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ON THE FUNDAMENTALS OF ANALYSIS

SIX PUBLIC LECTURES
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AT THE UNIVERSITY OF CALCUTTA

by the Hardinge Professor

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FOREWORD

In the following pages are reproduced in a revised form the contents of six public lectures delivered at the University of Calcutta in February 1938. For the convenience of the reader, the text has been subdivided systematically into sections. Starred sections may be omitted at the first reading. Most of the quotations will be found in the 'Critical and Historical Notes' toward the end of the book to which a list of references is added.

In placing these lectures before the public, I offer my heartiest thanks to the authorities of the Calcutta University for their support, particularly to the Vice-chancellor, the Hon'ble Khan Bahadur M. Aziz-ul-Huque, C.I.E., B.L., M.L.A., and to the Chairman of the Press and Publication Committee, Syamaprasad Mookerjee, Esq., M.A., B.L., D.Litt., Barrister-at-Law, M.L.A.

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Asutosh Building
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February 1939.

F. W. Levi.

INTRODUCTION

By 'Analysis' we understand nowadays that branch of Mathematics which is based on the notion of a continuous totality of numbers. The theories of real functions and of complex functions and their special branches as well as Differential Geometry, Hydrodynamics, the theories of Elasticity and of Electro-Magnetic fields, etc., belong to Analysis; in general, nearly every mathematical theory of Physics is an application of Analysis. At the end of the 19th century there was a general trend towards Analysis; the success of analytical methods was so stupendous that preference was given to them everywhere on nearly every occasion, and Analysis seemed to be the only genuine subject of Mathematics. Perhaps also the trend towards analytic ways of thought was so strong as to attract the most capable scholars to work in this line, and it may be for this reason that such a progress has been made in Analysis. It may be the task of a philosopher reviewing the culture of that time to find out what are the reasons and what are the consequences. As a matter of fact a few decades ago every mathematical proposition was considered from the point of view of Analysis, and still today many mathematicians are 'Analysts' in this sense. Even those who do not acknowledge Analysis as the only true object of mathematical investigation, are bound to recognize its very high importance, and it is especially men of this way of thinking who are mainly interested in the fundamentals of Analysis.

In these lectures the Analysis of our present time will be considered. A few theorems will be stated to be fundamental, and it will be shown how some parts of Analysis can be directly derived from them, whereas other theorems need the consideration of more specific properties of the system of the real numbers. From this point of view distinct layers will become visible, but these are quite different from the historical strata. The distinction between basic and higher portions of Analysis, as adopted here, will not be the usual one; the order of the investigations will sometimes appear to be reversed; thus history will play only a minor rôle in these lectures. However, it is hardly advisable to omit completely the historical development of these problems. The difficulty—due to the complex nature of Analysis—of constructing an appropriate basis for it, can only be fully appreciated by considering the different attempts made in ancient and recent times. These difficulties seem to lie in the nature of the subject. Of course Analysis is based on two opposite ideas: Number and continuity.—Numbers are generated by the process of enumeration, they are distinct elements, each provided with individual properties. The idea of continuity, however, is connected with the conception of a fluid entity which permits partition of every kind and which does not show any individual structure. Out of this foundation on two incompatible ideas arose in different ages critical situations of mathematics. The first crisis of mathematics originated from the discovery that the diagonal of a square is incommensurable to the side. This new knowledge was crucial for western philosophy as well as for the mathematics of that ancient time.

After the failure of the Hylozoists to give a satisfactory explanation of the Universe by a materialistic philosophy, the Pythagoreans had built up their doctrine on the idea of number. But, as the ratio of the simplest geometrical entities had been proved to be inexpressible by numbers, this idea was shown to be unfit to take the place of a leading principle of the universe, and Pythagorean philosophy was shocked to its foundations. We do not know anything about that discovery. Theodoros is said to have taken the next steps in the same direction by proving the irrationality of the square roots, of the non-squares from 3 to 17, but the discoverer of the irrationality of $\sqrt{2}$ was unknown to the posterity; his name has vanished and there was later a belief that he lost his life in a shipwreck as a punishment for his outrageous discovery, and that the mystery of irrationality should never have been disclosed to mankind. We have little knowledge of the attempts to reconstruct Mathematics after this shock; the combined investigations of the philosophers and mathematicians of today have not discovered what happened at that time and has lain forgotten during these 23 centuries. However, those modern scholars gave us enough to appreciate better the solution given by the Greek mathematicians.* This solution, which is supposed to be due to Eudoxos, has been codified by Euclid in his 5th book. It appears there as the theory of proportions, and it is interesting to state that this most important portion of Euclid's work has not been appreciated very much and has been sometimes unfavourably commented on by mathematicians of later times. The fundamental idea of this theory of proportions is the following: The ratio of two segments—or of any other pair of elements of an homogeneous entity—is not a number but an object of its own kind, let us say $A : B$. The equalities and inequalities between segments become defined in a geometrical manner, also the product $m\,a$, m being an integral number, has to be interpreted geometrically.

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* See Hasse and Scholz [20], furthermore Bonnesen [5] and Toeplitz [45].

Let A and B be two elements of an homogeneous entity, e.g. segments, and let C and D be two elements of an homogeneous entity which may or may not be different from the first. If there exist two numbers m, n such that $mA > nB$ $mC < nD$ holds, then $A : B > C : D$, but if for every pair m, n of numbers for which $mA \gtreqless nB$, the corresponding relation holds for C and D, then the ratios $A : B$ and $C : D$ are considered to be equal. In this manner a theory of proportions has been built up without developing a theory of irrational numbers; the ratios are defined by it only for pairs of elements of the same homogeneous entity, but ratios derived from pairs of elements of different entities become comparable. From this stage only one step is necessary to consider the ratios as elements of a field, but this step was not taken. Among the geometers, the tradition of calculating with homogeneous entities only has been observed during two millennia. The ideas which form the basis of the analytic Geometry of Descartes, Desargues and Fermat are different from those of our contemporary analytic Geometry. Nowadays we start by representing every geometrical object by the help of numbers, then we apply the methods of Algebra and Analysis to these numbers, and finally we interpret this result in a geometrical manner. In Descartes' Geometry every formula signifies a segment; addition, subtraction, multiplication, division, square-root . . . . . of segments is defined in a geometrical manner, the result being always a segment. It may be mentioned that this method again became important in modern times in Hilbert's calculus of segments.* In principle Descartes' point of view was the same as that of the ancient Geometers, but in praxi the difference between an Analysis of real segments and an Analysis of real numbers is not very important. Descartes' Geometry was published in 1637, twenty years after Napier had died. At that time irrational numbers were used by the Algebraists, but they were considered as 'surd' numbers, as mathematical beings without any real sense which—by a kind of miracle—are leading to true mathematical results. Thus there existed two different Analyses at this time, the classical geometrical Analysis of Descartes, Fermat, Desargues and other scholars educated in the spirit of the classical antiquity and the 'Analysis of the surd numbers' which happened to be also the Analysis of applied Mathematics. In the following century the classical Analysis faded away. The success of the Calculus and the progress of Mathematics due to the great mathematicians of that time secured the victory for the new Analysis which developed quickly and flourished, although the foundations were very unsafe. Never—except perhaps in the

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* See [24].

age of Pythagoras—was the public so much interested in mathematical and pseudo-mathematical ideas as at that time. The infinite and especially the infinitely small proved to be very attractive to the 'bel esprit', but many scholars were aware of the weakness of the basis. 'Allez en avant et la foi vous viendra'—was the device of that time, and so the rebuilding of the foundations was postponed to the 19th century. Although the development of Mathematics in the 19th century was not slower than that in the preceding period, it proved impossible to postpone that task to future times which may be quieter. Thus the foundation was laid in the 19th century. Two different stages may be distinguished. Firstly infinitesimal operations were replaced by limit-operations; this Analysis stands safe on the basis of the totality of the real numbers. The second class of critical investigations concerns this totality and shows how to build up the real numbers from the most general notion of set. At the beginning of the 20th century there was a general belief among the mathematicians that the general notion of set is a primitive idea of human thought and that it is possible to build up Analysis without any non-trivial axiom. This sense of absolute security has been deeply shattered by intuitionists. It would go far beyond the aim of these lectures to give a report on intuitionist philosophy and its results, but as a matter of fact the foundation of Analysis has again—to a certain extent—become problematical by these criticisms. The general idea of 'set' cannot be considered as a primitive idea of the human thought. Every branch of mathematics dealing with infinite entities needs some kind of special axioms. The complete system of axioms necessary for the general theory of sets is not needed for the foundation of each single branch of Analysis. But these lectures will not be of an axiomatic character; the converse way will be followed. Starting from Analysis as it is known nowadays in its main lines to every mathematician, some of the methods applied in it will be considered. Methods are instruments, often complicated instruments, say machines. We may examine a machine, enquiring either how it is constructed, or how it is working, or for what purpose it is meant and what effect we may expect of it. Similarly any mathematical method can be reviewed. We may simply describe it, or we may investigate the conclusiveness of the different steps, or we may review for what kind of mathematical operations the method is helpful. In these lectures the methods will be considered from these three points of view, but to the last one importance will be given.

Some parts of mathematics need only simple methods, other parts are inaccessible to them. As Analysis deals with infinite entities, we have to apply methods in which the notion of infinity occurs somewhere. Of course there is, e.g., the so-called 'Epsilontic', named after the preamble: To every positive ϵ there corresponds a positive δ (or an integer n) . . . By suppositions or propositions of this kind properties of an individual point and properties of an infinite set (the neighbourhood of this point) are connected. One common feature of all these statements is that no special property of this neighbourhood is important, it needs only to be a neighbourhood. Furthermore any statement of this kind does not concern one neighbourhood only, but an infinity of them corresponding to the different values of ϵ, and it relates therefore to an infinity of properties. Such a set of properties is said to be a property *im Kleinen*.* The property of a point that a function, say $f(x)$, has at that point a differential coefficient equal to a is, e.g., a property *im Kleinen*, and it is well known that if this property holds at every point of an interval, the function $f(x)$ can be expressed there by $ax+b$. By this property the constants a and b are connected with the whole interval, we apply to such properties the term *im Grossen*.* As in this example, the epsilontic is often applied to find out properties of a domain *im Grossen* when properties *im Kleinen* are supposed to hold in every point of this domain.

Sometimes, however, the investigations follow the reverse path. If a certain property *im Grossen* holds in a certain domain, it can be proved that in consequence of it, there exists a point in the domain where a certain property *im Kleinen* holds. The mean value theorem of the differential calculus is an instance of a conclusion of this kind; the theorem on the four cyclic points of an oval, Böhmer-Minkowski's theorem and many other theorems of modern Geometry and Analysis are of this type.

If a method is often applied in Mathematics, its essential part may be formulated into a short lemma. In the case where a property *im Grossen* is the consequence of a property *im Kleinen* holding at every point, Borel's lemma often fulfils this task. In the converse case Weierstrass' lemma on limiting points is very helpful in shortening the discussion. Of course in Weierstrass' lemma a closed interval is supposed to contain an infinite point-set M; this supposition obviously concerns a property *im Grossen*, and the proposition is, that at least one point of I has the property *im Kleinen* of being a limiting point of M.

This lemma is usually applied in the following way. If any property *im Grossen* holds, a suitable infinite point-set contained in a closed interval is shown to be connected with it; at the limiting points of that set, a particular

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* *Im Kleinen* = in sufficiently small parts. *Im Grossen* = in large parts. These German terms are usual in English papers.

property *im Kleinen* holds. In a similar manner Borel's lemma is applied to prove that a property *im Grossen* holds in a closed and bounded set if a suitable property *im Kleinen* holds in each of its points.

The applicability of these two lemmas depends on the nature of the properties which are to be considered. In §1 some classes of properties will be investigated, and it will be shown in §2 that for properties of this kind certain theorems hold of which the lemmas of Borel and Weierstrass are special cases. These theorems enable us to apply a direct method to prove fundamental theorems on continuous functions very briefly (§5). By this method the properties of continuous functions, and the proofs of those propositions become more accessible to intuition. Mathematical induction can often be avoided as the proofs are valid for any number of dimensions, but some knowledge about intervals in n-dimensional Euclidean space (§4) is necessary. It is often very helpful to restrict the consideration of sets to certain systems of them, called *admissible* sets (§3). As we have some liberty of choosing that system according to the nature of the particular problem, our method becomes more general. Further generalization will be obtained by considering more general spaces. It is not necessary to justify the introduction of such spaces, as these are largely used in Mathematics as well as in Physics. It has been proved that the main theorems of §2 hold in every compact metrical space, but in these lectures no reference to metrical spaces will be given. The fundamental properties of Hausdorff spaces will be given briefly, and an important theorem on the continuity of a real function in a Hausdorff space will be established (§6). The notion of *convergence* can be considered as a special case of continuity in a suitable Hausdorff space, and the theorem of §6, just mentioned, shows how to get simultaneously theorems on continuous functions, on sequences of functions and on double series as mere corollaries (§7). Finally (§8) it will be shown that the method applied in these lectures, can be generalized still further. It can be generalized to spaces in which fundamental properties of the point-sets considered in Analysis no longer hold; thus we are led to the limits of what may be considered as the fundamental portion of Analysis. In this connection Lindelöf's lemma will appear to be very important.

Treated in this manner, the character of the subject becomes similar to that of some other portions of modern Mathematics. The use of formulæ is restricted; calculation is often replaced by discussion. The application of Hausdorff spaces may be considered as a geometrization; on the other hand the opportunity of deriving several theorems from one by a suitable choice of a topology in a set, or of the system of admissible sets, sometimes reminds one of modern Algebra.

I do not know whether Newton and Euler rising from their graves would like this Analysis, or Lagrange, who was perhaps the first man to apply general spaces on important problems; we may expect the future to get a newer and finer Analysis, but I hope that some contemporary students of Mathematics in India and abroad may derive advantage from these lectures.

§1. *Fundamental notions and notations.* In these lectures we have to deal with *points*, *sets* and *properties*. The points are not necessarily points of any geometrical space. They may be mathematical entities of any kind which form certain collections, called sets.*

Let us furthermore consider mathematical statements concerning sets which may either hold or not hold † for any given set. Each of these statements is connected with a pair of properties, say

$$\mathbf{P} \text{ and } \overline{\mathbf{P}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1)$$

in this manner. If for a particular set S the statement is true, S is said to have the property $\mathbf{P}$, if it is not true, S has the property $\overline{\mathbf{P}}$. Thus every set has either the property $\mathbf{P}$ or $\overline{\mathbf{P}}$. For this reason $\overline{\mathbf{P}}$ is said to be the *negative* of $\mathbf{P}$; if a particular set has not the property $\overline{\mathbf{P}}$, it has the property $\mathbf{P}$. Hence ‡

$$\overline{\overline{\mathbf{P}}} = \mathbf{P}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2)$$

'To contain a particular point p' is, e.g., a property, ‡ say $\mathbf{P}$, of a point-set; then $\overline{\mathbf{P}}$ is the property 'Not to contain p'.

