unwetted tend to be forced apart. In Fig. 178 (c) the meniscus between the bodies rises only to a height B , which is lower than the height A to which the liquid outside rises, because it is being pulled down by the depression at C . Similarly, C is not so depressed as D because the meniscus between the bodies is being pulled up at B . By the same arguments as applied to cases (a) and (b) $p_2 > P > p_1$. Hence the bodies are forced apart.

Review of the various methods of measuring surface tension

The relative accuracy and importance of those methods of measuring surface tension already described will now be considered, and also the other main methods available.

(1) *Capillary tube method and its variants.* The capillary rise method is the ultimate standard and the values given in *International Critical Tables* are based upon it. Indeed, other methods are judged from the point of view of accuracy by how far they can reproduce the following results, obtained by the capillary rise method at 20° C. and under atmospheric pressure:

Surface tension of water = 72.75 ± 0.05 dyne per cm.

Surface tension of benzene = 28.88 ± 0.03 dyne per cm.

In the most accurate determinations with a capillary tube, corrections must be made for the departure of the meniscus from a perfect hemispherical form.

Ferguson's method is particularly suitable when only a small quantity of the liquid is available.

(2) *Jaeger's method.* This method has many advantages: it does not involve the angle of contact, there is particularly good temperature control, and contamination is avoided because a fresh surface is formed continually. Nevertheless, the method is unsuitable for absolute measurements since the bubble's maximum pressure does not seem to occur when its radius is exactly equal to that of the tube. The apparatus is usually calibrated with a liquid of known surface tension.

It is difficult to measure the depth of the tip of the tube below the surface (Fig. 168) to an accuracy greater than 0.1 mm., and allowance must be made for the curvature of the surface of the liquid in the vessel unless the diameter of the latter is over 3 cm. Sugden has avoided these difficulties by using two tubes of different radii r_1 and r_2 at exactly the same depth below the surface, finding the excess pressures p_1 and p_2 inside the bubbles over that of the surrounding liquid by two manometers:

$$p_1 - p_2 = 2\gamma \left(\frac{1}{r_1} - \frac{1}{r_2} \right).$$

(3) *Pulling a wire or plate out of the surface.* Instead of the glass plate described on p. 178, which is commonly used in the school laboratory, a fine ring of platinum-iridium wire hanging from a torsion wire has been employed for the most accurate determinations made by this method. Elementary theory gives the pull required to detach the wire from the surface as $4\pi r\gamma$, where r is the mean radius of the ring (since, if the wire is thin, both the internal and external circumferences of the ring can be taken as $2\pi r$), but correcting factors must be applied and there is uncertainty about the exact shape of the liquid film clinging to the ring when the latter breaks away. The method is not, therefore,

a good one for making absolute determinations, but it is a rapid and accurate way of making comparative measurements.

(4) *The drop weight method.* This method is the most popular of all since it is so easy to carry out. It consists of weighing drops formed slowly from a tube of known radius whose tip has been ground sharp and plane. The earliest and somewhat crude formula for the weight, mg , of a drop falling from a tube of radius r , was $mg = 2\pi r\gamma$. This assumes that the film round the rim of the tube is vertical and that all the drop falls, two unwarrantable assumptions, as can be seen from Fig. 179. Harkins and Brown in 1919 investigated the method thoroughly and obtained the expression $mg = k\gamma$, where k depends on the radius of the tube and the volume of the drop. As a result of their laborious standardisation experiments tables are now available for determining surface tension accurately by this method.

(5) *Other methods.* The foregoing are the most important methods but the following have also been used:

(a) Measurement of the height and diameter of a large drop, such as a drop of mercury on a glass plate.

(b) Measurement of the period of oscillation of jets from non-circular orifices. (A stream of water often has a twisted or screwed appearance due to the vibration of its surface.)

(c) Measurement of the wave length of ripples on the surface of the liquid.

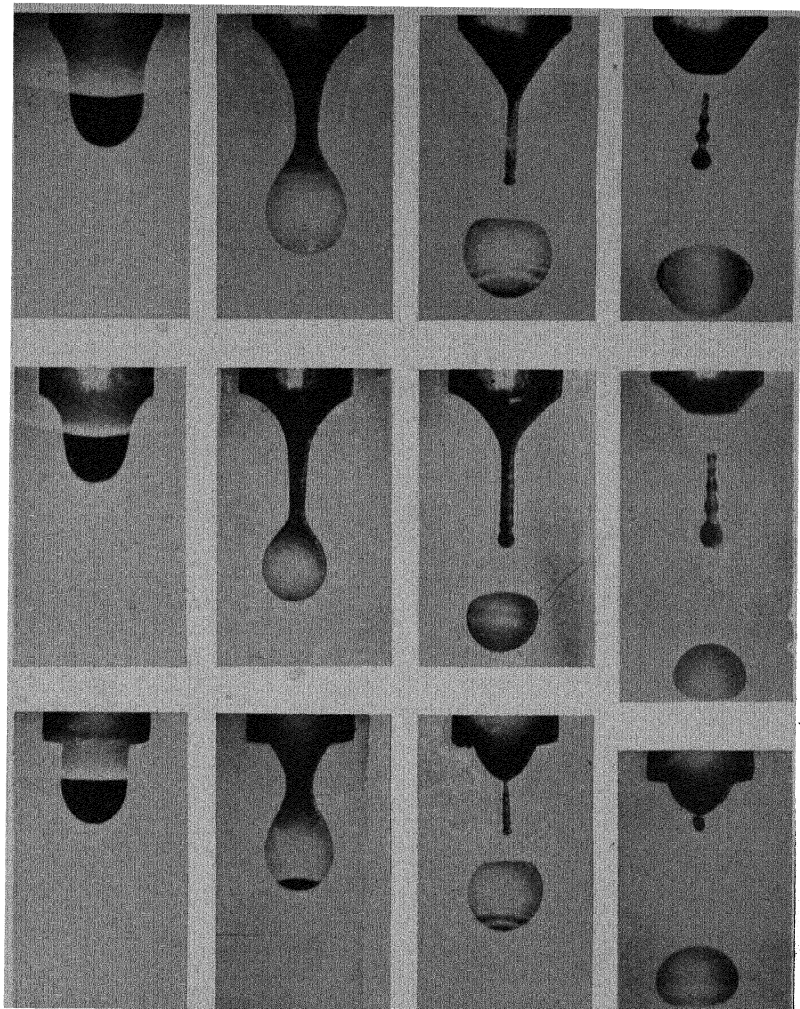
Variation of surface tension with temperature

The variation of the surface tension of a liquid with temperature may be investigated with a capillary tube, the U tube in Fig. 180 being a convenient form. If h is the difference in level between the liquid surfaces in the two arms of the U tube, r_1 and r_2 are the radii of the narrow and wide arms respectively, ρ is the density and γ the surface tension of the liquid, then

$$\frac{2\gamma}{r_1} - \frac{2\gamma}{r_2} = g\rho h.$$

The liquid in the U tube is heated in a water-bath, and corresponding values of h and temperature are recorded. Jaeger's method is also a convenient one for finding the variation of surface tension with temperature.

The surface tension of a liquid decreases with increasing temperature and must vanish at the critical temperature, i.e. the temperature at which the surface of the liquid disappears and liquid and vapour are



G. A. Goode and D. G. A. Dyson

Fig. 179. Formation of drops falling from a glass tube at the rate of about two per second. In the top line water was used. In the second line a little Kodak Wetting Agent was added to the water to lower its surface tension; note that the drop is considerably smaller. In the third line the end of the tube was waxed and the drop forms from the inside bore of the tube instead of from the outer (water being used). In each case there is formed a long cylindrical neck which is unstable when its length is greater than its circumference and which breaks up into small drops known as Plateau's spherules. The photographs were taken by illuminating the falling drops by intermittent flashes of light admitted through a slot in a rotating disc. A large number of random photographs were thus obtained from which the above sequences were selected.

indistinguishable. Ferguson discovered independently in 1916 the following formula, proposed in 1894 by van der Waals:

$$\gamma = \gamma_0(1 - bt)^n,$$

where γ_0 is the surface tension of the liquid at 0°C. , $t^\circ \text{C.}$ is the temperature, b is a constant for a given liquid and n is approximately 1.2 for all liquids. The critical temperature (for which $\gamma = 0$) is clearly $1/b$, and its value, obtained from surface tension measurements, agrees very well with direct measurements.

The formula applies only to unassociated liquids, i.e. liquids in which the molecules do not become attached to each other. There is some evidence that water at ordinary temperatures contains both single H_2O molecules and groups consisting of two attached H_2O molecules; thus water is an associated liquid at ordinary temperatures.

The removal of grease stains from a garment by placing it on blotting paper and pressing it with a hot iron depends on the reduction of surface tension with rise of temperature. The grease is drawn away from the hot iron into the cooler blotting paper.

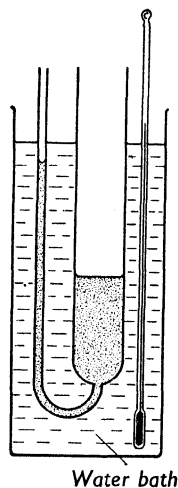


Fig. 180

Movement of camphor particles on the surface of water

Tiny scrapings of camphor dropped on to the surface of clean water dart about, spin round, and exhibit a restless agitation. This is due to the fact that a solution of camphor in water has a smaller surface tension than pure water. The tiny particles of camphor do not dissolve in the water equally in all directions, and they are drawn away by the stronger surface tension of the water from the direction in which they have dissolved most copiously.

Lord Rayleigh, in 1890, determined the minimum thickness of a layer of oil (which reduces the surface tension of water) necessary to stop the agitation of camphor particles on water. He found that a drop of olive oil, of mass 0.81 mgm., stopped the movement of the camphor particles over an area of 555 sq. cm., giving a value for the thickness of the oil layer of 1.6×10^{-7} cm. He suggested that the oil layer was then one molecule thick, and that its thickness represented the diameter of the oil molecule. This marked the beginning of the important modern

study of monomolecular surface films, from which much information has been obtained of the size, shape and structure of molecules.

The cause of surface tension

Surface tension is caused by the inward, unbalanced attraction on the surface molecules by the molecules below them. Every molecule attracts, and is attracted by, the molecules surrounding it, but this attraction has a restricted range, known as the 'sphere of molecular attraction'. It can be seen from Fig. 181, in which the spheres of molecular attraction of two molecules are drawn, that, while a molecule in the interior of a liquid is attracted equally in all directions, one in the surface is attracted downwards.

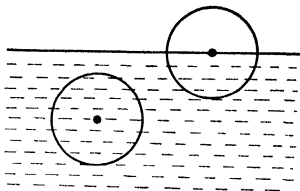


Fig. 181

The fact that the pressure is greater under a convex liquid surface than under a plane surface, and greater under a plane surface than under a concave surface, may be explained on similar lines. In Fig. 182 the spheres of attraction of molecules the same distance below each of the three shapes of surface are shown, being very much exaggerated in size for the sake of clearness. The dark shaded portions of the spheres

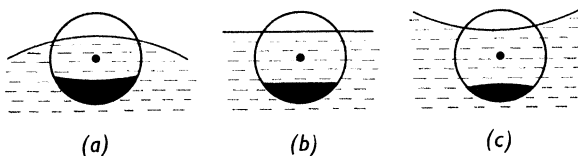


Fig. 182

are symmetrical with the portions above the liquid surface, and hence represent the resultant downward pull on the molecule. It can be seen that the downward pull is greatest on a molecule below a convex surface and least on one below a concave surface. The attractions between the molecules die away to a negligible value after such a short distance (of the order of a molecular diameter) that they cannot be gravitational. If the force were gravitational and obeyed an inverse square law 'the surface tensions of the ocean would be far greater than that of a cupful of water because the distant parts would act with sensible effect.'* Hawksbee, who in a book published in 1709, was the first to explain

* N. K. Adam, *Physics and Chemistry of Surfaces*. Oxford.

the rise of a liquid in a capillary tube as due to the attraction of the glass for the liquid, noted that a liquid does not rise higher in a thick-walled tube than in a thin one, and concluded that the attraction must be confined to the molecules near the surface of the glass. It is now realised that the forces between the molecules causing surface tension must be the same as those responsible for cohesion, solution and chemical reaction.

