TYPICAL MEANS

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1. TYPICAL MEANS. *By* K. Chandrasekharan & S. Minakshisundaram

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TO

K. ANANDA-RAU

PREFACE

THIS book deals with the theory of 'typical means' and its applications to Dirichlet series and Fourier series. More than forty years have now passed since 'typical means' were first introduced by M. Riesz for the summation of divergent series, and quite an extensive theory has developed during this period. We have attempted here to give a systematic account of this development. Readers of our account will hardly need to be told how much we owe to the Cambridge tract by Hardy and Riesz on the general theory of Dirichlet series.

We wish to acknowledge our indebtedness to Dr. L. S. Bosanquet, who has read the proofs and helped us to remove many errors and obscurities. His comments have stimulated us to improve the text in several places.

K. C.
S. M.

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CONTENTS

TYPICAL MEANS

I

FIRST THEOREM OF CONSISTENCY AND SOME CONVERSE THEOREMS

1.1. Introduction

IN this chapter we define the Riesz means of infinite series. The Riesz means have a certain *type* λ and a certain *order* k; correspondingly we define the *summability* (λ, k) of series, which reduces to convergence for $k = 0$. After establishing some relations between Riesz means of the same type but of different orders, we prove that if a series is summable (λ, k), $k \geqslant 0$, then it is also summable (λ, k') for $k' > k$. This is called the *first theorem of consistency*. We then study the converse problem. Knowing the order of magnitude of the Riesz mean (λ, k), we determine the order of magnitude of the Riesz mean (λ, r), $r < k$. This leads us to the fundamental theorem of M. Riesz, which imposes order-conditions on the Riesz means (λ, k) and $(\lambda, 0)$, and shows that they imply a restriction on the order of magnitude of the intermediate Riesz means (λ, r) for $0 < r < k$. We next use this theorem to prove Tauberian results. We assume that a series Σa_n is summable (λ, k), $k > 0$, and that its terms $\{a_n\}$ satisfy some appropriate order-condition, and deduce that Σa_n converges. We also define the notion of *absolute summability* (λ, k), which generalizes the notion of absolute convergence, and prove the analogue of the first theorem of consistency.

DEFINITION OF RIESZ SUMMABILITY. Let $\sum_{n=0}^{\infty} a_n$ be an infinite series, and let $\{\lambda_n\}$ be an arbitrary sequence of positive numbers such that

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty.$$

We write

$$A_n = a_0 + a_1 + \ldots + a_n\,,$$

and if $t > 0$, $\lambda_n \leqslant t < \lambda_{n+1}$, then

$$A_\lambda(t) \equiv A_n = a_0 + \ldots + a_n = \sum_{\lambda_\nu \leqslant t} a_\nu\,,$$

and for $k > 0$,

$$A_\lambda^k(t) = \sum_{\lambda_\nu \leqslant t} (t - \lambda_\nu)^k a_\nu$$

$$= k \int_0^t (t-\tau)^{k-1} A_\lambda(\tau) d\tau$$

$$= \int_0^t (t-\tau)^k dA_\lambda(\tau).$$

We define $A_\lambda^0(t) \equiv A_\lambda(t)$, and if $t < \lambda_0$, $A_\lambda^k(t) \equiv 0$ for every $k \geqslant 0$. $A_\lambda(t)$ is a discontinuous function, being constant in intervals, while $A_\lambda^k(t)$ is a continuous function of t for $k > 0$—in fact, absolutely continuous in every finite interval if $0 < k \leqslant 1$, and differentiable with continuous derivatives if $k > 1$. Actually we have

$$\frac{d}{dt}[A_\lambda^k(t)] = kA_\lambda^{k-1}(t), \ k > 1,$$

and if k is an integer, and $t \neq \lambda_n$,

$$\left(\frac{d}{dt}\right)^k A_\lambda^k(t) = k!\ A_\lambda(t) = \Gamma(k+1)\ A_\lambda(t).$$

If we write

$$C_\lambda^k(x) = x^{-k} A_\lambda^k(x),$$

then $C_\lambda^k(x)$ is called the *Riesz mean of order k and type λ*, while $A_\lambda^k(x)$ is called the *Riesz sum of order k and type λ* associated with the series Σa_n.

DEFINITION 1.11. *If $\lim_{x\to\infty} C_\lambda^k(x) = s$ exists, where s is finite, we say that Σa_n is summable by Riesz means of order k and type λ, or simply, summable $(R; \lambda, k)$ to the sum s.*

DEFINITION 1.12. *If $C_\lambda^k(x) = O(1)$, then Σa_n is said to be bounded $(R; \lambda, k)$.*

DEFINITION 1.13. *If $\int_h^\infty |dC_\lambda^k(x)| < \infty$, $h > 0$, we say that Σa_n is absolutely summable by Riesz means of order k and type λ, or simply, summable $|R; \lambda, k|$.*

When $k = 0$, Definitions 1.11 and 1.13 lead to convergence and absolute convergence respectively.

Since we are only concerned with Riesz summability in the following pages, we omit the 'R' in the definition, and speak of summability (λ, k) and $|\lambda, k|$ respectively.

1.2. Relations between different Riesz sums

We shall establish here some of the formulae used frequently in our later sections.

In the first place, we prove certain relations between Riesz sums of different *orders*.

If $k > 0$, $l > 0$, then

$$A_\lambda^{k+l}(x) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l)} \int_0^x (x-t)^{l-1} A_\lambda^k(t)\,dt. \tag{1.21}$$

If $k > 0$, $0 < l < 1$, $l < k$, or $l = k$ and $x \neq \lambda_n$, then

$$A_\lambda^{k-l}(x) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(1-l)} \int_0^x (x-t)^{-l}\,dA_\lambda^k(t). \tag{1.22}$$

Proof of (1.21).

$$\begin{aligned}\int_0^x (x-t)^{l-1} A_\lambda^k(t)\,dt &= k\int_0^x (x-t)^{l-1}\,dt \int_0^t (t-u)^{k-1} A_\lambda(u)\,du \\ &= k\int_0^x A_\lambda(u)\,du \int_u^x (x-t)^{l-1}(t-u)^{k-1}\,dt \\ &= \frac{k\,\Gamma(k)\,\Gamma(l)}{\Gamma(k+l)} \int_0^x (x-u)^{k+l-1} A_\lambda(u)\,du \\ &= \frac{\Gamma(k+1)\,\Gamma(l)}{\Gamma(k+l+1)} A_\lambda^{k+l}(x).\end{aligned}$$

Proof of (1.22).

$$A_\lambda^{k-l}(x) = \frac{1}{k-l+1}\left[\frac{d}{dx} A_\lambda^{k-l+1}(x)\right]$$

$$= \frac{\Gamma(k-l+2)}{(k-l+1)\Gamma(k+1)\Gamma(1-l)} \cdot \frac{d}{dx}\int_0^x (x-t)^{-l} A_\lambda^k(t)\,dt, \qquad \text{by (1.21)}.$$

By partial integration and differentiation, we obtain (1.22).

We next establish a relation between Riesz sums of different *types*. Let $\mu(\sigma)$ be a positive, non-decreasing function of σ diverging to infinity with σ, and let us set $\mu(\lambda_n) = \mu_n$, so that

$$0 < \mu_0 < \mu_1 < \mu_2 < \ldots < \mu_n \to \infty,$$

and

$$A^k_\mu(x) = \sum_{\mu_n \leqslant x} (x - \mu_n)^k a_n = k \int_0^x (x - t)^{k-1} A_\mu(t) dt,$$

where

$$A_\mu(t) = \sum_{\mu_m \leqslant t} a_m .$$

We observe that if $t = \mu(\tau)$, then $A_\mu(t) = A_\mu[\mu(\tau)] = A_\lambda(\tau)$, and if $\omega = \mu(x)$, then

$$A^k_\mu(\omega) = k \int_0^\omega (\omega - t)^{k-1} A_\mu(t)\, dt$$

$$= k \int_0^x [\mu(x) - \mu(\tau)]^{k-1} A_\lambda(\tau)\, d\mu(\tau)$$

$$= k \int_0^x [\mu(x) - \mu(\tau)]^{k-1} A_\lambda(\tau) \mu'(\tau)\, d\tau, \quad (1.23)$$

on assuming that $\mu'(\tau)$ exists.

1.3. Finite differences of Riesz sums

We now proceed to establish some formulae on finite differences which will be of frequent use in the sequel.

If $\zeta > 0$, and $F(x)$ is a function of x, and $m > 0$ is any integer, we set

$$\Delta^m_\zeta F(x) = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} F(x + m - \nu\zeta); \quad \Delta^0_\zeta F(x) = F(x); \quad (1.31)1$$

$$\Delta^m_{-\zeta} F(x) = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} F(x - \nu\zeta); \quad \Delta^0_{-\zeta} F(x) = F(x). \quad (1.31)2$$

If $0 < a < 1$, we set

$$\Delta_\zeta^a F(x) = a \int_x^{x+\zeta} (x + \zeta - t)^{a-1} F(t)dt, \tag{1.32)1}$$

$$\Delta_{-\zeta}^a F(x) = a \int_{x-\zeta}^{x} (x - t)^{a-1} F(t)dt. \tag{1.32)2}$$

Also, we define

$$\Delta_\zeta^{m+a} F(x) = \Delta_\zeta^a \, [\Delta_\zeta^m \, F(x)], \tag{1.33)1}$$

$$\Delta_{-\zeta}^{m+a} F(x) = \Delta_{-\zeta}^a \, [\Delta_{-\zeta}^m \, F(x)]. \tag{1.33)2}$$

It is easily seen that

$$\Delta_\zeta^a \, [\Delta_\zeta^m \, F(x)] = \Delta_\zeta^m \, [\Delta_\zeta^a \, F(x)], \tag{1.34)1}$$

$$\Delta_{-\zeta}^a \, [\Delta_{-\zeta}^m \, F(x)] = \Delta_{-\zeta}^m \, [\Delta_{-\zeta}^a \, F(x)]. \tag{1.34)2}$$

We then have, for $0 \leqslant m \leqslant h$, and m an integer,

$$\Delta_\zeta^m \, A_\lambda^h(x) = \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \int_x^{x+\zeta} dt_1 \int_{t_1}^{t_1+\zeta} dt_2 \ldots \int_{t_{m-1}}^{t_{m-1}+\zeta} A_\lambda^{h-m} (t_m)dt_m \, , \tag{1.35)1}$$

$$\Delta_{-\zeta}^m \, A_\lambda^h(x) = \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \int_{x-\zeta}^{x} dt_1 \int_{t_1-\zeta}^{t_1} dt_2 \ldots \int_{t_{m-1}-\zeta}^{t_{m-1}} A_\lambda^{h-m}(t_m)dt_m \, . \tag{1.35)2}$$

These formulae (1.35) are easily proved by induction. We have also

$$\Delta_\zeta^m \, A_\lambda^h(x) = \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \zeta^m A_\lambda^{h-m}(x) +$$

$$+ \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \int_x^{x+\zeta} dt_1 \ldots \int_{t_{m-1}}^{t_{m-1}+\zeta} [A_\lambda^{h-m}(t_m) - A_\lambda^{h-m}(x)] \, dt_m \, , \tag{1.36)1}$$

and

$$\Delta_\zeta^{m+a} A_\lambda^h(x) = a \int_x^{x+\zeta} (x + \zeta - t)^{a-1} \, \Delta_\zeta^m \, A_\lambda^h(t)dt \qquad \text{[cf. (1.33)1]}$$

$$= \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \zeta^{m+\alpha} A_\lambda^{h-m}(x) + \frac{\alpha\,\Gamma(h+1)}{\Gamma(h-m+1)} \times$$

$$\times \int_x^{x+\zeta} (x+\zeta-t)^{\alpha-1} dt \int_t^{t+\zeta} dt_1 \ldots \int_{t_{m-1}}^{t_{m-1}+\zeta} [A_\lambda^{h-m}(t_m) - A_\lambda^{h-m}(x)]\, dt_m$$

$$= \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \zeta^{m+\alpha} A_\lambda^{h-m}(x) + \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \times$$

$$\times \Delta_\zeta^\alpha \left[\int_x^{x+\zeta} dt_1 \ldots \int_{t_{m-1}}^{t_{m-1}+\zeta} [A_\lambda^{h-m}(t_m) - A_\lambda^{h-m}(x)] dt_m \right]. \quad (1.36)2$$

Similar to (1.36) we have

$$\Delta_{-\zeta}^m A_\lambda^h(x) = \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \zeta^m A_\lambda^{h-m}(x) -$$

$$- \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \int_{x-\zeta}^{x} dt_1 \cdots \int_{t_{m-1}-\zeta}^{t_{m-1}} [A_\lambda^{h-m}(x) - A_\lambda^{h-m}(t_m)] dt_m, \quad (1.37)1$$

and

$$\Delta_{-\zeta}^{m+\alpha} A_\lambda^h(x) = \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \zeta^{m+\alpha} A_\lambda^{h-m}(x) -$$

$$- \frac{\Gamma(h+1)}{\Gamma(h-m+1)} \Delta_{-\zeta}^\alpha \left[\int_{x-\zeta}^{x} dt_1 \cdots \int_{t_{m-1}-\zeta}^{t_{m-1}} [A_\lambda^{h-m}(x) - A_\lambda^{h-m}(t_m)] dt_m \right]. \quad (1.37)2$$

We could conveniently rewrite the above formulae thus :

If m is a positive integer, $r \geqslant 0$, $0 \leqslant \beta < 1$, then we obtain from (1.36)1 and (1.36)2,

$$\zeta^{m+\beta} A_\lambda^r(x) = \frac{\Gamma(r+1)}{\Gamma(r+m+1)} \Delta_\zeta^{m+\beta} A_\lambda^{r+m}(x) -$$

$$- \Delta_\zeta^\beta \left[\int_x^{x+\zeta} dt_1 \int_{t_1}^{t_1+\zeta} dt_2 \ldots \int_{t_{m-1}}^{t_{m-1}+\zeta} [A_\lambda^r(t_m) - A_\lambda^r(x)]\, dt_m \right]. \quad (1.38)$$

Similarly we obtain from (1.37)1 and (1.37)2,

$$\zeta^{m+\beta}A_\lambda^r(x) = \frac{\Gamma(r+1)}{\Gamma(r+m+1)}\,\Delta_{-\zeta}^{m+\beta}\,A_\lambda^{r+m}(x) + $$
$$+\,\Delta_{-\zeta}^{\beta}\left[\int_{x-\zeta}^{x} dt_1 \int_{t_1-\zeta}^{t_1} dt_2 \ldots \int_{t_{m-1}-\zeta}^{t_{m-1}} [\,A_\lambda^r(x) - A_\lambda^r(t_m)\,]\,dt_m\right]. \tag{1.39}$$

1.4. Two lemmas

We now prove two lemmas which play an important part in the proof of some of our later theorems.

Lemma 1.41. *If* $0 \leqslant \xi < x$, $k \geqslant 0$, $0 < l \leqslant 1$, *then*

$$|\,g(\xi,x)\,| \equiv \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l)}\left|\int_0^\xi A_\lambda^k(t)\,(x-t)^{l-1}dt\right| \leqslant \max_{0\leqslant t\leqslant\xi} |\,A_\lambda^{k+l}(t)\,|,$$

so that if $A_\lambda(t)$ *is real, we have*

$$\frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l)}\int_0^\xi A_\lambda^k(t)\,(x-t)^{l-1}\,dt = A_\lambda^{k+l}(\tau)\,,\quad 0 \leqslant \tau \leqslant \xi,$$

for some τ.

Proof. The case $l = 1$ being trivial, we observe that if $0 < l < 1$,

$$(x-t)^{l-1} = \int_t^\xi \theta_x(\nu)\,(\nu - t)^{l-1}d\nu,$$

where

$$\theta_x(\nu) = \frac{(x-\xi)^l\,(x-\nu)^{-1}\,(\xi-\nu)^{-l}}{\Gamma(l)\,\Gamma(1-l)}\,,$$

so that

$$\theta_x(\nu) > 0, \text{ and } \int_{-\infty}^{\xi} \theta_x(\nu)\,d\nu = 1.$$

Hence

$$g(\xi,x) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l)}\int_0^\xi A_\lambda^k(t)\,dt\int_t^\xi \theta_x(\nu)\,(\nu-t)^{l-1}\,d\nu$$

$$= \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l)} \int_0^{\xi} \theta_x(\nu)\,d\nu \int_0^{\nu} (\nu - t)^{l-1} A_\lambda^k(t)\,dt$$

$$= \int_0^{\xi} A_\lambda^{k+l}(\nu)\,\theta_x(\nu)\,d\nu. \tag{1.41}$$

Therefore

$$\min_{0 \leqslant t \leqslant \xi} A_\lambda^{k+l}(t) \leqslant g(\xi, x) \leqslant \max_{0 \leqslant t \leqslant \xi} A_\lambda^{k+l}(t).$$

LEMMA 1.42. *Let $\varphi(x)$ be a positive, non-decreasing function of x, defined for $x > 0$, and let $0 \leqslant \xi < x$, $0 < l < 1$, $k \geqslant 0$. Then*

$$A_\lambda^{k+l}(x) = o\,[\varphi(x)] \tag{1.42}$$

implies, uniformly in ξ,

$$g(\xi, x) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l)} \int_0^{\xi} (x-t)^{l-1} A_\lambda^k(t)\,dt = o\,[\varphi(x)]. \tag{1.43}$$

PROOF. From (1.41) we have

$$g\,(\xi, x) = \int_0^{\xi} A_\lambda^{k+l}(\nu)\,\theta_x(\nu)\,d\nu,$$

where

$$\theta_x(\nu) > 0,$$

$$\int_0^{h} \theta_x(\nu)\,d\nu = o\,(1),$$

as $x \to \infty$, for every fixed $h > 0$, and

$$\int_0^{\xi} \theta_x(\nu)\,d\nu < 1.$$

These properties of $\theta_x(\nu)$, together with the hypothesis (1.42), lead to (1.43).