Similarly 'To be contained in a set M' can be considered as a property $\mathbf{Q}$. Now let A be composed of two sets B and C. If A has the property $\mathbf{P}$ then at least one of the sets B, C has the property $\mathbf{P}$, and conversely if one of the sets B, C has the property $\mathbf{P}$, then A has $\mathbf{P}$; the same holds for the property $\overline{\mathbf{Q}}$. On the other hand if A has the property $\mathbf{Q}$ (or the property $\overline{\mathbf{P}}$) then both the sets B, C have the property $\mathbf{Q}$ (or the property $\overline{\mathbf{P}}$) and conversely. This observation leads to the consideration of some special classes of properties of which $\mathbf{P}$, $\mathbf{Q}$, $\overline{\mathbf{P}}$, $\overline{\mathbf{Q}}$ are instances and which are very important for the foundation of Analysis.

Notations. $a, b, c, \ldots$ will be used for *points*, no
$A, B, C \ldots$ for *sets*,
$\mathbf{A}, \mathbf{B}, \mathbf{C} \ldots$ for properties only.
α *or* β means: at least§ one of the entities α, β.
α & β means: both the entities α, β considered individually.

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* See Note V. † See Note II. ‡ See Note I.

§ There are therefore three different cases: (1) α, but not β; (2) β, but not α; (3) α & β.

$$S \subset A \qquad (3)$$

means that S is a subset of A, i.e. that every point of S is a point of A; thus every set is a subset of itself, and an empty set is a subset of every set. There will be no misinterpretation if we denote for the purpose of these lectures the set containing only one point, say a, by a. Hence

$$a \subset A \quad .. \quad .. \quad .. \quad (4)$$

means that a is contained in the set A.

The points contained in A & B form a set, the *meet*

$$A \cap B \quad .. \quad .. \quad .. \quad (5)$$

of A and B; the points contained in A *or* B form a set, the *join*

$$A \cup B \quad .. \quad .. \quad .. \quad (6)$$

of A and B.

1.1. *Repartitive Properties.* Let us consider now the statements (which may hold or may not hold) about any particular property

E is a property of A & B .. .. (&)
E ,, ,, ,, $A \cup B$ .. .. ($\cup$)
E ,, ,, ,, A *or* B .. .. (*or*)

These three statements are not independent, as obviously the third statement is a consequence of the first; we may express this fact * by

$$(\&) \longrightarrow (or) \quad .. \quad .. \quad .. \quad (1)$$

In general (1) is the only relation between the three statements, but in important particular cases there are also other relations. If, e.g., we put for **E** the property **P** ('to contain a particular point p') considered above, or the property $\overline{\mathbf{Q}}$, the statement ($\cup$) is a consequence of the statement (*or*), and conversely (*or*) is a consequence of ($\cup$). Hence for $\mathbf{E} = \mathbf{P}$, and for $\mathbf{E} = \overline{\mathbf{Q}}$ the relation

$$(or) \underset{\longrightarrow}{\longleftarrow} (\cup) \quad .. \quad .. \quad .. \quad (\delta)$$

holds. The formula (δ) does not hold for $\mathbf{E} = \mathbf{Q}$, nor for $\mathbf{E} = \overline{\mathbf{P}}$. In these cases, we have

$$(\&) \underset{\longleftarrow}{\longrightarrow} (\cup) \quad .. \quad .. \quad .. \quad (\kappa)$$

Properties satisfying (δ) are said to be *distributive*, and properties satisfying (κ) are said to be *collective*. Properties satisfying

$$(\&) \longrightarrow (\cup) \longrightarrow (or) \quad .. \quad .. \quad (\rho)$$

are said to be *repartitive*. From (1) it follows that (ρ) is a consequence of (δ), and also a consequence of (κ), thus every distributive property is reparti-

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* Read: (&) leads to (*or*).

tive and every collective property is repartitive, but there are repartitive properties which are neither distributive nor collective.

Let, e.g., the 'points' be the points of the real axis, and $f(x)$ be an arbitrary real function; let $F(M)$ be the upper limit of $f(x)$ in an arbitrary set M and a be a real number.

$$F(M)>a$$

is a property of the sets M, and this property is a distributive one. Similarly

$$F(M)<a$$

is a collective property, and

$$F(M) = a$$

is a repartitive property which is neither distributive nor collective. The formula (ρ) may be expressed in a different manner. Let **R** be any repartitive property, and $\overline{\mathbf{R}}$ be its negative,

$(\&)\longrightarrow(\cup)$ means: It is impossible for **R** to hold in A & B, and at the same time for $\overline{\mathbf{R}}$ to hold in $A \cup B$.

$(\cup)\longrightarrow(or)$ means: It is impossible for $\overline{\mathbf{R}}$ to hold in A & B. and at the same time for **R** to hold in $A \cup B$.

The two statements on the right are equivalent to the formulae on the left and form, therefore, the necessary and sufficient conditions in order that **R** should be repartitive. As the pair of statements is symmetric in **R** and $\overline{\mathbf{R}}$, it follows that if **R** is repartitive, $\overline{\mathbf{R}}$ is repartitive too. Let **D** be a distributive property; since **D** is repartitive so is $\overline{\mathbf{D}}$. If **D** holds in A *or* B, $\overline{\mathbf{D}}$ cannot hold in $A \cup B$; if therefore $\overline{\mathbf{D}}$ holds in $A \cup B$, it holds in A & B. Therefore $(\cup)\longrightarrow(\&)$ holds for $\overline{\mathbf{D}}$ in addition to (ρ). Hence $\overline{\mathbf{D}}$ is collective. In the same manner we may prove that the negative of a collective property is distributive. Hence

Theorem 1. Distributive and collective properties are repartitive; the negative of any repartitive property is repartitive; the negative of any distributive property is collective; the negative of any collective property is distributive.

The formulæ (δ), (κ), (ρ) can be generalized for joins of $n \geq 2$ sets

$$A = A_1 \cup \ldots\ldots \cup A_n.$$

If any repartitive property **R** holds for A_1 & ... & A_n, then **R** holds for

$$A_1 \cup A_2,\ (A_1 \cup A_2) \cup A_3, \ldots, A_1 \cup \ldots \cup A_n = A.$$

If conversely **R** holds in A, it is impossible that $\overline{\mathbf{R}}$ holds in

$$A_1 \,\&\, \ldots \,\&\, A_n,$$

viz. $\overline{\mathbf{R}}$ is also repartitive. Hence **R** holds in A_1 *or* . . . *or* A_n.

If any distributive property holds for A_1 *or* . . . *or* A_n it holds for at least one of these sets, say for A_k, and therefore for $A = A \cup A_k$. If any collective property holds for $A = A \cup A_j$ $(j = 1, \ldots, n)$, it holds for A_j. Hence

Theorem 2. Let **R** be any repartitive, **D** any distributive, **C** any collective property.

If **R** holds for $A_1 \,\&\, \ldots \,\&\, A_n$, it holds for $A_1 \cup \ldots \cup A_n$.

If **R** holds for $A_1 \cup \ldots \cup A_n$, it holds for A_1 *or* . . . *or* A_n .. (ρ')

If **D** holds for A_1 *or* . . . *or* A_n, it holds for $A_1 \cup \ldots \cup A_n$ and conversely .. .. (δ')

If **C** holds for $A_1 \,\&\, \ldots \,\&\, A_n$, it holds for $A_1 \cup \ldots \cup A_n$ and conversely .. .. (κ')

It may be mentioned that these formulæ cannot be generalized to the case of joins of an infinite number of sets.

If any distributive property **D** holds in the empty set E, then **D** holds in every set A, viz., $A = A \cup E$. From this remark follows

Theorem 2*a*. If the distributive property **D** holds in

$$A = A_1 \cup \ldots \cup A_n,$$

and A is not empty, then **D** holds in at least one A_k which is not an empty set.

Proof. From Theorem 2 it follows that **D** holds in at least one A_k; if this A_k were empty, **D** would hold in every A_j, and these cannot all be empty, for emptiness is a collective property, and A is not empty.

*** 1.2.** The operations & and *or* permit us to build up new properties from given ones. In particular it is possible to generate repartitive properties from distributive and collective ones. **A** & **B** signifies the property of a set that **A** and **B** both hold; similarly **A** *or* **B** means that at least one of these properties holds. The following formulæ are direct consequences of these definitions :—

$$\mathbf{A} \,\&\, \mathbf{B} = \mathbf{B} \,\&\, \mathbf{A} \qquad \mathbf{A} \text{ or } \mathbf{B} = \mathbf{B} \text{ or } \mathbf{A},$$

$$(\mathbf{A} \,\&\, \mathbf{B}) \,\&\, \mathbf{C} = \mathbf{A} \,\&\, (\mathbf{B} \,\&\, \mathbf{C}), \qquad (\mathbf{A} \text{ or } \mathbf{B}) \text{ or } \mathbf{C} = \mathbf{A} \text{ or } (\mathbf{B} \text{ or } \mathbf{C}),$$

$$(\mathbf{A} \text{ or } \mathbf{B}) \,\&\, \mathbf{C} = (\mathbf{A} \,\&\, \mathbf{C}) \text{ or } (\mathbf{B} \,\&\, \mathbf{C}).$$

These formulæ remind us of well-known fundamental formulæ of Algebra; we need only replace the operation *or* by addition, and & by multiplication. The connection becomes more distinct if we consider two properties to be equal if each is the consequence of the other one. Under this supposition the properties

$$\mathbf{A} \mathit{ or } \overline{\mathbf{A}} = 1 \text{ and } \mathbf{A} \,\&\, \overline{\mathbf{A}} = \mathbf{O}$$

are independent of the choice of **A**, viz., 1 holds for every set, and **O** for none. Considering these facts we get that for every property **P**

$$\mathbf{P} \mathit{ or } \mathbf{O} = \mathbf{P} \qquad \mathbf{P} \,\&\, \mathbf{O} = \mathbf{O}$$

$$\mathbf{P} \mathit{ or } 1 = 1 \qquad \mathbf{P} \,\&\, 1 = \mathbf{P}.$$

Thus **O** is the zero-element, while 1 is the unit-element (and in a certain sense an infinite-element) of that 'Algebra' in which *or* represents addition, and & multiplication. This Algebra is called a *Boolean** Algebra and it has acquired recently a certain importance, but we shall not apply Boolean Algebra in these investigations of the fundamentals of Analysis. Furthermore

$$\overline{\mathbf{A} \mathit{ or } \mathbf{B}} = \overline{\mathbf{A}} \,\&\, \overline{\mathbf{B}}$$

$$\overline{\mathbf{A} \,\&\, \mathbf{B}} = \overline{\mathbf{A}} \mathit{ or } \overline{\mathbf{B}}$$

$$\mathbf{A} \,\&\, \mathbf{A} = \mathbf{A} \mathit{ or } \mathbf{A}.$$

Without referring to the significance of the operations introduced here it is possible to deduce from the above formulæ in a purely formal manner

$$\mathbf{A} \,\&\, (\mathbf{A} \mathit{ or } \mathbf{B}) = \mathbf{A} \text{ or } (\mathbf{A} \,\&\, \mathbf{B}) = \mathbf{A}, \qquad \mathbf{A} \text{ or } \mathbf{B} = (\mathbf{A} \,\&\, \overline{\mathbf{B}}) \text{ or } \mathbf{B}.$$

The properties **O** and **A** are distributive and collective. If, on the other hand, a property is simultaneously distributive and collective, it holds either in all sets, or in none of them.

The letters **C**, **D**, **R** will be used for collective, distributive and repartitive properties respectively. From the definition of these properties it follows that

$$\mathbf{D}_1 \mathit{ or } \mathbf{D}_2 = \mathbf{D}$$

$$\mathbf{C}_1 \,\&\, \mathbf{C}_2 = \mathbf{C}$$

$$\mathbf{R} \mathit{ or } \mathbf{D} = \mathbf{R}_1$$

$$\mathbf{R} \,\&\, \mathbf{C} = \mathbf{R}_2$$

On putting **C** (or **D**) for **R**, we get

$$(\mathbf{C} \mathit{ or } \mathbf{D}) = \mathbf{R}_3$$

$$\mathbf{C} \,\&\, \mathbf{D} = \mathbf{R}_4$$

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* Named after G. Boole [6] who introduced this algebra for logistic purposes. Later Boolean algebra has been considered more from an algebraic point of view. For reference see [18]. Recent applications to theory of groups, see [36].

Let, e.g., **D** be the property $F(M)>a$, and **C** be the property $F(M)<a$, where $F(M)$ has the same significance as in 1.1, then

$$\overline{(\mathbf{C} \text{ or } \mathbf{D})} = \overline{\mathbf{C}} \mathbin{\&} \overline{\mathbf{D}} = \mathbf{R}$$

is the property $F(M) = a$.

The above formulæ allow us to construct an infinity of repartitive properties starting from collective and from distributive ones. All the repartitive properties known up to now are of this character. On combining repartitive properties by *or* and & we shall not in general get repartitive properties, but in special cases we do. E.g.

$$\mathbf{R} \mathbin{\&} \mathbf{C} = \mathbf{R}' \text{ and } \mathbf{D} \mathbin{\&} \mathbf{C} = \mathbf{R}''$$

are repartitive properties which can be supposed to be neither distributive nor collective.

$$(\mathbf{R}' \text{ or } \mathbf{R}'') = (\mathbf{R} \mathbin{\&} \mathbf{C}) \text{ or } (\mathbf{D} \mathbin{\&} \mathbf{C}) = \mathbf{C} \mathbin{\&} (\mathbf{R} \text{ or } \mathbf{D}) = \mathbf{C} \mathbin{\&} \mathbf{R}_1$$

is a repartitive property.

§ 2. *The main theorems.** Let C be a closed, bounded and non-empty point set in an n-dimensional Euclidean space. C is enclosed in an n-dimensional cube Γ; the length of the side of Γ will be chosen as the unity of measurement. The fundamental notions of the theory of point sets, like open set, closed set, limiting point, will be applied in the following in the usual manner. A subset of C will be said to be *open relatively to C* or briefly *open* (C) if it is the meet of an open point set and C.

2.1. *Distributive theorem.* Let C have a distributive property **D**. We divide Γ into 2^n closed cubes of side $\frac{1}{2}$; these cubes intersect C in 2^n subsets (these subsets may have common points, some of them may also be empty). From 1.1, Theorem 2_a, it follows that at least one of these subsets, say C_1, has the property **D** and is not empty.

$$C_1 = C \cap \Gamma_1,$$

where Γ_1 is a cube with side $\frac{1}{2}$. Proceeding with C_1 and Γ_1 in the same manner as was done with C and Γ, we get

$$C_2 = C_1 \cap \Gamma_2$$

where Γ_2 is a cube with side $\frac{1}{2^2}$, and C_2 is not empty and has the property **D**. On repeating this procedure an infinite number of times we get a sequence of non-empty sets

$$C \supset C_1 \supset C_2 \supset \dots$$

which all have the property **D** . $C_k \subset \Gamma_k$, where Γ_k is a cube with side $\frac{1}{2^k}$, and $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \dots$.

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* See Note III.

This sequence of cubes has one common point, say d. As none of the sets C_k is empty, we can select from each C_k one point, and this sequence converges to d. Therefore d is a limiting point of C, and as C is closed, d belongs to C. Every open set Ω containing d contains Γ_n for a suitably chosen n. Let now C^* be any subset of C which is open (C), then

$$C^* = \Omega \cap C, \quad \Omega \supset \Gamma_n, \quad C^* \supset C_0.$$

By this consideration we get the distributive theorem.