Surface energy

Suppose the film in Fig. 183 (considering a single surface only) is stretched isothermally, i.e. without change of temperature, a distance x .

$$\text{Work done} = \gamma l x$$

$$= \gamma \times \text{increase in area.}$$

Thus *the surface tension is numerically equal to the work done in enlarging the surface of the liquid by unit area under isothermal conditions.*

This work is necessary to draw molecules to the surface from the liquid below it against the downward attraction of the neighbouring molecules.

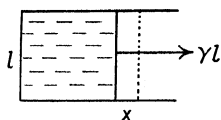


Fig. 183

When a surface is enlarged its temperature tends to fall because, as the molecules are drawn from the interior against the attractive force, their thermal agitation is retarded. This is an example of Le Chatelier's principle: 'If one of the factors of any system in equilibrium is changed, thus disturbing the equilibrium, the effect produced tends to restore that factor to its original value.' A lowering of the temperature of a liquid surface raises its surface tension and thereby increases the force opposing the enlargement of the surface. It can be shown (see Appendix, p. 207) that the heat that must be supplied to a surface when it is extended isothermally by unit area is $T \frac{d\gamma}{dT}$ (in ergs), where T is the absolute temperature and $\frac{d\gamma}{dT}$ is the rate of change of surface tension with temperature. Thus if E is the total energy supplied when a film is extended by unit area isothermally,

$$E = \gamma - T \frac{d\gamma}{dT}.$$

The negative sign is necessary because $\frac{d\gamma}{dT}$ is negative, so that $-T \frac{d\gamma}{dT}$ is a positive quantity which is added to γ . E is called the *total surface energy* and γ is called the *free surface energy*.

The surface tension for water at 15° C. is 74 dynes per cm.

$$\frac{d\gamma}{dT} = -0.148.$$

$$\therefore E = 74 + (288 \times 0.148) = 117 \text{ ergs per sq. cm.}$$

When changes in the area of a surface are made isothermally, and the heat which must be given to, or taken away from, the surface to ensure this is ignored, only the free surface energy γ need be considered; this is a great advantage because free surface energy corresponds to potential energy in a conservative system (p. 71).

The conception of γ as free surface energy per unit area is much closer to physical reality than the conception of force per unit length, since the surface molecules are being pulled inwards at right angles to the surface and there is no special force between the surface molecules parallel to the surface. The dimensions (see p. 225) of energy per unit area and of force per unit length are the same, MT^{-2} . Historically the conception of tension came first and it is always legitimate, and often convenient, to attack problems from the point of view of tension rather than of energy, as has been done so far in this chapter. There are, however, phenomena which are better understood from the point of view of energy, and some of these will be considered shortly.

The conception of free surface energy throws light on the phenomena already discussed. The surface area of a liquid tends to be a minimum because the potential energy of a system tends to be a minimum. The fact that the pressure is greater on the concave side than on the convex side of a curved liquid surface is seen to be due to the work that must be done to enlarge the area when the surface is displaced parallel to itself from the concave to the convex side, as when a spherical soap bubble is blown up.

The work done in blowing up a spherical soap bubble, assuming that it is done isothermally, is clearly equal to the total surface area of the bubble times the surface tension.

It is possible to apply the conception of free surface energy to find the excess pressure inside a spherical bubble or drop. Suppose a drop has a radius r , surface tension γ , and the excess pressure inside it is p . The area of the drop A is given by

$$A = 4\pi r^2.$$

$$\therefore \frac{dA}{dr} = 8\pi r.$$

By the principle of Virtual Work, when a body in equilibrium is displaced by a small amount, in accordance with the geometrical

conditions, the sum of the virtual works is zero. Let the radius of the drop increase by a small amount, δr , isothermally.

Work done by the excess pressure $= 4\pi r^2 p \delta r$.

Work done to increase the free surface energy $= \gamma \delta A$
 $= \gamma \times 8\pi r \delta r$.

$$\therefore 4\pi r^2 p \delta r = \gamma \cdot 8\pi r \delta r,$$

$$p = \frac{2\gamma}{r}.$$

This applies to a liquid drop with a single surface. The expression for a soap bubble, which has two surfaces, is $p = 4\gamma/r$.

The student should apply the same method to the case of a cylindrical drop or bubble.

Suppose a spherical drop is evaporating under isothermal conditions. Latent heat must be supplied from the surroundings but some energy comes from the surface energy of the drop whose area is decreasing. When the radius has decreased to a certain value, the change in surface energy is enough, and then more than enough, to supply the latent heat. Hence tiny drops cannot exist unless there is a very high degree of supersaturation; dust or other particles are needed for drops to form so as to cause them to exceed the critical size.

Spreading of one liquid over another

When two immiscible liquids are in contact, the boundary between them exhibits surface tension called the *interfacial tension*. This can be measured by the drop method, drops of the denser liquid being allowed to form in the lighter one. The interfacial tension is always less than that of the liquid with the higher surface tension because the molecules at the common surface experience attractions from both liquids. The resulting surface tension is therefore less than if air were on one side. Zero interfacial tension means complete miscibility.

A liquid A spreads over the surface of a liquid B with which it does not mix if the process involves a reduction in the free surface energy, i.e. if energy is released rather than absorbed. Suppose γ_A and γ_B are the surface tensions (or free surface energies) of the two liquids and γ_{AB} is the interfacial tension. If some of the liquid A spreads over unit area of the liquid B , the increase in the free surface energy is $\gamma_A - \gamma_B + \gamma_{AB}$. Spontaneous spreading will occur if the free surface energy is thereby reduced, and hence if $\gamma_A - \gamma_B + \gamma_{AB}$ is negative.

The values of this expression for liquids (*A*) with respect to water (*B*) in ergs per sq. cm. are as follows:

Benzene: -8.86 , spreads fairly well on water;

Oleic acid: -24.57 , spreads very well on water;

Carbon disulphide: $+7.26$, does not spread on water.

The molecule of oleic acid, which spreads very well, contains a COOH group which is soluble in water, and it is believed that this is attracted into the water while the rest of the molecule sticks out at right angles to the surface. Such liquids form a layer one molecule thick on water, while the excess liquid collects into lens-shaped drops on the surface. Globules of fat, too, float on water in equilibrium with a monomolecular layer of fat covering the rest of the surface.

Oils used for calming waves when making rescues at sea must have good spreading power, and fish oils, which comply with this requirement much better than mineral oils, are carried in ships and in life-boats. The action of the oil is to damp out the ripples which cause the breaking of the big waves. The effect on the amplitudes of the big waves or swell is inappreciable, but ships can ride a swell without danger if the waves do not break. The wind drives the oil forward leaving a clean water surface behind. Since energy is released when the oil spreads over water, energy must be absorbed when the opposite process takes place and this absorption of energy has a damping action.

Spreading of a liquid over a solid

Although it may be argued that a solid cannot have a surface tension, a solid must, nevertheless, have surface energy. The molecules in the surface must possess energy by virtue of their position. The free surface energy of a solid is not directly measurable, but the conception can be used and can, indeed, be replaced by the idea of a tension.

Consider a drop of a liquid *A* resting on a solid *B* (Fig. 184). Let the angle of contact be θ and let the surface tensions at the various interfaces be as shown. The three forces are in equilibrium and hence

$$\gamma_B = \gamma_{AB} + \gamma_A \cos \theta.$$

The same result can be obtained from a consideration of free surface energy. Suppose the liquid surface is displaced a very small distance parallel to itself, thereby increasing the area of *B* in contact with *A* by δa (Fig. 185). The increase in the free surface energy for a slight dis-

placement must be zero, since the system is in equilibrium (Principle of Virtual Work).

$$\therefore \gamma_{AB}\delta a + \gamma_A\delta a \cos \theta - \gamma_B\delta a = 0.$$

$$\therefore \gamma_B = \gamma_{AB} + \gamma_A \cos \theta,$$

$$\cos \theta = \frac{\gamma_B - \gamma_{AB}}{\gamma_A}.$$

The maximum value of $\cos \theta$ is 1 (when $\theta = 0$); hence if $\gamma_B - \gamma_{AB} > \gamma_A$, equilibrium is impossible and the liquid will spread over the solid.

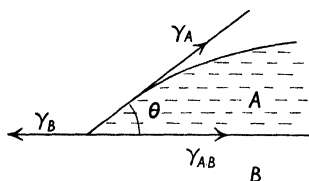


Fig. 184

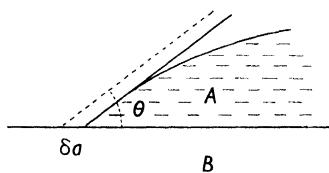


Fig. 185

Free surface energy and capillarity

The source of the energy which enables a liquid to rise in a capillary tube is the decrease of free surface energy.

Suppose a liquid of density ρ rises a distance h in a glass capillary tube of radius r . Let γ_A , γ_B and γ_{AB} be the free surface energies of the liquid-air, glass-air, and liquid-glass surfaces respectively, and the angle of contact be θ . The area of the glass-air surface is reduced, and that of the liquid-glass surface is increased, by $2\pi rh$.

$$\therefore \text{Decrease in free surface energy} = (\gamma_B - \gamma_{AB}) 2\pi rh.$$

But

$$\gamma_B - \gamma_{AB} = \gamma_A \cos \theta \quad (\text{see above}).$$

$$\therefore \text{Decrease in free surface energy} = \gamma_A \cos \theta 2\pi rh.$$

The centre of gravity of the liquid in the capillary tube has been raised by $\frac{1}{2}h$, and hence the gain in potential energy of the liquid is $\pi r^2 h \rho \frac{1}{2}hg$.

At first sight it might seem that the gain in potential energy of the liquid should be equal to the decrease in the surface energy; but the gain in potential energy of the liquid is, in fact, equal to only half the decrease in free surface energy. Some energy must be supplied for the necessary work which must be done against viscosity (the internal friction of the liquid) when the liquid moves up the tube. But why is exactly half of the free surface energy converted into the potential energy of the liquid?

As the liquid rises higher up the tube the work which must be done to supply its potential energy increases, but the free surface energy released per unit length remains constant. At the height h the two have become equal, and all the released free surface energy is utilised in increasing the potential energy of the liquid; at zero height a negligible part of the free surface energy is required for increasing the potential energy of the liquid. Thus, over the whole of the height h , on an average, only one-half of the free surface energy is converted to potential energy of the liquid.

$$\therefore \frac{1}{2}\gamma_A \cos \theta \cdot 2\pi rh = \pi r^2 h \rho \frac{1}{2}hg,$$

$$\gamma_A = \frac{r h \rho g}{2 \cos \theta}.$$

An alternative method of obtaining this expression is to apply the principle of virtual work. When standing at a height h , the liquid is in equilibrium. Hence if the height is changed by a small amount δh the changes in the free surface energy and in the potential energy of the liquid must be equal, i.e.

$$2\pi r \delta h (\gamma_B - \gamma_{AB}) = \pi r^2 \delta h \rho g h.$$

$$\therefore \gamma_B - \gamma_{AB} = \frac{r h \rho g}{2}.$$

$$\therefore \gamma_A = \frac{\gamma_B - \gamma_{AB}}{\cos \theta} = \frac{r h \rho g}{2 \cos \theta}.$$

Industrial applications of the theory of spreading

The theory of spreading has direct application to the industrial process of the flotation of mineral ores. Many millions of tons of ore are treated every year by the flotation process with the object of separating the heavy mineral, which sinks in water, from the other constituents of the powdered ore which stay at the water-air surface. In order that the mineral shall sink, the water must spread over each particle and wet it; special substances called depressants are added to the water and they act selectively on the mineral to ensure this. On the other hand, the particles of the constituents which are to float must not be wetted; substances called collectors, which are adsorbed at the surface of the particles, give them a finite angle of contact with the liquid and hence prevent them from being wetted. Substances called frothers are also added to the water to make foam and give a large water-air surface.

