

UNIVERSAL
LIBRARY

OU_156512

UNIVERSAL
LIBRARY

OSMANIA UNIVERSITY LIBRARY

Call No. 5153/P31P Accession No. 26132

Author Patterson B.C

Title Projective Geometry

This book should be returned on or before the date last marked below.

PROJECTIVE GEOMETRY

PROJECTIVE GEOMETRY

BY

BOYD CRUMRINE PATTERSON, PH. D.

Professor of Mathematics, Hamilton College

NEW YORK

JOHN WILEY & SONS, INC. -

LONDON: CHAPMAN & HALL, LIMITED

COPYRIGHT, 1937
BY BOYD C. PATTERSON

All Rights Reserved

*This book or any part thereof may not
be reproduced in any form without
written permission from the publishers.*

SECOND PRINTING, DECEMBER, 1946

Printed in U. S. A.

PREFACE

This book is the outgrowth of a set of notes used for the past five or six years in a semester course in projective geometry offered to juniors and seniors at Hamilton College. A primary motivation in writing the book was my feeling that, for a first course in the subject, it would be desirable to have a more complete discussion than is given in existing elementary textbooks. For then the need of supplementing the text would not arise so frequently; and the instructor would have at his disposal in the classroom more time for the analysis, solution, and discussion of problems, and for actual drawing board constructions. Thus, the underlying principle has been to present as complete a discussion of the subject matter as seemed consistent with an introductory course.

The method adopted is the method of synthetic geometry. The development of this method is intuitional and is based on the student's earlier acquaintance with the elementary geometry of the schools. It seems, indeed, to be a matter of general agreement that a postulational approach to the first study of projective geometry is not desirable. But, with due regard to rigor in its proper setting, an attempt has been made to be as rigorous as the circumstances would permit and, at all times, to be *reasonable*, for reasonableness appeals to the beginner when rigor does not.

The mathematical prerequisites to a study of this book are few. Elementary geometry is necessary; trigonometry and analytic geometry are helpful in that they broaden the student's background, but, in fact, only occasional reference is made to them. It is very probable that a general maturity in the student's mental processes is more important than a large mathematical technique.

Unquestionably, the content of projective geometry is valuable to the student who intends to continue his mathematical study in graduate work; and to the student who plans to teach mathematics in the schools, it offers an opportunity to broaden and to strengthen his mathematical foundation. But the value of the subject is not confined to the future specialist or teacher only; its cultural and practical aspects for the student of the humanities or the en-

gineering sciences or the fine arts are not to be minimized. To these latter it offers, without requiring too much previous mathematical training, an insight into one of the great developments of the human mind and an appreciation of the elegance of geometric reasoning. Indeed, as a part of the undergraduate mathematics curriculum the so-called synthetic projective geometry occupies a rather unique position.

Among the points of departure in this text from the usual elementary treatment, the following deserve explicit mention:

(1) A careful distinction is made between projective and non-projective ideas.

(2) The principle of duality is developed in some detail and is emphasized throughout the book.

(3) The proofs of some of the earlier theorems are separated into steps, a procedure familiar to the student from his study of elementary geometry.

(4) The proof of the fundamental theorem is built on the concept of a net of rationality on a line.

(5) The theory of the imaginary elements is introduced in some detail and is applied to various construction problems.

(6) A "focus" is introduced by the Plücker definition, i.e., as an ordinary point of intersection of tangents from the circular points.

(7) In the final chapter there is a brief discussion of the general projective group of transformations on a plane and of some of its subgroups, of which the most notable is, of course, the metric group. Thus we are able to exhibit the familiar geometry of Euclid as a special case of projective geometry.

Inasmuch as most instructors will have their own preferences as to conducting the course, the text has been arranged to permit a considerable degree of flexibility in its use. This degree of flexibility increases as the theory progresses. Chapters I to IV and Chapter VI constitute a basis upon which the rest of the theory stands; but even in these chapters certain items may be either touched upon briefly or omitted entirely. For example, in Art. 3.2, the ten examples of perspective primitive forms are not all important for the beginner. But, whether he studies them all in detail or not, to see them written out, and to read them, and to think about them, enlarges his grasp of the concept of perspectivity and,

also, of duality in a way which makes their inclusion very much worth while.

In later chapters this flexibility is even more in evidence. The involution, for example, is mentioned in Arts. 9.5, 10.2, and 12.11; but the theory of the involution is deferred until Chapter XIII, and in that chapter considerable detail may be omitted. Finally, the metric specializations of projective concepts are placed in separate chapters and articles, which will be easily identified by their titles. These metric specializations are in no way necessary to the development of the projective theory.

How much of such material the teacher may wish to consider, and the emphasis which he may wish to place upon that which he does consider, remain, for the most part, at his pleasure. In particular, he will find, after taking up Desargues' Theorem (Art. 3.3), that the portions of the text concerned with the geometry in the plane constitute an uninterrupted sequence, and may be separated from the remainder without difficulty.

In my own use of the text I have covered each semester (45 lessons) Chapters I, II, III, IV, VI, VII, and VIII rather completely; selected portions of Chapters V, IX, and X, varying the material chosen from year to year; and the general theory of Chapters XII, XIII, and XIV. It will be found, of course, that, to attain this end, quite liberal omissions in the last-mentioned chapters will be necessary. However, the interest which the student displays in the theory of the imaginary elements (Chapter XIV) amply repays the instructor for what he might feel has been lost by omission.

As a natural sequel to the chapter on the imaginary elements I have added a chapter on the focal properties of conics, a subject which is of interest to the undergraduate who has studied analytic geometry because it presents old, familiar results in a new light.

The justification for Chapter XVI; which is of necessity rather brief, lies in the fact that it affords an opportunity to introduce some fundamental ideas and concepts of higher geometry. It also leads naturally to the introduction of analytic methods in projective geometry.

The exercises have been grouped together at the ends of the chapters. It will be seen that the number of exercises is much larger in those chapters where omissions of the text are likely to be made. This will provide the instructor with sufficient drill material for whatever articles he may decide to use.

The book contains enough material for a full year's course in the subject. In case a full year is allotted to the course, an alternative plan would be to use the text for a semester and a half, and then, on the basis of the material in Chapter XVI, to introduce and develop analytic methods for the remainder of the second semester. It was only after much consideration that I finally decided not to include a chapter on analytic methods. For, because of limited space, it could have been an introduction only; and it seemed inadvisable to have as a closing note an introduction to a subject different in both method and aim from that of the whole.

Another question which proved quite bothersome was the question of how much, or how little, reference should be made in a text of this character to the history of the subject. I resolved the difficulty by deciding not to include any historical material, notwithstanding my own predilection for things historical. It is very probable that no field of intellectual endeavor has a history more venerable than that of projective geometry. Better, however, than any series of disconnected historical footnotes, which might have been scattered through the text, are the articles on geometry in the *Encyklopädie der mathematischen Wissenschaften*, to which I refer the instructor who may wish to follow such inquiry. He will find, also, in the introduction to Cremona's *Elements of Projective Geometry* (Oxford University Press, 3rd edition, 1913) and in Smith's *Source Book in Mathematics* (McGraw-Hill Book Co., 1929) some very useful historical information.

The few references to Euclid in the text are to the recent edition of Todhunter's *Elements of Euclid*, with an introduction by Sir Thomas L. Heath, published as No. 891 of Everyman's Library (E. P. Dutton & Co.). To the student who has only a recollection, perhaps rather faint, of his school geometry text, this little volume will be of more than passing interest, historically and mathematically, for he will find in it much that has new meaning in the light of his study of projective geometry.

Our debt to a long list of predecessors who have paved the way of geometry is without measure. We have but to consider the contributions of Euclid, Pappus, Desargues, Poncelet, Steiner, von Staudt, Chasles, Reye, and Cremona, to mention only a few from ancient to more recent times, to realize that what we study today has deep roots and enduring qualities. To these and to some others of the present century, notably Mathews, Veblen, Young,

and Baker, I would acknowledge a lasting obligation. I have, also, a particular sense of gratitude to Professor Frank Morley, who initiated, some time back, that which developed into a sustaining and ever-increasing interest in geometry.

In preparing this book, I have the privilege of making due acknowledgments to my colleague, Professor W. M. Carruth, and to my neighbor, Professor A. D. Campbell of Syracuse University, with both of whom I discussed the text while in manuscript form. Professors C. S. Atchison and H. L. Dorwart, of Washington and Jefferson College, were extremely diligent and helpful in reading and criticizing the entire text in proof; I wish to express my thanks to these gentlemen and to add, also, that much more is due them.

The figures illustrating a text on projective geometry are, to my mind, of great importance. For preparing the illustrations for the engraver, I wish to thank Mrs. S. W. Nile and Mr. H. R. Jones, who carried out my wishes in that respect in such a satisfactory way that I consider myself fortunate to have had their services.

B. C. P.

HAMILTON COLLEGE
CLINTON, NEW YORK
March, 1937

CONTENTS

CHAPTER I

INTRODUCTORY CONCEPTS

ARTICLE	PAGE
1.1 Introduction	1
1.2 Elements	1
1.3 Ideal elements	2
1.4 Axioms	5
1.5 Primitive forms	7
1.6 Central projection	7
1.7 Dimensions of the primitive forms	8
1.8 Separated pairs of elements	10
Exercises	11

CHAPTER II

THE PRINCIPLE OF DUALITY — SIMPLE AND COMPLETE FIGURES

2.1 A reciprocal relation existing among the axioms	13
2.2 Duality in space	13
2.3 Duality in the plane	14
2.4 Duality in the bundle	15
2.5 Summary	15
2.6 Simple figures in the plane	16
2.7 Complete figures in the plane	17
2.8 Simple and complete figures in space and in the bundle	18
2.9 The Desargues configuration	20
Exercises	21

CHAPTER III

PERSPECTIVITY — DESARGUES' THEOREM

3.1 One-to-one correspondence	23
3.2 Perspectivity	24
3.3 Desargues' Theorem	28
3.4 Converse of Desargues' Theorem	30
3.5 Perspective quadrangles	31
Exercises	32

CHAPTER IV

HARMONIC SETS

ARTICLE		PAGE
4.1	Definitions	34
4.2	Construction of the fourth harmonic	35
4.3	Conjugate pairs	36
4.4	Projection and section of an harmonic set	38
4.5	The order of elements in an harmonic set	39
4.6	Harmonically separated unlike pairs	40
4.7	Problem	40
	Exercises	41

CHAPTER V

METRIC PROPERTIES — THE DOUBLE RATIO

5.1	Projective and metric properties	42
5.2	Directed segments	43
5.3	The double ratio of four points	44
5.4	Projective character of the double ratio	45
5.5	The double ratio of four lines	48
5.6	Applications	50
5.7	The six double ratios	54
5.8	Coincident points	56
5.9	Metric properties of an harmonic range	56
5.10	The pair of points harmonically separating two given pairs	60
	Exercises	61

CHAPTER VI

PROJECTIVELY RELATED PRIMITIVE FORMS

6.1	Chain of perspectivities	64
6.2	Projectivity	65
6.3	Projective triads	68
6.4	Projective tetrads	69
6.5	Net of rationality	69
6.6	A theorem concerning the net of rationality	70
6.7	A fundamental assumption concerning a projectivity	71
6.8	The fundamental theorem of projectivity geometry	71
6.9	Projective forms which are perspective	72
6.10	Axis and center of homology	73
6.11	Double elements	77
	Exercises	79

CHAPTER VII

CONICS AND CONES

7.1	Definitions	82
7.2	Fundamental properties of conics	84
7.3	The construction of conics	87

CONTENTS

xi

ARTICLE	PAGE
7.4 Relations between conics and cones	89
7.5 Degenerate conics and cones	90
7.6 A metric classification of conics	91
Exercises	92

CHAPTER VIII

PASCAL'S THEOREM AND BRIANCHON'S THEOREM

8.1 The theorems	95
8.2 The converse theorems	97
8.3 Applications	98
8.4 Degenerate cases of the theorems	100
8.5 Alternative proof of the complete quadrangle theorem	104
8.6 Construction problems	105
8.7 Metric specializations	107
8.8 A relation between point and line conics	108
8.9 Chasles' Theorem	109
Exercises	110

CHAPTER IX

THE THEORY OF POLE AND POLAR

9.1 Definitions	111
9.2 Properties of pole and polar	112
9.3 The construction of pole and polar	114
9.4 Conjugate points and lines	115
9.5 Two theorems	116
9.6 Polar reciprocation	119
9.7 Common self-polar triangle of two conics	121
9.8 Two quadrangles or two quadrilaterals with a common diagonal triangle	123
9.9 Quadratic transformations	124
Exercises	128

CHAPTER X

METRIC PROPERTIES OF CONICS

10.1 The circle	130
10.2 The orthogonal involution	131
10.3 Bisectors of angles	133
10.4 Diameters and center of a conic	134
10.5 Conjugate diameters of central conics	135
10.6 The axes of a conic	136
10.7 Construction of the axes	138
10.8 Similarly projective point-rows	139
10.9 A construction of the hyperbola	140
10.10 Rectangular Cartesian equations of the conics	141
Exercises	146

CHAPTER XI

RULED SURFACES

ARTICLE	PAGE
11.1 The theory of cones	148
11.2 Reguli	150
11.3 Ruled quadrics	151
11.4 Tangent planes and tangent lines	153
11.5 Tangent cones and contact conics	154
11.6 Order and class	155
11.7 Theory of pole and polar	156
11.8 Metric considerations	159
Exercises	161

CHAPTER XII

EXTENDED THEORY OF PROJECTIVITY

12.1 The one-dimensional forms	163
12.2 Perspective one-dimensional forms	164
12.3 Harmonic sets of elements	166
12.4 Extended definition of projectivity	168
12.5 Construction of a projectivity between two forms	169
12.6 Third degree forms	172
12.7 Projectivity on a conic	174
12.8 Axis and center of projectivity	175
12.9 Projectivity on a one-dimensional form	177
12.10 Problems	178
12.11 Cyclic projectivities	180
Exercises	184

CHAPTER XIII

THE THEORY OF INVOLUTION — METRIC PROPERTIES

13.1 Definition and theorems	186
13.2 Special properties of involutions	187
13.3 Involutions on a primitive form	188
13.4 The construction of homologous points of an involution	190
13.5 Desargues' Theorem on conics	191
13.6 Involutions determined by a conic	193
13.7 Metric definition of an involution on a line	195
13.8 Metric construction of homologous pairs	197
Exercises	199

CHAPTER XIV

IMAGINARY ELEMENTS

14.1 The elliptic involution	203
14.2 Imaginary elements	204
14.3 The incidence relations	205

ARTICLE	PAGE
14.4 Imaginary lines of the second kind	209
14.5 Desargues' Theorem for complex triangles	210
14.6 Properties of imaginary elements	211
14.7 Common elements of a conic and a complex line or point	213
14.8 The construction of conics	217
14.9 A metric specialization	220
Exercises	222

CHAPTER XV

FOCI AND FOCAL PROPERTIES OF CONICS

15.1 Foci	224
15.2 Determination of the foci	225
15.3 Construction of the foci	227
15.4 Directrix, focal radius, and latus rectum	230
15.5 Focal properties	231
15.6 Metric properties of a conic	234
15.7 Focal properties, continued	235
15.8 A triangle theorem	238
15.9 Confocal conics	240
15.10 The construction of conics	241
Exercises	243

CHAPTER XVI

PLANAR COLLINEATIONS

16.1 Introduction	245
16.2 Perspective forms	245
16.3 The planar perspectivity	246
16.4 Projective forms	249
16.5 Groups of transformations	250
16.6 The fundamental theorem	253
16.7 Construction of a projectivity	253
16.8 The affine group	254
16.9 The homothetic group	256
16.10 The similarity group	257
16.11 The group of rigid motions	259
Exercises	261
SUPPLEMENTARY READING	263
APPENDIX A	265
APPENDIX B	267
INDEX	271

PROJECTIVE GEOMETRY

CHAPTER I

INTRODUCTORY CONCEPTS

1.1 Introduction. A logical structure reared on a foundation of certain undefined terms and undemonstrated theorems or assumptions, which we call axioms, describes what is ordinarily called a geometry. It must be kept in mind that the nature of a geometry depends entirely upon the undefined terms and axioms; and that the axioms are assumptions and are not demonstrable. Hence there must be a geometry for every set of axioms obeying certain logical requirements as to consistency, independence, etc. One example of a geometry is already familiar to us, viz., the so-called Euclidean geometry. Our purpose in the following pages is to become familiar with a much more general geometry than that of Euclid. We say that it is a more general geometry because it is based on concepts which are less restrictive. For example, this geometry may be developed without the use of the concept of "measure"—a concept which is fundamental in Euclidean geometry. As a matter of fact, we shall see, as we go into our subject more deeply, that this geometry includes Euclidean geometry as a special case.

The early writers on the subject called it by various names—Geometry of Position, Higher Geometry, Modern Geometry, etc. But, for various reasons, these names are no longer satisfactory, so today we give it the name Projective Geometry and thus indicate that we are interested primarily in those properties of figures which are preserved under all projections (a term which will be defined later more explicitly).

1.2 Elements. The undefined elements of projective geometry are three in number, namely, the plane, the line, and the point. We may, for the time being, think of them in much the same way as we did in our study of elementary Euclidean geometry. But

this, let us remind the reader, is only as a temporary convenience, for strictly speaking the elements are undefined and as such have no properties as yet. In Art. 1.4, however, we shall make some assumptions about these elements, and it will be seen then that in assuming certain things concerning the elements we are, to all intents and purposes, endowing them with certain properties.

We consider each kind of element as existing independently of the others; and the plane and line we consider to be unlimited, i.e., of indefinite extent. For convenience and uniformity we shall adopt the customary notation in regard to the elements. Points will be indicated by capital letters A, B, C, \dots ; lines by small letters a, b, c, \dots ; and planes by small Greek letters $\alpha, \beta, \gamma, \dots$.

1.3 Ideal Elements. In Euclidean geometry the following propositions are familiar:

(1) Two lines in a plane intersect in one point or they are parallel.

(2) Three lines in a plane ~~(a)~~ intersect in one point, or (b) are all parallel to one another, or (c) intersect in three points, or (d) are such that two of the three lines are parallel and each is cut by the third.

(3) Two planes intersect in one line or they are parallel.

Compare the first of these propositions with the Euclidean proposition that two points determine one line. From considerations of symmetry alone it would be convenient to be able to say, also, that two lines in a plane determine one point, their point of intersection. But parallel lines have no point of intersection; hence we must take account of this exception and state the proposition in the form of (1) above.

Exceptional cases of the kind just noted abound in the statements of Euclidean geometry, and the reader will remember many of them. They are, indeed, characteristic of Euclidean geometry and are responsible for a great lack of symmetry in the subject.

Projective geometry, on the other hand, possesses a remarkable symmetry of expression; and this is obtained merely by the introduction of a device by which the exceptional cases of the type mentioned above are changed into special cases. What is this simple device? It is the invention of a new kind of element which

we adjoin to the elements of Euclidean geometry. The elements of Euclidean geometry, which we shall call **ordinary**, together with these new elements, which we are about to introduce, are the elements of projective geometry.

Consider (Fig. 1) a line l and a point S not on l , and the plane π determined by the line l and the point S . Of the lines which pass through S and lie in the plane π each intersects the line l in one point with the exception of that one line through S which is parallel to l . In projective geometry we *assume* that the line through S and parallel to l intersects l in one point. This point of intersection is not a point according to the Euclidean meaning of the term; it is a point whose existence we postulate for the purpose of attaining a greater symmetry of expression in the statements of

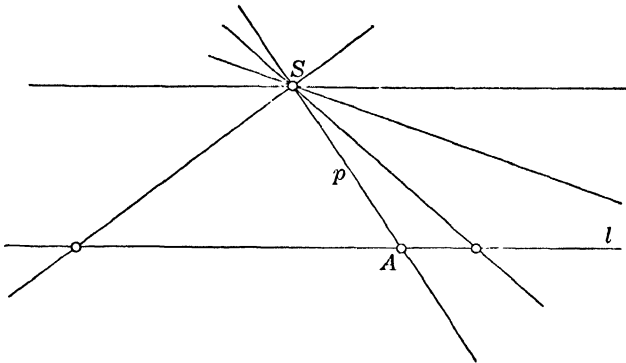


FIG. 1

projective geometry. To distinguish such a point from the ordinary points of a line we call it the **ideal point** of the line. We state this fundamental assumption of projective geometry more carefully:

On each line we assume the existence of one and only one ideal point, which is the point of intersection of that line with all lines parallel to it.

The term "line" now has a new significance. It is no longer the ordinary line of Euclidean geometry, for it contains one additional point. We shall call it an **augmented Euclidean line** when it is necessary to make a distinction; otherwise, we shall continue to call it an **ordinary line**. It is important to note, also, that we are assuming only one ideal point on each line. For if we should assume two ideal points on each line, we would then be confronted

with a situation in which two parallel lines intersect in two points, whereas two nonparallel lines intersect in only one point. This would be inconvenient.

If we consider parallelism in terms of direction, it is evident that two augmented lines which have the same direction have in common one ideal point; and two lines with different directions have in common one ordinary point, and their ideal points are distinct. Thus, we may continue to represent the ordinary (Euclidean) points on paper by dots; the newly invented ideal points we may represent by directions. It is important, however, to observe that either of two opposite directions represents the same ideal point.

We have now reached the stage where we can say that an augmented line is determined by two ordinary points or by one ordinary point and one ideal point (i.e., one direction). Can we say also that such a line is determined by two ideal points (i.e., two directions)? At the present time we cannot, obviously. For the assumption is that there is one and only one ideal point on each augmented Euclidean line

Let us examine now the set of all ideal points in a plane. That set of points constitutes the ideal region of the plane. The choice of what we shall say about that region is ours to make. Our decision, however, should be consistent with all that has been agreed upon in the earlier paragraphs of this article. For example, an augmented line, we have assumed, has in common with that region one and only one point, the ideal point of the line. When this is taken into account the ideal region cannot be a curve, for then each line would intersect it in more than one ideal point. Such considerations lead us to assume the existence, in each plane, of one and only one **ideal line**. It is the locus of all the ideal points of the plane, and it is also the line of intersection of the plane with all parallel planes. This new kind of line is now to be adjoined to the augmented Euclidean lines of the plane. The plane then becomes an **augmented Euclidean plane**, and we have an answer to the question of the preceding paragraph: Two given ideal points determine one line, an ideal line, which is the locus of all the ideal points of each plane containing the two given ideal points. All planes having a common ideal line are parallel to one another; thus a system of parallel planes is determined by two ideal points (i.e., two directions).

A similar discussion leads us to the consideration of the ideal region of space. It is such that each augmented line intersects it in one ideal point and each augmented plane intersects it in one ideal line. We assume, therefore, the existence of an **ideal plane** of space, the locus of all ideal points and ideal lines of space.

We are now prepared to point out a fundamental difference between Euclidean geometry and projective geometry: In projective geometry we have, adjoined to the ordinary elements of Euclidean geometry, the ideal elements of this article. There are no ideal elements in Euclidean geometry, and, on the other hand, in projective geometry no distinction is made between the ordinary and ideal elements. Thus, a projective line may consist of all the points of an Euclidean line with the addition of one ideal point, or it may consist entirely of ideal points — and no distinction need be made between these two kinds of lines.

The assumption of the existence of ideal elements enables us to introduce in the statements of projective geometry a brevity and conciseness which are quite impossible in the corresponding statements of Euclidean geometry. The principle of duality, which we shall take up in the next chapter, is a case in point. For the present we shall merely illustrate the foregoing discussion by restating the three propositions at the beginning of this article, this time, however, in terms of projective points, lines, and planes:

- (1') Two lines in a plane intersect in one point.
- (2') Three lines in a plane intersect either in one point or in three points.
- (3') Two planes intersect in one line.

The student will readily see that the content of these propositions has been greatly enlarged while the wording has been reduced to a minimum. In the sequel we shall, of course, always use the terms point, line, and plane in the projective sense unless explicitly stated otherwise.

1.4 Axioms. Without attempting to be logically rigorous, let us state certain propositions or assumptions about the undefined elements. The use of the so-called “on language” and of the terms “incident” and “nonincident” enables us to state these propositions in a concise and symmetrical form. The terminology of the “on language” will become apparent with use; however, we may explain it here by an example. The ideas involved

in the two statements " a point lies on a line " and " a line passes through a point " are, to some extent, much the same. Because of this, therefore, we shall frequently word the last statement in the form " a line lies on a point." As to the meaning of the term " incident," another example will suffice. If a point lies on a line, the line lies on the point, and we say that the point and line are " incident." It is evident that two elements not of the same kind are either incident or nonincident. On the other hand, if two elements of the same kind, say two points A and B , are such that one lies on the other, then these elements are said to be " coincident." .

The following propositions, which are sometimes referred to as incidence relations, are assumed to be valid with regard to the three elements, plane, line, and point, in space of three dimensions. They are not all independent, but they serve our purpose in several ways and provide us with a suitable beginning for our study of projective geometry.¹

(1) A plane is incident with an unlimited number of lines and points.

(2) A line is incident with an unlimited number of planes and points.

(3) A point is incident with an unlimited number of lines and planes.

(4) Two distinct planes α, β have one and only one line incident with both planes. This line is denoted by $\alpha\beta$.

(5) Two distinct points A, B have one and only one line incident with both points. This line is denoted by AB .

(6) Either a plane α and a line a are incident or else there is a single point αa incident with both.

(7) Either a point A and a line a are incident or else there is a single plane Aa incident with both.

(8) Two distinct lines a, b in general have no plane incident with both; but if they do, there is only one such plane, and the lines are then both incident with a single point ab .

(9) Two distinct lines a, b in general have no point incident with both; but if they do, there is only one such point, and the lines are then both incident with a single plane ab .

¹ For a logical development of the subject from a set of axioms possessing the proper logical characteristics see Veblen and Young, *Projective Geometry*, Ginn & Co., 1910.

1.5 Primitive Forms. From the first three of the propositions stated above it is evident that a point, a line, or a plane may be taken as the **base** or **support** of an unlimited number of elements of a different kind. The various sets of elements which thus arise we shall call **primitive forms**. In the sequel we shall consider the following ten primitive forms:

(1) The totality of points on a line is a **point-row** or a **range of points**. The elements of a point-row are points, and the base is a line.

(2) The totality of lines on one point and on one plane is a **flat pencil** or a **pencil of lines**. Some writers call it a **sheaf of lines** or a **sheaf of rays**, the term ray here being synonymous with line. In the plane the elements of a flat pencil are lines, and the support is a point called the **center** or **vertex** of the pencil.

(3) The totality of planes on a line is an **axial pencil** or a **pencil of planes** or a **sheaf of planes**. The elements of an axial pencil are planes, and the support is a line called the **axis** of the pencil.

(4) The totality of points lying on one plane is a **field of points**.

(5) The totality of lines lying on one plane is a **field of lines**.

(6) The totality of lines in space lying on one point is a **bundle of lines**.

(7) The totality of planes in space lying on one point is a **bundle of planes**.

(8) The totality of points in space is a **space of points**.

(9) The totality of planes in space is a **space of planes**.

(10) The totality of lines in space is a **space of lines**.

1.6 Central Projection. Consider a part of a field of points and lines, i.e., a figure consisting of points A, B, C, \dots , and lines $AB, BC, CA, \dots, a, b, c, \dots$, which lies on a plane π . Now from a point S not on π draw the lines SA, SB, SC, \dots and the planes $SAB, SBC, SCA, \dots, Sa, Sb, Sc, \dots$. These lines and planes, indefinitely extended of course, are called the **projectors** of the points and lines, respectively, which constitute the original figure. The projectors, we note, are part of a bundle of lines and planes. If these projectors are cut by a second plane π' in any position in space but not coincident with π nor incident with S , we obtain a **section** of the projectors consisting of points A', B', C', \dots , where SA, SB, SC, \dots cut π' , and of lines $A'B', B'C', C'A', \dots, a', b', c', \dots$, where $SAB, SBC, SCA, \dots, Sa, Sb, Sc, \dots$

intersect π' . The figure on π' thus obtained is the **central projection** from S , or simply the **projection**, of the original figure on π ; it is part of a new field of points and lines. The process of passing from the figure on π to that one on π' is a **central projection**. It consists, as we have seen, of two parts: first, a projection from a point or center S , and second, a plane section of the projectors so formed.

We have at hand two examples of a central projection: When we view a picture the lens of the eye acts as a center of projection and sends light rays to the retina on which we then have a projection of the picture; and the camera makes a projection of a landscape on the film, the lens of the camera in this case being the center of projection. When we examine a photograph we recognize at once that each point, line, and curve of the landscape has its **image** point, line, and curve in the photograph. Furthermore, two intersecting lines in the former have images in the latter which are intersecting lines, their point of intersection being the image of the point of intersection of the original lines in the landscape, and so on. We recognize, then, that there are certain properties of the original figure which appear again, after the central projection, as properties of the projection of the original. And also, we observe that there are other properties which are changed by the central projection; for instance, some lines which are parallel or perpendicular in the original may have images which are not parallel or perpendicular; and again, some circles may reappear as ellipses after the projection; and finally, the area photographed is not the same in size as the area shown in the photograph.

We can now amplify the statement made at the end of Art. 1.1. Projective geometry studies those properties of figures which are never changed by *any* central projection. We say that such properties are **invariant** under projections. This excludes, of course, the study of properties such as angle and distance, which have to do with the idea of measure, for we have seen that by a central projection the measure of an angle or an area may be changed.

1.7 Dimensions of the Primitive Forms. The assumption of ideal elements leads us to a classification of the primitive forms on the basis of a correspondence principle. Each line of a flat pencil, to illustrate, cuts any given line in the same plane which does not pass through the center of the pencil in one point. By associating with each line of the flat pencil that point of the point-row incident

with it, and conversely, we have a one-to-one correspondence between the elements of the two forms. Upon attempting to set up one-to-one correspondences between various pairs of the primitive forms, we find that the ten primitive forms fall into four groups:

(1) The point-row, flat pencil, and axial pencil may be put in one-to-one correspondence, each with either one of the others, and are said to be **one-dimensional** forms. For example, a section of an axial pencil by a line not intersecting its axis is a point-row, and we may set up a one-to-one correspondence between them such that to each point of the point-row there corresponds that plane of the axial pencil incident with it; and conversely, to each plane of the axial pencil there corresponds that point of the point-row incident with it.

(2) The field of points, field of lines, bundle of lines, and bundle of planes may be put in one-to-one correspondence, each with either of the others, and are said to be **two-dimensional** forms. For instance, we may set up such a correspondence between a bundle of lines and a field of points whose plane is not incident with the center of the bundle by pairing each line of the bundle with that point of the field incident with it, and conversely.

(3) The space of points and the space of planes are **three-dimensional** forms.

(4) The space of lines is a **four-dimensional** form.

In our study of projective geometry we shall be interested primarily in the one-dimensional forms. In this connection the reader should notice that the projectors of a point-row or a flat pencil, from a point not on the line of the point-row or on the plane of the pencil, form respectively a flat pencil or an axial pencil; and that the projectors of a point-row from a line² not in the same plane with the point-row form an axial pencil. On the other hand, a section of a flat pencil by a line in its plane but not passing through its center is a point-row; also a section of an axial pencil by a line not intersecting its axis is a point-row; and a section of an axial pencil by a plane not incident with its axis is a flat pencil.

²That is, the line or axis of projection determines with each point of the point-row a plane, the projector of that point from the given axis. A section of these projectors results in an **axial projection** of the original point-row (cf. Art. 1.6).

In concluding, we should also point out that the term "dimension" as used in this article refers to the above classification of the primitive forms by means of the correspondence principle. The term should not be confused with the dimensions of space in which the forms exist. Indeed, in this latter sense the point-row, flat pencil, and axial pencil occupy respectively one, two, and three spatial dimensions.

1.8 Separated Pairs of Elements. A flat pencil with center S is divided in two parts by two of its lines a, b . Each of these parts is called a **complete angle** and, when S is an ordinary point, consists of what are usually called two vertical angles. We shall indicate one of the complete angles by the symbol ab , and the other by $b.a$. A third line c , if it lies on S , must lie in one or the other of the two

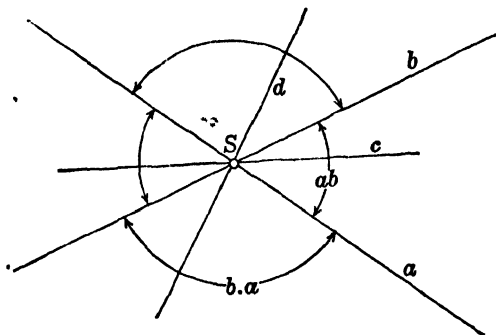


FIG. 2

complete angles $ab, b.a$. Suppose it lies in the complete angle ab (Fig. 2). Two **orders** or **senses** of turning about the point S are now determined, namely the senses acb and bca , which are opposite to each other. The sense acb may also be represented by cba or bac , while bca represents the same sense as cab or abc . We note that the line c cannot be said to separate the lines a and b . For while it is true that we cannot turn from line a to line b , or from b to a , in the complete angle ab without passing over line c , we can, nevertheless, turn from a to b , or from b to a , in the complete angle $b.a$ without passing over c . To separate a pair of lines a, b , therefore, a second pair of lines c, d is needed. Hence the

DEFINITION. The pair of lines a, b on a point S is **separated** by the pair of lines c, d on the same point S if c lies in the complete angle ab (or $b.a$) and d lies in the complete angle $b.a$ (or ab).

It follows that if the pair c, d separates the pair a, b , then the pair a, b separates the pair c, d .

Similarly, a point-row u is divided into two **segments** by a pair of points A, B . We shall indicate one of the segments by the symbol AB , and the other by $B.A$. A third point C , if it lies on u , must lie in one or the other of the two segments $AB, B.A$. Suppose it lies in the segment AB (Fig. 3). Two **orders** or **senses** are now determined on the line u ; we represent them by ACB , or CBA or BAC , and by BCA , or CAB or ABC . As in the case of two lines,



FIG. 3

we note that the point C cannot be said to separate the points A and B . For while it is true that we cannot pass from A to B , or from B to A , on the segment AB without passing over C , we can, nevertheless, pass from A to B , or from B to A , on the segment $B.A$ without passing over C . To separate a pair of points A, B , therefore, a second pair of points C, D is necessary. Hence the

DEFINITION. The pair of points A, B on a line u is **separated** by the pair of points C, D on the same line u if C lies in the segment AB (or $B.A$) and D lies in the segment $B.A$ (or AB).

It follows that if the pair C, D separates the pair A, B , then the pair A, B separates the pair C, D .

EXERCISES

1. Two coplanar lines a, b with an ordinary point of intersection P are projected from a point S , which is not on their common plane, into two lines a', b' on a plane π' . Describe this central projection. Describe a central projection which sends a, b into two parallel lines.

2. Describe a central projection which will send a quadrangle $ABCD$ into a parallelogram.

3. Project the two lines a, b of Ex. 1 into two perpendicular lines. Describe the central projection used.

4. Project a quadrangle $ABCD$ into a rectangle, and describe the central projection used. Can a rectangle be projected into a square?

5. The triangle ABC in plane π is projected from S into the triangle $A'B'C'$ in plane π' . Prove that $BC, B'C'$ and $CA, C'A'$ and $AB, A'B'$ intersect respectively in three points P, Q, R which are collinear. What is their line of collinearity?

6. Project a circle into (a) an ellipse; (b) an hyperbola; (c) a parabola.

7. A tetrahedron $ABCD$ is cut by a plane π not on any vertex. Describe the section. How must π be chosen so that the section will be a parallelogram?

8. Given a bundle of lines and a bundle of planes on distinct points; set up a correspondence such that each line of the first corresponds to that plane of the second which is perpendicular to it, and reciprocally. Is this a one-to-one correspondence?

9. Set up a one-to-one correspondence between a field of lines and a bundle of lines, and thus justify the statement that they are of like dimensions.

10. Given two skew (nonintersecting) lines a , b and a point P not on either line. Construct a line through P which intersects both a and b . How many such lines are there?

11. Construct a plane through P which is parallel to both the lines a , b of Ex. 10. How many such planes are there?

12. Draw figures illustrating the one-dimensional primitive forms.

13. Construct with cardboard and string a figure illustrating Ex. 5.

14. Draw a figure illustrating Ex. 7.

15. Construct a triangle having (a) one ideal vertex; (b) two ideal vertices.

CHAPTER II

THE PRINCIPLE OF DUALITY — SIMPLE AND COMPLETE FIGURES

2.1 A Reciprocal Relation Existing among the Axioms. The nine axioms which we listed in Art. 1.4 are space propositions in the sense that they are valid for all planes, lines, and points of three-dimensional space. The reader has probably noticed* that if the words "point" and "plane" are interchanged in these nine axioms no new propositions result; but instead (1) becomes (3) and, vice versa, (3) becomes (1); and the same is true of the pairs (4) and (5), (6) and (7), and (8) and (9). Axiom (2), on the other hand, is not altered by this interchange of "point" and "plane."

The phenomenon here observed to exist among those nine axioms will be in evidence throughout our study of projective geometry. We call it the **principle of duality**. Two statements such as (1) and (3) of Art. 1.4 are space duals of each other, while a statement such as (2) is self-dual, for it, as we have remarked, is unaltered by this interchange of words.

2.2 Duality in Space. Suppose that we were erecting a system of projective geometry in a logical and most rigorous manner from a set of undefined terms and axioms. Is it not evident that a principle of duality which exists in the axioms must necessarily be present in the theorems deduced from those axioms? This, in fact, is precisely the situation. And because of it, having proved a theorem we can state its dual *without proof*; for the proof of the dual theorem will be the dual of the proof of the original and, in the last analysis, will be based on the duals of the axioms from which the original theorem was deduced.

With this intuitive justification of space duality we notice first that the point and plane are **dual** or **reciprocal** elements in space. In the second place, we note that an incidence relation between elements gives rise to an incidence relation between their duals; consequently, there is a duality among the primitive forms. For

example, the point-row and axial pencil are dual forms, and the flat pencil is self-dual. The reader should satisfy himself that the dual of each primitive form is a primitive form by dualizing the contents of Art. 1.5.

As examples of dual theorems we state the following, in each case writing the theorem and its dual side by side:

1. Three points not on a line determine a plane. The three points and the three lines determined by them form a triangle on that plane.

2. If A, B, C, D are four points and the lines AB, CD are on one point, then the four points A, B, C, D are on one plane, and the lines AC and BD , and also the lines AD and BC , have a point in common.*

S1. Three planes not on a line determine a point. The three planes and the three lines determined by them form a trihedron on that point.

S2. If $\alpha, \beta, \gamma, \delta$ are four planes and the lines $\alpha\beta, \gamma\delta$ are on one plane,* then the four planes $\alpha, \beta, \gamma, \delta$ are on one point, and the lines $\alpha\gamma$ and $\beta\delta$, and also the lines $\alpha\delta$ and $\beta\gamma$, have a plane in common.

2.3 Duality in the Plane. Consider the projectors from a center S of a figure consisting of points and lines lying in one plane σ . This set of projectors will consist of lines and planes passing through S . The space dual of the projectors will consist of lines and points lying in σ' , a plane which is the space dual of the point S . We are thus led, by means of the space dual of the projector of the figure of points and lines in σ , to a figure of lines and points in σ' ; and similarly, from any theorem concerning the points and lines in σ , we are led to a theorem concerning the lines and points in σ' .

We might have made this passage from the theorem concerning the figure in σ to that in σ' more directly by interchanging the words "point" and "line." When the theorem is stated strictly in the "on language" no other interchange than that of "point" and "line" is necessary; when it is not so stated, however, certain obvious changes in phrasing are required.

The principle of plane duality is based on the fact that any proposition concerning points and lines on a plane remains valid when the words "point" and "line" are interchanged.

That is, in the plane the point and line are dual or reciprocal elements.

As examples of this type of duality we have:

1. Any two points of a plane determine a line.

2. If two coplanar triangles (3-points), ABC and $A'B'C'$, have AA' , BB' , CC' meeting at a point S , then the lines BC , $B'C'$ and CA , $C'A'$ and AB , $A'B'$ intersect respectively in three collinear points.

P1. Any two lines of a plane determine a point.

P2. If two coplanar trilaterals (3-lines), abc and $a'b'c'$, have aa' , bb' , cc' lying on a line s , then the points bc , $b'c'$ and ca , $c'a'$ and ab , $a'b'$ are joined respectively by three concurrent lines.

2.4 Duality in the Bundle. Just as we have figures in space and in the plane, so we have figures in the bundle, i.e., figures consisting of lines and planes on one point. The space dual of the italicized statement of Art. 2.3 tells us that *the principle of bundle (i.e., point) duality is based on the fact that any proposition concerning planes and lines on a point remains valid when the words "plane" and "line" are interchanged.* That is, in the bundle the plane and line are dual or reciprocal elements.

As an example of bundle duality:

1. The totality of lines (in a bundle) lying on one plane is a flat pencil.

B1. The totality of planes (in a bundle) lying on one line is an axial pencil.

2.5 Summary. In summarizing the principle of duality we note that it consists of three parts:

(1) Space duality, in which point and plane are dual elements and the line is self-dual;

(2) Plane duality (a consequence of the principle of space duality), in which point and line are dual elements; and

(3) Bundle duality (the space dual of the principle of plane duality), in which plane and line are dual elements.

Furthermore, we note that the space dual of a space, plane, or bundle proposition is respectively a space, bundle, or plane proposition; while the plane dual of a plane proposition is another plane proposition and the bundle dual of a bundle proposition is another bundle proposition. Obviously, a plane proposition has no bundle dual, nor has a bundle proposition a plane dual.

2.6 Simple Figures in the Plane.

DEFINITION. A **simple plane n -point** is a set of n points (vertices) of a plane taken in a definite order in which no three consecutive points are collinear, together with the n lines (sides) joining successive points.

DEFINITION. A **simple plane n -line** is a set of n lines (sides) of a plane taken in a definite order in which no three consecutive lines are concurrent, together with the n points (vertices) of intersection of successive lines.

These two definitions include the polygons of elementary geometry, but from a slightly different point of view. For here we consider the figure as a sequence of points and lines in a definite order, whereas in elementary geometry a polygon is a set of vertices and sides, and the sides are segments in the Euclidean sense.

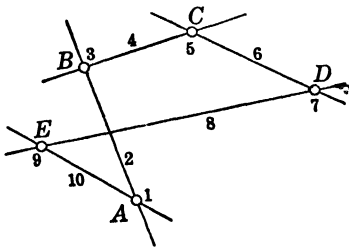


FIG. 4

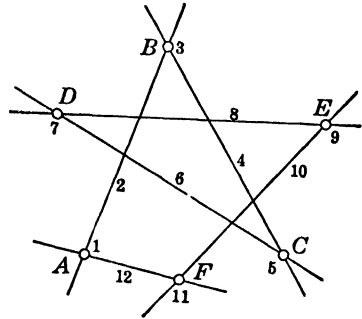


FIG. 5

In each such figure there are $2n$ elements which fall into n pairs of **opposite elements**. If n is even the opposite elements are of the same kind, i.e., both points or both lines; if n is odd they are not of the same kind. For example, in the simple 5-point of Fig. 4 we number the elements consecutively from 1 to 10 and find that vertex 3 and side 8 are opposite elements, as are side 6 and vertex 1, etc. And in the simple 6-point of Fig. 5 opposite pairs are vertices 1 and 7, sides 10 and 4, etc. In general, the elements k and $k + n$ are opposite if $k \leq n$, while if $k > n$ the opposite elements are k and $k - n$.

DEFINITION. The **diagonal lines** of a simple n -point are the $\frac{1}{2}n(n - 1) - n = \frac{1}{2}n(n - 3)$ lines which join pairs of non-consecutive vertices.

DEFINITION. The **diagonal points** of a simple n -line are the $\frac{1}{2}n(n - 1) - n = \frac{1}{2}n(n - 3)$ points of intersection of pairs of nonconsecutive sides.

2.7 Complete Figures in the Plane.

DEFINITION. A **complete plane n -point** is a set of n points (vertices) of a plane, no three of which are collinear, and the $n(n-1)/2$ lines (sides) which join every pair of these points.

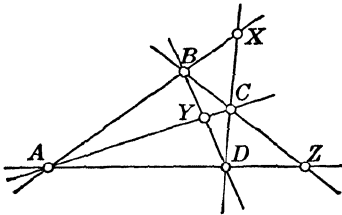


FIG. 6

DEFINITION. A **complete plane n -line** is a set of n lines (sides) of a plane, no three of which are concurrent, and the $n(n-1)/2$ points (vertices) of intersection of every pair of these lines.

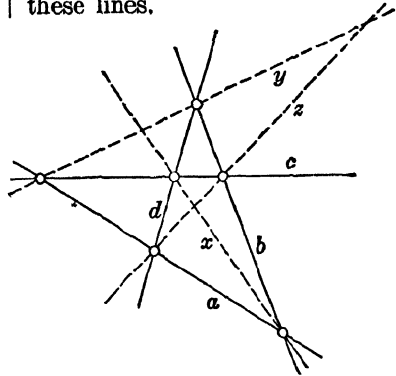


FIG. 7

The most important of the complete figures with which we shall be concerned are the quadrangle (4-point) and the quadrilateral (4-line). In particular,

✱ A **complete quadrangle** consists of four vertices, A, B, C, D , and the six lines joining them by pairs. The six sides fall into three pairs of **opposite sides**, the line joining two vertices and the line joining the other two forming such a pair (Fig. 6).

The three pairs of opposite sides intersect in three additional points, X, Y, Z , which are called **diagonal points** and which form the **diagonal triangle** (3-point) XYZ of the quadrangle.¹

A **complete quadrilateral** consists of four sides, a, b, c, d , and their six points of intersection by pairs. The six vertices fall into three pairs of **opposite vertices**, the point of intersection of two sides and the point of intersection of the other two forming such a pair (Fig. 7).

The three pairs of opposite vertices are joined by three additional lines, x, y, z , which are called **diagonal lines** and which form the **diagonal triangle** (3-line) xyz of the quadrilateral.

¹ For symmetry we shall ordinarily letter the figure so that the diagonal points X, Y, Z lie respectively on sides AB, AC, AD , and dually.

A triangle is self-dual and may be considered as a 3-point or a 3-line. Moreover, it is a simple as well as a complete figure. On the other hand, a complete quadrangle $ABCD$ contains the three simple quadrangles $ABCD$, $ACDB$, and $ACBD$; and in general a complete n -point (n -line) consists of $(n-1)!/2$ simple n -points (n -lines).

2.8 Simple and Complete Figures in Space and in the Bundle.

The definitions of the space figures analogous to those of Arts. 2.6 and 2.7 are as follows:

DEFINITION. A **simple space n -point** is a set of n points (vertices) taken in a definite order in which no four consecutive points are coplanar, together with the n lines (edges) joining pairs of successive points and the n planes (faces) determined by pairs of successive lines.

For example, the simple space 4-point consists of four points A, B, C, D together with the four lines AB, BC, CD, DA and the four planes ABC, BCD, CDA, DAB .

DEFINITION. A **complete space n -point** is a set of n points (vertices), no four of which are coplanar, together with the $n(n-1)/2$ lines (edges) joining them in pairs and the $n(n-1)(n-2)/6$ planes (faces) joining them in threes.

DEFINITION. A **simple space n -plane** is a set of n planes (faces) taken in a definite order in which no four consecutive planes are concurrent, together with the n lines (edges) of intersection of pairs of successive planes and the n points (vertices) determined by pairs of successive lines.

For example, the simple space 4-plane consists of four planes $\alpha, \beta, \gamma, \delta$ together with the four lines $\alpha\beta, \beta\gamma, \gamma\delta, \delta\alpha$ and the four points $\alpha\beta\gamma, \beta\gamma\delta, \gamma\delta\alpha, \delta\alpha\beta$.

DEFINITION. A **complete space n -plane** is a set of n planes (faces), no four of which are concurrent, together with the $n(n-1)/2$ lines (edges) of intersection in pairs and the $n(n-1)(n-2)/6$ points (vertices) of intersection in threes.

To a certain extent the tetrahedron in space plays the rôle of the triangle in the plane. For the tetrahedron is self-dual and may be considered equally well as a complete 4-point or a complete 4-plane. It is not, however, a simple space figure, for it has six edges, whereas a simple space 4-point or 4-plane can have only four.

The definitions of the simple and complete figures in the bundle are obtained directly as the space duals of the definitions of Arts. 2.6 and 2.7.

DEFINITION. A **simple n -plane in the bundle** is a set of n planes (faces) on a point taken in a definite order in which no three consecutive planes are coaxial, together with the n lines (edges) determined by pairs of successive planes.

DEFINITION. A **simple n -line in the bundle** is a set of n lines (edges) on a point taken in a definite order in which no three consecutive lines are coplanar, together with the n planes (faces) determined by pairs of successive lines.

We note that the space dual of a simple plane n -point and the projector of a simple plane n -line from a point outside its plane are both simple n -planes in the bundle. Also, that the space dual of a simple plane n -line and the projector of a simple plane n -point from a point outside its plane are both simple n -lines in the bundle. A like state of affairs obtains for the complete figures.

DEFINITION. A **complete n -plane in the bundle** is a set of n planes (faces) on a point, no three of which are coaxial, and the $n(n-1)/2$ lines (edges) of intersection by pairs.

DEFINITION. A **complete n -line in the bundle** is a set of n lines (edges) on a point, no three of which are coplanar, and the $n(n-1)/2$ planes (faces) determined by them by pairs.

The rôle of the triangle in the plane is taken here by the trihedron (3-plane or 3-line) which, considered as either a simple or complete figure, is self-dual.

We have purposely given this complete statement of the definitions of simple and complete figures, not that we shall use all of them in the sequel, but because they are valuable in exhibiting the three aspects of duality (space, plane, and bundle) and in showing the connections between those three aspects. It is suggested that Art. 2.5 be reread at this point.

It should be noticed that such a phrase as "the n lines of intersection of successive planes" becomes when dualized for the bundle "the n planes determined by successive lines." This and other differences of terminology disappear if the so-called "on language" is used; and the reader should satisfy himself that

this is so by restating each of the above definitions using the "on language" consistently throughout.

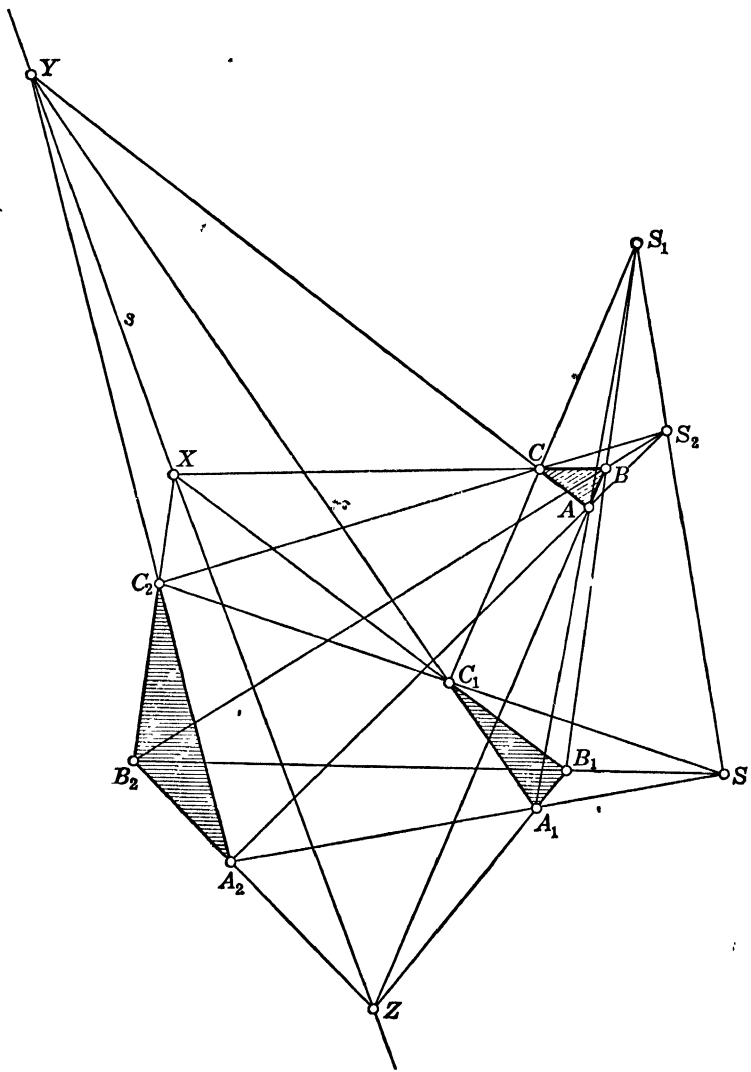


FIG. 8

2.9 The Desargues Configuration. A most important configuration for our work is obtained by taking a plane section of the complete space 5-point. The complete 5-point in space con-

tains the ten lines and the ten planes determined by the five points; furthermore, there are certain incidence relations among its points, lines, and planes. They are indicated frequently by the symbol

	points	lines	planes	
points	5	4	6	,
lines	2	10	3	
planes	3	3	10	

which is read by rows as follows:

Row 1. There are 5 points, 4 lines on each point, and 6 planes on each point.

Row 2. There are 2 points on each line, 10 lines, and 3 planes on each line.

Row 3. There are 3 points on each plane, 3 lines on each plane, and 10 planes.

A plane section not passing through a vertex cuts each line in one point and each plane in one line. It is called the **configuration of Desargues** and has the symbol

	points	lines	
points	10	3	,
lines	3	10	

which indicates that it is a figure consisting of 10 points with 3 lines on each point and 10 lines with 3 points on each line. Fig. 8 shows the complete 5-point S_1S_2ABC and its plane section made up of 10 points collinear in sets of 3 on 10 lines, which are concurrent in sets of 3 on the 10 points.

EXERCISES

1. On a plane π there is given a triangle consisting of the three points A , B , C and the three lines BC , CA , AB determined by them. Describe the
 - (a) space dual of triangle ABC ;
 - (b) projectors of ABC from a point S not on π ;

- (c) space dual of (b);
- (d) plane dual of ABC ;
- (e) space dual of (d);
- (f) bundle dual of (e);
- (g) section of (f) by a plane σ .

2. If, on one plane, the points A, B, C lie on a line u and the 'points A', B', C' lie on a second line u' , then the lines BC' and $B'C$, CA' and $C'A$, AB' and $A'B$ intersect respectively at the three points X, Y, Z which lie on a third line p .

Assuming the validity of this theorem, write its plane dual and also its space dual.

3. If, in one plane, a triangle varies in such a way that two of its vertices lie on two fixed lines and its three sides pass through three fixed collinear points, then the third vertex will lie on a line concurrent with the two fixed lines.

Assuming the validity of this theorem, state its plane dual.

4. State the space dual of each of the ten primitive forms. Which of the primitive forms have plane duals; which have bundle duals?

5. State and prove the space dual of Ex. 5, Chapter I. Compare your proof with the proof of that exercise.

6. Dualize the statement of Ex. 10, Chapter I, and solve.

7. Describe the simple 5-point in the plane; the complete 5-point in the plane. What are the plane and space duals?

8. Describe the simple 5-point in space; the complete 5-point in space. What are the space duals?

9. What is the configuration symbol for (a) the complete n -point in space; (b) a section of the complete n -point in space by a plane not on a vertex; (c) the complete n -point in the plane?

10. Describe the plane dual of Desargues' configuration.

11. Construct a complete quadrangle which has (a) two ideal vertices; (b) two ideal diagonal points.

12. Construct a simple 5-point in the plane which has two ideal vertices. Indicate those elements which are opposite to the ideal elements; indicate, also, the diagonal lines.

Note that the ideal vertices may or may not be consecutive; hence, two figures should be drawn.

13. Draw a figure illustrating the theorem of Ex. 2; and also a figure illustrating the plane dual of that theorem.

14. Construct a Desargues configuration as a plane section of a complete 5-point in space.

15. Describe the section of a complete 5-plane in space by a plane not on any vertex of the 5-plane. Illustrate with a drawing.

CHAPTER III

PERSPECTIVITY — DESARGUES' THEOREM

3.1 One-to-one Correspondence. A **one-to-one correspondence** is said to exist between the elements of two primitive forms, or between the elements of two simple or complete figures, if there is associated with each element of one form a unique element of the other form, and reciprocally, with each element of the last a unique element of the first. Elements so paired are called **homologous** and either one is said to **correspond** to the other.

We remind the reader that the classification of the primitive forms as given in Art. 1.7 was based upon this concept of a (1,1) correspondence.

Moreover, we note:

(1) That two primitive forms cannot be put in a (1,1) correspondence unless they are forms of the same dimension.

(2) That two simple or complete figures cannot be put in a (1,1) correspondence unless they are figures having the same number of elements.

(3) That if two elements E, F of one form correspond respectively to the elements E', F' of a second, we shall, in what follows, agree that the element EF of the first shall correspond to the element $E'F'$ of the second.

If two forms are each in (1,1) correspondence with a third, they are in (1,1) correspondence with each other. Hence, in Fig. 9, if a (1,1) correspondence is set up between the flat pencil $S(abcd \text{ ---})$, with center S and lines a, b, c, d, \dots , and the point-row $u(ABCD \text{ ---})$, with base u and points A, B, C, D, \dots , by pairing each line of S with that point in which it cuts u , and if $u(ABCD \text{ ---})$ is in (1,1) correspondence with $S'(a'b'c'd' \text{ ---})$, and $S'(a'b'c'd' \text{ ---})$ with $u'(A'B'C'D' \text{ ---})$ in the same way (i.e., by pairing incident elements), then the primitive forms $S(abcd \text{ ---})$ and $u'(A'B'C'D' \text{ ---})$ are in (1,1) correspondence and any line d of S is homologous to one point D' of u' , and conversely.

To mention another possible (1,1) correspondence between a flat pencil $S(abcd \text{ ---})$ and a point-row $u(ABCD \text{ ---})$ we refer

to Fig. 10, in which each line a of S is homologous to that point A in which u is cut by the line of S perpendicular to a , etc. In par-

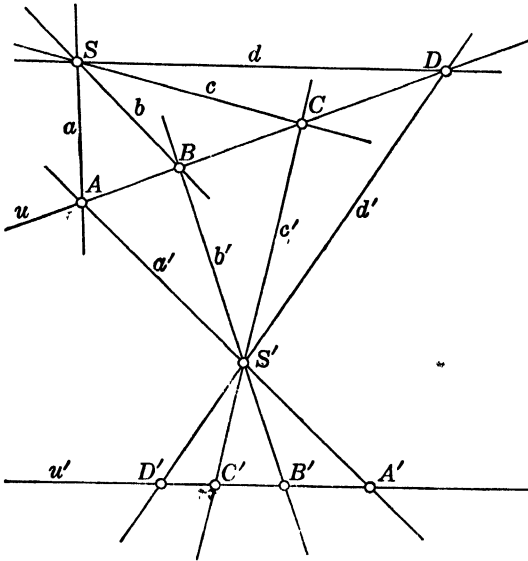


FIG. 9

ticular, the line i of S perpendicular to u is homologous to I , the ideal point of u . This correspondence, however, based as it is on the idea of perpendicularity involves metric considerations.

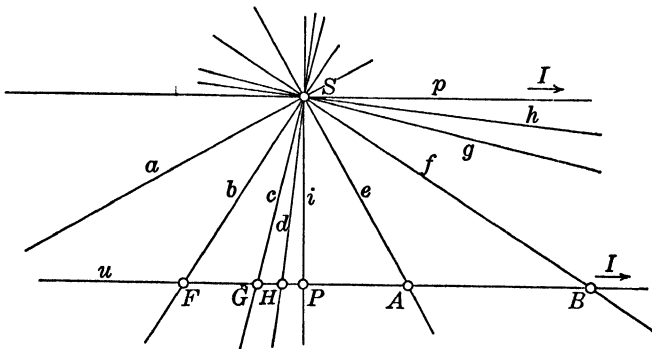


FIG. 10

3.2 Perspectivity. Two primitive forms consisting of the same kind of elements are **like** primitive forms; otherwise, they are **unlike**.

DEFINITION. Two unlike primitive forms are in **perspective position** ($\overline{\wedge}$), or are **perspective**, if they are in (1,1) correspondence and if homologous elements are incident.

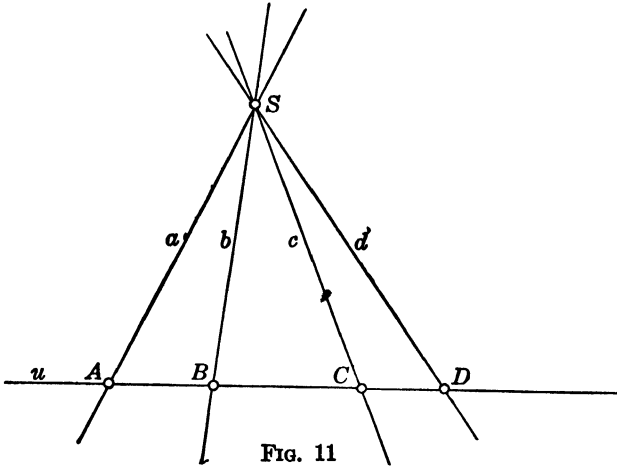


FIG. 11

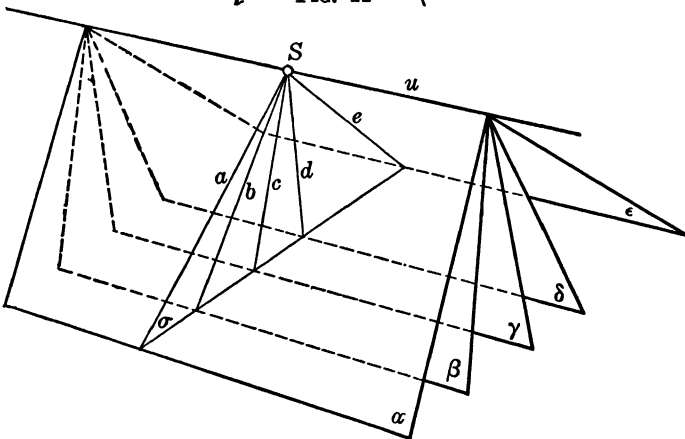


FIG. 12

For example,

1. A point-row $u(ABCD \text{---})$ and a flat pencil $S(abcd \text{---})$ are perspective if they are coplanar with nonincident supports and if they are in a (1,1) correspondence such that homologous elements, point and line, are incident (Fig. 11).

S1. An axial pencil $u(\alpha\beta\gamma\delta \text{---})$ and a flat pencil $\sigma(abcd \text{---})$ are perspective if they are copunctal with nonincident supports and if they are in a (1,1) correspondence such that homologous elements, plane and line, are incident (Fig. 12).

These examples are space duals. We have also the self-dual perspectivity:

2. A point-row $u(ABCD \text{ ---})$ and an axial pencil $v(\alpha\beta\gamma\delta \text{ ---})$ are perspective if their axes are skew and if they are in a (1,1) correspondence such that homologous elements, point and plane, are incident.

DEFINITION. Two like primitive forms are in **perspective position** ($\overline{\wedge}$), or are **perspective**, if they are in (1,1) correspondence and if homologous elements are both incident with the same element of a third primitive form unlike each of the two given forms.

That is, to continue with our examples,

3. Two point-rows with coplanar bases are perspective if they are in a (1,1) correspondence such that each pair of homologous elements, point and point, is incident with one line of a flat pencil. We say that they are **perspective from a point**, the center of the flat pencil.

4. Two flat pencils on the same plane are perspective if they are in a (1,1) correspondence such that each pair of homologous elements, line and line, is incident with one point of a point-row. We say that they are **perspective from a line**, the base of the point-row.

We note that Examples 3 and 4 are plane duals, and that S3 and S4 are bundle duals.

5. Two point-rows with skew bases are perspective if they are in a (1,1) correspondence such that each pair of homologous elements, point and point, is incident with one plane of an axial pencil.

S3. Two axial pencils with intersecting axes are perspective if they are in a (1,1) correspondence such that each pair of homologous elements, plane and plane, is incident with one line of a flat pencil. We say that they are **perspective from a plane**, the plane of the flat pencil.

S4. Two flat pencils on the same point are perspective if they are in a (1,1) correspondence such that each pair of homologous elements, line and line, is incident with one plane of an axial pencil. We say that they are **perspective from a line**, the axis of the axial pencil.

S5. Two axial pencils with skew bases are perspective if they are in a (1,1) correspondence such that each pair of homologous elements, plane and plane, is incident with one point of a point-row.

In each case the two forms are perspective from a line.

6. Two flat pencils which are neither coplanar nor concentric are perspective if they are in a (1,1) correspondence such that each pair of homologous elements, line and line, is incident with one point of a point-row or, what is the same thing, with one plane of an axial pencil.

For a thorough understanding of perspective primitive forms we should note the following facts:

(1) There are two types of perspectivity, referring to unlike and like primitive forms respectively.

(2) A perspectivity of either type requires (a) a (1,1) correspondence between the forms and (b) an incidence relation between homologous elements. The incidence relation varies with the type of perspectivity involved.

(3) If two forms are perspective, their duals are also perspective. This is an important observation, for it assures us that the principle of duality may be applied to all statements concerning perspectivities.

For the simple and complete figures we have the following:

DEFINITION. Two plane n -points are in **perspective position** if they are in a (1,1) correspondence such that pairs of homologous vertices are joined by lines concurrent at one point P , called the **center of perspectivity**.

We say that they are perspective from, or with, the center P .

DEFINITION. Two n -planes each in a bundle are in **perspective position** if they are in a (1,1) correspondence such that pairs of homologous faces intersect in lines coplanar on one plane π , called the **plane of perspectivity**.

We say that they are perspective from, or with, the plane π .

The reader should notice that we give on the right the space dual of the definition on the left; for we consider the two plane n -points on the left in their most general position, i.e., noncoplanar. If, on the other hand, the n -points are coplanar, the definition on the left admits a plane as well as a space dual. We shall give an explicit statement of the former but leave the latter for the student to write out and examine carefully.

DEFINITION. Two plane n -lines are in **perspective position** if they are in a (1,1) correspondence such that homologous sides intersect in points collinear on one line p , called the **axis of perspectivity**.

We say that they are perspective from, or with, the axis p .

There is one other definition which we should mention, namely, that of perspective plane n -lines which are not coplanar. We shall not, however, write it out here since its statement is identical with that of the definition last given.

3.3 Desargues' Theorem. Theorem I. *If two triangles are perspective from a point, they are perspective from a line.*

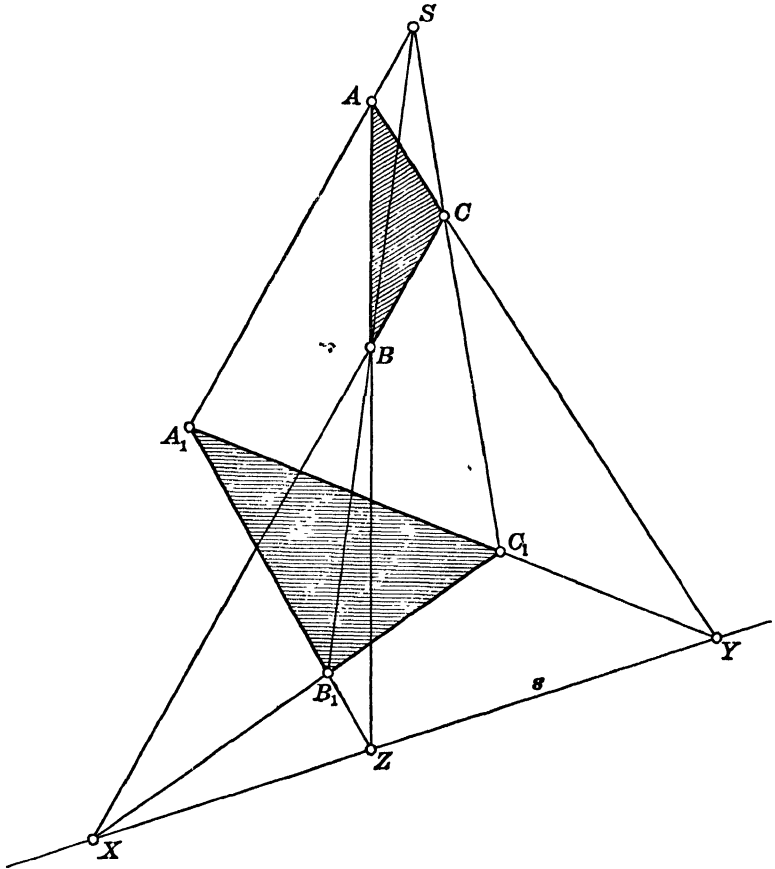


FIG. 13

Case 1. When the triangles are noncoplanar. Let the two triangles be ABC and $A_1B_1C_1$ with pairs of homologous vertices joined by lines AA_1 , BB_1 , CC_1 meeting at S (Fig. 13). To prove that BC , B_1C_1 and CA , C_1A_1 and AB , A_1B_1 intersect respectively at three collinear points X , Y , and Z .