1.5. First theorem of consistency

We shall now prove two elementary theorems on Riesz summability, concerning the relation between *different orders* of summability belonging to the *same type*. The first theorem in this direction says that the power of Riesz summability increases with the order. Thus we have

THEOREM 1.51. *If* Σa_n *is summable* (λ, k), $k \geqslant 0$, *to the sum* c, *then* Σa_n *is also summable* (λ, k'), *for* $k' > k$, *to the same sum.*

The proof follows easily from the formula :

$$C_\lambda^{k'}(x) = x^{-k'} A_\lambda^{k'}(x)$$
$$= \frac{\Gamma(k'+1)\, x^{-k'}}{\Gamma(k+1)\,\Gamma(k'-k)} \int_0^x (x-t)^{k'-k-1}\, t^k\, C_\lambda^k(t)\, dt. \qquad (1.51)$$

The above theorem is called the 'first theorem of consistency', and it follows therefrom that a convergent series is always summable (λ, k) to the same sum, for every $k > 0$, whatever the particular divergent sequence $\{\lambda_n\}$ may be. By using formula (1.51), we can also prove

THEOREM 1.52. *If* $W(x)$ *is a positive, non-decreasing function of* x, *then* $A_\lambda^k(x) = O[W(x)]$, $k \geqslant 0$, *implies* $A_\lambda^{k'}(x) = O[x^{k'-k}\, W(x)]$, $k' > k$.

1.6. Scope of Riesz summability

The scope of a method of summability can be roughly determined by an examination of the nature of series which are summable by that method. Thus we should know, first of all, what are the *necessary* implications of the statement that a given series is summable (λ,k). Some such knowledge is gained from the following

THEOREM 1.61. *If* $\varphi(x)$ *is a positive non-decreasing function of* x, *and if*

$$A^k(x) - c\,x^k = o\,[\varphi(x)],\ k > 0,$$

then for $\lambda_n \leqslant x < \lambda_{n+1}$, *and* $0 \leqslant r < k$, *we have*

$$A^r(x) - c\,x^r = \begin{cases} o\left[\dfrac{\varphi(\lambda_{n+1})}{(\lambda_{n+1}-\lambda_n)^{k-r}}\right], & \textit{if } r \textit{ is an integer or zero;} \\[2ex] o\left[\dfrac{\varphi(\lambda_n)}{(\lambda_n-\lambda_{n-1})^{k-r}}\right] + o\left[\dfrac{\varphi(\lambda_{n+1})}{(\lambda_{n+1}-\lambda_n)^{k-r}}\right], & \textit{if } r \textit{ is non-integral.} \end{cases}$$

REMARK. In this section, as in some later sections, we write $A^k(x)$ in place of $A^k_\lambda(x)$.

PROOF. We shall assume, without loss of generality, that $c = 0$; for, if $c \neq 0$, we set $B(x) = A(x) - c$, so that $B^k(x) = A^k(x) - c\,x^k$, and argue with $B(x)$ in place of $A(x)$.

Let $h = [k]$, where $[k]$ denotes the integral part of k, $(h+1)\zeta = \lambda_{n+1} - \lambda_n$; let l be an integer less than or equal to h, or zero; let $k = h + \beta$, so that $0 \leqslant \beta < 1$. Then we have

$$\Delta_\zeta^{l+\beta} A^h(\lambda_n) = o\,[\varphi(\lambda_{n+1})]. \tag{1.61}$$

For, if $\beta > 0$,

$$\begin{aligned}\Delta_\zeta^{l+\beta} A^h(\lambda_n) &= \Delta_\zeta^l \left[\int_{\lambda_n}^{\lambda_n+\zeta} (\lambda_n + \zeta - t)^{\beta-1} A^h(t)\,dt \right] \\ &= o\,[\varphi(\lambda_{n+1})],\end{aligned}$$

on account of the hypothesis on $A^k(x)$ and Lemma 1.42.

If $\beta = 0$, then $k = h$, and

$$\Delta_\zeta^{l+\beta} A^h(\lambda_n) = \Delta_\zeta^l A^h(\lambda_n) = o\,[\varphi(\lambda_{n+1})].$$

CASE (i). Let $r = 0$. Setting h for m, and λ_n for x, and 0 for r in formula (1.38), we obtain

$$\begin{aligned}\zeta^{h+\beta} A(\lambda_n) &= \frac{\Delta_\zeta^{h+\beta} A^h(\lambda_n)}{\Gamma(h+1)} - \\ &\qquad - \Delta_\zeta^\beta \left[\int_{\lambda_n}^{\lambda_n+\zeta} dt_1 \int_{t_1}^{t_1+\zeta} dt_2 \cdots \int_{t_{h-1}}^{t_{h-1}+\zeta} [A(t_h) - A(\lambda_n)]\,dt_h \right] \\ &= \frac{\Delta_\zeta^{h+\beta} A^h(\lambda_n)}{\Gamma(h+1)}, \qquad \text{since } A(t_h) = A(\lambda_n), \\ &= o[\varphi(\lambda_{n+1})],\end{aligned}$$

by (1.61). Hence

$$A(\lambda_n) = o\left[\frac{\varphi(\lambda_{n+1})}{(\lambda_{n+1} - \lambda_n)^k}\right]. \tag{1.62}$$

CASE (ii). Let r be a positive integer. We prove the result now by induction. The theorem has already been proved for $r = 0$. Assuming it to be true for $r - 1$, we prove that it is true for r. If

$$A^{r-1}(x) = o\left[\frac{\varphi(\lambda_{n+1})}{(\lambda_{n+1} - \lambda_n)^{k-r+1}}\right],$$

then

$$A^r(x) - A^r(\lambda_n) = r\int_{\lambda_n}^{x} A^{r-1}(t)\,dt = o\left[\frac{\varphi(\lambda_{n+1})}{(\lambda_{n+1} - \lambda_n)^{k-r}}\right]. \qquad (1.63)$$

Setting $h - r$ for m and λ_n for x in (1.38), we obtain

$$\zeta^{h-r+\beta} A^r(\lambda_n) = \frac{\Gamma(r+1)}{\Gamma(h+1)}\Delta_\zeta^{h-r+\beta} A^h(\lambda_n) -$$

$$- \Delta_\zeta^\beta\left[\int_{\lambda_n}^{\lambda_n+\zeta} dt_1 \int_{t_1}^{t_1+\zeta} dt_2 \ldots \int_{t_{h-r-1}}^{t_{h-r-1}+\zeta} [A^r(t_{h-r}) - A^r(\lambda_n)]\,dt_{h-r}\right].$$

The first expression on the right side is $o[\varphi(\lambda_{n+1})]$ on account of (1.61); while the second is $o[\varphi(\lambda_{n+1})]$ because of (1.63). Hence

$$A^r(\lambda_n) = o\left[\frac{\varphi(\lambda_{n+1})}{(\lambda_{n+1} - \lambda_n)^{k-r}}\right]. \qquad (1.64)$$

CASE (iii). Let r be non-integral; $s = [r]$, $r = s + a$, $0 < a < 1$; let p denote the greatest integer *less than* k, so that if k is non-integral, $p = h$, and if k is integral $p = k - 1$. By (1.21) we have

$$A^r(x) = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(a)}\int_0^x (x-t)^{a-1} A^s(t)\,dt$$

$$= \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(a)}\left[\int_0^{\lambda_{n-1}} + \int_{\lambda_{n-1}}^{\lambda_n} + \int_{\lambda_n}^{x}\right]$$

$$\equiv J_1 + J_2 + J_3,$$

say. Now, using the result obtained above in Case (ii), we obtain

$$J_3 = o\left(\frac{\varphi(\lambda_{n+1})}{(\lambda_{n+1} - \lambda_n)^{k-s}}\right)\int_{\lambda_n}^{x} (x-t)^{r-s-1}\,dt$$

$$= o\left[\frac{\varphi(\lambda_{n+1})}{(\lambda_{n+1} - \lambda_n)^{k-r}}\right], \qquad (1.65)$$

and

$$J_2 = o\left(\frac{\varphi(\lambda_n)}{(\lambda_n-\lambda_{n-1})^{k-s}}\right) \int_{\lambda_{n-1}}^{\lambda_n} (x-t)^{r-s-1}\,dt$$

$$= o\left[\frac{\varphi(\lambda_n)}{(\lambda_n-\lambda_{n-1})^{k-r}}\right]. \tag{1.66}$$

Integrating J_1 by parts, we obtain

$$J_1 = \sum_{\nu=1}^{p-s} \frac{\Gamma(r+1)}{\Gamma(s+\nu+1)\,\Gamma(r-s-\nu+1)} (x-\lambda_{n-1})^{r-s-\nu} A^{s+\nu}(\lambda_{n-1}) +$$

$$+ \frac{\Gamma(r+1)}{\Gamma(p+1)\,\Gamma(r-p)} \int_0^{\lambda_{n-1}} A^p(t)\,(x-t)^{r-p-1}dt$$

$$\equiv J_{1,1} + J_{1,2}, \tag{1.67}$$

say. Since $r-s-\nu = (\alpha-\nu) < 0$, we may majorize powers of $(x-\lambda_{n-1})$ in $J_{1,1}$ by the corresponding powers of $(\lambda_n - \lambda_{n-1})$; if we do this, and apply the result proved in Case (ii), we obtain

$$J_{1,1} = o\left[\frac{\varphi(\lambda_n)}{(\lambda_n-\lambda_{n-1})^{k-r}}\right];$$

$$J_{1,2} = c\int_0^{\lambda_{n-1}} (x-t)^{r-k}(x-t)^{k-p-1} A^p(t)dt, \qquad \text{where } c \text{ is a constant,}$$

$$= c\,(x-\lambda_{n-1})^{r-k} \int_{\xi}^{\lambda_{n-1}} (x-t)^{k-p-1}A^p(t)dt, \quad 0 < \xi < \lambda_{n-1},$$

by the second mean-value theorem.

Replacing x by λ_n in the expression outside the integral sign, and applying Lemma 1.42 to the integral, we obtain

$$J_{1,2} = o\left[\frac{\varphi(\lambda_n)}{(\lambda_n-\lambda_{n-1})^{k-r}}\right]. \tag{1.68}$$

Combining results (1.65)-(1.68), we prove the required result.

As a particular case of Theorem 1.61, we obtain the well-known

THEOREM 1.62. *If Σa_n is summable (λ, k) to the sum c, then for $0 \leqslant r < k$, and $\lambda_n \leqslant x < \lambda_{n+1}$, we have*

$$A^r(x) - cx^r = \begin{cases} o\left[\dfrac{\lambda_{n+1}^k}{(\lambda_{n+1}-\lambda_n)^{k-r}}\right], & \text{if } r \text{ is an integer or zero;} \\ o\left[\dfrac{\lambda_n^k}{(\lambda_n-\lambda_{n-1})^{k-r}}\right] + o\left[\dfrac{\lambda_{n+1}^k}{(\lambda_{n+1}-\lambda_n)^{k-r}}\right], & \text{if } r \text{ is non-integral.} \end{cases}$$

This is obtained by setting $\varphi(x) = x^k$ in Theorem 1.61.

COROLLARY 1.61. *If $\lambda_n = n$, Theorem 1.62 shows that if Σa_n is summable (n, k), then*

$$a_n = o\,(n^k),\ A_n = o\,(n^k),\quad .\ .\ .\ .\ ,\quad A^r(x) = o\,(x^k).$$

COROLLARY 1.62. *If $\lambda_{n+1} = O(\lambda_{n+1} - \lambda_n)$, or* $\liminf \dfrac{\lambda_{n+1}}{\lambda_n} > 1$, *then $A^r(x) - cx^r = o[\varphi(\lambda_{n+1})/\lambda_n^{k-r}]$.*

In particular, under the hypothesis of Corollary 1.62, a summable series is necessarily convergent. For example, *if $\lambda_n = 2^n$, then Riesz's method of summability will sum only convergent series.*

1.7. A theorem of M. Riesz

In Theorem 1.62, we assumed that a given series Σa_n was summable (λ, k), and deduced therefrom the order of magnitude of the Riesz sums $A^r(x)$ for $0 \leqslant r < k$. More generally, in Theorem 1.61, we proved that if $A^k(x)$ satisfies a certain order-condition, then $A^r(x)$, for $0 \leqslant r < k$, satisfies another order-condition. In the following theorem, due to M. Riesz, we assume that both $A^k(x)$ and $A(x)$ satisfy certain order-conditions (either of O or o type), and prove that $A^r(x)$, $0 < r < k$, will satisfy an order-condition related to the given conditions on $A^k(x)$ and $A(x)$. Thus we have

THEOREM 1.71. *Let $V(x)$ and $W(x)$ be two positive non-decreasing functions of x defined for $x > 0$. Let*

$$U_r(x) = [V(x)]^{1-r/k}\,[W(x)]^{r/k}.$$

Then

(A) $$|A^k(x)| < W(x) \text{ and } |A(x)| < V(x)$$

imply

$$|A^r(x)| < c\, U_r(x), \text{ for } 0 < r < k,$$

where c is a constant depending on r and k only, while

(B) $$|A^k(x)| < W(x) \text{ and } A(x) = o\,[\,V(x)\,]$$

imply

$$A^r(x) = o\,[U_r(x)],\ 0 < r < k,$$

and

(C) $$A^k(x) = o\,[W(x)] \text{ and } |A(x)| < V(x)$$

imply

$$A^r(x) = o\,[U_r(x)],\ 0 < r < k.$$

PROOF OF (A). The hypothesis $|A(x)| < V(x)$ implies $|A^r(x)| < x^r V(x)$, and so the theorem becomes trivial if

$$x^r \leqslant [W/V]^{r/k}\ , \text{ or } x \leqslant [W/V]^{1/k}.$$

We shall therefore assume that we can always determine $\xi > 0$ by the equation

$$x - \xi = [W(x)/V(x)]^{1/k}\ .$$

CASE (i). $0 < k \leqslant 1$.

$$A^r(x) = r\int_0^x (x-t)^{r-1} A(t)\,dt = r\left[\int_0^\xi + \int_\xi^x\right] \equiv J_1 + J_2, \text{ say.}$$

$$|J_2| = \left| r\int_\xi^x (x-t)^{r-1} A(t)\,dt \right| < (x-\xi)^r V(x) = U_r(x). \qquad (1.71)$$

$$J_1 = r\int_0^\xi (x-t)^{r-k}(x-t)^{k-1} A(t)\,dt$$

$$= r\,(x-\xi)^{r-k}\int_u^\xi (x-t)^{k-1} A(t)\,dt,\ 0 \leqslant u \leqslant \xi,$$

$$= \frac{r}{k}\cdot (x-\xi)^{r-k}\,[g(x,\xi) - g\,(x,u)],$$

in the notation of Lemma 1.41. Hence

$$|J_1| < (2r/k)\,(x-\xi)^{r-k} W(x) = (2r/k)\,U_r(x). \qquad (1.72)$$

Combining (1.71) and (1.72), we observe that

$$|A^r(x)| < (2r/k + 1)\, U_r(x).$$

CASE (ii). $k > 1$, $h = [k]$, $(h+1)\zeta = x - \xi$, $k = h + \beta$, $0 \leqslant \beta < 1$. We shall first prove that the result is true for integral r, by induction, the case $r=0$ being trivial. Suppose

$$A^{r-1}(x) = O\left[U_{r-1}(x)\right].$$

Then, for $\xi \leqslant x' < x$,

$$A^r(x) - A^r(x') = r \int_{x'}^{x} A^{r-1}(t)\, dt$$

$$= O\left[(x - x')\, U_{r-1}(x)\right]$$

$$= O\left[U_r(x)\right]. \tag{1.73}$$

Writing $h - r$ for m in (1.39), we obtain

$$\zeta^{h-r+\beta} A^r(x) = \frac{\Gamma(r+1)}{\Gamma(h+1)} \Delta_{-\zeta}^{h-r+\beta} A^h(x) +$$

$$+ \Delta_{-\zeta}^{\beta}\left[\int_{x-\zeta}^{x} dt_1 \int_{t_1-\zeta}^{t_1} dt_2 \ldots \int_{t_{h-r-1}-\zeta}^{t_{h-r-1}} \left[A^r(x) - A^r(t_{h-r})\right] dt_{h-r}\right].$$

The first expression on the right side is $O[W(x)]$ on account of the hypothesis on $A^k(x)$ and Lemma 1.42, while the second is $O[U_r(x)\,\zeta^{k-r}]$, on account of (1.73). Hence

$$\zeta^{k-r} A^r(x) = O[W(x)] + O[U_r(x)\,\zeta^{k-r}],$$

or

$$A^r(x) = O[U_r(x)]. \tag{1.74}$$

Next let us consider the case where r is non-integral. Let $s = [r]$, $r = s + \alpha$, $0 < \alpha < 1$. From (1.74) we then obtain $A^s(x) = O[U_s(x)]$. Also

$$A^{s+1}(x) = O[U_{s+1}(x)], \text{ if } s + 1 \leqslant k,$$

so that

$$A^r(x) = A^{s+\alpha}(x) = O[U_s^{1-\alpha} U_{s+1}^{\alpha}] = O[U_r(x)],$$

on account of the result proved in Case (i).

If, however, $s + 1 > k$, that is $s = h$, then $A^h(x) = O[U_h(x)]$, which together with $A^k(x) = O[W(x)]$ yields the required result, on an application of Case (i).