The washing of fabrics consists largely of the removal of particles of dirt held to the fabric by grease. The detergent solution must be able to spread over the fabric and displace the grease, thereby releasing the dirt.

The process of soldering involves the wetting of metals by liquid metals. Calling the solder A and the metal being soldered B , soldering can be done if $\gamma_A - \gamma_B + \gamma_{AB}$ is negative. Thus a good flux should clean the surface to increase γ_B and perhaps lower γ_{AB} . Spreading occurs most readily if γ_A is small, but for penetration into joints a high value for γ_A is desirable.

In some industrial processes it does not matter whether or not the liquid will spread over the solid spontaneously. In painting and spraying, for example, the requirement is that the liquid, when spread, shall maintain a stable layer. Active substances called wetting agents are added to the paint to prevent it from retreating from a surface over which it is spread.

Surface tension and vapour pressure

The vapour pressure above a convex liquid surface is greater than that above a plane surface, and the vapour pressure above a plane surface is greater than that above a concave surface. The effect is so

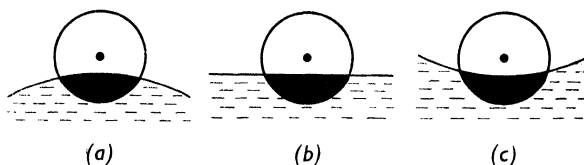


Fig. 186

small as to be negligible at ordinary curvatures, but it is of decisive importance at the great curvatures of minute drops which are just in process of formation.

It was explained on p. 195 that a molecule just inside a convex liquid surface is more strongly attracted inwards than a molecule just inside a plane surface (thus accounting for the excess pressure inside a convex surface). It might be thought therefore that the vapour pressure at a convex surface would be less than at a plane surface. But once molecules have escaped from the surface into the vapour above it they are less strongly attracted back into a convex surface than into a plane surface, and hence there is a larger saturation vapour pressure near the convex surface. Similarly, vapour molecules above a concave surface

are more strongly attracted to the surface than molecules over a plane surface. Fig. 186 represents three molecules, the same distance above the three types of surface, and their spheres of molecular attraction, greatly exaggerated for the sake of clearness, have been drawn. It can be seen that the molecule least strongly attracted to the liquid is that in (a) and the one most strongly attracted is in (c).

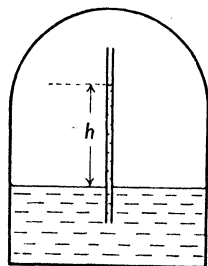


Fig. 187

The phenomenon can also be explained from the point of view of surface energy. Condensation of vapour on the convex surface of a drop increases the surface area and hence increases the free surface energy. The tendency in any system is for the free or potential energy to diminish. Thus surface tension opposes condensation and hence increases the effective vapour pressure. In the case of a plane surface the area is unchanged after condensation and in the case of a concave surface it is diminished.

The magnitude of the effect can be calculated by considering the capillary tube in the enclosed space in Fig. 187 which contains only liquid and vapour. The pressure above the concave surface in the capillary tube is less than that above the flat surface of the liquid by $g\sigma h$, where σ is the density of the vapour, owing to the fact that the surfaces differ in level by h . If the saturation vapour pressure at the concave meniscus were the same as that at the plane surface there would be continual evaporation at the concave meniscus and condensation at the flat surface, with a consequent continual flow up the capillary tube—perpetual motion without any external supply of energy. To fulfil the principle of the conservation of energy it must be assumed that the vapour pressure at the concave surface is less than that at the plane surface by $g\sigma h$. But

$$h = \frac{2\gamma}{r\rho g}, \quad (\text{see p. 187})$$

where γ is the surface tension, ρ the density of the liquid, and r is the radius of the tube.

$\therefore$ Decrease in vapour pressure over the concave meniscus

$$\begin{aligned} &= g\sigma h \\ &= \frac{2\gamma\sigma}{r\rho}. \end{aligned}$$

At 15°C . the saturated vapour pressure of water over a plane surface is 1.28 cm. of mercury. Before the change in the vapour pressure over

a curved surface can be calculated, the density of saturated water vapour at 15° C. must be found. Assuming that 18 gm. of water vapour occupy 22.4 litres at N.T.P.,

Volume occupied by 18 gm. of water vapour at 15° C. and 1.28 cm. of mercury

$$= \frac{22.4 \times 1000 \times 288 \times 76}{273 \times 1.28} \text{ c.c.}$$

$\therefore$ Density of water vapour

$$= \frac{18 \times 273 \times 1.28}{22.4 \times 1000 \times 288 \times 76} = 1.28 \times 10^{-5} \text{ gm. per c.c.}$$

The surface tension of water at 15° C. is 73 dynes per cm. and the density may be taken as 1 gm. per c.c.

Change in the vapour pressure over a surface of radius r cm.

$$\begin{aligned} &= \frac{2\gamma\sigma}{r\rho} \\ &= \frac{2 \times 73 \times 1.28 \times 10^{-5}}{r \times 1} \text{ dynes per sq. cm.} \\ &= \frac{2 \times 73 \times 1.28 \times 10^{-5}}{r \times 13.6 \times 981} \text{ cm. of mercury} \\ &= \frac{1.4 \times 10^{-7}}{r} \text{ cm. of mercury.} \end{aligned}$$

Thus in the case of a drop of radius 10^{-7} cm. the increase in the vapour pressure will be 1.4 cm. of mercury and the total vapour pressure will be more than double that over a plane surface. Hence, unless the air is supersaturated to this extent, the small drop must immediately evaporate. In other words such drops cannot exist in normal circumstances. If, however, the drops form on dust particles of radius 10^{-6} cm. the increase in vapour pressure will be only 0.14 cm., about 10% more than the vapour pressure over a plane surface, and with particles of larger radius the increase is still less. Drops having these radii can therefore persist. The soot particles which are plentiful in the atmosphere of industrial towns are a potent cause of fogs because they provide nuclei for the formation of drop which persist because they are in equilibrium with their surroundings.

In the Wilson cloud chamber, tracks of particles are made visible by the condensation of water vapour on gaseous ions formed along the track of the particle. The space inside the cloud chamber is dust free and saturated with water vapour. It is expanded suddenly and the consequent cooling causes supersaturation. If the volume expands 1.4 times causing a degree of supersaturation of about 12 times, dense

clouds are formed in the absence of any nuclei whatever. Condensation occurs on gaseous ions, however, at a smaller degree of supersaturation than this, as a result of their electric charge. The self-repulsion of an electric charge acts in opposition to the surface tension so that the effective surface tension is reduced. Consequently the effect on the vapour pressure is also reduced and re-evaporation is minimised.

Porous articles, like cloths of cotton and linen, become damp in a saturated atmosphere. Once water has begun to collect in the pores, the concave surfaces, at which the vapour pressure is reduced, encourage continuous condensation because, with respect to them, the atmosphere is supersaturated.

SUMMARY

Two parallel and numerically equal concepts, surface tension (defined as force per unit length) and free surface energy (defined as work per unit increase in area under isothermal conditions), are used for the consideration of liquid surfaces. The latter is more in accord with the cause of surface tension, which is the inward attraction of the surface molecules of a liquid by the molecules immediately beneath the surface; the former was earlier historically and provides a simpler treatment of some phenomena.

The pressure on the concave side of any spherical liquid surface is always greater than that on the convex side by an amount $2\gamma/r$ and this fundamental fact may be used to explain the rise of a liquid in a capillary tube.

The spreading of a liquid over another liquid surface or over a solid surface may be considered most conveniently from the point of view of surface energy, which provides also an alternative treatment of capillarity.

The vapour pressure above a convex liquid surface is greater than that above a plane surface, and the vapour pressure above a plane surface is greater than that above a concave surface. This is of great importance in the formation of drops by condensation of vapour.

Liquid	Surface tension (dynes per cm.)	Angle of contact with clean glass
Water	73 at 15° C.	0
Mercury	465 at 20° C.	140°
Methyl alcohol	23 at 20° C.	0
Ethyl alcohol	22 at 20° C.	0
Benzene	29 at 17.5° C.	0
Paraffin oil	26 at 25° C.	26°
Turpentine	27 at 15° C.	17°
Soap solutions	25 (approx.)	0

APPENDIX I

Proof of the expression for the difference in pressure across a curved surface due to surface tension, $p = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$.

Fig. 188 represents small areas $ABCD$ of the two main types of surface: in (a) the principal radii of curvature R_1 and R_2 are in the same sense and this is called a synclastic surface; in (b) the principal radii of curvature are in opposite senses and the surface is anticlastic.

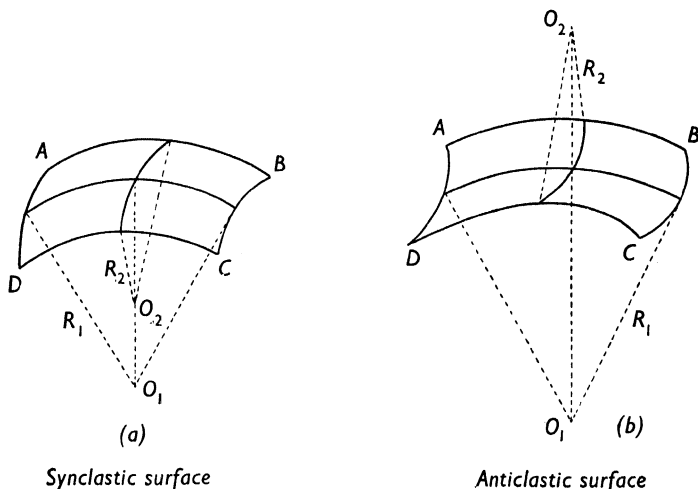


Fig. 188

Consider the synclastic surface of Fig. 189. Suppose the film subtends a very small angle $2\delta\theta$ at the centre of curvature O_1 , $AB = CD = \delta s$, $BC = AD = \delta x$ and the surface tension is γ . The rest of the film exerts forces $\gamma\delta x$ at right angles to BC and AD . The resultant of these two forces is along NO_1 .

$$\begin{aligned}
 \text{Resultant} &= 2\gamma\delta x \sin \delta\theta \\
 &= 2\gamma\delta x \delta\theta && \text{(since } \delta\theta \text{ is small)} \\
 &= \gamma\delta x \frac{\delta s}{R_1} && (\delta s = R_1 \cdot 2\delta\theta) \\
 &= \frac{\gamma}{R_1} \delta a, \quad \text{where } \delta a = \delta x \delta s = \text{area of } ABCD.
 \end{aligned}$$

The resultant of the similar forces on AB and CD

$$= \frac{\gamma}{R_2} \delta a.$$

These two forces are counteracted by the force due to the excess pressure on the concave side of $ABCD$.

$$\begin{aligned}
 \therefore p\delta a &= \frac{\gamma}{R_1} \delta a + \frac{\gamma}{R_2} \delta a, \\
 p &= \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right).
 \end{aligned}$$

A similar argument may be applied to Fig. 188 (b), but the pressure due to the two curvatures act in opposite directions; this is allowed for in the above equation by the fact that R_1 and R_2 are opposite in sign.