Distributive theorem. If any distributive property **D** holds in a closed, limited and non-empty set C, then there exists in C a point d, such that every set which includes d and is open (C) has also the property **D**.

This theorem has many applications in Analysis; it shows that a distributive property is linked up with certain points where it seems to be concentrated. A simple application of it is Weierstass' lemma.[†]

2.11. *Weierstrass' lemma.* If an infinite point-set M is contained in a closed bounded set C, then there exists a limiting point of M in C.

Proof. The property 'to contain an infinity of points of M' is distributive; hence there exists in M a point such as is required by the distributive theorem, and d is a limiting point of M.

Again let M be a subset of a closed and bounded set C. The following properties are obviously distributive properties of a set C. That $M \subset C$

(1) is a non-enumerable infinite set,
(2) has a particular infinite power,
(3) has an exterior measure >0,
(4) has a discrepancy (i.e. the difference of the exterior and the interior measure) >0.

The following theorems are immediate consequences of this remark :—

(1) If $M \subset C$ is non-enumerably infinite, then C contains a point of condensation of M.

(2) If $M \subset C$ is infinite, then C contains a *complete* limiting point d of M. (I.e. every cube of which d is an interior point, contains a subset of M which is of the same power as M.)

(3) If the exterior measure of $M \subset C$ is positive, then C contains a point d such that every open set containing d contains a subset of M which has a positive exterior measure.

(4) If the discrepancy of $M \subset C$ is positive, then C contains a point d such that every open set containing d contains a subset of M which has a positive discrepancy.

The last two theorems hold for both Jordan's and Lebesgue's measure.

[†] See Note IV.

From the distributive theorem the following important theorem can easily be derived as a converse theorem.

2.2. *Collective theorem.* Let C be a closed, bounded and non-empty set, and let **C** be a collective property. If every point of C is contained in a set which is open (C) and in which **C** holds, then **C** holds in C.

Proof. If **C** does not hold in C, then the distributive property $\overline{\mathbf{C}}$ holds in C. Hence there exists a point d in C, such that every point-set which is open (C) and contains d, has the property $\overline{\mathbf{C}}$. This result contradicts the supposition that every point is contained in a set open (C) in which **C** holds.

A direct consequence of the collective theorem is

2.21. *Borel's lemma.*[*] Let C be a closed, bounded and non-empty set, and let every point p of C be contained in an open set Ω_p, then C is contained in the join of a finite number of suitably chosen sets Ω_p.

Proof. 'To be overlapped by a finite number of sets Ω_p' is a collective property, say **C**. Every point p is contained in a set open (C) which has this property, e.g. in $\Omega_p \cap C$. Hence the lemma.

The distributive and the collective theorems are special cases of the following theorem:

2.3. *Repartitive theorem.* Let C be a closed, bounded and non-empty point-set, **R** any repartitive property. (1) If every point p of C is enclosed in a set $C_p \subset C$ which is open (C) and has the property **R**, then C has the property **R**. (2) If C has the property **R**, then there exists a point d in C such that every set which is open (C) and contains d, has the property **R**.

Proof. (1) From Borel's lemma it follows that C can be overlapped by a finite number of open point-sets $C \subset \Omega_1 \cup \ldots \cup \Omega_n$ such that $\Omega_k \cap C$ is a set C_p, for $k = 1, \ldots, n$. As **R** is supposed to hold in every C_p, it follows from 1.1, theorem 2, that **R** holds in **C**. (2) If no point d with this property exists, every point p of C will be contained in a $C^1{}_p$, which is open (C) and have the property $\overline{\mathbf{R}}$. As $\overline{\mathbf{R}}$ is repartitive, it follows from the first part of the theorem that $\overline{\mathbf{R}}$ holds in C, contrary to the supposition of (2).

The points d whose existence in C has been stated by the distributive theorem, form a set $D(C)$. Let p be a limiting point of $D(C)$. Every pointset which is open (C) and contains p, contains points of $D(C)$ and has therefore the property **D**. Hence p satisfies the necessary and sufficient conditions for being a point of $D(C)$, and therefore:

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[*] See Note IV.

Theorem. $D(C)$ is closed.

***2.4.** *Connection between* $\mathbf{D}$ *and* $D(C)$. Let $\mathbf{D}$ be a particular distributive property, C_1 be a closed and non-empty subset of C, and a be a point of $D(C_1)$. If A is a point-set which is open (C) and contains a, then $A \cap C_1 = A_1$ is open (C_1) and contains a. Hence A_1 has the property $\mathbf{D}$. As $A = A \cup A_1$, the set A has also the property $\mathbf{D}$. Therefore a belongs to $D(C)$. Hence $D(C) \supset D(C_1)$.

Let $C = C_1 \cup C_2$, the sets C_i being closed and non-empty. Then

$$D(C) \supset D(C_1) \cup D(C_2).$$

It will be shown that equality holds. Let p be a point of $D(C)$ which does not belong to $D(C_1) \cup D(C_2)$. If p belongs to $C_1 \cap C_2$, there are point-sets B_1 open (C_1) and B_2 open (C_2) which contain p, and in which $\overline{\mathbf{D}}$ holds. Let $B_1 = C_1 \cap \Omega_1$, $B_2 = C_2 \cap \Omega_2$, $\Omega_1 \cap \Omega_2 = \Omega$, where Ω_1, Ω_2, Ω are open. $A_1 = C_1 \cap \Omega$, $A_2 = C_2 \cap \Omega$, $A = A_1 \cup A_2 = C \cap \Omega$. The collective property $\overline{\mathbf{D}}$ holds therefore in A_1, A_2, and A, but as A is open (C) and contains the point a of $D(C)$, the property $\mathbf{D}$ must hold in A. Thus p can only belong to one of the sets C_1 and C_2, say to C_1. As C_2 is closed, there exists an open set Ω' containing p but no point of C_2. If $\overline{\mathbf{D}}$ were to hold in $B_1 = C_1 \cap \Omega'$ it would also hold in a set which contains p and is open (C), contrary to the supposition. Hence p cannot exist, i.e.

$$D(C) = D(C_1) \cup D(C_2).$$

Let $A \subset C$ have the property $\mathbf{D}$. On adding every limiting point of A to A_1 we get a closed set, the *closure* $[A]$ of A. In $[A] = [A] \cup A$ the property $\mathbf{D}$ holds, hence $D([A])$ is not empty. As $D([A])$ is a subset of $[A]$ and a subset of $D(C)$, the set $[A]$ contains a point of $D(C)$. Hence:

Theorem. If A has the property $\mathbf{D}$, the closure $[A]$ contains a point of $D(C)$.

The converse does not necessarily hold.

Let D be a closed subset of C, and $\mathbf{D}^*$ the property of any set A that $[A]$ contains a point of D. The property $\mathbf{D}^*$ is distributive, and from the distributive theorem it follows that there exists a certain set of points d, say $D^*(C)$. Every point of D belongs obviously to $D^*(C)$. Let p be a point of C not belonging to D. As D is closed, p is contained in an open set Ω such that $[\Omega]$ contains no point of D; then $\Omega \cap C$ has the property $\overline{\mathbf{D}}^*$; hence p is not a point of $D^*(C)$, and

$$D^*(C) = D.$$

The class Δ of all the properties corresponding to the set D contains therefore the property $\mathbf{D}^*$, on the other hand, if in any subset A of C any property of the class Δ holds, the property $\mathbf{D}^*$ holds also in A. The property $\mathbf{D}^*$ is therefore the 'maximum' property of the class Δ; it is the property which may be constructed by combining all the properties of the class Δ by *or*.

In general there will be no 'minimum'-property in the class Δ, as the following example shows:

Let D consist of a single point d of C, which is a limiting point of C. Let $\mathbf{D}_1$ be the property 'to contain d', and let $\mathbf{D}_2$ be the property 'to have d as a limiting point'. If a 'minimum'-property $\mathbf{D}_0$ exists in Δ, every set having the property $\mathbf{D}_0$ must have the properties $\mathbf{D}_1$ & $\mathbf{D}_2$, and every set A which is open (C) and contains d must have the property $\mathbf{D}_0$. But $A = d \cup (A - d)$ where d has not the property $\mathbf{D}_2$, and $A - d$ has not the property $\mathbf{D}_1$. Hence $\mathbf{D}_0$ is not distributive, and does not belong to Δ.

2.5. *Complementary properties.* There is one reciprocity between the notions of distributive and collective properties, and another reciprocity between join and meet. These reciprocities should not be confounded. If we replace the fundamental formula $(\cup)$ of 1.1 by a similar formula $(\cap)$, there will not result a mere interchanging of the significance of (δ) and (κ). Of course we should get the property 'to be the complement of a set which has a particular distributive (collective) property'. There is up to now no reason for a detailed study of these complementary properties.

§ 3. *Generalization of the main theorems.* The distributive, the collective and the repartitive theorem still hold under certain conditions if we restrict our consideration to a certain class of sets, called the *admissible* sets only. The class of admissible sets can be chosen in different ways but certain conditions have to be satisfied in order that the above main theorems may hold.

Let Σ be a system of sets. We suppose that

$$A,\ B \text{ and } A \cup B$$

are admissible sets and we define distributive, collective, repartitive properties by the same formulas as before. Then 1.1, theorem 1, holds. It may be that three admissible sets have a collective property, and that the join of those sets is admissible, but the join of every two of them is not admissible. In this case it may be that the join of the three sets has not the collective property, contrary to 1.1, theorem 2.

We now establish two conditions for the system Σ which are sufficient to ensure that the theorems 2, 2_a of 1.1 and the three main theorems of § 2 hold.

I. If A is an admissible set, there exists an infinite system of admissible sets

$A=A_1 \cup A_2,$
$A_1=A_{11} \cup A_{12}, \qquad A_2=A_{21} \cup A_{22},$
$A_{11}=A_{111} \cup A_{112}, \quad A_{12}=A_{121} \cup A_{122}, \quad A_{21}=A_{211} \cup A_{212}, \quad A_{22}=A_{221} \cup A_{222}$

. . . .

This system is supposed to have the property that if $i, j, k, \dots l \dots$ is an infinite sequence, either $A_{ijk\dots l}$ and the following sets are empty, or the sequence

$$A_i, A_{ij}, A_{ijk}, \dots, A_{ijk\dots l}, \dots$$

converges to a point.

II. If the join of n admissible sets is an admissible set, those n sets can be denoted in a suitable order by $A_1, A_2, \dots, A_n$ such that

$$A_1 \cup A_2 = B_2, \quad B_2 \cup A_3 = B_3, \dots, \quad B_{n-2} \cup A_{n-1} = B_{n-1}$$

are also admissible sets.

If we restrict the consideration to systems of admissible sets satisfying these conditions, the preceding theorems hold without any essential alteration of the proofs given above. The system of all bounded sets is obviously an admissible one; on the other hand the supposition that C is bounded becomes superfluous in the distributive, collective and repartitive theorems, as every admissible set is bounded.

§**4.** *Some properties of open intervals.* The advantages we obtain by using *admissible* sets, is based in several cases on the special properties of open intervals. A few of those properties will therefore be considered in this section.

An n-dimensional interval is defined by n inequalities of the coordinates x_j

$$a_j < x_j < b_j, \quad j = 1, \dots, n \qquad \dots \qquad \dots \quad (1)$$

4.1. If $n = 1$, the intervals are said to be *linear* open intervals (a, b). The following properties of linear open intervals are obvious:

(1) If I_1, I_2, and $I_1 \cup I_2$ are open intervals, then I_1 and I_2 are overlapping.

(2) Let $I_j = (a_j, b_j)$, $j = 1, \dots, m$, and let the join $I_1 \dots I_m$ be an interval. If $b_k \leqq b_{j \neq k}$, then the join of the $m-1$ intervals $I_{j \neq k}$ also forms an interval.

(3) The m intervals $I_1, \dots, I_m$ can be arranged in such a manner, that

$$I_{\nu_1} \cup \dots \cup I_{\nu_t}$$

is an interval for $1 \leqq t \leqq m$.

This proposition is a consequence of the preceding one.

(4) The open intervals on a straight line form a system of 'admissible' sets. The condition I of admissibility holds also for n-dimensional intervals; condition II is identical with the preceding proposition.

4.2. Let the number n of the dimension be arbitrary, and Ω be the join

$$\Omega = I_1 \cup \dots \cup I_m \qquad \dots \quad (1)$$

of m open intervals. The primes bounding these intervals generate a division of Ω into open intervals of dimensions n, $n-1 \dots 2, 1, 0$, (the vertices of the subdivision being considered as 0-dimensional intervals)

$$\Omega = J^n_1 \cup \dots \cup J^n_{m_0} \cup J^{n-1}_1 \cup \dots \cup J^{n-1}_{m_1} \cup \dots \cup J^0_1 \cup \dots \cup J^0_{m_n} \quad \dots \quad (2)$$

Different J^μ_ν have no common point. If a point of any particular J^μ_ν belongs to an interval I_k the whole interval J^μ_ν belongs to I_k. Those intervals (2) of dimension $>\mu$ which have J^μ_ν on their boundary, have common points with I_k and are therefore contained in I_k. Let δ be a positive number which is supposed to be less than the distances between the vertices J^0_s, and let L be any n-dimensional interval of length $\leq \delta$. Any two intervals J^ζ_σ, J^ζ_τ of the same dimension ζ are either abutting to a common interval of lower dimension, or they differ by more than δ in every pair of points for at least one of the co-ordinates. If therefore I contains points of J^ζ_σ and J^ζ_τ, these intervals are abutting to a particular J^κ_λ, $(\kappa < \zeta)$, and as I is an interval, it contains points of J^κ_λ. Let p be the minimum dimension of intervals (2) having common points with I, then there exists only one such interval (2), say J^p_q, which may be included in the interval I_k of the intervals (1). We consider the intervals (2) which have common points with I. Those of dimension $p+1$ are abutting to J^p_q; those of dimension $p+2$ are abutting to the preceding ones and therefore also to J^p_q, etc.; these intervals are all abutting to J^p_q. However it has been shown that if J^p_q is contained in I_k the intervals (2) of higher dimension abutting to J^p_q are also contained in I_k. Hence I is contained in I_k. In other words

Theorem. To every join Ω of a finite number of open intervals (1) there corresponds a positive number δ such that every interval I of length $<\delta$, which is a subset of Ω, is also a subset of a particular I_k.

§**5.** *Fundamental theorems of Analysis.* As Dedekind's lemma is often used to define the real numbers, and as this lemma is a very helpful expedient of Analysis, it may be interesting to show how it can be deduced by the methods explained above. In this proof the real numbers are supposed to form an ordered set, and it is supposed that the open intervals of real numbers, as defined by §4.1, are admissible sets.

Dedekind's lemma. Let the set of the real numbers be divided in two non-empty sets, A and B such that every element of A is less than every element of B, then there exists a real number d such that every number less than d belongs to A, and every number greater than d belongs to B.

