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PART II

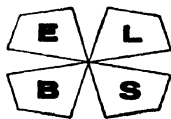
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DYNAMICS

BY
A. S. RAMSEY

PART II



THE ENGLISH LANGUAGE BOOK SOCIETY
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PREFACE

It may well be considered that some explanation should be offered for adding to the number of books on Dynamics. The book which I wrote some eight years ago for the use of scholarship candidates and First Year students at the Universities appeared at the time to fill a gap, and it was then all that I proposed to write on the subject. Since then, however, I have had a good many inquiries as to whether there was not more to follow, particularly as the *Treatise on Dynamics* by the late Dr Besant, the last edition of which I prepared for the press more than twenty years ago, has long been out of print. So this book has been written as a continuation of the earlier one, which will now have to be called Part I while this is Part II. It amplifies the Dynamics of a Particle of Part I, extending it to problems in three dimensions, and it contains a much fuller treatment of Rigid Dynamics than was possible in Part I, carrying the subject in three dimensions sufficiently far to meet the requirements of an Honours Degree Course, including the use of generalized co-ordinates, motion under no forces and the motion of tops and gyrostats.

It is hardly possible to ignore the present-day fashion of using vectors in Dynamics; and, to meet the needs of students who wish to know how to obtain the fundamental equations of dynamics by vectorial methods, I have introduced the subject and taken it thus far in an appendix. I doubt whether it is a fashion which has come to stay, and I am unwilling to require all readers of this book to use vectors. I tried to fit them in, in several places in the body of the book, but eventually decided that an appendix would be the best place for them.

Dr Besant's *Treatise on Dynamics* was published more than fifty years ago. It was a model of conciseness and lucid exposition. Through the kindness of Mr W. L. W. Besant and of the publishers, Messrs G. Bell and Sons, I have been allowed to make free use of this book. I have drawn upon it considerably in Chapters I, II and IV of my book, and for examples in all the chapters. I collected some four hundred additional examples for the last edition of Dr Besant's treatise and as the present output of examples, at Cambridge at any rate, is so very much less than thirty years ago, in order to

get an adequate number it is necessary to go back to old collections. It may not be generally known that at Cambridge, since the war, there has been but one Annual Intercollegiate Examination in Mathematics with but four papers. It is now called the Preliminary Examination in Mathematics and is conducted by the University. These four papers take the place of about eighty papers which, in the days of the *Old Tripos*, were set annually in College and Intercollegiate Examinations for Honours students in Mathematics.

The source of the examples, *Tripos*, College, Intercollegiate or Preliminary, has been indicated when possible.

Dr Besant's book used to be regarded as one of the easier text-books on Dynamics. Readers are warned that they are likely to find this book difficult by comparison, at any rate in the early chapters, unless they bear in mind that the easy parts of the subject are contained in Part I, and begin by studying it carefully. Thus Chapter I, on Central Orbits, in this volume, is supplementary to, and should be studied in conjunction with, Chapter XII in Part I.

The working of problems in advanced Dynamics generally presents difficulties to the average student. But the problems which I have inserted in the Chapters to illustrate the book-work have been chosen, when possible, from recent examination papers and, to that extent, they represent the kind of knowledge which present-day students are expected to possess; and the solutions of seventy of them are given in the text. Further, for the convenience of students working alone, by drawing a line in each of the sets of examples at the ends of the chapters, I have made a rough division of them into easier and harder problems.

The proofs have been read by my friend Dr S. Verblunsky and I wish to acknowledge with gratitude my debt to him for his careful and valuable criticism and the many improvements which he has suggested; and at the same time I wish to thank the printers and readers of the University Press for their excellent work in setting up the book and eliminating mistakes.

A. S. RAMSEY

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Chapter I

TWO-DIMENSIONAL PROBLEMS. CENTRAL ORBITS

1.1. As this book is to be regarded as a continuation of my book on Dynamics for first year students and others, it will be assumed that readers are acquainted with so much of the subject as is therein contained. So far as is practicable, repetition will be avoided and, when necessary, references will be given to the earlier volume, indicated by the letters **Pt. I.**

In the present chapter we shall consider some properties central orbits, particularly the problem of time in an elliptic orbit and the effects of small disturbances on elliptic motion.

1.2. To find the orbit for a given law of force. We have seen in **Pt. I, 12.21** that the force per unit mass directed to the origin under which a central orbit is described is given by the formulae

$$f = \frac{h^2}{p^3} \frac{dp}{dr} = h^2 u^2 \left(\frac{d^2 u}{d\theta^2} + u \right) \dots\dots\dots(1),$$

where $u = 1/r$, p is the perpendicular from the origin to the tangent and h is the constant moment of the velocity about the origin. We shall consider some applications of these formulae.

1.21. Examples. (i) Find the law of force when the orbit is

$$r = a \sin n\theta.$$

Here $dr/d\theta = na \cos n\theta$, and

$$\begin{aligned} \frac{1}{p^3} &= \frac{1}{r^3} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \\ &= \frac{n^2 a^2}{r^4} - \frac{n^2 - 1}{r^2}. \end{aligned}$$

Hence

$$\frac{1}{p^3} \frac{dp}{dr} = \frac{2n^2 a^2}{r^5} - \frac{n^2 - 1}{r^3}$$

and

$$f = \frac{h^2}{p^3} \frac{dp}{dr} = h^2 \left(\frac{2n^2 a^2}{r^5} - \frac{n^2 - 1}{r^3} \right).$$

(ii) *A particle moves under a central force ϕ/r^2 , where ϕ is a function of θ only. Shew that the orbit can be found by a quadrature.*

Prove that if $\phi = \mu (\cos 2\theta)^{-\frac{3}{2}}$ the possible orbits are conics touching two straight lines. [I. 192]

In this case from 1.2 (1) the equation for the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{\phi}{h^2} \dots\dots\dots(1),$$

where ϕ is a function of θ .

To find a particular integral, using D for $d/d\theta$, we have

$$\begin{aligned} u &= \frac{1}{D^2 + 1} \frac{\phi}{h^2} = \frac{1}{2i} \left(\frac{1}{D-i} - \frac{1}{D+i} \right) \frac{\phi}{h^2} \\ &= \frac{1}{2ih^2} \left\{ e^{i\theta} \int e^{-i\theta} \phi d\theta - e^{-i\theta} \int e^{i\theta} \phi d\theta \right\} \\ &= \frac{1}{2ih^2} \left\{ \int_0^\theta e^{i(\theta-\theta')} \phi(\theta') d\theta' - \int_0^\theta e^{-i(\theta-\theta')} \phi(\theta') d\theta' \right\} \\ &= \frac{1}{h^2} \int_0^\theta \sin(\theta - \theta') \phi(\theta') d\theta'. \end{aligned}$$

Therefore the complete solution is

$$u = A \cos \theta + B \sin \theta + \frac{1}{h^2} \int_0^\theta \sin(\theta - \theta') \phi(\theta') d\theta' \dots\dots\dots(2).$$

When $\phi = \mu (\cos 2\theta)^{-\frac{3}{2}}$, we have

$$\int_0^\theta \frac{\sin(\theta - \theta')}{\cos^{\frac{3}{2}} 2\theta'} d\theta' = \left[\frac{\sin \theta \sin \theta' - \cos \theta \cos \theta'}{\cos^{\frac{1}{2}} 2\theta'} \right]_0^\theta,$$

so that
$$\frac{1}{r} = u = A \cos \theta + B \sin \theta - \frac{\mu}{h^2} \cos^{\frac{1}{2}} 2\theta,$$

or
$$1 = Ax + By - \frac{\mu}{h^2} \sqrt{(x^2 - y^2)},$$

i.e.
$$(Ax + By - 1)^2 = \frac{\mu^2}{h^4} (x^2 - y^2),$$

and this equation represents conics touching the lines $x \pm y = 0$, where the lines $Ax + By - 1 = 0$ cut them.

(iii) *A particle which moves under an attraction to a fixed point O is projected from a point A in a direction perpendicular to OA with velocity v . It is given that, when the central acceleration is μ/r^n , the equation to its path is $u = 1/r = f(\theta, \alpha)$, where f is a function of θ and α , α is the distance OA and θ is measured from OA . If the central acceleration is changed to $\mu/r^n + \lambda/r^3$, the initial conditions remaining unchanged, prove that the equation to its path is*

$$u = kf(\gamma\theta, k\alpha),$$

where $k^{n-3} = \gamma^2 = 1 - \lambda/(a^2 v^2)$.

[M. T. 1935]

The differential equation of the path in the first case is

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^{n-2}}{h^2} \dots\dots\dots(1),$$

where, from the initial conditions, $h = av$.

This equation is satisfied by $u = f(\theta, a)$, so that

$$f''(\theta, a) + f(\theta, a) = \frac{\mu}{h^2} f(\theta, a)^{n-2} \dots\dots\dots(2),$$

and since the point A is an apse on the path we must have

$$\frac{1}{a} = f(0, a) \dots\dots\dots(3)$$

and

$$0 = f'(0, a) \dots\dots\dots(4).$$

In the second case the differential equation is

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^{n-2}}{h^2} + \frac{\lambda u}{h^2} \dots\dots\dots(5),$$

or

$$\frac{d^2u}{d\theta^2} + \gamma^2 u = \frac{\mu u^{n-2}}{h^2} \dots\dots\dots(6),$$

where

$$\gamma^2 = 1 - \lambda/h^2 = 1 - \lambda/(a^2 v^2).$$

A comparison of (6) and (1) suggests that we may find a solution of (6) if we put $\gamma\theta$ instead of θ in the solution of (1) and adjust the constants. We therefore substitute in (6) as a trial solution

$$u = kf(\gamma\theta, a') \dots\dots\dots(7),$$

where k, a' are constants. We find that we must have

$$\gamma^2 f''(\gamma\theta, a') + \gamma^2 f(\gamma\theta, a') = \frac{\mu k^{n-3}}{h^2} f(\gamma\theta, a')^{n-2};$$

and from (2) we see that this is true if $k^{n-3} = \gamma^2$.

But we have further to satisfy the conditions that A may be an apse on the locus (7), i.e. that

$$\frac{1}{a} = kf(0, a') \dots\dots\dots(8)$$

and

$$0 = k\gamma f'(0, a') \dots\dots\dots(9).$$

But from (3) $f(0, a') = \frac{1}{a'}$, so that (8) is satisfied if $a' = ka$; and (9)

follows from (4), so that the required equation of the path is

$$u = kf(\gamma\theta, ka).$$

1.22. Examples. 1. If the force in a central orbit is $\mu(u^2 + ku^3)$, shew that the equation of the orbit is

$$u = u_1 \cos^2 \left\{ \frac{\theta}{2} \sqrt{\left(1 - \frac{k}{l}\right)} \right\} + u_2 \sin^2 \left\{ \frac{\theta}{2} \sqrt{\left(1 - \frac{k}{l}\right)} \right\},$$

where $h^2 = \mu l$, the apsidal distances are $1/u_1$, $1/u_2$, and θ is measured from an apse line.

[M. T. 1927]

2. A particle moves in a field of force such that the acceleration is μ/r^2 towards a fixed point. It is projected with velocity V from a point at distance R , at an angle β to the radius vector. Prove that in the subsequent motion

$$r^3 = V^2 - \frac{2\mu}{R} + \frac{2\mu}{r} - \frac{R^3 V^2 \sin^2 \beta}{r^2}.$$

Hence find conditions satisfied by R , V and β , that r may (1) increase steadily to infinity, (2) decrease to a minimum and then increase to infinity, (3) oscillate between two fixed values. [I. 1933]

3. Prove that, in order that it may be possible for a particle to describe a circular orbit under the attraction of a centre of force situated at any given point within the circle, the law of force must be of the form $\mu r/(r^2 + b^2)^3$.

Prove that the group of possible circles in a field of this form consists of the orbits described when the particle is projected from any point distant b from the centre of force in any direction with the velocity from infinity, and that the periods vary as the cubes of the radii of the circles. [M. T. 1919]

4. Prove that if a particle describes a circle of radius b , under a force to a point at distance c ($< b$) from the centre, the force varies as

$$r(r^2 + b^2 - c^2)^{-3}.$$

Shew that the motion can be considered as the orthogonal projection of the motion of a particle describing an ellipse about a focus, and shew how the law of force for the circle could be deduced from the law for the ellipse. [I. 1929]

5. Shew that if the central force is $\mu r/x^3$ the orbit is a conic; and shew further that if the particle is projected at right angles to the radius vector at $(a, 0)$ with speed given by $2v^2 a = \mu$, the orbit is a circle. [M. T. 1909]

6. Shew that, if the law of force is $\mu(r-a)^{-2}$, and the particle is projected with the velocity from infinity from a point at a distance c , ($2a > c > a$), at an angle $2 \cos^{-1}(a/c)^{\frac{1}{2}}$, it will describe the orbit

$$\frac{1}{2}\theta = \tanh^{-1} \sqrt{\left(\frac{r-a}{a}\right)} - \tan^{-1} \sqrt{\left(\frac{r-a}{a}\right)}. \quad [\text{I. 1912}]$$

7. Find an approximation to the orbit described under a central force $Au^2 + Bu^3$, where B is so small that its square may be neglected. [I. 1908]

8. Two particles of masses M and m , attracting according to the law of gravitation, are at great distance apart and move towards one another with relative velocity V , p being the perpendicular distance between the directions of their velocities, which are parallel. Shew that

the least distance d between them during the subsequent motion is given by

$$\frac{1}{d} = \frac{\gamma(M+m)}{p^2 V^2} \left\{ 1 + \sqrt{1 + \frac{p^2 V^4}{\gamma^2 (M+m)^2}} \right\},$$

γ being the constant of gravitation. Find also the angle through which the velocity of m is turned by the encounter. [I. 1927]

9. A foreign body travelling at velocity V relative to the Sun's motion penetrates into the Solar System and approaches to a distance D from the Sun before receding again: prove that when it has left the system its direction of relative motion is deflected through an angle β , given by $\operatorname{cosec} \frac{1}{2}\beta = 1 + DV^2/D_0 V_0^2$, where V_0 is the orbital velocity of a planet revolving at distance D_0 . Calculate the deflection roughly when V is 20 miles per second and D is 67 million miles, the distance of Venus. [M. T. 1918]

10. The components, of masses m_1 and m_2 , forming a double star, describe elliptic orbits round each other in period T , their greatest and least distances apart being D_1 and D_2 : prove that the rotational momentum of the orbital motion of the system is

$$\frac{m_1 m_2}{m_1 + m_2} \frac{\pi}{T} (D_1 D_2)^{\frac{1}{2}} (D_1 + D_2). \quad [\text{M. T. 1918}]$$

11. Two gravitating particles of masses M and m are initially at a great distance apart; M is at rest and m has a velocity v in a direction which passes M at a perpendicular distance σ . Prove that after the encounter when m has again receded to a great distance, M will have acquired a velocity

$$\frac{2vGm}{\sqrt{\{G^2(M+m)^2 + \sigma^2 v^4\}}},$$

where G is the constant of gravitation.

[M. T. 1926]

1.3. Planetary motion. Time in an orbit. The time of passing from one point of a central orbit to another is in all cases determined by the equation

$$r^2 \dot{\theta} = h \dots\dots\dots (1),$$

which expresses the fact that the rate of describing area by the radius vector is constant.

(i) **Parabolic orbit.** The equation of the parabola being

$$2a/r = 1 + \cos \theta, \quad \text{or} \quad r = a \sec^2 \frac{1}{2}\theta,$$

we have

$$\begin{aligned} \frac{dt}{d\theta} &= \frac{r^2}{h} = \frac{r^2}{\sqrt{(2\mu a)}} \quad [\text{Pt. I, 12.51}] \\ &= \sqrt{\left(\frac{a^3}{2\mu}\right)} \sec^4 \frac{1}{2}\theta, \end{aligned}$$

so that, by integration, the time from the vertex is

$$t = \sqrt{\left(\frac{2a^3}{\mu}\right)} \left(\tan \frac{1}{2}\theta + \frac{1}{3} \tan^3 \frac{1}{2}\theta\right) \dots\dots\dots(1).$$

(ii) **Elliptic orbit.** Taking $l/r = 1 + e \cos \theta$ as the equation of the curve, we have

$$\begin{aligned} \frac{dt}{d\theta} &= \frac{r^2}{h} = \frac{r^2}{\sqrt{(\mu l)}} \quad [\text{Pt. I, 12·5}] \\ &= \sqrt{\left(\frac{l^3}{\mu}\right)} \frac{1}{(1 + e \cos \theta)^2} \dots\dots\dots(2). \end{aligned}$$

Let $1 + e \cos \theta = U$, then

$$\begin{aligned} \frac{d}{d\theta} \frac{e \sin \theta}{U} &= \frac{e \cos \theta}{U} + \frac{e^2 \sin^2 \theta}{U^2} \\ &= \frac{U-1}{U} + \frac{e^2 - (U-1)^2}{U^2} \\ &= \frac{e^2 - 1}{U^2} + \frac{1}{U}. \end{aligned}$$

Therefore

$$\begin{aligned} (1-e^2) \int \frac{d\theta}{(1+e \cos \theta)^2} &= -\frac{e \sin \theta}{1+e \cos \theta} + \int \frac{d\theta}{1+e \cos \theta} \\ &= -\frac{e \sin \theta}{1+e \cos \theta} + \frac{2}{\sqrt{1-e^2}} \tan^{-1} \left\{ \sqrt{\left(\frac{1-e}{1+e}\right)} \tan \frac{1}{2}\theta \right\}; \end{aligned}$$

and since $l = a(1-e^2)$, the time from the vertex is

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \left[2 \tan^{-1} \left\{ \sqrt{\left(\frac{1-e}{1+e}\right)} \tan \frac{1}{2}\theta \right\} - \frac{e \sqrt{1-e^2} \sin \theta}{1+e \cos \theta} \right] \dots\dots(3).$$

As in Pt. I, 12·5 the periodic time is $2\pi a^{\frac{3}{2}}/\sqrt{\mu}$, and this follows also from (3) by putting $\theta = \pi$ and doubling the result.

(iii) **Hyperbolic orbit.** In this case we still have equation (2), but e is now greater than unity, and the result of integration is

$$\begin{aligned} (e^2-1) \int \frac{d\theta}{(1+e \cos \theta)^2} &= \frac{e \sin \theta}{1+e \cos \theta} \\ &\quad - \frac{1}{\sqrt{e^2-1}} \log \frac{\sqrt{e+1} + \sqrt{e-1} \tan \frac{1}{2}\theta}{\sqrt{e+1} - \sqrt{e-1} \tan \frac{1}{2}\theta} \dots\dots(4). \end{aligned}$$

Thus, if n is the mean angular velocity and ϵ is the longitude of the D.M.P. at the instant from which time t is measured,

$$\text{the mean longitude} = nt + \epsilon$$

and

$$\text{the mean anomaly} = nt + \epsilon - \varpi.$$

The constant ϵ is called the **epoch**.

The longitude of the apse ϖ is of course a constant of the orbit and there are thus three variables, the longitude θ , the eccentric anomaly u and the mean longitude $nt + \epsilon$, which may be expressed in terms of one another.

1.41. To connect the true longitude θ with the eccentric anomaly u . We have

$$\begin{aligned} SP &= a - eCN \\ &= a(1 - e \cos u), \end{aligned}$$

but

$$\frac{l}{SP} = 1 + e \cos(\theta - \varpi),$$

so that

$$a(1 - e^2) = a(1 - e \cos u) \{1 + e \cos(\theta - \varpi)\},$$

or

$$\cos(\theta - \varpi) = \frac{\cos u - e}{1 - e \cos u};$$

whence we get

$$\tan^2 \frac{1}{2}(\theta - \varpi) = \frac{1 + e}{1 - e} \tan^2 \frac{1}{2}u.$$

1.42. To connect the mean longitude $nt + \epsilon$ with the eccentric anomaly u . Since angles described by the D.M.P. are proportional to the times, and therefore proportional to areas described by the true planet, we have

$$\begin{aligned} \frac{nt + \epsilon - \varpi}{2\pi} &= \frac{\text{area } ASP}{\pi ab} \\ &= \frac{b}{a} \cdot \frac{\text{area } ASQ}{\pi ab} \\ &= \frac{b}{a} \cdot \frac{\frac{1}{2}a^2(u - e \sin u)}{\pi ab}, \end{aligned}$$

so that

$$nt + \epsilon - \varpi = u - e \sin u.$$

1.43. To connect the true longitude θ with the mean longitude $nt + \epsilon$. We have

$$r^2 \theta = h = 2\pi ab \left/ \left(\frac{2\pi}{n} \right) \right. = nab.$$

But

$$\frac{a(1 - e^2)}{r} = 1 + e \cos(\theta - \varpi),$$

so that

$$\frac{d\theta}{dt} = \frac{n}{(1 - e^2)^{\frac{3}{2}}} \{1 + e \cos(\theta - \varpi)\}^{\frac{3}{2}};$$

and

$$nt + C = \int \frac{(1 - e^2)^{\frac{3}{2}} d\theta}{\{1 + e \cos(\theta - \varpi)\}^{\frac{3}{2}}}.$$

If we use the condition that the mean anomaly $nt + \epsilon - \varpi$ vanishes when $\theta = \varpi$, we get

$$nt + \epsilon - \varpi = \int_{\varpi}^{\theta} \frac{(1 - e^2)^{\frac{3}{2}} d\theta}{\{1 + e \cos(\theta - \varpi)\}^2}.$$

In practice the eccentricity e is small, and, if we expand the integrand in powers of e and neglect higher powers than e^2 , we have

$$\begin{aligned} nt + \epsilon - \varpi &= \int_{\varpi}^{\theta} (1 - \tfrac{3}{2}e^2) \{1 - 2e \cos(\theta - \varpi) + 3e^2 \cos^2(\theta - \varpi)\} d\theta \\ &= \int_{\varpi}^{\theta} \{1 - 2e \cos(\theta - \varpi) + \tfrac{3}{2}e^2 \cos 2(\theta - \varpi)\} d\theta \\ &= \theta - \varpi - 2e \sin(\theta - \varpi) + \tfrac{3}{4}e^2 \sin 2(\theta - \varpi) \dots\dots\dots(1). \end{aligned}$$

Then to express $\theta - \varpi$ in terms of t we reverse the series by successive approximations.

Thus, if we neglect e , we have, as a first approximation,

$$\theta - \varpi = nt + \epsilon - \varpi \dots\dots\dots(2);$$

then, retaining the first power of e in (1) and substituting (2) in the small term, we get as a second approximation

$$\theta - \varpi = nt + \epsilon - \varpi + 2e \sin(nt + \epsilon - \varpi) \dots\dots\dots(3);$$

and finally retaining in (1) the terms in e and e^2 , and substituting (3) in the term in e , and (2) in the term in e^2 , we have

$$\begin{aligned} \theta - \varpi &= nt + \epsilon - \varpi + 2e \sin\{nt + \epsilon - \varpi + 2e \sin(nt + \epsilon - \varpi)\} \\ &\quad - \tfrac{3}{4}e^2 \sin 2(nt + \epsilon - \varpi); \end{aligned}$$

and on expanding the term in e and neglecting higher powers than e^2 , we get

$$\theta - \varpi = nt + \epsilon - \varpi + 2e \sin(nt + \epsilon - \varpi) + \tfrac{5}{4}e^2 \sin 2(nt + \epsilon - \varpi) \dots\dots(4).$$

1.44. Examples. 1. Two small planets P_1 and P_2 describe coplanar orbits about the Sun, with the same period; the orbit of P_1 is a circle, and that of P_2 is an ellipse of eccentricity e . Treating the centre S of the Sun as fixed, find the inclination of P_2S to the apse line when the angle P_1SP_2 is stationary; and prove, neglecting e^2 , that the interval between this instant and perihelion is

$$\frac{1}{4} - \frac{5e}{8\pi},$$

the period being taken as the unit of time.

[I. 1930]

2. Shew that, if the eccentricity e is small and e^2 is neglected,

$$r = a(1 - e \cos nt + e^2 \sin^2 nt)$$

and

$$\omega = n(1 + \tfrac{1}{2}e^2 \cos 2nt),$$

where r is the radius vector, t the time measured from the nearer apse, $2\pi/n$ the period, a the semi-axis major and ω the angular velocity about the empty focus.

[I. 1905]

3. A body moves in an ellipse under a central force directed to a focus. Shew that by a suitable choice of units the cartesian graph of r , the radius vector from the focus, expressed as a function of the mean anomaly m is a trochoid which is the locus of the focus when the auxiliary circle rolls on a tangent to the circle. Hence or otherwise, find the position in the orbit at which the rate of increase of r is most rapid.

[M. T. 1913]

4. If an orbit can be described about two different centres of force in the same period, prove that the difference of the time of describing any arc is proportional to the difference of the perpendiculars drawn from the ends of the arc to the line joining the centres of force. [I. 1895]

1.5. Stability of an orbit. If, when a particle is slightly disturbed from its orbit, it continues to describe a path never deviating far from its original path, we may call the original motion a **stable** one.

In order to determine whether a given orbital motion is stable or not, we assume that the inverse radius u is equal to $u_0 + x$, where u_0 is the value in the undisturbed orbit and x is a small quantity. If the solution of the equation of motion, neglecting higher powers of x than the first, gives x a small periodic value, the assumption is justified and the motion is stable, otherwise it is unstable. The stability of a circular orbit is discussed in Pt. I, 17.6.

1.6. Orbits in resisting media.

(i) *Central orbit with resistance varying as the velocity.*

Taking the mass as unity, the resistance as ks , and resolving at right angles to the radius vector, we have

$$\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = -ksr \frac{d\theta}{ds}, \quad \text{or} \quad \frac{d}{dt} (r^2 \dot{\theta}) = -kr^2 \dot{\theta}.$$

Therefore $r^2 \dot{\theta} = he^{-kt}$ or $ps = he^{-kt}$ (1),

where h is a constant and p denotes the central perpendicular on the tangent to the path.

Again, if F denotes the central force per unit mass, by resolving along the normal

$$\frac{\dot{s}^2}{\rho} = F \frac{p}{r},$$

so that from (1) $\frac{h^2}{p^2} e^{-2kt} = Fp \frac{dr}{dp}$ (2).

But
$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2,$$

therefore
$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{p^3} \frac{dp}{d\theta} \bigg/ \frac{du}{d\theta} = -\frac{1}{p^3} \frac{dp}{du}$$

$$= \frac{r^2}{p^3} \frac{dp}{dr};$$

and, from (2),
$$\frac{d^2u}{d\theta^2} + u = \frac{F e^{2kt}}{h^2 u^2} \dots\dots\dots(3).$$

(ii) *Central orbit with resistance varying as the square of the velocity.*

With the same notation, taking the transverse acceleration

$$\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = -k \dot{s}^2 r \frac{d\theta}{ds} = -k \dot{s} r \dot{\theta},$$

or
$$\frac{1}{r^2 \dot{\theta}} \frac{d}{dt} (r^2 \dot{\theta}) = -k \dot{s},$$

and therefore
$$r^2 \dot{\theta} = h e^{-ks} \dots\dots\dots(4),$$

where h is a constant, or

$$p \dot{s} = h e^{-ks} \dots\dots\dots(5).$$

Then resolving along the normal we get

$$\frac{\dot{s}^2}{\rho} = F \frac{p}{r}, \quad \text{or} \quad \frac{h^2}{p^2} e^{-2ks} = F p \frac{dr}{dp} \dots\dots\dots(6).$$

But as in (i) above
$$\frac{d^2u}{d\theta^2} + u = \frac{r^2}{p^3} \frac{dp}{dr},$$

so that
$$\frac{d^2u}{d\theta^2} + u = \frac{F e^{2ks}}{h^2 u^2} \dots\dots\dots(7).$$

(iii) *To find the resistance necessary for the description of a given orbit with a given central force.*

Taking R as the resistance per unit mass, and with the foregoing notation and resolutions, we have

$$\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = -R \frac{p}{r} \quad \text{and} \quad \frac{\dot{s}^2}{\rho} = F \frac{p}{r},$$

or
$$\frac{d}{dt} (p \dot{s}) = -R p \quad \text{and} \quad \frac{d}{dt} (p \dot{s})^2 = -2 R p^2 \dot{s}.$$

Therefore
$$R = -\frac{1}{2p^2} \frac{d}{ds} (p^2 \dot{s}^2),$$

or
$$R = -\frac{1}{2p^2} \frac{d}{ds} \left(F p^3 \frac{dr}{dp} \right).$$

1.7. Orbit described under the action of a force with given radial and transverse components. If F , T denote the force to the origin and the transverse force, we have

$$\ddot{r} - r\dot{\theta}^2 = -F \quad \text{and} \quad \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = T \quad \dots\dots\dots(1),$$

or writing $r^2 \dot{\theta} = h$ (not a constant),

$$\dot{h} = Tr \quad \dots\dots\dots(2).$$

Hence $2h\dot{h} = 2Tr^3\dot{\theta}$, and therefore

$$\frac{dh^2}{d\theta} = \frac{2T}{u^3} \quad \dots\dots\dots(3).$$

Then since
$$\frac{d}{dt} = hu^2 \frac{d}{d\theta},$$

$$\begin{aligned} \frac{h^2}{r^3} - F = \ddot{r} &= hu^2 \frac{d}{d\theta} \left\{ hu^2 \frac{d}{d\theta} \left(\frac{1}{u} \right) \right\} \\ &= -hu^2 \frac{d}{d\theta} \left(h \frac{du}{d\theta} \right) \\ &= -h^2 u^2 \frac{d^2 u}{d\theta^2} - \frac{T}{u} \frac{du}{d\theta}, \text{ from (3);} \end{aligned}$$

therefore
$$h^2 u^2 \left(\frac{d^2 u}{d\theta^2} + u \right) = F - \frac{T}{u} \frac{du}{d\theta} \left\{ \dots\dots\dots(4). \right.$$

where
$$h^2 = 2 \int \frac{T d\theta}{u^3}$$

In terms of a field of force of potential energy V per unit mass at (r, θ) , we may put

$$F = -\frac{\partial V}{\partial r} = u^2 \frac{\partial V}{\partial u},$$

and
$$T = -\frac{1}{r} \frac{\partial V}{\partial \theta} = -u \frac{\partial V}{\partial \theta},$$

and (4) takes the form

$$\left. \begin{aligned} h^2 u^2 \left(\frac{d^2 u}{d\theta^2} + u \right) &= u^2 \frac{\partial V}{\partial u} + \frac{du}{d\theta} \frac{\partial V}{\partial \theta} \\ \text{where } h^2 &= -2 \int \frac{1}{u^2} \frac{\partial V}{\partial \theta} d\theta \end{aligned} \right\} \dots\dots\dots (5).$$

1.71. Examples. 1. If the apsidal distances are a and b ($a < b$), prove that the velocity at the former is given by $v_a^2 = \frac{2b^2}{b^2 - a^2} \int_a^b F dr$ and the velocity at any distance r is given by

$$v^2 = \frac{2a^2}{b^2 - a^2} \int_a^r F dr + \frac{2b^2}{b^2 - a^2} \int_r^b F dr,$$

where F is the central force.

[I. 1913]

2. A particle under no external forces is attached to a fixed point by an elastic string of natural length a . If the particle were describing a circle of radius ρ , shew that after a slight disturbance the apsidal angle would be $\pi \sqrt{\{(\rho - a)/(4\rho - 3a)\}}$.

[M. T. 1910]

3. A particle moves in a plane under an acceleration μ/r^2 towards a fixed point. The point is surrounded by a gas, which revolves about it in such a way that the velocity of the gas at any point is $\sqrt{(\mu/r)}$ at right angles to the radius vector. The resistance of the medium produces on the particle a retardation equal to k times the velocity of the particle relative to the medium in its neighbourhood, where k is a constant. If the particle is slightly disturbed from motion in a circle, shew that in the subsequent motion r is of the form

$$a \{1 + \lambda e^{-kt} \cos n(t - t_0)\},$$

where $2\pi/n$ is the period of revolution.

[M. T. 1928]

4. If the attractive force is always zero, and the particle moves in an equiangular spiral of angle α , prove that the transverse force T is given by $T = \mu r^{2 \sec^2 \alpha - 3}$, and that $h = (\mu \sin \alpha \cos \alpha)^{\frac{1}{2}} r^{\sec^2 \alpha}$. [M. T. 1901]

5. Shew that, with radial and transverse forces P and Q , each a function of r and θ , the differential equation of the orbit is

$$\frac{d}{d\theta} \left[\left(Pr + Q \frac{dr}{d\theta} \right) r^4 / \left\{ r^2 + 2 \left(\frac{dr}{d\theta} \right)^2 - r \frac{d^2 r}{d\theta^2} \right\} \right] = 2Qr^3.$$

[M. T. 1908]

6. A particle is describing an equiangular spiral in a resisting medium under a force to the pole and the rate of description of areas is uniformly retarded; prove that the force is $\mu r^{-3} - \lambda r^{-1}$, and find the law of resistance.

7. Shew that, with a central force μr^n , where n is a positive integer, if a particle is projected from an apse at distance a with velocity $\sqrt{(\mu a^{n+1})}$, the path is a circle, and that there is an apsidal angle $\pi/\sqrt{3+n}$.

8. Find the velocity necessary for the description of a circular orbit of radius a under a central force $2\mu a^2 u^5 - \mu u^3$. Shew that the orbit is unstable and that if a slight disturbance takes place inwards the path may be represented by $r = a \tanh \theta$. [I. 1914]

1·8. Disturbed elliptic motion. When a body moving in a central orbit is subject to the action of small impulsive forces the effect is to produce small instantaneous changes in the elements of the orbit. In the case of elliptic motion about a focus, the 'elements' are the length of the major axis, the eccentricity and the longitude of the apse. In planetary motion another element is the inclination of the plane of the orbit to a fixed plane in space, but we shall limit our considerations to disturbances in the plane of the orbit.

When the body is subject to small finite and continuous forces, such as the action of a resisting medium, the orbit is not an ellipse, but the motion can be represented by supposing that the body moves in an ellipse the elements of which are changing continuously. The ellipse is then called the instantaneous orbit, and it may be defined as the orbit which would be described after any instant of time, if the action of the disturbing forces were suspended at that instant.

1·81. Small tangential impulse. Let the velocity v in an elliptic orbit receive a small increment δv . Then since

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} \quad [\text{Pt. I, 12·5}] \dots\dots\dots(1),$$

and r undergoes no instantaneous change, we have

$$2v\delta v = \frac{\mu}{a^2} \delta a,$$

$$\text{or} \quad \delta a = \frac{2va^2}{\mu} \delta v \dots\dots\dots(2).$$

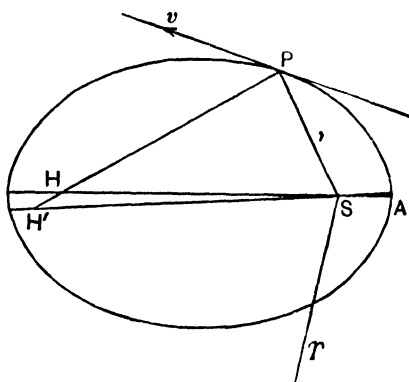
Since the direction of motion and the radius vector are unchanged by the impulse, therefore the increment of the major axis means an increment in the second focal distance PH , i.e. the second focus is displaced to H' where $HH' = 2\delta a$, so that the apse line is turned through an angle $H'SH$ or $\delta\varpi$, where ϖ is the longitude of the apse $\sphericalangle S A$.

Hence

$$\begin{aligned}\delta\omega &= \frac{HH'}{SH} \sin PHS = \frac{\delta a}{ae} \frac{r}{2a-r} \sin PSH \\ &= \frac{2av\delta v}{\mu e} \frac{r}{2a-r} \sin(\theta - \omega) \dots\dots\dots(3),\end{aligned}$$

where θ is the longitude γSP of the body. And from (1) this becomes

$$\delta\omega = \frac{2\delta v}{e} \sqrt{\left(\frac{ar}{\mu(2a-r)}\right)} \sin(\theta - \omega) \dots\dots\dots(4).$$



Again, in the usual notation, $h = vp$, where p is unaltered by the impulse, so that

$$\delta h = p \delta v = h \frac{\delta v}{v} \dots\dots\dots(5).$$

But

$$h^2 = \mu a (1 - e^2),$$

so that

$$2h \delta h = \mu (1 - e^2) \delta a - 2\mu a e \delta e;$$

therefore, from (2),

$$\begin{aligned}\mu a e \delta e &= v a^2 (1 - e^2) \delta v - \frac{h^2}{v} \delta v \\ &= b^2 \frac{\delta v}{v} \left(v^2 - \frac{\mu}{a} \right),\end{aligned}$$

or

$$\begin{aligned}\delta e &= \frac{2b^2}{\mu a e} \frac{\delta v}{v} \left(\frac{\mu}{r} - \frac{\mu}{a} \right) \\ &= \frac{2a(1 - e^2)}{e} \left(\frac{1}{r} - \frac{1}{a} \right) \frac{\delta v}{v} \dots\dots\dots(6).\end{aligned}$$

1.82. Continuous tangential force. The mass of the body being denoted by m , let it be acted upon by a continuous small tangential disturbing force mf . We can represent this force as the limit of a succession of small impulses producing a change of velocity δv at intervals of

time δt , so that $\dot{v} = f$. Then the small rates of change of the elements of the orbit are given by

$$\dot{a} = \frac{2va^2}{\mu} f \dots\dots\dots(1),$$

$$\dot{\omega} = \frac{2f}{e} \sqrt{\left(\frac{ar}{\mu(2a-r)}\right)} \sin(\theta - \varpi) \dots\dots\dots(2)$$

and

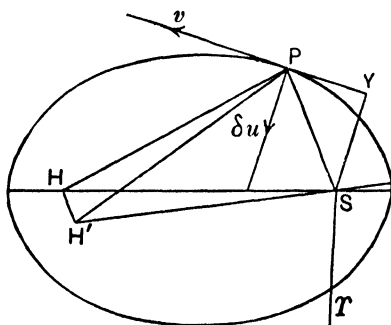
$$\dot{e} = \frac{2(1-e^2)}{e} \left(\frac{1}{r} - \frac{1}{a}\right) f \dots\dots\dots(3).$$

1·83. Small normal impulse. Let an impulse give the body a small velocity δu along the inward normal to the ellipse. To the first order of δu this leaves the square of the velocity unchanged, so since

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} \dots\dots\dots(1),$$

therefore

$$\delta a = 0 \dots\dots\dots(2).$$



But the direction of the velocity, i.e. the tangent to the path, is turned through a small angle $\delta u/v$. And the tangent is equally inclined to the focal distances PS, PH , so if H' is the new second focus we have $HPH' = 2\delta u/v$, and since $\delta a = 0$ therefore $H'P = HP$, and $HH' = \frac{2\delta u}{v} HP$.

$$\text{Hence} \quad \delta\varpi = HSH' = \frac{HH'}{HS} \cos SHP = \frac{\delta u}{ae v} HP \cos SHP$$

$$= \frac{\delta u}{ae v} \{SH + SP \cos(\theta - \varpi)\}, \quad \text{and from (1)}$$

$$= \frac{\delta u}{e} \{2ae + r \cos(\theta - \varpi)\} \sqrt{\left\{\frac{r}{\mu a(2a-r)}\right\}} \dots\dots\dots(3).$$

Again, since

$$h^2 = \mu a(1 - e^2),$$

therefore

$$h \delta h = -\mu a e \delta e \dots\dots\dots(4).$$

But δh is the increment in the moment of the velocity about S , so that

$$\begin{aligned}\delta h &= \delta u \cdot PY \\ &= \delta u r \frac{dr}{ds},\end{aligned}$$

where
$$\frac{l}{r} = 1 + e \cos(\theta - \varpi)$$

and
$$\frac{l}{r^2} \frac{dr}{ds} = e \sin(\theta - \varpi) \frac{d\theta}{ds};$$

but
$$r \frac{d\theta}{ds} = \sin SPY = \frac{p}{r} = \frac{b}{\sqrt{r(2a-r)}},$$

therefore
$$\frac{dr}{ds} = \frac{be \sin(\theta - \varpi)}{l} \sqrt{\left(\frac{r}{2a-r}\right)}$$

and
$$\delta h = \frac{\delta u be \sin(\theta - \varpi)}{l} \sqrt{\left(\frac{r^3}{2a-r}\right)}.$$

Thence, from (4), it follows that

$$\delta e = -\delta u \sin(\theta - \varpi) \sqrt{\left(\frac{r^3}{\mu a(2a-r)}\right)} \dots\dots\dots(5).$$

1.84. Continuous normal force. If f' denotes the accelerating effect of the normal force, we deduce that

$$\dot{a} = 0 \dots\dots\dots(1),$$

$$\dot{\varpi} = \frac{f'}{e} \{2ae + r \cos(\theta - \varpi)\} \sqrt{\left(\frac{r}{\mu a(2a-r)}\right)} \dots\dots\dots(2)$$

and
$$\dot{e} = -f' \sin(\theta - \varpi) \sqrt{\left(\frac{r^3}{\mu a(2a-r)}\right)} \dots\dots\dots(3).$$

1.85. Small radial impulses. Let an impulse in the direction SP give the body a small velocity δu . It is now convenient to represent the velocity in the undisturbed orbit by the components μ/h at right angles to SP and $e\mu/h$ along the ordinate NP [Pt. I, 12.54].

Let PN' be perpendicular to the new position of the apse line. The moment of the velocity about S is unaltered, i.e. h is unaltered, therefore the velocity in the new orbit is compounded of μ/h at right angles to SP and $e'\mu/h$ along $N'P$, where $e' = e + \delta e$. But these two components are equivalent to μ/h at right angles to SP , $e\mu/h$ along NP and δu along SP , so that $e'\mu/h$ along $N'P$ is the resultant of $e\mu/h$ along NP and δu along SP .

$$\begin{aligned}\text{Then we have } \delta\varpi &= NSN' = NPN' = \frac{\delta u \sin SPN}{e\mu/h} \\ &= -\frac{\delta u \cos(\theta - \varpi)}{e} \sqrt{\left(\frac{a(1-e^2)}{\mu}\right)} \dots\dots\dots(1).\end{aligned}$$

Again, by resolving in the direction NP ,

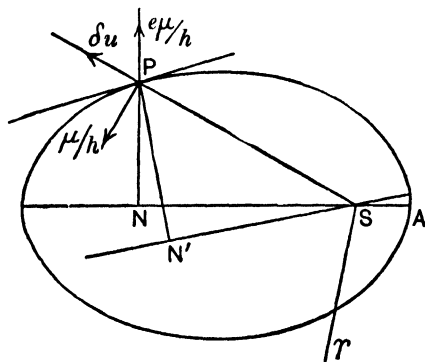
$$\frac{e'\mu}{h} \cos \delta\varpi = \frac{e\mu}{h} + \delta u \cos SPN,$$

whence

$$\frac{\mu}{h} \delta e = \delta u \sin(\theta - \varpi),$$

or

$$\delta e = \delta u \sin(\theta - \varpi) \sqrt{\left(\frac{a(1-e^2)}{\mu}\right)} \dots\dots\dots(2).$$



Lastly

$$h^2 = \mu a (1 - e^2),$$

therefore

$$0 = (1 - e^2) \delta a - 2ae \delta e$$

and

$$\delta a = 2e \delta u \sin(\theta - \varpi) \sqrt{\left(\frac{a^3}{\mu(1-e^2)}\right)} \dots\dots\dots(3).$$

1.86. Continuous radial force. If f_1 denotes the accelerating effect of the radial force, we deduce that

$$\dot{\varpi} = -\frac{f_1 \cos(\theta - \varpi)}{e} \sqrt{\left(\frac{a(1-e^2)}{\mu}\right)} \dots\dots\dots(1),$$

$$\dot{e} = f_1 \sin(\theta - \varpi) \sqrt{\left(\frac{a(1-e^2)}{\mu}\right)} \dots\dots\dots(2)$$

and

$$\dot{a} = 2ef_1 \sin(\theta - \varpi) \sqrt{\left(\frac{a^3}{\mu(1-e^2)}\right)} \dots\dots\dots(3).$$

1.87. Small transverse impulse. Let an impulse at right angles to SP increase the velocity component μ/h by a small amount δu . Then in the figure of 1.85, the components of velocity in the new orbit are μ/h' at right angles to SP and $e'\mu/h'$ along $N'P$, where $h' = h + \delta h$ and $e' = e + \delta e$, and the increment $\delta h = r \delta u$.

Resolving along and at right angles to SP , we have

$$\frac{e'\mu}{h'} \cos(SPN - \delta\varpi) = \frac{e\mu}{h} \cos SPN$$

and

$$\frac{\mu}{h'} - \frac{e'\mu}{h'} \sin(SPN - \delta\varpi) = \frac{\mu}{h} + \delta u - \frac{e\mu}{h} \sin SPN.$$

These equations reduce to

$$\delta e \cos SPN + e \delta \varpi \sin SPN - \frac{er}{h} \delta u \cos SPN = 0$$

and $-\delta e \sin SPN + e \delta \varpi \cos SPN - \delta u \left(\frac{h}{\mu} + \frac{r}{h} - \frac{er}{h} \sin SPN \right) = 0,$

giving $\delta e = \delta u \left[\frac{r}{h} \{e + \cos(\theta - \varpi)\} + \frac{h}{\mu} \cos(\theta - \varpi) \right] \dots\dots\dots(1)$

and $\delta \varpi = \frac{\delta u \sin(\theta - \varpi)}{\mu e} \left(h + \frac{\mu r}{h} \right) \dots\dots\dots(2).$

Then since $h^2 = \mu a (1 - e^2),$

therefore $2hr \delta u = \mu (1 - e^2) \delta a - 2\mu ae \delta e$

and this leads to $\delta a = \frac{2a^2}{h} \delta u \{1 + e \cos(\theta - \varpi)\} \dots\dots\dots(3).$

The effects of a continuous transverse force may now be written down as in 1.86.

1.88. Small change in the magnitude of the absolute force.

Let μ receive an increment $\delta\mu$, the velocity v remaining unchanged. Then since

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right),$$

therefore $\delta a = -\frac{a(2a-r)}{r} \frac{\delta\mu}{\mu} \dots\dots\dots(1).$

Since the direction of motion is unaltered, the new second focus H' lies on PH and is such that $HH' = 2\delta a$.

Then, using the figure of 1.81,

$$\delta \varpi = \frac{HH'}{SH} \sin PHS$$

$$= \frac{\delta a}{ae} \sin PHS$$

and from (1) $= -\frac{\delta\mu}{e\mu} \cdot \frac{PH}{PS} \sin PHS$

$$= -\frac{\delta\mu}{e\mu} \sin(\theta - \varpi) \dots\dots\dots(2).$$

Again, $h^2 = \mu a (1 - e^2)$ and h is unchanged; therefore

$$2\mu ae \delta e = (\mu \delta a + a \delta \mu) (1 - e^2),$$

which, from (1), gives $\delta e = \frac{1 - e^2}{e} \left(1 - \frac{a}{r} \right) \frac{\delta\mu}{\mu} \dots\dots\dots(3).$

If the change in μ is continuous at a rate $n\mu$ per unit time, we get

$$\dot{a} = -na(2a-r)/r \dots\dots\dots(4),$$

$$e\dot{\omega} = -n \sin(\theta - \omega) \dots\dots\dots(5)$$

and

$$\dot{e} = \frac{n(1-e^2)}{e} \left(1 - \frac{a}{r}\right) \dots\dots\dots(6).$$

1·89. Examples. 1. When a body moving in an ellipse about the focus is at an end of the latus rectum a small velocity u is communicated to it parallel to the axis. Prove that the change of eccentricity is ehu/μ , and that the apse line is turned through an angle $2hu/e\mu$.

2. In an elliptic orbit about the focus, if the direction of motion at any point is turned through a small angle $\delta\alpha$, shew that the apse line is turned through an angle $\frac{\delta\alpha}{e^2} \left(1 + e^2 - \frac{r}{a}\right)$.

3. A particle is describing an ellipse under a force to the focus when at a certain point the intensity (μ) of the force is increased by $(1/n)\text{th}$. Shew that the semi-major axis a is increased by $a^2v^2/n\mu$, where v is the velocity; and find the position of the point (1) if the eccentricity is unaltered, (2) if the position of the apse line is unaltered. Can both these conditions be fulfilled at one point?

4. A planet is describing an ellipse round the Sun, and is struck by a small projectile from the Sun. Shew that the distance of perihelion will be increased if the planet is at a point on its path at which the distance from the Sun is decreasing. Find expressions for this change and for that in the direction of the major axis. [I. 1920]

5. If the velocity at any given point in an ellipse described about a centre of force in the focus be suddenly increased by a small given quantity, prove that the alteration in the position of the transverse axis is proportional to the cosine of the angle which the direction of motion at the point makes with the radius vector.

6. If S, E denote the masses of the Sun and Earth, and the Sun receives a small increment of mass m , without change of velocity, when the Earth's angular distance from perihelion is θ , shew that the apse line is turned through an angle $m \sin \theta / e(S + E)$, where e denotes the eccentricity of the Earth's orbit. [M. T. 1911]

7. A body moving in an elliptic orbit about a centre of force in the focus is acted upon by a small impulse in a direction perpendicular to the radius vector, so that the area described in a unit of time is increased in the ratio $1+n:1$. Shew that the corresponding changes in the elliptic elements are given by the equations

$$\delta e = n(2 \cos v + e + e \cos^2 v),$$

$$e \delta \omega = n(2 \sin v + e \sin v \cos v),$$

$$\frac{\delta a}{a} = \frac{2n(1 + e \cos v)^2}{1 - e^2},$$

where v is the true anomaly of the body at the instant of disturbance.

EXAMPLES

1. Find the law of force to the pole when the path is the cardioid $r = a(1 - \cos \theta)$; and prove that, if F be the force at the apse, and v the velocity,

$$3v^2 = 4aF.$$

2. A body is describing an ellipse under the action of a force to a focus. When the body is at one extremity of the minor axis the law of force is changed without instantaneous change of the magnitude of the force or of the velocity; if the force now vary as the distance, prove that the periodic time is the same as before.

3. A particle describes a rectangular hyperbola under the action of a force to the centre; prove that if the radius vector, during the time t after leaving the apse, passes over the angle θ ,

$$\tan \theta = \tanh \sqrt{\mu} t,$$

μr being the force at the distance r on unit mass.

4. If any curve is described by a particle under the action of a force tending to a fixed point O , prove that the velocity of the foot of the perpendicular on the tangent to the path varies inversely as the chord of curvature through O .

5. OY is the perpendicular from a fixed point O on the tangent to a curve at any point P . If the curve is such that PY is constant, and a particle describes it under the action of a force to O , prove that the force varies as OP/OY^4 .

6. A curve described by a particle under the action of a central force is such that, if at any moment the component velocity along the radius vector be destroyed by an impulse along the radius vector, the particle will proceed to describe a circle. Prove that the curve is a reciprocal spiral.

7. Prove that the law of force under which the pedal of $p = f(r)$ can be described is

$$h^2 \left\{ \frac{2r^2}{p^5} - \frac{r}{p^4} \frac{dr}{dp} \right\}.$$

If p varies as r^n , the law of force under which the pedal can be described varies inversely as (distance) $^{5-\frac{2}{n}}$.

8. Prove that, if a particle moves in a smooth tube under the action of forces to centres, the pressure at any point of the tube will vary as

$$\frac{1}{\rho} \left\{ C - \Sigma \frac{1}{p} \frac{d}{dp} (p^3 F) \right\},$$

where dF/dr is the force per unit mass towards any one of the centres, and ρ is the radius of curvature; and hence shew that the pressure at any point of the tube will vary as the curvature, whenever the orbit is such as could be described freely under the action of each of the forces taken separately.

9. A particle, mass m , is acted upon by a central force which is some function of the distance r and by a transverse force $m\mu r$. Prove that, if the particle moves completely round any closed curve, the square of the velocity is increased by 4μ (area of curve).

10. A particle describes the curve $r^2 = a^2 \cos 2\theta$ under the action of a force to the pole. Prove that the square of the force varies as the seventh power of the angular velocity of the radius vector of the particle.

11. A particle P describes a central orbit, centre of force S , and through P is drawn a straight line at right angles to PS , which line touches its envelope in Q ; prove that the velocity of Q varies as

$$\frac{1}{r^2} \left(\frac{d^2 r}{d\theta^2} + r \right),$$

and is constant only when the path of P is a parabola, whose focus is S .

12. A particle is to be projected with given velocity from a given point under a central force varying inversely as the square of the distance so that the apse line shall make a given angle θ with the initial radius vector. Shew that there are two directions of projection making angles with this radius vector whose sum is

$$\theta + \frac{\pi}{2} \quad \text{or} \quad \theta + \frac{3\pi}{2}.$$

13. A particle m is projected from an apse with the velocity from infinity under the attraction of a force $\frac{m\mu}{r^3} \log \frac{r}{a}$ directed to a centre at a distance a ; find the equation of the orbit described.

14. A particle under the action of a central force $\mu(r+a)/r^3$ is projected from an apse at distance a with a velocity which is to that in a circle at the same distance as $1 : \sqrt{2}$; shew that the equation of the orbit is $r(2 + \theta^2) = 2a$, and that the particle will arrive at the origin in time $\pi\sqrt{(a^3/8\mu)}$.

15. A particle moves on a smooth table and is attached by a light inextensible string, which passes through a hole in the table, to an equal particle which hangs vertically. Find the differential equation of the path of the first particle, and prove that, if it is projected from an apse at a distance a from the hole with velocity $\sqrt{(\frac{2}{3}ag)}$, the particle below the table will rise through a distance a . [I. 1897]

16. Prove that if a particle describes an epicycloid under a centre of force at the centre of the fixed circle the force varies as r/p^4 .

17. A particle is projected at a distance a from a centre of force with a velocity $2\sqrt{\mu/a}$, and at an angle $\frac{1}{2}\pi$ with the radius vector, the force being $\mu \left(\frac{3}{r^3} + \frac{a^2}{r^5} \right)$; determine the orbit described, and shew that the time to the centre of force is $\frac{a^{\frac{3}{2}}}{\sqrt{(2\mu)}} \left\{ 2 - \frac{\pi}{2} \right\}$.

18. Two curves are inverse to one another with regard to a point O ; prove that, if they can be described under forces F and F' respectively tending to O , then

$$\frac{r^3 F}{h^2} + \frac{r'^3 F'}{h'^2} = \frac{2}{\sin^3 \phi},$$

where r and r' are corresponding radii vectores, ϕ the angle which r or r' makes with the tangent and h, h' are constants.

Hence find the force tending to a point on the circumference under which a circle can be described.

19. A body is describing an ellipse of eccentricity e under the action of a force tending to a focus and when at the nearer apse the centre of force is transferred to the other focus. Prove that the eccentricity of the new orbit is $e(3+e)/(1-e)$.

20. Two equal particles are revolving in the same direction in the same ellipse, under the action of a force tending to a focus; shew that, if they become rigidly connected when they are at the ends of a chord through the centre of force, they will afterwards move about their centre of gravity with an angular velocity which varies inversely as the length of the chord.

21. Two particles of equal mass moving in opposite directions collide and coalesce at a distance b from the centre of force round which they are moving under an attraction varying inversely as the square of the distance. At the time of their collision one particle is at the nearer and the other at the farther apse of two ellipses both of eccentricity e . Find the major axis of the orbit in which the joint mass moves.

22. A particle describes an ellipse of major axis $2a$ under the action of a force to a focus. When the particle is at an end of the minor axis the centre of force is suddenly moved a small distance αx towards the particle. Shew that the eccentricity e of the orbit is unaltered, and that the apse line is turned through a small angle $\alpha \sqrt{e^2 - 1}$.

23. If a body moving in an ellipse under a force to the centre undergoes a disturbance producing a normal acceleration g , find the variations in the axes of the instantaneous ellipse. Shew that their maxima are given by

$$\frac{\dot{a}}{b} = \frac{\dot{b}}{-a} = \frac{\pm g}{\sqrt{\mu(a+b)}}$$

and distinguish the alternative signs.

24. The co-ordinates $(x, y), (x_1, y_1)$ of the simultaneous position of two equal particles are given by the equations

$$\begin{aligned} x &= a\theta - 2a \sin \theta, & x_1 &= a\theta, \\ y &= a - a \cos \theta, & y_1 &= -a + a \cos \theta; \end{aligned}$$

prove that if they move under their mutual attractions, the law of force will be the inverse fifth power of the distance.

25. Two bodies of masses m_1 and m_2 move under their mutual gravitation; prove that if when m_1 is at rest the velocity of m_2 is at right angles to the line joining the masses and equal to $\sqrt{\gamma(m_1 + m_2)/d}$, where γ is the constant of gravitation and d the distance of the bodies, then the path of m_1 in space is a succession of cycloids, the body coming to rest at a cusp at equal intervals of time. [I. 1892]

26. Two particles of masses m and m' move under the action of their mutual gravitation ($\gamma mm'/r^2$); their momenta are equal and opposite constituting a couple of moment H ; the total energy, kinetic and potential, throughout the motion is E reckoned from a state of rest at an infinite distance apart; prove that the distance of nearest approach of the two particles is a root of

$$x^2 E + \gamma mm' x - \frac{1}{2} H^2 \left(\frac{1}{m} + \frac{1}{m'} \right) = 0,$$

the positive root if E be positive and the smaller positive root if E be negative. [I. 1906]

27. Two masses m, m' are connected by an inextensible string of length a . The end A to which m is attached is made to move with uniform acceleration f in a straight line under the action of a force P in the straight line, and the end B to which m' is attached is made to describe a circle round A with uniform angular velocity ω under the action of a force Q perpendicular to AB . Find P and Q , and prove that the least value of P is $mf - \frac{m'a^2\omega^4}{4f}$, provided $a\omega^2 < 2f$.

28. Two masses m, m' are connected by a light elastic string of natural length l_0 . m is held fixed while m' is whirled in a circle so that the string is stretched to a length l . m is then released. Shew that in the subsequent motion the length of the string never exceeds l , and that if

$$m/(m+m') = l_0^2/l(l+l_0)$$

the string will just become slack. [St John's Coll. 1905]

29. A particle describes an ellipse under a force to the focus S , and SY is a perpendicular upon the tangent to the orbit; shew that the angular velocity of SY is a minimum or maximum when the particle is at the farther apse according as the eccentricity is less or greater than $1/3$.

30. SN is a fixed line through a centre of force S ; PN is the ordinate at any point P of the path of a particle acted on by the central force; find the force when the path always bisects the angle SPN .

31. A particle m is projected from an apse at distance c with velocity $\sqrt{2\mu c^3}/\sqrt{3}$. The force to the centre being $\mu(r^5 - c^4r)$, prove that the orbit is $x^4 + y^4 = c^4$.

32. Two equal particles P and Q , on a smooth horizontal plane, are connected with each other by a stretched inelastic string of length l , which passes through a smooth ring O fixed to the plane; P being projected perpendicularly to PO , find the equation to its path, and shew that, when Q arrives at O , P will have described a right angle about O if the initial distance of P from O be equal to $l \cos \pi/2\sqrt{2}$.

33. A particle of mass m is attached to a fixed point on a smooth horizontal plane by an elastic string of natural length a , whose coefficient of elasticity is mg . It is projected with the velocity $\sqrt{ag}$ in a direction perpendicular to the string which is initially unstretched. Prove that the apsidal distances of its orbit are given by

$$a^2(r^2 - a^2) - r^2(r - a)^2 = 0.$$

34. A rod which is extensible in accordance with Hooke's law is made to rotate, with uniform angular velocity ω , about a line through one extremity perpendicular to its length, and is supposed to remain straight. If a be its natural length, m the mass of unit of length when unstretched, λ its modulus of elasticity, and l its length when rotating, shew that

$$l = \frac{1}{\omega} \sqrt{\frac{\lambda}{m}} \tan \left\{ a\omega \sqrt{\frac{m}{\lambda}} \right\}.$$

35. A heavy particle of mass m is lying on a smooth horizontal table at a given distance c from a small hole in the table and is connected with a heavy particle of mass m' by an inelastic string passing through the hole. The particle m is held at rest, supporting m' , also at rest; find the velocity with which m must be projected along the table, at right angles to the distance c , in order that it may move in the circle of which the hole is the centre.

If the circular orbit be slightly disturbed, prove that the apsidal angle of the nearly circular path of m is

$$\pi \left\{ \frac{m+m'}{3m} \right\}^{\frac{1}{2}}.$$

36. A particle is projected at a distance a from the origin perpendicular to the initial line with velocity $2\sqrt{\mu/a}$ in a field whose potential is $-\mu \cos \theta/r^2$; prove that the orbit is

$$r = a \sec \{ \sqrt{2} \log \tan \frac{1}{2}(\pi + \theta) \}.$$

37. Shew that the time average of the kinetic energy of a planet is equal to the value at the end of the minor axis of its orbit. Shew also that the time mean of the distance is $a(1 + \frac{1}{2}e^2)$. [St John's Coll. 1913]

Chapter II

TANGENTIAL AND NORMAL ACCELERATIONS. MOTION OF PARTICLES AND CHAINS

2.1. In this chapter we shall consider further two-dimensional problems including the application of tangential and normal accelerations to the motion of particles and chains under constraints, motion in rotating tubes and the free motion of a chain.

2.2. Motion of a particle on a rough plane curve. Let s be the arc measured from a fixed point A on the curve, in the direction of motion, to the position of the particle P whose mass is m . Let R be the pressure measured inwards, μ the coefficient of friction and T, N the components of the external force on the particle along the tangent and the outward normal. Then the equations of motion are

$$mv \frac{dv}{ds} = T - \mu R \dots\dots\dots(1)$$

and
$$\frac{mv^2}{\rho} = R - N \dots\dots\dots(2).$$

In the case in which a heavy particle is sliding upwards on a curve in a vertical plane, the equations are

$$mv \frac{dv}{ds} = -mg \sin \psi - \mu R \dots\dots\dots(3)$$

and
$$\frac{mv^2}{\rho} = R - mg \cos \psi \dots\dots\dots(4),$$

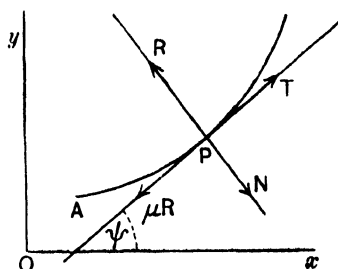
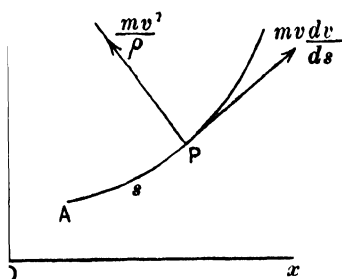
where ψ is the inclination of the tangent to the horizontal.

Eliminating R , we get

$$v \frac{dv}{ds} + \mu \frac{v^2}{\rho} = -g (\sin \psi + \mu \cos \psi),$$

or
$$\frac{dv^2}{ds} + 2\mu v^2 = -2g\rho (\sin \psi + \mu \cos \psi) \dots\dots\dots(5).$$

From the intrinsic equation of the curve, say $s=f(\psi)$, we get $\rho=f'(\psi)$, and the equation (5) has an integrating factor $e^{2\mu\psi}$.



In applying the method of this article it should be noticed that the friction must always oppose the motion, and that in the case of a two-sided constraint (e.g. bead on a wire), if R changes sign during the motion, the motion will cease to be represented by equation (5), because it will become necessary to change the sign of R in (4) but not in (3). This will be exemplified in the next article.

2.21. Example. A bead of mass m slides on a rough circular wire of radius a in a vertical plane, the coefficient of friction being $1/\sqrt{2}$. The bead is projected from the lowest point; shew that in order that it may just reach the highest point the velocity of projection must be

$$\sqrt{\frac{1}{2} \sqrt{6gae} v^2 \left(\frac{\pi}{2} + \alpha \right)},$$

where α is the acute angle $\tan^{-1} \sqrt{2}$.

[M. T. 1931]

If v is the velocity when θ is the angular distance of the bead from its lowest position, the equations of motion, in the early stages of the motion, are

$$mv \frac{dv}{ds} = -mg \sin \theta - \mu R \quad \dots (1)$$

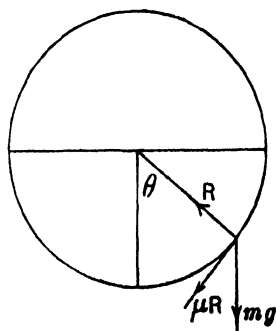
$$\text{and} \quad \frac{mv^2}{a} = R - mg \cos \theta \quad \dots \dots \dots (2)$$

leading to

$$\frac{dv^2}{d\theta} + 2\mu v^2 = -2ga (\sin \theta + \mu \cos \theta) \quad (3),$$

which gives, on integration,

$$v^2 e^{2\mu\theta} = C - 2gae^{2\mu\theta} \frac{3\mu \sin \theta + (2\mu^2 - 1) \cos \theta}{4\mu^2 + 1} \quad \dots (4).$$



For $\mu = \frac{1}{\sqrt{2}}$, this becomes

$$v^2 e^{\sqrt{2}\theta} = v_0^2 - \sqrt{2}gae^{\sqrt{2}\theta} \sin \theta \dots\dots\dots(5),$$

where v_0 is the value of v when $\theta = 0$.

According to the data v decreases steadily and vanishes when $\theta = \pi$, so there will be an angle θ between 0 and π for which $\cos \theta = -v^2/ag$, and from (2) for this value of θ the pressure R vanishes and changes sign. From this point onwards (3) ceases to represent the motion, because (1) still holds good, but instead of (2) we have

$$\frac{mv^2}{a} = -R - mg \cos \theta \dots\dots\dots(6),$$

which with (1) leads to

$$\frac{dv^2}{d\theta} - 2\mu v^2 = -2ga(\sin \theta - \mu \cos \theta) \dots\dots\dots(7),$$

giving, on integration,

$$v^2 e^{-2\mu\theta} = C' - 2gae^{-2\mu\theta} \frac{3\mu \sin \theta + (2\mu^2 - 1) \cos \theta}{4\mu^2 + 1} \dots\dots(8).$$

For $\mu = \frac{1}{\sqrt{2}}$, this becomes

$$v^2 e^{-\sqrt{2}\theta} = C' + \sqrt{2}gae^{-\sqrt{2}\theta} \sin \theta.$$

But v is to vanish when $\theta = \pi$, therefore $C' = 0$, and in the later stages of the motion

$$v^2 = \sqrt{2}ga \sin \theta \dots\dots\dots(9).$$

But the motion being continuous the values of v^2 given by (5) and (9) are the same at the point at which R vanishes, i.e. when v^2 is also $-ga \cos \theta$. Substituting this value in (9) gives

$$-\cos \theta = \sqrt{2} \sin \theta, \quad \text{or} \quad \frac{\cos \theta}{-\sqrt{2}} = \frac{\sin \theta}{1} = \frac{1}{\sqrt{3}},$$

so that $\theta = \frac{1}{2}\pi + \alpha$, where α is the acute angle $\tan^{-1}\sqrt{2}$. Then from (5) and (9) we get

$$v_0^2 = 2\sqrt{2}gae^{\sqrt{2}\theta} \sin \theta = \frac{2}{3}\sqrt{6}gae^{\sqrt{2}(\frac{1}{2}\pi + \alpha)}.$$

2.22. Resisting medium. *Motion of a heavy particle in a medium the resistance of which varies as the square of the velocity.*

Measuring ψ downwards from a horizontal line, the equations of motion are

$$v \frac{dv}{ds} = g \sin \psi - kv^2 \quad \text{and} \quad \frac{v^2}{\rho} = g \cos \psi.$$

The second equation shews that for a given velocity the curvature of the path is independent of the resistance, a theorem which is true for motion in any resisting medium under any forces.

Eliminating v we obtain

$$\cos \psi \frac{d\rho}{d\psi} - 3\rho \sin \psi + 2k\rho^2 \cos \psi = 0,$$

or
$$\frac{d}{d\psi} \left(\frac{1}{\rho} \right) + 3 \tan \psi \cdot \frac{1}{\rho} = 2k.$$

This equation has an integrating factor $\sec^3 \psi$ and gives on integration

$$\frac{\sec^3 \psi}{\rho} = k \{ \tan \psi \sec \psi + \log (\tan \psi + \sec \psi) \} + C.$$

This equation gives the curvature of the path in terms of its inclination to the horizontal.

If ρ_0 is the radius of curvature at the highest point, $C = 1/\rho_0$; and if ρ, ρ' are the radii of curvature at the two points at which the tangent to the path is inclined at the same angle ψ to the horizontal, we have

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{2 \cos^3 \psi}{\rho_0}.$$

2.23. Examples. 1. A ring which can slide on a rough circular wire in a vertical plane is projected from the lowest point with such velocity as will just take it to the horizontal diameter; if the ring returns to the lowest point, shew that its velocity on arrival is to its velocity on departure as $(1 - 2\mu^2 - 3\mu e^{-\mu\pi})^{\frac{1}{2}} : (1 - 2\mu^2 + 3\mu e^{\mu\pi})^{\frac{1}{2}}$,

where μ is the coefficient of friction.

2. A heavy particle slides, from a cusp, down the arc of a rough cycloid, the axis of which is vertical. Prove that its velocity at the vertex is to its velocity at the same point when the cycloid is smooth as

$$\sqrt{(e^{-\mu\pi} - \mu^2)} : \sqrt{(1 + \mu^2)},$$

where μ is the coefficient of friction.

3. A heavy particle moves on a smooth cycloid whose plane is vertical and vertex downwards in a medium resisting as the velocity. Discuss the possible types of motion. [I. 1908]

4. A particle of mass m is projected in a medium offering resistance $mg(w/w_0)^2$ when the velocity is w . If u denotes the horizontal component of velocity, s the length of arc traversed, and ψ the inclination of the direction of motion to the vertical, shew that

$$\frac{du}{ds} = -\frac{g}{w_0^2} u, \quad \frac{d\psi}{ds} = \frac{g}{u^2} \sin^3 \psi,$$

and determine the (s, ψ) equation of the path.

[M. T. 1929]

5. A particle is moving under gravity in a medium which produces a resisting force $f(v)$ per unit mass, where v is the velocity. If ψ is the

angle the direction of motion makes with the horizontal at any time, shew that

$$\frac{1}{v} \frac{dv}{d\psi} - \frac{f(v)}{g \cos \psi} - \tan \psi = 0.$$

If $f(v) = Av^3$, A being constant, find v in terms of ψ . [I. 1924]

6. For a particle moving under gravity in a medium such that the resistance per unit mass is κv^n , where v is the velocity, prove that the equation of the hodograph can be written

$$\left(\frac{1}{r \cos \theta} \right)^n = C - \frac{\kappa n}{g} \int \sec^{n+1} \theta d\theta.$$

A particle of mass m is projected with velocity u at an angle of elevation α , and the initial value of the resistance is known to be λmg , where λ is small; the velocity which the particle will have at the highest point of its path is calculated on the hypotheses that the resistance varies (i) as the cube, (ii) as the first power of the velocity. Prove that the ratio of the calculated values is approximately

$$1 + \frac{2}{3} \lambda \sin^3 \alpha. \quad [\text{I. 1930}]$$

7. If a particle of mass m be acted upon by equal constant forces mf in the directions of the tangent and normal to its path, and if the resistance be $mk^2 v^2 f$, prove that the intrinsic equation of the path is

$$e^{2k^2 s} - 1 = k^2 u^2 (e^{2\psi} - 1),$$

where u is the velocity of projection.

2·3. Two or more centres of force. To determine whether a given orbit can be described under the action of given forces acting simultaneously to two or more centres, it is only necessary to write down the tangential and normal equations of motion and investigate whether the value of v^2 obtained by integrating the tangential equation can be identified with the value of v^2 given by the normal equation.

For example, if the orbit is an ellipse and the forces per unit mass are μ/r^2 and μ'/r'^2 towards the two foci, we have

$$v \frac{dv}{ds} = -\frac{\mu}{r^2} \frac{dr}{ds} - \frac{\mu'}{r'^2} \frac{dr'}{ds} \quad \dots\dots\dots (1)$$

$$\text{and} \quad \frac{v^2}{\rho} = \frac{\mu}{r^2} \sin \psi + \frac{\mu'}{r'^2} \sin \psi \quad \dots\dots\dots (2),$$

where ψ is the inclination of the tangent to each radius vector.

Since in an ellipse $\rho \sin \psi$ is the projection of the radius of

curvature on the focal distance and equal to rr'/a , therefore the equations are equivalent to

$$v^2 = C + \frac{2\mu}{r} + \frac{2\mu'}{r'} \dots\dots\dots(3)$$

and

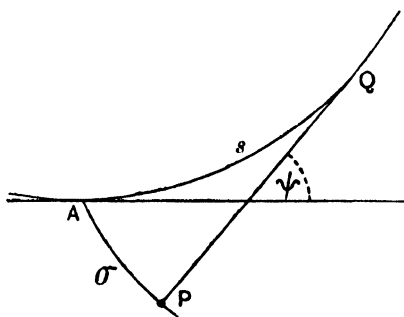
$$v^2 = \frac{\mu}{a} \frac{r'}{r} + \frac{\mu'}{a} \frac{r}{r'},$$

or since $r + r' = 2a$, $v^2 = \frac{2\mu}{r} + \frac{2\mu'}{r'} - \frac{\mu}{a} - \frac{\mu'}{a} \dots\dots\dots(4)$

and the values of v^2 are the same if we take

$$C = -\frac{\mu}{a} - \frac{\mu'}{a}.$$

2.4. Motion of a particle fastened to a light string which is wound round a fixed plane curve. When the string is completely wound up let the particle be at A , and let P be a subsequent position. Then the unwound portion of the string PQ is the tangent to the curve at Q , and $PQ = AQ = s$. Let ψ be the angle which PQ makes with the tangent at A . Let T be



the tension of the string, S and N the components of external force on the particle in the direction of its motion and along PQ . The path of P is an involute of the curve AQ and its radius of curvature is PQ or s , so if σ denotes the arc AP of the path and v is the velocity at P

$$mv \frac{dv}{d\sigma} = S \quad \text{and} \quad \frac{mv^2}{s} = T + N,$$

but $d\sigma = s d\psi$, so the equations are

$$mv \frac{dv}{d\psi} = Ss \quad \text{and} \quad \frac{mv^2}{s} = T + N.$$

2.41. Example. In the case just considered let the curve be a circle and the external force a repulsive force from the centre equal to μ times the distance.