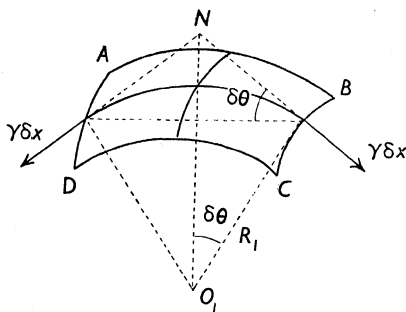


Fig. 189

APPENDIX II

Proof that the heat absorbed when a film is stretched isothermally is $T \frac{d\gamma}{dT}$ per unit area.

A surface film may be treated as a reversible heat engine and taken through the cycle ABCD in Fig. 190. Starting at A, let the film be stretched by an area a until it reaches the condition represented by B. Let the process take place very slowly in a constant temperature enclosure so that the absolute temperature T of the film remains constant. Let γ be the surface tension at temperature T and let the heat absorbed be h per unit area. The work done on the film is γa .

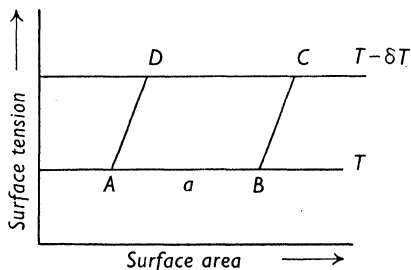


Fig. 190

Now let the film be caused to expand adiabatically from B to C so that the temperature falls a very small amount δT . Let the surface tension at the temperature $T - \delta T$ be $\gamma + \delta \gamma$. Continuing the cyclic process let the film contract isothermally by an area a from C to D, the work done by the film being $(\gamma + \delta \gamma) a$. To complete the cycle the film contracts adiabatically from D to A. The work done on the film from B to C is equal to the work done by the film from D to A.

Total work done by film in the course of the cycle $= (\gamma + \delta \gamma) a - \gamma a = a \delta \gamma$.

The efficiency of a reversible heat engine

$$= \frac{\text{Work done}}{\text{Heat absorbed from source}} = \frac{a\delta\gamma}{ha}.$$

But the efficiency of a reversible heat engine is known to be equal to

$$\frac{\text{Temperature difference}}{\text{Temperature of source}}, \text{ i.e. } \frac{\delta T}{T}.$$

$$\therefore \frac{a\delta\gamma}{ha} = \frac{\delta T}{T}.$$

$$\therefore h = T \frac{d\gamma}{dT}.$$

QUESTIONS

[Take $g=981$ cm. sec.⁻²]

1. Explain what is meant by surface tension, and describe one method which you have used for measuring it.

Suggest methods which might be suitable for measuring the surface tension of (a) liquid helium (temperature 4° K.), (b) molten sodium (temperature 150° C.).

(C. S.)

2. Describe in detail the experiments you would make, using a capillary tube method, to find the surface tension of a liquid which wets glass.

A circular ring of thin wire of 2 cm. mean radius, suspended horizontally by a thread passing through the 5 cm. mark on a metre rule pivoted at its centre, is balanced by a 5 gm. weight suspended from the 70 cm. mark. A beaker of liquid is then arranged so that the ring is just attached to the liquid surface when the rule is horizontal. It is found necessary to move the 5 gm. to the 80 cm. mark in order to detach the ring from the liquid surface. Find the surface tension of the liquid.

(N.)

3. A square glass plate, 0.1 cm. thick with edges 4.9 cm. long, is hung with its square faces vertical from one arm of a balance and counterpoised, the lower edge of the plate being horizontal. A dish of water is now gradually raised under the plate until the lower edge of the plate just touches the water. It is found that the plate is pulled downwards, and that an extra weight of 0.76 gm. must be added to the counterpoise to keep the edge of the plate just in the plane of the surface of the water. Calculate the surface tension of the water.

(N.)

4. A glass rod of circular section has a diameter of 2 mm. It is suspended from the arm of a balance so that it hangs vertically in water. Find the length of rod submerged when the apparent weight is equal to the weight of the rod in air.

(Surface tension of water = 73 dynes per cm.)

(B.)

5. A solid cylinder, of length L , radius r , and weight mg , is placed vertically with its lower end submerged to a depth h ($< L$) in a liquid of density ρ , surface tension T , and zero angle of contact. Find an expression for its apparent weight $m'g$, and sketch the form of the graph that would be obtained by plotting m' against h . Explain how the values of ρ and T could be deduced from the graph.

(O. & C.)

6. The stem of a common hydrometer is a circular cylinder of diameter 2 mm. It floats, with its stem wetted, in alcohol, whose specific gravity is 0.796, and surface tension 25.5 dynes per cm. Calculate how much deeper it floats than if the alcohol had had zero surface tension. (C.)

7. Determine the capillary error in the readings of a mercury cistern barometer, the tube of which has an internal diameter of 3 mm. The surface tension of mercury is 540 dynes per cm. and the angle of contact of mercury with glass is 140° .

(Density of mercury = 13.6 gm. per c.c.) (L.)

8. Discuss the rise of a liquid in a capillary tube. A U tube, whose ends are open and whose limbs are vertical, contains mercury. The diameter of one limb is 2 mm., and the diameter of the other is 0.6 mm. Find the difference in level of the mercury surfaces in the two limbs.

(Surface tension = 547 dynes per cm., angle of contact = 139° , specific gravity = 13.6.) (C. S.)

9. Define *surface tension* and find an expression for the excess pressure inside a spherical soap bubble of radius r .

A bubble of steam 1 mm. in diameter is formed at a depth of 10 cm. below the surface of some water in a beaker. If the barometric pressure is 75.10 cm. of mercury, what is the pressure of the water vapour inside the bubble?

(The density of the water may be taken as 0.96 gm. per c.c. and its surface tension as 60 c.g.s. units. The density of mercury is 13.6 gm. per c.c.) (N.)

10. If the maximum pressure of a certain man's lungs is equivalent to that of 2 metres of water, calculate the smallest diameter of the capillary tubing through which he can blow air from his lungs into water.

(Take the surface tension of water to be 70 dynes per cm.) (C.)

11. A spherical soap bubble is brought into contact with a larger soap bubble so that a part of the surface of one coincides with a part of the surface of the other. If the radius of curvature of the smaller bubble when in contact with the larger one is 2 cm., and that of the common part is 3 cm., what is the radius of curvature of the larger bubble? (N.)

12. Two soap bubbles each of radius a coalesce to form a bubble of radius b . Show that the surface tension of the soap solution is $B(b^3 - 2a^3)/4(2a^2 - b^2)$, where B is the atmospheric pressure. (O. S.)

13. Discuss whether uniformity of the bore of the tube is essential for an accurate determination, by capillary rise, of the surface tension of a liquid which wets glass.

A uniform capillary tube is held vertically with its lower end in a vessel of liquid which rises in the tube. Derive an expression for the pressure which must be applied to depress the meniscus of the liquid column to the lower end of the tube.

(C. S.)

14. Give four illustrations of the phenomenon of surface tension with a brief explanation in each case.

A capillary tube of 1 mm. internal diameter is immersed vertically in water with the upper end connected to the gas mains. It is found that when the gas is turned on the lower end of the tube has to be immersed to a depth of 20 cm. to stop the flow of bubbles. If the surface tension of water is 70 dynes per cm., estimate the excess pressure of the gas above atmospheric pressure, expressing your result in cm. of water. (N.)

15. Two plane glass plates, fixed at a small uniform distance d apart, are placed vertically and partly immersed in a liquid of density ρ . Find the height, above the liquid in the outer vessel, to which the liquid will rise between the plates. Assume that the surface tension of the liquid is T and its angle of contact 0° . (N.)

16. Two square sheets of glass measuring 8 cm. along each edge are held vertically so that their lower edges just dip into water. Two of their vertical edges are brought into contact, whilst the others are kept 2 mm. apart by a match-stalk. Calculate the height to which the water will rise between the plates above points marked off at regular intervals of 1 cm. along the bottom edges of the plates. Draw a diagram full size showing the surface of the water between the plates.

(Surface tension of water = 72 dynes per cm.) (D.)

17. A small volume v of liquid of density ρ and surface tension T is in an open tube having a uniform taper such that the diameter changes by 0.01 cm. per cm. of length. Find an expression for the length of the liquid column when the tube is held vertically. (O. S.)

18. Describe experiments to show:

- (a) that the surface tension of alcohol is less than that of water;
- (b) that the surface tension of a film does not vary with the thickness.

(C. S.)

19. Deduce an expression for the difference of pressure between the inside and outside of a non-spherical soap bubble.

A spherical soap bubble is blown in a space where the pressure is negligible. The bubble suffers a small isothermal radial expansion or contraction. Discuss the stability of the bubble (a) if it is isolated, (b) if it is connected to a source of gas at constant pressure. (C. S.)

20. Explain carefully with reference to the physical principles involved:

- (a) A column of water will remain in a vertical capillary tube open at both ends.
- (b) A large and a small soap bubble are blown at the ends of two tubes, which are then connected. What happens?

(c) Two glass plates with a drop of water between them tend to adhere. What happens if the water is replaced by a drop of mercury?

(d) Small pieces of broken porcelain are used to prevent 'bumping' in a boiling liquid.

21. Two circular glass plates, each 5 cm. radius, are separated by a film of water 0.01 mm. thick. Find in kg. wt. the minimum direct pull required to separate them. Give the theory underlying the calculation. The surface tension of water is 73.0 dynes per cm. (N.)

22. Explain the meaning of *surface tension* and *surface energy*.

A loop of cotton of length l is placed on a plane soap film of surface tension T , and the film inside the loop is then broken. Show that the loop will assume the form of a circle, and find the tension in the thread. (C.)

23. A uniform rod AB , of mass 1.59 gm., is suspended horizontally by light flexible vertical threads AD and BC from a horizontal bar DC . When a soap film is stretched on $ABCD$, the surface tension causes the threads to assume the form of arcs of

circles of radii 13.0 cm., and the horizontal distance between the midpoints of the threads is then 3.0 cm. Calculate (a) the surface tension of the soap film, (b) the tension in the threads.

(Note that, at the place where the width of the film is least, the weight of AB is supported by the vertical pull due to surface tension, together with the vertical tensions in the threads.) (O. & C.)

24. A vertical tube, communicating with the atmosphere through a small orifice at the bottom, and closed at the top by a spherical soap film, is kept filled with hydrogen. If the vertical height of the tube between the orifice at the bottom and the soap film at the top is 100 cm., the atmospheric pressure 760 mm. of mercury, and the temperature 17°C ., calculate the radius of curvature of the soap film, and indicate whether it is convex or concave towards the outside.

(Density of air at N.T.P. 0.00129 gm. per c.c.; of hydrogen 0.00009 gm. per c.c.; surface tension of soap solution 24 dynes per cm.) (O. & C.)

25. Deduce an expression for the difference of pressure between the inside and outside of a cylindrical soap film. Is it possible to have a film such that the pressures inside and outside are the same? Find the work done in blowing a spherical soap bubble of radius 2 cm. from soap solution of surface tension 20 dynes per cm. (C. S.)

26. Assuming the surface tension of petrol to be 26 dynes per cm., find the work done on 1 c.c. of petrol in a carburettor jet if 10^9 drops are formed. (C.)

27. Calculate the radius of the largest drop of water which might evaporate at 0°C . without heat being applied to it. The surface energy of water is 117 ergs per sq. cm. and the latent heat of evaporation 606 calories per gm. at 0°C . (1 calorie is equivalent to 4.2×10^7 ergs.) (C. S.)

28. Derive an expression showing the change in the saturation vapour pressure which is produced by curvature of a liquid surface. Discuss the effects of this change on meteorological phenomena. Calculate the change in the saturated vapour pressure due to curvature in the case of a water droplet diameter 0.002 mm. at 15°C . given the following data:

Surface tension of water at 15°C . = 74 dynes per cm.

Saturation vapour pressure of water at 15°C . = 1.3 cm. of mercury.

Density of air at 15°C . under standard pressure = 1.23 gm./litre.