Proof. (1) BC and B_1C_1 are coplanar and intersect at a point X . For the four points B, B_1, C, C_1 lie on one plane, since BB_1 and CC_1 intersect at S , by hypothesis.

(1') Similarly CA, C_1A_1 intersect at a point Y , and AB, A_1B_1 intersect at a point Z .

(2) The point X lies on s , the line of intersection of the planes of the two triangles. For X lies on BC and hence on the plane of ABC , and it likewise lies on B_1C_1 and hence on the plane of $A_1B_1C_1$.

(2') Similarly Y and Z lie on this same line s .

(3) Therefore, by definition, the triangles are perspective from a line (the line s).

Case 2. When the triangles are coplanar. Let the two triangles lying on the plane π be $A_1B_1C_1$ and $A_2B_2C_2$ with pairs of homologous vertices joined by lines A_1A_2, B_1B_2, C_1C_2 meeting at S (Fig. 8). To prove that B_1C_1, B_2C_2 and C_1A_1, C_2A_2 and A_1B_1, A_2B_2 intersect at three collinear points X, Y , and Z .

Proof. (1) The lines B_1C_1, B_2C_2 intersect at a point X . For the two triangles, of which these lines are homologous sides, are coplanar.

(1') Similarly the other two pairs of homologous sides C_1A_1, C_2A_2 and A_1B_1, A_2B_2 intersect at Y and Z .

(2) Project $A_1B_1C_1$ from S_1 and $A_2B_2C_2$ from S_2, S_1 and S_2 being any two points collinear with S but not on the plane of the two triangles.

(3) S_1A_1 and S_2A_2 are coplanar and intersect at a point A . For the four points A_1, A_2, S_1, S_2 lie on one plane, the line S_1S_2 having been constructed to meet A_1A_2 at S .

(3') Similarly S_1B_1 and S_2B_2 intersect at a point B , and S_1C_1 and S_2C_2 intersect at a point C .

(4) The noncoplanar triangles $A_1B_1C_1$ and ABC are perspective from the center S_1 . Hence, by Case 1, they are perspective from the line s which is common to their two planes. That is, B_1C_1 meets BC where the latter cuts the plane π , etc.

(4') Similarly the triangles $A_2B_2C_2$ and ABC are perspective from the center S_2 and, by Case 1, from the line s . That is, B_2C_2 meets BC where the latter cuts the plane π , etc.

(5) Therefore the intersection X of B_1C_1 and B_2C_2 is the point where BC cuts π ; and furthermore, X lies on s . For BC cuts π in only one point, which point is on line s .

(5') Similarly Y and Z lie on line s .

(6) Therefore the two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are perspective from the line s .

The proof just given is intimately connected with the configuration of Desargues (Art. 2.9). In fact, the figure formed by the coplanar triangles, their center, and their axis of perspectivity is a plane section of the complete space 5-point S_1S_2ABC and is, therefore, such a configuration. Any one of the ten points of that configuration is the center of perspectivity of two triangles whose vertices are also points of the configuration. This accounts for seven of the ten points and nine of the ten lines. The three remaining points lie on the tenth line, which is the axis of perspectivity of the same two triangles. The pair of perspective triangles, $A_1B_1C_1$ and $A_2B_2C_2$, of the above proof of Case 2 is evidently only one of ten such pairs in the configuration of Desargues.

3.4 Converse of Desargues' Theorem. *If two triangles are perspective from a line, they are perspective from a point.*

Case 1. When the triangles are noncoplanar. Let the two triangles be ABC and $A_1B_1C_1$ with pairs of homologous sides BC, B_1C_1 and CA, C_1A_1 and AB, A_1B_1 meeting respectively at X, Y, Z , three collinear points. To prove that AA_1, BB_1, CC_1 are concurrent lines.

Proof. (1) BC, B_1C_1 , since they intersect at X , determine a plane α .

(1') Similarly CA, C_1A_1 , intersecting at Y , and AB, A_1B_1 , intersecting at Z , determine planes β and γ .

(2) The three planes α, β, γ intersect at a point $S = \alpha\beta\gamma$.

(3) Now $AA_1 = \beta\gamma$, for the points A and A_1 lie on both β and γ ; AA_1 consequently passes through S .

(3') Similarly $BB_1 = \gamma\alpha$ and $CC_1 = \alpha\beta$ pass through S .

(4) Therefore the triangles ABC and $A_1B_1C_1$ are perspective from the point S .

Case 2. When the triangles are coplanar. We have here the plane dual of Theorem I, Case 2. Theorem I, Case 2, having been demonstrated in Art. 3.3, the validity of its converse is assured by the principle of duality.

The simplicity of the last statement will be a challenge to the careful reader. Let him attempt to dualize, in the plane, the proof of Case 2, Art. 3.3; he will observe immediately that this is impossible, for the proof as given is a space demonstration although the theorem which it proves is a plane proposition. Space dualiz-

ing, though possible, does not solve the difficulty, for the space dual of the proof of Theorem I, Case 2, will be the proof of the space dual of Theorem I, Case 2, and not of its converse. These remarks together with a rereading of Art. 2.3 will enable the reader to meet the challenge of the preceding paragraph.

A proof of the converse of Theorem I, Case 2, without recourse to the principle of duality can also be developed along the lines suggested by the proof of the direct theorem as given in Art. 3.3.

3.5 Perspective Quadrangles. Theorem II. *If two complete quadrangles are in (1,1) correspondence and so situated that five pairs of homologous sides intersect in points of the same straight line, then the point of intersection of the sixth pair of homologous sides is also on this line and the quadrangles are perspective to one another from a point and from a line.*

Given two complete (coplanar or noncoplanar) quadrangles, $KLMN$ and $K_1L_1M_1N_1$, in (1,1) correspondence, having five pairs of homologous sides intersecting at points on a line u , viz.,

KL and K_1L_1 intersecting at A ,
 KM and K_1M_1 intersecting at B ,
 KN and K_1N_1 intersecting at C ,
 MN and M_1N_1 intersecting at D ,
 NL and N_1L_1 intersecting at E .

To prove (a) that LM and L_1M_1 intersect at a point F also on u , and (b) that $KLMN \overline{\overline{K_1L_1M_1N_1}}$ from a point and from a line (Fig. 14).

Proof. (1) The triangles KLN and $K_1L_1N_1$ are perspective from the axis ECA , by hypothesis.

(2) Therefore KK_1 , LL_1 , NN_1 are concurrent at a point O , by the converse of Theorem I.

(1') The triangles KMN and $K_1M_1N_1$ are perspective from the axis DCB , by hypothesis.

(2') Therefore KK_1 , MM_1 , NN_1 are concurrent at the same point O , for KK_1 and NN_1 intersect at O , by (2).

(3) $LMN \overline{\overline{L_1M_1N_1}}$, by (2) and (2').

(4) $LMN \overline{\overline{L_1M_1N_1}}$ from a line, by (3) and Theorem I.

(5) MN and M_1N_1 meet at D , and NL and N_1L_1 at E , by hypothesis. Therefore LM and L_1M_1 meet at a point F which lies on DE . But DE is coincident with u ; hence F lies on u .

(6) By (2) and (2') $KL MN \overline{\wedge}^O K_1 L_1 M_1 N_1$, and by (5) $KL MN \overline{\wedge}^u K_1 L_1 M_1 N_1$.

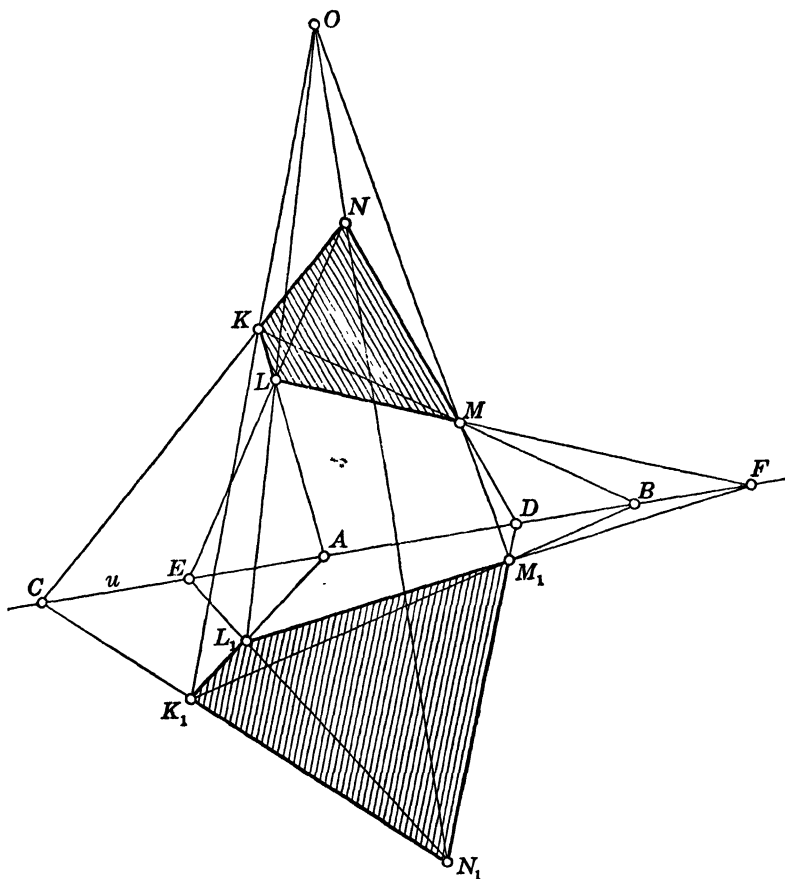


FIG. 14

To the reader we leave the task of dualizing the theorem of this article.

EXERCISES

1. If two point-rows u, u' are coplanar and $u \overline{\wedge} u'$, what is the dual figure in the plane; what is the dual figure in space? If u and u' are not coplanar and $u \overline{\wedge} u'$, what is the dual figure?

2. Prove the theorem: If two coplanar triangles are perspective from a line, they are perspective from a point.

3. State the space dual of Desargues' Theorem (two cases); and the space dual of its converse (two cases).

4. Indicate the ten pairs of perspective triangles in the Desargues configuration (Fig. 8), and indicate for each pair the center and axis of perspectivity.

5. Two lines a, b are coplanar but their common point $S = ab$ is not on the drawing paper. Given a point P on the drawing paper, it is required to construct the line PS .

6. Our straightedge is not long enough to draw the line s joining two points A, B . Given a line p coplanar with AB , it is required to construct the point ps .

7. If the points A, B, C of line u and the points A', B', C' of line u' are so situated that AA', BB', CC' are concurrent at a point P , then the intersections of BC' and $B'C, CA'$ and $C'A, AB'$ and $A'B$ are collinear on a line p which passes through the point uu' . Prove this theorem and state its plane dual.

The line p is called the *polar line* of the point P with respect to u and u' ; and P is called the *pole* of line p .

8. In the plane of a given triangle ABC the lines joining a point P to the vertices of ABC intersect the opposite sides at A', B', C' respectively, and the triangles ABC and $A'B'C'$ are perspective from the point P . Construct p , the axis of perspectivity of these two triangles.

This axis p is called the *polar line* of point P with respect to triangle ABC ; and P is called the *pole* of line p . Dualize in the plane.

9. Show that Theorem II remains valid in case the line u passes through one or two diagonal points of the quadrangles.

10. State Theorem II for the special case when line u is the ideal line of the plane.

11. There is given coplanar with a line l and a point P a fixed parallelogram. Using only a straightedge construct the line through P which is parallel to the line l .

12. Prove Theorem II when a vertex (or side) of one of the quadrangles coincides with a vertex (or side) of the other quadrangle.

13. Assuming the quadrangles of Theorem II to be coplanar, state and prove the plane dual of that theorem.

CHAPTER IV

HARMONIC SETS

4.1 Definitions.

1. Four collinear points A, B, C, D constitute an **harmonic range of points** $H(AC, BD)$ if it is possible to construct a complete quadrangle having A and C as two diagonal points while B and D lie on the two sides through the third diagonal point (Fig. 15).

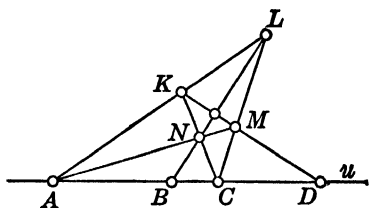


FIG. 15

P1. Four concurrent lines a, b, c, d constitute an **harmonic pencil of lines** $H(ac, bd)$ if it is possible to construct a complete quadrilateral having a and c as two diagonal lines while b and d pass through the two vertices lying on the third diagonal line (Fig. 16).

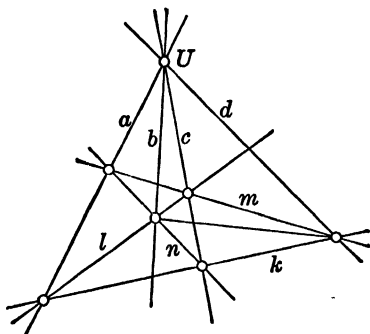


FIG. 16

In addition to the above two definitions, which are plane duals, we give a third, the space dual of the definition on the left:

S1. Four coaxial planes $\alpha, \beta, \gamma, \delta$ constitute an **harmonic pencil of planes** $H(\alpha\gamma, \beta\delta)$ if it is possible to construct a complete 4-plane in a bundle having α and γ as two diagonal planes while β and δ pass through the two edges lying on the third diagonal plane.

The reader should notice that the space dual of the definition on the right above gives a definition of an harmonic pencil of lines in the bundle. This is to be expected, for we have previously remarked (Art. 2.2) that the space dual of a flat pencil in a plane is a flat pencil in a bundle.

The harmonic range, pencil of lines, and pencil of planes are the harmonic sets of this chapter. With reference to the harmonic range $H(AC, BD)$ we note an apparent difference between the pairs A, C and B, D ; for the first pair A, C consists of two diagonal points of the complete quadrangle, while the second pair B, D consists of points common to the line AC and the third pair of opposite sides. We shall see later that this is not a real difference; that the two pairs A, C and B, D are, in fact, of equal importance and have similar properties. On the other hand, the points A and C enjoy like rôles in the definition and may be interchanged without disturbing the harmonic relation; and the same can be said of the points B and D .

Analogous statements are valid for the other two harmonic sets.

4.2 Construction of the Fourth Harmonic. If four points A, B, C, D on a line u form an harmonic range $H(AC, BD)$, defined by a complete quadrangle $KLMN$, then a second complete quadrangle $K_1L_1M_1N_1$, in the same plane with $KLMN$ or in any plane passing through u , which has one pair of opposite sides K_1L_1, M_1N_1 intersecting at A and a second pair K_1N_1, L_1M_1 intersecting at C , while the fifth side L_1N_1 passes through B , will have its sixth side K_1M_1 passing through D . That this is so is at once evident from Theorem II, Art. 3.5.

In other words, the pair of points A, C together with the point B determine uniquely a fourth point D , which forms with A, B, C an harmonic range $H(AC, BD)$. We term D the **fourth harmonic** of A, B, C and indicate its relation to these points by the symbol $D = H(AC, B)$.

To construct $D = H(AC, B)$, given the three points A, B, C on line u with A and C paired, we draw arbitrarily, in any plane through u , three nonconcurrent lines — one through each of the given points. A triangle KLN is thus formed whose sides KL, LN, NK meet u at A, B, C , respectively. Now join the paired points A and C to N and L , respectively; the lines AN and CL meet at a point, call it M . The complete quadrangle $KLMN$ thus constructed has A, C as diagonal points while the fifth side LN cuts AC at B . The sixth side KM , consequently, cuts u at the desired point D .

Moreover, Theorem II assures us that any other quadrangle $K_1L_1M_1N_1$ constructed in the same manner will determine on u the same point D .

The construction just given involves two important features which should be noted. The first is that three of the six sides of the quadrangle are drawn arbitrarily through A , B , C , and, when once drawn, they determine uniquely the other three sides. The second is that the interchange of A and C does not change the construction in any particular; hence $D = H(AC, B) = H(CA, B)$.

The plane dual of this construction is most important and frequently used; the student should become familiar with it now.

4.3 Conjugate Pairs. The symbol $H(AC, BD)$ expresses the existence of an harmonic relation between the four points A , B , C , D on a line u — an harmonic relation, in fact, in which A , C are two diagonal points of a complete quadrangle and B , D are the points where AC is met by the two sides of the quadrangle passing through the remaining diagonal point. As we have previously remarked, the pairs of points A , C and B , D have, apparently, different rôles in the set $H(AC, BD)$. But this, we have also noted, is not so. To show that the pairs A , C and B , D are of equal importance we prove the

THEOREM. *The relation $H(AC, BD)$ implies the relation $H(BD, AC)$.*

Given $H(AC, BD)$ or, what is the same thing, that there is a complete quadrangle $KLMN$ such that $KL, MN = A$; $KN, LM = C$; $LN, AC = B$; and $KM, AC = D$. To prove $H(BD, AC)$.

Proof. (1) Let KM and LN meet at O (Fig. 17). Draw BK and DL intersecting at R .

(2) The triangles RLK and OMN are perspective from the line ABD .

(3) By (2) and the converse of Theorem I, the triangles RLK and OMN are perspective from a point, and that point is C , the intersection of LM and KN . That is, RO , the third join of homologous vertices of these perspective triangles, passes through C .

(4) The complete quadrangle $RKOL$ defines the harmonic range $H(BD, AC)$ for $RK, OL = B$; $KO, RL = D$; $KL, BD = A$; and $RO, BD = C$.

The pairs A , C and B , D , which are thus shown to be alike in that either A , C or B , D may be taken as two diagonal points of a complete quadrangle, are observed to separate each other according to the definition of separated pairs in Art. 1.8. We say

that the pair A, C of the harmonic range $H(AC, BD)$ is **harmonically separated** by the pair B, D ; and this, by the above theorem, implies that the pair B, D is harmonically separated by the pair A, C . The two pairs A, C and B, D are called **conjugate pairs**, and either pair is said to be **conjugate** to the other. The fourth harmonic $D = H(AC, B)$ is the **harmonic conjugate** of B with respect to A and C ; $C = H(A, BD)$ is the harmonic conjugate of A with respect to B and D , etc.

If the pair A, C remains fixed while the point B moves from A to C on the segment AC , the harmonic conjugate of B with respect to A and C moves from A to C on the segment $C.A$. That is

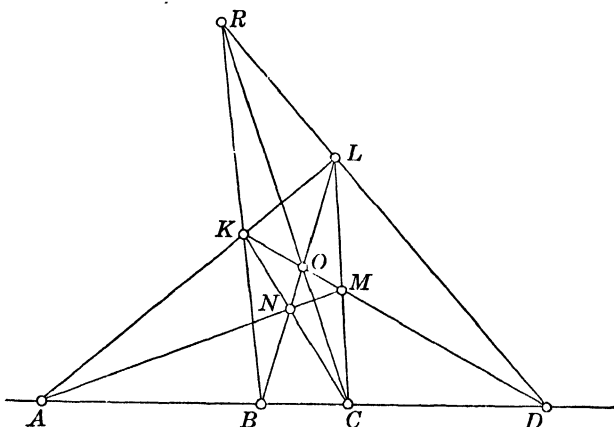


FIG. 17

in Fig. 17, as B moves to the right and approaches C , point $D = H(AC, B)$ moves to the left and also approaches C . For if A and C are fixed while B varies, the vertices K and L and the sides AL, CK , and CL of the complete quadrangle $KLMN$ will also remain fixed. And as B moves to the right on AC , the side LB of the quadrangle rotates about the fixed point L in a counter-clockwise sense; and N moves on the line KC from K to C . Hence AN rotates about A in a clockwise sense while M moves from L to C . The side KM , therefore, will rotate about K in a clockwise sense while D , the intersection of KM and AC , moves on $C.A$ to the left toward C . Needless to say that if B describes AC moving from C to A , then by the same kind of argument D describes $C.A$ moving from C to A .

In general, if B describes AC in a certain sense, then point $D = H(AC, B)$ describes CA in the opposite sense.

The contents of this article and of the one immediately preceding will be dualized for the harmonic pencil of lines and the harmonic pencil of planes by the careful reader before he continues further.

4.4 Projection and Section of an Harmonic Set.

1. The projector of the harmonic range $H(AC, BD)$ from a point S not on the support of the range is the harmonic pencil of lines $H(ac, bd)$, where $a = SA$, $b = SB$, etc.

S1. The section of the harmonic pencil of planes $H(\alpha\gamma, \beta\delta)$ by a plane σ not on the axis of the pencil is the harmonic pencil of lines $H(ac, bd)$, where $a = \sigma\alpha$, $b = \sigma\beta$, etc.

As to the proposition on the left, if we draw (Fig. 18) through A any line l cutting b at N and c at M , and then draw $CN = n$ to cut

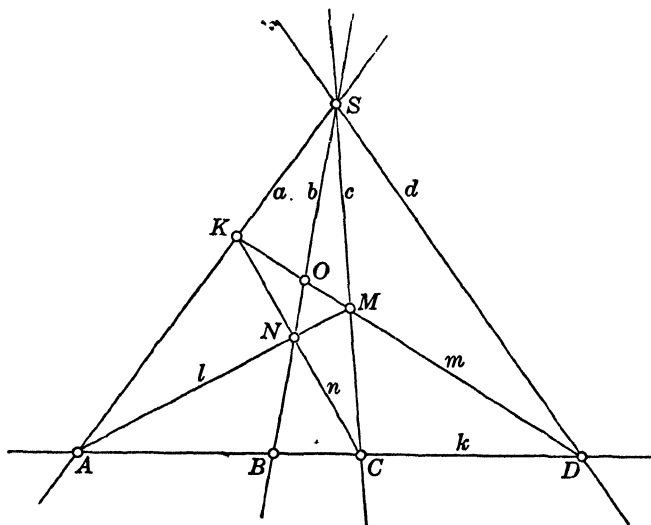


FIG. 18

a at K , we see that the complete quadrangle $KSMN$ has A and C as diagonal points and a fifth side passing through B ; therefore, the sixth side $KM = m$ passes through D , since $D = H(AC, B)$. If $AC = k$, then the complete quadrilateral $klmn$ has a and c as diagonal lines while b and d join $ac = S$ to the fifth and sixth vertices, $ln = N$ and $km = D$. Hence, $klmn$ defines $H(ac, bd)$.

The plane dual of the proposition on the left leads to:

- | | |
|--|--|
| <p>2. The section of the harmonic pencil of lines $H(ac, bd)$ by a line s on the plane of the pencil but not on the center of the pencil is the harmonic range $H(AC, BD)$, where $A = sa$, etc.</p> | <p>S2. The projector of the harmonic pencil of lines $H(ac, bd)$ from a line s on the center of the pencil but not on the plane of the pencil is the harmonic pencil of planes $H(\alpha\gamma, \beta\delta)$, where $\alpha = sa$, etc.</p> |
|--|--|

We remark that S2 is the bundle dual of S1. Finally,

- | | |
|--|---|
| <p>3. The section of the harmonic pencil of planes $H(\alpha\gamma, \beta\delta)$ by a line s which is skew to the axis of the pencil is the harmonic range $H(AC, BD)$, where $A = s\alpha$, etc.</p> | <p>S3. The projector of the harmonic range $H(AC, BD)$ from a line s which is skew to the support of the range is the harmonic pencil of planes $H(\alpha\gamma, \beta\delta)$, where $\alpha = sA$, etc.</p> |
|--|---|

The proof of 3 follows from S1 and 2, while the proof of S3 is a consequence of the principle of duality.

The combined results of the above six propositions give immediately:

Theorem III. *The projection or section of an harmonic set is an harmonic set.*

If we have n forms, $F_1, F_2, F_3, \dots, F_n$ each consisting of four elements, and if moreover

$$F_1 \overline{\wedge} F_2 \overline{\wedge} F_3 \overline{\wedge} \dots \overline{\wedge} F_n,$$

we conclude from Theorem III that if one of the forms F_i is an harmonic set then each of the others is also an harmonic set. This most important result follows from the fact that if two forms are perspective either the one is a section or projection of the other or they are both sections or projections of a third form; and hence, an harmonic relation in the one implies, by Theorem III, an harmonic relation in the other.

4.5 The Order of Elements in an Harmonic Set. In the symbol for the harmonic set $H(AC, BD)$, as defined in Art. 4.1, certain permutations of the letters are possible. In fact, as we have noted, the relation $H(AC, BD)$ implies $H(CA, BD)$, $H(AC, DB)$, and $H(CA, DB)$; by the theorem of Art. 4.3, $H(AC, BD)$ also implies $H(BD, AC)$ and hence $H(DB, AC)$, $H(BD, CA)$, and $H(DB, CA)$.

Of the twenty-four possible permutations of the four letters A, B, C, D in the symbol $H(AC, BD)$, eight represent harmonic sets. We observe that these eight permutations are those in which A is paired with C and B is paired with D , while in the remaining sixteen permutations A is paired with B (or D) and C is paired with D (or B).

To exhibit the equality of points A and C geometrically we project the harmonic range $H(AC, BD)$, for example, from S (Fig. 18) by the harmonic pencil $H(ac, bd)$. The harmonic pencil $H(ac, bd)$ is cut by m in the harmonic range $H(KM, OD)$ which, in turn, is projected on k through N in the harmonic range $H(CA, BD)$. Hence $H(AC, BD)$ implies $H(CA, BD)$, etc.

The dual statements should be considered by the reader.

4.6 Harmonically Separated Unlike Pairs.

DEFINITION. Two lines b, d are said to **separate harmonically** the two points A, C if the line AC cuts b and d , respectively, at two points B and D such that $H(AC, BD)$. We may then write $H(AC, bd)$.

DEFINITION. Two points B, D are said to **separate harmonically** the two lines a, c if the point ac is joined to B and D , respectively, by two lines b and d such that $H(ac, bd)$. We may then write $H(ac, BD)$.

With these definitions of harmonically separated pairs which consist of unlike elements the following propositions are evident:

Two diagonal points of a complete quadrangle are harmonically separated by the two sides on the third diagonal point.

Two diagonal lines of a complete quadrilateral are harmonically separated by the two vertices on the third diagonal line.

4.7 Problem. *Two given lines a and b are coplanar but their common point O is not on the paper. We wish to draw the line PO , where P is any given point in the plane of a and b .*

Through P draw two lines AB and A_1B_1 cutting a at A and A_1 , and b at B and B_1 . Let AB_1 and A_1B meet at R . Then the complete quadrangle AA_1BB_1 shows that P, R are harmonically separated by a and b , by the proposition of Art. 4.6. Through R now draw a third line cutting a at A_2 and b at B_2 and let A_1B_2 and A_2B meet at S . The complete quadrangle $A_1A_2B_2B$ exhibits R and S as harmonically separated by a and b . Hence P and S both lie on the harmonic conjugate of OR with respect to a and b . Therefore PS goes through O and is the required line.

EXERCISES

1. Given three points A, B, C of a point-row, it is required to construct the harmonic conjugate of A with respect to B and C , i.e., construct $A' = H(A, BC)$; construct also $B' = H(B, CA)$ and $C' = H(C, AB)$.
2. Write the plane dual of Ex. 1, and solve.
3. A complete quadrangle $ABCD$ has a diagonal triangle XYZ . Show that the six additional points common to the sides of the diagonal triangle and the complete quadrangle are the vertices of a complete quadrilateral. Dualize.
4. Prove that an harmonic pencil of lines is cut by any line on its plane but not on its center in an harmonic range of points.
5. Dualize in the plane the statement and the construction of the problem of Art. 4.7. (Cf. Exs. 5 and 6, Chapter III.)
6. A line turns about a fixed point P and intersects two fixed lines a and b , which are coplanar with it, at the variable points A and B respectively. What is the locus of the point $X = H(P, AB)$?
7. Show that the locus of the point X of Ex. 6 is the polar line of P with respect to the lines a, b . (Cf. Ex. 7, Chapter III.)
8. Discuss the plane duals of Exs. 6 and 7.
9. If line p is the polar of point P and line q is the polar of point Q with respect to two coplanar lines a and b , show that when P lies on q then Q lies on p .
10. A straight line intersects the sides of a triangle ABC at the points A_1, B_1, C_1 , and the harmonic conjugate, with respect to the two vertices on the same side, of each of these points is constructed; for example, $A_2 = H(A_1, BC)$, etc. Show that A_1, B_2, C_2 are collinear points, as are A_2, B_1, C_2 and A_2, B_2, C_1 . Show also that AA_2, BB_2, CC_2 are concurrent lines, as are AA_2, BB_1, CC_1 and AA_1, BB_2, CC_1 and AA_1, BB_1, CC_2 .

CHAPTER V

METRIC PROPERTIES — THE DOUBLE RATIO

5.1 Projective and Metric Properties. In Art. 1.6 we characterized projective geometry as the geometry which studies those properties of space which are invariant (i.e., unchanged) under all projections. Let us be more explicit.

DEFINITION. A **projective property** of a figure is a property which is preserved by all projections.

From our study of the more familiar geometry of Euclid, we are acquainted with many properties of figures which are not projective. For example, the distance between two points and the size of an angle are not invariant under projections. On the other hand, the reader will recognize that these properties are properties which are preserved when the figure is subjected to any rigid motion, i.e., when the figure is subjected to a change in position only. They are based upon the concept of measure and are called **metric properties**.

Metric properties are not invariant under projections. Projective properties, however, are invariant under rigid motions. For example, such projective properties as concurrence of lines and collinearity of points are evidently preserved by all rigid motions. What then is the distinction between projective and metric properties?

DEFINITION. A **metric property** is a property which is preserved by all rigid motions but is not preserved by all projections.

Thus, a theorem is a projective theorem if it is concerned with projective properties only; it is a metric theorem if it is concerned with metric properties alone or with metric properties and projective properties together.

Euclidean geometry, excluding that part devoted to the theory of similar figures, is strictly a metric geometry. Moreover, from one point of view it is a special case of projective geometry; and in order to be able later to show precisely in what way it is a special case of projective geometry we introduce the reader, in this chapter, to certain projective properties which are based upon the idea

of distance, a metric concept. He will find Chapters X and XV to be somewhat similar as to purpose.

It should be emphasized, however, that projective geometry can be developed without any reference whatsoever to metric considerations and was so developed by von Staudt in his *Geometrie der Lage*. On the other hand, it can be developed entirely from metric considerations such as we are about to introduce. The classical work of this kind is Chasles' *Traité de géométrie supérieure*.

5.2 Directed Segments. According to the discussion of Art. 1.8 two points A, B of a line u determine on u two segments AB and $B.A$. The points and line here referred to are projective points and projective line, of course; for we agreed that these terms were to be construed in the projective sense unless explicitly stated otherwise.

If now we assign to the ideal elements a special significance whereby they are to be distinguished from the ordinary elements we are, strictly speaking, no longer in the domain of projective geometry. There will then be two kinds of points, ordinary and ideal, and there will be two kinds of lines, the augmented Euclidean or ordinary lines and the ideal line; and it will be necessary in each instance to state which kind of element is intended. Such a specialization of ideal elements is basic to the developments of this chapter and must be kept in mind. Unless the context clearly indicates otherwise we shall adhere to the convention of using the terms "point" and "line" when unqualified in the strict projective sense.

With this specialization of the ideal point of an ordinary line u as an exceptional point on the line we see that two ordinary points A, B determine on such a line u two segments. One of these segments does not contain this exceptional point. We shall indicate it by AB , and, in this chapter, we shall refer to the number representing the distance between A and B as the **magnitude** of this segment AB .

As in Art. 1.8 there are on every such segment AB (as well as on the line u) two orders or senses which are opposite to each other. We shall refer to these opposite senses as **directions**, arbitrarily choosing one as positive and the other as negative.

DEFINITION. A **directed segment** is a portion of an ordinary line possessing the attributes of magnitude *and* direction.

Hence, two ordinary points A, B of a line u determine on u two directed segments, viz., the directed segment AB , whose direction is the order or sense from A to B , and the directed seg-

ment BA , which has the *same* magnitude as AB but the opposite direction, i.e., the order or sense is from B to A . These two directed segments are connected by the relation

$$AB = -BA,$$

or

$$AB + BA = 0.$$

Throughout this chapter the term segment will always mean *directed segment* unless the contrary is so stated.

Three ordinary points A, B, C on a line u determine three consecutive directed segments AB, BC, CA , which are connected by the relation

$$(1) \quad AB + BC + CA = 0.$$

This is an *identity* connecting the three segments, for its validity does not depend upon the relative positions of the three collinear points.

As a special case of the identity in the last paragraph we have, if O is a fixed point or origin on the line u and A, B the end points of a segment AB on the same line,

$$(1') \quad AB = OB - OA.$$

This follows immediately from (1) upon replacing therein C by O and rewriting as indicated. It is, as will be seen, a very useful relation in that it enables us to express a segment AB in terms of segments measured from a common origin O .

5.3 The Double Ratio of Four Points. If an ordinary point C is collinear with and distinct from the end points of a segment AB , we define the **ratio of division** of the segment AB by the point C to be AC/CB .

This ratio is a positive number when C lies between, and a negative number when C does not lie between, the points A and B .

Two pairs of ordinary points A, B and C, D which are distinct and collinear determine the two ratios AC/CB and AD/DB . They are the ratios in which the segment AB , whose end points are those of the first pair, is divided by the point C and the point D , respectively, of the second pair.

We define the ratio of these ratios

$$\frac{AC/AD}{CB/DB}$$

to be a **double ratio** or **cross ratio** of the four points A, B, C, D .

Taking into consideration the pairing of the four points, we symbolize this double ratio by (AB, CD) .

The double ratio

$$(2) \quad (AB, CD) = \frac{AC}{CB} / \frac{AD}{DB} = \frac{AC \cdot BD}{AD \cdot BC}$$

is a number; positive if C and D both lie on the segment AB or both lie outside the segment AB , and negative if C (or D) lies on, while D (or C) lies outside, the segment AB . In other words, the double ratio (AB, CD) is positive if the pair A, B is not separated by the pair C, D , and negative if the pair A, B is separated by the pair C, D .

It is well to note that the right member of (2) above is obtained directly from the symbol (AB, CD) by writing the letters A and B of the first pair in the *first* and *third* places, respectively, of both numerator and denominator; and then writing the letters C and D of the second pair, respectively, in the *second* and *fourth* places of the numerator and the *fourth* and *second* places of the denominator. This rule of thumb procedure is convenient and, if used, will prevent confusion later on, although the underlying definition of the double ratio must not be neglected.

5.4 Projective Character of the Double Ratio. THEOREM 1. *If the four ordinary points A, B, C, D on a line u are projected through a center S into the four ordinary points A', B', C', D' on a line u' , then $(AB, CD) = (A'B', C'D')$.*

Consider the triangles ASJ and $SA'I'$ (Fig. 19), where J is the projection on u of J' , the ideal point of u' , and I' is the projection on u' of I , the ideal point of u . The angles SAJ and $A'SI'$, ASJ and $SA'I'$ are equal since SI' is parallel to u and SJ is parallel to u' . Therefore, the triangles ASJ and $SA'I'$ are similar and

$$\frac{AJ}{JS} = \frac{SI'}{I'A'}$$

or
$$I'A' = \frac{JS \cdot SI'}{AJ}.$$

In like manner, from the similar triangles CSJ and $SC'I'$, we find

$$I'C' = \frac{JS \cdot SI'}{CJ}.$$

Now $JS \cdot SI'$ being a constant, say k , for this particular position of S , we can write

$$I'A' = \frac{k}{AJ}$$

and

$$I'C' = \frac{k}{CJ},$$

from which, making use of equation (1'), we have

$$A'C' = I'C' - I'A' = \frac{k(AJ - CJ)}{AJ \cdot CJ}$$

or

$$A'C' = \frac{k \cdot AC}{AJ \cdot CJ}.$$

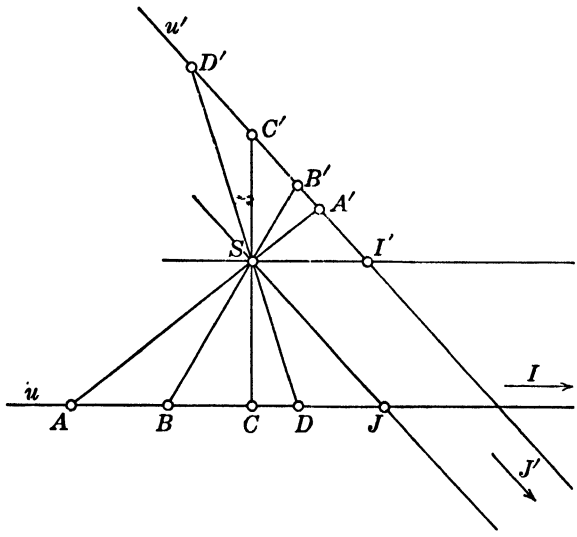


FIG. 19

Similar calculations give

$$B'D' = \frac{k \cdot BD}{BJ \cdot DJ},$$

$$A'D' = \frac{k \cdot AD}{AJ \cdot DJ},$$

$$B'C' = \frac{k \cdot BC}{BJ \cdot CJ}.$$

The product of the first two of these four relations divided by the product of the last two results in

$$\frac{A'C' \cdot B'D'}{A'D' \cdot B'C'} = \frac{AC \cdot BD}{AD \cdot BC}$$

or $(A'B', C'D') = (AB, CD)$.

This last equation establishes a *numerical* equality only. For the relations from which it was derived, such as

$$A'C' = \frac{k \cdot AC}{AJ \cdot CJ},$$

do not involve directed segments since the points A', C', A, C , etc., are not collinear. We have yet to show that the two members are equal in sign as well as in magnitude. That such is the case, however, is a consequence of the fact that the projection on u' of the pairs A, B and C, D consists of two pairs A', B' and C', D' which do or do not separate each other according as A, B and C, D do or do not separate each other. For example, if the pair A, B is separated by the pair C, D , then (AB, CD) is negative and so also is $(A'B', C'D')$. Hence, in the case under consideration, i.e., when the four points and their projections are ordinary points, we have established the *algebraic* equality of the double ratios, and proved the theorem.

From the above equations we may, by direct substitution, also establish the relation

$$\frac{A'C' \cdot B'I'}{A'I' \cdot B'C'} = (A'B', C'I') = \frac{AC}{BC}.$$

The four points A', B', C', I' are ordinary points of line u' ; they are in fact the projections on u' of the ordinary points A, B, C and the ideal point I of line u . Now, since the definition of double ratio in Art. 5.3 is in terms of ordinary points and accordingly attaches no meaning to the symbol (AB, CI) we proceed to give a meaning to this symbol. That is, we state the

DEFINITION. The **double ratio** (AB, CI) of the three ordinary points A, B, C and the ideal point of I of a line u is the single ratio AC/BC .

In case two, and therefore all, of the four points A, B, C, D are ideal* points, an additional definition is necessary.

DEFINITION. The **double ratio** (AB, CD) of the ideal points A, B, C, D is the double ratio $(A'B', C'D')$ of any¹ four ordinary points A', B', C', D' into which the four ideal points may be projected.

The purpose of the last two definitions, as the reader may surmise, is to enable us to exhibit the double ratio as a projective property of four points. And it cannot, of course, be a projective property if restricted to ordinary (i.e., Euclidean) points only. With these additional definitions the double ratio has meaning for any four projective points.

Consequently, by Theorem 1 and the definitions of this article, we have

THEOREM 2. *The double ratio of four points on a line is invariant under all projections.*

The reader will note that the terms of this theorem are to be construed in the projective sense and not in the augmented Euclidean sense of the earlier part of this chapter.

5.5 The Double Ratio of Four Lines. The projectors of a point-row $u(ABCD \text{ ---})$ from a point S not on u is a flat pencil $S(abcd \text{ ---})$, where $a = SA, b = SB$, etc. Conversely, or dually, the section of a flat pencil S by a line u not on S is a point-row u . In each case

$$u(ABCD \text{ ---}) \overline{\wedge} S(abcd \text{ ---}).$$

If $u(ABCD \text{ ---})$ and $u'(A'B'C'D' \text{ ---})$ are both sections of a flat pencil $S(abcd \text{ ---})$ then

$$u \overline{\wedge} S \overline{\wedge} u',$$

and since, by Theorem 2, Art 5.4, the double ratio of four points is invariant under projections we have, e.g.,

$$(AB, CD) = (A'B', C'D').$$

These considerations lead us naturally to the

DEFINITION. The **double ratio** of four concurrent lines $S(abcd)$ is the double ratio of the four collinear points $u(ABCD)$ in which any line u cuts $S(abcd)$. Symbolically,

$$(ab, cd) = (AB, CD).$$

¹ If the ideal points are projected into A', B', C', D' by one center of projection S' , and into A'', B'', C'', D'' by a second center S'' , it becomes evident, by a superposition of the flat pencils S' and S'' and by Theorem 1, that $(A'B', C'D') = (A''B'', C''D'')$.

The definition just given is applicable to any four concurrent lines. They may be ordinary lines with a common ordinary point of intersection or they may be ordinary lines with an ideal point of intersection; also, one of the lines may be the ideal line of the plane, in which case S is again an ideal point. The terms of the definition are projective.

In the case when S is an ordinary point, and consequently none of the lines is an ideal line, it is possible to express the double ratio (ab, cd) in terms of the angles which the lines form with one another.

With this in mind we first remark that, as in the analogous discussion of Art 5.2, two lines a, b on a point S determine on S two complete angles ab and $b.a$. Now in speaking of the measure of an angle between two lines it is, of course, highly important that there be no ambiguity as to what is meant. That is, we must, by some convention, agree as to the meaning of the "angle" formed by two lines. Toward this end then we arbitrarily choose, in the plane of S but not on S , a line u which meets a, b, c, \dots at A, B, C, \dots respectively; and by the angle between lines a and b we shall mean the angle ASB . Furthermore, the **directed angle** (ab) is defined to be the angle ASB measured from AS to SB . We shall choose arbitrarily one of the two senses of rotation about S as positive, and the other as negative.

Thus we write

$$(ab) + (ba) = 0,$$

and if c is a third line on S ,

$$(ab) + (bc) + (ca) = 0.$$

Taking h as the length of the perpendicular from S to line u , the following relations are at once evident from Fig. 20:

$$h \cdot AC = AS \cdot SC \cdot \sin ASC,$$

$$h \cdot BD = BS \cdot SD \cdot \sin BSD,$$

$$h \cdot AD = AS \cdot SD \cdot \sin ASD,$$

$$h \cdot BC = BS \cdot SC \cdot \sin BSC;$$

and they give rise to

$$\frac{AC \cdot BD}{AD \cdot BC} = \frac{\sin ASC \cdot \sin BSD}{\sin ASD \cdot \sin BSC}.$$

That this is a *numerical* equality only is of course evident from the fact that its derivation depends upon the quantities h , AS , SC , etc., which are necessarily undirected.

Since, however, the ratios AC/CB and $\sin(ac)/\sin(cb)$ are both positive if C lies between A and B , and both negative if C does not lie between A and B , and similarly for AD/DB and $\sin(ad)/\sin(db)$,

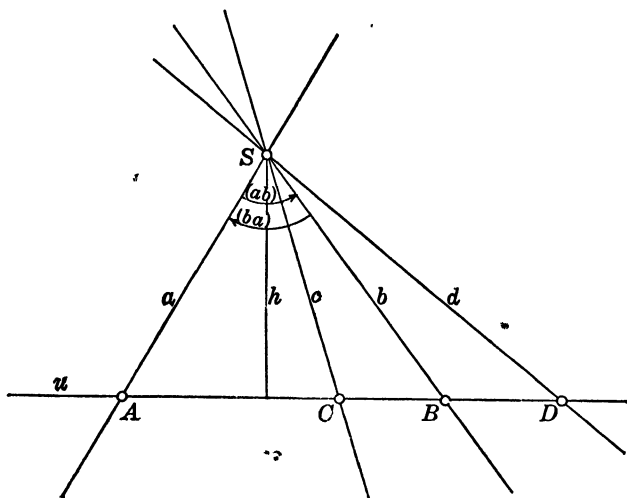


FIG. 20

we deduce from the numerical equality immediately above the *algebraical* equality

$$(AB, CD) = \frac{AC \cdot BD}{AD \cdot BC} = \frac{\sin(ac) \cdot \sin(bd)}{\sin(ad) \cdot \sin(bc)}.$$

Hence, by definition and the above argument, the double ratio of four lines on an ordinary point S :

$$(ab, cd) = \frac{\sin(ac) \cdot \sin(bd)}{\sin(ad) \cdot \sin(bc)}.$$

5.6 Applications. If I is the ideal point of a line u and F , G , H are three ordinary points on u , we have defined their double ratio (FG, HI) to be FH/GH . We shall use this to solve two important problems.

1. *Given three points A , B , C on a line u and a number t , to construct the point D such that $(AB, CD) = t$.*

If the three given points are ordinary points, we draw through C any line u' and take on u' two points A' and B' such that

$A'C/B'C = t$. If t is positive, we take A' and B' on the same side of C (Fig. 21a); and if t is negative, we take A' and B' on opposite sides of C (Fig. 21b). Now let AA' and BB' meet at a point S . The line through S parallel to u' cuts u at the required point D . For if I' is the ideal point of u' , we have A, B, C, D on u projected from S into A', B', C, I' of u' . But, by Theorem 2, Art. 5.4,

$$(AB, CD) = (A'B', CI') = \frac{A'C}{B'C} = t,$$

which proves the construction.

It is not necessary that line u' be drawn through C , the partner of the required point D , as in the above construction; the line u' may be drawn through any one of the three given points with but

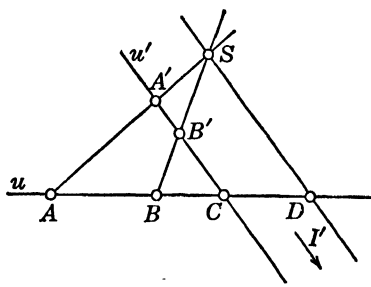


FIG. 21a

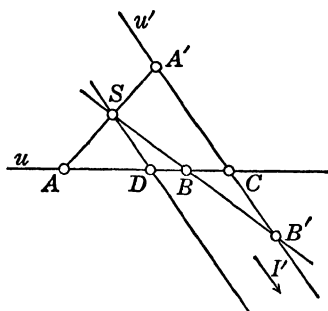


FIG. 21b

slight modification of the details, and it is suggested that the reader examine these variations of the construction.

That the point D thus found is the only point satisfying the requirements follows from the fact that if there were a second such point, say D_1 , we would have

$$(AB, CD) = t = (AB, CD_1)$$

or

$$\frac{AC \cdot BD}{AD \cdot BC} = \frac{AC \cdot BD_1}{AD_1 \cdot BC},$$

whence

$$\frac{BD}{AD} = \frac{BD_1}{AD_1};$$

and, since D and D_1 divide the segment BA in equal ratios, D and D_1 must coincide.

If one or three of the given points are ideal, they may be projected into ordinary points and the above construction carried

through. The fourth point, when found, is then the projection of the required point D .

2. If A, B, C, D are four collinear ordinary points, to prove that

$$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0$$

is an identity.

Project A, B, C, D on line u from a center S onto a line u' , so chosen as to be parallel to SD . Let A', B', C' be the projections of A, B, C respectively; the projection of D will be D' , the ideal point of u' . Now, by (1) of Art. 5.2,

$$A'B' + B'C' + C'A' = 0,$$

whence, dividing by $A'C'$,

$$\frac{B'A'}{C'A'} + \frac{B'C'}{A'C'} - 1 = 0.$$

Since D' is the ideal point on u' , this last equation may be translated into double ratios as follows:

$$(B'C', A'D') + (B'A', C'D') - 1 = 0.$$

But a double ratio is invariant under projections, hence

$$(BC, AD) + (BA, CD) - 1 = 0,$$

or

$$\frac{BA \cdot CD}{BD \cdot CA} + \frac{BC \cdot AD}{BD \cdot AC} - 1 = 0,$$

from which, upon multiplying by $-AC \cdot DB$, we have

$$(3) \quad AB \cdot CD + AC \cdot DB + AD \cdot BC = 0.$$

As an application of the theory of Art. 5.5, we submit the following:

3. If a, b, c, d are four lines having in common one ordinary point S , to prove that

$$\sin(ab) \cdot \sin(cd) + \sin(ac) \cdot \sin(db) + \sin(ad) \cdot \sin(bc) = 0$$

is an identity.

Let the four concurrent lines a, b, c, d meet a fifth line u not on S at A, B, C, D . Between the segments formed by these four points we have the identity (3) above, which may be written in the form

$$\frac{AB \cdot CD}{AD \cdot CB} + \frac{AC \cdot BD}{AD \cdot BC} - 1 = 0,$$

i.e.,
$$(AC, BD) + (AB, CD) - 1 = 0.$$

Now, by definition,

$$(ab, cd) = (AB, CD),$$

whence

$$\widetilde{(ac, bd)} + (ab, cd) - 1 = 0,$$

or

$$\frac{\sin(ab) \cdot \sin(cd)}{\sin(ad) \cdot \sin(cb)} + \frac{\sin(ac) \cdot \sin(bd)}{\sin(ad) \cdot \sin(bc)} - 1 = 0.$$

Multiplying this last by $-\sin(ad) \cdot \sin(bc)$, we now have

$$(4) \sin(ab) \cdot \sin(cd) + \sin(ac) \cdot \sin(db) + \sin(ad) \cdot \sin(bc) = 0.$$

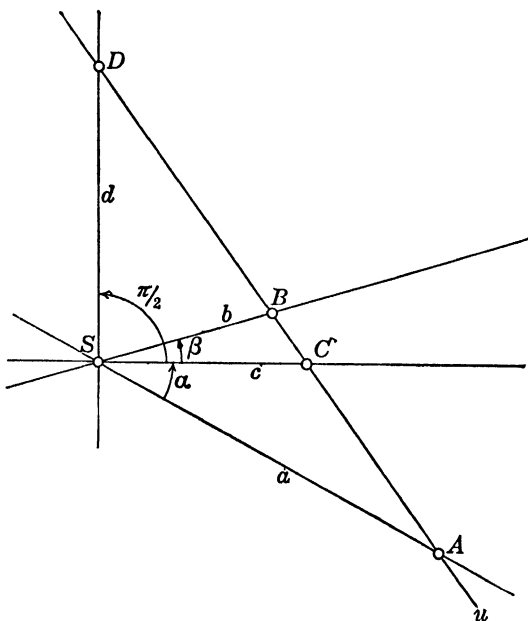


FIG. 22

Fig. 22 is a particular arrangement of four concurrent lines for which

$$(ac) = \alpha$$

$$(cb) = \beta$$

$$(ab) = \alpha + \beta$$

$$(cd) = \frac{\pi}{2}$$

$$(ad) = (ac) + (cd) = \alpha + \frac{\pi}{2}$$

$$(bd) = (bc) + (cd) = -\beta + \frac{\pi}{2}.$$

For this special case, equation (4) reduces to

$$\sin(\alpha + \beta) + \sin \alpha \sin\left(\beta - \frac{\pi}{2}\right) + \sin\left(\alpha + \frac{\pi}{2}\right) \sin(-\beta) = 0,$$

or

$$(5) \quad \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

Writing $\left(\frac{\pi}{2} - \alpha\right)$ for α and $(-\beta)$ for β in (5) gives

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

Summarizing the derivation of these trigonometric formulas (from which all the others follow) we see that they are special cases of (4) which was derived from (3) with the aid of the theory of double ratios. And by the same theory, (3) was derived from the relation

$$(1) \quad AB + BC + CA = 0,$$

an identity for three collinear ordinary points.

5.7 The Six Double Ratios. Four collinear points A, B, C, D (and, similarly, four concurrent lines) give rise to 24 double ratios, one for each permutation of the four letters in the symbol (AB, CD) . These 24 double ratios, however, are not all distinct but fall into six groups of four each, as follows:

$$\begin{aligned} (AB, CD) &= (BA, DC) = (CD, AB) = (DC, BA) \\ (AB, DC) &= (BA, CD) = (CD, BA) = (DC, AB) \\ (AC, BD) &= (BD, AC) = (CA, DB) = (DB, CA) \\ (AC, DB) &= (BD, CA) = (CA, BD) = (DB, AC) \\ (AD, CB) &= (BC, DA) = (CB, AD) = (DA, BC) \\ (AD, BC) &= (BC, AD) = (CB, DA) = (DA, CB). \end{aligned}$$

The four double ratios of each group have equal values by the following:

THEOREM 1. *The value of the double ratio remains unchanged if any two of the four letters are interchanged simultaneously with the other two.*

For assuming, as we may without loss of generality, that the four points are ordinary points, we have

$$(AB, CD) = \frac{AC \cdot BD}{AD \cdot BC} = \frac{BD \cdot AC}{BC \cdot AD} = (BA, DC),$$

and similar relations follow for the other double ratios.

THEOREM 2. *If the two letters of either pair of a double ratio are interchanged, the value of the double ratio is changed to its reciprocal.*

If $(AB, CD) = t$, we have, by definition,

$$t = (AB, CD) = \frac{AC}{CB} / \frac{AD}{DB},$$

and the interchange of C and D , for example, inverts the right member, giving

$$\frac{AD}{DB} / \frac{AC}{CB} = (AB, DC) = \frac{1}{t},$$

and similarly if A and B are interchanged.

THEOREM 3. *If the double ratio $(AB, CD) = t$, then the double ratio $(AC, BD) = 1 - t$.*

For upon dividing equation (3) by $AD \cdot BC$ we have

$$-(AC, BD) - (AB, CD) + 1 = 0,$$

and therefore,

$$(AC, BD) = 1 - t.$$

To summarize, if

$$(AB, CD) = t,$$

we have, by Theorem 2,

$$(AB, DC) = \frac{1}{t},$$

and, by Theorem 3,

$$(AC, BD) = 1 - t.$$

From this last and Theorem 2,

$$(AC, DB) = \frac{1}{1 - t},$$

which leads, because of Theorem 3, to

$$(AD, CB) = 1 - \frac{1}{1 - t} = \frac{t}{t - 1}$$

and this, in turn, by Theorem 2, gives

$$(AD, BC) = \frac{t - 1}{t}.$$

In concluding, we note that the values of all the double ratios of four points can be found if the value of one of their double ratios is known.

5.8 Coincident Points. If the four points A, B, C, D are not all distinct, we define the value of their double ratio (AB, CD) as follows:

(1) When A coincides with B

$$(AB, CD) = (AA, CD) = 1.$$

(2) When A coincides with C

$$(AB, CD) = (AB, AD) = 0.$$

(3) When A coincides with D

$$(AB, CD) = (AB, CA) = \infty.^2$$

Other cases of the coincidence of two of the points can be handled in a similar manner.

Conversely, if a double ratio has the value 1, 0, or ∞ , two of the four points are coincident. For example, if

$$(AB, CD) = \frac{AC \cdot BD}{AD \cdot BC} = 1,$$

then either

$$\frac{CA}{AD} = \frac{CB}{BD},$$

or

$$\frac{AC}{CB} = \frac{AD}{DB};$$

which indicates that either A and B divide the segment CD in equal ratios, and hence A coincides with B , or that C and D divide the segment AB in the same ratio, and hence C coincides with D . Similarly, if the double ratio has the value 0 (or ∞), the point A coincides with C (or D) or the point B coincides with D (or C).

5.9 Metric Properties of an Harmonic Range. The complete quadrangle $ABCD$ of Fig. 23 whose diagonal triangle XYZ has two ideal vertices (X and Z) is a parallelogram. By definition of an harmonic range of points (Art. 4.1), we have on the ideal line the harmonic set $H(ZX, JI)$, which projects through D into $H(AC, YI)$ on the line AC . But Y is the intersection of the

² The symbol ∞ , as used here, may be regarded as a convention whereby the equation $(AB, CA) = \infty$ is simply an abbreviation for the statement that "as A approaches D the value of (AB, CD) increases without limit."

diagonals AC and BD of the parallelogram and is, as we know from Euclid, the mid-point of the segment AC . Hence,

THEOREM 1. *The harmonic conjugate of the ideal point of a line with respect to two ordinary points A, C of that line is the mid-point of the segment AC .*

In general, any harmonic range of points $H(AC, BD)$ on a line u is projected through a center S into an harmonic range $H(A'C', B'D')$ on another line u' . If u' is parallel to SD , then D' is the ideal

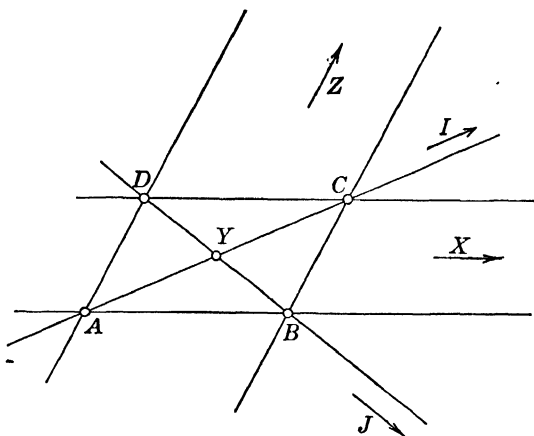


FIG. 23

point of u' and B' is the mid-point of $A'C'$. Consequently,

$$(AC, BD) = (A'C', B'D') = \frac{A'B'}{C'B'} = -1;$$

also

$$(AC, DB) = -1,$$

$$(AB, CD) = (AD, CB) = 2,$$

and

$$(AB, DC) = (AD, BC) = \frac{1}{2},$$

by the theorems of Art. 5.7. This proves

THEOREM 2. *The double ratio of four points of an harmonic range has the value -1 , or 2 , or $\frac{1}{2}$; and conversely, if the double ratio of four points is -1 , 2 , or $\frac{1}{2}$, the four points properly paired constitute an harmonic range.*

We leave the proof of the converse to the reader.

Since the double ratio of the harmonic range of ordinary points $H(AC, BD)$ is -1 , we have

$$(AC, BD) = \frac{AB \cdot CD}{AD \cdot CB} = -1;$$

in other words, either

$$\frac{AB}{BC} = -\frac{AD}{DC}$$

or

$$\frac{BA}{AD} = -\frac{BC}{CD}.$$

Thus we have

THEOREM 3 *In the harmonic range of ordinary points $H(AC, BD)$ the ratios of division of the segment AC (or BD) by B and D (or by A and C) are numerically equal but opposite in sign, and conversely.*

Again; if the ordinary points A, B, C, D constitute an harmonic range $H(AC, BD)$, we have, by Theorem 2 above,

$$\frac{AB \cdot CD}{AD \cdot CB} = -1$$

This, by (1') of Art. 5.2, may be rewritten as

$$AB(AD - AC) = -AD(AB - AC),$$

and then reduced to

$$\frac{2}{AC} = \frac{1}{AB} + \frac{1}{AD}.$$

This last, if put in the form

$$\frac{1}{AC} - \frac{1}{AB} = \frac{1}{AD} - \frac{1}{AC},$$

exhibits the sequence

$$\frac{1}{AB}, \frac{1}{AC}, \frac{1}{AD}$$

as an arithmetic progression, and, consequently, the sequence of reciprocals

$$AB, AC, AD$$

is an harmonic progression.³ This result, which is of historical interest in that it shows the probable origin of the term "harmonic," we state as

THEOREM 4. *In the harmonic range of ordinary points $H(AC, BD)$ the three segments AB, AC, AD are in harmonic progression, and conversely.*

³ An harmonic progression is defined as a progression whose terms are the reciprocals of an arithmetic progression.

Returning once more to Theorem 2 and taking M as the mid-point of segment BD of the harmonic range $H(AC, BD)$, we have

$$\frac{AB \cdot CD}{AD \cdot CB} = -1$$

and, by (1') of Art. 5.2,

$$\frac{(MB - MA)(MD - MC)}{(MD - MA)(MB - MC)} = -1,$$

which, since $BM = MD$, reduces to

$$MA \cdot MC = MB^2,$$

and we have

THEOREM 5. *If given $H(AC, BD)$, an harmonic range of ordinary points, and M the mid-point of BD , then $MA \cdot MC = MB^2$, and conversely.*

To continue, any circle (O) passing through the pair of points A, C of the harmonic range $H(AC, BD)$, as in Fig. 24, has from M a

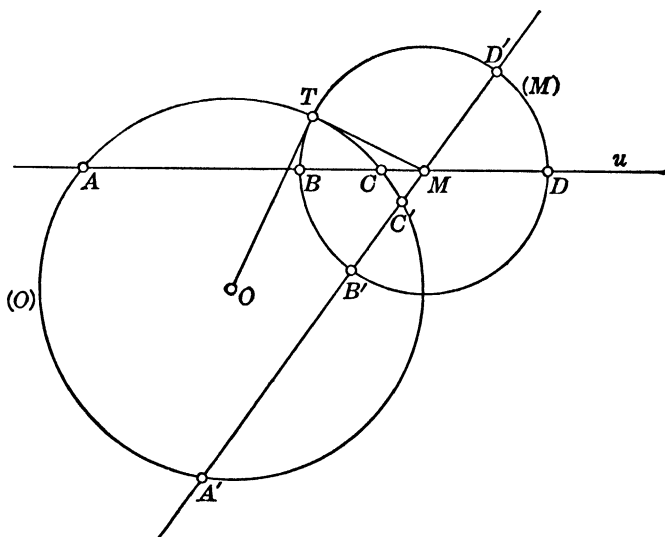


FIG. 24

tangent MT ; for the product $MA \cdot MC$ is positive ($= MB^2$), and hence M lies outside the segment AC and outside the circle (O).

Moreover,⁴

$$MT^2 = MA \cdot MC = MB^2,$$

⁴ Cf. Euclid III, Props. 35, 36, pp. 107, 110, Everyman's edition.

and the undirected segments MT and MB are equal. Also, radius OT is perpendicular to tangent MT and, consequently, the circle (M) on BD as diameter is orthogonal to (O), any circle through A and C . We state more precisely

THEOREM 6. *A circle passing through one conjugate pair of an harmonic range of ordinary points is cut orthogonally by the circle whose diameter is the segment determined by the other conjugate pair.*

Finally, on a line through M which cuts (O) at A', C' and (M) at B', D' we have

$$MA' \cdot MC' = MT^2 = MB^2 = MB'^2,$$

from which, by Theorem 5, follows the relation $H(A'C', B'D')$, and thus

THEOREM 7. *Two orthogonal circles are cut by any line through the center of one of them in four points of an harmonic range.*

5.10 The Pair of Points Harmonically Separating Two Given Pairs. *Given two pairs of points A, C and A', C' on a line u , to construct the pair harmonically separating both A, C and A', C' .*

Case 1. When A, C and A', C' are ordinary points and do not separate each other. Construct (Fig. 25) through A and C a circle (O) and through A', C' a second circle (O') intersecting (O) at P and Q . Draw PQ cutting u at M . The so-called **power** of M with respect to (O) is

$$MA \cdot MC = MP \cdot MQ = MT^2,$$

where MT is the length of the tangent from M to (O); and the power of M with respect to (O') is

$$MA' \cdot MC' = MP \cdot MQ = MT'^2,$$

MT' being the length of the tangent from M to (O'). The powers of M with respect to (O) and (O') are equal. Hence, the circle (M) with center M and radius $MT = MT'$ cuts u at B and D , the required points. The proof of the construction lies in the fact that

$$MA \cdot MC = MB^2 = MA' \cdot MC'$$

which, by Theorem 5, Art. 5.9, implies both $H(AC, BD)$ and $H(A'C', BD)$, since M is the mid-point of BD .

Case 2. When A, C and A', C' are ordinary points and separate each other. Construct, as in Case 1, the circles (O) and (O') intersecting at P and Q . Let PQ meet u at M . In this case, however,

M lies inside each of the circles (O) and (O') , for the powers of M with respect to (O) and (O') , while equal, are negative quantities; and there is no circle analogous to (M) of Case 1. In fact, there is no real common conjugate pair when the given pairs of points separate each other.

Case 3. When A, C, A', C' are not all ordinary points. We leave the development of this construction to the student. As a hint,

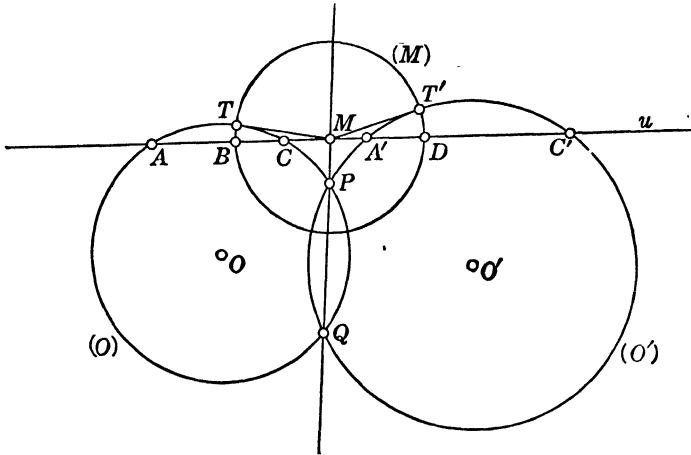


FIG. 25

however, we refer him to the last paragraph of the first application in Art. 5.6.

EXERCISES

1. If A, B, C, O are ordinary points on a line, show that the following relations are identities:

(a) $OA^2 + OB^2 = AB^2 + 2OA \cdot OB.$

(b) $OA^2 \cdot BC + OB^2 \cdot CA + OC^2 \cdot AB = -BC \cdot CA \cdot AB.$

(c) $OB \cdot OC \cdot BC + OC \cdot OA \cdot CA + OA \cdot OB \cdot AB = -BC \cdot CA \cdot AB.$

2. Given three concurrent lines a, b, c and a number t ; it is required to construct the line d such that $(ab, cd) = t.$

3. If $(AB, CP) = m$ and $(AB, CQ) = n$, show that

$$(AC, PQ) = \frac{n - 1}{m - 1}.$$

4. If ten points on a line are so related that

$$(AA', MN) = (BB', MN) = (CC', MN) = (DD', MN),$$

show that

$$(AB, CD) = (A'B', C'D').$$

5. If the ordinary points A, B are separated harmonically by the ordinary points C, D , and if M is the mid-point of AB , then

$$(a) \quad \frac{MD}{MB} = \frac{BD}{CB} = \frac{AD}{AC} = \frac{MB}{MC}$$

$$(b) \quad AC \cdot BD = CD \cdot MB.$$

6. Given the ordinary points A, B and a line l parallel to AB ; construct the mid-point of segment AB .

7. Given a segment AB and its mid-point M ; construct the line parallel to AB and passing through a given point P not collinear with A and B .

8. Given two coplanar ordinary lines a, b and an ordinary point M ; construct through M a line which cuts a, b respectively at two points A, B such that $AM = MB$; and another line through M such that $AB = BM$. Discuss.

9. The bisectors of the angles formed by two lines on an ordinary point harmonically separate the two lines forming the angles; and conversely, if two pairs of lines on an ordinary point harmonically separate each other and if one of the pairs consists of perpendicular lines, then that pair bisects the angles formed by the other pair.

10. If A is an ordinary vertex of the triangle ABC , the internal and external bisectors of the angle at A intersect the opposite side at two points harmonically separated by B and C .

11. If the triangle ABC has ordinary vertices, and if the bisector of angle A meets BC at A' and the feet of the perpendiculars from B and C onto AA' are B_1 and C_1 , respectively, then we have $(AA', B_1C_1) = -1$.

12. If triangle ABC has ordinary vertices, the harmonic conjugate of the median through A with respect to AB and AC is parallel to BC .

13. If a square $ABCD$ is inscribed in a given circle and P is any point on the circle, show that the four lines PA, PB, PC, PD when properly paired form an harmonic pencil.

14. If a square $abcd$ is circumscribed about a given circle and p is any tangent to the circle, show that the four points pa, pb, pc, pd when properly paired form an harmonic range.

15. If AB is any diameter of a circle and CD is any chord perpendicular to AB , show that the four lines projecting A, B, C, D from any point P of the circle form an harmonic pencil when properly paired.

16. If AB and CD are two diameters of a circle and if the ordinary points P, Q and R, S separate harmonically the pairs A, B and C, D , respectively, show that P, Q, R, S are concyclic, i.e., that they lie on one circle.

17. Given two pairs of lines a, c and a', c' on a point U ; it is required to construct the pair harmonically separating both a, c and a', c' . Discuss.

18. If the lines joining a point P to the vertices of a triangle ABC intersect the opposite sides at X, Y, Z , respectively, the triangles ABC and XYZ are perspective from a line p . If p intersects BC, CA, AB at X', Y', Z' , show that

$$(BC, XX') = (CA, YY') = (AB, ZZ') = -1.$$

(Cf. Ex. 8, Chapter III.)

19. If the altitudes of a triangle ABC meet the opposite sides at D, E, F , respectively, show that DE, DF are harmonically separated by DA, DB .

20. If the mid-points of the sides of triangle ABC are respectively A' , B' , C' , show that $A'C'$, $A'B'$ are harmonically separated by $A'A$, $A'C$.