The resolved parts of the external force along and at right angles to PQ are $-\mu s$ and μa , so that

$$mv \frac{dv}{d\sigma} = \mu a \quad \text{and} \quad \frac{mv^2}{s} = T - \mu s.$$

But $d\sigma = s d\psi$ and $s = a\psi$, so that if the motion start from rest at A

$$mv^2 = \mu a^2 \psi^2 = \mu s^2,$$

and $T = 2\mu s$.

2.42. Examples. 1. If a particle moves in an ellipse and has accelerations f_1, f_2 directed to the foci, prove that

$$\frac{df_1}{dr_1} + \frac{2f_1}{r_1} = \frac{df_2}{dr_2} + \frac{2f_2}{r_2},$$

where r_1, r_2 are the focal distances.

[I. 1915]

2. A particle describes the hyperbola $x^2 - y^2 = a^2$ under the action of an attractive force μ/r^2 towards the nearer focus and a repulsive force μ/r^2 from the other focus. Prove that

$$a^3 \phi^2 = 2\mu(1 + 2 \cosh^2 \phi) / \cosh^2 2\phi,$$

where

$$x = a \cosh \phi.$$

[I. 1908]

3. A particle of unit mass moves in a smooth tube in the form of the lemniscate $r_1 r_2 = a^2$, where r_1, r_2 are the distances from two fixed points A, B at a distance $2a$ apart. The particle is acted upon by attractive forces $\mu/r_1^2, \mu/r_2^2$ towards A, B , and its velocity at the node is $2\sqrt{(\mu/a)}$; shew that at any other point the square of the velocity is $2\mu(r_1 + r_2)/a^2$, and that the pressure on the tube at either apse is $6\mu/a^2$. [I. 1907]

4. A particle is free to move under the action of two central forces μ/r^2 directed towards a point S and away from a point H respectively. Prove that, if the particle starts from rest in a position equidistant from S and H , it will oscillate in a semi-ellipse of which S and H are foci. [I. 1906]

5. A string of length πa is wrapped round the circumference of a circle of radius a on a smooth horizontal table. One end of the string is fastened to a point A of the circumference and to the other end is attached a particle of mass m . The particle is projected with velocity V perpendicular to the string from the opposite end of the diameter through A so as to unwind the string from the circle. Prove that the

string is completely unwound after a time $a\pi^2/2V$ and that the tension of the string while unwinding at time t from the beginning of the motion is

$$mV^{\frac{1}{2}}/(2at)^{\frac{1}{2}}. \quad [\text{I. 1913}]$$

2.5. Motion of a particle in a smooth plane tube rotating about a point in its plane. Let the tube rotate about the origin O with angular velocity ω . Let A be a fixed point on the tube and P the position of the particle at time t . Let OA make an angle ϕ with a fixed direction and let $AOP = \theta$ and $OP = r$.

The accelerations of P are

$$\ddot{r} - r(\dot{\theta} + \dot{\phi})^2 \quad \text{along } OP$$

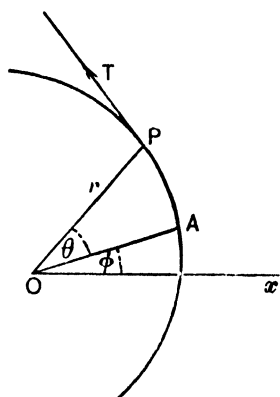
and
$$\frac{1}{r} \frac{d}{dt} \{r^2(\dot{\theta} + \dot{\phi})\}$$

perpendicular to OP .

Or, writing $\phi = \omega t$, the accelerations

are
$$\ddot{r} - r\dot{\theta}^2 - 2r\dot{\theta}\omega - r\omega^2$$

and
$$\frac{1}{r} \frac{d}{dt} (r^2\dot{\theta}) + 2\dot{r}\omega + r\dot{\omega}.$$



The terms independent of ω represent components of acceleration relative to the tube, and are therefore equivalent to $v dv/ds$ along the tangent and v^2/ρ along the inward normal, where v denotes the velocity of the particle relative to the tube and s denotes the arc AP .

The components $-2r\dot{\theta}\omega$ and $2\dot{r}\omega$ are 2ω times the polar components of velocity v turned through a right angle, and are therefore equivalent to $2\omega v$ along the normal inwards.

The term $-r\omega^2$ is equivalent to

$$-\omega^2 r \frac{dr}{ds} \quad \text{along the tangent}$$

and
$$\omega^2 r^2 \frac{d\theta}{ds} \quad \text{along the inward normal;}$$

and the term $r\dot{\omega}$ is equivalent to

$$\dot{\omega} r^2 \frac{d\theta}{ds} \quad \text{along the tangent}$$

and
$$\dot{\omega} r \frac{dr}{ds} \quad \text{along the inward normal.}$$

Hence the tangential and normal components of acceleration are

$$v \frac{dv}{ds} - \omega^2 r \frac{dr}{ds} + \dot{\omega} r^2 \frac{d\theta}{ds}$$

and
$$\frac{v^2}{\rho} + 2\omega v + \omega^2 r^2 \frac{d\theta}{ds} + \dot{\omega} r \frac{dr}{ds};$$

and the equations of motion are

$$m \left(v \frac{dv}{ds} - \omega^2 r \frac{dr}{ds} + \dot{\omega} r^2 \frac{d\theta}{ds} \right) = T \quad \dots\dots\dots(1),$$

$$m \left(\frac{v^2}{\rho} + 2\omega v + \omega^2 r^2 \frac{d\theta}{ds} + \dot{\omega} r \frac{dr}{ds} \right) = N + R \dots\dots\dots(2),$$

where T , N are the tangential and normal forces and R the pressure of the tube inwards.

These equations may also be written

$$mv \frac{dv}{ds} = T + m\omega^2 r \frac{dr}{ds} - m\dot{\omega} r^2 \frac{d\theta}{ds} \quad \dots\dots\dots(3),$$

$$m \frac{v^2}{\rho} + 2m\omega v = N - m\omega^2 r^2 \frac{d\theta}{ds} - m\dot{\omega} r \frac{dr}{ds} + R \dots\dots\dots(4),$$

showing that the motion relative to the tube is the same as if the tube were at rest and two extra forces were applied to the particle, viz.

$m\omega^2 r$ along the radius vector,

$m\dot{\omega} r$ perpendicular to the radius vector in the opposite sense to the rotation.

But if we adopt this method of finding the relative motion, we must observe that the pressure R as given by (4) exceeds by an amount $2m\omega v$ the value which would be obtained by regarding the tube as at rest and applying the extra forces.

2.51. Example. *A particle can move in a smooth plane tube in the form of a parabola and the tube is initially at rest with the particle at the vertex and then suddenly begins to rotate about the focus with uniform angular velocity. Prove that the pressure in any position is proportional to $\sqrt{r}(3\sqrt{a} - 4\sqrt{r})$, where r is the distance from the focus, and $4a$ is the latus rectum.*

[I. 1910]

Since the angular velocity ω is uniform and there are no external forces, the velocity v of the particle relative to the tube is given by

$$v \frac{dv}{ds} = \omega^2 r \frac{dr}{ds},$$

or

$$v^2 = \omega^2 r^2 + C.$$

But initially $r = a$, the particle is at rest and the point A of the tube has a velocity ωa relative to it; therefore $C = 0$ and

$$v = -\omega r,$$

the negative sign because initially $v = -\omega a$.

Then from 2.5 (4)

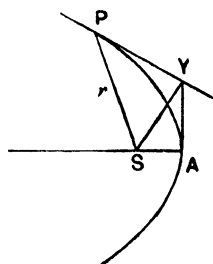
$$R = m \left(\frac{v^2}{\rho} + 2\omega v + \omega^2 r \sin \phi \right),$$

where ϕ is the angle between the radius vector and the tangent.

But in the parabola $p^2 = ar$, $r \sin \phi = p = \sqrt{ar}$ and $\rho = r dr/dp = 2rp/a$.

Hence

$$\begin{aligned} R &= m\omega^2 \left(\frac{ra}{2p} - 2r + p \right) \\ &= \frac{1}{2} m\omega^2 \sqrt{r} (3\sqrt{a} - 4\sqrt{r}). \end{aligned}$$



2.52. Examples. 1. A smooth circular wire is made to rotate about a point O of its circumference with uniform angular velocity ω in a horizontal plane. A small ring P slides on the wire and is attracted towards O by a force varying as its distance from this point. Shew that θ , the angle between OP and the diameter through O , satisfies an equation of the form $\theta = (\lambda - \omega^2) \sin \theta \cos \theta$,

where λ is constant.

[I. 1924]

2. A particle moves on a wire in the form of a circle of radius a , constrained to rotate in its own plane with angular velocity ω about a point of itself. The particle is initially at the extremity of the diameter through the centre of rotation and is projected with velocity $2b\omega$ relative to the wire. Shew that the particle describes a quadrant of the circle in time

$$\frac{\pi a}{4b\omega} \left[1 + \frac{1}{4} \frac{a^2}{b^2} \left(1 - \frac{2}{\pi} \right) \right]$$

approximately, if b is large.

[I. 1931]

3. A bead can move on a smooth circular wire and is initially at rest at a point A . The wire is made to rotate uniformly, in its own plane, with unit angular velocity, about the other end of the diameter through A . Shew that the pressure between the bead and the wire vanishes at a time $\log_e \{ (3 + \sqrt{5})/2 \}$ after the start.

[M. T. 1914]

4. A smooth circular tube is fixed at one point A and contains a particle which is initially at rest at the opposite end of the diameter

through A . The tube is then made to rotate in a horizontal plane with uniform angular velocity ω ; shew that the angle described by the particle about the centre of the tube in time t is $4 \tan^{-1}(\tanh \frac{1}{2}\omega t)$.

5. A particle P is free to move on a smooth circular wire whose centre C revolves with constant angular velocity in the plane of the wire about a fixed point O . Shew that if $CP = 3CO$ and the particle just makes complete revolutions, the pressure between the particle and the wire vanishes when CP makes with OC an angle $\sec^{-1} 3$.

6. A particle is free to move along a smooth curve $r^n = a^n \cos n\theta$, which revolves in its own plane round the origin with uniform angular velocity ω . Prove that, if the particle be initially at rest in the position $\theta = 0$, and there are no external forces, its distance from the origin at time t afterwards is $a \operatorname{sech}^{\frac{1}{n}} n \omega t$. Find also the equation of the path in space. [I. 1908]

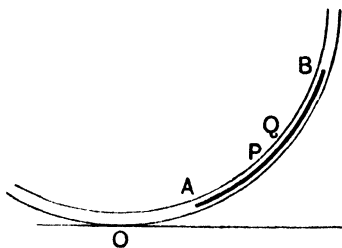
7. A bead is at rest at the vertex of a smooth wire in the shape of a cardioid $r = a(1 + \cos \theta)$. The wire begins to move about its pole with uniform angular velocity ω ; shew that the pressure will vanish when the bead has described an angle $2 \cos^{-1} \frac{1}{2}$ about the pole, relative to the wire, and that the distance of the bead from the pole at time t is

$$r = 2a \operatorname{sech}^2 \frac{1}{2} \omega t. \quad [\text{I. 1900}]$$

8. A smooth plane tube is made to rotate in its own plane about the origin with constant angular velocity ω . A particle moves in the tube under the action of an attraction $\omega^2 r$ per unit mass towards the origin. Prove that the pressure is given by $A + B/\rho$, where A, B are constants, and ρ is the radius of curvature of the tube at the point occupied by the particle. [M. T. 1906]

2.6. Motion of a string or fine chain, under the action of given forces, inside a fixed smooth plane tube. Let AB be the chain of length l , s the length OA of the axis of the tube measured from a fixed point O , and σ the length of any portion AP of the chain.

Let $m \delta \sigma$ denote the mass of a small element PQ of the chain; then if T and $T + \delta T$ denote the tensions at P and Q , $R \delta \sigma$ the pressure of the tube on the element and S, N the tangential and normal forces per unit length of chain, since the tensions on the element PQ are equivalent to forces δT along the tangent and $T \delta \psi$ along



the inward normal [Pt. I, 10.7], the equations of motion of the element PQ are

$$m d\sigma \cdot v \frac{dv}{ds} = \delta T + S \delta \sigma \dots\dots\dots(1)$$

and
$$m d\sigma \cdot \frac{v^2}{\rho} = T \delta \psi + N \delta \sigma + R \delta \sigma \dots\dots\dots(2),$$

where ψ is the inclination of the tangent at P to the tangent at O .

We observe that in these equations v is the velocity of every point of the chain, and has therefore the same value for all points from A to B , and the tension T vanishes at A and B , so by integrating (1) from A to B we get

$$ml v \frac{dv}{ds} = \int_0^l S d\sigma \dots\dots\dots(3).$$

Also $\rho = d\sigma/d\psi$, so that (2) may be written

$$R = m \frac{v^2}{\rho} - \frac{T}{\rho} - N \dots\dots\dots(4).$$

When S and N are given, v is determined by (3), and T is determined by (1) and (3), and then (4) determines R .

2.61. Examples. (i) *A smooth tube forms a quadrant of a vertical circle of radius a with the tangent at the lowest point vertical. A heavy chain of the same length as the tube slides down from rest in the position in which it fills the tube. Find its velocity in any position and shew that its velocity when it is just clear of the tube is*

$$\left\{ ag \left(\frac{\pi}{2} + \frac{4}{\pi} \right) \right\}^{\frac{1}{2}}.$$

Let AB be the chain and CD the tube, $COA = \theta$ and $AOP = \phi$, where PQ is a small element of the chain of mass $ma\delta\phi$. The equation of motion of the element PQ resolving along the tangent at P is

$$ma\delta\phi \cdot a\ddot{\theta} = ma\delta\phi \cdot g \sin(\theta + \phi) + \delta T \dots\dots\dots(1),$$

where T denotes the tension at P .

This equation holds good for all elements of the chain between A and D . Integrating (1) from A to P and noting that $T=0$ when $\phi=0$, we have

$$ma^2\phi\ddot{\theta} = T - mga \cos(\theta + \phi) + mga \cos \theta \dots\dots\dots(2);$$

so that if T_1 denotes the tension at D , where $\phi = \frac{1}{2}\pi - \theta$, we have

$$ma^2 \left(\frac{1}{2}\pi - \theta \right) \ddot{\theta} = T_1 + mga \cos \theta \dots\dots\dots(3).$$

Again, let δy be an element SR of the straight portion of the chain at a depth y below D ; then the equation of motion of this element is, in like manner,

$$m \delta y \cdot a \ddot{\theta} = mg \delta y + \delta T \quad \dots(4),$$

where T is now the tension at S .

This equation holds good for all elements between D and B , so that, by integrating from D to B and noting that $T = T_1$ at D and $T = 0$ at B , we get

$$ma\theta \cdot a \ddot{\theta} = mga\theta - T_1 \dots\dots(5).$$

Now eliminate T_1 between (3) and (5) and we have

$$\frac{1}{2}\pi a \ddot{\theta} = g(\theta + \cos \theta) \dots\dots(6).$$

This is the equation of motion of the whole chain, and, by multiplying by 2θ and integrating, we get

$$\frac{1}{2}\pi a \dot{\theta}^2 = g(\theta^2 + 2 \sin \theta) \dots\dots\dots(7),$$

since $\dot{\theta} = 0$ when $\theta = 0$.

When the chain is just clear of the tube $\theta = \frac{1}{2}\pi$ and the velocity is given by

$$a^2 \dot{\theta}^2 = ag \left(\frac{\pi}{2} + \pi \right) \dots\dots\dots(8).$$

The last result may easily be verified by equating the kinetic energy of the chain to the work done upon it by gravity as it descends from the curved to the straight position.

(ii) A smooth circular tube rotates uniformly in its own plane about a point in its circumference. Investigate the motion of a length of uniform fine chain initially at rest inside the tube in an unsymmetrical position.

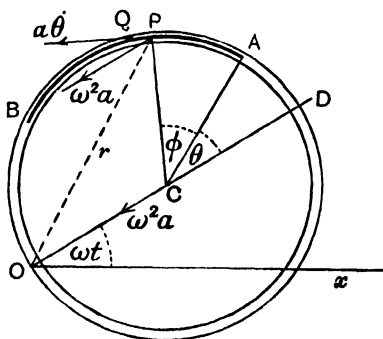
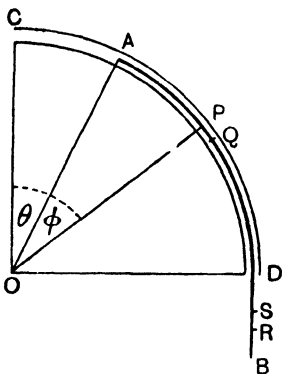
Let a be the radius of the tube and ω its angular velocity about O .

Let PQ be an element of the chain AB . Let OCD be the diameter through O making an angle ωt with a fixed direction Ox . Let $DCA = \theta$, $ACP = \phi$ and $OP = r$.

In determining the motion relative to the tube of the element PQ or $a \delta \phi$ of the chain we may proceed as in 2.5, regarding the tube as at rest if we impose on the element a force $ma \delta \phi \omega^2 r$ along OP , where m is the mass of unit length of chain; or we may make use of relative accelerations. Thus the acceleration of P relative to C is

$$a \frac{d^2}{dt^2} (\theta + \phi + \omega t) = a \ddot{\theta}$$

along the tangent at P , and the acceleration of C is $\omega^2 a$ along CO .



Hence, resolving along the tangent at P , we get

$$m\alpha \delta\phi \{a\ddot{\theta} + \omega^2 a \sin(\theta + \phi)\} = \delta T \quad \dots\dots\dots(1),$$

where T is the tension at P .

Then if AB subtends an angle α at C , we get by integrating (1) from A to B , i.e. from $\phi = 0$ to $\phi = \alpha$,

$$\alpha\ddot{\theta} = \omega^2 \{\cos(\theta + \alpha) - \cos\theta\} \quad \dots\dots\dots(2);$$

giving on integration

$$\alpha\dot{\theta}^2 = C + 2\omega^2 \{\sin(\theta + \alpha) - \sin\theta\} \quad \dots\dots\dots(3).$$

From (2) we see that $\ddot{\theta}$ is negative, so that if the chain starts from rest relative to the tube on one side of OD , the motion is initially towards D .

Further if, initially, $\theta = \beta$ and $\dot{\theta} = 0$, (3) becomes

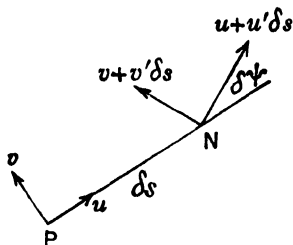
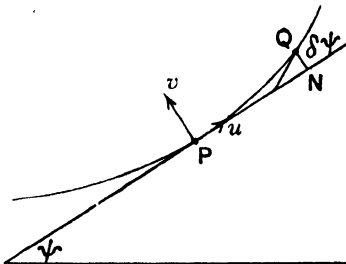
$$\alpha\dot{\theta}^2 = 2\omega^2 \{\sin(\theta + \alpha) - \sin\theta - \sin(\beta + \alpha) + \sin\beta\},$$

shewing that $\dot{\theta}$ also vanishes when $\theta = -(\alpha + \beta)$, so that the chain will perform symmetrical oscillations about D .

2·62. Examples. 1. A uniform chain is placed on the arc of a smooth cycloid whose axis is vertical and vertex upwards. Shew that so long as the chain is wholly in contact with the cycloid the tension at any point of the chain remains constant throughout the motion, and is a maximum at the middle point.

2. A uniform chain of length $2l$ and mass $2ml$ is in a smooth tube in the form of an equiangular spiral which revolves in a horizontal plane about its pole with uniform angular velocity ω . Prove that the tension at any point of the chain is $\frac{1}{2}m\omega^2 \cos^2 \alpha (l^2 - x^2)$, where α is the angle of the spiral and x the distance of the point from the middle point of the chain measured along the arc. [M. T. 1879]

2·7. Motion of a free inextensible chain in a plane under the action of given forces in the plane. At any instant let u, v denote the tangential and normal velocities of the point P of the chain. If s denotes the length of the chain measured from a chosen point up to P , then u, v are functions of s as well as of the time t . Denote a partial differentiation with respect to s by an accent, and a partial differentiation with respect to t by a dot.



Let $PQ = \delta s$ be an element of the chain and N the projection of Q on the tangent at P . Since the chain is inextensible, δs does not vary with the time. The motion of PQ may be identified with that of PN , because QN is always of the order $(\delta s)^2$. Then the motion of N relative to P is circular, with angular velocity ψ . This gives

$$u + u' \delta s - \delta \psi (v + v' \delta s) - u = 0,$$

$$\psi \delta s = v + v' \delta s + \delta \psi (u + u' \delta s) - v,$$

with errors of the second order. On dividing these equations by δs , and taking the limit as $\delta s \rightarrow 0$, we obtain

$$\frac{\partial u}{\partial s} - \frac{v}{\rho} = 0 \quad \dots\dots\dots(1),$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial v}{\partial s} + \frac{u}{\rho} \quad \dots\dots\dots(2).$$

To form the dynamical equations, we have, for the acceleration of P in the direction of the axis from which ψ is measured,

$$\frac{\partial}{\partial t} (u \cos \psi - v \sin \psi) = (\dot{u} - v\psi) \cos \psi - (\dot{v} + u\psi) \sin \psi,$$

where the dot denotes the operation $\partial/\partial t$.

But the direction of this axis is arbitrary, so by taking the values 0 and $\frac{1}{2}\pi$ for ψ , we find that the tangential and normal accelerations of P are

$$\dot{u} - v\psi \quad \text{and} \quad \dot{v} + u\psi.$$

Hence if $m\delta s$ denotes the mass of the elements δs , T the tension at P and $S\delta s$ and $N\delta s$ the components of the external force on the element in the direction of the tangent and normal, then since the effects of the tensions at the ends of an element are equivalent to forces δT along the tangent and $T\delta\psi$ along the inward normal [Pt. I, 10·7], we have

$$m\delta s (\dot{u} - v\psi) = \delta T + S\delta s$$

and

$$m\delta s (\dot{v} + u\psi) = T\delta\psi + N\delta s,$$

or

$$m (\dot{u} - v\psi) = \frac{\partial T}{\partial s} + S \quad \dots\dots\dots(3)$$

and

$$m (\dot{v} + u\psi) = \frac{T}{\rho} + N \quad \dots\dots\dots(4).$$

We can find an equation for the tension by eliminating u and v . Thus from (3) and (4) we have

$$\begin{aligned} \frac{\partial}{\partial s} \left(\frac{1}{m} \frac{\partial T}{\partial s} \right) + \frac{\partial}{\partial s} \left(\frac{S}{m} \right) - \frac{T}{m} \left(\frac{\partial \psi}{\partial s} \right)^2 - \frac{N}{m} \frac{\partial \psi}{\partial s} \\ = \frac{\partial}{\partial s} (\dot{u} - v\psi) - \dot{v} \frac{\partial \psi}{\partial s} - u\psi \frac{\partial \psi}{\partial s} \\ = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial s} - v \frac{\partial \psi}{\partial s} \right) - \psi \left(\frac{\partial v}{\partial s} + u \frac{\partial \psi}{\partial s} \right); \end{aligned}$$

whence, by using (1) and (2), we get

$$\frac{\partial}{\partial s} \left(\frac{1}{m} \frac{\partial T}{\partial s} \right) + \frac{\partial}{\partial s} \left(\frac{S}{m} \right) - \frac{T}{m} \left(\frac{\partial \psi}{\partial s} \right)^2 - \frac{N}{m} \frac{\partial \psi}{\partial s} + \psi^2 = 0 \quad \dots (5).$$

2.71. Invariable form. When an endless inextensible chain originally at rest under the action of conservative forces and the reactions of smooth curves is set in motion in such a way that each point of the chain begins to move in the direction of the tangent at the point, then the chain retains its form.

For if u denotes the velocity at any point, then u is independent of s , and the equations of motion can be satisfied by taking u to be constant. The tangential acceleration is then zero and the normal acceleration is u^2/ρ , and equations 2.7 (3) and (4) become

$$0 = \frac{\partial T}{\partial s} + S \quad \dots \dots \dots (1)$$

and

$$m \frac{u^2}{\rho} = \frac{T}{\rho} + N \quad \dots \dots \dots (2).$$

Since u is independent of s , these equations may be written

$$0 = \frac{\partial}{\partial s} (T - mu^2) + S \quad \dots \dots \dots (3)$$

and

$$0 = \frac{T - mu^2}{\rho} + N \quad \dots \dots \dots (4),$$

leading to

$$S - \frac{\partial}{\partial s} (\rho N) = 0 \quad \dots \dots \dots (5).$$

This is an equation giving the form of the curve when the string is in motion, S and N being given functions of position; but it is exactly the same equation as if the chain were at rest, i.e. as if $u = 0$. It follows that the chain retains its form, but that the tension at every point is increased by mu^2 .

When, for example, a heavy endless chain placed over a drum is made to revolve rapidly, it retains the form of the catenary in which it hangs when at rest, and if an indentation in the form is caused by a

blow from a heavy hammer, the indentation remains for a considerable time. This is because in rapid rotation the tension is so great that the action of gravity is of less importance, though in time the action of gravity removes the indentation and restores the original form.

2·72. Initial tension of an inextensible chain. Suppose that a chain is starting into motion under the action of given forces. The initial velocities are zero, and if α , β denote the initial accelerations of a point P of the chain in the direction of the tangent and normal and we use the notation of 2·7, then at time δt , $u = \alpha \delta t$, $v = \beta \delta t$. Hence, substituting in 2·7 (1) and dividing by δt , we obtain

$$\frac{\partial \alpha}{\partial s} - \frac{\beta}{\rho} = 0 \dots\dots\dots(1).$$

Also the initial forms of the equations of motion 2·7 (3) and (4) are

$$m\alpha = \frac{\partial T}{\partial s} + S \dots\dots\dots(2)$$

and
$$m\beta = \frac{T}{\rho} + N \dots\dots\dots(3).$$

If we eliminate α and β from (1), (2) and (3), we get

$$\frac{\partial}{\partial s} \left(\frac{1}{m} \frac{\partial T}{\partial s} \right) - \frac{T}{m\rho^2} = \frac{N}{m\rho} - \frac{\partial}{\partial s} \left(\frac{S}{m} \right) \dots\dots\dots(4).$$

This is the initial form of 2·7 (5) since initially $\psi = 0$. If the chain is uniform, the factor m may be removed.

The constants which appear in the integration of (4) are to be found from the boundary conditions. Thus, if the chain has a free end, the tension at that end is zero. If it has a fixed end, the tangential acceleration at that end is zero.

When gravity is the only external force, if ψ denotes the inclination of the tangent to the horizontal, we have

$$S = -mg \sin \psi, \quad N = -mg \cos \psi,$$

no matter what be the law of density, so that

$$\frac{\partial}{\partial s} \left(\frac{S}{m} \right) = -g \cos \psi \frac{d\psi}{ds} = \frac{N}{m\rho}$$

and (4) reduces to
$$\frac{\partial}{\partial s} \left(\frac{1}{m} \frac{\partial T}{\partial s} \right) - \frac{T}{m\rho^2} = 0 \dots\dots\dots(5).$$

Further, if the chain, of any law of density, is initially hanging at rest, by resolution along the normal for an element of mass $m \delta s$ in equilibrium, we have

$$T \delta \psi - mg \delta s \cos \psi = 0;$$

and, by resolving horizontally for any finite length of the chain, we have

$$T \cos \psi = \text{const.}$$

From these two equations we get the law of density corresponding to a prescribed form, namely

$$m = C \sec^2 \psi \frac{\partial \psi}{\partial s},$$

where C is a constant.

If we substitute this value for m in (5), it reduces to

$$\frac{\partial}{\partial \psi} \left(\frac{\partial T}{\partial \psi} \cos^2 \psi \right) = T \cos^2 \psi$$

or
$$\frac{\partial^2 T}{\partial \psi^2} \cos \psi - 2 \sin \psi \frac{\partial T}{\partial \psi} - T \cos \psi = 0.$$

By integration we get

$$\frac{\partial T}{\partial \psi} \cos \psi - T \sin \psi = C_1,$$

and therefore
$$T \cos \psi = C_1 \psi + C_2 \dots \dots \dots (6),$$

where C_1, C_2 are constants.

Subject to finding the special values of the constants, this equation therefore determines the initial tension in a heavy chain of any law of density when starting into motion from a position in which it has been hanging with two points fixed.

2.721. Examples. (i) *Find the initial tension at any point of a heavy uniform chain with fixed ends, when severed at the vertex.*

Here, in 2.72 (6), we may put $T = 0$ when $\psi = 0$, so that $C_2 = 0$ and

$$T \cos \psi = C_1 \psi \dots \dots \dots (1).$$

If $\psi = \gamma$ at the upper end of one portion, then since at that end there is no tangential acceleration, from 2.72 (2) we get

$$\frac{\partial T}{\partial s} - mg \sin \psi = 0, \quad \text{when } \psi = \gamma \dots \dots \dots (2).$$

But $s = c \tan \psi$, and from (1)

$$\frac{\partial T}{\partial \psi} = C_1 (\sec \psi + \psi \sec \psi \tan \psi) \dots\dots\dots (3).$$

Hence from (2) and (3) we get

$$C_1 = \frac{mgc \sin \gamma}{\cos \gamma + \gamma \sin \gamma}$$

and thus T is found.

(ii) *A uniform heavy chain, passing under and in contact with the arc of a fixed vertical circle, has its ends fixed at two points in contact with the circle each at the angular distance γ from the lowest point, the pressure at which is zero. Find the tension when the circle is removed.*

In this case 2·72 (5) becomes

$$\frac{\partial^2 T}{\partial \psi^2} - T = 0,$$

so that

$$T = A \cosh \psi + B \sinh \psi.$$

By symmetry the tangential acceleration vanishes at the lowest point as well as at $\psi = \gamma$. But from 2·72 (2)

$$m\alpha = \frac{\partial T}{a \partial \psi} - mg \sin \psi,$$

where a is the radius; therefore when $\psi = 0$, $\partial T / \partial \psi = 0$,

and when $\psi = \gamma$, $\partial T / \partial \psi = mga \sin \gamma$.

Hence $B = 0$ and $A \sinh \gamma = mga \sin \gamma$,

so that $T \sinh \gamma = mga \sin \gamma \cosh \psi$.

If the chain were severed at the lowest point at the same instant that the circle is removed, the conditions would be $T = 0$ when $\psi = 0$, and $\alpha = 0$ when $\psi = \gamma$, leading to

$$T \cosh \gamma = mga \sin \gamma \sinh \psi.$$

(iii) *A uniform inextensible chain rests in the form of a circle on the surface of a smooth sphere with all its links at an angular distance α from the summit. Shew that if it snaps at one point the broken ends move initially in directions inclined to the circle at an angle*

$$\cot^{-1} \{ \tanh (\pi \cos \alpha) \}.$$

Though this is a three-dimensional problem, since the motion of each point of the chain lies in a tangent plane to the sphere, we have only two equations of motion and the condition of inextensibility and the equations of motion can be obtained in a similar manner to the equations of 2·72.

Thus let PQ be an element of the chain which subtends an angle $\delta\theta$ at its centre C . Then if T denotes the tension at P , the tensions at P and Q are equivalent to δT along the tangent at P and $T\delta\theta$ along PO .

Let A, B denote the accelerations of P along the tangent to the chain and the meridian curve, and $A + \delta A, B + \delta B$ those of Q . Then since the acceleration of Q along the tangent at P is ultimately equal to the acceleration of P in the same direction,

$$(A + \delta A) \cos \delta \theta + (B + \delta B) \cos \alpha \sin \delta \theta = A,$$

or
$$\frac{\partial A}{\partial \theta} + B \cos \alpha = 0 \dots (1).$$

Taking $ma \delta \theta$ as the mass of the element PQ , its equations of motion are

$$ma \delta \theta A = \delta T$$

and
$$ma \delta \theta B = mga \delta \theta \sin \alpha - T \delta \theta \cos \alpha;$$

or
$$maA = \frac{\partial T}{\partial \theta} \dots \dots \dots (2)$$

and
$$maB = mga \sin \alpha - T \cos \alpha \dots \dots \dots (3).$$

Eliminating A and B by means of (1), we get

$$\frac{\partial^2 T}{\partial \theta^2} - (T - mga \tan \alpha) \cos^2 \alpha = 0,$$

so that
$$T = mga \tan \alpha + E \cosh (\theta \cos \alpha) + F \sinh (\theta \cos \alpha) \dots \dots (4).$$

Now suppose that at the ends of the string $\theta = \pm \pi$; then since T vanishes at the ends, we have

$$F = 0 \quad \text{and} \quad E \cosh (\pi \cos \alpha) + mga \tan \alpha = 0,$$

so that
$$T = mga \tan \alpha \{1 - \cosh (\theta \cos \alpha) \operatorname{sech} (\pi \cos \alpha)\}.$$

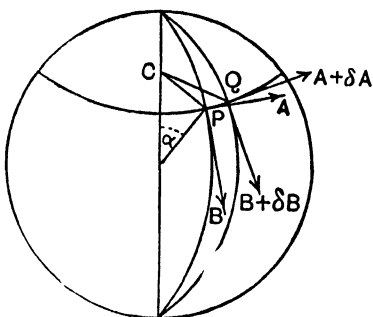
Then from (2) and (3)
$$A/B = \tanh (\theta \cos \alpha),$$

and the end at $\theta = \pi$ therefore moves initially in a direction inclined at an angle $\cot^{-1} \{ \tanh (\pi \cos \alpha) \}$ to the circle.

2.73. Sudden change of motion in a moving chain.

Suppose that the moving chain of 2.7 is subjected to a sudden change. Let ω be the value of ψ at the instant considered. Let α, β denote the initial tangential and normal accelerations of P . Then, initially, the motion of N relative to P is motion in a circle, radius δs , with angular velocity ω . Hence the acceleration of N relative to P , along PN , is $-\omega^2 \delta s$. On the other hand, this relative acceleration equals

$$\alpha + \alpha' \delta s - \delta \psi (\beta + \beta' \delta s) - \alpha.$$



Hence
$$\frac{\partial \alpha}{\partial s} - \frac{\beta}{\rho} = -\omega^2,$$

and the equations of motion take the forms 2.72 (2) and (3).

2.731. Example. *A circular chain on a smooth horizontal plane revolves uniformly about its centre. If the chain breaks, find the initial tension and the initial acceleration of its free ends.*

Taking a as the radius and ω as the angular velocity, and putting $\delta s = a \delta \theta$, the equations are

$$m\alpha = \frac{\partial T}{a \partial \theta}, \quad m\beta = \frac{T}{a} \quad \text{and} \quad \frac{\partial \alpha}{\partial \theta} - \beta = -a\omega^2.$$

Therefore
$$\frac{\partial^2 T}{\partial \theta^2} - T = -ma^2\omega^2,$$

so that
$$T = ma^2\omega^2 + A \cosh \theta + B \sinh \theta.$$

If the free ends are at $\theta = \pm \pi$, we have

$$B = 0 \quad \text{and} \quad ma^2\omega^2 + A \cosh \pi = 0,$$

so that
$$T = ma^2\omega^2 (1 - \cosh \theta / \cosh \pi)$$

and
$$\alpha = -a\omega^2 \sinh \theta / \cosh \pi, \quad \beta = a\omega^2 (1 - \cosh \theta / \cosh \pi).$$

Therefore the initial accelerations of the free ends are given by

$$\alpha = \mp a\omega^2 \tanh \pi, \quad \beta = 0,$$

so that they begin to separate with relative acceleration $2a\omega^2 \tanh \pi$.

2.74. Examples. 1. Shew that if a uniform chain falls under gravity the square of the angular velocity of the tangent at any point is equal to $\frac{1}{m} \left(\frac{T}{\rho^2} - \frac{\partial^2 T}{\partial s^2} \right)$, where m is the mass of unit length.

2. A uniform heavy chain is suspended by its ends from two points in the same horizontal line, the length of the chain being twice the parameter of the catenary in which it hangs. If one end is suddenly released, shew that the tension at the lowest point is immediately diminished in the ratio $2\pi + 4 : \pi$. [I. 1898]

3. The ends of a chain of any given law of density are fastened to fixed points; shew that, if the chain is severed at its lowest point, the vertical component of the initial acceleration of either of the severed ends is g , and the horizontal component is $g/(\psi + \cot \psi)$, where ψ is the inclination to the horizontal of the tangent at the corresponding fixed end. [M. T. 1902]

4. A uniform string hangs in equilibrium over two smooth pegs in the same horizontal line. If the string is cut at its middle point, prove that the initial tension at any point on the curved portion is

$$mgc\psi \sec \psi / \{ \gamma + \tan (\tfrac{1}{2}\pi - \tfrac{1}{2}\gamma) \},$$

where ψ is the angle the tangent at the point makes with the horizontal, γ is its value at one of the pegs, m is the mass of unit length of string and c is the parameter of the catenary. [I. 1906]

5. A heavy non-uniform chain hangs in equilibrium with its ends fixed. Shew that, if one end be suddenly released, the angular acceleration of the tangent at any point is initially proportional to $\kappa \cos \psi$, where ψ is the inclination of the tangent to the horizontal, and κ is the curvature at the point. [M. T. 1907]

2·8. Chain set in motion by an impulse. Let an inextensible chain resting on a smooth horizontal plane be set in motion by a tangential impulse at one or both ends.

If u, v denote the initial velocities of a point P and we consider a small element PQ of length δs and mass $m\delta s$, then as in 2·7 the condition of inextensibility gives a relation

$$\frac{\partial u}{\partial s} - \frac{v}{\rho} = 0 \quad \dots\dots\dots(1).$$

If T denotes the impulsive tension at P and $\delta\psi$ the angle between the tangents at P and Q , the effects of the tensions at P and Q on the element PQ may be represented by impulses δT along the tangent at P and $T\delta\psi$ along the inward normal, so that

$$m\delta s \cdot u = \delta T \quad \text{and} \quad m\delta s \cdot v = T\delta\psi,$$

$$\text{or} \quad mu = \frac{\partial T}{\partial s} \quad \text{and} \quad mv = \frac{T}{\rho} \quad \dots\dots\dots(2).$$

Therefore from (1)

$$\frac{\partial}{\partial s} \left(\frac{1}{m} \frac{\partial T}{\partial s} \right) - \frac{T}{m\rho^2} = 0 \quad \dots\dots\dots(3),$$

and this gives the impulsive tension.

If the chain be uniform, the factor m may be omitted.

2·81. Example. A chain in the form of a quadrant of a circle the density at any point of which varies as e^θ , where θ is the angular distance from one end, receives a tangential jerk at one end. Determine the initial tension at any point.

Putting $s = a\theta$ and $m \propto e^\theta$, 2·8 (3) becomes

$$\frac{\partial}{\partial \theta} \left(e^{-\theta} \frac{\partial T}{\partial \theta} \right) - e^{-\theta} T = 0$$

or

$$\frac{\partial^2 T}{\partial \theta^2} - \frac{\partial T}{\partial \theta} - T = 0.$$

The integral is

$$T = e^{\frac{1}{2}\theta} \left(A \cosh \frac{\sqrt{5}}{2} \theta + B \sinh \frac{\sqrt{5}}{2} \theta \right),$$

and the constants are determined by the conditions

$$T = 0 \text{ when } \theta = 0, \text{ making } A = 0,$$

and

$$T = T_1 \text{ when } \theta = \frac{\pi}{2},$$

so that

$$\frac{T}{T_1} = \frac{e^{\frac{1}{2}\theta} \sinh \frac{\sqrt{5}}{2} \theta}{e^{\frac{1}{2}\pi} \sinh \frac{\sqrt{5}}{4} \pi}.$$

2.82. Examples. 1. Prove that if a uniform chain lying on a smooth horizontal table in the form of the curve $r = ae^\theta$ from $\theta = 0$ to $\theta = \beta$ receives a tangential impulse T_0 at the end $\theta = 0$, the other end being free, the impulsive tension at any point is equal to

$$T_0 e^{\frac{1}{2}\theta} \sinh \frac{\sqrt{5}}{2} (\beta - \theta) \operatorname{cosech} \frac{\sqrt{5}}{2} \beta.$$

2. A uniform chain, of mass m per unit length, rests upon a smooth horizontal table in the form of the curve $r = ae^{3\theta/2}$ from $\theta = 0$ to $\theta = \beta$. A tangential impulse I is applied at the end $\theta = 0$, the other end being free. Shew that the kinetic energy generated is

$$\frac{1}{2\sqrt{13}} \frac{I^2}{am} \frac{e^{5\beta/2} + 4}{e^{5\beta/2} - 1}. \quad [\text{I. 1927}]$$

3. A string on a smooth horizontal plane is uniform and in the form of a circular arc subtending an angle α at the centre, and a given impulsive tension is applied at one end. Shew that the kinetic energy generated is $\alpha \coth \alpha$ times that generated by an equal impulse in a particle of the same mass. [M. T. 1903]

4. If a string is such that it would hang under gravity in the form of the catenary of equal strength [v. *Statics*, p. 243], and it is laid on a smooth table in this form and receives given tangential impulses at its ends, shew that the normal velocity of any point of the string is proportional to

$$(A\psi + B) \sin \psi$$

and that the impulsive tension is $\rho(A\psi + B)$, where A and B are constants, ρ is the radius of curvature, and ψ is the inclination of the tangent to an asymptote. [I. 1907]

EXAMPLES

1. If a particle moves under the action of forces $F, F', \dots$ to any number of fixed points, and if $q, q', \dots$ be the chords of curvature of the path in the directions of these forces, shew that

$$2mv^2 = \Sigma Fq.$$

2. Two elastic strings, the natural length of each of which is $\frac{1}{2}\pi a$, are fastened at a point P in a circular tube (radius a) of small bore; the strings are stretched in opposite directions, and their other extremities fastened to a particle of given weight. If the plane of the tube be horizontal, and the particle be displaced from its position of equilibrium through an angle less than $\pi/2$, shew that the time of an oscillation is independent of the extent of the displacement.

3. Within a smooth circular tube of radius a held fixed in a vertical plane lies a light string of length greater than half the circumference of the tube. The string carries equal weights at its ends, which balance within the tube, and the string subtends at the centre an angle $2(\pi - \alpha)$. If the weights be slightly disturbed, shew that the time of a small oscillation is the same as that of a simple pendulum of length $a \sec \alpha$.

4. Shew that the differential equation to the curve described under a force R inclined at an angle θ to the direction of motion is

$$d.(R\rho \sin \theta) - 2R \cos \theta . ds = 0,$$

ρ being the radius of curvature and ds the element of arc of the curve.

If R is always parallel to a fixed line, and the normal component constant, shew that the curve will be a catenary.

5. A particle, mass m , moves in a smooth circular tube of radius a under the action of a force, μm . (distance), to a point inside the circle at a distance c from its centre; if the particle be placed very nearly at its greatest distance from the centre of force, prove that it will pass over the quadrant ending at its least distance in the time

$$\sqrt{a} \log(\sqrt{2} + 1) / \sqrt{\mu c}.$$

6. A heavy particle is projected in a resisting medium. If v be the velocity at any time, ϕ the inclination to the vertical of the direction of motion, and f the retardation, prove that

$$\frac{1}{v} \frac{dv}{d\phi} + \cot \phi + \frac{f}{g \sin \phi} = 0.$$

If $f = \mu v^3$, find v in terms of ϕ .

7. Two equal particles, connected by a fine string, are placed in a circular tube, to one point of which they are attracted with a force varying inversely as the distance, the length of the string being less than a quadrant of the circle. One of the particles is initially at its greatest distance from the centre of force. If v, v' are the velocities

with which they successively pass through the point whose angular distance from the centre of force is 90° , shew that

$$e^{-v^3/\mu} + e^{-v^2/\mu} = 1.$$

8. A particle describes a parabola with an acceleration α to the focus, and an acceleration β parallel to the axis; prove that

$$\frac{d}{dr}(\alpha + \beta) + \frac{2\alpha}{r} = 0.$$

9. O, A, B are three collinear points on a smooth table such that $OA = a$ and $AB = b$. A string is laid along AB and to the end B is attached a particle. Shew that if the end A is made to describe a circle whose centre is O with uniform velocity v , the motion of the string relatively to the revolving radius OA is the same as that of a simple pendulum of length gab/v^2 , and further that the string will not remain taut unless $a > 4b$. [M. T. 1908]

10. A fine chain of given length is contained in a smooth circular tube which rotates uniformly in a horizontal plane about a fixed point in the circumference; if the chain subtends an angle 2α at the centre, and if one end be initially fastened to the tube at the end of the diameter through the fixed point, and be released, prove that the angular motion of the chain relative to the tube will be given by the equation

$$\alpha\theta^2 = 2\omega^2 \sin \alpha \{\cos(\theta + \alpha) - \cos \alpha\},$$

where θ is the angular displacement.

11. A string of varying density slides in a smooth cycloidal tube whose axis is vertical and vertex downwards. Shew that if the string be let fall from any position in which its whole length is within the tube, its centre of gravity will reach the vertex in the same time.

12. A particle moves in a fine smooth circular tube of radius a under the action of two centres of force, of magnitude μ . (distance), one attractive and the other repulsive at opposite ends of a diameter. Find the greatest velocity, if the pressure of the tube acts outwards over one-third of the circumference.

13. A uniform string falls freely in a vertical plane under the influence of gravity; prove that the angular acceleration of the tangent at any point is

$$\frac{2}{m\sqrt{\rho}} \frac{d}{ds} \left(\frac{T}{\sqrt{\rho}} \right),$$

where T is the tension at the point, ρ the radius of curvature, s the arc measured from a fixed point of the string, and m the mass of a unit of length.

14. A heavy particle of mass m slides down a smooth cycloid, whose axis is vertical and vertex upwards, in a medium whose resistance is $mv^2/2c$, and the distance of the starting point from the vertex is c . Prove that the time to the cusp is $\sqrt{\{8a(4a - c)/gc\}}$, where $2a$ is the diameter of the generating circle of the cycloid.

15. A small bead P of mass m is constrained to move along a smooth circular wire of radius a and centre O , and is attracted by the force μPQ to a point Q which revolves with constant angular velocity ω in a concentric circle of radius b . If the bead has an angular velocity Ω , when OPQ is a straight line, prove that the pressure on the wire will be a maximum, when the angle subtended by PQ at O is given by the equation

$$4\mu b \sin^2 \frac{1}{2}\theta = ma \{\Omega^2 - 2\Omega\omega + \frac{1}{2}\omega^2\}.$$

16. Two equal particles are connected by a string passing through a small hole in a smooth horizontal table, one particle hanging down, and the other held on the table at a distance c from the hole; if the latter be projected along the table at right angles to the string with the velocity $\sqrt{2gc}$, prove that the initial radius of curvature of its path is $4c/3$.

17. A string of infinite length is laid on a smooth table in the form of a portion of one branch of the curve $r^n \sin n\theta = a^n$, so that one extremity of the string is at a finite distance from the origin of polar co-ordinates; to this end a tangential impulse is applied, so that the initial direction of motion of each point of the string and the radius vector to the point are equally inclined to the corresponding tangent. Shew that the impulsive tension at any point varies as $r^{-(n-1)}$ and the density of the string varies as

$$r^{1-2n} (r^{2n} - a^{2n})^{(2n-1)/2n}.$$

18. A particle of mass m is attached by n equal elastic strings of natural length a to the fixed angular points of a regular polygon of n sides the radius of whose circumscribing circle is c . Shew that, if the particle be slightly displaced from its equilibrium position in the plane of the polygon, it will execute harmonic oscillations in a straight line, the length of the equivalent simple pendulum being $2mgac/n\lambda(2c-a)$, and that for oscillations perpendicular to the plane of the polygon the corresponding length will be $mgac/n\lambda(c-a)$, λ being the modulus of elasticity of each string. [M. T. 1900]

19. A particle slides in a smooth circular tube of radius a under the action of two equal centres of force of magnitude $\mu/(\text{distance})^3$ situated at the ends of a diameter. The particle is slightly disturbed from rest at a point equally distant from the centres of force. Prove that the pressure on the tube at any point is

$$\mu (2a^4 - r^2 r'^2) / a^3 r^2 r'^2,$$

where r, r' are the distances of the point from the centres of force.

[I. 1909]

20. An endless string on which runs a small smooth bead encloses an elliptic lamina, whose perimeter is less than the length of the string. The bead is projected in a direction which keeps the string in a state of tension. Shew that the velocity of the bead will be constant throughout the motion, and that the tension of the string will vary as

the sum of the reciprocals of the lengths of the straight parts of the string.

21. A particle moves under the action of a central force which is such that the normal acceleration of the particle is constant. Find a differential equation of the first order to the path of the particle, and shew that $r^3 \sin 3\theta = a^3$ is a particular integral.

22. On a wire in the form of a parabola with axis vertical and vertex downwards is a bead attached to the focus by an elastic string whose natural length is one-eighth of the latus rectum and whose modulus of elasticity is equal to the weight of the bead. Prove that the time of a small oscillation is $2\pi\sqrt{(a/g)}$, where $4a$ is the length of the latus rectum.

23. If there be two equal centres of force at S, S' such that SCS' is a straight line and $SC = S'C = c$, and each force has the value $\mu r/(a^2 + r^2 - c^2)^3$ at distance r , shew that a particle will describe a circle, centre C and radius a ($> c$), if projected from a point in the line CS distant a from C with velocity $\sqrt{\{\mu(a^2 + c^2)\}/2a(a^2 - c^2)}$ perpendicular to CS .

24. Two equal particles, P and Q , are connected by a fine string of length $2c$ which passes through a small hole in a smooth horizontal table. P is held on the table at the distance c from the hole and Q hangs at rest. If P be projected horizontally with the velocity v at right angles to the string, find the initial tension of the string, and prove that the initial radius of curvature of the path of P is

$$2cv^2/(gc + v^2).$$

If P be initially at rest and Q be projected horizontally with the velocity v , prove that the initial radius of curvature of the path of Q is

$$2cv^2/(gc - v^2).$$

25. A small ring of unit mass is free to move on a circular wire, which is constrained to rotate in its own plane with constant angular velocity about a point O on the circumference. The ring is also attached to O by a string of modulus of elasticity λ . Shew that the ring is in relatively stable equilibrium at the end A of the diameter through O , provided that in that position the pressure R of the wire on the ring is inwards; and that in this case the time of a small oscillation is $2\pi\sqrt{(d/R)}$, where d is the diameter. Shew also that, if the equilibrium at A is unstable, there are two oblique positions of equilibrium; and that, if one of these is equidistant from O and A , the period of a small oscillation is

$$2\pi(d\sqrt{2/\lambda})^{\frac{1}{2}}. \quad [\text{M. T. 1909}]$$

26. A small bead is strung upon a smooth circular wire of radius a , which is constrained to rotate with uniform angular velocity ω about a point on itself. The bead is initially at the extremity of the diameter through the centre of rotation, and is projected with velocity $2\omega b$

relative to the wire. Shew that the position of the bead at time t is given by the equation $\sin \theta = \sin b\omega t/a$ (modulus a/b), or $\sin \theta = (b/a) \sin \omega t$ (modulus b/a) according as $a <$ or $> b$, θ being the angle which the radius vector to the bead makes with the diameter of the circle through the centre of rotation. [M. T. 1900]

27. A piece of uniform chain, on a smooth horizontal table, receives a tangential jerk at one end.

(1) Find the form of the curve if the initial direction of motion of any point makes a constant angle with the tangent at that point.

(2) Find the impulsive tension at any point when the form is such that the line density at any point varies as the curvature.

(3) Prove that, if the impulsive tensions follow the same law as the tensions in a non-uniform string hanging in the same curve under the action of gravity, then

$$2\rho \frac{d^2\rho}{ds^2} = \left(\frac{d\rho}{ds}\right)^2 + 4.$$

(4) If all the particles of the chain start with equal velocities, prove that the form must be that of a straight line or of a catenary.

28. A uniform chain of length $2a\alpha$ is in motion inside a fine smooth circular tube of radius a in a vertical plane. Find the tension at any point of the chain, and shew that the sign of the tension at the middle point depends only on the position of this point.

29. If a particle moves on an ellipse under a force to the centre equal to $mr - nr \log \frac{c-r}{c+r}$, where $c^2 = a^2 + b^2$, and if R is the pressure on the curve, and ρ the radius of curvature, prove that

$$R\rho = 2ncr + \text{constant}.$$

Prove that if the velocity vanish at an extremity of the major axis and

$$\frac{m}{n} = \log \frac{c-a}{c+a}, \quad \text{then} \quad R\rho = 2nc(r-a).$$

30. A string is in equilibrium in the form of a circle under the action of a central repulsive force. If the string be cut at any point, prove that the tension at a point, the angular distance of which from the point of section is θ , is instantaneously changed in the ratio

$$\cosh \pi - \cosh (\pi - \theta) : \cosh \pi.$$

31. A string of variable density has its ends fixed at the points A and B and is allowed to hang freely. The tangents at A and B make angles α and β with the horizon, and ψ is the angle made with the horizontal by the tangent at a point P between the lowest point and B . Prove that, if the end A be released, the tension at P will be instantaneously changed in the ratio

$$(\psi - \alpha) \sin \beta : \cos \beta + (\beta - \alpha) \sin \beta.$$

32. A piece of uniform string hangs at rest with its two ends fastened to fixed points in the same horizontal line. If one end is released, prove that the tension at the other end is instantaneously changed in the ratio $2\theta \sin \theta : \cos \theta + 2\theta \sin \theta$,

where θ is the inclination to the horizontal of the tangent at each end.

33. The ends of a heavy heterogeneous chain are held in the same horizontal line, and the chain, when in equilibrium, takes the form of an arc of a circle less than a semicircle. If equal tangential jerks be applied simultaneously to the two ends, find the impulsive tension at any point, and prove that the initial normal velocities at the lowest point and at either end are in the ratio of $1 : \cos \theta$, where 2θ is the angle subtended by the arc at its centre.

Find also the direction in which either end begins to move.

34. The two ends of a heavy chain of mass $2M$ are attached to heavy rings, each of mass M' , which can slide on a smooth horizontal wire. The rings are held so that the ends of the chain are inclined at the same angle γ to the horizontal, and are then let go. Prove that the tension at the lowest point is instantaneously changed in the ratio

$$M' : M' + M \cot^2 \gamma.$$

35. A uniform string falls in a vertical plane with constant acceleration f , retaining an invariable form while the string advances along itself with a velocity which at any instant is the same for all points of the string. Shew that the angle ϕ which the tangent at any point of the string makes with the horizontal, considered as a function of the arc s measured up to this point from some fixed point of the string and the time t , satisfies the two partial differential equations

$$(f-g) \left[\frac{\partial^2 \phi}{\partial s^2} \cos \phi + 2 \left(\frac{\partial \phi}{\partial s} \right)^2 \sin \phi \right] + \frac{\partial^2 \phi}{\partial t^2} \frac{\partial \phi}{\partial s} - \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial s \partial t} = 0,$$

$$\frac{\partial \phi}{\partial s} \frac{\partial^2 \phi}{\partial s \partial t} - \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial s^2} = 0.$$

Chapter III

KINEMATICS

3.1. Displacement of a rigid body. So far we have limited our considerations for the most part to motion in a plane, but we shall now be concerned mainly with motion in three dimensions.

The fixing of one point of a rigid body leaves the body free to turn about every line through this point. The fixing of a second point of the body limits the possible motions to rotation about the line joining the two fixed points; and the fixing of a third point of the body, not collinear with the other two, fixes the position of the body. Hence the positions in space of any three non-collinear points of the body determine the position of the body. Nine co-ordinates are required to determine the positions of the three points, but they are connected by the three relations which express the distances between the three chosen points of the body, so that there are only six independent variables, and it follows that a rigid body free from all constraint has six degrees of freedom. A body with one point fixed has three degrees of freedom, and a body with two points fixed has only one degree of freedom.

3.11. Translation and rotation. A general displacement of a rigid body is one in which any three points O, P, Q of the body are displaced to other positions O', P', Q' . We shall shew that the displacement can be effected by a translation of the body as a whole, in which all points describe equal and parallel straight lines, together with a rotation about a definite axis. Let the translation be such as will move O to its final position O' , and let P, Q be thereby moved to P_1, Q_1 . We have then to turn the body about some axis through O' so as to bring P_1, Q_1 into their final positions, P', Q' . Imagine a sphere of centre O' fixed in the body, and let lines $O'P_1, O'Q_1, O'P', O'Q'$ cut the sphere in p_1, q_1, p', q' . Let planes through O' and the middle points m, n of the great circle arcs p_1p', q_1q' , at right angles to these arcs respectively, intersect in a straight

line $O'I$ meeting the sphere in I . Then $O'I$ is the required axis of rotation. For it follows from the construction that

$Ip_1 = Ip'$, and $Iq_1 = Iq'$;
and by hypothesis

$$p_1q_1 = p'q',$$

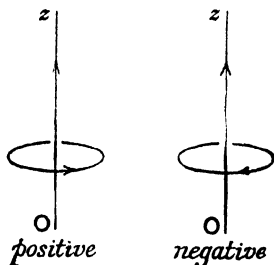
so that the spherical triangles Ip_1q_1 and $Ip'q'$

are congruent, and a rotation of the sphere about the diameter $O'I$ through an angle p_1Ip' will effect the required displacement.

It is evident that the order in which the operations of translation and rotation are performed is immaterial.

3·12. Instantaneous axis of rotation. A body turning about a fixed point O does not in general continue to turn about the same axis, but the position of the *instantaneous axis of rotation* at any instant is determined if the directions of motion of any two lines OA , OB in the body are known. For if AA' , BB' represent infinitesimal displacements, the axis of rotation must lie in a plane through OA perpendicular to AA' and also in a plane through OB perpendicular to BB' . It is therefore the intersection of these two planes.

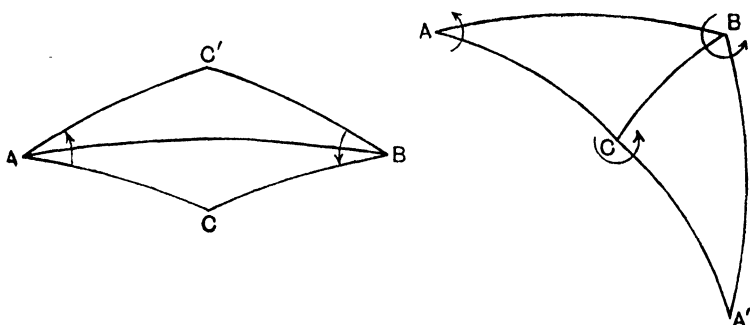
3·13. A rotation about an axis will be considered to be positive or negative according as its sense bears to the positive sense of the axis the relation of a right-handed or of a left-handed screw.



3·2. Composition of rotations.

Rotations through finite angles α , β about intersecting axes OA , OB are equivalent to a single rotation about an axis OC , whose position can be found thus. Let A , B be the points in which the axes cut a sphere of centre O . Draw the great circle arc AB and arcs AC , AC' , BC , BC' such that the angle $CAB = BAC' = \frac{1}{2}\alpha$ and $C'BA = ABC = \frac{1}{2}\beta$. Then a rotation about OA through an angle α will move C to C' , and a rotation about OB through an

angle β will move C' back to the original position C , so that the two successive rotations leave the position of C unchanged. Hence OC is the axis of the resultant rotation.



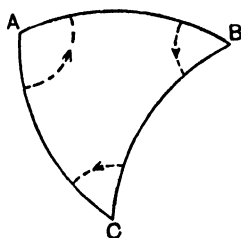
To find the angle of rotation about OC , consider the spherical triangle $BCA' \equiv BCA$, on the opposite side of BC from A . The rotation about OA leaves the point A unchanged, and the rotation about OB moves A to A' , since the angle ABA' is β . Hence A' is the final position of A and the rotation about OC is through the angle

$$\begin{aligned} ACA' &= 2(\pi - ACB) \\ &= -2ACB, \end{aligned}$$

since a rotation through 2π leaves positions unaltered.

3.21. Hamilton's Theorem. *Successive rotations about three concurrent lines fixed in space, through twice the angles between the planes formed by them, restore a body to its original position.*

Let the lines OA , OB , OC cut a sphere of centre O in the corners of a spherical triangle ABC . Then, as in 3.2, rotations about OA , OB through angles $2A$, $2B$ are equivalent to a rotation $-2C$ about OC . Hence the theorem follows.



3.22. Rotations about parallel axes. When the axes are parallel we may proceed as in 3.2, if we consider now that the point O is at an infinite distance and that the figure represents a plane section at right angles to the axes of rotation, with plane instead of spherical triangles.

The construction appears to fail when the angles of rotation

about the parallel axes through A and B are equal and opposite, because in this case the lines AC , BC are parallel, so that the resultant displacement is a rotation about a parallel axis at an infinite distance, i.e. a translation.

We can also deal with this case of equal and opposite rotations about parallel axes by another argument. Since there is no motion parallel to the axes we may limit our considerations to a plane perpendicular to the axes. Let L be a line in such a plane. A rotation of the figure through an angle α about A turns L through an angle α , and a rotation about B through an angle $-\alpha$ brings L back to its original direction, though not to its original position. Hence the resultant displacement leaves all lines in the plane unaltered in direction, so that the displacement is a translation in a direction at right angles to the axes of rotation.

If we write this:

rot. (α) about axis A + rot. ($-\alpha$) about parallel axis $B \equiv$
transl. perp. to the axes;

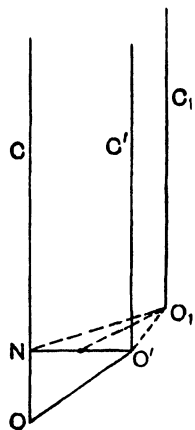
it is equivalent to

rot. (α) about axis $A \equiv$ rot. (α) about parallel axis B + transl.
perp. to the axes.

So that a rotation about any axis is equivalent to an equal rotation about a parallel axis together with a translation; and conversely.

3.23. Screw motions. We saw in 3.1 that any displacement of a rigid body is equivalent to a translation together with a rotation about an axis. We can now shew that the position of the axis of rotation can be so chosen that the translation is parallel to it.

Let O , O' be the initial and final positions of a point of the body. Then the displacement may be made by a translation OO' and a rotation about an axis $O'C'$. Let OC be parallel to $O'C'$, and let N be the projection of O' on OC . Then the translation OO' may be compounded of translations ON , NO' and the whole displacement consists of a translation ON parallel to the axis of rotation $O'C'$, a translation NO' perpendicular to $O'C'$ and a rota-



tion about $O'C'$. By 3·22 the last two displacements are equivalent to an equal rotation about some axis parallel to $O'C'$, i.e. parallel to ON , so that the whole displacement consists of a translation parallel to a certain axis and a rotation about it. Such a combination of translation and rotation is called a **screw**, and the **pitch** of the screw is the ratio of the distance of translation to the angle of rotation.

We have yet to indicate how to construct the position of the axis of the screw. This is simple if we bear in mind that O' is to be the final position of O produced by a rotation through a given angle, say θ , following on a translation which has transferred O to N . The required axis C_1O_1 is parallel to $C'O'$ and CO , and it must clearly lie in the plane which bisects NO' at right angles, and be at such a distance that NO' subtends an angle θ at the nearest point O_1 , on the axis, i.e. the distance of O_1 from the middle point of NO' is $\frac{1}{2}NO' \cot \frac{1}{2}\theta$. Then since a translation ON parallel to O_1C_1 and a rotation θ about O_1C_1 produces the required displacement of the body, therefore O_1C_1 is the axis of the screw.

3·24. Motion about a fixed point. When a body is in motion with one point fixed, the instantaneous axis of rotation always passes through this point but in general its position varies both in the body and in space. Its locus in the body is a cone and its locus in space is another cone. The two cones cut a unit sphere whose centre is at their common vertex in curves which correspond to the *pole curves* in plane motion [Pt. I, 5·43] and just as the motion of a plane lamina can be produced by the rolling of the pole curve fixed in the lamina on the pole curve fixed in space, so we are able to infer that the continuous motion of a body about a fixed point can be produced by the rolling of a cone fixed in the body on a cone fixed in space. We shall consider later the special case in which these cones are right circular.

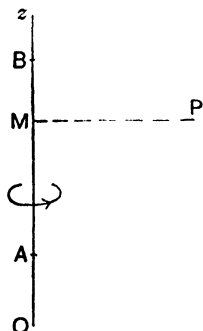
3·3. Angular velocity. The angular velocity of a body about an axis is the rate of increase of the angle between two planes through the axis one of which is fixed in the body while the other is fixed in space. It follows from 3·22 that an angular velocity about an axis is equivalent to an equal angular velocity about a parallel axis together with a velocity perpendicular to the axes, and conversely.

3·31. Angular velocity is a vector. Since the magnitude and sense of a rotation about an axis can be indicated by a

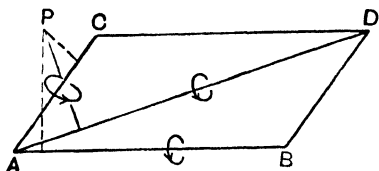
length measured in a definite sense along the axis, angular velocities can also be so represented; and it is then only necessary to shew that when they are so represented they obey the parallelogram law of composition, in order to demonstrate that an angular velocity (like a force) may be regarded as a vector localised in a line.

Let P be a point of a body turning about an axis Oz with an angular velocity ω . We can represent the angular velocity by a length AB equal to ω measured along Oz , and bearing a right-handed screw relation to the sense of ω .

Let PM be perpendicular to Oz . Then the velocity of P is $\omega \cdot MP$ or $AB \cdot MP$; i.e. what in Statics is called 'the moment of AB about P '.



Now let the body have angular velocities represented by AB, AC , two intersecting axes. Complete the parallelogram $ABDC$. Then since the sum of the moments about P of AB and AC is equal to the moment of AD , therefore the angular velocities represented by AB, AC are equivalent to an angular velocity represented by AD .



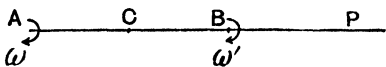
Consider next a body which has angular velocities ω, ω' in the same sense about parallel axes. Let the axes cut the plane of the paper at right angles in A, B . Let C be the point between A and B for which

$$\omega \cdot AC = \omega' \cdot CB \dots\dots\dots(1).$$

Then it is evident that C has no velocity, so that the motion is equivalent to a rotation about a parallel axis through C .

To find the resultant angular velocity, we have, for the velocity of any point P on AB produced,

$$\begin{aligned} \omega \cdot AP + \omega' \cdot BP &= \omega (AC + CP) + \omega' (CP - CB) \\ &= (\omega + \omega') CP, \text{ by (1).} \end{aligned}$$



Hence the resultant motion is due to an angular velocity $\omega + \omega'$ about a parallel axis through C .

Thus angular velocities are compounded, in every case, by the vector law of addition, and *angular velocity is a vector localised in a line* (the axis of rotation).

3.32. Composition of angular velocities.

We may also consider this matter analytically. Thus suppose that a body possesses simultaneously angular velocities $\omega_x, \omega_y, \omega_z$ about three rectangular axes Ox, Oy, Oz , and let OI be a line whose direction cosines are proportional to $\omega_x, \omega_y, \omega_z$. Then if x, y, z are the co-ordinates of a point P of the body, N the projection of P on OI , M its projection on the plane xOy and L the projection of M on Ox , and if u, v, w are the components of velocity of P , by considering velocities parallel to Ox , we have

$$u = \text{vel. of } P \text{ relative to } M + \text{vel. of } M \text{ relative to } L,$$

$$\left. \begin{array}{l} \text{or} \quad u = z\omega_y - y\omega_z; \\ \text{similarly} \quad v = x\omega_z - z\omega_x \quad \text{and} \quad w = y\omega_x - x\omega_y \end{array} \right\} \dots\dots\dots(1).$$

$$\text{Therefore} \quad u\omega_x + v\omega_y + w\omega_z = 0 \dots\dots\dots(2),$$

which proves that the direction of motion of P is at right angles to OI .

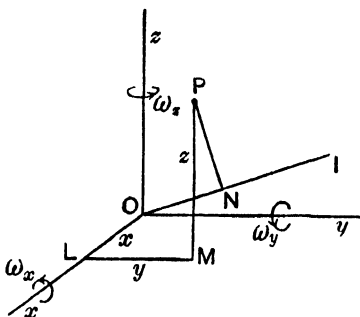
$$\begin{aligned} \text{Again,} \quad NP^2 &= x^2 + y^2 + z^2 - \frac{(x\omega_x + y\omega_y + z\omega_z)^2}{\omega_x^2 + \omega_y^2 + \omega_z^2} \\ &= \frac{(z\omega_y - y\omega_z)^2 + (x\omega_z - z\omega_x)^2 + (y\omega_x - x\omega_y)^2}{\omega_x^2 + \omega_y^2 + \omega_z^2}; \end{aligned}$$

so that from (1) the velocity of P is

$$\sqrt{(u^2 + v^2 + w^2)} = NP \sqrt{(\omega_x^2 + \omega_y^2 + \omega_z^2)},$$

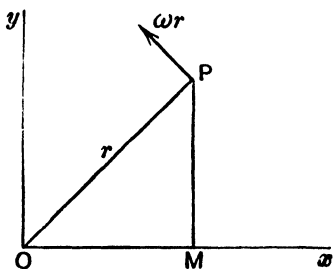
and this is the same as if the body were turning about OI with an angular velocity $\sqrt{(\omega_x^2 + \omega_y^2 + \omega_z^2)}$, shewing again that the angular velocities about the axes are to be compounded vectorially in the ordinary way.

3.4. Moving axes. It is often found convenient in the solution of dynamical problems to refer the motion of a body to a frame of reference which is itself moving in a known manner. The motion of the body referred to a fixed frame is then its motion referred to the moving frame compounded with the motion of that frame.



Let us consider first a two-dimensional motion referred to rectangular axes turning about a fixed origin with angular velocity ω .

Let P be a point whose co-ordinates are x, y referred to the moving axes, and let M be its projection on the axis Ox . Then $\dot{x}, \dot{y}$ are the velocity components of P referred to the moving frame, or the rates of departure of the moving point P from the point which we regard as rigidly connected to the frame at P . And because of the motion of the frame this point has a velocity ωr perpendicular to OP , where $OP = r$. Hence the velocity of P is compounded of $\dot{x}, \dot{y}$ and ωr , and if u, v denote its components parallel to Ox and Oy



$$u = \dot{x} - \omega y \quad \text{and} \quad v = \dot{y} + \omega x \quad \dots\dots\dots(1).$$

It is to be noted that u, v are the components of the velocity of P referred to axes fixed in space with which the moving axes are momentarily coincident.

Otherwise we may say that the velocity of P is its velocity relative to M compounded with the velocity of M . But the velocity of P relative to M has components $\dot{y}$ along MP and ωy parallel to xO , and the velocity of M has components $\dot{x}$ along OM and ωx parallel to Oy ; so that, as before

$$u = \dot{x} - \omega y \quad \text{and} \quad v = \dot{y} + \omega x.$$

Further, using the same figure, let OP represent any vector (e.g. velocity, momentum, moment of momentum) and let u, v denote its components OM, MP . Then by the same argument as above the rate of change of this vector referred to fixed axes with which the moving axes are momentarily coincident has components

$$\dot{u} - \omega v \quad \text{and} \quad \dot{v} + \omega u \quad \dots\dots\dots(2).$$

Taking the case in which the vector denotes velocity and substituting from (1) and (2), we find that the acceleration of P has components

$$\left. \begin{aligned} \ddot{x} - \dot{\omega}y - 2\omega\dot{y} - \omega^2x \\ \ddot{y} + \dot{\omega}x + 2\omega\dot{x} - \omega^2y \end{aligned} \right\} \quad \dots\dots\dots(3).$$

and

3.41. The formulae for velocity and acceleration in polar co-ordinates [Pt. I, 12.1] follow directly from 3.4 (1) and (2) if we put $x=r$ and $y=0$; for then

$$u=\dot{r} \quad \text{and} \quad v=\omega r;$$

$$\dot{u}-\omega v=\ddot{r}-\omega^2 r \quad \text{and} \quad \dot{v}+\omega u=\dot{\omega} r+2\omega \dot{r}.$$

Conversely, the results 3.4 (3) may be obtained by assuming the formulae for acceleration in polar co-ordinates and compounding the accelerations of P relative to M with the accelerations of M .

3.42. Example. Shew that if the axes of x and y rotate with angular velocities ω_1, ω_2 respectively, and ψ is the angle between them, the component accelerations of the point (x, y) parallel to the axes are

$$\ddot{x}-x\omega_1^2-(x\dot{\omega}_1+2\dot{x}\omega_1)\cot\psi-(y\dot{\omega}_2+2\dot{y}\omega_2)\operatorname{cosec}\psi$$

$$\text{and} \quad \ddot{y}-y\omega_2^2+(x\dot{\omega}_1+2\dot{x}\omega_1)\operatorname{cosec}\psi+(y\dot{\omega}_2+2\dot{y}\omega_2)\cot\psi.$$

Let P be the point (x, y) , where $ON=x$ and $NP=y$.

The acceleration of P relative to N has radial and transverse components R_2, T_2 , where

$$R_2=\ddot{y}-y\omega_2^2$$

$$\text{and} \quad T_2=y\dot{\omega}_2+2\dot{y}\omega_2 \quad \dots\dots(1)$$

and the acceleration of N has radial and transverse components R_1, T_1 , where

$$R_1=\ddot{x}-x\omega_1^2 \quad \text{and} \quad T_1=x\dot{\omega}_1+2\dot{x}\omega_1 \quad \dots\dots\dots(2).$$

But by resolving obliquely T_1 is equivalent to $-T_1 \cot \psi$ and $T_1 \operatorname{cosec} \psi$ parallel to Ox and Oy . Similarly T_2 is equivalent to $-T_2 \operatorname{cosec} \psi$ and $T_2 \cot \psi$. Hence the required components of acceleration are

$$R_1-T_1 \cot \psi-T_2 \operatorname{cosec} \psi \quad \text{and} \quad R_2+T_1 \operatorname{cosec} \psi+T_2 \cot \psi,$$

which, from (1) and (2), give the required results.