PROOF OF (B). Given $\varepsilon > 0$, choose x_0 such that $|A(x)| < \varepsilon V(x)$, for $x > x_0$. Let ξ be such that

$$x - \xi = \left(\frac{W}{\varepsilon V}\right)^{1/k}. \tag{1.75}$$

CASE (i). $0 < k \leqslant 1$. If $0 < x_0 < \xi$, then

$$A^r(x) = r\int_0^x (x - t)^{r-1} A(t)dt = r\left[\int_0^{\xi} + \int_{\xi}^{x}\right] \equiv J_1 + J_2, \text{ say.}$$

$$|J_1| = \left| r\,(x - \xi)^{r-k}\int_u^{\xi} (x - t)^{k-1} A(t)\,dt\right|, \ 0 \leqslant u \leqslant \xi,$$

by the second mean-value theorem. Thus

$$|J_1| \leqslant 2r\left(\frac{W}{\varepsilon V}\right)^{r/k-1}. \ W(x) = 2r.\ \varepsilon^{1-r/k}\ U_r(x),$$

on account of (1.75) and the hypothesis on $A^k(x)$. Again

$$|J_2| \leqslant \varepsilon V\,(x)\,(x - \xi)^r = \varepsilon^{1-r/k}\ U_r(x).$$

Hence

$$|A^r(x)| \leqslant |J_1| + |J_2| \leqslant (2r + 1)\ \varepsilon^{1-r/k}\ U_r(x). \tag{1.76}$$

If, however, $x > x_0 > \xi$, we write

$$A^r(x) = r\left[\int_0^{x_0} + \int_{x_0}^{x}\right] \equiv I_1 + I_2,$$

say. As in J_2, we have

$$|I_2| < \varepsilon^{1-r/k}\ U_r(x),$$

while

$$|I_1| = \left| r \int_0^{x_0} (x-t)^{r-1} A(t)\, dt \right|$$

$$< r(x - x_0)^{r-1} \int_0^{x_0} |A(t)|\, dt$$

$$= o(1),$$

as $x \to \infty$, since $r - 1 < 0$; hence

$$A^r(x) = o(1) + o[U_r(x)] = o[U_r(x)].$$

CASE (ii). $k > 1$. The hypotheses of Theorem (B) necessarily imply that

$$A^r(x) = O[U_r(x)], \tag{1.77}$$

on account of the previous result, Theorem (A). Actually we have got to show that

$$A^r(x) = o[U_r(x)].$$

This we prove by combining Case (i) of the present Theorem (B) with the previous Theorem (A). For, we have, by Case (i) of (B),

$$A^\beta(x) = o[U_\beta(x)], \quad 0 \leqslant \beta < 1;$$

and by (1.77), if $0 \leqslant \beta \leqslant k-1$,

$$A^{1+\beta}(x) = O[U_{1+\beta}(x)].$$

Applying Case (i), we deduce that

$$A^\gamma(x) = o[U_\gamma(x)], \ 0 \leqslant \gamma < 1 + \beta, \text{ or } \gamma < 2.$$

This, together with

$$A^{\gamma+1}(x) = O[U_{\gamma+1}(x)],$$

will again lead to

$$A^\delta(x) = o[U_\delta(x)], \quad 0 \leqslant \delta < 3,$$

and so on, until the result is proved for $0 < r < h = [k]$. Thus

$$A^{h-\varepsilon}(x) = o[U_{h-\varepsilon}(x)],$$

for every small positive ε, which, together with the hypothesis on $A^k(x)$, leads to the required result.

PROOF OF (C). Given $\varepsilon > 0$, choose x_0 such that

$$|g(\eta, x)| = \left| k \int_0^{\eta} (x-t)^{k-1} A(t)\, dt \right| < \varepsilon\, W(x),$$

for $x > x_0$ and $0 \leqslant \eta < x$, by Lemma 1.42. Determine ξ such that

$$x - \xi = \left(\frac{\varepsilon W}{V}\right)^{1/k}.$$

CASE (i). $0 < k \leqslant 1$. We write

$$A^r(x) = r \int_0^x (x - t)^{r-1} A(t)\, dt = r \left[\int_0^{\xi} + \int_{\xi}^{x} \right] \equiv J_1 + J_2,$$

say. First, let us suppose that $\xi > x_0$; then

$$|J_1| = \left| r\,(x - \xi)^{r-k} \int_u^{\xi} (x - t)^{k-1} A(t)\, dt \right|, \qquad 0 \leqslant u \leqslant \xi,$$

by the second mean-value theorem. Now applying Lemma 1.42, we get

$$\begin{aligned} |J_1| &< (r/k)\,(x - \xi)^{r-k}\, |g(\xi, x) - g(u, x)| \\ &< (2r/k)\, \varepsilon^{r/k-1}\, (W/V)^{r/k-1}.\ \varepsilon W(x). \\ &= (2r/k)\, \varepsilon^{r/k}\, U_r(x). \end{aligned} \tag{1.78}$$

$$\begin{aligned} |J_2| &= \left| r \int_{\xi}^{x} (x - t)^{r-1} A(t)\, dt \right| \\ &< (x - \xi)^r\, V(x) \\ &= \varepsilon^{r/k}\, U_r(x). \end{aligned} \tag{1.79}$$

From (1.78) and (1.79) we get

$$|A^r(x)| < (2r/k + 1)\, \varepsilon^{r/k}\, U_r(x).$$

If $\xi < x_0$, we write

$$A^r(x) = r \left[\int_0^{x_0} + \int_{x_0}^{x} \right] \equiv J_1 + J_2,$$

and argue as in Case (i) of Theorem (B).

CASE (ii). $k > 1$. The hypotheses of Theorem (C) necessarily imply that

$$A^r(x) = O\,[U_r(x)],$$

on account of Theorem (A). Choosing $r = k-1$, and applying Case (i) of this Theorem (C), we obtain

$$A^s(x) = o[U_s(x)], \quad k - 1 < s \leqslant k.$$

Again considering $k - 2 < s \leqslant k - 1$, we obtain

$$A^s(x) = o\,[U_s(x)], \quad k - 2 < s \leqslant k - 1\,;$$

and we proceed like that until we finally get the result.

COROLLARY 1.71. *If Σa_n is bounded (λ, k), then it is either summable (λ, k') for every $k' > k$, or not at all.*

PROOF. By hypothesis, $A^k(x) = O(x^k)$. If for some $l > k$ we have $\dfrac{A^l(x)}{x^l} \to c$, that is $A^l(x) - cx^l = o(x^l)$, then

$$A^{k'}(x) - cx^{k'} = o\,(x^{k'}), k < k' \leqslant l;$$

for by setting $B(x) = A(x) - c$, we observe, in the light of Theorem 1.71, that

$$B^k(x) = O\,(x^k) \quad \text{and} \quad B^l(x) = o\,(x^l)$$

together imply

$$B^{k'}(x) = o\,(x^{k'}), \quad k < k' \leqslant l.$$

COROLLARY 1.72. *If a series Σa_n with bounded partial sums is summable (λ, k) for some k, then it is summable (λ, k) for every $k > 0$.*

1.8. Tauberian theorems

In this section we aim at obtaining order-relations for $A^r(x)$ with a hypothesis on $A(x)$ different from the one in Theorem 1.71. The hypotheses which we consider here are similar to those satisfied by the so-called *slowly increasing* or *slowly oscillating* functions. We have called these theorems Tauberian, since we are concerned with deducing the behaviour of $A(x)$ as $x \to \infty$.

THEOREM 1.81. *Let $W(x)$ be a positive non-decreasing function of x, and $V(x)$ any positive function of x, both defined for $x > 0$. Then we have the following :*

(a) $A(x) - A(x-t) = O\,[t^\gamma V(x)\,], 0 < t = O\,[\{W/V\}^{1/(k+\gamma)}], \gamma > 0,$

and

$$A^k(x) = o\,[W(x)]\,, \quad k > 0,$$

together imply

$$A(x) = o\left[V^{k/(k+\gamma)} W^{\gamma/(k+\gamma)}\right].$$

If further $V^{k/(k+\gamma)} W^{\gamma/(k+\gamma)}$ *is non-decreasing, then*

$$A^r(x) = o\left[V^{(k-r)/(k+\gamma)} W^{(\gamma+r)/(k+\gamma)}\right], 0 \leqslant r \leqslant k.$$

(b) $A(x+t) - A(x) = O\left[t^\gamma V(x)\right]$, $0 < t = O\left[\{W/V\}^{1/(k+\gamma)}\right]$, $\gamma > 0$,

and

$$A^k(x) = o\left[W(x)\right], \; k > 0,$$

where

$$0 < \frac{W(x')}{W(x)} < H < \infty, \textit{for } 0 < x' - x = O\left(\frac{W}{V}\right)^{1/(k+\gamma)},$$

together imply

$$A(x) = o\left(V^{k/(k+\gamma)} W^{\gamma/(k+\gamma)}\right).$$

If further $V^{k/(k+\gamma)} W^{\gamma/(k+\gamma)}$ *is non-decreasing, then*

$$A^r(x) = o\left[V^{(k-r)/(k+\gamma)} W^{(\gamma+r)/(k+\gamma)}\right], \quad 0 \leqslant r \leqslant k.$$

PROOF OF (a). Let $h = [k]$; $k = h + \beta$, $0 \leqslant \beta < 1$. Given $\varepsilon > 0$, choose

$$\zeta = \left[\frac{W(x)\varepsilon}{V(x)}\right]^{1/(k+\gamma)}.$$

Writing h for m and 0 for r in (1.39), we obtain

$$\zeta^{h+\beta} A(x) = \frac{\Delta_{-\zeta}^{h+\beta} A^h(x)}{\Gamma(h+1)} +$$

$$+ \Delta_{-\zeta}^{\beta}\left[\int_{x-\zeta}^{x} dt_1 \int_{t_1-\zeta}^{t_1} dt_2 \dots \int_{t_{h-1}-\zeta}^{t_{h-1}} [A(x) - A(t_h)]\, dt_h\right]$$

$$\equiv J_1 + J_2, \tag{1.81}$$

say. By the hypothesis on $A^k(x)$ and Lemma 1.42, we have

$$|J_1| < \varepsilon\, W(x),$$

for x sufficiently large. By the hypothesis on $A(x)$, we have

$$|J_2| < c\, V(x)\, \zeta^{k+\gamma},$$

where $c > 0$ is a constant. Hence

$$|A(x)| < \frac{\varepsilon W(x)}{\zeta^k} + c\,\zeta^\gamma\, V(x),$$

i.e. $$A(x) = o\,(V^{k/(k+\gamma)}\;W^{\gamma/(k+\gamma)}). \tag{1.82}$$

If we now apply Theorem 1.71, we obtain

$$A^r(x) = o\,[\,V^{(k-r)/(k+\gamma)}\;W^{(\gamma+r)/(k+\gamma)}], \quad 0 \leqslant r \leqslant k.$$

PROOF OF (b). With the same notation as in the proof of (a), but using the relation

$$\zeta^{h+\beta} A(x) = \frac{\Delta_\zeta^{h+\beta} A^h(x)}{\Gamma(h+1)} -$$
$$- \Delta_\zeta^\beta \left[\int_x^{x+\zeta} dt_1 \int_{t_1}^{t_1+\zeta} dt_2 \ldots \int_{t_{h-1}}^{t_{h-1}+\zeta} [A^r(t_h) - A^r(x)]\, dt_h \right] \tag{1.83}$$

(which is obtainable from (1.38)) instead of (1.81), and arguing as in (a), we get the required result.

It may be noted that the extra hypothesis on $W(x)$ is used in proving that the first expression on the right of (1.83) is $o[W(x)]$.

REMARKS. Theorem 1.81 will remain valid if, in the hypothesis on $A(x)$, we replace the continuous variable x by the discrete variable λ_n. Thus we have the following

THEOREM 1.82 *Let $W(x)$ be a positive non-decreasing function of x, and $V(x)$ any positive function of x, both defined for positive values of the argument. Then*

(a) $A(\lambda_n) - A(\lambda_n - t) = O\,[t^\gamma V(\lambda_n)]$, $\gamma > 0$, $0 < t = O\,[\{W/V\}^{1/(k+\gamma)}]$,

and

$$A^k(x) = o[W(x)], \; k > 0,$$

together imply

$$A(\lambda_n) = o[\,V(\lambda_n)^{k/(k+\gamma)} W(\lambda_n)^{\gamma/(k+\gamma)}].$$

If further $V(x)^{k/(k+\gamma)} W(x)^{\gamma/(k+\gamma)}$ is non-decreasing, then

$$A^r(x) = o[\,V(x)^{(k-r)/(k+\gamma)} W(x)^{(\gamma+r)/(k+\gamma)}],\; 0 \leqslant r \leqslant k.$$

(b) $A(\lambda_n + t) - A(\lambda_n) = O[t^\gamma V(\lambda_n)], \gamma > 0, 0 < t = O[\{W/V\}^{1/(k+\gamma)}]$,

and

$$A^k(x) = o[W(x)],\ 0 < \frac{W(x')}{W(x)} < H < \infty \text{ for } 0 < x' - x = O\left(\frac{W}{V}\right)^{1/(k+\gamma)},$$

imply

$$A(\lambda_n) = o[V(\lambda_n)^{k/(k+\gamma)} W(\lambda_n)^{\gamma/(k+\gamma)}].$$

If further $V(x)^{k/(k+\gamma)} W(x)^{\gamma/(k+\gamma)}$ *is non-decreasing, then*

$$A^r(x) = o[V(x)^{(k-r)/(k+\gamma)} W(x)^{(\gamma+r)/(k+\gamma)}].$$

The following well-known Tauberian theorems are deducible from Theorem 1.82 (b).

COROLLARY 1.81. *If* $a_n = O[\lambda_n^\alpha(\lambda_n - \lambda_{n-1})]$,

and

$$A^k(x) = o(x^\beta),\ k > 0,\ 0 \leqslant \beta \leqslant \alpha + k + 1,$$

then

$$A(x) = o(x^{(\beta+\alpha k)/(k+1)}).$$

PROOF. The first hypothesis implies, for $t = O(\lambda_n)$,

$$A(\lambda_n + t) - A(\lambda_n) = O(\lambda_n^\alpha t);$$

and therefore, by Theorem 1.82 (b), we obtain the required result.

COROLLARY 1.82. *If* $p > 1$, $\alpha + 1 + 1/p \geqslant 0$,

$$\sum_{\nu=0}^{n} |a_\nu|^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n^{p(\alpha+1)+1}),$$

and

$$A^k(x) = o(x^\beta),\ k > 0,\ 0 \leqslant \beta \leqslant \alpha + k + 1,$$

then

$$A(x) = o\left[x^{\frac{(\alpha+1/p)k + \beta/q}{k+1/q}}\right],\ 1/p + 1/q = 1.$$

PROOF. We have only to observe that the first hypothesis implies, for $t = O(\lambda_n)$,

$$A(\lambda_n + t) - A(\lambda_n) = O(\lambda_n^{\alpha+1/p} t^{1/q}),$$

and then apply Theorem 1.82 (b).

Tauberian theorems (O_L type)

We now replace the two-sided hypotheses on $A(x \pm t) - A(x)$ by one-sided hypotheses.

Theorem 1.83. *Let $W(x)$ and $V(x)$ be as in Theorem* 1.81, *and let*

$$A^k(x) = o[W(x)],\ k > 0, \tag{1.84}$$

where $0 < \dfrac{W(x')}{W(x)} < H < \infty$, *for* $0 < x' - x = O[\{W/V\}^{1/(k+\gamma)}]$.

Let

$$A(x) - A(x-t) = O_L\,[t^\gamma V(x)],\ \gamma > 0, \tag{1.85}$$

$$A(x+t) - A(x) = O_L\,[t^\gamma V(x)], \tag{1.86}$$

where $0 < t = O\,[\{W/V\}^{1/(k+\gamma)}]$. *Then we have*

$$A(x) = o[V(x)^{k/(k+\gamma)} W(x)^{\gamma/(k+\gamma)}]. \tag{1.87}$$

Proof. We require formulae (1.81) and (1.83). Using the hypothesis (1.85) in (1.81), we obtain, for $\varepsilon > 0$,

$$A(x) > -\,\varepsilon\,\frac{W(x)}{\zeta^k} + O_L\,(\zeta^\gamma V),$$

i.e.

$$-\,A(x) < o[V^{k/(k+\gamma)} W^{\gamma/(k+\gamma)}]. \tag{1.88}$$

Using the hypothesis (1.86) in (1.83), we obtain

$$A(x) < o[V^{k/(k+\gamma)} W^{\gamma/(k+\gamma)}]. \tag{1.89}$$

Now (1.88) and (1.89) lead to (1.87).

N.B.—Remarks similar to those at the end of Theorem 1.81 apply here as well.

Theorem 1.84. *Let $W(x)$ be a positive non-decreasing function of x, and $V(x)$ be any positive function of x, both defined for $x > 0$ and such that, if $k > 1$ and $\zeta = (W/V)^{1/k}$, there exist constants h, H and K such that, for $x - K\zeta \leqslant x' \leqslant x + K\zeta$, we have*

$$0 < h < \frac{V(x')}{V(x)} < H < \infty,$$

$$0 < h < \frac{W(x')}{W(x)} < H < \infty.$$

Then

$$A^k(x) = o[W(x)],\ \textit{and}\ A(x) = O_L\,[V(x)]\ \textit{or}\ O_R\,[V(x)],$$

together imply

$$A^r(x) = o[U_r(x)], \quad 1 \leqslant r \leqslant k,$$

where $U_r(x)$ is as in Theorem 1.71 *and is non-decreasing for* $r \geqslant 1$.

PROOF. This theorem follows easily from Theorem 1.83, if we observe that the hypotheses on $A(x)$ and $V(x)$ imply

$$A^1(x+t) - A^1(x) = \int_x^{x+t} A(u)\,du = O_L\,[t\,V(x)],$$

$$(t \leqslant K\zeta)$$

$$A^1(x) - A^1(x-t) = \int_{x-t}^{x} A(u)\,du = O_L\,[t\,V(x)].$$

REMARK. The O-versions of Theorems 1.81-1.84 are also valid, where O replaces o both in the hypotheses on $A^k(x)$ and the conclusions about $A(x)$.

COROLLARY 1.83. *A series Σa_n whose partial sums are positive (or bounded on one side) is summable $(\lambda, 1)$, if it is summable (λ, k) for any $k > 1$.*

Converse theorems on summability

We now state conditions under which a summable series is convergent. Although these results could have been included as corollaries in previous sections, we have preferred to collect them here in the order of increasing generality, as these are of special interest.

Let us set

$$b_n = \lambda_n a_n, \quad B(\omega) = \sum^{n} \lambda_\nu a_\nu, \quad \lambda_n \leqslant \omega < \lambda_{n+1},$$

$$B^k(\omega) = \sum^{n} (\omega - \lambda_\nu)^k \lambda_\nu a_\nu, \quad k > 0.$$

We then have

$$B(x) - B(\omega) = \int_\omega^x t\,dA(t) = xA(x) - \omega\,A(\omega) - \int_\omega^x A(t)dt,$$

$$\frac{A^k(\omega)}{\omega^k} - \frac{A^{k+1}(\omega)}{\omega^{k+1}} = \frac{B^k(\omega)}{\omega^{k+1}}.$$

From this last formula we deduce

THEOREM 1.85. *A necessary and sufficient condition that a series Σa_n summable or bounded $(\lambda, k+1)$ should be summable or bounded (λ, k) is that $B^k(\omega) = o(\omega^{k+1})$ or $O(\omega^{k+1})$ respectively.*

COROLLARY 1.84. *If* $B^k(\omega) = o(\omega^{k+1})$, *then* Σa_n *is either summable* (λ, k) *or is never summable.*

For if Σa_n is summable $(\lambda, k'+1)$ for $k' > k$, then it is summable (λ, k'), since $B^k(\omega) = o(\omega^{k+1})$ implies $B^{k'}(\omega) = o(\omega^{k'+1})$ for every $k' > k$. Hence etc.