21. *Ceva's Theorem.* If the points X , Y , Z on the sides, respectively, of a triangle ABC are joined to the vertices opposite by concurrent lines, then

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1,$$

the relative senses of collinear segments being taken into consideration; and conversely.

Suggestion. If the parallels to AX through B and C intersect AC and AB respectively at B' and C' , we have $(CA, YB') = (BA, ZC')$; hence,

$$\frac{CY \cdot AZ}{AY \cdot BZ} = -\frac{CX}{BX}, \text{ etc.}$$

22. *Menelaus' Theorem.* If the points X' , Y' , Z' on the sides, respectively, of a triangle ABC are collinear, then

$$\frac{BX'}{CX'} \cdot \frac{CY'}{AY'} \cdot \frac{AZ'}{BZ'} = 1,$$

the relative senses of collinear segments being taken into consideration; and conversely. (Cf. Ex. 10, Chapter IV.)

CHAPTER VI

PROJECTIVELY RELATED PRIMITIVE FORMS

6.1 Chain of Perspectivities. In this chapter, unless explicitly stated otherwise, we shall be concerned with one-dimensional primitive forms — the point-row, the flat pencil, and the axial pencil. And since any one of these three forms can be obtained from either of the other two by a projection or section, we shall, when concerned with like primitive forms, confine our discussion to point-rows. The reader should not neglect, however, to extend the argument to include the other one-dimensional forms. This he can easily do with the aid of the principle of duality.

Consider in a single plane the sequence of primitive forms $u_1, S_1, u_2, S_2, \dots, u_{n-1}, S_{n-1}, u_n$, the small letters u_i signifying point-rows and the capital letters S_k signifying flat pencils. If these forms are so related that each is perspective to the one immediately preceding it and the one immediately following it in the sequence, i.e., if

$$u_1 \overline{\wedge} S_1 \overline{\wedge} u_2 \overline{\wedge} S_2 \overline{\wedge} \dots \overline{\wedge} u_{n-1} \overline{\wedge} S_{n-1} \overline{\wedge} u_n,$$

we say that the $2n - 1$ forms constitute a **chain of perspectivities**.

Two observations are important:

1. To each element of any one of the forms there is associated (Art. 3.2) one and only one element of each of the other forms.
2. To each harmonic set in any one of the forms there is associated (Theorem III) an harmonic set in each of the other forms.

We further remark that the supports of the forms constituting the chain of perspectivities need not all be distinct. In particular the point-rows u_1 and u_n might have the same base but still be distinct point-rows, e.g., if the point-row A_1, B_1, C_1, \dots and the point-row A_n, B_n, C_n, \dots should lie on the same line while having homologous points A_1 different from A_n , B_1 different from B_n , etc. Two such point-rows are said to be **superposed**. If they are superposed and, in addition, each element of the one point-row coincides with its homologous element in the other point-row, i.e., $A_1 = A_n, B_1 = B_n$, etc., the point-rows are said to be **identical**.

6.2 Projectivity. DEFINITION. Two one-dimensional primitive forms are **projectively related** ($\bar{\wedge}$), or **projective** to one another, if they are any two members of a chain of perspectivities.

A projectivity, we observe, establishes a (1,1) correspondence between the forms.

This definition leads at once to the following propositions:

(1) If two forms F_1 and F_2 are projectively related, i.e., if $F_1 \bar{\wedge} F_2$, then to every harmonic set of elements in F_1 there corresponds an harmonic set of elements in F_2 .

(2) If $F_1 \bar{\wedge} F_2$, then $F_2 \bar{\wedge} F_1$.

(3) If $F_1 \bar{\bar{\wedge}} F_2$, then $F_1 \bar{\wedge} F_2$. (The converse is not, in general, valid.)

(4) If $F_1 \bar{\wedge} F_2$ and $F_2 \bar{\wedge} F_3$, then $F_1 \bar{\wedge} F_3$.

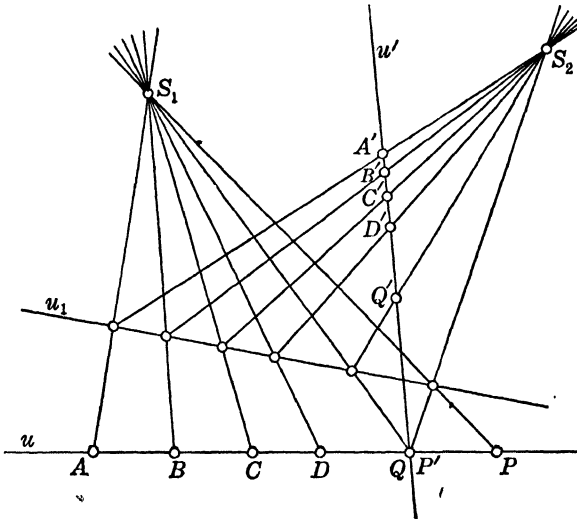


FIG. 26

The definition of a projectivity given above is Poncelet's definition.¹ It is the one we shall adopt. Von Staudt,¹ however, based his theory on the following definition, which we give here in order to provide a more comprehensive background for our future investigations.

DEFINITION. Two one-dimensional primitive forms are **projectively related** ($\bar{\wedge}$), or **projective** to one another, if their

¹ Poncelet, *Traité des propriétés projectives des figures*, Paris, 1822. Von Staudt, *Geometrie der Lage*, Nürnberg, 1847.

elements are in a (1,1) correspondence such that to every harmonic set of elements in the one there corresponds an harmonic set of elements in the other.

For our purposes, the two definitions are equivalent.

To understand more fully this concept of projectivity we include here several illustrative examples. In Fig. 26, for instance, we have

$$u(ABCD - PQ \text{ ---}) \bar{\wedge} S_1 \bar{\wedge}^{u_1} S_2 \bar{\wedge} u'(A'B'C'D' - P'Q' \text{ ---}),$$

and, therefore,

$$u(ABCD \text{ ---} ; PQ \text{ ---}) \bar{\wedge} u'(A'B'C'D' - P'Q' \text{ ---}).$$

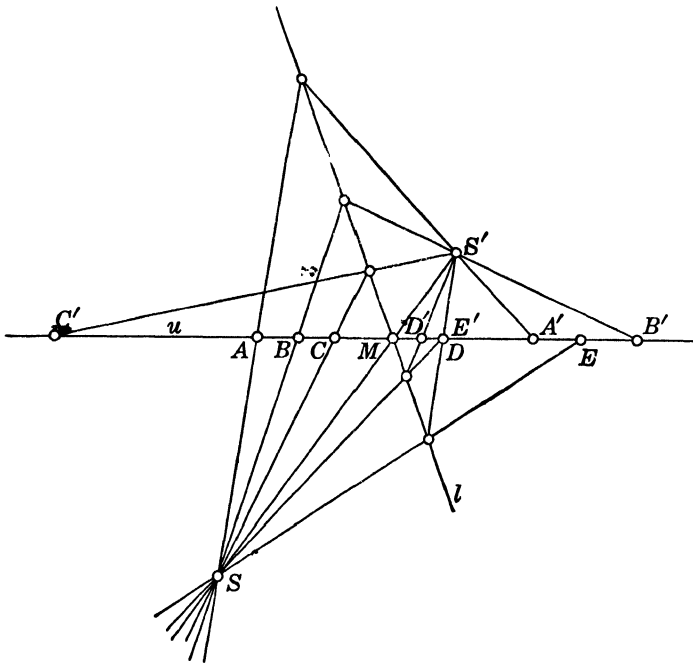


FIG. 27

The (1,1) correspondence between the point-rows u and u' is determined by the two perspective flat pencils S_1 and S_2 having u_1 as axis of perspectivity; u and u' , therefore, are the first and last members of a chain of perspectivities and are projectively related. We note in particular that the point of intersection of u and u' has two names; $uu' = Q$ when considered as a point of u , and $uu' = P'$ when considered as a point of u' . In general

Q' is a point on u' other than P' , and P is a point on u other than Q . It is possible, however, to have a projectivity between two forms such that the common element of the two forms is **self-corresponding**. We shall prove later that when such is the case the forms are not only projective but also perspective.

In Fig. 27 we have an example of two superposed point-rows which are projective, i.e.,

$$u(ABCDE \text{ ---}) \bar{\wedge} u(A'B'C'D'E' \text{ ---}).$$

Each point of u has two names; e.g., D, E' are the two labels for one and the same point. D is a point of the point-row $u(ABC \text{ ---})$

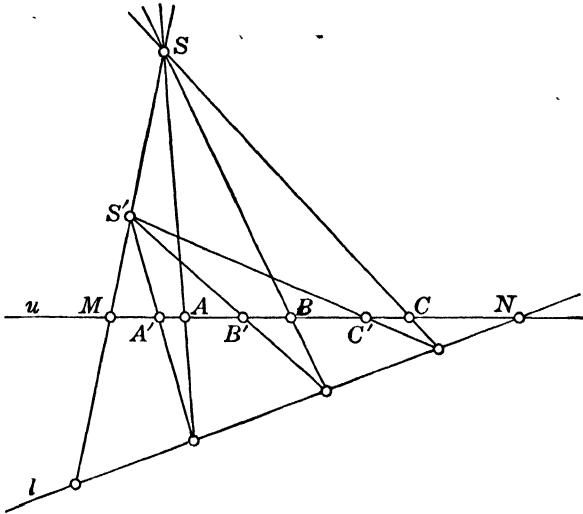


FIG. 28

and as such corresponds to D' of the other point-row; but the same point, when considered as belonging to $u(A'B'C' \text{ ---})$ is called E' and is homologous to the point E of the other point-row. When a point corresponds to itself in a correspondence between two superposed point-rows we call it a **double point**. The projectivity illustrated in Fig. 27 has M as a double point.

Fig. 28 is illustrative of a projectivity on a line which has two double points. For we have

$$u(MABCN \text{ ---}) \bar{\wedge} S(MABCN \text{ ---}) \stackrel{l}{\bar{\wedge}} S'(MA'B'C'N \text{ ---}) \bar{\wedge} u(MA'B'C'N \text{ ---}).$$

Therefore,

$$u(MABCN \text{ ---}) \bar{\wedge} u(MA'B'C'N \text{ ---}),$$

a projectivity in which A, B, C correspond respectively to A', B', C' and M, N are double points.

6.3 Projective Triads. Theorem IV. *Three points A, B, C of a line u may be projected into any three points A', B', C' of another line u' by means of two centers of perspectivity.*

The theorem is concerned with two triads, i.e., two sets of three points each, on two distinct lines which may or may not be coplanar. We wish to prove that the one set, A, B, C on u , may be projected into any given set, A', B', C' on u' .

Should one of the pairs of corresponding points, say A and A' , consist of coincident points (i.e., u and u' are coplanar and $uu' = A = A'$) the triad A, B, C may be projected into the triad A', B', C' by a single perspectivity — namely, the perspectivity having its center at the intersection of BB' and CC' .

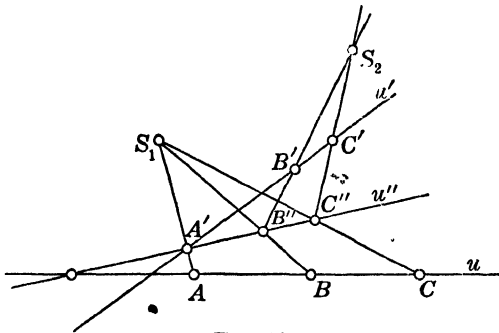


FIG. 29

If such is not the case, however, we draw (Fig. 29) through A' any third line u'' which has a point in common with u , and on u'' project the triad A, B, C from an arbitrary point S_1 on the join AA' . Let S_1B and S_1C meet u'' at B'' and C'' , respectively. Now from S_2 , the intersection of $B'B''$ and $C'C''$, project $A'B''C''$ of u'' on line u' . We have

$$ABC \bar{\wedge}^{S_1} A'B''C'' \bar{\wedge}^{S_2} A'B'C',$$

which proves the theorem.

COROLLARY 1. *Three points A, B, C of a line u may be projected into any three points A', B', C' of the same line u by means of three centers of perspectivity.*

For if A', B', C' are projected into A_1, B_1, C_1 on a line u_1 , distinct from u , by one center of perspectivity S_3 , then by Theorem IV the points A, B, C may be projected into A_1, B_1, C_1 by two centers of perspectivity.

COROLLARY 2. *The harmonic set $H(AC, BD)$ on a line u may be projected into any harmonic set $H(A'C', B'D')$ on u' .*

By Theorem IV we can always project A, B, C into A', B', C' . Hence, by Theorem III, Art. 4.4, $D = H(AC, B)$ projects into $D' = H(A'C', B')$.

The two conjugate pairs A, C and B, D of the harmonic range $H(AC, BD)$ separate each other (Art. 4.3) and, according to Corollary 2, project into two pairs A', C' and B', D' which are also conjugate pairs of an harmonic set $H(A'C', B'D')$. Hence the pairs A', C' and B', D' also separate each other.

6.4 Projective Tet-

rads. If A, B, C, D are four distinct points on a straight line u , the tetrad $ABCD$ may be projected into the tetrad $CDAB$, obtained by interchanging simultaneously two pairs of the original tetrad. For, by projecting $ABCD$ from any center M not on u

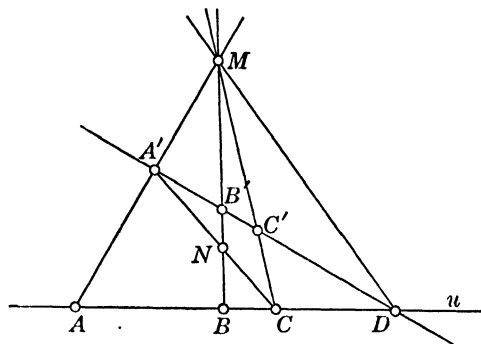


FIG. 30

and cutting the flat pencil $M(ABCD)$ by any line through D , we have, as in Fig. 30,

$$ABCD \overline{\overline{M}} A'B'C'D' \overline{\overline{C}} NB'MB \overline{\overline{A'}} CDAB.$$

Similarly, we can show that $ABCD$ may be projected into $BADC$ or $DCBA$, as well as $CDAB$. Hence, the²

THEOREM. *Any tetrad of elements on a one-dimensional primitive form may be projected into any tetrad obtained from the original by a simultaneous interchange of elements in pairs.*

6.5 Net of Rationality. Consider the sequence of collinear points $A, B, C, P_1, P_2, P_3, \dots, P_i, \dots, P_n$ constructed as follows: Beginning with three distinct points A, B, C on the line we construct P_1 , the harmonic conjugate of A with respect to B and C , then P_2 , the harmonic conjugate of B with respect to A and C , and so on. Each point P_i of the sequence is the harmonic conjugate of one of the points preceding it in the sequence with respect to

² Cf. Art. 5.4 and Theorem 1, Art. 5.7.

two other points preceding it in the sequence; for example, we might have $P_7 = H(P_1, BP_6)$. We say that any point X of the sequence is **harmonically related** to the three points A, B, C .

DEFINITION. The **net of rationality** $[ABC]$ on a line u consists of all points on u which are harmonically related to three distinct points A, B, C .

The net of rationality $[ABC]$ on a line u does *not* include all the points of u . But it has one most important characteristic, namely, that, although it does not include all the points of u , it is nevertheless a set of points which is **everywhere dense**. The meaning of the phrase "everywhere dense" is best explained by an example. If K and L are any two distinct points of the net $[ABC]$ then either

(1) on each of the segments KL and LK there are points of $[ABC]$, or

(2) on one of the segments there are no points of $[ABC]$.

The points of a net $[ABC]$ are everywhere dense if on each of the segments determined by any two of its points there are to be found other points of the net. Therefore, to show that $[ABC]$ is everywhere dense we shall show that (2) above is impossible.

If K and L are any two distinct points of the net and one of the segments, say KL , is free of points belonging to $[ABC]$, then, since the net contains points other than K and L , these other points must all lie on the segment LK , for KL contains none of them. Let X be one of the points of $[ABC]$ on LK . The harmonic conjugate of X with respect to K and L is also a point of $[ABC]$ by definition — call it Y . Hence the four points K, Y, L, X form an harmonic set $H(KL, XY)$, and the pair X, Y separates the pair K, L . But since X lies on LK the point Y lies on KL , a contradiction of the hypothesis that KL contains no point of the net, and evidence of the impossibility of (2) above.

6.6 A Theorem Concerning the Net of Rationality. We have seen that it is possible (Art. 6.3) to project any three points of one line into any three points of another, or the same, line by a finite number of centers of perspectivity; two centers are sufficient if the two sets of three points lie on distinct lines, while three centers are required if the two sets lie on the same line. A query now presents itself: Can four points of one line be projected into any four points of another or the same line? We know (Theorem IV, Corollary 2) that if the two sets of four points are each harmonic sets then it is possible to project the one into the other. On the other

hand, if one set of four points is an harmonic set and the second set of four points is not an harmonic set, then, of course, it is not possible to project the first set into the second. In general, therefore, the answer to the above question is, No.

The line of inquiry suggested in the paragraph just preceding is fundamental in the study of projective geometry. It will be developed in some detail in this and the two following articles.

Theorem V. *A projectivity between two superposed point-rows which has three double points has every point of the net of rationality defined by these three points as a double point.*

Let A , B , and C be the three double points of this projectivity. Then any point P of the net $[ABC]$ is harmonically related to A , B , C . Since a projectivity preserves the harmonic relation, which is the basis of the definition of $[ABC]$, the point P' , the homologue of P , is coincident with P . The projectivity under consideration, therefore, possesses an everywhere dense set of double points, namely, the points of $[ABC]$.

Let it be particularly noted that Theorem V is about the points of a net of rationality and not about the totality of points on the line. Concerning this totality of points an assumption is necessary.

6.7 A Fundamental Assumption Concerning a Projectivity. *A projectivity between two superposed point-rows which has three double points has every point of the line as a double point, and the point-rows are identical.*

In conjunction with Theorem V, an examination of this assumption reveals that we are simply assuming that a projectivity which leaves invariant every point of a net of rationality on a line (which is an everywhere dense set of points) leaves every point of the line invariant.

6.8 The Fundamental Theorem of Projective Geometry. *If A, B, C, D are any four points of one point-row and A', B', C' are any three points of another or the same point-row, and if by one projectivity $ABCD \bar{\wedge} A'B'C'D'$ and by another projectivity $ABCD \bar{\wedge} A'B'C'D'_1$, then we have $D' = D'_1$.*

A projectivity, by definition, is the resultant of a sequence of perspectivities; and, we know, the resultant of two projectivities is a projectivity. Hence, since

$$ABCD \bar{\wedge} A'B'C'D'$$

and

$$ABCD \bar{\wedge} A'B'C'D'_1,$$

we have

$$A'B'C'D' \bar{\wedge} A'B'C'D'_1.$$

But this last projectivity is a projectivity between two superposed point-rows having three double points. Therefore, by Art. 6.7, the point-rows are identical and, hence, $D' = D'_1$.

More generally, with the aid of duality, the theorem may be stated as follows:

Theorem VI. *A projectivity between two one-dimensional forms is uniquely determined when three pairs of homologous elements are given.*

COROLLARY 1. *If two coplanar point-rows on distinct lines are projective and their common point is self-corresponding, they are perspective.*

COROLLARY P1. *If two coplanar flat pencils on distinct points are projective and their common line is self-corresponding, they are perspective.*

For, as regards two point-rows, if

$$u(ABC \text{ --- }) \bar{\wedge} u'(AB'C' \text{ --- }),$$

we have a projectivity determined by the three pairs A and A , B and B' , C and C' , which is equivalent to a perspectivity whose center is $S = BB', CC'$.

6.9 Projective Forms which are Perspective. The following dual theorems will be found useful in the further applications of projectivities between one-dimensional forms.

1. *If two projective point-rows lying in the same plane are so situated that the joins of three pairs of homologous points are concurrent, then all the lines joining pairs of homologous points will likewise be concurrent at the same point, and the common point of the two point-rows is self-corresponding.*

P1. *If two projective flat pencils lying in the same plane are so situated that the intersections of three pairs of homologous lines are collinear, then all the points of intersection of homologous lines will likewise be collinear on the same line, and the common line of the two flat pencils is self-corresponding.*

On the left, the point of concurrency of the joins of three pairs of homologous points is the center of two superposed flat pencils which are projectively related, since each is perspective to one of the point-rows. But these two superposed projective flat pencils have three double lines and, hence, are identical. That is, each

line of the superposed flat pencils joins a pair of homologous points of the two point-rows.

Moreover, the point-rows are perspective to the pencil and to each other, and, therefore, their common point is self-corresponding.

The space duals of these theorems are of sufficient importance to deserve an explicit statement:

2. *If two projective axial pencils whose axes intersect are so situated that the intersections of three pairs of homologous planes are coplanar, then all the lines of intersection of pairs of homologous planes will likewise be coplanar on the same plane, and the common plane of the two axial pencils is self-corresponding.*

The two axial pencils are the projectors of the same flat pencil and are perspective from a plane (Art. 3.2).

B2. *If two projective flat pencils which are concentric are so situated that the planes determined by three pairs of homologous lines are coaxial, then all the planes determined by pairs of homologous lines will likewise be coaxial on the same line, and the common line of the two flat pencils is self-corresponding.*

The two flat pencils are the sections of the same axial pencil and are perspective from a line (Art. 3.2).

6.10 Axis and Center of Homology. Given two one-dimensional forms which are projectively related, it shall be understood that either we know a chain of perspectivities connecting the two forms or we know three pairs of homologous elements of the two forms. We have seen (Theorem IV) that three points of one line may be projected into any three points of a second line by two centers of perspectivity; we know also (Theorem VI) that three pairs of homologous elements determine a projectivity uniquely. From this we conclude that knowing three pairs of homologous elements of two projective forms is equivalent to knowing a chain of perspectivities connecting the two forms.

A convenient method of setting up a projectivity between two like one-dimensional forms is contained in the following dual theorems:

Theorem VII. *If $u(ABCD—)$ and $u'(A'B'C'D'—)$ are any two projective point-rows in the same plane but on distinct sup-*

Theorem P VII. *If $U(abcd—)$ and $U'(a'b'c'd'—)$ are any two projective flat pencils in the same plane but on distinct supports,*

ports, there exists a line s such that, if A, A' and B, B' are any two pairs of homologous points, the lines AB' and $A'B$ meet on s .

The line s is called the **axis of homology** of the two projective point-rows.

there exists a point S such that, if a, a' and b, b' are any two pairs of homologous lines, the points ab' and $a'b$ are joined by a line through S .

The point S is called the **center of homology** of the two projective flat pencils.

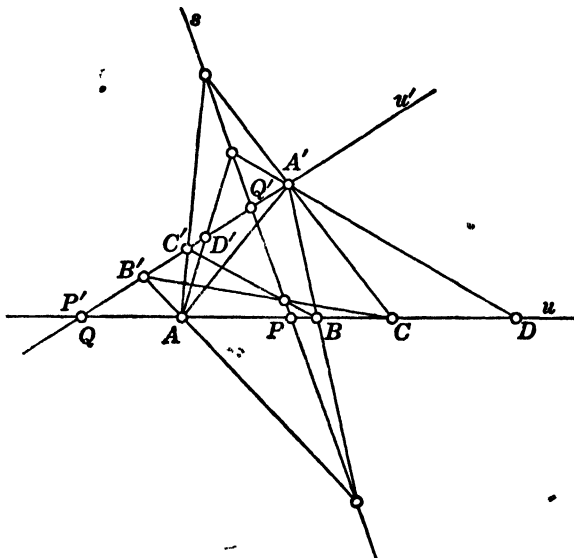


FIG. 31

We shall prove the theorem on the left. Since

$$u(ABCD \text{ --- }) \bar{\wedge} u'(A'B'C'D' \text{ --- }),$$

we have (Fig. 31)

$$A'(ABCD \text{ --- }) \bar{\wedge} u(ABCD \text{ --- }) \bar{\wedge} u'(A'B'C'D' \text{ --- }) \\ \bar{\wedge} A(A'B'C'D' \text{ --- }),$$

whence

$$A'(ABCD \text{ --- }) \bar{\wedge} A(A'B'C'D' \text{ --- }).$$

Moreover, by Corollary P1 of Art. 6.8,

$$A'(ABCD \text{ --- }) \bar{\wedge} A(A'B'C'D' \text{ --- }).$$

Hence, pairs of homologous lines $AB', A'B$; $AC', A'C$; etc., meet on a line s .

The proof of the theorem, however, is not complete. We have yet to show that any two pairs of homologous points such as B, B' and C, C' are so situated that BC' and $B'C$ meet on this same line s . In fact, the line s intersects u at P and u' at Q' , where P and Q' are respectively the points homologous to $P' = uu'$, considered as a point of u' , and $Q = uu'$, considered as a point of u . For $AP' = u$ and $A'P$ are homologous lines of the flat pencils A and A' , and hence meet on s ; and similarly for AQ' and $A'Q = u'$. We conclude, therefore, that if B and B' (or any other pair of homologous points) had been taken for the centers of the pencils projecting $u(ABCD \text{ --- })$ and $u'(A'B'C'D' \text{ --- })$ instead of A and A' , as in Fig. 31, the axis of perspectivity of those two flat pencils would necessarily be the same line s . For, this choice of B and B' as centers instead of A and A' would not alter the fact that P and Q' , which lie on and, hence, determine s , are the homologues of $P' = uu'$ and $Q = uu'$ in the given projectivity.

That is to say, we should then have

$$B'(ABCD \text{ --- }) \overset{s}{\bar{\wedge}} B(A'B'C'D' \text{ --- }),$$

and $BC', B'C; BD', B'D$; etc., meet on the line s —which completes the proof.

Throughout this proof it is, of course, assumed that u is not perspective to u' ; for if $u \bar{\wedge} u'$, the common point uu' is self-corresponding.

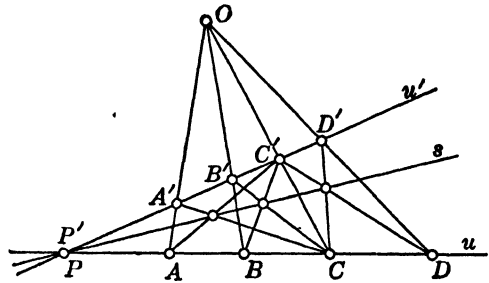


FIG. 32

In this case, however, we find (Fig. 32) that if

$$u(ABCD \text{ --- }) \overset{O}{\bar{\wedge}} u'(A'B'C'D' \text{ --- }),$$

then the triangles ABC' and $A'B'C$ are perspective through the center O , and hence, by Desargues' Theorem, homologous sides $BC', B'C$ and $C'A, CA'$ and $AB (= u), A'B' (= u')$ intersect on a line s passing through the point uu' , etc. This proves the validity of Theorem VII for the special case when the point-rows are not only projective but also perspective.

In the proof of Theorem VII we have also proved the following corollaries.

COROLLARY. *If $u \bar{\wedge} u'$ but not perspective, the axis of homology s of this projectivity cuts u, u' respectively at P, Q' which are the homologues of $P' = Q = uu'$.*

COROLLARY. *If $U \bar{\wedge} U'$ but not perspective, the center of homology S of this projectivity is joined to U, U' respectively by p, q' which are the homologues of $p' = q = UU'$.*

The axis (and dually the center) of homology plays an important rôle in determining homologous pairs of points of two intersecting projective point-rows when the projectivity is defined by three

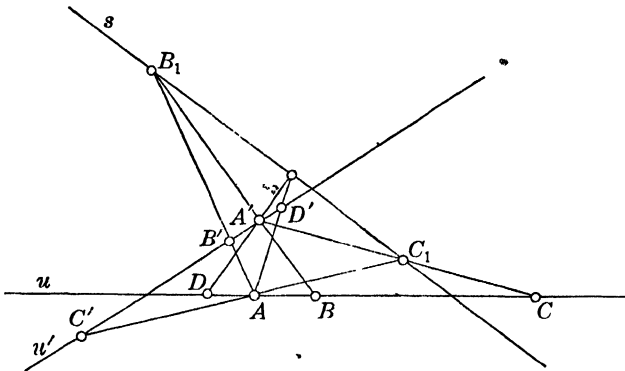


FIG. 33

pairs of homologous points. For example, suppose (Fig. 33) that $u \bar{\wedge} u'$ and the points A, B, C of u correspond respectively to A', B', C' of u' . If D is a point on u , how can we determine D' , its homologue on u' ? The lines $AB', A'B$ meet at B_1 and $AC', A'C$ meet at C_1 ; $B_1C_1 = s$, the axis of homology. The lines $A'D$ and AD' also meet on s . Hence, the line joining the point of intersection of $A'D$ and s to the point A will cut u' at D' , the required point.

This exceedingly important construction, together with its duals (plane and space), will be of such frequent use in the sequel as to make a thorough understanding of it now very much worth while.

Interesting, in this connection, is the **configuration of Pappus**. It arises as a special case of Theorem VII when we consider, instead of two point-rows, two triads of points on distinct lines. Thus, if ABC and $A'B'C'$ are two such

triads, we may always project the one into the other (Theorem IV); the points of intersection, then, of the cross joins, i.e., the points $A'' = BC'$, $B'C$ and $B'' = CA'$, $C'A$ and $C'' = AB'$, $A'B$ are collinear (Fig. 34).

The configuration of Pappus consists of the nine points and nine lines as illustrated; on each line there are three points and on each point, three lines. We represent it by the configuration symbol

9	3
3	9

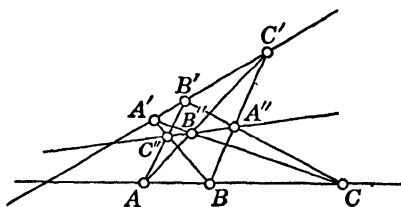


FIG. 34

6.11 Double Elements. A projectivity between two superposed one-dimensional forms which are not identical cannot have three double elements. It may, however, have less than three double elements. If it has two distinct double elements, we call it an **hyperbolic** projectivity (Fig. 28); if only one double element, it is a **parabolic** projectivity (Fig. 27); and if the projectivity has no double elements, it is an **elliptic** projectivity. For the present we shall not be concerned with elliptic projectivities other than to say that they do exist.

Concerning the hyperbolic and parabolic projectivities the following theorem is interesting in that it not only shows how to construct such projectivities but it also reveals something of the nature of the parabolic projectivity.

THEOREM 1. *If a projectivity between two superposed one-dimensional primitive forms has one double element, it has in general a second, which may, however, coincide with the first.*

To prove the theorem for two superposed point-rows let M be the given double point and let B, B' and C, C' be two pairs of homologous points (Fig. 35). Since M is the equivalent of one pair of homologous points, we have here sufficient data to determine the projectivity. From two points S and S' , which are collinear with M , project MBC and $MB'C'$ respectively. The two flat pencils $S(MBC \text{ ---})$ and $S'(MB'C' \text{ ---})$ are perspective, for (Theorem VI, Corollary P1)

$S(MBC \text{ ---}) \bar{\wedge} u(MBC \text{ ---}) \bar{\wedge} u(MB'C' \text{ ---}) \bar{\wedge} S'(MB'C' \text{ ---})$,
that is,

$$S(MBC \text{ ---}) \bar{\wedge} S'(MB'C' \text{ ---}),$$

and

$$SM = S'M,$$

by construction. If $SB, S'B'$ meet at B_1 and if $SC, S'C'$ meet at C_1 , then the axis of perspectivity of these two flat pencils is the line

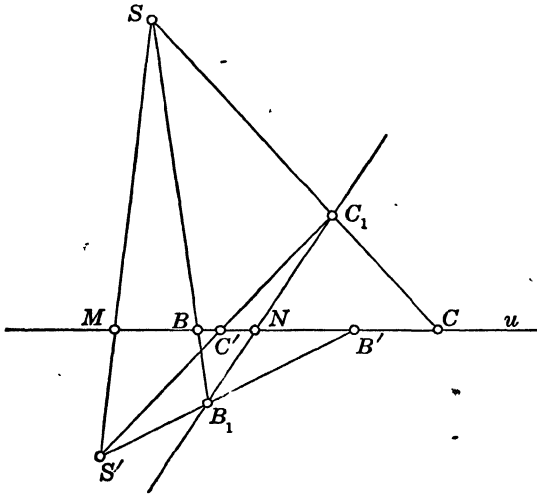


FIG. 35

B_1C_1 . The point N where B_1C_1 intersects the line u is the second double point of the projectivity. In general, the point N is distinct from M but it will coincide with M if B_1C_1 should pass through M .

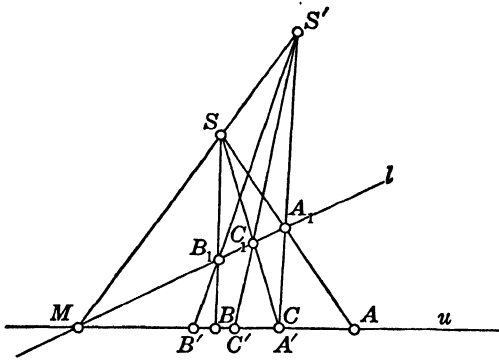


FIG. 36

The construction of a parabolic projectivity on a line u follows from the above theorem. If M is the double point of a parabolic projectivity (and hence equivalent to two pairs of homologous points) and A, A' is a pair of homologous points, draw a line

through M and on it take two points S and S' (Fig. 36). Now let $SA, S'A'$ meet at A_1 , and join A_1 to M by a line l . Any point B on u has as its partner in this parabolic projectivity a point B' which is the intersection of $S'B_1$ with u , B_1 being the meet of SB and l . For we have

$$u(MAB \text{ ---}) \bar{\wedge} S(MAB \text{ ---}) \stackrel{l}{\bar{\wedge}} S'(MA'B' \text{ ---}) \bar{\wedge} u(MA'B' \text{ ---}),$$

and therefore

$$u(MAB \text{ ---}) \bar{\wedge} u(MA'B' \text{ ---}).$$

If A' is considered as a point of the first point-row and labeled C , we find its homologue C' as above. Let C_1 on l be the intersection of SC and $S'C'$. The complete quadrangle $S'A_1C_1S$ defines the harmonic set $H(MA', AC')$. Thus,

THEOREM 2. *If, in a parabolic projectivity with double point M , A' corresponds to A and C' to $C (= A')$, then the pair M, A' is harmonically separated by the pair A, C' .*

EXERCISES

1. State and prove the plane dual of Theorem IV.
2. If a, b, c, d are four distinct lines on one point U , project the tetrad $abcd$ into the tetrad $badc$.
3. Choose arbitrarily on a line u three points A, B, C . Construct on this line u a number of points of the net of rationality $[ABC]$.
4. If u, u' are two distinct coplanar lines and A, B, C are any three points on u while A', B', C' are any three points on u' , construct the axis of homology of the projectivity

$$u(ABC \text{ ---}) \bar{\wedge} u'(A'B'C' \text{ ---}).$$

Construct the point Q' on u' which corresponds to uu' considered as a point of u ; and the point P on u which corresponds to uu' considered as a point of u' . If I, J' are the ideal points respectively on u, u' , construct the corresponding points I' and J . If D, E' are any points on u, u' respectively, construct D', E .

5. If $u(ABC \text{ ---}) \bar{\wedge} u(A'B'C' \text{ ---})$ is a projectivity between two superposed point-rows and if D is any point of the first point-row, construct its homologous point D' in this projectivity. If the same point D is considered as belonging to the second point-row and is labeled E' , construct its homologous point E . If the ideal point of u is I (or J'), construct I' (or J).

6. If $U(abc \text{ ---}) \bar{\wedge} U'(a'b'c' \text{ ---})$ is a projectivity between two distinct coplanar flat pencils, construct the center of homology. Construct the line q' on U' , which corresponds to line UU' considered as a line on U ; and the line p on U which corresponds to UU' considered as a line on U' . If d, e' are any two lines on U, U' respectively, construct d', e .

7. If $U(abc \text{ ---}) \bar{\wedge} U(a'b'c' \text{ ---})$ is a projectivity between two superposed flat pencils and if d is a line of the first flat pencil, construct its homolo-

gous line d' . If the line d is considered as belonging to the second flat pencil and is labeled e' , construct its homologous line e .

8. Discuss and illustrate with a figure the plane dual of the configuration of Pappus.

9. Given two pairs of homologous points A, A' and B, B' and a double point M of a projectivity between two superposed point-rows, construct the points corresponding to C , considered as a point of the first point-row, and D' , considered as a point of the second point-row, when (a) C and D' are distinct points; (b) C and D' are coincident points. Construct, also, the second double point of this projectivity.

10. State the plane dual of Ex. 9, and solve.

11. Given one pair of homologous elements and the double elements of an hyperbolic projectivity on a line; construct other pairs of homologous elements. Dualize.

12. Given one pair of homologous elements and the double element of a parabolic projectivity on a line; construct other pairs of homologous elements. Dualize.

13. In a projectivity between two superposed point-rows,

$$u(ABC \text{ ---}) \bar{\wedge} u(A'B'C' \text{ ---}),$$

construct other pairs of homologous elements when (a) A' and C coincide; (b) A' and C and also C' and A coincide.

In (b) above, the elements A, A' are said to be *doubly corresponding*.

14. In an hyperbolic projectivity M is a double point and A, A' is a pair of doubly corresponding points. Construct the second double point; and also, other pairs of homologous points.

15. If, in a projectivity $u(ABC \text{ ---}) \bar{\wedge} u(A'B'C' \text{ ---})$, one pair of homologous elements corresponds doubly, every pair of homologous elements corresponds doubly. Prove this theorem.

16. Dualize, in the plane, Exs. 14 and 15.

17. Set up a projectivity between two distinct coplanar point-rows u and u' and construct at least twenty-five pairs of homologous points A, A' and the lines $a = AA'$ determined by each pair. How many lines joining homologous points pass through an arbitrary point of the plane?

18. Carry out the plane dual of the construction of Ex. 17.

19. Two projectively related flat pencils which are coplanar but not concentric have not more than two pairs of parallel homologous lines unless every pair consists of parallel lines. Prove this theorem.

20. In a plane we have given two fixed lines u, u' intersecting at O and two fixed points S, S' collinear with O . If now a line x turns about a fixed point P and intersects u at X and u' at X' , show that the locus of the intersection of SX and $S'X'$ is a line v . Under what conditions will this line v pass through O ?

21. State and prove the plane dual of Ex. 20.

22. Given a projectivity between two distinct coplanar point-rows u, u' where the ordinary points A, B correspond respectively to the ordinary points A', B' and the ideal point C corresponds to the ordinary point C' . Construct at least twenty-five pairs of homologous points and the lines joining each

23. Given a projectivity between two distinct coplanar point-rows determined by the homologous ordinary points A, A' and B, B' and the homologous ideal points C, C' . Construct at least twenty-five joins of homologous points.

24. Given the axis of homology s and one pair of homologous points A, A' of the projectivity $u \bar{\wedge} u'$; construct D' and E , the points homologous respectively to a point D on u and a point E' on u' , when A, A' are ordinary points and s

- (a) intersects u and u' at their common point uu' ;
- (b) intersects u and u' at two distinct ordinary points;
- (c) is parallel to u (or to u');
- (d) is the ideal line.

Consider other metric specializations of this problem.

25. Two flat pencils $U(abcd \text{ ---})$ and $U'(a'b'c'd' \text{ ---})$ are in a (1,1) correspondence such that the angle between any two lines a, b of the first is equal to the angle between the homologous lines a', b' of the second. Show that the flat pencils are projectively related.

If the flat pencils are superposed under what conditions will the projectivity have double lines?

CHAPTER VII

CONICS AND CONES

7.1 Definitions.

1. The points of intersection of homologous lines of two projective flat pencils which are coplanar but are neither concentric nor perspective constitute a **point conic** or a **point-row of second order**.

The point conic has not more than two points in common with any straight line of its plane. For if three pairs of homologous lines of the flat pencils meet at three collinear points, the flat pencils are perspective (Art. 6.9).

A line t in the plane of a point conic on which lies one and only one point T of this point conic is called the **tangent** to the point conic at T .

P1. The lines joining homologous points of two projective point-rows which are coplanar but are neither cobasal nor perspective constitute a **line conic** or a **pencil of lines of second class**.

The line conic has not more than two lines passing through any point of its plane. For if three pairs of homologous points of the point-rows are joined by three concurrent lines, the point-rows are perspective (Art. 6.9).

A point T in the plane of a line conic through which passes one and only one line t of this line conic is called the **point of contact** of the line conic on t .

In each of the above dual statements we have, in the second paragraphs, a justification (for the present) of the use of the term "conic," a term already familiar to the reader in another connection. Later we shall completely justify this terminology by identifying the point conic with the conic section of analytic geometry, and the line conic with the system of tangents to that same curve.

2. The planes determined by homologous lines of two projective flat pencils which are concentric but are neither coplanar nor perspective constitute a

B2. The lines of intersection of homologous planes of two projective axial pencils whose axes intersect but which are neither coaxial nor perspective consti-

cone of planes or a **pencil of planes of second class** in a bundle. The common center of the two flat pencils is the **vertex** of the cone of planes.

Through any line on the vertex there are not more than two planes of the cone; a line on the vertex on which there is one and only one plane of the cone is called the **line of contact** on that plane.

Note that Definitions 2 and B2 are respectively the space duals of Definitions 1 and P1, and hence the cone of planes and the cone of lines are bundle forms.

The terms "order" and "class" as used in the above definitions are in agreement with the customary usage as regards nondegenerate¹ forms; to wit, in the plane,

The **order** of a point conic is the maximum number of points which it may have in common with any line of its plane.

And in the bundle,

The **class** of a cone of planes is the maximum number of planes which it may have in common with any line of its bundle.

tute a **cone of lines** or a **pencil of lines of second order** in a bundle. The intersection of the axes of the two axial pencils is the **vertex** of the cone of lines.

On any plane through the vertex there are not more than two lines of the cone; a plane on the vertex on which there is one and only one line of the cone is called the **tangent plane** on that line.

The **class** of a line conic is the maximum number of lines which it may have in common with any point of its plane.

The **order** of a cone of lines is the maximum number of lines which it may have in common with any plane of its bundle.

This second set consists of the space duals of the first set, and the terms "order" and "class" are dual terms in the plane, in the bundle, and in space.

The four forms defined above are **one-dimensional forms of second degree**.² Collectively, we shall refer to the one-dimensional forms of second degree which lie in a plane as conics, and to those which lie in a bundle as cones.

¹ The "degenerate" forms will be introduced in Art. 7.5.

² A fifth one-dimensional form of second degree, the regulus, will be discussed in Chapter XI.

7.2 Fundamental Properties of Conics. In discussing the properties of conics which follow we shall adopt the notations here indicated:

If two projective flat pencils, $U(abc \text{ ---}) \bar{\wedge} U'(a'b'c' \text{ ---})$, are coplanar but are neither concentric nor perspective, the intersections $A = aa'$, $B = bb'$, etc., of homologous lines constitute a point conic.

If two projective point-rows, $u(ABC \text{ ---}) \bar{\wedge} u'(A'B'C' \text{ ---})$, are coplanar but are neither cobasal nor perspective, the joins $a = AA'$, $b = BB'$, etc., of homologous points constitute a line conic.

We shall prove, now, some properties of point conics and leave to the reader the proofs of the duals. And it would be an excellent exercise for him to state and prove the analogous properties of cones.

1. *The centers of the two projective flat pencils which define the point conic are themselves points of the point conic.*

P1. *The bases of the two projective point-rows which define the line conic are themselves lines of the line conic.*

For if $UU' = p'$ is considered as a line of the flat pencil U' , its homologous line p of U meets p' at U . Hence $pp' = U$ is a point of the point conic, by definition, and similarly for U' .

2. *That line of the one flat pencil which is homologous to the common line of the two pencils, considered as belonging to the second flat pencil, is a tangent to the point conic at the center of the first flat pencil.*

P2. *That point of the one point-row which is homologous to the common point of the two point-rows, considered as belonging to the second point-row, is a point of contact of the line conic on the base of the first point-row.*

Any line a of the pencil U has two points in common with the point conic, namely, the points U and $A = aa'$. These points are distinct, except when line a is the homologue of UU' considered as a line of U' . That is, the line p of the pencil U , which is the homologue of $p' = UU'$ considered as a line of the pencil U' , has in common with the point conic only one point, namely, the point $pp' = U$. This line p , moreover, is the only line on U which has only one point in common with the point conic; for, since a pro-

jectivity is a (1,1) correspondence, p' has but one homologue. By definition, p is the tangent to the point conic at U .

Similarly, if UU' is considered as a line of pencil U and is called q , then q' is the tangent at U' .

If U is an ordinary point of the plane, we may also define the tangent line p , of the above paragraph, as the limiting position of the secant $a = UA$ as A , a point of the point conic, varies and approaches U . For as A approaches U , the line a' approaches $p' = UU'$ and the line a approaches p , the homologue of p' .

This alternative definition is probably a more familiar one to the student who has studied the calculus. But in order to use it as a projective definition it would be necessary to delete the adjective "ordinary" and, after that, to make some agreement concerning the meaning of the phrase "as A approaches U " when U is an ideal point.

Dually, for the line conic defined by two projective point-rows,

$$u(ABC \text{ ---}) \overline{\wedge} u'(A'B'C' \text{ ---}),$$

the point of contact P on line u is the limiting position of the point $A = ua$ as a , a line of the line conic, varies and approaches u . For as a approaches u , the point A' approaches $P' = uu'$ and the point A approaches P , the homologue of P' .

That there is a tangent at each point of a point conic, or dually a point of contact on each line of a line conic, is an immediate consequence of the properties just discussed and the following:

<p>3. A point conic is projected from any two of its points by two projective flat pencils, homologous lines of which intersect on the point conic.</p>	<p>P3. A line conic is cut by any two of its lines in two projective point-rows, homologous points of which are joined by lines of the line conic.</p>
---	--

On the left, let the point conic be defined by the projective flat pencils

$$U(ABC \text{ ---}) \overline{\wedge} U'(ABC \text{ ---}),$$

and let P be any point of this point conic (Fig. 37). Then

$$U(ABCP \text{ ---}) \overline{\wedge} U'(ABCP \text{ ---}),$$

and if we take a section of pencil U by CP , viz., $KLCP \text{ ---}$, and a section of U' by AP , viz., $AMNP \text{ ---}$, we have

$$AMNP \text{ ---} \overline{\wedge} U'(ABCP \text{ ---}) \overline{\wedge} U(ABCP \text{ ---}) \overline{\wedge} KLCP \text{ ---},$$

and hence,

$$AMNP \text{ ---} \overline{\wedge} KLCP \text{ ---}.$$

But these projective point-rows have their common point P corresponding to itself; we may write, then,

$$AMNP \text{ --- } \overline{\wedge} KLCP \text{ ---};$$

whence we conclude that the lines AK , ML , and NC are concurrent at a point, which we call S .

Now let P be a variable point of the point conic while the points U , U' , A , B , C remain fixed. Since $S = AK$, NC or, what is the same thing, $S = UA$, $U'C$, we see that S too is fixed. Also, since P is a variable point, the lines AP and CP are variable lines of two flat pencils whose centers are respectively the fixed points A and C . At the same time, the points M and L are variable points of two point-rows on $U'B$ and UB , respectively. These point-rows are perspective because, for each position of the variable points M and L , the line ML goes through the fixed point S ; that is, for each position of M and L ,

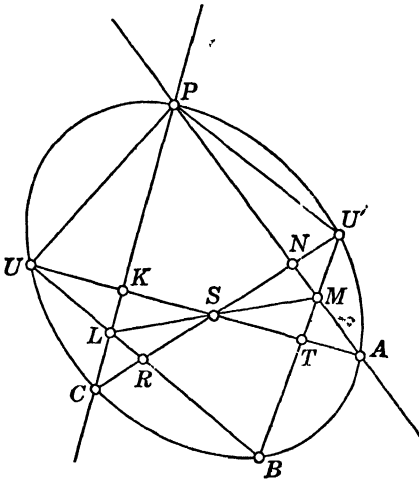


FIG. 37

$$BU'MT \text{ --- } \overline{\wedge}^S BRLU \text{ ---},$$

where $T = U'B$, UA and $R = UB$, $U'C$.

From the chain of perspectivities

$$A(BU'MT \text{ ---}) \overline{\wedge} BU'MT \text{ --- } \overline{\wedge}^S BRLU \text{ --- } \overline{\wedge} C(BRLU \text{ ---}),$$

it follows that

$$A(BU'MT \text{ ---}) \overline{\wedge} C(BRLU \text{ ---}),$$

or

$$A(BU'PU \text{ ---}) \overline{\wedge} C(BU'PU \text{ ---}).$$

Therefore, the points of the point conic, represented here by the variable point P , are projected from A and C by two projective flat pencils. But, for A and C we may substitute any two points of the point conic. Hence the proof of Property 3 above.

4. A point conic is uniquely determined by five points or, what is equivalent, four points and the tangent at one of them or three points and the tangents at two of them. It is assumed that no three of the given points are collinear, and that a given tangent does not lie on two of the given points.

P4. A line conic is uniquely determined by five lines or, what is equivalent, four lines and the point of contact on one of them or three lines and the points of contact on two of them. It is assumed that no three of the given lines are concurrent, and that a given point of contact does not lie on two of the given lines.

For example, if five points are given, we may choose any two of them as the centers of two flat pencils which have, as an homologous pair, each pair of lines joining the centers to one of the remaining given points. For, in discussing Property 3, we have shown that the conic described by the point P may be defined either by the projectivity

$$U(ABC \text{ ---}) \bar{\wedge} U'(ABC \text{ ---})$$

or by the projectivity

$$A(BU'U \text{ ---}) \bar{\wedge} C(BU'U \text{ ---}).$$

In either case the projectivity, having three pairs of homologous lines given, is uniquely determined (Art. 6.8).

When four points and a tangent at one of them are given, we take the point at which the tangent is known as one of the centers. Any one of the other three points may then be chosen as the second center, and the argument continues as before. Likewise, when three points and the tangents at two of them are known, we choose as centers those two points at which the tangents are given. In each case we have, because of Property 2, three pairs of homologous lines, and the projectivity is determined uniquely.

7.3 The Construction of Conics. The properties of the preceding article together with the theorems of Art. 6.10 enable us to carry out the following constructions:

To construct a point conic when given (1) five points; (2) four points and the tangent at one of them; (3) three points and the tangents at two of them.

To construct a line conic when given (1') five lines; (2') four lines and the point of contact on one of them; (3') three lines and the points of contact on two of them.

Taking first (1'), let the five given lines be u, u', a, b, c . If a cuts u and u' respectively at A and A' , b cuts u, u' at B, B' , and c cuts at C, C' (Fig. 38), the problem reduces to that of finding pairs of homologous points D, D' on the two projective point-rows $u(ABC \text{ ---})$ and $u'(A'B'C' \text{ ---})$. By Art. 6.10, the points $AB', A'B; AC', A'C; BC', B'C$ lie on the axis of homology s ; we construct this axis by joining any two of these three points. Again as in Art. 6.10, for any point D on u the homologous point D' is the intersection of u' with AD_1 , D_1 having been found as the point common to s and $A'D$. Now $d = DD'$ is a line of the line conic; and

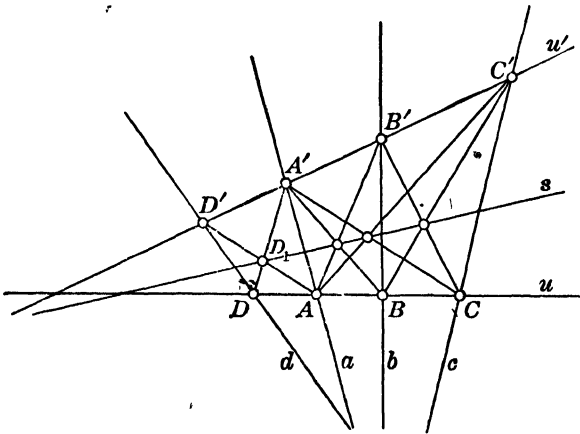


FIG. 38

in the same way any number of its lines may be drawn. A dual construction solves (1) on the left above.

The solution of (2) is very similar to that of (1). Let the four points be U, U', A, B and let p be the tangent at U . Homologous lines are $UA = a$ and $U'A = a'$, $UB = b$ and $U'B = b'$, and finally p and $p' = UU'$, by Property 2, Art. 7.2. The line p , by the corollary of Art. 6.10, meets the line joining $a'b$ and ab' at S , the center of homology (Fig. 39). Any line d of U has a homologous line d' of U' such that the join of $a'd$ and ad' is a line through S ; and the point $D = dd'$ is a point of the point conic. Thus we may construct any number of points of the point conic.

Various special cases arise which will be considered later. For the present we simply point out

(1) that two of the points, or one of the points, or one of the lines, may be ideal; and

(2) that the definitions of Art. 7.1 require that no three of the points (lines) are collinear (concurrent), and no tangent (point of contact) lies on two of the points (lines).

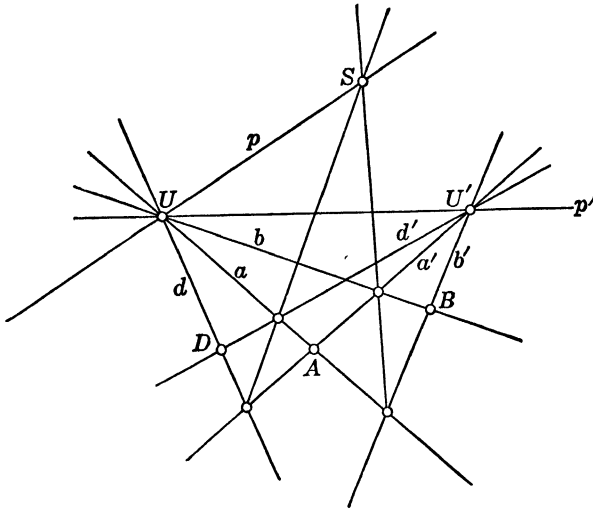


FIG. 39

7.4 Relations between Conics and Cones. Relations between the conics and cones other than the relations of duality are exhibited in the following theorems:

The projector of a point conic from a point not on its plane is a cone of lines.

For the projectors of the two projective flat pencils, which define the point conic, from a point not in their common plane are two projective axial pencils whose axes intersect. These axial pencils define a cone of lines. Furthermore, the projectors of two homologous lines of the flat pencils, which meet in a point of the point conic, are two homologous planes of the axial pencils, which intersect

The section of a cone of planes by a plane not on its vertex is a line conic.

For the sections of the two projective flat pencils, which define the cone of planes, by a plane not on their common center are two projective point-rows whose bases are coplanar. These point-rows define a line conic. Furthermore, the sections of two homologous lines of the flat pencils, which determine a plane of the cone of planes, are two homologous points of the point-rows, which

in a line of the cone of lines; and this line is the projector of a point on the point conic.

The section of a cone of lines by a plane not on its vertex is a point conic.

are joined by a line of the line conic; and this line is the section of a plane of the cone of planes.

The projector of a line conic from a point not on its plane is a cone of planes.

We leave to the reader the proofs of these last theorems.

7.5 Degenerate Conics and Cones. The cases excluded from the definitions of Art. 7.1, namely, the projective primitive forms which are also perspective, are covered by the following definitions:

1. The points of intersection of homologous lines of two perspective flat pencils which are coplanar but not concentric constitute a **degenerate point conic** consisting of two point-rows, namely, the point-row on the axis of perspectivity and the point-row on the common line of the two flat pencils.

The axis of perspectivity is obviously the support of points of the degenerate point conic.

Moreover, the common line UU' is self-corresponding; and any point A on UU' is a point common to homologous lines UA and $U'A$. Hence, all such points are to be regarded as points of the degenerate point conic.

2. The planes joining homologous lines of two perspective flat pencils which are concentric but not coplanar constitute a **degenerate cone of planes** consisting of two axial pencils, namely, the axial pencil on the

P1. The lines joining homologous points of two perspective point-rows which are coplanar but not cobasal constitute a **degenerate line conic** consisting of two flat pencils, namely, the flat pencil on the center of perspectivity and the flat pencil on the common point of the two point-rows.

The center of perspectivity is obviously the support of lines of the degenerate line conic.

Moreover, the common point uu' is self-corresponding; and any line a on uu' is a line common to homologous points ua and $u'a$. Hence, all such lines are to be regarded as lines of the degenerate line conic.

B2. The lines of intersection of homologous planes of two perspective axial pencils whose axes intersect but which are not coaxial constitute a **degenerate cone of lines** consisting of two flat pencils, namely, the flat

axis of perspectivity and the axial pencil on the common line of the two flat pencils.

The axis of perspectivity, we note, is the axis of the axial pencil of which the two perspective flat pencils are sections.

pencil on the plane of perspectivity and the flat pencil on the common plane of the two axial pencils.

The plane of perspectivity, we note, is the plane of the flat pencil of which the two perspective axial pencils are projectors.

We shall find in the following chapters certain theorems concerning elements on *proper* conics which are equally valid for elements on degenerate conics. The usefulness of the above definitions lies in that direction.

7.6 A Metric Classification of Conics.

The two projective flat pencils, $U(abc \text{ ---}) \bar{\wedge} U'(a'b'c' \text{ ---})$, which define a point conic, determine on any line in the plane of the point conic a projectivity; this projectivity is hyperbolic, parabolic, or elliptic according as the line has two, one, or no points in common with the point conic.

For the two projective flat pencils cut the line in two superposed projective point-rows. If the line has a point in common with the point conic, that point is the intersection of two homologous lines of U and U' , and is, therefore, a double point of the projectivity on the line.

The two projective point-rows, $u(ABC \text{ ---}) \bar{\wedge} u'(A'B'C' \text{ ---})$, which define a line conic, determine on any point in the plane of the line conic a projectivity; this projectivity is hyperbolic, parabolic, or elliptic according as two, one, or no lines of the line conic pass through the point.

For the two projective point-rows are projected from the point by two superposed projective flat pencils. If there is one line of the line conic through the point, that line is the join of two homologous points of u and u' , and is, therefore, a double line of the projectivity on the point.

A metric classification of point conics is possible on the basis of the theorem on the left. By assigning to the ideal line a special significance,³ whereby it is considered as an exceptional line of

³ We here call attention to the important fact that all metric specializations in projective geometry involve the ideal line. In this connection reread Art. 5.2.

the plane, we call a point conic an **hyperbola** if the flat pencils defining it determine on the ideal line an hyperbolic projectivity; a **parabola** if the projectivity on the ideal line is parabolic; and an **ellipse** if that projectivity is elliptic. Otherwise stated, an hyperbola is a point conic two of whose points are ideal; a parabola has one ideal point (or two coincident ideal points); an ellipse has no ideal points. The tangents to an hyperbola at its two ideal points are called **asymptotes**. The tangent to a parabola at its one ideal point is the ideal line of the plane of the parabola.

Let us repeat that this is a metric classification of point conics, and not a projective classification. An ellipse, for example, can be projected into an hyperbola or parabola as follows: Let E represent an ellipse on a plane π and let l be any line which cuts the ellipse in two (one) points. A central projection from a point S , not on π , onto a plane π' which is parallel to the plane Sl gives E' , the image of E , and l' , the image of l . E' has two (one) points in common with l' , and is therefore, since l' is the ideal line of π' , an hyperbola (parabola). Thus the projection of an ellipse is not necessarily another ellipse; and the classification here given is not a projective classification.

With a specialization of the ideal line the duality of projective forms is lost. Hence, the metric classification of this article cannot be dualized for line conics. However, we should observe that two projective point-rows lying in the same plane and having their ideal points as one homologous pair define a line conic which includes among its lines the ideal line of the plane. The line conic in this case is a "line parabola" and has one ideal point of contact. The line conics corresponding to the hyperbola and ellipse have, respectively, two ideal points of contact (one on each asymptote) and no ideal point of contact.

The various metric specializations of the constructions of Art. 7.3 are listed in Appendix A.

EXERCISES

1. Construct the point conic $(ABCDE)$ determined by five given points.⁴
2. Construct the line conic $(abcdA)$ determined by four given lines a, b, c, d and the point of contact A on the line a .

⁴ Cf. Art. 7.6 and Appendix A. We introduce here a convenient symbolism for stating the data. In particular, if E is an ideal point, we attach an asterisk to it, e.g., $(ABCDE^*)$. Thus $(ABCD^*d)$ represents three ordinary points A, B, C and an asymptote d of an hyperbola.

3. Using the data of Ex. 1 construct the tangent to the point conic at the point A

4. Using the data of Ex. 2 construct the point of contact of the line conic on line b .

5. A point conic ($ABCab$) is determined by three of its points A, B, C and the tangents a, b at A, B respectively. Construct

(a) the second point of intersection with the point conic of a line passing through the given point A ;

(b) the second point of intersection with the point conic of a line passing through the given point C ;

(c) the tangent to the point conic at C .

Suggestion. Use the theorems of Arts. 6.11 and 7.6 in (b) and (c).

6. Write the plane dual of Ex. 5, and solve.

7. In Ex. 20, Chapter VI, what is the locus of the intersection of SX and $S'X'$ if S and S' are not collinear with O ?

8. If the vertices of a complete plane quadrangle $ABCD$ are projected from a point S in the same plane by four harmonic lines, show that the locus of S is a point conic.

9. The vertices A, B, C of a triangle move on three fixed lines a, b, c , respectively, in such a way that side BC always lies on a fixed point S_1 and side CA always lies on a fixed point S_2 . Show that side AB either generates a line conic or passes through one point.

10. Dualize Ex. 9.

11. Discuss the metric specialization of Ex. 9 which results when S_1 and S_2 are ideal points.

12. A variable triangle ABC with fixed vertex A and constant angle A moves so that B and C lie on two fixed lines b and c in the plane of the triangle. Show that side BC generates a line conic. Discuss the metric specialization arising when b (or c) is the ideal line. (Cf. Ex. 25, Chapter VI.)

13. Show that the points of a circle are points of a point conic in accordance with the definition of Art. 7.1. Show also that the tangents to a circle are lines of a line conic.

14. If, in Definition 2 of Art. 7.1, the common center of the projective flat pencils is an ideal point, what metric specialization results? Consider, from a similar point of view, Definition B2 when the axial pencils have parallel axes.

15. Given a projectively related flat pencil $S(abc \text{ ---})$ and point-row $u(A'B'C' \text{ ---})$; draw through every point P' on u the line p' which is parallel to p , the homologue of P' in $S(abc \text{ ---})$. Show that the lines p' are lines of a "line parabola" or they are concurrent.

16. Given five points of a point conic ($ABCDE^*$), one of which is an ideal point; construct the tangent to this point conic at the given ideal point. Is the point conic an hyperbola or a parabola?

17. Given four points of a point conic ($ABCD^*d^*$), one of which is an ideal point, and the ideal line as a tangent to this point conic at the given ideal point; construct the second point of intersection with the point conic of a line through its ideal point.

18. Of a "line hyperbola" ($abcdD^*$) we are given four lines and on one of them (the asymptote) the point of contact is the ideal point. Construct that line of this line conic which passes through a given point on the asymptote.

19. Given two ordinary lines of a "line parabola" (abc^*AB) and the points of contact on those lines; construct that line of this line conic which has a given direction. (Cf. Ex. 5(b).)

20. State all possible metric specializations of the data of Ex. 5.

CHAPTER VIII

PASCAL'S THEOREM AND BRIANCHON'S THEOREM

8.1 The Theorems. Our further study of the curves of second degree is based on two famous theorems, named after their discoverers, B. Pascal (1623-1662) and C. J. Brianchon (1785-1864). We have seen (Art. 7.2) that five points in a plane, no three of which are collinear, completely determine a point conic. Six points in a plane, no three of which are collinear, will not, in general, lie on one and the same point conic; but if these six points do lie on one point conic there is a relation existing between them. Pascal's Theorem expresses this relation in precise language; and Brianchon's Theorem, the dual relation existing between six lines which are lines of one line conic.

Theorem VIII — Pascal's Theorem. *If the vertices of a simple hexagon (6-point) are points of a point conic, the three intersections of pairs of opposite sides of the 6-point are collinear.*

That is, if the 6-point is $ABCDEF$, the points $P_1 = AB, DE$ and $P_2 = BC, EF$ and $P_3 = CD, FA$ lie on a line $p = P_1P_2P_3$. This line is called the **Pascal line** of the simple 6-point (Fig. 40).

Choosing A and C as the centers of two flat pencils projecting the points B, D, E, F , we have, by Property 3, Art. 7.2,

$$A(BDEF \text{---}) \bar{\wedge} C(BDEF \text{---}).$$

The section of $A(BDEF)$ by DE and the section of $C(BDEF)$

Theorem P VIII — Brianchon's Theorem. *If the sides of a simple hexagon (6-line) are lines of a line conic, the three joins of pairs of opposite vertices of the 6-line are concurrent.*

That is, if the 6-line is $abcdef$, the lines $p_1 = ab, de$ and $p_2 = bc, ef$ and $p_3 = cd, fa$ lie on a point $P = \bar{p}_1p_2p_3$. This point is called the **Brianchon point** of the simple 6-line (Fig. 41).

Choosing a and c as the bases of two point-rows cutting the lines b, d, e, f , we have, by Property P3, Art. 7.2,

$$a(bdef \text{---}) \bar{\wedge} c(bdef \text{---}).$$

The projector of $a(bdef)$ from de and the projector of $c(bdef)$

by EF are projective. That is, if $AF, DE = K$ and $CD, EF = L$,

$$P_1DEK \text{ --- } \overline{\wedge} P_2LEF \text{ ---}.$$

But these two projective point-rows have their common element E as self-corresponding; they are, therefore, also perspective (Theorem VI, Corollary 1), i.e.,

$$P_1DEK \text{ --- } \overline{\wedge} P_2LEF \text{ ---}.$$

The center of this perspective is the intersection of $DL (= CD)$ and $KF (= FA)$ which is, by hypothesis, the point P_3 . Hence, the join of the pair of homologous points P_1, P_2 passes through P_3 ; and P_1, P_2, P_3 are collinear.

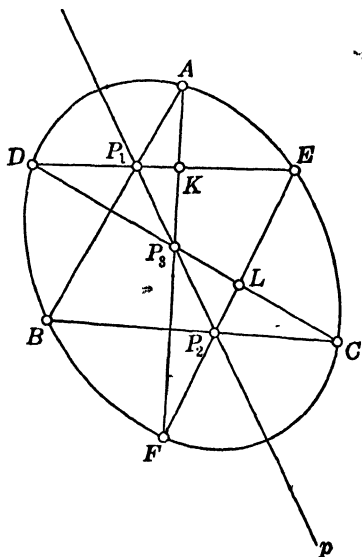


FIG. 40

from ef are projective. That is, if $af, de = k$ and $cd, ef = l$,

$$p_1dek \text{ --- } \overline{\wedge} p_2lef \text{ ---}.$$

But these two projective flat pencils have their common element e as self-corresponding; they are, therefore, also perspective (Theorem VI, Corollary P1), i.e.,

$$p_1dek \text{ --- } \overline{\wedge} p_2lef \text{ ---}.$$

The axis of this perspective is the join of $dl (= cd)$ and $kf (= fa)$ which is, by hypothesis, the line p_3 . Hence, the intersection of the pair of homologous lines p_1, p_2 lies on p_3 ; and p_1, p_2, p_3 are concurrent.

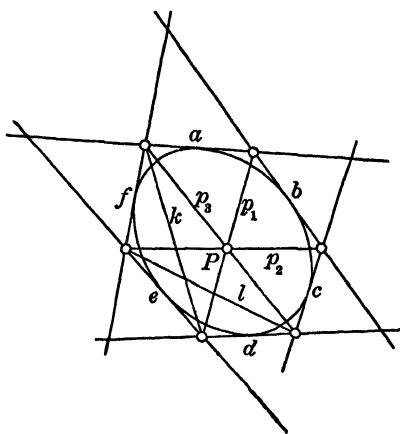


FIG. 41

As an aid in beginning the proof of the theorem on the left the student should observe that any two vertices of the hexagon which have a common adjacent vertex may be chosen as the

centers of the flat pencils. The two centers having thus been chosen there remain but two sides of the hexagon which do not pass through either of these centers. These are the sides which will be the bases of the two perspective point-rows. The perspective point-rows are so chosen that two of the three points P_1, P_2, P_3 appear as a pair of homologous points. For instance, in the above proof, it was necessary to cut $A(BDEF \text{ ---})$ by DE and $C(BDEF \text{ ---})$ by EF so as to have P_1 and P_2 as one pair of homologous points. Similar remarks apply to the proof of Brianchon's Theorem.