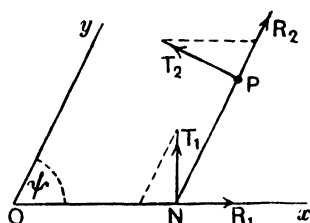
Alternatively, let f_1, f_2 denote the required accelerations, u, v the velocities parallel to the axes, and let the axes make angles θ, ϕ with a fixed direction OX . Then, by projecting displacements, velocities and accelerations on OX and a perpendicular direction, we get

$$u \cos \theta + v \cos \phi = \frac{d}{dt} (x \cos \theta + y \cos \phi)$$

$$= \dot{x} \cos \theta + \dot{y} \cos \phi - x\omega_1 \sin \theta - y\omega_2 \sin \phi$$

$$\text{and} \quad u \sin \theta + v \sin \phi = \frac{d}{dt} (x \sin \theta + y \sin \phi)$$

$$= \dot{x} \sin \theta + \dot{y} \sin \phi + x\omega_1 \cos \theta + y\omega_2 \cos \phi.$$



Similarly

$$\begin{aligned} f_1 \cos \theta + f_2 \cos \phi &= \frac{d}{dt} (u \cos \theta + v \cos \phi) \\ &= (\ddot{x} - x\omega_1^2) \cos \theta + (\ddot{y} - y\omega_2^2) \cos \phi \\ &\quad - (x\dot{\omega}_1 + 2\dot{x}\omega_1) \sin \theta - (y\dot{\omega}_2 + 2\dot{y}\omega_2) \sin \phi \dots (3) \end{aligned}$$

and

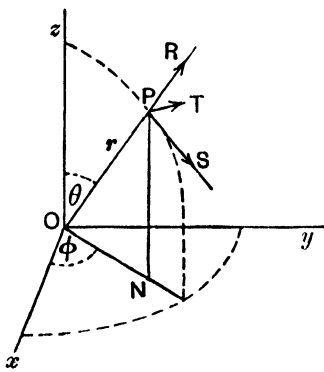
$$\begin{aligned} f_1 \sin \theta + f_2 \sin \phi &= \frac{d}{dt} (u \sin \theta + v \sin \phi) \\ &= (\ddot{x} - x\omega_1^2) \sin \theta + (\ddot{y} - y\omega_2^2) \sin \phi \\ &\quad + (x\dot{\omega}_1 + 2\dot{x}\omega_1) \cos \theta + (y\dot{\omega}_2 + 2\dot{y}\omega_2) \cos \phi \dots (4). \end{aligned}$$

By solving these equations for f_1 , f_2 and putting $\phi - \theta = \psi$, we get the same results as before.

3.5. Motion in three dimensions. Polar co-ordinates.

Let r, θ, ϕ be the polar co-ordinates of a point P , measured as in the figure; and let N be the projection of P on the plane Oxy .

The velocities of P in the plane OPz are $\dot{r}$ along OP , and $r\dot{\theta}$ at right angles to OP , i.e. tangent to the meridian curve; and the velocity of P at right angles to the plane OPz is clearly the same as the velocity of N in the same direction, i.e. $ON \cdot \dot{\phi}$ or $r \sin \theta \cdot \dot{\phi}$.



The accelerations of P are compounded of $\ddot{z}$ or $\frac{d^2}{dt^2} (r \cos \theta)$ parallel to Oz , and its accelerations parallel to the plane Oxy , and these latter are the same as the accelerations of N ; viz.

$$\frac{d^2}{dt^2} (ON) - \dot{\phi}^2 \cdot ON \text{ parallel to } ON$$

and $\frac{1}{ON} \frac{d}{dt} (\dot{\phi} \cdot ON^2)$ perpendicular to the plane OPN ,

or $\frac{d^2}{dt^2} (r \sin \theta) - \dot{\phi}^2 \cdot r \sin \theta$ parallel to ON

and

$$\frac{1}{r \sin \theta} \frac{d}{dt} (\dot{\phi} \cdot r^2 \sin^2 \theta) \text{ perpendicular to the plane } OPN.$$

Hence if R , S , T denote the components of acceleration along OP , along the meridian curve and at right angles to the plane OPN , we find, by resolving the foregoing components in these directions, that

$$\left. \begin{aligned} R &= \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2 \theta \\ S &= r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin \theta \cos \theta \\ \text{and } T &= r\ddot{\phi} \sin \theta + 2\dot{r}\dot{\phi} \sin \theta + 2r\dot{\phi}\dot{\theta} \cos \theta \end{aligned} \right\} \dots\dots(1).$$

These expressions take simpler forms in special cases. Thus

(i) *For the motion of a particle on a right circular cone we put $\theta = \alpha$ and obtain*

$$\ddot{r} - r\dot{\phi}^2 \sin^2 \alpha$$

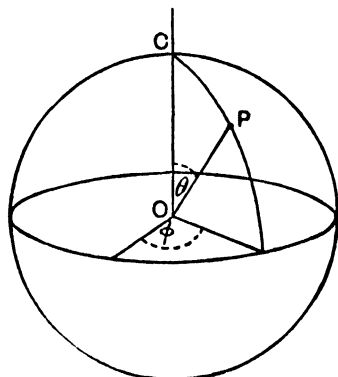
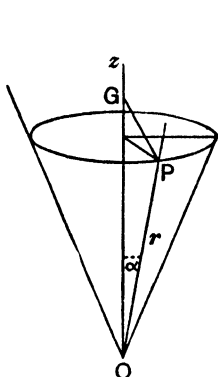
for the acceleration along the generator OP ,

$$r\dot{\phi}^2 \sin \alpha \cos \alpha$$

along the inward normal PG and

$$r\ddot{\phi} \sin \alpha + 2\dot{r}\dot{\phi} \sin \alpha$$

at right angles to the plane OPG .



(ii) *For motion on the surface of a sphere we put $r = a$ and obtain*

$$a(\ddot{\theta} - \dot{\phi}^2 \sin^2 \theta)$$

for the acceleration towards the centre, i.e. along PO ;

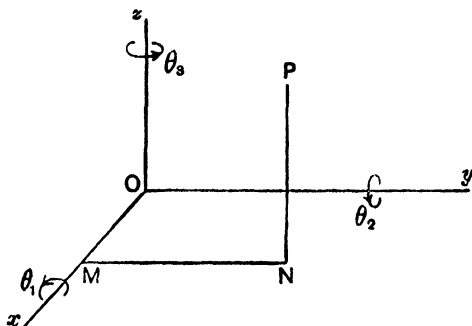
$$a(\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta)$$

along the meridian curve CP , and

$$a(\ddot{\phi} \sin \theta + 2\dot{\phi}\dot{\theta} \cos \theta)$$

perpendicular to the plane COP .

3.6. Moving axes in three dimensions. Let rectangular axes $Oxyz$ be turning about some line through the fixed point O with an angular velocity whose components are $\theta_1, \theta_2, \theta_3$



about the lines fixed in space with which the axes are momentarily coincident. Let P be the point (x, y, z) , N, M its projections on the plane xOy and on Ox . Let u, v, w be the velocities of P referred to the fixed axes with which the moving axes are momentarily coincident.

Then considering velocities parallel to Ox ,

P 's velocity relative to N is $z\theta_2$,

N 's M is $-y\theta_3$

and M 's velocity is $\dot{x}$;

so that $u = \dot{x} - y\theta_3 + z\theta_2$
 similarly $v = \dot{y} - z\theta_1 + x\theta_3$
 and $w = \dot{z} - x\theta_2 + y\theta_1$ (1).

Again, if OP be a vector representing a physical magnitude such as velocity, angular velocity, momentum, etc., and u, v, w denote its components, the rates of displacement of the point P represent the rates of change of the components of this vector, and by similar reasoning they are

$$\left. \begin{aligned} \dot{u} - v\theta_3 + w\theta_2 \\ \dot{v} - w\theta_1 + u\theta_3 \\ \dot{w} - u\theta_2 + v\theta_1 \end{aligned} \right\} \dots\dots\dots (2).$$

If the point O is in motion, the expressions (1) and (2) give the components of velocity and acceleration respectively in the directions Ox , Oy , Oz relative to O .

3.61. Angular velocity referred to moving axes. Let a body be turning about some instantaneous axis through a fixed point O with an angular velocity Ω , and let the components of Ω about a fixed set of axes be ω_x , ω_y , ω_z and about the rotating set of axes ω_1 , ω_2 , ω_3 . At the instant when the two sets of axes coincide we have

$$\omega_x, \omega_y, \omega_z = \omega_1, \omega_2, \omega_3,$$

but in general $\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z \neq \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3$;

because, if as in 3.6 the vector OP denotes the angular velocity Ω , then

$$\left. \begin{aligned} \dot{\omega}_x &= \dot{\omega}_1 - \omega_2 \theta_3 + \omega_3 \theta_2 \\ \dot{\omega}_y &= \dot{\omega}_2 - \omega_3 \theta_1 + \omega_1 \theta_3 \\ \dot{\omega}_z &= \dot{\omega}_3 - \omega_1 \theta_2 + \omega_2 \theta_1 \end{aligned} \right\} \dots\dots\dots (1).$$

and

If however the moving axes are fixed in the body, then Ω is also the angular velocity of the rotating frame, so that in this case

$$\theta_1, \theta_2, \theta_3 \equiv \omega_1, \omega_2, \omega_3;$$

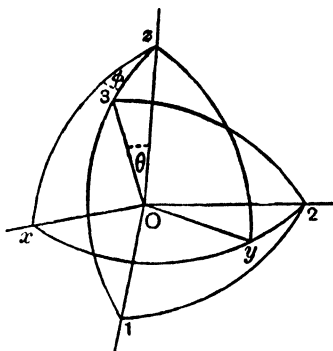
and in this case only, (1) reduces to

$$\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z = \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3 \dots\dots\dots (2).$$

In what follows, when employing moving axes we shall, as here, use suffixes x, y, z to denote components of a vector with regard to fixed axes and suffixes 1, 2, 3 to denote components with regard to moving axes.

3.62. Polar co-ordinates. We may obtain the results of 3.5 by using moving axes. Let r, θ, ϕ be the polar co-ordinates, and take the moving axes so that $O3$ lies along the radius vector and $O1$ lies in the plane $zO3$. Since this plane $z31$ cuts xOy at right angles, the pole of the circle $z31$ lies in the plane xOy , i.e. the axis $O2$ lies in this plane.

Then θ is the angular velocity about $O2$ and the angular velocity ϕ about



Oz may be resolved into components along $O1$ and $O3$ so that the angular velocities of the moving axes are

$$\theta_1 = -\dot{\phi} \sin \theta, \quad \theta_2 = \dot{\theta}, \quad \theta_3 = \dot{\phi} \cos \theta.$$

Now in 3.6 (1) put $x=0, y=0, z=r$, i.e. bring the moving and fixed axes into coincidence, and we have for the velocities

$$u = r\dot{\theta}, \quad v = r\dot{\phi} \sin \theta, \quad w = \dot{r};$$

and for the acceleration

$$\ddot{u} - v\theta_3 + w\theta_2 = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin \theta \cos \theta \text{ along the meridian,}$$

$$\ddot{v} - w\theta_1 + u\theta_3 = r\ddot{\phi} \sin \theta + 2\dot{r}\dot{\phi} \sin \theta + 2r\dot{\theta}\dot{\phi} \cos \theta \text{ perpendicular to the meridian plane, and}$$

$$\ddot{w} - u\theta_2 + v\theta_1 = \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2 \theta \text{ along the radius vector.}$$

3.63. Geometrical applications. (i) To find the angle between two positions of a moving line at a short interval of time δt , where l, m, n are the direction cosines of the line referred to moving axes.

Draw from the origin a line OP of unit length in the direction (l, m, n) at time t . Let OP' be the position of the line at time $t + \delta t$. Then POP' is the required angle $\delta\chi$, and l, m, n being the co-ordinates of P , the projections of PP' on fixed axes which coincide with the moving axes at time t are

$$\delta l = (\dot{l} - m\theta_3 + n\theta_2) \delta t,$$

$$\delta m = (\dot{m} - n\theta_1 + l\theta_3) \delta t$$

and

$$\delta n = (\dot{n} - l\theta_2 + m\theta_1) \delta t.$$

Also $(\delta\chi)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$ gives the angle required.

(ii) *Frenet's formulae for curvature and torsion of a twisted curve.*

Let the direction cosines of the tangent, principal normal and binormal at a point on a curve be l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 referred to axes fixed in space. The corresponding triad of lines at a small distance ds along the arc is obtained by making a twist $d\tau$ about the tangent and a twist $d\epsilon$ about the binormal, so that if $\theta_1, \theta_2, \theta_3$ denote the components of angular velocity of these moving axes, we may put $\theta_1 dt = d\tau, \theta_2 dt = 0, \theta_3 dt = d\epsilon$.

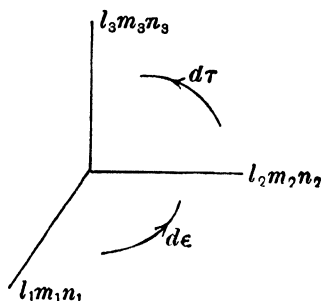
Since the direction cosines of the axis of x referred to the moving axes are l_1, l_2, l_3 and this axis is fixed in space, we have

$$0 = \frac{dl_1}{dt} - l_2\theta_3 + l_3\theta_2,$$

$$0 = \frac{dl_2}{dt} - l_3\theta_1 + l_1\theta_3$$

and

$$0 = \frac{dl_3}{dt} - l_1\theta_2 + l_2\theta_1,$$



or

$$0 = dl_1 - l_2 d\epsilon,$$

$$0 = dl_2 - l_3 d\tau + l_1 d\epsilon$$

and

$$0 = dl_3 + l_2 d\tau.$$

From which we get at once **Frenet's formulae**

$$\frac{dl_1}{ds} = \frac{l_2}{\rho}, \quad \frac{dl_2}{ds} = \frac{l_3}{\sigma} - \frac{l_1}{\rho}, \quad \frac{dl_3}{ds} = -\frac{l_2}{\sigma},$$

where ρ , σ denote radii of curvature and torsion.

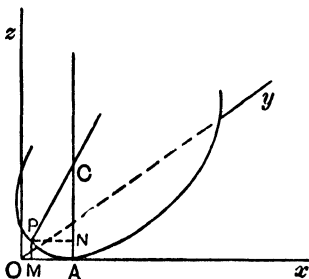
3.64. Example. A rigid body is rolling without sliding on a plane, and at any instant its angular velocity has components ω_1 , ω_2 along the tangents to the lines of curvature at the point of contact, and ω_3 along the normal; shew that the point of the body which is at the point of contact has component accelerations

$$-R_2 \omega_1 \omega_3, \quad -R_1 \omega_2 \omega_3, \quad R_1 \omega_2^2 + R_2 \omega_1^2,$$

where R_1 , R_2 are the principal radii of curvature of the surface of the body at the point of contact.

Let Ox , Oy be tangents to the lines of curvature at the point of contact O of the body and the plane.

In a short time dt the body undergoes a displacement compounded of three rotations $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$, and we may consider the effects of these in any order. Since the normals at neighbouring points on a line of curvature intersect, the body may roll through a small angle $\omega_2 dt$ along Ox without any twisting about the normal. Let A be the point of contact at the end of this displacement, the original point of contact on the body moving from O to P . Let PM , PN be perpendicular to Ox and to the normal at A .



In this position the body has angular velocities ω_1 , ω_2 , ω_3 about Ox , a parallel to Oy through A , and the normal AC , neglecting infinitesimal changes which would introduce higher powers of dt than we require. Hence the components parallel to the axes of the velocity of P are

$$\omega_2 MP, \quad -\omega_3 PN - \omega_1 MP, \quad \omega_2 PN.$$

But $PC = R_1$ and the angle $PCA = \omega_2 dt$, so that, to the second power of dt , $MP = \frac{1}{2} R_1 (\omega_2 dt)^2$, and $PN = R_1 \omega_2 dt$, and to the first power of dt the components of velocity are

$$0, \quad -R_1 \omega_2 \omega_3 dt, \quad R_1 \omega_2^2 dt.$$

If we now take into account the simultaneous displacement $-\omega_1 dt$ of rolling in the direction Oy , we see that in time dt the point P also acquires velocities

$$-R_2 \omega_1 \omega_3 dt, \quad 0, \quad R_2 \omega_1^2 dt.$$

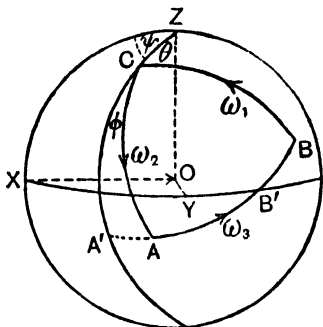
Hence the total velocity acquired by P in time dt has components

$$-R_2\omega_1\omega_3dt, \quad -R_1\omega_2\omega_3dt, \quad (R_1\omega_2^2 + R_2\omega_1^2)dt$$

to the first power of dt , and the component accelerations are

$$-R_2\omega_1\omega_3, \quad -R_1\omega_2\omega_3, \quad R_1\omega_2^2 + R_2\omega_1^2.$$

3.7. Euler's Geometrical Equations. When a body is free to turn about a fixed point O it is often convenient to refer its position to a standard position by means of three angular coordinates θ, ϕ, ψ . We take a set of rectangular axes OA, OB, OC fixed in the body and let θ be the angle which OC makes with an axis OZ fixed in space, ψ the angle which the plane ZOC makes with a fixed plane ZOX and ϕ the angle which the plane COA fixed in the body makes with the plane ZOC . The figure represents the intersections of the planes through O with a unit sphere of centre O . Let the angular velocity of the body have components $\omega_1, \omega_2, \omega_3$ about the moving axes OA, OB, OC ; we shall express these in terms of θ, ϕ, ψ . Let the arc AB cut the arc ZC in A' ; then $CA' = CA = \frac{1}{2}\pi$, so that $A'A = \phi$.



By considering the motion of the point C along and perpendicular to ZC we get

$$\dot{\theta} = \omega_1 \sin \phi + \omega_2 \cos \phi \quad \dots\dots\dots(1)$$

$$\text{and} \quad \dot{\psi} \sin \theta = \omega_2 \sin \phi - \omega_1 \cos \phi \quad \dots\dots\dots(2),$$

from which it follows that

$$\omega_1 = \dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi \quad \dots\dots\dots(3)$$

$$\text{and} \quad \omega_2 = \dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi \quad \dots\dots\dots(4).$$

Again, $ZA' = \frac{1}{2}\pi + \theta$, so that A' has a velocity $\dot{\psi} \sin(\frac{1}{2}\pi + \theta)$ or $\dot{\psi} \cos \theta$ at right angles to ZA' , and A has a velocity $\dot{\phi}$ relative to A' , so that

$$\omega_3 = \dot{\phi} + \dot{\psi} \cos \theta \quad \dots\dots\dots(5).$$

Equations (3), (4) and (5) are generally known as **Euler's Geometrical Equations**.

3·71. We may express the direction-cosines of the moving axes OA, OB, OC with regard to fixed axes OX, OY, OZ thus: if the arc $A'A'B' = \frac{1}{2}\pi$, then B' is the pole of $A'CZ$ and therefore lies on the great circle XY of which Z is the pole. Also since Z is the pole of XY and C is the pole of $A'B'$, therefore the angle $A'B'X = CZ = \theta$, and $B'B = A'A = \phi$. Also $XB' = \psi + \frac{1}{2}\pi$, and $YB' = \psi$, so that by using spherical triangles we have

$$\begin{aligned}\cos XA &= \cos XB' \cos AB' + \sin XB' \sin AB' \cos XB'A \\ &= -\sin \psi \sin \phi + \cos \psi \cos \phi \cos \theta;\end{aligned}$$

$$\begin{aligned}\cos YA &= \cos YB' \cos AB' + \sin YB' \sin AB' \cos YB'A \\ &= \cos \psi \sin \phi + \sin \psi \cos \phi \cos \theta;\end{aligned}$$

$$\begin{aligned}\cos ZA &= \cos AC \cos ZC + \sin AC \sin ZC \cos ACZ \\ &= -\sin \theta \cos \phi;\end{aligned}$$

$$\begin{aligned}\cos XB &= \cos XB' \cos BB' + \sin XB' \sin BB' \cos XB'B \\ &= -\sin \psi \cos \phi - \cos \psi \sin \phi \cos \theta;\end{aligned}$$

$$\begin{aligned}\cos YB &= \cos YB' \cos BB' + \sin YB' \sin BB' \cos YB'B \\ &= \cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta;\end{aligned}$$

$$\begin{aligned}\cos ZB &= \cos BC \cos ZC + \sin BC \sin ZC \cos BCZ \\ &= \sin \theta \sin \phi;\end{aligned}$$

$$\begin{aligned}\cos XC &= \cos XB' \cos CB' + \sin XB' \sin CB' \cos XB'C \\ &= \cos \psi \sin \theta;\end{aligned}$$

$$\begin{aligned}\cos YC &= \cos YB' \cos CB' + \sin YB' \sin CB' \cos YB'C \\ &= \sin \psi \sin \theta;\end{aligned}$$

$$\cos ZC = \cos \theta.$$

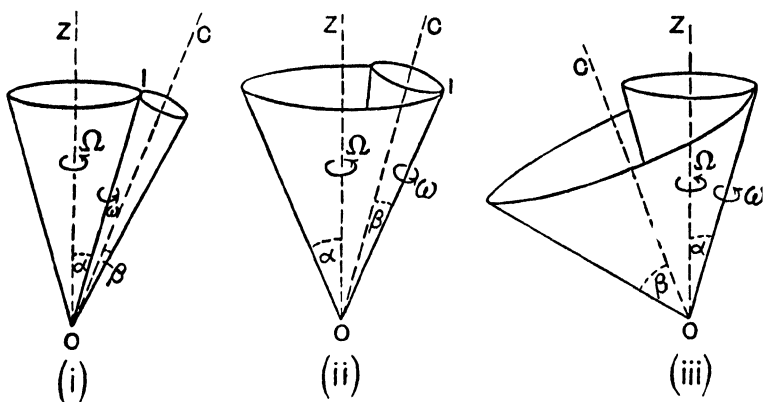
It is sometimes necessary to express the angular velocities $\omega_x, \omega_y, \omega_z$ of the body about the fixed axes OX, OY, OZ in terms of θ, ϕ, ψ , and we may make use of the foregoing direction cosines for this purpose;

$$\begin{aligned}\text{thus} \quad \omega_x &= \omega_1 \cos XA + \omega_2 \cos XB + \omega_3 \cos XC \\ &= -\dot{\theta} \sin \psi + \dot{\phi} \cos \psi \sin \theta, \\ \text{and similarly} \quad \omega_y &= \dot{\theta} \cos \psi + \dot{\phi} \sin \psi \sin \theta \\ \omega_z &= \dot{\phi} \cos \theta + \dot{\psi}\end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \dots\dots\dots(1);$$

or we may arrive at the same result if we impress on space and on the body an angular velocity equal and opposite to the resultant angular velocity of the body. The frame OX, OY, OZ then has an angular velocity with components $-\omega_x, -\omega_y, -\omega_z$, and OA, OB, OC become fixed; and the values of $-\omega_x, -\omega_y, -\omega_z$ are got from 3·7 (3), (4) and (5) by changing θ into $-\theta$, ϕ into $-\phi$ and ψ into $-\psi$.

3·8. Rolling cones. We have explained in 3·24 that the motion of a body about a fixed point can be produced by the rolling of a cone fixed in the body on a cone fixed in space. In

the case where the cones are right circular, let OZ be the axis of the cone fixed in space and OC that of the cone fixed in the body, α and β their semi-angles. The generator OI along which they touch one another is the instantaneous axis of rotation.



Let OI travel round the fixed axis OZ with angular velocity Ω . This is called the angular velocity of *precession*. Let the body turn about the instantaneous axis OI with angular velocity ω . There are three cases; the rolling cone may be convex and rolling on the (i) outside or (ii) inside of the fixed cone, or (iii) it may be concave and rolling outside the fixed cone. Because the cones touch along OI , therefore the lines OC , OI , OZ are always coplanar, so that a point P on OC describes a circle about OZ , and has a velocity

$$\Omega \cdot OP \sin(\alpha + \beta), \quad \Omega \cdot OP \sin(\alpha - \beta), \quad \Omega \cdot OP \sin(\beta - \alpha)$$

in cases (i), (ii) and (iii) where Ω is taken in the right-handed sense about OZ . And if we take the positive sense of the angular velocity ω to be right-handed about OI , by taking $\omega \cdot OP \sin \beta$ as the velocity of P and equating with proper regard to signs, we get

$$\text{in case (i)} \quad \Omega \sin(\alpha + \beta) = \omega \sin \beta,$$

$$\text{in case (ii)} \quad \Omega \sin(\alpha - \beta) = -\omega \sin \beta,$$

$$\text{and in case (iii)} \quad \Omega \sin(\beta - \alpha) = \omega \sin \beta.$$

Again, if we want to use the angular velocity of the instantaneous axis about the axis OC of the cone which it describes in the body and denote it by ω' , also taken positive in the right-handed sense about OC , we get by equating small arcs which come into contact in time δt

$$\omega' \delta t \cdot OI \sin \beta = -\Omega \delta t \cdot OI \sin \alpha,$$

or
$$\omega' \sin \beta = -\Omega \sin \alpha \quad \text{in case (i),}$$

and similarly

$$\omega' \sin \beta = \Omega \sin \alpha \quad \text{in cases (ii) and (iii).}$$

EXAMPLES

1. Prove that, if a spherical polygon turns about its angular points in succession, always remaining on the same spherical surface, and the angle turned through in each case is twice the angle of the polygon at the point, the polygon is restored to its original position.

2. Prove that a body, after successive rotations represented by doubles of the successive sides of a spherical polygon taken in order, is restored to its original position.

3. A right circular cone of semi-vertical angle α rolls without sliding on a plane. Shew that, if ω is the angular velocity of the line of contact, the angular velocity of the cone is $\omega \cot \alpha$.

4. A circular disc of radius a rolls on a plane at a constant inclination α and its centre describes a circle of radius c with angular velocity Ω . Shew that the angular velocity of the disc about its axis is $\Omega c/a$.

5. Two wheels, radii a and $2a$, are freely mounted at the ends of an axle, at right angles to it, and their rims rest on a rough floor on which they roll, the distance apart of the points of contact being b . Relative to a frame of reference rotating about the vertical with the axle, the angular velocities of the wheels are ω in opposite senses. Prove that the vertical plane containing the axle rotates with angular velocity $3a\omega/b$. [M. T. 1931]

6. Shew that, if the instantaneous axis of rotation of a body free to move about a fixed point is fixed in the body, it is also fixed in space.

7. Shew that the acceleration of a particle moving in a plane is equivalent to components $h^2 r/p^3 \rho$ along the radius vector to the origin and $\frac{h}{p^2} \frac{dh}{ds}$ along the tangent to the path, where p is the perpendicular from the origin on the tangent to the path and h is the moment of the velocity about the origin.

Find similar expressions for the components in the direction of the radius vector and the normal to the path.

8. A particle moves with constant velocity V along a straight line fixed in a plane, while the plane turns round a fixed axis perpendicular to itself, with angular velocity ω ; prove that the path of the particle is the curve

$$V\theta = V \cos^{-1}(a/r) + \omega \sqrt{(r^2 - a^2)},$$

where a is the least distance of the particle from the axis. Shew also that the accelerations along and at right angles to a perpendicular on the axis are respectively

$$\omega(2aV + r^2\omega)/r \quad \text{and} \quad 2V\omega\sqrt{(r^2 - a^2)}/r. \quad [\text{I. 1899}]$$

9. If the axes Ox , Oy revolve with constant angular velocity ω , and the component velocities of the point (x, y) parallel to the axes are

$$\omega y(a^2 - b^2)/(a^2 + b^2) \quad \text{and} \quad \omega x(a^2 - b^2)/(a^2 + b^2),$$

prove that the point describes relatively to the axes an ellipse in the periodic time

$$\frac{\pi}{\omega} \frac{a^2 + b^2}{ab}.$$

10. If the motion of a point in a plane be referred to two axes one of which is fixed while the other revolves about the origin so that the line joining the origin to the point is equally inclined at an angle $\frac{1}{2}\theta$ to the axes, shew that the component acceleration parallel to the fixed axis (ξ) is

$$\xi - (2\xi\dot{\theta} + \xi\ddot{\theta}) \operatorname{cosec} \theta,$$

and find the other component.

11. If the perpendiculars from a point P on axes Ox , Oy of which Ox is fixed and Oy revolves uniformly are ξ , η , prove that the accelerations of P parallel to the axes at the instant when they are perpendicular are

$$\ddot{\eta} - 2\xi\dot{\omega} + \eta\omega^2 \quad \text{and} \quad \ddot{\xi},$$

the angular velocity of Oy being ω .

12. The centre C of an elliptic wire is moved with uniform velocity along a fixed line CY in its own plane whilst the wire is in contact at P with a fixed line PY perpendicular to CY ; shew that the acceleration perpendicular to PY of the point of the wire in contact at P varies as $x^{-2}y^{-2}p^{-5}$, where $CY = p$, and x, y are the co-ordinates of P referred to the axes of the ellipse.

13. A point P moves so that its velocity is compounded of two constant velocities, one of which is in a fixed direction and the other is perpendicular to the line joining P to a fixed point. Find the orbit described by P .

14. A plane is moving about an axis perpendicular to it, and a point is moving in a given curve traced on the plane; in any position ω is the angular velocity of the plane, v the velocity of the particle relative to

the plane, r its distance from the axis, p the perpendicular on the tangent, s the arc described along the plane; prove that the acceleration along the tangent to the curve is

$$v \left(\frac{dv}{ds} + p \frac{d\omega}{ds} \right) - \omega^2 r \frac{dr}{ds}.$$

15. The position of a point in a plane is determined by the co-ordinates x, r , where r is its distance from the origin and x is measured parallel to a fixed direction. If u, v denote the component velocities in the directions x, r , express u, v in terms of x, r and their derivatives, and shew that the components of acceleration in the same direction are

$$\dot{u} + uv/r \quad \text{and} \quad \dot{v} - xuv/r^2.$$

16. The position of a point is given by the co-ordinates x, y, r , where x, y, z, r have their usual signification relative to rectangular axes. Shew that, if u, v, w denote the component velocities in the directions x, y, r , then the corresponding accelerations are

$$\dot{u} + uw/r, \quad \dot{v} + vw/r \quad \text{and} \quad \dot{w} - uwx/r^2 - vwy/r^2.$$

17. The component velocities u and v are measured along two families of orthogonal curves; prove that the corresponding accelerations are

$$\dot{u} - \kappa_1 uv + \kappa_2 v^2 \quad \text{and} \quad \dot{v} - \kappa_2 uv + \kappa_1 u^2,$$

where κ_1, κ_2 are the curvatures of the curves meeting at the particle considered positive in the directions of the velocities. [I. 1897]

18. A point moves in a plane with an angular velocity ω , and the plane is turning round the radius vector with an angular velocity ω' ; prove that the accelerations in the plane are $\ddot{r} - \omega^2 r$, and $r\dot{\omega} + 2\dot{r}\omega$, and that the acceleration perpendicular to the plane is $r\omega\omega'$.

19. A point P moves on a straight line OP which is made to describe uniformly a right circular cone about an axis OA , while OA sweeps out uniformly a right circular cylinder; find an expression for the acceleration of the point P in the direction OP .

20. The velocity of a point moving in a plane is the resultant of two velocities v and v' along radii r, r' measured from two fixed points at a distance a apart. Prove that the corresponding accelerations are

$$\dot{v} + vv'(r^2 - r'^2 + a^2)/2r^2r' \quad \text{and} \quad \dot{v}' + vv'(r'^2 - r^2 + a^2)/2r'^2r.$$

21. Shew that for the general motion of a rigid body the acceleration of any point P is expressible as the resultant of two components; of which one is along and proportional to the perpendicular drawn from P to a certain straight line; and the other is proportional to the perpendicular drawn from P to another line, and is perpendicular to the plane through P and that line. [M. T. 1907]

22. Prove that the meridian acceleration of a point moving on the surface of the anchor ring,

$$x = (c + a \sin \theta) \cos \phi, \quad y = (c + a \sin \theta) \sin \phi, \quad z = a \cos \theta,$$

is $a\ddot{\theta} - (c + a \sin \theta) \cos \theta \dot{\phi}^2$; and determine the normal and transverse accelerations. [I. 1894]

23. The position of a moving point is determined by the radii $1/\xi, 1/\eta, 1/\zeta$ of the three spheres which pass through it, and touch three fixed rectangular co-ordinate planes at the origin. Find the component velocities u, v, w of the point in the directions of the outward normals to the spheres, and shew that the component accelerations in the same directions are $\dot{u} + v(\eta u - \xi v) - w(\xi w - \zeta u)$, and two similar expressions. [I. 1896]

24. The centre of a smooth spherical bowl, of radius a , revolves with uniform angular velocity ω about a vertical axis at a distance c from the centre. Shew that the meridian acceleration of a particle moving on the surface of the sphere is

$$a\ddot{\theta} - a \sin \theta \cos \theta (\dot{\phi} + \omega)^2 - c\omega^2 \cos \phi \cos \theta,$$

where θ is the angular distance of the particle from the lowest point of the sphere, and ϕ is the angle between a vertical plane through the particle and the centre of the bowl and a plane through the centre and the axis. Determine also the normal and transverse accelerations. [I. 1898]

25. A particle moving in the sun's gravitational field is slightly disturbed from a circular orbit of radius a , described with angular velocity n . Shew that the equations of motion are

$$\begin{aligned} \xi - 2n\eta - 3n^2\xi &= X \\ \ddot{\eta} + 2n\dot{\xi} &= Y \end{aligned}$$

referred to rectangular axes whose origin moves in the circle of radius a with constant angular velocity n , the axis of ξ being away from the sun; X and Y are the components of acceleration due to disturbing bodies other than the sun.

If $X = \lambda \cos 2n't$, $Y = -\lambda \sin 2n't$, find the periodic parts of ξ and η .

[I. 1932]

26. A point O moves along a twisted curve with velocity v , and the co-ordinates of P , referred to the principal axes of the curve at O , are ξ, η, ζ . Shew that the velocities of P parallel to the moving axes are

$$v + \xi - \frac{\eta v}{\rho}, \quad \dot{\eta} - \frac{\zeta v}{\sigma} + \frac{\xi v}{\rho}, \quad \dot{\zeta} + \frac{\eta v}{\sigma},$$

where ρ and σ are the radii of curvature and torsion.

Prove that, if P lies at a constant distance c from O along the principal normal, OP will be the binormal to the locus of P if

$$\sigma^2 \rho = c(\rho^2 + \sigma^2). \quad [\text{M. T. 1919}]$$

Chapter IV

MOTION OF A PARTICLE IN THREE DIMENSIONS

4.1. Equation of energy. When a particle moves in free space or in contact with a smooth curve or surface, the equation of motion in the direction of the tangent to the path is

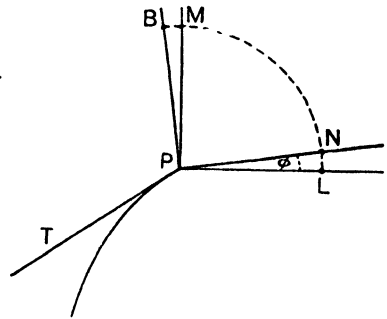
$$mvdv/ds = S \dots\dots\dots(1),$$

where m is the mass of the particle and S the component of the external force in the direction of the tangent. This gives

$$\frac{1}{2}m(v^2 - u^2) = \int S ds \dots\dots\dots(2),$$

where u, v denote the velocities at the ends of the range of integration, representing that the increase in kinetic energy is equal to the work done by external forces, or to the loss in potential energy [Pt. I, 4.4, and 6.4]. In such a case the reaction of a smooth curve or surface does no work.

4.11. Reaction of a smooth surface on which a particle is moving. Let P be the particle, PT, PN, PB the tangent, principal normal and binormal to the path, PL the normal to the surface and PM the tangent line to the surface at right angles to the path, so that the lines PL, PM, PN, PB are coplanar, being all at right angles to the path. Let the angle $LPN = \phi$.



If ρ' is the radius of curvature of the path and ρ that of the normal section through the same tangent line, then by Meunier's Theorem $\rho' = \rho \cos \phi$.

Now the acceleration along the normal PN to the path is v^2/ρ' , and this has components

$v^2 \cos \phi / \rho'$ or v^2 / ρ along PL , the normal to the surface,
and $v^2 \sin \phi / \rho'$ or $v^2 \tan \phi / \rho$ along PM .

Hence if R is the pressure of the surface measured inwards and L, M are the components of external force in the directions PL, PM , we have

$$mv^2/\rho = L + R \quad \dots\dots\dots(1)$$

and $mv^2 \tan \phi / \rho = M \quad \dots\dots\dots(2).$

Assuming the velocity to be found from the equation of energy, (1) gives the normal reaction and (2) serves to determine the position of the osculating plane at any point of the path.

If θ be the inclination of the direction of motion to the direction of least curvature of the surface and ρ_1, ρ_2 denote the greatest and least radii of curvature of normal sections at the point P of the surface, then by Euler's Theorem

$$\frac{1}{\rho} = \frac{\cos^2 \theta}{\rho_1} + \frac{\sin^2 \theta}{\rho_2},$$

and (1) takes the form

$$mv^2 \left(\frac{\cos^2 \theta}{\rho_1} + \frac{\sin^2 \theta}{\rho_2} \right) = L + R \quad \dots\dots\dots(3).$$

Thus in the case of a cylindrical surface ρ_1 is infinite, and

$$mv^2 \sin^2 \theta / \rho_2 = L + R \quad \dots\dots\dots(4),$$

where ρ_2 is the radius of curvature of the cross-section of the cylinder and θ is the angle at which the path cuts the generator.

4.12. Geodesic path. When a particle moves on a smooth surface, under the action of no forces save the reaction of the surface, then the path is a geodesic on the surface, for from 4.11 (2), $\tan \phi = 0$, so that the osculating plane at every point of the path contains the normal to the surface.

We notice that this result is also true when the surface is rough, for the friction, if any, acts along the path, and the above result is obtained by resolving at right angles to the path and is therefore unaffected by friction.

4.2. Motion of a heavy particle on a smooth surface of revolution the axis of which is vertical. Using cylindrical co-ordinates and measuring z vertically upwards, we have an equation of energy

$$\frac{1}{2}m(\dot{z}^2 + \dot{r}^2 + r^2\dot{\theta}^2) = C - mgz \quad \dots(1).$$

The only forces acting on the particle are the force of gravity parallel to the axis and the normal reaction intersecting the axis, therefore the moment of momentum about the axis is constant, that is

$$r^2\dot{\theta} = h \quad \dots\dots\dots(2).$$

If the equation of the surface is

$$z = f(r) \quad \dots\dots\dots(3),$$

we can now reduce (1) to the form

$$\frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \{f'(r)^2 + 1\} + \frac{1}{r^2} = C - \frac{2g}{h^2} f(r) \quad \dots\dots\dots(4),$$

which is the differential equation of the projection of the path on a horizontal plane.

The same result may be deduced from the equations of motion, thus: the components of acceleration are

$$\ddot{z}, \quad \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta}),$$

and therefore, resolving along the tangent PT to the meridian curve, we have

$$\ddot{z} \sin \psi + (\ddot{r} - r\dot{\theta}^2) \cos \psi = -g \sin \psi \quad \dots\dots\dots(5),$$

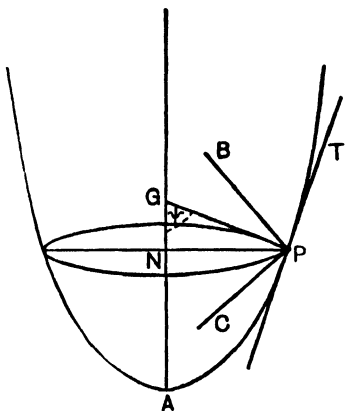
where ψ is the inclination of the normal PG to the axis; and, since there is no force at right angles to the meridian plane,

$$r^2\dot{\theta} = h \quad \dots\dots\dots(6),$$

so that $\dot{r} = \frac{h}{r^2} \frac{dr}{d\theta}$, and from (3) $\dot{z} = f'(r) \frac{h}{r^2} \frac{dr}{d\theta}$; also $\tan \psi = dz/dr = f'(r)$,

so that (5) reduces to

$$\frac{1}{r^4} \{1 + f'(r)^2\} \frac{d^2r}{d\theta^2} + \left[\frac{f'(r)f''(r)}{r^4} - \frac{2}{r^5} \{1 + f'(r)^2\} \right] \left(\frac{dr}{d\theta} \right)^2 - \frac{1}{r^3} = -\frac{g}{h^2} f'(r) \quad \dots\dots\dots(7).$$



Multiplying by $2dr/d\theta$ and integrating, we get as before

$$\frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \{1 + f'(r)^2\} + \frac{1}{r^2} = C - \frac{2g}{h^2} f(r) \quad \dots\dots\dots(8).$$

The normal pressure in the direction PG is found by resolving in that direction, thus

$$m\{\ddot{z} \cos \psi - (\ddot{r} - r\dot{\theta}^2) \sin \psi\} = R - mg \cos \psi \quad \dots\dots(9).$$

The condition that the path may be a circle of radius a is obtained from (7), namely

$$h^2 = ga^3 f'(a) \quad \dots\dots\dots(10).$$

This also follows from the consideration that the acceleration v^2/a in the circular path is maintained by gravity and the normal reaction, i.e. that $mv^2/a = R \sin \psi$ and $mg = R \cos \psi$, so that

$$v^2/ag = \tan \psi = f'(a), \quad \text{or} \quad h^2 = ga^3 f'(a).$$

4.21. Motion in limited zones. The velocity v in any position is given by the energy equation

$$v^2 = 2g(c - z) \quad \dots\dots\dots(1),$$

where the constant c depends on the initial conditions. Also the constant moment h of the velocity about the axis must lie between zero and rv . It would be the former if the motion were in a vertical plane, and the latter if v were horizontal. Hence, for an assigned value of h , the particle can only occupy positions on the surface for which

$$rv \geq h,$$

$$\text{i.e., from (1),} \quad 2gr^2(c - z) \geq h^2.$$

Hence, if we construct the cubic surface of revolution

$$2gr^2(c - z) = h^2 \quad \dots\dots\dots(2),$$

the r of the particle for an assigned value of z must be greater than or equal to the r of this surface.

The cubic surface (2) in general divides the given surface $z = f(r)$ into zones separated by horizontal circles, and the particle can only move with the given initial conditions on a zone of the given surface which is more remote from the axis

than the corresponding part of the cubic surface. And the motion of the particle must clearly be confined to the zone on which it starts, so that it rises and falls between two horizontal circles.

4.22. When the particle is projected horizontally to determine whether it will rise or fall. In the figure of 4.2 let PC , PB be the principal normal and binormal to the path in the initial position of the particle. Since the path is horizontal at P , PC and PB are both in the vertical plane through P . Let $CPG = \phi$, and let ρ' be the radius of curvature of the path; then since PG is the radius of curvature of the normal section at right angles to the meridian plane (as the figure revolves round the axis the normal at P continues to pass through G), therefore by Meunier's Theorem

$$\rho' = PG \cos \phi = r \operatorname{cosec} \psi \cos \phi \dots\dots\dots(1),$$

where $NP = r$.

The only acceleration of the particle in the vertical plane is v^2/ρ' along the normal PC to its path; therefore the acceleration along TP is $v^2 \sin \phi / \rho'$, so that

$$v^2 \sin \phi / \rho' = g \sin \psi$$

or

$$v^2 \tan \phi = gr \dots\dots\dots(2).$$

Now the particle rises or falls according as the osculating plane of its path is above or below PN ; i.e. according as

$$\phi < \text{or} > \frac{1}{2}\pi - \psi,$$

i.e. according as

$$\tan \phi < \text{or} > \cot \psi,$$

or as

$$gr < \text{or} > v^2 \cot \psi \dots\dots\dots(3).$$

Eliminating ϕ between (1) and (2) gives the initial radius of curvature of the path, namely

$$\rho' = \frac{v^2 r}{\sin \psi \sqrt{(v^4 + g^2 r^2)}} \dots\dots\dots(4).$$

4.23. To find the apsidal angle when the projection of the path on a horizontal plane is nearly circular. When the particle is describing a horizontal circle on the smooth surface

of revolution we can imagine the motion slightly disturbed by a small impulse in the meridian plane through the particle so that h is unaltered.

We put $r = a + x$, where x is small and neglect $(dx/d\theta)^2$, then the differential equation 4.2 (7) becomes

$$\{1 + f'(a)^2\} \frac{d^2x}{d\theta^2} - a - x = -\frac{g}{h^2} (a+x)^4 f'(a+x),$$

which from 4.2 (10)

$$= -\frac{1}{a^3 f'(a)} (a^4 + 4a^3x) \{f'(a) + xf''(a)\},$$

or
$$\{1 + f'(a)^2\} \frac{d^2x}{d\theta^2} + \left\{3 + \frac{af''(a)}{f'(a)}\right\} x = 0 \dots\dots\dots(1);$$

so that the apsidal angle is

$$\pi \{[1 + f'(a)^2] / [3 + af''(a)/f'(a)]\}^{\frac{1}{2}}.$$

This also shews that the motion is stable if

$$3 + af''(a)/f'(a) \text{ is positive.}$$

4.24. Motion of a heavy particle on a smooth sphere. Let O be the centre and a the radius of the sphere. The position of the particle can be defined by polar co-ordinates θ, ϕ , θ being measured from the downward vertical diameter.

The horizontal velocity is $a\phi \sin \theta$, so that

$$a^2 \phi^2 \sin^2 \theta = h^2 \dots\dots\dots(1);$$

and the energy equation is

$$a^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) = C + 2ga \cos \theta \dots\dots(2).$$

Whence, by eliminating ϕ , we get

$$a^2 \dot{\theta}^2 + \frac{h^2}{a^2 \sin^2 \theta} = C + 2ga \cos \theta \dots\dots\dots(3).$$

The constant C depends on the initial conditions. Suppose that the particle is projected with horizontal velocity u from a position $\theta = \alpha$. Then

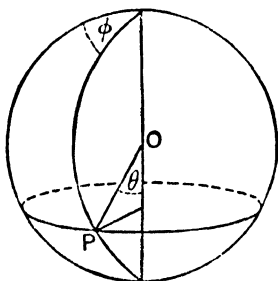
$$u^2 = C + 2ga \cos \alpha \dots\dots\dots(4)$$

and

$$h = ua \sin \alpha \dots\dots\dots(5),$$

so that (3) may be written

$$a^2 \dot{\theta}^2 + u^2 \left(\frac{\sin^2 \alpha}{\sin^2 \theta} - 1 \right) = 2ga (\cos \theta - \cos \alpha) \dots\dots\dots(6).$$



To find the greatest and least values of θ , we put $\theta = 0$ and get either $\theta = \alpha$ or

$$u^2 (\cos \theta + \cos \alpha) = 2ga \sin^2 \theta \quad \dots\dots\dots(7).$$

This quadratic in $\cos \theta$ may be written

$$f(\cos \theta) \equiv 2ga \cos^2 \theta + u^2 \cos \theta + u^2 \cos \alpha - 2ga = 0 \quad \dots\dots(8),$$

so that

$$f(\cos 0) = u^2 (1 + \cos \alpha) \text{ is positive,}$$

$$f(\cos \alpha) = 2(u^2 \cos \alpha - ga \sin^2 \alpha),$$

$$f(\cos \pi) = u^2 (-1 + \cos \alpha) \text{ is negative.}$$

Hence there is a value of θ for which θ vanishes and it lies between 0 and α or between α and π according as

$$ga \sin \alpha > \text{ or } < u^2 \cot \alpha \quad \dots\dots\dots(9).$$

This verifies that the particle is confined to a zone between two horizontal circles, and (9) gives the criterion as to whether the particle initially falls below or rises above the circle $\theta = \alpha$ in accordance with 4.22.

Further, if we pursue the method of 4.21 and put $r = a \sin \theta$ and $z = a(1 - \cos \theta)$, we find that the cubic surface of revolution has for equation what we get by putting $\theta = 0$ in (6); so that the intersections of the cubic surface with the sphere are the circle $\theta = \alpha$ and the other circle determined by (8) as above.

To find the pressure of the sphere on the particle, if we use cylindrical co-ordinates and measure z downwards from the centre we have

$$m\{\ddot{z} \cos \theta + (\ddot{r} - r\dot{\phi}^2) \sin \theta\} = mg \cos \theta - R \quad \dots\dots\dots(10),$$

and, putting $z = a \cos \theta$ and $r = a \sin \theta$, this gives

$$R = mg \cos \theta + ma(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta),$$

or, using (2) and (4),

$$R = mg(3 \cos \theta - 2 \cos \alpha) + mu^2/a \quad \dots\dots\dots(11).$$

We have obtained this result without considering the position of the osculating plane of the path, but we notice that it accords with 4.11 (1), since in this case

$$v^2 = u^2 + 2ga(\cos \theta - \cos \alpha).$$

To determine the position of the osculating plane at any point of the path, say when the path cuts the meridian circle at an angle ψ , we make use of 4.11 (2). Let ϕ_1 denote the angle between the principal normal to the path and the normal PO to the sphere, then

$$(mv^2 \tan \phi_1)/a = M,$$

where M is the component of the weight mg resolved at right angles to the path in the tangent plane to the sphere, i.e. $M = mg \sin \theta \sin \psi$, so that

$$\tan \phi_1 = ga \sin \theta \sin \psi / v^2 \quad \dots\dots\dots(12).$$

4.25. Examples. 1. A spherical pendulum is started horizontally at the level of the centre. Prove the equations of motion in azimuth and depth, viz.

$$(i) (l^2 - z^2) \dot{\psi} = V l, \quad (ii) l^2 \dot{z}^2 = z \{2g(l^2 - z^2) - V^2 z\},$$

and verify that, if $z = l \cos \theta$,

$$\sin \theta \sin (\dot{\psi} - V t / 2l) = (V / \sqrt{2gl}) \sqrt{\cos \theta}. \quad [C. 1906]$$

2. A particle moves under gravity on the surface of a smooth sphere of 1 metre radius between two horizontal circles at depths 40 centimetres and 50 centimetres below the centre. Shew that the velocity of the particle ranges between 404 and 428 centimetres per second.

[M. T. 1919]

3. A particle moves under gravity on a smooth spherical surface whose centre is O . If A be a point at which the velocity is horizontal, prove that the path is symmetrical with respect to the vertical plane through OA ; and hence infer that the whole path is included between two horizontal circles which it alternately touches.

If those circles be nearly coincident, at an angular distance α from the lowest point of the sphere, find the apsidal angle. [M. T. 1911]

4. A heavy particle P moves on the smooth inner surface of a fixed sphere of centre O , and the angle between OP and the downward vertical is denoted by θ . The particle is projected horizontally on the surface from a point at which $\theta = \alpha$ ($< \frac{1}{2}\pi$). Prove that, whatever be the velocity of projection, θ will not exceed $\pi - \alpha$ in the subsequent motion, and that, if $\sin \alpha > \frac{1}{2}$, the particle will not leave the surface.

[M. T. 1922]

5. A particle tied to a fixed point by an inextensible string is projected under gravity in any manner so that the string remains taut in the subsequent motion. Find the differential equation of the projection of the path on a horizontal plane and prove that the (p, r) equation of this curve is of the form

$$l^2/p^2 = 1 + A(l^2 - r^2) + B(l^2 - r^2)^{\frac{3}{2}},$$

where l is the length of the string, and A, B are constants.

Prove that the curve has no points of inflexion.

[M. T. 1913]

6. A particle is projected horizontally under gravity with velocity V from a point on the inner surface of a smooth sphere at an angular distance α from the lowest point. Prove that the pressure on the surface when it is at an angular distance θ from the lowest point is

$$mg(3 \cos \theta - 2 \cos \alpha + V^2/ag),$$

where m is the mass of the particle and a the radius of the sphere.

Prove that in the subsequent motion the particle will leave the surface if $3 \sin \alpha < 1$ and $2V^2/ag - 7 \cos \alpha$ lies between $\pm 3\sqrt{1 - 9 \sin^2 \alpha}$.

[I. 1912]

7. A heavy particle of mass m moves on the smooth inner surface of a sphere of radius a and its greatest and least depths below the centre are $\frac{1}{2}a$ and $\frac{1}{4}a$. Shew that when the depth below the centre is z , the normal reaction of the sphere is $3mg(z + \frac{1}{2}a)/a$.

Shew also that the time from maximum to minimum depth is

$$\sqrt{\left(\frac{a}{g}\right)} \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1 - \frac{1}{8} \sin^2 \phi)}}. \quad [\text{M. T. 1933}]$$

8. A heavy particle is projected horizontally along the inner surface of a smooth sphere, with a velocity due to a fall from the level of the centre to the point of projection. Shew that the radius of curvature of its path when it is at an angular distance θ from the lowest point of the sphere is

$$a/(1 + \frac{1}{4} \sin^2 \alpha \cos \alpha \sec^3 \theta)^{\frac{1}{2}},$$

where α is the initial value of θ , and a is the radius of the sphere.

[I. 1898]

9. A particle is projected with velocity V along the inside of a smooth sphere of radius a along a parallel whose latitude is λ . It is repelled from the point upon the sphere whose latitude is 90° by a force per unit mass equal to μ times the distance from that point. Find the rates of increase of latitude and longitude of the particle when its latitude is θ and shew that its path cuts the equator at

$$\tan^{-1} \left\{ \tan \lambda \left(1 + \frac{2a^2\mu}{V^2} \operatorname{cosec} \lambda \right)^{\frac{1}{2}} \right\}. \quad [\text{I. 1926}]$$

10. A heavy particle moves on the smooth surface of a paraboloid of revolution whose axis is vertical and vertex downwards, being projected horizontally with velocity $\sqrt{2gh_2}$ at a height h_1 above the vertex. Shew that the orbit touches alternately the two horizontal circles at heights h_1, h_2 above the vertex, and that the least value of the angle at which the orbit cuts the meridian is

$$\tan^{-1} \frac{2\sqrt{(h_1 h_2)}}{|h_1 - h_2|}. \quad [\text{M. T. 1925}]$$

11. A heavy particle is projected horizontally with velocity u along the inner surface of a smooth vertical circular cylinder of radius a . Shew that the radius of curvature at any point of the path is

$$av^3/u\sqrt{(u^2v^2 + g^2a^2)},$$

where v is the velocity at the point.

[I. 1908]

12. The surface of a smooth funnel is given by the equation

$$z = c^2(x^2 + y^2)^{-\frac{1}{2}},$$

the positive axis of z being vertically downwards. A particle is projected horizontally, along the inner surface, at the level $z = c$, with such a velocity that it is again moving horizontally at the level $z = 2c$. Prove that the principal normal, at the highest point of the path, makes an angle $\tan^{-1} \frac{1}{2}$ with the horizontal.

[I. 1929]

13. A particle is moving under gravity in contact with the inside of the smooth surface of revolution $z=f(r)$, the positive z axis being vertical and upwards. Originally the particle is describing a horizontal circle of radius a , its angular momentum about the z axis being h . It is then slightly disturbed so that the angular momentum in the subsequent motion is $h + \delta h$. Prove that the particle oscillates with period $2\pi/n$, where

$$n^2 = \frac{a}{g} \cdot \frac{3f'(a) + af''(a)}{1 + f'^2(a)},$$

about the circle $r = a + \delta a$, where

$$\delta a = \frac{2\delta h a f'(a)}{h \{3f'(a) + af''(a)\}}. \quad [\text{I. 1924}]$$

4.26. Motion of a heavy particle on the surface of a smooth right circular cone with axis vertical and vertex downwards.

Let PP' be an element of the path, O the vertex of the cone and α its semi-vertical angle. Let the generator OP' cut the circular section through P in Q . Using cylindrical co-ordinates r, θ, z ,

$$OC = z, \quad CP = r, \quad PCQ = d\theta.$$

Taking the acceleration along the generator, we have

$$(\ddot{r} - r\dot{\theta}^2) \sin \alpha + \ddot{z} \cos \alpha = -g \cos \alpha,$$

or, since $z = r \cot \alpha$,

$$\ddot{r} - r\dot{\theta}^2 \sin^2 \alpha = -g \sin \alpha \cos \alpha \dots (1).$$

We have also $r^2 \dot{\theta} = h$, and if we write $1/u$ for r , we find that (1) transforms to

$$\frac{d^2 u}{d\theta^2} + u \sin^2 \alpha = \frac{g \sin \alpha \cos \alpha}{h^2 u^2} \dots \dots \dots (2)$$

and this is the differential equation of the projection of the path on a horizontal plane.

Further, if r' denotes distance from the vertex and u' its reciprocal, and if $d\theta'$ denotes the angle POP' , we have

$$r = r' \sin \alpha \quad \text{and} \quad r d\theta = PQ = r' d\theta',$$

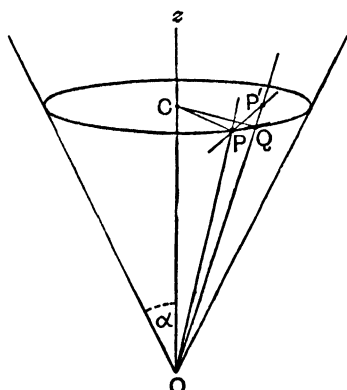
so that $u = u' \operatorname{cosec} \alpha$ and $d\theta = d\theta' \operatorname{cosec} \alpha$

$$\text{and (2) becomes} \quad \frac{d^2 u'}{d\theta'^2} + u' = \frac{g \sin^2 \alpha \cos \alpha}{h^2 u'^2} \dots \dots \dots (3).$$

It follows that if the cone be developed into a plane the trace of the path on the surface will be the same as if a particle moved in a plane under the action of a constant central force.

In the general case there is an energy equation

$$\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 + 2gz = C,$$



where $r = z \tan \alpha$; and $r^2 \dot{\theta} = h$, a constant; so that

$$\dot{z}^2 \sec^2 \alpha + \frac{h^2 \cot^2 \alpha}{z^2} + 2gz = C \dots\dots\dots (4)$$

gives the vertical motion.

4.27. Examples. 1. Prove that if particles move on a right circular cone under no forces, the projections of their paths on a plane perpendicular to the axis are similar curves of the type $r \sin n\theta = c$, whatever be their initial velocities. [C. 1913]

2. A heavy particle moves on a smooth right circular cone with axis vertical and vertex downwards, being projected with the velocity due to a fall from a height z_0 above the vertex to the point of projection. Prove that, if z_1, z_2 are the greatest and least heights attained, then

$$z_1^3 + z_1 z_2 + z_2^3 = z_0 (z_1 + z_2). \quad [\text{I. 1901}]$$

3. A particle is projected horizontally along the inner surface of a smooth cone whose axis is vertical and vertex upwards. Find the pressure at any point in terms of the depth below the vertex; and shew that the particle leaves the cone at a depth $(V^2 h^2 / g \tan^2 \alpha)^{\frac{1}{2}}$, where h is the initial depth, V the initial velocity and α the semi-angle of the cone. [M. T. 1912]

4. A heavy particle moves on the external surface of a smooth circular cone, which is fixed with its axis vertical and vertex upwards. The velocity of the particle is that which it would acquire by sliding from the vertex. Shew that the path of the particle, when developed into a plane, has a polar equation of the form

$$r^3 = a^3 \sec^2 \frac{3}{2} \theta,$$

referred to the vertex as origin.

[M. T. 1935]

5. A particle moves along the smooth surface of a right circular cone under the action of a force parallel to the axis of the cone and proportional to the distance of the particle from the axis, its initial velocity being that which it would acquire in moving from the vertex. Prove that its path when the cone is developed into a plane is a rectangular hyperbola. [I. 1907]

6. A particle moves on a smooth right circular cone under the action of a force from the vertex, the law of repulsion being

$$\mu \left(\frac{a \cos^2 \alpha}{r^2} - \frac{1}{2r^2} \right),$$

where α is the semi-vertical angle of the cone. Prove that, if it be projected from an apse at distance a with velocity $\sqrt{(\mu/a)} \sin \alpha$, the path will be a parabola. [I. 1899]

4.3. Motion of a particle on a rough surface. It was shewn in 4.12 that *when a particle moves on a rough surface under the*

action of no forces but the reaction of the surface its path is a geodesic.

The method of solving problems of the motion of a particle on a rough surface will be illustrated by an example.

A particle moves on the surface of a rough circular cone under the action of no forces. It is projected with velocity V at right angles to a generator at a distance d from the vertex. Shew that, when it has moved through a distance s , its velocity v is given by

$$\log(V/v) = \mu s / \{\tan \alpha (s^2 + d^2)^{\frac{1}{2}}\},$$

where μ is the coefficient of friction and α the half-angle of the cone.

[M. T. 1921]

Let P be the particle after moving a distance s . Let the normal to the cone at P meet the axis in G , and let r be the distance of P from the vertex. Then if R denotes the normal reaction and μR the friction, we have equations of motion, in the direction of the tangent,

$$mv \frac{dv}{ds} = -\mu R \dots\dots\dots(1)$$

and in the direction of the normal, as in 4.11 (1),

$$mv^2/\rho = R \dots\dots\dots(2),$$

where ρ is the radius of curvature of the normal section of the cone through the tangent line to the path. There is also the equation obtained by resolving at right angles to the path in the tangent plane, which proves, as stated above, that the path is a geodesic. But the equations of a geodesic on any surface of revolution

$$x^2 + y^2 = f(z)$$

are
$$\frac{d^2x}{ds^2} \bigg/ 2x = \frac{d^2y}{ds^2} \bigg/ 2y = \frac{d^2z}{ds^2} \bigg/ -f'(z).$$

Hence, by integration,

$$x \frac{dy}{ds} - y \frac{dx}{ds} = \text{constant};$$

or, in cylindrical co-ordinates r', θ ,

$$r'^2 \frac{d\theta}{ds} = \text{constant}.$$

Therefore if ψ be the inclination of the curve to the generator, in this case, we have

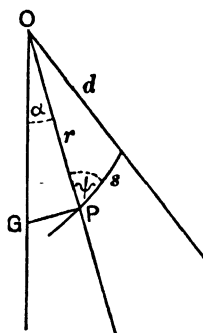
$$r \sin \psi = \text{constant} = d \dots\dots\dots(3).$$

Again,

$$dr/ds = \cos \psi = \sqrt{1 - d^2/r^2},$$

from which we find, since $r = d$, when $s = 0$, that

$$r^2 = d^2 + s^2.$$



Again, by Euler's theorem on curvature,

$$\frac{1}{\rho} = \frac{\cos^2 \psi}{\infty} + \frac{\sin^2 \psi}{PG} = \frac{d^2}{r^3 \tan \alpha},$$

so that, from (1) and (2), $\frac{dv}{v} + \frac{\mu d^2 \cdot ds}{r^3 \tan \alpha} = 0$,

or
$$\frac{dv}{v} + \frac{\mu d^2 \cdot ds}{(s^2 + d^2)^{\frac{3}{2}} \tan \alpha} = 0.$$

Therefore
$$\log v + \frac{\mu s}{\tan \alpha (s^2 + d^2)^{\frac{1}{2}}} = C,$$

but $v = V$ when $s = 0$, so that

$$\log (V/v) = \mu s / \{ \tan \alpha (s^2 + d^2)^{\frac{1}{2}} \}.$$

4.31. Examples. 1. A particle is moving under gravity in contact with the inside of a rough circular cylinder with vertical generators. At time t the particle is moving with velocity V in a direction that makes an acute angle ϕ with the downward vertical; establish the equation

$$\frac{1}{V} \frac{dV}{d\phi} = \frac{\mu V^2}{ga} \sin \phi - \cot \phi,$$

where a is the radius of the cylinder and μ the coefficient of friction between the particle and the cylinder. If the velocity of the particle was U when it was moving horizontally, prove that at time t the horizontal component, u , of its velocity is given by the equation

$$\frac{1}{U^2} - \frac{1}{u^2} = \frac{2\mu}{ga} \log \tan \frac{1}{2} \phi. \quad [\text{M. T. 1934}]$$

2. A particle moves on a rough circular cylinder under the action of no external forces. Prove that the space described in time t is

$$(a/\mu) \sec^2 \alpha \log (1 + \mu V t \cos^2 \alpha / a),$$

where the particle has initially a velocity V in a direction making an angle α with the transverse plane of the cylinder. [M. T. 1888]

3. A heavy particle moves on a rough vertical circular cylinder and is projected horizontally with a velocity V . Prove that at the point where the path cuts the generator at an angle ϕ , the velocity v is given by

$$ag/v^2 \sin^2 \phi = ag/V^2 + 2\mu \log (\cot \phi + \operatorname{cosec} \phi),$$

and that the azimuthal angle θ and vertical descent z are given by $ag\theta = \int v^2 d\phi$ and $gz = \int v^2 \cot \phi d\phi$, the limits being $\frac{1}{2}\pi$ to ϕ . [M. T. 1888]

4.4. Motion of a particle in a smooth plane tube which revolves about an axis in its plane. Take the axis of rotation as the axis of z and use cylindrical co-ordinates. Let ψ be the inclination of the normal to the axis, and let S , N , T be the components of the external force on the particle along the

tangent and normal to the curve and perpendicular to its plane, and R, R' the pressures of the tube on the particle in the two latter directions. The equations of motion are

$$m\{(\ddot{r} - r\dot{\theta}^2)\cos\psi + \ddot{z}\sin\psi\} = S \dots (1),$$

$$m\{-(\ddot{r} - r\dot{\theta}^2)\sin\psi + \ddot{z}\cos\psi\} = R + N \dots (2),$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = R' + T \dots (3).$$

We notice that (1) is the same as

$$mv dv/ds = S + mr\omega^2 \cos\psi \dots (4)$$

if v denotes the velocity of the particle relative to the tube and ω the angular velocity of the tube. So that in the case of uniform angular velocity the motion of the particle relative to the tube is the same as if the tube were at rest and an additional radial force $m\omega^2 r$ were applied to the particle; but this hypothesis will not be sufficient for the determination of the reaction R' .

4.41. Example. Let the tube be parabolic with its vertex downwards, and revolving uniformly about its axis which is vertical.

In this case $S = -mg\sin\psi$, $N = -mg\cos\psi$, $T = 0$,

$$\dot{\theta} = \omega, \quad r^2 = 4az \quad \text{and} \quad \tan\psi = r/2a.$$

Hence (4) may be written

$$v \frac{dv}{ds} = -g \frac{dz}{ds} + r\omega^2 \frac{dr}{ds},$$

or

$$v^2 = C - 2gz + r^2\omega^2,$$

i.e.

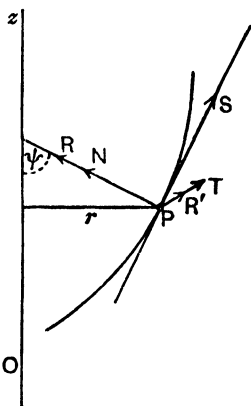
$$v^2 = C + r^2(\omega^2 - g/2a).$$

If initially the particle is at relative rest in the tube where $r = c$, then

$$v^2 = (\omega^2 - g/2a)(r^2 - c^2).$$

This shews that if $2a\omega^2 = g$ the particle remains at relative rest; and that otherwise it rises or falls according as $2a\omega^2 > \text{or} < g$.

4.42. Motion of a heavy uniform chain inside a fine smooth tube which is revolving uniformly about a vertical axis in its own plane. Let PQ be an element $\delta\sigma$ of the chain



AB . Let $AP = \sigma$, and let s be the arc of the tube measured from a fixed point to A . Take the axis of rotation as the axis of z and let r, θ, z be the cylindrical co-ordinates of P . Let the normal at P be inclined to the vertical at an angle ψ . Then if T is the tension at P , the equation of motion of the element PQ in the direction of the tangent is

$$m\delta\sigma\{\ddot{z}\sin\psi + (\ddot{r} - \omega^2 r)\cos\psi\} = \delta T - mg\delta\sigma\sin\psi \dots(1).$$

Since σ is not a function of the time, we have

$$\dot{z} = \dot{s}\sin\psi \quad \text{and} \quad \dot{r} = \dot{s}\cos\psi \dots(2),$$

$$\text{so that} \quad \ddot{z}\sin\psi + \ddot{r}\cos\psi = \ddot{s} \dots\dots\dots(3).$$

This also follows from the fact that the two terms on the left are what the tangential acceleration would be if there were no rotation. Therefore (1) may be written

$$m\delta\sigma(\ddot{s} - \omega^2 r\cos\psi) = \delta T - mg\delta\sigma\sin\psi$$

$$\text{or} \quad m\delta\sigma.\ddot{s} = \delta T + m\omega^2 r\delta r - mg\delta z \dots\dots\dots(4).$$

Hence if l be the length of the chain AB , by integrating along AB we get

$$l\ddot{s} = \frac{1}{2}\omega^2(r_1^2 - r_2^2) - g(z_1 - z_2) \dots\dots\dots(5),$$

where r_1, z_1 refer to B and r_2, z_2 to A .

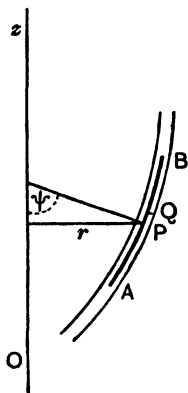
The pressure of the tube on the chain, R per unit length, may be found by a normal resolution; thus

$$m\delta\sigma\{\ddot{z}\cos\psi - (\ddot{r} - \omega^2 r)\sin\psi\} = R\delta\sigma + T\delta\psi - mg\delta\sigma\cos\psi \dots\dots(6);$$

and by making use of (2) this becomes

$$m\dot{s}^2/\rho + m\omega^2 r\sin\psi = R + T/\rho - mg\cos\psi \dots\dots(7).$$

To proceed further in a particular case, we observe that in (5), r_1, r_2, z_1, z_2 are functions of s , so that $\ddot{s}^2$ can be found by integrating (5), and T can be found by eliminating $\ddot{s}$ between (4) and (5), then from (7) we can find R .



4.43. Example. Let the tube be circular, of radius a , and let the chain AB subtend an angle 2α at the centre and be at relative rest in its lowest position. To determine whether the equilibrium is stable or not.

Let the chain be slightly displaced as in the figure. Then 4.42 (5) takes the form

$$2a^2\alpha\ddot{\theta} = \frac{1}{2}\omega^2a^2\{\sin^2(\alpha+\theta) - \sin^2(\alpha-\theta)\} - ga\{\cos(\alpha-\theta) - \cos(\alpha+\theta)\},$$

or

$$2a\alpha\ddot{\theta} = \frac{1}{2}\omega^2a\sin 2\alpha \sin 2\theta - 2g\sin\alpha \sin\theta \dots\dots\dots(1).$$

For small values of θ this reduces to

$$a\alpha\ddot{\theta} + \theta\sin\alpha(g - a\omega^2\cos\alpha) = 0 \dots\dots\dots(2),$$

giving an oscillatory motion or not, according as

$$a\omega^2\cos\alpha < \text{or} > g.$$

But if $a\omega^2\cos\alpha = g$, then (1) becomes

$$a\alpha\ddot{\theta} = g\sin\alpha \sin\theta(\cos\theta - 1) \dots\dots\dots(3),$$

so that $\ddot{\theta}$ is negative and this implies that the position of relative equilibrium is stable.

In problems of this kind the test of stability might be defined in two ways: either the system receives a small impulse causing it to move from its position of relative equilibrium, or the system receives a definite small displacement and is then set free, and in either case, for stability, it should move towards the equilibrium position.

In the case considered, if the chain be given a relative motion expressed by θ_0 when $\theta=0$, by integrating (3) and adjusting the constant of integration we get

$$a\alpha\dot{\theta}^2 = a\alpha\dot{\theta}_0^2 - g\sin\alpha(1 - \cos\theta)^2 \dots\dots\dots(4),$$

which shews that, if $\dot{\theta}_0$ is small, $\dot{\theta}$ will certainly vanish and the negative value of $\ddot{\theta}$ given by (3) will then reverse the motion.

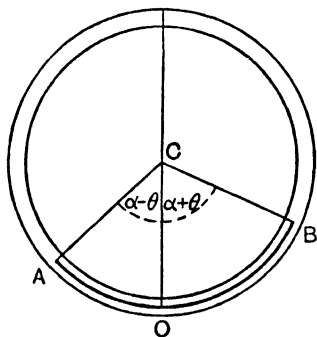
On the other hand if we imagine the chain to start from relative rest in the position $\theta = \beta$, then by integrating (3) and adjusting the constant, we get

$$a\alpha\dot{\theta}^2 = g\sin\alpha(\cos\beta - \cos\theta)(\cos\beta + \cos\theta - 2) \dots\dots\dots(5).$$

The last factor on the right is negative, therefore the motion must be such that $\cos\theta > \cos\beta$, or $\theta < \beta$, i.e. towards the position of relative equilibrium; so that by either definition the equilibrium is stable.

4.44. Examples. 1. A circular wire of radius a rotates with uniform angular velocity ω about its vertical diameter. Prove that a small smooth ring on the wire placed at the lowest point is in stable equilibrium if $a\omega^2 < g$, and investigate carefully the case in which $a\omega^2 = g$.

[M. T. 1902]



2. A smooth circular wire of radius a rotates with uniform angular velocity ω round the tangent at the end A of a diameter AB of the wire. A small ring of unit mass is free to slide on the wire under the action of a repulsive force μr from A , r being the distance from A . If $\omega^2 = 7\mu$ and the ring is initially at B with a velocity $4a\sqrt{2\mu}$ relative to the wire, shew that after a time t its angular distance from B is

$$2 \tan^{-1} (2\sqrt{2} \sinh t\sqrt{\mu}). \quad [\text{I. 1907}]$$

3. A smooth thin circular tube of radius a rotates uniformly at a rate ω about a vertical diameter and within it are two equal particles connected by a light string along the arc below them. Find the tension of the string when the particles are at rest relative to the tube with the radii to the particles at equal angles α on opposite sides of the downward vertical.