Proof. Let $a \subset A$, $b \subset B$, and let the admissible subsets of the closed interval $C = [a, b]$ be the meets of any open interval with C. For these admissible sets, the property 'to contain points of A as well as of B' is a distributive property, for it is impossible to subdivide an admissible set into two admissible sets $A' \subset A$, and $B' \subset B$, as A' and B' are overlapping. Hence there exists a point $d \subset C$ such that every set open (C) which contains d, contains points of A and of B. Let $b' > d$, $b' \subset A$, then $(a, b') \subset A$ which is impossible. Hence every $b' > d$ belongs to B, and similarly every $a' < d$ belongs to A.

Let S be a bounded set of real numbers, B the set of the real numbers exceeding all the numbers of S, and A the complementary set; by applying Dedekind's lemma it follows that S has an *upper limit.* E.g. S may be the set of values taken by any bounded function $f(p)$ which may exist for an arbitrary set of points p. Hence

Theorem 1. Any bounded real function has an upper limit.

The same holds for the lower limit. Again let C be a closed bounded and non-empty n-dimensional point-set, $f(p)$ any bounded real function of the points of C, and let a be the upper limit of $f(p)$ in C. The property of any subset M of C that the upper limit of $f(p)$ in M is equal to a is a repartitive property. Hence

Theorem 2. There exists in C a point d such that in every subset which is open (C) and contains d, the upper limit of $f(p)$ is equal to a.

Theorem 3. Let E be a set of positive numbers ϵ, and let the lower limit be equal to 0; to every ϵ let a set A_ϵ correspond, which is contained in an interval of length ϵ, let $A_{\epsilon_\nu} \subset A_{\epsilon_\mu}$ if $\epsilon_\nu < \epsilon_\mu$; then there exists one and only one number d such that every open interval containing d contains an A_ϵ,

Proof. If the join of two open linear intervals is an interval

$$I = I_1 \cup I_2,$$

the intervals I_1 and I_2 are overlapping by an interval of length, say δ. If therefore I contains A_ϵ, and $\epsilon' < \epsilon$ & δ, then $A_{\epsilon'}$ is completely contained in I_1 *or* I_2. The property 'to contain any A_ϵ' is therefore distributive for open intervals. Thus it follows from the distributive theorem that a point d as proposed exists. An interval containing any A_ϵ must contain points of every A_ϵ. If now $d' = d \pm 2k$, the interval $(d'-k, d'+k)$ contains no point of A_k. Hence there exists only one point with the properties proposed for d.

5.1. *Continuous functions.* Definition: $f(p)$ is said to be continuous at the point q in the set A if to every $\epsilon > 0$ there exists a set $\Omega \supset q$ which is open (A), such that

$$| f(p_1) - f(p_2) | < \epsilon$$

if p_1 & $p_2 \subset \Omega$. If $f(p)$ is continuous at every point in C, it is said to be continuous in C. The property 'to be bounded' is collective. From the definition of continuity and the collective theorem it follows therefore:

Theorem 1. If $f(p)$ is continuous * in C, it is bounded in C.

At the point d (see 5, theorem 1) the continuous function $f(p)$ takes the value $f(d) = a$, for if $f(d) = a - 2\alpha$, there would be a point-set containing d and open (C) in which $f(d) - f(p) < \alpha$ and therefore the upper limit of $f(p)$ must be less than $a - \alpha$ contrary to 5, theorem 1. Hence

Theorem 2. If $f(p)$ is continuous * in C, it has a maximum in C. A corresponding result holds obviously for the minimum.

To prove the theorem of uniform continuity, we consider as *admissible* subsets * of C the meet of C with any join of a finite number of open intervals, and we examine the property $\mathbf{C}_\epsilon$ of admissible sets 'that there exists some $\delta > 0$ such that $| f(p_1) - f(p_2) | < \epsilon$ if the points p_1 & p_2 of C are situated in an (n-dimensional) interval of length $< \delta$'. If $f(p)$ is continuous, $\mathbf{C}_\epsilon$ holds in a set open (C) containing any particular point p of C.

If $\mathbf{C}_\epsilon$ holds in two admissible sets $A = C \cap \Omega_1$ and $B = C \cap \Omega_2$, the corresponding δ-values being δ_1 and δ_2, we can represent Ω_1 (or Ω_2) as a join of a finite number of open intervals of length $< \delta_1$ (or δ_2). From the theorem of §4 it follows that there exists a number $\delta > 0$ such that every open interval $I \subset \Omega_1 \cup \Omega_2$ of length $< \delta$ is completely enclosed in one of those intervals. Hence $\mathbf{C}_2$ is collective. Applying the collective theorem we get

Theorem 3. If $f(p)$ is continuous * in C, it is uniformly continuous, i.e. to every $\epsilon > 0$ there corresponds a $\delta > 0$ such that in the meet of C with any open interval of length $< \delta$, the inequality $| f(p_1) - f(p_2) | < \epsilon$ holds.

The notion of continuous function has been generalized to the case when the values of the function are not real numbers but points of a particular set. For the purpose of these lectures it is sufficient to consider the following case: The independent variable is a real number, say $0 \leqq t \leqq 1$, $\phi(t)$ is a point of

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* C as usual is supposed to be bounded, closed and non-empty.

an n-dimensional space Σ and to every t, and every open set $\Omega \supset \phi(t)$ there exists a number $\delta > 0$ such that $\phi(\tau) \subset \Omega$ if $| t - \tau | < \delta$. The values $\phi(\tau)$ are said to form an *arc* G joining $\phi(0)$ and $\phi(1)$. If $f(p)$ is a continuous real function of the points $p \subset G$, then $f(p) = f(\phi(t) = \mathrm{F}(t))$ is a continuous real function of the real variable t.

A point-set A is said to be *connected* if any pair of points of A can be joined by an arc which is a subset of A.

Theorem 4. Let $f(p)$ be continuous in a connected set A, let $a<b<c$, and let $f(p)$ take the values a and c in A, then there exists a point $q \subset A$, such that $f(q) = b$.

Proof. Two points in which $f(p)$ takes the values a and c can be joined by an arc J. Along J, $f(p) = \mathrm{F}(t)$, $0 \leqq t \leqq 1$, where $\mathrm{F}(t)$ is continuous. We consider $C = [0, 1]$, and the meets of C with any open interval as admissible point-sets. The property that $\mathrm{F}(t)$ takes in an admissible interval values $\geqq b$ and values $\leqq b$ is distributive. Let d be the point existing according to the distributive theorem. As $\mathrm{F}(t)$ is continuous $\mathrm{F}(d) = b$.

The fundamental theorems of classical Analysis have been proved here without any reference to the dimension of the underlying point-set, i.e. to the number of the independent variables. These proofs would not be simplified by sticking to the case of one variable only. The only exception is theorem 3 as the geometrical theorem of §4 which is applied in the proof of theorem 3 becomes easier to prove if we confine ourselves to the one-dimensional case. It may be mentioned that the importance of theorem 3 (which certainly is a fundamental one) seems to have been overestimated for a long time. There was e.g. a general belief that the Riemann-integrability of continuous functions could not be proved without the help of that theorem. This opinion has been shown to be ill founded *. On the other hand it will become obvious later that the notion of uniform convergence is only a special case of uniform continuity and that theorem 3 contains certain important theorems on convergence, but the discovery of Lebesgue-integration has impaired the importance of uniform convergence to a certain extent.

§6. *Hausdorff spaces.*† The general considerations of §1 have been restricted by §2 to investigations in n-dimensional Euclidean spaces. Of course, notions like *open sets, closed sets, limiting points, sets converging to a point* have been utilized. These notions are all invariant under continuous

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* See [28].

† See Note VI.

transformations, they relate to the connection between a point and its surroundings; properties which are specifically metric have been applied in §4 and §5 only. For this reason it seems likely that the main theorems could be generalized to a more general class of spaces than n-dimensional Euclidean spaces. These considerations will be helpful later in building up the fundamentals of Analysis in such a manner that the general theory of series is included in the theory of continuous functions as a special case. As only a small portion of the theory of Hausdorff spaces is needed for our purposes, the definitions will be given here in a simplified form, but the spaces so defined are equivalent to Hausdorff spaces.

A set H is said to be a *Hausdorff space* if a particular class of subsets is distinguished as being *open sets*, and if the points of H and the open sets are connected together by the three conditions as stated below. An open set containing a particular point a is said to be a *neighbourhood* $N(a)$ of a; if a set A contains any neighbourhood of a, then a is said to be an *inner point* of A. The system of the open sets is supposed to satisfy the following conditions :—

1. The meet of two open sets is open.
2. The join of any (finite or infinite) number of open sets is open.
3. If $a \neq b$, there exist neighbourhoods $N(a)$ and $N(b)$ without a common point. If the space H contains less than two points, every subset of H is an open set.

By introducing a system of open sets satisfying these conditions, the point-set gets a *topology*. In general a topology can be introduced in more than one manner; thus there are different Hausdorff spaces formed by the same set of points. So long as such cases are not considered, we may denote the Hausdorff space by the same letter as the set of its points. From the above conditions the following theorems follow directly:

Theorem 1. Every point of H has a neighbourhood.—Consequence of 3.

Theorem 2. H is an open set.—Consequence of 2 and th. 1.

Theorem 3. An empty set is open.—Consequence of 3 and 1.

Definition. If the set a (formed by the point a only—see §1) is open, the point a is said to be an *isolated* point.

Theorem 4. If an open set contains only a finite number of points, these points are isolated ones.—Consequence of 3.

In other words:

Theorem 4′. If a is not isolated, every $N(a)$ contains an infinity of points.

Definition. Let $S \subset H$, and let every neighbourhood of a particular point a contain points of S different from a, then a is said to be a *limiting point* of A.

Theorem 5. An isolated point cannot be a limiting point. A non-isolated point is a limiting-point of each of its neighbourhoods.

Proof. If a is isolated, the neighbourhood $N(a)$ of a contains no point different from a. If a is non-isolated, $N(a) \cap N_1(a)$ is a neighbourhood of a and contains therefore an infinity of points. Hence every neighbourhood $N(a)$ contains an infinity of points of the particular $N_1(a)$. Therefore a is a limiting point of $N_1(a)$.

Definition. The join of A and all its limiting points is the *closure* $[A]$ of A (cf. 2.1). If $A = [A]$, then A is said to be *closed.*

Theorem 6. C is closed if and only if its complement $H-C$ is open.

Proof. If $H-C$ is open, it is the neighbourhood of each of its points a; as this neighbourhood does not contain any point of C, no point a can be a limiting point of C. Hence C is closed. If C is supposed to be closed, every point $a \subset H-C$ has a neighbourhood $N(a) \subset H-C$. The join of all these open sets is open, but this join is equal to $H-C$.

Theorem 7. $[A]$ is closed.

Proof. Let $b \subset H-[A]$, then there exist an $N(b)$ which does not contain a point of A, and contains therefore no limiting point of A, thus it contains no point of $[A]$. $H-[A]$ is the join of these neighbourhoods, and therefore open. Hence $[A]$ is closed.

6.1. *Examples of Hausdorff spaces.* The n-dimensional Euclidean space is a Hausdorff space, as the conditions 1, 2, 3 hold for the sets which are open in the usual sense. Other examples:

6.11. Let H be an arbitrary set and let every subset of H be an open set. Then every point of H is an isolated one, there are no limiting points, and every subset is closed. On the other hand if every point of a space is isolated, every non-empty subset is the join of open sets and is therefore open. In the case of finite sets there is no alternative.

6.12. Let H be an ordered set of points x, and let—apart from the empty sets—those and only those sets be considered to be open which are joins of a finite or of an infinite number of sets of the types

$$x<a, \quad a<x<b, \quad b<x.$$

The definition satisfies the conditions 1, 2, 3. The linear Euclidean space is an instance of a space of this type.

6.2. Starting from given Hausdorff spaces it is possible to construct others by the following methods of construction.

6.21. *Method of direct addition.* Let H_1 and H_2 be Hausdorff spaces, $a_1, \ldots$ be the points of H_1, and $b_1, \ldots$ be the points of H_2. Let H be the set of all pairs $\{a_\mu, b_\nu\}$. The non-empty open sets of H will be defined in the following manner: Let A' be any open subset of H_1, and $a_1', \ldots$ its points; similarly $b_1', \ldots$ be the points of an open subset $B' \subset H_2$; the pairs $\{a_\nu', b_\nu'\}$ form a set $\{A', B'\}$. A non-empty subset of H is considered to be an open set, if and only if it is the join of any finite or infinite number of sets of the type $\{A', B'\}$. The conditions 1, 2, 3 for open point-sets are satisfied. H is said to be the direct sum of H_1 and H_2. Repeating this procedure we get the direct sum of n Hausdorff spaces. The n-dimensional Euclidean space is for instance the direct sum of n linear spaces of the type of the real axis.

Similarly the direct sum of an infinity of Hausdorff spaces may be constructed.

6.22. *Method of meet.* Let H be a Hausdorff space and $H' \subset H$; the meet of every open set of H with H' is considered as an open subset of H'. The necessary conditions for open subsets are obviously satisfied.

6.23. *Method of join.* Let $H_1, H_2, \ldots$ be Hausdorff spaces, no point being common to two of them. The join H of the spaces can be considered as a Hausdorff space, if every join of open sets of $H_1, H_2, \ldots$ is considered as an open set of H. If there are points common to some spaces H_k, it may be possible to apply this method along with the following method.

***6.24.** *Method of representation.* Let H be a Hausdorff space, $a, a_1, a_2, \ldots$ its points. We represent every a by a point b of a particular set H'. The points of H which are represented by the same point b are said to be equivalent and form a class (a). By $N((a))$ we denote a neighbourhood of a which is composed of complete classes. The subsets of H' representing sets $N((a))$ are the open sets of H'. This system of sets satisfies the conditions for Hausdorff spaces if (1) to every pair of non-equivalent points a, b there exist sets $N((a))$ and $N((b))$ without common points, (2) the meet of $N_1((a))$ and $N_2((a))$ is also an $N((a))$. In the case of (1, 1) representation these conditions obviously hold. The method of representation is often used in the form of *identification*. In this case $H' \subset H$; the points of H belonging to H' are represented by themselves, and the points of $H-H'$ become *identified* with some of them by representation. This method is very familiar in Topology and its applications, e.g. the theory of Riemann surfaces. These Hausdorff spaces are built up by *simplexes*, say triangles which—by the method of meet—can be considered as Hausdorff spaces

H_1, H_2, . . . ; these are *joined*; and in the join the boundaries are *identified* in a certain manner.

***6.25.** *Example.* By the above methods the same Hausdorff space can be generated in different ways. By the method of meet any subset of the Euclidean plane—e.g. a circle—can be made into a Hausdorff space. The addition of two circle-spaces generates a ring-space.

The same ring-space can be obtained from the surface of a solid ring in the 3-dimensional space by the method of meet. A third method of getting this space is to start from the Euclidean plane. Two points (x_1, y_1) and (x_2, y_2) are represented by the same point $p \subset H'$ if and only if $x_1 - x_2$ and $y_1 - y_2$ are integral numbers. The space H' is a ring-space. Furthermore, this ring-space may be generated with the help of a square by identifying the four corners and also every pair of corresponding points on opposite sides of the square (i.e. pasting together opposite sides).