COROLLARY 1.85. *If* Σa_n *is summable* (λ, l), *then a necessary and sufficient condition that* Σa_n *should converge is that* $B(\omega) = o(\omega)$.

This is a restatement of Corollary 1.84 with $k = 0$.

THEOREM 1.86. *If* $\sum^{n} \lambda_\nu a_\nu = O(\lambda_n)$, Σa_n *is either summable* (λ, k) *for every* $k > 0$, *or is never summable.*

For if Σa_n is summable (λ, l), then $B^l(\omega) = o(\omega^{l+1})$. This result together with the hypothesis $B(\omega) = O(\omega)$ implies $B^k(\omega) = o(\omega^{k+1})$ by Theorem 1.71(B). And so the result follows from Corollary 1.84.

THEOREM 1.87. *If* $B(\omega) = O_L(\omega)$, *and* Σa_n *is summable or bounded* (λ, l), *then* Σa_n *is summable or bounded* $(\lambda, 1)$.

Summability or boundedness (λ, l) implies $B^l(\omega) = o(\omega^{l+1})$ or $O(\omega^{l+1})$, which, in conjunction with the hypothesis $B(\omega) = O_L(\omega)$, implies $B^1(\omega) = o(\omega^2)$ or $O(\omega^2)$. We have now only to use Theorem 1.85.

THEOREM 1.88. *If* $\lambda_n \leqslant \omega < \lambda_{n+1}$, *and*

$$\liminf_{\omega \to \infty} \min_{\omega < \lambda_r \leqslant (1+\delta)\omega} \sum_{n+1}^{r} a_r \geqslant -\varphi(\delta),$$

where $0 < \varphi(\delta)$ *tends to* 0 *as* $\delta \to 0$, *then* Σa_n *converges whenever it is summable* (λ, k).

We prove the result in two stages: (α). The hypothesis that

$$\liminf_{\omega \to \infty} \min_{\omega < t \leqslant (1+\delta)\omega} [A(t) - A(\omega)] \geqslant -\varphi(\delta),$$

for fixed $\delta > 0$, implies $B(\omega) = O_L(\omega)$, so that by Theorem 1.87 summability (λ, l) implies summability $(\lambda, 1)$. (β). Summability $(\lambda, 1)$ and the hypothesis that $\varphi(\delta) \to 0$ as $\delta \to 0$ imply convergence.

PROOF OF (α).

$$B(x) - B(\omega) = \int_\omega^x t dA(t) = xA(x) - \omega A(\omega) - \int_\omega^x A(t)dt$$

$$= \omega[A(x) - A(\omega)] + \int_\omega^x [A(x) - A(t)]dt.$$

If $\omega < x \leqslant (1 + \delta)\omega = \lambda\omega$, we observe that

$$\liminf_{\omega \to \infty} \min \frac{B(x) - B(\omega)}{\omega} \geqslant -(1 + \delta)\varphi(\delta),$$

and hence

$$\liminf_{\omega \to \infty} \frac{B(\omega)}{\omega} = \liminf_{\omega \to \infty} \left[\frac{1}{\omega} \sum_{r=0}^{\infty} B\left(\frac{\omega}{\lambda^r}\right) - B\left(\frac{\omega}{\lambda^{r+1}}\right)\right]$$

$$\geqslant -\frac{(1 + \delta)}{\delta} \varphi(\delta),$$

which proves (α).

Aliter. That $A^1(\omega) - s\omega = o(\omega)$ can be seen otherwise from (1.89) and Theorem 1.84, if we observe that the hypothesis can be put in the form

$$A(\omega + t) - A(\omega) = O_L[\varphi(\delta)] = O_L(t\,\omega^{-1}),$$

for $t \leqslant \delta\omega$, δ fixed.

PROOF OF (β). Let $A^1(\omega)/\omega \to s$. We have the following formulae:

$$A(\omega) = \frac{1 + \delta}{\delta} \frac{A^1(x)}{x} - \frac{1}{\delta} \frac{A^1(\omega)}{\omega} - \frac{1}{\delta\omega} \int_\omega^x [A(t) - A(\omega)]dt,$$

$$x = (1 + \delta)\omega;$$

$$A(\omega) = \frac{1 + \delta}{\delta} \frac{A^1(x)}{x} - \frac{1}{\delta} \frac{A^1(\omega)}{\omega} + \frac{1}{\delta\omega} \int_\omega^x [A(x) - A(t)]dt.$$

Fixing δ and letting $\omega \to \infty$, we have

$$\limsup_{\omega \to \infty} A(\omega) \leqslant \frac{1 + \delta}{\delta} s - \frac{s}{\delta} + \varphi(\delta) = s + \varphi(\delta),$$

$$\liminf_{\omega \to \infty} A(\omega) \geqslant s - \varphi(\delta),$$

and now letting $\delta \to 0$, we obtain the result.

THEOREM 1.89. *If $\lambda_n a_n = O_L(\lambda_n - \lambda_{n-1})$, then Σa_n has bounded partial sums if it is bounded (λ, l), so that it is summable (λ, k) for every $k > 0$, or never summable. If further $\frac{\lambda_n}{\lambda_{n-1}} \to 1$, then Σa_n is convergent whenever it is summable.*

We have $B(\omega) = \Sigma\, O_L(\lambda_n - \lambda_{n-1}) = O_L(\omega)$. Hence, by Theorem 1.87, Σa_n is summable or bounded $(\lambda, 1)$ according as it is summable or bounded (λ, l).

To prove further that $A(\omega) = O(1)$, we note that

$$A(\omega) = \frac{A^1(\omega)}{\omega} + \frac{B(\omega)}{\omega}$$

implies $A(\omega) = O_L(1)$, since $B(\omega) = O_L(\omega)$; while

$$A(\lambda_n) = \frac{1+\delta}{\delta}\frac{A^1(x)}{x} - \frac{A^1(\lambda_n)}{\delta\lambda_n} - \frac{1}{\delta\lambda_n}\int_{\lambda_n}^{x}[A(t) - A(\lambda_n)]dt,$$

$$\lambda_n \leqslant x = (1+\delta)\lambda_n,$$

implies $A(\lambda_n) = O_R(1)$, since $A(t) - A(\lambda_n) = O_L\{(t-\lambda_n)\,\lambda_n^{-1}\}$, $\lambda_n < t \leqslant x$.

Boundedness of $A(\omega)$ implies summability (λ, k) for every $k > 0$, if it is summable at all. On the other hand, the further assumption that $\frac{\lambda_n}{\lambda_{n-1}} \to 1$ will enable us to prove that

$$A(x) - A(\omega) = O_L(\delta), \quad \omega \leqslant x \leqslant (1+\delta)\omega,$$

which, by Theorem 1.88, will prove all that is required.

REMARK. The same conclusion as in Theorem 1.89 will result from the following hypothesis on a_n:

$$\sum^{n} (|a_\nu| - a_\nu)^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n),\ p > 1.$$

1.9. Absolute summability

The notion of absolute Riesz summability defined in § 1.1 bears the same relation to ordinary summability as, for instance, absolute convergence does to ordinary convergence. The preceding sections show that ordinary summability has been the subject of intensive study, one consequence of which has been the development of a satisfactory Tauberian theory; absolute Riesz summability, however, has not received the same degree of attention, and very little is known about Tauberian conditions which would enable one to infer absolute convergence from absolute summability. The analogues of several theorems in the ordinary case have not been explicitly proved. We therefore have to content ourselves with proving here only the analogue of the first theorem of consistency.

THEOREM 1.91. *If the series Σa_n is summable $|\lambda, k|$, $k \geqslant 0$, then it is also summable $|\lambda, l|$ for $l > k$.*

PROOF. We recall the fact that $|\lambda, 0|$ summability is equivalent to absolute convergence, and prove the theorem in two parts, according as $k = 0$ or $k > 0$.

(a). $k = 0$. We have to prove that $\int_h^\infty |dC_\lambda^l(x)| < \infty$, where we may assume without loss of generality that $0 < h < \lambda_0$. If we set

$$S_{l-1}(x) = \sum_{\lambda_n \leqslant x} a_n \lambda_n (x - \lambda_n)^{l-1},$$

then we have to prove that

$$I = \int_h^\infty x^{-l-1} \, |S_{l-1}(x)| \, dx < \infty.$$

However,

$$I \leqslant \int_h^\infty x^{-l-1} \Big[\sum_{\lambda_n \leqslant x} |a_n| \, \lambda_n \, (x - \lambda_n)^{l-1} \Big] \, dx.$$

Interchanging the order of integration and summation, we have

$$I \leqslant \sum_{n=0}^\infty \int_{\lambda_n}^\infty |a_n| \, \lambda_n \, (x-\lambda_n)^{l-1} x^{-l-1} dx = \frac{1}{l} \sum_{n=0}^\infty |a_n| < \infty.$$

(b). $k > 0$. Here let us again consider two cases, namely, $k \geqslant 1$ and $0 < k < 1$.

(b$_1$). If $k \geqslant 1$ and $l = k+m$, we have

$$S_{l-1}(x) = c \int_h^x S_{k-1}(u) \, (x-u)^{m-1} du,$$

where $c = \dfrac{\Gamma(l)}{\Gamma(k)\,\Gamma(m)}$, $h > 0$. If $y > h$, we also obtain

$$\int_h^y x^{-k-m-1}\ |S_{l-1}(x)|\ dx \leqslant c \int_h^y x^{-k-m-1}dx \int_h^x |S_{k-1}(u)|\ (x-u)^{m-1}du$$

and

$$\int_h^y x^{-k-m-1}dx \int_h^x |S_{k-1}(u)|\ (x-u)^{m-1}du$$

$$= \int_h^y |S_{k-1}(u)|\ du \int_u^y x^{-k-m-1}(x-u)^{m-1}dx$$

$$< \int_h^y |S_{k-1}(u)|\ du \int_u^\infty x^{-k-m-1}(x-u)^{m-1}dx.$$

But

$$\int_u^\infty x^{-k-m-1}(x-u)^{m-1}dx = \frac{u^{-1-k}\,\Gamma(k+1)\,\Gamma(m)}{\Gamma(k+m+1)},$$

so that

$$l \int_h^\infty x^{-l-1}\ |S_{l-1}(x)|\ dx \leqslant k \int_h^\infty u^{-1-k}\ |S_{k-1}(u)|\ du, \tag{1.91}$$

which proves the required result.

(b_2). If $0 < k < 1$, set

$$I_m(x) = c \int_h^x S_{k-1}(u)\ (x+a-u)^{m-1}du, \quad a > 0.$$

We then have

$$\int_h^x x^{-k-m-1}\ |I_m(x)|\ dx \leqslant c \int_h^x |S_{k-1}(u)|\ du \int_u^x v^{-k-m-1}(v+a-u)^{m-1}dv.$$

Now if $a \to 0$, the left side tends to

$$\int_h^x x^{-k-m-1}\ |S_{k+m-1}(x)|\ dx,$$

and the right side tends to

$$\int_h^x |S_{k-1}(u)|\, du \int_h^x v^{-k-m-1}(v-u)^{m-1}dv,$$

so that (1.91) holds for $l > k > 0$.

NOTES ON CHAPTER I

§1.1. The term 'typical means' is interchangeable with 'Riesz means' except that when we deal with Dirichlet series, as we do in Chapter III, we use two types of Riesz means (λ, k) or (l, k) for the two types of series $\Sigma a_n e^{-\lambda_n s}$ or $\Sigma a_n l_n^{-s}$, and in this context, the word 'typical' has a special signification.

It is assumed throughout that the terms of the series Σa_n are real, unless there is an indication to the contrary. If the terms are complex, the real and imaginary parts can be separately discussed.

The letter λ in '(λ, k)' is supposed to refer to the given sequence $\{\lambda_n\}$. When $\lambda_n = n$ or n^2, however, we allow ourselves the liberty of writing '(n, k)' or '(n^2, k)' as the case may be.

For an account of Riesz means see G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series*, Cambridge (1915). We shall refer to this as the *Tract*. See also E. Kogbetliantz, *Sommation des séries et intégrales divergentes par les moyennes arithmétiques et typiques*, Mémorial des sciences mathématiques, Fascicule 51 (1931) and M. Riesz, *Acta Mathematica*, 81 (1949), 1-223.

It is possible to define the summability of integrals instead of series. We suppose that $\lambda(t)$ is a positive and continuous function of t, tending steadily to infinity with t, with $\lambda(0) = 0$, and set

$$A_\lambda(t) = \int_{\lambda(u)<t} a(u)\, du = \int_0^{\lambda^{-1}(t)} a(u)\, du,$$

$$A_\lambda^k(t) = k \int_0^t A_\lambda(s)\, (t-s)^{k-1}\, ds.$$

Then, if $t^{-k} A_\lambda^k(t) \to c$ as $t \to \infty$, we say that the integral

$$\int_0^\infty a(u)\, du$$

is summable (λ, k) to the sum c. This definition may then be applied to study integrals of the type

$$\int_0^\infty a(u)\, e^{-s\lambda(u)}\, du.$$

Absolute Riesz summability was defined first by N. Obrechkoff, *Math. Zeitschrift*, 30 (1929), 375-386, on the lines of absolute Cesàro summability defined earlier by M. Fekete and E. Kogbetliantz, *Math. és Termész. Ert.* 29 (1911), 719-726, 32 (1914), 389-425, and *Bull. Sc. Math.* (2) 49 (1925), 234-256.

The equivalence between summability (n, k) and Cesàro summability of order k, for various values of k, was announced by M. Riesz, *Comptes Rendus*, 22 Nov. 1909, 12 June 1911, *Proc. London Math. Soc.* 22 (1923-24), 418. See R. P. Agnew, *Trans. American Math. Soc.* 35 (1932), 532-548. An unpublished proof by A. E. Ingham is referred to by G. H. Hardy in his *Divergent series*, Oxford (1949), 119. For a different approach see J. J. Gergen, *Duke Math. Journal*, 3 (1937), 133-148. For further work on this topic see B. Kuttner, *Proc. London Math. Soc.* (2) 45 (1939), 398.

§1.2. Many of the results we prove in this section remain valid if $A_\lambda(t)$ is a function of bounded variation in every finite interval instead of being a step-function.

Formulae (1.21) and (1.22) are proved in the *Tract*, pp.27-28. For formula (1.23) see G. H. Hardy, *Proc. London Math. Soc.* 15 (1916), 72-88.

§1.3. Finite differences were first introduced into this theory by H. D. Kloosterman, *Jour. London Math. Soc.* 15 (1940), 91-96, and their use was systematized by L. S. Bosanquet, *Jour. London Math. Soc.* 18 (1943), 239-248.

It should be noted that $\Delta_\zeta^\alpha A(x)$, $0 < \alpha < 1$, in our notation would, in Bosanquet's notation, be $\Delta_\zeta^\alpha A_\alpha(x)/\Gamma(\alpha + 1)$. Our notation has been chosen for convenience.

The integrals defining $\Delta_\zeta^\alpha F(t)$ are assumed to exist.

§1.4. The proof of Lemma 1.41 as given here is different from that in the *Tract*. This proof was communicated to us by Prof. M. Riesz.

Use is made of this proof in Lemma 1.42 as well. In Lemma 1.42, it will be sufficient if $\varphi(x)$ possesses the stated property beyond a definite stage, since we are concerned only with the behaviour at ∞. Thus $\varphi(x)$ may be any logarithmico-exponential function. See G. H. Hardy, *Orders of infinity*, Cambridge (1910).

Lemma 1.42 remains valid if 'o' is replaced by 'O' in both (1.42) and (1.43).

§1.5. The word 'consistency' is not considered appropriate nowadays. Theorem 1.51 proves that Riesz summability is 'regular'; however, we have chosen to keep to the usage in the *Tract*. The o-version of Theorem 1.52 is obvious.

§1.6. Theorem 1.61 with x instead of $\varphi(x)$ is proved in the *Tract*; but the more general form in which we have stated it here requires no special artifice. See Theorems 21 and 22 of the *Tract*.

§1.7. Theorem 1.71 may be considered as a convexity theorem in a certain sense. Convexity theorems were initiated by G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* 11(1913), 411-478. Theorem 1.71 was proved by M. Riesz, *Acta Szeged,* 1 (1923), 114-126. It seems to have been obtained independently by K. Ananda-Rau in a Smith's Prize Essay (Cambridge, 1918) which was published, in part, only several years later, *Proc. London Math. Soc.* 34 (1932), 414-440.

§1.8. For information about Tauberian conditions of various types, o, O, O_L and O_R, see G. H. Hardy, *Divergent series*, Oxford (1949), 149; and for definitions of slowly decreasing and slowly oscillating functions, ibid., 124, 286.

Theorem 1.82(b) is an extension of a theorem of Bosanquet, and by the same method. See *Jour. London Math. Soc.* 18 (1943), 239-248. For other theorems of this type, see S. Minakshisundaram and C. T. Rajagopal, *Quarterly Jour. Math.* (Oxford), 17 (1946), 153-161, and *Proc. London Math. Soc.* 50 (1948), 242-255.

For special cases of Corollary 1.81 see K. Ananda-Rau, *Proc. London Math. Soc.* 34 (1932), 414-440, and for the corresponding cases of Corollary 1.82, see V. Ganapathy Iyer, *Annals of Math.* 36 (1935), 100-116. The results of Ananda-Rau and of Ganapathy Iyer were extended by S. Minakshisundaram, *Jour. Indian Math. Soc.* 2 (1936) 147-155. A further generalization of Ananda-Rau's result was made by Bosanquet, loc. cit. Corollaries 1.81 and 1.82 are valid for $\beta > a + k + 1$ as well. Cf. Ananda-Rau, Ganapathy Iyer, and Bosanquet, loc. cit. In Corollary 1.82 it should be noted that if $a + 1 + 1/p < 0$, then all the a_ν vanish. Here we make the convention that $\lambda_{-1} = 0$. The hypothesis on a_ν could also take the form

$$\sum_{\lambda_n < \lambda_\nu < (1+\delta)\lambda_n} |a_\nu|^p \quad (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n^{pa+1}),\ \delta > 0,\ p > 1,$$

which is in line with the hypotheses formulated by Otto Szász, *Trans. American Math. Soc.* 39 (1936), 117-130.