If the string breaks, shew that the maximum angular distance from the vertical attained by the radii to the particles is $\cos^{-1} \left(\frac{2g}{a\omega^2} - \cos \alpha \right)$.
[I. 1925]

4. A fine smooth tube in the form of a circle of radius a is made to revolve with angular velocity $\sqrt{g/a}$ about a vertical axis in its plane distant μa from its centre. Shew that if a smooth particle in the tube oscillates so that it just reaches the lowest point of the tube, the time t measured from this instant to the instant where the central radius to the particle makes an angle θ with the downward vertical is given by

$$\cot \frac{1}{2}\theta = \wp \left(\sqrt{\frac{\mu g}{4a}} t, -4, \frac{4}{\mu} \right). \quad [\text{M. T. 1904}]$$

5. A heavy bead slides on a smooth elliptic wire whose major axis is vertical, and the wire is made to rotate round the vertical axis with uniform angular velocity ω . Find the unsymmetrical position of relative equilibrium, and shew that the time of a small oscillation about it is

$$\frac{2\pi a}{b\omega} \sqrt{\left(\frac{b^4\omega^4 - a^2e^2g^2}{b^4\omega^4 - a^2g^2} \right)},$$

where a, b are the semi-axes of the ellipse.

[I. 1899]

6. A smooth wire in the form of an ellipse of semi-axes a and b ($a > b$) is spun with angular velocity ω about its major axis which is vertical. A bead is slightly displaced from rest at the lowest point of the wire. Prove that it returns to the lowest point or departs from it according as ω^2 is less or greater than ga/b^3 , and that in the latter case it attains a maximum distance ϖ from the axis of rotation given by

$$\varpi^2 = \frac{4ga}{\omega^2} \left(1 - \frac{ga}{b^2\omega^2} \right). \quad [\text{M. T. 1928}]$$

7. A thin smooth tube in the form of a parabola of latus rectum $4a$ rotates with constant angular velocity ω about its axis, which is vertical. A particle is at rest at a height x_0 above the vertex. Shew that,

if the angular velocity is reduced to ω' , the particle will oscillate from the level x_0 in the one limb to the same level in the other limb in time

$$\frac{2}{\sqrt{(\omega^2 - \omega'^2)}} \int_0^{\frac{1}{2}\pi} \sqrt{\left(1 + \frac{x_0}{a} \sin^2 \theta\right)} d\theta. \quad [\text{M. T. 1927}]$$

8. A smooth wire, in the form of that part of the curve

$$y = b \cos \frac{x}{a}$$

which lies between $x = -a\pi$ and $x = +a\pi$, a and b being positive numbers, rotates with constant angular velocity ω about the axis of y , which is vertically downwards. Prove that, if $a^2\omega^2 < bg$, there are three positions of relative equilibrium for a bead which can slide on the wire, and that two of these are unstable. [I. 1930]

4.5. Motion of a particle on a revolving tortuous smooth curve. It is convenient, in this case, to take the axis of rotation as the axis of z and either take axes of x and y fixed relatively to the curve and revolving with it, or use cylindrical co-ordinates r, θ, z fixed relatively to the curve and revolving with it. If we adopt the former method and suppose the curve to rotate about Oz with uniform angular velocity ω , then using the components of acceleration given in 3.4 (3) and resolving along the tangent we have for the tangential acceleration

$$(\ddot{x} - \omega^2 x - 2\omega \dot{y}) \frac{dx}{ds} + (\ddot{y} - \omega^2 y + 2\omega \dot{x}) \frac{dy}{ds} + \ddot{z} \frac{dz}{ds}.$$

The terms independent of ω must be equivalent to $\ddot{s}$; and if r denotes distance from Oz , so that $r^2 = x^2 + y^2$, the terms in ω^2 are $-\omega^2 r dr/ds$, and the remaining terms vanish; so that, if T represents the tangential force on the particle, the equation of motion is

$$m(\ddot{s} - \omega^2 r dr/ds) = T$$

or

$$m\ddot{s} = T + m\omega^2 r dr/ds \dots\dots\dots(1).$$

It follows that the motion of the particle relative to the curve is the same as if the curve were at rest and the particle were subject to an additional force $m\omega^2 r$ from the axis. If we put $v dv/ds$ for $\ddot{s}$, then (1) has an integral which is an equation of energy

$$\frac{1}{2}mv^2 = C + \int T ds + \frac{1}{2}m\omega^2 r^2 \dots\dots\dots(2).$$

To find the reaction of the curve on the particle we must resolve along the principal normal and binormal; thus if ρ be the radius of curvature and dashes denote differentiations with respect to s , the acceleration along the principal normal is

$$(\ddot{x} - \omega^2 x - 2\omega \dot{y}) \rho x'' + (\ddot{y} - \omega^2 y + 2\omega \dot{x}) \rho y'' + \ddot{z} \rho z''.$$

The terms independent of ω must now be equivalent to $\dot{s}^2/\rho$, so that we get an equation

$$m \left\{ \frac{\dot{s}^2}{\rho} - \omega^2 \rho (xx'' + yy'') + 2\omega \dot{s} \rho (x'y'' - y'x'') \right\} = N + R$$

.....(3),

where N , R represent the components of the external force and the reaction of the curve in the direction of the principal normal.

The acceleration along the binormal is

$$(\ddot{x} - \omega^2 x - 2\omega \dot{y}) \rho (y'z'' - z'y'') + (\ddot{y} - \omega^2 y + 2\omega \dot{x}) \rho (z'x'' - x'z'') + \ddot{z} \rho (x'y'' - y'x'').$$

Since there is no acceleration along the binormal when ω is zero, the terms independent of ω vanish; and, by using

$$\dot{x} = x'\dot{s}, \quad \ddot{x} = x''\dot{s}^2 + x'\ddot{s}$$

and similar relations, the remaining terms can be combined; so that if we resolve along the binormal we find an equation

$$m [-\omega^2 \rho \{x(y'z'' - z'y'') + y(z'x'' - x'z'')\} - 2\omega \dot{s} \rho z''] = B + R'$$

.....(4),

where B , R' denote like components in the direction of the binormal.

We will illustrate the alternative method of using cylindrical co-ordinates by considering *the case of a heavy particle moving on a smooth curve in the form of a helix, having its axis vertical and revolving uniformly about its axis with angular velocity ω .*

Measuring θ from the foot of the helix, the cylindrical co-ordinates of the particle are a , θ , z , where $z = a\theta \tan \alpha$, but the radius has an angular velocity $\dot{\theta} + \omega$.

By resolving along the tangent, we get

$$a\ddot{\theta} \cos \alpha + \ddot{z} \sin \alpha = -g \sin \alpha,$$

or, if v is the velocity relative to the helix, so that $a\dot{\theta} = v \cos \alpha$ and $\dot{z} = v \sin \alpha$, then

$$\dot{v} = -g \sin \alpha \quad \text{.....(5);}$$

so that the particle moves along the curve with constant acceleration.

A resolution along the principal normal, i.e. perpendicular to the axis of rotation, gives $ma(\dot{\theta} + \omega)^2 = R$,

or $m(v \cos \alpha + a\omega)^2/a = R$ (6);

and along the binormal

$$m(\ddot{z} \cos \alpha - a\dot{\theta} \sin \alpha) = R' - mg \cos \alpha,$$

leading to $R' = mg \cos \alpha$ (7).

It is easy to deduce (5), (6) and (7) from (1), (3) and (4) above.

4.51. Motion of a particle on a smooth surface which revolves about a fixed axis. Take axes rigidly connected with the revolving surface, let the zx plane contain the axis about which the surface revolves and let the axis Oz make an angle α with this fixed axis. Then if ω is the angular velocity, in the notation of 3.6, we have

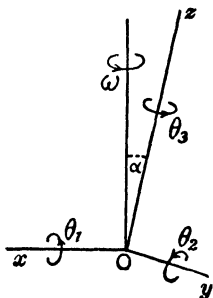
$$\theta_1 = \omega \sin \alpha, \quad \theta_2 = 0, \quad \theta_3 = \omega \cos \alpha,$$

and therefore the velocities are

$$u = \dot{x} - y\omega \cos \alpha,$$

$$v = \dot{y} - z\omega \sin \alpha + x\omega \cos \alpha,$$

$$w = \dot{z} + y\omega \sin \alpha;$$



and the accelerations

$$f_1 = \ddot{x} - v\omega \cos \alpha,$$

$$f_2 = \ddot{y} - w\omega \sin \alpha + u\omega \cos \alpha,$$

$$f_3 = \ddot{z} + v\omega \sin \alpha.$$

It follows that if the forces acting on the particle are given the equations of motion can be written down.

4.52. Example. A smooth plane revolves with uniform angular velocity ω about a fixed vertical axis which intersects it at the point O , at which a heavy particle is placed at rest. Shew that during the subsequent motion $v^2 = r^2 \omega^2 + 2gh$, where h is the depth of the particle below O , r is its distance from the axis and v is its velocity relative to the plane.

Taking axes as in 4.51, with the plane xy as the revolving plane, we have $z = 0$, and, by resolving parallel to Ox and Oy ,

$$\ddot{x} - 2\omega \cos \alpha \dot{y} - \omega^2 \cos^2 \alpha x = -g \sin \alpha$$

and

$$\ddot{y} + 2\omega \cos \alpha \dot{x} - \omega^2 y = 0.$$

Therefore $\ddot{x}\dot{x} + \dot{y}\dot{y} - \omega^2 x\dot{x} \cos^2 \alpha - \omega^2 y\dot{y} = -g \sin \alpha \dot{x}$,

or $\dot{x}^2 + \dot{y}^2 - \omega^2 (x^2 \cos^2 \alpha + y^2) = -2gx \sin \alpha$,

with no constant of integration since the particle starts from rest at the origin. This is equivalent to

$$v^2 = r^2 \omega^2 + 2gh.$$

4.53. Examples. 1. A straight fine smooth tube turns about a vertical axis, which it intersects at an angle α , with uniform angular velocity ω . A particle slides down the tube starting from rest at its intersection with the vertical axis. Prove that the distance described in time t is

$$2g \cos \alpha \sinh^2 \left(\frac{1}{2} \omega t \sin \alpha \right) / \omega^2 \sin^2 \alpha.$$

2. A smooth plane revolves with uniform angular velocity ω about a vertical axis which meets it at a point O , the normal to the plane always making an angle $\cos^{-1} \frac{1}{3}$ with the vertical. A particle is projected from O horizontally along the plane with velocity V . Prove that, if Ox and Oy are axes fixed in the plane, Ox down the line of greatest slope and Oy horizontal, the position of the particle at time t is given by

$$x = \frac{6\sqrt{2}g}{\omega^2} \left(\cosh \frac{\omega t}{\sqrt{3}} - 1 \right) + \left(V - \frac{2\sqrt{2}g}{\omega} \right) \frac{t}{\sqrt{3}} \sinh \frac{\omega t}{\sqrt{3}},$$

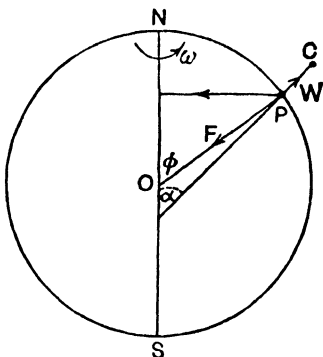
$$y = \frac{2\sqrt{6}g}{\omega^2} \sinh \frac{\omega t}{\sqrt{3}} + \left(V - \frac{2\sqrt{2}g}{\omega} \right) \left(\frac{2}{\sqrt{3}\omega} \sinh \frac{\omega t}{\sqrt{3}} + \frac{t}{3} \cosh \frac{\omega t}{\sqrt{3}} \right).$$

3. A smooth plane revolves about a vertical axis with uniform angular velocity ω , the normal to the plane making an angle α with the vertical. Prove that for a particle projected as in the preceding example it is illegitimate to neglect terms in ω^2 even if ω is small.

[M. T. 1936]

4.6. Motion relative to the earth. In defining the weight of a body as the force with which the earth attracts it [Pt. I, 4.17] we have neglected the earth's rotation. If, however, we agree to measure the weight by the pull exerted on a spring balance, or by the tension of a supporting string, when the body is at rest relative to the earth, and take the rotation of the earth into account, then, since the body is describing a circle about the earth's axis, it has an acceleration toward the axis, and this acceleration is produced by the joint effect of the attraction of the earth and the tension of the supporting string. It follows that the weight is not equal to the earth's attraction.

Let P be the bob of a pendulum of weight W and C its point of support. CP is of course small when compared with the radius OP of the earth. The direction of the 'plumb line' PC is called the vertical at P . Let F be the earth's attraction on the body at P . Let m be the mass of the body, a the earth's radius and let CP , PO make angles α , ϕ with the earth's axis, then $ma\omega^2 \sin \phi$ at right angles to the axis is the resultant of F along PO and W along PC , so that



$$W : F : ma\omega^2 \sin \phi = \cos \phi : \cos \alpha : \sin (\phi - \alpha) \quad \dots(1).$$

The angle $\phi - \alpha$ is the deviation of the plumb line, i.e. the angle between the plumb line and the earth's radius, due to the earth's rotation.

In Pt. I, 4.17 we connected W with the mass m of the body by the relation $W = mg$, assuming that g , the acceleration of a falling body, is constant in the region considered, and we shall investigate to what extent it may be necessary to modify this definition when the earth's rotation is taken into account. We return to this point in 4.72. In the meantime, putting mg for W , we have, from (1),

$$\sin (\phi - \alpha) = (a\omega^2/g) \sin \phi \cos \phi \quad \dots\dots\dots(2).$$

Taking the earth's radius as 4000 miles, $a\omega^2$ in foot-second units is approximately $1/9$ and $a\omega^2/g$ is roughly $1/290$, so that the deviation of the plumb line is small in all latitudes and approximately equal to

$$(a\omega^2/g) \sin \phi \cos \phi \quad \dots\dots\dots(3).$$

Referring again to the figure, by resolving at right angles to CP and along it, we get

$$F \sin (\phi - \alpha) = ma\omega^2 \sin \phi \cos \alpha \quad \dots\dots\dots(4),$$

$$F \cos (\phi - \alpha) = mg + ma\omega^2 \sin \phi \sin \alpha \quad \dots\dots\dots(5).$$

4.7. Axes fixed relatively to the earth and moving with it. Taking the polar axis of the earth as directed from South to North, the angular velocity ω is in the right-handed screw sense in relation to the axis.

With any point O on the surface as origin, take the upward vertical (plumb line) as axis of z , Ox at right angles to Oz in the meridian plane directed South, and Oy perpendicular to the meridian plane towards the East. Let Oz make an angle α with the axis. Then ω is the angular velocity of this frame of reference and regarding it as a vector in the direction SN , its components about the axes Ox , Oy , Oz are

$$\theta_1 = -\omega \sin \alpha, \quad \theta_2 = 0, \quad \theta_3 = \omega \cos \alpha \dots\dots(1);$$

and the origin O has velocity components

$$0, \quad a\omega \sin \phi, \quad 0,$$

where a is the earth's radius, and ϕ is the angle which the arc ON subtends at the centre as in the figure of 4.6. Hence, if x, y, z are the co-ordinates of a particle, from 3.6 (1) and (2) its components of velocity and acceleration in the directions Ox, Oy, Oz relative to O are

$$\dot{x} - y\omega \cos \alpha,$$

$$\dot{y} + z\omega \sin \alpha + x\omega \cos \alpha,$$

$$\dot{z} - y\omega \sin \alpha,$$

$$\text{and} \quad \ddot{x} - 2\dot{y}\omega \cos \alpha - z\omega^2 \sin \alpha \cos \alpha - x\omega^2 \cos^2 \alpha,$$

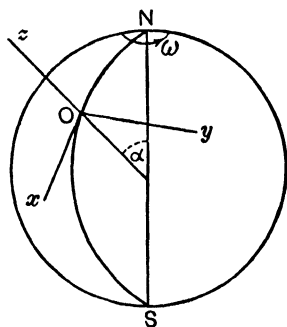
$$\ddot{y} + 2\dot{z}\omega \sin \alpha + 2\dot{x}\omega \cos \alpha - y\omega^2,$$

$$\ddot{z} - 2\dot{y}\omega \sin \alpha - z\omega^2 \sin \alpha - x\omega^2 \sin \alpha \cos \alpha.$$

But the component velocities and accelerations of O in the directions Ox, Oy, Oz are

$$0, \quad a\omega \sin \phi, \quad 0$$

$$\text{and} \quad -a\omega^2 \sin \phi \cos \alpha, \quad 0, \quad -a\omega^2 \sin \phi \sin \alpha.$$



Hence the component velocities u, v, w of the particle and its component accelerations f_1, f_2, f_3 are given by

$$\left. \begin{aligned} u &= \dot{x} - y\omega \cos \alpha \\ v &= \dot{y} + z\omega \sin \alpha + x\omega \cos \alpha + a\omega \sin \phi \\ w &= \dot{z} - y\omega \sin \alpha \end{aligned} \right\} \dots\dots\dots(2)$$

and

$$\left. \begin{aligned} f_1 &= \ddot{x} - 2\dot{y}\omega \cos \alpha - z\omega^2 \sin \alpha \cos \alpha - x\omega^2 \cos^2 \alpha - a\omega^2 \sin \phi \cos \alpha \\ f_2 &= \ddot{y} + 2\dot{z}\omega \sin \alpha + 2\dot{x}\omega \cos \alpha - y\omega^2 \\ f_3 &= \ddot{z} - 2\dot{y}\omega \sin \alpha - z\omega^2 \sin^2 \alpha - x\omega^2 \sin \alpha \cos \alpha - a\omega^2 \sin \phi \sin \alpha \end{aligned} \right\} \dots\dots\dots(3).$$

Suppose that the particle is acted upon by the earth's attraction F as in 4·6 and some other force whose components are X, Y, Z . Then its equations of motion are

$$\left. \begin{aligned} m(\ddot{x} - 2\dot{y}\omega \cos \alpha - z\omega^2 \sin \alpha \cos \alpha - x\omega^2 \cos^2 \alpha - a\omega^2 \sin \phi \cos \alpha) \\ m(\ddot{y} + 2\dot{z}\omega \sin \alpha + 2\dot{x}\omega \cos \alpha - y\omega^2) = Y \\ m(\ddot{z} - 2\dot{y}\omega \sin \alpha - z\omega^2 \sin^2 \alpha - x\omega^2 \sin \alpha \cos \alpha - a\omega^2 \sin \phi \sin \alpha) \end{aligned} \right\} \begin{aligned} &= -F \sin(\phi - \alpha) + X \\ &= -F \cos(\phi - \alpha) + Z \end{aligned} \dots\dots\dots(4)$$

and from 4·6 (4) and (5) these reduce to

$$\left. \begin{aligned} m(\ddot{x} - 2\dot{y}\omega \cos \alpha - z\omega^2 \sin \alpha \cos \alpha - x\omega^2 \cos^2 \alpha) &= X \\ m(\ddot{y} + 2\dot{z}\omega \sin \alpha + 2\dot{x}\omega \cos \alpha - y\omega^2) &= Y \\ m(\ddot{z} - 2\dot{y}\omega \sin \alpha - z\omega^2 \sin^2 \alpha - x\omega^2 \sin \alpha \cos \alpha) &= -mg + Z \end{aligned} \right\} (5);$$

and we notice that the terms in F have cancelled the terms arising from the motion of the origin, leaving on the left only terms relative to the origin and on the right the applied force and a force mg along the downward plumb line.

4·71. Foucault's Pendulum. Consider the motion of a simple pendulum making small oscillations about its position of relative equilibrium. Let l be the length of the string, T its tension, m the mass of the bob and let the origin be the point of suspension. Then we may

put $z = -l$ and neglect x^2 and y^2 . We may also neglect $\omega^2 x$, $\omega^2 y$, $\omega^2 z$ in comparison with $\omega^2 a$ or g . Equations 4.7 (5) become

$$\left. \begin{aligned} \ddot{x} - 2\dot{y}\omega \cos \alpha &= -\frac{T}{m} \frac{x}{l} \\ \ddot{y} + 2\dot{x}\omega \cos \alpha &= -\frac{T}{m} \frac{y}{l} \end{aligned} \right\} \dots\dots\dots(1).$$

and

$$-2\dot{y}\omega \sin \alpha = -g + \frac{T}{m}$$

Eliminating T and retaining only terms of the same order, we get

$$\left. \begin{aligned} \ddot{x} - 2\dot{y}\omega \cos \alpha + n^2 x &= 0 \\ \ddot{y} + 2\dot{x}\omega \cos \alpha + n^2 y &= 0 \end{aligned} \right\} \dots\dots\dots(2),$$

and

where $n^2 = g/l$.

From (2) it follows that

$$x\ddot{y} - y\ddot{x} + 2\omega \cos \alpha (x\dot{x} + y\dot{y}) = 0,$$

therefore

$$x\dot{y} - y\dot{x} + \omega \cos \alpha (x^2 + y^2) = C,$$

or

$$r^2 \dot{\theta} + \omega r^2 \cos \alpha = C \dots\dots\dots(3).$$

It follows that, if the pendulum is oscillating in a plane through the vertical, then $C = 0$ and $\dot{\theta} = -\omega \cos \alpha \dots\dots\dots(4).$

But $\dot{\theta}$ is the angular velocity relative to the axes fixed to the earth of the plane in which the pendulum swings, and $\omega \cos \alpha$ is the component of the earth's angular velocity about the vertical at the point of suspension. Hence (4) shows that the plane of motion turns round the vertical at the point of suspension with an angular velocity equal and opposite to that of the earth. An equivalent statement is that the plane of oscillation of the pendulum remains fixed in direction in space and the earth turns round under the pendulum. Looking down on the earth as in the figure of 4.7, if we ascribe the relative motion to the earth it is a counter-clockwise rotation in the sense from Ox to Oy , or if we ascribe the relative motion to the plane of oscillation of the pendulum, it is clockwise in the sense from Oy to Ox .

The experimental verification of this result is one of the most conclusive of the proofs of the earth's rotation about its axis.

Returning to the general case, equations (2) may be solved by writing $\zeta = x + iy$, so that $\ddot{\zeta} + 2i\dot{\zeta}\omega \cos \alpha + n^2 \zeta = 0$.

The ordinary process of solution leads to

$$\zeta e^{i\omega t \cos \alpha} = A e^{i\mu t} + B e^{-i\mu t} \dots\dots\dots(5),$$

where $\mu^2 = n^2 + \omega^2 \cos^2 \alpha$, and A, B are arbitrary complex constants.

If we take axes Ox' , Oy' revolving round Oz with angular velocity $-\omega \cos \alpha$, we have

$$x' + iy' = (x + iy) e^{i\omega t \cos \alpha} = A e^{i\mu t} + B e^{-i\mu t} \dots\dots\dots(6),$$

or

$$x' = C \cos \mu t, \quad y' = D \sin \mu t \dots\dots\dots(7).$$

These equations give the projection of the path of the bob of the pendulum on a horizontal plane, and (7) shews that it may be described as an ellipse which revolves about the vertical with angular velocity $-\omega \cos \alpha$. This is in agreement with what we have already proved in the special case in which the pendulum oscillates in a plane through the vertical.

4.72. Falling body. When a body falls from a considerable height under the influence of gravity, and the earth's rotation is taken into account, its equations of motion are, from 4.7 (5),

$$\left. \begin{aligned} \ddot{x} - 2\dot{y}\omega \cos \alpha - z\omega^2 \sin \alpha \cos \alpha - x\omega^2 \cos^2 \alpha &= 0 \\ \ddot{y} + 2\dot{z}\omega \sin \alpha + 2\dot{x}\omega \cos \alpha - y\omega^2 &= 0 \\ \ddot{z} - 2\dot{y}\omega \sin \alpha - z\omega^2 \sin^2 \alpha - x\omega^2 \sin \alpha \cos \alpha &= -g \end{aligned} \right\} \dots\dots(1).$$

By taking the origin at the point from which the body falls, we see at once that the *initial* acceleration of the falling body is g in the direction of the downward plumb line, and this is the modification in the definition of g when the rotation of the earth is taken into account, to which reference was made in 4.6.

Now suppose that the body falls from a height h vertically above the origin. We get a first approximation to the solution of the equations by noting that $\dot{x}$, $\dot{y}$ are small in comparison with $\dot{z}$ and neglecting the terms in ω^2 , so that

$$\ddot{x} = 0, \quad \ddot{y} + 2\dot{z}\omega \sin \alpha = 0, \quad \ddot{z} = -g \dots\dots\dots(2).$$

Since the initial velocities are zero and the body starts from $(0, 0, h)$, therefore

$$x = 0, \quad z = h - \frac{1}{2}gt^2 \quad \text{and} \quad y = \frac{1}{3}gt^3\omega \sin \alpha \dots\dots\dots(3).$$

Thus, to a first approximation, the time of falling from a height h is $\sqrt{(2h/g)}$, and there is an easterly deviation from the vertical of amount $\frac{1}{3}\omega \sin \alpha \sqrt{(8h^3/g)}$.

We can now obtain a second approximation by substituting from (3) in (1) and neglecting terms in ω^3 ; the equations thus become

$$\left. \begin{aligned} \ddot{x} - \frac{2}{3}gt^2\omega^2 \sin \alpha \cos \alpha - h\omega^2 \sin \alpha \cos \alpha &= 0 \\ \ddot{y} - 2gt\omega \sin \alpha &= 0 \\ \ddot{z} - \frac{2}{3}gt^2\omega^2 \sin^2 \alpha - h\omega^2 \sin^2 \alpha &= -g \end{aligned} \right\} \dots\dots\dots(4).$$

and

These give on integration

$$\left. \begin{aligned} x &= \frac{1}{6}gt^4\omega^2 \sin \alpha \cos \alpha + \frac{1}{2}ht^2\omega^2 \sin \alpha \cos \alpha \\ y &= \frac{1}{3}gt^3\omega \sin \alpha \\ z &= h + \frac{1}{3}gt^4\omega^2 \sin^2 \alpha + \frac{1}{2}ht^2\omega^2 \sin^2 \alpha - \frac{1}{2}gt^2 \end{aligned} \right\} \dots\dots\dots(5).$$

and

By putting $z = 0$, we find that a second approximation to the time of falling from a height z is given by

$$t^2 = \frac{2h}{g} + \frac{3h^2\omega^2 \sin^2 \alpha}{g^3} \dots\dots\dots(6),$$

and then by substituting in the expressions for y and x we get the easterly deviation as before and a southerly deviation

$$3h^2\omega^2 \sin \alpha \cos \alpha / 2g.$$

4.73. Body projected vertically upwards. Using the equations of 4.72, let the body be projected upwards from the origin with velocity w . The integration of 4.72 (2) then gives, for a first approximation,

$$x=0, \quad z=wt-\frac{1}{2}gt^2 \quad \text{and} \quad y=(\frac{1}{3}gt^3-wt^2)\omega \sin \alpha \dots\dots(1).$$

It follows that the time of ascent and descent is $2w/g$, and this makes $y = -\frac{4}{3}\frac{w^3}{g^2}\omega \sin \alpha$, shewing a deviation to the West of amount $\frac{4}{3}\frac{w^3}{g^2}\omega \sin \alpha$ when the particle returns to the level of projection. We see also from (1) that y vanishes at time $t=3w/g$, so that if the body continues to fall below the level of projection it will cross the vertical and ultimately have an easterly deviation.

For a second approximation, we return to 4.72 (1) and, by using the first approximation above, we obtain

$$\left. \begin{aligned} \ddot{x} + 3\left(wt - \frac{1}{2}gt^2\right)\omega^2 \sin \alpha \cos \alpha &= 0 \\ \ddot{y} + 2\left(w - gt\right)\omega \sin \alpha &= 0 \\ \ddot{z} + 3\left(wt - \frac{1}{2}gt^2\right)\omega^2 \sin^2 \alpha &= -g \end{aligned} \right\} \dots\dots\dots(2),$$

$$\begin{aligned} \text{leading to} \quad x &= -\left(\frac{1}{2}wt^3 - \frac{1}{8}gt^4\right)\omega^2 \sin \alpha \cos \alpha \\ y &= -\left(wt^2 - \frac{1}{3}gt^3\right)\omega \sin \alpha \\ \text{and} \quad z &= wt - \frac{1}{2}gt^2 - \left(\frac{1}{2}wt^3 - \frac{1}{8}gt^4\right)\omega^2 \sin^2 \alpha \end{aligned} \left. \right\} \dots\dots\dots(3).$$

4.74. Motion of a projectile. The equations of motion in the general case are the equations 4.72 (1), and we may suppose that the projectile starts from the origin with a velocity whose components are u, v, w .

Neglecting the terms in ω^2 , we have

$$\ddot{x} - 2\dot{y}\omega \cos \alpha = 0 \dots\dots\dots(1),$$

$$\ddot{y} + 2\dot{z}\omega \sin \alpha + 2\dot{x}\omega \cos \alpha = 0 \dots\dots\dots(2)$$

$$\text{and} \quad \ddot{z} - 2\dot{y}\omega \sin \alpha = -g \dots\dots\dots(3).$$

The integration of (1) and (3) gives

$$\dot{x} = 2y\omega \cos \alpha + u \quad \text{and} \quad \dot{z} = 2y\omega \sin \alpha - gt + w \dots\dots(4);$$

and the substitution of these values in (2) gives, to the first power of ω ,

$$\ddot{y} + 2\left(w - gt\right)\omega \sin \alpha + 2u\omega \cos \alpha = 0;$$

$$\text{so that} \quad \dot{y} + 2\left(wt - \frac{1}{2}gt^2\right)\omega \sin \alpha + 2ut\omega \cos \alpha = v$$

$$\text{and} \quad y = vt - \left(u \cos \alpha + w \sin \alpha\right)t^2\omega + \frac{1}{3}gt^3\omega \sin \alpha \dots\dots\dots(5).$$

The substitution of this value in (4) then gives, to the first power of ω ,

$$\dot{x} = u + 2vt\omega \cos \alpha \quad \text{and} \quad \dot{z} = w - gt + 2vt\omega \sin \alpha \dots\dots(6),$$

$$\text{leading to} \quad x = ut + vt^2\omega \cos \alpha \quad \text{and} \quad z = wt - \frac{1}{2}gt^2 + vt^2\omega \sin \alpha \dots\dots(7).$$

Taking u, v to be positive components in the directions South and East, these results shew that the earth's rotation causes an increase in height, a deviation to the South and a deviation which is either to the West or East according as t is $<$ or $> 3(u \cos \alpha + w \sin \alpha)/g \sin \alpha$.

In the case of a rifle bullet the vertical velocity is small in comparison with the horizontal, and if V denotes the horizontal component and the vertical plane of projection is in a direction θ East of South, we may put $u = V \cos \theta$, $v = V \sin \theta$, and the displacement to the right of this vertical plane looking in the direction of projection is then

$$(x - ut) \sin \theta - (y - vt) \cos \theta = Vt^2 \omega \cos \alpha + (wt^3 - \frac{1}{3}gt^3) \omega \sin \alpha \cos \theta.$$

4.75. Body projected along a smooth horizontal plane.* In this case the equations of motion are got from 4.7 (5) by putting $z = 0$, $X = 0$, $Y = 0$, and $Z = R$, where R represents the reaction of the smooth plane. The equations are

$$\ddot{x} - 2\dot{y}\omega \cos \alpha - x\omega^2 \cos^2 \alpha = 0 \quad \dots\dots\dots(1),$$

$$\ddot{y} + 2\dot{x}\omega \cos \alpha - y\omega^2 = 0 \quad \dots\dots\dots(2),$$

$$-2\dot{y}\omega \sin \alpha - x\omega^2 \sin \alpha \cos \alpha = -g + R/m \quad \dots\dots\dots(3).$$

The path is determined by (1) and (2), and it is important to notice that though in the preceding articles we have obtained approximate solutions by neglecting the terms $x\omega^2, y\omega^2$ in comparison with g , it is not legitimate to neglect these terms in solving (1) and (2) unless the solution so obtained makes the terms retained in (1) and (2) large when compared with the terms neglected.

If we solve (1) and (2) neglecting ω^2 , and suppose that the body is projected from the origin with a velocity V in a direction β East of South, we get by integration

$$\dot{x} - 2y\omega \cos \alpha = V \cos \beta \quad \dots\dots\dots(4)$$

$$\text{and} \quad \dot{y} + 2x\omega \cos \alpha = V \sin \beta \quad \dots\dots\dots(5).$$

Eliminating $\dot{y}$ from $\dot{x} - 2y\omega \cos \alpha = 0$ and (5) gives

$$\ddot{x} + 4x\omega^2 \cos^2 \alpha = 2V\omega \sin \beta \cos \alpha \quad \dots\dots\dots(6).$$

The solution of this differential equation with the given initial conditions is

$$2x\omega \cos \alpha = V \sin \beta \{1 - \cos (2\omega t \cos \alpha)\} \\ + V \cos \beta \sin (2\omega t \cos \alpha) \quad \dots\dots\dots(7).$$

Then from (4), (6) and (7) we see that $\dot{x}\omega, \ddot{x}$ and $x\omega^2$ are all of order $V\omega$, and similarly $\dot{y}\omega, \ddot{y}$ and $y\omega^2$ are also of order $V\omega$, so that it is not permissible to neglect the terms $x\omega^2$ and $y\omega^2$ in solving (1) and (2).

An exact solution may be found by the ordinary process of solving linear differential equations. It appears that the solution contains periodic terms when $3 \cos \alpha > 1$.

* The mistake in earlier versions of this article was detected by Mr E. Cunningham.

For an approximate solution in powers of ωt , it is simpler to proceed thus: substitute in (1) and (2)

$$x = p_1 \omega t + p_2 \omega^2 t^2 + p_3 \omega^3 t^3 + p_4 \omega^4 t^4 + \dots$$

$$y = q_1 \omega t + q_2 \omega^2 t^2 + q_3 \omega^3 t^3 + q_4 \omega^4 t^4 + \dots$$

and equate to zero the coefficients of powers of t .

Then it appears that the equations are satisfied by

$$x = p \omega t + q \omega^2 t^2 \cos \alpha - \frac{1}{2} p \omega^3 t^3 \cos^2 \alpha + \frac{1}{2} q \omega^4 t^4 (\cos \alpha - 3 \cos^3 \alpha) + \dots$$

$$y = q \omega t - p \omega^2 t^2 \cos \alpha + \frac{1}{2} q \omega^3 t^3 (1 - 4 \cos^2 \alpha) - \frac{1}{2} p \omega^4 t^4 (\cos \alpha - 3 \cos^3 \alpha) + \dots,$$

where from the initial conditions

$$p \omega = V \cos \beta, \quad q \omega = V \sin \beta.$$

The deviation to the right of the direction of projection looking from the origin is $x \sin \beta - y \cos \beta$, or

$$V \omega t^2 \{ \cos \alpha - \frac{1}{2} \omega t \sin \beta \cos \beta \sin^2 \alpha + \frac{1}{2} \omega^2 t^2 (\cos \alpha - 3 \cos^3 \alpha) \}.$$

4.76. Horizontal pressure on railway lines. Let M be the mass of a train travelling on straight rails in a direction θ East of South. Then if R denotes the horizontal pressure of the rails measured in the sense θ North of East, using equations 4.7 (5), we have

$$M (\ddot{x} - 2\dot{y}\omega \cos \alpha - x\omega^2 \cos^2 \alpha) = -R \sin \theta$$

and

$$M (\ddot{y} + 2\dot{x}\omega \cos \alpha - y\omega^2) = R \cos \theta.$$

If the velocity relative to the rails is uniform and equal to V so that $\dot{x} = V \cos \theta$, and $\dot{y} = V \sin \theta$, and we put $x = 0$, $y = 0$ we find that

$$R = 2M V \omega \cos \alpha,$$

indicating a pressure on the rails to the right, looking in the direction of motion.

4.77. Examples. 1. A train of mass m is travelling with uniform velocity V along a parallel of latitude. Prove that the difference between the pressure on the rails when it travels East and when it travels West is $4mV\omega \cos \lambda$, where λ is the latitude and ω is the angular velocity of the earth.

2. A particle is projected in latitude λ with velocity V at an elevation γ in a direction θ East of South. Prove that owing to the earth's rotation with angular velocity ω the time of flight T is increased by

$$2TV (\omega/g) \cos \gamma \cos \lambda \sin \theta;$$

that the particle falls a distance

$$4\omega \sin^2 \gamma (\cos \gamma \sin \lambda + \frac{1}{2} \sin \gamma \cos \lambda \cos \theta) V^2/g^2$$

to the right of the vertical plane of projection; and that the range is increased by

$$4\omega \sin \gamma \cos \lambda \cos \theta (\cos^2 \gamma - \frac{1}{2} \sin^2 \gamma) V^2/g^2 \text{ approximately.}$$

3. In latitude 45° N. a gun is fired due north at an object distant 20 kilometres, this being the maximum range of the gun. Shew that, if the earth's rotation has not been allowed for in aiming, the shell should fall about 44 metres East of the mark. Shew also that if the

shell is fired South under similar conditions the deviation will be twice as great and towards the West. [Air-resistance is neglected.]

[M. T. 1916]

4. A shot is fired from a point on the earth's surface, the angle between the meridian and the plane of projection being θ , measured from North to East. Shew that, when the shot again reaches the earth's surface, its deviation from the plane of projection, due to the rotation of the earth, is to the right or to the left according as

$$3 \tan \lambda - \tan \alpha \cos \theta$$

is positive or negative, where λ is the latitude and α is the angle of projection. The square of the angular velocity of the earth is neglected, and the earth's surface within range is taken to be plane. [P. 1934]

EXAMPLES

1. A particle is projected horizontally with velocity $\sqrt{(2ga)}$ along the smooth surface of a sphere of radius a at the level of the centre. Prove that the motion is confined between two horizontal planes at a distance $\frac{1}{2}(\sqrt{5}-1)a$ apart. [I. 1914]

2. A heavy particle moves on the inside of a smooth spherical shell; shew that, if the velocity be due to falling from the level of the centre, the pressure on the surface will vary as the depth below the centre.

3. A heavy particle is moving on the interior surface of a smooth sphere with velocity due to falling from the level of the centre, and its motion is horizontal at a depth c below the centre; shew that the radius of curvature of its path at that point is $\frac{2ac}{\sqrt{a^2+3c^2}}$, where a is the radius of the sphere.

4. A particle, mass m , is projected along the surface of a paraboloid of revolution, of latus rectum $4a$, with a velocity $4\sqrt{\mu a}$, in the plane of the latus rectum, and is acted upon by a force to the focus, $m\mu$. (distance); prove that the initial osculating plane of its path is inclined to the axis at an angle $\cot^{-1} 3/5$, and that the initial pressure is $3m\mu a\sqrt{2}$.

5. If a particle be moving on a smooth circular cone under a force to the vertex varying inversely as the square of the distance, prove that if the cone be developed on to a tangent plane the path will be developed into a conic having the vertex of the cone for one focus.

6. A particle is projected from a given point with a given velocity and is acted upon by a given force to a fixed point. Shew that the centre of curvature of the orbit at the point of projection lies on a fixed plane. [M. T. 1905]

7. A particle is placed between two smooth co-axial circular cylinders of nearly equal radii, whose common axis is inclined to the

vertical, and slides down under gravity. If β be the angular distance of its initial position from the lowest point of the cross-section through that position, shew that the particle will never press the inner surface if $2\beta < \pi$; but that if $2\beta > \pi$, the particle will pass from the inner to the outer surface, and back again, and so on, when ϕ , its angular distance at any point from the lowest point of the cross-section through that point, takes the successive values given by $3 \cos \phi = 2 \cos \beta$.

8. A particle slides on a smooth helix of radius a and angle α under the action of a force to a fixed point in the axis equal to μ times the distance. Investigate the motion, and prove that the pressure cannot vanish unless the greatest velocity of the particle be $\sqrt{\mu a} \sec \alpha$.

9. The force on a particle moving on the surface of a sphere is along the meridian. The particle is projected from a point on the equator and its path is a loxodrome*. Determine the law of force.

10. A heavy bead moves along a vertical circular wire which revolves uniformly about a vertical straight line in its own plane. Find the time of a small oscillation, and the pressure on the wire.

11. A smooth surface is generated by the revolution of the curve $x^2y = c^3$ about the axis of y which is vertically downwards, and a heavy particle is projected along the surface with velocity due to falling from the horizontal plane through the origin. Prove that its path on the surface is a loxodrome.

12. A surface of revolution is such that if it be held with its axis vertical, and a heavy particle be projected along it with suitable velocity at any point in any direction, its path will cut every meridian of the surface at a constant angle. Shew that the surface may be generated by the revolution round the axis of y of the curve

$$h(x^2 - a^2) + x^2y = 0.$$

13. A point describes a loxodrome on a sphere in such a way that its longitude increases uniformly; prove that the resultant acceleration varies as the cosine of the latitude, and that its direction makes with the normal to the sphere an angle equal to the latitude.

14. A particle falls down a smooth tube in the shape of a helix. Find the pressure when it has fallen through a height h , if the axis of the helix is vertical and the particle is descending with uniform velocity v .

15. A heavy uniform string of given length is enclosed in a smooth straight tube, which is made to revolve uniformly about a vertical axis, so as to describe a right circular cone; determine the motion of the string, and the tension at any point.

* A loxodrome, or rhumb line, is a curve on a surface of revolution cutting the meridians at a constant angle.

16. A heavy particle moves on a curve which revolves uniformly about a vertical axis; prove that the time of an oscillation of the particle about a position of relative equilibrium will be

$$\frac{2\pi}{\omega} \left(\frac{\rho \sin \alpha}{k - \rho \sin \alpha \cos^2 \alpha} \right)^{\frac{1}{2}},$$

ρ being the radius of curvature at the point of equilibrium, α the angle made by the normal at that point with the vertical, k the distance of the point from the axis of revolution, and ω the angular velocity of the curve.

17. A particle of mass m is attached to one end of an elastic string, the other end of which is fastened to the vertex of a smooth cone of vertical angle 2α , having its axis vertical and vertex upwards; prove that the particle can move with a constant velocity v round the surface of the cone and with the string stretched to double its natural length, provided that the modulus of elasticity $> mg \cos \alpha$, and that

$$v^2 \cos \alpha < 2ag \sin^2 \alpha.$$

If the particle be slightly disturbed in the direction of the string, find the time of a small oscillation.

18. Two particles move on smooth helices

$x = a_1 \cos \theta_1$, $y = a_1 \sin \theta_1$, $z = c\theta_1$, and $x = a_2 \cos \theta_2$, $y = a_2 \sin \theta_2$, $z = c\theta_2$ respectively. The particles repel one another with a force which varies directly as their distance apart. Prove that

$$\phi^2 = A\phi^3 - 2H \cos \phi + B,$$

where $\phi = \theta_1 - \theta_2$ and A, H, B are constants.

[M. T. 1906]

19. A heavy particle moves upon a surface of revolution, axis vertical, formed by the revolution of a parabola of latus rectum $4a$ about the tangent at the vertex; prove that the differential equation of the projection of the path upon a horizontal plane is

$$(1 + au) \frac{d^2 u}{d\theta^2} + \frac{a}{2} \left(\frac{du}{d\theta} \right)^2 + u = \frac{\lambda}{u} \left(\frac{a}{u} \right)^{\frac{1}{2}},$$

where λ is a constant.

20. A smooth hollow ellipsoid of revolution is fixed with its axis ($2a$) vertical, and a particle is projected from a point in the horizontal plane through the centre and on the inside surface with velocity $\sqrt{(2ga)}$ and inclination α to the horizon. Find the value of α in order that the greatest depth below the centre may be $2a/3$, and find in that case the greatest height reached.

21. A particle moves on the inside of a smooth circular cone, vertical angle 2α , under the action of a force to the vertex varying inversely as the square of the distance. It is projected from an apse at a distance c from the axis with the velocity which bears to the

velocity requisite for circular motion the ratio of $\sqrt{3}:\sqrt{2}$. Prove that the projection of the path on a plane perpendicular to the axis is

$$3c = 2r + r \cos(\theta \sin \alpha),$$

that the time from one apse to the next is $\pi(2c \operatorname{cosec} \alpha)^{\frac{3}{2}}/\sqrt{\mu}$, and that the pressure of the particle on the surface of the cone is inversely proportional to the cube of its distance from the vertex.

22. Two particles of masses m and m' are connected by a string passing through a small hole at the vertex of a cone having its axis vertical and vertex uppermost. If m' hangs vertically, find the condition that m may describe a circle of radius c on the cone, and shew that if the particle be slightly disturbed it will oscillate about the circular path in the time

$$2\pi \sqrt{\left\{ \frac{c(m+m')}{3g(m'-m \cos \alpha) \sin \alpha} \right\}}.$$

23. A particle is revolving on a smooth plane about a centre of force, the accelerating effect to the centre being $\mu \times$ distance, and when the body arrives at an apse the plane begins to revolve with an angular velocity $\frac{1}{2}\sqrt{3\mu}$ about the apsidal line; shew that the subsequent orbit described on the plane will be a portion of a parabola; and that, when the particle leaves the plane, its velocity will be $\sqrt{3} \times$ velocity at the vertex. [M. T. 1868]

24. A small bead slides on a smooth circular ring of radius a , which is made to revolve with uniform angular velocity ω round a vertical axis passing through its centre, the plane of the ring being inclined at a constant angle α to a horizontal plane. Shew that the law of angular motion of the bead on the ring is the same as that of a bead on a ring of radius $a \operatorname{cosec} \alpha$ revolving round a vertical diameter with angular velocity $\omega \sin \alpha$.

25. A smooth wire, in the form of a parabola, latus rectum l , revolves about its axis which is vertical, the vertex being uppermost, with uniform angular velocity $\sqrt{g/l}$. A string passing through a fixed ring at the focus carries, at one end, a small ring, mass m , which slides on the wire, and at the other end a particle, mass m' , which hangs freely. Given the velocity, V , of the ring at the vertex, determine the rate at which the ring describes the parabola at any point.

If $m = 4m'$, and $2V^2 = gl$, prove that, at a time t after the ring has passed the vertex, the angle θ between the two parts of the string is given by the equation

$$\cot \frac{\theta}{2} = \sqrt{\frac{2g}{l}} t.$$

26. Three masses m_1, m_2, m_3 are fastened to a string which passes through a ring, and m_1 describes a horizontal circle as a conical pendulum while m_2 and m_3 hang vertically.

If m_3 drops off, prove that the instantaneous change of tension of the string is

$$\frac{gm_1 m_2}{m_1 + m_2}.$$

27. A particle connected to a fixed point O by an inextensible string is in stable equilibrium under the action of a conservative system of forces. Prove that if the particle be slightly disturbed from its position A , the two periods of independent small oscillations are

$$2\pi\sqrt{\left(\frac{a\rho_1}{f(a+\rho_1)}\right)} \quad \text{and} \quad 2\pi\sqrt{\left(\frac{a\rho_2}{f(a+\rho_2)}\right)},$$

where a is the length of the string, f the force per unit mass, ρ_1, ρ_2 the principal radii of curvature of the equipotential surface at A . The last three are estimated positive in the direction OA . [M. T. 1903]

28. A particle is free to move along a helix whose axis is vertical, under the action of a centre of force whose accelerating effect is $\mu \times$ distance, on the axis of the helix. The particle is so placed as to be in equilibrium, and the centre of attraction then begins to move vertically upwards with a velocity V ; prove that, after a time t ,

$$\mu \sin^2 \alpha (s \sin \alpha - Vt)^2 + (\dot{s} \sin \alpha - V)^2 = V^2,$$

s being the arc of the helix measured from the position of equilibrium, and α the angle which the helix makes with the horizontal. Hence determine s in terms of t .

29. A circular tube of smooth bore has its centre fixed above a rough horizontal plane and is made to roll uniformly in contact with the plane. Shew that the motion of a particle of unit mass within the tube is given by

$$a\ddot{\phi} - a\Omega^2 \sin^2 \alpha \sin \phi \cos \phi + g \sin \alpha \sin \phi = 0,$$

and the pressures towards the centre and perpendicular to the plane of the tube are determined by

$$a(\dot{\phi} + \Omega \cos \alpha)^2 + a\Omega^2 \sin^2 \alpha \sin^2 \phi + g \sin \alpha \cos \phi = R,$$

$$2a\dot{\phi}\Omega \sin \alpha \cos \phi + a\Omega^2 \sin \alpha \cos \alpha \cos \phi - g \cos \alpha = S,$$

where Ω is the angular velocity of the point of contact round the vertical and α the inclination of the plane of the tube to the horizon.

30. A particle is projected horizontally from a point on the surface in the horizontal plane containing the lower focus of the spheroid

$$x^2 + y^2 = (a^2 - z^2)(1 - e^2),$$

where the axis of z is vertical. Shew that if V is the initial velocity when the path is a circle, and v_1 is the initial velocity when the particle is projected so as to rise to the level of the centre and move horizontally at that level with velocity v_2 , then

$$v_2^2 = v_1^2(1 - e^2) = 2V^2.$$

Prove also that the time taken by the particle when projected with velocity v_1 to rise to the level of the centre is

$$\int_0^{ae} \left\{ \frac{e(a - ez)}{2gz(ae - z)} \right\}^{\frac{1}{2}} dz. \quad [\text{I. 1913}]$$

31. A heavy particle is projected horizontally along a smooth surface of revolution whose equation in cylindrical co-ordinates is $8z^3 = 27ar^2$, the positive axis of z being vertically upwards. Shew that, if the normal at the point of projection is inclined to the vertical at an angle $\frac{1}{3}\pi$ and the particle leaves the surface at a point where the normal makes an angle $\frac{1}{3}\pi$ with the vertical, the velocity of projection must be $\sqrt{(ga/525)}$. [I. 1908]

32. A particle moves on a rough surface under no forces but the reaction of the surface. The surface is one of revolution whose meridian curves are catenaries with the axis for directrix, and the particle is projected from the equator in a direction making an angle α with the meridian. Prove that the velocity v of the particle, when it crosses the meridian at an angle ϕ , satisfies the equation

$$\left(\frac{dv}{d\phi}\right)^2 = \frac{2\mu^2 v^2 \cos^2 2\phi}{\cos 2\phi - \cos 2\alpha},$$

μ being the coefficient of friction. [I. 1911]

33. A particle moves on a developable surface whose edge of regression is a helix, and is under the action of a force towards the axis. If the particle starts from rest, shew that it will describe a geodesic; and that, if the force $= \mu \cdot (\text{distance})$, its position at time t is given by the equation $r^2 \cos^4 \alpha = c^2 \cos^4 \alpha \cos^2 (\sqrt{\mu} t \cos \alpha) - a^2 \sin^2 (\sqrt{\mu} t \cos \alpha)$,

where r is the distance from the helix measured along a generator, c the initial value of r , and a the radius of the helix. [M. T. 1907]

34. A heavy particle is attached to one end of a light string of length l , the other end of which is fastened to a point on a perfectly rough right circular cylinder of radius a , whose axis is vertical. The string is drawn tight and is initially a horizontal tangent to the cylinder, and the particle is projected horizontally at right angles to the string with velocity $\sqrt{(2gh)}$ so that the string becomes wrapped round the cylinder. Prove that the curve formed by the string on the cylinder satisfies the equation

$$\frac{d}{d\psi} (a^2 \rho^{-2} \sec^2 \psi + \tan^2 \psi) = \frac{l-s}{h} \cos \psi,$$

where s is the arc measured from the fixed point, ψ is the angle which the tangent makes with the horizontal and $\rho = ds/d\psi$. [I. 1906]

Chapter V

MOMENTS OF INERTIA

5.1. In Pt. I, Ch. XIII we have defined moments of inertia and proved the theorem of parallel axes, and some other theorems relating more particularly to two-dimensional problems.

In this chapter we shall consider material systems in three dimensions and shall adopt the following notation: if m is the mass of a typical particle of a system and x, y, z are its co-ordinates,

$$A = \Sigma m (y^2 + z^2), \quad B = \Sigma m (z^2 + x^2), \quad C = \Sigma m (x^2 + y^2),$$

$$D = \Sigma m yz, \quad E = \Sigma m zx, \quad F = \Sigma m xy.$$

5.2. To express the moment of inertia of a body about any line through the origin in terms of A, B, C, D, E, F .

Let l, m, n be the direction cosines of the line; m' the mass of an element at P ; x, y, z the co-ordinates of P , and PN perpendicular to the line.

Then the required moment of inertia I is given by

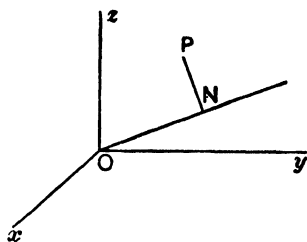
$$I = \Sigma m' (NP)^2$$

$$= \Sigma m' \{x^2 + y^2 + z^2 - (lx + my + nz)^2\}$$

$$= \Sigma m' \{(x^2 + y^2 + z^2)(l^2 + m^2 + n^2) - (lx + my + nz)^2\}$$

$$= l^2 \Sigma m' (y^2 + z^2) + m^2 \Sigma m' (z^2 + x^2) + n^2 \Sigma m' (x^2 + y^2) \\ - 2mn \Sigma m' yz - 2nl \Sigma m' zx - 2lm \Sigma m' xy$$

$$= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm.$$



5.3. Momental Ellipsoid. If we construct the quadric

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = M\epsilon^4 \dots (1),$$

where M denotes a mass and ϵ a length, and if l, m, n are the direction cosines of a radius vector of length r , then, from (1),

$$Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm = M\epsilon^4/r^2 \dots (2),$$

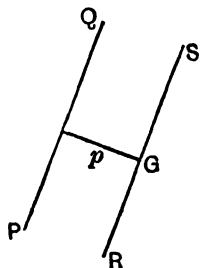
so that the moment of inertia about any line from the origin is $M\epsilon^4/r^2$, where r is the length of the line cut off by this quadric. It follows that every radius vector of the quadric is real, so that it is an ellipsoid and it is called the **Momental Ellipsoid** of the body at the point O .

Since it is always possible to choose co-ordinate axes so that the coefficients of the product terms in the equation of an ellipsoid vanish, therefore at every origin there is a set of rectangular axes for which the products of inertia D, E, F vanish. These axes are called the **Principal Axes** of the body at the point O , and in this case A, B, C are called its **Principal Moments of Inertia**.

5.4. Equipomental bodies. *Bodies are said to be equipomental when they have equal moments of inertia about every line in space.*

This is clearly the case if the bodies have the same mass, the same centre of gravity, the same principal axes at the centre of gravity and the same principal moments of inertia.

That the converse is also true may be proved thus: bearing in mind the theorem of parallel axes, the moment of inertia of a body about an axis through its centre of gravity is less than its moment about any parallel axis. Let two bodies of masses M, M' and centres of gravity G, G' be equipomental. Then at any point their axes of greatest moment of inertia coincide, and



their axes of least moment coincide; and, from what precedes, their moments about any two parallel lines through G, G' respectively are each less than their moments about any other

parallel lines; therefore these lines must coincide, whatever their direction may be; therefore G , G' coincide. Then the axes of greatest moment at G are the same, as also are the axes of least moment, and an axis perpendicular to both of these is the third principal axis for both bodies.

Lastly, to prove the masses equal, let p be the distance of G from any line PQ and let RS be a parallel line through G . Then

$$\begin{aligned} Mp^2 + \text{mt. of } M \text{ about } RS \\ &= \text{mt. of } M \text{ about } PQ \\ &= \text{mt. of } M' \text{ about } PQ \\ &= M'p^2 + \text{mt. of } M' \text{ about } RS; \end{aligned}$$

but M and M' have equal moments about RS , therefore

$$Mp^2 = M'p^2$$

or

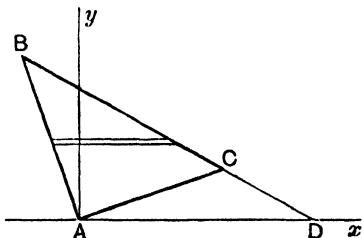
$$M = M'.$$

5.41. Example. To find a system of particles equimomental with a plane triangular lamina.

Let Ax , Ay be rectangular axes through the corner A of the triangle ABC , and let BC meet Ax in D . Let β , γ be the distances of B , C from Ax , and let $AD = a$.

The moment of inertia of the triangle ABD about Ax

$$= \int_0^\beta a \left(1 - \frac{y}{\beta}\right) y^2 dy = \frac{1}{12} a \beta^3.$$

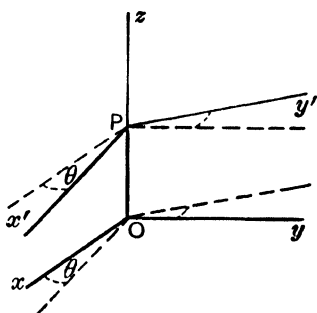


The like moment of the triangle ACD is $\frac{1}{12} a \gamma^3$, and the area ABC is $\frac{1}{2} a (\beta - \gamma)$. Denoting this area by M , the moment of inertia about Ax of the triangle ABC is $\frac{1}{6} M (\beta^2 + \beta\gamma + \gamma^2)$.

This may be written $\frac{1}{6} M \left\{ \left(\frac{\beta}{2}\right)^2 + \left(\frac{\gamma}{2}\right)^2 + \left(\frac{\beta + \gamma}{2}\right)^2 \right\}$, which is also the moment of inertia about Ax of three particles, each of mass $\frac{1}{3} M$, placed at the middle points of the sides of the triangle. Also these systems have the same mass and the same centre of gravity G , and therefore the same moment of inertia about an axis Gx' parallel to Ax . But the direction of this line was arbitrary, therefore the systems have equal moments of inertia about all lines through G in the plane, and, by the theorem of parallel axes, about all lines in the plane; and since for a plane body $O = A + B$, therefore for all axes at right angles to the plane. Hence the systems are equimomental.

5.5. Principal axes. *To find at what point, if any, of a straight line it is a principal axis of a material system.*

Take the straight line as axis of z , some point O on it as origin, and let P be the point at which it is a principal axis and $OP = h$. Let Px' , $P y'$ be the other two principal axes at P , and θ the angle between Px' and Ox .



The necessary and sufficient conditions that Oz may be a principal axis at P are

$$\Sigma m y' z' = \Sigma m z' x' = \Sigma m x' y' = 0 \quad \dots\dots\dots(1),$$

where

$$x' = x \cos \theta + y \sin \theta,$$

$$y' = y \cos \theta - x \sin \theta$$

and

$$z' = z - h.$$

Hence the three conditions to be satisfied are

$$\cos \theta \Sigma m y z - \sin \theta \Sigma m z x - h (\cos \theta \Sigma m y - \sin \theta \Sigma m x) = 0 \quad \dots(2),$$

$$\cos \theta \Sigma m z x + \sin \theta \Sigma m y z - h (\cos \theta \Sigma m x + \sin \theta \Sigma m y) = 0 \quad \dots(3)$$

$$\text{and} \quad \sin \theta \cos \theta \Sigma m (y^2 - x^2) + \cos 2\theta \Sigma m x y = 0 \quad \dots(4).$$

The elimination of h between (2) and (3) leads to

$$\Sigma m z x \Sigma m y = \Sigma m y z \Sigma m x \quad \dots\dots\dots(5)$$

as the condition that Oz should be a principal axis at some point of its length; and then the result of substituting from (5) in (2) gives

$$h = \Sigma m y z / \Sigma m y = \Sigma m z x / \Sigma m x \quad \dots\dots\dots(6),$$

thus determining the position of the point required.

We have not yet made use of (4). It may be written

$$\tan 2\theta = 2F / (B - A) \quad \dots\dots\dots(7)$$

in our former notation, and it serves to determine the position of the other principal axes at P .

5.6. Ellipsoid of Gyration. Let the momental ellipsoid (5.3 (1)) at a point O be referred to the principal axes at O as co-ordinate axes so that its equation is

$$Ax^2 + By^2 + Cz^2 = M\epsilon^4 \dots\dots\dots(1).$$

If l, m, n are the direction cosines of the perpendicular ON from the origin to the tangent plane at a point P on the surface, we have

$$\frac{ON^2}{M\epsilon^4} = \frac{l^2}{A} + \frac{m^2}{B} + \frac{n^2}{C} \dots\dots\dots(2).$$

If on ON we take a point P' such that $OP' \cdot ON = \epsilon^2$, then the locus of P' is the reciprocal of the locus of P , and it is an ellipsoid

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{1}{M} \dots\dots\dots(3),$$

called the **Ellipsoid of Gyration**, at the point O , of the material system.

Let l', m', n' be the direction cosines of the perpendicular from O to a tangent plane to the ellipsoid of gyration and p' its length, then from (3)

$$Mp'^2 = Al'^2 + Bm'^2 + Cn'^2 \dots\dots\dots(4);$$

so that the moment of inertia about this perpendicular is Mp'^2 . This is the characteristic property of an ellipsoid of gyration, viz. that the moment of inertia about the central perpendicular on a tangent plane is equal to the mass times the square of the length of this perpendicular.

5.7. Arrangement of principal axes. *Given the principal axes and the principal moments of inertia at the centre of gravity O of the body, to find the principal axes at a point P .*

For simplicity denote the mass of the system by unity. Construct a quadric confocal with the ellipsoid of gyration and passing through the point P . Let its equation be

$$\frac{x^2}{A+\lambda} + \frac{y^2}{B+\lambda} + \frac{z^2}{C+\lambda} = 1 \dots\dots\dots(1).$$

Let l, m, n be the direction cosines of the perpendicular from O to the tangent plane at P . The moment of inertia about this

perpendicular is $Al^2 + Bm^2 + Cn^2$; i.e. if this perpendicular be an axis Oz'

$$\Sigma m (x'^2 + y'^2) = Al^2 + Bm^2 + Cn^2;$$

therefore the moment of inertia with regard to a plane through O parallel to the tangent plane is

$$\begin{aligned}\Sigma m z'^2 &= \Sigma m (x'^2 + y'^2 + z'^2) - \Sigma m (x'^2 + y'^2) \\ &= \frac{1}{2} (A + B + C) - (Al^2 + Bm^2 + Cn^2);\end{aligned}$$

and the moment with regard to the tangent plane is found by adding the square of the perpendicular on the tangent plane, viz.

$$(A + \lambda)l^2 + (B + \lambda)m^2 + (C + \lambda)n^2,$$

and is therefore $\frac{1}{2} (A + B + C) + \lambda$,

and is constant for tangent planes at all points on the quadric on which P lies.

Again, if (1) denotes an ellipsoid through P , the tangent plane to this ellipsoid at P will have a greater moment of inertia than any other plane through P . For considering any other plane α passing through P , α is nearer to the origin than is the tangent plane to (1) parallel to α . Therefore the moment of inertia with regard to α is less than the moment of inertia with regard to a tangent plane to (1), i.e. less than the moment of inertia with regard to the tangent plane at P . Hence the tangent plane at P to the confocal ellipsoid through P is the plane of greatest moment. Similarly the tangent plane at P to the confocal hyperboloid of two sheets is the plane of least moment, and the third principal plane must therefore be tangent to the confocal hyperboloid of one sheet through P .

Thus the principal axes at any point P are the normals to the quadrics passing through P , confocal with the ellipsoid of gyration at the centre of gravity of the system.

Also the moment of inertia about the point P is

$$\frac{1}{2} (A + B + C) + OP^2,$$

and the moment with regard to a tangent plane to the quadric (1) is

$$\frac{1}{2} (A + B + C) + \lambda,$$

therefore the moments about the principal axes at P are

$$OP^2 - \lambda_1, \quad OP^2 - \lambda_2, \quad OP^2 - \lambda_3,$$

where $\lambda_1, \lambda_2, \lambda_3$ are the parameters of the quadrics passing through P confocal with the ellipsoid of gyration at O .

5·71. The arrangement of the principal axes of inertia of a system is thus allied to the properties of confocal quadrics. The condition that a line may be a principal axis at some point of its length is the condition that it may be a normal at some point to one of a family of confocal quadrics.

EXAMPLES

1. Prove that the product of inertia of a body with regard to any two intersecting axes is equal to the product with regard to parallel axes through the centre of gravity together with the product of the whole mass collected at the centre of gravity.

2. Obtain the moments and products of inertia for the corner of a uniform cube, taking the cube edges as axes. [M. T. 1932]

3. Prove that the moment of inertia of a body of mass M about a straight line whose equations are $(x-a)/l = (y-b)/m = (z-c)/n$ referred to any rectangular axes through the centre of gravity is

$$Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm \\ + M \{(nb - mc)^2 + (lc - na)^2 + (ma - lb)^2\},$$

if l, m, n are the direction cosines of the line.

4. A uniform square lamina is bounded by the axes of x and y and the lines $x = 2c, y = 2c$, and a corner is cut off it by the line $x/a + y/b = 2$. Shew that, if $c > a$ and $c > b$, the two principal axes at the centre of the square which are in its own plane are inclined to the axis of x at angles given by

$$\tan 2\theta = \frac{ab - 2c(a+b) + 3c^2}{(a-b)(a+b-2c)}. \quad [\text{I. 1901}]$$

5. Particles of equal mass are placed at the corners of a regular polygon. Prove that the squares of the principal radii of gyration at any point O in the plane of the polygon are $\frac{1}{2}a^2$ and $\frac{1}{2}a^2 + h^2$, where a is the radius of the circumscribing circle of the polygon and h is the distance of O from the centre of this circle. [I. 1913]

6. A triangular plate has sides of length 3, 4 and 5 feet. Find the equation of the momental ellipse at the centre of inertia of the plate referred to axes through the centre of inertia parallel to the perpendicular sides of the triangle. [I. 1908]

7. Prove that the moment of inertia of a homogeneous anchor ring about a diameter of its largest circular section is $\frac{1}{8}M(4c^2 + 5a^2)$, where M is the mass and a, c are the radii of the generating circle and the path of its centre. [M. T. 1911]

8. Prove that, if lines passing through a fixed point are such that a body has the same moment of inertia about each of them, the lines lie on a cone of the second degree which has the same circular sections as the momental ellipsoid at the point.

9. Prove that, if

$$S \equiv Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{const.}$$

be the equation of the momental ellipsoid at the centre of gravity O of a body referred to any rectangular axes through O , then the momental ellipsoid at the point (a, b, c) is

$$S + M\{(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2\} = \text{const.},$$

where M is the mass of the body.

10. Prove that at the centre of any regular solid the momental ellipsoid is a sphere.

11. Find the equation of the momental ellipsoid at the vertex of a uniform solid right circular cone, of mass M , height h , and semi-vertical angle α .

Find four particles, one at the vertex and three in a plane perpendicular to the axis, which form a system equimomental with the cone. [I. 1930]

12. Prove that a uniform tetrahedron is equimomental with particles each one-twentieth of its mass at its four corners and a particle of four-fifths of its mass at its centre of gravity.

13. Prove that a line which is a principal axis of a body at its centre of gravity is also a principal axis at every point of its length.

14. Prove that, if a line is a principal axis of a body at more than one point, it must pass through the centre of gravity of the body and be a principal axis at every point of its length.

15. Prove that an edge of a tetrahedron can only be a principal axis at some point of its length if it is perpendicular to the opposite edge.

16. Each of three equal uniform rods AB, BC, CD is of mass m and of length $2a$, and each is perpendicular to the other two. Prove that the principal moments of inertia at the centre of inertia of the system are $ma^2, \frac{1}{3}ma^2, 4ma^2$. [M. T. 1907]

17. A uniform solid circular cone of half angle α and height h is cut in half by a plane through its axis. Shew that the principal moments of inertia at the vertex for one of the halves are

$$\frac{3}{8}Mh^2(1 + \frac{1}{4}\tan^2\alpha)$$

and $\frac{3}{10}Mh^2\left[1 + \frac{3}{4}\tan^2\alpha \pm \sqrt{\left\{(1 - \frac{1}{4}\tan^2\alpha)^2 + \frac{64}{9\pi^2}\tan^2\alpha\right\}}\right],$

where M is the mass of the body. Indicate how the axes of inertia lie. [M. T. 1921]

18. Prove that the moment of inertia of an elliptic disc of mass M and equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

about the diameter parallel to the axis of x , is $-\frac{1}{4}aM\Delta/(ab - h^2)^2$, where

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2.$$

19. The principal moments of inertia at the centre of gravity G of a body are A, B, C . The six co-ordinates of a line referred to the principal axes at G are $l, m, n, \lambda, \mu, \nu$. Shew that the moment of inertia of the body, mass M , about the line is

$$Al^2 + Bm^2 + Cn^2 + M(\lambda^2 + \mu^2 + \nu^2).$$

If $l', m', n', \lambda', \mu', \nu'$ are the co-ordinates of a line perpendicular to the given line and intersecting it, shew that the product of inertia for the pair of lines is

$$-\{All' + Bmm' + Cnn' + M(\lambda\lambda' + \mu\mu' + \nu\nu')\}. \quad [\text{M. T. 1926}]$$

20. From any point a tangent cone is drawn to the ellipsoid of gyration at the centre of gravity of a body. Prove that the moment of inertia about every generator of the reciprocal cone is the same.

[M. T. 1895]

21. A quadric confocal with the ellipsoid of gyration at the centre of gravity of a body intersects a concentric sphere. Prove that at all points of the curve of intersection one of the principal moments of inertia has the same value.

22. Prove that, if A, B, C are the principal moments of inertia of a body at its centre of gravity, the plane $lx + my + nz = 1$ is a principal plane at the point where it intersects the line

$$\frac{x}{l} - A = \frac{y}{m} - B = \frac{z}{n} - C.$$

Chapter VI

MISCELLANEOUS PROBLEMS IN RIGID DYNAMICS MAINLY TWO-DIMENSIONAL

6.1. In the later chapters of **Pt. I** we established the equations of motion of a rigid body moving in two dimensions, together with the principles of conservation of energy and momentum, and applied them to the solution of a variety of problems of an elementary kind. The present chapter contains further applications of the same dynamical principles to a wider range of problems.

6.11. Rolling spheres and cylinders. Examples on Friction.

(i) *A uniform thin hollow cylinder of mass M and radius a is free to rotate about its axis, which is horizontal. A uniform solid cylinder of mass m and radius b rolls without slipping on the inner surface of the first cylinder. Shew that, if the plane containing the axes of the cylinders oscillates through an angle α , its period is the same as that of a simple pendulum of length*

$$(a-b) \frac{3M+m}{2M+m}$$

oscillating through the same angle α .

[P. 1934]

Let an axial plane fixed to the first cylinder make an angle ϕ and the common axial plane an angle θ with the vertical.

Let the friction F act on the first cylinder in the sense in which θ increases, and let ω be the angular velocity of the second cylinder.

The equations of motion are, for the first cylinder,

$$Ma^2\ddot{\phi} = -Fa \quad \dots\dots(1)$$

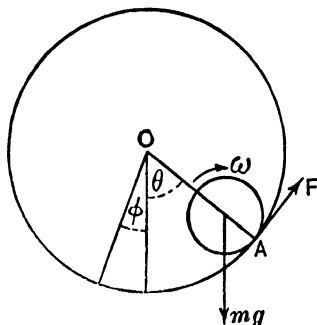
and, for the second cylinder,

$$\frac{1}{2}mb^2\dot{\omega} = Fb \quad \dots\dots(2)$$

and $m(a-b)\ddot{\theta} = -F - mg \sin \theta \dots\dots(3).$

There is also the condition for rolling, i.e. that the point A of both cylinders has the same velocity, or

$$(a-b)\dot{\theta} - b\omega = -a\dot{\phi} \quad \dots\dots(4).$$



By eliminating F , ω and ϕ from these four equations, we find that

$$(a-b) \frac{3M+m}{2M+m} \ddot{\theta} + g \sin \theta = 0,$$

so that if the common axial plane has an oscillatory motion it behaves like a simple pendulum of length $(a-b) \frac{3M+m}{2M+m}$.

(ii) *A uniform hollow cylinder of radius a is at rest inside a cylinder of internal radius b , the axes of both cylinders being horizontal. The system is started impulsively from rest so that the cylinder of radius b moves, in a horizontal direction at right angles to its axis, with velocity V , and it is then forced to continue moving horizontally with the same velocity, and without rotation. The cylinder of radius a rolls without slipping on the inner surface of the cylinder of radius b . Prove that it makes complete revolutions right round that surface, if $V^2 > 12g(b-a)$. [M. T. 1935]*

Let ω be the angular velocity of the inner cylinder, and θ the angle which the common axial plane makes with the vertical axial plane.

The normal pressure R and the friction F are shown in the figure in the sense in which they act on the inner cylinder. Its equations of motion are

$$m(b-a) \ddot{\theta} = -F - mg \sin \theta \dots (1),$$

$$m(b-a) \dot{\theta}^2 = R - mg \cos \theta \dots (2),$$

$$ma^2 \dot{\omega} = Fa \dots (3);$$

and we have a condition for rolling, viz.

$$(b-a) \dot{\theta} - a\omega + V \cos \theta = V \cos \theta,$$

or $a\omega = (b-a) \dot{\theta} \dots (4).$

By eliminating F and ω from (1), (3) and (4), we get

$$2(b-a) \ddot{\theta} = -g \sin \theta,$$

which gives $(b-a) \dot{\theta}^2 = (b-a) \dot{\theta}_0^2 - g(1 - \cos \theta) \dots (5),$

where $\dot{\theta}_0$ is the initial value of $\dot{\theta}$.