6.3. *Main theorems for Hausdorff spaces.* A sequence B_1, B_2, . . . of subsets in a space is said to *converge* to a point d if every $N(d)$ contains nearly all B_j. As now all the terms used in §2 and §3 have been defined for Hausdorff spaces, we can introduce *admissible* sets satisfying the conditions I and II of §3. For systems of admissible sets *the theorems of* §2 *and* §3 *hold without any essential alteration of the proofs for arbitrary Hausdorff spaces.* As it has been stated in §3, it is not necessary to suppose that C is bounded; in arbitrary Hausdorff spaces the term 'bounded' has no sense.

The theorems 1 and 2 of §5 hold for Hausdorff spaces without any essential alteration of the proofs given above.

The same is true for the theorems 1, 2 and 4 of §5.1 if we extend the definitions of *continuous function* and of *arc* given there to Hausdorff spaces without altering the wording of those definitions. The notion of *uniform continuity* has no sense in general Hausdorff spaces.

§7. *Convergence.* The notion of convergence appears in Analysis in different connections; e.g. convergence of a function and convergence of a sequence. The same word is used for different things because there is an obvious similarity in the definitions as well as in some theorems and in their proofs. It will be shown here that there is more than a bare similarity; in fact there exists a common source of every kind of convergence, and different theorems on functions, sequences, double series, series of functions can easily be derived from one very simple theorem on Hausdorff spaces. Consider a few well-known definitions.

1. A real function $f(x)$ converges to the value b as $x \to a$, if and only if to every $\epsilon > 0$ there exists a δ such that

$$| f(x) - b | < \epsilon \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

for

$$| x - a | < \delta \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

According as (2) is replaced by

$$0 < x - a < \delta, \quad \ldots \quad \ldots \quad \ldots \quad (2^r)$$

or by

$$0 < a - x < \delta \quad \ldots \quad \ldots \quad \ldots \quad (2^l)$$

$f(x)$ is said to converge *from the right* or *from the left* to b.

2. A real function $f(x)$ converges to b for $x \longrightarrow +\infty$, if and only if to every $\epsilon > 0$ there exists a value ξ, such that (1) holds if $x > \xi$.

3. A sequence of real numbers $f_1, f_2, \ldots$ converges to b, if and only if to every $\epsilon > 0$ there exists an integer n, such that

$$| f_m - b | < \epsilon \quad \text{if } m > n.$$

If in the first case we put $f(a) = b$, the condition for convergence becomes identical with the condition for continuity at a on the real axis.

Similarly the condition for convergence from the right is identical with the condition for continuity on the Hausdorff space $x \geqq a$ at a, etc. Thus we can replace the definition 1 by the following. Consider the Hausdorff spaces

$$H : x \lesseqgtr a \qquad H_1 : x \leqq a \qquad H_2 \;\; a \leqq x$$
$$H' : x \neq a \qquad H_1' \;\; x < a \qquad H_2' \;\; a < x.$$

If on putting $f(a) = b$ we extend the function $f(x)$ defined in H' (or H_1', or H_2' respectively) to a function defined in H (or H_1, or H_2 respectively), and $f(x)$ becomes continuous at a, then $f(x)$ converges (or converges from the right, or converges from the left respectively) to b at a.

The second case can be treated in a similar manner: By adding the point $+\infty$ we extend the space H to the space $\overline{H}$, and the sets $a < x$ to the neighbourhoods $N_a(+\infty)$ of $+\infty$ in $\overline{H}$. Put $f(+\infty) = b$. 'Convergence' means the continuity of $f(x)$ in $\overline{H}$ at the point $+\infty$.

In the third case we have to consider the space H^* generated from $\overline{H}$ and its subset $1, 2 \ldots +\infty$ by the method of meet. In this space every point except $+\infty$ is an isolated one. Put $f(n) = f_n$, $f(+\infty) = b$. 'Convergence' of the sequence f_n means the continuity of $f(x)$ in H^* at $+\infty$. It may be mentioned that $f(x)$ is continuous at every point n, for these points are isolated and every function is continuous at an isolated point.

In all these cases we have the following phenomenon:

A function f is given in a certain space. This space is extended by adding certain limiting points. If by assigning suitable values for f at these limiting points, f becomes continuous at these points, then f is said to be *convergent* to these values.

It may be objected that by the above explanation convergence has not been uniquely defined, as by adding particular limiting points to a given Hausdorff space we do not get a Hausdorff space until the neighbourhoods of the newly added points are defined in a manner tallying with the axioms for Hausdorff spaces. The definition of the neighbourhoods may be given in different ways; thus in any particular problem there may be convergence or not according to the topology adopted. Of course there are different kinds of convergence occurring in mathematics—e.g. uniform convergence, simply uniform convergence—and it will be shown later that some of these distinctions can be completely explained by the difference in the topologies adopted in the extended space. It may also be asked whether in a particular case there exist different possibilities of choosing the values of f in the limiting points. This question will be answered in the negative by the theorem which follows.

7.1. *General criterion of convergence.* The main criteria of convergence of sequences, functions, double sequences, sequences of continuous functions, etc. can be considered as special cases of the following theorem :

Theorem 1. Let $a \not\subset M$ be a limiting point of M, and let $f(p)$ exist for $p \subset M$, then there exists an extension of $f(p)$ such that $f(p)$ is continuous at a in $M \cup a$ if and only if to every $\epsilon > 0$ there exists a neighbourhood $N_\epsilon(a)$ such that

$$|f(p_1) - f(p_2)| < \epsilon \qquad \cdots \qquad \cdots \quad (1)$$

for p_1 & $p_2 \subset N(a) \cap M$. The extension, if any, is unique.

Proof. The condition is obviously necessary. Let the condition hold, and let A_ϵ be the set of the real values which $f(p)$ takes in $N(a) \cap M$. If $f(p)$ can be extended to the point a, so that it becomes continuous at a, every interval of real numbers containing $f(a)$ must contain A_ϵ for sufficiently small ϵ, and conversely by an extension of this kind, $f(p)$ becomes continuous at a. From §5, theorem 3, it follows that there exists one and only one value $f(a)$ with this property. Hence the theorem.

Theorem 2. Let $f(p)$ be continuous in M, and let the suppositions of theorem 1 hold for every point a of $[M]-M$, then $f(p)$ can be extended in one and only one manner to a function continuous in M.

Proof. If $f(p)$ is continuous in $[M]$ it is also continuous in $M \cup a$; hence there cannot be any other continuous extension of $f(p)$ different from the extension given by theorem 1. Let $f(p)$ be extended in this manner and let a be an arbitrary point of $[M]$. For every point p of $N_\epsilon(a) \cap M$ the inequality $|f(a)-f(p)| < \epsilon$ holds; we prove that the corresponding formula holds for any point b of $N_{\epsilon'}(a) \cap M$, where $0<\epsilon'<\epsilon$. To every $\delta>0$ there exists an $N_\delta(b)$. Let

$$q \subset N_\delta(b) \subset N_\epsilon(a) \cap M,$$

this set not being empty. Hence $|f(a)-f(q)| < \epsilon'$, $|f(b)-f(q)| < \delta$; hence $f(a)-f(b) < \epsilon' + \delta$, and as this formula holds for every positive δ, $|f(a)-f(b)| \leqq \epsilon' < \epsilon$. Hence $f(p)$ is continuous in $[M]$ at a.

7.2. *One-dimensional problems of convergence.* On applying theorem 1 of 7.1 to the spaces denoted by H, $\overline{H}$, H^* at the beginning of §7, we get immediately the famous general criteria of convergence.

1. $f(x)$ converges as $x \to a$ if and only if to every $\epsilon>0$ there corresponds a number δ, such that

$$|f(p_1)-f(p_2)| < \epsilon \text{ if } 0 < |a-p_k| < \delta \text{ for } k = 1, 2.$$

2. $f(x)$ converges as $x \to +\infty$ if and only if to every $\epsilon>0$ there corresponds an integral number N, such that

$$|f(p_1)-f(p_2)| < \epsilon \text{ if } p_k > N \text{ for } k = 1, 2.$$

3. $f_1, f_2, \ldots$ converges, if and only if to every $\epsilon>0$ there exists an integer N, such that

$$|f_m - f_n| < \epsilon \text{ if } m>N,\ n>N.$$

To transform a criterion on sequences to a criterion on series, put $f_1 = a_1$, $f_k - f_{k-1} = a_k$ for $k>2$. Hence

3′. The infinite sum Σa_j converges if and only if to every $\epsilon>0$ there exists an integer N, such that $|a_m + a_{m+1} + \ldots + a_{m+k}| < \epsilon$ if $m>N$.

7.21. *Absolute convergence and functions of bounded variation.* By the last remark the question of convergence of series has been connected with the considerations on continuity of functions in a Hausdorff space, and the question arises whether it is possible to treat the whole theory of series from this point of view. Defining—as above—the function $f(p)$ in the space H^* by $f(n) = f_n$ the total variation of $f(p)$ is expressed by

$$\Sigma\, |f(n)-f(n-1)| = \Sigma\, |a_n|.$$

The notion of *function of bounded variation* which is usually only applied to real functions of a real variable existing in an interval, can easily be extended to real functions which exist in any ordered set.

Definition. Let $f(x)$ be a real function defined on an ordered set S, then $f(x)$ is said to be of *bounded variation* in a subset $S' \subset S$, if there exists an upper limit for the finite sums

$$\sum_{k=1}^{m} | f(x_k) - f(x_{k-1}) | , \qquad \ldots \qquad (1)$$

where $m = 1, 2 \ldots$ and $x_0 < \ldots < x_m$ belongs to S'.

The definition is generally applied to subsets S' of the following types

$$S_a : a \leqq \xi, \quad S_a^b : a \leqq \xi \leqq b, \quad S^b : \xi \leqq b.$$

The upper limit of (1) in S_a^x is said to be the total variation on S_a^x and is a monotonic increasing function $t(x)$. If we restrict the summation in (1) only to positive or to negative values of $f(x_k) - f(x_{k-1})$ we get two other monotonic increasing functions $p(x)$ and $-n(x)$ respectively. If $f(x)$ is of bounded variation in S_a, it is of bounded variation in every S_a^x. Under this supposition the following propositions can be proved in the same way as in the theory of functions of a real variable

1. $f(x)$ converges for $x \to \infty$.
2. $f(x) = f(a) + p(x) - n(x)$, $t(x) = p(x) + n(x)$.

Applied to the space H^* these propositions mean

1'. Absolutely convergent series are convergent.

2'. If b_ν are the positive and $-c_\mu$ the negative elements of an absolutely convergent series Σa_j then

$$\Sigma a_j = \Sigma b_\nu - \Sigma c_\mu, \quad \Sigma \, | \, a_j \, | = \Sigma b_\nu + \Sigma c_\mu.$$

By 2' the problem of permutation in absolutely convergent series has been reduced to the same problem in positive convergent series where the problem is trivial.

***7.22.** *Series and integrals.* The sequence corresponding to a particular series is not invariant for permutations of the elements of the series. Problems concerning permutation of elements cannot therefore be expected to be completely solved by the methods applied here. But these problems may also be treated from a more general point of view by considering series as special cases of integrals. If the value of any series is independent of the order of the elements, that value depends only on the *number* of times a certain value occurs in the series, and is independent of the *positions* in which it occurs. Speaking in terms of integrals, this means: the value of the integral depends only on the *measure* of the point-set in which the function takes a certain value. Thus the theorems mentioned at the end of 7.21 are special cases of the following theorems on Lebesgue-integrals:

$$\int_0^\infty f(x)\, dx = \int_{-\infty}^{+\infty} \mathrm{M}\ (f \geqq y)\ dy$$

if $f(x)$ is measurable and if the integral on the left is absolutely convergent (in particular if $f(x)$ is positive and the integral converges).

7.3. *The two-dimensional case.* In different general problems of Analysis the following situation occurs. M and N are Hausdorff spaces, $f(p) = f(x, y)$ is a real function existing for the points $p = (x, y)$ of the *direct sum* of M and N; furthermore $a \subset [M]-M$ and $b \subset [N]-N$. Under these suppositions questions of the following type are often considered.

Is $f(x, y)$ convergent as $x \to a$ to some function $G(y)$ of $y \subset N$? (I.e. can $f(x, y)$ for every particular y be extended to a function in $M \cup a$, continuous at a?)

Is $f(x, y)$ convergent as $y \to b$ to any function $F(x)$ of $x \subset M$?

Is $F(x)$ convergent as $x \to a$, is $G(y)$ convergent for $y \to b$, and are these limits equal?

Can $f(x, y)$ be extended to a function defined in the direct sum of $M \cup a$ and $N \cup b$ which is continuous in this set at $\{a, b\}$?

How are these problems connected together if special suppositions for M, N and $f(x, y)$ are made?

7.31. *Typical examples of two-dimensional fundamental problems.* (1) Let $f(x, y)$ be a real function of two real variables, let $\operatorname*{Lim}_{x \to a} f(x, y) = G(y)$ exist for $y \neq b$ and also $\operatorname*{Lim}_{y \to b} f(x, y) = F(x)$ for $x \neq a$. To establish the conditions for which $\operatorname*{Lim}_{x \to a} F(x)$ and $\operatorname*{Lim}_{y \to b} G(y)$ exist and are equal.

(2) Let $f_1(x), f_2(x), \ldots$ exist in an interval $[c, d]$ and be continuous at a; let $\operatorname*{Lim}_{n \to \infty} f_n(x) = F(x)$ exist for $c \leqq x \leqq d$. To establish the conditions for $F(x)$ to be continuous at a; to establish furthermore the conditions for $F(x)$ to be continuous in $[c, d]$ if the functions $f_n(x)$ are continuous in the same interval.

(3) Let $\sum_{\nu} a_{m, \nu} = S_m$ for $m = 1, 2, \ldots$, $\Sigma a_{\mu, n} = \sigma_n$ for $n = 1, 2, \ldots$

To establish the conditions for Σs_m and $\Sigma \sigma_n$ to exist and to be equal.

All these problems can be solved by Theorem 1 of 7.1. A function existing in the direct sum of two Hausdorff spaces has to be extended to certain limiting points and we have to arrange the topology of the extended space in such a manner that the function is continuous in those limiting points if and only if the required conditions of convergence hold.

7.32. *Uniform convergence.* If in the first case $f(x, y)$ can be extended to $\{a, b\}$ in such a manner that $f(x, y)$ is a continuous function on the

Euclidean plane (or in a subset of it) at $\{a, b\}$, then $F(x)$ and $G(y)$ converge to the same value $f(a, b)$. Therefore a sufficient condition for

$$\lim_{x=a} F(x) = \lim_{y=b} G(y)$$

is that to every $\epsilon > 0$ there corresponds a positive δ such that $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$ if $|x_1 - a|$, $|x_2 - a|$, $|y_1 - b|$ and $|y_2 - b|$ are less than δ.