For Tauberian theorems of O_L type on Riesz summability see S. Minakshisundaram and C. T. Rajagopal, loc. cit. For one-sided general Tauberian theorems see, for instance, S. Bochner and K. Chandrasekharan, *Fourier transforms*, Princeton (1949), Th. 92.

In connexion with the Remark at the end of §1.8, see Otto Szász, loc. cit.

§1.9. Converse theorems of a comparatively simple kind are known in the case of absolute Abel summability which J. M. Whittaker defined. See *Proc. Edinburgh Math. Soc.* 2 (1930-31), 4. For example, J. M. Hyslop has proved that if Σa_n is summable $|A|$ and $\Sigma |b_n - b_{n-1}|$ converges, where $b_n = na_n$, then Σa_n converges absolutely, *Jour. London Math. Soc.* XII (1937), 180; and A. Zygmund has shown that a lacunary series which is summable $|A|$ converges absolutely. See *Trans. American Math. Soc.* 55 (1944), 194.

The equivalence of summability $|n, k|$ and $|C, k|$ has been established by J. M. Hyslop, *Proc. Edinburgh Math. Soc.* 5 (1937-1938), 46.

Summability $|\lambda, k|$ obviously implies (λ, k), but not conversely, (Ex. $1 - 1 + 1 - 1 + \dots$). Also, convergence (of a Fourier series at a point for instance) need not imply absolute Abel summability (see J. M. Whittaker, *Proc. Edinburgh Math. Soc.* 2 (1930-31), 4) ; nor does absolute Abel summability necessarily imply convergence (see B. N. Prasad, *Proc. Edinburgh Math. Soc.* 2 (1930-31), 134). Since absolute Cesàro summability implies absolute Abel summability (M. Fekete, *Proc. Edinburgh Math. Soc.* 2 (1933), 132) it follows that convergence does not necessitate summability $|n, k|$.

Theorem 1.91 is due to N. Obrechkoff, *Math. Zeitschrift*, 30 (1929), 375-386.

II

SECOND THEOREM OF CONSISTENCY

2.1. Introduction

By the first theorem of consistency (Th.1.51), we can say that a series which is summable (λ, k) is also summable (λ, k') for $k' > k$; in other words, we can say that the power of Riesz summability increases with an increase in the order, the type remaining the same. Now the question arises as to what can be said about summability (λ, k) when the order k is kept fixed, and the type λ is varied. The general result in this direction, which is called the *second theorem of consistency,* is that the power of summability (λ, k) increases when the rate of growth of λ_n decreases, the order k remaining constant. On this topic we have actually a number of theorems closely related to one another, and it is our object to present them here.

2.2. An account of the results

The first theorem bearing on our topic was proved by G. H. Hardy and M. Riesz. It runs as follows : *if a series Σa_n is summable (λ, k), where $\lambda_n = e^{\mu_n}$, then it is also summable (μ, k) to the same sum.* In particular, if a series is summable by arithmetic means $(\lambda_n = n)$, then it is also summable by logarithmic means $(\lambda_n = \log n)$ of the same order. Later, Hardy gave a generalization of this theorem, and using Hardy's method of proof, A. Zygmund and K. A. Hirst gave further generalizations.

Before we state Hardy's theorem, it is necessary to recall the definition of a logarithmico-exponential function or, briefly, L-function.

A logarithmico-exponential function $f(x)$ is a real, one-valued function of the real variable x defined, for all sufficiently large values of x, by a finite combination of the ordinary algebraical symbols, $+, -, \times, \div, \sqrt[n]{\ }$, and the functional symbols, $\log(\ldots)$, $e^{(\ldots)}$, operating on x and on real constants.

Hardy's theorem runs as follows :

(H_1). *If* (i) *the series* Σa_n *is summable* (λ, k) *to the sum* s, *and* (ii) μ *is a logarithmico-exponential function of* λ *tending to* $+\infty$ *with* λ, *such that* $\mu = O(\lambda^{\Delta})$, $\Delta > 0$ *being a constant however large, then the series* Σa_n *is summable* (μ, k) *to the same sum* s.

This theorem not only tells us that the efficiency of Riesz summability increases when the type decreases (the order remaining constant), but asserts the equivalence of any two types of summability, when those types are related to each other in a certain 'regular' fashion, for example when both are powers of n.

A. Zygmund has completed Hardy's theorem by proving

(H_2). *If* $\lambda^{\delta} \prec \mu(\lambda) \prec \lambda^{\Delta}$, *for certain* $\delta, \Delta > 0$, *then summability* (λ, k) *and summability* (μ, k) *are equivalent.*

The proof of (H_2) results from the fact that $\bar{\mu}(x)$ which is the inverse of $\mu(x)$ [that is, $x = \bar{\mu}(y)$ if $y = \mu(x)$], though not necessarily an L-function, satisfies all the requirements on μ which materially enter into Hardy's proof of (H_1).

(H_1) and (H_2) are supplemented by

(H_3). *If* $1 \prec \mu(\lambda) \prec \lambda^{\delta}$, *for every* $\delta > 0$, *then the method* (μ, k) *is more powerful than* (λ, k).

$\mu(\lambda) = \log \lambda$ will serve as an example. Theorems (H_1)-(H_3) also give rise to another line of questioning. Suppose that Σa_n is not summable (λ, k) but only bounded (λ, k) [cf. Defn. 1.12] ; in that case, we cannot *a priori* assert that Σa_n is summable by a process which is more powerful than (λ, k) : for example, by $(\log \lambda, k)$. But if we assume not only (i) that Σa_n is bounded (λ, k), but also (ii) that Σa_n is summable (λ, l) for some $l > k$, then by a theorem of Zygmund, we can say that Σa_n is indeed summable $(\log \lambda, k)$. Thus we have

(Z_1). *If* Σa_n *is bounded* (λ, k), *and summable* (λ, l), *for some* $l > k$, *then* Σa_n *is summable* $(\log \lambda, k)$.

Now summability $(\log \lambda, k)$ is just one of several methods which are more powerful than (λ, k) ; in fact we stated in H_3 that summability (μ, k), where μ is any function satisfying the condition

$$1 \prec \mu(\lambda) \prec \lambda^{\delta}$$

for every $\delta > 0$, is more powerful than (λ, k); it is therefore proper to ask whether we can replace $\log \lambda$ in (Z_1) by such a function μ. The answer is in the affirmative, and we have

(Z_2). *If* Σa_n *is bounded* (λ, k), *and summable* (λ, l), *for* $0 < k < l$, *then* Σa_n *is also summable* (μ, k), *where* $\mu = \mu(\lambda)$ *satisfies the condition*

$$1 \prec \mu(\lambda) \prec \lambda^{\delta},$$

for every $\delta > 0$.

Actually a more general theorem was proved by Zygmund, namely

(Z_3). *If* Σa_n *is summable* (λ, l), *and*

$$A_{\lambda}^{k}(x) = o\left[\left(\frac{\mu(x)}{\mu'(x)}\right)^{k}\right],$$

where $\dfrac{\mu(x)}{x\,\mu'(x)} \succ 1$, $0 < k < l$, *then* Σa_n *is summable* (μ, k).

The above theorem could also be put in a slightly different form, namely

(Z_4). *If* Σa_n *is summable* (λ, l), *and*

$$A_{\lambda}^{k}(x) = o\left[\{x\psi(x)\}^{k}\right],\ k > 0,$$

where $\psi(x)$ *is an L-function tending to* $+\infty$ *with* x, *the series* Σa_n *is summable* (μ, k), *where*

$$\mu(x) = \exp\left[\int_a^x \frac{dt}{t\,\psi(t)}\right],$$

with the proviso that the last integral diverges as $x \to \infty$.

The content of theorems (Z_1)-(Z_4) is this: if a series Σa_n is summable (λ, l), and if the function $C_{\lambda}^{k}(x) = A_{\lambda}^{k}(x)\,x^{-k}$, $0 < k < l$, while it does *not* tend to a finite limit as $x \to \infty$, is however *not very large*, then Σa_n is indeed summable by a method (μ, k) [of a lower order than that of (λ, l), but of a different type] which is more powerful than (λ, k), and which is such that the order of infinity of $\mu(x)$ depends on that of $C_{\lambda}^{k}(x)$.

On the other hand, let us suppose that Σa_n is summable (λ, k) to the sum zero. Then $C_\lambda^k(x) = o(1)$ as $x \to \infty$. We may then ask the question : if the expression $C_\lambda^k(x)$ tends very rapidly to zero, can we assert that the series Σa_n is indeed summable by a method which is less powerful than (λ, k), namely by a method (μ, k) where $\mu(\lambda) \succ \lambda^\Delta$ for every $\Delta > 0$. The answer to the above question is furnished by the following theorem.

(Z_5). *If, for the series* Σa_n, *we have*

$$A_\lambda^k(x) = o\left[\left(\frac{\mu(x)}{\mu'(x)}\right)^k\right],$$

where (i) $\mu(\lambda)$ *satisfies the condition* $\lambda^\Delta \prec \mu(\lambda) \prec e^{\delta\lambda}$ *for every* δ, $\Delta > 0$, *and* (ii) $\mu(\lambda)$ *possesses for* $\lambda > \lambda_0 > 0$ *a sufficient number of positive derivatives, then* Σa_n *is summable* (μ, k).

Theorem (Z_5) can be put into a slightly different form :

(Z_6). *If, for the series* Σa_n, *we have*

$$A_\lambda^k(x) = o\left[\{x\,\psi(x)\}^k\right],$$

where $\psi(x)$ *is an L-function and* $1/x \prec \psi(x) \prec 1$, *the series is summable* (μ, k), *where* $\mu(x) = \exp\left[\int_a^x \frac{dt}{t\psi(t)}\right]$, *provided that a sufficient number of derivatives of the function* $\mu(x)$ *are positive for* $x \geq \lambda_0$.

All the theorems stated so far are concerned with the relationship between two methods of summability, (λ, k) and (μ, k), whose respective types λ and μ are such that μ is an L-function of λ tending to $+\infty$ with λ, and subject to appropriate additional restrictions. What we make use of is the fact that L-functions have a certain regularity in growth ; it should be possible to axiomatize the required properties, without assuming that we deal only with L-functions. K. A. Hirst carried out this idea, as far as Hardy's theorem (H_1) is concerned ; Hirst's theorem may be stated thus:

(H_4). *If* Σa_n *is summable* (λ, k), *then it is summable* (μ, k) *to the same sum, where* $\mu = \varphi(\lambda)$, *and* $\varphi(t)$ *is a function which increases steadily to* $+\infty$ *with* t, *and satisfies the following conditions*:

(i) *when k is an integer,*

(a) $$\int_0^x t^r \, |\varphi^{(r+1)}(t)| \, dt = O\{\varphi(x)\}, \quad r = 1, 2, 3, \ldots, k;$$

(ii) *when k is not an integer,*

(a) $$\int_0^x t^r \, |\varphi^{(r+1)}(t)| \, dt = O\{\varphi(x)\}, r = 1, 2, \ldots, h + 1, h = [k],$$

and EITHER

(b) $\varphi'(t)$ *is a monotonic increasing function,*

OR

(c_1) $\varphi'(t)$ *is a monotonic decreasing function*

and

(c_2) $$t\,\varphi''(t) = O\{\varphi'(t)\}.$$

These conditions on φ are, of course, satisfied by a class of functions which is larger than that of L-functions, and hence (H_4) is a generalization of (H_1).

It is thus clear that (H_1) and (H_4) are the important theorems of the (H) set, while (Z_3) and (Z_5) are the important ones of the (Z) set. In our presentation, we shall try to combine (H_1) and (Z_3) into a single result, from which a slightly restricted version of (H_4) also follows ; next, we shall prove (Z_5) with like generality.

2.3. Some preliminary lemmas

LEMMA 2.31. *The n-th derivative of* $\{f(x)\}^m$ *is the sum of a number of terms of the form*

$$c\,\{f(x)\}^{m-r}\,\{f'(x)\}^{a_1}\,\{f''(x)\}^{a_2} \ldots \{f^{(n)}(x)\}^{a_n},$$

where c is a constant, $r \leqslant n$, *and the a's are positive integers or zero, such that* $\sum_{p=1}^{n} a_p = r$, $\sum_{p=1}^{n} p\,a_p = n$. *Further if m is a positive integer, then* $r \leqslant m$.

LEMMA 2.32. *If* $f(x)$ *is a positive decreasing function of x, and if* $x f'(x) = O\{f(x)\}$, *then there exists a constant* $c > 1$ *such that*

$$\frac{f(cx)}{f(x)} > \tfrac{1}{2}, \textit{ for all } x > x_0.$$

PROOF. Clearly $\dfrac{f(cx)}{f(x)} \leqslant 1$. Also, by hypothesis, there exists an x_0 such that $|xf'(x)/f(x)| < k$, for $x > x_0$, $k > 0$. Thus

$$\left|\frac{f(cx)}{f(x)} - 1\right| = \frac{(c-1)\,x\,|f'(\xi)|}{f(x)}, \quad x < \xi < cx,$$
$$\leqslant \frac{(c-1)\,\xi\,|f'(\xi)|}{f(\xi)} < k(c-1), \text{ for } x > x_0,$$
$$< \tfrac{1}{2}, \text{ if } c < 1 + 1/2k.$$

LEMMA 2.33. *Let $\varphi(t)$ be a monotonic increasing function, and $t\varphi''(t) = O\{\varphi'(t)\}$. Then there exists a constant $c > 1$, such that, if $\varphi'(t)$ is monotonic decreasing, we have $1 \geqslant \dfrac{\varphi'(ct)}{\varphi'(t)} > \frac{1}{2}$, for all $t > t_0$; and if $\varphi'(t)$ is monotonic increasing, $c > 1$ may be chosen such that $2 > \dfrac{\varphi'(ct)}{\varphi'(t)} \geqslant 1$, for $t > t_0$.*

PROOF. The first part follows from Lemma 2.32. To prove the second part, we choose $f(x) = 1/\varphi'(x)$, so that

$$xf'(x) = -\frac{x\varphi''(x)}{\{\varphi'(x)\}^2} = O\left[\frac{1}{\varphi'(x)}\right] = O\{f(x)\}.$$

Hence, for $t > t_0$,

$$1 \geqslant \frac{\varphi'(t)}{\varphi'(ct)} > \tfrac{1}{2}.$$

LEMMA 2.34. *If $\varphi(t)$ is a monotonic increasing function, and $\varphi'(t)$ is monotonic, then $\dfrac{\varphi(x)-\varphi(t)}{x-t}$, where x is fixed and $t < x$, is a monotonic function of t, which increases or decreases according as $\varphi'(t)$ increases or decreases.*

PROOF. We have

$$\frac{d}{dt}\left[\frac{\varphi(x)-\varphi(t)}{x-t}\right] = \frac{\varphi(x)-\varphi(t)-(x-t)\,\varphi'(t)}{(x-t)^2}$$
$$= \frac{(x-t)\,\varphi'(\xi)-(x-t)\,\varphi'(t)}{(x-t)^2}, \quad t \leqslant \xi \leqslant x;$$
$$= \frac{\varphi'(\xi)-\varphi'(t)}{x-t},$$

which is positive or negative according as φ' increases or decreases.

LEMMA 2.35. *If $\varphi(x)$ is as in Lemma 2.33, then there exist two positive constants α, β such that, for all $x > x_0$,*

$$\alpha\, x\, \varphi'(x) \leqslant \varphi(cx) - \varphi(x) \leqslant \beta\, x\, \varphi'(x),$$
$$\alpha\, x\, \varphi'(x) \leqslant \varphi(x) - \varphi(x/c) \leqslant \beta\, x\, \varphi'(x).$$

PROOF. Let $c > 1$ be as in Lemma 2.33. Then $\varphi(cx) - \varphi(x) = (c - 1)\, x\varphi'(\eta), x \leqslant \eta \leqslant cx$. If φ' is monotonic increasing,

$$\varphi'(x) \leqslant \varphi'(\eta) \leqslant \varphi'(cx) < 2\, \varphi'(x), x > x_0,$$

and if φ' is monotonic decreasing,

$$\varphi'(x) \geqslant \varphi'(\eta) \geqslant \varphi'(cx) > \tfrac{1}{2}\, \varphi'(x), x > x_0.$$

2.4. The main theorem

We now assume that

$\varphi(t)$ is a positive non-decreasing function of t diverging to $+\infty$, having $N + 1$ derivatives, where N is sufficiently large; (2.41)

$\varphi'(t)$ is monotonic; (2.42)

$t^r\, \varphi^{(r+1)}(t) = O[\varphi'(t)],\ r = 1, 2, \ldots, N$; (2.43)

we also assume that

$\psi(t)$ is a positive non-decreasing function of t. (2.44)

THEOREM 2.41. *Suppose that*

(a) $A_\lambda^k(x) = o[\{x\, \psi(x)\}^k], \quad k \geqslant 0;$

(b) $A_\lambda^l(x) = o\,(x^l)$, *for some* $l > k$.

Then we have

$$A_\varphi^k(x) = o[\{\varphi(x)\}^k] + o[\{x\, \varphi'(x)\, \psi(x)\}^k],$$

where

$$A_\varphi^k(x) = A_{\varphi(\lambda)}^k\{\varphi(x)\} = \int_0^x A(t)\, \frac{d}{dt}\,[\{\varphi(x) - \varphi(t)\}^k]\, dt. \quad \text{[cf. (1.23)]}$$

We remark that the first theorem of consistency permits us to suppose, without loss of generality, that l is an integer in hypothesis (b).

We also observe that, when $\varphi'(t) > 0$, the condition

$$t^r\, \varphi^{(r+1)}(t) = O\{\varphi'(t)\},\ r = 0, 1, 2, \ldots,$$

is more restrictive than Hirst's condition

$$\int_0^t x^r \,|\varphi^{(r+1)}(x)|\, dx = O\{\varphi(t)\},\ r = 0, 1, 2 \dots,$$

which, in turn, is more restrictive than

$$t^r \varphi^{(r)}(t) = O\{\varphi(t)\},\ r = 0, 1, 2, \dots .$$

All the three conditions are, however, equivalent for L-functions.