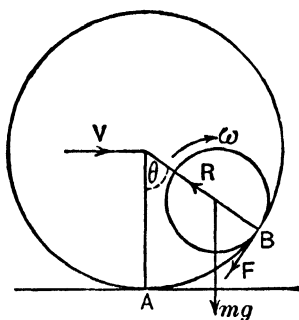
Initially the point of contact is at A and the inner cylinder is set in motion by an impulsive action acting through the point A . Hence the moment about A of the momentum of the inner cylinder immediately after the impulse is the same as it was before the impulse, i.e. zero. This gives an equation $ma^2 \omega_0 + ma(b-a) \dot{\theta}_0 + maV = 0,$

where ω_0 denotes the initial value of ω . But, as in (4),

$$a\omega_0 = (b-a) \dot{\theta}_0,$$

therefore

$$2(b-a) \dot{\theta}_0 = -V.$$



Hence, from (5), we get

$$(b-a)\dot{\theta}^2 = \frac{V^2}{4(b-a)} - g(1 - \cos \theta);$$

so that from (2)
$$\frac{R}{m} = \frac{V^2}{4(b-a)} - g(1 - 2\cos \theta).$$

The inner cylinder will make complete revolutions if R is positive when $\theta = \pi$, i.e. if $V^2 > 12g(b-a)$.

(iii) *A uniform sphere of radius a is gently placed on the top of a thin vertical pole of height h ($> a$) and then allowed to fall over. Shew that, however rough the pole may be, the sphere will slip on the pole before it finally falls off it.* [I. 1931]

Let θ be the angle which the radius to the point of contact O makes with the vertical, F the friction, R the normal reaction. So long as rolling lasts the centre G has acceleration $a\ddot{\theta}$ at right angles to OG and $a\dot{\theta}^2$ along GO , and the sphere has an angular acceleration $\ddot{\theta}$. Therefore

$$ma\ddot{\theta} = mg \sin \theta - F \dots\dots\dots(1),$$

$$ma\dot{\theta}^2 = mg \cos \theta - R \dots\dots\dots(2),$$

$$\frac{2}{3}ma^2\ddot{\theta} = Fa \dots\dots\dots(3).$$

From (1) and (3) we get

$$\frac{7}{5}a\ddot{\theta} = g \sin \theta;$$

or, on integration,

$$\frac{7}{5}a\dot{\theta}^2 = -2g \cos \theta + \text{const.}$$

If we assume that the sphere is slightly disturbed from its position of unstable equilibrium, we may put $\dot{\theta} = 0$ when $\theta = 0$; but it should be noticed that a disturbance however slight is necessary to start the motion since $\dot{\theta}$ vanishes with θ .

Thus we have
$$a\dot{\theta}^2 = \frac{1}{7}g(1 - \cos \theta) \dots\dots\dots(4),$$

and then, from (2),
$$R = \frac{1}{7}mg(17 \cos \theta - 10) \dots\dots\dots(5),$$

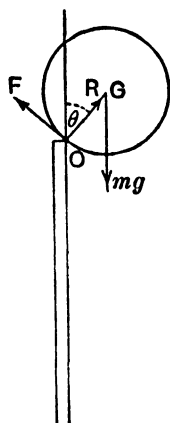
while from (1) and (3)
$$F = \frac{2}{7}mg \sin \theta \dots\dots\dots(6).$$

Let μ denote a finite coefficient of friction, and let

$$\begin{aligned} f(\theta) &= F - \mu R \\ &= \frac{1}{7}mg \{2 \sin \theta - \mu(17 \cos \theta - 10)\}. \end{aligned}$$

Then
$$f'(\theta) = \frac{1}{7}mg(2 \cos \theta + 17\mu \sin \theta),$$

so that for acute angles $f'(\theta)$ is positive and $f(\theta)$ an increasing function. But $f(\theta)$ is negative when $\theta = 0$ and positive when $\cos \theta = \frac{1}{17}\mu$, therefore $f(\theta)$ vanishes and changes sign for a value of θ between 0 and $\cos^{-1} \frac{1}{17}\mu$. It follows that, for this value of θ , the friction necessary to maintain rolling would exceed the maximum friction. Therefore sliding takes place no matter how large the coefficient of friction provided it is finite.



(iv) A uniform rod is held nearly vertically with its lower end resting on an imperfectly rough plane. It is released from rest and falls forwards. The inclination to the vertical at any instant is θ . Prove (i) that if the coefficient of friction is less than a certain finite amount the lower end of the rod will slip backwards before $\sin^2 \frac{1}{2}\theta = \frac{1}{3}$; (ii) that however great the coefficient of friction may be the lower end will begin to slip forwards at a value of $\sin^2 \frac{1}{2}\theta$ between $\frac{1}{3}$ and $\frac{1}{2}$. [M. T. 1928]

Let m be the mass and $2a$ the length of the rod. In the beginning of the motion the rod turns about its lower end O which we take as an origin, so that with horizontal and vertical axes the co-ordinates of the centre G of the rod are

$$x = a \sin \theta, \quad y = a \cos \theta,$$

and, if F , R denote the friction and the reaction of the plane, we have

$$m\ddot{x} = F, \quad m\ddot{y} = R - mg,$$

and, by moments about O ,

$$\frac{1}{2}ma^2\ddot{\theta} = mga \sin \theta,$$

or $a\ddot{\theta} = \frac{2}{3}g \sin \theta$, and $a\dot{\theta}^2 = \frac{2}{3}g(1 - \cos \theta)$,

assuming, as in the preceding example, that the rod is slightly disturbed from its position of unstable equilibrium.

Hence we get $F = ma(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta)$

$$= \frac{2}{3}mg \sin \theta (3 \cos \theta - 2),$$

and $R = mg - ma(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta)$

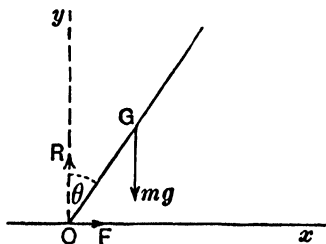
$$= \frac{1}{3}mg(1 - 3 \cos \theta)^2.$$

It appears that F vanishes when $\theta = 0$ and when $\cos \theta = \frac{2}{3}$ and is positive for intermediate values of θ , and being a continuous function has a maximum value between these limits, say F_1 ; also R does not exceed mg within this range of values for θ . There is a finite μ for which $F_1 > \mu mg$, and for this μ sliding will take place before θ reaches the value $\cos^{-1} \frac{2}{3}$, i.e. before $\sin^2 \frac{1}{2}\theta = \frac{1}{3}$.

Secondly, let the coefficient of friction be large; as θ passes the value $\cos^{-1} \frac{2}{3}$, F becomes negative, i.e. the direction of the friction is reversed, and the end of the rod slips forwards when

$$3 \sin \theta (2 - 3 \cos \theta) > \mu (1 - 3 \cos \theta)^2.$$

But as θ increases from $\cos^{-1} \frac{2}{3}$ to $\cos^{-1} \frac{1}{2}$ the expression on the left increases continuously from 0 to $\sqrt{8}$, while the factor of μ on the right decreases continuously from 1 to zero. It follows that however great μ may be, provided it is finite, there will be a value of θ within this range for which the condition is satisfied; i.e. the end of the rod slips forwards at a value of $\sin^2 \frac{1}{2}\theta$ between $\frac{1}{3}$ and $\frac{1}{2}$.



(v) A heavy inelastic uniform rod of length $2a$ is allowed to fall without rotation so that one end strikes a smooth horizontal plane. Prove that, if α be the inclination to the vertical, u the velocity of the rod at the instant of impact and

$$9u^2 \sin^2 \alpha \cos \alpha > ga(1 + 3 \sin^2 \alpha)^2,$$

then contact is broken at once.

Using the figure of the preceding example, to find the pressure on the plane after the impact, we have equations

$$m\ddot{y} = R - mg \quad \text{and} \quad \frac{1}{3}ma^2\ddot{\theta} = aR \sin \theta,$$

where $y = a \cos \theta$. Therefore, eliminating R and y ,

$$\frac{1}{3}a\ddot{\theta}(1 + 3 \sin^2 \theta) + a\dot{\theta}^2 \sin \theta \cos \theta = g \sin \theta$$

$$\text{and} \quad R(1 + 3 \sin^2 \theta) = m(g - a\dot{\theta}^2 \cos \theta) \dots\dots\dots(1).$$

Let u' be the velocity of the centre and ω the angular velocity at the instant after impact. The change in linear momentum is $m(u' - u)$ and in moment of momentum $\frac{1}{3}ma^2\omega$; but the moment of momentum about the point of impact is unaltered, therefore

$$m(u' - u)a \sin \alpha + \frac{1}{3}ma^2\omega = 0,$$

where, since ω is the initial value of $\dot{\theta}$ and $-u'$ is the initial value of $\dot{y}$, $u' = a\omega \sin \alpha$. Hence

$$a\omega(\sin^2 \alpha + \frac{1}{3}) = u \sin \alpha.$$

Then if we substitute the initial values in (1), it is evident that contact is broken at once if $a\omega^2 \cos \alpha > g$, i.e. if

$$9u^2 \sin^2 \alpha \cos \alpha > ga(1 + 3 \sin^2 \alpha)^2.$$

6.12. Examples. 1. A uniform circular cylinder of mass M is free to rotate around its axis which is smooth and horizontal and about which its radius of gyration is equal to its radius. A uniform solid sphere of mass m is placed with its lowest point in contact with the highest generator of the cylinder, both sphere and cylinder being initially at rest. The sphere is then slightly disturbed and rolls down the cylinder, all points of the sphere moving in planes perpendicular to the axis of the cylinder. Shew that slipping takes place before the sphere leaves the cylinder, and begins when

$$2M \sin \theta = \mu[(17M + 6m) \cos \theta - (10M + 4m)],$$

μ being the coefficient of friction between the sphere and the cylinder, and θ the inclination to the vertical of the plane through the axis of the cylinder and the centre of the sphere. [M. T. 1933]

2. A uniform cylinder of mass M and radius a lies on a smooth horizontal plane, and a second uniform cylinder of mass m and radius b lies in contact with and outside the first along a generator. The surfaces between the two cylinders are rough. The system is allowed to move

from rest in this position. Shew that as long as the cylinders remain in contact

$$\frac{3M + 2m \sin^2 \theta}{2(M + m)} (a + b) \dot{\theta}^2 = 2g (\cos \theta_0 - \cos \theta),$$

where θ is the angle between the vertical and the plane through the axes of the cylinders, and θ_0 is its initial value.

Find an expression for the time to roll from $\theta = \theta_0$ to $\theta = \theta_1$ when both θ_0 and θ_1 are small. [M. T. 1924]

3. A uniform thin hollow cylinder of mass M and radius b moves on a smooth horizontal plane. A second uniform thin hollow cylinder of mass m and radius a ($< b$) rolls, without slipping, on the inner surface of the first cylinder. If at time t during the motion the plane containing the axes makes an angle θ with the vertical, and the first cylinder has turned through an angle $\theta + \phi$ from a standard position, shew that

$$(bM + am) \dot{\theta} + b(M + m) \dot{\phi}$$

remains constant.

Shew further that, if the motion starts from rest when $\theta = \alpha$, then

$$\dot{\theta}^2 = \frac{g}{b-a} \frac{2(M+m)}{2M + m \sin^2 \theta} (\cos \theta - \cos \alpha). \quad [\text{M. T. 1931}]$$

4. A uniform solid sphere rolling with angular velocity ω_0 along a horizontal plane strikes a rough inelastic plane inclined at an angle α to the horizontal, so as to run up a line of greatest slope. Assuming that there is no impulsive frictional reaction on impact and that the conditions are such that the sphere ceases to slide before beginning to descend, shew that the time between the impact and the moment of rest is

$$a\omega_0 \left(\frac{2}{3} + \cos \alpha \right) / g \sin \alpha. \quad [\text{M. T. 1927}]$$

5. A golf ball of radius a rolls directly towards the centre of a hole of diameter c with velocity v . Assuming that the ball strikes the opposite edge of the hole and that no slipping takes place, shew that if the ball overshoots the hole

$$\left\{ \left(\frac{k^2}{a} + a \cos \alpha \right) v - \frac{ga \sin \alpha (c - a \sin \alpha)}{v} \right\}^2 > 2ga (1 - \cos \alpha) (k^2 + a^2),$$

where $\cos \alpha = 1 - \frac{g(c - a \sin \alpha)^2}{2av^2}$,

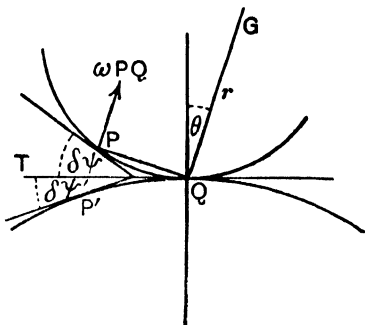
and k is the radius of gyration of the ball about a diameter. [M. T. 1926]

6.2. Motion of a plane lamina with a curved boundary rolling on a fixed curve under the action of given forces. Let the curve PQ roll on the fixed curve $P'Q$. Let P, P' be in contact at time t , and let Q be the point of contact after a short interval δt ; let the arc $PQ = P'Q = \delta s$, and let ρ, ρ' be the radii of curvature and $\delta\psi, \delta\psi'$ the angles between the tangents at

P, P' and the tangent TQ at Q . Then if ω is the angular velocity of the rolling curve

$$\omega \delta t = \delta \psi + \delta \psi' = \delta s \left(\frac{1}{\rho} + \frac{1}{\rho'} \right) \dots\dots\dots (1).$$

To find the acceleration of the point of contact, we note that Q is the instantaneous centre of rotation, so that in time δt the point P has acquired a velocity ωPQ at right angles to PQ , and this has normal and tangential components $\omega PQ \cos PQT$ and $\omega PQ \sin PQT$. But $\lim PQ/\delta s = 1$, $\lim \cos PQT = 1$ and $\lim \sin PQT = 0$, so that we have



$$\text{normal accel.} = \lim \omega \frac{PQ}{\delta s} \frac{\delta s}{\delta t} \cos PQT = \frac{\omega^2 \rho \rho'}{\rho + \rho'} \dots (2),$$

and

$$\text{tangential accel.} = \lim \omega \frac{PQ}{\delta s} \frac{\delta s}{\delta t} \sin PQT = 0 \dots\dots\dots (3).$$

Now let G be the centre of gravity of the lamina, m its mass, $QG = r$ and θ the angle between QG and the normal.

Then by taking moments about the instantaneous centre of rotation Q [Pt. I, 16.6 (1)], the equation of motion is

$$\frac{1}{2\omega} \frac{d}{dt} \{m(r^2 + k^2) \omega^2\} = L \dots\dots\dots (4),$$

where L is the moment of the external forces about Q .

This may be written

$$m(r^2 + k^2) \dot{\omega} + m r \dot{r} \omega = L \dots\dots\dots (5);$$

but

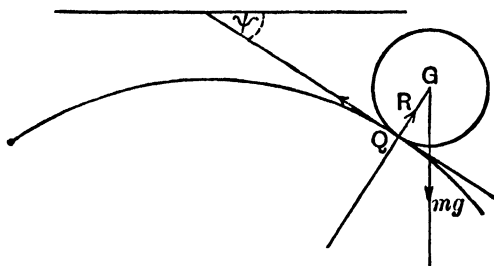
$$\begin{aligned} \dot{r} &= \dot{s} \frac{dr}{ds} = -\dot{s} \sin \theta \\ &= -\frac{\omega \rho \rho'}{\rho + \rho'} \sin \theta, \text{ from (1).} \end{aligned}$$

Therefore
$$m(r^2 + k^2)\dot{\omega} - \frac{m\rho\rho'}{\rho + \rho'} r\omega^2 \sin \theta = L \quad \dots\dots\dots(6)$$

is the equation determining the motion.

To determine the normal pressure R and the friction F on the rolling curve, we note that, relative to Q , G has accelerations $\ddot{r} - r\omega^2$ along QG and $2\dot{r}\omega + r\dot{\omega}$ at right angles to QG . If these are compounded with the acceleration of Q given by (2), we can then write down equations of motion for normal and tangential directions which will serve to determine R and F in terms of the other data. But in actual problems there are usually data which simplify the solution.

6·21. Example. *A uniform circular disc of mass m and radius a rolls down a curve $s = c \tan \psi$ placed in a vertical plane with its vertex uppermost. Find an equation to determine where the disc leaves the curve.*



Let ψ denote the inclination to the horizontal of the tangent at the point of contact; then in the notation of 6·2 $\rho = a$, $\rho' = c \sec^2 \psi$, and $r = a$; so that from 6·2 (1)

$$\omega = \dot{s} \left(\frac{1}{\rho} + \frac{1}{\rho'} \right)$$

or
$$a\omega = \dot{s} (a + c \sec^2 \psi) \quad \dots\dots\dots(1).$$

Also 6·2 (4) gives
$$\frac{3}{2}ma^2\dot{\omega} = mga \sin \psi,$$

therefore
$$a^2\omega\dot{\omega} = \frac{2}{3}g\psi \sin \psi (a + c \sec^2 \psi) \quad \dots\dots\dots(2);$$

and, if we suppose that the disc is slightly disturbed and rolls down from the highest point of the curve, we get on integration

$$a^2\omega^2 = \frac{4}{3}g(a - a \cos \psi + c \sec \psi - c) \quad \dots\dots\dots(3).$$

To find the pressure R we require the normal acceleration of G ; this can be got by combining its acceleration $\omega^2 a$ along GQ relative to Q with the acceleration of Q along QG , which from 6·2 (2) is

$$\omega^2 a c \sec^2 \psi / (a + c \sec^2 \psi),$$

or from the simpler consideration that G is moving with velocity $a\omega$

on a parallel curve of radius of curvature $a + c \sec^3 \psi$, so that its acceleration along the inward normal to its path is $a^2 \omega^2 / (a + c \sec^3 \psi)$.

Thence we have
$$\frac{ma^2 \omega^2}{a + c \sec^3 \psi} = mg \cos \psi - R \dots\dots\dots(4).$$

Hence R vanishes when

$$a^2 \omega^2 = g(a \cos \psi + c \sec \psi),$$

or, from (3), when

$$4(a - a \cos \psi + c \sec \psi - c) = 3(a \cos \psi + c \sec \psi),$$

i.e.
$$7a \cos^2 \psi - 4(a - c) \cos \psi - c = 0 \dots\dots\dots(5),$$

and the positive root of this equation determines the required value of ψ .

6·22. Examples. 1. A uniform circular disc of radius a rolls without slipping on a wire in a vertical plane. The wire is in the form of the curve $s = f(\psi)$, s being the length of the curve from a fixed point to any point P , and ψ the angle that the tangent at P makes with the horizontal. Shew that the ψ co-ordinate of the point of the curve in contact with the disc at any time satisfies the equation

$$\frac{d^2}{dt^2} \{f(\psi) - a\psi\} = -\frac{2}{3}g \sin \psi. \quad [\text{I. 1924}]$$

2. A uniform circular disc of radius r rolls, in a vertical plane, from rest at the highest point of a cycloidal wire placed in a vertical plane with its vertex upwards—the parameter of the cycloid being a ; if the wire be sufficiently rough to prevent sliding, shew that the disc will leave the wire when the common normal makes an angle θ with the vertical given by

$$3(4a \cos \theta + r) \cos \theta = 8(a \sin^2 \theta + r \sin^2 \theta/2). \quad [\text{I. 1927}]$$

3. A uniform circular disc of radius a is free to move in a vertical plane and rolls under gravity on a curve in that plane. Shew that the angular velocity of the disc is $\frac{1}{a} \sqrt{\frac{4}{3}gz}$, where z is the depth of the centre of the disc below a certain level. [M. T. 1924]

6·3. Sudden changes and Initial motions. In Pt. I, 15·41 we solved some problems on the effects of prescribed impulsive changes in the motion of a body, and in Pt. I, 16·3 attention was called to the simplification of the equations of motion when a body or system starts from rest and only the initial values of the accelerations and stresses are required. As a further example of the first kind let us consider the change of motion produced in a lamina with a curved boundary by its impact on a fixed rough inelastic curve in its plane.

Referring to the figure of 6·2, let Q be the point of impact, ω , ω' the angular velocities of the lamina just before and after the impact, and v the component velocity of G at right angles to GQ just before impact. Then since the moment of momentum about Q is unaltered by the blow, the equation

$$m(k^2 + r^2)\omega' = mk^2\omega + mvr$$

determines the angular velocity with which the lamina begins to roll.

6·31. Initial curvature. *To determine the initial curvatures of the paths of assigned points of a system, when the system is set in motion in a given manner, or when it begins to move from rest owing to the giving way of a constraint.*

In the former case, if the velocity of an assigned point is known in magnitude and direction, since the normal acceleration v^2/ρ can be found from the equation of motion, therefore the curvature $1/\rho$ is determinate.

In the latter case, if we can find the initial direction of motion, and also find the small displacements which take place in a short interval of time along and perpendicular to the direction of motion, then we may be able to find the curvature by using Newton's formula

$$\frac{1}{\rho} = \lim \frac{2y}{x^2}.$$

Otherwise it may be necessary to take such an expression as

$$\frac{1}{\rho} = \pm \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}} \dots\dots\dots(1),$$

which gives the curvature of the path of the point (x, y) at time t , and find the limit of this expression as $t \rightarrow 0$; or, if polar co-ordinates (r, θ) are used, the limit of

$$\frac{1}{\rho} = \pm \frac{r^2\dot{\theta}^3 + 2\dot{r}^2\dot{\theta} - r\ddot{r}\dot{\theta} + r\dot{r}\ddot{\theta}}{(\dot{r}^2 + r^2\dot{\theta}^2)^{\frac{3}{2}}} \dots\dots\dots(2).$$

Consider (1), and denote by $\dot{x}_0, \dots$ the value of $\dot{x}, \dots$ at time $t=0$. We have $\dot{x}_0 = \dot{y}_0 = 0$. Then, by Maclaurin's Theorem,

$$\dot{x}^2 + \dot{y}^2 = t^2 (\ddot{x}_0^2 + \ddot{y}_0^2) + \dots \quad \dots\dots\dots(3),$$

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = \frac{t^2}{2} (\ddot{x}_0\ddot{y}_0 - \ddot{x}_0\ddot{y}_0) + \frac{t^3}{3} (\ddot{x}_0\ddot{y}_0 - \ddot{x}_0\ddot{y}_0) + \dots \quad \dots\dots(4).$$

The values of $x_0, y_0, \dot{x}_0, \dot{y}_0$ are known. The values of $\ddot{x}_0, \ddot{y}_0$ are obtained from the equations of motion; and the initial values of the higher derivatives are obtained by successive differentiations of the equations of motion. Usually, the coefficient of t^2 in (3) is positive, while the coefficient of t^2 in (4) is zero, and the coefficient of t^3 is not zero. Then the initial curvature is

$$\pm \frac{\ddot{x}_0\ddot{y}_0 - \ddot{x}_0\ddot{y}_0}{3(\ddot{x}_0^2 + \ddot{y}_0^2)^{\frac{3}{2}}} \quad \dots\dots\dots(5).$$

Similarly, in polar co-ordinates, the initial curvature is

$$\pm \frac{3r_0^2\ddot{\theta}_0^3 + 6\ddot{r}_0^2\ddot{\theta}_0 + r_0\ddot{r}_0\ddot{\theta}_0 - r_0\ddot{r}_0\ddot{\theta}_0}{3(r_0^2\ddot{\theta}_0^2 + \ddot{r}_0^2)^{\frac{3}{2}}} \quad \dots\dots\dots(6).$$

6.32. Examples. (i) *Two particles A and B are connected by a fine string; A rests in limiting equilibrium on a rough horizontal table and B hangs vertically at a distance a below the edge of the table. The string being entirely in a plane at right angles to the edge, B is projected horizontally in this plane with velocity V. Prove that the initial acceleration of A is $\mu V^2/(\mu+1)a$, and that the initial curvature of the path of B is $1/a(\mu+1)$.*

Let m, m' denote the masses of A, B. Let the position of B be defined by polar co-ordinates r, θ referred to an origin at the edge of the table and such that initially $r=a, \theta=0$ and $r\dot{\theta}=V$.

For limiting equilibrium we must have $m'g = \mu mg$.

If T is the tension, then, in the initial position, the equations of motion are

$$m\ddot{r} = T - \mu mg$$

and

$$m'(\ddot{r} - r\dot{\theta}^2) = m'g - T;$$

whence by addition

$$(m + m')\ddot{r} = m'r\dot{\theta}^2 = m'V^2/a$$

or

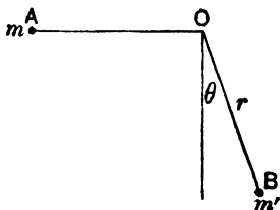
$$\ddot{r} = \mu V^2/(\mu+1)a.$$

Again, if ρ be the initial radius of curvature of the path of B,

$$\frac{V^2}{\rho} = -(\ddot{r} - r\dot{\theta}^2) = -\frac{\mu V^2}{(\mu+1)a} + \frac{V^2}{a} = \frac{V^2}{(\mu+1)a},$$

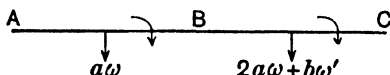
so that

$$\rho = (\mu+1)a.$$



(ii) Two uniform rods AB , BC are smoothly jointed at B and can turn about the end A which is fixed. Initially they are held so that ABC is a horizontal straight line and then they are let go. Find the initial curvature of the path of C .

Let m , m' be the masses and $2a$, $2b$ the lengths of AB , BC , and ω , ω' their initial angular accelerations.



By taking moments about B for BC and about A for the system, we get

$$\frac{1}{3}m'b^2\omega' + m'b(2a\omega + b\omega') = m'gb$$

and $\frac{1}{3}ma^2\omega + \frac{1}{3}m'b^2\omega' + m'(2a\omega + b\omega')(2a + b) = mga + m'g(2a + b)$,

leading to

$$\frac{\omega}{\omega'} = -\frac{b}{a} \frac{m' + 2m}{m}.$$

Suppose that, after a short time t , θ , ϕ are the inclinations of AB , BC to the horizontal; then the horizontal and vertical displacements of C are $2a(1 - \cos \theta) + 2b(1 - \cos \phi)$ and $2a \sin \theta + 2b \sin \phi$, or, approximately, $a\theta^2 + b\phi^2$ and $2a\theta + 2b\phi$.

The initial tangent to the path of C is vertical, so that, if ρ is the radius of curvature,

$$2\rho = \lim \frac{4(a\theta + b\phi)^2}{a\theta^2 + b\phi^2}.$$

But the first approximations to θ and ϕ are

$$\theta = \frac{1}{2}\omega t^2, \quad \phi = \frac{1}{2}\omega' t^2,$$

and hence we obtain $\rho = \frac{2ab(m + m')^2}{b(m + 2m')^2 + am^2}.$

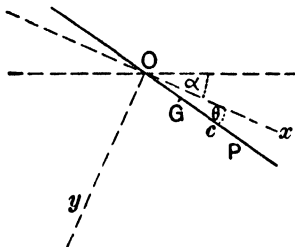
(iii) A uniform thin rod of length $2a$ is held at an inclination α to the horizontal in contact with a smooth peg at its middle point. Prove that when the rod is let go the initial radius of curvature of the path of a point at a distance c from the middle point is $(a^2/c) \tan \alpha$.

Let r , θ be the co-ordinates of P , so that $r - c$, θ are the co-ordinates of G . Then the equations of motion are

$$\ddot{r} - (r - c)\dot{\theta}^2 = g \sin(\alpha + \theta) \dots (1),$$

$$(r - c)[(r - c)\ddot{\theta} + 2\dot{r}\dot{\theta}] + \frac{1}{3}a^2\ddot{\theta}$$

$$= g(r - c) \cos(\alpha + \theta) \dots (2).$$



Then $r_0 = c$, $\dot{r}_0 = 0$, $\theta_0 = 0$, $\dot{\theta}_0 = 0$. By (1), (2), $\ddot{r}_0 = g \sin \alpha$, $\ddot{\theta}_0 = 0$. We wish to differentiate (1) and (2), twice. We can therefore neglect small quantities of the 3rd order in powers of t . Since $\dot{r}_0 = 0$, $r - c$ is of the second

order; and similarly, θ is of the 3rd order. Hence to the required degree of approximation, (1) and (2) become

$$\ddot{r} = g \sin \alpha, \quad a^2 \ddot{\theta} = 3g(r - c) \cos \alpha,$$

giving $\ddot{r}_0 = \dot{r}_0 = 0, \quad \ddot{\theta}_0 = 0, \quad \dot{\theta}_0 = 3g^2 \sin \alpha \cos \alpha / a^2.$

Hence, by 6.31 (6), the initial radius of curvature is $a^2 \tan \alpha / c$.

6.33. Examples. 1. A uniform circular disc of radius a , falling in a vertical plane without rotation, impinges on the rough edge of a fixed inelastic circular disc of radius b in the same plane, the common radius at the point of impact making an angle α with the vertical. Prove that the discs separate when the common normal makes an angle θ with the vertical given by

$$7 \cos \theta = 4 \cos \alpha + 4V^2 \sin^2 \alpha / 3(a + b)g,$$

where V is the velocity of the falling disc at the moment of impact, provided this gives a real value.

2. Two equal uniform rods AB, BC , each of weight W , are smoothly jointed at B , and the end A is fixed. The system is held so that A and C are in the same horizontal line and the angle ABC is a right angle, and C is then released. Shew that the initial reaction at B is $\frac{1}{2}\sqrt{\frac{2}{3}}W$.

[I. 1899]

3. A uniform rod of length $2a$ and mass m , carrying at one end a particle of mass m , has that end connected by a weightless inextensible string of length a to a fixed point. There is a smooth peg fixed at a distance a from the point, in the same horizontal line. The rod is placed at rest with its middle point against the peg, so that the string is stretched, the particle downwards and the whole system in a vertical plane. Shew that the rod begins to move with angular acceleration $15g/44a$, and the string with angular acceleration $15g/22a$.

[M. T. 1902]

4. Obtain the expression of 6.31 (6), viz.

$$\rho = \frac{3(r_0^2 \ddot{\theta}_0^2 + \dot{r}_0^2)^{\frac{3}{2}}}{3r_0^2 \ddot{\theta}_0^3 + 6\ddot{r}_0^2 \ddot{\theta}_0 + r_0 \ddot{r}_0 \dot{\theta}_0 - r_0 \dot{r}_0 \ddot{\theta}_0},$$

for the initial radius of curvature of the path of a point in polar co-ordinates.

5. Three particles A, B, C are connected by two strings AB, AC and placed in a line on a smooth table. The extreme particles B and C are then projected at right angles to the strings with velocities u, v . Prove that the initial curvatures of the paths of the extreme particles are respectively

$$\frac{(q+m)u^2b + qv^2a}{(p+q+m)u^2ab} \quad \text{and} \quad \frac{(p+m)v^2a + pu^2b}{(p+q+m)v^2ab},$$

m, p, q being the masses of the particles, and a, b the lengths of the strings.

6. Two particles of masses m, m' are tied to the ends of a string which passes through a bead of mass μ , and the whole system is placed on a smooth table with m, m' at the acute angles and μ at the right angle of a right-angled triangle. If the particles are projected with velocities u, v and at right angles to the respective portions of string, the lengths of which are a and b , prove that, if ρ, ρ' are the initial radii of curvature of their paths,

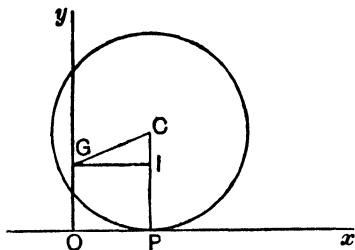
$$\frac{mu^2}{\rho} = \frac{m'v^2}{\rho'} = \left(\frac{u^2}{a} + \frac{v^2}{b} \right) / \left(\frac{1}{m} + \frac{2}{\mu} + \frac{1}{m'} \right).$$

7. A plane lamina, which is moveable about a horizontal axis in its plane, passing through its centre of gravity, is horizontal when a particle of mass m is placed upon it at a distance c from the axis. Shew that the initial curvature of the path of the particle is $I/3mc^3$, where I is the moment of inertia of the lamina about the axis.

6.4. Oscillations of a heterogeneous sphere on a horizontal plane. Let a be the radius of the sphere, m its mass and c the distance of the centre of gravity G from the centre C .

(a) *When the plane is smooth.*

Since there is no horizontal force on the sphere, the motion of G is in a vertical straight line Oy , and the motion of the point of contact P is one of sliding along the horizontal Ox . Hence the instantaneous centre I is the intersection of a horizontal through G with the radius PC , and if $GCP = \theta$, the equation of moments about I is



$$\frac{1}{2\theta} \frac{d}{dt} \{ (k^2 + c^2 \sin^2 \theta) \dot{\theta}^2 \} = -gc \sin \theta \dots\dots\dots(1).$$

For small oscillations about the equilibrium position this reduces to

$$k^2 \ddot{\theta} + gc\theta = 0 \dots\dots\dots(2),$$

making the length of the equivalent simple pendulum k^2/c .

(b) *When the plane is rough.* When there is rolling, P is the instantaneous centre, and the corresponding equation of moments about P is

$$\frac{1}{2\theta} \frac{d}{dt} \{ (k^2 + a^2 + c^2 - 2ac \cos \theta) \dot{\theta}^2 \} = -gc \sin \theta \dots\dots\dots(3),$$

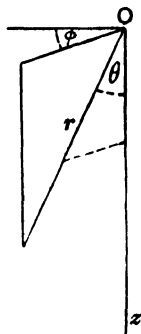
giving for small oscillations about the position of equilibrium

$$\{ k^2 + (a - c)^2 \} \ddot{\theta} + gc\theta = 0 \dots\dots\dots(4),$$

so that in this case the length of the equivalent simple pendulum is

$$\{ k^2 + (a - c)^2 \} / c.$$

6.41. Motion of a uniform heavy rod which is free to turn about an end which is fixed. Let $2a$ be the length of the rod, O the fixed end and Oz vertical, M the whole mass and m the mass of a small element at a distance r from the fixed end; θ the inclination to the vertical and ϕ the angle which the vertical plane through the rod makes with a fixed vertical plane.



Let us find the condition that the rod may rotate uniformly so that ϕ and θ may have constant values ω and α .

The moment of momentum about Oz is constant, since the weight and the reaction at O have no moment about Oz . Hence

$$\Sigma (mr^2 \sin^2 \theta \cdot \dot{\phi}) = \text{const.},$$

$$\text{or} \quad \dot{\phi} \sin^2 \theta = \omega \sin^2 \alpha \quad \dots\dots\dots(1),$$

for Σmr^2 is a constant $\frac{4}{3}Ma^2$.

Again, there is an energy equation

$$\frac{1}{2} \Sigma \{mr^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta)\} - Mga \cos \theta = \text{const.},$$

$$\text{or} \quad \frac{2}{3}Ma^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) - Mga \cos \theta = \frac{2}{3}Ma^2 \omega^2 \sin^2 \alpha - Mga \cos \alpha \quad \dots\dots\dots(2).$$

If we eliminate $\dot{\phi}$ between (1) and (2), we get

$$\frac{2}{3}a \left(\dot{\theta}^2 + \frac{\omega^2 \sin^4 \alpha}{\sin^2 \theta} \right) - g \cos \theta = \frac{2}{3}a \omega^2 \sin^2 \alpha - g \cos \alpha \quad (3).$$

Differentiating and dividing by $\dot{\theta}$, we find that

$$4a (\ddot{\theta} - \omega^2 \sin^4 \alpha \cot \theta \operatorname{cosec}^2 \theta) + 3g \sin \theta = 0 \quad \dots\dots(4).$$

For steady motion this must be satisfied by $\theta = \alpha$, so that the required condition is

$$4a \omega^2 \cos \alpha = 3g \quad \dots\dots\dots(5).$$

For small oscillations about this state of steady motion, we put $\theta = \alpha + \chi$, where χ is small, in (4), and retain only the first power of χ . This substitution is most simply effected by observing that equation (4) is of the form

$$\ddot{\theta} + f(\theta) = 0 \quad \dots\dots\dots(6),$$

which is equivalent to $\ddot{\chi} + f(\alpha + \chi) = 0$,

$$\text{or} \quad \ddot{\chi} + f(\alpha) + \chi f'(\alpha) = 0,$$

and $f(\alpha)$ vanishes because of (5), so that (6) reduces to

$$\ddot{\chi} + \chi f'(\alpha) = 0;$$

or, in this case, $\ddot{\chi} + \omega^2(1 + 3 \cos^2 \alpha) \chi = 0$,

which represents an oscillatory motion with period

$$2\pi/\omega\sqrt{1 + 3 \cos^2 \alpha}.$$

6.42. Application of the Principle of Virtual Work.

Since in any configuration of a moving system the effective forces are the exact equivalent of the impressed forces, it follows that if we consider any possible virtual displacement of the system from this configuration the virtual work of the effective forces must be equal to that of the impressed forces; provided that we include in this equation of virtual work every force whether internal or external which does work in the displacement.

Consider for example a single rigid body. Suppose that it possesses a rate of change of linear momentum F in an assigned direction and that δs is the virtual displacement of the centre of gravity in that direction, then a term $F \delta s$ enters the equation of virtual work. Similarly, if it possesses a rate of change of moment of momentum H about an assigned axis through its centre of gravity and $\delta \phi$ is the virtual rotation about that axis, the corresponding term in the equation of virtual work is $H \delta \phi$.

6.43. Example. *A uniform elastic ring of mass m , length $2\pi c$ and modulus of elasticity mg is placed horizontally on a smooth paraboloid of revolution of latus rectum $4a$ with axis vertical and vertex uppermost. Find the equation of motion, the position of equilibrium and the time of a small oscillation about it.*

Let $y^2 = 4ax$ be the equation of the meridian section of the paraboloid, T the tension of the ring when its radius is y . If s denotes an arc of the meridian curve, any point of the ring has accelerations $\ddot{s}$ and $\ddot{s}^2/\rho$, and if we consider a virtual displacement of the ring downwards we have an equation of virtual work

$$m \ddot{s} \delta s = mg \delta x - T \delta (2\pi y),$$

$$\text{or} \quad \ddot{s} = g \frac{dx}{ds} - 2\pi g \frac{y - c}{c} \frac{dy}{ds} \dots\dots\dots (1).$$

This equation might have been found by considering an element of the string, or by writing down the equation of energy and differentiating it.

Expressing $\ddot{s}$ and x in terms of y , we get

$$\ddot{y}(y^2 + 4a^2) + y\dot{y}^2 = 2agy - 8\pi ga^2(y - c)/c \dots\dots\dots(2).$$

Hence the equilibrium position is given by

$$y_0 = 4\pi ac/(4\pi a - c) \dots\dots\dots(3).$$

For small oscillations we put $y = y_0 + \eta$, where η is small, and retain only the first power of η . Thus

$$\ddot{\eta}(y_0^2 + 4a^2) = 2a\eta g \left(1 - \frac{4\pi a}{c}\right) \dots\dots\dots(4).$$

Since (3) requires $4\pi a - c$ to be positive, (4) represents an oscillatory motion of period

$$2\pi \left[\frac{2ca \{4\pi^2 c^2 + (4\pi a - c)^2\}^{\frac{1}{2}}}{g(4\pi a - c)^3} \right]^{\frac{1}{2}}.$$

6.44. Examples. 1. A rod of length $2a$ moves under gravity about one end which is smoothly jointed to a fixed point, and has its other end joined by an elastic string, of natural length a and modulus of elasticity equal to the weight of the rod, to a second point at a distance a vertically below the first. Prove that the angular velocity with which the rod must rotate about the vertical in order that the string may be stretched to twice its natural length is $\sqrt{(6g/a)}$, and that the time of a small oscillation about the steady motion is $16\pi\sqrt{(2a/957g)}$. [I. 1906]

2. Four equal uniform rods AB, BC, CD, DA , each of length $2a$, are smoothly hinged so as to form a rhombus, and are suspended by the hinge at A . The hinges B, D are connected by an elastic string whose modulus is twice the weight of a rod, and whose natural length is $a\sqrt{2}$. The whole system revolves round the vertical diagonal AC . Determine the angular velocity ω when, in a state of steady motion, the rods form a square; and shew that if C be slightly disturbed vertically, the period of a small oscillation is $2\pi/\omega$. [I. 1898]

3. An elliptic cylinder of semi-axes a and b and radius of gyration k about the line of centres rolls on a horizontal plane.

Prove that $(r^2 + k^2)\omega^2 + 2gp = \text{constant}$,

where r is the distance of the point of contact from the line of centres, ω is the angular velocity, and p the height of the line of centres above the plane. Shew also that ω and p are related to ϕ , the eccentric angle of the point of contact, by the equations

$$\omega = -\frac{ab}{a^2 \sin^2 \phi + b^2 \cos^2 \phi} \dot{\phi},$$

$$p = \frac{a^2 b^2}{a^2 \sin^2 \phi + b^2 \cos^2 \phi}.$$

Explain briefly why in this case the rate of change of the angular momentum about the line of contact is not equal to the moment of the forces about that line.

If the cylinder executes small oscillations about a position of equilibrium, shew that the length of the equivalent simple pendulum is

$$b(b^2 + k^2)/(a^2 - b^2). \quad [\text{I. 1932}]$$

4. A smooth curve has its concavity upwards, is symmetrical about the vertical, and the tangent at its lowest point is horizontal. A rod of length $2a$ passes through a smooth ring situated at a distance b measured inwards on the normal at the lowest point. Shew that, if the rod be slightly displaced, the length of the equivalent simple pendulum is

$$\frac{1}{3}\rho\{a^2 + 3(b-a)^2\}/(b^2 - a\rho),$$

where ρ is the radius of curvature at the lowest point.

5. A uniform rod AB , of length l and mass m , is smoothly pivoted at the end A , and the end B is connected by an elastic string, of negligible mass, natural length l and modulus $5mg$, to a point C at a distance $2l$ vertically above A . The arrangement rotates freely in steady motion with angular velocity Ω about AC , the angle CAB being α . If $\cos \alpha = \frac{1}{2}$, prove that $\Omega^2 = 54g/l$.

Prove further that the period of a small oscillation about this state of steady motion is $2\pi/n$, where $n^2 = \frac{2277}{32} \frac{g}{l}$. [M. T. 1935]

EXAMPLES

1. A uniform sphere is placed on the top of a rough circular cylinder whose generators are horizontal. Shew that, if slightly displaced, it will roll on the cylinder until it reaches a place where the inclination θ of the tangent plane to the horizon is given by

$$2 \sin \theta = \mu (17 \cos \theta - 10),$$

μ being the coefficient of friction.

Shew also that the subsequent velocity v of the centre of the sphere, so long as the latter is in contact with the cylinder, is given by a formula of the type

$$v^2 = A \cos \theta + B \sin \theta + Ce^{2\mu\theta},$$

where A, B, C are independent of θ .

[M. T. 1920]

2. A solid uniform sphere of radius a and mass M is free to turn about a horizontal diameter as a fixed axis. A second sphere of mass m is placed on it at a point whose latitude measured from the vertical equatorial plane is α and whose longitude measured from the vertical plane through the axis is β . The spheres are perfectly rough. Shew that the initial acceleration of the centre of the free sphere makes an angle with the meridian plane whose tangent is

$$\frac{\tan \beta}{\sin \alpha} \cdot \frac{M + m \cos^2 \alpha}{M + \frac{1}{2}m \cos^2 \alpha}$$

and that the initial angular acceleration of the other sphere is

$$\frac{mg \sin \beta \cos \alpha}{a \left(\frac{1}{2}M + m \cos^2 \alpha \right)}. \quad [\text{M. T. 1904}]$$

3. A thin hollow cylinder of radius a and mass M is free to turn about its axis, which is horizontal, and a similar cylinder of radius b and mass m rolls inside it without slipping, the axes of the two cylinders being parallel. Shew that when the plane of the two axes is inclined at an angle θ to the vertical the angular velocity of the larger cylinder is given by

$$a^2 (M + m) (2M + m) \Omega^2 = 2gm^2 (a - b) (\cos \theta - \cos \alpha),$$

provided both cylinders are at rest when $\theta = \alpha$.

[I. 1908]

4. A rough hollow cylinder of mass M , with external and internal radii a, b , contains a rough sphere of mass m and radius c , and rolls up a rough inclined plane of inclination α ; shew that the centre of the sphere may describe a straight line parallel to the inclined plane and at a distance $a - (b - c) \cos (\alpha + \beta)$ from it, provided a constant force parallel to the inclined plane be applied to the cylinder, and prove that the acceleration of translation of the system is

$$g \sin \beta / \{ bk^2/ac^2 + \cos (\alpha + \beta) \} \equiv f,$$

and that the applied force is

$$(M + m) g \sin \alpha + \{ M (K^2/a^2 + 1) + m (b^2k^2/a^2c^2 + 1) \} f.$$

[M. T. 1906]

5. A smooth circular wire of mass M and of radius a can turn freely about a diameter which is fixed vertically. A small ring of mass m can slide freely along the wire. The ring being at rest at the lowest point of the wire, shew that if the wire be set in rotation with angular velocity n , and left to itself, the ring will not remain at the lowest point of the wire unless $n < (g/a)^{\frac{1}{2}}$; and that if n is greater than $\left\{ \frac{2g}{a} \left(1 + \frac{2m}{M} \right) \right\}^{\frac{1}{2}}$ the ring will rise in its oscillations to a level above that of the centre of the circular wire.

Prove also that the ring can rest in relative equilibrium at a point whose angular distance α from the lowest point of the wire is $\cos^{-1} (g/an^2)$, whilst the wire is made to rotate with angular velocity $n (> (g/a)^{\frac{1}{2}})$; and that if, the wire being now left free, the ring be slightly disturbed, the ring will perform small oscillations of period

$$2\pi \left\{ \frac{a \cos \alpha}{g \sin^3 \alpha} \cdot \frac{M + 2m \sin^2 \alpha}{M + 2m \sin^2 \alpha + 8m \cos^2 \alpha} \right\}^{\frac{1}{2}}. \quad [\text{M. T. 1910}]$$

6. A heavy particle of mass m moves in a fine smooth tube of mass M in the form of a circular helix ($x = a \cos \theta$, $y = a \sin \theta$, $z = c\theta$). The tube is free to move round its axis which is vertical. Prove that the vertical acceleration of the particle is

$$g / \left\{ 1 + \frac{Ma^2}{(M + m)c^2} \right\}.$$

Shew also that if the system starts from rest the radial component of the reaction of the tube on the particle is

$$2mM^2gaz/[(M+m)\{(M+m)c^2+Ma^2\}],$$

where z is the vertical distance through which the particle has fallen.

[M. T. 1932]

7. A rough circular cylinder of mass m and radius a with its centre of gravity distant c from its axis rests on the surface of a board of mass M whose upper surface is rough and lower surface is smooth and resting on a smooth table. If x be the distance moved over by the board and θ the angle turned through by the cylinder, prove that the equations of linear momentum and angular momentum about the point of contact are respectively

$$\frac{d}{dt}[M\dot{x} + m\{\dot{x} + (a - c \cos \theta) \dot{\theta}\}] = 0$$

$$\text{and} \quad \frac{d}{dt}[k^2\dot{\theta} + c(c - a \cos \theta) \dot{\theta} + (\dot{x} + a\dot{\theta})(a - c \cos \theta)] \\ - c \sin \theta (\dot{x} + a\dot{\theta}) \dot{\theta} = -gc \sin \theta,$$

where k is the radius of gyration of the cylinder about its axis.

Deduce that

$$\frac{1}{2}\dot{\theta}^2 \left[k^2 + a^2 - 2ac \cos \theta + c^2 - \frac{m}{M+m} (a - c \cos \theta)^2 \right] = gc \cos \theta + \text{const.}$$

[St John's Coll. 1911]

8. A spherical shell of mass m , whose outer surface is rough and of radius a , has its inner surface smooth and of radius b . A particle of mass m moves inside while the shell rolls on a rough table. Shew that if the excursions of the particle be α on either side of the vertical, then

$$[M(a^2 + k^2) + ma^2 \sin^2 \theta] b \dot{\theta}^2 = 2g[M(a^2 + k^2) + ma^2][\cos \theta - \cos \alpha],$$

where k is the radius of gyration of the shell about a diameter.

9. A circular disc (radius c) is placed within a vertical circle (radius a) so as to be in contact with it at the extremity of a horizontal diameter, and is then projected vertically. The interior being perfectly rough, find the initial angular velocity, and the point at which the disc will leave the curve, if its angular velocity on leaving it is $\sqrt{g(a-c)}/c$.

10. A cylinder, whose surface is smooth, stands on a smooth horizontal plane on one of its circular ends; find the impulse at a given point of its surface which will cause it to fall over.

11. The nut of a smooth screw rests on a smooth horizontal plane, over a hole cut so as to allow a free passage for the screw, and the screw descends through the nut by its own weight; determine the motion, (a) when the nut is fixed, (b) when it is moveable.

12. A circular ring, mass M and radius a , lies on a smooth horizontal plane, and a fly, mass m , alighting upon it starts off and crawls round the ring, with a velocity v , which is uniform relative to the ring. Prove that the angular velocity of the ring is $mv/(M + 2m)a$.

13. If the ring in the previous question be vertical and moveable about its centre of gravity, and if the fly starts off and as before moves uniformly relative to the ring, find its angular velocity in any position, supposing the fly to start from the lowest point. Also find the least ratio of the masses in order that the fly may reach the highest point of the ring.

14. A circular hoop of radius a and mass $3m$ is free to move about its centre in a horizontal plane. An insect of mass m stands on the hoop, the system being initially at rest. The insect leaps with velocity w relative to the hoop, and alights again on the hoop. Shew that when it alights the system comes to rest.

Shew further that, provided $w^2 \geq \frac{1}{2}ag$, the greatest angle through which the hoop can be turned in this way is one-third of a radian.

[M. T. 1931]

15. Four equal particles, connected by inelastic strings forming the sides of a square, mutually repel each other with a force $m\mu \cdot (\text{distance})$; if one string be cut, shew that the angle θ that either string makes with its original position is given by

$$\theta^2 (2 - \sin^2 \theta) = 4\mu \sin \theta (2 + \sin \theta).$$

16. A thin uniform circular ring is set rotating when hot about its centre, and its radius diminishes in cooling by a fraction proportional to the time. If no forces act, shew that the angle through which the ring has revolved in time t is $\Omega at/r$, where a, Ω are the initial radius and velocity, and r the final radius.

17. A rod falls in a vertical plane but remains in contact with a fixed smooth horizontal peg to which it is perpendicular. Write down sufficient equations to determine the motion and obtain one integral of them.

If the rod is at rest when horizontal, prove that the differential equation of the path of its centre of gravity is

$$r(r^2 + k^2) \cos \theta = \frac{d}{d\theta} \left\{ \frac{r(r^2 + k^2)^2 \sin \theta}{(dr/d\theta)^2 + r^2 + k^2} \right\}. \quad [\text{I. 1901}]$$

18. A smooth spherical shell of mass M rests on an inclined plane, being fastened to a point of the plane by a string; a particle m rests inside the sphere; prove that if the string be cut, the ratio of the initial pressure between the sphere and the particle to that between the sphere and the plane is $m \cos \alpha : M + m$,

where α is the inclination of the plane to the horizontal.

19. A smooth plane of mass M is freely moveable about a horizontal axis lying in it and passing through its centre of gravity, the radius of gyration of the plane about the axis being k . The plane being inclined at an angle α to the vertical, a sphere of mass m is placed gently upon it. If initially the centre of the sphere be in a vertical through the axis of

the plane, and if h be its initial height above that axis, shew that the angle ϕ , which the initial direction of motion of the centre makes with the vertical, is given by

$$(Mk^2 + mh^2) \tan \phi = Mk^2 \tan \alpha. \quad [\text{M. T. 1879}]$$

20. A cube is rotating with angular velocity ω about a diagonal, when one of its edges which does not meet that diagonal becomes fixed; shew that the angular velocity about this edge will be $\omega/4\sqrt{3}$.

21. Three equal particles at rest, tied together by three equal strings of length a , so as to form an equilateral triangle, repel each other with a constant force mf . If one string be cut, shew that the equation to determine the motion is

$$a\theta^2(3 - 2\sin^2\theta) = 3f(2\sin\theta - 1),$$

2θ being the angle between the strings.

22. A straight uniform rod slides freely with its ends on two smooth wires which intersect at right angles and are in the same vertical plane equally inclined to the vertical. Determine the motion and the time of a small oscillation about the position of stable equilibrium. [I. 1907]

23. A heavy elastic string, in the form of a ring, is placed, with its plane horizontal, over a smooth cone, the axis of which is vertical; find the position of equilibrium. Also, if the string be held in contact with the cone at its natural length, and let go, find how far it will descend, and the greatest vertical velocity during the motion.

24. On a smooth surface formed by the revolution of a parabola about its directrix is placed a uniform elastic ring of natural length less than $2\pi a$, under no forces but the reaction of the surface and its own tension. If the plane of the ring be perpendicular to the directrix, shew that the time of a small oscillation is $2\sqrt{Ma\pi/T}$, where M is the mass of the ring, T the tension in the position of equilibrium, and $4a$ the latus rectum of the parabola.

25. A heavy uniform rod is supported against a smooth fixed sphere by a horizontal string fastened to its upper end, and also to the highest point of the sphere; if the string be cut, prove that the pressure on the sphere is changed in the ratio of $\cos^2\alpha : 1 + 3\sin^4\alpha$, where α is the angular distance from the vertex of the point of contact.

26. A uniform circular ring moves on a rough curve in the same plane under the action of no forces, the curvature of the curve being everywhere less than that of the ring. Shew that, if the ring be projected without rotation from a point A of the curve and begin to roll at a point B , the angle between the normals at A and B is $(\log 2)/\mu$, where μ is the coefficient of friction.

27. A perfectly rough rod is gently placed with one end upon another rod of equal mass and in the same vertical plane, moving with the velocity $\sqrt{2gc}$ on a smooth table. If the initial inclination of the first rod to the horizon be α , and its length $2a$, shew that it will just rise to a vertical position if

$$2a(1 - \sin \alpha)(5 + 3 \cos^2 \alpha) = 3c \sin^3 \alpha.$$

28. A rigid pendulum is hanging at rest when its point of support is suddenly started into motion so that its subsequent horizontal displacement is given by $A \sin pt$, where A is small and $2\pi/p$ is not equal to the free period of the pendulum. Find the motion.

If there is a small amount of damping, shew that in the final state there will be one point of the pendulum nearly at rest if $2\pi/p$ is less than the free period of the pendulum, and the pendulum is sufficiently long.
[I. 1920]

29. The extremities of a uniform heavy rod of length $2c$ slide on a smooth wire in the form of the parabola $x^2 - 4ay = 0$, the axis of the parabola being vertical, and $c > 2a$. If the rod be slightly displaced from its position of stable equilibrium, prove that the time of a small oscillation is

$$2\pi \{2a(c^2 + 12a^2)/3g(c^2 - 4a^2)\}^{\frac{1}{2}}.$$

30. The middle point of a uniform rod is fixed midway between two centres of force, which attract with a force varying inversely as the square of the distance. Prove that the time of a small oscillation is

$$\pi(a^2 - c^2)\sqrt{M}/\sqrt{3\mu ac},$$

where M is the mass of the rod, $2c$ its length, $2a$ the distance between the centres of force, and $\mu \delta x/r^2$ the attraction on an element δx of the rod at distance r .

31. A ring rests upon two smooth horizontal bars which in the position of equilibrium subtend an angle 2α at the centre; shew that, if the ring be disturbed by twisting it through a small angle about its vertical diameter, the length of the equivalent simple pendulum will be $\frac{1}{2}c \cot \alpha \operatorname{cosec} \alpha$.

32. A man walks on a large rough ball so as to make the ball roll straight up an inclined plane of inclination α , keeping himself at an angular distance β from the highest point of the ball; if the weights of the man and the ball are equal, prove that the acceleration of each is

$$5g(\sin \beta - 2 \sin \alpha)/(12 + 5 \cos(\alpha + \beta)).$$

33. A perfectly rough sphere of radius a is placed quite close to the intersection of the highest generating lines of two fixed equal horizontal right cylinders of radius c , the axes being inclined at an angle 2α to each other, and is allowed to roll down between them. Prove that the vertical velocity of its centre in any position will be

$$\sin \alpha \cos \phi \{10g(a+c)(1 - \sin \phi)\}^{\frac{1}{2}}/(7 - 5 \cos^2 \phi \cos^2 \alpha)^{\frac{1}{2}},$$

where ϕ is the inclination to the horizontal of the radii to the points of contact. [M. T. 1877]

34. An insect alights on a horizontal circular disc, rotating about its centre, which is fixed, and crawls with uniform relative velocity, so as always to approach the centre without altering the motion of the disc; prove that it will reach the centre before the disc has made more than a quarter of a revolution.

35. A uniform rod is placed with one extremity at the middle point of the line joining two equal centres of force attracting inversely as the square of the distance, and at right angles to that line; find the velocity with which the centre of the rod will reach the line.

36. If a bullet of mass m be fired with velocity u perpendicular to the face of a block of wood of mass M , placed on a smooth horizontal plane, and remains just embedded, prove that the angular velocity acquired by the block is

$$mbu/\{(M+m)k^2 + m(a^2 + b^2)\},$$

where a is the distance of the centre of gravity of the block from the face struck, and b is the distance of the point struck from the foot of the perpendicular drawn from the centre of gravity of the block to the face.

37. A fine circular tube, radius c , lies on a smooth horizontal plane, and contains two equal particles connected by an elastic string in the tube, the natural length of which is equal to half the circumference. The particles are in contact and fastened together, the string being stretched round the tube.

If the particles become disunited, prove that the velocity of the tube when the string has regained its natural length is

$$\sqrt{\frac{2\pi\lambda mc}{M(M+2m)}},$$

where M , m are the masses of the tube and each particle, and λ is the modulus of elasticity.

38. If in the previous question one of the particles be fixed in the tube, and if the tube be moveable about the point at which the particle is fixed, prove that when the string has regained its natural length the angular velocity of the tube is

$$\sqrt{\frac{\pi\lambda m}{M(M+2m)c}}.$$

39. Two equal rods AC, CB , hinged at C and having their extremities A, B connected by a fine thread so that ACB is a right angle, are revolving in their own plane about A , which is fixed, with uniform angular velocity. Prove that if the string be cut the stress at the hinge is instantaneously changed in the ratio $\sqrt{5}:4$.

40. Three equal and similar rods moveable about one common extremity are held at right angles to each other so that the three other extremities are in a horizontal plane. Shew that if they be dropped upon a smooth inelastic horizontal plane their vertical velocity will be diminished one half.

41. AB , BC , CD are three equal uniform rods freely jointed together and moveable about the extremity A ; the rods fall from a horizontal position of rest: prove that the initial radius of curvature of the path of the extremity D of the further rod is $81a/131$, where a is the length of each rod. Prove also that the initial stresses at C , B and A are in the ratio of 1, 4 and 15.

42. A rod, the radius of gyration of which about its centre of gravity is k , touches a smooth fixed circular disc of radius a . Initially the rod has an angular velocity ω , and the centre of gravity, which is the point of contact, has a velocity $(a^2 - k^2)\omega/2a$ along the rod, towards the end which is turning towards the disc. Shew that the distance between the centre of gravity and the point of contact increases at a uniform rate $2n^2a\omega$, and that in time t the rod turns through an angle θ given by $\tan n\theta = n\omega t$, where $4n^2 = (a^2 + k^2)/a^2$. [M. T. 1907]

43. On a smooth table there lies a straight bar with one end pivoted to a point of the table. A uniform disc of radius a rests on the table in contact with the bar, the point of contact being at a distance b from the pivot. The disc can slide freely on the table, but friction prevents any slipping between the disc and the bar. The bar starts rotating about the pivot with a constant angular velocity. Shew that if $b^2 = \frac{3}{2}a^2$, the point of contact will describe an equiangular spiral. [M. T. 1920]

44. An inelastic rod is set spinning on a smooth table with its centre at rest and strikes a smooth peg fixed in the table, the point of impact being at a distance c from the centre of the rod, whose length is $2a$. Determine the motion immediately after impact and subsequently; and shew that when the end of the rod slips off the peg, the rod will have turned through an angle θ given by $\tan \theta = c/a$, mod $\{a/(a^2 + 3c^2)^{\frac{1}{2}}\}$. [M. T. 1909]

45. A rigid lamina can turn in its own plane, which is horizontal, about a fixed point; when in motion it is acted on by a constant frictional couple G , opposing the motion. It is attached to a spring which exerts a couple $\lambda\theta$, where θ is the angle turned through by the lamina from a given position. The lamina, when in this position, is set rotating with angular velocity Ω ; it is found to turn through an angle θ_1 before coming to rest, and then to move backwards and come to rest again when $\theta = \frac{1}{2}\theta_1$. Prove that if the spring were removed, and the lamina were started with the same angular velocity Ω , it would turn through an angle $\frac{3}{2}\theta_1$ before it was brought to rest by the couple G . [I. 1930]

46. A rod of length $4a$ leans in a vertical plane against a fixed smooth cylinder, of radius a , lying on the ground with its axis horizontal. One end of the rod is on the smooth ground at a distance from the point of contact of the cylinder greater than $2a$ and the inclination of the rod to the horizontal is then 2α ; the rod is kept in position by a block on the ground. If the block is removed, find what the instantaneous pressure of the rod on the cylinder becomes. [I. 1928]

47. A uniform hollow sphere of radius R and mass M rests on a smooth horizontal plane; a particle of mass m is placed at a point on the lower half of the smooth inner surface of the sphere, and is given an initial velocity V parallel to the horizontal tangent at that point. If, in the subsequent motion, P denotes the position of the particle, O the centre of the sphere and A the point of contact of the sphere and the plane, shew, assuming that the particle does not leave the sphere, that the angle POA oscillates between its original value α_0 and a value of α given by

$$2g \sin^2 \alpha = \frac{M}{m+M} \frac{V^2}{R} (\cos \alpha + \cos \alpha_0). \quad [\text{M. T. 1934}]$$

48. A number of uniform straight rods of the same weight and length, freely jointed end to end, are supported in a horizontal straight line, with the extreme end of the last rod fixed. If the supports are all removed at once, obtain equations to determine the initial angular accelerations of the different rods, and prove that if ω_n , ω_{n+1} and ω_{n+2} be those of any three adjacent rods,

$$\omega_n + 4\omega_{n+1} + \omega_{n+2} = 0. \quad [\text{M. T. 1903}]$$

Chapter VII

EQUATIONS OF MOTION AND APPLICATIONS IN THREE DIMENSIONS

7·1. In this chapter we shall obtain the general equations of motion of a system of particles, or of a rigid body, referred to rectangular axes by extending to three dimensions the methods used for the two-dimensional case in **Pt. I**, Chapters ix and xv; and, by using the results of 3·6 for moving axes, we shall obtain, in their most general form, the equations of motion of a rigid body with a moving frame of reference.

We shall then apply the equations of motion to the solution of a number of problems, particularly to the motion of a sphere on a variety of surfaces.

An alternative method of obtaining the equations of motion by the use of vectors will be found in the appendix.

7·11. Motion of a system of particles. We take it as axiomatic [**Pt. I**, 4·161] that the vector which represents the acceleration of a particle multiplied by its mass and the vector which represents the resultant of the forces acting on the particle are the same. We also assume the law of reaction, viz. that action and reaction are equal and opposite, and then, by the method used for the two-dimensional case [**Pt. I**, 9·2], we are able to obtain the general equations of a system of particles moving in three dimensions in the form

$$\Sigma m\ddot{x} = \Sigma X, \quad \Sigma m\ddot{y} = \Sigma Y, \quad \Sigma m\ddot{z} = \Sigma Z \quad \dots\dots(1),$$

$$\left. \begin{aligned} \Sigma m(y\ddot{z} - z\ddot{y}) &= \Sigma(yZ - zY) \\ \Sigma m(z\ddot{x} - x\ddot{z}) &= \Sigma(zX - xZ) \\ \Sigma m(x\ddot{y} - y\ddot{x}) &= \Sigma(xY - yX) \end{aligned} \right\} \dots\dots\dots(2),$$

where x, y, z are the co-ordinates of the particle of mass m and X, Y, Z are the corresponding components of the *external* forces acting upon it.

These equations embody the fundamental facts that

(i) *the rate of change of momentum of a system in any direction is equal to the resolved part of the external forces in that direction;* and

(ii) *the moment of the rate of change of momentum about any fixed axis is equal to the sum of the moments of the external forces about that axis.*

7·12. Proceeding as in **Pt. I, 9·2** we can deduce at once the principles of conservation of linear momentum and moment of momentum; and proceeding as in **Pt. I, 9·3** we can shew that *the motion of the centre of gravity G is the same as if all the mass were collected into a particle at G and all the external forces were moved parallel to themselves to act at G , or that the equations 7·11 (1) are equivalent to*

$$M\ddot{\bar{x}} = \Sigma X, \quad M\ddot{\bar{y}} = \Sigma Y, \quad M\ddot{\bar{z}} = \Sigma Z \quad \dots\dots\dots(1),$$

where M denotes the whole mass and $(\bar{x}, \bar{y}, \bar{z})$ its centre of gravity.

Similarly, as in **Pt. I, 9·3**, we may deduce that if x', y', z' are the co-ordinates of m referred to parallel axes through G the equations 7·11 (2) can be reduced to

$$\left. \begin{aligned} \Sigma m (y' \ddot{z}' - z' \ddot{y}') &= \Sigma (y' Z - z' Y) \\ \Sigma m (z' \ddot{x}' - x' \ddot{z}') &= \Sigma (z' X - x' Z) \\ \text{and} \quad \Sigma m (x' \ddot{y}' - y' \ddot{x}') &= \Sigma (x' Y - y' X) \end{aligned} \right\} \quad \dots\dots\dots(2),$$

shewing that *the motion relative to the centre of gravity is independent of the motion of the centre of gravity.*

It also follows as in **Pt. I, 9·5** that *the kinetic energy of the system is equal to the kinetic energy of the whole mass moving with the velocity of the centre of gravity together with the kinetic energy of the particles in their motion relative to the centre of gravity.*

7·2. Moving axes. We shall now consider how the equations of motion, 7·11 (1) and (2), are to be modified when the axes are rotating with an angular velocity whose components about themselves are $\theta_1, \theta_2, \theta_3$, while the origin has a velocity whose components are α, β, γ .

Firstly, let us keep the axes fixed in direction and see how the expressions for rates of change of momentum and moment of momentum are affected by the motion of the origin. Let p and h denote the resultant linear momentum and moment of momentum about the axes, with components p_x, p_y, p_z and h_x, h_y, h_z , then, if v_x, v_y, v_z denote the components of velocity of the particle of mass m , we have

$$p_x, p_y, p_z = \Sigma m v_x, \Sigma m v_y, \Sigma m v_z \dots\dots\dots(1),$$

where $v_x = \alpha + \dot{x}, v_y = \beta + \dot{y}, v_z = \gamma + \dot{z} \dots\dots\dots(2);$

and $h_x, h_y, h_z = \Sigma m (y v_z - z v_y), \Sigma m (z v_x - x v_z), \Sigma m (x v_y - y v_x) \dots\dots\dots(3).$

It is clear that the equations expressing the rate of change of linear momentum 7·11 (1) are unaltered in form and may be written

$$\dot{p}_x, \dot{p}_y, \dot{p}_z = \Sigma X, \Sigma Y, \Sigma Z \dots\dots\dots(4).$$

Again $\dot{h}_x = \Sigma m (y \dot{v}_z - z \dot{v}_y) + \Sigma m (\dot{y} v_z - \dot{z} v_y),$
so that from (2)

$$\dot{h}_x = \Sigma m (y \dot{v}_z - z \dot{v}_y) + \Sigma m (-\beta \dot{v}_z + \gamma \dot{v}_y);$$

and this may be written

$$\dot{h}_x + p_z \beta - p_y \gamma = \Sigma m (y \dot{v}_z - z \dot{v}_y) \dots\dots\dots(5).$$

Now the expression on the right is the total moment about Ox of the rate of change of momentum, and corresponds to the left-hand side of 7·11 (2); so that, if we use L, M, N to denote the moments of the external forces about the axes, the equations 7·11 (2) take the form

$$\left. \begin{aligned} \dot{h}_x + p_z \beta - p_y \gamma &= L \\ \dot{h}_y + p_x \gamma - p_z \alpha &= M \\ \dot{h}_z + p_y \alpha - p_x \beta &= N \end{aligned} \right\} \dots\dots\dots(6).$$

We have now to take into account the rotation of the axes, and if we use suffixes 1, 2, 3 to denote components referred to the moving axes, as in 3·6, the equations (4) and (6) become

$$\left. \begin{aligned} \dot{p}_1 - p_2 \theta_3 + p_3 \theta_2 &= \Sigma X \\ \dot{p}_2 - p_3 \theta_1 + p_1 \theta_3 &= \Sigma Y \\ \dot{p}_3 - p_1 \theta_2 + p_2 \theta_1 &= \Sigma Z \end{aligned} \right\} \dots\dots\dots(7);$$

and

$$\left. \begin{aligned} \dot{h}_1 - h_2\theta_3 + h_3\theta_2 + p_3\beta - p_2\gamma &= L \\ \dot{h}_2 - h_3\theta_1 + h_1\theta_3 + p_1\gamma - p_3\alpha &= M \\ \dot{h}_3 - h_1\theta_2 + h_2\theta_1 + p_2\alpha - p_1\beta &= N \end{aligned} \right\} \dots\dots\dots (8).$$

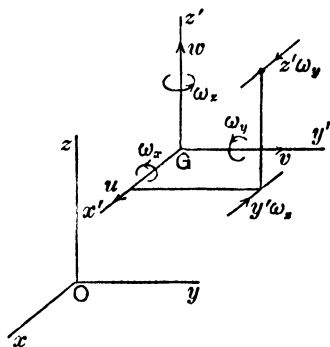
These are the equations of motion in their most general form. They may be simplified in special cases. For example, if the origin is at the centre of gravity of the system, then

$$p_1, p_2, p_3 = M\alpha, M\beta, M\gamma,$$

so that the terms in α, β, γ in (8) disappear, and the equations of moments are the same in form as if the origin were at rest.

7·3. Momentum of a rigid body. It follows from Chapter III that the motion of a rigid body can be specified by the linear velocity of some one point of it together with the angular velocity about an axis through that point; also that the magnitude of the angular velocity and direction of its axis are independent of the particular point chosen.

Let u, v, w denote the components of velocity of the centre of gravity G of a body referred to fixed axes, and $\omega_x, \omega_y, \omega_z$ the components of the angular velocity of the body. Then, as in 7·11 and 7·12, taking x, y, z as the co-ordinates of a particle of mass m , by considering relative velocities we easily see that



$$\dot{x} = u - y'\omega_z + z'\omega_y,$$

$$\dot{y} = v - z'\omega_x + x'\omega_z$$

and

$$\dot{z} = w - x'\omega_y + y'\omega_x.$$

Hence, since $\Sigma mx' = \Sigma my' = \Sigma mz' = 0$,

we have for the components of linear momentum

$$\Sigma m\dot{x}, \Sigma m\dot{y}, \Sigma m\dot{z} \equiv Mu, Mv, Mw \dots\dots\dots (1);$$

and, for the moment of momentum about Ox ,

$$\begin{aligned}\Sigma m (y\dot{z} - z\dot{y}) &= \Sigma m \{(\bar{y} + y')(w - x'\omega_y + y'\omega_x) - (\bar{z} + z')(v - z'\omega_x + x'\omega_z)\} \\ &= M(\bar{y}w - \bar{z}v) + \omega_x \Sigma m (y'^2 + z'^2) - \omega_y \Sigma m x'y' - \omega_z \Sigma m x'z' \\ &\quad \dots\dots(2)\end{aligned}$$

and similar expressions for the moments of momentum about Oy and Oz .

In (2) the terms in M are the moments about the axes of a vector localized at G with components Mu , Mv , Mw ; i.e. the linear momentum. The remaining terms are the moments of momentum in the motion of the body relative to G about parallel axes through G , and we note that they are independent of the motion of G . These terms may be written

$$\left. \begin{aligned}A\omega_x - F\omega_y - E\omega_z \\ B\omega_y - D\omega_z - F\omega_x \\ C\omega_z - E\omega_x - D\omega_y\end{aligned} \right\} \dots\dots\dots(3),$$

where A , B , C , D , E , F are the moments and products of inertia of the body about the axes through G .

Thus it appears as in **Pt. I, 14·3** that the whole momentum of a body may be regarded as composed of a vector, the linear momentum, localized at G , together with a *spin couple* whose components are the expressions (3); and the moment of momentum about any axis is the moment of the linear momentum vector together with the resolved part of the spin couple.

It should be noted that if, as above, we refer a moving body to axes fixed in space, its moments and products of inertia about such axes vary with the time, so that in most cases it is more convenient to take axes fixed in the body. Further, if ω_1 , ω_2 , ω_3 denote the components of angular velocity of the body about axes through its centre of gravity fixed in itself, and we take the fixed axes Ox , Oy , Oz above momentarily coincident in direction with the moving axes,

then $\omega_x, \omega_y, \omega_z = \omega_1, \omega_2, \omega_3$, the foregoing argument holds good and the components of the moment of momentum are

$$\left. \begin{aligned} A\omega_1 - F\omega_2 - E\omega_3 \\ B\omega_2 - D\omega_3 - F\omega_1 \\ C\omega_3 - E\omega_1 - D\omega_2 \end{aligned} \right\} \dots\dots\dots(4),$$

where now A, B, C, D, E, F have constant values.

Though we have in this article analysed the momentum of a rigid body into a vector localized at the centre of gravity and a spin couple, and in the solution of many problems it is advantageous to adopt this method, yet if a body is turning about a fixed point it is simpler not to consider the separate motion of the centre of gravity, but take equations of moments about axes through the fixed point, when, as appears in the next article, the moments of momentum have the same form as above.

7·4. Motion of a rigid body about a fixed point. Euler's Equations. Taking the fixed point as origin and using axes fixed in the body and rotating with angular velocities $\omega_1, \omega_2, \omega_3$, the equations of motion 7·2 (8) become

$$\left. \begin{aligned} \dot{h}_1 - h_2\omega_3 + h_3\omega_2 &= L \\ \dot{h}_2 - h_3\omega_1 + h_1\omega_3 &= M \\ \dot{h}_3 - h_1\omega_2 + h_2\omega_1 &= N \end{aligned} \right\} \dots\dots\dots(1),$$

where h_1, h_2, h_3 are the components of the moment of momentum.