8.2 The Converse Theorems.

<p><i>If the three intersections of pairs of opposite sides of a simple 6-point are collinear, the vertices of the 6-point are points of a point conic.</i></p>	<p><i>If the three joins of pairs of opposite vertices of a simple 6-line are concurrent, the sides of the 6-line are lines of a line conic.</i></p>
---	--

We shall prove only the theorem on the left. Let $AF, DE = K$ and $CD, EF = L$ (Fig. 40). On the lines DE and EF we have the perspectivity

$$P_1DK \text{ ---} \overline{\frac{P_3}{\lambda}} P_2LF \text{ ---},$$

and E , the common point of the two supports, is self-corresponding. Hence

$$P_1DEK \text{ ---} \overline{\frac{P_3}{\lambda}} P_2LEF \text{ ---}.$$

Projecting these point-rows from A and C , respectively, we have

$$A(P_1DEK \text{ ---}) \overline{\wedge} C(P_2LEF \text{ ---})$$

or, what is the same,

$$A(BDEF \text{ ---}) \overline{\wedge} C(BDEF \text{ ---}).$$

Now the point conic through the five points A, B, C, D, F is completely determined by the projectivity

$$A(BDF \text{ ---}) \overline{\wedge} C(BDF \text{ ---}).$$

But AE and CE are homologous lines of this projectivity, for we have shown $A(BDEF \text{ ---})$ to be projective to $C(BDEF \text{ ---})$. Therefore E , the point of intersection of AE and CE , is a point of the point conic through A, B, C, D , and F . The six vertices, consequently, are points of the same point conic.

A special case, which deserves notice, arises when the simple 6-point has three nonconsecutive vertices, say $B, D,$ and $F,$ which are collinear. The projectivity of the last paragraph is then a perspectivity, i.e.,

$$A(BDF \text{ ---}) \overline{\wedge} C(BDF \text{ ---}),$$

and AE and CE are homologous lines projecting the same point, call it $R,$ on $BDF.$ Since AE and CE have the two points E and R in common we must, therefore, conclude that A, C, E are also collinear. The point conic is degenerate and consists of the two lines BDF and ACE (Art. 7.5); these, with the hexagon $ABCDEF$ and the Pascal points and line form a configuration of Pappus (Art. 6.10).

Pascal's Theorem and its converse state the necessary and sufficient condition that six points are points of a point conic, and Brianchon's Theorem and its converse state the necessary and sufficient condition that six lines are lines of a line conic.

8.3 Applications. We shall apply the theorems of the preceding articles to the following construction problems.

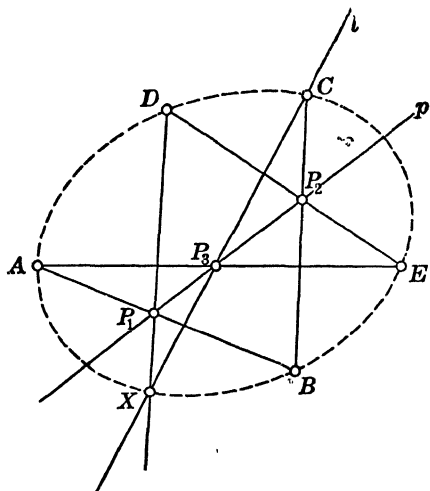


Fig. 42

- AB and XD meeting at $P_1,$
- BC and DE meeting at $P_2,$
- CX and EA meeting at $P_3.$

1. Given five points of a point conic and a line through one of them; to find the second point which the given line has in common with the point conic.

Let A, B, C, D, E be the given points and l the given line which passes through one of the given points, say C (Fig. 42). If X is the second point which l has in common with the point conic (C being the first), then the simple hexagon $ABCXDE$ has the opposite sides

Now we can determine P_2 and P_3 and hence $p,$ the Pascal line of the hexagon, for $CX = l.$ P_1 is the intersection of AB and $p,$ and P_1D cuts l at the desired point $X.$ We suggest the following symbolic statement of this construction, which is self-explanatory:

The simple 6-point is $ABCXDE$, where $CX = l$ is known. Hence,

$$\left. \begin{array}{l} AB, XD = \\ BC, DE = P_2 \\ l, EA = P_3 \end{array} \right\} p, AB = P_1; P_1D, l = X.$$

Another construction which will lead to the same point X uses the simple 6-point $ABXCDE$, where $XC = l$ is known. Hence,

$$\left. \begin{array}{l} AB, CD = P_1 \\ BX, DE = \\ l, EA = P_3 \end{array} \right\} p, DE = P_2; P_2B, l = X.$$

Let the reader satisfy himself that, by this construction, the same point X is obtained. He can do this by carrying out both constructions with the same five given points and the same given line. He should notice, however, that the simple hexagons $ABCXDE$ and $ABXCDE$ are not the same; hence the lines labeled p in the two constructions will not be the same.

In fact, the six points A, B, C, D, E, X give rise to $(6!/2 \cdot 6) = 60$ different simple hexagons and hence 60 different Pascal lines. The 60 Pascal lines connected with six points of a point conic constitute a remarkable configuration known as the **mystic hexagram**.¹ In the problem under discussion it is evident that the number of hexagons which may be used is somewhat less than 60; for if l is to be a side of the hexagon, C and X must be consecutive vertices. The number of such hexagons is $2(5!/2 \cdot 5) = 24$.

2. *Given five lines of a line conic and a point on one of them; to find the second line which the given point has in common with the line conic.*

Let a, b, c, d, e be the given lines and S the given point which lies on one of the given lines, say d (Fig. 43). If x is the second line of the line conic through S (d being the first), then the simple 6-line $abcdxe$ has the opposite vertices

$$\begin{array}{l} ab \text{ and } dx \text{ joined by } p_1, \\ bc \text{ and } xe \text{ joined by } p_2, \\ cd \text{ and } ea \text{ joined by } p_3. \end{array}$$

¹ Salmon, *Conic Sections*, London, 1879, p. 379. For a collection of interesting figures the reader is referred to the thesis of Anne and Elizabeth Linton, *Pascal's Mystic Hexagram*, Philadelphia, 1921.

Now we can determine p_1 and p_3 and hence P , the Brianchon point, for $dx = S$. The line p_2 is the join of P and bc , and p_2e is joined to S by the desired line x . We may indicate the construction thus:

The simple 6-line is $abcdxe$, where $dx = S$ is known. Hence,

$$\left. \begin{array}{l} ab, S = p_1 \\ bc, xe = \\ cd, ea = p_3 \end{array} \right\} P, bc = p_2; p_2e, S = x.$$

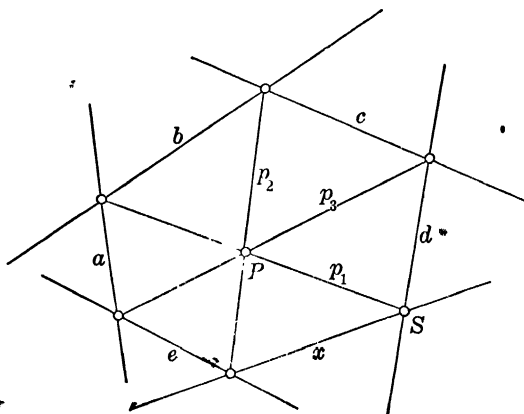


FIG. 43

8.4 Degenerate Cases of the Theorems. Poncelet, in his *Traité des propriétés projectives des figures* (1822), formulated the **principle of continuity**. The principle of continuity says, in substance, that any property of a geometric figure which continues to be a property of the figure, as the figure varies in a prescribed manner, will be a property of the figure in its limiting position. This principle applied to the theorems of the present chapter leads immediately to several important **degenerate** cases.

It should be remarked, however, that proofs of these so-called degenerate cases can readily be given without an appeal to the principle of continuity. And for one who prefers not to use the limiting process of the principle of continuity, there are, in Appendix B, other proofs of these same theorems.

In particular, with regard to Pascal's Theorem the continuity principle asserts that, since the intersections of opposite sides of a simple hexagon are collinear as long as the six vertices are points on a point conic, the theorem is valid also when the hexagon takes

the limiting position which it approaches as two of its vertices approach coincidence. Thus, in the notation of this chapter, as F approaches A while remaining on the point conic, the simple hexagon $ABCDEF$ approaches as its limiting position the simple pentagon (5-point) with a tangent to the conic at one of its vertices. Concerning this simple 5-point and the tangent to the point conic at one vertex, we have

THEOREM 1. *If the vertices of a simple 5-point are points of a point conic, the tangent to the point conic at one of the vertices meets the side opposite that vertex at a point which is collinear with the points of intersection of the other two pairs of nonadjacent sides.*

THEOREM P1. *If the sides of a simple 5-line are lines of a line conic, the point of contact of the line conic on one of the sides is joined to the vertex opposite that side by a line which is concurrent with the lines joining the other two pairs of nonadjacent vertices.*

For the theorem on the left, and dually for that on the right, we introduce a convenient notation. Let the vertices of the 5-point

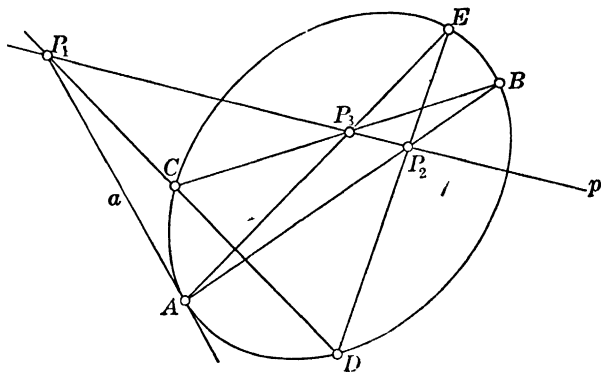


FIG. 44

be A, B, C, D, E , and let a be the tangent to the point conic at A (Fig. 44). We shall indicate the degenerate hexagon, i.e., the pentagon and the tangent at one vertex, by $AABCDE$. Thus, according to the theorem,

$$\begin{aligned} AA &= a \text{ meets } CD \text{ at } P_1, \\ AB &\text{ meets } DE \text{ at } P_2, \\ BC &\text{ meets } EA \text{ at } P_3, \end{aligned}$$

and P_1, P_2, P_3 are collinear. We note that CD is the side opposite vertex A of the simple 5-point (Art. 2.6), and that the four remaining sides AB, BC, DE, EA fall into the two pairs of non-adjacent sides AB, DE and BC, EA .

Let us again apply the principle of continuity, this time to the hexagon $ABECDF$ which varies in such a way that two of its vertices, F and E , always remaining on the point conic, approach the vertices A and B , respectively. The limiting position of the hexagon is a simple quadrangle $ABCD$ with tangents to the point conic at A and B , and we have

THEOREM 2. *If the vertices of a simple quadrangle $ABCD$ are points of a point conic, the tangent at A and the side BC , the tangent at B and the side DA , and the pair of sides AB and CD meet in three collinear points.*

THEOREM P2. *If the sides of a simple quadrilateral $abcd$ are lines of a line conic, the point of contact on a and the vertex bc , the point of contact on b and the vertex da , and the pair of vertices ab and cd are joined by three concurrent lines.*

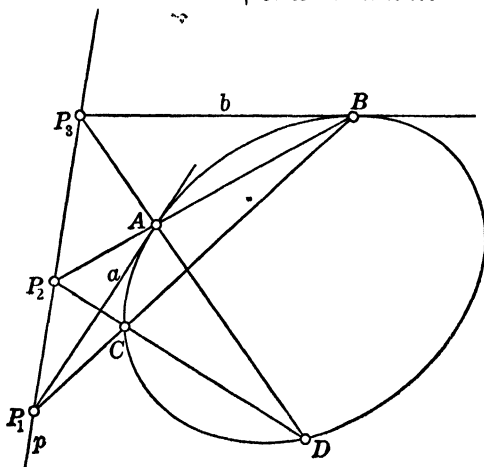


FIG. 45

That is, to use the suggested notation, the degenerate hexagon $AABBCD$ has its opposite sides (Fig. 45)

- $AA' = a$ and BC meeting at P_1 ,
- AB and CD meeting at P_2 ,
- $BB' = b$ and DA meeting at P_3 ,

which three points are collinear.

THEOREM 3. *If the vertices of a complete quadrangle are points of a point conic, the tangents at a pair of vertices meet at a point collinear with those diagonal points of the quadrangle which do not lie on the side joining that pair of vertices.*

THEOREM P3. *If the sides of a complete quadrilateral are lines of a line conic, the points of contact on a pair of sides are joined by a line concurrent with those diagonal lines of the quadrilateral which do not lie on the point of intersection of that pair of sides.*

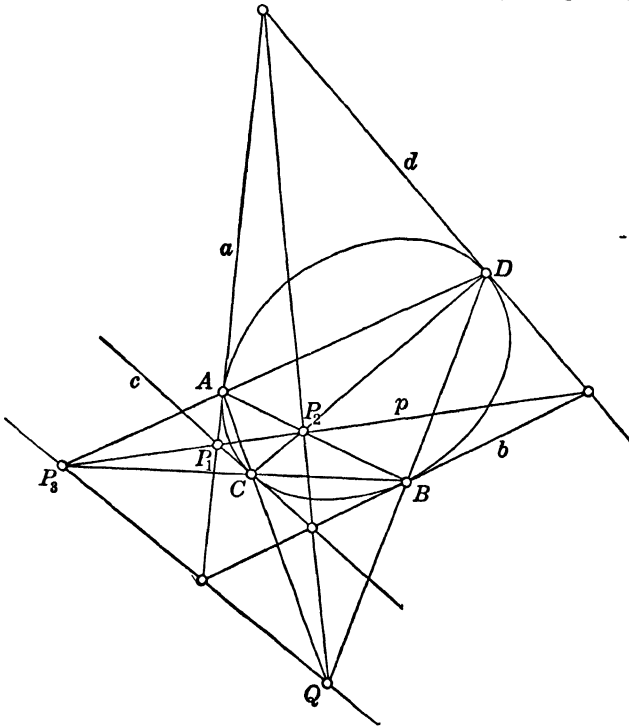


FIG. 46

The simple quadrangle \$ABCD\$ (Fig. 46) with tangents \$a\$ and \$c\$, at \$A\$ and \$C\$, respectively, is the limiting position of the hexagon \$ABECDF\$ when vertices \$F\$ and \$E\$ vary and approach \$A\$ and \$C\$, respectively. The sides \$AF\$ and \$CE\$ then approach \$a\$ and \$c\$, the tangents at \$A\$ and \$C\$. Hence, the degenerate hexagon \$AABCCD\$ has opposite sides

$$\begin{aligned}
 AA &= a \text{ and } CC = c \text{ meeting at } P_1, \\
 AB &\text{ and } CD \text{ meeting at } P_2, \\
 BC &\text{ and } DA \text{ meeting at } P_3,
 \end{aligned}$$

and P_1, P_2, P_3 are collinear points. But $ABCD$, considered as a complete quadrangle, has opposite sides

$$\begin{aligned} AB \text{ and } CD \text{ meeting at } P_2, \\ AC \text{ and } BD \text{ meeting at } Q, \\ AD \text{ and } BC \text{ meeting at } P_3. \end{aligned}$$

That is, P_2, Q, P_3 are diagonal points of the complete quadrangle $ABCD$. Of these diagonal points, Q lies on side AC joining the vertices at which a and c are tangents. Consequently, $ac = P_1$ is collinear with P_2 and P_3 , the other two diagonal points. By the same theorem, b and d , the tangents at B and D , intersect on the line P_2P_3 also; and a, b meet at a point on QP_3 , etc. We state these results in a slightly different form.

COROLLARY. *A complete quadrangle whose vertices are points of a point conic and the complete quadrilateral whose sides are tangents to the point conic at these vertices have the same diagonal triangle.*

THEOREM 4. *If the vertices of a triangle (3-point) are points of a point conic, the points of intersection of each side of the triangle with the tangent to the point conic at the opposite vertex are collinear.*

THEOREM P4. *If the sides of a triangle (3-line) are lines of a line conic, the joins of each vertex of the triangle with the point of contact of the line conic on the opposite side are concurrent.*

The reader can easily supply those proofs of the above theorems which are lacking.

We have called attention to the fact that each of these theorems may be proved directly without recourse to the concept of limit. To illustrate the spirit of such a proof we shall give, in the next article, an alternative proof of Theorem 3, page 103.

8.5 Alternative Proof of the Complete Quadrangle Theorem.

Let the complete quadrangle be $ABCD$ (Fig. 46) and let a, c be the tangents at A, C respectively. Choosing A and C as centers of two flat pencils projecting the point conic, we have (Art. 7.2)

$$A(P_1BCD \text{ ---}) \bar{\wedge} C(ABP_1D \text{ ---}),$$

where $P_1 = ac$. Sections of these flat pencils by CD and DA , respectively, are projective; i.e., if $K = AP_1, CD$ and $L = CP_1, DA$,

$$KP_2CD \text{ ---} \bar{\wedge} AP_3LD \text{ ---}.$$

But these point-rows have their common point D as self-corresponding and, consequently,

$$KP_2CD \text{ --- } \overline{\wedge} AP_3LD \text{ ---}.$$

The center of this perspectivity is the intersection of KA ($= a$) and CL ($= c$), i.e., it is P_1 . Therefore, the homologous pair P_2, P_3 is collinear with P_1 .

As in Theorem 3, Art. 8.4, the complete quadrangle $ABCD$ has diagonal points P_2, Q, P_3 , and P_1 is collinear with P_2 and P_3 , the diagonal points which do not lie on side AC .

The proof of the dual theorem we omit. The reader will have no difficulty in proving it for himself.

8.6 Construction Problems. The degenerate cases discussed in Art. 8.4 enable us to carry out the following constructions:

1. Given five points of a point conic; construct the tangent to the point conic at one of the given points.

2. Given four points of a point conic and the tangent to the point conic at one of the points;

a. Construct additional points of the point conic.

b. Construct the tangent to the point conic at one of the given points.

3. Given three points of a point conic and the tangents to the point conic at two of the points;

a. Construct additional points of the point conic.

b. Construct the tangent to the point conic at the third given point.

P1. Given five lines of a line conic; construct the point of contact of the line conic on one of the given lines.

P2. Given four lines of a line conic and the point of contact of the line conic on one of the lines;

a. Construct additional lines of the line conic.

b. Construct the point of contact of the line conic on one of the given lines.

P3. Given three lines of a line conic and the points of contact of the line conic on two of the lines;

a. Construct additional lines of the line conic.

b. Construct the point of contact of the line conic on the third given line.

Each of the above problems may be solved by the method of Art. 7.3, which involves setting up the projectivity and finding the center or axis of homology of that projectivity. Should the reader

compare this method with the one to follow, he will without doubt agree that the method now to be presented is much less involved. It depends, as we have said, on the theorems of Art. 8.4.

We shall solve, now, three of these problems.

1. Let the five points be A, B, C, D, E . To construct a , the tangent at A , we consider the degenerate hexagon $AABCDE$, the side AA being the desired tangent a . Expressing the construction symbolically, as in Art. 8.3, we have:

The degenerate 6-point is $AABCDE$, where $AA = a$. Hence,

$$\begin{array}{l} \bar{a}, CD = \\ AB, DE = P_2 \\ BC, EA = P_3 \end{array} \left. \vphantom{\begin{array}{l} \bar{a}, CD = \\ AB, DE = P_2 \\ BC, EA = P_3 \end{array}} \right\} p, CD = P_1; P_1A = a.$$

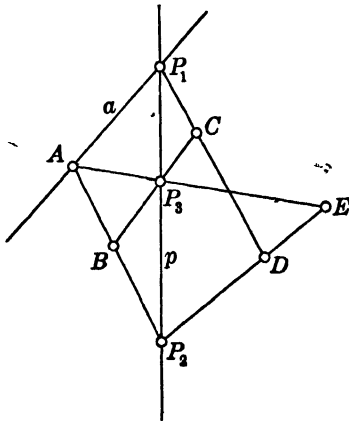


FIG. 47

Verbally, this construction of a is: $P_2 = AB, DE$ is joined to $P_3 = BC, EA$ by the Pascal line p . Now p meets CD at P_1 , which, by Theorem 1, Art. 8.4, lies on a . Therefore P_1A is the desired tangent at A (Fig. 47).

P2a. Let the four lines be a, b, c, d and A the point of contact on a . To construct a line of the line conic we take any point S on line a and find through S the second line x (a being the first) of the line conic.

The degenerate 6-line is $xaabcd$, where $xa = S$ and $aa = A$. Hence,

$$\begin{array}{l} S, bc = p_1 \\ A, cd = p_2 \\ ab, dx = \end{array} \left. \vphantom{\begin{array}{l} S, bc = p_1 \\ A, cd = p_2 \\ ab, dx = \end{array}} \right\} P, ab = p_3; p_3d, S = x.$$

Verbally, we determine p_1 as the join of S and bc , and p_2 as the join of A and cd . The meet of p_1, p_2 is the Brianchon point P , which is joined to ab by p_3 . Now p_3 and d and the unknown line x are concurrent; therefore, x is the join of p_3d and S . This construction follows from Theorem P1, Art. 8.4. Repetition of the construction for a different point S (on a or on any other of the given lines) gives another line of the line conic. And thus may be found as many lines of the line conic as one desires.

3a. Let A, B, C be the three points, and a, b the tangents at A, B . If l is a line through C which cuts the point conic a second time at X , we can determine X as follows:

The degenerate 6-point is $AABBCX$ where $AA = a, BB = b, CX = l$. Hence,

$$\begin{aligned} a, BC = P_1 & \rangle p, b = P_3; P_3A, l = X. \\ AB, l = P_2 & \\ b, XA = & \end{aligned}$$

This construction is based on Theorem 2, Art. 8.4.

On the other hand, if l is a line through A , we have:

The degenerate 6-point $AAXBBC$ where $AA = a, AX = l, BB = b$. Hence,

$$\begin{aligned} a, b = P_1 & \rangle p, CA = P_3; P_3B, l = X. \\ l, BC = P_2 & \\ XB, CA = & \end{aligned}$$

This construction is an application of Theorem 3, Art. 8.4.

8.7 Metric Specializations. The problems of the preceding article admit of many variations. These arise when we consider one or two of the given points or one of the given lines as ideal. With such specializations, however, the principle of duality is no longer valid.² For example, consider the following:

Given a point A and two asymptotes of an hyperbola; to construct the tangent at A .

This is a metric specialization of Problem 3b, Art.

8.6; the three given points are A and the two ideal points of the asymptotes, and the two given tangents are these asymptotes. Let

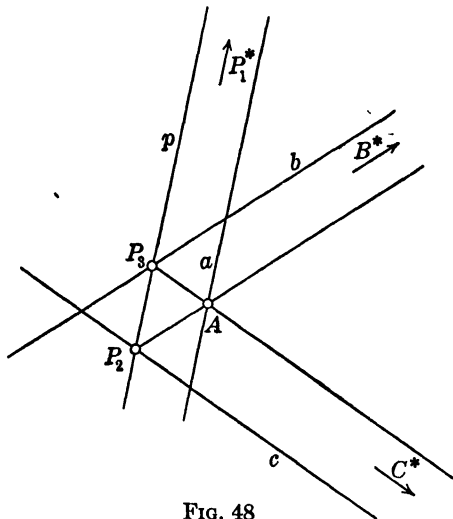


FIG. 48

² See Art. 7.6 and Appendix A. In metric specializations of this kind it is convenient to represent ideal elements by letters with an asterisk. Thus, the data of the problem to follow may be indicated by the symbol (AB^*C^*bc) .

the ideal points be B^* and C^* , and the tangents thereat, b and c . To construct a , the tangent at A , we have (Fig. 48):

The degenerate 6-point $AA B^* B^* C^* C^*$, where $AA = a$, $B^* B^* = b$, $C^* C^* = c$, and $B^* C^* = l^*$, the ideal line. Hence,

$$\left. \begin{array}{l} a, l^* = \\ AB^*, c = P_2 \\ b, C^* A = P_3 \end{array} \right\} p, l^* = P_1^*; P_1^* A = a.$$

8.8 A Relation between Point and Line Conics.

Theorem IX. *The tangents to a point conic constitute a line conic.*

Theorem P IX. *The points of contact of a line conic constitute a point conic.*

For a proof of the theorem on the left, consider the fixed points A, B, C of the given point conic and the tangents a, b, c at those

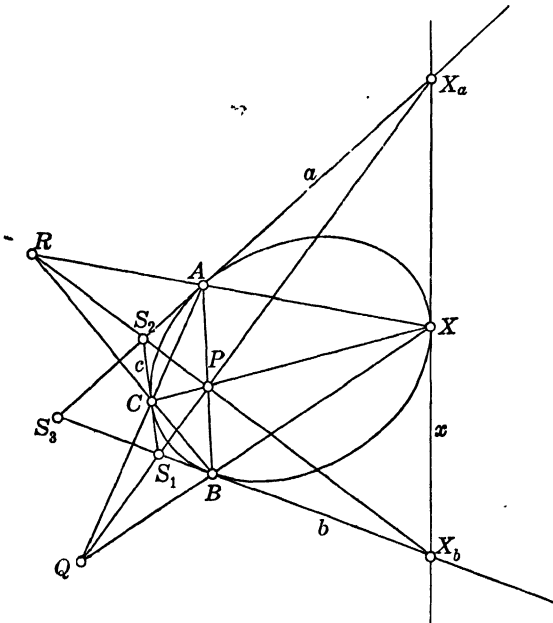


FIG. 49

points respectively (Fig. 49). From another point of view these three points and any two of the tangents suffice to determine completely the point conic; the tangent at the third vertex may then be constructed by Theorem 4, Art. 8.4.

Now let X be a variable fourth point on the point conic and x the tangent at that point. The four points determine a complete quadrangle $ABCX$ with diagonal points $P = AB, CX$; $Q = AC, BX$; and $R = AX, BC$. To this figure we apply Theorem 3, Art. 8.4, with the following results: $bc = S_1$ and $xa = X_a$ are collinear with P and Q ; $ca = S_2$ and $xb = X_b$ are collinear with R and P .

Consider now the projectivity set up on the lines a and b by the three pairs of homologous elements A and $S_3 = ab$, S_2 and S_1 , S_3 and B . The axis of homology of this projectivity is the join of AS_1, S_2S_3 (or A) and S_2B, S_1S_3 (or B); that is, it is the line AB . According to the procedure of Art. 6.10, the homologue of any point X_a on a is determined by finding P , the meet of S_1X_a and AB , and joining P to S_2 ; the line S_2P cuts b at the point homologous to X_a . But this point is X_b . Therefore the totality of lines $x = X_aX_b$ constitutes the line conic defined by the projectivity

$$a(AS_2S_3 \text{ ---}) \overline{\wedge} b(S_3S_1B \text{ ---}).$$

The term "conic" introduced in Art. 7.1 we now define more precisely as a point conic together with its tangents or, dually, as a line conic together with its points of contact. In this sense "conic" is a self-dual term.

8.9 Chasles' Theorem. Theorem X. *If A is any point and b any tangent of a conic and if X is a variable point of the conic and x its tangent, the flat pencil $A(X \text{ ---})$ is projective to the point-row $b(x \text{ ---})$.*

This follows immediately from the preceding theorem. For (Fig. 49).

$$A(X \text{ ---}) \overline{\wedge} C(X \text{ ---}),$$

by Property 3, Art. 7.2. The section of $C(X \text{ ---})$ by AB gives

$$\hat{C}(X \text{ ---}) \overline{\wedge} P \text{ ---},$$

the right member of this perspectivity being a point-row on AB . But

$$P \text{ ---} \overline{\wedge}^{S_2} b(x \text{ ---}).$$

Hence,

$$A(X \text{ ---}) \overline{\wedge} b(x \text{ ---}),$$

and the theorem is demonstrated.

EXERCISES

1. Prove Theorem 1, Art. 8.4, without using the principle of continuity. Dualize.
2. Prove Theorem 2, Art. 8.4, without using the principle of continuity. Dualize.
3. Prove Theorem 4, Art. 8.4, using Theorem 2, Art. 6.11. Dualize.
4. State the converses of the theorems of Art. 8.4.
5. A point-row and a coplanar flat pencil are projectively related, i.e., $u(ABC \text{---}) \bar{\wedge} S(a'b'c' \text{---})$. Show that the perpendicular lines p from points P of u onto the homologous lines p' of S are tangents to a parabola or are concurrent.
6. A conic ($ABCDE^*$) is given by five of its points, and one of them is an ideal point. Find the second point of intersection with the conic of a line on one of the given ordinary points; and, also, of a line on the given ideal point.
7. Using the data of Ex. 6 find the second point common to the conic and the ideal line.
8. A parabola ($abcde^*$) is given by four of its ordinary tangents. Construct the second tangent to the parabola from a point on one of the given ordinary lines; and, also, from an ideal point.
9. Using the data of Ex. 8 construct the ideal point of the parabola.
10. An hyperbola (AB^*C^*ab) is given by one of its ordinary points, the tangent thereat, one asymptote, and the direction of the second asymptote. Construct the second point of intersection with the hyperbola of a line on the given ordinary point; and, also, of a line parallel to the given asymptote.
11. Using the data of Ex. 10 construct the second asymptote of the hyperbola.
12. An hyperbola ($abcBC^*$) is given by two of its tangents, the point of contact on one of them, and an asymptote. Construct the tangent to the hyperbola from a point on the given asymptote; and also, that tangent to the hyperbola which is parallel to the given tangent whose point of contact is not known.
13. Using the data of Ex. 12 construct the point of contact on the given tangent.
14. Using the data of Ex. 12 construct the second asymptote of the hyperbola.
15. Construct by the methods of this chapter an ellipse, considered as (a) a point conic; (b) a line conic.
16. Construct by the methods of this chapter an hyperbola, considered as (a) a point conic; (b) a line conic.
17. Construct by the methods of this chapter a parabola, considered as (a) a point conic; (b) a line conic.

CHAPTER IX

THE THEORY OF POLE AND POLAR

9.1 Definitions.

The **polar** with respect to a given conic of a point X not on the conic is the line x joining the two diagonal points of a complete quadrangle whose vertices are points of the conic and whose third diagonal point is X (Fig. 50).

The **polar** x of a point X on the given conic is the tangent to the conic at X .

The **pole** with respect to a given conic of a line x not touching the conic is the point of intersection X of the two diagonal lines of a complete quadrilateral whose sides are lines of the conic and whose third diagonal line is x (Fig. 50).

The **pole** X of a line x touching the given conic is the point of contact of the conic on x .

At first, the reader may think that the polar of a point (and dually, the pole of a line) with respect to a fixed conic is not unique. Or, in other words, since an unlimited number of quadrangles $ABCD$ having X as one diagonal point may be drawn, he may think a different polar of X is determined by each quadrangle.

Consider, however, the complete quadrangle of Fig. 50 whose vertices are points of the conic and whose diagonal points are X , Y , and Z . If X is the intersection of the sides AB and CD and if a and b are the tangents to the conic at A and B , then the point ab lies on YZ ($= x$), by Theorem 3, Art. 8.4. Moreover, if X_1 is the meet of AB and x , the complete quadrangle $CDYZ$ exhibits the four points X , A , X_1 , B on line AB as an harmonic set $H(AB, XX_1)$ — this by the definition of Art. 4.1. Now the line x ($= YZ$) may be determined without reference to the points C and D . For x is the join of ab and X_1 . Furthermore, if the secant ABX remains fixed while the secant CDX turns about X , we have, for each position of the variable secant, two points definitely determined which lie on x . They are cd , the intersection of the tangents at C and D , and X_2 , the harmonic conjugate of X with respect to C and D . Hence, for each point X there is one and only one polar line x with respect to the conic.

Another question arises: If x is the polar of X with respect to a given conic, is X the pole of x with respect to the same conic? That the answer to this is in the affirmative, when X is not on the conic, is evident from Fig. 50 and the corollary to Theorem 3, Art. 8.4. For the quadrilateral $abcd$ and the quadrangle $ABCD$ have the same diagonal triangle, and the point yz is X . And when

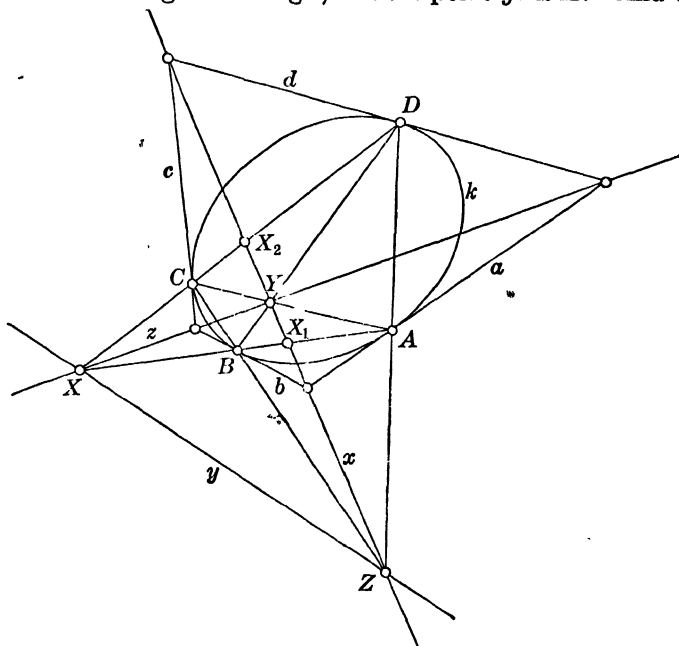


FIG. 50

X is on the conic, the answer is still in the affirmative, by the definitions. Hence, the

THEOREM. *If x is the polar line of a point X with respect to a conic, then X is the pole of x with respect to the same conic.*

9.2 Properties of Pole and Polar. As immediate consequences of the above definitions of pole and polar and the discussion as to their uniqueness we have:

1. If X is a point in the plane of a given conic, the tangents to the conic at two curve points collinear with X intersect on the polar of X with respect to the conic.

P1. If x is a line in the plane of a given conic, the points of contact of the conic on two curve lines concurrent with x are joined by a line on the pole of x with respect to the conic.

2. If X is a point in the plane of a given conic, the harmonic conjugate of X with respect to any pair of curve points collinear with X lies on the polar of X with respect to the conic.

P2. If x is a line in the plane of a given conic, the harmonic conjugate of x with respect to any pair of curve lines concurrent with x lies on the pole of x with respect to the conic.

If T is the point of contact on a tangent to the conic from a point X , and C, D are two curve points collinear with X , the tangents c, d to the conic at C, D intersect the lines TD, TC at S, U respectively (Fig. 51). Now if $cd = R$, the complete quadrangle $RSTU$ has C and D as diagonal points, and a fifth side SU passes through X , by Theorem 4, Art. 8.4. The sixth side RT , therefore, lies on $X_2 = H(X, CD)$, and is, by Properties 1 and 2, the polar line of X with respect to the conic. This proves

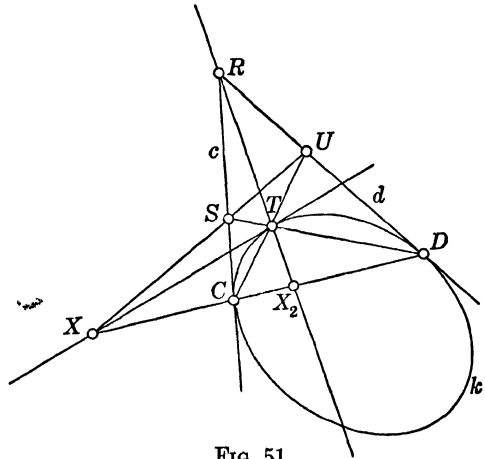


FIG. 51

3. If X is a point in the plane of a given conic, the points of contact on the tangents from X , if such tangents exist, lie on the polar of X with respect to the conic.

P3. If x is a line in the plane of a given conic, the tangents at the curve points on x , if such curve points exist, lie on the pole of x with respect to the conic.

The conditional clause in Property 3 and its dual, "if such . . . exist," is seen intuitively to be necessary. It leads to the following considerations. We may consider a conic as a closed curve in the sense that it is possible for a variable point, by moving from any given initial position in either one of two ways but always remaining on the curve, to traverse the whole, and finally arrive at its original position again. By such a curve the plane is divided into two regions, customarily called the inside and outside.

Two points A and B are said to lie in the same region if it is possible to move from A to B without crossing the curve. Otherwise the points A and B lie in different regions. We must now decide which of the two regions of a conic is to be called the inside and which the outside.

| We define the **inside** of a conic to be that region of the plane in which, if any point be chosen, all lines through the point cut the conic in two points. That is, every line through an inside point is a secant, and no tangents can be drawn to a conic from such a point.

On the other hand, the **outside** of a conic is defined to be that region of the plane in which, if any point be chosen, some lines through the point cut the conic and some do not. Hence, from a point outside the conic two tangents may be drawn to the curve.

We observe, therefore, that the polar p of a point P outside the conic cuts the curve in two points, the points of contact of the tangents from P ; and if P lies inside the conic, its polar p lies entirely outside, for each point on p is, in this case, harmonically separated from P by two points of the conic.

Similarly, the pole P of a line p which cuts the conic has two tangents passing through it, the tangents at the curve points on p , and lies outside the conic; and if p lies entirely outside, its pole P lies inside, for each line on P is, in this case, harmonically separated from p by two tangents of the curve.

9.3 The Construction of Pole and Polar. The problem of constructing the polar of a point, or the pole of a line, with respect to a given conic is easily solved when the conic is fully drawn by direct application of the definitions and properties of the preceding articles. For example, the polar of a point X with respect to the conic of Fig. 50 can be determined in a number of ways. Through X draw any two secants ABX and CDX . The polar x is then uniquely determined as the join of any two of the six points ab, cd, X_1, X_2, Y, Z .

In case the conic is not fully drawn but is given by any one of the sets of conditions of Art. 7.3, the polar of a given point, or, dually, the pole of a given line, follows in a straightforward manner. We shall illustrate the various ramifications of the problem by two examples.

1. *Given five points A, B, C, D, E of a conic k and a sixth point X not on the conic, to construct x , the polar of X with respect to k .*

We draw XA and XB and find on these two lines the points A' and B' where they cut the conic k a second time. This involves the method of Art. 8.3. Now the complete quadrangle $AA'B'B'$ has X as one diagonal point; the other two diagonal points, $Y = AB$, $A'B'$ and $Z = A'B'$, $A'B$, determine the polar line $x = YZ$, by definition.

We call attention to the fact that in this problem the given elements are essentially point conic k and point X . If the given elements should be a combination of points and lines, such as in the example which follows, the resulting construction is more involved.

2. *Given four lines a, b, c, d and point A , the point of contact on a , of a conic k and a point X not on the conic, to construct x , the polar of X with respect to k .*

In this problem the given elements are the conic k , which is essentially a line conic, and a point X . As in Problems P2b and P3b, Art. 8.6, we construct the points of contact B, C on lines b, c , respectively. The conic k may now be looked upon as a point conic determined by three of its points A, B, C and the tangents at two of them. Now find A', B' , where XA, XB cut the conic a second time. The given point X is one diagonal point of the quadrangle $AA'B'B'$ whose vertices are points of k ; and the required polar is the join of the other two diagonal points.

9.4 Conjugate Points and Lines. Theorem XI. *If a point P lies on the polar q of a second point Q with respect to a given conic k , then Q lies on the polar p of P .*

Case 1. When Q lies inside the conic. In this case the polar q and, consequently, P lie outside the conic. Now the line PQ cuts the conic k at two points A and B , which are harmonically separated by P and Q . Hence $Q = H(P, AB)$, and, since the harmonic conjugate of P with respect to two curve points collinear with P lies on the polar of P , Q lies on p .

Case 2. When Q lies on the conic. The point P lies on the tangent to k at Q , and is outside the conic. But p , the polar of P , passes through the points of contact of the tangents from P . Since Q is one of these points of contact, Q lies on p .

Case 3. When Q lies outside the conic. In this case P may be either inside or outside k , for q cuts the conic. But, in either case, the tangents at the curve points on any secant through P (and q is such a secant) intersect at a point on the polar of P .

We know that the tangents at the curve points on q meet at Q . Hence Q lies on p , the polar of P .

The dual of Theorem XI we leave to the reader for statement and proof.

DEFINITION. Two points, in the plane of a conic k , are **conjugate points** with respect to k if each point lies on the polar (with respect to k) of the other.

DEFINITION. Two lines, in the plane of a conic k , are **conjugate lines** with respect to k if each line lies on the pole (with respect to k) of the other.

By Theorem XI, it is evident that P and Q are conjugate points if Q lies on p , the polar of P . For then, by the theorem, P lies on q . We speak of each point as being **conjugate** to the other with respect to the conic.

9.5 Two Theorems. A given point P , in the plane of a fixed conic k , is conjugate to all points on p , its polar line. Or, in other words, any point on p and P itself are conjugate points. It follows, therefore, that with each line q through P there is associated one point Q , the pole of the line q , which lies on p , and P and Q are conjugate points with respect to k . Conversely, with each point on p there is associated one line through P , namely, the polar of the point on p .

In general, we may say that in the plane of a given conic k there is associated with each flat pencil $X(y \text{ ---})$ a point-row $x(Y \text{ ---})$, where x is the polar of X and Y is the pole of y with respect to k .

THEOREM 1. *In the plane of a fixed conic k a point-row $x(Y \text{ ---})$ is projectively related to its associated flat pencil of polar lines $X(y \text{ ---})$.*

In Fig. 50, consider the point X and its polar x with respect to the given conic k . Let XAB be a fixed secant cutting k at A and B , and let XCD be a variable secant turning about X and cutting k at C and D . Now for each position of XCD we have on x two points, Y and Z , definitely determined; they are, in fact, diagonal points of the complete quadrangle $ABCD$. Furthermore, they are conjugate points with respect to k .

On this line x , consequently, we have two superposed point-rows in a (1,1) correspondence such that homologous points Y, Z are conjugate points with respect to k . Is this correspondence a projective one? To answer this query we note, first, that the flat

pencils $A(C \text{ ---})$ and $B(C \text{ ---})$ are projectively related, A and B being two fixed points and C a variable point of k . Secondly,

$$x(Y \text{ ---}) \overline{\wedge} A(C \text{ ---}) \overline{\wedge} B(C \text{ ---}) \overline{\wedge} x(Z \text{ ---}).$$

Hence,

$$x(Y \text{ ---}) \overline{\wedge} x(Z \text{ ---}),$$

and the correspondence is a projective one.

But, since

$$x(Z \text{ ---}) \overline{\wedge} X(y \text{ ---}),$$

it follows that

$$x(Y \text{ ---}) \overline{\wedge} X(y \text{ ---}),$$

which proves the theorem when x is not tangent to k .

In case the line x of the theorem is tangent to k , its pole X is the point of contact of k on x . Then, if P is a variable point of k and p is the tangent at P , and if q is any fixed tangent, we have

$$X(P \text{ ---}) \overline{\wedge} q(p \text{ ---}),$$

by Theorem X, Art. 8.9, and

$$q(p \text{ ---}) \overline{\wedge} x(p \text{ ---}),$$

by Property P3, Art. 7.2. Hence,

$$x(p \text{ ---}) \overline{\wedge} X(P \text{ ---})$$

or, what is the same,

$$x(Y \text{ ---}) \overline{\wedge} X(y \text{ ---}),$$

where $Y = px$ and its polar with respect to k is $y = PX$. The proof of the theorem is now complete.

We have previously discussed projectivities on a one-dimensional form (Arts. 6.2, 6.11), and we classified them according to the number of their double elements. It will be seen upon rereading those articles that we have, in the second paragraph of the proof just given, a special type of projectivity. Ordinarily, in a projectivity on a one-dimensional form each element has two names or labels; e.g., the element A of the first form will have the label B' as an element of the second form, while the homologue of A (in the first form) will be A' (in the second form). Note that A and B' are two different names for the same element, and A and A' are the names of two homologous elements. When A and A' coincide, i.e., when $B' = A'$, the point A is a double point. In general, if A' is the homologue of A in a projectivity on a one-

THEOREM 2. *If the vertices of a triangle are points on a conic, any line conjugate to one side, with respect to the conic, cuts the other two sides in a pair of conjugate points, and conversely.*

THEOREM P2. *If the sides of a triangle are tangents to a conic, any point conjugate to one vertex, with respect to the conic, is joined to the other two vertices by a pair of conjugate lines, and conversely.*

On the left, let ABC (Fig. 52) be the given triangle inscribed in a conic k , and let S be the pole of BC with respect to k . The tangents to k at B and C are BS and CS . If now $R = AB$, CS and $T = AC$, BS , and if any line l through S' (and hence conjugate to BC) cuts AB at P and AC at Q , we have the perspective point-rows

$$ATCQ \text{ --- } \overline{\overline{S}} \wedge ABRP \text{ ---},$$

which are projected from B and C respectively by two projective flat pencils; for

$$B(ATCQ \text{ ---}) \overline{\overline{S}} \wedge ATCQ \text{ --- } \overline{\overline{S}} \wedge ABRP \text{ --- } \overline{\overline{S}} \wedge C(ABRP \text{ ---}),$$

or

$$B(ATCQ \text{ ---}) \overline{\overline{S}} \wedge C(ABRP \text{ ---}).$$

We rewrite this last in the equivalent form

$$B(ASCQ \text{ ---}) \overline{\overline{S}} \wedge C(ABSP \text{ ---}),$$

which exhibits $D = BQ, CP$ as a point of the conic defined by the projectivity

$$B(ASC \text{ ---}) \overline{\overline{S}} \wedge C(ABS \text{ ---}).$$

This conic is identical with k for it goes through the points A, B, C and has BS and CS as tangents at B and C . Hence D is a point of k .

To conclude, we note that P and Q are two diagonal points of the complete quadrangle $ABCD$ with vertices on k ; that is, the polar of Q goes through P , and P and Q are conjugate points with respect to k .

We leave for the reader the proofs of the converses.

9.6 Polar Reciprocation. The (1,1) correspondences which we have studied have been of two kinds, namely, those correspondences which pair like elements and those which pair unlike elements. The former we call **collineations**, and the latter **corre-**

lations. A fixed conic k determines a correlation between the points and lines of its plane by associating with each point of the plane its polar with respect to k , and with each line its pole with respect to k .

The projective plane, i.e., the plane of projective geometry, is an Euclidean plane to which have been adjoined the ideal line and the ideal points thereof on an equal basis with all other lines and points of the plane. Now, in such a plane, a fixed conic k determines a **polar system** or a correlation whereby each point of the plane is paired with one line, its polar with respect to k , and, reciprocally, each line is paired with one point, its pole. A consequence of this is that each flat pencil of the plane is paired with one point-row, the center of the flat pencil and the base of the point-row being in the relation of pole and polar with respect to k . Moreover, the flat pencil and its associated point-row are projectively related (Theorem 1, Art. 9.5). Two projectively related flat pencils U and V define, by the intersections X of homologous lines, a point conic G ; and in the polar system the **correlative** of this is a pair of projectively related point-rows u and v whose homologous points are joined by lines x of a line conic g . The line conic g is called the **correlative** or the **polar reciprocal** with respect to k of the point conic G .

The correlation defined by the polar system of the conic k is a special case of the principle of duality. It is special in the sense that, whereas the principle of duality merely *states* the existence of a dual figure, the theory of the polar system *determines definitely* the position in the plane of the polar reciprocal and, by the constructions of pole and polar (Art. 9.3), enables us to construct this polar reciprocal.

For example, if the conic G has a point A in common with k , the polar reciprocal of A is the line a which, since A is a point of k , must be the tangent to k at A . But a is a line of g . Hence, for each point common to G and k we have, correlatively, a line common to g and k .

As another example, let us consider the theorems of Pascal and Brianchon. By polar reciprocation with respect to a conic k , the hexagon (6-point) $ABCDEF$ with vertices on k becomes the hexagon (6-line) $abcdef$ with sides touching k . The Pascal line joining the three collinear points AB, DE and BC, EF and CD, FA becomes the Brianchon point of intersection of the three concur-

rent lines ab , de and bc , ef and cd , fa . From this it is evident that the Pascal line of an inscribed 6-point is the polar of the Brianchon point of the 6-line whose sides are tangents to the conic at the vertices of the 6-point.

A figure which coincides with its polar reciprocal is said to be **self-polar**. The triangle XYZ (or xyz) of Fig. 50 is self-polar with respect to the conic k ; for the polar of any vertex is the side opposite that vertex, and the pole of any side is the vertex opposite that side.

For a given conic k we may construct an unlimited number of self-polar triangles. For we may choose arbitrarily any point P in the plane of k ; on p , the polar of P , choose a second point Q and construct its polar q , which will pass through P . The intersection $R = pq$ is determined, and the triangle PQR is self-polar with respect to k . Only one of the points P , Q , R can lie inside k , a statement which we leave to the reader to prove. Let him also investigate the dual of the above construction.

9.7 Common Self-polar Triangle of Two Conics. By polar reciprocation or, if one prefers, by the corollary of Theorem 3, Art. 8.4, the diagonal triangle XYZ of the complete quadrangle $ABCD$ (Fig. 50), whose vertices are points of a conic k , and the diagonal triangle xyz of the complete quadrilateral $abcd$, whose sides are respectively the tangents to k at those vertices, are one and the same triangle. Four points do not determine a conic, and so there are conics other than k passing through A , B , C , D ; and, also, conics other than k which are tangent to a , b , c , d .² But all such conics have XYZ (or xyz) as a self-polar triangle.

Consider two conics k_1 and k_2 (Fig. 53) which have four points A , B , C , D in common. These conics also have four common tangents k , l , m , n , and a question immediately arises as to their common self-polar triangle: Is the self-polar triangle determined by the diagonal triangle of the complete quadrangle $ABCD$ the same as the self-polar triangle determined by the complete quadrilateral $klmn$? We answer this affirmatively, by the

THEOREM. *Two conics having four points and four tangents in common have one self-polar triangle in common, which may be determined as the diagonal triangle of either the complete 4-point or the complete 4-line common to the conics.*

² There is of course only one conic, viz., k , which passes through A , B , C , D and touches at the same time a , b , c , d .

For, let p, q, r be the diagonal triangle of the quadrilateral $klmn$, and therefore a self-polar triangle with respect to each of the conics k_1 and k_2 . The vertex $qr = P$ is the pole of the line p with respect to each of the conics. If the line PA cuts p at M and k_1, k_2 at B_1, B_2 , respectively, we have $B_1 = H(A, PM)$ and

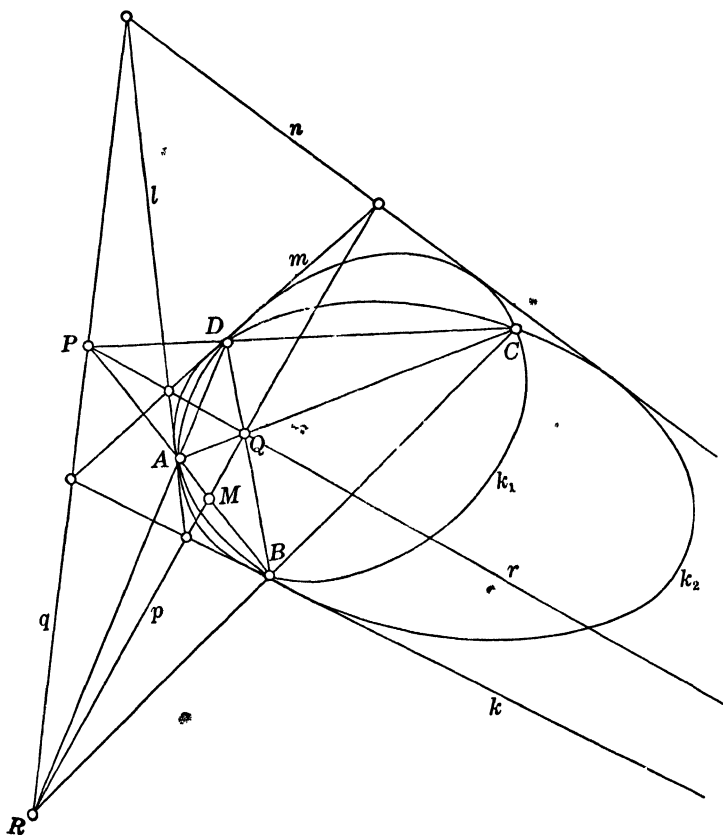


FIG. 53

$B_2 = H(A, PM)$; i.e., $B_1 = B_2$, and PA cuts k_1, k_2 a second time at one of their common points, say B . The point P is thus shown to be collinear with A and B ; and, in a similar manner, we may show that P, C , and D are collinear points, and hence $P = AB, CD$. Likewise $rp = Q$ is the intersection of AC and BD , and $pq = R$ is the intersection of AD and BC . Hence, PQR is the diagonal triangle of the complete quadrangle $ABCD$. In other words,

the self-polar triangle of two conics having four common points and four common tangents may be determined

(1) as the diagonal triangle of their common inscribed quadrangle, or

(2) as the diagonal triangle of their common circumscribed quadrilateral. These two triangles are identical.

• The student can easily show that k_1 and k_2 cannot have two common self-polar triangles.

9.8 Two Quadrangles or Two Quadrilaterals with a Common Diagonal Triangle.

THEOREM 1. *If two complete quadrangles have the same diagonal triangle, their eight vertices either lie in fours on two lines or else they all lie on one conic.*

THEOREM P1. *If two complete quadrilaterals have the same diagonal triangle, their eight sides either lie in fours on two points or else they all touch one conic.*

To prove the theorem on the left let $ABCD$ and $A'B'C'D'$ be the two quadrangles having the same diagonal triangle PQR . We may, without loss of generality, so letter the figure that

$$\begin{aligned} AB, CD \text{ and } A'B', C'D' \text{ all meet at } P, \\ AC, BD \text{ and } A'C', B'D' \text{ all meet at } Q, \\ AD, BC \text{ and } A'D', B'C' \text{ all meet at } R. \end{aligned}$$

Case 1. When three of the vertices are collinear. No three of the four points A, B, C, D can be collinear, nor can three of the four points A', B', C', D' (Art. 2.7); but it may be that one of the vertices of one quadrangle lies on a side of the other. For example, suppose that A' lies on AB . We see at once that B' must also lie on AB , for $A'B'P$ and ABP are one and the same line. Furthermore, since each of the two sets of four lines

$$\begin{aligned} AB, CD, PQ, PR \\ A'B', C'D', PQ, PR \end{aligned}$$

are projectors of harmonic ranges (Art. 4.1), they are harmonic pencils of lines. Consequently, if sides AB and $A'B'$ coincide, so also must sides $C'D'$ and CD . Hence the eight vertices lie in fours on two lines.

Case 2. When no three of the vertices are collinear. Five of the vertices, say A, B, C, D, A' , determine a conic k , and our prob-

lem is to show that the three remaining ones are points of this conic. Let K be the meet of PA' and QR (Fig. 54). The complete quadrangle $QC'RD'$ defines the harmonic set $H(A'B', PK)$, and therefore $B' = H(A', PK)$; and, since QR is the polar of P with respect to k , it follows, if PA' cuts k a second time at B'_1 , that $B'_1 = H(A', PK)$. Hence $B' = B'_1$, and B' is a point on

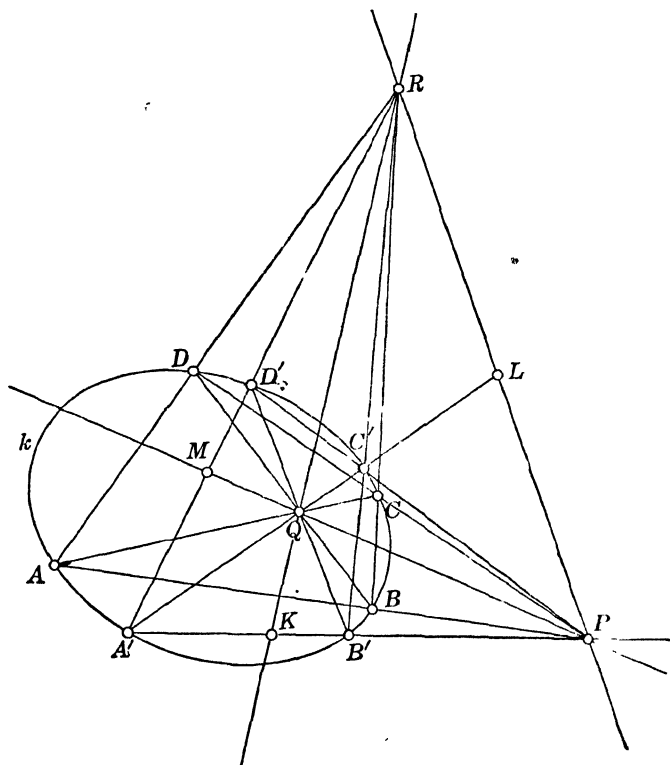


FIG. 54

k . A similar reasoning applied to QA' and again to RA' exhibits C' and D' as points of k also, and the proposition is demonstrated.

The proof of the dual theorem is, of course, unnecessary.

9.9 Quadratic Transformations. A (1,1) correspondence between two geometric forms is sometimes most conveniently thought of as a **transformation**, in the sense that it carries over, or transforms, one of the forms into the other form by carrying over, or transforming, each element of the one form into the homologous

element of the other. From this point of view, for example, one of two projectively related primitive forms is said to be transformed into the other by the projectivity. A projectivity between two superposed forms transforms the common support into itself; and if the projectivity is an identity (i.e., if it has more than two double elements), it transforms the common support into itself *identically*. Again, the correlation of a polar system (Art. 9.6) transforms a given form into a second form which is the polar reciprocal of the first with respect to the fixed conic of the correlation.

The correspondences heretofore studied have all been of the type which transforms a primitive form into a primitive form or a conic into a conic. There are numerous other types of transformations the study of which must be deferred for the present. We shall, however, give two examples of the so-called **quadratic transformations**, which differ from the transformations previously studied in that they transform a one-dimensional primitive form into a one-dimensional form of second degree.

<p>1. Given, in a plane, a fixed conic k and a fixed point P; the point corresponding to any point A of the plane is the intersection A' of PA and a, the polar of A with respect to k.</p>	<p>P1. Given, in a plane, a fixed conic k and a fixed line p; the line corresponding to any line a of the plane is the join a' of pa and A, the pole of a with respect to k.</p>
---	--

It is evident, is it not, that A and A' are conjugate points with respect to k and are, moreover, collinear with the fixed point P . The correspondence is involutory (Art. 9.5), for if a point B coincides with A' , then B' coincides with A .

There are, however, certain exceptional points of the plane which require closer investigation. In the first place, suppose that A is any point on p , the polar of P ; A' then coincides with P . On the other hand, if A coincides with P , PA and, therefore, A' are indeterminate.

The homologue of any point on p is P , and the homologue of P is indeterminate.

In the second place, suppose that P is outside k and that the tangents from P to k are PT_1 and PT_2 ; if T_1 and T_2 are the points of contact on these tangents, what are their homologues T'_1 , T'_2 ? Since T_1 , T_2 are points on p we would expect T'_1 , T'_2 to

coincide with P . But our expectations do not go far enough. The polar of T_1 , for example, is PT_1 , and this coincides with the line joining P to T_1 ; so we must say either that the homologue of T_1 is any point on PT_1 or that the homologue of T_1 is indeterminate, and the latter is probably the better statement. Similarly, the homologue of any point on PT_1 is T_1 , etc. Enough has been said to indicate the nature of this quadratic correspondence. Let us continue.

If the point A describes a line q which is not the polar of P , then A' , the homologue of A in the Transformation 1 above, generates a conic through P , and also through T_1, T_2 if P lies outside k .

If the line a turns about a point Q which is not the pole of p , then a' , the homologue of a in the Transformation P1 above, generates a conic touching p , and also touching t_1, t_2 if p cuts k .

Theorem 1, Art. 9.5, provides the clue to the theorem on the left. For if Q is the pole of q (Fig. 55), the chain

$$P(A \text{ ---}) \overleftrightarrow{\bar{\lambda}} \hat{q}(A \text{ ---}) \bar{\lambda} Q(a \text{ ---})$$

leads to

$$P(A \text{ ---}) \bar{\lambda} Q(a \text{ ---}),$$

and in this projectivity the line PA of the flat pencil on P is homologous to the line a of the flat pencil on Q . Whence $A' = PA$, a is a point of the conic defined by this projectivity, and the conic passes through P and Q .

There is another reason why P is a point of this conic. The line q cuts p at K , let us say. But K' (which is a point on the conic) coincides with P since K lies on p . Similarly, if L and M are the intersections of q with PT_1 and PT_2 , their homologues L' and M' coincide respectively with T_1 and T_2 , if, of course, P lies outside k . Hence, T_1 and T_2 are also points of the conic.

The proof is now complete, but to make our analysis complete we must recognize the special case which arises when PQ is tangent to k , or what is the same, when q passes through T_1 (or T_2). The projectivity

$$P(A \text{ ---}) \bar{\lambda} Q(a \text{ ---})$$

then has PQ as self-corresponding line and is, consequently, a perspectivity. The conic is, therefore, degenerate (Art. 7.5).

2. Given two conics k_1 and k_2 in the same plane; the point corresponding to any point A of the plane is A' , the meet of a_1 and a_2 which are the polars of A with respect to k_1 and k_2 .

P2. Given two conics k_1 and k_2 in the same plane; the line corresponding to any line a of the plane is a' , the join of A_1 and A_2 which are the poles of a with respect to k_1 and k_2 .

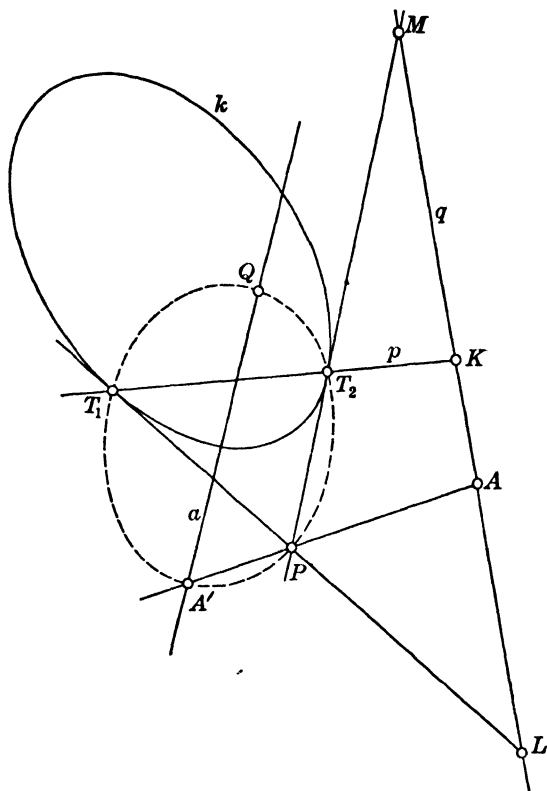


FIG. 55

Homologous points A and A' are conjugate points with respect to each of the given conics. Certain exceptions to the uniqueness of the correspondence arise when k_1 and k_2 have a common self-polar triangle PQR . For then, if A coincides with P , its homologue A' is any point on QR (i.e., indeterminate). Conversely, however, if A is any point on QR , A' coincides with P . A similar argument applies when A coincides with Q or R , or when A lies on PR or PQ .

If the point A describes a line s , then A' , the homologue of A in the Transformation 2 above, generates a conic through the vertices of the common self-polar triangle of k_1 and k_2 , if such exists.

If the line a turns about a point S , then a' , the homologue of a in the Transformation P2 above, generates a conic touching the sides of the common self-polar triangle of k_1 and k_2 , if such exists.

The discussion of this is so much like that of the preceding theorem that we shall leave it as an exercise for the reader. May we remind him, however, not to neglect the exceptional cases.

EXERCISES

1. A conic ($ABCDd$) is given by four of its points and the tangent at one of them. Construct the polar line with respect to this conic of a given point.
2. A conic ($abcAB$) is given by three of its tangents and the points of contact on two of them. Construct the pole with respect to this conic of a given line.
3. Using the data of Ex. 1 construct the pole with respect to that conic of a given line.
4. Using the data of Ex. 2 construct the polar with respect to that conic of a given point.
5. Two points, one asymptote, and the direction of the other asymptote of an hyperbola (ABC^*D^*d) are given. Construct (a) the polar of a given point; (b) the pole of a given ordinary line; and (c) the pole of the ideal line with respect to this hyperbola.
6. A parabola ($abcd^*A$) is given by three ordinary tangents and the point of contact on one of them. Construct (a) the polar of a given point; and (b) the pole of a given line with respect to this parabola.
7. If four collinear points form an harmonic set, their polars with respect to a given conic are concurrent and form an harmonic set.
8. If two tangents to a conic k vary in such a way that their chord of contact is tangent to a second conic k_1 , then their point of intersection will lie on a third conic k_2 .
9. If x is a variable tangent to a conic and X is its point of contact, its intersections with two fixed tangents a, b are harmonically separated by X and the intersection of x with the polar of point ab .
10. If two conics have only two points in common, show how to construct one vertex and the side opposite of their common self-polar triangle.
11. Construct the common self-polar triangle of two conics which have no common points.
12. If two triangles are each self-polar with respect to a conic k , their six vertices lie on a conic k_1 and their six sides touch a conic k_2 .
13. If two conics intersect in four points, the eight tangents at these four points either lie in fours on two points or they all touch a conic. Dualize.

14. If PQR is a self-polar triangle of a conic k and if the triangle ABC has its vertices on k and sides CA , AB passing through Q , R respectively, the side BC will pass through P . Furthermore, the vertices on each side of ABC are harmonically separated by the vertex of the self-polar triangle on that side and the point common to that side and the opposite side of the self-polar triangle. Triangles PQR and ABC are also perspective.

15. We are given three points A , B , C of a conic k , a point P not on k , and the polar p of P with respect to k ; that is, the data³ are $(ABC; Pp)$. Construct the conic k . Is the conic uniquely determined?

16. If in Ex. 15 we are given two points of k and the tangent at one of them, i.e., $(ABa; Pp)$, instead of three points, construct the conic k .

17. State and solve the plane duals of Exs. 15 and 16.

18. State and solve some metric specializations of Exs. 15, 16, and 17.

19. We are given one point A of a conic and two points P , Q whose polars with respect to the conic are respectively two given lines p , q ; i.e., $(A; PQpq)$. Investigate (a) the construction of the conic; (b) the plane dual; and (c) the metric specializations.

20. Given two points of a conic k and a triangle which is self-polar with respect to k ; construct the conic k .

21. Instead of two points of k , as in Ex. 20, we are given a point of k and the tangent at that point. Discuss the construction of k .

22. Investigate the plane duals of Exs. 20 and 21.

23. What metric specializations result from Exs. 20, 21, and 22?

³ We introduce an extension of the symbolic representation of data. The semicolon separates elements of the conic from elements not of the conic. In the data $(ABa; Pp)$ the elements A and a are incident but P and p are not incident. However, if P is a point of the conic its polar p is incident with it, and we write, as in Chapter VII, $(ABPap)$, i.e., three points of the conic and the tangents at two of them.

CHAPTER X

METRIC PROPERTIES OF CONICS

10.1 The Circle. In our study of conics we have quite neglected the circle. It is very easy to construct a circle, provided we allow ourselves the use of a compass. But we have avoided constructions necessitating instruments other than a straightedge, which is probably one reason that the circle has heretofore been

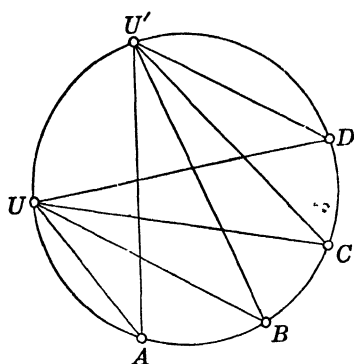


FIG. 56

slighted. A more fundamental reason than this, however, is that the circle is a *metrically specialized* conic.

To show this we recall a well-known property of the circle; namely, that equal angles on the circumference are subtended by equal arcs. If, in Fig. 56, the points A, B, C, D, \dots of a circle are projected from two points U and U' on the circle, the angles AUB and $AU'B$, CUD and

$CU'D$, etc., are equal or supplementary, and the flat pencils $U(ABCD \text{ ---})$ and $U'(ABCD \text{ ---})$ are projective.

That is, in the projectivity

$$U(ABC \text{ ---}) \bar{\wedge} U'(ABC \text{ ---}),$$

the homologue of line UD is line $U'D$; if this were not so, the double ratio (Art. 5.5) of the four lines $U(ABCD)$ would not be equal to the corresponding double ratio of the four lines $U'(ABCD)$, as it must be since corresponding angles are known to be equal or supplementary. Hence, the circle is a curve of second order according to Definition 1, Art. 7.1.

The metric specialization lies in the fact that the angles, such as AUB and $AU'B$, are equal or supplementary. Furthermore, the circle determines on the ideal line¹ a projectivity which, having

¹ See footnote, Art. 7.6.

no double points, is elliptic. For if the projectivity determined on the ideal line possessed a double point M , the lines UM and $U'M$ would be parallel; but this is not possible since UM and $U'M$, being homologous lines of the two projective pencils, intersect on the circle.