If $\psi(x) = \dfrac{\varphi(x)}{x\,\varphi'(x)} \succ 1$, where φ is an L-function, then we deduce Zygmund's theorem (Z_3). If $\varphi(t)$ is an L-function such that $\varphi(t) = O(t^{\Delta})$, $\Delta > 0$, then $1/\psi(x) = O(1)$, and we deduce Hardy's theorem (H_1).

2.5. Proof of Theorem 2.41

Let h stand for the greatest integer less than k, and D stand for the differential operator $\dfrac{d}{dt}$. Then $A^k_\eta(x)$ is, except for a constant factor,

$$\int_0^x A^h_\lambda(t).\, D^{h+1}[\{\varphi(x) - \varphi(t)\}^k]\, dt = \int_0^{x_1} + \int_{x_1}^{x} \equiv I_1 + I_2,$$

say, where $x_1 = x/c$, $c > 1$ being chosen as in Lemma 2.33. We shall show that

$$I_1 = o[\{\varphi(x)\}^k] + o[\{x\,\varphi'(x)\,\psi(x)\}^k], \tag{2.51}$$

while

$$I_2 = o[\{x\,\varphi'(x)\,\psi(x)\}^k]. \tag{2.52}$$

Integrating I_1 by parts $l - h$ times, we have

$$\begin{aligned} I_1 &= \sum_{r=h+1}^{l} P_r H_r - P_l \int_0^{x_1} A^l_\lambda(t).D^{l+1}[\{\varphi(x) - \varphi(t)\}^k]\, dt \\ &\equiv I_{1,1} + I_{1,2}, \end{aligned} \tag{2.53}$$

say, where $\{P_r\}$ stand for some constants, and

$$H_r = A^r_\lambda(x_1)\,[D^r\{\varphi(x) - \varphi(t)\}^k]_{t=x_1}.$$

We now estimate $I_{1,1}$. By Lemma 2.31, H_r is a sum of constant multiples of expressions of the form

$$A_\lambda^r(x_1)\,[\{\varphi(x)-\varphi(x_1)\}^s\{\varphi'(x_1)\}^{s_1}\ldots\{\varphi^{(r)}(x_1)\}^{s_r}],$$

where $s+s_1+\ldots+s_r=k$, $s_i \geqslant 0$ are integers for $i=1,2,\ldots,$ and $s_1+2s_2+\ldots+rs_r=r$. Now, by hypothesis (a) and Theorem 1.52, we have

$$A_\lambda^r(x_1)=o[x_1^r\{\psi(x_1)\}^k]=[\{x^r\psi(x)\}^k], \tag{2.54}$$

for $r \geqslant h+1$. Again, using Lemma 2.35, we have, for $s \gtrless 0$,

$$\{\varphi(x)-\varphi(x_1)\}^s=O[\{x\,\varphi'(x)\}^s]. \tag{2.55}$$

Next, by the assumptions on φ, we have

$$[\varphi'(x_1)]^{s_1}\ldots[\varphi^{(r)}(x_1)]^{s_r}=O\left[\frac{\{\varphi'(x_1)\}^{s_1+\ldots+s_r}}{x_1^{s_2+2s_3+\ldots+(r-1)s_r}}\right]$$

$$=O\left[\frac{\{\varphi'(x)\}^{k-s}}{x^{r-k+s}}\right], \tag{2.56}$$

by Lemma 2.33.

Combining (2.54), (2.55) and (2.56), we have

$$H_r=o[\{x\,\varphi'(x)\,\psi(x)\}^k],\quad r \geqslant h+1, \tag{2.57}$$

and hence

$$I_{1,1}=o[\{x\,\varphi'(x)\,\psi(x)\}^k]. \tag{2.58}$$

We next consider $I_{1,2}$ which is a sum of constant multiples of integrals of the form

$$\int_0^{x_1} A_\lambda^l(t)\,\{\varphi(x)-\varphi(t)\}^p\,\{\varphi'(t)\}^{p_1}\ldots\{\varphi^{(l+1)}(t)\}^{p_{l+1}}\,dt,$$

where

$$p+p_1+\ldots+p_{l+1}=k,$$

$$p_1+2p_2+\ldots+(l+1)p_{l+1}=l+1,$$

and $p_i \geqslant 0$ are integers. Here we observe, by Lemma 2.35, that, for $0 \leqslant t \leqslant x_1$,

$$\varphi(x) \geqslant \varphi(x)-\varphi(t) \geqslant \varphi(x)-\varphi(x_1) \geqslant ax\varphi'(x).$$

Thus $I_{1,2}$ is a sum of terms of the form

$$o\left[\int_0^{x_1}\frac{t^l\,\{\varphi(x)-\varphi(t)\}^p\,\{\varphi'(t)\}^{k-p}}{t^{l+1-k+p}}\,dt\right]$$

$$= o\left[\int_0^{x_1} t^{k-p-1}\,\{\varphi(x)-\varphi(t)\}^p\,\{\varphi'(t)\}^{k-p}\,dt\right]$$

$$= \begin{cases} o\left[\{\varphi(x)\}^p \int_0^{x_1} t^{k-p-1}\{\varphi'(t)\}^{k-p}\,dt\right] & (p > 0) \\ o\left[\{x\,\varphi'(x)\}^p \int_0^{x_1} t^{k-p-1}\,\{\varphi'(t)\}^{k-p}\,dt\right] & (p < 0) \end{cases}$$

$$= o\left[\{\varphi(x)\}^p \int_0^{x_1} \{\varphi(t)\}^{k-p-1}\,\varphi'(t)\,dt\right]$$

$$= o\,[\{\varphi(x)\}^k], \tag{2.59}$$

since $k - p \geqslant 1$, because $l + 1 \geqslant 1$, and $t\,\varphi'(t) = O\,\{\varphi(t)\}$, by hypothesis. (2.58) and (2.59) serve to establish (2.51).

We have now to estimate I_2, which is a sum of constant multiples of integrals of the form

$$\int_{x_1}^{x} A_\lambda^h(t)\,\{\varphi(x)-\varphi(t)\}^q\,\{\varphi'(t)\}^{q_1}\ldots\{\varphi^{(h+1)}(t)\}^{q_{h+1}}\,dt, \qquad (2.59)1$$

where

$$q + q_1 + \ldots + q_{h+1} = k,$$
$$q_1 + 2\,q_2 + \ldots + (h+1)\,q_{h+1} = h + 1.$$

In all these integrals *except one*, we have $q > 0$. The only case when $q < 0$ occurs is when $q = k - h - 1$; this implies that $q_1 = h+1$ [for $q_1 + \ldots + q_{h+1} = q_1 + 2q_2 + \ldots + (h+1)\,q_{h+1} = h+1$]. We then have, using Lemmas 2.33-2.35, the second mean-value theorem, and Lemma 1.41,

$$\int_{x_1}^{x} A_\lambda^h(t)\,(x-t)^{k-h-1}\left[\frac{\varphi(x)-\varphi(t)}{x-t}\right]^{k-h-1}\{\varphi'(t)\}^{h+1}\,dt$$

$$= O[\{\varphi'(x)\}^{k-h-1}].\,\{\varphi'(x)\}^{h+1}\int_{\xi_1}^{\xi_2} A_\lambda^h(t)\,(x-t)^{k-h-1}\,dt, x_1 \leqslant \xi_1 < \xi_2 \leqslant x;$$

$$= O[\{\varphi'(x)\}^k].\;o[\{x\,\psi(x)\}^k] = o[\{x\,\varphi'(x)\,\psi(x)\}^k]. \qquad (2.59)2$$

On the other hand, we have integrals of the type (2.59)1, with $q > 0$. If we integrate any of these once more by parts, we have

$$\left[\frac{A_\lambda^{h+1}(t)}{h+1}\{\varphi(x)-\varphi(t)\}^q \varphi'(t)^{q_1}\dots\right]_{x_1}^{x} +$$

$$+\frac{q}{h+1}\int_{x_1}^{x} A_\lambda^{h+1}(t)\{\varphi(x)-\varphi(t)\}^{q-1}\varphi'(t)\{\varphi'(t)\}^{q_1}\dots dt +$$

$$+\sum Q_r \int_{x_1}^{x} A_\lambda^{h+1}(t)\{\varphi(x)-\varphi(t)\}^q\{\varphi'(t)\}^{r_1}\{\varphi''(t)\}^{r_2}\dots dt$$

$$\equiv J_1 + J_2 + J_3, \tag{2.59)3}$$

say, where Q_r is a constant depending on the r's and

$$q + r_1 + r_2 + \dots = k, \qquad r_1 + 2r_2 + \dots = h + 2.$$

We now estimate J_1. By hypothesis (a) of Theorem 2.41, and Theorem 1.52, $A_\lambda^{h+1}(x_1) = o[x^{h+1}\{\psi(x)\}^k]$. By Lemma 2.35, $\{\varphi(x)-\varphi(x_1)\}^q = O[\{x\,\varphi'(x)\}^q]$. By the hypothesis on $\varphi^{(r)}$,

$$\{\varphi'(x_1)\}^{q_1}\{\varphi''(x_1)\}^{q_2}\dots = O[\{\varphi'(x_1)\}^{q_1+q_2+\dots}x_1^{-q_2-2q_3-\dots}]$$
$$= O[\{\varphi'(x)\}^{k-q}x^{k-q-h-1}],$$

by Lemma 2.33. Hence

$$J_1 = o[\{x\,\varphi'(x)\,\psi(x)\}^k]. \tag{2.59)4}$$

Again

$$J_2 = o\left[\int_{x_1}^{x} t^{h+1}\{\psi(t)\}^k\{\varphi(x)-\varphi(t)\}^{q-1}\varphi'(t)\frac{\{\varphi'(t)\}^{q_1+q_2+\dots}}{t^{q_2+2q_3+\dots}}dt\right]$$

$$= o\left[\int_{x_1}^{x}\{t\psi(t)\}^k t^{-q}\{\varphi'(t)\}^{k-q}\{\varphi(x)-\varphi(t)\}^{q-1}\varphi'(t)dt\right]$$

$$= o\left[\{x\psi(x)\}^k x^{-q}\{\varphi'(x)\}^{k-q}\int_{x_1}^{x}\{\varphi(x)-\varphi(t)\}^{q-1}\varphi'(t)\,dt\right]$$

$$= o[\{x\psi(x)\}^k x^{-q}\{\varphi'(x)\}^{k-q}\{\varphi(x)-\varphi(x_1)\}^q], \qquad \text{since } q > 0,$$

$$= o[\{x\varphi'(x)\,\psi(x)\}^k]. \tag{2.59)5}$$

Similarly $$J_3 = o[\{x\varphi'(x)\,\psi(x\}^k]. \tag{2.59)6}$$

From (2.59)2-(2.59)6 it follows that $I_2 = o[\{x\varphi'(x)\,\psi(x\}^k]$, and thus (2.52) is established, which completes the proof of the theorem.

2.6. An auxiliary theorem

We shall now prove theorem (Z_5) in a slightly modified form. We assume that

$\varphi(t)$ is a positive non-decreasing function of t diverging to $+\infty$, having N derivatives, where N is sufficiently large ; (2.61)

$$\varphi^{(r)}(x) > 0 \quad for\, x > \lambda_0, \quad r = 1, 2, \dots, N ; \tag{2.62}$$

$\dfrac{\varphi^{(r)}(x)}{\varphi\ (x)}\left(\dfrac{\varphi(x)}{\varphi'(x)}\right)^r$ *is monotonic and bounded for large x,* $r = 1, 2, \dots, N$;

(2.63)

if

$$\psi(x) = \frac{\varphi(x)}{x\varphi'(x)},$$

then

$$\frac{1}{x} \prec \psi(x) \prec 1 ; \tag{2.64}$$

and, for every $\delta > 0$,

$$\frac{\varphi^{1-\delta}}{\varphi'},\ \frac{\varphi'}{\varphi} \text{ and } \frac{\varphi''}{\varphi'} \text{ are monotonic decreasing.} \tag{2.65}$$

Before we proceed to prove the theorem, we need the following

LEMMA 2.61. *Let* $\xi(t) = \dfrac{\varphi'(t)\,(x-t)}{\varphi(x) - \varphi(t)}$, *where* $\lambda_0 \leqslant t < x$. *Then* $\xi(t)$ *is an increasing function of t, if* $\dfrac{\varphi''(t)}{\varphi'(t)}$ *is a decreasing function.*

PROOF. We have

$$\left[\frac{\varphi(x) - \varphi(t)}{x - t}\right]^2 \xi'(t)$$

$$= \varphi''(t).\ \frac{\varphi(x) - \varphi(t)}{x - t} - \varphi'(t).\ \frac{-\varphi'(t)\,(x-t) + \{\varphi(x) - \varphi(t)\}}{(x-t)^2}.$$

In order that $\xi'(t) \geqslant 0$, it is necessary and sufficient that

$$\frac{\varphi''(t)}{\varphi'(t)} \geqslant \frac{\{\varphi(x) - \varphi(t)\} - \varphi'(t)\,(x-t)}{(x-t)\,\{\varphi(x) - \varphi(t)\}}$$

$$= \frac{\varphi'(t_1) - \varphi'(t)}{\varphi(x) - \varphi(t)},\ t < t_1 < x.$$

This will be satisfied if

$$\frac{\varphi''(t)}{\varphi'(t)} \geqslant \frac{\varphi'(t_1) - \varphi'(t)}{\varphi(t_1) - \varphi(t)} = \frac{\varphi''(t_2)}{\varphi'(t_2)}, \; t < t_2 < t_1;$$

that is, if $\dfrac{\varphi''}{\varphi'}$ is decreasing.

THEOREM 2.61. *If* $\varphi(x)$ *and* $\psi(x)$ *satisfy conditions* (2.61)–(2.65), *and*

$$A_\lambda^k(x) = o\left[\{x\psi(x)\}^k\right], \quad k > 0,$$

then

$$A_\varphi^k(x) = o\left[\{\varphi(x)\}^k\right],$$

where A_φ^k *is defined as in Theorem* 2.41.

PROOF. Let h be the greatest integer less than k. Then $A_\varphi^k(x)$ is, except for a constant factor,

$$\left[\int_0^{x_1} + \int_{x_1}^{x}\right] A_\lambda^h(t)\, D^{h+1}\{\varphi(x) - \varphi(t)\}^k dt, \equiv I_1 + I_2,$$

say, $x_1 = x/c$, $c > 1$. To estimate I_1, let us integrate it by parts once more, so that

$$I_1 = \left[\frac{A_\lambda^{h+1}(t)}{h+1} D^{h+1}\{\varphi(x) - \varphi(t)\}^k\right]_{t=0}^{t=x_1} -$$

$$- \frac{1}{h+1}\int_0^{x_1} A_\lambda^{h+1}(t)\, D^{h+2}\{\varphi(x) - \varphi(t)\}^k\, dt$$

$$= I_{1,1} + I_{1,2}, \text{ say.}$$

$$I_{1,1} = \sum o\left[x_1^{h+1}\{\psi(x_1)\}^k \{\varphi(x) - \varphi(x_1)\}^s \{\varphi'(x_1)\}^{s_1} \{\varphi''(x_1)\}^{s_2} \cdots\right],$$

where the summation runs over s's such that

$s + s_1 + s_2 + \ldots + s_{h+1} = k,\; s_1 + 2s_2 + 3s_3 + \ldots + (h+1)s_{h+1} = h + 1.$

Now

$$\frac{\varphi(x_1)}{\varphi(x)} = e^{\log \varphi(x_1) - \log \varphi(x)} = e^{-x_1(c-1)\varphi'(x_2)/\varphi(x_2)}, \; x_1 \leqslant x_2 \leqslant x,$$

$$\leqslant e^{-x_1(c-1)\varphi'(x)/\varphi(x)}. \tag{2.66}$$

Hence

$$I_{1,1} = \sum o\left[x_1^{h+1}\{\psi(x_1)\}^k\ \{\varphi(x)\}^s \frac{\{\varphi'(x_1)\}^{s_1+2s_2+\dots}}{\{\varphi(x_1)\}^{s_2+2s_3+\dots}}\right]$$

$$= \sum o\left[x_1^{h+1}\{\psi(x_1)\}^k\{\varphi(x)\}^s\{\varphi'(x_1)\}^{h+1}\frac{\{\varphi(x_1)\}^{k-s}}{\{\varphi(x_1)\}^{h+1}}\right]$$

$$= \sum o\left[\{\varphi(x)\}^k\left(\frac{\varphi(x_1)}{\varphi(x)}\right)^{k-s}\{\psi(x_1)\}^k\left(\frac{x_1\varphi'(x_1)}{\varphi(x_1)}\right)^{h+1}\right],$$

where $k - s \geqslant 0$. Substituting for $\psi(x_1)$, we obtain

$$I_{1,1} = o\left[\{\varphi(x)\}^k\left(\frac{\varphi(x_1)}{\varphi(x)}\right)^{k-s}\left(\frac{x_1\varphi'(x_1)}{\varphi(x_1)}\right)^{h+1-k}\right]. \qquad (2.67)$$

If k is an integer, since φ is increasing and $k - s \geqslant 0$, the proof that $I_{1,1} = o[\{\varphi(x)\}^k]$ is obvious. If k is not an integer, the result follows from (2.66).