Relative density of water vapour to air = 0.625. (N.)

29. By considering the equilibrium of a liquid which has risen in a fine capillary tube, obtain a relationship between the surface tension, the vapour pressure over a flat surface, and that over a curved surface whose radius of curvature is r .

Hence calculate the radius of a capillary tube from which water at 15°C . will not evaporate when the humidity is 99.9%.

(Surface tension of water = 80 dynes per cm. Gas constant = 2 cal. per $^{\circ}\text{C}$. per mole. Mechanical equivalent of heat = 4.2 joule per cal.) (Modified O. S.)

30. Surface tension is the energy per unit area of a surface. Discuss this definition. Discuss the effect of (a) dust, (b) charged nuclei in facilitating condensation. Briefly mention any application of the latter. (C. S.)

CHAPTER 9

VISCOSITY AND LUBRICATION

SOME liquids can be poured more quickly from a vessel than others; water pours more quickly than treacle and petrol than engine oil. Treacle and engine oil are said to be more *viscous* than water and petrol, the internal friction in a liquid being called *viscosity*.

When a liquid flows over a solid surface, the layer in contact with the surface remains stationary because of adhesion. If this were not so the banks of rivers would be worn away rapidly. There is therefore a velocity gradient in a moving liquid, which can be observed when oil is poured from a can (Fig. 191), or in a river. At slow speeds of flow the particles

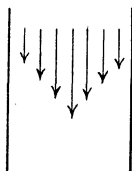


Fig. 191. Flow of a liquid.

of a liquid follow definite paths called stream lines, but above a certain speed, called the *critical velocity*, which varies for different liquids, the liquid is thrown into turbulence (Fig 193). Thus there is stream-line flow at the side of a ship which is gliding slowly through the water, whereas the flow is turbulent when the ship travels at speed.

Viscosity

A liquid flows only when caused to do so by a difference in pressure, and there must be a tangential stress across the layers which are moving relatively to each other. In a liquid flowing with stream-line motion, a section which is rectangular at one instant will be sheared as shown in Fig. 192 after a short interval of time. The ratio of the shearing stress to the *rate* of shearing strain is found to be a constant and is called the *viscosity*, η (compare with the definition of rigidity on p. 161).

Thus

$$\eta = \frac{F/A}{\frac{dv}{dx}},$$

where F = tangential force on area A , and dv = difference of velocity of two layers at distance dx apart.

That dv/dx is the rate of shearing strain can be seen from the fact that

$$\theta = \frac{dv}{dx} \cdot t, \quad \text{i.e.} \quad \frac{dv}{dx} = \frac{\theta}{t}.$$

Dimensions of viscosity

The viscosity is a tangential force per unit area per unit velocity gradient. Hence its dimensions (see p. 225) are as follows:

$$[\eta] = \frac{[\text{Force}]}{[\text{Area}] \left[\frac{[\text{Velocity}]}{[\text{Length}]} \right]} = \frac{MLT^{-2}}{L^2LT^{-1}L^{-1}} \\ = ML^{-1}T^{-1}.$$

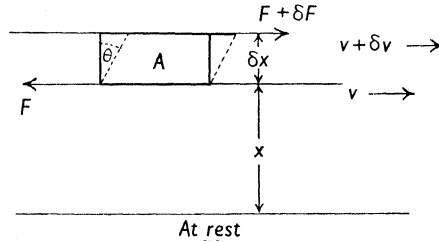


Fig. 192

Flow through capillary tube by method of dimensions

The simplest and the most accurate method of finding the viscosity of a liquid is by measuring its rate of flow through a capillary tube. If the reasonable assumption is made that the volume of liquid flowing per second depends on the viscosity, the pressure gradient along the tube and the radius of the tube, the relation between these quantities can be obtained by the method of dimensions. Suppose

$$\frac{\text{Volume}}{\text{Time}} \propto (\text{Viscosity})^x \left(\frac{\text{Pressure}}{\text{Length}} \right)^y (\text{Radius})^z. \\ \therefore L^3T^{-1} = (ML^{-1}T^{-1})^x (ML^{-2}T^{-2})^y L^z.$$

Equating dimensions, $3 = -x - 2y + z,$
 $-1 = -x - 2y,$
 $0 = x + y,$

whence $x = -1, y = 1, z = 4.$

$$\therefore V = k \frac{pr^4}{l\eta},$$

where V = the volume flowing per second,

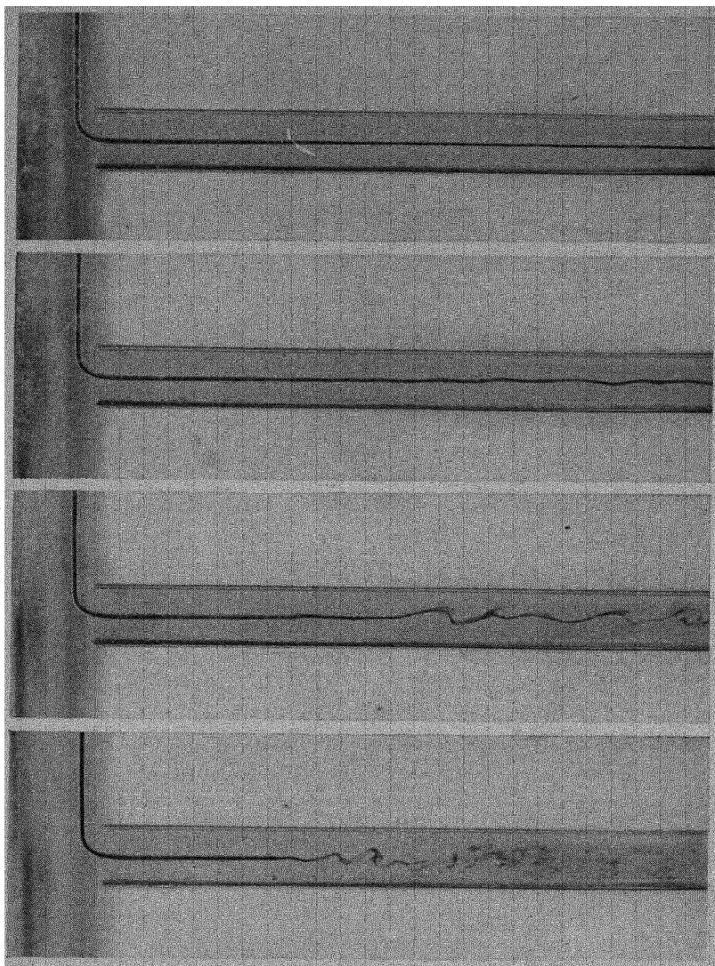
p = the difference of pressure between the ends of the tube,

r = radius of the tube,

l = the length of the tube,

η = the viscosity,

and k is a constant.



D. G. A. Dyson

Fig. 193. Streamline and turbulent flow in a liquid. The wide horizontal glass tube which is seen in each photograph is situated in the middle of a large rectangular glass tank kept full of water to overflowing. On the right, not showing in the photographs, the wide tube is bent up and down again in the form of an inverted U so that water is also being siphoned out of the tank through the tube, and an adjustable clip allows this to be done at a controllable rate. Coloured water is injected into the middle of the wide tube by the fine bent capillary on the left which ends a short way inside the wider tube. When the water is moving slowly, as in the first picture, the flow is streamline, the liquid moving in a series of concentric tubes, the outer one fixed to the wall of the tube, and the others successively increasing in speed to the innermost one, the jet of coloured fluid, which is seen not to mix with that surrounding it. At a greater speed, the injected jet is seen in the second picture to waver slightly, indicating the very beginning of the onset of turbulence. The last two pictures show turbulence fully established, and a very short way from the orifice of the jet, the coloured liquid mixes with its surroundings. (The photographs were taken with an exposure of $1/1000$ sec., through the side of the tank, the background, of white card, being brightly illuminated by photoflood lamp.)

Poiseuille's formula

The value of the constant k is not revealed by the method of dimensions and, unless it is known, no more than the comparison of the viscosities of different liquids by the capillary tube method is possible. Hence it is necessary to derive the above equation by a different method, due to Poiseuille, which enables the value of k to be found.

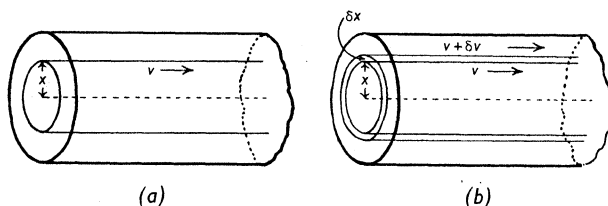


Fig. 194

Suppose the liquid is flowing through a horizontal capillary tube, of length l and radius r , and suppose that the velocity of the liquid at distance x from the axis is v (Fig. 194 (a)).

Consider a cylinder of liquid of radius x . The velocity of the liquid in this cylinder varies from a maximum at the axis to a minimum at the surface, but no part of it is accelerated after the steady state has been reached. Hence there is equilibrium between the viscous drag on its surface and the force between its ends. In other words, the total force acting on the cylinder is zero.

The viscous drag on the surface of the cylinder $= -2\pi x l \eta dv/dx$. The minus sign is necessary because dv/dx is negative.

The force between the ends of the cylinder $= \pi x^2 p$.

$$\therefore -2\pi x l \eta \frac{dv}{dx} = \pi x^2 p.$$

Integrating,

$$\int dv = \int -\frac{px}{2l\eta} dx,$$

$$v = -\frac{px^2}{4l\eta} + C,$$

where C is a constant of integration. Since the liquid in contact with the sides of the capillary tube is at rest, $v=0$ when $x=r$.

$$\therefore 0 = -\frac{pr^2}{4l\eta} + C.$$

$$\therefore v = \frac{p}{4l\eta} (r^2 - x^2).$$

The volume of liquid passing per second through a cylindrical shell of radius x and thickness δx (Fig. 194 (b))

$$= 2\pi x \delta x v.$$

$\therefore$ Total volume of liquid passing through the whole cross-section of the tube per second

$$\begin{aligned} &= \int_0^r 2\pi x v dx \\ &= \int_0^r 2\pi x \frac{p}{4l\eta} (r^2 - x^2) dx \\ &= \frac{p\pi}{2l\eta} \left[\frac{1}{2} r^2 x^2 - \frac{1}{4} x^4 \right]_0^r \\ &= \frac{p\pi r^4}{8l\eta}. \end{aligned}$$

No allowance has been made in the derivation of this expression for the kinetic energy of the liquid as it leaves the capillary tube. It is essential, therefore, to use a long and thin tube so that the water drops out of the end and does not issue with an appreciable kinetic energy.

Determination of the viscosity of water

A convenient form of apparatus for determining the viscosity of water is shown in Fig. 195. A capillary tube about 50 cm. long and 1–2 sq. mm. in bore is surrounded for most of its length by a water jacket. This is important because the viscosities of liquids vary rapidly with temperature, that of water changing by 3% per °C. in the neighbourhood of 15° C.

The overflow pipe enables a constant head of water to be maintained. The arrangement is rather different from the horizontal capillary tube considered when deriving Poiseuille's formula. The pressure at the lower end of the capillary tube is atmospheric and that at the upper end is atmospheric pressure + $g\rho h$, where ρ is the density of water and h is the depth of the upper end of the capillary tube. The viscous drag supports the weight of the water in the capillary tube, and hence the effective pressure difference between the ends of $g\rho(h+l)$, where l is the length of the capillary tube. It is essential, as explained earlier, that the water should drip out of the capillary tube and not issue with appreciable speed.

The volume V of the water passing through the tube in a time t sec. is given by

$$V = \frac{g\rho(h+l)\pi r^4}{8l\eta} t.$$

The radius r of the tube may be determined by inserting a thread of mercury into the dried capillary tube and measuring its length and mass.