10.2 The Orthogonal Involution. In a projectivity on a one-dimensional form,

$$. u(ABC \text{ ---}) \bar{\wedge} u(A'B'C' \text{ ---}),$$

to each element A there corresponds one and only one element A' . We have twice before (Arts. 6.2, 9.5) observed that A' , considered as an element of the first form and labeled D , has as homologue an element D' of the second form, and D' is not, in general,

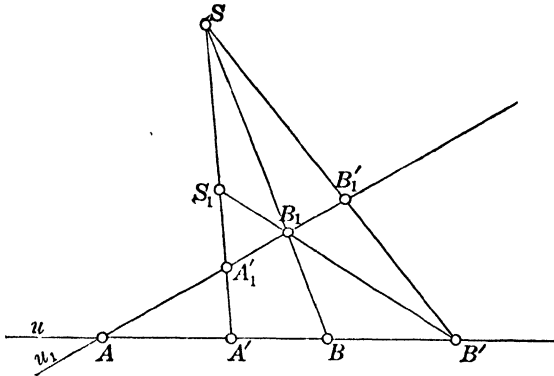


FIG. 57

coincident with A . When D' and A are the same element, that is, when the projectivity is such that it transforms A into A' and $A' (= D)$ into $A (= D')$, the pair of homologous elements A , A' is said to **correspond doubly**.²

If, in a projectivity

$$u(ABA' \text{ ---}) \bar{\wedge} u(A'B'A \text{ ---}),$$

in which the pair A , A' is doubly corresponding, the element B' is the homologue of B , then B , B' is also a doubly corresponding pair, and

$$u(ABA'B' \text{ ---}) \bar{\wedge} u(A'B'AB \text{ ---}).$$

For, from a point S (Fig. 57), we may project $ABA'B'$ on u into

² Do not confuse a doubly corresponding pair with a double point.

$AB_1A_1B_1'$ on a line u_1 passing through A . If, now, \hat{S}_1 is the intersection of SA' and B_1B' , we have the chain of perspectivities

$$u(ABA'B' \text{ ---}) \xrightarrow{\frac{S}{\bar{\lambda}}} u_1(AB_1A_1B_1' \text{ ---}) \xrightarrow{\frac{B'}{\bar{\lambda}}} A'S_1A_1'S \text{ ---} \\ \xrightarrow{\frac{B_1}{\bar{\lambda}}} u(A'B'AB \text{ ---}),$$

whence,³

$$u(ABA'B' \text{ ---}) \bar{\lambda}. u(A'B'AB \text{ ---}).$$

Since B and B' may be any pair of homologous elements, we conclude,

THEOREM 1. *If, in a projectivity on a one-dimensional form, one pair of homologous elements corresponds doubly, every pair of homologous elements corresponds doubly.*

DEFINITION. A projectivity on a one-dimensional form in which

every pair of homologous elements corresponds doubly is an **involution** (cf. Art. 9.5).

Hence, by the above and by Theorem VI, Art. 6.8,

THEOREM 2. *An involution is completely determined when two pairs of homologous elements are given.*

The projectivity determined on the ideal line by the projective

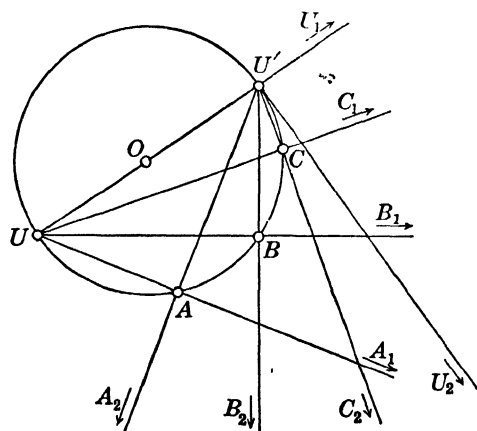


FIG. 58

flat pencils U and U' of Fig. 56 is not, in general, an involution. However, if U and U' , the centers of the flat pencils, are collinear with the center of the circle (Fig. 58), homologous lines $UA, U'A$ of the two pencils are perpendicular, and pairs of homologous elements, such as A_1 and A_2 , determined on the ideal line by the projectivity $U \bar{\lambda} U'$ are doubly corresponding. The involution thus determined on the ideal line by the projective flat pencils U and U' is an involution of conjugate points (Art. 9.4) with respect to

³ Compare this with Art. 6.4.

the circle. Quite apart from its connection with the circle, however, this involution on the ideal line, homologous points of which are the ideal points of two perpendicular lines, is so important that we give to it a name and a label of its own. We shall, in the sequel, refer to it as the **orthogonal involution I** .

The projectors from any ordinary point P of the orthogonal involution I constitute an involution in a flat pencil on P which we call a **circular involution**. Or, otherwise stated, pairs of perpendicular lines in a flat pencil are pairs of homologous lines of the circular involution on the center of the flat pencil.

The concept of the orthogonal involution enables us to relate perpendicular lines to the ideal line in a manner analogous to that of parallel lines. For example,

(1) Two ordinary lines are parallel if their ideal points are coincident; and two ordinary lines are perpendicular if their ideal points are homologous points of the orthogonal involution.

(2) Through any ordinary point there is one and only one line parallel to a given ordinary line; and one and only one line perpendicular to a given ordinary line.

(3) A line perpendicular to one of two parallel lines is perpendicular to the other; and a line parallel to one of two perpendicular lines is perpendicular to the other.

10.3 Bisectors of Angles. **THEOREM.** *The bisectors of the angles formed by two lines on an ordinary point harmonically separate the two lines forming the angles; and conversely, if two pairs of lines on an ordinary point harmonically separate each other and if one of the pairs consists of perpendicular lines, then that pair bisects the angles formed by the other pair.*

The angles formed by two lines a, b on an ordinary point $S = ab$ are bisected by two lines c, d on S . The lines c, d are, we know, perpendicular to each other. Consequently, if a line l which is parallel to d but not on S intersects a, b, c, d at A, B, C, D ,

respectively, the point C is the mid-point of the segment AB , and D is the ideal point of l (Fig. 59). Hence, by Theorem 1, Art. 5.9, we have $H(AB, CD)$, and, by Theorem III, Art. 4.4, $H(ab, cd)$.

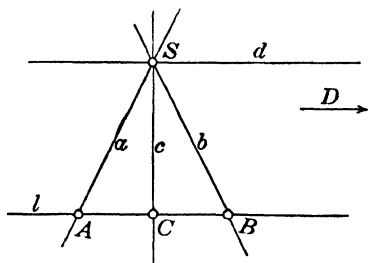


FIG. 59

Conversely, if we are given $H(ab, cd)$ with c perpendicular to d , then $H(AB, CD)$ follows, and the angles ASC and CSB are equal, since C is the mid-point of AB . The lines c, d bisect, therefore, the angles formed by a and b .

10.4 Diameters and Center of a Conic. If a line l cuts a conic at two distinct ordinary points A, B we define the undirected segment AB (Art. 5.2) on line l to be a **chord** of the conic.

DEFINITION. A **diameter** of a conic is an ordinary line whose pole with respect to the conic is an ideal point.

According to the metric classification of conics of Art. 7.6, the ideal line cuts an hyperbola in two points, is tangent to a parabola, and does not cut an ellipse. Consequently, some of the diameters of an hyperbola intersect the curve and some do not; all the diameters of a parabola, since they have one ideal point in common, are parallel; and all the diameters of an ellipse cut the curve.

From the theory of Chapter IX it follows that any line l which cuts a conic at the ordinary points A, B meets the polar line d of its ideal point at a point which is harmonically separated from A and B by that ideal point. Or, the diameter d bisects those chords of the conic which are on lines conjugate to d . The lines conjugate to d are all parallel to one another, for they all have one ideal point in common, namely, the pole of d .

When a diameter cuts a conic in two ordinary points we shall refer to those two points as the **extremities** of the diameter; and

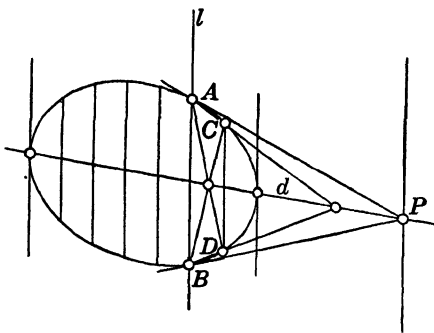


FIG. 60

to the length of the chord defined on the diameter by its extremities, as the **length** of the diameter.

The extremities, if any, of a diameter d are the points of contact of tangents to the conic from the pole of d . These tangents are parallel to the system of parallel chords bisected by d . If one of these chords cuts the conic

at A and B , the tangents to the conic at A and B intersect at a point on d . And if a second such chord cuts the conic at C and D , then not only the tangents at C and D but also the lines AC and BD and the lines AD and BC intersect on d (Fig. 60).

If on a diameter d we take an ordinary point P outside the conic, the harmonic conjugate of d with respect to the tangents to the conic from P is a line parallel to the system of parallel chords bisected by d .

DEFINITION. The center of a conic is an ordinary point whose polar line with respect to the conic is the ideal line.

Again in accordance with the metric classification of Art. 7.6, the center of an hyperbola lies outside the curve; the center of an ellipse lies inside; and a parabola has no center. Thus, we call hyperbolas and ellipses **central conics**.

A conic has an unlimited number of diameters; if it is a central conic it has but one center. All diameters of a central conic pass through the center, and all lines through the center are diameters. Moreover, the center is equidistant from the extremities, if any, of a diameter, for it is the harmonic conjugate with respect to those extremities of an ideal point.

10.5 Conjugate Diameters of Central Conics. Each diameter of a conic has an associated system of parallel lines which are con-

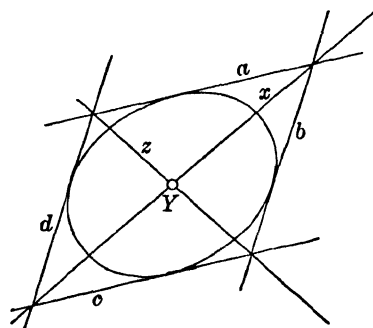


FIG. 61

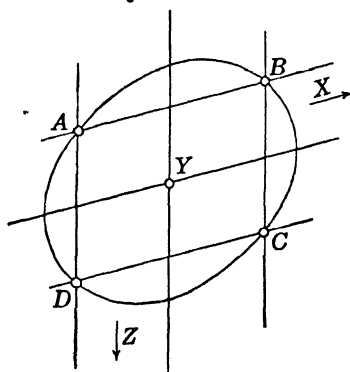


FIG. 62

jugate to it with respect to the conic. On some, or all, of these lines there are chords of the conic, which are bisected by the diameter; if the conic is a central conic, two of the lines may be tangents to the conic, but if it is a parabola, only one is a tangent; that one of these conjugate lines which passes through the center, if there is a center, is also a diameter of the conic. Two diameters which are conjugate lines with respect to a central conic are called **conjugate diameters** of the conic. Hence,

THEOREM 1. *Each of two conjugate diameters bisects all chords of the conic parallel to the other.*

A complete quadrilateral which has an ideal diagonal line is a parallelogram. If such a quadrilateral (Fig. 61) has its sides touching a central conic, one side of its diagonal triangle (3-line) is the ideal line and the vertex opposite that side is the center of the conic. The triangle is self-polar with respect to the conic, and the sides of the triangle through the center are conjugate diameters. Hence,

THEOREM 2. *The diagonals of a parallelogram whose sides touch a conic are conjugate diameters with respect to the conic.*

A similar argument applied to an inscribed quadrangle (Fig. 62) leads to

THEOREM 3. *The sides of a parallelogram whose vertices lie on a conic are parallel to a pair of conjugate diameters with respect to the conic.*

THEOREM 4. *The lines joining any point on a conic to the extremities of a diameter are parallel to a pair of conjugate diameters.*

We direct the reader's attention to the fact that the conic of the theorems of this article is necessarily a *central conic*; otherwise, the term "conjugate diameter" has no meaning.

10.6 The Axes of a Conic. 1. *Central Conics.* A conic determines an involution of conjugate lines on any point of its plane (Art. 9.5). Pairs of conjugate diameters of a central conic are homologous lines of an involution of conjugate lines on the center of the conic as support. Such an involution is not, in general, a circular involution. It is, in fact, a projection of the involution on the ideal line of conjugate points with respect to the conic. Now, if the conic is a circle (Art. 10.2), the involution of conjugate points on the ideal line is the orthogonal involution, and its projection is then, and only then, a circular involution. Consequently, the involution of conjugate diameters is a circular involution only when the conic is a circle. Or, in other words, the involution of conjugate diameters is a circular involution if it has two pairs of homologous lines in common with the circular involution on the center (Theorem 2, Art. 10.2), i.e., if it has two pairs of perpendicular conjugate diameters. In general, the involution of conjugate diameters does not have two pairs of perpendicular conjugate diameters.

DEFINITION. The *axes* of a central conic are the perpendicular conjugate diameters of the conic.

By this definition and Theorem 1, Art. 10.5, the chords of a central conic which are parallel to one axis are perpendicular to and bisected by the other axis. Hence, a central conic is symmetric

with respect to each axis and with respect to the intersection of the axes, i.e., the center of the conic.

An involution of conjugate lines with respect to a conic is hyperbolic or elliptic according as the support of the involution lies outside or inside the conic. Thus, the involution of conjugate diameters with respect to an hyperbola is hyperbolic, and with respect to an ellipse it is elliptic.

In the case of the hyperbola, the double lines of the involution of conjugate diameters are the asymptotes, for they are the tangents to the curve at its ideal points and, therefore, pass through the center and are self-conjugate. Moreover, any pair of ideal conjugate points are separated harmonically by the pair of ideal points on the hyperbola. Hence, by a projection from the center, we have the theorem which follows.

THEOREM 1. *Any pair of conjugate diameters of an hyperbola is harmonically separated by the asymptotes.*

As a consequence of this and the theorem of Art. 10.3, we have

THEOREM 2. *The axes of an hyperbola bisect the angles formed by the asymptotes.*

DEFINITION. A **vertex** of a central conic is an extremity of an axis.

Since the center of an hyperbola lies outside the curve and since its axes are separated by the asymptotes, it is evident that only one of the axes cuts the curve. Consequently, an hyperbola has two vertices. An ellipse, however, is cut by both axes and has four vertices.

By the **length** of an axis we shall mean the length of the chord determined on it by the vertices of the conic. The lengths of the axes of an ellipse are unequal and the axis of greater length is called the **transverse axis**; the other is the **conjugate axis**. The chord on the transverse axis is frequently called the **major axis**; the chord on the conjugate axis is called the **minor axis**. Only one axis of an hyperbola cuts the curve and it is called the **transverse axis**; the other axis is the **conjugate axis**, and with it the term "length" has no meaning.

2. *Parabolas.* The diameters of a parabola form a system of parallel lines. Each diameter has associated with it a system of conjugate chords, and each chord of this system is bisected by the diameter. The diameter is not, in general, perpendicular to its system of conjugate chords.

DEFINITION. The **axis** of a parabola is that diameter which is perpendicular to its associated system of conjugate chords.

It follows that a parabola is symmetric with respect to its axis.

DEFINITION. The **vertex** of a parabola is the ordinary point of intersection of the axis and the curve.

A parabola has only one vertex.

10.7 Construction of the Axes. In the preceding article we assumed the existence of axes of a conic. In order to show that a conic does have an axis, or axes, we shall actually construct the same for a given conic which is completely drawn.

Case 1. When the conic is a central conic. Let A, B be the extremities of a diameter of the given conic. Now construct the circle which has AB as a diameter. This circle will cut the conic again at two distinct points C, D , or it will have common tangents with the conic at A and B .

To consider the latter case first, the diameter AB of the conic is perpendicular to the tangents at A and B , and these tangents are

conjugate lines to AB with respect to the conic. That is, the chords, and the diameter, which are parallel to these tangents are conjugate and perpendicular to the diameter AB . Hence, AB is one axis, and its perpendicular bisector is the other.

In the former case, when the circle cuts the conic at four distinct points (Fig. 63), the four points form a rectangle $ACBD$ (its diagonal

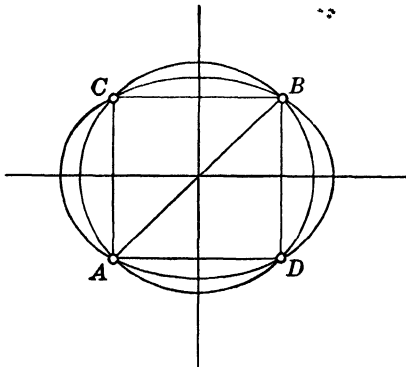


Fig. 63

AB is a diameter of a circle). By Theorem 3, Art. 10.5, the sides of the rectangle are parallel to a pair of conjugate diameters of the conic. These conjugate diameters are perpendicular and are, therefore, the axes of the conic.

If the conic has two pairs of perpendicular conjugate diameters, let AB be any other diameter cutting the conic at A and B . Then the segment AB is the common diagonal of two rectangles each of which has vertices on the conic and sides which are parallel, respectively, to a pair of perpendicular conjugate diameters. Conse-

quently, we have six points of the conic which are also points on a circle, the circle on AB as diameter. But there cannot be two distinct conics through six points. Hence, the given conic coincides with the circle. This agrees, of course, with an earlier result.

Case 2. When the conic is a parabola. Let A be an ordinary point on the parabola and I its ideal point; AI is then a diameter of the curve. If M is the mid-point of any chord PQ which is perpendicular to AI , the diameter MI , i.e., the line through M and parallel to AI , is the axis of the parabola (Fig. 64). This follows from the definition, for, since all diameters of a parabola are parallel, the chords perpendicular and conjugate to the axis are also perpendicular to the other diameters.

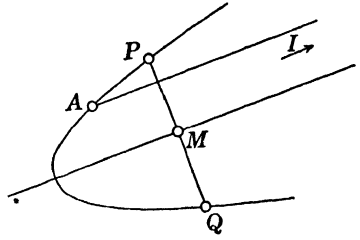


FIG. 64

10.8 Similarly Projective Point-rows. On two ordinary lines u_1 , u_2 consider the projectively related point-rows $u_1(A_1B_1C_1J_1 \text{ ---})$ and $u_2(A_2B_2C_2J_2 \text{ ---})$ which have their ideal points J_1 , J_2 as one homologous pair. By Theorem 2, Art. 5.4, the double ratio of any four points of u_1 is equal to the double ratio of the four homologous points of u_2 ; hence

$$(A_1B_1, C_1J_1) = (A_2B_2, C_2J_2),$$

or, since J_1 and J_2 are ideal points,

$$\frac{A_1C_1}{B_1C_1} = \frac{A_2C_2}{B_2C_2},$$

which may be written,

$$\frac{A_1C_1}{A_2C_2} = \frac{B_1C_1}{B_2C_2}.$$

Because of this proportionality of homologous segments, such point-rows are said to be **similarly projective**.

Of the lines joining homologous points of two similarly projective point-rows, one is the ideal line. Consequently, the joins of homologous points of two similarly projective point-rows form a system of tangents to a parabola. Moreover, by Property P3, Art. 7.2, any two tangents at ordinary points of this parabola are cut by the other tangents in projectively related point-rows; and these point-

rows, having their ideal points as homologues, are similarly projective. This leads to the

THEOREM. *On two fixed ordinary tangents to a parabola, corresponding segments cut off by the other tangents are proportional.*

10.9 A Construction of the Hyperbola. If a line l cuts an hyperbola (Fig. 65) at the points P and Q , and the asymptotes at P_1 and Q_1 , the diameter conjugate to l cuts l at M , the mid-point of the

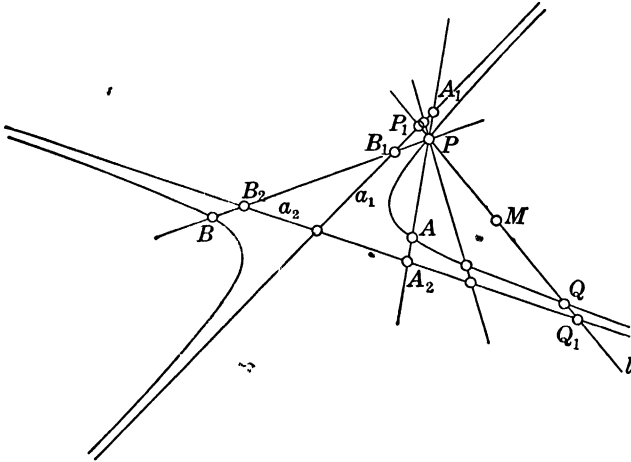


FIG. 65

segment PQ (Art. 10.4). The diameter parallel to l and the diameter conjugate to l are harmonic conjugates with respect to the asymptotes (Theorem 1, Art. 10.6). Hence, M is also the mid-point of the segment P_1Q_1 . Now, since

$$P_1M = MQ_1$$

and

$$PM = MQ,$$

we have

$$P_1P = QQ_1.$$

THEOREM. *On any secant of an hyperbola the two segments between the curve and its asymptotes are equal.*

We recall that an hyperbola is completely determined when its asymptotes and one point of the curve are given. A very simple and easily executed construction of an hyperbola under these conditions follows from the above theorem — *provided we allow ourselves the use of a pair of dividers.*

Let the asymptotes be a_1 and a_2 , and let P be one point of the

curve (Fig. 65). On a line through P which cuts a_1 at A_1 and a_2 at A_2 , lay off the segment A_2A equal to PA_1 . The point A thus determined is a point of the hyperbola; and in a similar manner other points of the curve may be found.

10.10 Rectangular Cartesian Equations of the Conics. Making use of a system of rectangular coordinate axes, such as one uses in elementary analytic geometry, we are able to derive algebraic equations representing the various metric specializations of the conic. The derivation of each of these equations is based on the theory of this chapter and the so-called complete quadrangle theorem (Theorem 3, Art. 8.4).

The importance of this article lies in the fact that herein we have the promised identification of the conic, as defined in projective geometry (Art. 7.1), with the familiar conic section of analytic geometry. Other evidence that the two curves have identical properties will be given in Chapter XV in connection with the definition and study of their foci.

1. *The Ellipse.* In the derivation of the equation of any curve it is, of course, imperative to specify precisely the reference frame or system of coordinate axes to which the curve is referred. We shall choose our coordinate axes to be the axes of the ellipse; the horizontal or X -axis shall coincide with the transverse axis of the ellipse, whose length we take to be $2a$, and the vertical or Y -axis shall coincide with the conjugate axis, whose length we take to be $2b$. The origin of our XY -coordinate system is now the center of the ellipse, and the vertices of the ellipse have the coordinates $(\pm a, 0)$, $(0, \pm b)$, the customary conventions as to signs being adopted.

The student will observe that the ellipse is completely determined when the positions of its axes and their lengths are given. This information, in fact, determines for us four points of the ellipse and the tangent at one of them.

The ellipse (Fig. 66) cuts its axes at the four vertices A, B, C, D , and lies entirely within the rectangle formed by the tangents at these vertices. Hence, any point $P(x, y)$ on the curve has its abscissa x ($= OM$) numerically less than a . The complete quadrangle $APBC$ has two diagonal points at S and T , and a third on the line AB . Therefore, the tangents at A and B and the line ST are concurrent (Theorem 3, Art. 8.4). But, since the tangents are parallel, this point of concurrency is their common ideal point; it

or

$$\frac{a^2 - x^2}{y^2} = \frac{a^2}{b^2},$$

which we may write

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In a circle two conjugate diameters are always perpendicular (Art. 10.6). Consequently, by Theorem 4, Art. 10.5, the angle APB is a right angle for all points on the circle; in particular, ACB is an isosceles right triangle, and

$$b = OC = OB = a.$$

The equation of the circle is then

$$x^2 + y^2 = a^2.$$

2. *The Hyperbola.* The axes of an hyperbola do not both cut the curve. Making a choice of coordinate axes similar to that for the ellipse, let the X -axis coincide with the transverse axis of the hyperbola, whose length we take to be $2a$, and let the Y -axis coincide with the conjugate axis. The vertices A and B have, then, the coordinates $(\pm a, 0)$. If we construct the tangents to the hyperbola at A and B , they are parallel to the Y -axis, and the equal segments intercepted on them by the asymptotes are bisected by the X -axis at A and B (Theorem 2, Art. 10.6). If we let $2b$ be the length of each of these segments, the coordinates of their end-points are $(\pm a, \pm b)$. The rectangle formed by these end-points has the asymptotes as diagonals. We label the points where this rectangle intersects the Y -axis C and D , and their coordinates are $(0, \pm b)$.

The student will observe that the hyperbola is completely determined when the positions of its axes and the lengths $2a$ and $2b$ are given. This, in fact, determines for us three points of the curve and the tangents at two of them; e.g., the two asymptotes and the point A , or the points A and B with their tangents and one ideal point.

The hyperbola lies between the asymptotes and entirely outside the rectangle formed by the tangents at A and B and the parallels to the X -axis through C and D (Fig. 67). Hence, any point $P(x, y)$ on the curve has its abscissa $x (= OM)$ numerically greater than a . Now let I be the ideal point on one of the asymptotes. The quadrangle $PABI$ has two diagonal points at S and T , and a

and

$$\frac{BH}{HS} = \frac{BO}{OC} = \frac{a}{b};$$

and, finally,

$$\frac{AH \cdot BH}{HT \cdot HS} = \frac{a^2}{b^2}.$$

That is,

$$\frac{x^2 - a^2}{y^2} = \frac{a^2}{b^2},$$

which we write

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

That hyperbola for which a and b are equal has its asymptotes perpendicular to each other, and is called a **rectangular hyperbola**. Its equation is

$$x^2 - y^2 = a^2.$$

3. *The Parabola.* The parabola has only one axis and one vertex. The tangent to the parabola at its vertex is perpendicular to

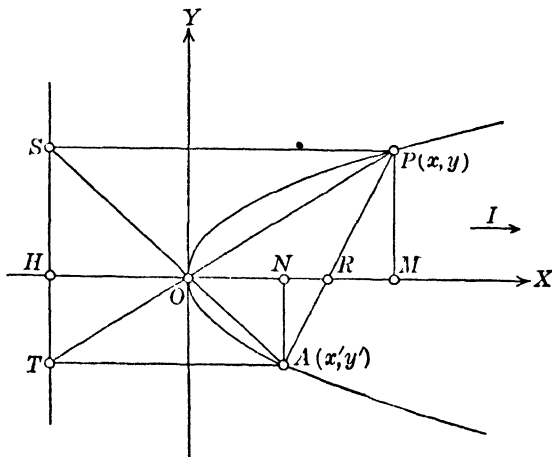


FIG. 68

the axis. We choose for X -axis the axis of the parabola, and for Y -axis the tangent at the vertex. The vertex is then at the origin of the reference system. We have tacitly assumed so far that the axis, the vertex, and the tangent at the vertex are known. This is

⁴ Note that in Fig. 67 BO and OC are each negative but their ratio is positive, as is the ratio BH/HS .

the equivalent of two points and the tangents at those two points; viz., the vertex O and its tangent, and the ideal point I of the parabola and its tangent (the ideal line). Of course, these data are not sufficient to determine the parabola completely and, consequently, not sufficient to enable us to derive its equation. We assume then that one other point $A(x', y')$ of the parabola is known.

If, now, $P(x, y)$ is a variable point on the parabola (Fig. 68), the complete quadrangle $PAIO$ has diagonal points R, S, T . The line ST is concurrent with the tangents at O and I (Theorem 3, Art. 8.4); that is, ST is parallel to the Y -axis.

From the similar right triangles OHT and OMP , we have,

$$\frac{HT}{OH} = \frac{MP}{OM},$$

or, since $HT = NA = y'$,

$$(1) \quad y \cdot OH = y'x.$$

And, from the similar right triangles OHS and ONA ,

$$\frac{HS}{OH} = \frac{NA}{ON};$$

or, since $HS = MP = y$,

$$(2) \quad \frac{y}{OH} = \frac{y'}{x'}.$$

Multiplying equalities (1) and (2) we have, for each position of $P(x, y)$ on the parabola,

$$y^2 = \frac{y'^2 x}{x'}.$$

Since $A(x', y')$ is a fixed point on the curve, we may replace the constant y'^2/x' by the constant $2p$ and write the equation of the parabola in the more familiar form

$$y^2 = 2px.$$

EXERCISES

1. The polar line of a point P with respect to a circle is perpendicular to the diameter through P .

2. A conic ($ABCab$) is given by three of its points and the tangents at two of them. Construct a diameter and the center, if any, of this conic.

3. A parabola ($abcde^*$) is given by four of its ordinary tangents. Construct that diameter of this parabola which is conjugate to one of the given ordinary tangents.

4. If A, B are the extremities and M the mid-point of a chord of a parabola, show that P , the pole of AB , is joined to M by a diameter of the parabola and that the segment PM is bisected by the parabola.

5. On any line parallel to an asymptote of an hyperbola the segment determined by an arbitrary point and its polar with respect to the hyperbola is bisected by the curve.

6. A conic k fully drawn and a point M inside this conic are given. Construct that chord of k which has M as its mid-point.

7. Construct a conic when given a pair of conjugate diameters and (a) two points of the conic; or (b) two tangents of the conic; or (c) a point of the conic and the tangent thereat.

8. A conic is given by its center and (a) three of its points; or (b) three of its tangents. Construct the conic.

9. A diameter of a parabola and (a) three of its ordinary points or (b) three of its ordinary tangents are given. Construct the curve.

10. A conic k has AB and CD as conjugate diameters. If P is any point on the conic and l is any line parallel to CD , show that AP and BP intersect l at conjugate points with respect to the conic.

11. If in Ex. 10 the diameters parallel to AP and BP intersect the tangents at A, B in the points A', B' , respectively, show that $A'B'$ is tangent to the conic.

12. Chords of a given conic which are bisected by a given chord u lie on lines of a parabola.

13. If S is an ordinary point in the plane of a given central conic and if the perpendiculars p from S onto the diameters of this conic intersect the conjugate diameters at points P , show that these points P lie on an hyperbola.

14. If S is an ordinary point in the plane of a given central conic and if the parallels p through S to the diameters of this conic intersect the conjugate diameters at points P , show that these points P lie on a conic.

15. Construct a parabola by the method of Art. 10.8.

16. Construct an hyperbola by the method of Art. 10.9.

17. The triangle formed by the asymptotes and a variable tangent of an hyperbola has a constant area.

Suggestion. Use Theorem P3, Art. 8.4.

18. Using a system of oblique coordinate axes, show that the equation of an hyperbola referred to its asymptotes as axes is of the form $xy = a$ constant.

19. Choosing a pair of conjugate diameters as oblique coordinate axes, derive the equation of (a) an ellipse; (b) an hyperbola.

20. Choosing a diameter and the ordinary tangent at its extremity as oblique coordinate axes, derive the equation of a parabola.

CHAPTER XI

RULED SURFACES

11.1 The Theory of Cones. We have, in Chapters VII to IX, the development of the theory of conics, beginning, of course, with the definitions of point conic and line conic. This whole theory is a part of the geometry of the plane and has, as such, a counterpart, the theory of cones, in the geometry of the bundle; the one is the space dual of the other. That is, the space dual of the theory of conics is the theory of cones.

In Art. 7.1 we defined the cone of planes and the cone of lines as space duals of the point conic and line conic, respectively. Thus, we have the starting point of the theory of cones. In the same chapter (Art. 7.4), we pointed out, also, that the cone of lines and the cone of planes are, respectively, the projectors of a point conic and a line conic. This relationship, too, provides a method of building the theory of cones.

The connection between the idea of a cone of planes, for example, as the space dual of a point conic and the idea of it as the projector of a line conic is to be found, as one might expect, in the principles of plane and bundle duality (Arts. 2.3, 2.4). For the space dual of a plane proposition is the same as the bundle dual of the proposition obtained from the plane proposition by a projection from a point.

Consequently, the principle of space duality applied to Chapters VII to IX results in a theory of cones quite analogous to the theory of conics as set forth therein. In particular, we have the space duals of the theorems of Art. 8.8.

Theorem S IX. *The lines of contact of a cone of planes constitute a cone of lines.*

Theorem SP IX. *The tangent planes to a cone of lines constitute a cone of planes.*

These theorems enable us to define more precisely the term "cone," introduced in Art. 7.1, as a cone of planes together with its lines of contact or, dually (in the bundle), as a cone of lines together

with its tangent planes. In this sense "cone" is a self-dual term in the bundle.

Following this, the next step would be the consideration of the space duals of the theorems of Pascal and Brianchon. The statements of these we leave for the student to make.

Again, the space duals of the definitions of Art. 9.1 define for us, respectively, the **pole-line** x with respect to a cone of a plane ξ (on the vertex of the cone), and the **polar-plane** ξ with respect to a cone of a line x (on the vertex). From these definitions follow, in a similar manner, properties of cones analogous to the pole-polar properties of conics.

The section of a cone by a plane not on the vertex of the cone is a conic; or, in other words, a cone and a plane not on its vertex have in common a conic. On the other hand, a cone and a plane on its vertex have in common one point, one line (repeated), or two lines, according as the plane contains no line, is a tangent plane, or contains two lines of the cone.

A cone whose vertex is an ideal point we call a **cylinder**. Cylinders are metrically specialized¹ cones in that they are cones whose lines are parallel to one another.

The section of a cone by the ideal plane of space is, in general, an **ideal conic**. A cylinder and the ideal plane may have in common no line, one line, or two lines. According as the cylinder has no ideal line, one ideal line, or two ideal lines, it is called **elliptic**, **parabolic**, or **hyperbolic**. The section of an elliptic, parabolic, or hyperbolic cylinder by a plane not on its ideal vertex is an ellipse, parabola, or hyperbola, respectively. For such a section of an elliptic cylinder, for example, is a conic which has no ideal points, i.e., an ellipse.

DEFINITION. A surface generated by a straight line moving in space is a **ruled surface**.

The generating line, in any one of its positions, lies entirely on the ruled surface.

We shall not consider the plane as a ruled surface since a straight line generating it moves in only two dimensions. A cone, however, is a ruled surface of a special type. It is one of a class of ruled surfaces called **developable** surfaces, and can, by a suitable cut, be rolled out, or developed, on a plane without tearing or stretching.

¹ Again we point out the connection of metric specializations with ideal elements. (Cf. Arts. 5.2, 7.6, 10.1.)

This chapter will be devoted to the study of another, and a more general, type of ruled surface.

11.2 Reguli. In each of the definitions of Art. 7.1 we have two homologous elements of two projective one-dimensional primitive forms of like kind (in the same plane or in the same bundle) determining a third element, which is an element of a one-dimensional form of second degree (conic or cone).

To complete the enumeration of forms defined by two projective one-dimensional primitive forms we must consider:

Two projective point-rows whose bases are skew.

DEFINITION. The lines joining homologous points of two projective point-rows on skew supports constitute a **regulus**.

No two lines of the regulus intersect. For if two lines should intersect, the two pairs of homologous points determining them would be coplanar. Thus, the supports of the projective point-rows would be coplanar, contrary to the demands of the definition.

Two projective axial pencils whose axes are skew.

DEFINITION. The lines of intersection of homologous planes of two projective axial pencils on skew supports constitute a **regulus**.

No two lines of the regulus intersect. For if two lines should intersect, the two pairs of homologous planes determining them would be concurrent. Thus, the supports of the projective axial pencils would be intersecting, contrary to the demands of the definition.

The lines joining homologous points of two projective point-rows whose bases are skew are, at the same time, the lines of intersection of homologous planes of the two projective axial pencils obtained by an axial projection of each point-row from the support of the other point-row. For if the point-rows are projective,

$$u(ABC \text{ ---}) \bar{\wedge} u'(A'B'C' \text{ ---}),$$

then the axial pencils $u'(ABC \text{ ---})$ and $u(A'B'C' \text{ ---})$ are projective, since

$$u'(ABC \text{ ---}) \bar{\bar{\wedge}} u(ABC \text{ ---}) \bar{\wedge} u'(A'B'C' \text{ ---}) \\ \bar{\bar{\wedge}} u(A'B'C' \text{ ---}).$$

But homologous planes of the projective axial pencils, such as $u'A$, uA' and $u'B$, uB' , etc., intersect in the lines AA' , BB' , etc., which

are the joins of homologous points of the projective point-rows. That is,

THEOREM 1. *The space dual of a regulus is a regulus.*

THEOREM 2. *The projector of a regulus from a point not lying on a line of the regulus is a cone of planes.*

For the projectors of the point-rows defining the regulus are two concentric flat pencils which are projective and which, therefore, define a cone of planes; and every plane of the cone is the projector of a line of the regulus.

THEOREM S2. *The section of a regulus by a plane not lying on a line of the regulus is a point conic.*

For the sections of the axial pencils defining the regulus are two coplanar flat pencils which are projective and which, therefore, define a point conic; and every point of the conic is the section of a line of the regulus.

In addition to the combinations of primitive forms used in the definitions of Art. 7.1 and of the present article, one might be led to consider others. But should the reader take the time to examine other combinations he will find one of two things, as illustrated in the following paragraphs.

In the first place, consider, for example, a flat pencil and a point-row projective to it but not in the same plane. Each point of the point-row determines with its homologous line of the flat pencil a plane. Now this plane is also the plane determined by a pair of homologous lines of two projective flat pencils; one of the flat pencils is the given pencil, the other is the projector of the point-row from the center of the given flat pencil. The plane, then, is one of a cone of planes.

In the second place, consider a point-row and an axial pencil projective to it; pairs of homologous elements here consist of a point and a plane. Since a point and a plane do not determine a third element (point, line, or plane), the two primitive forms, in this combination, do not define a third form.

In fact, any other combination of two projective one-dimensional primitive forms gives rise to no new form. The conics, cones, and regulus complete the list of one-dimensional forms of second degree.

11.3 Ruled Quadrics. **DEFINITION.** The ruled surface supporting a regulus is called a **ruled quadric**.

THEOREM 1. *A ruled quadric upon which lies a regulus (v) is the support of a second regulus (u); and each line of either regulus intersects all the lines of the other.*

To prove the theorem, let the regulus (v) be defined by the projective point-rows

$$u_1(A_1B_1C_1 \text{ ---}) \overline{\wedge} u_2(A_2B_2C_2 \text{ ---}),$$

where $A_1A_2 = v_1$, $B_1B_2 = v_2$, $C_1C_2 = v_3$, etc. Now consider the axial pencil on the line v_3 . Sections of this axial pencil by the skew lines v_1 and v_2 are two point-rows which are projectively related, since each is perspective to the same axial pencil.² The joins of homologous points of v_1 and v_2 are, by definition, lines of a regulus which we shall call (u). The plane v_3u_1 of the axial pencil on v_3 cuts v_1 at A_1 and v_2 at B_1 . This particular pair of homologous points A_1, B_1 determines a line of the regulus (u), which is none other than u_1 . Similarly, the line u_2 is also a line of the regulus (u). This proves the existence of a second regulus (u); that (u) lies on the same quadric with (v) is a consequence of the proof of the second part of the theorem.

To prove this we shall show that every line of regulus (u) meets three lines of regulus (v), and consequently meets all lines of (v). Each line of (u) is the join of two points, one point on v_1 and one on v_2 , and lies in one plane of the axial pencil on v_3 ; hence, each line of (u) meets v_1, v_2 , and v_3 . Now let u_i be any one of the u -lines, and from u_i project the point-rows $u_1(A_1B_1C_1 \text{ ---})$ and $u_2(A_2B_2C_2 \text{ ---})$; we thus have on u_i two axial pencils, and

$$u_i(A_1B_1C_1 \text{ ---}) \overline{\wedge} u_i(A_2B_2C_2 \text{ ---}).$$

This projectivity between two superposed axial pencils on u_i has, moreover, three double planes, viz.,

$$u_iA_1 = u_iv_1 = u_iA_2,$$

$$u_iB_1 = u_iv_2 = u_iB_2,$$

$$u_iC_1 = u_iv_3 = u_iC_2,$$

and is, by Art. 6.7, an identity; that is, each plane of this superposed projectivity is a double plane. But this is only another way of saying that each line of the regulus (v) lies in one plane of the axial pencil on u_i . Hence each line of (v) meets the line u_i or,

² The point-rows are axially perspective (cf. Example 5, Art. 3.2).

what is the same, u_i meets all lines of (v) . And similarly, since each line of regulus (v) meets u_1, u_2 , and u_i of regulus (u) , it meets all lines of (u) .

Any u -line is met by all lines of (v) and consists of points of the ruled quadric which supports (v) . The regulus (u) , therefore, lies with (v) on one and the same ruled quadric, and the demonstration is complete.

The two reguli (u) and (v) on a ruled quadric are termed **conjugate reguli**. Every line or **ruling** of one of two conjugate reguli is called a **directrix** of the other regulus. Hence, if u_1, u_2, u_3 are three directrices of the regulus (v) , this regulus may be described, or generated, by a variable line v_i moving so as to intersect all three of the directrices. For, since v_i intersects u_1, u_2, u_3 , it is always the join of a pair of homologous points in the (1,1) correspondence set up between the point-rows u_1 and u_2 by the axial pencil on u_3 , to which axial pencil each point-row is perspective. That is,

$$u_1 \overline{\wedge} u_3 \overline{\wedge} u_2,$$

and, therefore,

$$u_1 \overline{\wedge} u_2,$$

a projectivity which defines the regulus (v) . Hence,

THEOREM 2. *The system of lines meeting three arbitrary skew lines is a regulus.*

Again, if u_1, u_2, u_3 are any three directrices of a regulus (v) , then each line v_i determines with u_3 , let us say, a plane u_3v_i . The regulus (v) , then, determines on u_3 an axial pencil $u_3(v_1v_2v_3 \text{ ---})$. The point-rows on u_1 and u_2 are each perspective to this axial pencil; hence, they are projectively related, and we have

THEOREM 3. *The sections of a regulus by any two of its directrices are two point-rows projectively related.*

THEOREM S3. *The projectors of a regulus from any two of its directrices are two axial pencils projectively related.*

11.4 Tangent Planes and Tangent Lines. At every point T of a ruled quadric a line of the regulus (u) meets a line of the regulus (v) . The plane τ determined by these two lines is a **tangent plane** to the surface at the point T . The point of intersection T of this u -line and this v -line is the **point of contact** of the surface on τ .

A tangent plane cuts the ruled quadric in two lines, a u -line and a v -line; or, in other words, the points which a tangent plane has in common with the surface are the points of the u -line and the v -line through its point of contact. The tangent plane has no other points in common with the surface. To prove this let us suppose that a tangent plane τ , determined by the rulings u_i and v_k , has in common with the surface a point P which is not a point of u_i nor of v_k . Now consider the flat pencil on τ whose center is P . Each line l of this flat pencil cuts u_i at some point, say Q , and v_k at some other point, say R . Hence, l has three points in common with the ruled quadric; which means that l cuts three lines of either the regulus (u) or the regulus (v). If so, then l is a line of either the regulus (v) or the regulus (u). But this is clearly an impossibility for τ already contains one line of (u) and one line of (v), and no two lines of either regulus can be coplanar.

Thus, on a tangent plane τ , with point of contact T , any line not on T has but two points in common with the quadric, namely, those points where it cuts u_i and v_k . Any line through the point of contact T and on τ is called a **tangent line**; it has no point other than T in common with the surface unless it is one of the rulings. Hence, the

THEOREM. *At every point of a ruled quadric there is one tangent plane and one flat pencil of tangent lines, and the tangent plane is the plane of the flat pencil of tangent lines.*

11.5 Tangent Cones and Contact Conics.

THEOREM 1. *The projector of a ruled quadric from a point not on a ruling is a cone of planes.*

For the projector of each regulus is a cone of planes (Art. 11.2). Moreover, every plane of the cone is the projector of two rulings of the quadric, one of each regulus, and is, therefore, the tangent plane to the quadric at the point of intersection of these two rulings.

THEOREM S1. *The section of a ruled quadric by a plane not on a ruling is a point conic.*

For the section of each regulus is a point conic (Art. 11.2). Moreover, every point of the conic is the trace of two rulings of the quadric, one of each regulus, and is, therefore, the point of contact with the quadric of the tangent plane determined by these two rulings.

We are thus led to consider:

THEOREM 2. *The points of contact of a ruled quadric on the planes of its projector from a point not on a ruling are points of a conic.*

THEOREM S2. *The tangent planes to a ruled quadric at points of its section by a plane not on a ruling are planes of a cone.*

Let us prove the theorem on the left. If P is the center of projection and α, β, γ are three distinct planes of the projector of the quadric from P , these planes are tangent planes to the quadric. The points of contact A, B, C on the tangent planes α, β, γ , respectively, must determine a plane. We say these three points *must* determine a plane and cannot be collinear. For if they were collinear their common line would be a ruling of the quadric (Art. 11.4); the projecting planes would then coincide with the plane common to this line and P ; and, since the projecting planes are tangent planes to the quadric, the three lines of the conjugate regulus which pass through the three points A, B, C would lie in the plane common to this line and P . But this is contrary to the definition of Art. 11.2. Hence, the three points determine a plane, call it π , whose section with the quadric is a point conic (Theorem S1, above). This point conic contains the points A, B, C and has as tangents at those points the lines $\pi\alpha, \pi\beta, \pi\gamma$. Also, the section by π of the cone of planes projecting the quadric from P is a line conic (Art. 7.4) containing the lines $\pi\alpha, \pi\beta, \pi\gamma$, and it has, on those lines respectively, the points of contact A, B, C . But these two conics in the plane π coincide, for they have in common three points A, B, C and the tangents thereat. Consequently, the ruled quadric and the cone of tangent planes projecting it from a point have a conic in common; this conic contains the points of contact on the planes of the cone, and the theorem is demonstrated.

DEFINITION. The projector of a ruled quadric from a point P not on the surface is called the **tangent cone** to the quadric from P , and the conic which this tangent cone has in common with the quadric is called its **contact conic**.

11.6 Order and Class. Analogous to the order and the class of a plane or bundle form, as given in Art. 7.1, we have the order and class of a space form. As applied to the surfaces of this chapter, we use the terms as follows:

The **order** is the maximum number of points which the surface may have in common with any line not a ruling of the surface.

The **class** is the maximum number of (tangent) planes which the surface may have in common with any line not a ruling of the surface.

Thus, the ruled quadric is a surface of second order and second class, for

a line, not lying entirely on the surface, can have not more than two points in common with the surface.

If a line l cuts the ruled quadric in more than two points, it meets three lines of one of the reguli (u) or (v), and hence is itself a ruling of the quadric.

a line, not lying entirely on the surface, can have not more than two (tangent) planes in common with the surface.

If a line l has in common with the ruled quadric more than two tangent planes, it meets three lines of one of the reguli (u) or (v), and hence is itself a ruling of the quadric.

To complete the discussion on the right above, we note that a line l which cuts the quadric in two points, A and B , meets at A one u -line u_a and one v -line v_a , and similarly at B , one u -line u_b and one v -line v_b . There are, then, two tangent planes to the quadric through l ; they are lu_a and lu_b which contain, respectively, the lines v_b and v_a , since each v -line intersects every u -line, and these particular v -lines intersect l also.

If the line l , however, is tangent to the quadric at a point T , there is but one tangent plane through l . It is (Art. 11.4) the plane determined by the two rulings through T . And, finally, if the line l has no points in common with the quadric, there are no tangent planes through l ; for l is skew to every ruling and has no plane in common with any ruling.

11.7 Theory of Pole and Polar.

DEFINITION. The **polar** with respect to a given ruled quadric of a point X not on the quadric is the plane ξ of the contact conic of the tangent cone whose vertex is X .

DEFINITION. The **pole** with respect to a given ruled quadric of a plane ξ not tangent to the quadric is the vertex X of the tangent cone whose contact conic lies on ξ .

The **polar** ξ of a point X on the given quadric is the tangent plane to the quadric at X .

The **pole** X of a plane ξ tangent to the given quadric is the point of contact of the quadric on ξ .

From these definitions and the theorems of Art. 11.5, it is evident that, if ξ is the polar plane of a point X with respect to a ruled quadric, then X is the pole of the plane with respect to the same quadric.

THEOREM 1. *If X is a point not on a given ruled quadric, the harmonic conjugate of X with respect to any pair of surface points collinear with X lies on the polar of X with respect to the quadric.*

THEOREM S1. *If ξ is a plane not tangent to a given ruled quadric, the harmonic conjugate of ξ with respect to any pair of tangent planes coaxial with ξ lies on the pole of ξ with respect to the quadric.*

Consider a ruled quadric and a point X not on the quadric. Any line through X which cuts the quadric at two points A and B will cut ξ , the polar of X with respect to the quadric, at a point X_1 . Let σ be a plane on this secant XAB ; it cuts the quadric in a conic and the polar ξ in a line. The tangent cone from X is cut by σ in two lines, which are tangents to the conic cut from the quadric by σ at the points where the line $\sigma\xi$ cuts this conic. Hence, by Property 3, Art. 9.2, in the plane σ the polar line of X with respect to the conic is $\sigma\xi$, and XAB cuts this line and, consequently, the plane ξ at the harmonic conjugate of X with respect to A and B , i.e., $X_1 = H(X, AB)$.

THEOREM 2. *The polars with respect to a ruled quadric of the points X on a line l are the planes ξ on a line l' , and $l(X \text{ ---}) \bar{\wedge} l'(\xi \text{ ---})$.*

Case 1. When l is not a ruling of the quadric. Let σ_1 and σ_2 be two planes on l (Fig. 69). The plane σ_1 has in common with the quadric a conic, and with respect to that conic the points of l have polar lines which form a flat pencil S_1 . Moreover, the point-row on l is projective to the flat pencil on S_1 , by Theorem 1, Art. 9.5. Similarly, in σ_2 there is a flat pencil of polar lines S_2 projectively related to the point-row on l . But any two homologous lines of S_1 and S_2 in the projectivity thus determined between them,

$$S_1 \bar{\wedge} l \bar{\wedge} S_2,$$

are coplanar, since such homologous lines are the traces in σ_1

and σ_2 of the same plane, namely, the polar plane ξ with respect to the quadric of a point X on l . Hence, the polar planes of

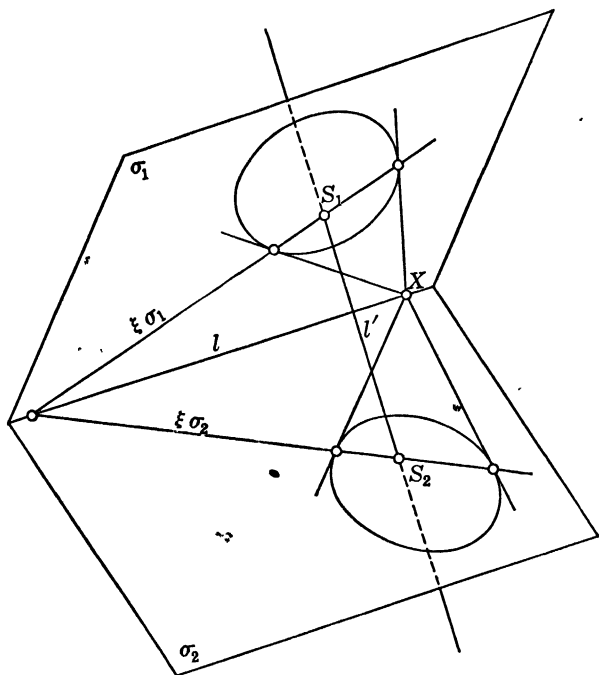


FIG. 69

points on l are coaxial, their axis being the line $S_1S_2 = l'$. Now from the chain of perspectivities,

$$l(X \text{ ---}) \bar{\wedge} S_1 \bar{\wedge} l'(\xi \text{ ---}),$$

we have

$$l(X \text{ ---}) \bar{\wedge} l'(\xi \text{ ---}),$$

and the theorem is demonstrated.

Case 2. When l is a ruling of the quadric. Let l be a line of the regulus (u) and, therefore, a directrix of regulus (v). The polars ξ of points X on l constitute an axial pencil on l consisting of the planes lv_1, lv_2 , etc. The lines of (v) determine, on l and l_1 , a second directrix, two projective point-rows, by Theorem 3, Art. 11.3. But each point of the point-row on l_1 is incident with the polar plane of its homologous point on l . Therefore,

$$l(X \text{ ---}) \bar{\wedge} l_1 \bar{\wedge} l(v_1v_2v_3 \text{ ---}),$$

and

$$l(X \text{ ---}) \bar{\wedge} l(\xi \text{ ---}).$$

In Case 2 the lines l and l' are coincident.

From the content of this theorem we are led to the following definitions.

DEFINITION. The **polar line** with respect to a given ruled quadric of a line l not a ruling of the quadric is the axis l' of the axial pencil of polar planes of points on l .

DEFINITION. The **polar line** with respect to a given ruled quadric of a ruling l of the quadric is the ruling l itself.

11.8 Metric Considerations. As we have previously remarked, any metric specialization in projective geometry is intimately

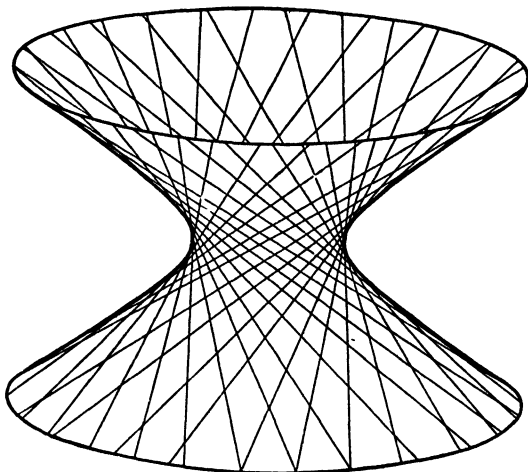


FIG. 70

connected with the ideal region. A metric classification of the ruled quadrics of this chapter results from a study of the possible sections of these quadrics by the ideal plane. We recall that any plane section of a ruled quadric is either a conic or two intersecting lines (rulings of conjugate reguli). In the latter case the plane of section is a tangent plane to the quadric. Hence,

DEFINITION. A ruled quadric which has in common with the ideal plane of space a proper conic is called a **hyperboloid of one sheet** or an **unipartite hyperboloid** (Fig. 70).

DEFINITION. A ruled quadric which has the ideal plane of space as a tangent plane and, therefore, has in common with it two lines is called an **hyperbolic paraboloid** (Fig. 71).

The section of an hyperboloid of one sheet by a plane π is an

ellipse, parabola, or hyperbola, according as π has no ideal point, one ideal point, or two ideal points in common with the hyperboloid, or, what is the same, according as π has in common with the ideal conic of the hyperboloid no point, one point, or two points.

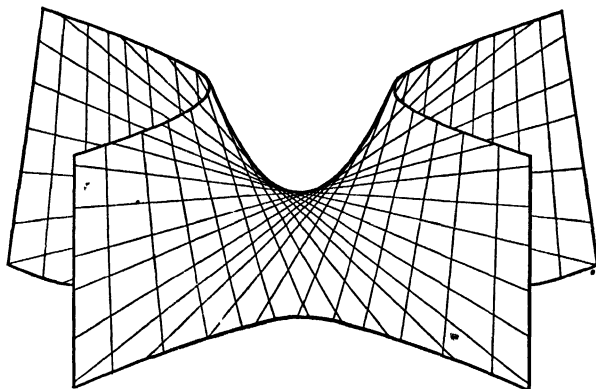


FIG. 71

Thus, if a plane π cuts all rulings of the hyperboloid in ordinary points, it has in common with the hyperboloid an ellipse; if π is parallel to a tangent plane with an ideal point of contact,³ the conic of intersection is a parabola; and, finally, if π is parallel to a tangent plane with an ordinary point of contact, the conic of intersection is an hyperbola.

The pole of the ideal plane with respect to an hyperboloid of one sheet is called the **center** of the hyperboloid. The tangent cone from the center is the **asymptotic cone**, and its contact conic is the ideal conic of the hyperboloid.

On the other hand, an hyperbolic paraboloid has in common with the ideal plane two lines, and the ideal plane is a tangent plane. Any other plane section of such a quadric has one or two ideal points. A section by a plane which contains the ideal point of contact of the ideal plane is a parabola; a section which does not contain this ideal point of contact cuts the two ideal rulings of the surface in distinct points, and is an hyperbola. Evidently, a plane section of this surface cannot be an ellipse.

From another point of view, the ruled quadric supporting a regulus (v), defined by two projective point-rows whose ideal

³ The plane π , in this case, is parallel to the ruling and the directrix through the ideal point of contact.

points are homologous (cf. Art. 10.8), is an hyperbolic paraboloid, since it contains one ideal v -line and, hence, one ideal u -line also.

If in Theorem 2, Art. 11.3, one of the three arbitrary skew lines is ideal, we have the

THEOREM. *The system of lines meeting two given skew lines and parallel to a given plane, which is not parallel to either of the given lines, is a regulus (v) on an hyperbolic paraboloid.*

The conjugate regulus (u) contains one ideal line, and may be described as the system of lines meeting two v -lines and parallel to the plane containing the ideal v -line. When the two ideal lines of the reguli (u) and (v) lie in planes which are perpendicular, the hyperbolic paraboloid is said to be **rectangular**.

The ruled quadrics of this chapter are not the only quadrics or surfaces of second degree — but they are the only ruled quadrics. Other quadrics with which the reader is probably familiar are the sphere, ellipsoid, elliptic paraboloid, and hyperboloid of two sheets. The study of these surfaces, however, belongs to a more advanced part of projective geometry.

EXERCISES

1. If P and P' are two points on a ruled quadric and their join is not a ruling of the surface, the rulings u, v through P intersect the rulings u', v' through P' in two points Q and Q' , which form with P, P' a tetrahedron which is both inscribed and circumscribed to the quadric. Show that the tetrahedron $PP'QQ'$ is uniquely determined by the choice of P and P' . Discuss and then dualize.

2. Given two skew lines u_1, u_2 and three points A, B, C of which no two are coplanar with either of the given lines; show how to construct the ruled quadric containing these elements.

3. Discuss the dual of Ex. 2.

4. If four lines of a regulus are cut by one line of the conjugate regulus in four harmonic points, they are so cut by every line of the conjugate regulus. Dualize.

5. If a point X lies on the polar η with respect to a ruled quadric of a second point Y , then Y lies on the polar ξ of X . The points X and Y are *conjugate* with respect to the quadric.

6. If l' is the polar line of l with respect to a ruled quadric, then l is the polar line of l' .

7. If l and l' are polar lines with respect to a ruled quadric, every point on l' is conjugate to every point on l .

8. The harmonic conjugate of a tangent line t to a ruled quadric with respect to the ruling and the directrix on its point of contact is the polar line of t with respect to the quadric.

9. If u_1, u_2, u_3 are three lines of a regulus and v_1, v_2, v_3 are three lines of its conjugate regulus, then the joins of the pairs of points u_2v_3 and u_3v_2 , u_3v_1 and u_1v_3 , and u_1v_2 and u_2v_1 meet at a point P . The polar of P is the plane containing the lines of intersection of the pairs of tangent planes at the points u_2v_3 and u_3v_2 , etc.

10. If $u(ABC \text{ ---}) \overline{\wedge} S(abc \text{ ---})$ and u is not on the plane of S nor parallel to it, show that lines drawn through points of u parallel to the homologous lines of S are lines of a regulus on an hyperbolic paraboloid.

CHAPTER XII

EXTENDED THEORY OF PROJECTIVITY

12.1 The One-dimensional Forms. The theory of the projective relationship as applied to the one-dimensional primitive (i.e., first degree) forms in the foregoing chapters will, in this chapter, be extended and applied to the one-dimensional forms of second degree. The term "form" as used in the following discussion will be understood to include the eight one-dimensional forms heretofore defined, viz., the three primitive forms — point-row, flat pencil, and axial pencil — and the five forms of second degree — point conic, line conic, cone of planes, cone of lines, and regulus.

The important difference between the primitive forms and the second degree forms is in the nature of their supports. The supports of the former are elements (point, line, or plane), and the supports of the latter are not elements. The supports of second degree forms are conics, cones, or ruled quadrics.

The important characteristic which the primitive forms and the second degree forms have in common is that they are all one-dimensional. As in Art. 1.7, when we say that two forms are of the same dimensions we mean that it is possible to set up a (1,1) correspondence between the elements of the two forms. The point-row, flat pencil, and axial pencil were defined to be one-dimensional forms. Hence, the second degree forms are also one-dimensional inasmuch as their definitions (Arts. 7.1, 11.2) show clearly a (1,1) correspondence between the elements of each of the primitive forms defining the second degree form and the elements of the second degree form itself.

Consider, for example, two projective flat pencils $U(abc\text{---})$ and $U'(a'b'c'\text{---})$. Homologous lines a, a' intersect at a point A of a point conic k (or point-row of second order), which we shall symbolize by $k\{ABC\text{---}\}$ — using braces here rather than parentheses, for we shall agree to reserve the latter for first degree forms. The (1,1) correspondence between the elements of $U(abc\text{---})$ and the elements of $k\{ABC\text{---}\}$, in which each line

of the flat pencil is paired with that point of the point conic incident with it, justifies us in applying the term "one-dimensional" to point conics. Similar examples may be presented for the other second degree forms.

The (1,1) correspondence between the elements of the flat pencil and the point conic of the preceding paragraph recalls to mind dual statements such as (in a plane):

The projector of a point-row from a point is a flat pencil.	The section of a flat pencil by a line is a point-row.
---	--

This leads us to extend our concept of projector and section to include cases such as are illustrated in the following paragraphs.

The projector of a point conic from a point on the conic is a flat pencil.	The section of a line conic by a line of the conic is a point-row.
---	---

The section of a flat pencil by a coplanar conic through the center of the flat pencil is a point conic or point-row of second order.	The projector of a point-row from a coplanar conic touching the support of the point-row is a line conic or pencil of lines of second class
--	--

12.2 Perspective One-dimensional Forms. The definitions of perspectivity as stated in Art. 3.2 we now extend to one-dimensional forms. This extension is made by simply deleting the word "primitive" from those definitions.

Let us give in detail several particular examples.

1. A point conic and a flat pencil in the same plane are perspective if the center of the pencil is a point of the conic and if the forms are in a (1,1) correspondence such that homologous elements, point and line, are incident.	P1. A line conic and a point-row in the same plane are perspective if the base of the point-row is a line of the conic and if the forms are in a (1,1) correspondence such that homologous elements, line and point, are incident.
--	--

The center of the flat pencil considered as a point of the conic is homologous to the tangent to the conic at that center.	The base of the point-row considered as a line of the conic is homologous to the point of contact of the conic on that base.
--	--

2. A cone of planes and a flat pencil in the same bundle are perspective if the plane of the pencil is a plane of the cone and if the forms are in a (1,1) correspondence such that homologous elements, plane and line, are incident.

B2. A cone of lines and an axial pencil in the same bundle are perspective if the axis of the pencil is a line of the cone and if the forms are in a (1,1) correspondence such that homologous elements, line and plane, are incident

Examples 2 and B2, the reader will observe, are respectively the space duals of Examples 1 and P1.

3. A regulus and a point-row are perspective if the base of the point-row is a directrix of the regulus and if the forms are in a (1,1) correspondence such that homologous elements, line and point, are incident.

S3. A regulus and an axial pencil are perspective if the axis of the pencil is a directrix of the regulus and if the forms are in a (1,1) correspondence such that homologous elements, line and plane, are incident.

The two forms in each of the above examples are unlike as to their elements. Perspective forms of like elements are illustrated by the following:

4. A point conic and a point-row in the same plane are perspective if the forms are in a (1,1) correspondence such that each pair of homologous elements, point and point, is incident with one line of a flat pencil whose center is a point of the conic.

The conic and the point-row are each perspective to the same flat pencil.

5. A cone of planes and an axial pencil in the same bundle are perspective if the forms are in a (1,1) correspondence such that each pair of homologous

P4. A line conic and a flat pencil in the same plane are perspective if the forms are in a (1,1) correspondence such that each pair of homologous elements, line and line, is incident with one point of a point-row whose base is a line of the conic.

The conic and the flat pencil are each perspective to the same point-row.

B5. A cone of lines and a flat pencil in the same bundle are perspective if the forms are in a (1,1) correspondence such that each pair of homologous

elements, plane and plane, is incident with one line of a flat pencil whose plane is a plane of the cone.

The cone and the axial pencil are each perspective to the same flat pencil.

6. Two point conics not in the same plane are perspective if the forms are in a (1,1) correspondence such that each pair of homologous elements, point and point, is incident with one line of a cone.

The point conics are sections of the same cone of lines.

elements, line and line, is incident with one plane of an axial pencil whose axis is a line of the cone.

The cone and the flat pencil are each perspective to the same axial pencil.

S6. Two cones of planes not on the same vertex are perspective if the forms are in a (1,1) correspondence such that each pair of homologous elements, plane and plane, is incident with one line of a conic.

The cones of planes are projectors of the same line conic.

These twelve examples will suffice to illustrate perspectivities between two one-dimensional forms in the extended sense. The first five pairs exhibit perspectivities between a form of the first degree and a form of the second degree; the last pair, perspectivities between two second degree forms.

12.3 Harmonic Sets of Elements. The harmonic sets of Chapter IV consist in each case of four elements of a primitive form. We extend the concept now to include harmonic sets of four elements on a form of second degree. The extension is based on Theorem III, the definitions of Art. 7.1, and the preceding article.

DEFINITION. Four points of a conic are **harmonic** if they are projected from a fifth point of the conic by an harmonic pencil of lines.

DEFINITION. Four lines of a conic are **harmonic** if they are cut by a fifth line of the conic in an harmonic range of points.

That four harmonic points of a conic are projected from *all* points of the conic by harmonic pencils follows immediately from Property 3, Art. 7.2, and dually for four harmonic lines of a conic.

Considering a conic as a self-dual form in the sense of Art. 8.8, we have

THEOREM 1. *If four points of a conic are harmonic, the tangents at those points are harmonic.*

For, if the four points are A, B, C, D , and U is any fifth point of the conic, then $U(ABCD)$ is an harmonic pencil. And if a, b, c, d are the tangents at those four points and l is a fifth tangent (not necessarily the tangent at U) we have, by Theorem X, Art. 8.9,

$$l(abcd \text{ ---}) \overline{\wedge} U(ABCD \text{ ---}).$$

Hence, $l(abcd)$ is an harmonic range, since it is projective to an harmonic pencil. Consequently, a, b, c, d are harmonic lines, by the definition of this article.

We shall not prove the converse, for it is the plane dual of the direct theorem.

THEOREM 2. *The lines joining conjugate pairs of four harmonic points of a conic are conjugate lines with respect to the conic.¹*

THEOREM P2. *The points of intersection of conjugate pairs of four harmonic lines of a conic are conjugate points with respect to the conic.¹*

To prove the theorem on the left, let A, C and B, D (Fig. 72) be conjugate pairs of the harmonic set $H\{AC, BD\}$. By definition, the four lines projecting A, B, C, D from any point of the conic, and in particular the four lines $D(ABCP)$, where DP is the tangent at D , form an harmonic pencil. Hence, if R is the intersection of AC and BD and if P is the intersection of AC and the tangent DP , we have on AC the harmonic range $H(AC, RP)$. But the polar of P with respect to the conic is $DR (= DB)$, by Properties 2 and 3 of Art. 9.2. That is, AC passes through the pole of BD and, by the dual of Theorem XI, the lines AC and BD are conjugate lines with respect to the conic.

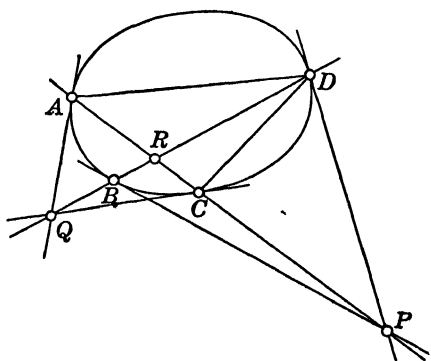


FIG. 72

¹ Note particularly that the term "conjugate" in this theorem is used in two ways (cf. Arts. 4.3 and 9.4).

Conversely,

If two conjugate lines with respect to a conic have in common with it the pairs of points A, C and B, D , these four points form an harmonic range $H\{AC, BD\}$ on the conic.

If two conjugate points with respect to a conic have in common with it the pairs of lines a, c and b, d , these four lines form an harmonic pencil $H\{ac, bd\}$ on the conic.

For if AC and BD are conjugate lines with respect to the conic, then AC passes through P , the pole of BD . And, if $R = AC, BD$, then we have $H(AC, RP)$, and the pencil $D(ARCP)$ is harmonic. Hence, the relation $H\{AC, BD\}$.

The space duáls of the definitions at the beginning of this article define for us the harmonic sets of elements on a cone. And, of course, the above theorems also have duals in space. For the most part the definitions and theorems of this chapter will be stated for conics only, that is, for plane forms. But the student must not forget that we have the principle of duality in the background, and it will, when necessary, provide further extensions of any results we may obtain.

The definition of harmonic elements of a regulus requires separate statement. It obviously cannot be obtained from earlier definitions by the principle of duality, for the regulus is a self-dual form in space. Thus, the

DEFINITION. Four lines of a regulus are **harmonic** if they are cut by a directrix of the regulus in an harmonic range of points.

DEFINITION. Four lines of a regulus are **harmonic** if they are projected from a directrix of the regulus by an harmonic pencil of planes.