As in 7·3, if v_1, v_2, v_3 denote the components of velocity of the particle of mass m at (x, y, z) ,

$$v_1 = z\omega_2 - y\omega_3, \quad v_2 = x\omega_3 - z\omega_1, \quad v_3 = y\omega_1 - x\omega_2$$

and $h_1 = \sum m (yv_3 - zv_2)$

$$= \omega_1 \sum m (y^2 + z^2) - \omega_2 \sum m xy - \omega_3 \sum m xz,$$

or

$$h_1 = A\omega_1 - F\omega_2 - E\omega_3$$

and, similarly,

$$h_2 = B\omega_2 - D\omega_3 - F\omega_1 \dots\dots\dots(2),$$

and

$$h_3 = C\omega_3 - E\omega_1 - D\omega_2$$

where A, B, C, D, E, F are the moments and products of inertia of the body with regard to the axes fixed in it.

If the axes chosen are the principal axes of the body at the origin, $D = E = F = 0$, and

$$h_1, h_2, h_3 = A\omega_1, B\omega_2, C\omega_3,$$

so that the equations of motion take the form

$$\left. \begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= L \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= M \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= N \end{aligned} \right\} \dots\dots\dots (3).$$

These are known as **Euler's Equations for the motion of a body about a fixed point.**

7.41. Moments about the instantaneous axis. In Pt. I, 16.6 we obtained an equation of moments about the instantaneous centre of rotation, for use in two-dimensional problems. We shall now shew that the same equation holds good for moments about the instantaneous axis of rotation in three-dimensional problems in which a body is turning about a fixed point.

We make use of Euler's Equations, taking the fixed point as origin and the principal axes of the body at that point for axes of reference, so that A, B, C remain constant.

Let λ, μ, ν be the direction cosines of the instantaneous axis and ω the resultant angular velocity, so that

$$\omega_1, \omega_2, \omega_3 = \lambda\omega, \mu\omega, \nu\omega.$$

Then Euler's Equations 7.4 (3) take the form

$$A\lambda\dot{\omega} + A\dot{\lambda}\omega - (B - C)\mu\nu\omega^2 = L,$$

$$B\mu\dot{\omega} + B\dot{\mu}\omega - (C - A)\nu\lambda\omega^2 = M,$$

$$C\nu\dot{\omega} + C\dot{\nu}\omega - (A - B)\lambda\mu\omega^2 = N.$$

Multiplying these equations by λ, μ, ν and adding, we obtain the equation

$$(A\lambda^2 + B\mu^2 + C\nu^2)\dot{\omega} + (A\lambda\dot{\lambda} + B\mu\dot{\mu} + C\nu\dot{\nu})\omega = \lambda L + \mu M + \nu N;$$

or if I denotes the moment of inertia about the instantaneous axis and G the moment of external force about that axis

$$I\dot{\omega} + \frac{1}{2}\dot{I}\omega = G,$$

i.e.
$$\frac{1}{2\omega} \frac{d}{dt}(I\omega^2) = G \dots\dots\dots(1).$$

It follows, as in **Pt. I, 16·6**, that

- (i) if the body is turning about an axis fixed in the body,
 - (ii) in an initial motion,
 - (iii) in a small oscillation about a position of equilibrium,
- equation (1) reduces to the simpler form

$$I\dot{\omega} = G \dots\dots\dots(2).$$

7·42. Kinetic energy of a rigid body. In the notation of 7·3 the kinetic energy T of the body is given by

$$\begin{aligned} 2T &= \Sigma m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \Sigma m \{ (u - y'\omega_x + z'\omega_y)^2 + (v - z'\omega_x + x'\omega_z)^2 \\ &\quad + (w - x'\omega_y + y'\omega_x)^2 \}. \end{aligned}$$

Since $\Sigma mx' = \Sigma my' = \Sigma mz' = 0$, therefore the foregoing expression reduces to

$$\begin{aligned} 2T &= M(u^2 + v^2 + w^2) + \omega_x^2 \Sigma m(y'^2 + z'^2) + \omega_y^2 \Sigma m(z'^2 + x'^2) \\ &\quad + \omega_z^2 \Sigma m(x'^2 + y'^2) \\ &\quad - 2\omega_y \omega_z \Sigma m y' z' - 2\omega_z \omega_x \Sigma m z' x' - 2\omega_x \omega_y \Sigma m x' y', \end{aligned}$$

or
$$2T = M(u^2 + v^2 + w^2) + A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_y \omega_z - 2E\omega_z \omega_x - 2F\omega_x \omega_y \dots(1),$$

where u, v, w are the components of velocity of the centre of gravity and A, B, C, D, E, F are moments and products of inertia about the axes of reference.

If $\omega_1, \omega_2, \omega_3$ are the components of angular velocity of the body about axes through its centre of gravity fixed in itself and we take fixed axes momentarily coinciding in direction with the moving axes, so that $\omega_x, \omega_y, \omega_z = \omega_1, \omega_2, \omega_3$, then

$$\begin{aligned} 2T &= M(u^2 + v^2 + w^2) + A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \\ &\quad - 2D\omega_2 \omega_3 - 2E\omega_3 \omega_1 - 2F\omega_1 \omega_2 \dots(2), \end{aligned}$$

where $A, B, \dots$ are now constants.

Further, if the body be turning about a fixed point, and we take this point as origin, and axes fixed in the body rotating with angular velocities $\omega_1, \omega_2, \omega_3$, then, as in 7.4, the components of velocity of a particle m at (x, y, z) are

$$z\omega_2 - y\omega_3, \quad x\omega_3 - z\omega_1 \quad \text{and} \quad y\omega_1 - x\omega_2;$$

so that the kinetic energy T is given by

$$\begin{aligned} 2T &= \sum m \{ (z\omega_2 - y\omega_3)^2 + (x\omega_3 - z\omega_1)^2 + (y\omega_1 - x\omega_2)^2 \} \\ &= A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2D\omega_2\omega_3 - 2E\omega_3\omega_1 - 2F\omega_1\omega_2 \\ &\quad \dots\dots(3). \end{aligned}$$

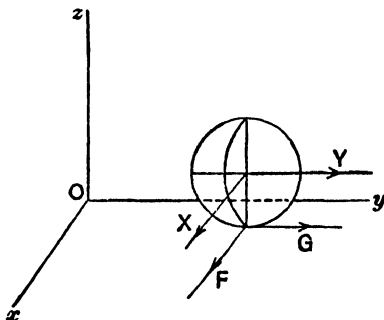
The expressions (2) and (3) are clearly simplified by choosing for axes the principal axes of the body at the specified origin.

7.5. Examples of the use of fixed axes.

(i) *Motion of a sphere on a rough plane, under the action of forces the resultant of which passes through the centre of the sphere.* Take axes Ox, Oy fixed in the plane and Oz at right angles to it. If x, y, a are the co-ordinates of the centre of the sphere and m is its mass, the linear equations of motion are

$$m\ddot{x} = X + F, \quad m\ddot{y} = Y + G, \quad 0 = Z + R \dots\dots\dots(1),$$

where X, Y, Z are the components of the external forces and F, G are the components of the friction, and R is the normal reaction.



If $\omega_x, \omega_y, \omega_z$ denote the components of the angular velocity, the equations of moments about axes through the centre parallel to the co-ordinate axes are

$$\frac{2}{5}ma^2\dot{\omega}_x = Ga, \quad \frac{2}{5}ma^2\dot{\omega}_y = -Fa, \quad \frac{2}{5}ma^2\dot{\omega}_z = 0 \dots\dots(2).$$

The last equation gives $\omega_z = \text{const.}$, so that the spin of the sphere about an axis normal to the plane remains unaltered.

We have now to express the conditions for rolling, assuming that the friction prevents sliding. They are that the point of contact of the sphere with the plane is at rest, i.e. that

$$\dot{x} - a\omega_y = 0, \quad \dot{y} + a\omega_x = 0 \dots\dots\dots(3).$$

From (1), (2) and (3) we find that

$$m\ddot{x} = \frac{5}{2}X \quad \text{and} \quad m\ddot{y} = \frac{5}{2}Y \dots\dots\dots(4),$$

determining the motion of the centre of the sphere.

We might have avoided the introduction of F and G by taking moments about lines parallel to the axes through the point of contact, thus

$$-ma\ddot{y} + \frac{2}{5}ma^2\dot{\omega}_x = -Ya, \quad \text{and} \quad ma\ddot{x} + \frac{2}{5}ma^2\dot{\omega}_y = Xa \dots\dots(5),$$

which are deducible from (1) and (2).

(ii) *If the plane be made to revolve about Oz with uniform angular velocity Ω* , provided we retain the same fixed axes, the equations of motion of the sphere are unaltered, but instead of (3) we have equations expressing the condition that the points of contact of the sphere and the plane have the same velocity, i.e. that

$$\dot{x} - a\omega_y = -\Omega y, \quad \dot{y} + a\omega_x = \Omega x \dots\dots\dots(6).$$

In the case in which there are no forces save the reactions of the plane, we get from (5) and (6)

$$\ddot{x} + \frac{2}{5}\Omega\dot{y} = 0, \quad \ddot{y} - \frac{2}{5}\Omega\dot{x} = 0,$$

which are reducible to

$$\ddot{x} + n^2(x-d) = 0, \quad \ddot{y} + n^2(y-e) = 0,$$

where $n = \frac{2}{5}\Omega$ and d, e are constants.

On integrating and eliminating t , we find that the path of the centre of the sphere referred to fixed axes is a circle, of which the point (d, e) is centre.

The result might also be obtained by referring the motion to axes fixed in the plane and revolving with it, but the reader will find that in this case the use of fixed axes greatly simplifies the working.

7·6. Examples of the use of one fixed and two moving axes.

(i) *Motion of a rigid body about a fixed axis.* Take the fixed axis as axis of z and a perpendicular axis through the centre of gravity G as the axis Ox . Let $OG = h$ and let ω be the angular velocity about Oz and m the mass of the body; then the components of linear momentum are

$$p_1, p_2, p_3 = 0, m\omega h, 0 \dots\dots\dots(1)$$

and the expressions for the moments of momentum are, by 7·4 (2),

$$h_1, h_2, h_3 = -E\omega, -D\omega, C\omega \dots\dots\dots(2);$$

also the angular velocities of the axes are

$$\theta_1, \theta_2, \theta_3 = 0, 0, \omega.$$

Hence the rates of change of linear momentum are

$$\dot{p}_1 - p_2 \theta_3 + p_3 \theta_2 = -m\omega^2 h,$$

$$\dot{p}_2 - p_3 \theta_1 + p_1 \theta_3 = m\dot{\omega} h,$$

$$\dot{p}_3 - p_1 \theta_2 + p_2 \theta_1 = 0,$$

and of moment of momentum, in like manner,

$$-E\dot{\omega} + D\omega^2, -D\dot{\omega} - E\omega^2, C\dot{\omega}.$$

Suppose that the body is acted upon by external forces whose components and moments are X, Y, Z, L, M, N , and that it is connected with the fixed axis at two points at distances c, c' from O , where R_1, R_2, R_3 and R_1', R_2', R_3' are the components of the reactions of the axis on the body, then the equations of motion are

$$-m\omega^2 h = X + R_1 + R_1',$$

$$m\dot{\omega} h = Y + R_2 + R_2',$$

$$0 = Z + R_3 + R_3',$$

$$-E\dot{\omega} + D\omega^2 = L - cR_2 - c'R_2',$$

$$-D\dot{\omega} - E\omega^2 = M + cR_1 + c'R_1',$$

$$C\dot{\omega} = N.$$

The last equation determines ω , and there are then four equations to determine R_1, R_2, R_1', R_2' .

(ii) *Motion of a heavy sphere on the interior rough surface of a vertical circular cylinder.* Take axes through the centre C of the sphere, 1 vertically upwards, 3 to intersect the axis of the cylinder and 2 horizontal forming a right-handed screw system. Let a, b be the radii of the sphere and cylinder, and θ the azimuth of the centre, i.e. the angle which the plane through the axis of the cylinder and the centre of the sphere makes with a fixed vertical plane. Then the angular velocities of the axes are

$$\theta_1, \theta_2, \theta_3 = \dot{\theta}, 0, 0 \dots\dots\dots(1).$$

If x be measured vertically upwards along the axis of the cylinder, the components of velocity of C are

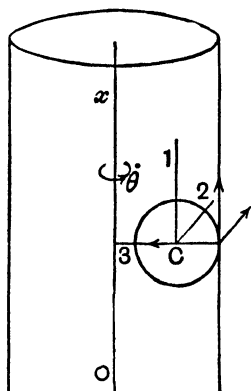
$$u, v, w = \dot{x}, (b-a)\dot{\theta}, 0 \dots\dots\dots(2);$$

and, if $\omega_1, \omega_2, \omega_3$ are the components of angular velocity of the sphere, the equations of no sliding of the point of contact are

$$u - a\omega_2 = 0, \quad v + a\omega_1 = 0,$$

or

$$a\omega_2 = \dot{x}, \quad a\omega_1 = -(b-a)\dot{\theta} \dots\dots\dots(3).$$



In order to avoid introducing the forces of friction we may take moments about axes parallel to 1, 2, 3 through the point of contact. Thus we have

$$-ma(\dot{v} - w\theta_1 + u\theta_3) + A(\dot{\omega}_1 - \omega_2\theta_3 + \omega_3\theta_2) = 0 \quad \dots\dots\dots(4),$$

$$ma(\dot{u} - v\theta_3 + w\theta_2) + A(\dot{\omega}_2 - \omega_3\theta_1 + \omega_1\theta_3) = -mga \quad \dots\dots\dots(5),$$

$$A(\dot{\omega}_3 - \omega_1\theta_2 + \omega_2\theta_1) = 0 \quad \dots\dots\dots(6),$$

where $A = \frac{2}{5}ma^2$.

On substituting from (1), (2) and (3) in (4), (5) and (6), the latter reduce to

$$\theta = 0 \quad \dots\dots\dots(7),$$

$$\frac{2}{5}\ddot{x} - \frac{2}{5}a\omega_3\dot{\theta} = -g \quad \dots\dots\dots(8)$$

and

$$a\omega_3 + \dot{x}\dot{\theta} = 0 \quad \dots\dots\dots(9).$$

From (7) we have $\dot{\theta} = \text{const.} = n$ say, then from (9)

$$a\omega_3 + nx = a\mu \quad \dots\dots\dots(10),$$

where μ is a constant.

$$\text{Then (8) reduces to } \ddot{x} + \frac{2}{5}n^2x = \frac{2}{5}a\mu n - \frac{5}{2}g \quad \dots\dots\dots(11),$$

so that the sphere rises and falls in the cylinder with a vertical simple harmonic motion of period $\sqrt{14\pi/n}$.

If initially $x = 0$, $\dot{x} = 0$, $\dot{\theta} = n$ and $\omega_3 = \mu$, which means that the sphere has a spin μ about its radius drawn to the axis of the cylinder and its centre has a horizontal velocity $(b-a)n$, then the solution of (11) is

$$x = \left(\frac{a\mu}{n} - \frac{5}{2} \frac{g}{n^2} \right) \left(1 - \cos \sqrt{\frac{2}{7}} nt \right),$$

so that the sphere rises above or descends below its initial level according as $a\mu n >$ or $< \frac{5}{2}g$.

7·61. Examples. 1. A homogeneous circular cylinder is divided by an axial plane and kept in shape by a band round it. If the cylinder is placed on a smooth horizontal plane with the plane of separation vertical and the band is then cut, prove that the pressure on the plane is instantaneously reduced by the fraction $32/27\pi^2$ of itself. [I. 1914]

2. A uniform sphere is projected horizontally on an inclined plane the surface of which is perfectly rough; shew that its centre describes a parabola.

3. A uniform solid sphere rolls without slipping on an inclined plane which rotates about an axis normal to itself with constant angular velocity ω . Shew that, if the centre of the sphere is initially at rest relative to fixed axes along and at right angles to a line of greatest slope, the co-ordinates of the centre at time t can be written in the form

$$x = x_0 + \frac{35g}{4\omega^2} \sin \alpha (1 - \cos \frac{2}{5}\omega t),$$

$$y = y_0 + \frac{35g}{4\omega^2} \sin \alpha (\frac{2}{5}\omega t - \sin \frac{2}{5}\omega t),$$

where α is the inclination of the plane to the horizontal.

[P. 1934]

4. A uniform solid sphere rolls on the inside of a fixed rough circular cylinder whose axis is vertical. Prove that the curve traced out on the cylinder by the point of contact is

$$x = a \cos \sqrt{\frac{7}{4}} \theta,$$

where a depends on the circumstances of projection.

[I. 1923]

5. Shew that in the preceding example, if the cylinder is made to rotate about its axis with uniform angular velocity ω , the period is

$$\sqrt{14\pi}/(n + \omega);$$

also prove that if the cylinder is free to rotate about its axis its angular velocity is constant.

6. A uniform solid sphere rolls on the inside of a rough circular cylinder whose axis is vertical. Shew that, if the cylinder be made to rotate with constant angular velocity ω about a generating line in the plane through the axis from which the azimuth θ is measured, the azimuthal motion of the centre of the sphere relative to the cylinder is given by

$$7(b - a)\dot{\theta} = 5b\omega^2 \sin \theta,$$

where a , b are the radii of the sphere and the cylinder.

7. A uniform solid sphere of mass m rolls inside a rough circular cylinder and is constrained by a force P parallel to the axis to move so that its centre describes a helix. Prove that, if n is the angular velocity round the axis of the cylinder and x the distance described parallel to the axis, then $5P = 2mn^2x$.

8. A uniform solid sphere of radius a rolls without sliding inside a fixed circular cylinder of radius b whose axis is inclined at an angle α to the horizon. Obtain the equations of motion

$$(b - a)\dot{\theta}^2 = \frac{1}{7}g \cos \alpha \cos \theta + \text{const.},$$

$$\dot{x}^2 + \frac{7}{4}a^2\omega_s^2 = \frac{1}{2}g x \sin \alpha + \text{const.},$$

$$a\theta \left(\frac{d^2\omega_s}{d\theta^2} + \frac{7}{4}\omega_s \right) = \frac{5}{4}g \sin \alpha,$$

where x is the distance the sphere has moved parallel to the axis of the cylinder, θ is the angle between the axial plane through the centre and the vertical axial plane and ω_s is the angular velocity of the sphere about the radius to the point of contact. [M. T. 1933]

9. A rough material plane turns with constant angular velocity ω about a horizontal axis fixed in itself. At the instant when the plane is horizontal a uniform solid sphere of radius a having an angular velocity Ω about a vertical diameter is gently placed on the plane, touching it at a point on the axis of rotation. Shew that so long as the sphere

remains in contact with the plane the displacement of the point of contact after time t is given by

$$x = \sqrt{\frac{7}{5}} \left(\frac{5}{12} \frac{g}{\omega^2} - a \right) \sinh \sqrt{\frac{5}{7}} \omega t - \frac{5}{12} \frac{g}{\omega^2} \sin \omega t,$$

$$y = \frac{\Omega a}{\omega} \left(1 - \cos \sqrt{\frac{2}{7}} \omega t \right),$$

where x is measured perpendicular and y parallel to the axis of rotation.

[M. T. 1925]

7·7. Three moving axes. A heavy sphere moving on the inner rough surface of a right circular cone with its axis vertical. The centre C of the sphere moves on an equal cone of angle 2α . Let r be the distance of C from the vertex of this cone and ψ the azimuthal angle. Take moving axes 1, 2, 3 at the centre of the sphere as in the figure.

Then $\theta_1, \theta_2, \theta_3 = \psi \cos \alpha, 0, \psi \sin \alpha$ } (1).
and $u, v, w = \dot{r}, r\dot{\psi} \sin \alpha, 0$

The conditions for no sliding are

$$u - a\omega_2 = 0, \quad v + a\omega_1 = 0$$

or $a\omega_2 = \dot{r}, \quad a\omega_1 = -r\dot{\psi} \sin \alpha$ (2).

The equations of motion obtained by taking moments about lines parallel to 1, 2, 3 through the point of contact are

$$-ma(\dot{v} - w\theta_1 + u\theta_3) + A(\dot{\omega}_1 - \omega_2\theta_3 + \omega_3\theta_2) = 0 \quad \text{..... (3),}$$

$$ma(\dot{u} - v\theta_3 + w\theta_2) + A(\dot{\omega}_2 - \omega_3\theta_1 + \omega_1\theta_3) = -mga \cos \alpha \quad \text{..... (4),}$$

$$A(\dot{\omega}_3 - \omega_1\theta_2 + \omega_2\theta_1) = 0 \quad \text{..... (5),}$$

where $A = \frac{2}{5}ma^2$.

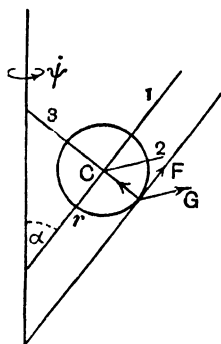
On substituting from (1) and (2) in (3), (4) and (5), the latter reduce to

$$(r\ddot{\psi} + 2\dot{r}\dot{\psi}) \sin \alpha = 0,$$

or $r^2\dot{\psi} \sin \alpha = \text{const.} = h$, say (6),

$$\frac{7}{5}(\ddot{r} - r\dot{\psi}^2 \sin^2 \alpha) - \frac{2}{5}a\omega_3\dot{\psi} \cos \alpha = -g \cos \alpha \quad \text{..... (7)}$$

and $a\dot{\omega}_3 + \dot{r}\dot{\psi} \cos \alpha = 0$ (8).



Hence
$$a\dot{\omega}_3 + h\dot{r} \cot \alpha / r^2 = 0,$$

or
$$a\omega_3 = h \cot \alpha / r + \text{const.} \dots\dots\dots (9).$$

The equations are therefore reduced to

$$r^2 \ddot{\psi} \sin \alpha = \hbar$$

and
$$\ddot{r} - r\dot{\psi}^2 \sin^2 \alpha = \text{a function of } r,$$

which are the same in form as the equations for the motion of a heavy particle on the surface of a smooth cone (4·26), and it follows as in 4·26 that if the cone on which the centre of the sphere moves is developed into a plane by putting $d\psi \sin \alpha = d\theta$, the trace of the path of the centre will be that of a particle moving under a central force.

The frictional forces F , G along and at right angles to a generator can be found by resolving along the axes 1 and 2, thus

$$m(\dot{u} - v\theta_3) = F - mg \cos \alpha$$

and
$$m(\dot{v} + u\theta_3) = G.$$

If the cone be made to rotate about its axis with uniform angular velocity ω , the equations of no sliding become

$$u - a\omega_2 = 0,$$

$$v + a\omega_1 = \omega(r \sin \alpha + a \cos \alpha)$$

and the other equations are as before.

If the cone be free to rotate about its axis, ω is not a constant, but we have an additional equation

$$I\dot{\omega} = -G(r \sin \alpha + a \cos \alpha),$$

where I is the moment of inertia of the cone about its axis.

7·71. Examples. 1. A perfectly rough right circular cone of semi-vertical angle α has its axis vertical and vertex upwards and is made to rotate about its axis with uniform angular velocity Ω . A uniform solid sphere of radius a is gently placed on a point of the cone at a distance $a \cot \alpha$ from the vertex. Shew that the axial plane through the centre of the sphere rotates with angular velocity $\frac{1}{2}\Omega$ and that the distance of the point of contact of the sphere from the vertex of the cone at time t is given by the equation

$$r = a \cot \alpha + \frac{5}{2}g \cos \alpha (1 - \cos \mu t) / \mu^2,$$

where

$$343\mu^2 = \Omega^2 (2 + 5 \sin^2 \alpha). \quad [\text{M. T. 1900}]$$

2. A sphere of radius a rolls inside a circular cone of angle 2α with its axis vertical and vertex downwards. Shew that the equations of motion have first integrals

$$\varpi^2 \phi = h,$$

$$a\omega_s - \frac{h}{\varpi \tan \alpha} = an,$$

$$\frac{7}{2}\dot{\varpi}^2 + \left(2g\varpi + \frac{4}{5}\frac{nah}{\varpi}\right) \cos \alpha \sin \alpha + \left(\frac{2}{5} + \sin^2 \alpha\right) \frac{h^2}{\varpi^2} = E,$$

where ϖ is the distance of the centre of the sphere from the axis, ω_s the spin of the sphere about the common normal, ϕ the azimuth of the centre of the sphere, and h, n and E are constants. [M. T. 1928]

3. A uniform circular disc, of radius a , rolls without slipping on the inside of a right circular cone of semi-angle α which is fixed with its axis vertical and vertex downwards. The motion is steady; the centre of the disc describes a horizontal circle of radius c with angular velocity ω , and the plane of the disc is inclined always at an angle β to the vertical. Shew that

$$a \cos \beta + 6c \cot \beta = 4g/\omega^2.$$

Shew also that to prevent slipping the angle of friction must not be less than $|\alpha - \gamma|$, where γ is the acute angle $\tan^{-1} g/c\omega^2$. [M. T. 1930]

4. A plane inclined at an angle α to the horizontal rotates with uniform angular velocity ω about a vertical axis. A rough heavy uniform sphere rolls on the plane. Shew that the equations determining the motion of the point of contact (x, y) in the plane are

$$7\ddot{x} - 12\omega\dot{y} \cos \alpha - 5\omega^2 x = 0,$$

$$7\ddot{y} + 12\omega\dot{x} \cos \alpha - (7 \cos^2 \alpha - 2) \omega^2 y = 0,$$

where the axis of y is taken up the line of greatest slope through the point where the axis of rotation meets the plane, and the origin is suitably chosen in relation to the initial conditions. [M. T. 1930]

5. A sphere of radius a rolls inside a cone of semi-vertical angle α , which is made to rotate about a vertical axis with angular velocity Ω . If ω_s is the spin of the sphere about the radius to the point of contact, shew that

$$a\omega_s + \frac{\cos \alpha}{7} \Omega r = \frac{A}{r} + B,$$

where $r + a \cot \alpha$ is the distance of the point of contact from the vertex and A, B are constants. [I. 1928]

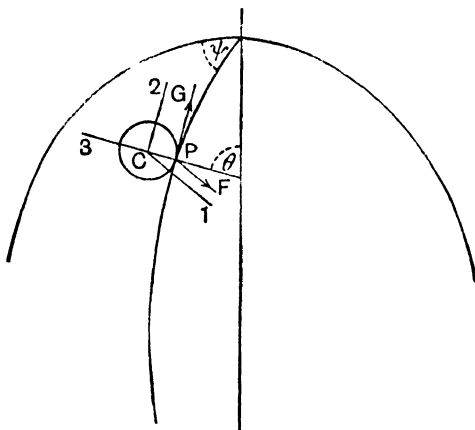
7·8. Motion of a sphere on a rough surface of revolution.

Let m be the mass and a the radius of the sphere, θ the angle the common normal makes with the axis and ψ the azimuthal angle. Take axes 1, 2, 3 at the centre of the sphere, where 3 is along the common normal outwards, 2 is parallel to the tangent to the meridian curve at the point of contact P , and 1 is parallel to the tangent to the circular section through P .

Then we have $\theta_1, \theta_2, \theta_3 = \theta, \psi \sin \theta, \psi \cos \theta \dots\dots\dots(1).$

Also, with the usual notation, we have, as the conditions for no sliding at P ,

$$u - a\omega_2 = 0, \quad v + a\omega_1 = 0; \quad \text{also } w = 0 \dots\dots\dots(2).$$



Then the accelerations of the centre C are

$$\left. \begin{aligned} \dot{u} - v\theta_3 + w\theta_2 &= a\dot{\omega}_2 + a\omega_1\psi \cos \theta \\ \dot{v} - w\theta_1 + u\theta_3 &= -a\dot{\omega}_1 + a\omega_2\psi \cos \theta \\ \dot{w} - u\theta_2 + v\theta_1 &= -a\dot{\omega}_2\psi \sin \theta - a\omega_1\dot{\theta} \end{aligned} \right\} \dots\dots\dots(3).$$

Let L, M, N denote the moments of the external forces about lines through P parallel to the axes 1, 2, 3; then the equations of moments about these lines are

$$mk^2(\dot{\omega}_1 - \omega_2\psi \cos \theta + \omega_3\psi \sin \theta) - ma(-a\dot{\omega}_1 + a\omega_2\psi \cos \theta) = L \dots\dots\dots(4),$$

$$mk^2(\dot{\omega}_2 - \omega_3\dot{\theta} + \omega_1\psi \cos \theta) + ma(a\dot{\omega}_2 + a\omega_1\psi \cos \theta) = M \dots\dots\dots(5),$$

$$mk^2(\dot{\omega}_3 - \omega_1\psi \sin \theta + \omega_2\dot{\theta}) = N \dots\dots\dots(6),$$

where k is the radius of gyration of the sphere about a diameter.

When the axis is vertical and gravity is the only external force, we have $L = mga \sin \theta, M = 0, N = 0$. For a steady motion to be possible in which the centre of the sphere

describes a horizontal circle of radius r , we must have $\theta = \alpha$, a constant, and $\dot{\psi} = \Omega$, a constant, so that

$$a\omega_1 = -v = 0, \quad a\omega_2 = u = r\Omega.$$

Then (6) gives $\dot{\omega}_3 = 0$, so that $\omega_3 = n$, a constant, (5) is satisfied identically and (4) gives

$$(k^2 + a^2)r\Omega^2 \cot \alpha - k^2 an\Omega + ga^2 = 0;$$

or taking $k^2 = \frac{2}{5}a^2$ for a solid sphere

$$7r\Omega^2 \cot \alpha - 2an\Omega + 5g = 0;$$

and this only gives a real value for Ω if

$$a^2 n^2 \geq 35rg \cot \alpha.$$

7·81. Motion of a sphere on a sphere.

(i) *A sphere rolling and spinning on a fixed sphere.*

If in 7·8 the surface of revolution be a sphere of radius b , then

$$u = (a+b)\dot{\psi} \sin \theta \quad \text{and} \quad v = -(a+b)\dot{\theta},$$

so that $a\omega_2 = (a+b)\dot{\psi} \sin \theta \quad \text{and} \quad a\omega_1 = (a+b)\dot{\theta} \dots\dots\dots(7).$

Taking gravity to be the only external force, equations (4), (5), (6) may be written

$$(k^2 + a^2)(\dot{\omega}_1 - \omega_2 \dot{\psi} \cos \theta) + k^2 \omega_3 \dot{\psi} \sin \theta = ga \sin \theta \dots\dots(8),$$

$$(k^2 + a^2)(\dot{\omega}_2 + \omega_1 \dot{\psi} \cos \theta) - k^2 \omega_3 \dot{\theta} = 0 \dots\dots\dots(9)$$

and $\dot{\omega}_3 = 0 \dots\dots\dots(10).$

Hence $\omega_3 = \text{const.}$, or the spin of the sphere about the normal to the surface has a constant value n . Then substituting from (7) in (8) and (9), we get

$$(a+b)(\ddot{\theta} - \dot{\psi}^2 \sin \theta \cos \theta) + \frac{k^2}{a^2 + k^2} an \dot{\psi} \sin \theta = \frac{ga^2}{a^2 + k^2} \sin \theta \dots\dots(11)$$

and $(a+b) \frac{1}{\sin \theta} \frac{d}{dt} (\dot{\psi} \sin^2 \theta) - \frac{k^2}{a^2 + k^2} an \dot{\theta} = 0 \dots\dots\dots(12).$

The integral of (12) is

$$(a+b) \dot{\psi} \sin^2 \theta + \frac{k^2}{a^2 + k^2} an \cos \theta = H \dots\dots\dots(13).$$

If we multiply (11) by $2\dot{\theta}$ and (12) by $2\dot{\psi} \sin \theta$ and add and integrate, we get

$$(a+b)(\dot{\theta}^2 + \dot{\psi}^2 \sin^2 \theta) = \frac{2ga^2}{a^2 + k^2} (1 - \cos \theta) + \text{const.} \dots\dots(14).$$

But the latter equation might have been obtained directly as the equation of energy, beginning in the form

$$\frac{1}{2}m(u^2 + v^2) + \frac{1}{2}mk^2(\omega_1^2 + \omega_2^2 + \omega_3^2) = mg(a+b)(1 - \cos \theta) + \text{const.}$$

Equations (13) and (14) determine the motion.

The reactions at the point of contact can now be found by resolving parallel to the axes in turn.

(ii) *A sphere rolling and spinning on a sphere which is free to rotate about its centre.*

Let I denote the moment of inertia of the latter sphere about a diameter; $\Omega_1, \Omega_2, \Omega_3$ its angular velocities about axes through its centre parallel to the axes 1, 2, 3 already defined; F, G the frictional forces taken as acting positively in the senses 1, 2 on the sphere of radius a , and in the contrary sense on the sphere of radius b ; and R the normal reaction.

With the same notation as before the geometrical equations are

$$u - a\omega_2 = b\Omega_2, \quad v + a\omega_1 = -b\Omega_1, \quad w = 0 \quad \dots\dots\dots(15);$$

and we have as before

$$u = (a+b)\psi \sin \theta, \quad v = -(a+b)\dot{\theta} \quad \dots\dots\dots(16).$$

The dynamical equations are

$$\left. \begin{aligned} m(\ddot{u} - v\dot{\psi} \cos \theta) &= F \\ m(\ddot{v} + u\dot{\psi} \cos \theta) &= G - mg \sin \theta \\ m(-u\dot{\psi} \sin \theta + v\dot{\theta}) &= R - mg \cos \theta \end{aligned} \right\} \quad \dots\dots\dots(17),$$

$$\left. \begin{aligned} mk^2(\dot{\omega}_1 - \omega_2\dot{\psi} \cos \theta + \omega_3\dot{\psi} \sin \theta) &= Ga \\ mk^2(\dot{\omega}_2 - \omega_3\dot{\theta} + \omega_1\dot{\psi} \cos \theta) &= -Fa \\ mk^2(\dot{\omega}_3 - \omega_1\dot{\psi} \sin \theta + \omega_2\dot{\theta}) &= 0 \end{aligned} \right\} \quad \dots\dots\dots(18)$$

and

$$\left. \begin{aligned} I(\dot{\Omega}_1 - \Omega_2\dot{\psi} \cos \theta + \Omega_3\dot{\psi} \sin \theta) &= Gb \\ I(\dot{\Omega}_2 - \Omega_3\dot{\theta} + \Omega_1\dot{\psi} \cos \theta) &= -Gb \\ I(\dot{\Omega}_3 - \Omega_1\dot{\psi} \sin \theta + \Omega_2\dot{\theta}) &= 0 \end{aligned} \right\} \quad \dots\dots\dots(19).$$

From the last equations of (18) and (19) we get

$$a\dot{\omega}_3 + b\dot{\Omega}_3 - (a\omega_1 + b\Omega_1)\dot{\psi} \sin \theta + (a\omega_2 + b\Omega_2)\dot{\theta} = 0,$$

but from (15) and (16)

$$a\omega_1 + b\Omega_1 = (a+b)\dot{\theta}, \quad \text{and} \quad a\omega_2 + b\Omega_2 = (a+b)\dot{\psi} \sin \theta;$$

therefore

$$a\dot{\omega}_3 + b\dot{\Omega}_3 = 0,$$

or

$$a\omega_3 + b\Omega_3 = \text{const.} = cn \text{ say.}$$

We proceed to eliminate G from the dynamical equations. From (18) and (19) we have

$$a\dot{\omega}_1 + b\dot{\Omega}_1 - (a\omega_2 + b\Omega_2)\dot{\psi} \cos \theta + cn\dot{\psi} \sin \theta = G \left(\frac{a^2}{mk^2} + \frac{b^2}{I} \right);$$

and from (15) this may be written

$$-\dot{v} - u\dot{\psi} \cos \theta + cn\dot{\psi} \sin \theta = G \left(\frac{a^2}{mk^2} + \frac{b^2}{I} \right),$$

then substituting for G from (17) and for u, v from (16) we get

$$\left(1 + \frac{a^2}{k^2} + \frac{mb^2}{I} \right) (a+b) (\dot{\theta} - \dot{\psi}^2 \sin \theta \cos \theta) + cn\dot{\psi} \sin \theta = \left(\frac{a^2}{k^2} + \frac{mb^2}{I} \right) g \sin \theta \quad \dots\dots\dots(20).$$

In like manner the elimination of F leads to

$$\left(1 + \frac{a^2}{k^2} + \frac{mb^2}{I}\right) \frac{a+b}{\sin \theta} \frac{d}{dt} (\psi \sin^2 \theta) - cn\theta = 0 \quad \dots\dots\dots(21).$$

These equations are identical in form with (11) and (12) for a sphere rolling and spinning on a fixed sphere and the solution may be completed in the same way.

(iii) *A sphere rolling and spinning on a sphere which is made to rotate with uniform angular velocity about its vertical diameter.*

Let Ω denote the uniform angular velocity of the sphere of radius b . In this case the geometrical equations are

$$u - a\omega_2 = b\Omega \sin \theta, \quad v + a\omega_1 = 0, \quad w = 0 \quad \dots\dots\dots(22).$$

The effect is to add an extra term to each of the accelerations of C given in (3), so that they are, in this case,

$$\begin{aligned} & a\dot{\omega}_2 + b\Omega\dot{\theta} \cos \theta + a\omega_1\dot{\psi} \cos \theta, \\ & -a\dot{\omega}_1 + a\omega_2\dot{\psi} \cos \theta + b\Omega\dot{\psi} \sin \theta \cos \theta, \\ & -a\omega_2\dot{\psi} \sin \theta - b\Omega\dot{\psi} \sin^2 \theta - a\omega_1\dot{\theta}. \end{aligned}$$

The dynamical equations in this case reduce to

$$(a+b)(\ddot{\theta} - \dot{\psi}^2 \sin \theta \cos \theta) + \frac{k^2}{a^2 + k^2} (\Omega b \cos \theta + a\omega_3) \dot{\psi} \sin \theta = \frac{ga^2}{a^2 + k^2} \sin \theta \quad \dots\dots\dots(23),$$

$$(a+b) \frac{1}{\sin \theta} \frac{d}{dt} (\dot{\psi} \sin^2 \theta) - \frac{k^2}{a^2 + k^2} (\Omega b \cos \theta + a\omega_3) \dot{\theta} = 0 \quad \dots\dots\dots(24)$$

$$\text{and} \quad a\dot{\omega}_3 - \Omega b\dot{\theta} \sin \theta = 0 \quad \dots\dots\dots(25).$$

The last equation gives

$$a\omega_3 + \Omega b \cos \theta = an \quad \dots\dots\dots(26)$$

and the substitution of this in (23) and (24) brings them back to the equations (11) and (12), i.e. the same as if the revolving sphere were fixed. Hence the general motion is the same as in case (i).

(iv) *A sphere rolling and spinning on a sphere free to rotate about its vertical diameter.*

We give it as an exercise for the reader to shew, with the same notation, that

$$I\Omega + mb(a+b)\dot{\psi} \sin^2 \theta = \text{const.},$$

$$\dot{\theta}^2 + \dot{\psi}^2 \sin^2 \theta + \frac{mk^2b^2}{I(a^2+k^2)} \dot{\psi}^2 \sin^4 \theta = \frac{2a^2}{a^2+k^2} \frac{g}{a+b} (1 - \cos \theta) + \text{const.},$$

$$(a+b)\dot{\psi} \sin^2 \theta + \frac{k^2}{a^2+k^2} (a\omega_3 \cos \theta - b\Omega \sin^2 \theta) = \text{const.},$$

$$\Omega^2 \left(I + \frac{k^2b^2}{a^2+k^2} \sin^2 \theta \right) + \frac{k^2a^2}{a^2+k^2} \omega_3^2 = \text{const.}$$

7·82. Examples. 1. A homogeneous heavy sphere of radius a rolls on the outside of a rough fixed sphere of radius b . Shew that, if the spheres are imperfectly rough, steady motion is possible, in which the

point of contact describes a small circle of angular radius α on the fixed sphere in time $2\pi/\Omega$, provided that

$$(a+b)\Omega^2/g < \operatorname{cosec} \alpha \tan(\epsilon - \alpha),$$

where ϵ is the angle of friction.

[I. 1903]

2. A homogeneous sphere rolls in any manner under the action of gravity on a perfectly rough fixed sphere whose centre is O . Shew that

(1) the velocity of the centre of the moving sphere is that due to five-sevenths of its depth below a fixed horizontal plane, and

(2) the moving sphere will leave the fixed sphere when the altitude of its centre above O is ten-seventeenths of the altitude of the fixed plane above the same point.

[I. 1926]

3. Find the equations for the motion of an isotropic sphere of radius b mass M and moment of inertia I , rolling without slip on the surface of a fixed sphere of radius $c-b$, in terms of the inclination θ to the vertical of the common radius, and the azimuthal angle ψ of the vertical plane containing this radius.

Shew that, if In is the constant moment of spin of the upper sphere about the common radius and if

$$cL = \frac{1}{2}(I + Mb^2)c(\dot{\theta}^2 + \dot{\psi}^2 \sin^2 \theta) + Inb\dot{\psi} \cos \theta - Mgb^2 \cos \theta,$$

the equations of motion are of the form

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\psi}} \right) - \frac{\partial L}{\partial \psi} = 0. \quad [\text{M. T. 1918}]$$

4. A rough sphere of radius a and radius of gyration K capable of moving about its centre is initially at rest; another sphere of $1/n$ th the mass and of radius b and radius of gyration k is placed gently on it having initially an angular velocity ω about the common normal which makes an angle α with the vertical drawn upwards. Prove that the second sphere will not roll off provided

$$\omega^2 > \frac{2\mu(a+b)g}{(3\mu+1)b^2} \{ (3\mu+1)^2 - 4\mu^2 \cos^2 \alpha \} \sec \alpha,$$

where

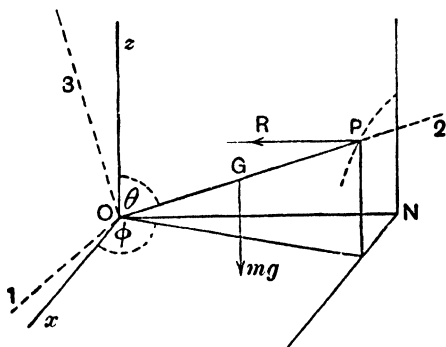
$$\mu = a^2/nK^2 + b^2/k^2. \quad [\text{M. T. 1888}]$$

5. A perfectly rough sphere of radius a is made to rotate about a vertical diameter which is fixed with constant angular velocity n . A uniform sphere of radius b is placed on it at a point distant $a\alpha$ from the highest point; investigate the motion and determine in any position the angular velocity of the sphere. Shew that the sphere will leave the rotating sphere when the point of contact is at an angular distance θ from the vertex, where

$$\cos \theta = \frac{1}{2} \cos \alpha + \frac{4a^2 n^2 \sin^2 \alpha}{119(a+b)g}. \quad [\text{M. T. 1889}]$$

7.9. Motion of a uniform rod one end of which is fixed while the other end slides on a smooth vertical plane. Let O be the fixed end of the rod OP , a its length and m its mass; c the

distance of O from the vertical plane. The end P describes a circular arc on the vertical plane whose centre N is the foot of the perpendicular from O . The direction of OP is defined by the angle θ which it makes with the vertical Oz , and the angle ϕ which the plane zOP makes with a vertical plane parallel to the given one. It is convenient to take moving axes through O , the axis 2 along the rod, the axis 1 horizontal and at right angles to the plane zOP , and the axis 3 at right angles to the rod in the plane zOP .



Then since these axes are fixed to the rod, we have

$$\omega_1, \omega_2, \omega_3 = -\dot{\theta}, \dot{\phi} \cos \theta, \dot{\phi} \sin \theta \dots\dots\dots(1).$$

The rod has no products of inertia and its moments of inertia about the axes are $A = C = \frac{1}{3}ma^2$, $B = 0$.

The forces acting on the rod which have moments about the axes are its weight mg and the reaction R of the vertical plane, parallel to NO ; and the components of the latter parallel to the axes 1, 2, 3 are $R \cos \phi$, $-R \sin \phi \sin \theta$, $R \sin \phi \cos \theta$.

Hence Euler's Equations are

$$\left. \begin{aligned} A\dot{\omega}_1 + C\omega_2\omega_3 &= Ra \sin \phi \cos \theta - \frac{1}{2}mga \sin \theta \\ C\dot{\omega}_3 - A\omega_1\omega_2 &= -Ra \cos \phi \end{aligned} \right\} \dots\dots\dots(2);$$

the equation of moments about the axis 2 vanishing identically.

Therefore

$$\frac{1}{3}ma^2(\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta) = \frac{1}{2}mga \sin \theta - Ra \sin \phi \cos \theta \dots\dots(3)$$

$$\text{and} \quad \frac{1}{3}ma^2(\ddot{\phi} \sin \theta + 2\dot{\phi}\dot{\theta} \cos \theta) = -Ra \cos \phi \dots\dots\dots(4).$$

But the co-ordinates are subject to the condition

$$a \sin \theta \sin \phi = c \dots\dots\dots(5),$$

so that

$$\dot{\theta} \cos \theta \sin \phi + \dot{\phi} \sin \theta \cos \phi = 0 \dots\dots\dots(6).$$

If we multiply (3) by $\dot{\theta}$ and (4) by $\dot{\phi} \sin \theta$ and add, the coefficient of R vanishes by (6), and the remaining terms integrate giving

$$\frac{1}{3}ma^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) = mga(\cos \alpha - \cos \theta) \dots\dots\dots(7),$$

where α is the initial value of θ , assuming the rod to start moving from rest. This last equation might have been written down as the equation of energy.

To find R we may proceed thus: by differentiating (6) we get

$$\ddot{\theta} \cos \theta \sin \phi + \dot{\phi} \sin \theta \cos \phi - (\dot{\theta}^2 + \dot{\phi}^2) \sin \theta \sin \phi + 2\dot{\theta}\dot{\phi} \cos \theta \cos \phi = 0 \quad \dots\dots(8).$$

Eliminate $\ddot{\theta}$ and $\dot{\phi}$ from (3), (4) and (8). The result is

$$\begin{aligned} \frac{1}{3}ma(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) \sin \theta \sin \phi \\ = \frac{1}{2}mg \sin \theta \cos \theta \sin \phi - R(\sin^2 \phi \cos^2 \theta + \cos^2 \phi) \quad \dots\dots(9). \end{aligned}$$

Then from (5), (7) and (9),

$$\begin{aligned} R(\sin^2 \phi \cos^2 \theta + \cos^2 \phi) &= \frac{1}{2}mg \frac{c}{a} \cos \theta - mg \frac{c}{a} (\cos \alpha - \cos \theta) \\ &= \frac{mgc}{2a} (3 \cos \theta - 2 \cos \alpha); \end{aligned}$$

so that the rod leaves the plane when $\cos \theta = \frac{2}{3} \cos \alpha$.

7·91. Example of Initial Motion. *A uniform solid ellipsoid is held in contact with a smooth horizontal plane and released from rest, to find the initial pressure of the plane.*

Let $\omega_1, \omega_2, \omega_3$ denote angular velocities about the axes of the ellipsoid; x, y, z the co-ordinates of the point of contact, p the perpendicular distance of the centre from the given plane; l, m, n the direction cosines of the normal at (x, y, z) ; M the mass, A, B, C principal moments of inertia and a, b, c semi-axes of the ellipsoid, and R the pressure of the plane. Then

$$l, m, n = px/a^2, py/b^2, pz/c^2 \quad \dots\dots\dots(1),$$

and the equations of moments about the axes are

$$\left. \begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= -R(ny - mz) \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= -R(lz - nx) \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= -R(mx - ly) \end{aligned} \right\} \quad \dots\dots\dots(2);$$

or, since initial velocities are zero, and by using (1),

$$\left. \begin{aligned} A\dot{\omega}_1 &= -R(b^2 - c^2)mn/p \\ B\dot{\omega}_2 &= -R(c^2 - a^2)nl/p \\ C\dot{\omega}_3 &= -R(a^2 - b^2)lm/p \end{aligned} \right\} \quad \dots\dots\dots(3).$$

By resolving vertically we have

$$M\ddot{p} = R - Mg \quad \dots\dots\dots(4).$$

But

$$p^2 = a^2 l^2 + b^2 m^2 + c^2 n^2 \quad \dots\dots\dots(5)$$

and, since l, m, n are direction cosines of a fixed direction,

$$\left. \begin{aligned} l - m\omega_3 + n\omega_2 &= 0 \\ \dot{m} - n\omega_1 + l\omega_3 &= 0 \\ \dot{n} - l\omega_2 + m\omega_1 &= 0 \end{aligned} \right\} \quad \dots\dots\dots(6).$$

Whence, since initial velocities are zero, we have from (5) and (6)

$$p\ddot{p} = a^2\ddot{l} + b^2m\ddot{m} + c^2n\ddot{n} \dots\dots\dots(7)$$

and

$$\left. \begin{aligned} \ddot{l} - m\dot{\omega}_3 + n\dot{\omega}_2 &= 0 \\ \ddot{m} - n\dot{\omega}_1 + l\dot{\omega}_3 &= 0 \\ \ddot{n} - l\dot{\omega}_2 + m\dot{\omega}_1 &= 0 \end{aligned} \right\} \dots\dots\dots(8).$$

Then by eliminating the accelerations from (3), (4), (7) and (8) we get

$$\begin{aligned} p \left(\frac{R}{M} - g \right) &= - \frac{a^2 l R}{p} \left\{ \frac{l m^2 (a^2 - b^2)}{C} - \frac{l n^2 (c^2 - a^2)}{B} \right\} \\ &\quad - \frac{b^2 m R}{p} \left\{ \frac{m n^2 (b^2 - c^2)}{A} - \frac{m l^2 (a^2 - b^2)}{C} \right\} \\ &\quad - \frac{c^2 n R}{p} \left\{ \frac{n l^2 (c^2 - a^2)}{B} - \frac{n m^2 (b^2 - c^2)}{A} \right\} \end{aligned}$$

$$\text{or } p^3 \left(\frac{R}{M} - g \right) = -R \left\{ \frac{m^2 n^2 (b^2 - c^2)^2}{A} + \frac{n^2 l^2 (c^2 - a^2)^2}{B} + \frac{l^2 m^2 (a^2 - b^2)^2}{C} \right\},$$

so that

$$R = p^2 g / \left\{ \frac{p^2}{M} + \frac{m^2 n^2 (b^2 - c^2)^2}{A} + \frac{n^2 l^2 (c^2 - a^2)^2}{B} + \frac{l^2 m^2 (a^2 - b^2)^2}{C} \right\}.$$

EXAMPLES

1. A rod, of mass $3m$ and length $2a$, is moveable in a vertical plane about its middle point, and carries at one end a particle of mass m ; if the vertical plane be made to revolve, with uniform angular velocity ω , about the vertical through the middle point, prove that the equation of motion of the rod is

$$2a\ddot{\theta} - 2a\omega^2 \sin \theta \cos \theta + g \sin \theta = 0.$$

2. One end of a heavy rod rests on a horizontal plane and against the foot of a vertical wall, the other end rests against a parallel vertical wall, all the surfaces being smooth. Shew that if it slips down, the angle ϕ through which it turns round the common normal to the vertical walls is given by the equation

$$\phi^2 (1 + 3 \cos^2 \phi) = C - (6g \sin \phi) / \sqrt{(a^2 - b^2)},$$

where $2a$ is the length of the rod, and $2b$ the distance between the walls.

3. A smooth plate inclined at an angle ϕ to the horizon is made to rotate about a vertical axis AB with uniform angular velocity ω . A rod of mass m is compelled by guides to be always vertical, and at a distance c from AB , while it rests with one end in contact with the plate, sliding up and down as the latter rotates. Shew that, if the rod be initially in its lowest position, the pressure on the plate at the end of the time t will be

$$m(g \cos \phi + c\omega^2 \cos \omega t \sin \phi) \sec^2 \phi.$$

4. A heavy sphere moves on a rough horizontal plane which can revolve about a fixed vertical axis. The system being set in motion in any manner, prove that the curve described in space by the centre of the sphere is given by equations which can be put in the form

$$x = c \int \cos \frac{1}{2} (\phi - \phi') \frac{d\phi}{\phi} + b \cos \frac{1}{2} (\phi - \alpha),$$

$$y = c \int \sin \frac{1}{2} (\phi - \phi') \frac{d\phi}{\phi} + b \sin \frac{1}{2} (\phi - \alpha),$$

where ϕ is the angle turned through by the plane, and ϕ' is put equal to ϕ after integration. [M. T. 1888]

5. A perfectly rough vertical plane revolves with a uniform angular velocity μ about an axis perpendicular to itself, and also with a uniform angular velocity Ω about a vertical axis in its own plane, which meets the former axis. A heavy uniform sphere, of radius c , is placed in contact with the plane; prove that the position of its centre, at any time t , will be determined by the equations

$$7\xi - 5\Omega^2\xi = 2\mu\dot{z}, \quad 7\ddot{z} + 2\Omega^2\dot{z} + 2\mu(\xi + \Omega^2\xi) = 0,$$

z denoting the distance of the centre from the horizontal plane through the horizontal axis of revolution, and ξ that from the plane through the two axes.

Prove also that if a and b be the initial values of ξ and z , u and v those of $\dot{\xi}$ and $\dot{z}$, $7u = 7c\Omega + 2\mu b$, $7v + 2\mu a = 0$. [M. T. 1872]

6. A perfectly rough plane, inclined at a fixed angle to the vertical, rotates about a vertical line with uniform angular velocity; shew that the path of a sphere which is placed upon it is given by two equations of the form

$$\dot{y} + ax + by = 0, \quad \ddot{x} - a\dot{y} + b'\dot{x} = c,$$

the origin being the point where the vertical line meets the plane, and the axis of y being the straight line in the plane which is always horizontal. [M. T. 1865]

7. A rough vertical plane is made to rotate with uniform angular velocity ω about a vertical axis in itself and strikes a heavy uniform inelastic sphere of radius a at rest, the point of contact being at a distance $\sqrt{\frac{1}{2}}a$ from the axis. Shew that after a time t the distance of the point of contact from the axis is $\sqrt{\frac{1}{2}}a \exp(\sqrt{\frac{1}{2}}\omega t)$ and that contact will be maintained throughout the motion. Shew also that the vertical distance of the centre of the sphere below its initial position is

$$(5g/\omega^2) \sin^2(\omega t/\sqrt{14}). \quad [\text{M. T. 1932}]$$

8. A uniform sphere of mass m rolls without slipping on one face of a wedge of mass M and angle α , the other face of the wedge being in contact with a smooth horizontal plane. Rotation of the wedge, and translational motion of the wedge parallel to the line of intersection of the faces, are prevented by frictionless guides. Prove that, in

general, the point of contact of the sphere traces out a parabola on the inclined face of the wedge, and that the acceleration of the wedge is

$$\frac{5mg \sin \alpha \cos \alpha}{7M + 2m + 5m \sin^2 \alpha}. \quad [\text{M. T. 1934}]$$

9. A uniform solid ellipsoid, of semi-axes a, b, c ($a > b > c$), is mounted so that it can turn freely about an axle lying along the axis $2a$. This axle is made to rotate in a horizontal plane with uniform angular velocity ω . Prove that for steady motion one of the axes of the ellipsoid must be vertical, and that for small oscillations about steady motion the period is

$$\frac{2\pi}{\omega} \left(\frac{b^2 + c^2}{b^2 - c^2} \right)^{\frac{1}{2}}. \quad [\text{I. 1930}]$$

10. A heavy circular wire can rotate freely about a vertical diameter, and a uniform rod of length equal to the side of an inscribed equilateral triangle moves with its ends on the wire. If a is the radius of the circle and the rod is of mass m , prove that the kinetic and potential energies of the system are

$$\frac{1}{2} \left[(I + \frac{1}{2}ma^2) \dot{\phi}^2 + \frac{1}{2}ma^2\dot{\theta}^2 \right] \quad \text{and} \quad -\frac{1}{2}mga \cos \theta + C$$

respectively, where I is the moment of inertia of the wire about a diameter, ϕ is its angular velocity and θ is the inclination to the vertical of the line joining the centres of the rod and wire. Find the equations of finite motion and thence state in words the precise character of the motion. [St John's Coll. 1909]

11. A spherical ball spinning about its vertical diameter is gently placed at the extreme edge of a table and rolls off without slipping. Shew that if the radius of gyration has a certain value the centre of the ball describes a portion of a helix while contact lasts. [M. T. 1908]

12. Two rough rods OA, OB are fixed in a horizontal plane and make an angle 2α with each other. A heavy uniform sphere of radius a rolls down between the rods slightly disturbed from rest with the centre C vertically above O . Prove that so long as the sphere is in contact with the rods, its angular velocity is given by

$$\omega^2 (a^2 \sin^2 \alpha + z^2 \cos^2 \alpha + 2a^2/5) = 2g(a - z),$$

where z is the height of C above the plane OAB .

[I. 1913]

13. A heavy sphere is held in contact with a rough circular wire which is fixed in a horizontal plane, and a horizontal impulse is then applied to the sphere, causing it to roll round steadily. If c is the radius of the ring and b that of the sphere, and if α is the constant inclination to the vertical of the radius through the point of contact, prove that the angular velocity Ω of the point of contact is given by the equation

$$7\Omega^2 (c - b \sin \alpha) = 5g \tan \alpha,$$

and that the impulse required to produce this motion is such as would impart to the sphere, if it were free, the velocity

$$\frac{7}{2}\Omega(c-b\sin\alpha). \quad [\text{M. T. 1881}]$$

14. A heavy circular plate is pivoted at its centre O and upon its upper surface is a smooth circular groove, with its centre at O , in which a particle of mass m moves, so that the system moves steadily about the vertical with angular velocity λ . Shew that if $a\lambda^2\sin\alpha < g$, the particle must lie on the line of greatest slope through O , and that

$$\lambda^2(A - ma^2)\cos\alpha - C\lambda\Omega + mga\cot\alpha = 0,$$

where A, A, C are the moments of inertia of the plate, a is the radius of the groove, Ω is the angular velocity of the plate about a normal, and α is the inclination of the plate to the horizontal. [M. T. 1909]

15. The end of a uniform rod of length l is mounted on an axle so that the rod can turn freely in the plane perpendicular to the axle. If the axle is made to rotate in a horizontal plane with angular velocity ω , the point where the rod joins the axle remaining fixed, prove that θ , the angle which the rod makes with the downward vertical at time t , satisfies the equation

$$\ddot{\theta} = \omega^2 \sin\theta \cos\theta - (3g/2l)\sin\theta.$$

If ω is constant, examine the stability of the motions in which θ is constant. [M. T. 1933]

16. A flail is composed of two bars jointed together at an end of each; the other end of one of them is fixed and they are free to move on a horizontal plane. Prove that, if θ is the inclination of the first anchored bar to a fixed direction, and ψ the inclination of the second bar to the first, the angular velocities of their free motion are determined by equations of the form

$$A\dot{\theta} + B(\dot{\theta} + \dot{\psi}) + mab\cos\psi(2\dot{\theta} + \dot{\psi}) = G_0,$$

$$A\dot{\theta}^2 + B(\dot{\theta} + \dot{\psi})^2 + 2mab\cos\psi.\dot{\theta}(\dot{\theta} + \dot{\psi}) = 2T_0.$$

Find what motions are possible for which ψ remains constant; and examine their stability. [M. T. 1919]

17. A uniform rod is constrained to move on the surface of a smooth hyperboloid of revolution, there being no forces except the constraint. Shew that, however the motion be started, the component velocity of the centre of the rod parallel to the axis of revolution cannot vanish more than once in the subsequent motion; and that, unless this component velocity is always zero, the motion tends asymptotically to uniform motion along a generating line. [M. T. 1903]

18. A uniform circular disc is rotating in its own plane with very large angular velocity about its centre O , which is fixed. Prove that if a tap be given to the disc in a direction perpendicular to its plane, at a

point A , the diameter through A will approximately describe a plane slightly inclined to the original position of the plane of the disc, while the diameter at right angles to it will describe the same plane as before.

[M. T. 1903]

19. A sphere is rolling on the rough surface of a cylinder, the cross-section of which is the curve $3r = a\sqrt{2} \cdot \exp \theta\sqrt{(2/7)}$; prove that, if there be no forces, the path of the point of contact becomes, when the cylinder is developed into a plane, a curve of the form

$$y = (\alpha + \beta x) \cos(\log x/c) + (\gamma + \delta x) \sin(\log x/c).$$

20. A sphere moves under the action of gravity on the inside of a rough cylindrical surface, of which the generating lines are inclined at an angle α to the horizon, and the transverse section perpendicular to the generating lines is a cycloid with its vertex at the lowest generating line. The sphere is projected initially with a velocity V along the generating line at which the curvatures of the sphere and cycloid are equal. Prove that the motion will be comprised within a length

$$14a \left(\frac{2V^2}{5ga \cos \alpha} + 4 \tan^2 \alpha \right)^{\frac{1}{2}}$$

of the cylinder, and that the time between successive instants of the sphere reaching the original generating line is

$$8 \left(\cos^{-1} \frac{c}{4a} \right) \sqrt{\left(\frac{7a}{5g \cos \alpha} \right)},$$

where a is the radius of the generating circle of the cycloid and c is the radius of the sphere.

21. A solid with a smooth plane face inclined at an angle α to the horizontal can rotate freely about a fixed vertical axis. A light string of length a is attached to the axis at the point where it cuts the plane face and a particle of mass m is attached to the other end. Prove that, if the system is released from rest when the string lies on the plane face making an angle β with the line of greatest slope, the initial angular acceleration of the solid is

$$\frac{mg \sin \alpha \cos \alpha \sin \beta}{I + ma^2 \sin^2 \alpha \sin^2 \beta},$$

where I is the moment of inertia of the solid about the axis.

[I. 1912]

22. A body in the form of a hollow circular cone of semi-vertical angle α spins about its axis which is fixed and vertical, the vertex being the lowest point. Shew that, if a sphere of uniform density and of unit mass be placed on the interior of the cone which is rough,

$$(I + \frac{2}{5}a^2 \sin^2 \alpha) \omega = I\Omega_0,$$

where Ω_0 is the initial angular velocity of the cone, ω its angular velocity immediately after the sphere has been put on, I the moment

of inertia of the cone about its axis and a the distance of the point of contact from the vertex.

Prove that, if Ω be the angular velocity at any subsequent period of the motion and r the distance of the point of contact,

$$(I + \frac{2}{3}r^2 \sin^2 \alpha) (I + \frac{2}{3}a^2 \sin^2 \alpha) \Omega^2 = I^2 \Omega_0^2. \quad [\text{M. T. 1882}]$$

23. A uniform rough heavy circular disc of radius a has its edge touching a horizontal table, and rests against the pointed top of a peg of vertical height h fixed in the table. In the position of equilibrium its plane makes an angle α with the table. Shew that the length of the simple equivalent pendulum for a small oscillation in which there is no slipping is

$$\frac{ah \sec \alpha \tan \alpha}{4(h - a \sin \alpha)}. \quad [\text{M. T. 1904}]$$

24. A rigid body of mass M can rotate freely about a point O , and axes $O1, O2, O3$ are fixed in it. The point O is constrained to move with an acceleration whose components are (f_1, f_2, f_3) . Prove that

$$\frac{dh_1}{dt} + M(\bar{x}_2 f_3 - \bar{x}_3 f_2) = L_1,$$

where h_i are the components of the part of the angular momentum about O due to the rotation alone, $\bar{x}_i$ are the co-ordinates of the centre of mass, and L_i are the moments of the external forces about $O1, O2, O3$.

Two perpendicular rods $O1, O3$ are hinged together so that $O1$ can rotate freely about $O3$. The direction of $O3$ is given by Eulerian angles (θ, ψ) with respect to the vertical, and that of $O1$ by a further Eulerian angle ϕ . $O3$ is moved about in an arbitrary manner; prove that ϕ is given by the differential equation

$$I(\dot{\omega}_3 + \omega_1 \omega_2) + Mh f_2 = -Mgh \sin \theta \sin \phi,$$

where h is the distance of the centre of mass of the rod $O1$ from O , M the mass of the rod and I its moment of inertia about O . Hence shew that if θ and ψ make small oscillations about mean values α and 0 , where α is a small angle, $\phi + \psi$ is equally affected by an acceleration f_2 of O and a tilt f_2/g of $O3$ about $O1$. [I. 1933]

25. A circular cylinder is fixed with its axis horizontal, and a rough uniform sphere is projected horizontally from a point of the lowest generator in a direction making an angle α with this generator. Prove that for the sphere to come in contact with the highest generator the

initial velocity must be greater than $\sqrt{\frac{27}{7} \frac{(b-a)g}{\sin^2 \alpha}}$, where a and b are the radii of the sphere and cylinder respectively. [I. 1924]

26. A uniform solid circular cylinder of mass M is free to turn about its axis, which is horizontal. The cylinder is at rest, when a rough uniform solid sphere, of mass m and radius a , is placed in contact with the highest generator, the sphere having initially a spin n about its vertical diameter. Shew that, if the system is slightly disturbed, the sphere will

leave the cylinder when the plane through the axis of the cylinder and the centre of the sphere makes an angle α with the vertical, where $\cos \alpha = \frac{10M + 8m}{17M + 12m}$; and that, at the instant of separation, the component velocity of the centre of the sphere in the direction of the axis of the cylinder is $\sqrt{\frac{7}{2}}an \sin \sqrt{\frac{7}{2}}\alpha$. [M. T. 1929]

27. A uniform sphere of radius a rolls in a horizontal groove whose transverse section is an arc of a circle of radius b . Prove that the motion across the groove is similar to that of a simple pendulum of length $\frac{1}{2}(b-a)$, and that if the amplitude of the transverse motion is small the velocity along the groove is given by

$$\dot{x} = U + \frac{1}{2}a\omega\theta,$$

where U is a constant, θ is the inclination of the common normal to the vertical, and ω is the mean value of the spin about the common normal. [I. 1933]

28. A uniform sphere of radius a rolls on the inside of a rough horizontal cylindrical tube of radius b . It is set rolling at the bottom of the tube with a velocity V along the tube, a small spin n about a vertical axis and a small velocity u_0 across the tube. Determine its rate of progression along the tube, and shew that the time-average value is

$$V \left[1 - \frac{u_0^2}{10g(b-a)} \right],$$

correct to the second order in u_0 and n . [M. T. 1924]

29. A cylinder of internal radius c , whose axis is horizontal, is made to rotate about its axis with uniform angular velocity Ω . A uniform sphere of radius a , spinning about its vertical diameter with angular velocity n , is placed on the lowest generator of the cylinder and there is sufficient friction to prevent sliding. Prove that, if $c^2\Omega^2 > 35g(c-a)$, the sphere will roll completely round the cylinder. Shew also that the velocity of the sphere along the cylinder is given by

$$\dot{x} = an\sqrt{\frac{7}{2}} \sin \sqrt{\frac{7}{2}}\theta,$$

where θ is the angle which the plane through the centre of the sphere and the axis of the cylinder makes with the vertical.

30. A hollow circular cylinder of radius b is fixed with its axis horizontal, and a rough uniform sphere of radius a is placed inside the cylinder with a point on the lowest generator. It is projected horizontally with velocity V in a direction making an angle α with this generator. Shew that if V lies between

$$\{10(b-a)g/7\}^{\frac{1}{2}} \operatorname{cosec} \alpha \quad \text{and} \quad \{27(b-a)g/7\}^{\frac{1}{2}} \operatorname{cosec} \alpha,$$

the sphere will leave the cylinder at a point at which the radius makes an angle ϕ with the downward vertical given by

$$17 \cos \phi = 10 - 7 \frac{V^2}{(b-a)g} \sin^2 \alpha. \quad [\text{I. 1931}]$$

31. A rough heavy sphere, radius c , rolls on a fixed rough surface, of the form generated by the revolution of a circle, radius b , about a vertical axis in its own plane, distant a from the centre, a being greater than $b + c$. Prove that if, at the time t , ϕ be the angle through which the plane through the vertical axis and centre of the sphere has turned, θ the inclination to the vertical of the common normal, ω_s the angular velocity about that common normal, α and λ the initial values of θ and ϕ , and if $\dot{\theta}$ and ω_s be initially zero, and $l = b + c$, then

$$c\dot{\omega}_s = a\dot{\theta}\dot{\phi},$$

$$l^2\dot{\theta}^2 + (a - l\sin\theta)^2\dot{\phi}^2 - \lambda^2(a - l\sin\alpha)^2 + \frac{1}{2}c^2\omega_s^2 = \frac{1}{2}gl(\cos\alpha - \cos\theta),$$

$$\frac{d}{d\theta} \left\{ (a - l\sin\theta)^2 \frac{d\omega_s}{d\theta} \right\} + \frac{1}{2}a\omega_s(a - l\sin\theta) = 0.$$

32. The lower of two equal perfectly rough spheres, of radius a , in contact with each other, moves on a smooth horizontal plane. If ω , ω' be the angular velocities of the two spheres about their common diameter, θ the inclination of this diameter to the vertical, and ϕ the azimuth of the vertical plane through it, prove that throughout the motion under the action of gravity alone the following quantities are constant

(i) $\omega + \omega'$,

(ii) $7\phi\sin^2\theta + (\omega + \omega')\cos\theta$,

(iii) $(7 + 5\sin^2\theta)\dot{\theta}^2 + 7\dot{\phi}^2\sin^2\theta + 10(g/a)\cos\theta$. [M. T. 1896]

Chapter VIII

IMPULSIVE MOTION

8.1. The equations of motion for impulsive forces for systems of particles and rigid bodies, moving in two dimensions, obtained in **Part I**, 11.2 and 15.3, are completed for the general cases of motion in three dimensions by adding to the linear equations in x and y a similar equation in z ; and to the equation of moments similar equations of moments about the axes of x and y . And the equations embody nothing more than the facts stated in **Part I**, 11.2, that

(1) *the instantaneous increase in the linear momentum in any direction is equal to the sum of the externally applied impulsive forces in that direction; and*

(2) *the instantaneous increase in the moment of momentum about any axis is equal to the sum of the moments about that axis of the externally applied impulsive forces.*

8.11. To find the motion of a free single body acted on by a given impulse. Let an impulse J act in the line whose equations are $(x-f)/l = (y-g)/m = (z-h)/n$, referred to axes through the centre of gravity of the body, where l, m, n are the direction cosines of the line. This is equivalent to an impulse with components lJ, mJ, nJ at the centre of gravity, and couples $(gn - hm)J, (hl - fn)J, (fm - gl)J$. Hence, if $u, v, w, \omega_x, \omega_y, \omega_z$ denote the components of velocity of the centre of gravity and the components of angular velocity just before the impulse, and dashes denote values immediately after the impulse, we have

$$M(u' - u) = lJ, \quad M(v' - v) = mJ, \quad M(w' - w) = nJ \dots (1),$$

$$\left. \begin{aligned} A(\omega_x' - \omega_x) - F(\omega_y' - \omega_y) - E(\omega_z' - \omega_z) &= (gn - hm)J \\ B(\omega_y' - \omega_y) - D(\omega_z' - \omega_z) - F(\omega_x' - \omega_x) &= (hl - fn)J \\ \text{and} \\ C(\omega_z' - \omega_z) - E(\omega_x' - \omega_x) - D(\omega_y' - \omega_y) &= (fm - gl)J \end{aligned} \right\} (2),$$

where M is the mass of the body, and A, B, C, D, E, F are its moments and products of inertia about the axes.

Equations (1) and (2) determine the new components of velocity.

8·12. A body free to turn about a fixed point is set in motion by an impulsive couple. If we take the principal axes at the fixed point for axes of co-ordinates, and let H_1, H_2, H_3 denote the components of the impulsive couple H , and $\omega_1, \omega_2, \omega_3$ the components of angular velocity, we have

$$A\omega_1, B\omega_2, C\omega_3 = H_1, H_2, H_3 \dots\dots\dots(1).$$

The plane of the couple is parallel to $H_1x + H_2y + H_3z = 0$, or, from (1), $A\omega_1x + B\omega_2y + C\omega_3z = 0 \dots\dots\dots(2)$,

which is the diametral plane of the momental ellipsoid conjugate to the line whose direction cosines are proportional to $\omega_1, \omega_2, \omega_3$. Hence the instantaneous axis of rotation is the diametral line of the plane of the couple with regard to the momental ellipsoid.

If r be the radius of this ellipsoid in the direction (l, m, n) of the instantaneous axis and p the central perpendicular on the tangent plane, the moment of inertia about the instantaneous axis is $Al^2 + Bm^2 + Cn^2$, or, in the notation of 5·3, $M\epsilon^4/r^2$. Hence the moment of momentum about the instantaneous axis is $M\epsilon^4\omega/r^2$. But the resultant moment of momentum is H and its resolved part about the instantaneous axis is Hpr/r , therefore

$$\omega = Hpr/M\epsilon^4 \dots\dots\dots(3).$$

Again, the kinetic energy T is given by

$$\begin{aligned} 2T &= A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \quad (7·42) \\ &= \omega^2 (Al^2 + Bm^2 + Cn^2) \\ &= \omega^2 M\epsilon^4/r^2, \text{ or from (3)} \\ &= H^2 p^2 / M\epsilon^4 \dots\dots\dots(4). \end{aligned}$$

Hence, for a given value of H , the kinetic energy created by the impulse is greatest or least according as p is greatest or least, i.e. according as the axis of the couple is the axis of least or of greatest principal moment of the body.