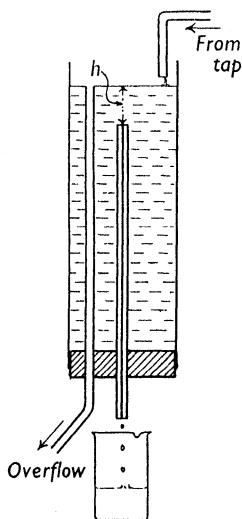


Fig. 195

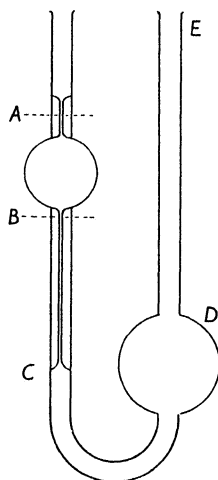


Fig. 196

Comparison of viscosities

Fig. 196 represents a viscometer, originally due to Ostwald, which, in numerous modified forms, has been widely used for making comparative measurements of viscosities. A given volume of liquid is allowed to flow under its own weight through the capillary tube BC , and the time taken by the upper surface of the liquid to fall between the marks A and B is measured. The liquid is introduced into the apparatus from a pipette through the tube E , and it is raised by suction into the left-hand arm so that its original level is above A .

The same formula as on p. 216 applies:

$$V = \frac{g\rho(h+l)\pi r^4}{8l\eta} t,$$

but in this case h represents the mean depth of liquid above the top of the capillary tube during the time t . Since V , h , l and r are always the same,

$$\eta \propto \rho t.$$

Thus if t_1 and t_2 represent the times of flow for equal volumes of two liquids of viscosities η_1 and η_2 and densities ρ_1 and ρ_2 ,

$$\frac{\eta_1}{\eta_2} = \frac{\rho_1 t_1}{\rho_2 t_2}.$$

The instrument should be immersed in a constant-temperature bath, and it can be calibrated with a liquid of known viscosity. Its dimensions are chosen to give convenient times for the liquids it is desired to test. Its popularity is due to the simplicity and rapidity of its operation.

Viscosity of glycerine

A method of finding the viscosity of a very viscous liquid, such as glycerine, is to time the fall of a small steel ball bearing through the liquid after the ball has acquired its steady terminal velocity.

Stokes deduced that the resisting force on a spherical body moving through a sea of liquid is given by $F = 6\pi\eta rv$, where η is the viscosity of the liquid, r is the radius of the ball, and v its velocity. When the ball has reached its terminal velocity the resisting force due to viscosity is equal to the weight of the ball less the upthrust of the liquid, i.e.

$$6\pi\eta rv = \frac{4}{3}\pi r^3(\rho_1 - \rho_2)g,$$

where ρ_1 and ρ_2 are the densities of steel and of glycerine respectively.

To prevent the walls of the tube from affecting the resisting force, the dimensions of the ball must be small compared with the width of the tube containing the liquid. Thus if a wide measuring cylinder is used, the ball should have a diameter of no more than $\frac{1}{8}$ in. It is convenient to gum two strips of paper about one-third and two-thirds of the way down the outside of the tube and to measure the time t taken by a number of similar steel balls to fall through the distance s between the two marks. Then $v = s/t$. The upper mark should be at a sufficient depth below the surface of the liquid for the ball to attain its terminal velocity before reaching the mark, and the lower mark should not be too near the bottom of the vessel.

The densities of steel and of glycerine may be found by means of a specific gravity bottle. From the value obtained of the density of steel, the radius of the steel balls may be calculated.

Uses of viscosity measurements

Viscosity measurements have been much used in the investigation of solutions and of the colloidal state, and they may, in the future, provide data for a theory of the liquid state comparable to the kinetic theory of gases. But, apart from their theoretical interest, they have a considerable practical value in the choice and blending of lubricants, without which the vast majority of modern machines could not be kept

running. The rest of this chapter will be devoted to a brief account of friction and lubrication and the following will be considered in turn:

Unlubricated surfaces: dry or solid friction.

Partially lubricated surfaces: boundary friction.

Completely lubricated surfaces: fluid or viscous friction.

Dry friction

The laws of solid friction (see p. 10) were discovered by Amontons (1699) and Coulomb (1779), but recent experiments have thrown considerable light on their meaning and on the mechanism of friction.

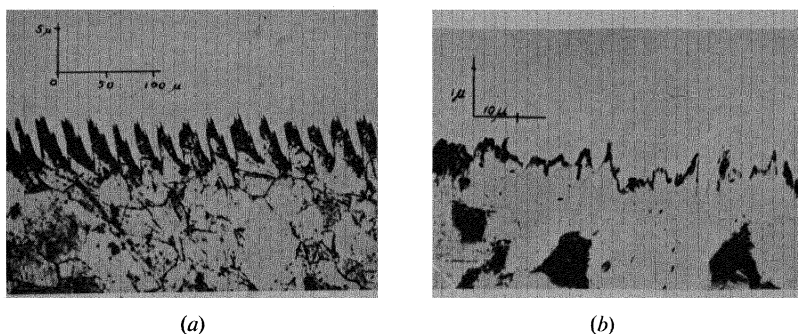


Fig. 197. Surfaces seen under a high-power microscope. (a) is a finely turned surface of copper, the turning marks being obvious; vertical magnification $\times 1800$, horizontal magnification $\times 210$. (b) is a steel surface ground with carborundum paper (grade 150); vertical magnification $\times 10,000$, horizontal magnification $\times 1000$. In the scales at the top, μ stands for a micron which is 10^{-4} cm.

Even highly polished surfaces are seen, under a high-power microscope, to possess projections and depressions which are large compared with a molecular diameter (Fig. 197), and by measuring the electrical resistance between two metals in contact it is possible to estimate the area of the surfaces which are actually touching. This is found to be very much less than the apparent area of contact. For flat steel surfaces the true area of contact may be only $\frac{1}{10,000}$ th of the apparent area; it is almost independent of the size and roughness of the surfaces and directly proportional to the load, suggesting that projections on the surfaces are crushed down until the true area of contact is sufficient to support the load. Amontons's law, that the force of friction is independent of the area, is explained by the fact that changing the apparent area has no effect on the true area of contact for the same load.

Since the true area of contact is so small, the pressure at the points of contact must be intense; in the case of mild steel it may be of the order of 100 tons per sq. in. This suggests that the temperatures at the points of contact, during sliding, must become very high. The temperature may be estimated by sliding one metal over a dissimilar metal and using the two metals as a thermocouple. High temperatures of 1000°C . at the 'hot spots' or points of contact, lasting only a few ten-thousandths of a second, have been measured by applying the generated thermo-E.M.F. to the deflecting plates of a cathode-ray oscillograph after suitable amplification. The temperatures reached depend on the load, the speed of sliding, the thermal conductivity and the melting-

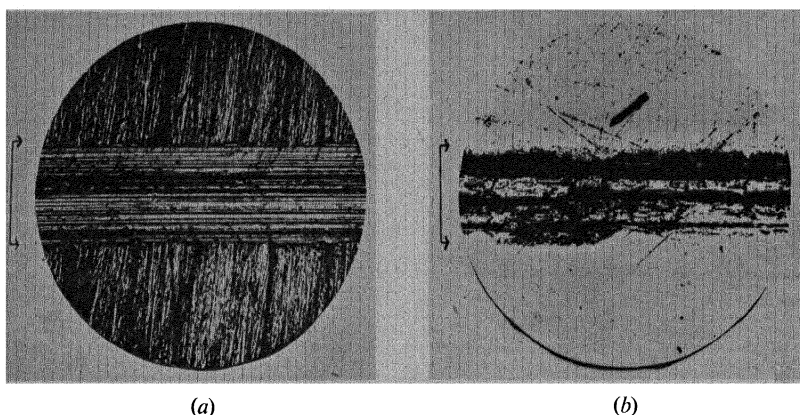


Fig. 198. (a) Track formed when steel slides on copper. The steel slider has ploughed and torn the copper surface. Magnification $\times 20$. (b) Track formed when nickel slides on nickel. The damage to the surface is very heavy. Magnification $\times 20$.

point of the metals. The following metals (whose melting-points are given), gallium 32°C ., Wood's metal 72°C ., lead 327°C ., constantan 1290°C ., when caused to slide on steel, attain maximum temperatures no higher than their melting-points in each case, indicating that during sliding melting, and therefore flow of metal, must occur.

With poor thermal conductors it would be expected that high-surface temperatures would be reached more easily. The thermocouple method is usually not applicable to these materials, but other simple methods have been devised. When polished glass or quartz surfaces, for example, are rubbed together in the dark, tiny stars of light can be seen, reddish in colour at low speeds and whiter at higher speeds.

Thus during sliding some surface damage always occurs; when a soft metal slides against a hard metal the surface of the soft metal is ploughed

and torn by the projections on the hard metal, while the surface of the hard metal has soft metal welded on to it. The most serious damage occurs when similar metals slide one against the other (Fig. 198), and a combination of similar metals always gives the highest coefficient of friction. When the surfaces are lubricated by even a thin layer of lubricant the surface damage is much reduced.

It is believed that the force of friction between unlubricated metals is due to two distinct causes, the shearing of junctions of the two metals which have become welded together at points of contact, and the ploughing of the surface of the softer metal by projections of the harder metal, as has just been described. The shearing force, which is usually considerably larger than the ploughing force, is proportional to (a) the real area of contact, and (b) the shear strength of the softer metal. Thus, to reduce friction, it is necessary to make (a) and (b) as small as possible. If a hard metal is made to slide, say, on steel, (a) is small but (b) is high; if the hard metal is replaced by a soft metal then (b) is small but (a) is high. For this reason the coefficient of friction of most metals is about the same, between 0.6 and 1.0.

It is possible, however, to reduce friction by smearing a thin film of a soft metal on a harder one. The shear strength is then small and the area of contact does not become unduly large. A very thin film of iridium, a metal softer than lead, on steel reduces the coefficient of friction between this surface and steel to 0.04. Bearings made of copper containing particles of lead, which becomes smeared on the surface, or silver smeared with lead or iridium are commonly used in aircraft engines.

All metal surfaces in air, even though apparently perfectly clean, are covered with a thin film of oxide or other contaminant, and this greatly reduces friction because of its low shear strength. Friction between perfectly clean, smooth metal surfaces *in vacuo* is much greater than normal and is due to adhesion, the forces being chemical rather than physical; the surfaces seize under the slightest load. It is, indeed, fortunate that metals are covered with a film of contamination when in air; otherwise, in the words of Dr Bowden, the world would be a very sticky place.*

Boundary friction

Boundary friction occurs when there is a very thin film of a lubricant or contaminant between the surfaces; as already pointed out, normal friction is really of this type.

* See Dr Bowden's article on friction in *Science News*, IV, Penguin Books, from which the above account is taken.

Sir William Hardy found that the strength of the boundary film depends on the chemical nature of the metal and of the lubricant. The molecules of many oils are long and thin, and at one end there is a polar group (COOH) which is attracted to the metal much more strongly than the non-polar end. As a result the layer of molecules of the lubricant against the surface of the metal is oriented, each molecule standing out at right angles to the surface with its polar end inwards (cf. p. 199). On this layer, further double layers with polar ends adjacent, may form, and sliding occurs between pairs of layers over non-polar ends. The value of a lubricant which adheres tenaciously to metal surfaces, thereby reducing wear, gives great practical importance to the study of boundary films.

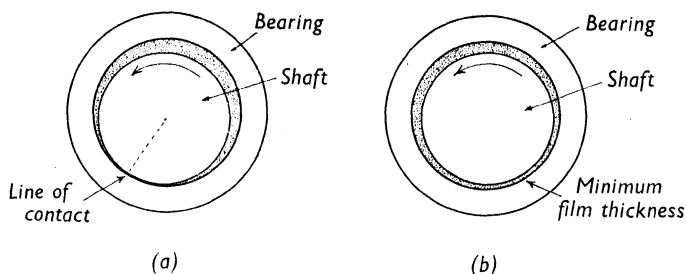


Fig. 199. (a) Shaft rotating slowly. Imperfect lubrication.
(b) Shaft rotating at high speed. Perfect lubrication.