A sufficient condition in the second case can easily be established by putting

$$f_n(x) = f\left(x, \frac{1}{n}\right), \quad F(x) = f(x, 0)$$

$F(x)$ is continuous at $x = a$ if $f(x, y)$ is continuous at $\{a, 0\}$ in the subset of the Euclidean space where it has been defined. Hence

$F(x)$ is continuous at $x = a$, if to every $\epsilon > 0$ there exists an integer n and a positive number δ such that

$$|f_p(x_1) - f_q(x_2)| < \epsilon \text{ if } p \;\&\; q > n \text{ and } |x_1 - a| \;\&\; |x_2 - a| < \delta.$$

Let the functions $f_n(x)$ be continuous in the interval $[c, d]$; $F(x)$ is therefore continuous if $f(x, y)$ is continuous throughout the whole subset of the Euclidean space where it has been defined. As this set is closed, $f(x, y)$ is uniformly continuous, and therefore:

Theorem. If the functions $f_n(x)$ are continuous in $[c, d]$ and to every $\epsilon > 0$ there exists an integer n, such that $|f_p(x_1) - f_q(x_2)| < \epsilon$ if $p \;\&\; q > n$, then $F(x)$ is continuous in $[c, d]$. This case is known as *uniform convergence.*

To treat the third problem, the double series should be transformed into a double sequence by

$$f_{mn} = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} a_{\mu,\nu}, \quad a_{m,n} = f_{m,n} + f_{m-1,n-1} - f_{m,n-1} - f_{m-1,n}.$$

Hence $$\lim_{n \to \infty} f_{m,n} = s_1 + \dots + s_m; \quad \lim_{m \to \infty} f_{m,n} = \sigma_1 + \dots + \sigma_n$$

To investigate the conditions for $\lim_{m\to\infty} \lim_{n\to\infty} f_{m,n} = \lim_{n\to\infty} \lim_{m\to\infty} f_{m,n}$ put

$f_{m,n} = f\left(\frac{1}{m}, \frac{1}{n}\right)$, and consider the continuity of $f(x, y)$ at $(x, y) = (0, 0)$.

A sufficient condition for the continuity is that to every $\epsilon > 0$ there exists an integer N such that $f_{m,n} - f_{p,q} < \epsilon$ if the four indices are not less than N. This inequality may also be replaced by $|f_{m,n} - f_{N,N}| < \epsilon$, and therefore by $\left|\sum_{N}^{m} \sum_{N}^{n} a_{\mu,\nu}\right| < \epsilon$.

7.33. *Convergence to continuous functions.* The sufficient conditions established in **7.32** are not necessary as the following example shows:

Let $| x-y | = \lambda$; for $\lambda<1$, $f(x, y) = 1-\lambda$ $\qquad a = b = +\infty$.

for $\lambda \geqq 1$ $f(x, y) = 0$

$F(x)$ and $G(y)$ are equal to 0; thus $F(a) = G(b) = 0$. In every two-dimensional neighbourhood of $(+\infty, +\infty)$ there are values $+1$ and 0 of $f(x, y)$; hence the above condition does not hold.

Putting $f(x, n) = f_n(x)$ we get $\operatorname*{Lim}_{n \to \infty} f_n(x) = 0$, and putting $f(m, n) = f_{m, n}$ (i.e. $a_{11} = 1$, $a_{22} = a_{33} = \ldots = 2$, $a_{n, n-1} = a_{n-1, p} = -1$), we get $\Sigma s_m = \Sigma \sigma_n = 0$.

Also in these cases the conditions of **7.32** do not hold.

To find out the necessary and sufficient conditions as required in **7.31**, we consider the general problems of **7.3**. $f(x, y)$ converges for every particular $y \subset N$, as $x \to a$ if for every particular $y = y_0$, and every $\epsilon>0$

$$| f(x_1, y_0)-f(x_2, y_1) | <\epsilon \qquad \ldots \qquad \ldots (1)$$

x_1 and x_2 being points of a neighbourhood of a in M, i.e.

$$x_1 \;\&\; x_2 \subset \Omega_{\epsilon, 0} = N(a) \cap M.$$

The sets $\Omega_{\epsilon, 0}$ for a particular ϵ and different y_0 may be different and the meet of these subsets may be empty. E.g. if y is a real variable, these sets can be chosen as open intervals, and the lower limit of the length may be equal to 0. Corresponding results hold for the convergence of $f(x, y)$ for $y \to b$.

Let $\operatorname*{Lim}_{x \to a} f(x, y) = G(y)$ for $y \subset N$, and $\operatorname*{Lim}_{y \to b} = F(x)$ for $x \subset M$, then

$$\operatorname*{Lim}_{x \to a} F(x) = \operatorname*{Lim}_{y \to b} G(y) \qquad \ldots \qquad \ldots (2)$$

if and only if to every ϵ there exist neighbourhoods of a in M, and of b in N, say $\Omega'_\epsilon \subset M$ and $\Omega''_\epsilon \subset N$ such that $| f(x_1, y_1)-f(x_2, y_2) | <\epsilon/3$ if

$$\text{either } \begin{matrix} x_k = a, \\ y_k \subset \Omega''_\epsilon - b \end{matrix} \qquad \text{or } \begin{matrix} x_k \subset \Omega_\epsilon' - a \\ y_k = b, \end{matrix} \qquad \text{for } k = 1, 2.$$

This necessary and sufficient condition means that $\{a, b\}$ has a *neighbourhood* containing an x-neighbourhood M_ϵ of points $\{\Omega'_\epsilon, b\}$ and a y-neighbourhood N_ϵ of points $\{a, \Omega''_\epsilon\}$ in which the oscillation of $f(x, y)$ is less than $\frac{\epsilon}{3}$. To each point of M_ϵ there exists a y-neighbourhood [$x \subset \Omega'_\epsilon$ being constant] in which the oscillation is less than $\frac{\epsilon}{3}$ and similarly to each point of N_ϵ, an x-neighbourhood with oscillation $< \frac{\epsilon}{3}$. Joining all these points we get a set Ω_ϵ in which the oscillation is $<\epsilon$. Thus the necessary and sufficient condition that the two limits may be equal, is that $f(x, y)$ can be extended in a continuous manner to $\{a, b\}$, where any neighbourhood of $\{a, b\}$ is the join of $\{a, b\}$ and sets of the type Ω_ϵ. To get a better intuition of the structure of these neighbourhoods, consider the case of two real variables x, y.

The neighbourhoods of points

$x \neq a$, $y = b$ consist of linear x-neighbourhoods.

$x = a$, $y \neq b$ „ „ y-neighbourhoods.

$x = a$, $y = b$ consist of a linear x-neighbourhood and the join of linear y-neighbourhoods of each of its points, and a linear y-neighbourhood and the join of x-neighbourhoods of each of its points.

To these sets arbitrary open subsets of $x \neq a$, $y \neq b$ may be joined. The neighbourhoods of the points $(x \neq a, y \neq b)$ is of no importance for the problem of convergence in $\{a, b\}$; we may e.g. consider all these points as isolated ones. The topology established here tallies obviously with the axioms for open point-sets. It is the topology which corresponds to the problems of double convergence in a single point because it furnishes necessary and sufficient conditions, whereas the simpler topology in **7.2** furnishes only sufficient ones. If the meet of the y-neighbourhoods corresponding to the different points of M_ϵ contains a neighbourhood of $b \subset N$ (or if the corresponding assumptions for x hold), then Ω_ϵ also contains a neighbourhood in the sense of **7.2**. The essential difference between that special case and the general case is that for some ϵ there may exist only such Ω_ϵ for which the meet of those subsets of N contains no neighbourhood of $b \subset N$. In the example given above this holds for every $\epsilon < 1$.

To every point $\{a, y_0\} \subset N_\epsilon$ there corresponds an x-neighbourhood of a for which the points $\{x, y_0\}$ are contained in Ω_ϵ. The meet $N'_{0,\epsilon}(a)$ of that neighbourhood and Ω'_ϵ is a neighbourhood of a and contains points $\neq a$, as a is not an isolated point.

Then for
$$x \subset N'_{0,\epsilon}(a) \qquad \ldots \qquad (3)$$
$$|f(x, y_0) - F(x)| < \epsilon \qquad \ldots \qquad (4)$$
holds. On the other hand this condition is sufficient for (2) as will be shown in the following theorem.

Theorem. The necessary and sufficient condition for (2) is that for every $\epsilon > 0$ in every neighbourhood $N(b)$, there exist points $y_0 \neq b$, and to every y_0 there corresponds an x-neighbourhood $N'_{0,\epsilon}(a)$ of a, such that for x satisfying (3), the formula (4) also holds.

Proof. That the condition is necessary, has been shown above. Let the condition hold, and let x_1 & $x_2 \subset \Omega_{\epsilon,0} \cap N'_{0,\epsilon}(a)$. (This point-set contains an infinity of points, for a is not isolated.)

$$| F(x_1)-F(x_2) | \leqq | f(x_2, y_0)-F(x_2) | + | F(x_1)-f(x_1, y_0) |$$
$$+ | f(x_1, y_0)-f(x_2, y_0) | < 3\epsilon.$$

Hence $F(x)$ is continuous at $x = a$.

7.331. *Simply uniform convergence.* By applying the preceding theorem to a sequence of functions, we get as a corollary the following theorem:

Theorem. Let $\lim\limits_{n\to\infty} f_n(p) = F(p)$ and let every $f_n(p)$ be continuous at $p = a$. The necessary and sufficient condition for $F(p)$ to be continuous at $p = a$ is: To every $\epsilon>0$ there exists an integer $m(\epsilon)$ such that for every $m'>m(\epsilon)$ there should exist $n>m'$, and for every such n there corresponds a neighbourhood $N(a)$, such that

$$| f_n(p)-F(p) | <\epsilon \text{ if } p \subset N(a).$$

In the case where the points p are real numbers, the convergence established here is called *simply uniform convergence.*

7.332. *Convergence by segments.* When the functions $f_n(p)$ are continuous in a closed set C, the condition can be transformed * as follows:

Theorem. Let $\lim\limits_{n\to\infty} f_n(p) = F(p)$, and let every $f_n(p)$ be continuous in a closed set C of a space in which the collective theorem holds. The necessary and sufficient condition that $F(p)$ may be continuous in C is: To every $\epsilon>0$ there exists an integer $m(\epsilon)$, to every $m'>m(\epsilon)$ there exists a finite number of open sets overlapping C, say $\Omega_1 \cup \dots \cup \Omega_t$ and to every Ω_k there corresponds an n_k such that

$$\left| f_{n_k}(p)-F(p) \right| <\epsilon \text{ if } p \subset \Omega_k. \quad .. \qquad .. \quad (1)$$

Proof. If the condition holds, it follows from the preceding theorem that $F(p)$ is continuous at every point $a \subset C$. On the other hand let $F(p)$ be continuous in C, then it follows from the preceding theorem, that every point of C is overlapped by a neighbourhood for which (1) holds. The property of being overlapped by a finite number of open sets Ω_k, is a collective one. Hence the theorem follows from the collective theorem.

In the case where C is a closed interval of real numbers, the open sets Ω_k can be chosen as open linear intervals (segments). Thus this kind of convergence is called *convergence by segments.*

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* Compare e.g. [46].

7.333. *Convergence of double series.* Let $\sum_{\nu} a_{m,\nu} = s_m$ and $\sum_{\mu} a_{\mu,n} = \sigma_n$ for m & $n = 1, 2\ldots$ putting (see 7.32)

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} a_{\mu,\nu} = f_{m,n} = f\left(\frac{1}{m}, \frac{1}{n}\right); \sum_{1}^{m} s_\mu = F\left(\frac{1}{m}\right), \sum_{1}^{n} \sigma_\nu = G\left(\frac{1}{n}\right),$$

we get the following theorem directly from 7.33 :

Theorem. The necessary and sufficient condition that Σs_m and $\Sigma \sigma_n$ may exist and be equal is: To every $\epsilon > 0$ there exists an integer $m(\epsilon)$, to every $m' > m(\epsilon)$ there exist integers $n > m'$ and to every n there corresponds an integer K such that

$$\left| \sum_{\mu=1}^{k} s_\mu - f_{k,n} \right| < \epsilon \quad \text{if } k > K.$$

In terms of $a_{\mu,\nu}$ the left side of this inequality is expressed by

$$\sum_{\mu=1}^{k} s_\mu - f_{k,n} = \sum_{\mu=1}^{k} \sum_{\nu > k} a_{\mu,\nu}.$$

This form of the condition shows a certain connection with Markoff's well-known theorem.

7.34. *Absolute convergence.* Let N be an ordered set, b its upper limit, M an arbitrary Hausdorff space, $x \subset M$, $y \subset N$, and $f(x, y)$ a real function which for every value of x is a *monotonic* function of y. Again $\operatorname*{Lim}_{y=b} f(x, y) = F(x)$.

If $| f(x, y_0) - F(x) | < \epsilon$ for $x \subset N(a)$, then $| f(x, y_1) - f(x, y_2) | < \epsilon$ for y_1 & $y_2 < b$. From **7.2** and **7.3** it follows that in this case there is no difference between the general convergence of **7.3** and the special convergence of **7.2**. The same holds if $f(x, y)$ is for every x a sum of monotonic functions, i.e. a function of *bounded variation.* Let y take the values $\frac{1}{n}$ only, put

$$f\left(x, \frac{1}{0}\right) = 0, \quad f\left(x, \frac{1}{n}\right) - f\left(x, \frac{1}{n-1}\right) = a_n(x);$$

then $f(x, y)$ is of bounded variation as a function of y if and only if $\Sigma a_n(x)$ is absolutely convergent. Thus we get the following theorems :

1. If $f(x, y)$ is a function of bounded variation for $y < b$ and each particular x, and if it converges as $y \to b$ to a function $F(x)$ which is continuous at $x = a$, then to every ϵ there corresponds a $\delta \subset N$ and a neighbourhood $N_\epsilon(a)$ such that $| F(a) - f(x, y) | < \epsilon$ if $x \subset N(a)$ and $b < y + \delta$. In particular let x be a real variable and $F(x)$ be continuous in a closed set C, then $f(x, y)$ converges *uniformly* in C.

2. If $a_1(x)\ a_2(x) \ldots$ are continuous functions in any closed set C of real numbers x, and $\Sigma a_n(x)$ converges absolutely to a continuous function, the sum converges uniformly.

3. If—on applying the notations of **7.333**— $\Sigma s_m = \Sigma \sigma_n = s$, and the series $\Sigma a_{m,\nu}$ converges absolutely, then there corresponds to every ϵ a pair of integers m, n such that $|\ s - a_{\mu,\nu}\ | < \epsilon$ if $\mu < m,\ \nu < n$.

7.341. *Doubly monotonic functions.* Let M and N be ordered sets, and let $f(x, y)$ be a monotonic increasing real function of x for each particular value of y, and a monotonic increasing function of y for each particular value of x. Let the upper limit of $f(x, y)$ be equal to c for $x < a,\ y < b$. Then there exist

$$\lim_{y\to b} f(x, y) = F(x) \text{ and } \lim_{x\to a} f(x, y) = G(y) \text{ and } c = \lim_{x\to a} F(x) = \lim_{y\to b} G(y).$$

On the other hand if $F(x)$ exists and converges to c, then c is equal to the upper limit of $f(x, y)$ and therefore equal to the limit of $G(y)$. From the first theorem of **7.34** it follows that for every $\epsilon > 0$, there exist points ξ and η such that $0 < c - f(x, y) < \epsilon$ if $\xi < x < a,\ \ \eta < y < b$.

For double series it follows that if every $a_{\mu,\nu} \geqq 0$ and $\Sigma s_\mu = s$, the proposition of **7.333**, theorem 3 holds. From this fact it follows in the same way as in the case of simple series, that in a positive double series arbitrary permutations are admissible.