Next let us consider $I_{1,2}$.

$$I_{1,2} = \sum o\left[\{\varphi(x)\}^s.\int_0^{x_1} t^{h+1}\{\psi(t)\}^k\{\varphi'(t)\}^{s_1}\{\varphi''(t)\}^{s_2}\dots dt\right],$$

where $s + s_1 + s_2 + \dots = k$, and $s_1 + 2s_2 + \dots = h + 2$. Now

$$I_{1,2} = o\left[\{\varphi(x)\}^s.\ x_1.\left(\frac{x_1^{h+1}\{\psi(x_1)\}^k\{\varphi'(x_1)\}^{h+2}\{\varphi(x_1)\}^{k-s}}{\{\varphi(x_1)\}^{h+2}}\right)\right]$$

$$= o[\{\varphi(x)\}^k] \qquad (2.69)$$

as in $I_{1,1}$. Now consider I_2, which is a sum of multiples of

$$\int_{x_1}^{x} A_\lambda^h(t)\{\varphi(x)-\varphi(t)\}^s\{\varphi'(t)\}^{s_1}\{\varphi''(t)\}^{s_2}\dots dt,$$

where

$$s + s_1 + \dots = k,\ s_1 + 2s_2 + \dots = h + 1.$$

We have $k - 1 \geqslant s \geqslant k - h - 1$, and if we set

$$W \equiv (\varphi')^{s_1}(\varphi'')^{s_2}\dots,\ V \equiv \frac{(\varphi')^{h+1}}{\varphi^{h+1-k+s}},$$

then, by (2.63), W/V is monotonic, and applying the second mean-value theorem, we have

$$\int_{x_1}^{x} A_\lambda^h(t)\,\{\varphi(x)-\varphi(t)\}^s\,\{\varphi'(t)\}^{s_1}\dots\,dt$$

$$= O(1).\int_{x_2}^{x_3} A_\lambda^h(t)\,\{\varphi(x)-\varphi(t)\}^s\,\frac{\{\varphi'(t)\}^{h+1}}{\{\varphi(t)\}^{h+1-k+s}}\,dt,\quad x_1 \leqslant x_2 < x_3 \leqslant x,$$

$$= O(1).\int_{x_2}^{x_3} A_\lambda^h(t)\,\{\varphi(x)-\varphi(t)\}^{(s-k+h+1)+(k-h-1)}.\,\frac{\{\varphi'(t)\}^{1+h}}{\{\varphi(t)\}^{h+1-k+s}}\,dt$$

$$= O[\{\varphi(x)\}^{s-k+h+1}].\int_{x_2}^{x_4} A_\lambda^h(t)\left(\frac{\varphi'(t)\,(x-t)}{\varphi(x)-\varphi(t)}\right)^{h+1-k}\times$$

$$\times\,\frac{(x-t)^{k-h-1}}{\{\varphi(t)\}^{h+1-k+s}}\,\{\varphi'(t)\}^k\,dt,\ x_2 \leqslant x_4 < x_3;$$

$$= O[\{\varphi(x)\}^{s-k+h+1}].\int_{x_5}^{x_4} A_\lambda^h(t)\,(x-t)^{k-h-1}\,\frac{\{\varphi'(t)\}^k}{\{\varphi(t)\}^{h+1-k+s}}\,dt$$

(cf. Lemma 2.61)

$$= O\left(\frac{\{\varphi(x)\}^{h+1-k+s}\,\{\varphi'(x)\}^k}{\{\varphi(x)\}^{h+1-k+s}}\right).\int_{x_6}^{x_7} A_\lambda^h(t)\,(x-t)^{k-h-1}\,dt,$$

$$x_5 \leqslant x_6 < x_7 \leqslant x_4,$$

by (2.65). And now if we use Lemma 1.41, we obtain

$$I_2 = o\,[\{\varphi(x)\}^k].$$

2.7. Absolute summability

Questions analogous to those answered in the foregoing sections arise also in absolute Riesz summation. One would expect companion-theorems following (H_1)-(H_4) and (Z_1)-(Z_6). But explicit proof of such analogues is not available, except in the case of (H_1) where we have the following.

(C_1). *If* Σa_n *is summable* $|\lambda, k|$ *to the sum* s, *and* μ *is an L-function of* λ *such that* $\mu = O(\lambda^\Delta)$, $\Delta > 0$, *then* Σa_n *is summable* $|\mu, k|$ *to the same sum.*

The method that has to be employed in such cases, however, should be clear from the theorems which we have established here.

NOTES ON CHAPTER II

§2.1. The words 'consistency' and 'second theorem' (cf. Notes on Chapter I) are not particularly appropriate, but we have chosen to keep to Hardy's usage.

§2.2. The theorem of G. H. Hardy and M. Riesz we refer to is Theorem 17 of the *Tract.* The following theorem is an interesting companion: *if* $\lambda_0 > 0$, *and* Σa_n *is summable* (λ, k), *then* $\Sigma a_n \lambda_n^{-k}$ *is summable* (l, k), $l_n = e^{\lambda_n}$. See p. 33 of the *Tract.*

For an account of logarithmico-exponential functions, reference may be made to G. H. Hardy, *Orders of infinity*, Cambridge (1910). According to Hardy's notation, $\lambda^\delta \prec \mu(\lambda)$ means that $\lambda^\delta/\mu(\lambda) \to 0$ as $\lambda \to \infty$.

For Theorem (H_1) see G. H. Hardy, *Proc. London Math. Soc.* (2) 15 (1916), 72-88.

Theorems (H_2), (H_3), (Z_1)-(Z_6) are all contained in a paper by A. Zygmund, *Bull. Acad. Polonaise*, A(1925), 265-287.

In this section, unless otherwise specified, μ stands for $\mu(\lambda)$.

For an L-function $\mu(\lambda)$, the condition that $1 \prec \mu(\lambda) \prec \lambda^\delta$ for every $\delta > 0$, is equivalent to: $\dfrac{\mu'(x)}{\mu(x)} \to 0$ as $x \to \infty$. See *Orders of infinity*, loc. cit., Theorems 19, 21.

Zygmund suggests (loc. cit., 268, foot-note 13) that it is perhaps sufficient to suppose in (Z_3) that $\lim_{\sigma \to +0} \Sigma a_n \exp(-\lambda_n \sigma)$ exists instead of summability (λ, l).

While it is obvious that (Z_3) follows from (Z_4), it is not equally obvious that (Z_4) can be deduced from (Z_3), unless $\mu(x)$ as defined in (Z_4) is an L-function, which is not necessarily the case; however, the derivatives of μ can be proved to satisfy the required conditions. See Zygmund, loc. cit., 268.

In (Z_5) plainly there is no loss of generality in supposing that the sum of the series is zero; if the sum is $s \neq 0$, then the hypothesis would be:

$$A_\lambda^k(x) = s\,(x - \lambda_0)^k + o[\{\mu(x)/\mu'(x)\}^k].$$

The relationship of (Z_6) to (Z_5) is similar to that of (Z_4) to (Z_3).

For (H_4) see K. A. Hirst, *Proc. London Math. Soc.* (2) 33 (1932), 355-366. Hirst explains the difficulty in replacing his conditions on φ in (H_4) by the less exacting ones: $t^r \varphi^{(r)}(t) = O\{\varphi(t)\}$. In this connexion, he points out a minor gap in Hardy's argument, and shows that it could be suitably amended.

B. Kuttner has recently shown that Hirst's condition, when k is an integer, is in fact necessary, *Jour. London Math. Soc.* 26 (1951), 104-111. Dr. Bosanquet informs us that more recently still, Kuttner has obtained a necessary and sufficient condition for the case k fractional, and that his proof will appear in the *Jour. London Math. Soc.* 27 (1952).

§2.3. Lemma 2.31 is reproduced from Hirst, loc. cit. But it can also be found in a text-book like Ch.-J. de la Vallée Poussin's *Cours d' analyse infinitesimale* I, ed. 5, 89.

Lemmas 2.32 and 2.34 are explicitly proved by Hirst.

§2.4. It is perhaps possible to prove Theorem 2.41 with Hirst's conditions on φ; such a proof, one would expect, will require some more attention to details and will not need any substantial change in the argument.

For the behaviour of the derivatives of L-functions see G. H. Hardy, *Orders of infinity*, loc. cit., 38-39, and *Proc. London Math. Soc.* 15 (1916), 75.

§2.6. Theorem 2.61 is a slight modification of Theorem Z_5 of Zygmund, loc. cit., 272.

§2.7. For Theorem C_1 see K. Chandrasekharan, *Jour. Indian Math. Soc.* (2) (1942), 168-180. The proof given needs drastic revision if k is non-integral.

That this theorem (and its companions) could be proved in a more general form, which dispenses with L-functions, will be evident from the earlier sections.

In conclusion, we may refer to the work of B. Kuttner on the positivity (instead of convergence) of Riesz means (n^a, k), for varying a, of the Fourier series of a positive function. In his work, unlike in the theorems of this chapter, there is a distinction in behaviour between $a \geqslant 2$ and $a = 1$. See *Jour. London Math. Soc.* 18 (1943), 148, and 19 (1944), 77.

III

APPLICATIONS TO DIRICHLET SERIES

3.1. Introduction

In this chapter we shall discuss some applications of the results on Riesz means obtained in the foregoing chapters to the study of Dirichlet series. It will appear from the applications that Riesz means furnish an appropriate tool for studying the summability of Dirichlet series.

We shall deal with two types of series, either of the form $\Sigma a_n e^{-\lambda_n s}$ or $\Sigma a_n l_n^{-s}$, according to convenience, where $\{\lambda_n\}$ is an increasing sequence of positive real numbers diverging to $+\infty$, and $l_n = e^{\lambda_n}$. We shall first prove a few theorems on the *abscissae of summability* of Dirichlet series and the functions represented by them, which will show how summability by typical means helps in tackling the problem of analytic continuation of functions represented by Dirichlet series in their half-plane of convergence. We then prove a few converse theorems on abscissae of summability, which will show the very close connexion that exists between certain properties of functions represented by Dirichlet series and the summability of such series by Riesz means. We next prove some Tauberian theorems, and conclude the chapter with some results on the Dirichlet product of summable series.

3.2. Notations

We introduce here certain notations which will be used in the rest of this chapter. If Σa_n is a given series, and $\{\lambda_n\}$ is an increasing sequence of positive numbers diverging to $+\infty$, we denote the Riesz sum of type λ and order k of the series Σa_n by $A_\lambda^k(\omega)$, $k \geqslant 0$. In conformity with our notation in § 1.1, we have, if $k > 0$,

$$A_\lambda^k(\omega) = k\int_0^\omega (\omega - t)^{k-1} A_\lambda(t)\,dt,\ A_\lambda(t) \equiv A_\lambda^0(t) = \sum_{\lambda_r \leqslant t} a_r,$$

and if $k > 0$, we have

$$A_\lambda^k(\omega) = \int_0^\omega (\omega - t)^k dA_\lambda(t).$$

Correspondingly, $B_\mu^k(t)$ denotes the Riesz sum of type μ and order k of the series Σb_n. We define

$$\bar{A}_\lambda^k(\omega) = \sum_{\lambda_r \leqslant \omega} (\omega - \lambda_r)^{k-1}\lambda_r a_r. \qquad (k > 0)$$

After applying Abel's method of partial summation to the sum on the right, whose general term may be considered as the product of $(\omega-\lambda_r)^{k-1}\lambda_r$ and a_r, we see that

$$\bar{A}_\lambda^k(\omega) = -\int_0^\omega A_\lambda(t).\frac{d}{dt}[(\omega - t)^{k-1}t]\,dt \qquad (k > 1)$$

$$= \int_0^\omega (\omega - t)^{k-1}t\, dA_\lambda(t). \qquad (k \geqslant 1)$$

Thus if $k \geqslant 1$, we obtain

$$\bar{A}_\lambda^k(\omega) = \omega A_\lambda^{k-1}(\omega) - A_\lambda^k(\omega). \tag{3.21}$$

We use s to denote a complex number : $s = \sigma + i\tau$, where σ and τ are real. Correspondingly we use

$$s_0 = \sigma_0 + i\tau_0, s_1 = \sigma_1 + i\tau_1, \dots, s^* = \sigma^* + i\tau^*.$$

If $k > 1$, we denote by h the greatest integer less than k ; if $0 < k \leqslant 1$, we define $h = 0$, and $h = -1$ if $k = 0$. $[k]$ will denote the integral part of k. If $\frac{d}{dt}$ stands for the ordinary differential operator, we write

$$D = \frac{d}{dt}, D^h = \frac{d^h}{dt^h}.$$

$c, c_1, c_2, \dots$ stand for numerical constants, not necessarily having the same value in all occurrences.

3.3. Abelian theorems on abscissae of summability

Given a Dirichlet series $\Sigma a_n e^{-\lambda_n s}$ and a method of summation (λ, k), it is possible that the series is summable by that method for all values of s, or some values of s, or no value of s. If we know that it is summable (λ, k) for a certain value of s, we would like to know for what other values of s it is so summable. This section answers that question. In the first place, we show that if the series is summable (λ, k) for $s = s^*$, then it is summable for all values of s such that $\sigma > \sigma^*$. It follows from this proposition that there exists a number σ_k, called the ***abscissa of summability*** (λ, k), such that the series is summable (λ, k) for $\sigma > \sigma_k$ and not so summable for $\sigma < \sigma_k$, the case $\sigma = \sigma_k$ being undecidable, in general. The line $\sigma = \sigma_k$ is called the ***line of summability*** (λ, k). The region defined by $\sigma > \sigma_k$ is called the ***half-plane of summability*** (λ, k). It may happen of course that $\sigma_k = +\infty$ or $-\infty$. If $\sigma_k > 0$, we prove an explicit formula for σ_k. The line of summability for Dirichlet series reminds one of the circumference of the circle of convergence for power series; the 'abscissa' corresponds to the 'radius of convergence' and the 'half-plane' corresponds to the 'interior of the circle of convergence.' We next show that if a Dirichlet series is summable, then the sum-function is regular in the half-plane of summability. Finally we examine the regions of *uniform* summability; we show on the one hand that the series is uniformly summable in any finite region for all points of which $\sigma \geqslant \sigma_k + \varepsilon > \sigma_k$, and on the other hand we show that if the series is summable for a certain $s = s^*$, it is uniformly summable in the angular region defined by θ, where

$$|\operatorname{am}(s - s^*)| \leqslant \theta < \pi/2.$$

We conclude the section with some analogues for Dirichlet series of the form $\Sigma a_n l_n^{-s}$, associated with summability of type l and order k.

We start with a formula for the Riesz mean of a Dirichlet series.

Lemma 3.31. *If Σb_n is a given infinite series, $\{\lambda_n\}$ an increasing sequence of positive numbers diverging to ∞, and s is a complex number, then for $k \geqslant 0$, we have*

$$\omega^{-k} \sum_{\lambda_r \leqslant \omega} (\omega - \lambda_r)^k b_r e^{-\lambda_r s} = e^{-\omega s} \omega^{-k} B_\lambda^k(\omega) +$$

$$+ \frac{\Gamma(k+1)}{\Gamma(h+2)\,\Gamma(k-h-1)} \omega^{-k} \int_0^\omega B_\lambda^{h+1}(t)\,(e^{-ts} - e^{-\omega s})\,(\omega - t)^{k-h-2} dt +$$

$$+ \frac{\omega^{-k}}{\Gamma(h+2)} \times$$

$$\times \sum_{r=1}^{h+1} s^r \binom{h+2}{r} \frac{\Gamma(k+1)}{\Gamma(k-h+r-1)} \int_0^\omega B_\lambda^{h+1}(t)\, e^{-ts}(\omega-t)^{k-h+r-2}\, dt +$$

$$+ \frac{s^{h+2}}{\Gamma(h+2)}\, \omega^{-k} \int_0^\omega B_\lambda^{h+1}(t)\, e^{-ts}(\omega-t)^k\, dt. \tag{3.31}$$

PROOF.

$$\omega^{-k} \sum_{\lambda_r \leqslant \omega} (\omega - \lambda_r)^k\, b_r e^{-\lambda_r s}$$

$$= \omega^{-k}\, e^{-\omega s} \sum_{\lambda_r \leqslant \omega} (\omega - \lambda_r)^k\, b_r + \omega^{-k} \sum_{\lambda_r \leqslant \omega} (\omega - \lambda_r)^k\, b_r\, (e^{-\lambda_r s} - e^{-\omega s})$$

$$= \omega^{-k}\, e^{-\omega s}\, B_\lambda^k(\omega) - \omega^{-k} \int_0^\omega B_\lambda(t)\, \frac{d}{dt} \{(e^{-ts} - e^{-\omega s})(\omega - t)^k\}\, dt$$

$$= \omega^{-k}\, e^{-\omega s}\, B_\lambda^k(\omega) +$$

$$+ \frac{(-)^h\, \omega^{-k}}{\Gamma(h+2)} \int_0^\omega B_\lambda^{h+1}(t)\; D^{h+2} \{(e^{-ts} - e^{-\omega s})\, (\omega - t)^k\}\, dt.$$

The second step in this chain of equalities requires Abel's method of partial summation, and the subsequent steps require repeated integrations by parts.

By elementary rules of differentiation, however,

$$D^{h+2} \{(e^{-ts} - e^{-\omega s})\, (\omega - t)^k\}$$

$$= (-)^{h+2} \Big[\frac{\Gamma(k+1)}{\Gamma(k-h-1)}\, (e^{-ts} - e^{-\omega s})\, (\omega - t)^{k-h-2} +$$

$$+ \sum_{r=1}^{h+1} s^r \binom{h+2}{r} \frac{\Gamma(k+1)}{\Gamma(k-h+r-1)}\, e^{-ts}\, (\omega - t)^{k-h+r-2} +$$

$$+ s^{h+2} e^{-ts}\, (\omega - t)^k \Big].$$

Substituting this in the last formula, we prove the lemma.

THEOREM 3.31. *If the series $\Sigma a_n e^{-\lambda_n s}$ is summable (λ, k), or bounded (λ, k), $k > 0$, for $s = s^*$, $\sigma^* \geqslant 0$, then, for $\sigma > \sigma^*$,*

$$A_\lambda^k(\omega) = o\, (\omega^k e^{\omega\sigma}). \tag{3.32}$$

Conversely, if (3.32) *is satisfied for* $\sigma = \sigma^* \geqslant 0$, *then* $\Sigma a_n e^{-\lambda_n s}$ *is summable* (λ, k) *for every value of* s *such that* $\sigma > \sigma^*$, *to the sum* $f(s)$, *where*

$$f(s) = \frac{s^{k+1}}{\Gamma(k+1)} \int_0^\infty A_\lambda^k(t)\, e^{-ts}\, dt. \tag{3.33}$$

PROOF. To prove the first part of the theorem, we choose $s = -s^*$ and set $b_n = a_n e^{-\lambda_n s^*}$ in Lemma 3.31. Then the left side of (3.31) gives $\omega^{-k} A_\lambda^k(\omega)$, while the right side is the sum of $h+4$ terms each of which is $o(e^{\omega\sigma})$. For the first term is $O(e^{\omega\sigma^*})$ which is $o(e^{\omega\sigma})$ for $\sigma > \sigma^*$, since the hypothesis implies that $B_\lambda^k(\omega) = O(\omega^k)$. In the second term we have a factor whose modulus is

$$|e^{ts^*} - e^{\omega s^*}| = \left| s^* \int_t^\omega e^{ts^*}\, dt \right| \leqslant |s^*|\, (\omega - t) e^{\omega\sigma^*},$$

so that the absolute value of the second term will be

$$\leqslant c\, |s^*|\, \omega^{-k} e^{\omega\sigma^*} \int_0^\omega |B_\lambda^{h+1}(t)| \,.\, (\omega - t)^{k-h-1}\, dt$$

$$\leqslant c\, |s^*|\, \omega . e^{\omega\sigma^*} = o(e^{\omega\sigma}),$$

since $B_\lambda^{h+1}(t) = O(t^{h+1})$. The next $(h+2)$ terms are similarly seen to be $o(e^{\omega\sigma})$.