Fluid friction

Fluid friction occurs when the film of lubricant is sufficiently thick for most of its molecules to be outside the range of the adhesive forces of the solid, and hence for viscosity to come into play.

In 1886 Osborne Reynolds put forward the fundamental theory of lubrication, based on the work of Beauchamp Tower, who showed it to be an experimental fact that an oil film is formed between a lubricated bearing and a revolving shaft (Fig. 199 (b)). Reynolds suggested that the film is thick enough for fluid friction to apply and explained, on hydrodynamical principles, how the film of oil can be maintained despite the enormous pressures between the surfaces. He showed that it is essential for the maintenance of the film that the radii of the shaft and of the bearing should be different (the correct allowance is about $\frac{1}{1000}$ th of the diameter of the shaft), so that the surfaces of the shaft and of the bearing can become eccentric, causing a wedge of oil to be formed, whose pressure increases as the wedge narrows. The oil is

carried round by the adhesion of the layer nearest to the shaft and by viscous forces on more distant layers.

At low speeds, before the thickness of the oil film has been built up, there may be boundary friction in this simple type of bearing (Fig. 199 (a)), and thus it is during starting and stopping that wear occurs.

The fundamental principle of lubrication, then, is that two surfaces must be kept apart when in relative motion, and viscosity in a lubricant is essential because it makes this possible. On the other hand, too high a viscosity is undesirable because it causes resistance to motion. Generally speaking, since the resisting force due to viscosity increases with speed, an oil of lower viscosity is needed for high speeds than for low. Other important factors must be considered by the engineer, such as the decrease in viscosity with rise in temperature, and the increase in viscosity with increase in pressure. Lubrication is a wide subject, and it has been possible merely to mention some of its elements in this brief account.

SUMMARY

The viscosity of a liquid is the ratio of the shearing stress to the rate of development of shearing strain. It can be determined most accurately by measuring the flow of the liquid through a capillary tube.

Viscosity plays an essential part in lubrication because it enables two surfaces in relative motion to be kept apart by a film of liquid, and in this way wear and tear of running machinery is minimised.

QUESTIONS

1. Explain what is meant by the *coefficient of viscosity* (or viscosity) of a liquid. Find its dimensions in mass, length and time.

A liquid has a coefficient of viscosity of 1.5×10^{-2} expressed in c.g.s. units. What would be the value in foot-pound-second units?

(1 lb. = 454 gm. 1 ft. = 30.5 cm.) (C.)

2. What do you understand by the viscosity of a fluid?

The viscosity η of a liquid may be obtained by measuring the volume V of the liquid flowing in time t through a tube of radius r and length l from the equation $V = \frac{\pi r^4 t}{8\eta l}$, where p is the pressure difference between the ends of the tube. Describe such an experiment, pointing out what precautions you would take to obtain an accurate result.

Assuming that any direct measurement of p , t and V can be made exactly, discuss the errors which will arise in the determination of η when the radius of the tube is measured (a) by means of a travelling microscope, (b) by filling the tube with mercury and weighing the mercury. (C. S.)

3. When a liquid is driven by pressure through a narrow tube it moves without acceleration. Explain carefully the reason for this, and describe how you would study experimentally the nature of the motion of the liquid.

A boiling tube which stands vertically has sealed into its lower end a short length of capillary tube which is horizontal. It is filled with glycerine of density 1.26 gm. per c.c. and viscosity 8.50 c.g.s. units to a certain mark, and it takes 45 sec. to fall to a second mark by escaping through the capillary tube. With castor oil of density 0.97 gm. per c.c. the time taken to fall from the one mark to the other is 68 sec. Determine the viscosity of castor oil. (L.)

4. Explain the term *viscosity*, and describe how you would compare the viscosities of water and alcohol. (The mathematical theory of the experiment is not required.)

A large bottle is fitted with a siphon made of capillary glass tubing. Compare the times taken to empty the bottle when it is filled (a) with water, (b) with petroleum of sp. gr. 0.8. The viscosities of water and petroleum are 0.01 and 0.02 respectively. (C.)

5. Describe the change in the mode of the flow of water through a long straight cylindrical tube of uniform bore as the velocity increases. Assuming that the velocity at which the character of the flow changes depends only on the coefficient of viscosity of the fluid η , on its density ρ and the diameter of the tube a , prove by the method of dimensions that this velocity is proportional to $\frac{\eta}{\rho a}$. (C. S.)

6. By the method of dimensions, or otherwise, show that the retarding force acting on a sphere of radius r moving with velocity v through a medium of viscosity η is proportional to $(rv\eta)$.

Show, on this assumption, that for a given rate of rainfall (in inches per hour) the pressure on the ground varies as the square of the radius of the drops.

How would you attempt to measure the drop size in practice? (O. S.)

7. By the method of dimensions, or otherwise, show that the viscous couple exerted on a sphere of radius a rotating with angular velocity ω in a fluid of viscosity η is $k\eta\omega a^3$, where k is a dimensionless constant.

A sphere of mass 100 gm. and radius 2 cm. is set rotating in a larger vessel containing water (viscosity 0.018 c.g.s. units). Its angular velocity is observed to fall to one-half the initial value in 61.2 sec. Calculate the value of the constant k . (O. S.)

8. Give an account of the principles underlying the use of the theory of dimensions in the solution of physical problems.

When the liquid in a stationary cylindrical vessel of given shape is stirred and allowed to come to rest, the time required for the angular momentum to fall to half of its original value depends on the size of the vessel and the density and viscosity of the liquid. If the required time is 15 sec. for a cup of tea 8 cm. in diameter, how long will it be for a similarly shaped vat of marmalade 2 metres in diameter?

(Viscosity of tea = 0.010 c.g.s. unit. Viscosity of marmalade = 50 c.g.s. units. Density of tea = 1 gm. per c.c. Density of marmalade = 1.4 gm. per c.c.) (C. S.)

APPENDIX

DIMENSIONS

PHYSICAL quantities may be divided into two categories, primary and secondary. Those usually selected as primary quantities are mass, length and time, and these are regarded as of irreducible simplicity. They are measured by comparing them directly with quantities of the same kind. Secondary quantities, on the other hand, are determined by measuring primary quantities and combining them. Thus a velocity is obtained by dividing the number measuring a length by the number measuring a time. We can write

$$[\text{Velocity}] = [\text{Distance}] \div [\text{Time}] = LT^{-1}.$$

Similarly $[\text{Acceleration}] = [\text{Velocity}] \div [\text{Time}] = LT^{-2},$

$$[\text{Force}] = [\text{Mass}] \times [\text{Acceleration}] = MLT^{-2},$$

$$[\text{Work}] = [\text{Force}] \times [\text{Distance}] = ML^2T^{-2}.$$

The powers of M , L and T are called the dimensions of a quantity. From them it is possible to check or even to derive an equation since each term must have the same dimensions. If the terms had different dimensions a change in the fundamental units of mass, length and time would affect each term differently and the equation would not hold for all systems of units.

Derivation of the formula for the period of a simple pendulum

The method of dimensions may be used to determine the period of a simple pendulum. The period, t , may depend on the length, l , the acceleration due to gravity, g , the mass of its bob, m , and the angular amplitude of swing, α . The equation connecting these quantities may be written

$$t = kl^x g^y m^z \alpha^w,$$

where k , x , y , z , w are unknown constant numbers. Inserting the dimensions of each term,

$$T = L^x (LT^{-2})^y M^z,$$

the dimensions of α being zero.

Equating dimensions of T , L , M respectively on the two sides of the equation,

$$1 = -2y. \quad \therefore y = -\frac{1}{2}.$$

$$0 = x + y. \quad \therefore x = \frac{1}{2}.$$

$$0 = z.$$

$$\therefore t = k \sqrt{\frac{l}{g}}.$$

The method gives no information as to the value of k nor of the way α enters into the equation.

Dimensions are arbitrary and conventional

There are two independent relations between force and mass,

$$F = ma \quad \text{and} \quad F = G \frac{m_1 m_2}{r^2} \quad (\text{pp. 56 and 135}).$$

Substituting in the second equation the dimensions of force, MLT^{-2} , derived from the first equation,

$$MLT^{-2} = [G] \frac{M^2}{L^2}.$$

$$\therefore [G] = M^{-1}L^3T^{-2}.$$

G is called a dimensional constant, and its numerical value changes according to the system of units employed.

Force might equally well have been defined from the second equation, making G unity and dimensionless, when a dimensional constant would have been required in the first equation. This would change the dimensions of force and hence of most physical quantities. Thus dimensions are arbitrary and conventional. Their value lies in providing a means of checking equations connecting physical quantities

ANSWERS TO QUESTIONS

CHAPTER 1 (page 19).

1. (a) $6\frac{2}{3}$ lb. wt. (b) 10 lb. wt.
2. Reaction at $A=250$ lb. wt.; horizontal and vertical components at $D=250$ and 200 lb. wt. respectively.
3. 140 lb. wt., 1.2 ft. from B on the side remote from A .
4. $\frac{b\mu Mg}{a + \mu\sqrt{(b^2 - a^2)}}$.
5. $\sqrt{3}$.
6. $171\frac{3}{7}$ lb. wt.
7. 36 lb. wt., 31.75 lb. wt.
8. (i) 13.33 lb. wt., (ii) 8.03 lb. wt.
9. 1200 gm. wt., 1044 gm. wt. at $5^\circ 30'$ to horizontal.
11. $5\frac{5}{8}$ ft. lb.
12. 0.25.
13. 61.3 lb. wt. at $78^\circ 10'$ to ground; 0.21.
15. 13 ft.
16. 0.58.
17. $\frac{W(\mu \cos \theta + \sin \theta)}{\sqrt{(\mu^2 + 1)}}$, $\phi = \tan^{-1} \mu$.
18. $\sin^{-1} \mu \cot \theta$.
19. $\frac{a}{\sqrt{(u^2 - V^2)}}$.
20. (a) 22.36 knots, N. $63^\circ 26'$ W. (b) 2.85 sea miles, 2.55 min. past 10 a.m.
21. 69.3 m.p.h. at 30° to road in an upwards or in a downwards direction.
23. He will escape $\frac{v}{\sqrt{(V^2 - v^2)}} > \frac{a}{b}$, assuming that he moves in the optimum direction at an angle $\cos^{-1} \frac{v}{V}$ to the kerb.
25. $2w$.
26. $\frac{6w}{\sqrt{7}}$.
28. (a) 9, (b) $8\frac{2}{3}$, (c) $1\frac{1}{3}$.
29. $46\frac{2}{3}$ lb. wt., $373\frac{1}{3}$ ft. lb., $32\frac{1}{3}$ lb. wt.
30. $466\frac{2}{3}$ lb. wt.
31. 2530 lb. wt.
32. 1 in 6.56.
33. 20 in.
34. $\frac{2}{5}$ in. from centre of disc.
35. $5\frac{1}{4}$ in. from each of the complete sides of the square.
36. $1\frac{1}{2}$ cm. and 1 cm. from sides of length 3 and 4 cm. respectively.
38. On axis, 0.399*a* from base.
39. $34^\circ 22'$.
40. Slides.
41. $\tan^{-1} \frac{\mu}{4}$.
43. $\frac{3}{8}a$ from centre of sphere of which hemisphere is a part.
45. 9.837 gm., 1.0005.
46. The c.o.g. of the beam is 0.29 cm. below its centre.
47. 0.84 cm.
48. $\frac{1}{n^3} : 1$.

CHAPTER 2 (page 45).