That four harmonic lines of a regulus are cut by *all* directrices of the regulus in harmonic ranges follows from this definition and Theorem 3, Art. 11.3; and dually, four harmonic lines of a regulus are projected from *all* directrices by harmonic pencils of planes.

12.4 Extended Definition of Projectivity. **DEFINITION.** Two one-dimensional forms are **projectively related** ($\bar{\wedge}$), or **projective** to one another, if they are any two members of a chain of perspectivities.

The reader will observe that this definition is the definition of

Art. 6.2 with the omission of the word "primitive." Moreover, the individual members of the "chain of perspectivities" mentioned in this definition need not all be primitive forms (Art. 12.2). Thus the content of the definition is now much larger, for the forms referred to are the one-dimensional forms of this chapter.

It will also be noticed that, by Art. 12.3, if two forms are projectively related, to every harmonic set of elements in one there corresponds an harmonic set of elements in the other. That is, a projectivity preserves harmonic relationships.

Finally, a perspectivity between two forms is a projectivity, but not conversely.

The fundamental theorem of Art. 6.8 we now restate:

THEOREM. *A projectivity between two one-dimensional forms is uniquely determined when three pairs of homologous elements are given.*

The wording is the same as in Art. 6.8 but the content of the theorem is much greater; for, by the extensions of this chapter, the term "form" is more inclusive than it was in Chapter VI.

12.5 Construction of a Projectivity between Two Forms. A projectivity between two forms is completely determined by three pairs of homologous elements. Other pairs of homologous elements of this projectivity may be determined, in general, by setting up a projectivity between two *primitive* forms perspective to the original forms. We illustrate by an example.

Let the projectivity between two conics k_1 and k_2 be such that the points A_1, B_1, C_1 on k_1 correspond, respectively, to the points A_2, B_2, C_2 on k_2 . To determine other pairs of homologous points on k_1 and k_2 we project k_1 and k_2 , respectively, from two points, S' on k_1 and S'' on k_2 , and then determine homologous lines of the projectivity

$$S'(A_1B_1C_1 \text{ ---}) \bar{\wedge} S''(A_2B_2C_2 \text{ ---}),$$

by the usual method (Art. 6.10). The centers S' and S'' may be any two points on the conics k_1 and k_2 . In practice, however, we choose, as in Fig. 73, S' and S'' so that the flat pencils are perspective. That is, we draw A_1A_2 , say, and take S' and S'' at the points where A_1A_2 cuts k_1 and k_2 a second time; $S'A_1$ is identical with $S''A_2$, now, and the projectivity $S' \bar{\wedge} S''$ is a perspectivity. Hence, the chain of perspectivities

$$k_1\{A_1B_1C_1\text{---}\} \bar{\wedge} S'(A_1B_1C_1\text{---}) \bar{\wedge} S''(A_2B_2C_2\text{---}) \bar{\wedge} k_2\{A_2B_2C_2\text{---}\}.$$

Corresponding, now, to any point P_1 of k_1 we have the line $S'P_1$ of S' ; its homologue in the flat pencil S'' is a line cutting k_2 at P_2 , and the pair P_1, P_2 is a pair of homologous points in the projectivity, i.e.,

$$k_1\{A_1B_1C_1 \text{ --- } P_1 \text{ ---}\} \bar{\wedge} k_2\{A_2B_2C_2 \text{ --- } P_2 \text{ ---}\}.$$

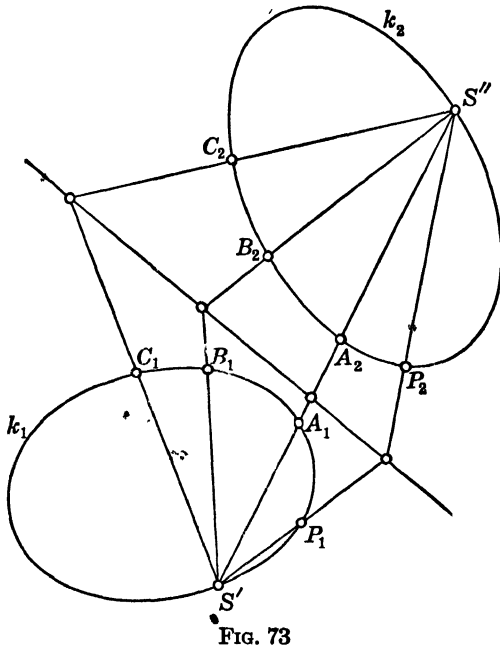


FIG. 73

In Fig. 73, illustrating this example, the conics k_1 and k_2 do not have any points in common. Should they have a point in common, such as in Fig. 74, that point will have two names, say E_1 and F_2 . The homologue of E_1 , that is E_2 , will not in general coincide with E_1 . When E_1 and E_2 do coincide we say that the common element of the two conics is **self-corresponding**.

Let us examine some projectivities between two conics which have self-corresponding elements.

In Fig. 75, we have one such projectivity illustrated. Each conic is perspective to a flat pencil whose center S is a point common to the conics. The point S , considered as a point of k_1 and called D_1 , is homologous to D_2 , the point of k_2 which lies on the tangent to k_1 at S . Similarly, E_2 ($=S$) is homologous to E_1 . The

point S is *not* a self-corresponding point. But the other common point of the conics, $F_1 = F_2$, is self-corresponding. In this projectivity, therefore, there is *one* self-corresponding point.

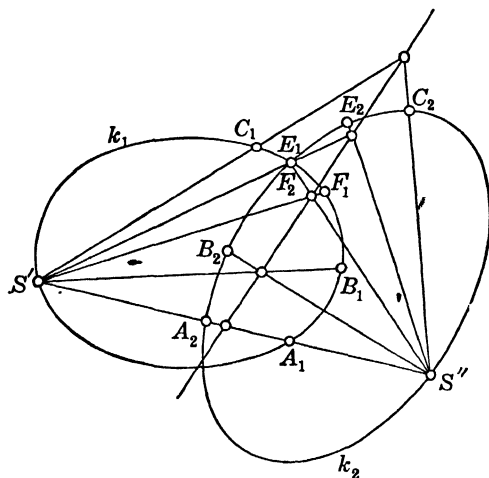


FIG. 74

In Fig. 76, the two conics have in common three points and a tangent at one of the points. Each conic is perspective to the flat pencil whose center S is the common point of contact on their com-

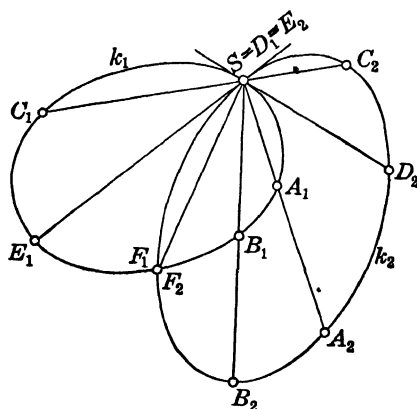


FIG. 75

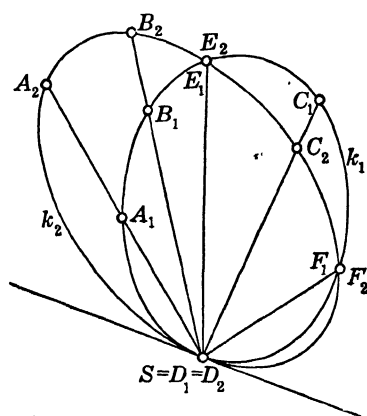


FIG. 76

mon tangent. The point S , considered as a point of k_1 and labeled D_1 , is self-corresponding, for the tangent to k_1 at S is also the tangent to k_2 at S ; that is, D_2 coincides with D_1 . The other two points

common to k_1 and k_2 are also self-corresponding points, and this projectivity has *three* self-corresponding points.

If a projectivity between two conics has more than three self-corresponding points, the two conics must have in common at least four points and a tangent at one of them — they are, therefore, identical (Property 4, Art. 7.2).

The discussion of this article leads to the following dual theorems:

THEOREM 1. *If two conics are projectively related and have more than three self-corresponding points, the conics are identical and all points are self-corresponding.*

THEOREM P1. *If two conics are projectively related and have more than three self-corresponding lines, the conics are identical and all lines are self-corresponding.*

A similar argument applied to a projectivity between a conic and a cone (or a regulus) results in a theorem to the effect that, if the projectivity has more than three corresponding elements which are incident, the conic and the cone (or regulus) are in perspective position.

12.6 Third Degree Forms.

THEOREM 1. *If a point-row lies in the plane of a line conic to which it is projective but not perspective, there is at least one pair of homologous elements which are incident, and at most three pairs.*

THEOREM P1. *If a flat pencil lies in the plane of a point conic to which it is projective but not perspective, there is at least one pair of homologous elements which are incident, and at most three pairs.*

We shall prove the theorem on the right. If (Fig. 77) the flat pencil $S(abc \text{ ---})$ is projective to the point conic $k\{ABC \text{ ---}\}$ and if S' is a point on k , we have

$$S(abc \text{ ---}) \bar{\wedge} k\{ABC \text{ ---}\} \bar{\wedge} S'(ABC \text{ ---}),$$

and therefore,

$$S(abc \text{ ---}) \bar{\wedge} S'(ABC \text{ ---}).$$

Now let k' be the conic defined by these two projective flat pencils. It has in common with k the point S' and at least one other point, which may coincide with S' , or at most three other points. The chain

$$k \bar{\wedge} S' \bar{\wedge} k'$$

exhibits the conics k and k' as projectively related through a flat pencil whose center is a common point of the two conics. Hence, as in the examples of the last article, k and k' have in common at

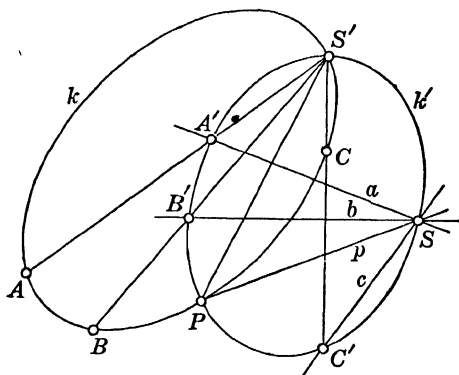


FIG. 77

least one point which is self-corresponding, and at most three such points. If P is a self-corresponding point of the projectivity $k \bar{\wedge} k'$, then SP and P are homologous elements of the projectivity

$$S(abc \text{ ---}) \bar{\wedge} k\{ABC \text{ ---}\},$$

and since SP and P are incident, the theorem is demonstrated.

DEFINITION. The points of intersection of homologous lines of a flat pencil and a projectively related line conic which are coplanar but not perspective constitute a **point cubic** or a **point-row of third order**.

The point cubic has at least one and at most three points in common with any line of its plane.

DEFINITION. The lines joining homologous points of a point-row and a projectively related point conic which are coplanar but not perspective constitute a **line cubic** or a **pencil of lines of third class**.

The line cubic has at least one and at most three lines in common with any point of its plane.

In the second paragraphs of the above we have our justification for speaking of the forms thus defined as of the third degree. To show that the line cubic has at least one line, and at most three lines, in common with any point of its plane, let us suppose that the line cubic is generated by the projectivity

$$u_1(A_1B_1C_1 \text{ ---}) \bar{\wedge} k\{ABC \text{ ---}\}.$$

We note that any point S of the plane projects the point-row u_1 by a flat pencil $S(abc \text{ ---})$. Then

$$S \overline{\wedge} u_1 \overline{\wedge} k,$$

and hence

$$S \overline{\wedge} k.$$

Now, by Theorem P1, there is at least one line of S which is incident with its homologous point on k , and at most three such lines. But such a line of S , call it p , is not only incident with its homologue P on k but is also incident with its homologue P_1 on u_1 . This line $p = PP_1$ is, therefore, a line of the line cubic; furthermore, S is any point of the plane. Hence, on any point of the plane there is at least one line of the line cubic, and at most there are three such lines.

The above rather brief discussion refers to the third degree forms *in general*. The various special types of third degree forms and their degenerate cases offer an interesting subject for investigation, but it is one which we shall not undertake in this book.

In general, two projectively related forms generate a form of higher degree if each pair of homologous elements of the projective forms determines a third element which varies with the pair. Two projective forms of the first degree (primitive forms) generate a form of second degree; two projective forms, one of the first and the other* of the second degree, generate a form of third degree, etc.

12.7 Projectivity on a Conic. Two forms consisting of like elements may have the same support as, for example, two ranges of points on the same conic. Two such forms are said to be **superposed** (cf. Art. 6.1).

Two superposed forms which are projectively related constitute a projectivity on the common support. In particular, two projective ranges of points of second order on the same conic, or, dually, two projective pencils of lines of second class on the same conic, constitute a projectivity on the conic.

In the projectivity

$$k\{ABC \text{ ---}\} \overline{\wedge} k\{A'B'C' \text{ ---}\}$$

on a conic k , each point of k has two names or labels, — such as P , when considered as a point of the first range, and Q' , when considered as a point of the second. Ordinarily, of course, the homologue P' of P in the projectivity will not coincide with P . When, however, a point P and its homologue P' do coincide we call P a **double**

point of the projectivity on the conic. If all points of a projectivity are double points, the projectivity is an identity. And thus, by Art. 6.7 and its dual, the

THEOREM. *If a projectivity on a conic has three double elements, the projectivity is an identity.*

For example, a projectivity on a conic k which has three double points is projected from any point S of k by a projectivity between two superposed flat pencils which has three double lines, and the flat pencils are, therefore, identical. Hence, every point of k is a double point.

We have seen in Art. 12.4 that three pairs of homologous elements suffice to determine a projectivity uniquely. This is also valid, of course, when the forms are superposed.

12.8 Axis and Center of Projectivity.

Theorem XII. *If $k\{ABC \text{---}\}$ and $k\{A'B'C' \text{---}\}$ are two projective ranges of points on the same conic k , there exists a line u such that, if A, A' and B, B' are any two pairs of homologous points, the lines AB' and $A'B$ meet on line u .*

The line u is called the **axis of the projectivity on the conic k** .

Theorem P XII. *If $k\{abc \text{---}\}$ and $k\{a'b'c' \text{---}\}$ are two projective pencils of lines on the same conic k , there exists a point U such that, if a, a' and b, b' are any two pairs of homologous lines, the points ab' and $a'b$ are joined by a line through point U .*

The point U is called the **center of the projectivity on the conic k** .

To prove the theorem on the left, consider the chain of perspectivities (Fig. 78)

$$A'(ABC \text{---}) \overline{\wedge} k\{ABC \text{---}\} \overline{\wedge} k\{A'B'C' \text{---}\} \\ \overline{\wedge} A(A'B'C' \text{---}),$$

whence,

$$A'(ABC \text{---}) \overline{\wedge} A(A'B'C' \text{---}).$$

The common line of these projective flat pencils is self-corresponding; hence

$$A'(ABC \text{---}) \overline{\wedge} A(A'B'C' \text{---}),$$

and pairs of homologous lines $A'B, A'B'$ and $A'C, AC'$, etc., are collinear on a line u .

Moreover, u is the Pascal line of the 6-point $AB'CA'BC'$, for it

is the join of the points common to the opposite sides AB' , $A'B$ and AC' , $A'C$. Hence, the intersection of $B'C$ and BC' , the third pair of opposite sides, also lies on u (Art. 8.1). In other words, the proof of the theorem does not depend upon the fact that we chose A and A' as the centers of the projecting pencils; if we had chosen B and B' instead, the same line u would have been determined.

If, now, D is any point of $k\{ABC \text{ ---}\}$, its homologue D' in the projectivity on k is found as the intersection with k of the line AD_1 , D_1 being the meet of u and $A'D$. For we know that $A'D$ and AD' intersect on the axis u . Moreover, any point M common to the conic k and this axis u is a double point of the projectivity of k .

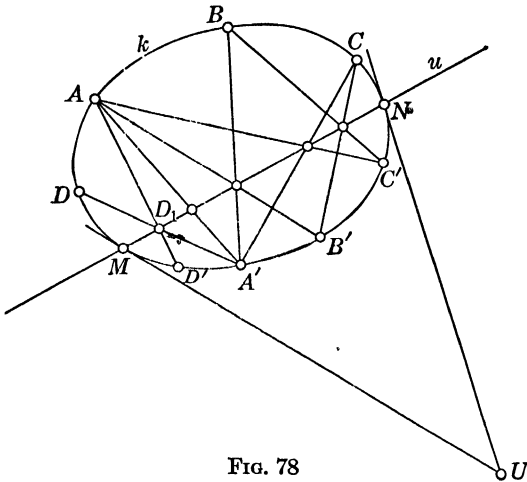


FIG. 78

The reader should now dualize the above proof and the construction which follows it.

As in Art. 6.11, a projectivity on a conic is termed **hyperbolic**, **parabolic**, or **elliptic**, according as it has two, one, or no double elements.

In general, the theorem on the right above is the plane dual of the theorem on the left, and the conics k of the two theorems are not necessarily the same. In particular, however, if the conics k are the same and if a is the tangent at A , etc., the theorem on the right above is the polar reciprocal with respect to k of the theorem on the left (Art. 9.6); and the center of the projectivity is then the pole of the axis of the projectivity.

To summarize, we may have on any conic k (in the sense of Art. 8.8)

a projectivity between two ranges of points, which has the axis u (Theorem XII). The center of this projectivity is the pole of u with respect to k .

a projectivity between two pencils of lines, which has the center U (Theorem P XII). The axis of this projectivity is the polar of U with respect to k .

Finally, by Art. 9.2, we state

THEOREM 1 *A projectivity on a conic is hyperbolic, parabolic, or elliptic, according as the center of the projectivity lies outside the conic, on the conic, or inside the conic.*

12.9 Projectivity on a One-dimensional Form. A projectivity on a one-dimensional form can be replaced, by the operations of projection and section, by a projectivity on a conic in case the support of the original projectivity is not a conic. The double elements, if any, of a projectivity on a conic are easily found by the theorems of the preceding article, and we are thereby provided with a method for constructing the double elements of a projectivity on any one-dimensional form.

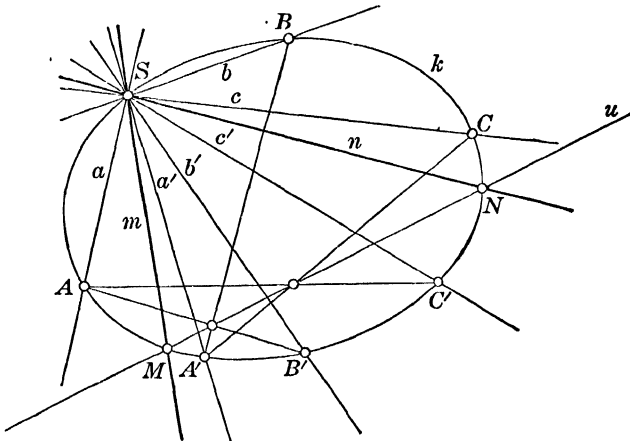


FIG. 79

For example, we have (Fig. 79) on a point S a projectivity between two superposed flat pencils,

$$S(abc \text{ ---}) \bar{\wedge} S(a'b'c' \text{ ---}).$$

To construct the double elements of this projectivity, we project it onto a conic k which passes through S . Since

$$k\{ABC \text{ ---} \} \bar{\wedge} S(abc \text{ ---}) \bar{\wedge} S(a'b'c' \text{ ---}) \bar{\wedge} k\{A'B'C' \text{ ---} \},$$

we have on k two projective ranges of points. The points of intersection of the pairs of lines AB' , $A'B$ and AC' , $A'C$ determine u , the axis of this projectivity. If u and k have in common the points M and N , these points are the double points of the projectivity on the conic. Hence, the lines $m = SM$ and $n = SN$ are the double lines of the original projectivity on S . (In practice, we usually take for the conic k a circle [Art. 10.1] as being more convenient to draw.)

The dual problem, to construct the double points of a projectivity

$$p(ABC \text{ ---}) \bar{\wedge} p(A'B'C' \text{ ---})$$

on a line p , may be solved by a dual argument. However, it is usually more practical to project the points A, B, C on p from some point S into a projectivity on a flat pencil; the double lines of the projectivity on S are then found as in the preceding paragraph, and these double lines are the projectors from S of the double points of the projectivity on the line p .

12.10 Problems.

To construct the points common to a given line p and a given conic k , if such points exist, when the conic is not completely drawn but is determined by five points, or four points and the tangent at one of them, or three points and the tangents at two of them.

To construct the tangents from a given point P to a given conic k , if such tangents exist, when the conic is not completely drawn but is determined by five lines, or four lines and the point of contact on one of them, or three lines and the points of contact on two of them.

The student is asked to reread Art. 7.3 at this time.

To solve the problem on the left when five points V, V', A, B, C of k are given, consider (Fig. 80) the projectivity determined on the line p by the projectively related flat pencils

$$V(ABC \text{ ---}) \bar{\wedge} V'(ABC \text{ ---}).$$

The double points, if any, of this projectivity on p ,

$$p(A_1B_1C_1 \text{ ---}) \bar{\wedge} p(A_2B_2C_2 \text{ ---}),$$

are the required points. To construct these double points (Art. 12.9), a second conic k' must be drawn; for convenience this conic k' may be a circle.

The particular case of this problem arising when p is the ideal line provides a method for classifying metrically (Art. 7.6) any conic defined by five conditions.

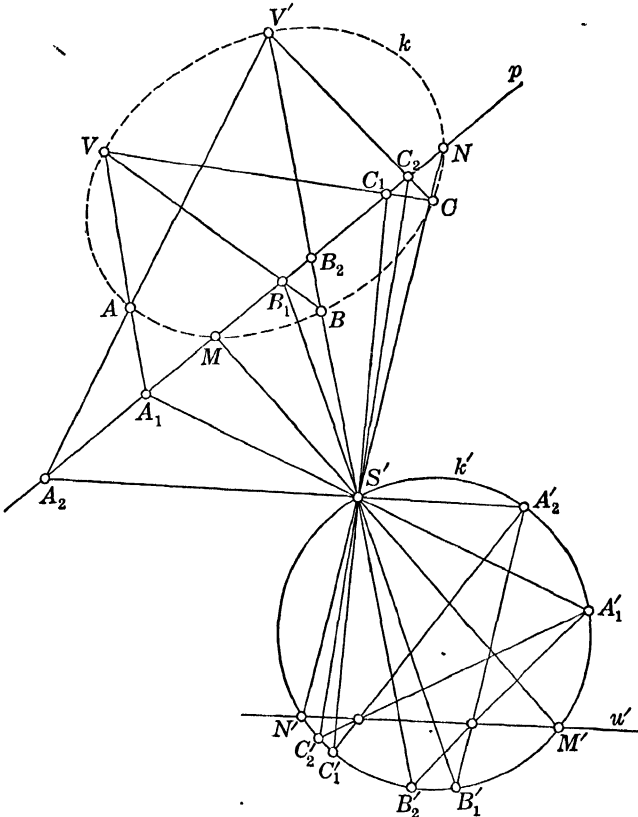


FIG. 80

We have, in this article, an example of a kind of projective problem fundamentally different from most of the problems of the earlier chapters. The solution of this problem consists in constructing *two* points common to the given line and conic, and in doing this we find it necessary to use an auxiliary conic, so to speak, i.e., the conic k' of the above discussion. It is evident, too, that the problem may have no solution, as is the case when p does not

cut k ; and, again, the solution may consist of two coincident points, as when p is tangent to k , for the projectivity on p is then parabolic. (Cf. the problems of Arts. 5.10 and 10.7 in this connection.)

Construction problems of this type are called **second degree** problems. Their solutions require the use of a straightedge and an instrument for drawing the auxiliary conic. This latter instrument, however, may be a compass, for a circle (which we can draw with the compass) is equivalent projectively to a conic. Moreover, if but *one* circle is given in a plane, any second degree problem in that plane can be solved using a straightedge alone.

On the other hand, we recall problems in the earlier chapters whose solutions required only a straightedge; that is, no auxiliary conic (or circle) was necessary. And in each case, the solution consisted of the determination of a single element. Such problems are called **first degree** or **linear geometric** problems.

As an example of a first degree problem we have, in Art. 6.11, the construction of the second double point of an hyperbolic projectivity on a line when one double point and two pairs of homologous points are given. For other examples, we direct the reader's attention to the problems of Arts. 4.2, 8.3, 8.6, etc.

12.11 Cyclic Projectivities. We shall find it convenient in this article to consider a projectivity on a conic as a transformation (Art. 9.9). The projectivity transforms or sends every point P , considered as a point of one of the projective ranges, into its homologous point P' of the other range. In other words, every point of the conic is transformed, in general, into some other point of the conic. *The conic, nevertheless, is transformed into itself.*

Any point of the conic which is transformed into itself is a double point. If the projectivity has more than two double points, then all points of the conic are double points. In this trivial case the projectivity transforms the conic **identically** and it is called an **identity transformation**.

Consider now, on a conic k , the projectivity

$$k\{PP_1P_2P_3 \text{ --- } P_{n-1} \text{ ---}\} \bar{\wedge} k\{P_1P_2P_3P_4 \text{ --- } P_n \text{ ---}\}.$$

We have changed our customary notation slightly, but that is all. The notation here used is intended to indicate that the projectivity transforms the point P , considered as of the first range, into a point P_1 , considered as of the second range; that it transforms P_1 , as of

Similarly, if any other point E of k is transformed into F , then F is transformed into E , and we have

$$k\{ABCDEF \text{ ---} \} \bar{\wedge} k\{BADCFE \text{ ---} \}.$$

Moreover, u joins two diagonal points of the complete quadrangle $ABEF$, and U , therefore, is on line EF . Again, by Property 1, Art. 9.2, the tangents to k at each pair of homologous points intersect on the axis u .

These considerations prove the theorems:

THEOREM 1. *If, in a projectivity on a one-dimensional form, one pair of homologous elements corresponds doubly, every pair of homologous elements corresponds doubly.*

THEOREM 2. *In a cyclic projectivity of period 2 on a conic the joins of homologous points are concurrent at the center of the projectivity, and the intersections of tangents at homologous points are collinear on the axis of the projectivity.*

The first of these theorems is identical in statement to Theorem 1 of Art. 10.2. It has been demonstrated here, however, quite independently of that discussion and is, in fact, an extension of that theorem to the one-dimensional forms of the present chapter.

THEOREM 3. *In the plane of a given conic k those lines of a flat pencil U (with center U not on k) which cut the conic determine on it a cyclic projectivity of period 2, homologous points of which are collinear with the center U .*

To prove this let two lines of the pencil U , such as AB and CD , cut the conic at pairs of points A, B and C, D (Fig. 81). These points on k are so situated that the lines AD and BC , and also the lines AC and BD , intersect on u , the polar of U . The pairs of points A, B and C, D , and also the pairs B, A and C, D , are, therefore, pairs of homologous points in the projectivity whose axis is u . Moreover, the points of each pair are collinear with U , which is the center of the projectivity. Thus, the projectivity is a cyclic projectivity of period 2.

Four points A, B, C, D arbitrarily chosen on a conic k determine three cyclic projectivities of period 2 on k , viz.,

$$k\{ABCD \text{ ---} \} \bar{\wedge} k\{BADC \text{ ---} \} \text{ with center } U \text{ and axis } u,$$

$$k\{ABCD \text{ ---} \} \bar{\wedge} k\{CDAB \text{ ---} \} \text{ with center } V \text{ and axis } v,$$

$$k\{ABCD \text{ ---} \} \bar{\wedge} k\{DCBA \text{ ---} \} \text{ with center } W \text{ and axis } w.$$

Thus, we have

$$k\{ABCD\text{---}\} \bar{\wedge} k\{BADC\text{---}\} \bar{\wedge} k\{CDAB\text{---}\} \\ \bar{\wedge} k\{DCBA\text{---}\};$$

and the center and axis of the projectivity between any two members of this chain are, respectively, a vertex and an opposite side of the diagonal triangle of the complete quadrangle $ABCD$.

In anticipation of the next chapter we now state the

DEFINITION. An **involution** is a cyclic projectivity of period 2.

2. *Cyclic Projectivities of Period 3.* Three points A, B, C arbitrarily chosen on a conic k determine one cyclic projectivity of period 3 on k ,² viz.,

$$k\{ABC\text{---}\} \bar{\wedge} k\{BCA\text{---}\}.$$

If a, b, c are the tangents to k at A, B, C , respectively (Fig. 82), we know (Theorem 4, Art. 8.4) that the two triangles abc and ABC are perspective from a line u . This line u contains the points of intersection of $a = AA$ and BC , $b = BB$ and CA , $c = CC$ and AB , and is, therefore, the axis of the projectivity on k , by Theorem XII. Moreover, the triangles abc and ABC are perspective from a point U , which is the pole of u with respect to k (Theorem P4, Art. 8.4). This point U is the center of the projectivity on k .

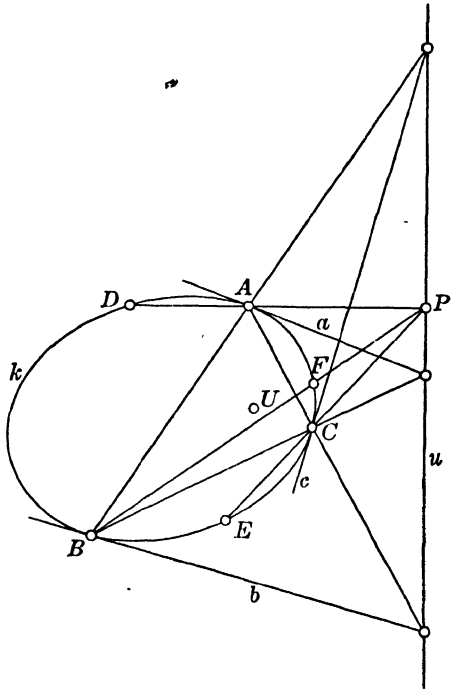


FIG. 82

By this same projectivity any fourth point D is transformed into a point E ; the point E is transformed into a point F ; F is transformed into D ; and the triad D, E, F is also a cycle of

² We note that three points are not sufficient to determine a cyclic projectivity of period 2; four points are necessary, and then three such projectivities are determined.

the projectivity. We prove that F is transformed into D as follows: Suppose F is transformed not into D but into a point F' . The projectivity as it now stands is

$$k\{ABCDEF \text{---}\} \bar{\wedge} k\{BCAEFF' \text{---}\}.$$

Since C, A and D, E are two pairs of homologous points, the lines CE and AD meet at a point P on the axis u . Similarly, since B, C and E, F are pairs of homologous points, we know that CE and BF meet on u , and at the same point P ; and because A, B and F, F' are pairs, that BF meets AF' on u , again at the same point P . That is, the lines ADP and $AF'P$ coincide and cut the conic once at A . Hence, their second points of intersection with k are coincident, i.e., $F' = D$. Therefore, F is transformed into D .

Having shown, by these examples, the existence of cyclic projectivities of period 2 (involutions) and of period 3, we content ourselves with mere statement of the fact that there are cyclic projectivities of periods other than two and three. For our work those of period 2 are the important ones.

The dual of this article follows readily and will be omitted.

EXERCISES

1. Given two pairs of points A, B and C, D on a conic, it is required to find the pair of points P, Q which separates harmonically both A, B and C, D . Under what circumstances will there be such a pair (cf. Art. 5.10)?
2. Given two pairs of points on a line, construct by the methods of this chapter the pair which harmonically separates both the given pairs (cf. Art. 5.10).
3. State and solve the plane duals of Exs. 1 and 2.
4. If a flat pencil is projectively related to a point conic coplanar with it and if more than three pairs of homologous elements, line and point, are incident, then the center of the flat pencil lies on the conic and the two forms are in perspective position. Prove this theorem and state its dual.
5. Construct a point cubic by the method indicated in Art. 12.6.
6. Discuss the construction of the point curve consisting of the points of intersection of homologous lines of two projectively related coplanar line conics.
7. Given a projectivity of points on a conic, construct the homologues D', E' of the two points D, E on the conic (a) when D and E are distinct points; and (b) when D and E are coincident points. Consider the projectivity as determined (1) by three pairs of homologous points; and then (2) by two pairs of homologous points and a double point; and finally (3) by one pair of homologous points and the two double points.
8. State and solve the plane dual of Ex. 7.

9. A conic ($ABCDd$) is determined by four of its points and the tangent at one of those points. Find the points common to this conic and a given line p , if there are such points.

10. A conic ($abcdD$) is determined by four of its tangents and the point of contact on one of those tangents. Find the points common to this conic and a given line p , if such points exist.

11. A conic ($abcAB$) is determined by three of its tangents and the points of contact on two of those tangents. Find the tangents to this conic from a given point P , if such tangents exist.

12. A conic ($ABCab$) is determined by three of its points and the tangents at two of those points. Find the tangents to this conic from a given point P , if such tangents exist.

13. Two points of an hyperbola (ABC^*D^*a), the tangent at one of those points, and the directions of the asymptotes are given. If p is any line of the plane, construct the points of intersection of p and the hyperbola, if there are such points.

14. One ordinary tangent, the axis, and the vertex B (which is a point on the axis) of a parabola (abc^*BC^*) are given. If P is any point of the plane, construct the tangents to the parabola from P , if such tangents exist.

15. Choose arbitrarily in one plane four ordinary points and a line on one of them; and then determine what type of conic passes through the four points and touches the given line — hyperbola, parabola, or ellipse.

16. A projectivity on a line is determined uniquely by three pairs of homologous points A and A' , B and B' , C and C' . Construct the double points of the projectivity (a) if the six given points are distinct; (b) if A coincides with B' but B does not coincide with A' ; (c) if A coincides with B' and B coincides with A' ; and finally (d) if A coincides with A' (cf. Art. 6.11).

17. Discuss the plane dual of Ex. 16.

18. The double point and one pair of homologous points of a parabolic projectivity on a line are known. Construct other pairs of homologous points of this projectivity. Transfer this projectivity to a conic and construct its axis and center.

19. If M, N are the double elements of an hyperbolic involution, show that the involution is uniquely determined, and construct other pairs of homologous elements of this involution.

20. If five points of a conic k are known, construct on a given line u of the plane an involution of conjugate points with respect to k . Is the involution as constructed hyperbolic or elliptic?

21. On a line u construct a cyclic projectivity of period 3. Show that a cyclic projectivity of period 3 is always elliptic.

CHAPTER XIII

THE THEORY OF INVOLUTION — METRIC PROPERTIES

13.1 Definition and Theorems. We refer to Art. 12.11 for the

DEFINITION. An **involution** is a cyclic projectivity of period 2.

Any two homologous elements of an involution on a one-dimensional form are doubly corresponding (Arts. 9.5, 10.2, 12.11), and the elements of the form upon which the involution exists are said to be paired in **involution**. We have proved, also,

THEOREM 1. *If, in a projectivity on a one-dimensional form, one pair of homologous elements corresponds doubly, every pair of homologous elements corresponds doubly.*

Since an involution is a special projectivity, and since its pairs of homologous elements correspond doubly, we have, from the theorem of Art. 12.4,

THEOREM 2. *An involution is completely determined by two pairs of homologous elements.*

We shall now state some other theorems concerning involutions. These theorems are all either repetitions or restatements of results obtained from earlier investigations.

Thus, from Theorem 2, Art. 12.11, we have

THEOREM 3. *The center of an involution of points on a conic is collinear with each pair of homologous points; and the tangents at each pair of homologous points intersect on the axis of the involution. The center and the axis of the involution are pole and polar with respect to the conic.*

A dual theorem may be stated for an involution of lines on a conic.

The classification of projectivities according to the number of their double elements (Art. 12.8), and the pole-polar relation of the center and axis of projectivity, give rise to the three theorems which follow.

THEOREM 4. *If the axis of an involution on a conic cuts the conic or, what is the same, if the center of the involution lies outside the conic, the involution is hyperbolic; and the points common to the axis and the conic are double points of the involution of points on the conic,*

while the tangents to the conic from the center of the involution are double lines of the associated involution of lines on the conic.

THEOREM 5. *If the axis of an involution on a conic touches the conic, the involution is parabolic; and the point of contact of the conic on the axis and the axis itself are respectively the double point and the double line of the involutions of points and of lines on the conic (cf. Art. 13.2).*

THEOREM 6. *If the axis of an involution on a conic does not cut the conic or, what is the same, if the center of the involution lies inside the conic, the involution is elliptic.*

Finally, the theory of Art. 12.9 enables us to state

THEOREM 7. *An involution on any one-dimensional form can be replaced, by the operations of projection and section, by an involution on a conic.*

13.2 Special Properties of Involution. The theorems of this article although demonstrated for involutions of points on a conic are, by Theorem 7 of the preceding article, valid for involutions on any one-dimensional form.

1. *Hyperbolic Involution.* If A, A' is any pair of homologous points of an hyperbolic involution on a conic, the line AA' passes through the center U of the involution, and is, therefore, a conjugate line to the axis u with respect to the conic. If u cuts the conic at M and N (Fig. 83), it follows from the converse of Theorem 2, Art. 12.3, that the pairs A, A' and M, N form an harmonic set $H\{AA', MN\}$. Hence,

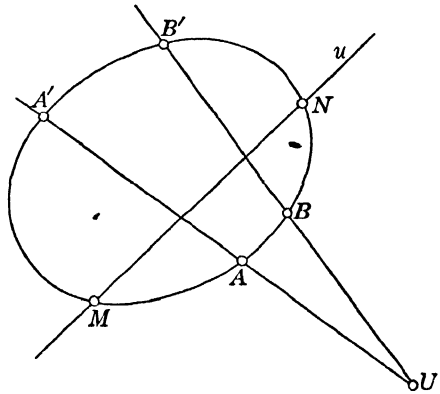


FIG. 83

THEOREM 1. *In an hyperbolic involution the double elements of the involution form with each pair of homologous elements an harmonic set.*

The pairs A, A' and M, N of the harmonic set $H\{AA', MN\}$ separate each other. Pairs of points separating each other are said to be **interlaced**. We observe (Fig. 83) that pairs of homologous points, such as A, A' and B, B' , of an hyperbolic involution

do not separate each other and are not interlaced, for the lines AA' and BB' intersect at a point U outside the conic.

2. *Parabolic Involutions.* A parabolic involution is trivial, and we shall find it plays no important part in the general theory. Since the center U of such an involution lies on the conic it is incident with the axis u . Moreover, U is itself one member of every pair of homologous points. Since, conversely, every point of the conic is paired in involution with U , the parabolic involution is hardly deserving of the name "one-to-one correspondence" (cf. Art. 3.1).

3. *Elliptic Involutions.* If A, A' is any pair of homologous points of an elliptic involution on a conic (Fig. 84), the line AA' passes through U , the center of the involution, and the pole P of the line AA' is joined to U by a line conjugate to AA' with respect to the conic. This line PU , and no other line, cuts the conic in two points B, B' which are homologous points of the involution and which, at the same time, form with

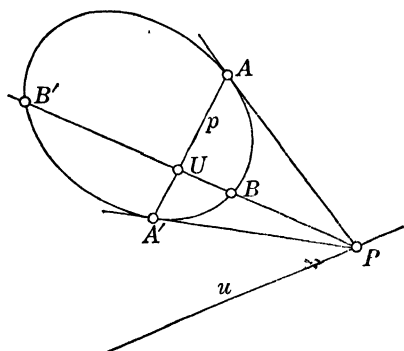


FIG. 84

A, A', B, B' an harmonic set $H\{AA', BB'\}$. Hence,

THEOREM 2. *In an elliptic involution there is one, and only one, pair of homologous elements which forms with a given pair an harmonic set.*

An elliptic involution has no double elements. Moreover, since its center lies inside the conic, any two pairs of homologous points separate each other and are interlaced in the sense described above.

13.3 Involutions on a Primitive Form.

THEOREM 1. *The three pairs of opposite sides of a complete quadrangle are cut by any line not on a vertex in three pairs of homologous points of an involution.*

THEOREM P1. *The three pairs of opposite vertices of a complete quadrilateral are projected from any point not on a side by three pairs of homologous lines of an involution.*

To prove the theorem on the left, let u be any line cutting the opposite sides of the complete quadrangle $KLMN$ at A and A' , B and B' , C and C' , as in Fig. 85. We wish to prove now that the projectivity

$$u(ABC \text{ ---}) \bar{\wedge} u(A'B'C' \text{ ---})$$

is an involution, and we shall do this by showing one pair of homologous points of the projectivity to be doubly corresponding; it then follows, from Theorem 1, Art. 13.1, that all pairs are doubly corresponding.

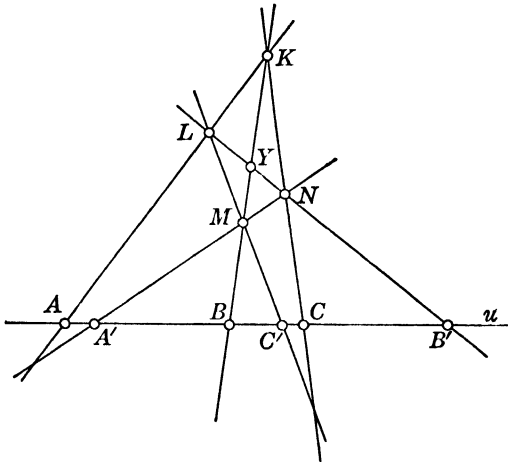


FIG. 85

If the diagonal point Y is the intersection of sides KM and LN of the complete quadrangle, we have

$$ABCB' \text{ ---} \bar{\wedge}^K LYNB' \text{ ---} \bar{\wedge}^M C'BA'B' \text{ ---};$$

hence,

$$ABCB' \text{ ---} \bar{\wedge} C'BA'B' \text{ ---}.$$

But also, by the theorem of Art. 6.4,

$$C'BA'B' \text{ ---} \bar{\wedge} A'B'C'B \text{ ---}.$$

Consequently, we have on u

$$u(ABCB' \text{ ---}) \bar{\wedge} u(A'B'C'B \text{ ---}),$$

and the pair B, B' is doubly corresponding. The projectivity is then an involution, and the theorem is demonstrated.

In connection with the theorem on the left, and dually for that on the right, the reader will notice that the involution determined on line u by the pairs of opposite sides of the complete quadrangle may be elliptic or hyperbolic according to the position of u . (If u contains one vertex of the quadrangle, the involution is parabolic. Cf. Art. 13.2.) If u passes through a diagonal point, the involution is hyperbolic and has the diagonal point as one double point; and if u contains two diagonal points, those are the double points of the involution, and the remaining pair is harmonically separated by them (cf. Art. 4.1 and Theorem 1, Art. 13.2).

13.4 The Construction of Homologous Points of an Involution.

An involution on a line u is determined by two pairs of homologous points A, A' and B, B' . Given a fifth point C on u , it is required to construct its homologue C' in the involution.

An involution on a point U is determined by two pairs of homologous lines a, a' and b, b' . Given a fifth line c on U , it is required to construct its homologue c' in the involution.

By transferring the involution to a conic the problem on the left (and its dual) can be solved by the method of Art. 12.8. But the problem is fundamentally a first degree problem, and as such the introduction of an auxiliary conic is not necessary. A linear construction follows from the theory of the preceding article.

Thus, we draw, in any plane on u , three nonconcurrent lines through A, B' , and C , respectively (Fig. 85). A triangle KLN is now formed whose sides KL, LN, NK meet u at A, B', C . Now join the other given points A' and B to N and K , respectively. The lines $A'N$ and BK meet at M , say, and the line LM will intersect u at the required point C' , the partner of C in the given involution, by Theorem 1, Art. 13.3.

Any other complete quadrangle $K'L'M'N'$ which has $K'L'$ and $M'N'$, $K'M'$ and $L'N'$, and $K'N'$ cutting u , respectively, at A and A' , B and B' , and C will have its sixth side $L'M'$ cutting u at the same point C' , by Theorem II of Art. 3.5.

We ask the reader to compare this construction with a similar one in Art. 4.2, and also to consider its dual.

13.5 Desargues' Theorem on Conics.

THEOREM 1. *The points of a conic on any line cutting the conic are homologous points in the involution determined on that line by pairs of opposite sides of any complete quadrangle whose vertices lie on the conic.*

THEOREM P1. *The tangents to a conic from any point outside the conic are homologous lines in the involution determined on that point by pairs of opposite vertices of any complete quadrilateral whose sides touch the conic.*

On the left, let the line u cut the conic at P and P' , and the pairs of opposite sides of the complete quadrangle $KLMN$ at A and A' , B and B' , C and C' (Fig. 86). Since the four points

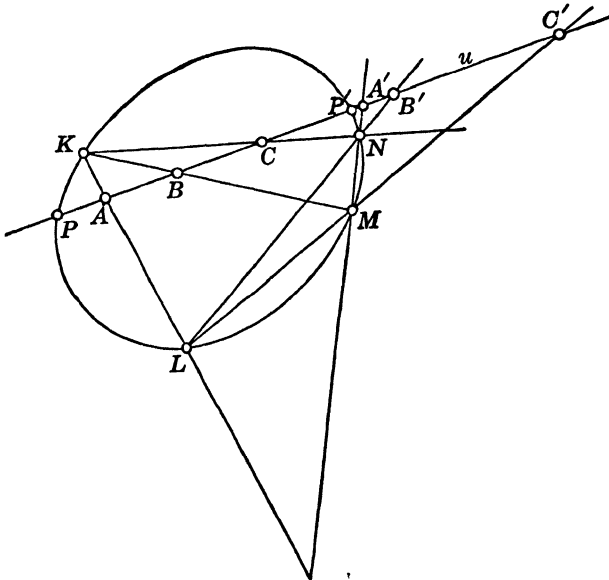


FIG. 86

P, K, P', N lie on the conic, and since L and M are two points on the same conic, we have

$$L(PKP'N \text{ ---}) \bar{\wedge} M(PKP'N \text{ ---}), \bullet$$

and also

$$u(PAP'B' \text{ ---}) \bar{\wedge} L(PKP'N \text{ ---}) \bar{\wedge} M(PKP'N \text{ ---}) \\ \bar{\wedge} u(PBP'A' \text{ ---});$$

whence,

$$u(PAP'B' \text{ ---}) \bar{\wedge} u(PBP'A' \text{ ---}).$$

But, by Art. 6.4,

$$u(PBP'A' \text{ ---}) \bar{\wedge} u(P'A'PB \text{ ---}),$$

and, therefore,

$$u(PAP'B' \text{ ---}) \bar{\wedge} u(P'A'PB \text{ ---}).$$

Now, in this last projectivity, the pair P, P' is a doubly corresponding pair, and the projectivity is an involution in which P, P' and A, A' and B, B' (and also C, C' , by Theorem 1, Art. 13.3) are pairs. Hence, the theorem is established.

DEFINITION. The totality of conics passing through the vertices of a complete quadrangle is called a **pencil of conics**.

DEFINITION. The totality of conics touching the sides of a complete quadrilateral is called a **range of conics**.

Consider the pencil of conics having in common the four points K, L, M, N of Theorem 1. Any line u cuts each conic of this pencil, if at all, in two points P, P' which are paired in involution with the points common to u and pairs of opposite sides of the complete quadrangle $KLMN$. The question arises: Given four points and a line u in the same plane, can a conic be constructed passing through the four points and tangent to the line u ?

To answer this question we must first inquire as to the involution determined on u by the complete quadrangle $KLMN$; is that involution hyperbolic or elliptic (Art. 13.3)?

If the involution determined on u by $KLMN$ is hyperbolic, it has two double points X and Y . The conic determined by the five points K, L, M, N, X (or the five points K, L, M, N, Y) touches u at X (or Y). For this conic, the points P and P' of Fig. 86 have become coincident at X (or Y). In this case, then, there are two conics touching u .

On the other hand, if the involution is elliptic there are no double points; hence, there is no conic through K, L, M, N and touching u .

Finally, if line u passes through one vertex¹ of the quadrangle, say K , there is only one conic through K, L, M, N and tangent to u at K (Art. 7.2).

¹ There can be no conic tangent to u if u passes through two of the four given points.

We thus arrive at

THEOREM 2. *There are, at most, two conics passing through four given points, of which no three are collinear, and touching a given line.*²

THEOREM P2. *There are, at most, two conics touching four given lines, of which no three are concurrent, and passing through a given point.*

When we consider the given line of the theorem on the left to be the ideal line of the plane, a metric specialization of that theorem follows. We then have

THEOREM 3. *There are, at most, two parabolas passing through four given points of which no three are collinear.*

And similarly, from Theorem P2,

THEOREM 4. *There are, at most, two parabolas touching three given nonconcurrent ordinary lines and passing through a given point.*

13.6 Involutions Determined by a Conic. As a consequence of the discussion resulting from Theorem 1, Art. 9.5, a conic determines

on each line in its plane an involution of points conjugate (Art. 9.4) with respect to the conic.

This involution of conjugate points on a line is hyperbolic, parabolic, or elliptic, according as the line cuts the conic, is tangent to the conic, or does not cut the conic.

For if the involution has a double point, that point is self-conjugate with respect to the conic and is, therefore, a point on the conic.

THEOREM 1. *An involution on a line u of conjugate points with respect to a conic is projected from a point S of the conic by a flat pencil, which determines on the conic an involution of points whose axis is the line u of the original involution.*

on each point in its plane an involution of lines conjugate (Art. 9.4) with respect to the conic.

This involution of conjugate lines on a point is hyperbolic, parabolic, or elliptic, according as the point lies outside the conic, on the conic, or inside the conic.

For if the involution has a double line, that line is self-conjugate with respect to the conic and is, therefore, a tangent to the conic.

THEOREM P1. *An involution on a point U of conjugate lines with respect to a conic is cut by a tangent s of the conic in a point-row, which determines on the conic an involution of lines whose center is the point U of the original involution.*

² In connection with Theorems 1 and 2 of this article we direct the reader's attention to Appendix B.

Let (Fig. 87) A_1, A_2 be any pair of the involution on u of conjugate points with respect to a conic k , and let B_1, B_2 be any other pair. Then,

$$u(A_1B_1A_2 \text{ --- }) \bar{\wedge} u(A_2B_2A_1 \text{ --- }),$$

and A, A' , the projections on k of A_1, A_2 from a point S of the conic, are homologous points of an involution on k , viz.,

$$k\{ABA' \text{ --- } \} \bar{\wedge} k\{A'B'A \text{ --- } \}.$$

That this last projectivity is an involution is evident, of course, from the fact that A, A' are doubly corresponding points on k .

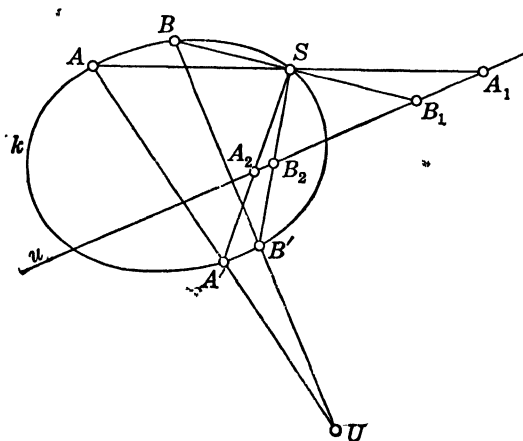


FIG. 87

Now consider the triangle SAA' , which has the line u cutting two of its sides at A_1 and A_2 , conjugate points with respect to k . By Theorem 2, Art. 9.5, the line u is conjugate to AA' or, what is the same, the line AA' passes through U , the pole of u with respect to k . Hence, U is the center of the involution on k , u is the axis of the involution, and the theorem is demonstrated.

THEOREM 2. *If the projector of an involution of points on a conic from a point of the conic is cut by the axis of the involution, there is determined on that axis an involution of conjugate points with respect to the conic.*

THEOREM P2. *If the section of an involution of lines on a conic by a tangent to the conic is projected from the center of the involution, there is determined on that center an involution of conjugate lines with respect to the conic.*

These last theorems, which are the converses of those just preceding, we leave to the reader to prove and illustrate with figures. As supplementary to these theorems, we add the following:

If the projector of an involution on a conic k from a point on the conic is cut by a line p not the axis of the involution, there is determined on p an involution — but not an involution of conjugate points with respect to k . Moreover, the double points, if any, of the involution on p are not the points common to line p and conic k .

If the section of an involution on a conic k by a tangent to the conic is projected from a point P not the center of the involution, there is determined on P an involution — but not an involution of conjugate lines with respect to k . Moreover, the double lines, if any, of the involution on P are not the tangents from point P to conic k .

An involution is completely determined by two pairs of homologous elements, or by one pair and a double element, or by its two double elements. Hence,

THEOREM 3. *Pairs of homologous points of an hyperbolic involution on a line are conjugate points with respect to every conic cutting the line at the double points of the involution.*

THEOREM P3. *Pairs of homologous lines of an hyperbolic involution on a point are conjugate lines with respect to every conic tangent to the double lines of the involution.*

For if M, N are the double points of the involution, then any pair A, A' of the involution forms with the points M, N an harmonic range $H(AA', MN)$, by Theorem 1, Art. 13.2. Hence, the polar of A with respect to a conic through M and N passes through A' (Property 2, Art. 9.2), and A, A' is a pair of conjugate points with respect to k .

13.7 Metric Definition of an Involution on a Line. In an involution of points on a line each point of the line is paired in involution with its homologue. If the line is an ordinary line of the plane, then the ideal point O' , say, is paired with one ordinary point O of the line, or it is a double point.

DEFINITION. If, in an involution on an ordinary line, the ideal point is not a double point, then its homologue is called the **center** of the involution.

We must caution the reader not to confuse the center of an involution *on a line* and the center of an involution *on a conic*. They are quite distinct and independent concepts.

An involution on a line is completely determined by its center and one pair of homologous points (Theorem 2, Art. 13.1). Therefore, if O is the center and A, A' is one pair of homologous points in an involution on a line u , we have

$$u(AA'OO' \text{ ---}) \bar{\wedge} u(A'AO'O \text{ ---}),$$

O' being the ideal point of u . Furthermore, if B, B' is any pair of the involution, we have

$$u(ABOO' \text{ ---}) \bar{\wedge} u(A'B'O'O \text{ ---}),$$

and hence, by Theorem 2, Art. 5.4,

$$(AB, OO') = (A'B', O'O);$$

also,

$$(AB, O'O) = (B'A', OO'),$$

by Theorem 1, Art. 5.7. But O' is the ideal point of u , hence

$$\therefore \frac{AO}{BO} = \frac{B'O}{A'O},$$

or

$$OA \cdot OA' = OB \cdot OB'.$$

Now, since B, B' is any pair and A, A' is a fixed pair of the involution on u , this product $OB \cdot OB'$ has the same value for all pairs, i.e.,

$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \dots = \text{a constant.}$$

DEFINITION. A pairing of ordinary points X, X' on a line u such that

$$OX \cdot OX' = k,$$

where k is any constant different from zero and O is a fixed ordinary point of u , is an **involution** with center O which is hyperbolic or elliptic according as k is greater than or less than zero.

Case 1. When $k > 0$. Any pair of points X, X' of the involution lies on the same side of the point O , for the product $OX \cdot OX'$ is a positive number; i.e., the directed segments OX and OX' are either both positive or both negative. Moreover, since k is positive, there is a point M , such that

$$OM = \sqrt{k},$$

and a point N , such that

$$ON = -\sqrt{k},$$

which correspond to themselves in the pairing; consequently, M and N are double points, and the involution is hyperbolic.

Moreover, O is the mid-point of the segment MN and

$$OM^2 = ON^2 = OA \cdot OA' = \dots = OX \cdot OX' = \dots = k.$$

From this last relation and Theorem 5, Art. 5.9, we have $H(XX', MN)$ which is in agreement, as it should be, of course, with Theorem 1, Art. 13.2.

Case 2. When $k < 0$. Any two homologous points X, X' of the involution are such that the point O lies between them. For, since k is negative, the segments OX and OX' are oppositely directed. There are no points M, N such as in Case 1, but the relation

$$OA \cdot OA' = \dots = OX \cdot OX' = \dots = k$$

is still valid.

Having no double points the involution is elliptic.

13.8 Metric Construction of Homologous Pairs. An involution of ordinary points on a line u is determined by two pairs A, A' and B, B' . Let C be any fifth point on u . To construct C' , the partner of C in the involution, draw two circles, (A_1) passing through A and A' and (B_1) passing through B and B' , which have in common two points P and Q .

The line PQ cuts u at O , the center of the involution, since

$$OA \cdot OA' = OP \cdot OQ = OB \cdot OB'.$$

The constant k of this involution is equal to the product $OP \cdot OQ$, and is the power³ of O with respect to each of the circles (A_1) and (B_1) . When the pair A, A' does not separate B, B' , the power k is positive and the involution is hyperbolic (Fig. 88); and when A, A' separates B, B' , the power k is negative and the involution is elliptic (Fig. 89).

Any ordinary point \mathcal{C} , on the line u and distinct from O , determines with P and Q a circle (C_1) which cuts u again at C' . Now

$$OC \cdot OC' = OP \cdot OQ = OA \cdot OA',$$

and C' , therefore, is the homologue of C in the involution.

³ Cf. Art. 5.10.

When the involution is hyperbolic, the circle whose center is O and whose radius is $r = \sqrt{k}$ cuts u at M and N , the double points of the involution.

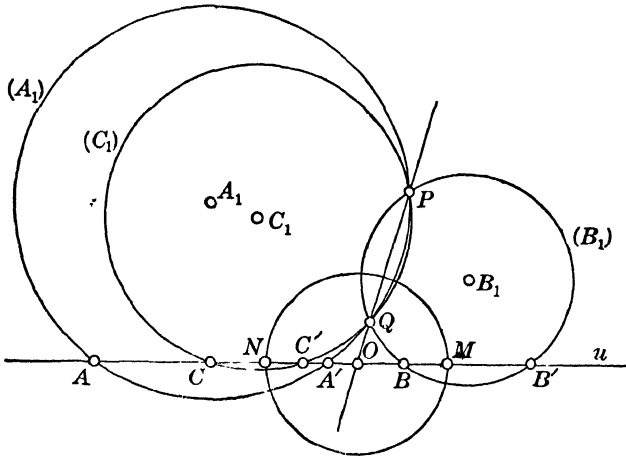


FIG. 88

Finally, consider the elliptic involution, and in drawing the circles (A_1) and (B_1) choose their centers on u so that AA' and BB' are diameters, respectively, as in Fig. 90. The line PQ is

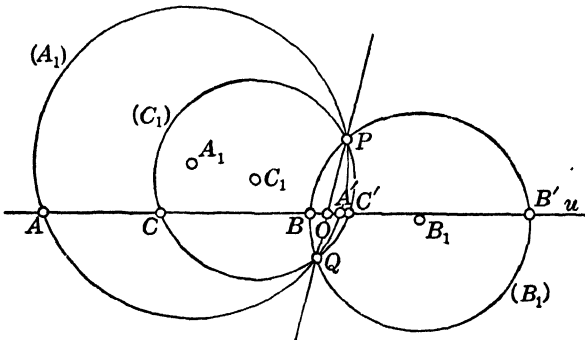


FIG. 89

then perpendicular to u , and O is the mid-point of the segment PQ . Any circle (C_1) through P and Q cuts u at a pair of homologous points C, C' of the elliptic involution. The projector of this involution from the point P (or Q) is an elliptic involution of lines

on P (or Q), homologous lines of which are perpendicular. This involution,

$$P(AA'B \text{ ---}) \bar{\wedge} P(A'AB' \text{ ---}),$$

is then the circular involution on P (Art. 10.2), and the section

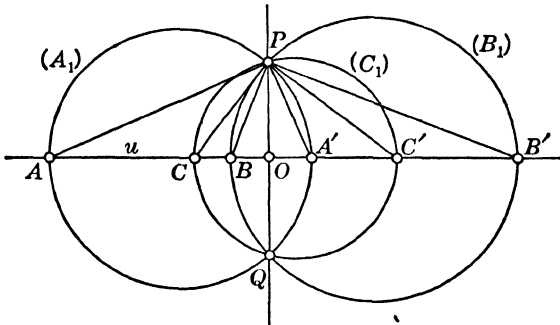


FIG. 90

by the ideal line of this circular involution is the orthogonal involution I . Hence, the

THEOREM. *In the plane of a given elliptic involution of ordinary points on a line, there are two ordinary points from which the given involution is projected by a circular involution.*

Thus, any elliptic involution of ordinary points on a line may be projected into the orthogonal involution from two ordinary points of the plane.

EXERCISES

1. On a line t tangent to a given conic k there is given an involution of points. Show that tangents to k from homologous points of this involution intersect in collinear points. Dualize.

2. Tangents to a conic from points which harmonically separate two fixed points, whose join touches the conic, intersect on a fixed line. Dualize.

3. The point of intersection of perpendicular tangents to a parabola lies on a fixed line, and the chord of contact passes through a fixed point.

4. The sides of a variable right angle whose vertex is a fixed point on a given conic intersect the conic at two points whose join passes through a fixed point.

5. If the axis of an hyperbolic involution on a conic is a diameter of the conic, the tangent at one extremity of this diameter is cut by tangents to the conic at homologous points of the given involution in homologous points of a second involution. What are the double points of this second involution?

6. Given two involutions on a one-dimensional form; construct that pair of homologous elements which is common to both involutions when (a) the involutions are both elliptic; (b) one involution is elliptic and the other hyperbolic; (c) the involutions are both hyperbolic. When do two hyperbolic involutions not have a common pair?

7. Given an involution on the ideal line; construct the pair common to this involution and the orthogonal involution (Art. 10.2).

8. Given an involution on a flat pencil; construct the homologous lines which are perpendicular to each other.

9. Given a pair of homologous points A, A' and one double point M of an hyperbolic involution on a line u ; construct the second double point N by the method of Art. 6.11. Construct another pair of homologous points B, B' , and use the resulting figure to prove Theorem 1, Art. 13.2. Dualize.

10. Prove Theorem 1, Art. 13.2, using the theory of Art. 5.7.

Suggestion. $MNBB' \bar{\wedge} MNB'B$.

11. Given an elliptic involution on a line u which has A, A' and B, B' as pairs of homologous elements; construct that pair of the involution which separates harmonically the pair A, A' . Dualize.

12. If in Ex. 11 the elliptic involution is the orthogonal involution, construct the pair harmonically separating A, A' .

13. If pairs of opposite sides of a complete quadrangle are cut by a line in the pairs of ordinary points A and A', B and B', C and C' , the segments determined by these points are connected by the relation

$$AB' \cdot BC' \cdot CA' + A'B \cdot B'C \cdot C'A = 0,$$

and conversely.

Suggestion. Use Art. 5.4.

14. Derive a relation analogous to that of Ex. 13 for three pairs of concurrent lines in involution.

15. A line u cuts the opposite sides of a complete quadrangle in three pairs of points in involution. Construct the double points, if any, of this involution on u .

16. Given an involution on a line, $u(ABA' \text{ ---}) \bar{\wedge} u(A'B'A \text{ ---})$; construct the homologue in this involution of any point C on the line. Dualize.

17. Given an hyperbolic involution on a line, $u(AA'M \text{ ---}) \bar{\wedge} u(A'AM \text{ ---})$, of which M is a double point; construct the homologue in this involution of any point C on the line. Dualize.

18. Given four points and a line not on any one of them; construct the points, if any, on the given line at which a conic through the given points can touch that line.

19. Construct the axes of the parabolas which pass through four given points. When do four points not lie on a parabola?

20. Given four lines and a point not on any one of them; determine whether a conic can be constructed touching the given lines and passing through the given point.

21. Given five points of a conic ($ABCDE$); show how to construct the conic using Theorem 1, Art. 13.5. Dualize.

22. If the vertices of a triangle ABC are points on a conic and u is any line cutting the conic at P and P' , these points P, P' are homologues in the involution determined on u by the intersections of u with AB and AC and the intersections of u with BC and the tangent to the conic at A . Prove this theorem and state its dual.

23. Given three points A, B, C and a line a on A , there are, at most, two conics through A, B, C and touching a at A which are tangent to a given line u . Prove this theorem and state its dual.

24. Given four points of a conic and the tangent to the conic at one of the given points; show how to construct the conic using the theorem of Ex. 23. Dualize.

25. Given a conic k and a point P not on k ; construct the conjugate lines on P which are at right angles to each other. In particular, if k is a central conic and P its center, what are these perpendicular conjugate lines?

26. A conic passes through three given noncollinear points A, B, C and determines on a given line p (not on any of the given points) an hyperbolic involution of conjugate points of which E, E' and F, F' are two pairs of homologues. Construct the conic.

27. If in Ex. 26 the given involution of conjugate points with respect to the conic is elliptic, construct the conic.

28. If in Ex. 27 the given elliptic involution of conjugate points is the orthogonal involution, construct the conic.

29. Investigate the plane duals of Exs. 26 and 27.

30. If the ideal point of a line is one double point of an hyperbolic involution on that line, if M is the other double point, and if X, X' is any pair of homologous points, then M is the mid-point of the segment XX' , that is, $MX = -MX'$.

31. Given an elliptic involution on a line u ; project it into the orthogonal involution. Consider line u as (a) an ordinary line of the plane; and (b) as the ideal line of the plane.

32. If two pairs of opposite sides of a complete quadrangle consist of perpendicular lines, show that the lines of the third pair are also perpendicular. Use this to prove that the altitudes of a triangle are concurrent. The point common to the altitudes is called the *orthocenter* of the triangle.

33. All conics through the vertices and orthocenter of a triangle are rectangular hyperbolas (Art. 10.10).

34. There is one rectangular hyperbola passing through the vertices of any quadrangle.

35. The lines joining the orthocenter of a triangle to a vertex and the ideal point of the side opposite are homologues in the circular involution on the orthocenter.

36. If a parabola touches the sides of a triangle, the tangents to the parabola from the orthocenter of the triangle are perpendicular. Also, the orthocenters of all triangles whose sides touch a parabola lie on a fixed line (cf. Ex. 3).

37. The orthocenters of the four triangles formed by the sides of a complete quadrilateral are collinear.

38. If the sides of a rectangle touch a conic, the tangents to this conic from any point on the circumscribing circle of the rectangle are perpendicular.

39. The point of intersection of perpendicular tangents to a central conic lies on a fixed circle.

40. Given a conic k and three points U, V, W not on k ; construct a triangle whose vertices lie on k and whose sides pass respectively through the three given points. Discuss and dualize.

Suggestion. If U, V, W are the centers of three involutions on k which pair respectively the points P and P_1, P_1 and P_2, P_2 and P_3 on k , then $k\{P\text{---}\}\bar{\wedge}k\{P_3\text{---}\}$. If M is a double point of this projectivity on k , M_3 coincides with M , and MM_1M_2 is a triangle satisfying the requirements.

CHAPTER XIV

IMAGINARY ELEMENTS

14.1 The Elliptic Involution. Two pairs of homologous elements E, E' and F, F' on a one-dimensional form determine an involution on that form. If the two pairs are interlaced, the involution is elliptic and has no double elements; if the pairs are not interlaced, the involution is hyperbolic and has two double elements. In each case the problem of constructing other pairs of homologous elements of the involution has been completely solved (Art. 13.4).

We have seen that the double elements of an hyperbolic involution form with every pair of homologous elements an harmonic set (Theorem 1, Art. 13.2). An elliptic involution, on the other hand, has no double elements; but each pair of homologous elements of such an involution is, we know, harmonically separated by one other pair of homologous elements. This fact is of such importance in the investigations of this chapter that we repeat here Theorem 2, Art. 13.2:

THEOREM. *In an elliptic involution there is one, and only one, pair of homologous elements which forms with a given pair an harmonic set.*

That is, if A, A' is any given pair of homologous elements in the elliptic involution

$$u(EFE' \text{ ---}) \bar{\wedge} u(E'F'E \text{ ---}),$$

there is one pair of this involution, say B, B' , which forms with A, A' an harmonic set $H(AA', BB')$. The construction of this pair B, B' has been given in Art. 13.2.

DEFINITION. An **harmonic representation** of an elliptic involution consists of two pairs of elements which are at once pairs of the involution and pairs of an harmonic set.

It follows, of course, that each element of the form may be taken as the beginning of an harmonic representation of a given elliptic involution on that form.

Two involutions on a one-dimensional form have a common pair of homologous elements if one of the involutions, at least, is elliptic.

Thus, in the case of two involutions of points on a conic, the common pair of homologous elements is that pair of points common to the conic and the join of the two centers of the involutions. For, since at least one of the involutions is elliptic, one center lies inside the conic; and the line joining it to the other center cuts the conic in the two points of the common pair.

If both involutions are hyperbolic they may or may not have a common pair of homologous elements, depending of course upon whether or not the join of their centers cuts the conic.

14.2 Imaginary Elements. The hyperbolic involution

$$u(EE'F \text{ ---}) \overline{\wedge} u(E'EF' \text{ ---})$$

on a line u defines two points M, N on u ; they are the double points of the involution. We know that M, N separate harmonically each pair of the involution; and, in particular, we have $H(EE', MN)$ and $H(FF', MN)$. The two points thus defined we label

$$M = EE'FF'$$

and

$$\sim N = FF'EE',$$

by a convention of notation which assigns the label $EE'FF'$ to that one of the double points which, with either F or F' , separates the pair E, E' . That is, the first pair E, E' in the symbol $EE'FF'$ is separated by the pair M, F (or by the pair M, F'). Similarly, according to our notation F, F' is separated by N, E (or by N, E').