All these considerations hold also for functions which are differences of doubly monotonous functions and similarly for double series which are differences of converging positive series. In such double series every sum $\sum_\nu |\ a_{\mu,\nu}\ | = \lambda_\nu$ and $\sum_\nu \lambda_\nu = a$ must converge.

On the other hand it follows easily from the considerations on absolute convergence of simple series that these conditions are sufficient for the double series to be the difference of two positive or negative elements. The 'General theorem of permutation of double series' is an immediate consequence of this fact.

§8. *Generalization of the method, and limits of its application.* The consideration of repartitive properties when applied to Hausdorff spaces has been shown to be a powerful method of building up the fundamentals of Analysis, and discovering the connections between them. In several cases different fundamental theorems of Analysis have been stated to be particular forms only of one theorem on Hausdorff spaces. We cannot expect this method to be sufficient in such cases where the specific properties of the set of the real numbers (linear continuum) are important. To find the limitation of the method, we have to state an important theorem of Analysis which does not hold in certain Hausdorff spaces in which the repartitive theorem is valid.

The considerations concerning the repartitive theorem started in these lectures (§2) with investigations on bounded sets in n-dimensional Euclidean space. Step by step, the method and its objects have been generalized (§3, §6) as far as it was necessary for the purpose of these lectures. The conditions I and II for the validity of the repartitive theorem are therefore sufficient, but not necessary as it will be shown later. The relative importance of the two conditions is very different. The second condition is satisfied if every subset is considered to be admissible, or even every subset of any particular subset C, or even every subset open (C), etc.; this holds independently of the special properties of the space which has to be considered. This condition means that the system of admissible subsets should not be a too restricted one. The first condition on the contrary restricts the admissible sets. E.g. in the case of a Euclidean space, only bounded sets can satisfy that condition; this restriction is necessary, as the repartitive theorem does not hold for unbounded sets. Furthermore, sets of a power higher than the continuum cannot satisfy condition I (see **8.1**) but this restriction is not necessary (see 8.221) for the validity of the repartitive theorem. Up to now condition I has only been applied to prove the distributive theorem; from this theorem the collective one has been derived and Borel's lemma is a consequence of it. From Borel's lemma and condition II, the repartitive theorem follows. Thus *we may replace condition I by Borel's lemma*; of course Borel's lemma is a necessary condition for the repartitive theorem, as it is a consequence of it.

8.1. *Lindelöf's lemma.* Let a be a point of a set A for which the condition I of §3 holds. Then there exists a sequence of sets

$$A_i,\ A_{i,k},\ \ldots,\ A_{i,k,\ldots l}\ldots \qquad \ldots \qquad \ldots \quad (1)$$

containing a which converges to a point, say b. As every neighbourhood of b contains at least one set of this sequence, it contains a, and from the third axiom on open point-sets (§6) it follows that $a = b$. Hence

Theorem. To every point a of a point-set satisfying the condition I of §3, there corresponds a sequence (1) converging to a.

As the system of indices possible for (1) forms a set of the same power as the continuum, it follows

Corollary. The power of any set for which the condition I of §3 holds, is at most the power of the continuum.

The preceding theorem enables us to prove an important lemma established by Lindelöf for point-sets in a Euclidean space.

Lindelöf's lemma. Let A be a set for which the condition I of §3 holds, $S \subset A$, and let there correspond to every point a of S a particular neighbourhood $N(a)$, then there exists an enumerable set of these neighbourhoods, which overlaps S.

Proof. From the preceding theorem it follows that to every $N(a)$ there correspond sets of the type (1) satisfying $a \subset A_{i, k, \ldots, l} \subset N(a)$. These $A_{i, k, \ldots l}$ overlap S, and their number is at most enumerable. Hence we can select an enumerable set of those neighbourhoods $N(a)$ such that the join of them overlaps S.

Lindelöf's lemma is very important as the theorem of Cantor-Bendixon is a nearly immediately consequence of it. Here the connection between this lemma and Borel's lemma may be discussed.

Let H be a Hausdorff space in which for every subset the proposition of Lindelöf's lemma holds. Let C be a closed and compact set in H. (A set is compact if every infinite subset has a limiting point in the space.) As Lindelöf's lemma is supposed to hold, to every point of C there corresponds a particular neighbourhood, and it is possible to select an enumerable set of these neighbourhoods overlapping C.

$$C \subset N_1 \cup N_2 \cup \ldots \quad .. \quad .. \quad (2)$$

The join on the right hand will not be altered if every N_k, which is completely contained in the join of the preceding ones, is struck out. Hence there is no loss of generality in supposing that in every N_k there exists a point q_k which is not contained in any $N_{j<k}$. If the number of these N_k is infinite, the number of the q_k will also be infinite, and as C is supposed to be compact and closed, a limiting point p must exist in C. As p is overlapped by a neighbourhood (2), say N_t, this N_t must contain an infinity of points $q_{s_1}, q_{s_2}, \ldots$; but as q_{s_j} is supposed not to be contained in any $N_{k<s_j}$, the index t should be higher than every s_j, and that is impossible. Hence C can be overlapped by a finite number of neighbourhoods N_k. Thus we get the result that in a Hausdorff space where Lindelöf's lemma holds for every set, Borel's lemma holds for the sets which are closed and compact. Hence compactness is equivalent to Weierstrass' lemma. Thus in these spaces Borel's lemma is a consequence of Weierstrass' lemma, whereas in general the converse holds.

8.2. *The repartitive theorem in ordered sets.* The notions of *open interval* and *closed interval* should be defined in ordered sets by the same inequalities as in real numbers. Any subset of a closed interval is said to be a *bounded set.* A subset of an ordered set is considered to be open if and only if it is the join of open intervals finite or infinite in number. By these statements an ordered set becomes a Hausdorff space. In the following, two cases will be considered where Borel's lemma holds for every closed and bounded set, but Lindelöf's lemma is not valid.

8.21. *Well-ordered sets. Theorem.* In any well-ordered set, Borel's lemma holds for every closed and bounded subset.

Proof. Any closed and bounded subset C of a well-ordered set has a last element, say a_1. Let $N(a_1)$ be the corresponding neighbourhood; then $C_1 = C-(C \cap N(a_1))$ is closed and bounded. If a_2 is the last element of C_1, let $C_2 = C_1-(C_1 \cap N(a_2))$, etc. As $a_1 > a_2 > \dots$ is a decreasing sequence in a well-ordered set, it is finite; let a_m be its last element, then C_m is empty, and $C \subset N(a_1) \cup \dots \cup N(a_m)$. Hence the theorem.

Thus in a well-ordered set every system of bounded sets is an admissible one if it satisfies the condition II of §3. E.g. the system of all bounded subsets is admissible, whence it follows that the power of the admissible sets may be higher than the power of the continuum. As furthermore a non-enumerable well-ordered set contains a non-enumerable number of isolated points, it is obvious that Lindelöf's lemma does not hold in such sets.

8.22. *D-sets.* An ordered set, which is dense everywhere and in which every bounded subset has an upper and a lower limit, will for abbreviation be called a *D-set.* (The letter *D* should remind one of *Dedekind.*)

Theorem. Borel's lemma holds for every closed and bounded non-empty subset of a D-set.

Proof. Let C be a closed and bounded subset of a D-set D, and let c_1 be the lower, c_2 the upper limit of C. In any D-set every neighbourhood of a point contains a closed interval containing that point. Hence there exist closed intervals $[c_1, b]$ such that $C_b = [c_1, b] \cap C$ is overlapped by a finite number of the selected neighbourhoods. Let d be the upper limit of the join of all these C_b. As C is closed, $d \subset C$, and $N(d)$ contains a point f, such that C_f is overlapped by a finite number of selected neighbourhoods, say $N_1, \dots, N_m$. Then $N_1 \cup \dots \cup N_m \cup N(d)$ overlaps C_d. If $c_2 \neq d$, then $C-C_d$ must have a lower limit $e \geqq d$, such that $N(e)$ contains a point $g \subset C-C_d$. Then C_g would be overlapped by a finite number $m+2$ of selected neighbourhoods, contrary to the supposition. Hence $d = c_2$, and therefore $C = C_d$. Hence the theorem.

8.221. *Example of a D-set in which Lindelöf's lemma does not hold.* Let x take all real values, and let y take the values of $[0, 1]$. The points $P = \{x, y\}$ will be ordered in the following manner:

$P_1 = (x_1, y_1) < P_2 = (x_2, y_2)$ if either $x_1 < x_2$ or $x_1 = x_2$, $y_1 < y_2$.

By this statement the set becomes an ordered set π; it will be proved that π is a D-set. Let B be a bounded subset and $P_j = (x_j, y_j)$ its points. The set of the real numbers x_j is also bounded, and it has therefore an upper limit, say x. If for every j, $x_j < x$, then $\{x, 0\}$ is the upper limit of B. If B contains $\{x, y_k\}$, the numbers y_k have an upper limit $y \leqq 1$, and $\{x, y\}$ is the upper limit of B. Similarly it follows that B has a lower limit. Hence π is a D-set, and Borel's lemma holds for every closed and bounded subset.

Let B be the set of the points $a_z = \{z, \frac{1}{2}\}$, $0 \leqq z \leqq 1$, and $N(a_z)$ be the interval $x = z$, $0 < y < 1$, then B cannot be overlapped by an enumerable set of neighbourhoods $N(a_z)$. Hence Lindelöf's lemma does not hold.

8.3. *Direct sums of ordered sets.* The proof given in **8.22** is nearly identical with the original proof of Borel's lemma for sets of real numbers. In §**2** this proof has been replaced by a simpler proof of a much more general theorem, but the original proof still preserves its interest besides from the historical point of view. This is true also for the original extension of that proof to two and more dimensions; this method will be applied now to prove a theorem which furnishes a large class of spaces in which the repartitive theorem holds for every closed and 'bounded' set, although Lindelöf's lemma may not hold.

Let A and B be ordered sets, and let a topology be introduced in these sets as described in **8.2**. By the method of direct addition, another Hausdorff space $\{A, B\}$ is generated (see **6.21**). The direct sum of open (closed) intervals of A and B is said to be an open (closed) interval of $\{A, B\}$. Every neighbourhood of $\{a, b\}$ contains an open interval $\{N(a), N(b)\}$. Any subset of a closed interval is said to be bounded.

Theorem. Let Borel's lemma hold for every bounded and closed subset of the ordered sets A and B, then it holds also for every bounded and closed subset of $\{A, B\}$.

Proof. Let C be a bounded and closed subset of $\{A, B\}$; hence $C \subset \{I, J\}$ where I and J are closed intervals in A and B respectively. To every point

$$c = \{a_\mu, b_\nu\} \subset C \qquad \ldots \qquad \ldots \quad (1)$$

there corresponds a neighbourhood, and there is no loss of generality in supposing it to be

$$N(c) = \{I_c, J_c\} \qquad .. \qquad .. \quad (2)$$

where I_c and J_c are open intervals in A and B respectively.

Let $a \subset A$ be different from every a_μ of (1). The set $\{a, J\}$ contains no point of C, and as C is closed, to every point $d = \{a, b\}$ of this set, there corresponds an interval

$$\{I'_d(a), J'(b)\} \; .. \qquad .. \qquad .. \quad (3)$$

containing no point of C. By (3) to every $b \subset J$ a neighbourhood $J'(b)$ has been allotted, and as Borel's lemma holds for J, this interval can be overlapped by a finite number of intervals $J'(b)$. Hence $\{a, J\}$ can be overlapped by a finite number of intervals (3), say by

$$\{I'_1(a), J'(b_1)\}; \; \ldots, \; \{I'_m(a) \; J'(b)\}.$$

As the meet of m open intervals is an open interval,

$$I'_1(a) \cap \ldots \cap I'_m(a) = I'(a),$$

and $\{I'(a), J\}$ contains no point of C. Hence the points a different from the points a_μ of (1) form an open set, and therefore the set A' of all points a_μ is closed.

In the same manner it can be shown that the closed set $\{a_\mu \; J\} \cap C$ can be overlapped by a finite number of intervals (2)

$$\{a_\mu, J\} \subset \{I_1(a_\mu), J_1\} \cup \ldots \cup \{I_n \, (a_\mu), J_n\} = \Omega_\mu \qquad .. \quad (4)$$

put $I_1(a_\mu) \cap \ldots \cap I_n(a_\mu) = I(a_\mu)$, then

$$\{a_\mu \; J\} \subset \{(I(a_\mu), J\} \text{ and } \{(I(a_\mu), J\} \cap C \subset \Omega_\mu. \qquad .. \quad (5)$$

Assign the interval $I(a_\mu)$ to every $a_\mu \subset A'$. As $A' \subset I$ is a bounded subset of A and as it has been proved to be closed, Borel's lemma can be applied. Hence

$A' \subset I(a_1) \cup \ldots \cup I(a_t)$, and from (5) it follows that

$C \subset \Omega_1 \cup \ldots \cup \Omega_t$.

From this formula and (4) the theorem follows.

CRITICAL AND HISTORICAL NOTES.

I. *Statements and Properties.*

For abbreviation a *property* is sometimes denoted in the text by the *statement* connected with it. There will hardly be any misinterpretation but a sentence like: '**P** is the property of containing p' is not absolutely correct. An exact wording would be: The statement that a set A contains a point p is connected with a pair of properties $\mathbf{P}$, $\overline{\mathbf{P}}$ such that $\mathbf{P}$ holds if A contains p and $\overline{\mathbf{P}}$ holds if A does not contain p'. In principle, *statements* have to be discriminated from *properties*, but as long as confusion cannot be expected, the shorter and more suggestive kind of wording will be used. A student of mathematics asked the author whether the property: 'the real number a is not less than one' was different from the property: $a \geqq 1$. To answer this question, one should remember that the two sentences are *statements*. A definition of equality of statements has not been given, and is also unnecessary for mathematics. The question is similar to the question whether a statement in English and its translation into Bengalee (or into any other language) are equal statements. In mathematics we have to deal with *properties* only, and these are mathematical entities which will be considered to be equal only if they are *identical*. Thus two properties $\mathbf{P}_1$ and $\mathbf{P}_2$ may be different although they are both connected in the positive sense with the same statement. This notion of equality is not without precedent in other branches of mathematics. E.g. in the theory of groups certain entities of a special kind, called *operators* are considered. These operators are connected with certain automorphisms or endomorphisms of the elements of the groups, but it is not useful to identify the operators completely with the corresponding endomorphisms, thus different operators may be connected with the same endomorphism. This is analogous to the case considered here. There exists also the possibility of proceeding in a different manner, and introducing classes of equivalent properties. The most natural definition would be to consider two properties $\mathbf{P}$ and $\mathbf{Q}$ as equivalent if $\mathbf{P}$ holds in every set in which $\mathbf{Q}$ holds and conversely. This equivalence is necessary to connect these observations with the idea of Boolean Algebra (see 1.2). For the foundation of Analysis the equivalence has no importance; therefore it was not necessary to discuss it in the text, moreover there exist some arguments against that equivalence. Firstly it is often difficult to decide whether two properties are equivalent or not, e.g. properties connected with the following two statements: (1) an integer n is > 2. (2) n is a positive integral number such that no triple of positive integers x, y, z exist for which $x^n+y^n = z^n$ holds. Of course the equivalence of the two properties means that Fermat's last theorem is true. Many mathematical propositions can be enunciated in the form of equivalence of two properties. Another argument is that the equivalence of two properties depends on the definition of *set*. Of course there are different ways of defining sets, and these may not lead to the same result. If we use the notion of set in the larger sense, there may exist a set which has the property **P** but has not property **Q**, whereas for all sets in the restricted sense these two properties might be equivalent. Thus the notion of equivalence of properties breaks down in topics of *metamathematics*, whereas it seems not to be useful for ordinary mathematics. These considerations will be illustrated by an example. Let I, II . . . be a system of axioms on sets formulated in such a manner that at least two of them, say I and II, are statements on sets; let **P** be the property connected with I, and **Q** the property connected with II. Then **P** and **Q** are both true for every set. Hence **P** and **Q** are equivalent.