1. 17.8 lb. wt. per sq. in.
2. 12.5 lb. wt.
3. 9.66 lb. wt. per sq. in.
6. 69 gm. wt., 8.6.
7. The volumes are equal.
9. 998.9 gm. wt.
10. 9.766 gm.
11. 1.24 cm.

12. 2.36 cm.
 13. 2.40.
 14. (a) Half, (b) $\frac{1}{3}$ weight of plank.
 15. $\frac{Al^2(\rho - \sigma)^2g}{2\rho}$, $\frac{Al^2(\rho - \sigma)^2g}{2\rho} \frac{B-A}{B}$.
 16. 5.7 ft.
 18. 1.2 watts.
 19. 177 lb. wt., 20.8 lb. wt.
 20. 14,100 lb. ft.
 21. 1410 lb. wt., 234 lb. wt.
 23. At vertical depth of $\frac{2a}{3} \cos \theta$ in. and $\frac{2(h^2 + ah \cos \theta + \frac{1}{2}a^2 \cos^2 \theta)}{2h + a \cos \theta}$ in.
 24. 49.8 metres.
 25. 81 cu. metres.
 26. 1.20×10^8 kg., 1.11×10^8 kg.
 30. 20 km., 34.5 km.
 32. 75.0 cm. of mercury.
 33. 29.69 in.
 34. 758.8 mm.
 36. 1:0.824.
 37. 25.
 39. $p = \frac{Pd}{v} + P \left(\frac{V}{V+v} \right)^m \left(1 - \frac{d}{v} \right)$, where p is the pressure after m strokes. When $m = \infty$, limiting pressure $= \frac{Pd}{v}$.

$$\text{Time} = \frac{\log \frac{d}{v-d}}{n \log \frac{V}{V+v}} \text{ sec.}$$

CHAPTER 3 (page 72).

1. 4 sec.
 2. 400 ft.
 3. 983 cm. sec.⁻².
 4. 125 c.p.s.
 5. 1 mile beyond station.
 6. East 3.28 cm.
 7. $51\frac{1}{3}$ sec., $2581\frac{1}{3}$ ft.
 8. 477 ft., 176 ft. sec.⁻¹.
 12. 87.9 ft. sec.⁻¹ at $9^\circ 27'$ to ground.
 13. $\sqrt{(u^2 + 2gh)}$, $\tan^{-1} \frac{1}{u} \sqrt{(2gh)}$.
 14. 30° , not nearer than $240 \sqrt{3}$ ft. to gun or to target.
 15. $\frac{1}{2}gt^2$ ft. below balloon; ut ft.
 16. 3180 ft. sec.⁻¹, 15 miles.
 17. 5280 ft.
 18. 6900 ft.
 21. 119 lb. wt.
 22. 0.299 g.
 23. 62.6 cm. sec.⁻¹.
 24. 25 cm. sec.⁻², 50 cm.
 25. 196 cm. sec.⁻² upwards.
 26. 50 lb. wt., 2.45 sec.
 28. 14.3 sec., 14.0 cm. sec.⁻¹, 4.90×10^5 dynes; 17.0 sec., 11.8 cm. sec.⁻¹, 3.47×10^5 dynes.
 29. $\frac{(\mu_1 - \mu_2)s}{\mu_2(\mu_1 + 1)}$.
 30. 2172 ft. lb., P.E. = 1800 ft. lb.
 31. (i) 5 ft. sec.⁻¹, (ii) $15\frac{3}{8}$ tons wt.
 32. 50 lb. wt.
 33. 32.7 ft. sec.⁻¹.
 34. 102 kg.
 36. 2400 ft. sec.⁻¹.
 37. 540 lb. wt.
 38. 4570 lb. wt., 2860 ft. lb.
 39. 20 ft. sec.⁻¹, 750 lb. wt.
 40. $\frac{M+m}{m} d \sqrt{\frac{g}{l}}$, $\frac{M}{M+m}$.
 42. $\frac{M'}{M+M'} \sqrt{(2gh)}$, $\frac{M'}{M+M'} \sqrt{(2gh)}$,

$$\frac{M'}{M-M'} \sqrt{\frac{8h}{g}}$$
.
 43. $41\frac{1}{3}$ ft. sec.⁻¹, $10\frac{5}{8}$ ft. sec.⁻¹, each opposite to the original direction.
 44. (a) $\frac{8}{5}V$, (b) 64%.
 45. $\frac{m^2u^2}{2ngM(m+M)}$.
 46. 0.63, 6.52 poundal sec., 0.63 ft.

47. (a) $-10 \text{ cm. sec.}^{-1}$, 10 cm. sec.^{-1} ;
 (b) $-20 \text{ cm. sec.}^{-1}$, 0 cm. sec.^{-1} ; in each case the velocity of the smaller ball is given first and the positive direction is that of the original velocity of 20 cm. sec.^{-1} .
48. $\sqrt{\left(\frac{2h}{g}\right) \frac{1+e}{1-e}}$.
49. 7.2 cm.
51. $216 \text{ cm. sec.}^{-1}$ at 30° with initial direction, $2.81 \times 10^5 \text{ ergs.}$
52. $\frac{1}{2}m(e+1)(u+U)\{u(1-e)-U(1+e)\}$, $nm(e+1)(u+U)$.
54. $115 \text{ cm. sec.}^{-1}$.
55. 1.23 lb. wt.
56. $2.55 \times 10^6 \text{ gm. wt.}$
57. (a) 12.6 ft. (b) 17.8 ft.
58. 83.0% , 0.050 .
60. 15.1 h.p. , 16.6% .
61. 12 h.p.
62. 120 lb. wt. , 21.4 m.p.h.
63. (a) 0.35 , (b) 1.9 h.p.

CHAPTER 4 (page 100).

1. 1.85 lb. wt.
2. 30 m.p.h.
3. 0.41 .
4. 5 ft.
5. 0.33 ton wt. , 0.084 h.p.
6. (i) 9.87 lb. wt. , (ii) 3.4 lb. wt.
7. When it has described an arc subtending $48^\circ 11'$.
8. $982 \text{ cm. sec.}^{-2}$ at angle of $5'$ to vertical.
9. $9^\circ 45'$.
10. $18^\circ 35'$.
12. 13 m.p.h. , 3.1 tons wt. , 0.62 .
13. 19.3 m.p.h.
16. $70^\circ 32'$.
18. 11.3 .
20. $1:2$.
21. 8 ft. sec.^{-2} .
22. 243 ergs.
23. $243 \text{ rad. sec.}^{-1}$.
24. 8.9 and $17.8 \text{ rev. sec.}^{-1}$.
25. (a) $3.3 \times 10^5 \text{ ergs}$,
 (b) $1.7 \times 10^6 \text{ gm. cm.}^2$
26. $3.59 \times 10^5 \text{ gm. cm.}^2$.
27. $89.2 \text{ cm. sec.}^{-2}$, 90.9 gm. wt. , $29.7 \text{ rad. sec.}^{-2}$.
29. (a) $934 \text{ cm. sec.}^{-2}$, (b) $980.7 \text{ cm. sec.}^{-2}$.
30. $\sqrt{\frac{3g(1-\cos \theta)}{l}}$, $\frac{3g \sin \theta}{2l}$.
31. $a^4 m \left(\frac{32}{3} - \frac{17\pi}{32} \right)$, where $m = \text{mass per unit area}$.

CHAPTER 5 (page 121).

1. 0.82 sec.
3. (a) 70 cm. sec.^{-1} , (b) 10 cm.
4. 9.1×10^{13} .
5. 1.00 sec.
6. 0.45 sec.
7. 4 cm. in water , 1.92 sec.
10. $16\pi \text{ cm. sec.}^{-1}$.
11. 5 cm. , 2 sec.
12. 16 cm. , 80 gm.
13. $0 \text{ hr. } 56 \text{ min. } 57 \text{ sec.}$
14. 0.18 sec.
18. $980 \text{ cm. sec.}^{-2}$.
22. 48.4 cm. , $4 \text{ positions } 19.5 \text{ cm. and } 10 \text{ cm. from centre of gravity on each side.}$
23. 2.67 ft.
24. 0.9824 sec.
25. (a) 2.22 sec. , (b) 1.92 sec.
26. 0.037 cm. , 6 min.
27. 0.229 l.

CHAPTER 6 (page 151).

1. 5.3 gm. cm.^{-3} .
2. $2.01 \times 10^{33} \text{ gm.}$
3. 3760 sec.
4. 11.9 years.
5. 0.1% .
6. $0.38 : 1.$
7. $27.8 : 1.$
8. $1.06 \times 10^{-9} \text{ ft.}^3 \text{ lb.}^{-1} \text{ sec.}^{-2}.$
9. $3.9 \times 10^5 \text{ km.}$
10. $0.019 \text{ gm. cm.}^{-3}.$
12. $2.28 \times 10^{-3} \text{ radian.}$
13. 31.6 mg.
14. $1.12 \times 10^6 \text{ cm. sec.}^{-1}.$
15. $2.38 \times 10^5 \text{ cm. sec.}^{-1}.$
17. 2580 sec.
18. $T = \text{const.} \sqrt{\frac{\rho r^3}{\gamma}}$, where ρ is the density, r the radius and γ the surface tension.

CHAPTER 7 (page 173).

1. $1.0 \times 10^6 \text{ ergs.}$
2. (a) $2.0 \times 10^{12} \text{ dyne cm.}^{-2}$, (b) $4.7 \times 10^5 \text{ ergs.}$
3. $1.51 \times 10^{12} \text{ dyne cm.}^{-2}.$
4. 10.1 kg. wt.
5. $4 \times 10^6 \text{ dynes, } 7.4 \times 10^6 \text{ dynes.}$
7. $\frac{1}{2}.$
8. $27.3 \text{ kg. wt. on copper and } 36.35 \text{ kg. wt. on steel; } 0.178 \text{ cm.}$
9. $\frac{mgl}{2\pi r^2 y}, \frac{y^2}{2l^2}, \frac{mgl^3}{\pi r^2 y^3}.$
10. $31.44 \text{ tons wt., } 0.097 \text{ ton wt. per sq. in.}$
11. 136° C.
12. 9.6 atm.
14. 2.2 km.
16. $(7.5 \pm 0.9) \times 10^{11} \text{ dyne cm.}^{-2}.$
17. $1.27 \times 10^5 \text{ ergs.}$
18. 785 kw.
19. $0.11\%.$

CHAPTER 8 (page 208).

2. $43.4 \text{ dyne cm.}^{-1}.$
3. $74.6 \text{ dyne cm.}^{-1}.$
4. 1.5 cm.
5. $m'g = mg - \pi r^2 h \rho g + 2\pi r T.$
6. 6.5 mm.
7. 4.1 mm.
8. 1.44 cm.
9. $1.013 \times 10^6 \text{ dyne cm.}^{-2}.$
10. $1.4 \times 10^{-3} \text{ cm.}$
11. 6 cm.
14. $22.9 \text{ cm. of water.}$
17. $\frac{100v\rho g}{\pi T}.$
21. 11.7 kg. wt.,
22. $\frac{lT}{\pi}.$
23. (a) $26.9 \text{ dyne cm.}^{-1}$, (b) 699 dynes.
24. $0.87 \text{ cm., convex.}$
25. 2000 ergs.
26. $1.26 \times 10^5 \text{ ergs.}$
27. $9.2 \times 10^{-9} \text{ cm.}$
28. $19.5 \text{ dyne cm.}^{-2}.$
29. $1.2 \times 10^{-4} \text{ cm.}$

CHAPTER 9 (page 223).

1. $1.01 \times 10^{-3} \text{ ft.}^{-1} \text{ lb. sec.}^{-1}.$
3. $9.9 \text{ cm.}^{-1} \text{ gm. sec.}^{-1}.$
4. $2 : 5.$
7. $12.6.$
8. 2.6 sec.

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