On the line u , the support of the elliptic involution

$$u(EFE' \text{ ---}) \overline{\wedge} u(E'F'E \text{ ---}),$$

there are two senses or orders which we indicate by the symbols $EFE'F'$ and $EF'E'F$. An elliptic involution when endowed with the additional attribute of sense is called an **oriented elliptic involution**. Thus, the single elliptic involution gives rise to the two oriented elliptic involutions $EFE'F'$ and $EF'E'F$, which differ in that their senses are opposite.

We define, after the manner of von Staudt,¹

1. An imaginary point is defined by, or is represented by, an oriented elliptic involution of points on a line.	S1. An imaginary plane is defined by, or is represented by, an oriented elliptic involution of planes on a line.
---	--

¹ Von Staudt, *Beiträge zur Geometrie der Lage*, Nürnberg, 1856.

That is, on the left, the elliptic involution

$$u(EFE' \text{ ---}) \bar{\wedge} u(E'F'E \text{ ---})$$

defines on the line u two **conjugate imaginary points** $X = EFE'F'$ and $\bar{X} = EF'E'F$.

On the right above we have given the space dual of Definition 1; we have also a plane dual.

P1. An **imaginary line of the first kind** is defined by, or is represented by, an oriented elliptic involution of coplanar lines on a point.

That is, on a flat pencil $U(efe'f' \text{ ---})$ the elliptic involution

$$U(efe' \text{ ---}) \bar{\wedge} U(e'f'e \text{ ---})$$

defines on the real point U and on the real plane of the flat pencil two **conjugate imaginary lines** $x = efe'f'$ and $\bar{x} = e'f'e'f$.

We observe, returning to Definition 1 above, that X and \bar{X} are distinct points and, in accordance with our notation,

$$X = EFE'F' = FE'F'E = E'F'EF = F'EFE',$$

and

$$\bar{X} = EF'E'F = F'E'FE = E'FEF' = FEF'E'.$$

The duals of this statement are at once evident.

14.3 The Incidence Relations. By the definitions of the preceding article we have introduced a new kind of element. These new elements are called "imaginary" in contradistinction to the old, which are called "real."

The meaning of the term "incident" when applied to a real element and an imaginary element is explained by the following definitions.

1. An imaginary point is incident with one and only one real line, the support of the involutory point-row defining the imaginary point, and with all the real planes on this line.

S1. An imaginary plane is incident with one and only one real line, the support of the involutory axial pencil defining the imaginary plane, and with all the real points on this line.

2. An imaginary line of the first kind is incident with one and only one real point and one and only one real plane, viz., the center and the plane of the involutory flat pencil defining the imaginary line.

As regards the meaning of the term "incident" when applied to two imaginary elements, we shall confine our discussion to the elements point and line. The student as an exercise should state the analogous definitions for point and plane and for line and plane.

3. An imaginary point $X = EFE'F'$, defined by

$$u(EFE' \text{ ---}) \bar{\wedge} u(E'F'E \text{ ---}),$$

and an imaginary line $y = ghg'h'$, defined by

$$S(ghg' \text{ ---}) \bar{\wedge} S(g'h'g \text{ ---}),$$

are incident when the involutory point-row $u(EFE'F' \text{ ---})$ is perspective to the involutory flat pencil $S(ghg'h' \text{ ---})$.

This simply means that if

$$u(EFE'F' \text{ ---}) \bar{\wedge} S(efe'f' \text{ ---}),$$

where $e = SE, f = SF$, etc., and if the pairs e, e' and f, f' define on S the same involution as the pairs g, g' and h, h' , then and only then will the imaginary point X be incident with the imaginary line y .

On the basis of these definitions we have the following theorems.

THEOREM 1. *A real point A and an imaginary point X have one and only one line incident with both points. This line is denoted by AX .*

THEOREM S1. *A real plane α and an imaginary plane ξ have one and only one line incident with both planes. This line is denoted by $\alpha\xi$.*

On the left, let $X = EFE'F'$ be defined by an oriented elliptic involution on a line u . Now if A lies on this line u , then line AX is real and is coincident with u . But if A does not lie on u , let $AE = e, AF = f$, etc., then the pairs e, e' and f, f' define on A two conjugate imaginary lines $efe'f'$ and $ef'e'f$. The first of these,

$$efe'f' = A(EFE'F'),$$

is incident also with the point X , by Definition 3 above; it is therefore AX . The second, i.e., the conjugate imaginary line to AX , is $A\bar{X} = ef'e'f$.

We leave with the reader the task of showing that there is only one line incident with both A and X .

The space dual of this discussion follows readily.

THEOREM 2. *Two imaginary points X and Y have one and only one line incident with both points. This line is denoted by XY .*

THEOREM S2. *Two imaginary planes ξ and η have one and only one line incident with both planes. This line is denoted by $\xi\eta$.*

Again restricting our discussion to the proposition on the left, let us dispose of the simpler situation first, namely, the case when the involutions defining X and Y are cobasal. The line XY is this common base and is real. Furthermore, it is the only line incident to both X and Y . If there were a second line it would be imaginary, by Definition 1, and the involutory flat pencil defining it would be perspective to each of the involutory point-rows defining X and Y — an evident impossibility if X and Y are distinct points on the same real line.

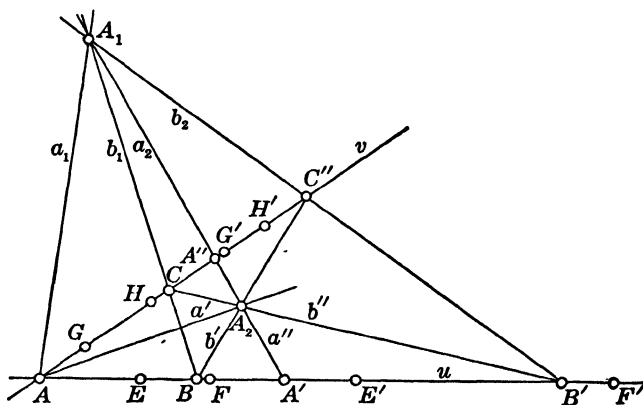


FIG. 91

On the other hand, if $X = EFE'E'$ and $Y = GHG'H'$ are not cobasal but lie respectively on the real lines u and v , we postulate first that u and v are coplanar. Now (Fig. 91) let $A = uv$ be their common point, and construct on u the harmonic representation $H(AA', BB')$ of the involution

$$I_1: u(EFE'E' \text{ ---}) \bar{\cap} u(E'F'E \text{ ---});$$

and, similarly, on v the harmonic representation $H(AA'', CC'')$ of the involution

$$I_2: v(GHG' \text{ ---}) \bar{\cap} v(G'H'G \text{ ---}).$$

We may now write

$$X = EFE'F' = ABA'B',$$

and

$$Y = GHG'H' = ACA''C''.$$

Because of the harmonic relationships involved, we have

$$u(ABA'B' \text{ ---}) \overline{\frac{A_1}{\lambda}} v(ACA''C'' \text{ ---}),$$

whence the involutory flat pencil

$$I_3: A_1(a_1b_1a_2 \text{ ---}) \overline{\lambda} A_1(a_2b_2a_1 \text{ ---}),$$

where $a_1 = \dot{A}_1A$, $b_1 = BC$, $a_2 = A'A''$, $b_2 = B'C''$.

The involution I_3 on the real point A_1 defines two conjugate imaginary lines on A_1 ; the line $XY = a_1b_1a_2b_2$, which is incident to both X and Y , and its conjugate imaginary line $\overline{XY} = a_1b_2a_2b_1$, which is incident to \overline{X} and \overline{Y} .

For later convenience we call attention at this time to a second perspectivity between the two harmonic representations, namely,

$$u(ABA'B' \text{ ---}) \overline{\frac{A_2}{\lambda}} v(AC''A''C \text{ ---}),$$

whence the involutory flat pencil

$$I_4: A_2(a'b'a'' \text{ ---}) \overline{\lambda} A_2(a''b''a' \text{ ---}),$$

where $a' = A_2A$, $b' = BC''$, $a'' = A'A''$, $b'' = B'C$. This involution I_4 defines the conjugate imaginary lines $X\overline{Y} = a'b'a''b''$ and $\overline{XY} = a'b''a''b'$ on the real point A_2 .

The real points A , A_1 , A_2 are thus seen to be the diagonal points of the complete quadrangle $X\overline{X}Y\overline{Y}$.

THEOREM 3. *A real line a and an imaginary line x which are coplanar are both incident with a single point ax .*

THEOREM 4. *Two imaginary lines x and y which are coplanar are both incident with a single point xy .*

THEOREM S3. *A real line a and an imaginary line x which intersect are both incident with a single plane ax .*

THEOREM S4. *Two imaginary lines x and y which intersect are both incident with a single plane xy .*

We omit the proofs of this last set of propositions, for they are quite analogous to those already given for Theorems 1 and 2.

The definitions and propositions concerning incidence relations between the elements, real and imaginary, as set forth in this article make no claim to completeness. But what we have given are sufficient to show how one would go about proving the consistency of the definitions of Art. 14.2. For, to be logically rigorous, having introduced these new elements it is necessary to show that they are in no disagreement with the fundamental assumptions of Art. 1.4, which were made, of course, for real elements. In other words, the definitions of the imaginary elements and their incidence properties must be consistent with what we earlier assumed concerning the elements of projective geometry.

With one yet unfilled gap, the definitions and theorems of this article enable us to extend all but two of the axioms of Art. 1.4 to include imaginary as well as real elements. For example, consider the following, from Art. 1.4:

Two distinct points have one and only one line incident with both points.

This assumption was originally stated for two real points; then their join is a real line. But, by Theorem 1 of this article, one of the points may be imaginary; then their join is either a real line or an imaginary line. Or, by Theorem 2 above, both points may be imaginary; and their join is, again, either real or imaginary.

The two axioms whose extensions cannot, as yet, be justified are (6) and (7); the justifications can easily be provided, however, by the introduction of appropriate definitions after the manner of this article. Consequently, we shall consider, when necessary, that the axioms have been so extended.

14.4 Imaginary Lines of the Second Kind. The lacuna referred to in the above article exists in our proof of Theorem 2, Art. 14.3, in the italicized phrase. That proposition, the reader should note, has been proved for two imaginary points X, Y only *when their supports u and v are coplanar*. When their supports are not coplanar, the proof of the theorem depends upon the introduction of still another imaginary element. To this end, then, we have the

DEFINITION. An **imaginary line of the second kind** is defined by, or is represented by, an oriented elliptic involution of lines on a regulus.

If a, a' and b, b' are two pairs of lines of a regulus, the elliptic involution

$$aba' \text{ --- } \bar{\wedge} \text{ --- } a'b'a \text{ ---}$$

defines two conjugate imaginary lines of the second kind,

$$x = aba'b' \text{ and } \bar{x} = ab'a'b.$$

An imaginary line of the second kind is incident with no real point and no real plane. It may be shown, also, that the join of two imaginary points defined by involutory point-rows on skew supports, or, dually, the intersection of two imaginary planes defined by involutory axial pencils on skew axes, is an imaginary line of the second kind.

14.5 Desargues' Theorem for Complex Triangles. A list of the basic theorems, the fundamentals of our theory, which we have discussed in this text would begin with that theorem of Desargues (Art. 3.3) concerning two triangles — if two triangles are perspective from a point they are perspective from a line, and conversely. We called it Theorem I; we used it to prove Theorem II (Art. 3.5). And Theorem II enabled us to prove the uniqueness of the fourth harmonic; and the harmonic relation was important in developing the theory of projectivity; and so on through the theory of conics, pole and polar, involution, etc.

The portion of projective geometry which has primarily engaged our attention has been the geometry on the plane. It would not be correct, however, to say that our geometry is a plane geometry; that it is inherently a space or three-dimensional geometry and that we have been interested in but a portion of the whole is the proper point of view. In support of this viewpoint we note:

(1) The axioms (Art. 1.4) are assumptions concerning the elements in a 3-space; duality, consequently, is a property of such a 3-space.

(2) The proof of Theorem I, Case 2, is three-dimensional.²

(3) All theorems which, directly or indirectly, depend on Theorem I are based fundamentally on a three-dimensional proof.

This exhibits, briefly, the important rôle played by Desargues' Theorem in the study of **real geometry**, that is, the geometry in which the elements are real elements.

To the real elements of projective space we now adjoin the imaginary elements as defined in this chapter. The real and imaginary elements constitute the **complex elements**, and the

² Theorem I cannot be proved on the basis of a set of axioms *in plano*. Cf. Baker, *Principles of Geometry*, v. I, pp. 8, 30.

geometry of a space in which the elements are complex is called a **complex geometry**.

Thus, a complex element is either real or imaginary; and the student should recognize that the terms "complex" and "imaginary" do not have the same connotation — the former includes the latter. On the other hand, the terms "real" and "imaginary" are mutually exclusive and are both included in "complex."

The part played by Desargues' Theorem in real geometry is analogous to the part which it plays in complex geometry. It is, again, fundamental. The complete proof of Desargues' Theorem for complex triangles, were we to give it, would be seen to involve the various kinds of complex elements which have been mentioned — planes, points, and lines, both real and imaginary.

We shall not prove Desargues' Theorem for complex triangles but, having indicated the direction in which one would turn to make a study of complex geometry, we shall henceforth assume the validity of the theorem for complex triangles, and return now to other considerations.

14.6 Properties of Imaginary Elements. The following properties of imaginary elements, which we shall here discuss with reference to imaginary points, are interesting in themselves and, at the same time, further clarify the rôle of the imaginary entity.

THEOREM 1. *If X and \bar{X} are conjugate imaginary points defined by the elliptic involution*

$$u(EFE' \text{ ---}) \bar{\wedge} u(E'F'E \text{ ---})$$

on a line u , then X and \bar{X} are double points³ of this involution, that is,

$$u(EFE'X\bar{X} \text{ ---}) \bar{\wedge} u(E'F'EX\bar{X} \text{ ---}).$$

We shall prove this property by constructing, after the manner of Art. 13.4, the homologue X' of the imaginary point X . If X' is coincident with X , the property is established.

³The reader will observe that this is the first time we have mentioned the double points of an *elliptic involution*. In real geometry an elliptic involution has no double points, but in complex geometry it has two conjugate imaginary double points. In other words, one difference between complex and real geometry is that in the former all involutions have double points, but in the latter such is not the case.

On line u the involution defining the imaginary point $X = EFE'F'$ is determined by two pairs of homologous points E, E' and F, F' . To construct the homologue X' of the point X we draw, in any plane on u , three nonconcurrent lines through E, F' , and X , respectively (Fig. 92). A triangle KLN is now formed whose sides KL, LN, NK meet u at E, F', X . It is evident, since X is imaginary, that side NK and vertex N of this triangle are imaginary elements; the imaginary point N , moreover, is defined by an oriented elliptic involution on $F'L$, i.e., $N = LF_1L_1F'$, and side LN is real. Now join the given points E' and F to N and K , re-

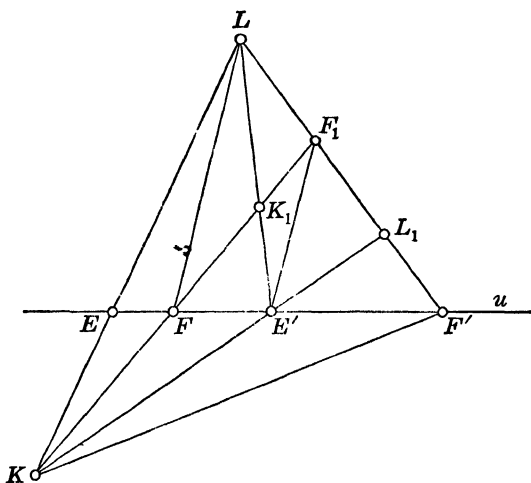


FIG. 92

spectively. The imaginary line $E'N$ and the real line FK meet at the imaginary point $M = K_1F_1KF$, defined on line FK ; and the join of L and M , i.e., the imaginary line $L(K_1F_1KF)$, meets u at the imaginary point $E'F'EF$. But this point $X' = E'F'EF$ is the same as $X = EFE'F'$. Hence, X corresponds to itself in the involution and is, though imaginary, properly designated a double point — and similarly for \bar{X} .

THEOREM 2. *If X and \bar{X} are conjugate imaginary points defined by the involution*

$$u(EFE' \text{ --- }) \bar{\wedge} u(E'F'E \text{ --- })$$

on a line u ; and the real points A, A' are any homologous points of

this involution, the pair X, \bar{X} separates harmonically the pair A, A' , i.e., $H(AA', X\bar{X})$.

The reader is asked to compare this theorem with Theorem 1, Art. 13.2.

Let $H(AA', BB')$ be the harmonic representation beginning with A of the point $X = EFE'F'$; that is,

$$X = EFE'F' = ABA'B',$$

$$\bar{X} = EF'E'F = AB'A'B.$$

And now, on any real line v through A , construct an harmonic set $H(AA'', CC'')$, thus defining on line v the imaginary points $Y = ACA''C''$ and $\bar{Y} = AC''A''C$. The complete quadrangle (see Fig. 91) $A_1A_2Y\bar{Y}$ has its opposite sides

$$A_1Y, A_2\bar{Y} \text{ meeting at } X,$$

$$A_1\bar{Y}, A_2Y \text{ meeting at } \bar{X},$$

and

$$A_1A_2, Y\bar{Y} \text{ meeting at } A''.$$

The two sides of the quadrangle through the diagonal point A'' cut the line $X\bar{X}$ joining the other two diagonal points at A and A' . Hence, by the definition of Art. 4.1, $H(AA', X\bar{X})$.

14.7 Common Elements of a Conic and a Complex Line or Point. In the light of the discussion of this chapter we extend some of the results of Art. 13.6. For simplicity, we shall consider only conics with real traces, i.e., conics which have real points. The term "conic" shall thus be understood throughout the sequel.

An involution, with axis u and center U , of real points on a conic k defines on k two points, the double points of the involution on k . These points are also the double points of the involution on u of conjugate points with respect to k . Furthermore, they are real and distinct or conjugate imaginary, according as the involution is hyperbolic or elliptic.

An involution, with center U and axis u , of real lines on a conic k defines on k two lines, the double lines of the involution on k . These lines are also the double lines of the involution on U of conjugate lines with respect to k . Furthermore, they are real and distinct or conjugate imaginary, according as the involution is hyperbolic or elliptic.

Hence, we have:

A conic k and a real line u , not touching k , have in common two points, which are real and distinct or conjugate imaginary according as the involution on u of conjugate points with respect to k is hyperbolic or elliptic.

That is, when the involution which conic k determines on line u is hyperbolic, its double points are the real points common to k and u (Art. 13.6); and when the involution is elliptic, its double points are, by definition, the conjugate imaginary points common to k and u .

THEOREM 1. *An imaginary line x on a real point P of a conic k has a second point Q in common with k . This second point is imaginary.*

With one exception all lines on P , a real point of the conic k , have a second point, distinct from P , in common with k . The exception is the tangent line to k at P , and this line is real. Hence, the second point common to k and an imaginary line on P is imaginary, by Definition 2, Art. 14.3.

To construct the point Q , i.e., to set up the oriented elliptic involution defining Q , we transfer the elliptic involution which defines $x = efe'f'$,

$$P(efe' \text{ ---}) \overline{\wedge} P(e'f'e \text{ ---}),$$

to an elliptic involution on k (Fig. 93), viz.,

$$k\{EFE' \text{ ---} \} \overline{\wedge} k\{E'F'E \text{ ---} \},$$

with center U and axis u . This latter involution projects through P into an involution of conjugate points on u (Theorem 2, Art. 13.6), namely,

$$u(E_1F_1E_2 \text{ ---}) \overline{\wedge} u(E_2F_2E_1 \text{ ---}),$$

A conic k and a real point U , not on k , have in common two lines, which are real and distinct or conjugate imaginary according as the involution on U of conjugate lines with respect to k is hyperbolic or elliptic.

That is, when the involution which conic k determines on point U is hyperbolic, its double lines are the real lines common to k and U (Art. 13.6); and when the involution is elliptic, its double lines are, by definition, the conjugate imaginary lines common to k and U .

THEOREM P1. *An imaginary point X on a real line p of a conic k has a second line q in common with k . This second line is imaginary.*

which defines on u the imaginary point $Q = E_1F_1E_2F_2$ common to line x and conic k .

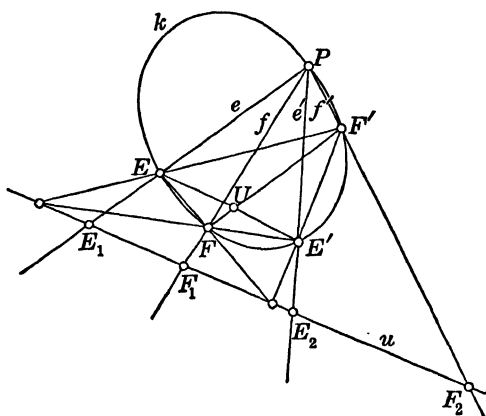


FIG. 93

We leave the reader to develop the proof that Q is unique, and also the plane dual of the above construction.

THEOREM 2. *An imaginary line x on a real point S has in common with a conic k , not on S , two imaginary points P and Q . If x is tangent to k , the points P and Q coincide.*

THEOREM P2. *An imaginary point X on a real line s has in common with a conic k , not touching s , two imaginary lines p and q . If X is on k , the lines p and q coincide.*

The discussion of the theorem on the right will be more useful in our later work. This theorem we shall prove by showing how to construct the two lines p, q which are common to the imaginary point X and the conic k . That is, we shall solve the problem:

To construct the tangents to a given conic k from a given imaginary point X .

The conic k determines on s an involution of conjugate points with respect to k , and this involution has in common with the elliptic involution defining X one pair of homologous points A, A' (Art. 14.1). These two points and S , the pole of s , are the vertices of a self-polar triangle SAA' with respect to k . Thus we have on s (Fig. 94) two involutions; the one,

$$s(ABA' \text{ --- }) \bar{\wedge} s(A'B_0A \text{ --- }),$$

is the involution on s of conjugate points with respect to k , and the other,

$$s(ABA' \text{ --- }) \bar{\wedge} s(A'B'A \text{ --- }),$$

defines the imaginary point X . That is, $X = ABA'B'$.

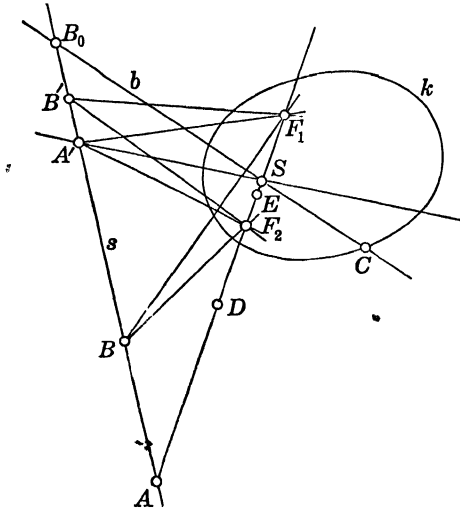


FIG. 94

Now if z is any real line on B and Z is its pole with respect to k , we know that Z lies on b , the polar of B , and also (Theorem 1, Art. 9.5) that

$$B(z \text{ --- }) \bar{\wedge} b(Z \text{ --- }).$$

Hence, since B' is not on b , the flat pencil $B'(Z \text{ --- })$ is perspective to the point-row $b(Z \text{ --- })$, and we have the projectivity

$$P_1: B(z \text{ --- }) \bar{\wedge} B'(Z \text{ --- }).$$

This projectivity between the two flat pencils on B and B' defines a conic k' . It is not necessary to construct this conic k' , but we note that

- (1) BS is the tangent to k' at B ; for when $z = BS$, the pole of BS with respect to k is B_0 , and its homologue in the projectivity P_1 is $B'B_0 = s$, the join of the centers of the flat pencils defining k' .
- (2) $B'S$ is tangent to k' at B' , by a similar argument.
- (3) C , a point common to line b and conic k , is also a point of the conic k' . For BC is tangent to k at C , and its pole is C .

Consequently, BC and $B'C$ are homologues in the projectivity P_1 and meet at a point of k' , i.e., at C .

The projectivity P_1 has provided us with three points of k' and the tangents at two of them; hence P_1 is equivalent to the projectivity

$$P_2: B(SB'C \text{ --- }) \bar{\wedge} B'(BSC \text{ --- }).$$

To continue, we see that one of the two lines SA , SA' cuts k' in two real points; for these lines are separated by the two tangents to k' from S . Let SA be that line and let F_1 and F_2 be its real points of intersection with k' .⁴

The lines p and q common to conic k and point X are the two imaginary lines F_1X and F_2X . It is evident of course that each of these lines passes through X ; but we have yet to show that each is a tangent to k .

To show that p is a tangent to k we note that

$$p = F_1X = F_1(ABA'B')$$

is an imaginary line defined by an oriented elliptic involution on F_1 ; that A' is the pole of F_1A ($= SA$) and, consequently, F_1A and F_1A' are conjugate lines with respect to k ; and, finally, that F_1B and F_1B' are conjugate lines with respect to k , by the definition of the projectivity P_1 . Therefore, by Theorem 2 of Art. 13.1, the involution defining the line p is identical to the involution on F_1 of conjugate lines with respect to k . Hence, the line p and, similarly, the line q are tangents to k , and the construction is justified.

The reader will find it interesting as well as instructive to consider the modifications of this construction when s , for example, cuts or touches k , or when X is an imaginary point on k , or when X is a real point defined by an hyperbolic involution on s , etc.

14.8 The Construction of Conics. The problem of constructing a conic when five of its elements are given was first introduced in Art. 7.3. The following problems will illustrate the method when certain pairs of the given elements consist of conjugate imaginary elements.

⁴The projectivity P_2 is cut by SA and determines on SA the involution

$$SAD \text{ --- } \bar{\wedge} ASE \text{ --- }.$$

The double points of this involution are F_1 and F_2 . (Cf. Arts. 7.6, 12.9 to 12.11.)

$C' = H(C_1, AC)$ and C_2 is the homologue of C_1 in the involution I_1 . Since A_1 and C_1 are points on u we know that U , the pole of u with respect to k , is the intersection of a_1 and c_1 .

For the second part of the construction let BC_1 intersect c_1 at D' ; then the harmonic conjugate D of B with respect to C_1 and D' is a point of the desired conic k . And if CA_1 intersects a_1 at E' , then $E = H(C, A_1E')$ is also a point of k . We now have five real points of k ; and the construction of k is reduced to the construction of a conic through five real points.

2. Given one real point and two pairs of conjugate imaginary points; it is required to construct the conic through these five points when no three of them are collinear.

P2. Given one real line and two pairs of conjugate imaginary lines; it is required to construct the conic touching these five lines when no three of them are concurrent.

Let the five points be the real point P , the conjugate imaginary points $X = E_1F_1E_2F_2$ and $\bar{X} = E_1F_2E_2F_1$ on line u , and the conjugate imaginary points $Y = E'F'E''F''$ and $\bar{Y} = E'F''E''F'$ on line v . If $uv = A$, we construct the harmonic representations beginning with A of the imaginary points; that is,

$$X = AB_1A_2B_2$$

and

$$Y = AB'A''B'',$$

and similarly for their conjugates \bar{X} and \bar{Y} . Then, as in Theorem 2, Art. 14.3, the diagonal triangle ABC of the complete quadrangle $X\bar{X}Y\bar{Y}$ is real and is a self-polar triangle with respect to the desired conic k .

Since BC is the polar of A with respect to k we find P' , the meet of PA and BC , and then construct $P_1 = H(P, AP')$. Similarly, $P_2 = H(P, BP'')$ is constructed, P'' being the intersection of PB and CA . The three real points P, P_1, P_2 lie on the conic k , and these with the conjugate imaginary points X, \bar{X} (or Y, \bar{Y}) enable us to construct the conic k by the method of Problem 1 of this article.

We suggest that the reader draw a figure illustrating this problem.

3. Given a real point, two conjugate imaginary points, and two conjugate imaginary lines incident respectively with the given imaginary points; it is required to construct the conic through these three points and touching these two lines when the real point is not on either of the given lines nor collinear with the given imaginary points.

P3. Given a real line, two conjugate imaginary lines, and two conjugate imaginary points incident respectively with the given imaginary lines; it is required to construct the conic touching these three lines and through these two points when the real line is not on either of the given points nor concurrent with the given imaginary lines.

Let the real point be A , the imaginary points $X = E_1F_1E_2F_2$ and $\bar{X} = E_1F_2E_2F_1$ on the real line u , and the complex lines $x = U(E_1F_1E_2F_2)$ and $\bar{x} = U(E_1F_2E_2F_1)$ on the real point U . Since x and \bar{x} are tangents to the desired conic k at X and \bar{X} , their intersection U is the pole of u with respect to k . Also, if AU meets u at G_1 , then $B = H(A, UG_1)$ is a point on k . And again, if H_1 is any point on u and H_2 is its homologue in the involution defining X, \bar{X} , we can construct the polar of H_1 with respect to k as the join of U and H_2 . Hence, if AH_1 meets this polar at H' then $C = H(A, H_1H')$ is a third real point on k . The construction of k now follows by the method of Problem 1.

The space duals and the metric specializations of the problems discussed and indicated herein are suggested as worthy of investigation. We shall consider one most important metric specialization in the next article.

14.9 A Metric Specialization. In Arts. 10.2 we defined the orthogonal involution, a special involution on the ideal line. We have since studied in more detail the properties of involutions in general and so can now interpret that article in a more comprehensive fashion. The results of such an interpretation may be summarized in one sentence:

All circles in a plane have in common with the ideal line of the plane a single pair of conjugate imaginary points, the double points of the orthogonal involution.

DEFINITION. The **circular points** of a plane are the conjugate imaginary points defined on the ideal line by the orthogonal involution. We shall label them Ω and $\bar{\Omega}$.

Thus, every circle passes through the two circular points.

Conversely, as a little reflection will show, every conic which passes through Ω and $\bar{\Omega}$ is a circle. For the involution of conjugate diameters on the center of such a conic is a circular involution, i.e., an involution in which homologous lines are perpendicular. Hence, since any two conjugate diameters of the conic are perpendicular, the conic is a circle.

DEFINITION. In a given plane an **isotropic line** on an ordinary point P is the imaginary line joining P to one of the circular points of the plane.

Evidently the two isotropic lines on an ordinary point P are $P\Omega$ and $P\bar{\Omega}$, the double lines of the circular involution on P .

The reader is now asked to construct the conic through the three real points A, B, C and the circular points $\Omega, \bar{\Omega}$; or, in other

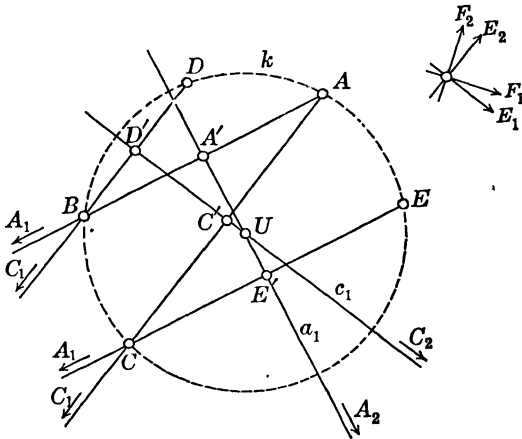


FIG. 96

words, the circle through A, B, C . The adjoined Fig. 96 illustrates this construction and should be compared with Fig. 95. We note:

- (1) the lines a_1 and c_1 of the latter become the perpendicular bisectors of segments AB and AC in Fig. 96;
- (2) $U = a_1c_1$ is the pole of the ideal line, i.e., U is the center of the conic;
- (3) U is equidistant from points A, B, C, D, E on the conic;
- (4) the conic has five points in common with a circle;
- (5) hence, the conic through $A, B, C, \Omega, \bar{\Omega}$ is a circle; and this metric specialization of the solution of Problem 1, Art. 14.8,

leads to the familiar construction of elementary geometry for a circle through three points.

It is evident, since all circles have the points $\Omega, \bar{\Omega}$ in common, that any two distinct circles can intersect in not more than two real points.

In concluding we shall state and leave with the reader for investigation one other metric specialization, a variation of Theorem P2, Art. 14.7.

THEOREM. *The isotropic lines through the center of a circle are the imaginary tangents to the circle at the circular points.*

EXERCISES

1. State definitions for the incidence of point and plane and of line and plane analogous to Definition 3 of Art. 14.3.
2. Given two imaginary lines of the first kind which are coplanar; construct their point of intersection. Observe that the common plane may be real or imaginary.
3. Given two imaginary lines of the first kind which intersect; construct the plane determined by them. Observe that the common point may be real or imaginary.
4. Show how to construct the join of two imaginary points defined by involutory point-rows on skew supports.
5. On a real line u there are given two involutions I_1 and I_2 , of which the second, viz., I_2 , is elliptic. If X is a double point of I_2 , construct its homologue X' in the involution I_1 .
6. Given two involutions on a real line u ; construct the common pair of homologous elements. Show that the elements of this common pair are real except when the given involutions are both hyperbolic with interlaced double elements, in which case the elements of the common pair are conjugate imaginary.
7. An imaginary point X on a real line p of a conic k has in common with k a second line q , which is imaginary. Construct the line q .
8. If instead of the imaginary point X of Ex. 7 we are given on p a real point M defined by an hyperbolic involution on p , construct the second line common to k and M . Compare this construction with that of Ex. 7.
9. As a metric specialization of Ex. 7, construct the imaginary tangents to a parabola from the circular points.
10. Construct the imaginary points common to a given imaginary line x on a real point S and a given conic k which does not pass through S .
11. As a metric specialization of the dual of Ex. 10 construct the tangents to a given central conic from the circular points.
12. Construct the conic touching five given lines, two of which are conjugate imaginary lines and no three of which are concurrent.
13. Construct the conic through two given real points, tangent to a given real line on one of these points, and passing through two given conjugate imaginary points.

14. Show how to construct the conic which touches the isotropic lines on each of two given real points and (a) which passes through a given real point; (b) which touches a given real line. In each case how many conics are there?

15. There are, at most, four conics touching the sides of a given triangle and passing through each of two given points. Prove this theorem (a) when the given points are real; and (b) when they are conjugate imaginary.

16. As a metric specialization of Ex. 15 consider the following: There are four circles touching three nonconcurrent ordinary lines.

CHAPTER XV

FOCI AND FOCAL PROPERTIES OF CONICS

15.1 Foci. Following the plan of Chapters V and X we devote this chapter to a study of certain metric properties of conics. The reader will observe that we are, throughout this investigation, considering the ideal line of the plane as of special significance, thus setting it apart from all other lines of the plane.

Consequently, we must distinguish between those points of the plane which are ordinary points and those points which are ideal points; and, for similar reasons, we must distinguish between conics in their relations to the ideal line. An hyperbola cuts the ideal line in two real ideal points, a parabola touches the ideal line, and an ellipse has in common with the ideal line two conjugate imaginary ideal points. If the two real points common to the ideal line and an hyperbola are homologues in the orthogonal involution, the hyperbola is a rectangular hyperbola; if the two conjugate imaginary points common to the ideal line and an ellipse are the circular points (Art. 14.9), the ellipse is a circle.

With these introductory remarks we now state the

DEFINITION. A **focus** of a conic is an ordinary point of intersection of tangents to the conic from the circular points.

From each of the two circular points there are two tangents to a given central conic. This, the student will observe, is a metric specialization of Theorem P2, Art. 14.7. Moreover, if t and \bar{u} are the tangents to a conic k from Ω , then the tangents to k from $\bar{\Omega}$ are the conjugate imaginary lines to t and u , viz., the lines \bar{t} and \bar{u} . These four tangents form a complete quadrilateral $t\bar{t}u\bar{u}$ whose sides touch k ; two of its six vertices are the ideal points $\Omega = tu$ and $\bar{\Omega} = \bar{t}\bar{u}$, and the other four vertices are ordinary points. These four ordinary points are the foci of the conic, by the above definition, and we shall label them as follows:

$$F_1 = t\bar{t}, F_2 = u\bar{u}, G = \bar{t}u, \bar{G} = t\bar{u}.$$

Thus, a central conic has two real foci and two conjugate imaginary foci. In the sequel we shall be primarily concerned with

the real foci F_1, F_2 ; and we shall understand the term "focus" when unqualified to refer to a real focus.

The above argument needs revision for a central conic passing through the circular points, i.e., for a circle. For then, since $\Omega, \bar{\Omega}$ are points on the circle, there is only one tangent to the circle at Ω and only one tangent at $\bar{\Omega}$. These tangents are conjugate imaginary lines and they intersect at a real point, the center of the circle (cf. Theorem, Art. 14.9). Hence the center of a circle is also its focus.¹ Because of this peculiarity of a circle many focal properties of central conics are not applicable to circles (cf. Art. 15.4 and Theorem 5, Art. 15.5).

In the case of a parabola the ideal line $\Omega\bar{\Omega}$ touches the curve. There is, then, but one imaginary tangent from Ω to the parabola and but one from $\bar{\Omega}$, by Theorem P1, Art. 14.7. If t is the tangent from Ω , the tangent from $\bar{\Omega}$ will be \bar{t} . Consequently, a parabola has but one focus $F' = t\bar{t}$, and it is real.

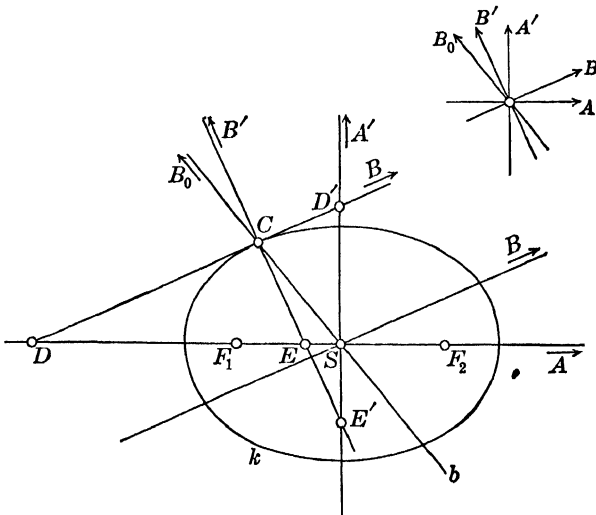


FIG. 97

15.2 Determination of the Foci. 1. Central Conics. We have illustrated in Fig. 97 a special case of Theorem P2, Art. 14.7. The imaginary point X and the real line s of that theorem are here the circular point Ω and the ideal line, respectively; and the

¹ In general, when a curve passes through $\Omega, \bar{\Omega}$ an ordinary point of intersection of its tangents thereat is called a **singular focus**.

conic k is here a central conic. Hence, on the ideal line s we have two involutions; viz., the orthogonal involution

$$s(ABA' \text{ ---}) \overline{\wedge} s(A'B'A \text{ ---}),$$

which defines Ω and $\overline{\Omega}$, and the involution

$$s(ABA' \text{ ---}) \overline{\wedge} s(A'B_0A \text{ ---}),$$

which k determines on s . These involutions have a common pair of homologous points A, A' — they are the ideal points on the axes of k . Moreover, the pole of s is S , the center of the conic k .

Now, if one tangent to k from B touches k at C , let BC meet SA at D and SA' at D' ; and let $B'C$, the normal to k at C , meet SA at E and SA' at E' . The foci F_1, F_2 are the double points of the hyperbolic involution

$$SAD \text{ ---} \overline{\wedge} ASE \text{ ---}$$

on the axis SA , for F_1 and F_2 are respectively the intersections of the tangents $F_1\Omega, F_1\overline{\Omega}$ and $F_2\Omega, F_2\overline{\Omega}$.

But these four tangents from Ω and $\overline{\Omega}$ have two other points of intersection; namely, the point G common to $F_1\Omega$ and $F_2\overline{\Omega}$, and the point \overline{G} common to $F_1\overline{\Omega}$ and $F_2\Omega$. These two points G, \overline{G} are the imaginary foci of the conic k . The reader can show without difficulty that G, \overline{G} are conjugate imaginary points defined on the axis SA' by the elliptic involution

$$SA'D' \text{ ---} \overline{\wedge} A'SE' \text{ ---},$$

which is projected from each real focus, F_1 or F_2 , by a circular involution (cf. Theorem, Art. 13.8).

We note, in passing, that the center (Art. 13.7) of the hyperbolic involution on SA which has F_1, F_2 as double points is S . Therefore, S is the mid-point of the segment F_1F_2 . The length of the segment F_1F_2 , which we call the **focal distance**, we shall represent by $2c$; then, $F_1S = SF_2 = c$.

2. *Parabolas.* Similarly to the above, we have illustrated in Fig. 98 a special case of Theorem P1, Art. 14.7. If $\Omega = ABA'B'$ is the point from which a tangent is to be drawn to a given parabola k , we transfer the orthogonal involution, which defines Ω , to an involution of lines on k , i.e.,

$$k\{aba' \text{ ---}\} \overline{\wedge} k\{a'b'a \text{ ---}\}.$$

This involution determines on its center F an involution of conjugate lines with respect to k ; viz., the involution

$$F(ABA' \text{---}) \bar{\cap} F(A'B'A \text{---}).$$

Hence, $F(ABA'B')$ and $F(AB'A'B)$ are tangents to the parabola from Ω and $\bar{\Omega}$, respectively, and the point F is the focus of the parabola.

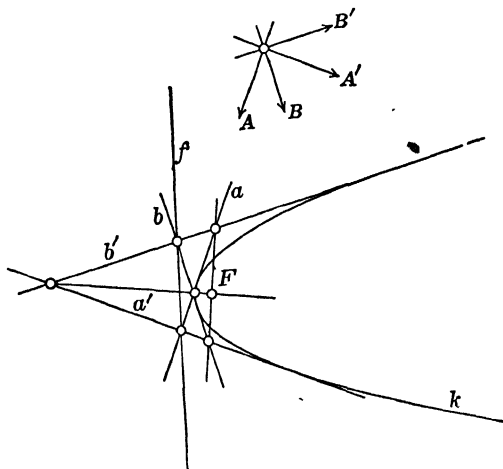


FIG. 98

15.3 Construction of the Foci. It is evident from the discussion in the earlier articles of this chapter that the tangents to a conic from a focus are the isotropic lines through the focus. These isotropic tangent lines are the double lines of the circular involution on the focus. Hence, for the real foci of a conic, an

ALTERNATIVE DEFINITION. A **focus** of a conic is a point on which the conic determines a circular involution of conjugate lines.

Obviously, this definition is the equivalent of that of Art. 15.1.

Since the involution of conjugate lines on a focus is elliptic, a focus lies inside the conic. Moreover, a diameter of the conic which passes through a focus is perpendicular to its conjugate chord through the focus. Consequently, such a diameter is an axis (Art. 10.6) of the conic or, to state it otherwise, a focus lies on an axis.

1. *The Ellipse.* Consider the ellipse with transverse axis V_1V_2 and conjugate axis CD , and let v_1, v_2, c be the tangents to the ellipse at V_1, V_2, C (Fig. 99). If now $v_1c = P$ and $v_2c = Q$, any

point on V_1V_2 , which is the polar of point v_1v_2 with respect to the ellipse, is joined to P and Q by conjugate lines with respect to the ellipse (Theorem P2, Art. 9.5). In particular, the points F_1 and F_2 common to V_1V_2 and the circle on PQ as diameter are each joined to P and Q by perpendicular conjugate lines; and through each of these same points there is a second pair of perpendicular conjugate lines, namely, the axis V_1V_2 and the chord conjugate to V_1V_2 .

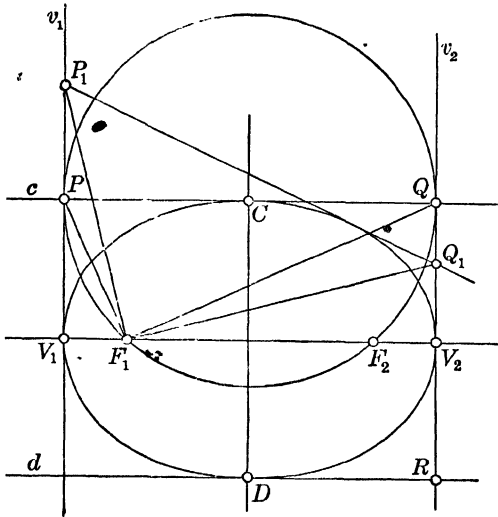


FIG. 99

Hence, since the involution of conjugate lines on F_1 contains two pairs of perpendicular homologues, the involution of conjugate lines on F_1 is circular; and F_1 is a focus, as is F_2 also.

Thus may we construct the foci of an ellipse. For the real foci lie on the transverse axis and on the circle with center C and diameter equal to V_1V_2 . Consequently, representing the lengths of the transverse and conjugate axes by $2a$ and $2b$, respectively, we have the focal distance (Art. 15.2) equal to $2\sqrt{a^2 - b^2}$.

If d is the tangent to the ellipse at D and $v_2d = R$, it is evident that the circle on QR as diameter does not intersect the conjugate axis CD .

More generally, we have

THEOREM 1. *If v_1, v_2 are the tangents at the extremities of the major axis of an ellipse, the segment intercepted on any other tangent to the ellipse by v_1 and v_2 subtends a right angle at a focus.*

That is, in Fig. 99, the angles $P_1F_1Q_1$ and $P_1F_2Q_1$ are right angles. The proof of this theorem follows readily in the manner of the preceding argument, and we omit it.

2. *The Hyperbola.* We have shown that a focus of a conic lies on an axis and inside the curve. In the case of an hyperbola only the transverse axis cuts the curve; the foci, then, lie on that axis.

The construction of the foci of an hyperbola follows from a theorem analogous to Theorem 1 above. It is

THEOREM 2. *If v_1, v_2 are the tangents to an hyperbola at its vertices, the segment intercepted on any other tangent to the hyperbola by v_1 and v_2 subtends a right angle at a focus.*

The proof of this theorem, too, we omit, for the student can easily prove it himself with the aid of Theorem P2, Art. 9.5.

As an application, however, we note that if the intersections of v_1, v_2 with an asymptote are P, Q , the circle on PQ as diameter cuts the transverse axis at the foci F_1, F_2 .

Furthermore, if the length of the transverse axis is $2a$ and if the length of the segment determined on tangent v_2 by the asymptotes is $2b$ (Art. 10.10), we have, for the hyperbola, the focal distance equal to $2\sqrt{a^2 + b^2}$.

3. *The Parabola.* If V is the vertex of a parabola and F the focus, the line VF is, we know, the axis. Now (Fig. 100) let v be the tangent at V, a any other tangent with point of contact A , and l the ideal line of the plane.

The triangle alv has its sides touching the parabola and hence, by Theorem P2 of Art. 9.5, any point on the axis is a conjugate point to $R = lv$ and is joined to the other two vertices, $P = av$ and $Q = al$, of the triangle by conjugate lines with respect to the parabola. These conjugate lines are not, in general, perpendicular. However, if the line perpendicular to the tangent a

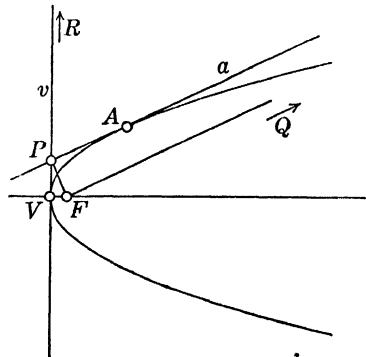


FIG. 100

at P cuts the axis at F , then the lines FP and FQ are perpendicular and conjugate with respect to the parabola. This is not the only pair of perpendicular conjugate lines through F , for we know that the axis and its conjugate chord is a second

such pair. Hence, the involution of conjugate lines on F is a circular involution and F is a focus.

Since there is but one perpendicular to line a at point P , there is only one point such as F ; and a parabola has but one focus.

The methods of this article are very useful in a practical way for constructing foci; but they do not, of course, exhibit the inherent characteristic of a focus as do the constructions of Art. 15.2.

15.4 Directrix, Focal Radius, and Latus Rectum. DEFINITION.

In any conic the polar line of a focus with respect to the conic is called a **directrix** of the conic.

Thus, corresponding to each focus F_i of a central conic there is a directrix f_i , and the directrices corresponding to the real foci are real lines perpendicular to the transverse axis. Neither of these directrices cuts the conic in real points. Furthermore, if V_1, V_2 are the vertices of the conic on the axis F_1F_2 and if the directrices f_1, f_2 meet this axis at D_1, D_2 respectively, then the pair V_1, V_2 separates harmonically each of the pairs D_1, F_1 and D_2, F_2 . Hence, if O is the center of the conic, we have

$$OF_1 \cdot OD_1 = OV_1^2 = OV_2^2 = OF_2 \cdot OD_2,$$

by Theorem 5, Art. 5.9; and from this, since

$$OF_2 = -OF_1 = c,$$

it follows that

$$OD_2 = -OD_1 = \frac{a^2}{c}.$$

A parabola, having only one focus, has only one directrix. This directrix is perpendicular to the axis of the parabola and intersects the axis at a point D , which is the harmonic conjugate of the focus with respect to the two curve points on the axis. The two curve points on the axis of a parabola are the vertex and the ideal point of the curve. Evidently, the vertex of a parabola bisects the segment of the axis between the focus and directrix, i.e., $DV = VF$.

DEFINITION. In the plane of a given conic a line passing through a focus is called a **focal radius** of the conic.

Since a focus lies inside the conic a real focal radius cuts the conic in two real points; these points, when they are ordinary points, are the extremities of a **focal chord** of the conic. And on each focus there is one focal chord perpendicular to the axis through

that focus or, what is the same, parallel to the directrix corresponding to that focus. Hence, the

DEFINITION. The chord passing through a focus of a conic and parallel to the corresponding directrix is called a **latus rectum** of the conic.

Since the conic is symmetric with respect to an axis it follows that the mid-point of a latus rectum coincides with a focus.

15.5 Focal Properties. Since a property of conics which involves foci is fundamentally a metric property, we shall find it necessary to indicate, in the statement of such a property, whether it is one of conics in general or whether it is a property of a particular kind of conic. In this article we shall give several important focal properties of conics in general.

THEOREM 1. *The line joining a focus of a conic to the point of intersection of two tangents to the conic makes equal angles with the lines joining the focus to the points of contact of the two tangents.*

To prove this theorem, let F_1 be the focus and T the intersection of the tangents p and q to the conic at P and Q , respectively (Fig. 101). Now the directrix f_1 , which is the polar of F_1 , and the line PQ , which is the polar of T , intersect at a point K whose polar is F_1T . Moreover, F_1K is a conjugate line to F_1T with respect to the conic and, since F_1 is a focus, these conjugate lines are perpendicular. The point L , common to PQ and F_1T , is the harmonic conjugate of K with respect to P and Q . Hence, the perpendicular lines F_1K , F_1L separate harmonically the lines F_1P , F_1Q and, by the theorem of Art. 10.3, are the bisectors of the angles formed by F_1P and F_1Q . But this, otherwise stated, says that F_1L (i.e., F_1T) makes equal angles with F_1P and F_1Q ; thus, the demon-

stration of the theorem.

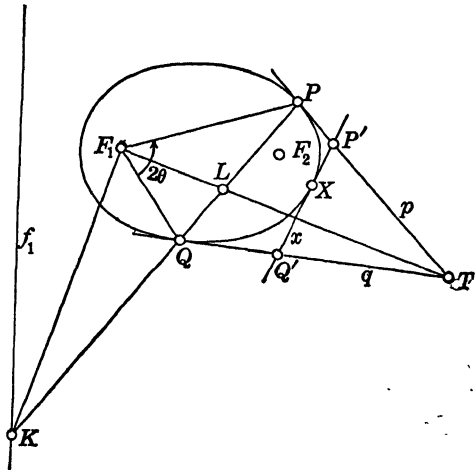


FIG. 101

As an immediate consequence of Theorem 1 we have the following, which is a generalization of the theorems of Art. 15.3.

THEOREM 2. *Two fixed tangents to a conic intersect a variable tangent to the conic at two points whose joins with a focus include a constant angle.*

We shall indicate by 2θ the measure of the smaller positive angle between the lines F_1P and F_1Q (Fig. 101). Now if a variable tangent x , with point of contact X ; intersects the fixed tangents p, q at P', Q' , respectively, we have, by Theorem 1,

$$Q'F_1X + XF_1P' = \frac{1}{2}(QF_1X + XF_1P) = \theta.$$

Hence,

$$Q'F_1P' = \theta;$$

i.e., one of the angles between the variable lines F_1P' and F_1Q' is θ , which is half the constant angle included between the fixed lines F_1P and F_1Q .

The reader should satisfy himself that, for all positions of the variable tangent x , one of the angles between the lines F_1P' and F_1Q' is equal to θ .

We are now prepared to consider

THEOREM 3. *A variable tangent to a conic determines on two fixed tangents to the conic two projectively related point-rows, which are projected from a focus of the conic by two projectively related superposed flat pencils between homologous lines of which a constant angle is included.*

That the point-rows determined on the fixed tangents are projective is known, by Property P3 of Art. 7.2; the remainder of the theorem follows immediately from Theorem 2.

THEOREM 4. *The segment determined on any tangent to a conic by its point of contact and its intersection with a directrix subtends a right angle at the focus corresponding to that directrix.*

For, if line p is tangent to a conic at P and meets a directrix f at T , the polar line of T is FP , where F is that focus of the conic corresponding to f ; hence, FP and FT are conjugate lines with respect to the conic, and, since they pass through a focus, they are perpendicular.

THEOREM 5. *The ratio of the undirected distances of any ordinary point on a conic from a focus and from the directrix corresponding to that focus is a constant.*

As in Fig. 102, let F_1 be a focus, f_1 the corresponding directrix, and P, Q two points on the conic, at which points the tangents are p, q . If $pq = T$ and PQ meets f_1 at K , we know, by Theorem 1, that F_1T and F_1K are bisectors of the angle at F_1 of triangle PF_1Q . Therefore,²

$$\frac{PK}{QK} = \frac{F_1P}{F_1Q}.$$

Moreover, if P_1 and Q_1 are respectively the feet of the perpendiculars from P and Q on the directrix f_1 , we have

$$\frac{PK}{QK} = \frac{PP_1}{QQ_1};$$

whence

$$\frac{F_1P}{PP_1} = \frac{F_1Q}{QQ_1}.$$

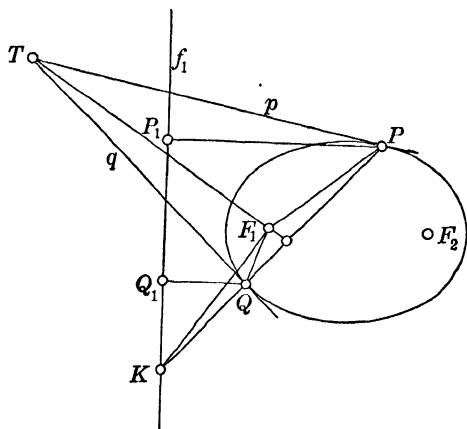


FIG. 102

Consequently, if Q is fixed and P is a variable point on the conic, the ratio F_1P/PP_1 is constant. This constant ratio is called the **eccentricity** of the conic, and we shall hereafter denote it by e .

The eccentricity e of a central conic is the same for either focus, that is,

$$\frac{F_1P}{PP_1} = \frac{F_2P}{PP_2},$$

where P_2 is the foot of the perpendicular from P onto the directrix f_2 . To show this we recall that

$$OF_2 = -OF_1 = c,$$

$$OD_2 = -OD_1 = \frac{a^2}{c},$$

$$OV_2 = -OV_1 = a.$$

Therefore, since V_1 is a point on the curve,

$$\frac{F_1P}{PP_1} = \frac{F_1V_1}{V_1D_1} = \frac{OV_1 - OF_1}{OD_1 - OV_1} = \frac{c}{a},$$

² Cf. Euclid VI, Prop. A, p. 179, Everyman's edition.

and, in like manner,

$$\frac{F_2P}{PP_2} = \frac{F_2V_2}{V_2D_2} = \frac{c}{a}.$$

15.6 Metric Properties of a Conic. The reader will have noticed that Theorem 5 of the preceding article states a property of conics which is often made the basis of the definition of that curve in elementary analytic geometry. In fact, the definition is the converse of that theorem. We shall derive in this article two other properties with which the student is no doubt familiar.

The eccentricity of a conic is a number associated with the conic. The eccentricity of a central conic, moreover, is the same for both foci; it is equal to c/a .

For an ellipse $c = \sqrt{a^2 - b^2}$ (Art. 15.3) and is less than a ; consequently, $e = c/a$ is less than 1. For an hyperbola, however, $c = \sqrt{a^2 + b^2}$ and is greater than a ; e , then, is greater than 1.

A parabola has but one focus, and the value of its eccentricity is FP/PP_1 , P being any point on the curve. In particular, the vertex is such a point; hence,

$$e = \frac{FP}{PP_1} = \frac{FV}{VD} = 1,$$

by Art. 15.4.

The results deduced here we state as

THEOREM 1. *The eccentricity of an ellipse, an hyperbola, or a parabola is a positive number which is respectively less than, greater than, or equal to 1.*

If r_1, r_2 are the distances of an ordinary point P on a central

conic from the foci F_1, F_2 , respectively, and d_1, d_2 its distances from the directrices, we have (Fig. 103)

$$e = \frac{r_1}{d_1} = \frac{r_2}{d_2} = \frac{r_1 \pm r_2}{d_1 \pm d_2}.$$

If the central conic is an ellipse, we take the positive sign in the above equality, and then

$$r_1 + r_2 = e(d_1 + d_2) = 2a,$$

since $e = c/a$ and $d_1 + d_2 = 2a^2/c$ (Art. 15.4). The negative sign

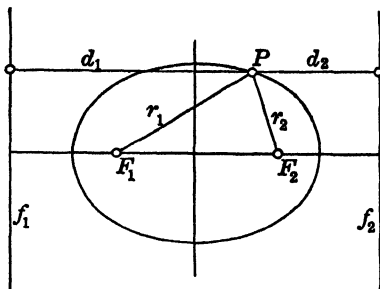


FIG. 103

is taken when the conic is an hyperbola; in that case,

$$r_1 - r_2 = e(d_1 - d_2) = 2a,$$

since $e = c/a$ and $d_1 - d_2 = 2a^2/c$. This gives

THEOREM 2 *In an ellipse the sum, and in an hyperbola the difference, of the distances of any ordinary point on the curve from the foci is constant and equal to the length of the transverse axis.*

15.7 Focal Properties, Continued. We continue, in this article, the study of focal properties of conics. They will be of two kinds, namely, properties of central conics and properties of parabolas.

THEOREM 1. *The tangent and normal at an ordinary point of a central conic are the bisectors of the angles formed by the focal radii through that point.*

If P is any ordinary point on the conic (Fig. 104) and p the tangent thereat, the point T , common to this tangent p and the tangent to the conic at a vertex V_1 on the transverse axis, is joined to F_1 by a line making equal angles with F_1V_1 and F_1P , by Theorem 1, Art. 15.5. Similarly, F_2T makes equal angles with F_2V_1 and F_2P . The point T , consequently, is equidistant from the sides of the triangle F_1PF_2 ; and PT , the tangent at P , is a bisector of one of its angles at P .

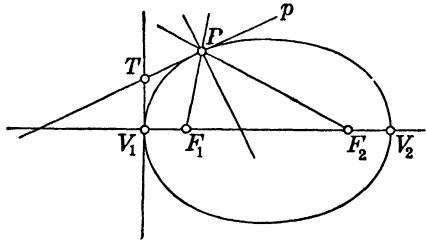


FIG. 104

It follows, therefore, that the normal to the conic at P is the bisector of the other angle at P of the triangle, and the proof is complete.

If the conic is an ellipse, the point T is the center of one of the escribed circles of the triangle F_1PF_2 ; when the conic is an hyperbola, T is the center of the inscribed circle of the triangle F_1PF_2 . The reader should satisfy himself of the truth of these last statements.

An alternative proof of Theorem 1 follows immediately from Art. 15.2 where it was shown that the tangent and normal at a point C (Fig. 97) of the central conic intersect the transverse axis at points D and E , respectively, which are homologues in the hyperbolic involution

$$SAD \text{ --- } \bar{\wedge} ASE \text{ ---},$$

whose double points are the foci F_1, F_2 . Hence, by Theorem 1, Art. 13.2, we

have the harmonic range $H(DE, F_1F_2)$, which is projected from C by an harmonic pencil of lines one pair of which, the tangent and normal, consists of perpendicular lines; and this pair, by Art. 10.3, bisects the angles formed by the other pair.

Analogous to the above theorem, we have for the parabola

THEOREM 1'. *The tangent and normal at an ordinary point of a parabola are the bisectors of the angles formed by the focal radius and the diameter through that point.*

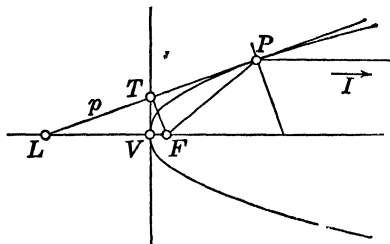


FIG. 105

The tangent p to the parabola at a point P intersects the tangent at the vertex at T and the axis at L (Fig. 105). Thus, by Theorem 1, Art. 15.5, the focus F is joined to T by a line FT which bisects the angle LFP ; this line FT , moreover, is per-

pendicular to the tangent PT (Art. 15.3). The triangle LFP is, in this way, shown to be isosceles; and the tangent p , therefore, makes equal angles with the axis LF and the focal radius FP . Or, since the diameter PI is parallel to LF , the tangent at P bisects one of the angles formed by the lines FP and PI , and the normal at P bisects the other angle. The theorem is demonstrated.

THEOREM 2. *The feet of the perpendiculars from the foci of a central conic upon any tangent lie on a circle which is concentric with the conic and has the length of transverse axis of the conic as diameter.*

If the foci of an ellipse are F_1, F_2 (Fig. 106) and if the perpendicular from F_1 onto the tangent at any point P of the ellipse intersects that tangent at M , the focal radii F_1M and F_2P will intersect at a point R , and the segments F_1M and MR will be equal. For

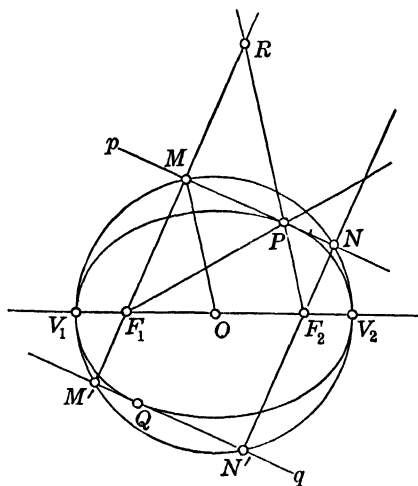


FIG. 106

PM is perpendicular to F_1R and, by Theorem 1, bisects the angle F_1PR . Also, $F_1P = PR$; that is F_2R is in length equal to $2a$, the length of the transverse axis of the ellipse, by Theorem 2, Art. 15.6. Now we know that O , the center of the ellipse, and M are the mid-points, respectively, of two sides of the triangle F_1F_2R ; hence $OM = \frac{1}{2}F_2R = a$ for all positions of P on the ellipse. Therefore, the locus of M is a circle of radius a and concentric with the ellipse.

We leave the reader to adjust this argument for the case when the conic is an hyperbola.

The locus of the foot M of a perpendicular from a focus onto a variable tangent is called the **major auxiliary circle** of the conic.

The corresponding theorem for the parabola has been discussed in Art. 15.3; it is

THEOREM 2'. *The foot of the perpendicular from the focus of a parabola upon any ordinary tangent lies on the tangent to the parabola at its vertex.*

THEOREM 3. *The product of the distances from the foci of a central conic to any tangent is constant.*

If the perpendiculars from the foci of an ellipse onto a tangent are F_1M and F_2N (Fig. 106), and if they intersect the major auxiliary circle again at M' and N' , respectively, evidently $M'F_1 = F_2N$. Hence,

$$F_1M \cdot F_2N = F_1M \cdot M'F_1 = F_1V_1 \cdot V_2F_1 = b^2,$$

where b is half the length of the conjugate axis.

For the hyperbola this constant product, as the student can easily show, is $-b^2$, the negative sign indicating that the foci are on opposite sides of the tangent line.

THEOREM 4. *The point of intersection of perpendicular tangents to a central conic lies on a circle concentric with the conic.*

If PT and QT are perpendicular tangents, as in Fig 107,

and if the feet of the perpendiculars from the foci on these tangents are M, N and M', N' , respectively, we know that these four points

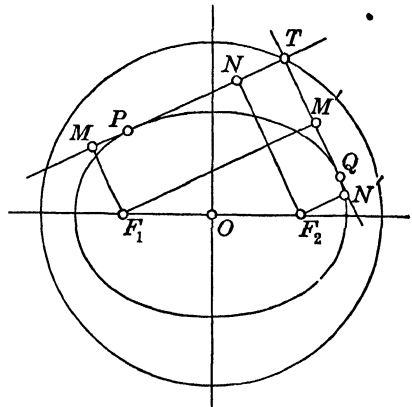


FIG. 107

lie on the major auxiliary circle of the ellipse. Consequently, the power (Art. 5.10) of T with respect to this circle is

$$TM \cdot TN = TM' \cdot TN' = OT^2 - a^2.$$

But, by Theorem 3,

$$F_1M \cdot F_2N = F_1M' \cdot F_2N' = b^2,$$

and since $F_1M = M'T$, etc.,

$$M'T \cdot N'T = MT \cdot NT = b^2,$$

whence

$$OT^2 - a^2 = b^2.$$

That is,

$$OT = \sqrt{a^2 + b^2},$$

and T is a constant distance from the center O .

This proves the theorem for the ellipse. For the hyperbola we find

$$OT = \sqrt{a^2 - b^2},$$

from which we see that an hyperbola will have pairs of perpendicular tangents only when $a > b$.

The locus of T is a circle, called the **director circle** of the conic. It has a real trace when the conic is an ellipse, or when the conic is an hyperbola with $a > b$.

THEOREM 4'. *The point of intersection of perpendicular tangents to a parabola lies on the directrix.*

The proof of this theorem, which follows readily from Theorem 2', we leave as an exercise (cf. Art. 15.2).

15.8 A Triangle Theorem. There is an interesting focal property of the parabola which leads to a well-known theorem in the geometry of the triangle. To arrive at this theorem we first prove

THEOREM 1. *The line joining the focus and the point of intersection of two ordinary tangents to a parabola makes with one of these tangents an angle equal to one of the angles which the other tangent makes with the axis of the parabola.*

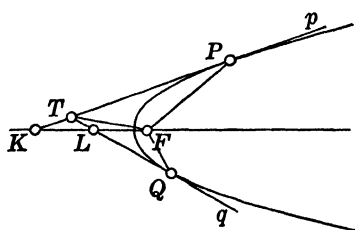


FIG. 108

In proving this theorem we shall measure all the angles in the same direction. Then we have, in Fig. 108,

$$FTP + TPF = FQT + QTF,$$

since each member of this equality is the supplement, respectively, of the equal angles PFT and TFQ (Theorem 1, Art. 15.5). But, by Theorem 1', Art. 15.7, we have

$$TPF = FKP, \quad FQT = QLF;$$

whence

$$QTF - FTP = FKP - QLF.$$

Moreover, from the triangle KLT ,

$$QTF + FTP = FKP + QLF;$$

and from these last two equations, by addition and subtraction, we obtain

$$QTF = FKP$$

and

$$FTP = QLF;$$

and the theorem is demonstrated.

As an immediate consequence of Theorem 1, we have

THEOREM 2. *If the three sides of a triangle are tangents to a parabola, the circumscribed circle of the triangle passes through the focus of the parabola.*

Let PQR be a triangle whose sides touch the parabola with focus F . The vertices of this triangle must all be ordinary points. Then, by Theorem 1, the angle FQP is equal to one of the angles between QR and the axis of the parabola and, similarly, FRP is equal to one of those angles. Consequently, the angles FQP and FRP are either equal or supplementary. In either case, however, the four points P, Q, R, F are concyclic; i.e., F lies on the circumscribed circle of triangle PQR .

We recall that there is one parabola touching four ordinary lines of which no three are concurrent; and, also, that there are at most two parabolas touching three nonconcurrent ordinary lines and passing through a given point (Theorem 4, Art. 13.5). Hence, there is an unlimited number of parabolas touching the sides of a triangle whose vertices are ordinary points. Concerning this system of parabolas inscribed in a triangle we have

COROLLARY 1. *The foci of all parabolas touching the sides of a given triangle lie on the circle circumscribing the triangle.*

COROLLARY 2. *The circumscribed circles of the four triangles formed from a complete quadrilateral by taking the sides in threes have one point in common.*

It is necessary to assume in the statement of this corollary that none of the six vertices of the complete quadrilateral is an ideal point. For if such were the case the term "circumscribed circle" would not always have a meaning. On the other hand, when all vertices of the quadrilateral are ordinary points, there is but one parabola touching the four lines. The point common to the four circles of the corollary is, then, the focus of that parabola.