II. *Intuitionism.*

In classical mathematical logic every statement S which has a meaning, is considered to be *either true or not true.* By the hypothesis that S should have a meaning, we exclude the case when S has the grammatical form of a statement, but is meaningless; symbols in S are allowed to denote entities of a certain class only. E.g. the statement 'a intersects the line at infinity' has a meaning if a is a curve, but it becomes meaningless by putting any number for a. The statement 'vector a is orthogonal to vector b' is neither true nor it is not true if a and b are vectors over fields of a different characteristic; but if there exists a vector-space containing a and b, the statement either holds, or it does not hold.

This kind of reflection, which is very familiar in classical mathematics, has been wholly rejected by the intuitionists. They recognize it to be legitimate only in the case of *solvable* problems, but not every mathematical problem is *a priori* considered to be solvable. E.g. let an infinite sequence of digits be given. We may think that this sequence is either periodic or non-periodic; the sequence a_1, a_2 . . . is considered to be given if each a_n can be determined, however we cannot decide whether the sequence is periodic or not by examining these digits subsequently. If we know the rule by which the sequence is constructed (e.g. decimal or continued fraction) the problem may be solvable in particular cases, but as long as no procedure is known by which the answer can be found out, the third alternative, namely that the problem may be undecidable, has to be considered. The apprehension that every mathematical problem has a determinate solution independently of the possibility of discovering it by the means of human brains, is opposed to the fundamental ideas of intuitionist philosophy.

Intuitionism was inaugurated in 1906 by L. E. J. Brouwer. The public attention given to intuitionist ideas has been enhanced by a pamphlet of H. Weyl (1918) [49] and the opposite theories propounded by D. Hilbert. [25]. For references on intuitionism see Heyting's report [23] and its very large bibliography. In India intuitionism has not been considered at all for a long time, but recently (March 1938) Dr. Vaidyanathaswami lectured on formalism and intuitionism in inaugurating the 10th conference of the Indian Mathematical Society at Lucknow [45^a].

In these lectures the intuitionist point of view has not been adopted. A large portion of classical analysis does not exist in intuitionist mathematics. It may be interesting to confront a few topics of these lectures with intuitionist ideas.

The abstract theory of properties of sets seems to be safe against intuitionistic objections, as it is postulated by definition that a statement connected with a property should be true or not true for every particular set. This definition involves the obligation to prove the decidability in every particular case. From the intuitionist point of view these proofs will be impossible even for very simple properties, e.g. for the property of a point p to be contained in a set A. If on the other hand p is contained in $A = B \cup C$, then it is impossible that neither B nor C contains p, but it is not proved from the intuitionist point of view that at least one of the sets B, C contains p, as the statement may be undecidable for both the sets. We may consider a set A to be *admissible* only if it is decidable whether any particular point is contained in A. For systems of admissible sets of this kind the statement: 'A contains p' is connected with a pair of properties **P**, $\overline{\mathbf{P}}$ where **P** is distributive and $\overline{\mathbf{P}}$ is collective. Thus by the introduction of admissible sets, the difficulty has been removed, but a new difficulty arises, namely to show that particular classes of sets can be considered to be admissible. We cannot expect to get an intuitionist foundation of classical analysis by considerations of this kind as we know that important portions of analysis have no meaning in intuitionist mathematics. However, such investigations may not be useless as they may gradually help to replace the general hypothesis: 'every mathematical problem has a uniquely determined solution' by some hypotheses of a more special character. But they cannot lead to a compromise between the classical and the intuitionist point of view as in intuitionist philosophy there is no place for such an 'opportunist' compromise; a reduction of the general hypothesis to special hypotheses may have a value from the classical point of view only.

III. *Main theorems.*

The distributive theorem has been established by G. Cantor [10]. He used the term of *quality* Y, and proved Weierstrass' lemma in this manner, but he did not lead this method to further results. Peano [38] proved the collective theorem; he used the terms *distributive* and *antidistributive.* Zermelo [51] introduced the term *collective* and showed that Borel's lemma is a corollary of the collective theorem. He also gave interesting applications of this method on the theory of Lebesgue's measure. A portion of his results seems to have been published in a thesis (inaccessible to the author) [4] of one of Zermelo's disciples. The notions of *admissible* sets, *repartitive* properties, and the *repartitive theorem* are due to the author [32].

IV. *Lemmas.*

Whereas a theorem gives some knowledge about mathematical notions and their mutual connection, a *lemma* is only equivalent to a method. We expect a theorem to be *instructive*, but a lemma to be *useful.* Certainly there exists no strict distinction between these two classes of statements, as in mathematics every operation may itself be considered as an object of investigation, and on the other hand every item of knowledge may be utilized for a new method, but it seems to the author that the statement on overlapping intervals named after Heine, Borel and Lebesgue is a standard example of a lemma. Some remarks on the history of this lemma may be useful.

In his lectures on the theory of definite integrals delivered in 1854 G. Lejeune-Dirichlet had used the method of the overlapping intervals to prove the theorem of uniform continuity. Although these lectures have only been published fifty years later (see [31]) there is no doubt of their authenticity. The editor had in his hand detailed notes of the lectures taken down by himself and he was anxious to reproduce the lectures with a kind of religious exactitude using as far as possible Dirichlet's words. Of course we cannot imagine how Dirichlet could have proved the theorem of uniform continuity in a different manner, and this theorem was necessary as a basis for his theory of integration. For about eighty years it has been considered necessary for any exact foundation of the integral calculus; only in 1935 did E. Landau (see [28]) show an alternative way. Dirichlet's lectures had a deep influence on the mathematicians of his time, and they became familiar with the notion of uniform continuity; Weierstrass also used it in his lectures delivered after 1860; thus it is not surprising that this theorem of Dirichlet and its proof can be found at the end of a paper of E. Heine [see [22]) published in 1871. The transformation of the method of overlapping intervals into a lemma was effected by E. Borel (1894); he established in his thesis (see [7]) that if a closed interval is overlapped by an enumerable number of open intervals, it is also overlapped by a finite number of them. This proof is based on the same idea as the proof of the theorem of uniform continuity, and the same proof can be found nowadays in most of the text-books. An alternative proof of E. Borel (see [8]) is very near to the idea of Zermelo's proof (see [50]) which has been given in these lectures; in fact, if we express Borel's 2nd proof in modern terms, it means that the property of an interval 'not to be overlapped by a finite number of selected intervals' is a distributive one. H. Lebesgue (see [29]) has shown that the set of overlapping intervals needs not be enumerable. For this reason French writers like to distinguish between Borel's theorem (the overlapping intervals being supposed to be enumerable) and Borel-Lebesgue's theorem (without that restriction). Schoenflies (see [41]) showed that the theorem holds for every n-dimensional closed and limited set; he also realized its close connection with the method applied in the proof of the theorem of uniform continuity; as Dirichlet's lectures were not yet published at that time, he gave the lemma the name of Heine-Borel's theorem, and from his report the name has been spread all over the world. Especially in English textbooks this term is common, whereas German writers mostly use the name 'Borel'scher Überdeckungssatz' (see [11], [50], [52]).

Borel's lemma later was gradually extended to metric and topological spaces (for references see Nr. 14 in [43], and Nr. 26 in [52]). The most general theorem was proved by Alexandroff and Urysohn [3].

It may finally be mentioned that the lemma holds also in intuitionist Mathematics if the 'space' as well as the 'closed point set' are *catalogued compact species* (see [9]). The significance of that term has been explained in another paper of Brouwer published in the same volume as [9] a few pages earlier.

Lindelöf's enunciation (see [33]) differs from Borel's lemma by a slight (but important) variation of the hypothesis and the proposition, and may therefore also be considered as a lemma.

The collective theorem is correlated to Borel's lemma in much the same way as the distributive theorem is related to the statement, named after *Bolzano* and *Weierstrass.* (Bolzano-Weierstrass' theorem, Weierstrass' theorem on limiting points). As this enunciation is the counterpart of Borel's lemma, it seems appropriate to call it also a lemma although it is formulated as a theorem of existence, and is both instructive and useful. The correspondence between Borel's and Weierstrass' lemmas becomes complete if we replace the notion of limiting point by *complete* limiting point; this fact has been established by Alexandroff and Urysohn (see [3]).

Dedekind's lemma was established by Dedekind [12] on November 24th 1858, not as a theorem but as a *definition* of real numbers. For the connection of this definition and Euclid's theory of proportions see the preface of [13]. Dedekind's definition was the starting point of the intuitionist criticisms of the foundation of Analysis, but doubts as to the admissibility of this kind of definition had also been expressed in earlier times (see [26]). In these lectures no particular definition of real numbers has been given; the basic properties of real numbers are supposed to be known. Supposing that the real numbers form an ordered set, and that the intervals satisfy the conditions I and II of §3, we get Dedekind's enunciation as corollary of the distributive theorem (see §5).

V. *Sets.*

The notion of set has been conceived in three different ways, and of course, there are three different theories of sets: the naive theory, the axiomatic theory, and the intuitionist theory.

In the naive theory a set is considered to be the result of a collection. By collecting certain entities,—e.g. those entities which satisfy certain conditions—a new entity is created which is called a set. The theory of sets, as it has been developed at the end of the 19th and in the early 20th century, is completely based on this 'principle of comprehension' which seems to be very intuitive, but its unrestricted application led to the famous antinomies (Russell, Richard). It would be very desirable to find out to what extent the creation of sets by comprehension is admissible; this task meets with enormous difficulties, and it has practically been given up. Many mathematicians are convinced that the actual application of that principle in mathematics remains inside the limits of admissibility. Of course, the application is actually restricted in such a manner that for the purpose of mathematical investigations the principle of comprehension can be replaced by a system of axioms on sets (see [51], [37], [23], [14], [34]).

From this point of view, a set is not a collection, nor is there any definition of 'set', but a set is a mathematical entity of a special kind; the only essential thing is that the sets satisfy certain conditions as announced in the axioms. When Zermelo [51] started these investigations and gave a complete system of axioms, he considered it only as an expedient which should be replaced, as soon as possible, by a satisfactory *definition* of the notion of set. He also expected a proof, that his axioms could not lead to any contradiction, to be possible. The development of this branch of mathematics in the last few decades, has shown that the historical merit of Zermelo's paper goes far beyond his own expectation; the classical theory of sets is now based on axioms. In consequence of the axioms, there exists a (1, 1)—correspondence between any particular set A and the collection of those points (or elements) which are said to be contained in A. The statement: A contains p, describes a purely formal relation between A and p, but the (1, 1)—correspondence between A and the collection of the points p satisfying that condition, enables us to use the familiar notations of the naive theory of sets.

The axiomatic conception of set expressed in terms of the naive theory also underlies to these lectures, but it seems to the author that the theory of sets is not necessary for every branch of mathematics in its most extended form. It is often better, to restrict consideration to special classes of sets, say *admissible* sets, as has e.g. been done in these lectures.

The intuitionist conception of set is very different from the axiomatic one. For reference, see [23] especially §5, No. 3 and 5.

VI. *General spaces.*

Gradually and only reluctantly mathematicians have generalized the notion of space. From the modern point of view it is very difficult to understand why Lagrange's dynamics did not lead to a development of n-dimensional geometry, but the spirit of that time was very far from ideas of this kind. For a very long period Non-Euclidean Geometry, n-dimensional Geometry, etc. have been considered to be geometrical *facons de parler* only of algebraic and analytic investigations. It is well known that *Cayley's* papers contain the essence of non-Euclidean Geometry without mentioning it. That the idea of different, non-equivalent geometries, enjoying the same right in mathematics has spread among mathematicians, is to a great extent due to Felix Klein [27], but geometers of the older type deeply disliked this view of Geometry. The author, who had the privilege of being a student of one of the most distinguished geometers of the classical type, can only endorse the very suggestive description of that kind of thought given in an obituary of this scholar (see [44] especially pp. 192-193). Historians of later times may think that in the first years of the 20th century the creation of a theory of general spaces was overdue. Of course there existed at that time—apart from n-dimensional Euclidean Geometry, Projective Geometry and Non-Euclidean Geometry as mentioned above—a very extended differential Geometry of intrinsic properties. Furthermore the notion of Riemann surface was generally used in the theory of functions, although this notion becomes clear and simple only by considering a Riemann surface as a topological space. At that time there existed a general desire to formulate suitable definitions and to investigate of the characteristic properties of the rather vague notions of curve, surface, dimension, etc. This desire met with the important results which H. Poincaré obtained since 1899 in combinatorial topology and led to a fresh interest in this science which before Poincaré existed only as a programme and as a collection of curiosities, and which is now-a-days a very important branch of Mathematics (for ref., see [1], [30]). In higher Analysis the importance of the use of an infinite number of variables became obvious from the theory of integral equations and this development found its continuation in the theory of functionals (for ref. see [47] and the very large bibliography given there). The common need of all these* branches of mathematics was a more general notion of space! Perhaps the 'mock' spaces used in the axiomatic investigations inaugurated by Hilbert's [24] book helped to pave the way, but the psychological resistance was still very strong and was only gradually overcome by the systematic investigations on general spaces which started from Frechet's thesis [15] in 1906. It seems characteristic that Frechet avoided the use of geometrical terms and that he called the generalized spaces simply *classes*; this term is still used by French mathematicians. Frechet's theory starts from sets in which a binary positive function is defined (metrical spaces) but in his later papers topological spaces have also been considered. This generalization is due to R. E. Root [39], [40] and to Hausdorff [21]. For the further development

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* This list does not claim to be complete; e.g. Sophus Lie's theory could also be considered from this point of view.

of the theory the papers of Alexandroff and Urysohn [2], [3] and of Menger [34] are of outstanding importance.

In a standard work on the theory of real functions [11] published twenty years ago general spaces were not mentioned, but nowadays the fundamental importance of this theory has been generally recognized. For reference see the articles in the *Encyklopädie der Mathematik* [43] and [52] Nr. 26. A sketch of the results up to 1922 in [16]. Fuller reports [21], [30], [35], [19].

In these lectures a very small portion only of the theory has been given. Only Hausdorff spaces (this term is usual in English literature) have been considered, but the axioms have not been given in the original form. In accordance with the special task of these lectures an idea of Tietze [42] has been utilized, which has also been successfully adopted in Hahn's book [19].

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* Starred sections may be omitted at a first reading.