8.13. Example. *A uniform solid falls and strikes a smooth horizontal inelastic plane, to find the change in its kinetic energy produced by the impact.*

Take the principal axes at the centre of gravity for axes of co-ordinates. Let u, v, w be the velocities of the centre of gravity after impact and $\omega_1, \omega_2, \omega_3$ the angular velocities set up by the impact. Let J be the impulse; l, m, n the direction cosines of the normal at the point of impact, whose co-ordinates are f, g, h ; and let V be the velocity of falling just before impact. Then if M is the mass and A, B, C are the principal moments of inertia,

$$\begin{aligned} M(u-lV) &= -lJ, & M(v-mV) &= -mJ, & M(w-nV) &= -nJ \\ A\omega_1 &= -(ng-mh)J \\ B\omega_2 &= -(lh-nf)J \\ C\omega_3 &= -(mf-lg)J \end{aligned} \quad \begin{aligned} & \dots\dots(1), \\ & \dots\dots\dots(2). \end{aligned}$$

We have also a kinematical condition that the normal velocity of the point of contact is zero, that is

$$l(u-g\omega_3+h\omega_2)+m(v-h\omega_1+f\omega_3)+n(w-f\omega_2+g\omega_1)=0 \dots(3).$$

For the kinetic energy we multiply equations (1) and (2) by $u, v, w, \omega_1, \omega_2, \omega_3$ and add, and in virtue of (3) we get

$$\begin{aligned} \frac{1}{2}M(u^2+v^2+w^2) + \frac{1}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2) &= \frac{1}{2}MV(lu+mv+nw) \\ &= \frac{1}{2}V(MV-J), \text{ by substituting from (1).} \end{aligned}$$

Hence the loss of kinetic energy is $\frac{1}{2}JV$, in accordance with the general theorem on the change of energy due to impulses. (Pt. I, 11.5 or 8.4 below.)

To find J we write (3) in the form

$$lu+mv+nw+\omega_1(ng-mh)+\omega_2(lh-nf)+\omega_3(mf-lg)=0.$$

and substitute from (1) and (2), so that

$$V - \frac{J}{M} - J \left\{ \frac{(ng-mh)^2}{A} + \frac{(lh-nf)^2}{B} + \frac{(mf-lg)^2}{C} \right\} = 0,$$

and the loss of energy

$$\frac{1}{2}JV = \frac{1}{2}V^2 \left/ \left\{ \frac{1}{M} + \frac{(ng-mh)^2}{A} + \frac{(lh-nf)^2}{B} + \frac{(mf-lg)^2}{C} \right\} \right.$$

8.14. Examples. 1. A solid rectangular parallelepiped with edges of lengths a, b, c is acted on by instantaneous couples with axes parallel to these edges and of moments in the ratio $p:q:r$; shew that the direction cosines of the instantaneous axis of rotation are in the ratio

$$\frac{p}{b^2+c^2} : \frac{q}{c^2+a^2} : \frac{r}{a^2+b^2}.$$

2. A rectangular parallelepiped is dropped on to a smooth floor so that one angular point first comes in contact; if the edges be $2a, 2b, 2c$ and equally inclined to the vertical at the instant of striking, find the impulse sustained by the floor.

3. A rigid body moveable about a fixed point is struck by a blow of given magnitude at a given point; if the angular velocity thus impressed upon the body be the greatest possible, prove that a, b, c being the co-ordinates of the given point in relation to the principal axes through the fixed point, and l, m, n being the direction cosines of the couple produced,

$$al + bm + cn = 0$$

and
$$\frac{a}{l} \left(\frac{1}{B^2} - \frac{1}{C^2} \right) + \frac{b}{m} \left(\frac{1}{C^2} - \frac{1}{A^2} \right) + \frac{c}{n} \left(\frac{1}{A^2} - \frac{1}{B^2} \right) = 0,$$

A, B, C being the moments of inertia of the body about the principal axes at the fixed point.

4. If an octant of an ellipsoid bounded by three principal planes be rotating about the axis a with angular velocity ω , and if this axis suddenly becomes free, and the axis b is fixed, shew that the new angular velocity is

$$2ab\omega/\pi(a^2 + c^2).$$

5. A body is moving in any manner. Find the condition that it may be brought to rest by a single impulse suitably applied.

A moving body satisfies this condition at any instant and is moving under no forces. Shew that in general the condition cannot remain satisfied during the subsequent motion; but that if the body is symmetrical about an axis through its centre of gravity, it will do so provided the angular velocity about that axis is zero. [St John's Coll. 1905]

6. The positive octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has the origin fixed. Shew that if an impulsive couple in the plane

$$ax + by = \pi z(a^2 + b^2)/2c$$

acts upon the octant, it will begin to revolve about the axis of z .

[I. 1903]

7. A uniform rectangular block, of height $2c$, stands on a horizontal plane; the sides of its base are of lengths $2a$ and $2b$. A vertical impulse upwards is applied at one of the corners; prove that the block leaves the plane at once if

$$c^2 > a^2 + b^2 + \{a^4 + 7a^2b^2 + b^4\}^{\frac{1}{2}}. \quad [\text{I. 1929}]$$

8. Prove that, if the uniplanar motion of a body be changed impulsively,

$$I'\omega'^2 - I\omega^2 = L'\omega' + L\omega,$$

where L and L' are respectively the moments of the impulses about the original and final instantaneous centres, and I, I' are the moments of inertia about these centres. [I. 1915]

9. A rigid system whose principal moments at its mass-centre G are each equal to I is rotating about GZ . Prove that if a particle m at (x, y, z) is instantaneously displaced by internal action through a small space (α, β, γ) the axis of rotation will be turned through angles

$$m(\alpha z + \gamma x)/I, \quad m(\beta z + \gamma y)/I,$$

towards GX, GY respectively.

[M. T. 1923]

10. A rigid lamina at rest but free to rotate about a fixed point in its plane receives an impulse P perpendicular to the plane at a point whose co-ordinates referred to the principal axes at the fixed point are $(x, y, 0)$. Prove that, in the subsequent motion, the component of angular velocity parallel to the plane of the lamina is constant and equal to $P\sqrt{(x^2/B^2 + y^2/A^2)}$, where A, B are the moments of inertia about the axes of x and y respectively.

Prove also that, if the plane through the instantaneous axis perpendicular to the lamina makes an angle ϕ with the plane zx ,

$$(A + B) \phi^2 = P^2 (A - B) \left(\frac{y^2}{A^2} \sin^2 \phi - \frac{x^2}{B^2} \cos^2 \phi \right). \quad [\text{M. T. 1932}]$$

8·2. Impact of inelastic bodies. When the bodies which impinge on one another are inelastic and smooth, there is a kinematical condition that the velocities, immediately after impact, of the points of the bodies in contact with each other are the same in the direction of the common normal to their surfaces.

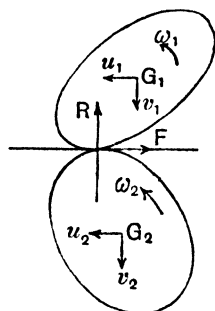
But if the bodies are inelastic and sufficiently rough to prevent sliding, the velocities of the points of contact immediately after impact are the same in every direction.

8·21. Elastic bodies. In Part I, 11·32 the case of the impact of smooth spheres was considered and it was shewn that Newton's Rule, *that the relative velocities, before and after impact, of the points of two smooth spheres which come into contact resolved along the common normal at the point of contact are in the ratio 1 : -e*, is equivalent to Poisson's Hypothesis *that the impulse during restitution is e times the impulse during compression*.

It is not difficult to establish the same equivalence in the case of two-dimensional problems of impact of any bodies whose surfaces are either smooth or not sufficiently rough to prevent sliding at the point of contact, so that we may assume a definite relation between the impulsive friction and the normal reaction. A method of procedure in such cases is to write down the equations of impulsive motion on the hypothesis that the bodies are inelastic and thus determine the normal impulse R up to the instant of greatest compression, then the whole changes in velocity, allowing for elasticity, can

be found by writing $(1+e)R$ in place of R in the equations of motion. In the case of impact of elastic bodies whose surfaces are sufficiently rough to prevent sliding we have no justification for a hypothesis as to the impulsive frictions during restitution and compression, and the only method of procedure is to assume Newton's rule in its generalized form [Pt. I, 11.35].

8.22. To establish the statements made about two-dimensional cases in 8.21, consider the impact of two discs moving in the same plane, taking the case where there is a coefficient of restitution e and the edges are sufficiently rough to prevent sliding. Let G_1 be the centre of gravity of one of the discs, u_1, v_1 its velocities just before impact parallel to the tangent and normal at the point of contact and ω_1 the angular velocity, m_1 the mass, $m_1 k_1^2$ the moment of inertia about G_1 , and x_1, y_1 the distances of G_1 from the common normal and tangent to the discs. Let the suffix 2 denote similar quantities for the second disc, and let dashes be used to denote values of the velocities immediately after impact. Let R, F denote the impulsive pressure and the friction, shown in the figure in the sense in which they act on m_1 .



The equations of impulsive motion are

$$m_1(u_1 - u_1') = F, \quad m_1(v_1 - v_1') = R, \quad m_1 k_1^2(\omega_1 - \omega_1') = Rx_1 - Fy_1 \quad \dots\dots(1);$$

$$m_2(u_2 - u_2') = -F, \quad m_2(v_2 - v_2') = -R, \quad m_2 k_2^2(\omega_2 - \omega_2') = -Rx_2 - Fy_2 \quad \dots\dots(2).$$

Newton's rule connecting the relative normal velocities of the points of contact gives

$$(v_1' + x_1 \omega_1') - (v_2' + x_2 \omega_2') = -e \{(v_1 + x_1 \omega_1) - (v_2 + x_2 \omega_2)\} \quad \dots(3),$$

and the condition of no sliding at the point of contact gives

$$u_1' - y_1 \omega_1' = u_2' + y_2 \omega_2' \quad \dots\dots\dots(4).$$

If we substitute in (3) and (4) the values of $u_1', v_1', \omega_1', u_2', v_2', \omega_2'$ obtained from (1) and (2), we get

$$R \left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{x_1^2}{m_1 k_1^2} + \frac{x_2^2}{m_2 k_2^2} \right) - F \left(\frac{x_1 y_1}{m_1 k_1^2} - \frac{x_2 y_2}{m_2 k_2^2} \right) \\ = (1+e)(v_1 + x_1 \omega_1 - v_2 - x_2 \omega_2) \quad \dots\dots(5)$$

$$\text{and} \quad R \left(\frac{x_1 y_1}{m_1 k_1^2} + \frac{x_2 y_2}{m_2 k_2^2} \right) + F \left(\frac{1}{m_1} + \frac{1}{m_2} - \frac{y_1^2}{m_1 k_1^2} - \frac{y_2^2}{m_2 k_2^2} \right) \\ = u_1 - y_1 \omega_1 - u_2 - y_2 \omega_2 \quad \dots\dots(6);$$

and these equations determine the impulsive pressure and friction. Equations (1) and (2) then determine the velocities after impact.

It is clear that the elimination of F from (5) and (6) will not in this case give a value for R containing $1+e$ as a factor, as it should if Poisson's hypothesis were deducible from Newton's rule.

But if the edges are smooth we have to put $F=0$ in (1) and (2); (3) remains unaltered and leads to (5) without the term in F , so that in this case R does contain $(1+e)$ as a factor. Also if the edges are rough but not sufficiently rough to prevent sliding we have equations (1), (2) and (3) as above, and, instead of (4), an equation $F=\mu R$. Then (5) follows as above and when we put $F=\mu R$, we find that in this case too R contains $1+e$ as a factor, so that in this case also Poisson's hypothesis is equivalent to Newton's rule.

8.23. Example. A disc of any form moving in its plane without rotation and with velocity V at right angles to a fixed plane strikes the plane so that the distances of its centre of gravity from the point of impact and from the plane are r and p . Assuming the plane to be elastic and sufficiently rough to prevent sliding, find the impulsive pressure and friction.

Let u, v be the velocities of the centre of gravity G parallel and at right angles to the plane just after impact, ω the angular velocity, R, F the impulsive pressure and friction. Then, if m is the mass, we have

$$m(V-v) = R \dots\dots\dots(1),$$

$$mu = F \dots\dots\dots(2),$$

$$mk^2\omega = R\sqrt{(r^2-p^2)} - Fp \dots\dots(3).$$

Also since there is no sliding

$$u - p\omega = 0 \dots\dots\dots(4),$$

and Newton's rule gives

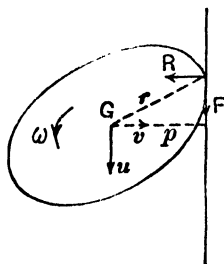
$$v - \omega\sqrt{(r^2-p^2)} = -eV \dots\dots\dots(5).$$

Eliminating u, v and ω , we find that

$$R = m(1+e)V(p^2+k^2)/(r^2+k^2)$$

and

$$F = m(1+e)Vp\sqrt{(r^2-p^2)}/(r^2+k^2).$$



8.24. Examples. 1. A circular disc of mass m , radius a , and moment of inertia about the centre mk^2 , is spinning with angular velocity ω on a smooth horizontal plane and impinges normally on the middle point of a rough inelastic rod of mass m' lying on the plane. Prove that the angular velocity immediately after impact is

$$(m+m')k^2\omega/\{m'a^2+(m+m')k^2\}.$$

2. Two uniform circular discs with rough edges lie flat on a smooth horizontal plane; one of them, of radius a and mass m , is in motion along the line of centres with velocity V and angular velocity Ω so as to impinge on the other, whose mass is m' . Shew that if the discs are so

rough and inelastic that the relative velocity of the points of impact is zero just after the impact, the line of motion of the centre of the moving disc is deflected through an angle whose tangent is $m'a\Omega/(3mV)$.
[M. T. 1912]

3. A circular disc of radius R and mass M can turn freely about its centre which is fixed in a smooth plane, another disc of radius r and mass m moves in the plane with velocity v without rotation, and impinges on the former. If both discs are inelastic and have milled edges, shew that the kinetic energy lost by the impact is

$$\frac{1}{2}mv^2 \cos^2 \alpha + \frac{1}{2}v^2 \sin^2 \alpha \left/ \left[\frac{1}{m} \left(1 + \frac{r^2}{k^2} \right) + \frac{R^2}{MK^2} \right] \right.,$$

where α is the angle of incidence and K, k are the radii of gyration of the discs.
[St John's Coll. 1904]

4. A cricket ball strikes the ground with horizontal velocity V and downward velocity W ; the angular velocities before impact are $\Omega_1, \Omega_2, \Omega_3$ about axes such that Ox is in the horizontal direction of travel, and Oz vertically upwards. Find the velocities of translation after impact on the supposition that the coefficient of restitution is e and the friction enough to destroy slipping; and give the lower limit to the coefficient of friction in order that the latter condition may hold. (The ball may be assumed homogeneous.)
[I. 1933]

8.3. Applications of the principle of virtual work. Let m_r denote the mass of a typical particle of a system. Let a number of impulses be applied to the system simultaneously, (X_r, Y_r, Z_r) denoting the impulse applied to the particle m_r and $(u_r, v_r, w_r), (u_r', v_r', w_r')$ the velocities of this particle just before and just after the application of the impulses. The connections, if any, between the particles will set up impulsive actions and reactions between them, but provided the connections are not violated the work done by mutual actions and reactions will be equal and opposite and can be omitted from the equation of virtual work. Consider a virtual displacement of the system which does not violate the geometrical connections, in which $\delta x_r, \delta y_r, \delta z_r$ are the components of the displacement of the typical particle m_r . Since the change of momentum in any direction is the measure of the impulse in that direction, we have

$$\begin{aligned} \Sigma m_r \{ (u_r' - u_r) \delta x_r + (v_r' - v_r) \delta y_r + (w_r' - w_r) \delta z_r \} \\ = \Sigma (X_r \delta x_r + Y_r \delta y_r + Z_r \delta z_r) \dots (1). \end{aligned}$$

We shall use this equation of virtual work for establishing some general properties of the motion by making suitable choice of the arbitrary displacements.

8·31. Self-contained system with no externally applied impulses. Carnot's Theorem.

Consider a system of inelastic bodies. Immediately after impact the points of contact have a common velocity along the common normal so that the mutual impulsive action and reaction are on the whole doing no work; consequently, if we take the virtual displacement to be the actual displacement immediately after the collision, i.e. if we put

$$\delta x_r : \delta y_r : \delta z_r = u_r' : v_r' : w_r',$$

and there are no external impulses, the right-hand side of 8·3 (1) is zero and

$$\sum m_r \{ (u_r' - u_r) u_r' + (v_r' - v_r) v_r' + (w_r' - w_r) w_r' \} = 0,$$

$$\text{or} \quad \sum m_r (u_r'^2 + v_r'^2 + w_r'^2) = \sum m_r (u_r u_r' + v_r v_r' + w_r w_r') \quad (1).$$

$$\text{Since} \quad u_r u_r' = \frac{1}{2} u_r^2 + \frac{1}{2} u_r'^2 - \frac{1}{2} (u_r - u_r')^2,$$

therefore

$$\frac{1}{2} \sum m_r (u_r'^2 + \dots) = \frac{1}{2} \sum m_r (u_r^2 + \dots) - \frac{1}{2} \sum \{ m_r (u_r - u_r')^2 + \dots \} \dots\dots(2);$$

shewing that *the kinetic energy of the system after the collision is less than the kinetic energy before the collision by the kinetic energy of the relative motion before and after*. This is the first part of *Carnot's Theorem*.

8·32. Kinetic energy increased by an internal explosion.

In the same circumstances, if we consider two particles which are about to be separated by the explosion, we may assume that they have the same velocity at the instant before the explosion, so that if we choose for virtual displacement the actual motion in a short interval just before the explosion it will be one in which the mutual impulsive reactions will do equal and opposite amounts of work; i.e. if we take

$$\delta x : \delta y : \delta z = u_r : v_r : w_r,$$

the right-hand side of 8.3 (1) being zero in this case also, we have

$$\Sigma m_r \{(u_r' - u_r) u_r + \dots\} = 0,$$

or

$$\Sigma m_r (u_r' u_r + \dots) - \Sigma m_r (u_r^2 + \dots) = 0,$$

which is equivalent to

$$\frac{1}{2} \Sigma m_r (u_r'^2 + \dots) - \frac{1}{2} \Sigma m_r (u_r^2 + \dots) = \frac{1}{2} \Sigma m_r \{(u_r' - u_r)^2 + \dots\} \dots\dots(1);$$

shewing that *after an explosion the kinetic energy is increased by the kinetic energy of the relative motion before and after*. This is the second part of *Carnot's Theorem*.

8.33. The effect of imposing frictionless constraints on a self-contained system is to decrease its kinetic energy by an amount equal to the kinetic energy of the relative motion before and after the imposition of the constraints. Also in any other dynamically possible motion compatible with the constraints, the kinetic energy is less than in the actual constrained motion; and in any other geometrically possible motion compatible with the constraints the relative kinetic energy is greater.

Referring to the general equation of virtual work 8.3 (1), there are in this case no externally applied impulses, and the constraints are such as do no work in the actual motion following their imposition. Hence, if u_r' , v_r' , w_r' denote the velocities of m_r immediately after the imposition of the constraints, we may put $\delta x : \delta y : \delta z = u_r' : v_r' : w_r'$, so that

$$\Sigma m_r \{(u_r' - u_r) u_r' + \dots\} = 0 \dots\dots\dots(1),$$

and as in 8.31 this leads to

$$\frac{1}{2} \Sigma m_r (u_r'^2 + \dots) = \frac{1}{2} \Sigma m_r (u_r^2 + \dots) - \frac{1}{2} \Sigma \{m_r (u_r' - u_r)^2 + \dots\} \dots\dots(2),$$

thus proving the first part of the proposition.

Now suppose that in any other motion compatible with the constraints the velocity of m_r becomes (u_r'', v_r'', w_r'') . Then since this motion also gives a possible virtual displacement whichever change of motion is under consideration, therefore we have

$$\Sigma m_r \{(u_r'' - u_r) u_r'' + \dots\} = 0 \dots\dots\dots(3)$$

and

$$\Sigma m_r \{(u_r' - u_r) u_r'' + \dots\} = 0 \dots\dots\dots(4).$$

From (3) and (4) we get

$$\Sigma m_r \{(u_r'' - u_r') u_r'' + \dots\} = 0 \dots\dots\dots (5).$$

$$\begin{aligned} \text{But } (u_r'' - u_r') u_r'' &= \frac{1}{2} (u_r'' - u_r') (u_r'' + u_r' + u_r'' - u_r') \\ &= \frac{1}{2} u_r''^2 - \frac{1}{2} u_r'^2 + \frac{1}{2} (u_r'' - u_r')^2; \end{aligned}$$

and therefore, from (5),

$$\frac{1}{2} \Sigma m_r (u_r''^2 + \dots) = \frac{1}{2} \Sigma m_r (u_r'^2 + \dots) - \frac{1}{2} \Sigma m_r \{(u_r'' - u_r')^2 + \dots\} \dots\dots (6),$$

proving that in any other motion compatible with the constraints the kinetic energy is less than in the constrained motion.

Now let (u_r'', v_r'', w_r'') denote the velocity of m_r in any other geometrically possible motion compatible with the constraints, then (4) is still true, and from (1) and (4) we get

$$\Sigma m_r \{(u_r' - u_r) (u_r' - u_r'') + \dots\} = 0 \dots\dots\dots (7)$$

and we have identically

$$(u_r' - u_r) (u_r' - u_r'') = \frac{1}{2} (u_r' - u_r)^2 - \frac{1}{2} (u_r'' - u_r)^2 + \frac{1}{2} (u_r' - u_r'')^2;$$

and therefore, from (7),

$$\begin{aligned} \frac{1}{2} \Sigma m_r \{(u_r' - u_r)^2 + \dots\} &= \frac{1}{2} \Sigma m_r \{(u_r'' - u_r)^2 + \dots\} \\ &\quad - \frac{1}{2} \Sigma m_r \{(u_r' - u_r'')^2 + \dots\} \dots\dots (8), \end{aligned}$$

showing that the kinetic energy of the relative motion in the latter case is greater than in the case of the constrained motion, the difference again being the energy of the difference of the two motions compared.*

8.4. External impulses applied to a moving system. *If any impulses are applied to a moving system, the change in the kinetic energy is equal to the sum of the products of the impulses and the mean of the velocities of their points of application before and after the action of the impulses resolved in the direction of the impulses.*

Using the notation of 8.3 and the equation of virtual work in the form 8.3 (1), it is clear that the actual motions of the system just before and just after the action of the impulses

* For this proof I am indebted to Professor H. Jeffreys.

both constitute displacements which do not violate the geometrical connections between the particles, so that in 8·3 (1) we may take

$$\delta x_r : \delta y_r : \delta z_r = u_r : v_r : w_r$$

and also

$$\delta x_r : \delta y_r : \delta z_r = u_r' : v_r' : w_r',$$

giving

$$\Sigma m_r \{ (u_r' - u_r) u_r + \dots \} = \Sigma (X_r u_r + \dots)$$

and

$$\Sigma m_r \{ (u_r' - u_r) u_r' + \dots \} = \Sigma (X_r u_r' + \dots).$$

Then, by adding these results we get

$$\Sigma m_r \{ (u_r'^2 - u_r^2) + \dots \} = \Sigma \{ X_r (u_r + u_r') + \dots \},$$

or

$$\begin{aligned} \frac{1}{2} \Sigma m_r (u_r'^2 + \dots) - \frac{1}{2} \Sigma m_r (u_r^2 + \dots) \\ = \Sigma \{ X_r \frac{1}{2} (u_r + u_r') + \dots \} \dots\dots(1), \end{aligned}$$

which proves the proposition.

8·41. Bertrand's Theorem. *If a system in motion is acted upon by any impulses, the kinetic energy of the subsequent motion is greater than if this system were subject to any additional constraints (of the kind which do no work in a small displacement) and acted upon by the same impulses.*

We may take (u_r, v_r, w_r) and (u_r', v_r', w_r') to denote the velocities of the particle m_r just before and just after the action of the impulses and (u_r'', v_r'', w_r'') to denote what the velocity would be if additional constraints were imposed.

Then 8·3 (1) is the equation of virtual work for the actual motion; but the constrained motion gives a possible displacement, so that

$$\Sigma m_r \{ (u_r' - u_r) u_r'' + \dots \} = \Sigma (X_r u_r'' + \dots) \dots\dots(1).$$

Applying the same principle to the constrained motion and noting that it is produced by the same set of impulses, we have

$$\Sigma m_r \{ (u_r'' - u_r) u_r'' + \dots \} = \Sigma (X_r u_r'' + \dots) \dots\dots(2).$$

Therefore

$$\Sigma m_r \{ (u_r'' - u_r) u_r'' + \dots \} = \Sigma m_r \{ (u_r' - u_r) u_r'' + \dots \}$$

or

$$\Sigma m_r (u_r''^2 + \dots) = \Sigma m_r (u_r' u_r'' + \dots) \dots\dots(3),$$

which is equivalent to

$$\begin{aligned} \frac{1}{2} \Sigma m_r (u_r'^2 + \dots) = \frac{1}{2} \Sigma m_r (u_r''^2 + \dots) + \frac{1}{2} \Sigma m_r \{ (u_r' - u_r'')^2 + \dots \} \\ \dots\dots(4) \end{aligned}$$

and proves the proposition.

8·42. Kelvin's Theorem. *If any points of a connected system are suddenly set in motion with prescribed velocities, the kinetic energy of the resulting motion is less than that of any other kinematically possible motion in which the given points have the prescribed velocities.*

For simplicity we will drop the suffixes and use Σ to denote a summation extending to all the particles of the system, and Σ' to denote a summation extending to those particles only whose motion is prescribed, X, Y, Z denoting the components of the external impulse applied to any one of these latter particles.

Let u, v, w denote the velocity components of a particle in the actual motion, including in this specification the prescribed velocities for the assigned points or particles. Let u', v', w' denote velocity components in any other kinematically possible motion with the proviso that for the assigned particles

$$u', v', w' = u, v, w.$$

For the actual motion we have

$$\Sigma m (u \delta x + v \delta y + w \delta z) = \Sigma' (X \delta x + Y \delta y + Z \delta z) \dots (1),$$

where $\delta x, \delta y, \delta z$ denote any geometrically possible virtual displacements. Since the actual motion is such a displacement, we may put

$$\delta x : \delta y : \delta z = u : v : w,$$

giving
$$\Sigma m (u^2 + v^2 + w^2) = \Sigma' (Xu + Yv + Zw) \dots (2).$$

But we may also take a displacement in the direction of the second or comparison motion and put

$$\delta x : \delta y : \delta z = u' : v' : w',$$

remembering that for the particles on which the external impulses act $u', v', w' = u, v, w$, so that

$$\Sigma m (uu' + vv' + ww') = \Sigma' (Xu + Yv + Zw) \dots (3).$$

From (2) and (3), it follows that

$$\begin{aligned} \Sigma m (u^2 + v^2 + w^2) &= \Sigma m (uu' + vv' + ww') \\ &= \frac{1}{2} \Sigma m (u^2 + \dots) + \frac{1}{2} \Sigma m (u'^2 + \dots) \\ &\quad - \frac{1}{2} \Sigma m \{(u - u')^2 + \dots\}. \end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{2}\Sigma m(u^2 + v^2 + w^2) &= \frac{1}{2}\Sigma m(u'^2 + v'^2 + w'^2) \\ &\quad - \frac{1}{2}\Sigma m\{(u - u')^2 + (v - v')^2 + (w - w')^2\} \dots (4),\end{aligned}$$

which proves the theorem.

8.43. Robin's Theorem. *If in the notation of 8.3 we put $\bar{u}_r = u_r - u'_r$, etc., so that $\frac{1}{2}\Sigma m_r(\bar{u}_r^2 + \bar{v}_r^2 + \bar{w}_r^2)$ may be called the kinetic energy of the relative motion, then if frictionless constraints are suddenly imposed on a moving system simultaneously with the application of impulses of the type (X_r, Y_r, Z_r) , the values of the velocities of type u'_r, v'_r, w'_r immediately afterwards will be such as to give a stationary value to the expression*

$$\frac{1}{2}\Sigma m_r(\bar{u}_r^2 + \bar{v}_r^2 + \bar{w}_r^2) + \Sigma (X_r \bar{u}_r + Y_r \bar{v}_r + Z_r \bar{w}_r).$$

To prove this theorem we have to shew that the actual velocities are such as to satisfy

$$\Sigma m_r(\bar{u}_r \delta \bar{u}_r + \dots) + \Sigma (X_r \delta \bar{u}_r + \dots) = 0,$$

$$\text{or} \quad \Sigma m_r(\bar{u}_r \delta u'_r + \dots) + \Sigma (X_r \delta u'_r + \dots) = 0 \quad \dots (1),$$

where $\delta u'_r, \delta v'_r, \delta w'_r$ denote any variations in the velocities compatible with the constraints.

The equation of virtual work is

$$-\Sigma m_r(\bar{u}_r \delta x_r + \dots) - \Sigma (X_r \delta x_r + \dots) = 0 \quad \dots (2),$$

where $\delta x_r, \delta y_r, \delta z_r$ represent the displacements of the particles m_r in any virtual displacement compatible with the constraints. But the actual motion represented by u'_r, v'_r, w'_r gives such a displacement, and we may take $u'_r + \delta u'_r, v'_r + \delta v'_r, w'_r + \delta w'_r$ to give another if $\delta u'_r, \delta v'_r, \delta w'_r$ have the meanings assigned above. If we substitute these expressions in turn for $\delta x_r, \delta y_r, \delta z_r$ in (2) and take the difference of the results, we are left with equation (1) and thus the theorem is proved.

In the more restricted case in which there are no applied impulses but merely the imposition of a constraint, it follows from this theorem that the new velocities must be such as to give a *stationary value* to the kinetic energy of the relative motion. But we have already shewn in 8.33 that the kinetic energy of the relative motion is in such a case a *minimum*.

8·5. Gauss's Principle of Least Constraint. Though the theorem which follows relates to motion in general and not simply to impulsive actions, yet as a proof of it is really contained in 8·33 this is a convenient place to introduce it. It is a theorem of a very general kind and constitutes one of the attempts which have been made from time to time to formulate a single law as a basis for the whole of mechanics, akin to the way in which, for example, Lagrange made the Principle of Virtual Work the foundation of his *Mécanique Analytique*.

In the foregoing theorems on impulsive action we have used the word *constraint* in a general way as something which puts a check upon motion. Later on we shall have occasion to regard a constraint explicitly as imposing a linear relation upon velocities. But in this theorem of Gauss's it has a special meaning. The *Principle of Least Constraint* may be stated in general terms thus: the actual motion of a system of connected particles is, subject to the connections, as nearly as possible what it would be if the particles were free: or, more explicitly thus: let a be the position of a typical particle of mass m at time t , b the position it would acquire at time $t + dt$ under the action of the external forces if the connections ceased at time t ; and c its position at time $t + dt$ in the actual motion. Then $\Sigma m(bc)^2$ is defined to be the *constraint* to which the system is subject at time t , and Gauss's Principle asserts that this constraint is less for the actual motion than for any other motion which is geometrically possible with the same connections.

To prove the truth of the principle we have to shew that if c' be the position of the particle m at time $t + dt$ in any other motion possible with the same connections, then $\Sigma m(bc)^2$ is less than $\Sigma m(bc')^2$. Now bc/dt is the relative velocity of the two motions, and the constraint, as defined above, is measured by

$$(dt)^2 \Sigma m \{(u - u')^2 + (v - v')^2 + (w - w')^2\},$$

where we are comparing a free motion (u, v, w) with a motion (u', v', w') of the same system when connected. And if we compare this with a varied motion (u'', v'', w'') compatible with the connections, so that while (u, v, w) corresponds to displacement

from a to b and (u', v', w') from a to c , (u'', v'', w'') corresponds to a displacement from a to c' , then this is precisely the comparison made in 8.33, where it was shewn that

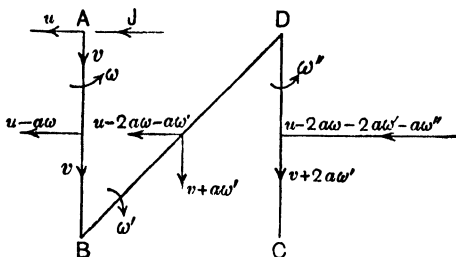
$$\Sigma m \{(u - u')^2 + \dots\} = \Sigma m \{(u - u'')^2 + \dots\} - \Sigma m \{(u' - u'')^2 + \dots\},$$

or
$$\Sigma m (bc)^2 = \Sigma m (bc')^2 - \Sigma m (cc')^2,$$

and thus the theorem is established.

8.6. Examples. (i) *Three uniform rods AB , BD , DC of equal mass, smoothly jointed at B and D , are at rest, forming two opposite sides and a diagonal of a square $ABCD$. A blow J is applied at A in the direction DA . Prove that the kinetic energy imparted is $185J^2/114m$, where m is the mass of a rod.* [M. T. 1902]

Let $2a$ be the length of a side of the square, u, v the velocities of A in the directions DA, AB , $\omega, \omega', \omega''$ the angular velocities of the rods in the senses indicated in the figure. Then the velocities of the middle points of the rods are easily expressible in terms of $u, v, \omega, \omega', \omega''$ as shewn in the figure.



By resolving parallel to DA and AB , we have

$$3u - 5a\omega - 3a\omega' - a\omega'' = J/m \quad \dots\dots\dots(1)$$

and

$$3v + 3a\omega' = 0 \quad \dots\dots\dots(2).$$

Constraints might be imposed by clamping either or both joints, or imposing any linear relation on the angular velocities, and by Bertrand's Theorem the actual motion has the greatest kinetic energy. It follows that the motion can be found by choosing the angular velocities so as to make T , the kinetic energy, a maximum.

Thus

$$\begin{aligned} \frac{2T}{m} = & (u - a\omega)^2 + v^2 + \frac{1}{3}a^2\omega^2 + (u - 2a\omega - a\omega')^2 + (v + a\omega')^2 + \frac{2}{3}a^2\omega'^2 \\ & + (u - 2a\omega - 2a\omega' - a\omega'')^2 + (v + 2a\omega')^2 + \frac{1}{3}a^2\omega''^2; \end{aligned}$$

and
$$\frac{1}{ma} \frac{\partial T}{\partial \omega} = -5u + \frac{2}{3}a\omega + 6a\omega' + 2a\omega'' = 0 \dots\dots\dots(3),$$

$$\frac{1}{ma} \frac{\partial T}{\partial \omega'} = -3u + 6a\omega + \frac{2}{3}a\omega' + 2a\omega'' = 0 \dots\dots\dots(4),$$

$$\frac{1}{ma} \frac{\partial T}{\partial \omega''} = -u + 2a\omega + 2a\omega' + \frac{1}{3}a\omega'' = 0 \dots\dots\dots(5);$$

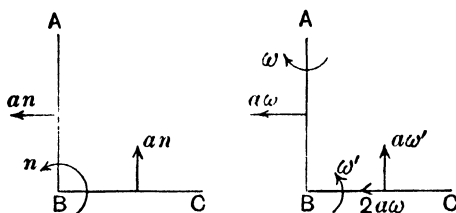
where we have simplified the expressions by using (2) after differentiation.

The solution of the equations (1), (3), (4), (5) is found to give

$$u = \frac{185}{57} \frac{J}{m},$$

so that the energy created $\frac{1}{2}Ju = 185J^2/114m$.

(ii) Two equal rods AB , BC smoothly hinged at B rotate round B with angular velocity n , the rods being at right angles. If A is suddenly fixed, shew that the rods begin to close up at a rate $9n/8$. [I. 1907]



This is a simple illustration of the principle of least constraint. If $2a$ denotes the length of a rod and m its mass and ω , ω' the angular velocities of AB , BC immediately after A is fixed, the velocities before and after the fixing are as indicated in the figures, so that the kinetic energy of the relative motion is given by

$$2T = ma^2(\omega - n)^2 + \frac{1}{3}ma^2(\omega + n)^2 + ma^2(\omega' - n)^2 + 4ma^2\omega^2 + \frac{1}{3}ma^2(\omega' - n)^2.$$

Then ω , ω' must be such as to make T a minimum, so that

$$0 = \frac{\partial T}{\partial \omega} = \omega - n + \frac{1}{3}(\omega + n) + 4\omega \quad \text{or} \quad \omega = \frac{1}{8}n,$$

and
$$0 = \frac{\partial T}{\partial \omega'} = \omega' - n + \frac{1}{3}(\omega' - n) \quad \text{or} \quad \omega' = n.$$

And the rate at which the angle ABC begins to close is $\omega + \omega'$ or $9n/8$, since ω , ω' are measured in opposite senses in the figure.

(iii) Four equal uniform rods AB , BC , CD , DA , each of mass M and length $2a$, are smoothly jointed together, and form a rhombus $ABCD$. A particle of mass M is in the line CA produced, and is connected to A by an inelastic inextensible string, which is originally slack. The rhombus

is moving, on a smooth horizontal table, on which lie also the string and the particle, with uniform velocity V in the direction AC . The angle BAC is equal to α . Prove that when the string tightens the loss of kinetic energy is

$$\frac{2MV^2}{5+3\sin^2\alpha}. \quad [\text{M. T. 1935}]$$

Let u be the velocity of the particle and of the point A when the string has tightened, and ω the angular velocity of the rods. If E, F are the middle points of AB, BC , since the distances of E, F from A measured along and at right angles to AC are

$$a \cos \alpha, a \sin \alpha$$

and $3a \cos \alpha, a \sin \alpha$,

therefore the velocities of E, F are

$$u - a\omega \sin \alpha, a\omega \cos \alpha, \quad \text{and} \quad u - 3a\omega \sin \alpha, a\omega \cos \alpha.$$

Hence the relative kinetic energy is given by

$$T = \frac{1}{2}Mu^2 + M \{ (V - u + a\omega \sin \alpha)^2 + a^2\omega^2 \cos^2 \alpha + \frac{1}{3}a^2\omega^2 \\ + (V - u + 3a\omega \sin \alpha)^2 + a^2\omega^2 \cos^2 \alpha + \frac{1}{3}a^2\omega^2 \}$$

or

$$T = \frac{1}{2}Mu^2 + 2M(V - u)^2 + 8M(V - u)a\omega \sin \alpha + 8Ma^2\omega^2 \sin^2 \alpha + \frac{8}{3}Ma^2\omega^2 \dots (1).$$

Using the theorem of 8·43 or 8·5, the values of u and ω must be such as to make T stationary. Hence

$$\frac{1}{M} \frac{\partial T}{\partial u} \equiv u - 4(V - u) - 8a\omega \sin \alpha = 0 \dots (2)$$

$$\text{and} \quad \frac{1}{Ma} \frac{\partial T}{\partial \omega} \equiv 8(V - u) \sin \alpha + 16a\omega \sin^2 \alpha + \frac{16}{3}a\omega = 0 \dots (3).$$

From (2) and (3) we get

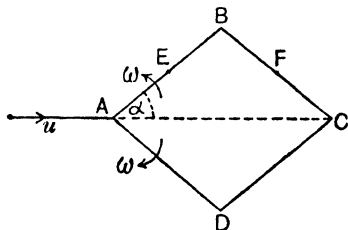
$$\frac{u}{8} = \frac{a\omega}{-3 \sin \alpha} = \frac{V}{2(5 + 3 \sin^2 \alpha)};$$

and substituting these values in (1) gives

$$T = 2MV^2 / (5 + 3 \sin^2 \alpha);$$

and by the theorem 8·33 this kinetic energy of the relative motion is the kinetic energy lost as the result of the tightening of the string.

The result may also be obtained by putting in the unknown impulsive reactions at B and D , taking moments about A for AB , and about C for BC ; resolving parallel to AC for BC, CD , and combining these equations with one which represents the constancy of momentum in the direction AC , i.e. equation (2) above.



8·61. Examples. 1. Two uniform rods, AB and BC , of equal masses and of lengths $4a$ and $2a$ respectively, lie in a straight line on a smooth horizontal table; BC is smoothly hinged at C to a vertical axis. Comparing the motions generated by a horizontal impulse applied at A , perpendicular to the rods,

(i) when they are smoothly jointed together at B ,

(ii) when they are rigidly fastened together there,

prove that the kinetic energy in the first case is decreased by $7/16$ of itself by the clamping of the joint at B if the same impulse is applied at A , and increased by $7/9$ of itself if the velocity communicated to A is the same in the two cases. [I. 1929]

2. Four equal uniform rods of mass m are freely jointed together to form a square. An impulse J is applied at the mid-point of one of the rods along a line which makes an angle α with the rod and lies in the plane of the square. Shew that the kinetic energy created is

$$\frac{J^2}{16m} (5 - 3 \sin^2 \alpha). \quad [\text{P. 1934}]$$

3. Five equal uniform rods AB, BC, CD, DE, EF are connected by smooth joints at B, C, D, E and are placed on a smooth table so as to form five sides of a regular hexagon. A blow is applied to the middle point of CD in a direction perpendicular to this rod. Prove that the angular velocities with which AB and BC begin to move are in the ratio $5:3$. [I. 1913]

4. Compare the kinetic energies imparted to an elliptic disc by a blow J in its plane through a focus at right angles to the major axis (i) when the disc is free, (ii) when the centre of the disc is fixed. [I. 1913]

5. Four like uniform rods, each of mass m , are smoothly jointed, forming a square. An impulse is applied at the end of one of the rods so that the end begins to move with velocity V in the direction of the rod. Shew that the kinetic energy of the motion is $4mV^2/5$. [I. 1906]

6. A corner of a uniform rectangular plate at rest is set in motion with velocity V perpendicular to the plane of the plate. Shew that the initial velocity of the centre of the plate is $V/7$ in the same direction. [I. 1924]

7. Three equal uniform rods AB, BC, CD freely jointed at B and C are placed so as to form three sides of a square $ABCD$. If A is suddenly jerked parallel to the direction CB , shew that the angular velocity of AB is initially seven times that of CD . [M. T. 1911]

EXAMPLES

1. AB , BC , CD , three equal uniform rods forming three sides of a square and smoothly jointed at the points B and C , lie on a smooth horizontal table. An impulse J is applied at A in the direction CA . Prove that the kinetic energy set up is $55J^2/56m$, where m is the mass of each rod. [I. 1909]

2. Two rods are smoothly jointed at A and B to a circular lamina, AB being a diameter of which the rods are continuations. The system lies at rest on a smooth horizontal table. Shew that a horizontal blow P applied normally to the rim of the lamina will produce in the latter no angular velocity, provided that

$$m_1 k_1^2 / (a_1^2 + k_1^2) = m_2 k_2^2 / (a_2^2 + k_2^2),$$

where m_1, m_2 are the masses of the rods, a_1, a_2 are the distances of their centres of gravity from A and B respectively, and k_1, k_2 are their radii of gyration about their centres of gravity. Determine the initial angular velocities of the rods. [I. 1911]

3. Two equal uniform rods AB, BC , smoothly jointed at B , and each of mass m , lie making an angle α with one another on a horizontal table, and pass at their middle points through smooth fixed rings. To the free end A is given a velocity v in a direction towards and perpendicular to BC . Prove that the kinetic energy of the motion is

$$\frac{1}{2}mv^2(2 - \cos \alpha \cos 3\alpha),$$

and that it is greater by $\frac{1}{24}mv^2 \cos^2 2\alpha$ than it would be if there were no ring on BC . [M. T. 1904]

4. A system of equal rods hinged end to end in a straight line lies at rest on a smooth horizontal plane and an impulse I is applied perpendicularly to the rods at one of the free ends of the system. Shew that as the number of rods given is increased, the energy of the motion set up tends to the limit $I^2\sqrt{3}$, the mass of a rod being the unit of mass. [M. T. 1909]

5. A uniform circular disc of mass m , initially at rest but free to move in any manner, is suddenly set in motion by an impulse applied to a point A of its circumference. The initial velocity of A has components U along the radius of the circle through A , V along the tangent to the circle at A and W along a third perpendicular; prove that T , the initial kinetic energy of the disc, is given by

$$2T = m(U^2 + \frac{1}{3}V^2 + \frac{1}{3}W^2). \quad [\text{M. T. 1934}]$$

6. A particle of mass m is attached by an inextensible string to the corner A of a uniform square $ABCD$ of mass M ; the system lies on a smooth horizontal plane, so that the string is straight and its direction

is along the side BA produced. The corner D is then started in motion with velocity V in a direction towards the centre. Find with what velocity the particle begins to move. [I. 1928]

7. A corner of a homogeneous cube at rest is given a velocity V perpendicular to and away from one of the faces meeting in that corner.

Shew that the initial velocity of the centre of the cube is $\frac{V}{11}\sqrt{43}$ and find its direction. [I. 1926]

8. A long light string hangs over three smooth pegs. Masses m_1 , m_2 , m_3 are attached to the ends, and smooth rings of masses m_2 , m_3 slide on the parts of the string hanging down between the pegs. The pegs are so placed that the strings are sensibly vertical throughout the motion. The system is instantaneously at rest when the mass m_1 is seized and moved downwards with velocity v . Prove that the mass m_4 begins to rise with velocity

$$\frac{v}{1 + 4 \frac{m_4(m_2 + m_3)}{m_2 m_3}}. \quad [\text{I. 1923}]$$

9. Two equal uniform rods AB , BC in the same line with a smooth hinge at B move on a smooth horizontal plane in a direction at right angles to the line of the rods. If the end A is suddenly stopped, shew that the rods begin to turn with angular velocities whose ratio is -3 .

[I. 1925]

10. A chain of three equal uniform rods AB , BC , CD , freely jointed at B and C , lies on a smooth table with ABC in line and CD perpendicular to ABC . The end A is struck a blow in a direction making an angle α with AB (aiming at a point in CD or CD produced). The end D begins to move at an angle β with CD produced. Prove that

$$\tan \alpha \cdot \tan \beta = 2. \quad [\text{I. 1923}]$$

11. Two uniform rods AB , BC , of masses $9m$ and $5m$ respectively, are freely jointed together at B , and rest in line. The system is set in motion by parallel impulses J , K , perpendicular to the line of the rods, applied at A , C respectively. Determine the velocities with which A , C start to move.

Uniform rods AB , BC , CD , ..., freely jointed together at B , C , D , ... to form an infinite chain, rest in line. The mass of each of the rods AB , CD , ... is $9m$, and the mass of each of the rods BC , DE , ... is $5m$. The system is set in motion by an impulse J , perpendicular to the line of the rods, applied at A . Shew that the kinetic energy communicated to the system is $J^2/5m$. [M. T. 1929]

12. A rectangular plate $ABCD$ of mass m has the corner A fixed. An impulse I is applied at the opposite corner C at right angles to the plate. Shew that the plate begins to rotate about a line parallel to BD and that the kinetic energy developed is $12I^2/7m$. Compare this with the value had A been free. [M. T. 1927]

13. A uniform circular disc is rotating about a diameter, when a point P of its edge is suddenly fixed. Prove that the velocity of the centre of the disc immediately afterwards is $\frac{1}{2}$ of the velocity of P before it was fixed. [I. 1925]

14. Four equal uniform rods are smoothly jointed in the form of a square $ABCD$ which is rotating with angular velocity ω about the perpendicular to the plane of the square through its centre. A point in one of the rods at a distance c from its middle point is suddenly fixed; shew that the angular velocity of that rod becomes $2a^2\omega/(3c^2 + 2a^2)$, and that of an adjacent rod becomes $\frac{2}{3}\omega$, immediately after the point is fixed, where $2a$ is the length of a rod. [M. T. 1926]

15. A uniform cube is spinning freely with angular velocity ω_0 about a diagonal through a corner O , when suddenly the diagonal OP of one of the faces through O becomes fixed. Shew that the new angular velocity is $2\sqrt{2}\omega_0/5\sqrt{3}$.

Shew that the same instantaneous change of motion could be brought about by fixing a single point in OP ; determine the point and the impulse which has to be applied to it. [M. T. 1932]

16. AB is a diameter of the base of a solid circular cylindrical column of radius a and height $2a$. An impulse is applied at A so that A has a given velocity V in a direction perpendicular to AB in the plane of the base; shew that the initial velocity of the centre of the column is $\frac{7}{13}V$. If the impulse is followed immediately by a second impulse applied at B such that B is given a velocity equal and opposite to that just given to A , shew that the velocity of the centre is changed to $-\frac{3}{1085}V$. [M. T. 1921]

17. Two equal uniform rods AB and BC , each of length $2a$, are smoothly jointed at B , and move in a vertical plane, AB being attached at A to a smooth fixed hinge. At an instant when the inclinations of AB and BC to the downward vertical are 30° and 60° respectively, and are increasing, AB hits a fixed inelastic peg. Find the new angular velocity of BC , in terms of the angular velocities Ω_1 and Ω_2 just before the impact, and prove that AB leaves the peg immediately if

$$\left(\Omega_2 + \frac{3\sqrt{3}}{4}\Omega_1\right)^2 < \frac{3}{8}\frac{g}{a}. \quad [\text{I. 1930}]$$

18. A uniform rod of length $2a$ and mass m , pivoted at its centre, is attached freely at one end to the circumference of a uniform circular disc of radius b and mass M and the whole system rotates with angular velocity ω as a rigid body, in a horizontal plane, the direction of the rod passing through the centre of the disc. If the free end of the rod be

suddenly fixed and the centre set free, shew that the resulting angular velocity ω' of the rod is given by

$$\frac{\omega'}{\omega} = \frac{1}{4} \frac{m + 2M}{m + M};$$

and find the angular velocity of the disc.

[I. 1927]

19. A rigid body of mass M is moving in any manner, h_1, h_2, h_3 are its moments of momentum about axes $Oxyz$ and u, v, w are the velocities of its centre of gravity G . Shew that its moment of momentum about a line of co-ordinates $(l, m, n, \lambda, \mu, \nu)$, reference system $Oxyz$, is

$$lh_1 + mh_2 + nh_3 + M(\lambda u + \mu v + \nu w).$$

A uniform ellipsoid of mass M and semi-axes a, b, c is rotating freely about the c -axis with spin ω and advancing along that axis with velocity v . The line of points

$$x = \frac{1}{2}a, \quad y/b = z/c$$

is suddenly fixed. What is the subsequent angular velocity of the body?

[M. T. 1924]

20. A uniaxial body spinning about its axis with angular velocity n and pivoted at its centre of gravity receives an impulse I along a line intersecting the axis at right angles at a distance c from the centre of gravity. Shew that, if subsequently the axis has an inclination θ to its initial position and an azimuth ϕ round it, $A^2\theta^2 = I^2c^2 - C^2n^2\tan^2\frac{1}{2}\theta$, where A, A, C are the principal moments of inertia at the centre of gravity.

Shew that $\tan\frac{1}{2}\theta \operatorname{cosec}\phi$ is constant during the motion, where ϕ is measured from the plane through the initial position of the axis and the line of action of the impulse.

[M. T. 1926]

21. A rhombus of mass M , formed of four equal rods jointed together, is moving in the direction of a diagonal with velocity u , and suddenly a particle of mass m becomes affixed to one end of the diagonal; prove that, if $2a$ be the length of each rod, the angular velocity ω suddenly acquired by each rod is such that

$$2a\omega\{M + m(1 + 3\sin^2\alpha)\} = 3mu\sin\alpha,$$

and that the kinetic energy lost is

$$\frac{1}{2}Mmu^2\{M + m(1 + 3\sin^2\alpha)\}.$$

[I. 1887]

22. A tetrahedron having its opposite edges equal to one another is turning with uniform angular velocity about one edge when suddenly the opposite edge becomes fixed. Shew that the angular velocity is reduced in the ratio

$$4c^2(a^2 \sim b^2) : 5c^2(a^2 + b^2) + a^2(b^2 + c^2) + b^2(c^2 + a^2),$$

where a, b, c are the shortest distances between pairs of opposite edges, c being that between the old and new axes of rotation.

23. The motion of a body A of mass M is constrained with regard to a body B by a smooth screw of pitch λ attached to it moving in a nut attached to B , while B is free to rotate about an axis coinciding with that of the screw. The relative motion is suddenly arrested, when A is moving with angular velocity ω and B is at rest, by the end of the screw impinging directly on a smooth inelastic plane forming part of the surface of B . Prove that the impulse on this surface is equal to $\{\lambda^{-1}AB(A+B)^{-1} + M\lambda\} \omega$, the moments of inertia of the two bodies about the axis of the screw being A and B .

24. An impulse of given magnitude R is applied at a given point P of a body at rest. Shew how to find the direction of the blow in order that the energy generated may be the greatest possible. Prove that the direction must be perpendicular to GP where G is the centre of inertia and that the maximum energy is $\frac{1}{2}R^2(K + 1/M)$, where K is the greater root of the equation

$$ABCK^2 - \{BC(y^2 + z^2) + CA(z^2 + x^2) + AB(x^2 + y^2)\}K + (x^2 + y^2 + z^2)(Ax^2 + By^2 + Cz^2) = 0,$$

where M is the mass of the body, A, B, C are the moments of inertia about the principal axes at G and x, y, z are the co-ordinates of P referred to these axes. [M. T. 1900]

25. A regular hexagon $ABCDEF$ composed of six equal rods, each of mass m and length $2a$, freely jointed at their extremities, lies on a smooth horizontal table, and an impulse I is applied at the point A in the direction DA . Shew that the initial angular velocity of the rods BC and EF is

$$I/ma\sqrt{3},$$

and find the initial angular velocities of the other pairs of rods.

[I. 1904]

Chapter IX

GENERALIZED CO-ORDINATES

9.1. When the position of a body or material system is determined by the values of a number of independent variables, it has the same number of degrees of freedom and these independent variables are called the **generalized co-ordinates** of the body or system.

Thus a free rigid body possesses six degrees of freedom and its position is determined by the three co-ordinates x, y, z of an assigned point of the body together with three angles, e.g. the Eulerian angles θ, ϕ, ψ of 3.7, which determine its orientation. These six variables might be taken as the generalized co-ordinates of the body. It is evident that generalized co-ordinates can be chosen in a great variety of ways.

The imposition of constraints reduces the number of degrees of freedom of a body and therefore also reduces the number of its independent generalized co-ordinates. Analytically a constraint may be imposed by requiring the otherwise independent generalized co-ordinates to satisfy an algebraical relation; one such relation for every constraint which is imposed. Thus in the case of the single rigid body, if a particular point of the body is constrained to lie on a given surface, this reduces by one its degrees of freedom, and imposes a relation, viz. the equation of the surface, which the co-ordinates of the point must satisfy. If a particular point of the body is required to lie on a given line, this reduced its degrees of freedom by two, and there are two relations to be satisfied, namely the equations of the line.

Though it is possible for a material system to possess an infinite number of degrees of freedom—e.g. a flexible string—we shall confine our considerations in what follows to systems with a finite number of degrees of freedom, and assume that the rectangular co-ordinates x, y, z of a typical particle of mass

m of the system are expressible in terms of a finite number n of generalized co-ordinates, which we shall denote by $q_1, q_2, \dots q_n$, and we shall not exclude the possibility that the geometrical relations may contain the time explicitly, i.e. that

$$\left. \begin{aligned} x &= f(q_1, q_2, \dots q_n, t) \\ y &= g(q_1, q_2, \dots q_n, t) \\ z &= h(q_1, q_2, \dots q_n, t) \end{aligned} \right\} \dots\dots\dots (1).$$

We shall see, in what follows, that much simplification results when these geometrical relations are independent of the time.

9.11. Holonomic and non-holonomic systems. When the geometrical relations of a dynamical system are such that a displacement represented by arbitrary infinitesimal changes in *all* its independent co-ordinates is a possible displacement the system is said to be **holonomic**; otherwise the system is **non-holonomic**.

This distinction is of great importance, for it will be seen in what follows that one of the most important forms of the equations of motion viz., Lagrange's equations, can only be applied (without modification) to holonomic systems.

A special case will make the distinction clearer. Consider the motion of a sphere of radius a which is to remain in contact with a fixed plane. Take fixed rectangular axes of which Ox, Oy are in the plane so that the co-ordinates of the centre of the sphere are x, y, a . Take a set of rectangular axes fixed in the sphere and define their position relative to axes through the centre parallel to the fixed axes by the Eulerian angles θ, ϕ, ψ , so that there are five variable co-ordinates x, y, θ, ϕ, ψ . If the plane is smooth it is possible to make a displacement of the sphere to correspond to simultaneous arbitrary changes $\delta x, \delta y, \delta \theta, \delta \phi, \delta \psi$ in the co-ordinates and the system is holonomic.

But if the plane is so rough that sliding is not possible, there are relations between the velocities expressing this fact. Thus if the angular velocity of the sphere has components $\omega_x, \omega_y, \omega_z$ about parallels to the fixed axes, the conditions for rolling are

$$\dot{x} - a\omega_y = 0 \quad \text{and} \quad \dot{y} + a\omega_x = 0,$$

$$\begin{aligned} \text{or, from 3.71 (1),} \quad & \dot{x} - a\dot{\theta} \cos \psi - a\dot{\phi} \sin \theta \sin \psi = 0 \\ \text{and} \quad & \dot{y} - a\dot{\theta} \sin \psi + a\dot{\phi} \sin \theta \cos \psi = 0 \end{aligned} \left\} \dots\dots\dots (1).$$

This means that a displacement of the sphere in which $\delta x, \delta y, \delta \theta, \delta \phi, \delta \psi$

are *arbitrary* is not possible, because by (1) these infinitesimal changes must conform to the relations

$$\left. \begin{aligned} \delta x - a \delta \theta \cdot \cos \psi - a \delta \phi \cdot \sin \theta \sin \psi &= 0 \\ \delta y - a \delta \theta \cdot \sin \psi + a \delta \phi \cdot \sin \theta \cos \psi &= 0 \end{aligned} \right\} \dots\dots\dots(2).$$

These are non-integrable relations. They effectively reduce the freedom of the sphere to three degrees, for they express possible values of δx , δy in terms of $\delta \theta$, $\delta \phi$ while $\delta \theta$, $\delta \phi$, $\delta \psi$ remain arbitrary. But because they are non-integrable they do not reduce the number of independent co-ordinates which are necessary to specify the position of the sphere. Because of the existence of relations of this kind between small increments in the co-ordinates the system is said to be non-holonomic.

9·2. Kinetic energy. Consider a dynamical system in which the co-ordinates of all particles are expressible in terms of n independent generalized co-ordinates by relations of the form 9·1 (1). The velocity components of a particle of mass m are given by

$$\dot{x} = \frac{\partial x}{\partial q_1} \dot{q}_1 + \frac{\partial x}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial x}{\partial q_n} \dot{q}_n + \frac{\partial x}{\partial t} \dots\dots\dots(1)$$

and similar expressions for $\dot{y}$ and $\dot{z}$.

Therefore the kinetic energy T of the system is given by

$$\begin{aligned} 2T &= \Sigma m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= a_{11} \dot{q}_1^2 + 2a_{12} \dot{q}_1 \dot{q}_2 + a_{22} \dot{q}_2^2 + \dots + a_{nn} \dot{q}_n^2 \\ &\quad + 2a_1 \dot{q}_1 + 2a_2 \dot{q}_2 + \dots + 2a_n \dot{q}_n + a \dots\dots\dots(2), \end{aligned}$$

where

$$\left. \begin{aligned} a_{rr} &= \Sigma m \left\{ \left(\frac{\partial x}{\partial q_r} \right)^2 + \left(\frac{\partial y}{\partial q_r} \right)^2 + \left(\frac{\partial z}{\partial q_r} \right)^2 \right\} \\ a_{rs} &= \Sigma m \left\{ \frac{\partial x}{\partial q_r} \frac{\partial x}{\partial q_s} + \frac{\partial y}{\partial q_r} \frac{\partial y}{\partial q_s} + \frac{\partial z}{\partial q_r} \frac{\partial z}{\partial q_s} \right\} \\ a_r &= \Sigma m \left\{ \frac{\partial x}{\partial q_r} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial q_r} \frac{\partial y}{\partial t} + \frac{\partial z}{\partial q_r} \frac{\partial z}{\partial t} \right\} \\ \text{and} \quad a &= \Sigma m \left\{ \left(\frac{\partial x}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial t} \right)^2 + \left(\frac{\partial z}{\partial t} \right)^2 \right\} \end{aligned} \right\} \dots\dots\dots(3),$$

and the summations extend to all the particles of the system.

It appears that the kinetic energy is a quadratic function of the generalized velocity components, in which the coefficients—the a 's—are functions of the generalized co-ordinates and

impulse acting on the particle m , and the system is subject to simultaneous impulses of this type, we may call

$$\Sigma (X'\delta x + Y'\delta y + Z'\delta z)$$

the virtual moment of the impulses, and by substituting from (1) this takes the form

$$Q_1'\delta q_1 + Q_2'\delta q_2 + \dots + Q_n'\delta q_n,$$

$$\text{where } Q_r' = \Sigma \left(X' \frac{\partial x}{\partial q_r} + Y' \frac{\partial y}{\partial q_r} + Z' \frac{\partial z}{\partial q_r} \right), \quad r = 1, 2, \dots, n$$

.....(5)

and $Q_1', Q_2', \dots, Q_n'$ may be called the generalized components of impulse.

If this system of impulses is one which would at time t create the given state of motion in the system, it must be the exact equivalent of the momentum. We have in fact for all permissible displacements

$$\Sigma m (\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z) = \Sigma (X'\delta x + Y'\delta y + Z'\delta z),$$

so that

$$p_1\delta q_1 + p_2\delta q_2 + \dots + p_n\delta q_n = Q_1'\delta q_1 + Q_2'\delta q_2 + \dots + Q_n'\delta q_n,$$

and since the q 's are independent

$$p_1, p_2, \dots, p_n = Q_1', Q_2', \dots, Q_n';$$

i.e. in generalized co-ordinates components of momentum are identical with the components of impulse which would produce the motion instantaneously from rest.

9·3. Generalized components of the effective and the applied forces. Lagrange's equations. For any displacements compatible with the geometrical conditions the virtual work of the effective forces and of the impressed forces must be the same, so that we have an equation

$$\Sigma m (\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = \Sigma (X\delta x + Y\delta y + Z\delta z) \dots (1),$$

where X, Y, Z are components of an impressed force.

If we substitute for $\delta x, \delta y, \delta z$ their values in terms of generalized co-ordinates as given by 9·21 (1), the left-hand side

takes a form linear in $\delta q_1, \delta q_2, \dots \delta q_n$ in which the coefficient of δq_r is

$$\Sigma m \left(\ddot{x} \frac{\partial x}{\partial q_r} + \ddot{y} \frac{\partial y}{\partial q_r} + \ddot{z} \frac{\partial z}{\partial q_r} \right) \dots\dots\dots(2),$$

and this may be regarded as a **generalized component of effective force**. It may be transformed thus: we have from 9·21

$$p_r = \Sigma m \left(\dot{x} \frac{\partial \dot{x}}{\partial \dot{q}_r} + \dot{y} \frac{\partial \dot{y}}{\partial \dot{q}_r} + \dot{z} \frac{\partial \dot{z}}{\partial \dot{q}_r} \right) = \Sigma m \left(\dot{x} \frac{\partial x}{\partial q_r} + \dot{y} \frac{\partial y}{\partial q_r} + \dot{z} \frac{\partial z}{\partial q_r} \right),$$

so that
$$\frac{dp_r}{dt} = \Sigma m \left\{ \ddot{x} \frac{\partial x}{\partial q_r} + \dot{x} \frac{d}{dt} \frac{\partial x}{\partial q_r} + \dots \right\}.$$

But
$$\begin{aligned} \frac{d}{dt} \frac{\partial x}{\partial q_r} &= \frac{\partial^2 x}{\partial q_1 \partial q_r} \dot{q}_1 + \frac{\partial^2 x}{\partial q_2 \partial q_r} \dot{q}_2 + \dots + \frac{\partial^2 x}{\partial q_n \partial q_r} \dot{q}_n + \frac{\partial^2 x}{\partial t \partial q_r} \\ &= \frac{\partial}{\partial q_r} \left(\frac{\partial x}{\partial q_1} \dot{q}_1 + \frac{\partial x}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial x}{\partial q_n} \dot{q}_n + \frac{\partial x}{\partial t} \right) \\ &= \frac{\partial \dot{x}}{\partial q_r}, \text{ from 9·2 (1).} \end{aligned}$$

Hence

$$\frac{dp_r}{dt} = \Sigma m \left\{ \ddot{x} \frac{\partial x}{\partial q_r} + \ddot{y} \frac{\partial y}{\partial q_r} + \ddot{z} \frac{\partial z}{\partial q_r} + \dot{x} \frac{\partial \dot{x}}{\partial q_r} + \dot{y} \frac{\partial \dot{y}}{\partial q_r} + \dot{z} \frac{\partial \dot{z}}{\partial q_r} \right\};$$

and therefore the generalized component of effective force (2) is equal to

$$\frac{dp_r}{dt} - \frac{\partial}{\partial q_r} \frac{1}{2} \Sigma m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2),$$

or, since $p_r = \partial T / \partial \dot{q}_r$, it is

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} \dots\dots\dots(3).$$

Again, the same substitution for $\delta x, \delta y, \delta z$ in the right-hand side of (1) transforms it to

$$Q_1 \delta q_1 + Q_2 \delta q_2 + \dots + Q_n \delta q_n,$$

where $Q_1, Q_2, \dots Q_n$ are called **generalized components of force**. Hence equation (1) is transformed to

$$\left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} - Q_1 \right\} \delta q_1 + \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} - Q_2 \right\} \delta q_2 + \dots = 0$$

.....(4);

and, provided the co-ordinates are independent, the coefficients of each of the δq 's must vanish separately, so that there are as many equations as there are independent co-ordinates, of the form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r, \quad (r = 1, 2, \dots n) \quad \dots\dots\dots(5)$$

and these are known as **Lagrange's equations**.

If the external forces form a conservative system and have a potential function $-V$, we have

$$-\delta V = Q_1 \delta q_1 + Q_2 \delta q_2 + \dots + Q_n \delta q_n \quad \dots\dots\dots(6),$$

so that $Q_r = -\partial V / \partial q_r$, and the Lagrange equations may be written

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = -\frac{\partial V}{\partial q_r} \quad (r = 1, 2, \dots n) \quad \dots\dots\dots(7);$$

or if we put $L = T - V$, L is called the **Lagrangian function**, and since V does not contain the velocities the equations become

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = 0 \quad (r = 1, 2, \dots n) \quad \dots\dots\dots(8).$$

Before proceeding further with the general theory, we shall illustrate the use of generalized co-ordinates by some simple examples, in which the special utility of Lagrange's equations will become apparent, viz. that it enables us to obtain a solution of a problem without reference to internal reactions.

It should be noted that the Q 's, which are conveniently called generalized components of force, are not necessarily of the dimensions of force. When δq_r is a length then Q_r is of the dimensions of force, but if δq_r is an angle then Q_r is of the dimensions of couple. Similar considerations apply to the Q 's of 9.21.

9.31. Examples. (i) *To determine the accelerations of masses hung by light strings passing over smooth pulleys as in the figure.*

We may take as generalized co-ordinates the depth q_1 of the mass m_1 below the centre of the pulley A , and the depth q_2 of the mass m_2 below the centre of the pulley B ; for if x_1, x_2, x_3 denote the depths of the three masses below the centre of A , we have

$$x_1 = q_1, \quad x_2 = l_1 - q_1 + q_2, \quad x_3 = l_1 + l_2 - q_1 - q_2 \quad \dots\dots(1),$$

where l_1, l_2 denote the lengths of the vertical parts of the two strings. Hence, neglecting the inertia of the pulleys, we have

$$2T = m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + m_3 \dot{x}_3^2 \\ = (m_1 + m_2 + m_3) \dot{q}_1^2 - 2(m_2 - m_3) \dot{q}_1 \dot{q}_2 + (m_2 + m_3) \dot{q}_2^2,$$

and

$$-V = (m_1 x_1 + m_2 x_2 + m_3 x_3) g + \text{const.} \\ = (m_1 - m_2 - m_3) g q_1 + (m_2 - m_3) g q_2 + \text{const.}$$

The Lagrange equations are

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} = - \frac{\partial V}{\partial q_1}$$

and

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} = - \frac{\partial V}{\partial q_2},$$

$$\text{or} \quad (m_1 + m_2 + m_3) \ddot{q}_1 - (m_2 - m_3) \ddot{q}_2 = (m_1 - m_2 - m_3) g \quad \dots\dots(2)$$

$$\text{as} \quad -(m_2 - m_3) \ddot{q}_1 + (m_2 + m_3) \ddot{q}_2 = (m_2 - m_3) g \quad \dots\dots\dots(3).$$

The reader will have no difficulty in verifying these results by elementary dynamics.

Further, we shall see presently that, when the geometrical equations do not contain the time, Lagrange's equations furnish a proof of the principle of conservation of energy, and that is the case in this example. The geometrical equations (1) do not contain the time, and if we multiply (2) by $\dot{q}_1$ and (3) by $\dot{q}_2$ and add and integrate, we get

$$\frac{1}{2} (m_1 + m_2 + m_3) \dot{q}_1^2 - (m_2 - m_3) \dot{q}_1 \dot{q}_2 + \frac{1}{2} (m_2 + m_3) \dot{q}_2^2 \\ = (m_1 - m_2 - m_3) g q_1 + (m_2 - m_3) g q_2 + \text{const.}$$

$$\text{or} \quad T + V = \text{const.}$$

(ii) *Equations of motion of a particle in polar co-ordinates.*

Using the notation of 3.5, the kinetic energy of a particle of mass m whose co-ordinates are r, θ, ϕ is given by

$$2T = m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta) \quad \dots\dots\dots(1),$$

and if mF, mG, mH denote the components of force in the direction of the elements $\delta r, r \delta \theta, r \sin \theta \delta \phi$, the potential energy is given by

$$-\delta V = m (F \delta r + Gr \delta \theta + Hr \sin \theta \delta \phi) \quad \dots\dots\dots(2).$$

Hence we obtain from Lagrange's equations

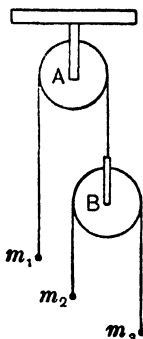
$$\ddot{r} - r \dot{\theta}^2 - r \dot{\phi}^2 \sin^2 \theta = F,$$

$$r \ddot{\theta} + 2 \dot{r} \dot{\theta} - r \dot{\phi}^2 \sin \theta \cos \theta = G$$

and

$$r \ddot{\phi} \sin \theta + 2 \dot{r} \dot{\phi} \sin \theta + 2 r \dot{\theta} \dot{\phi} \cos \theta = H.$$

The left-hand sides of these equations are the expressions obtained for the accelerations in 3.5 (1).



(iii) *Motion of a uniform heavy rod turning about one end which is fixed.*

This problem was solved in 6·41 by the principles of conservation of energy and moment of momentum. With the notation of 6·41 we have

$$2T = \frac{1}{3}Ma^2(\dot{\theta}^2 + \phi^2 \sin^2 \theta)$$

and

$$-V = Mga \cos \theta + \text{const.}$$

Lagrange's equations give

$$\frac{1}{3}a(\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta) = -g \sin \theta \quad \dots\dots\dots(1)$$

and

$$\frac{d}{dt}(\dot{\phi} \sin^2 \theta) = 0 \quad \dots\dots\dots(2).$$

The latter equation gives $\dot{\phi} \sin^2 \theta = \text{const.}$, which is the equation of moment of momentum 6·41 (1); the elimination of $\dot{\phi}$ from (1) then gives a differential equation for θ .

We notice that the ϕ equation, (2), takes a simple form integrable at once, because neither T nor V contains ϕ explicitly, and we shall refer to this later.

We shall consider next an example in which the time t enters explicitly into the geometrical equations.

(iv) *Motion of an insect which crawls at a uniform rate on a uniform heavy rod which is turning about a fixed end.*

With the same notation let m be the mass of the insect and u the speed at which it crawls, and suppose that it is at the middle point of the rod when $t=0$ and crawls downwards. Then its co-ordinates x, y, z at time t are given by

$$x = (a + ut) \sin \theta \cos \phi,$$

$$y = (a + ut) \sin \theta \sin \phi,$$

$$z = (a + ut) \cos \theta,$$

and the square of its velocity is

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = u^2 + (a + ut)^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta).$$

In this case therefore

$$2T = \frac{1}{3}Ma^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + m\{u^2 + (a + ut)^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta)\},$$

$$-V = Mga \cos \theta + mg(a + ut) \cos \theta + \text{const.}$$

There are then two Lagrange equations

$$\frac{d}{dt}[\frac{1}{3}Ma^2\dot{\theta} + m(a + ut)^2\dot{\theta}] - [\frac{1}{3}Ma^2 + m(a + ut)^2]\dot{\phi}^2 \sin \theta \cos \theta$$

$$= -\{Ma + m(a + ut)\}g \sin \theta \quad \dots\dots(1)$$

and

$$\frac{d}{dt}[\{\frac{1}{3}Ma^2 + m(a + ut)^2\}\dot{\phi} \sin^2 \theta] = 0 \quad \dots\dots\dots(2).$$

We observe that (2) integrates at once, giving an equation which represents the constancy of the moment of momentum about the vertical; also that we can then eliminate $\dot{\phi}$ from (1) and obtain a differential equation for θ .

Further, in Example (iii) it is quite easy to deduce the equation of energy from the equations of motion (1) and (2), but in this example there is no equation of energy; analytically, because the geometrical equations contain the time explicitly, and physically, because the insect is doing work in controlling its motion relative to the rod.

(v) *Motion of a uniform heavy rod inside a smooth sphere.*

For simplicity we will take the case in which the rod is of length $\sqrt{3}a$, where a is the radius of the sphere. Let G be the middle point of the rod PQ , only half of which is shown in the figure. Take moving axes OA parallel to the rod, OB through G and a third axis OC , and let the positions of these axes be defined by the Eulerian angles θ, ϕ, ψ as in 3.7.

The principal moments of inertia of the rod at G are

$$A, B, C = 0, \frac{1}{4}ma^2, \frac{1}{4}ma^2 \dots (1).$$

The co-ordinates of G are $(0, \frac{1}{2}a, 0)$, and if $\omega_1, \omega_2, \omega_3$ are the angular velocities of the rod (and the axes), the velocities of G are

$$u, v, w = -\frac{1}{2}a\omega_3, 0, \frac{1}{2}a\omega_1 \dots (2).$$

Therefore
$$2T = m(u^2 + v^2 + w^2) + A\omega_1^2 + B\omega_2^2 + C\omega_3^2$$

$$= \frac{1}{4}ma^2\{\dot{\theta}^2 + \dot{\psi}^2 \sin^2 \theta + 2(\dot{\phi} + \dot{\psi} \cos \theta)^2\} \dots (3),$$

using the expressions in 3.7 for $\omega_1, \omega_2, \omega_3$.