We conclude this article with

THEOREM 3. *The feet of the three perpendiculars upon the sides of a triangle from a point on its circumscribed circle are collinear.*

The parabola which touches the sides of the given triangle and has the given point as focus has, at its vertex, a tangent on which, by Theorem 2' of Art. 15.7, lies the foot of the perpendicular from the focus upon any tangent. Thus, the theorem follows immediately. The line of collinearity of the feet of the three perpendiculars is called the **Simson line**, or the **pedal line**, of the point with respect to the triangle.

15.9 Confocal Conics. Two central conics which have the same foci are said to be **confocal**. Two confocal central conics have four tangents in common, namely, the isotropic lines through the two foci. Consequently, a system of confocal conics is a range of conics (Art. 13.5), and there are, at most, two conics of the system passing through a given point.

Now if F_1, F_2 are the foci common to such a system of conics and P is any ordinary point, through which two conics of the system pass, the tangent and normal to one of the conics at P bisect the angles between the focal radii of P and are, respectively, the normal and tangent to the other conic at P . Hence,

THEOREM 1. *Confocal central conics which intersect cut one another at right angles.*

Two parabolas which have the same focus are said to be **confocal**. Two confocal parabolas have three tangents in common, namely, the isotropic lines through the focus and the ideal line. If two confocal parabolas have in addition a common ideal point, they are also **coaxal**.

There is an unlimited number of parabolas having a focus and an axis in common; they have in common three tangents and the point of contact on one of them. If two parabolas of a system of confocal and coaxal parabolas intersect, the angles formed by the focal radius and the diameter through the common

point have as bisectors the tangent and normal to one of the parabolas and these are, respectively, the normal and tangent to the other parabola. That is,

THEOREM 1'. *Confocal and coaxial parabolas which intersect cut one another at right angles.*

15.10 The Construction of Conics. In earlier chapters we have discussed the construction of a conic when five of its elements are given. Such a problem has many variations, as we have seen. For example, one conic is determined when five of its tangents are known (Art. 7.3); on the other hand, four tangents and a point not on any one of these tangents determine, at most, two conics (Art. 13.5). Also as a variation, we have the problem of constructing a conic when certain pairs of the determining elements consist of conjugate imaginary elements (Art. 14.8).

We wish to call attention now to several construction problems stated in terms of foci. Consider, for example, the following:

1. *To construct the conic which has a given point as focus and which touches each of three given nonconcurrent lines.*

This problem is simply a metric specialization of Problem P1, Art. 14.8. For the assigning of a particular point F as focus is equivalent to saying that the conic touches the conjugate imaginary lines $F\Omega$ and $F\bar{\Omega}$, i.e., the isotropic lines through F .

The actual construction of a conic determined by three tangents and a focus is easily carried out as an application of Theorem 2, Art. 15.5. The conic will be an ellipse, hyperbola, or parabola, according to the relative position of the focus with regard to the triangle formed by the tangents. We know, indeed, that if the focus lies on the circle circumscribing that triangle, the conic is a parabola (Art. 15.8).

2. *To construct the central conic which has two given points as foci and which touches a given line not incident with either focus.*

Since a focus is equivalent to a pair of conjugate imaginary tangents, it follows immediately that there is one and only one conic having two given points as foci and touching a given line; for the conic is completely determined by these five tangents (cf. Problem P2, Art. 14.8).

To construct the conic having F_1 and F_2 as foci and p as a tangent, we make use of Theorem 2, Art. 15.7. That is, if O is the mid-point of segment F_1F_2 , and M, N the feet of the perpendiculars from the foci onto tangent p , the circle with center O and radius

OM ($= ON$) intersects the focal axis F_1F_2 at the vertices V_1, V_2 of the conic. The tangents v_1, v_2 at these vertices are, moreover, perpendicular to F_1F_2 , and we draw them. The problem now is to construct the conic when three real tangents and the points of contact on two of them are given (Art. 7.3).

3. *To construct the central conic which has two given points as foci and which passes through a third given point.*

This problem is closely related to the one preceding it. There are, at most, two conics having F_1, F_2 as foci and passing through a point P (Art. 13.5). Moreover, if P is an ordinary point, the bisectors of the angles formed by the focal radii F_1P, F_2P are, respectively, the tangents to those conics (Art. 15.9) at P . It is evident, is it not, that of the two conics through P which have F_1, F_2 as foci one is an ellipse and the other an hyperbola. The tangent at P to the ellipse intersects the focal axis at a point outside the segment F_1F_2 , while the tangent to the hyperbola at the same point meets the focal axis at a point within the segment F_1F_2 .

If, on the other hand, the point P is an ideal point, the solution of the problem is somewhat different, and we leave it as an exercise.

4. *To construct the parabola which has a given point as focus and which touches a given ordinary line, not incident with the focus, at a given point on that line.*

The conic is completely determined since we are given four tangents and the point of contact on one of them. If F is the focus, two of the tangents are the conjugate imaginary lines $F\Omega, F\bar{\Omega}$; the ideal line is a third tangent; and the given line, call it p , is a fourth tangent, and on p the point of contact is the given point P , which must be an ordinary point.

There are two methods of solving this problem. One method is a variation of the construction indicated for Problem P1, Art. 14.8, the other is based on Theorems 1' and 2' of Art. 15.7; and the reader will have no difficulty in applying either method.

5. *To construct the parabola which has a given line as axis and a given point on that line as focus, and which passes through a second given ordinary point.*

We are given in this problem three tangents, the point of contact on one of them, and a point on the curve. There are two parabolas, at most, satisfying these conditions (Art. 15.9), and the reader will observe certain points of similarity with Problem 3 of this article. Thus we leave him to provide the solution.

EXERCISES

1. If P is any point not on an axis of a given central conic k , the pair of perpendicular lines on P which are also conjugate lines with respect to k intersect an axis of k in two points which are homologues in the involution defining the foci on that axis. Hence, two perpendicular conjugate lines with respect to k intersect an axis in two points harmonically separated by the foci.

Suggestion. Let the ideal points of the perpendicular conjugate lines on P be B and B' , and then consider the projectivity $B(SB'C -) \bar{\wedge} B'(BSC -)$, which cuts the axis in the involution defining the foci (cf. Theorem P2, Art. 14.7).

2. Discuss the converse of the theorem of Ex. 1.

3. Two tangents to a central conic make equal angles with the lines joining their point of intersection to the foci.

4. The circle through the real foci and any point P of a central conic intersects the conjugate axis at two points, one of which lies on the tangent and the other on the normal to the conic at P .

5. Using the results of Ex. 4, prove Theorem 1 of Art. 15.7.

6. State and prove the theorems for the parabola which are analogous to those of Exs. 1 and 3. (Cf. Theorem 1, Art. 15.8.)

7. The tangent and normal at any point of a parabola determine on the axis a segment which is bisected by the focus.

8. The semi-latus rectum of a central conic is an harmonic mean between the segments determined by the focus on any focal chord. Prove this theorem and then show that the length of the latus rectum is equal to $2b^2/a$, in our usual notation.

9. Show that the length of the latus rectum of a parabola is equal to twice the distance between focus and directrix.

10. The foci of an hyperbola and the intersections with the asymptotes of any tangent to the curve are four coneyclic points.

11. The locus of the center of a circle which touches two fixed circles consists of an ellipse and an hyperbola if the fixed circles intersect; and of two ellipses or two hyperbolas if the fixed circles do not intersect. Discuss this problem.

12. The locus of the center of a circle which touches a fixed line and a fixed circle consists of two parabolas. Discuss this problem for various positions of the fixed line and circle.

13. The angles formed by pairs of tangents from a given point to the individual conics of a confocal system are bisected by a fixed pair of lines. These fixed lines are the tangents to those conics of the system passing through the given point.

14. A conic is determined by one focus, two tangents, and the point of contact on one of the given tangents. Construct the conic.

15. A parabola is determined by its focus, one ordinary tangent, and the direction of a diameter. Construct the parabola.

16. A central conic is determined by its vertices on the transverse axis and one real focus, which is collinear with the given vertices. Construct the conic.

17. If in Ex. 16 an imaginary focus is given instead of a real focus, it lies on the perpendicular bisector of the segment joining the given vertices. Show how to construct the conic determined by these data.

18. Construct a central conic having given one focus, the corresponding directrix, and (a) one point of the curve; (b) one tangent of the curve.

19. Investigate the problem of constructing a central conic when given one directrix, the conjugate imaginary points common to the conic and this directrix, and one point of the curve. Is the conic uniquely determined?

20. Given the foci and the length of the transverse axis of a central conic; construct the tangents to the conic from a given point.

CHAPTER XVI

PLANAR COLLINEATIONS

16.1 Introduction. Up to this point we have confined our study of projective geometry to the one-dimensional forms. The reader will observe that in Chapter VI we introduced the concept of projectively related one-dimensional primitive forms and later, in Chapter XII, extended the concept to include all the one-dimensional forms. In this chapter we wish to extend the projectivity concept in another direction, so to speak. We shall then be in a position to exhibit the relation between projective geometry and Euclidean geometry in a more systematic manner than has been possible heretofore.

The extension which we are now undertaking will be, of necessity, much less detailed than that in the earlier chapters, for our purpose is merely to introduce the reader to certain ideas which are fundamental in the study of higher geometry and, at the same time, to do so in a limited number of pages.

The extension of the projective relationship to the two-dimensional primitive forms will be the basis of our investigations henceforth. The reader will recall that the two-dimensional primitive forms are the field of points, the field of lines, the bundle of planes, and the bundle of lines (Art. 1.7). The last two are respectively the space duals of the first two; and because of this duality we shall restrict our statements for the most part to the field of points and the field of lines. Also, since two points determine a line, it is frequently convenient to consider in conjunction with a field of points its associated field of lines, and, dually, in conjunction with a field of lines its associated field of points. Thus the **planar field** is the totality of points and lines on one plane. Its space dual is called a **bundle**, and is the totality of planes and lines on one point.

16.2 Perspective Forms. For our purposes it will be sufficient to note the following examples of perspective two-dimensional forms:

1. Two fields of points on distinct supports are perspective if they are in a (1,1) correspondence such that each pair of homologous elements, point and point, is incident with one line of a bundle of lines. We say they are perspective from a point.

S1. Two bundles of planes on distinct supports are perspective if they are in a (1,1) correspondence such that each pair of homologous elements, plane and plane, is incident with one line of a field of lines. We say they are perspective from a plane.

2. A planar field and a bundle are perspective if they have nonincident supports and if they are in a (1,1) correspondence such that homologous elements, point and line or line and plane, are incident.

In the perspectivities of Examples 1 and S1, the elements common to the two perspective forms are **self-corresponding** elements. That is, every point on the line of intersection of the supports of two perspective fields of points is a self-corresponding point; and, dually, every plane, on the line joining the centers of two perspective bundles of planes is a self-corresponding plane. Moreover, two perspective fields of points have the same projector from one point, the center of perspectivity; and two perspective bundles of planes have the same section by one plane, the plane of perspectivity.

Similar to the last, the planar field of Example 2 is a section of the bundle perspective to it, and the bundle is a projector of the planar field.

The reader can easily supply analogous statements for two perspective fields of lines or two perspective bundles of lines.

16.3 The Planar Perspectivity. Consider, as in Fig. 109, two distinct planes α , α' whose line of intersection is o , and a line l not on either plane. If any point P' of α' is projected from a point S_1 on l into point P_1 of α , and then if the same point P' is projected from another point S_2 on l into a point P_2 of α , we have, by the perspectivities

$$\alpha(P_1 \text{ ---}) \frac{S_1}{\lambda} \alpha'(P' \text{ ---}) \frac{S_2}{\lambda} \alpha(P_2 \text{ ---}),$$

a (1,1) correspondence between the points of the plane α , provided of course the centers of projection S_1 , S_2 are not on either of the planes α , α' .

It is obvious that P_1, P_2 are distinct points when P' does not lie on o ($= \alpha\alpha'$); and that P_1, P_2 are coincident points when P' does lie on o . That is, the points of o are self-corresponding points in each of the perspectivities between α and α' and are, therefore,

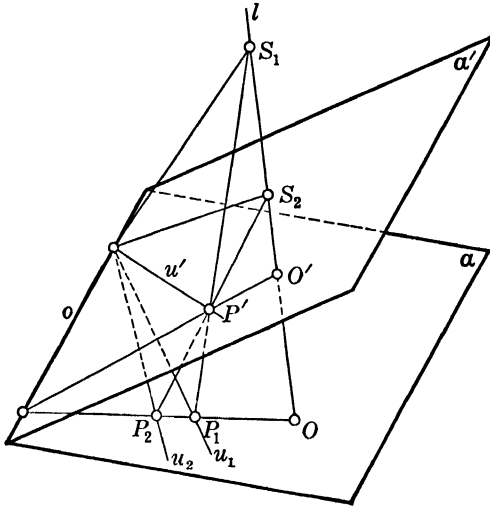


FIG. 109

self-corresponding points in the (1,1) correspondence between the points of α .

Again, if l meets α' at O' and α at O , pairs of homologous points such as P_1 and P_2 are collinear with O , and OP_1 ($= OP_2$) meets $O'P'$ at a point on o .

Similarly, if u' is a line on α' , it is projected on α ; from S_1 into u_1 , and from S_2 into u_2 . The correspondence above mentioned is thus seen to be a correspondence not only between the points of α but also between the lines of α . Two homologous lines u_1, u_2 are distinct when u' does not pass through O' ; otherwise u_1, u_2 are not distinct. Moreover, two homologous lines u_1, u_2 are concurrent with o .

To summarize, the (1,1) correspondence thus established between the elements of α is such that points correspond to points and lines correspond to lines; and every point on one line o is a self-corresponding point, and every line on one point O is a self-corresponding line.

To be more precise we state the

DEFINITION. A **planar perspectivity** is a (1,1) correspondence between the elements of a plane which has like elements as homologues and which leaves invariant every point on one line and every line on one point.

The support o of the invariant point-row is called the **axis of perspectivity**, and the center O of the invariant flat pencil is called the **center of perspectivity**.

A planar perspectivity whose axis and center are not incident is called a **planar homology**. If the axis and center are incident the planar perspectivity is called a **planar elation**.

An **involutory** or **harmonic** homology is an homology in which homologous pairs are doubly corresponding.

Theorem XIII. *A planar perspectivity is uniquely determined if the center, axis, and one pair of homologous points are given, provided that the homologous points are not incident with the axis and are collinear with the center.*

To prove this theorem we shall show that, when the center O , axis o , and homologous points A, A' are given, we can construct the homologue of any point B of the plane.

Consider, first, the construction of the homologue of B when B is not on the line $OA (= OA')$. This homologue, call it B' , lies on OB . Moreover, if AB cuts o at R , we know that R is an invariant point; hence $A'B'$ must intersect o at R . The homologue B' is, then, the intersection of OB and $A'R$, and as such is uniquely determined.

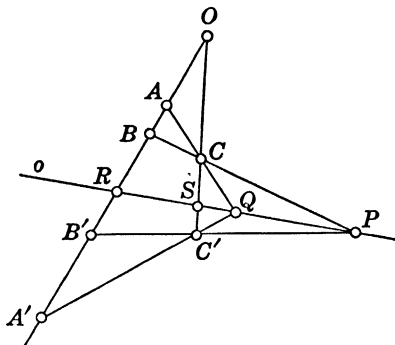


FIG. 110

On the other hand, if B is a point on OA , we construct any pair of homologous points C, C' not on OA , as explained in the preceding paragraph. Then, by the same procedure but using the pair C, C' instead of A, A' , we construct the homologue B' .

That the construction of B' is independent of the particular pair C, C' which we chose, follows immediately, in the case of a planar homology, from Fig. 110. For if A, A' meets o at R and if C, C'

meets o at S , we have

$$AA'OR \overline{\wedge}^Q CC'OS \overline{\wedge}^P BB'OR,$$

and hence,

$$AA'OR \overline{\wedge} BB'OR.$$

Now this is an hyperbolic projectivity on AA' and is completely determined by its double points O and R and the pair A, B (Art. 6.11). Consequently, B' is unique, for it is the partner of A' in that projectivity.

In case the planar perspectivity is a planar elation, the argument is much the same. Then, O coincides with R , the projectivity of the preceding paragraph is parabolic, and B' , the partner of A' , is unique.

By the construction of a unique homologue of a given point we have shown that the given data define not more than one planar perspectivity. To show that there is one such perspectivity we construct a pair of space perspectivities, as in Fig. 109, which satisfy the conditions of the theorem. This we shall leave as an exercise for the student.

We note that Theorem XIII has a plane dual; the statement and proof of that dual should be investigated.

16.4 Projective Forms. DEFINITION. Two two-dimensional primitive forms are **projectively related**, or are **projective** to one another, if there is a (1,1) correspondence between their elements such that to every one-dimensional primitive form in the one there corresponds a projective one-dimensional primitive form in the other.

In general there are two types of (1,1) correspondences between the elements of two two-dimensional forms. If a (1,1) correspondence between two forms is such that homologous elements are like elements and if to every one-dimensional primitive form there corresponds a one-dimensional primitive form, we call the correspondence a **collineation**. The planar perspectivity of the preceding article is a collineation between two superposed planar fields.

The other type of (1,1) correspondence is such that homologous elements are not of the same kind; it is called a **correlation**. We have in the pole-polar correspondence of Art. 9.6 a particular kind of correlation between two superposed planar fields. To every point there corresponds a line, its polar with respect to a given conic, and to every flat pencil there corresponds a point-row. This

correlation is a particular kind of correlation in two respects. In the first place, it is projective, for it associates with each flat pencil a point-row *projective* to it; and in the second place, it is involutory, for homologous elements are doubly corresponding.

We have indicated that all (1,1) correspondences of the types mentioned above are not projective correspondences, necessarily. It is true, however, in the domain of real projective geometry that all collineations and correlations are projective. Hence, since we shall restrict ourselves in what follows to real geometry, we shall understand the term "collineation" when unqualified to mean a projective collineation.

The remainder of this book will be devoted primarily to a study of collineations in the plane.

For an immediate illustration of a projectivity between two-dimensional forms we return now to the planar perspectivity of Art. 16.3. We have shown there that to every point-row $u(ABC \text{ ---})$ there corresponds a point-row $u'(A'B'C' \text{ ---})$. Moreover, if u and u' are not incident with O , the center, nor coincident with o , the axis, we have $u \overline{\wedge} u'$; if u is incident with O , u' is also incident with O , the point-rows are cobasal, and $u \overline{\wedge} u'$; and finally, if u coincides with o , the axis of the perspectivity, the point-rows are cobasal and identical, i. e., every point is a double point. In each case, therefore, the point-rows are projectively related.

Similarly, in a planar perspectivity to every flat pencil there corresponds a projectively related flat pencil.

Thus, a planar perspectivity is a projective collineation in accordance with the above definition.

16.5 Groups of Transformations. We introduced in Art. 9.9 the idea of a correspondence between two forms as a transformation which changes one form into the other by transforming each element of the one form into the homologous element of the other. When dealing with more than one transformation it is a convenience to be able to represent this idea symbolically.

Thus, if the elements of a form F are transformed into the elements of a form F' by a transformation T , we write $T(F) = F'$; and if a is any element of F and a' its corresponding element under this transformation T , we write $T(a) = a'$.

Two transformations T_1 , T_2 have a product T . That is, if $T_1(F) = F'$ and $T_2(F') = F''$, the transformation T which sends F into F'' directly is termed the **resultant** or the **product** of T_1 and

T_2 in the order indicated. Thus, since

$$F'' = T_2(F') = T_2(T_1(F)) = T_2T_1(F),$$

the product T of the transformations T_1, T_2 is T_2T_1 , and we write $T = T_2T_1$. The order of the symbols in this product T_2T_1 is important; for ordinarily T_2T_1 is not the same as T_1T_2 . If two transformations T_1, T_2 are such that $T_2T_1 = T_1T_2$, they are said to be **commutative**.

In a similar manner we may define the product or resultant of more than two transformations.

If a form F is transformed into F' by a given transformation T , the transformation which transforms F' into F is called the **inverse** of T , and we represent it by T^{-1} . Thus, $T(F) = F'$ and $F = T^{-1}(F')$. The product of a transformation by its inverse is a transformation which leaves unchanged every element of a form; it is called the **identity transformation**. We denote it by the symbol 1 ; hence, $TT^{-1} = T^{-1}T = 1$ and $1T = T1 = T$.

Consider now a set of transformations S, T, R, \dots such that the product of any two transformations of this set is a transformation of the set, and such that the inverse of each transformation is also contained in the set. Since the product of S and S^{-1} must be a transformation of the set, the identity transformation belongs to the set. It is evident, also, that the transformations of this set are **associative** as to products, i.e., we may write $S(TR) = (ST)R = STR$. The transformations of a set such as we have described constitute a **group of transformations**.

As an immediate example of a group of transformations we mention the set of *all* projective transformations (including the identity) of a one-dimensional primitive form which transform that form into itself. For it is evident, from the definition of a projectivity (Art. 6.2), that the resultant of any two such projectivities is a projectivity of the set, and the inverse of any one such projectivity is likewise a projectivity of the set.

Again, the set, including the identity transformation, of all collineations on a plane is a group. But, since the product of two planar correlations is a collineation, we see that the set of all correlations on a plane is not a group. On the other hand, the set of all collineations and all correlations on a plane is a group; it is called the **general projective group** of transformations on a plane.

Among the transformations of the general projective group on a plane we have all planar perspectivities; and among the planar perspectivities we have the set of all homologies having a common axis o . Consider two homologies of that set, H_1 with center O_1 and H_2 with center O_2 : What is the product H_2H_1 ?

If A is any point of the plane (Fig. 111) and $H_1(A) = A_1$ and $H_2(A_1) = A_2$, then $H_2H_1(A) = A_2$, while at the same time, if $O_1O_2 = l$, $H_2H_1(l) = l$; also $H_1(o) = o$, $H_2(o) = o$, and $H_2H_1(o) = o$. It is evident from these considerations that the corre-

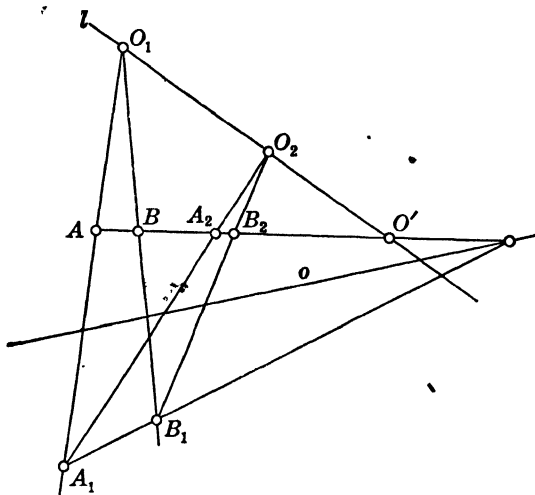


FIG. 111

spondence H_2H_1 leaves invariant the point O' , common to AA_2 and l , and the line o , and it transforms A into A_2 . Thus, as we have shown in Art. 16.3, H_2H_1 is a planar perspectivity. If AA_2 intersects l at a point O' not on the line o , then H_2H_1 is an homology, but if O' lies on o , H_2H_1 is an elation. This proves

THEOREM 1. *The product of two homologies with distinct centers and a common axis is an homology or an elation.*

Consequently, the set of all planar homologies with a common axis is not a group.

The reader can easily prove, furthermore,

THEOREM 2. *The product of two elations with distinct centers and a common axis is an elation.*

THEOREM 3. *The set of all homologies and all elations having a common axis is a group.*

16.6 The Fundamental Theorem. Corresponding to Theorem VI, Art. 6.8, we have for two fields of points:

Theorem XIV. *A projective collineation between two planes is uniquely determined when four points, no three collinear, of one plane are assigned as homologues of four points, no three collinear, of the other plane.*

The two planes of the theorem may be distinct or superposed.

To prove this theorem let us suppose there are two collineations Π_1, Π_2 each of which transforms A, B, C, D of plane α into A', B', C', D' , respectively, of plane α' . Then we have

$$\Pi_1(ABCD \text{ ---}) = (A'B'C'D' \text{ ---})$$

and

$$\Pi_2(ABCD \text{ ---}) = (A'B'C'D' \text{ ---});$$

from which we conclude that

$$\Pi_2^{-1}\Pi_1(ABCD \text{ ---}) = (ABCD \text{ ---}).$$

That is, the projectivity $\Pi_2^{-1}\Pi_1$ leaves invariant four points A, B, C, D of the plane α . But if it leaves these four points invariant it must also leave invariant each point of intersection of the sides of the complete quadrangle $ABCD$. Thus, the projectivity leaves invariant three and, therefore, every point on each side of this complete quadrangle (Theorem VI). Furthermore, each line of the plane intersects the sides of the quadrangle at invariant points and is, consequently, itself invariant under $\Pi_2^{-1}\Pi_1$. Finally, each point of the plane is invariant, for it is the intersection of two invariant lines. This whole argument leads to one important fact, namely, the transformation $\Pi_2^{-1}\Pi_1$ is an identity transformation, i.e., $\Pi_2^{-1}\Pi_1 = 1$, or $\Pi_1 = \Pi_2$. Our assumption that there are two collineations fulfilling the conditions of the theorem is thus contradicted, and the theorem is demonstrated.

The analogous theorem for two fields of lines can be supplied easily enough by the student — it will be useful in a later connection

16.7 Construction of a Projectivity. When given four pairs of homologous elements defining a projectivity Π (Theorem XIV) between two planes α, α' , we can construct the homologue in α' of any point P of α by the following procedure.

If A, B, C, D in α correspond respectively to A', B', C', D' in

α' , we have, by the definition of a projectivity,

$$A(BCD \text{ ---}) \bar{\wedge} A'(B'C'D' \text{ ---})$$

and

$$B(ACD \text{ ---}) \bar{\wedge} B'(A'C'D' \text{ ---}).$$

Each of these projectivities between flat pencils is uniquely determined, since in each three pairs of homologous lines are known. Consequently, if P is any point of α we can construct m' , the line through A' which is homologous to $m = AP$; similarly, we can construct n' , the line through B' homologous to $n = BP$. Obviously, P' is the intersection of m' and n' .

We leave with the reader the formulation of the analogous construction of the homologue in α' of a given line l of α . In particular, let him construct the homologue of the ideal line i of α , and the homologue of the ideal line j' of α' . If these homologues, i' of α' and j of α , are ordinary lines they are called the **vanishing lines**, respectively, of the planes α' and α .

16.8 The Affine Group. Among the projective collineations which transform a plane into itself there is an important group of transformations which transform a given line of the plane into itself. This exceptional line of the plane is transformed into itself and is invariant; but the points of this exceptional line are not necessarily invariant points.

DEFINITION. The set of all projective collineations which leave invariant a fixed line of the plane is a group of transformations called the **affine group** of the plane.

Those properties of figures which remain invariant under this group of transformations but do not remain invariant under the more general group of collineations are called **affine properties**, and the geometry which studies such properties is called **affine geometry**.

The affine group is contained in the collineation group and is called a **subgroup** of the collineation group.

The invariant line of the affine group of transformations may be any line of the projective plane; we call it the **ideal line**. Points on this exceptional line are **ideal points**, and all other points are **ordinary points**; all other lines are **ordinary lines**, and two ordinary lines which meet at an ideal point are **parallel lines**. The ordinary points and lines constitute the **Euclidean plane**.

The reader will observe that in the last paragraph we have introduced, in a systematic way by definition, the Euclidean plane as contained in the projective plane. The references in the earlier

parts of this book to nonprojective ideas required, of course, a distinction, which we based upon intuition, between ordinary and ideal elements. The reader will now note that such an intuitive basis was not necessary, for the introduction of nonprojective ideas might have been postponed until the present chapter. Under such a plan they would then be introduced by definition, after the manner of the preceding paragraph.

Any figure or diagram which we may hereafter draw to illustrate ideas connected with this Euclidean plane, as defined above, will have the same appearance as one traditionally associates with such a figure or diagram *only if* the ideal line, the exceptional line of the plane, is considered to be identical with the line called by that name throughout the earlier parts of this text. In the sequel, we shall so consider it. It should be emphasized, however, that the ideal line of an affine transformation may be *any* line of the projective plane; but, under such an interpretation, we must put aside tradition when we view a figure or diagram.

THEOREM 1. *There is a unique affine transformation which transforms the ordinary points A, B, C into the ordinary points A', B', C' , respectively.*

That this is a consequence of Theorem XIV is at once evident if the reader will replace the "points" of that theorem by "lines." Then it is seen that a collineation is uniquely determined when four pairs of homologous elements are given; and the four pairs of homologous elements here are the three pairs of sides of the two triangles, ABC and $A'B'C'$, and the ideal line, which corresponds to itself.

THEOREM 2. *Parallel lines are transformed into parallel lines by an affine transformation.*

This follows, of course, from the fact that ideal points are transformed into ideal points.

THEOREM 3. *The ratio of division of a segment AB by a point C remains invariant under an affine transformation.*

For the line l containing A, B, C is transformed into a line l' containing A', B', C' . Moreover, if I, I' are respectively the intersections of l, l' with the ideal line, we have (Art. 5.4)

$$(AB, CI) = (A'B', C'I'),$$

that is

$$\frac{AC}{CB} = \frac{A'C'}{C'B'} = \text{a constant.}$$

16.9 The Homothetic Group. DEFINITION. Any elation having the ideal line as its axis is called a **translation**. If l is an ordinary line through the center of the translation, the translation is said to be **parallel** to l .

From the general theory of the elation we know that the set of all translations is a group (Theorem 2, Art. 16.5), and this group is a subgroup of the affine group.

Since an elation is uniquely determined when its center, axis, and one pair of homologous points are known (Theorem XIII), it follows that there is one and only one translation sending a given point A into a given point A' . The center of this translation is, of course, the ideal point of the line AA' . A translation transforms every line not on its center into a parallel line and leaves invariant every line passing through its center.

DEFINITION. Any homology having the ideal line as its axis is called a **dilation** or a **stretch**. If O is the center of the stretch, the stretch is said to be **about** O .

From the considerations of Art. 16.5, the set of all stretches on a plane is not a group; for the product of two stretches is either a stretch or a translation.

DEFINITION. The set of all translations and all stretches on a plane is a group of transformations called the **homothetic group** of the plane.

The homothetic group is a subgroup of the affine, and a property

invariant under the former but not under the latter group is called an **homothetic property**. The corresponding geometry is called **homothetic geometry**. Homothetic figures are more familiar to the student, perhaps, as figures which are "similar and similarly placed."

The transformations of the homothetic group are translations and stretches. Let T be a translation with center O_1 , and S

a stretch with center O_2 (Fig. 112). It is evident from the figure that if

$$T(A) = A_1, \quad S(A_1) = A_2,$$

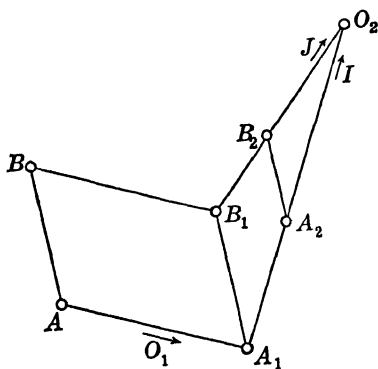


FIG. 112

then

$$T(B) = B_1, \quad S(B_1) = B_2,$$

and

$$T(AB) = A_1B_1, \quad S(A_1B_1) = A_2B_2,$$

and

$$ST(AB) = A_2B_2.$$

Hence, ABB_1A_1 is a parallelogram, and the segments AB and A_1B_1 are equal. Moreover, under the stretch S , the segments A_1B_1 and A_2B_2 are parallel and the ratio of their lengths is a constant μ for the transformation S . For, since S is determined when O_2 , A_1 , A_2 are known, we have

$$A_1A_2O_2I \overline{\wedge} B_1B_2O_2J,$$

the center of this perspectivity being the ideal point common to A_1B_1 and A_2B_2 . Consequently,

$$(A_1A_2, O_2I) = (B_1B_2, O_2J),$$

and

$$\frac{O_2A_1}{O_2A_2} = \frac{O_2B_1}{O_2B_2} = \text{a constant } \mu.$$

Hence,

$$\frac{AB}{A_2B_2} = \frac{A_1B_1}{A_2B_2} = \frac{O_2A_1}{O_2A_2} = \mu.$$

The constant μ is called the **stretch factor** of the transformation.

These results we state as a

THEOREM. *Under an homothetic transformation homologous segments are parallel and the ratio of their lengths is constant.*

16.10 The Similarity Group. An affine transformation leaves the ideal line invariant, but the points on that line are not invariant points. An homothetic transformation, on the other hand, leaves invariant the ideal line and every ideal point. We wish to consider in this article another subgroup of the affine group. To this end we choose on the ideal line a fixed, but nevertheless arbitrary, elliptic involution. In what follows we call it the **absolute** or the **orthogonal involution I** , and two ordinary lines whose ideal points are homologues in this involution are said to be **perpendicular lines**.

The orthogonal involution, as here defined, may be *any* elliptic involution on the ideal line. However (cf. Art. 16.8), only when it is considered to be identical with the involution called by that name in Art. 10.2, will our illustrations have the appearances which, by tradition, we usually associate with them.

DEFINITION. The set of all projective collineations which leave invariant the orthogonal involution is a group of transformations called the **similarity group** of the plane.

A property invariant under this group but not under the more general affine group is said to be a **similarity property**, and the corresponding geometry is called **similarity geometry**. Similarity geometry is, in fact, a part of the geometry of Euclid in its larger sense, i.e., the geometry of Euclid which includes the theory of similar figures as well as the theory of congruent figures.

Although we are restricting ourselves to real geometry, in which the orthogonal involution has no double points, it will save both time and words if we agree that the orthogonal involution has, in the sense of Chapter XIV, two conjugate imaginary double points, the circular points Ω , $\bar{\Omega}$. The transformations of the similarity group, then, leave invariant the pair of circular points.

Let the orthogonal involution I be defined by two pairs of homologous points A, B and C, D ; i.e., the involution I is the projectivity

$$ABC\bar{\Omega} \text{---} \bar{\Omega} BAD \text{---}.$$

If now Σ is a similarity transformation such that $\Sigma(A) = A'$, $\Sigma(B) = B'$, etc., it follows that the involution

$$A'B'C' \text{---} \bar{\Omega} B'A'D' \text{---}$$

is none other than I , for I is invariant under Σ . Consequently, homologous points of I are transformed by Σ into homologous points of I . By this and Theorem 2, Art. 16.8, we have

THEOREM 1. *Perpendicular lines are transformed into perpendicular lines and parallel lines are transformed into parallel lines by a similarity transformation.*

There are, in fact, two kinds of similarity transformations. A **direct** similarity transformation Σ transforms each circular point into itself, i.e., $\Sigma(\Omega) = \Omega$ and $\Sigma(\bar{\Omega}) = \bar{\Omega}$. An **indirect** similarity transformation Σ' transforms each circular point into the other, i.e., $\Sigma'(\Omega) = \bar{\Omega}$ and $\Sigma'(\bar{\Omega}) = \Omega$. It is evident from this that all direct similarity transformations form a group; but the set of all indirect similarity transformations is not a group, for the product of two indirect similarity transformations is a direct similarity transformation.

Under either kind of similarity transformation, however, the orthogonal involution remains invariant, and we have

THEOREM 2. *Circles are transformed into circles by a similarity transformation.*

16.11 The Group of Rigid Motions. Consider an involutory or harmonic homology (Art. 16.3) which has center O , axis o , and the distinct points A, A' as homologues. Since A is transformed into A' , and since the pair A, A' is doubly corresponding, it follows that A' is transformed into A . Therefore, if AA' intersects the axis o at R , we have, as in Art. 16.3,

$$AA'OR \overline{\wedge} A'AOR,$$

and

$$(AA', OR) = (A'A, OR).$$

But also, by Theorem 2, Art. 5.7,

$$(AA', OR) = \frac{1}{(A'A, OR)};$$

consequently,

$$(AA', OR) = \pm 1.$$

The positive sign is discarded, for then A and A' would not be distinct points. Hence, using the negative sign, we conclude that the pair A, A' harmonically separates the pair O, R . This proves the

THEOREM. *In an involutory or harmonic homology pairs of homologous points are harmonically separated by the center of the homology and the intersection of their join with the axis of homology.*

DEFINITION. Any harmonic homology whose center is an ideal point and whose axis intersects the ideal line at the point homologous to the center in the orthogonal involution is called an **orthogonal line reflection**.

From this definition and the theorem, it is evident that any pair of homologous ordinary points A, A' in an orthogonal line reflection L determines a segment AA' which is perpendicular to and bisected by the axis of L . If under L the ideal points I, I' are homologous points, if P is the ideal point on the axis of L , and if O is the center of L , we have the harmonic relation $H(II', OP)$, and conversely. In particular, by Theorem 2, Art. 14.6, $H(\Omega\bar{\Omega}, OP)$; that is, $L(\Omega) = \bar{\Omega}$ and $L(\bar{\Omega}) = \Omega$.

Again, under an orthogonal line reflection, the homologous segments AB and $A'B'$ are equal in length; and homologous angles ABC and $A'B'C'$ are equal in magnitude *but opposite in sign*. That is, the distance between two points and the magnitude of the angle between two lines are invariants under such a transformation.

DEFINITION. The product of an even number of orthogonal line reflections is called a **displacement** or a **rigid motion**.

The set of all rigid motions is a group of transformations called the **metric group** of the plane. This group is a subgroup of the similarity group. Figures which are homologous under a transformation of the metric group are said to be **congruent**; a property invariant under this group but not under the similarity group is called a **metric property**; and the geometry which studies such properties is called **metric geometry**.

The distance between two points and the magnitude *and sign* of the angle between two lines remain invariant under a metric transformation.

The reader will observe that, in accordance with the investigations of this chapter, a metric property is one which remains invariant under the group of rigid motions but not under the more general similarity group. This is of course a stricter interpretation of the term "metric" than that introduced in Chapter V. That is, in earlier chapters, we have frequently classified conics as ellipses, hyperbolas, or parabolas, and called this a metric classification. According to the notions developed in this chapter, however, such a classification is now more properly termed an affine classification. It is true, on the other hand, that *all* the metric specializations of the earlier chapters are invariant under the group of rigid motions but not under the affine group. For example, a circle is transformed into a circle by a rigid motion; but by an affine transformation it may be transformed into an ellipse. And similarly, in Chapter XV, foci and focal properties are invariant under metric but not affine transformations.

In the group of rigid motions we have those transformations which leave invariant one ordinary point. Hence,

DEFINITION. The set of all rigid motions which leave invariant one ordinary point O is a group of transformations called the **group of rotations** about O .

In particular, the product of two orthogonal line reflections whose axes intersect at an ordinary point O is a **rotation** about O .

It can be shown that any displacement or rigid motion is the resultant of a rotation about a given point and a translation.

Thus the metric group, the group of rigid motions, is exhibited as a special case of the general projective group. In doing so we have outlined, in brief, the fundamental characteristics of **other**

subgroups of the general projective group; but it should not be supposed that the groups and associated geometries mentioned in this chapter exhaust the possibilities. There are others; but enough has been said to indicate the all-inclusive nature of projective geometry and the significance of Cayley's remark, in his *Sixth Memoir on Quantics*, that projective geometry is *all* geometry.

EXERCISES

1. We are given the center, the axis, and a pair of homologous lines (concurrent with the axis) of a planar homology. Construct (a) several pairs of homologous lines in this homology; (b) several pairs of homologous points; (c) the vanishing lines; and (d) the homologues of the ideal points on the given lines.

2. Investigate the constructions of Ex. 1 when the planar perspectivity is an elation.

3. We are given the center O , the axis o , and a vanishing line j of a planar perspectivity. Show how to construct pairs of homologous elements when the perspectivity is (a) an homology; (b) an elation.

4. A planar homology is determined by its center O , its axis o , and the vanishing line j . Construct the figure K' which is homologous to a given circle K cutting the given line j in two distinct points A, B . The tangents to the circle at A, B are homologous to the tangents to K' at the ideal points A', B' .

5. If in Ex. 4 the circle K touches j instead of cutting it in two distinct points, construct K' .

6. If in Ex. 4 the circle K does not cut j , construct K' .

7. Draw figures illustrating Exs. 4, 5, and 6 when the perspectivity is an elation.

8. Given a projective collineation between two planes; show that a conic in either plane corresponds to a conic in the other. If the correspondence between two planes is a correlation, what is the figure homologous to a conic in one of the planes?

9. If the points A, B, C, D of a plane correspond respectively to B, A, D, C under a projective transformation, show that the transformation is involutory.

10. A plane and a bundle are projectively related. State for these forms the theorem analogous to Theorem XIV.

11. If a plane is projectively related to a bundle, what is the homologue of a conic in the plane?

12. Two bundles are in a (1,1) correspondence such that to every element (line or plane) of one there corresponds that element (plane or line) of the other which is perpendicular to it. Is this correspondence a projective one? What is the locus of the points of intersection of homologous elements? What is the section of this locus by a plane of either bundle? In particular, what is the section by that plane of one bundle which corresponds to the line joining the centers of the bundles?

13. If P and P' are any two homologous points in a given planar perspectivity, they are collinear with the center O and their join cuts the axis o at a point R . Show that the double ratio (PP', OR) is an invariant of this perspectivity.

14. Show that the product of two planar perspectivities, in the same plane and with a common center, is a perspectivity having the same center and an axis concurrent with the axes of the two original perspectivities. Also, show that the invariant double ratio of the resultant is the product of the invariant double ratios of the originals.

15. Show that two homologies having the same center and the same axis are commutative. Prove a like theorem for two elations.

16. Show that the set of all homologies in a plane which have the same center and the same axis is a group. Prove a like theorem for the set of all elations with the same center and axis.

17. Show that corresponding point-rows under an affine transformation are similarly projective (Art. 10.8).

18. Using Ex. 17 show that the ratio of the areas of corresponding parallelograms is an invariant under an affine transformation. Hence, the areas of corresponding triangles are in a constant ratio.

19. Under an affine transformation an hyperbola corresponds to an hyperbola, and the asymptotes of the one correspond to the asymptotes of the other.

20. Given an homothetic transformation in the plane; construct the figure corresponding to a given conic.

21. Given a displacement in the plane; construct the figure corresponding to a given triangle.

22. The product of an odd number of orthogonal line reflections is called a *symmetry*. Do the transformations of symmetry form a group? Construct a figure *symmetrical* to a given triangle.

23. Consider the following transformation of the points of a plane; it is somewhat different from those studied in the text: Given, in a plane, a fixed circle k with center P ; the point corresponding to any point A of the plane is the intersection A' of PA and a , the polar of A with respect to k . (Cf. 1, Art. 9.9.) Homologous points A, A' of this transformation are called *inverse points* with respect to the circle k , and the transformation is called an *inversion*. Corresponding figures under an inversion are said to be *inverse figures*, the one of the other. Show that the product $PA \cdot PA'$ is constant and equal to the square of the radius of the circle of inversion k . Is the transformation involutory? Does it have double points?

24. Show that the inverse (Ex. 23) of a line not on P is a circle; and the inverse of a circle not on P is a circle. What is the inverse of a line on P ; of a circle on P ?

25. Construct the inverse of a conic with respect to a circle k whose radius is r and whose center is a focus of the conic. Consider the three types of conics separately.

SUPPLEMENTARY READING

The following list of titles is arranged chronologically. It is not an exhaustive list, but it does cover a wide variety of books on projective geometry. Some of the books are of interest historically; some of them are general treatises on the subject; and others are more specialized in their treatments.

- PONCELET, *Traité des propriétés projectives des figures*, Paris, 1822.
STEINER, *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander*, Berlin, 1832; see also his *Gesammelte Werke*, Bd. 1.
VON STAUDT, *Geometrie der Lage*, Nürnberg, 1847.
CHASLES, *Traité de géométrie supérieure*, Paris, 1852.
VON STAUDT, *Beiträge zur Geometrie der Lage*, Nürnberg, 1856.
CHASLES, *Traité des sections coniques*, Paris, 1865.
REYE, *Die Geometrie der Lage*, Hannover, 1866; English translation of first volume by HOLGATE, 1898 (out of print).
CREMONA, *Elements of Projective Geometry*, London, 1885; original Italian edition, 1872.
ENRIQUES, *Leçons de géométrie projective*, Paris, 1930; original Italian edition, 1898.
WHITEHEAD, *The Axioms of Projective Geometry*, Cambridge (England), 1906.
VEBLEN and YOUNG, *Projective Geometry*, Boston, 1910 (v. 1) and 1918 (v. 2).
MILNE, *An Elementary Treatise on Cross-ratio Geometry*, Cambridge (England), 1911.
HATTON, *The Principles of Projective Geometry*, Cambridge (England), 1913.
BAKER, *Principles of Geometry*, Cambridge (England), 1922.
YOUNG, *Projective Geometry*, Chicago, 1930.

APPENDIX A

METRIC SPECIALIZATIONS

The metric specializations referred to in Arts. 7.6 and 8.7 are obtained by designating as ideal one of the given lines (tangents) or one or two of the given points in the data determining the conic. It is evident that of the five given elements defining a conic not more than two may be ideal. We shall follow the custom introduced earlier (Ex. 1, Chapter VII, and Art. 8.7) and indicate the ideal elements by a letter with an asterisk attached.

A.1 Point Conics. The 60 specializations of the construction problems connected with point conics are:

	GIVEN DATA	TYPE OF CONIC ¹	REQUIRED
1.	$(ABCDE)$	NS	I, III
2.	$(ABCDE^*)$	HP	I, III, IV, VI, VII
3.	$(ABCD^*E^*)$	H	I, III, IV, VI
4.	$(ABCDd)$	NS	I, II, III
5.	$(ABCD^*d)$	H	I, III, V, VIII
6.	$(ABCD^*d^*)$	P	I, III, IX
7.	(ABC^*Dd)	HP	I, II, III, IV, VI, VII
8.	(ABC^*D^*d)	H	I, III, IV, V, VI
9.	(ABC^*D^*a)	H	I, II, III, IV, VI
10.	$(ABCbc)$	NS	I, II, III
11.	(ABC^*bc)	H	I, II, III, V, VIII
12.	(AB^*C^*bc)	H	I, III, V
13.	(ABC^*bc^*)	P	I, II, III, IX
14.	(ABC^*ab)	HP	II, IV, VI, VII
15.	(AB^*C^*ab)	H	II, IV, V, VI

¹ NS, conic not specified; HP, hyperbola or possibly a parabola; H, hyperbola; P, parabola.

The Roman numerals in the third column of the above indicate the element to be constructed. Thus, it is required to find the

I. Intersection with conic of a line through an ordinary point whose tangent is not given.

II. Intersection with conic of a line through an ordinary point whose tangent is given.

III. Tangent to the conic at one of the given ordinary points.

IV. Intersection with hyperbola of a line parallel to an asymptote whose direction only is given. Or, in case the conic is a parabola, see IX below.

V. Intersection with hyperbola of a line parallel to one of the given asymptotes.

VI. Tangent (asymptote) to the hyperbola at one of its given ideal points.

VII. Direction of the second asymptote of the hyperbola when the direction only of the first is given.

VIII. Direction of the second asymptote of the hyperbola when the first asymptote is given.

IX. Intersection with parabola of a line (diameter) through its given ideal point.

A.2 Line Conics. The 46 specializations of the construction problems connected with line conics are:

	GIVEN DATA	TYPE OF CONIC	REQUIRED
1.	$(abcde)$	NS	I, VI, VIII
2.	$(abcde^*)$	P	I, II, VI, VII
3.	$(abcdD)$	NS	I, III, VI, VIII, IX
4.	$(abcd^*A)$	P	I, II, III, VI, VII
5.	$(abcd^*D^*)$	P	I, IV, VI
6.	$(abcdD^*)$	H	I, V, VI, VIII
7.	$(abcAB)$	NS	I, III, VI, VIII, IX
8.	(abc^*AB)	P	II, III, VII
9.	(abc^*BC^*)	P	I, III, IV, VI
10.	$(abcBC^*)$	H	I, III, V, VI, VIII, IX
11.	$(abcB^*C^*)$	H	I, V, VI, VIII

In these problems it is required to find the

I. Tangent to the conic from a point on an ordinary tangent whose point of contact is not given.

II. Tangent to the parabola from an ideal point when the ideal point of contact of the parabola is not given.

III. Tangent to the conic from a point on an ordinary tangent whose point of contact is given.

IV. Tangent to the parabola from an ideal point when the ideal point of contact of the parabola is given.

V. Tangent to the hyperbola from a point on a given asymptote.

VI. Point of contact on an ordinary tangent.

VII. Direction of the axis of the parabola.

VIII. Tangent to the conic which is parallel to a given ordinary tangent whose point of contact is not given.

IX. Tangent to the conic which is parallel to a given ordinary tangent whose point of contact is given.

A.3 Other Specializations. There are also metric specializations of problems involving poles and polars (Exs. 1 and 15, Chapter IX, for example) to be considered. We suggest that the student make a list of such problems.

APPENDIX B

SPECIAL CASES OF PASCAL'S THEOREM

The proofs of the theorems of Art. 8.4 need not depend upon the principle of continuity. They may be proved directly by a method much the same as that of Art. 8.5. Here, however, we shall give another proof of Pascal's Theorem; and upon that proof we shall base the proofs of the theorems of Art. 8.4 as *special cases* of Pascal's Theorem, rather than as *limiting cases* as in Art. 8.4.

The duals we omit.

B.1 Pascal's Theorem. Consider the conic k defined by the projectivity

$$U(abc \text{ ---}) \bar{\wedge} U'(a'b'c' \text{ ---})$$

between two flat pencils (Fig. 113). The centers U, U' and the intersections $A = aa', B = bb', C = cc'$ are points of this conic k .

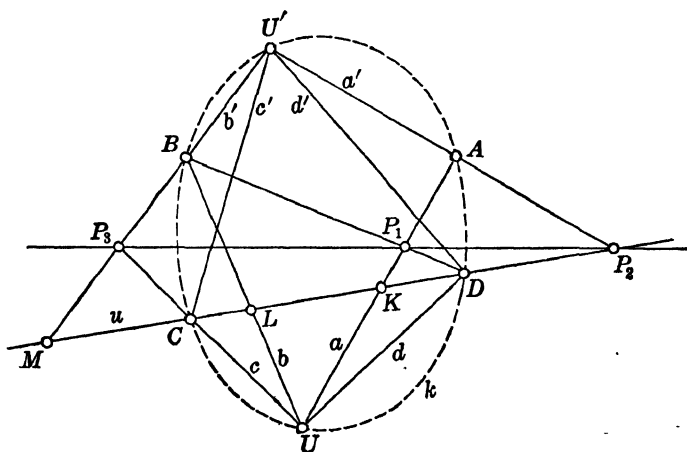


FIG. 113

If, now, u is any line on C , the projective flat pencils U and U' determine on u a projectivity having C as a double point. Thus, as in Fig. 113,

$$u(KLC \text{ ---}) \bar{\wedge} U(abc \text{ ---}) \bar{\wedge} U'(a'b'c' \text{ ---}) \bar{\wedge} u(P_2MC \text{ ---}),$$

and

$$* \quad u(KLC \text{ ---}) \bar{\wedge} u(P_2MC \text{ ---}).$$

This projectivity on u has a second double point D , by Theorem 1, Art. 6.11; and D will be the second point common to line u and conic k . If u is tangent to k , D coincides with C .

The construction of this double point D has been given in Art. 6.11. That is, if P_3 is the intersection of UC and $U'B$, we project the point-row $u(KLC \text{ ---})$ from U and the point-row $u(P_2MC \text{ ---})$ from P_3 by two perspective (Corollary P1, Art. 6.8) flat pencils, i.e.,

$$U(KLC \text{ ---}) \bar{\lambda} P_3(P_2MC \text{ ---}).$$

If UK and P_3P_2 meet at P_1 , the axis of perspectivity of these flat pencils is seen to be the line BP_1 , and this axis intersects u at the required point D .

Thus, the six points U, U', A, B, C, D on k have the joins

$$\begin{array}{l} \cdot \quad \quad \quad UA \text{ and } BD \text{ intersecting at } P_1, \\ \cdot \quad \quad \quad AU' \text{ and } DC \text{ intersecting at } P_2, \\ \cdot \quad \quad \quad U'B \text{ and } CU \text{ intersecting at } P_3; \end{array}$$

and the points P_1, P_2, P_3 are collinear by construction.

The preceding paragraph, stated otherwise, says that the intersections of opposite sides of the simple 6-point $UAU'BDC$, whose vertices lie on k , are collinear. We recognize this to be Pascal's Theorem.

B.2 Simple Pentagon Theorem. If the line u is tangent to the conic k at C , we call it c , and D coincides with C . The above construction then shows that

$$\begin{array}{l} UA \text{ and } BC \text{ intersect at } P_1, \\ AU' \text{ and } c \text{ intersect at } P_2, \\ U'B \text{ and } CU \text{ intersect at } P_3; \end{array}$$

and P_1, P_2, P_3 are collinear.

Hence we have Theorem 1, Art. 8.4, concerning the simple pentagon $UAU'BC$, whose vertices lie on k , and the tangent to k at the vertex C .

The student should construct a figure.

B.3 Simple Quadrangle Theorem. If the student will draw a figure for Art. B.2 and in it take the line a of the flat pencil $U(abc \text{ ---})$ to be tangent to k at U , he will note that a' is then the line UU' (Property 2, Art. 7.2). The construction, when applied to this case, shows that

$$\begin{array}{l} a \text{ and } BC \text{ intersect at } P_1, \\ UU' \text{ and } c \text{ intersect at } P_2, \\ U'B \text{ and } CU \text{ intersect at } P_3; \end{array}$$

and P_1, P_2, P_3 are collinear.

Hence, Theorem 2 of Art. 8.4 concerning the simple quadrangle $UU'BC$, with vertices on k , and the tangents to k at U and C .

B.4 Complete Quadrangle Theorem. Again, drawing a figure for Art. B.2 in which line a' is tangent to k at U' , we see that a is then the line UU' . The construction for this special case exhibits

$$\begin{array}{l} UU' \text{ and } BC \text{ intersecting at } P_1, \\ a' \text{ and } c \text{ intersecting at } P_2, \\ U'B \text{ and } CU \text{ intersecting at } P_3; \end{array}$$

and P_1, P_2, P_3 are collinear.

Hence, Theorem 3 of Art. 8.4 concerning the complete quadrangle $UU'BC$, with vertices on k , and the tangents to k at U' and C .

B.5 Triangle Theorem. Finally, we specialize the figure for Art. B.3 by taking b' to be the tangent at U' . Then, since a is the tangent at U , we have $a' = b = UU'$, and

$$\begin{aligned} a & \text{ and } U'C \text{ meet at } P_1, \\ UU' & \text{ and } c \text{ meet at } P_2, \\ b' & \text{ and } CU \text{ meet at } P_3; \end{aligned}$$

and P_1, P_2, P_3 are collinear.

This is Theorem 4, Art. 8.4, concerning the triangle $UU'C$, with vertices on k , and the tangents to k at those vertices.

B.6 Special Cases of Desargues' Theorem on Conics. As an exercise we suggest the investigation of the special cases of Theorems 1 and 2, Art. 13.5, along the lines of the above discussion. In this connection Exs. 22 and 23 of Chapter XIII should be examined.

INDEX

- Affine geometry, 254
 Angle, bisectors of, 133
 complete, 10
 directed, 49
 Appendices, A, 265
 B, 267
 Applications, of double ratios, 50
 of Pascal's and Brianchon's Theorems, 98, 99, 265, 266
 Assumption, fundamental for projective geometry, 71
 Asymptotes, 92
 theorem on, 137
 Asymptotic cone, 160
 Axial projection, 9
 Axioms, 6
 extended to imaginary elements, 209
 Axis, of a conic, 136, 138
 of homology, 74
 of involution on a conic, 186, 187
 of planar perspectivity, 248
 of projectivity, 175
 Brianchon's Theorem, 95
 constructions based on, 99, 105, 106, 266
 metric specializations of, 266
 converse of, 97
 degenerate cases of, 100-104
 polar reciprocal of, 120
 Bundle, 7, 15, 245
 Center, of a central conic, 135
 of homology, 74
 of hyperboloid, 160
 of involution on a conic, 186
 of involution on a line, 195
 of planar perspectivity, 248
 of projectivity, 175
 Central projection, 7
 Chasles' Theorem, 109
 Circle, 130
 as a conic through the circular points, 221
 director, 238
 equation of, 143
 major auxiliary, 237
 Circular involution, 133
 Circular points, 220
 Class, 83, 156, 173
 Collineation, 119, 249
 Conc, 83, 90, 148
 asymptotic, 160
 definitions of, 82
 degenerate, 90
 of lines, 83
 of planes, 83
 pole and polar with respect to, 149
 tangent, 155
 theory of, 148
 vertex of, 83
 Configuration, of Desargues, 20
 of Pappus, 76, 98
 Confocal conics, 240
 Conic, axis of, 136, 138
 center of, 135
 central, 135
 construction of, 87, 105, 217, 241
 contact, 155
 definitions of, 82, 109
 degenerate, 90
 determines an involution, 193, 214
 diameter of, 134
 director circle of, 238
 directrix of, 230
 eccentricity of, 233, 234
 focal distance of, 226, 228, 229
 focal properties of, 231-238
 focal radius of, 230
 focus of, 224, 227
 ideal, 149, 160

- Conic, latus rectum of, 231
 major auxiliary circle of, 237
 metric classification of, 91
 metric properties of, 130-146, 234
 metric specializations of, 265, 266
 normal of, 226
 projective properties of, 84, 95,
 108, 191
 vertex of, 137
 Conics, confocal, 240
 Desargues' Theorem on, 191
 equations of, 141-146
 pencil of, 192
 range of, 192
 Conics and cones, relations between,
 83, 89, 148
 Conjugate diameters, 135
 Conjugate elements, 115
 involutions of, 193, 213, 214
 Conjugate imaginary elements, 205
 Conjugate pairs of an harmonic set, 36
 construction of, 60
 Contact conic, 155
 Continuity, principle of, 100
 Correlation, 119, 249
 Cross ratio, *see* Double ratio.
 Cubic, 173
 Cyclic projectivity, 180-184
 of period 2, 181
 of period 3, 183
 period of, 181
 Cylinder, 149

 Desargues' configuration, 20
 Desargues' Theorem,
 on conics, 191
 special cases of, 269
 on triangles, 28
 in complex geometry, 210
 Diagonal triangle, 17
 common to quadrangle and quad-
 rilateral, 104, 123
 of complete quadrangle, 17
 of complete quadrilateral, 17
 Diameter, of a conic, 134
 Diameters, conjugate, 135
 Dilation, 256
 Dimensions of primitive forms, 8

 Directrix, of a conic, 230
 of a regulus, 153
 Displacement, 260
 Double elements, 67, 77, 174, 211
 Double ratio, applications of, 50
 of coincident points, 56
 of four ideal points, 48
 of four lines, 48
 of four ordinary points, 44
 of three ordinary points and one
 ideal point, 47
 projective character of, 45
 Double ratios, relations between, 54
 Doubly corresponding elements, 131,
 181
 Duality, principle of, 13
 in space, 13
 in the bundle, 15
 in the plane, 14

 Eccentricity, 233, 234
 Elation, planar, 248
 Elements, common to conic and line
 or point, 213-217
 complex, 210
 conjugate, of harmonic set, 36
 conjugate, with respect to a conic,
 115
 double, 67, 77, 174, 211
 doubly corresponding, 131, 181
 dual, 13-15
 ideal, 2-5, 43, 220, 254
 imaginary, 204, 205, 209
 properties of, 211
 ordinary, 3, 43, 254
 real and complex, 211
 self-corresponding, 67, 170, 246, 247
 separated pairs of, 10, 11
 undefined, 1
 Ellipse, 92
 axes of, 137
 equation of, 141
 Elliptic involution, 118, 187, 188, 203,
 211
 double elements of, 211
 harmonic representation of, 203
 Equations of conics, 141-146
 Euclidean plane, 120, 254

- Figures, in affine geometry, 255
 in similarity geometry, 257
 in space, 18
 in the bundle, 19
 in the plane, 16, 17
 First degree forms, 174
 First degree problems, 180
 Focal distance, 226, 228, 229
 Focal radius, 230
 Focus, 224
 alternative definition of, 227
 construction of, 227-230
 determination of, 225-227
 properties of, 231-238
 Forms, of second degree, 83, 150, 163
 of third degree, 172
 one-dimensional, 9, 83, 150, 163
 primitive, 7
 two-dimensional, 9, 245
 Fundamental theorem, 71
 extended, 175
 for two-dimensional forms, 253

 Geometric problems, degree of, 180
 Geometry, affine, 254
 complex and real, 210
 Euclidean, 1, 5, 42, 258
 homothetic, 256
 metric, 42, 260
 nature of, 1
 projective, 1, 5, 42, 251
 similarity, 258
 Group of transformations, 250, 251
 affine, 254
 general projective, 251
 homothetic, 256
 rigid motion, 260
 rotation, 260
 similarity, 257

 Harmonic, fourth, 35
 Harmonic pencil, 34
 Harmonic range, 34
 metric properties of, 56-60
 Harmonic representation, of elliptic
 involution, 203
 Harmonic sets, 34

 Harmonic sets, on a conic, 166
 on a regulus, 168
 order of elements in, 39
 projection and section of, 39
 Hexagram, mystic, 99
 Homologue, 71
 Homology, axis of, 74
 center of, 74
 harmonic, 248
 planar, 248
 Homothetic geometry, 256
 Hyperbola, 92
 asymptotes of, 92, 137
 axes of, 137
 equation of, 143
 metric construction of, 140
 rectangular, equation of, 145
 Hyperbolic paraboloid, 159
 rectangular, 161
 Hyperboloid of one sheet, 159
 asymptotic cone of, 160
 center of, 160

 Ideal conic, 149, 160
 Ideal elements, 2-5, 43, 220, 254
 Incidence relations, 6
 for imaginary elements, 205
 Inside, of a conic, 114
 Invariant, 8, 42, 254, 256, 258, 260
 Involution, 118, 132, 183, 186
 absolute, 257
 center of, 186, 195
 circular, 133
 construction of homologous ele-
 ments of, 190, 197, 198
 definition of, 132, 183, 186
 determined by a conic, 193, 214
 elliptic, 118, 187, 188, 203, 211
 harmonic representation of, 203
 oriented, 204
 hyperbolic, 118, 186, 187
 metric construction of, 197
 metric definition of, 196
 metric properties of, 195-199
 on a primitive form, 188
 orthogonal, 133, 257
 parabolic, 118, 188
 Isotropic lines, 221

- Latus rectum, 231
 Line, augmented Euclidean, 3
 ideal, 4, 254
 ordinary, 3, 254
 Line and point conics, 82
 a relation between, 108
 Linear geometric problems, 180
 Line cubic, 173
 Line of contact, 83
 Lines, isotropic, 221
 parallel, 3, 254
 perpendicular, 133, 257
 vanishing, 254

 Metric geometry, 42, 260
 Metric specializations, 43, 56, 91,
 107, 220
 of conics, 107, 265, 266
 Mystic hexagram, 99

n-line, in the bundle, 19
 in the plane, 16, 17
n-plane, in space, 18
 in the bundle, 19
n-point, in space, 18
 in the plane, 17
 Net of rationality, 69
 Normal, 226

 "On language," 5
 One-dimensional forms, 163
 of first degree, 174
 of second degree, 83, 150
 One-to-one correspondence, 9, 23
 Order, of a point conic, 83
 of a point cubic, 173
 of a ruled quadric, 156
 on a primitive form, 10, 11
 Ordinary elements, 3, 43, 254
 Oriented elliptic involution, 204
 Orthogonal involution, 133, 257
 Orthogonal line reflection, 259
 Outside, of a conic, 114

 Pappus configuration, 76, 98
 Parabola, 92
 axis of, 138, 139
 diameters of, 134

 Parabola, equation of, 145
 vertex of, 138
 Paraboloid, hyperbolic, 159
 Pascal's Theorem, 95
 constructions based on, 98, 105,
 107, 265
 metric specializations of, 107, 265
 converse of, 97
 degenerate cases of, 100-104
 polar reciprocal of, 120
 special cases of, 104, 267-269
 Pedal line, 240
 Pencil of conics, 192
 Perpendicular lines, 133, 257
 Perspectivities, chain of, 64
 Perspectivity, axis of, 26, 27, 248
 between figures, 27
 between like forms, 26
 between one-dimensional forms, 26,
 27, 164
 between two-dimensional forms,
 246
 between unlike forms, 25
 center of, 26, 27, 248
 examples of, 25-27, 164-166, 246
 in extended sense, 164, 246
 planar, 246
 Planar field, 245
 Planar perspectivity, 246
 axis and center of, 248
 Plane, augmented Euclidean, 4
 ideal, 5
 Point, ideal, 3, 254
 ordinary, 3, 43, 254
 Point and line conics, 82
 a relation between, 108
 Point cubic, 173
 Point of contact, 82, 153
 Points, circular, 220
 Polar line, of a line, 159
 of a point, 111
 Polar-plane, of a line, 149
 Polar plane, of a point, 156
 Polar reciprocal, 120
 Polar reciprocation, 119
 Polar system, 120
 Pole, of a line, 111
 of a plane, 156

- Pole and polar, construction of, 114
 properties of, 112, 113
 with respect to a cone, 149
 with respect to a conic, 111
 with respect to a ruled quadric, 156
 Pole-line, of a plane, 149
 Poncelet, 65, 100, 263
 Primitive forms, 7
 dimensions of, 8
 involution on, 188
 projectivity on, 67, 77, 177
 superposed, 64
 Principle of continuity, 100
 Problems, geometric, 180
 Product, of transformations, 250
 Projection, axial, 9
 central, 7
 Projective tetrads, 69
 Projective triads, 68
 Projectivity, axis of, 175
 between one-dimensional forms, 65,
 168, 169
 between two conics, 169
 between two-dimensional forms,
 249
 center of, 175
 classified, 77, 176
 construction of, 76, 169, 253
 cyclic, 180-184
 definition of, 65
 extended, 168, 249
 examples of, 66, 67, 77, 169-172
 on a conic, 174
 on a one-dimensional form, 67, 77,
 174, 177
 superposed, 67, 77, 174, 177
 Projector, 7, 164
 Property, affine, 254
 homothetic, 256
 metric, 42, 260
 projective, 8, 42
 similarity, 258
 Quadrangle, complete, 17
 diagonal triangle of, 17
 Quadrangles, perspective, 31
 Quadratic geometric problems, 180
 Quadric, 161
 ruled, 151
 contact conic of, 155
 metric classification of, 159
 order and class of, 156
 point of contact of, 153
 pole and polar of, 156-159
 tangent cone to, 155
 tangent line to, 154
 tangent plane to, 153
 Quadrilateral, complete, 17
 diagonal triangle of, 17
 Range of conics, 192
 Reguli, conjugate, 153
 Regulus, 150
 Resultant, of transformations, 71, 250
 Rigid motion, 42, 260
 Rotation, 260
 Ruled quadric, *see* Quadric, ruled.
 Ruled surface, 149
 Ruling, of a regulus, 153
 Second degree forms, 83, 150, 163
 Second degree problems, 180
 Section, 7, 164
 Segment, 11
 directed, 43
 Self-corresponding elements, 67, 170,
 246, 247
 Self-polar figures, 121
 Self-polar triangle, 121
 common to two conics, 121
 Sense, 10, 11
 see Order.
 Similarity geometry, 258
 Similarly projective point-rows, 139
 Simson line, 240
 Stretch, 256
 Supplementary reading, 263
 Surface, developable, 149
 ruled, 149
 Tangent cone, 155
 Tangent line, 82, 154
 imaginary, 215
 Tangent plane, 83, 153
 Tetrads, projective, 69
 Third degree forms, 172

- Transformations, dilation, 256
 group of, 250, 251
 homothetic, 256
 metric, 260
 product of, 250
 quadratic, 124-128
 resultant of, 250
 rotation, 260
 similarity, 258
 stretch, 256
 translation, 256
 Translation, 256
 Triads, projective, 68
- Triangle, pedal line of, 240
 self-polar, 121
 Simson line of, 240
 theorems, 28, 104, 119, 238-240
 Trigonometric formulas, 53
 Two-dimensional forms, 9, 245
 perspective, 246
 projective, 249
- Vanishing lines, 254
 Vertex, of a cone, 83
 of a conic, 137
 von Staudt, 65, 204, 263

PRINCIPAL THEOREMS

THEOREM	PAGE	THEOREM	PAGE
I.	28	VIII.	95
II.	31	IX.	108, 148
III.	39	X.	109
IV.	68	XI.	115
V.	71	XII.	175
VI.	72	XIII.	248
VII.	73	XIV.	253