Again, $\cos ZB = \cos CZ \cos CB + \sin CZ \sin CB \cos ZCB$
 $= \sin \theta \sin \phi,$

therefore
$$V = \frac{1}{2}mga \sin \theta \sin \phi + \text{const.} \dots (4).$$

Then the Lagrange equations are

$$\ddot{\theta} - \dot{\psi}^2 \sin \theta \cos \theta + 2(\dot{\phi} + \dot{\psi} \cos \theta) \dot{\psi} \sin \theta = -\frac{2g}{a} \cos \theta \sin \phi \dots (5),$$

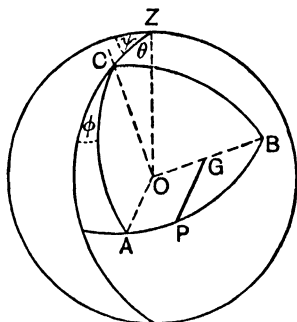
$$2 \frac{d}{dt} (\dot{\phi} + \dot{\psi} \cos \theta) = -\frac{2g}{a} \sin \theta \cos \phi \dots (6),$$

and
$$\frac{d}{dt} \{\dot{\psi} \sin^2 \theta + 2(\dot{\phi} + \dot{\psi} \cos \theta) \cos \theta\} = 0 \dots (7).$$

Noting that $\dot{\phi} + \dot{\psi} \cos \theta = \omega_3$, equation (7) gives

$$\dot{\psi} \sin^2 \theta + 2\omega_3 \cos \theta = D \dots (8).$$

There is of course an equation of energy $T + V = \text{const.}$ and it is easy to deduce it by multiplying (5) and (6) by $2\dot{\theta}$ and $2\dot{\phi}$ and adding, making use of (8) and integrating.



For a steady motion to be possible, in which $\psi = \Omega$, $\theta = \alpha$, $\phi = \beta$, we find from (5) that the necessary condition is

$$\Omega^2 \sin \alpha \cos \alpha = -\frac{2g}{a} \cos \alpha \sin \beta \dots\dots\dots(9).$$

The solution $\alpha = \frac{1}{2}\pi$ implies that the plane AOB through the rod and the centre of the sphere is vertical throughout the motion. The alternative is $\Omega^2 \sin \alpha = -\frac{2g}{a} \sin \beta$.

(vi) *To deduce Euler's equations from Lagrange's equations.*

In the case of a body turning about a fixed point, we have

$$2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \dots\dots\dots(1),$$

where, from 3.7,

$$\omega_1 = \dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi,$$

$$\omega_2 = \dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi,$$

$$\omega_3 = \dot{\phi} + \dot{\psi} \cos \theta.$$

Taking θ , ϕ , ψ as the generalized co-ordinates, we have

$$\frac{\partial T}{\partial \dot{\phi}} = C\omega_3 \frac{\partial \omega_3}{\partial \dot{\phi}} = C\omega_3,$$

and
$$\frac{\partial T}{\partial \dot{\phi}} = A\omega_1 \frac{\partial \omega_1}{\partial \dot{\phi}} + B\omega_2 \frac{\partial \omega_2}{\partial \dot{\phi}}$$

$$= A\omega_1 (\dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi) - B\omega_2 (\dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi)$$

$$= (A - B) \omega_1 \omega_2;$$

so that the ϕ -Lagrange equation is

$$C\dot{\omega}_3 - (A - B) \omega_1 \omega_2 = -\partial V / \partial \phi = N,$$

where N is the moment of the external forces about the principal axis $O3$. This is the third of Euler's equations, and the other two follow from the fact that the axes $O1$, $O2$, $O3$ may be cyclically interchanged.

9.32. Examples. 1. A light string passes over a fixed pulley and to each end is attached another pulley, while over these two moveable pulleys there also pass light strings supporting given masses. Employ Lagrange's equations to determine the constant angular acceleration of the upper pulley, taking into account the inertia of the pulleys but assuming that there is no slipping between any string and the pulley over which it passes. [M. T. 1901]

2. Apply Lagrange's equation to the case of a given sphere of mass m rolling down the rough surface of a given wedge of mass M which can move on a smooth horizontal plane; everything being symmetrical about a vertical plane.

3. One end A of a uniform rod AB can slide along a smooth vertical wire Oz and the other end B is attached to O by a string whose length is the same as that of the rod. The co-ordinates of the middle point of OB

are x, y, z referred to rectangular axes Ox, Oy, Oz , the latter being drawn vertically downwards. Prove that the equations of motion can be put in the form

$$\frac{\ddot{x}}{x} = \frac{\ddot{y}}{y} = \left(7\ddot{z} - \frac{9g}{4}\right) / z.$$

Prove further that a state of steady motion is possible in which B describes a horizontal circle at a depth $2c$ below O with angular velocity $\frac{3}{2}\sqrt{g/c}$. [I. 1913]

4. A particle moves on a smooth horizontal circular wire, of radius a , which is free to rotate about a vertical axis through a point O , distant c from the centre C . If θ is the angle between CO and the radius to the particle, shew that

$$a\ddot{\theta} + \dot{\omega}(a - c \cos \theta) = c\omega^2 \sin \theta,$$

where ω is the angular velocity of the wire. [I. 1928]

5. A rod AB of length $2l$ is attached to an equal rod $A'B'$ by strings AA', BB' each of length $2l$. The rod $A'B'$ is constrained to move about its centre O in a horizontal plane with uniform angular velocity ω , and the centre of AB is constrained to remain vertically beneath O . Shew that the equation giving the motion of AB relative to $A'B'$ is independent of the uniform rotation, and that the angle θ between AB and the vertical plane through $A'B'$ is given by the equation

$$\left(k^2 + l^2 \sin^2 \frac{\theta}{2}\right) \ddot{\theta} - 4gl \cos \frac{\theta}{2} = \text{constant},$$

k being the radius of gyration of AB about its centre. [I. 1923]

6. A solid circular cylinder of radius a rests on a horizontal plane, and a plank of length λa rests across the cylinder with one end resting on the plane, the axis of the cylinder being perpendicular to the long edges of the plank. The surfaces in contact are smooth except that between the cylinder and the plane there is sufficient friction to prevent slipping. Shew that, when the end of the plank comes into contact with the cylinder, the ratio of the angular velocity of the former to that of the latter is $(3n+2)(\lambda^2+1)/(\lambda^4+1)$, where the mass of the cylinder is n times that of the plank. [M. T. 1921]

7. A long flat ruler, of mass m_1 , lies on a horizontal table and is held fixed, its length being perpendicular to the edge of the table. An inextensible string, fastened to the ruler, passes over the edge, and supports a hanging mass m_2 , both parts of the string being in a plane p through the centre of mass of the ruler and perpendicular to the edge of the table. A uniform sphere of mass M is placed on the ruler, and is started rolling along it, so that its centre moves with velocity v_0 away from the edge, in the plane p . Assuming that the friction between the sphere and the ruler is sufficient to prevent slipping, and neglecting friction elsewhere, prove that if the ruler is released the mass m_2

descends with constant acceleration, and that the centre of the sphere comes to rest when m_2 has fallen through a distance

$$7v_0^2(2M + 7m_1 + 7m_2)/8m_2g. \quad [\text{I. 1929}]$$

9·4. To deduce the equation of energy from Lagrange's equations. In this and the following articles, until the contrary is expressly stated, we shall assume that *the geometrical equations do not contain the time explicitly*. It follows from 9·2 that the kinetic energy is a homogeneous quadratic function of the velocities, viz.

$$2T = a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2 + \dots + a_{nn}\dot{q}_n^2 \dots (1).$$

If we multiply the Lagrange equations 9·3 (7) by $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ and add, we get

$$\sum_{r=1}^n \left\{ \dot{q}_r \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \dot{q}_r \frac{\partial T}{\partial q_r} \right\} = - \sum_{r=1}^n \dot{q}_r \frac{\partial V}{\partial q_r},$$

or
$$\sum_{r=1}^n \left\{ \frac{d}{dt} \left(\dot{q}_r \frac{\partial T}{\partial \dot{q}_r} \right) - \left(\ddot{q}_r \frac{\partial T}{\partial \dot{q}_r} + \dot{q}_r \frac{\partial T}{\partial q_r} \right) \right\} = - \frac{dV}{dt} \dots (2).$$

But since T is a homogeneous quadratic function of the velocities,

$$\dot{q}_1 \frac{\partial T}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial T}{\partial \dot{q}_2} + \dots + \dot{q}_n \frac{\partial T}{\partial \dot{q}_n} = 2T \dots (3);$$

hence (2) becomes
$$2 \frac{dT}{dt} - \frac{dT}{dt} = - \frac{dV}{dt},$$

so that
$$T + V = \text{const.}$$

9·41. Reciprocal relations. Referring to 9·21, when the geometrical equations do not contain the time, the components of momentum are

$$\left. \begin{aligned} p_1 &= a_{11}\dot{q}_1 + a_{12}\dot{q}_2 + \dots + a_{1n}\dot{q}_n \\ p_2 &= a_{21}\dot{q}_1 + a_{22}\dot{q}_2 + \dots + a_{2n}\dot{q}_n \\ &\dots\dots\dots \\ p_n &= a_{n1}\dot{q}_1 + a_{n2}\dot{q}_2 + \dots + a_{nn}\dot{q}_n \end{aligned} \right\} \dots (1),$$

where $a_{rs} = a_{sr}$;

or
$$p_1, p_2, \dots, p_n = \frac{\partial T}{\partial \dot{q}_1}, \frac{\partial T}{\partial \dot{q}_2}, \dots, \frac{\partial T}{\partial \dot{q}_n} \dots (2).$$

Then from (7)

$$\sum_{r=1}^n \left(\frac{\partial T}{\partial \dot{q}_r} \delta \dot{q}_r + \frac{\partial T}{\partial q_r} \delta q_r \right) + \sum_{r=1}^n \left(\frac{\partial T_p}{\partial p_r} \delta p_r + \frac{\partial T_p}{\partial q_r} \delta q_r \right) \\ = \sum_{r=1}^n (p_r \delta \dot{q}_r + \dot{q}_r \delta p_r) \dots\dots(8).$$

But by (2) and (6) equation (8) reduces to

$$\sum_{r=1}^n \left(\frac{\partial T}{\partial q_r} + \frac{\partial T_p}{\partial q_r} \right) \delta q_r = 0 \dots\dots\dots(9),$$

and it is open to us to assign arbitrary values to the variations δq_r so that

$$\frac{\partial T}{\partial q_r} + \frac{\partial T_p}{\partial q_r} = 0 \dots\dots\dots(10).$$

9.42. Now let $\dot{q}_1', \dot{q}_2', \dots \dot{q}_n', p_1', p_2', \dots p_n'$ refer to any other possible state of motion of the system in the configuration specified by $q_1, q_2, \dots q_n$; so that

$$\left. \begin{aligned} p_1' &= a_{11}\dot{q}_1' + a_{12}\dot{q}_2' + \dots + a_{1n}\dot{q}_n' \\ p_2' &= a_{21}\dot{q}_1' + a_{22}\dot{q}_2' + \dots + a_{2n}\dot{q}_n' \\ &\text{etc.} \end{aligned} \right\} \dots\dots\dots(1),$$

where $a_{rs} = a_{sr}$.

It follows from these equations and 9.41 (1) that

$$p_1'\dot{q}_1 + p_2'\dot{q}_2 + \dots + p_n'\dot{q}_n = p_1\dot{q}_1' + p_2\dot{q}_2' + \dots + p_n\dot{q}_n',$$

or

$$\Sigma p_r'\dot{q}_r = \Sigma p_r\dot{q}_r' \dots\dots\dots(2).$$

In the case of motion starting instantaneously from rest, we have seen in 9.21 that the p 's are the impulses which would produce the motion. If we apply the impulse p_r alone, then by 9.41 (4), $\dot{q}_s = A_{rs}p_r$. And if we apply the impulse p_s' alone, then $\dot{q}_r = A_{sr}p_s'$. Hence if $p_s' = p_r$, it follows that $\dot{q}_s = \dot{q}_r'$; i.e. while an impulse p_r in the r co-ordinate produces a velocity $\dot{q}_s$ in the s co-ordinate an equal impulse p_s' in the s co-ordinate will produce the same velocity $\dot{q}_r'$ in the r co-ordinate.

9.43. **Theorems of Bertrand and Kelvin.** The theorems of Bertrand and Kelvin on the kinetic energy of systems whose motion is changed impulsively are capable of simple proof in generalized co-ordinates.

Let a system with n degrees of freedom be acted upon by *given impulses* $p_1, p_2, \dots, p_k$ in the first k of its co-ordinates, being otherwise free, i.e. $p_{k+1} = p_{k+2} = \dots = p_n = 0$; and let $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ denote the consequent velocities. Then, with the same given impulses in the first k co-ordinates, let frictionless constraints be imposed, so that the system acquires a different motion. These constraints imply that impulses are applied to some or all of the remaining $n-k$ co-ordinates, and without loss of generality we may suppose that the effect is to make the corresponding velocities zero. Hence in the constrained motion we have velocities $\dot{q}_1', \dot{q}_2', \dots, \dot{q}_n'$ whereof $\dot{q}_{k+1}', \dots, \dot{q}_n'$ are zero, and impulses $p_1', p_2', \dots, p_n'$ whereof $p_1', p_2', \dots, p_k' = p_1, p_2, \dots, p_k$.

Then if T, T' denote the kinetic energies of the two motions,

$$\begin{aligned} T - T' &= \frac{1}{2} \sum_{r=1}^n p_r \dot{q}_r - \frac{1}{2} \sum_{r=1}^n p_r' \dot{q}_r' \\ &= \frac{1}{2} \sum_{r=1}^n (p_r - p_r') (\dot{q}_r - \dot{q}_r') + \frac{1}{2} \sum_{r=1}^n p_r \dot{q}_r' + \frac{1}{2} \sum_{r=1}^n p_r' \dot{q}_r - \sum_{r=1}^n p_r' \dot{q}_r'. \end{aligned}$$

But by 9.42 (2) the second and third terms are equal, therefore

$$T - T' = \frac{1}{2} \sum_{r=1}^n (p_r - p_r') (\dot{q}_r - \dot{q}_r') + \sum_{r=1}^n (p_r - p_r') \dot{q}_r';$$

and in the last sum, for values of r from 1 to k , $p_r' = p_r$ and, for values from $k+1$ to n , $\dot{q}_r' = 0$, so this sum vanishes; in fact $p_r' - p_r$ is the type of the impulse which causes the constraints and $\sum_{r=1}^n (p_r' - p_r) \dot{q}_r' = 0$ merely expresses the fact that the impulses which cause the constraints do no work so long as the motion is guided by the constraints. Hence

$$T - T' = \frac{1}{2} \sum_{r=1}^n (p_r - p_r') (\dot{q}_r - \dot{q}_r')$$

and the result shews that the kinetic energy of the constrained motion is less than that of the unconstrained by the kinetic energy of the difference of the two motions. This is *Bertrand's theorem*.

Secondly, let the system be set in motion as before by impulses in the first k co-ordinates, being otherwise free, but now let the velocities $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_k$ be prescribed; i.e. the initial motion is such that

$$\dot{q}_1, \dot{q}_2, \dots, \dot{q}_k = \alpha_1, \alpha_2, \dots, \alpha_k \quad \text{and} \quad p_{k+1} = p_{k+2} = \dots = p_n = 0.$$

Then let frictionless constraints be applied and the system set in motion by another set of impulses so that

$$\dot{q}_1', \dot{q}_2', \dots, \dot{q}_k' = \alpha_1, \alpha_2, \dots, \alpha_k \quad \text{and} \quad \dot{q}_{k+1}' = \dot{q}_{k+2}' = \dots = \dot{q}_n' = 0.$$

$$\begin{aligned} \text{Then } T - T' &= \frac{1}{2} \sum_{r=1}^n p_r \dot{q}_r - \frac{1}{2} \sum_{r=1}^n p_r' \dot{q}_r' \\ &= \sum_{r=1}^n p_r \dot{q}_r - \frac{1}{2} \sum_{r=1}^n p_r \dot{q}_r' - \frac{1}{2} \sum_{r=1}^n p_r' \dot{q}_r - \frac{1}{2} \sum_{r=1}^n (p_r - p_r') (\dot{q}_r - \dot{q}_r'). \end{aligned}$$

But by 9.42 (2) the second and third terms are equal so that

$$T - T' = \sum_{r=1}^n p_r (\dot{q}_r - \dot{q}_r') - \frac{1}{2} \sum_{r=1}^n (p_r - p_r') (\dot{q}_r - \dot{q}_r');$$

and in the first sum, for values of r from 1 to k , $\dot{q}_r' = \dot{q}_r$, and, for values from $k+1$ to n , $p_r = 0$, so that this sum is zero and T is less than T' by the kinetic energy of the relative motion. This is *Kelvin's theorem*.

9.44. Connection between Bertrand's and Kelvin's theorems. It has been shewn by Professor G. I. Taylor that '*The reduction in energy due to the imposition of constraints in the Bertrand case is less than the increase due to the imposition of the same constraints in the Kelvin case*'.*

We consider the constraints to be the same in the two cases if, as in 9.43, their effect is to reduce to zero the velocities in the co-ordinates $q_{k+1}, q_{k+2}, \dots q_n$. But the impulses necessary to effect this result will be different in the two cases according as we take the impulses $p_1, p_2, \dots p_k$ to be prescribed (Bertrand) or the velocities $\dot{q}_1, \dot{q}_2, \dots \dot{q}_k$ to be prescribed (Kelvin).

To establish the result we note that we are concerned in all with five different states of motion. There is first the original motion (a) in which velocities $\dot{q}_1, \dot{q}_2, \dots \dot{q}_n$ are produced by impulses $p_1, p_2, \dots p_k$ with $p_{k+1} = p_{k+2} = \dots = p_n = 0$. Let T_a denote the energy of this motion.

Secondly, there is the Bertrand comparison motion (b) started by impulses $p_1', p_2', \dots p_n'$ and subject to the conditions that

$$p_1', p_2', \dots p_k' = p_1, p_2, \dots p_k \quad \text{and} \quad \dot{q}'_{k+1} = \dot{q}'_{k+2} = \dots = \dot{q}_n' = 0.$$

If T_b denotes the energy of this motion, Bertrand's theorem proves that

$$T_a - T_b = T_{a-b},$$

where the reduction of energy T_{a-b} is the energy of the motion got by subtracting the motion (b) from the motion (a) both as regards the impulses and the velocities they produce; i.e. the third motion (a-b) is one in which there are no impulses in the first k co-ordinates, and the velocities in the remaining $n-k$ co-ordinates are the velocities of the motion (a), viz.

$$\dot{q}_{k+1}, \dot{q}_{k+2}, \dots \dot{q}_n.$$

Fourthly, there is the Kelvin comparison motion (c) started by impulses which we will call $p_1'', p_2'', \dots p_n''$ and such that as regards velocities

$$\dot{q}_1'', \dot{q}_2'', \dots \dot{q}_k'' = \dot{q}_1, \dot{q}_2, \dots \dot{q}_k$$

(because in the Kelvin case the first k velocities are prescribed) and

$$\dot{q}''_{k+1} = \dot{q}''_{k+2} = \dots = \dot{q}_n'' = 0$$

(because the constraints are to be the same in both comparison cases).

Denoting the energy of this motion by T_c , Kelvin's theorem proves that $T_a - T_c = T_{a-c}$, where T_{a-c} is the energy of the motion got by taking the difference of the motion (a) and the motion (c). And the

* *Proceedings of the London Mathematical Society*, Series 2, vol. 21, p. 413.

increase in energy in the Kelvin case is therefore that of a motion ($a-c$) in which there are no velocities in the first k co-ordinates (though there are impulses), and the velocities in the remaining k co-ordinates are the velocities of the motion (a), viz. $\dot{q}_{k+1}, \dot{q}_{k+2}, \dots \dot{q}_n$.

Finally when we come to compare the motions ($a-b$) and ($a-c$), we see that both are motions set up impulsively in which the velocities of the last $n-k$ co-ordinates are prescribed, that in ($a-b$) the remaining k co-ordinates are unconstrained, but in ($a-c$) they are subject to impulses; but this is precisely the comparison made in Kelvin's theorem and therefore the energy T_{a-b} is less than the energy T_{a-c} .

Professor Taylor points out that, while the reduction of energy in the Bertrand case cannot exceed the whole energy of the unconstrained motion, there is no upper limit to the increase of energy in the Kelvin case, and illustrates this by the example of a straight rod set in motion by an impulse at right angles to its length applied at a point A . A constraint may be applied by fixing a point P of the rod. Then if the impulse at A is given, the energy of the resulting motion can be reduced towards vanishing point by making P approach A ; but if the velocity of A be given, the effect on the energy of the constraint at P will be to increase it indefinitely as P approaches A .

9.5. Lagrange's equations for impulsive motion. If simultaneous impulses are applied to a system already in motion, the sum of the virtual moments of the components of the change of momentum is equal to the sum of the virtual moments of the applied impulses, so that we have an equation

$$\begin{aligned} \Sigma m \{ (\dot{x} - \dot{x}_0) \delta x + (\dot{y} - \dot{y}_0) \delta y + (\dot{z} - \dot{z}_0) \delta z \} \\ = \Sigma (X' \delta x + Y' \delta y + Z' \delta z) \dots\dots(1), \end{aligned}$$

where the suffix indicates values of the velocities just before the impulses are applied.

As in 9.21 this may be transformed into

$$\left(\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial \dot{q}_{10}} \right) \delta q_1 + \left(\frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial \dot{q}_{20}} \right) \delta q_2 + \dots = \Sigma (Q_1' \delta q_1 + Q_2' \delta q_2 + \dots) \dots\dots(2),$$

so that, provided the co-ordinates are independent, we have n equations

$$\left. \begin{aligned} \frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial \dot{q}_{10}} &= Q_1' \\ \frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial \dot{q}_{20}} &= Q_2' \\ \text{etc.} \end{aligned} \right\} \dots\dots\dots(3),$$

which simply express the fact that the instantaneous change in a generalized component of momentum is equal to the impulse in the corresponding co-ordinate.

9.51. Examples of impulsive motion. (i) *A heavy uniform rod rotating in a vertical plane falls and strikes a smooth inelastic horizontal plane. To find the impulse.*

Let u, ω be the linear and angular velocities just before impact, y the distance of the centre of gravity from the horizontal plane and θ the inclination of the rod to the vertical. Then y, θ may be taken as the generalized co-ordinates.

Let P be the required impulsive reaction of the plane. The virtual displacement of the end of the rod which strikes the plane, consequent on changes $\delta y, \delta \theta$ in the co-ordinates, when resolved in the direction of P is $\delta y + a \sin \theta \delta \theta$, so that we have for the virtual moment of the impulse

$$\delta U = P \delta y + Pa \sin \theta \delta \theta \quad \dots\dots\dots(1).$$

Also
$$2T = m(\dot{y}^2 + \frac{1}{3}a^2\dot{\theta}^2) \quad \dots\dots\dots(2),$$

where m is the mass and $2a$ the length of the rod.

The dynamical equations are

$$\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial \dot{y}_0} = \frac{\partial U}{\partial \dot{y}} \quad \text{and} \quad \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \dot{\theta}_0} = \frac{\partial U}{\partial \dot{\theta}}$$

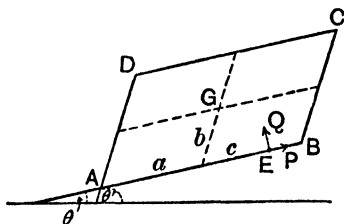
or
$$m(\dot{y} - \dot{y}_0) = P \quad \text{and} \quad \frac{1}{3}ma(\dot{\theta} - \dot{\theta}_0) = P \sin \theta.$$

But $\dot{y}_0 = -u, \dot{\theta}_0 = -\omega$, and since the plane is inelastic the end which strikes the plane loses its vertical velocity, or $\dot{y} + a\dot{\theta} \sin \theta = 0$. We find on substitution that

$$P(1 + 3 \sin^2 \theta) = m(u + a\omega \sin \theta).$$

(ii) *A framework is formed of four uniform rods smoothly jointed at their ends so as to form a parallelogram. It is turning about its centre with angular velocity ω when a point in one of the rods is suddenly fixed. To find the instantaneous change in the motion.*

Let m, m' be the masses of the rods, $2a, 2b$ their lengths; and let E , the point which is fixed, be at a distance c from the middle point of the rod AB of length $2a$. Let P, Q be the components of the impulse at E along and at right angles to AB . Let the sides AB, AD make angles θ, θ' with a fixed direction, let α be the angle BAD at the instant of fixture, and let $\delta x, \delta y$ denote small displacements of the centre of gravity G parallel and perpendicular to AB .



The virtual moment of the impulse is

$$\delta U = P(\delta x + b \delta \theta' \sin \alpha) + Q(\delta y + c \delta \theta - b \delta \theta' \cos \alpha);$$

and the kinetic energy T is given by

$$2T = 2(m + m')(\dot{x}^2 + \dot{y}^2) + 2m(k^2\dot{\theta}^2 + b^2\dot{\theta}'^2) + 2m'(k'^2\dot{\theta}'^2 + a^2\dot{\theta}^2),$$

where $k^2 = \frac{1}{3}a^2$ and $k'^2 = \frac{1}{3}b^2$.

Before the fixture we have $\dot{x} = \dot{y} = 0$ and $\theta = \theta' = \omega$. Therefore the Lagrange equations give

$$2(m + m')\dot{x} = P, \quad 2(m + m')\dot{y} = Q,$$

$$2(mk^2 + m'a^2)(\dot{\theta} - \omega) = Qc$$

$$\text{and} \quad 2(mb^2 + m'k'^2)(\dot{\theta}' - \omega) = Pb \sin \alpha - Qb \cos \alpha.$$

But we have also the conditions that the point E is reduced to rest, viz.

$$\dot{x} + b\dot{\theta}' \sin \alpha = 0, \quad \text{and} \quad \dot{y} + c\dot{\theta} - b\dot{\theta}' \cos \alpha = 0;$$

and by eliminating $\dot{x}$, $\dot{y}$, P and Q the equations for $\dot{\theta}$, $\dot{\theta}'$ become

$$(mk^2 + m'a^2)(\dot{\theta} - \omega) + (m + m')(c^2\dot{\theta} - bc\dot{\theta}' \cos \alpha) = 0$$

$$\text{and} \quad (mb^2 + m'k'^2)(\dot{\theta}' - \omega) + (m + m')(b^2\dot{\theta}' - bc\dot{\theta} \cos \alpha) = 0.$$

The same results can of course be obtained by making the kinetic energy of the relative motion before and after the fixture a minimum, subject to the condition that the point E is reduced to rest.

(iii) *Twelve equal uniform rods are smoothly jointed at their ends so as to form a cubical framework, which is suspended from a point by a string tied to one corner and kept in shape by a light string occupying the position of a vertical diagonal. To discuss possible motions.*

Let m be the mass and $2a$ the length of a rod, and in a general symmetrical position let θ be the inclination of each rod to the vertical.

(α) *To find the initial motion if the string OA is cut.*

The horizontal projection of the six middle rods is a hexagon, so that the distance from the vertical diagonal OA of the middle point of one of these rods is $2a \sin \theta \cos 30^\circ$ or $\sqrt{3}a \sin \theta$, and the depth of the middle point below the highest point O is $3a \cos \theta$. For one of the three lower rods the corresponding distances are $a \sin \theta$ and $5a \cos \theta$.

Hence we have for the kinetic energy of the framework

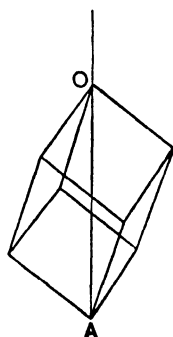
$$2T = 3m(k^2 + a^2)\dot{\theta}^2 + 6m(k^2 + 9a^2 \sin^2 \theta + 3a^2 \cos^2 \theta)\dot{\theta}^2 \\ + 3m(k^2 + 25a^2 \sin^2 \theta + a^2 \cos^2 \theta)\dot{\theta}^2,$$

where the three terms refer to the three upper rods, the six middle rods and the three lower rods, and $k^2 = \frac{1}{3}a^2$.

$$\text{Therefore} \quad 2T = 12ma^2\dot{\theta}^2 \left(\frac{1}{3} + 9 \sin^2 \theta \right) \dots \dots \dots (1).$$

The potential energy is given by

$$-V = 12mg \cdot 3a \cos \theta = 36mga \cos \theta \dots \dots \dots (2).$$



For an initial motion velocities are zero and the Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = - \frac{\partial V}{\partial \theta}$$

gives $(7 + 27 \sin^2 \theta_0) \ddot{\theta}_0 = -9 \frac{g}{a} \sin \theta_0$.

But, for a cube, $\sin \theta_0 = \sqrt{2/3}$,

therefore $\ddot{\theta}_0 = -\frac{9}{2\sqrt{3}} \sqrt{2/3} \frac{g}{a} \dots\dots\dots(3)$.

Also the initial acceleration of the centre of gravity of the framework

$$\begin{aligned} &= -3a \sin \theta_0 \cdot \ddot{\theta}_0 \\ &= \frac{1}{2} \frac{8}{5} g \dots\dots\dots(4). \end{aligned}$$

(β) Suppose that, instead of the string OA , the string supporting the framework is cut so that it falls and strikes a smooth inelastic horizontal plane.

Let u, u' be the velocities of the centre of gravity just before and just after the blow, y the height of the centre of gravity above the plane, and P the impulsive reaction of the plane. Then the vertical virtual displacement of the lowest point A consequent on increments $\delta y, \delta \theta$ in the co-ordinates y, θ is $\delta y + 3a \sin \theta \cdot \delta \theta$, so that the virtual moment of P is

$$\delta U = P (\delta y + 3a \sin \theta \cdot \delta \theta) \dots\dots\dots(5),$$

and by taking the kinetic energy to be that of the whole mass moving with the velocity of the centre of gravity, together with the kinetic energy of the motion relative to the centre of gravity, it is easy to shew that in this case

$$2T = 12m \left(\frac{1}{3} a^2 \dot{\theta}^2 + \dot{y}^2 \right) \dots\dots\dots(6).$$

Hence we have

$$12m (\dot{y} - \dot{y}_0) = P \quad \text{and} \quad 28ma (\dot{\theta} - \dot{\theta}_0) = 3P \sin \theta_0.$$

But $\dot{y}_0 = -u, \dot{y} = -u', \dot{\theta}_0 = 0$ and $\sin \theta_0 = \sqrt{2/3}$,

and the lowest point is reduced to rest, so that

$$\dot{y} + 3a \sin \theta \cdot \dot{\theta} = 0.$$

These equations give $u' = \frac{1}{2} \frac{8}{5} u$ and $P = \frac{8}{5} \frac{1}{2} mu$.

(γ) If the framework lies flat on a smooth horizontal plane, to find the upward impulse at the uppermost central joint which will make the centre move with velocity v .

Let P be the upward impulse required, and suppose that the lower central joint is fixed to the plane which sustains an impulsive pull Q .

With the same notation

$$2T = 12m \left(\frac{1}{3} a^2 \dot{\theta}^2 + \dot{y}^2 \right) \dots\dots\dots(6)$$

and $\delta U = P (\delta y - 3a \sin \theta \cdot \delta \theta) - Q (\delta y + 3a \sin \theta \cdot \delta \theta) \dots\dots(7)$.

Hence we have

$$12m \dot{y} = P - Q \quad \text{and} \quad 28ma \dot{\theta} = -3(P + Q) \sin \theta.$$

But we are assuming the lower end to be fixed so that

$$\dot{y} + 3a \sin \theta \cdot \dot{\theta} = 0.$$

Putting $\dot{y} = v$ and $\theta = 90^\circ$, we find that

$$P = \frac{3}{2}mv \quad \text{and} \quad Q = -\frac{1}{2}mv.$$

The negative value for Q implies that the attachment of the lower central joint to the plane is unnecessary as there is actually an impulsive pressure of the framework on the plane of amount $\frac{1}{2}mv$.

9.52. Examples. 1. A rhombus of smoothly jointed uniform rods is rotating in its plane about one corner, which is fixed, when a string joining the two adjacent corners suddenly becomes tight; shew that the impulsive tension in the string is proportional to the difference of the angular velocities of the sides and find the loss of kinetic energy.

[I. 1910]

2. A rhombus of smoothly jointed equal uniform rods of length $2a$ lies on a smooth horizontal table. It is struck by a blow P perpendicularly to one of its sides at a point distant x from the middle point; shew that the angular velocity of that side instantly becomes $3Px/2ma^2$, where m is the mass of the four rods.

3. A square of side $2a$ formed by four equal rods hinged at the ends lies flat on a smooth horizontal table and rotates with angular velocity n in its own plane with its centre at rest. If a corner A is suddenly fixed, shew that the angular velocity of each rod is instantly reduced to $2n/5$, and further that, if A and the opposite corner are connected by a slack string of length $4a \cos \alpha$, ($\alpha < \pi/4$), the rhombus formed by the rods after the string tightens in the subsequent motion revolves round A with angular velocity $n/(1 + 3 \cos^2 \alpha)$.

[M. T. 1908]

4. Three equal uniform rods AB , BC , CD each of mass m are freely jointed at B , C and are momentarily at rest when their directions are mutually perpendicular. Equal and parallel impulses I perpendicular to BC but in opposite senses act on the system at A and D . Prove that the kinetic energy after the action of the impulses is $2I^2/m$.

[M. T. 1933]

5. A framework in the form of a regular hexagon of six uniform rods smoothly jointed at the corners rests on a smooth table, and a string tied to the middle point of one side is jerked in the direction of that side; shew that the velocities along the sides of the centre of this side and of the centre of the opposite side are in opposite directions, and in the ratio 59:4. [Take as independent velocities the component velocities of the centres of the sides in the directions of those sides respectively.]

[M. T. 1906]

6. Two uniform circular discs, each of mass $2m$ and radius a , lie on a horizontal plane. The ends A and B of a uniform rod, of mass m

and length $2b$, are smoothly jointed to the two discs at the centres, and an equal rod is smoothly jointed at its ends C and D to points on the rims of the two discs respectively, so that $ABCD$ is a parallelogram. The whole system can turn about a fixed vertical axis through A , and is initially at rest with the angle CAB equal to 45° . Prove that if the rod AB receives a horizontal impulse P perpendicular to AB at a distance $\frac{3}{4}b$ from A , the initial angular velocity of this rod is $P/7mb$.

[I. 1930]

7. If the upper central corner of the framework of 9.51 (iii) is attached to a vertical string which passes over a smooth pulley and carries a mass M' at its other end, prove that the initial acceleration of M' is $(2M' - M)g/(\frac{1}{2}M + 2M')$, where M is the mass of the cubical framework.

9.6. Ignorance of co-ordinates. The Routhian function. In some problems certain of the generalized co-ordinates do not occur explicitly in the Lagrangian function L but only enter as velocity components. If q_r denotes such a co-ordinate, since $\partial L/\partial q_r = 0$, the corresponding Lagrangian equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) = 0 \quad \dots\dots\dots (1)$$

and this integrates at once in the form

$$\frac{\partial L}{\partial \dot{q}_r} = \beta_r \text{ (a constant)} \quad \dots\dots\dots (2).$$

Suppose that the last $n - k$ co-ordinates are of this type and that we denote them by $\theta_1, \theta_2, \dots \theta_{n-k}$, and the corresponding momenta by $u_1, u_2, \dots u_{n-k}$ so that $u_r = \partial T/\partial \dot{\theta}_r = \partial L/\partial \dot{\theta}_r$.

$$\text{Let} \quad R = L - u_1 \dot{\theta}_1 - u_2 \dot{\theta}_2 - \dots - u_{n-k} \dot{\theta}_{n-k} \quad \dots\dots\dots (3).$$

This is called the Routhian function after Dr E. J. Routh who first used it.

Since, by hypothesis, the co-ordinates $\theta_1, \theta_2, \dots \theta_{n-k}$ do not occur explicitly in the $n - k$ equations $\partial L/\partial \dot{\theta}_r = \beta_r$, therefore we can express the $n - k$ velocities $\dot{\theta}_1, \dot{\theta}_2, \dots \dot{\theta}_{n-k}$ in terms of $q_1, q_2, \dots q_k, \dot{q}_1, \dot{q}_2, \dots \dot{q}_k$ and $\beta_1, \beta_2, \dots \beta_{n-k}$, and suppose that these values are substituted in (3) so that R is expressed in terms of the latter quantities only.

Now let δ denote a variation due to arbitrary infinitesimal changes $\delta q_1, \delta q_2, \dots \delta q_k, \delta \dot{q}_1, \delta \dot{q}_2, \dots \delta \dot{q}_k, \delta \beta_1, \delta \beta_2, \dots \delta \beta_{n-k}$.

Since

$$R = L - \sum_{r=1}^{n-k} \theta_r \frac{\partial L}{\partial \theta_r} \dots\dots\dots(4),$$

therefore

$$\delta R = \delta L - \delta \left(\sum_{r=1}^{n-k} \theta_r \frac{\partial L}{\partial \theta_r} \right).$$

But

$$\delta L = \sum_{r=1}^k \frac{\partial L}{\partial q_r} \delta q_r + \sum_{r=1}^k \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r + \sum_{r=1}^{n-k} \frac{\partial L}{\partial \theta_r} \delta \theta_r$$

and, since $\partial L / \partial \theta_r = \beta_r$,

$$\delta \left(\sum_{r=1}^{n-k} \theta_r \frac{\partial L}{\partial \theta_r} \right) = \sum_{r=1}^{n-k} \theta_r \delta \beta_r + \sum_{r=1}^{n-k} \frac{\partial L}{\partial \theta_r} \delta \theta_r.$$

Hence

$$\delta R = \sum_{r=1}^k \frac{\partial L}{\partial q_r} \delta q_r + \sum_{r=1}^k \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r - \sum_{r=1}^{n-k} \theta_r \delta \beta_r \dots\dots\dots(5),$$

but the infinitesimals on the right-hand side are arbitrary and independent, so that we get the following sets of equations:

$$\frac{\partial R}{\partial q_r} = \frac{\partial L}{\partial q_r}, \quad r = 1, 2, \dots k \dots\dots\dots(6),$$

$$\frac{\partial R}{\partial \dot{q}_r} = \frac{\partial L}{\partial \dot{q}_r}, \quad r = 1, 2, \dots k \dots\dots\dots(7),$$

$$\frac{\partial R}{\partial \theta_r} = -\theta_r, \quad r = 1, 2, \dots n-k \dots\dots\dots(8).$$

Substituting in Lagrange's equations we have

$$\frac{d}{dt} \left(\frac{\partial R}{\partial \dot{q}_r} \right) - \frac{\partial R}{\partial q_r} = 0, \quad r = 1, 2, \dots k \dots\dots\dots(9),$$

where R is a function of $q_1, q_2, \dots q_k, \dot{q}_1, \dot{q}_2, \dots \dot{q}_k$ and the constants $\beta_1, \beta_2, \dots \beta_{n-k}$. The original problem has therefore been transformed into one in which there are only k independent co-ordinates, and when this has been solved the remaining variables of the original problem, viz. $\theta_1, \theta_2, \dots \theta_{n-k}$, can be found by integrating equations (8) in the form

$$\theta_r = - \int \frac{\partial R}{\partial \theta_r} dt, \quad r = 1, 2, \dots n-k \dots\dots\dots(10).$$

The process by which the function R is written down in (3) is called *modifying the Lagrangian function with regard to some of its co-ordinates*, and the process by which we get rid of

co-ordinates which do not occur explicitly in the Lagrangian function is called *Ignorance of co-ordinates*.

9.61. As examples of 9.6 we notice that in each of the examples 9.31 (iii), (iv) and (v) there is an ignorable co-ordinate. Thus in (iii) the co-ordinate ϕ does not occur explicitly in T or V ; equation (2) gives $\phi \sin^2 \theta = \beta$, and the Routhian function is

$$R = L - \phi \frac{\partial L}{\partial \phi},$$

or $R = \frac{1}{2} Ma^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + Mga \cos \theta - \frac{1}{2} Ma^2 \dot{\phi}^2 \sin^2 \theta$,
from which ϕ is to be eliminated, giving

$$R = \frac{1}{2} Ma^2 (\dot{\theta}^2 - \beta^2 \operatorname{cosec}^2 \theta) + Mga \cos \theta.$$

Then the equation $\frac{d}{dt} \left(\frac{\partial R}{\partial \dot{\theta}} \right) - \frac{\partial R}{\partial \theta} = 0$

gives $\ddot{\theta} - \beta^2 \cot \theta \operatorname{cosec}^2 \theta = -\frac{3g}{4a} \sin \theta$,

which is the equivalent of 9.31 (iii) (1) after ϕ has been eliminated.

9.62. **Example.** A gyrostat (solid of revolution) symmetrical about its axis of rotation is suspended from a fixed point by a string of length a fastened to a point on the axis of rotation. Prove that if the gyrostat is moving steadily with its axis horizontal the angle α which the string makes with the vertical is given by the equation

$$C^2 n^2 \tan \alpha = M^2 g h^2 (h + a \sin \alpha),$$

where n is the angular velocity of the gyrostat, h the distance of its centre of gravity from the point of attachment of the string, M its mass and C its moment of inertia about its axis of rotation. [M. T. 1888]

Let the axis of z be taken vertically downwards. Let the generalized co-ordinates be the angular co-ordinates θ_1, ψ_1 of the string OA and the angular co-ordinates θ_2, ψ_2, ϕ of the gyrostat, the latter being the angle which a plane through its axis fixed in the body makes with the vertical plane through its axis. Let G be the centre of gravity and $AG = h$.

Let A, A, C be the principal moments of inertia at G . Then, using Euler's expressions for angular velocities we find that the kinetic energy of the rotational motion about the centre of gravity is

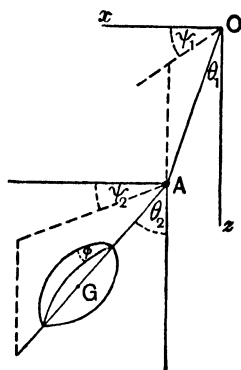
$$\frac{1}{2} A (\dot{\theta}_2^2 + \dot{\psi}_2^2 \sin^2 \theta_2) + \frac{1}{2} C (\dot{\phi} + \dot{\psi}_2 \cos \theta_2)^2.$$

The co-ordinates of G are

$$x = a \sin \theta_1 \cos \psi_1 + h \sin \theta_2 \cos \psi_2,$$

$$y = a \sin \theta_1 \sin \psi_1 + h \sin \theta_2 \sin \psi_2,$$

$$z = a \cos \theta_1 + h \cos \theta_2.$$



Differentiating these to find the velocities of G , squaring and adding, we get for the total kinetic energy

$$\begin{aligned}
 2T = & A(\dot{\theta}_2^2 + \dot{\psi}_2^2 \sin^2 \theta_2) + C(\dot{\phi} + \dot{\psi}_2 \cos \theta_2)^2 \\
 & + Ma^2(\dot{\theta}_1^2 + \dot{\psi}_1^2 \sin^2 \theta_1) + Mh^2(\dot{\theta}_2^2 + \dot{\psi}_2^2 \sin^2 \theta_2) \\
 & + 2Mah\dot{\theta}_1\dot{\theta}_2\{\cos \theta_1 \cos \theta_2 \cos(\psi_1 - \psi_2) + \sin \theta_1 \sin \theta_2\} \\
 & - 2Mah\dot{\theta}_2\dot{\psi}_1 \sin \theta_1 \cos \theta_2 \sin(\psi_1 - \psi_2) \\
 & - 2Mah\dot{\theta}_1\dot{\psi}_2 \cos \theta_1 \sin \theta_2 \sin(\psi_1 - \psi_2) \\
 & + 2Mah\dot{\psi}_1\dot{\psi}_2 \sin \theta_1 \sin \theta_2 \cos(\psi_1 - \psi_2) \dots\dots\dots(1).
 \end{aligned}$$

Also $-V = Mg(a \cos \theta_1 + h \cos \theta_2) + \text{const.} \dots\dots\dots(2).$

Since neither T nor V contains the co-ordinate ϕ , it is convenient to 'ignore' this co-ordinate. Thus we have

$$\frac{\partial T}{\partial \phi} = \text{const.}, \quad \text{or} \quad C(\dot{\phi} + \dot{\psi}_2 \cos \theta_2) = Cn,$$

n being the spin of the gyrostat about its axis; and we put

$$R = T - V - Cn\phi,$$

$$\begin{aligned}
 \text{or } 2R = & (A + Mh^2)(\dot{\theta}_2^2 + \dot{\psi}_2^2 \sin^2 \theta_2) + Cn^2 + Ma^2(\dot{\theta}_1^2 + \dot{\psi}_1^2 \sin^2 \theta_1) \\
 & + 2U\dot{\theta}_1\dot{\theta}_2 + 2V\dot{\theta}_2\dot{\psi}_1 + 2W\dot{\theta}_1\dot{\psi}_2 + 2X\dot{\psi}_1\dot{\psi}_2 + 2Mg(a \cos \theta_1 + h \cos \theta_2) \\
 & - 2Cn(n - \dot{\psi}_2 \cos \theta_2) + \text{const.},
 \end{aligned}$$

where the coefficients U, V, W, X are as in (1).

Before writing down the Lagrangian equations applied to the function R , we note that we want a steady motion in which

$$\theta_1 = \alpha, \theta_2 = \frac{1}{2}\pi, \psi_1 = \psi_2 = \omega t, \quad \text{so that} \quad \dot{\theta}_1 = \dot{\theta}_2 = 0, \text{ and } \dot{\psi}_1 = \dot{\psi}_2 = \omega.$$

Then we have

$$\frac{d}{dt} \left(\frac{\partial R}{\partial \dot{\theta}_1} \right) = \frac{d}{dt} (Ma^2\dot{\theta}_1 + U\dot{\theta}_2 + W\dot{\psi}_2) = 0$$

$$\text{and} \quad \frac{d}{dt} \left(\frac{\partial R}{\partial \dot{\theta}_2} \right) = \frac{d}{dt} \{ (A + Mh^2)\dot{\theta}_2 + U\dot{\theta}_1 + V\dot{\psi}_1 \} = 0.$$

Therefore the first two Lagrangian equations reduce, in the steady motion, to

$$\frac{\partial R}{\partial \dot{\theta}_1} = 0 \quad \text{and} \quad \frac{\partial R}{\partial \dot{\theta}_2} = 0,$$

$$\text{or} \quad Ma^2\omega^2 \sin \alpha \cos \alpha + \frac{\partial X}{\partial \dot{\theta}_1} \omega^2 - Mga \sin \alpha = 0$$

$$\text{and} \quad -Mgh - Cn\omega = 0.$$

And since in the steady motion $\partial X / \partial \dot{\theta}_1 = Mah \cos \alpha$,

$$\text{therefore} \quad g \tan \alpha = \omega^2 (a \sin \alpha + h)$$

$$\text{and} \quad Mgh = -Cn\omega;$$

$$\text{whence} \quad C^2 n^2 \tan \alpha = M^2 gh^2 (a \sin \alpha + h).$$

9·63. Examples. 1. A system is in motion under no forces and its kinetic energy is

$$\frac{1}{2}(A\dot{\theta}^2 + B\dot{\phi}^2),$$

where A and B are functions of θ alone; shew that a steady motion is possible in which $\theta = \alpha$ and $\dot{\phi} = \omega$, provided that, in the position $\theta = \alpha$, B is a maximum or minimum, and further that this state of motion is stable if B is a maximum and unstable if B is a minimum.

2. A uniform rod of mass $3m$ and length $2l$ has its middle point fixed, and a particle of mass m attached at one extremity. The rod when in a horizontal position is set rotating about a vertical axis through its centre with angular velocity $\sqrt{(2ng/l)}$. Shew that the heavy end of the rod will fall till the inclination of the rod to the vertical is

$$\cos^{-1}\{\sqrt{(n^2 + 1)} - n\},$$

and will then rise again.

9·64. Motion of a system constrained to rotate uniformly about an axis. Provided the system is holonomic we may apply Lagrange's equations. We have seen (4·4) that in the case of the motion of a particle in a tube constrained to rotate with uniform angular velocity ω , the relative motion is not altered if we suppose the tube to be at rest and the particle acted on by a radial force $\omega^2 r$ per unit mass directed away from the axis. It is easy to shew that this theorem also follows from Lagrange's equations.

Taking the axis of rotation as axis of z , using cylindrical co-ordinates and measuring θ from a plane which revolves about Oz with the prescribed angular velocity ω , we have

$$\begin{aligned} 2T &= \sum m \{ \dot{z}^2 + \dot{r}^2 + r^2 (\dot{\theta} + \omega)^2 \} \\ &= \sum m (\dot{z}^2 + \dot{r}^2 + r^2 \dot{\theta}^2) + 2\omega \sum m r^2 \dot{\theta} + \omega^2 \sum m r^2 \dots (1). \end{aligned}$$

Now suppose that the cylindrical co-ordinates r , θ , z of every particle are expressible in terms of generalized co-ordinates $q_1, q_2, \dots, q_n$. Then $\sum m r^2 \dot{\theta}^2$ will be a linear function of $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ in which the coefficients are functions of $q_1, q_2, \dots, q_n$, say

$$\sum m r^2 \dot{\theta}^2 = H = \alpha_1 \dot{q}_1 + \alpha_2 \dot{q}_2 + \dots + \alpha_n \dot{q}_n,$$

and $\sum m r^2$ is simply a function of $q_1, q_2, \dots, q_n$, say I , and the terms independent of ω in (1) are what the kinetic energy would be if the rotation ω were ignored, say $2T_1$; so that

$$2T = 2T_1 + 2\omega H + \omega^2 I \dots \dots \dots (2).$$

We may also suppose that the virtual work of the external forces is $Q_1 \delta q_1 + Q_2 \delta q_2 + \dots + Q_n \delta q_n$, where $Q_1, Q_2, \dots, Q_n$ depend only on the co-ordinates $q_1, q_2, \dots, q_n$.

Then the Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r$$

becomes

$$\frac{d}{dt} \left(\frac{\partial T_1}{\partial \dot{q}_r} \right) - \frac{\partial T_1}{\partial q_r} + \frac{d}{dt} \left(\omega \frac{\partial H}{\partial \dot{q}_r} \right) - \omega \frac{\partial H}{\partial q_r} - \frac{1}{2} \omega^2 \frac{\partial I}{\partial q_r} = Q_r \dots (3).$$

But

$$\begin{aligned} & \frac{d}{dt} \left(\omega \frac{\partial H}{\partial \dot{q}_r} \right) - \omega \frac{\partial H}{\partial q_r} \\ &= \omega \left(\dot{q}_r - \frac{\partial \alpha_1}{\partial q_r} \dot{q}_1 - \frac{\partial \alpha_2}{\partial q_r} \dot{q}_2 - \dots - \frac{\partial \alpha_n}{\partial q_r} \dot{q}_n \right) \\ &= \omega \left\{ \left(\frac{\partial \alpha_r}{\partial q_1} - \frac{\partial \alpha_1}{\partial q_r} \right) \dot{q}_1 + \left(\frac{\partial \alpha_r}{\partial q_2} - \frac{\partial \alpha_2}{\partial q_r} \right) \dot{q}_2 \right. \\ & \quad \left. + \dots + \left(\frac{\partial \alpha_r}{\partial q_n} - \frac{\partial \alpha_n}{\partial q_r} \right) \dot{q}_n \right\} \\ &= \beta_{r1} \dot{q}_1 + \beta_{r2} \dot{q}_2 + \dots + \beta_{rn} \dot{q}_n, \end{aligned}$$

where

$$\beta_{rs} = \omega \left(\frac{\partial \alpha_r}{\partial q_s} - \frac{\partial \alpha_s}{\partial q_r} \right) = -\beta_{sr}.$$

Therefore (3) takes the form

$$\frac{d}{dt} \left(\frac{\partial T_1}{\partial \dot{q}_r} \right) - \frac{\partial T_1}{\partial q_r} + \beta_{r1} \dot{q}_1 + \beta_{r2} \dot{q}_2 + \dots + \beta_{rn} \dot{q}_n = Q_r + \frac{1}{2} \omega^2 \frac{\partial I}{\partial q_r} \dots (4).$$

There are two special cases worth noting: (i) when $\theta = 0$ for every particle, so that in this case $H = 0$; and (ii) when there is only *one* generalized co-ordinate q . H is then of the form $f(q) \dot{q}$, and the terms in H in the one Lagrange equation are $\omega \frac{d}{dt} \{f(q)\} - \omega f'(q) \dot{q}$, i.e. zero. So that in both these cases the linear terms in the velocities in (4) disappear, and the only effect of ω is to add a term $\frac{1}{2} I \omega^2$ to the work function, where $I = \sum m r^2$.

Lagrange's method is to multiply the equations (3) by $\lambda_1, \lambda_2, \dots \lambda_m$ respectively and add them to (2), so that

$$\sum_{r=1}^n \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} - Q_r + \lambda_1 \frac{\partial f_1}{\partial q_r} + \lambda_2 \frac{\partial f_2}{\partial q_r} + \dots + \lambda_m \frac{\partial f_m}{\partial q_r} \right\} \delta q_r = 0 \quad \dots\dots(4).$$

Then give to the λ 's such values as will make the coefficients of $\delta q_1, \delta q_2, \dots \delta q_m$ in (4) vanish. The remaining $n-m$ co-ordinates being independent, the coefficients of $\delta q_{m+1}, \delta q_{m+2}, \dots \delta q_n$ must vanish also, so that we have n equations of the form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} - Q_r + \lambda_1 \frac{\partial f_1}{\partial q_r} + \lambda_2 \frac{\partial f_2}{\partial q_r} + \dots + \lambda_m \frac{\partial f_m}{\partial q_r} = 0 \quad (r=1, 2, \dots n) \quad \dots\dots(5);$$

and these n equations together with the m equations (1) are sufficient to determine the m λ 's and the n co-ordinates $q_1, q_2, \dots q_n$.

9·71. Non-holonomic systems. In the holonomic system just considered, the m equations (1) might have been used for reducing the number of co-ordinates to equality with the number of degrees of freedom; but in a *non-holonomic system* the number of independent co-ordinates *necessary* to specify the position of the system is a definite number greater than the number of its degrees of freedom, because of the existence of constraints of the kind which do no work and are expressible by *non-integrable* relations

$$A_{1s} \dot{q}_1 + A_{2s} \dot{q}_2 + \dots + A_{ns} \dot{q}_n = 0 \quad (s=1, 2, \dots m) \quad \dots(1),$$

where the A 's are functions of $q_1, q_2, \dots q_n$; *non-integrable* relations because an integrable relation would serve to reduce the number of independent co-ordinates necessary to specify the position of the system, but by hypothesis this is a fixed number n .

Taking the equation of virtual work in the form 9·7 (2), the possible virtual displacements $\delta q_1, \delta q_2, \dots \delta q_n$ are now connected by the m equations

$$A_{1s} \delta q_1 + A_{2s} \delta q_2 + \dots + A_{ns} \delta q_n = 0 \quad (s=1, 2, \dots m) \quad \dots(2),$$

and adopting the method of 9·7 we obtain n equations of the form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} - Q_r + \lambda_1 A_{r1} + \lambda_2 A_{r2} + \dots + \lambda_m A_{rm} = 0$$

$$(r = 1, 2, \dots n) \dots\dots(3),$$

which with the m equations (1) are sufficient for the determination of the m λ 's and the n co-ordinates $q_1, q_2, \dots q_n$.

The additional terms in the equations (3) can be identified with the constraints. Thus suppose that the system is free from the constraints imposed by equations (1) but compelled by additional external forces to take its actual motion; and let the work done by these additional forces in an arbitrary displacement be denoted by

$$Q_1' \delta q_1 + Q_2' \delta q_2 + \dots + Q_n' \delta q_n.$$

The system is now by hypothesis holonomic and its motion is given by ordinary Lagrange equations provided we include the last expression in the virtual work of the external forces, so that the equations are

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r + Q_r' \quad (r = 1, 2, \dots n) \dots\dots(4).$$

We have now to determine the forces $Q_1', Q_2', \dots Q_n'$ and we can do so from the consideration that in any displacement consistent with the original constraints represented by (1) these forces would do no work, so that we have an equation

$$Q_1' \delta q_1 + Q_2' \delta q_2 + \dots + Q_n' \delta q_n = 0,$$

true for all values of the ratios $\delta q_1 : \delta q_2 : \dots : \delta q_n$ which satisfy

$$A_{1s} \delta q_1 + A_{2s} \delta q_2 + \dots + A_{ns} \delta q_n = 0 \quad (s = 1, 2, \dots m);$$

and consequently

$$Q_r' + \lambda_1 A_{r1} + \lambda_2 A_{r2} + \dots + \lambda_m A_{rm} = 0 \quad (r = 1, 2, \dots n),$$

leading again to equations (3).

9·72. Example. *Motion of a uniform sphere on a rough horizontal plane.* As explained in 9·11 this is a non-holonomic problem. With the

notation of 7·5 (i), where the problem was solved by a direct method, let us follow the procedure of 9·71. We have, from 3·71,

$$\left. \begin{aligned} \omega_x &= -\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi \\ \omega_y &= \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi \\ \omega_z &= \dot{\phi} \cos \theta + \dot{\psi} \end{aligned} \right\} \dots\dots\dots(1),$$

and, taking x, y, θ, ϕ, ψ as the generalized co-ordinates,

$$\begin{aligned} 2T &= m(\dot{x}^2 + \dot{y}^2) + mk^2(\omega_x^2 + \omega_y^2 + \omega_z^2) \\ &= m(\dot{x}^2 + \dot{y}^2) + mk^2(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta) \end{aligned} \left\} \dots\dots(2),$$

and $\delta U = X\delta x + Y\delta y$

where X, Y are the external forces.

The conditions for rolling are

$$\dot{x} - a\omega_y = 0, \quad \dot{y} + a\omega_x = 0 \dots\dots\dots(3),$$

so that possible displacements have to satisfy

$$\left. \begin{aligned} \delta x - a\delta\theta \cos \psi - a\delta\phi \sin \theta \sin \psi &= 0 \\ \delta y - a\delta\theta \sin \psi + a\delta\phi \sin \theta \cos \psi &= 0 \end{aligned} \right\} \dots\dots\dots(4).$$

Corresponding to 9·71 (3) we therefore have five equations

$$m\ddot{x} - X + \lambda_1 = 0 \dots\dots\dots(5),$$

$$m\ddot{y} - Y + \lambda_2 = 0 \dots\dots\dots(6),$$

$$mk^2(\ddot{\theta} + \dot{\phi}\dot{\psi}\sin\theta) - \lambda_1 a \cos \psi - \lambda_2 a \sin \psi = 0 \dots\dots(7),$$

$$mk^2(\ddot{\phi} + \dot{\psi}\cos\theta - \dot{\theta}\dot{\psi}\sin\theta) - \lambda_1 a \sin \theta \sin \psi + \lambda_2 a \sin \theta \cos \psi = 0 \dots\dots(8),$$

$$mk^2 \frac{d}{dt}(\dot{\psi} + \dot{\phi}\cos\theta) = 0 \dots\dots\dots(9).$$

From (9) we get $\dot{\psi} + \dot{\phi}\cos\theta = \text{const.}$, say n(10),

and this is the spin about the vertical.

A comparison of (5) and (6) with 7·5 (i) (1) shows that $-\lambda_1, -\lambda_2$ are the frictional forces in the directions of the axes. And by solving (7) and (8) for λ_1, λ_2 with the help of (10), we are able to reduce (5) and (6) to the forms

$$m\left(1 + \frac{k^2}{a^2}\right)\ddot{x} = X \quad \text{and} \quad m\left(1 + \frac{k^2}{a^2}\right)\ddot{y} = Y.$$

We notice that by using (3) and (1) it is possible to obtain the kinetic energy in terms of three co-ordinates and their derivatives, viz.

$$2T = m(a^2 + k^2)(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + mk^2(\dot{\phi} \cos \theta + \dot{\psi})^2,$$

but we cannot obtain correct equations of motion by applying Lagrange's equations to this expression because the co-ordinates θ, ϕ, ψ are insufficient to define the position of the body.

This example is only given to illustrate the way in which the Lagrange multipliers are connected with the constraining forces, as it is clear that the method of 7·5 (i) is a much simpler way of approaching the problem.

9·73. Examples. 1. A heavy rod of length $2a$ slips down with its ends in contact with a smooth vertical wall and a smooth horizontal plane; the rod not being initially in a plane at right angles to the wall. Prove that, if θ be the inclination of the rod to the vertical and ψ the inclination of its horizontal projection to the intersection of the planes, then

$$4 \frac{d^2}{dt^2} (\sin \theta \sin \psi) = \tan \psi \frac{d^2}{dt^2} (\sin \theta \cos \psi)$$

and
$$4 \frac{d^2}{dt^2} (\cos \theta) = \cot \theta \sec \psi \frac{d^2}{dt^2} (\sin \theta \cos \psi) - \frac{3g}{a}.$$

2. Prove that, if l, m, n are the direction cosines with respect to fixed axes of a rod moving in any manner in space, and if V be the potential energy,

$$\frac{1}{l} (I\ddot{l} + \partial V / \partial l) = \frac{1}{m} (I\ddot{m} + \partial V / \partial m) = \frac{1}{n} (I\ddot{n} + \partial V / \partial n),$$

where I is the moment of inertia of the rod about an axis through its centre perpendicular to its length.

9·8. Hamilton's Equations of Motion. Considering a holonomic system with conservative forces and n independent co-ordinates $q_1, q_2, \dots, q_n$, let L denote the Lagrangian Function $T - V$, so that L is a quadratic function of $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ in which the coefficients are functions of $q_1, q_2, \dots, q_n$ and t .

As in 9·21 we may put

$$p_r = \frac{\partial L}{\partial \dot{q}_r} \quad (r = 1, 2, \dots, n) \dots \dots \dots (1),$$

and these equations will also serve to determine $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ in terms of the p 's and q 's, so that L may also be regarded as a function of the p 's, the q 's and t .

We now put
$$H = -L + \sum_{r=1}^n p_r \dot{q}_r \dots \dots \dots (2)$$

and call H the Hamiltonian Function.

Then
$$\delta H = -\delta \left\{ L - \sum_{r=1}^n p_r \dot{q}_r \right\}$$

$$= - \sum_{r=1}^n \left\{ \frac{\partial L}{\partial q_r} \delta q_r + \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r - p_r \delta \dot{q}_r - \dot{q}_r \delta p_r \right\} \dots \dots \dots (3).$$

But by Lagrange's equations,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = 0,$$

we have $\dot{p}_r = \frac{\partial L}{\partial q_r}$;

and, using (1), (3) becomes

$$\delta H = - \sum_{r=1}^n (\dot{p}_r \delta q_r - \dot{q}_r \delta p_r) \dots\dots\dots (4).$$

Hence if we regard H as a function of $q_1, q_2, \dots q_n, p_1, p_2, \dots p_n$ and t , we have

$$\dot{q}_r = \frac{\partial H}{\partial p_r}, \quad \dot{p}_r = - \frac{\partial H}{\partial q_r} \quad (r=1, 2, \dots n) \dots\dots\dots (5).$$

These $2n$ equations are *Hamilton's canonical equations of motion*. They involve $2n$ independent variables, the p 's and the q 's, but they are differential equations of the first order, whereas the equations of Lagrange are of the second order.

9·81. When the Lagrangian function L does not contain the time explicitly the Hamiltonian equations give an equation of energy. For we have

$$\frac{dH}{dt} = \sum_{r=1}^n \left(\frac{\partial H}{\partial q_r} \dot{q}_r + \frac{\partial H}{\partial p_r} \dot{p}_r \right) = 0,$$

from 9·8 (5). Therefore $H = \text{const.}$

This is also evident from the fact that T is a homogeneous quadratic function of the velocities, so that from (2)

$$H = -L + 2T = T + V = \text{const.}$$

9·82. Examples. (i) *The motion of a spherical pendulum.*

A particle of mass m is suspended by a fine string of length l , and we take q_1 as the inclination of the string to the vertical and q_2 as the azimuth of the vertical plane through the string. Then

$$2T = ml^2 (\dot{q}_1^2 + \dot{q}_2^2 \sin^2 q_1)$$

and

$$-V = mgl \cos q_1,$$

so that

$$L = \frac{1}{2} ml^2 (\dot{q}_1^2 + \dot{q}_2^2 \sin^2 q_1) + mgl \cos q_1 \dots\dots\dots (1),$$

and therefore

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = ml^2 \dot{q}_1 \quad \text{and} \quad p_2 = \frac{\partial L}{\partial \dot{q}_2} = ml^2 \dot{q}_2 \sin^2 q_1 \dots\dots\dots (2).$$

Hence

$$\begin{aligned} H &= -L + p_1 \dot{q}_1 + p_2 \dot{q}_2 \\ &= \frac{1}{2} \frac{p_1^2}{ml^2} + \frac{1}{2} \frac{p_2^2}{ml^2 \sin^2 q_1} - mgl \cos q_1 \dots\dots\dots (3). \end{aligned}$$

Then the four Hamiltonian equations are

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{ml^2}, \quad \dot{q}_2 = \frac{\partial H}{\partial p_2} = \frac{p_2}{ml^2 \sin^2 q_1} \dots\dots\dots (4),$$

$$\dot{p}_1 = - \frac{\partial H}{\partial q_1} = \frac{p_2^2 \cos q_1}{ml^2 \sin^3 q_1} - mgl \sin q_1, \quad \dot{p}_2 = - \frac{\partial H}{\partial q_2} = 0 \dots\dots\dots (5).$$

We note that (4) are the same equations as (2), that the last of (5) gives

$$p_2 = \text{const.} = ml^2 c, \text{ say; or } \dot{q}_2 \sin^2 q_1 = c \dots\dots\dots (6),$$

and the remaining equation then gives

$$\dot{q}_1 = \frac{c^2 \cos q_1}{\sin^3 q_1} - \frac{g}{l} \sin q_1 \dots\dots\dots (7).$$

On integration we get

$$\dot{q}_1^2 + \frac{c^2}{\sin^2 q_1} - 2 \frac{g}{l} \cos q_1 = \text{const.},$$

and it is easy to see that this is the same as the equation of energy $H = \text{const.}$, where the constant value of p_2 is substituted. In fact, since there are no geometrical equations containing the time explicitly, we might have written down the equation $H = \text{const.}$ at once.

(ii) *To deduce Euler's equations from Hamilton's equations.*

In the case of the motion of a rigid body about a fixed point, there are no geometrical equations containing the time explicitly, so that, as in 9·81, $H = T + V$.

Using the notation of 9·31 (vi), we have

$$2T = (A\omega_1^2 + B\omega_2^2 + C\omega_3^2) \dots\dots\dots (1),$$

where

$$\omega_1 = \dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi,$$

$$\omega_2 = \dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi,$$

$$\omega_3 = \dot{\phi} + \dot{\psi} \cos \theta.$$

Taking θ, ϕ, ψ as the generalized co-ordinates q_1, q_2, q_3 , we have

$$p_1 = \frac{\partial T}{\partial \dot{\theta}} = A\omega_1 \sin \phi + B\omega_2 \cos \phi,$$

$$p_2 = \frac{\partial T}{\partial \dot{\phi}} = C\omega_3$$

$$\text{and } p_3 = \frac{\partial T}{\partial \dot{\psi}} = -A\omega_1 \sin \theta \cos \phi + B\omega_2 \sin \theta \sin \phi + C\omega_3 \cos \theta.$$

$$\text{Therefore } A\omega_1 = p_1 \sin \phi - p_3 \frac{\cos \phi}{\sin \theta} + C\omega_3 \cot \theta \cos \phi$$

$$\text{and } B\omega_2 = p_1 \cos \phi + p_3 \frac{\sin \phi}{\sin \theta} - C\omega_3 \cot \theta \sin \phi$$

Hence

$$H = T + V = \frac{1}{2A} \left(p_1 \sin \phi - p_3 \frac{\cos \phi}{\sin \theta} + C\omega_3 \cot \theta \cos \phi \right)^2 + \frac{1}{2B} \left(p_1 \cos \phi + p_3 \frac{\sin \phi}{\sin \theta} - C\omega_3 \cot \theta \sin \phi \right)^2 + \frac{p_2^2}{2C} + V$$

$$\text{Then take the equation } \dot{p}_2 = -\frac{\partial H}{\partial \phi}.$$

$$\text{It gives } C\dot{\omega}_3 = \frac{1}{A} \cdot A\omega_1 \cdot B\omega_2 - \frac{1}{B} \cdot B\omega_2 \cdot A\omega_1 - \frac{\partial V}{\partial \phi}$$

or

$$C\dot{\omega}_3 - (A - B) \omega_1 \omega_2 = N.$$

(iii) *Motion of a particle in a plane referred to moving axes.*

Taking $\dot{x} - \omega y$, $\dot{y} + \omega x$ as the components of velocity, we may put

$$L = \frac{1}{2}m\{(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\} - V \dots\dots\dots(1),$$

so that

$$p_1 = \frac{\partial L}{\partial \dot{x}} = m(\dot{x} - \omega y)$$

and

$$p_2 = \frac{\partial L}{\partial \dot{y}} = m(\dot{y} + \omega x).$$

Then $H = -L + p_1\dot{x} + p_2\dot{y}$

$$\begin{aligned} &= -\frac{p_1^2}{2m} - \frac{p_2^2}{2m} + V + p_1\left(\frac{p_1}{m} + \omega y\right) + p_2\left(\frac{p_2}{m} - \omega x\right) \\ &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + p_1\omega y - p_2\omega x + V. \end{aligned}$$

The Hamiltonian equations give

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial H}{\partial x} = p_2\omega - \frac{\partial V}{\partial x}, & \dot{x} &= \frac{\partial H}{\partial p_1} = \frac{p_1}{m} + \omega y, \\ \dot{p}_2 &= -\frac{\partial H}{\partial y} = -p_1\omega - \frac{\partial V}{\partial y}, & \dot{y} &= \frac{\partial H}{\partial p_2} = \frac{p_2}{m} - \omega x. \end{aligned}$$

From these it follows that

$$m(\ddot{x} - 2\omega\dot{y} - \omega^2x) = -\partial V/\partial x$$

and

$$m(\ddot{y} + 2\omega\dot{x} - \omega^2y) = -\partial V/\partial y.$$

But Lagrange's equations applied to (1) would have given the same results immediately.

9.9. Variational methods. Hamilton's Principle. Principle of Least Action. Consider a conservative system whose configuration is defined by n independent generalized co-ordinates. Let $L = T - V$, where V is the potential energy, and suppose that the system has a definite motion, which we may speak of as its actual motion, between times $t = t_0$ and $t = t_1$. Consider also a varied motion in which the paths of the particles differ slightly from their paths in the actual motion. Thus if we were concerned with a single particle we might suppose it constrained by a smooth tube to move in a path differing slightly from its actual path. Let δ denote a variation in any function of the co-ordinates which does not involve a change in t , and let Δ denote a general variation, so that if the actual motion is from configuration A at time $t = t_0$ to configuration B at time $t = t_1$, a varied motion is from C at $t = t_0 + \Delta t_0$ to D at $t = t_1 + \Delta t_1$; and, if P, Q are corresponding configurations in

the two motions at times t and $t + \Delta t$, and q_r , $q_r + \Delta q_r$ are corresponding co-ordinates in P and Q , then

$$\Delta q_r = \delta q_r + \dot{q}_r \Delta t \dots\dots\dots(1)$$

and the operations δ and d/dt are commutative. Then since the value of the Lagrangian function at time t on the varied path is $L + \delta L$, we have

$$\begin{aligned} \Delta \int_{t_0}^{t_1} L dt &= \int_{t_0 + \Delta t_0}^{t_1 + \Delta t_1} (L + \delta L) dt - \int_{t_0}^{t_1} L dt \\ &= L_1 \Delta t_1 - L_0 \Delta t_0 + \int_{t_0}^{t_1} \delta L dt \\ &= \left[L \Delta t \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum_{r=1}^n \left(\frac{\partial L}{\partial q_r} \delta q_r + \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r \right) dt. \end{aligned}$$

$$\begin{aligned} \text{But } \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r dt &= \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_r} \frac{d}{dt} \delta q_r dt \\ &= \left[\frac{\partial L}{\partial \dot{q}_r} \delta q_r \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) \delta q_r dt. \end{aligned}$$

Therefore

$$\Delta \int_{t_0}^{t_1} L dt = \left[L \Delta t + \sum_{r=1}^n \frac{\partial L}{\partial \dot{q}_r} \delta q_r \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum_{r=1}^n \left\{ \frac{\partial L}{\partial q_r} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) \right\} \delta q_r dt \dots\dots(2);$$

and the co-ordinates being independent, it follows from Lagrange's equations that the terms under the sign of integration vanish, and, using (1), we have

$$\begin{aligned} \Delta \int_{t_0}^{t_1} L dt &= \left[L \Delta t + \sum_{r=1}^n \frac{\partial L}{\partial \dot{q}_r} (\Delta q_r - \dot{q}_r \Delta t) \right]_{t_0}^{t_1} \\ &= \left[\left\{ L - \sum_{r=1}^n \frac{\partial L}{\partial \dot{q}_r} \dot{q}_r \right\} \Delta t + \sum_{r=1}^n \frac{\partial L}{\partial \dot{q}_r} \Delta q_r \right]_{t_0}^{t_1}. \end{aligned}$$

The coefficient of Δt is $-H$, where H is the Hamiltonian function of 9.8; and thus we have

$$\Delta S = \left[-H \Delta t + \sum_{r=1}^n \frac{\partial L}{\partial \dot{q}_r} \Delta q_r \right]_{t_0}^{t_1} \dots\dots\dots(3),$$

where $S = \int_{t_0}^{t_1} L dt$ is called *Hamilton's Principal Function*.*

* Sir W. R. Hamilton, *Phil. Trans.* 1835, p. 95.