To prove the second part of the theorem, we choose $b_n = a_n$ in Lemma 3.31. Of the $h+4$ terms that occur on the right side of (3.31), each of the first $(h+3)$ terms will be seen to be $o(1)$ as $\omega \to \infty$, if $\sigma > \sigma^*$, by an argument similar to the above, so that we finally have

$$\lim_{\omega\to\infty} \omega^{-k} \sum_{\lambda_r \leqslant \omega} (\omega - \lambda_r)^k a_r e^{-\lambda_r s}$$

$$= \lim_{\omega\to\infty} \frac{s^{h+2}}{\Gamma(h+2)}\, \omega^{-k} \int_0^\omega A_\lambda^{h+1}(t)\, e^{-ts}\, (\omega - t)^k\, dt$$

$$= \lim_{\omega\to\infty} \frac{s^{h+2}}{\Gamma(h+2)}\, \omega^{-k} \int_0^\omega (\omega - t)^k d\varphi(t), \tag{3.34}$$

where

$$\varphi(t) = \int_0^t A^{h+1}(u)\, e^{-us} du.$$

Since $A_\lambda^k(u) = o(u^k e^{u\sigma^*})$ and $\sigma^* \geqslant 0$, we have, by the o-version of Theorem 1.52, $A_\lambda^{h+1}(u) = o(u^{h+1}\, e^{u\sigma^*})$, so that $\lim_{t\to\infty} \varphi(t)$ exists for $\sigma > \sigma^*$. Hence the limit on the right of (3.34) also exists, and equals $\dfrac{s^{h+2}}{\Gamma(h+2)} \lim_{t\to\infty} \varphi(t)$. Thus, for $\sigma > \sigma^*$,

$$\lim_{\omega\to\infty} \omega^{-k} \sum_{\lambda_r \leqslant \omega} (\omega - \lambda_r)^k a_r e^{-\lambda_r s} = \frac{s^{h+2}}{\Gamma(h+2)} \int_0^\infty A_\lambda^{h+1}(u)\, e^{-us}\, du$$

$$= \frac{s^{h+2}}{\Gamma(h+2)} \cdot \frac{\Gamma(h+2)}{\Gamma(k+1)\,\Gamma(h+1-k)} \int_0^\infty e^{-ts} dt \int_0^t (t-u)^{h-k} A_\lambda^k(u)\, du$$

$$= \frac{s^{h+2}}{\Gamma(k+1)\,\Gamma(h+1-k)} \int_0^\infty A_\lambda^k(u)\, du \int_u^\infty e^{-ts}(t-u)^{h-k}\, dt$$

$$= \frac{s^{k+1}}{\Gamma(k+1)} \int_0^\infty A_\lambda^k(u) e^{-us}\, du.$$

From Theorem 3.31 we can easily deduce

THEOREM 3.32. *If $\sum a_n e^{-\lambda_n s}$ is summable (λ, k), or bounded (λ, k), for $s = s^*$, then the series is summable (λ, k) for all values of s such that $\sigma > \sigma^*$.*

Theorem 3.32 and the classical argument associated with Dedekind's section for a real number yield the following

THEOREM 3.33. *There exists a number σ_k such that the series $\sum a_n e^{-\lambda_n s}$ is summable (λ, k) for $\sigma > \sigma_k$ and not summable (λ, k) for $\sigma < \sigma_k$.*

We may have $\sigma_k = -\infty$ or $+\infty$. This number σ_k is called the *abscissa of summability* (λ, k) of the series $\sum a_n e^{-\lambda_n s}$. It is clear that σ_k is a decreasing function of k.

The line $\sigma = \sigma_k$ is called the ***line of summability*** **(λ, k)**, and the region $\sigma > \sigma_k$ is called the ***half-plane of summability*** (λ, k). We proceed to give a formula for σ_k in case $\sigma_k > 0$.

THEOREM 3.34. *Let σ_k be the abscissa of summability (λ,k) of $\Sigma a_n e^{-\lambda_n s}$. If $\sigma_k > 0$, then*

$$\sigma_k = \limsup_{\omega \to \infty} \frac{\log |A_\lambda^k(\omega)|}{\omega}.$$

PROOF. Let

$$\limsup_{\omega \to \infty} \frac{\log |A_\lambda^k(\omega)|}{\omega} = \sigma_a.$$

Then

$$A_\lambda^k(\omega) = o\left\{e^{\omega(\sigma_a + \varepsilon)}\right\}, \ \varepsilon > 0,$$

and hence, by Theorem 3.31, $\Sigma a_n e^{-\lambda_n s}$ is summable (λ, k) if $\sigma > \sigma_a$. That is to say, $\sigma_a \geqslant \sigma_k$. On the other hand, since $\Sigma a_n e^{-\lambda_n s}$ is summable (λ, k) for $\sigma > \sigma_k$, it follows from the first part of Theorem 3.31 that

$$A_\lambda^k(\omega) = o\left\{e^{\omega(\sigma_k + \varepsilon)}\right\};$$

that is

$$\limsup_{\omega \to \infty} \frac{\log |A_\lambda^k(\omega)|}{\omega} \leqslant \sigma_k + \varepsilon,$$

or $\sigma_a \leqslant \sigma_k$, since ε is arbitrary.

Functions represented by Dirichlet series

We now show that the sum of a Dirichlet series, in its half-plane of summability, is an analytic function whose behaviour for large values of the argument can be stated with some precision.

THEOREM 3.35. *If $\Sigma a_n e^{-\lambda_n s}$ is summable (λ, k) for $s = s^*$, and $f(s)$ denotes its sum, then, uniformly for $\sigma \geqslant \sigma^* + \varepsilon > \sigma^*$, we have*

$$f(s) = o\left(|\tau|^{k+1}\right). \tag{3.35}$$

PROOF. We may assume, without loss of generality, that $s^* = 0$ so that $A_\lambda^k(t) = O(t^k)$. For, if we write b_n in place of $a_n e^{-\lambda_n s^*}$ and put $s' = s - s^*$, then the series $\Sigma b_n e^{-\lambda_n s'}$ is summable (λ, k) for $s' = 0$.

We start from the formula (3.33), and estimate the order of $f(s)$ in the region $\sigma \geqslant \varepsilon$. Given a number a such that $0 < a < \pi/2$, the half-plane defined by the relation $\sigma \geqslant \varepsilon$ can be considered as the set-union of two regions: the first region is defined by the relations $\sigma \geqslant \varepsilon$, $|\text{am } s| \leqslant a < \pi/2$, and the second by the relations $\sigma \geqslant \varepsilon$, $a < |\text{am } s| \leqslant \pi/2$.

We prove the result for the two regions separately. Formula (3.33) gives

$$|f(s)| \leqslant c \frac{|s|^{k+1}}{\Gamma(k+1)} \int_0^\infty t^k e^{-t\sigma}\, dt$$

$$= c \left(\frac{|s|}{\sigma}\right)^{k+1}$$

$$= c\, (\sec a)^{k+1} = O(1) = o\,(\tau^{k+1}),$$

in the region $|\text{am } s| \leqslant a < \pi/2$, which actually includes the first region defined above.

In the second region, $\operatorname{cosec} \theta$, $\theta = |\text{am } s|$ is finite, and we write

$$\int_0^\infty A_\lambda^k(t)\, e^{-ts}\, dt = \int_0^{\omega_0} + \int_{\omega_0}^\infty \equiv I_1 + I_2,$$

say. Since $A_\lambda^k(t) = O(t^k)$, we have, if δ is any positive number,

$$|I_2| \leqslant c \int_{\omega_0}^\infty t^k e^{-t\sigma}\, dt,$$

$$< \frac{c\delta}{\sigma^{k+1}},$$

for ω_0 sufficiently large, and

$$I_1 = \int_0^{\omega_0} A_\lambda^k(t)\, e^{-ts}\, dt$$

$$= -\frac{A_\lambda^k(\omega_0)\, e^{-\omega_0 s}}{s} + \frac{1}{s} \int_0^{\omega_0} e^{-ts}\, dA_\lambda^k(t)$$

$$= O\{1/|s|\}$$

for any fixed ω_0, and uniformly in $\sigma > 0$. Hence we obtain, from (3.33),

$$\begin{aligned} f(s) &= O(|s|^k) + O\{(\delta|s/\sigma|)^{k+1}\} \\ &= O(|s|^k) + O(\delta|s|^{k+1}), \qquad \text{for } \sigma \geqslant \varepsilon. \end{aligned}$$

Thus $f(s) = o(|\tau|^{k+1})$, since $|s/\tau| \leqslant \operatorname{cosec} \alpha < \infty$, and the theorem follows.

THEOREM 3.36. *Let σ_k be the abscissa of summability (λ, k) of $\Sigma a_n e^{-\lambda_n s}$. If D is any finite region for all points of which $\sigma \geqslant \sigma_k + \delta > \sigma_k$, the series $\Sigma a_n e^{-\lambda_n s}$ is uniformly summable (λ, k) throughout D, and its sum represents a branch of an analytic function regular throughout D. Further, for any non-negative integral r, $\Sigma \lambda_n^r a_n e^{-\lambda_n s}$ is uniformly summable throughout D to the value $(-1)^r f^{(r)}(s)$.*

PROOF. Uniform summability of $\Sigma a_n e^{-\lambda_n s}$ follows from Theorem 3.31, if we observe that the estimates for the $h + 4$ summands involved in the proof of that theorem are valid uniformly. That the sum is analytic follows from Weierstrass's theorem, since it is the uniform limit of analytic functions.

To prove the summability of $\Sigma \lambda_n^r a_n e^{-\lambda_n s}$, we observe that

$$\Sigma a_n e^{-\lambda_n (\sigma_k + \delta/2)}$$

is summable (λ, k), and hence

$$\sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^k \lambda_n^r a_n e^{-\lambda_n (\sigma_k + \delta/2)} = O(\omega^{r+k}),$$

for r integral, as can be seen from (3.21). By Theorem 3.31, this implies that $\Sigma \lambda_n^r a_n e^{-\lambda_n s}$ is summable (λ, k) for $\sigma > \sigma_k + \delta$. Hence, in D, by termwise differentiation of

$$f(s) = \lim_{\omega \to \infty} \omega^{-k} \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^k a_n e^{-\lambda_n s},$$

we get

$$(-1)^r f^{(r)}(s) = \lim_{\omega \to \infty} \omega^{-k} \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^k \lambda_n^r a_n e^{-\lambda_n s}.$$

Summability in an angle

We shall now prove a theorem which says a little more than Theorem 3.36 as regards the uniform summability of $\Sigma a_n e^{-\lambda_n s}$.

THEOREM 3.37. *If $\Sigma a_n e^{-\lambda_n s}$ is summable (λ, k) for $s = s^*$, then it is uniformly summable for all s in the angle defined by $|\mathrm{am}\,(s - s^*)| \leqslant \theta < \pi/2$, where θ is any fixed positive number less than $\pi/2$. If $f(s)$ denotes the sum, then $f(s) \to f(s^*)$ as $s \to s^*$ within this angle, and $f(s) = O(1)$ as $s \to \infty$ in the angle.*

PROOF. We may assume without loss of generality that $s^* = 0$, and that the sum at $s = s^*$ is also zero, for if $c \neq 0$ is the sum of Σa_n, then $(a_0 - c) + a_1 + a_2 + a_3 + \ldots$ is a series whose sum is zero, while the series $c + 0 + 0 + \ldots$ is uniformly summable in the angle to the sum c, and we can proceed with the former.

Let us consider formula (3.31) in Lemma 3.31 with $b_n = a_n$. We observe that

$$\omega^{-k} \Sigma (\omega - \lambda_n)^k a_n e^{-\lambda_n s} = \sum_{r=1}^{h+4} I_r ,$$

where I_r is known for each r, from (3.31), and

$$I_1 = e^{-\omega s}\, \omega^{-k} A_\lambda^k(\omega) = o(1),$$

uniformly in the angle $|\mathrm{am}\, s| \leqslant \theta < \pi/2$, by hypothesis.

$$|I_2| = \left| \frac{\Gamma(k+1)}{\Gamma(h+2)\,\Gamma(k-h-1)}\, \omega^{-k} \times \right.$$

$$\left. \times \int_0^\omega A_\lambda^{h+1}(t)\,(e^{-ts} - e^{-\omega s})\,(\omega - t)^{k-h-2} dt \right|$$

$$\leqslant c\,\omega^{-k} |s| \int_0^\omega |A_\lambda^{h+1}(t)|\,.\, e^{-\sigma t}(\omega - t)^{k-h-1} dt$$

$$\leqslant c\,\omega^{-k} \frac{|s|}{\sigma} \int_0^\omega \frac{|A_\lambda^{h+1}(t)|}{t}\,.\,\sigma t.\, e^{-\sigma t}(\omega - t)^{k-h-1} dt$$

$$\leqslant c\,\omega^{-k} \sec\theta \int_0^\omega o(t^h)\,(\omega - t)^{k-h-1} dt$$

$$= o(1),$$

since $\sigma t\, e^{-\sigma t} \leqslant 1$. For $1 \leqslant r \leqslant h+1$, we observe that

$$|I_{2+r}| = \left| c\, s^r\, \omega^{-k} \int_0^\omega A_\lambda^{h+1}(t)\, e^{-ts}\,(\omega - t)^{k-h+r-2} dt \right|$$

$$\leqslant c\,|s|^r\,\omega^{-k}\int_0^\omega |A_\lambda^{h+1}(t)|\,e^{-t\sigma}(\omega-t)^{k-h+r-2}dt$$

$$\leqslant c\,(\sec\theta)^r\,\omega^{-k}\int_0^\omega \frac{|A_\lambda^{h+1}(t)|}{t^r}(\sigma t)^r\,e^{-t\sigma}(\omega-t)^{k-h+r-2}dt$$

$$\leqslant c\,r\,!\,(\sec\theta)^r\,\omega^{-k}\int_0^\omega \frac{|A_\lambda^{h+1}(t)|}{t^r}(\omega-t)^{k-h+r-2}\,dt$$

$$=\omega^{-k}\int_0^\omega o\,(t^{h+1-r})\,(\omega-t)^{k-h+r-2}dt$$

$$=o\,(1).$$

Finally

$$I_{h+4}=\frac{s^{h+2}}{\Gamma(h+2)}\,\omega^{-k}\int_0^\omega A_\lambda^{h+1}(t)\,e^{-ts}\,(\omega-t)^k dt$$

tends uniformly (in the angle) to the convergent integral

$$\frac{s^{h+2}}{\Gamma(h+2)}\int_0^\infty A_\lambda^{h+1}(t)\,e^{-ts}\,dt,$$

since $A_\lambda^{h+1}(t)=o\,(t^{h+1})$, and this is equal to

$$\frac{s^{k+1}}{\Gamma(k+1)}\int_0^\infty A_\lambda^k(t)\,e^{-ts}\,dt,$$

and hence the first part of the theorem. To prove that $f(s)\to 0$ uniformly as $s\to 0$ in the angle, we observe that

$$f(s)=\frac{s^{k+1}}{\Gamma(k+1)}\int_0^\infty A_\lambda^k(t)\,e^{-ts}dt=\frac{s^{k+1}}{\Gamma(k+1)}\left[\int_0^{\omega_0}+\int_{\omega_0}^\infty\right]$$

$$\equiv\varphi_1+\varphi_2,\text{ say.}$$

If we choose ω_0 such that for $t>\omega_0$, we have $|A_\lambda^k(t)|<\delta t^k$, then

$$|\varphi_2|<\frac{|s|^{k+1}}{\Gamma(k+1)}\,\delta\int_{\omega_0}^\infty t^k\,e^{-t\sigma}dt\leqslant\frac{\delta}{\Gamma(k+1)}\cdot\frac{|s|^{k+1}}{\sigma^{k+1}}\leqslant\frac{(\sec\theta)^{k+1}\,\delta}{\Gamma(k+1)}=o\,(1),$$

since $\delta > 0$ is arbitrary. Again

$$|\varphi_1| \leqslant \frac{|s|^{k+1}}{\Gamma(k+1)} \int_0^{\omega_0} |A_\lambda^k(t)|\, dt$$
$$= O(|\sigma|^{k+1})$$
$$= o(1),$$

since $\frac{|s|}{\sigma} \leqslant \sec\theta$, and $\sigma \to 0$ as $s \to 0$. Since $f(s) = \varphi_1 + \varphi_2$, it follows that $f(s) = o(1)$ as $s \to 0$, uniformly.

That $f(s) = O(1)$ as $s \to \infty$ in the angle, is included in the proof of Theorem 3.35. We recall that

$$|f(s)| = \frac{|s|^{k+1}}{\Gamma(k+1)} \left| \int_0^{\infty} A_\lambda^k(t) e^{-t\sigma} dt \right|$$
$$= O\{(|s|/\sigma)^{k+1}\}$$
$$= O\{(\sec\theta)^{k+1}\} = O(1).$$

Summability (l, k)

We shall now prove analogous results for Dirichlet series of the form $\Sigma a_n l_n^{-s}$, where $1 \leqslant l_0 < l_1 < \ldots < l_n \to \infty$, with which summability (l, k) will be associated. Corresponding to Lemma 3.31, we have

LEMMA 3.32. *If* $k \geqslant 0$, *then*

$$\omega^{-k} \sum_{l_n \leqslant \omega} (\omega - l_n)^k a_n l_n^{-s} = \omega^{-k-s} A_l^k(\omega) + \frac{\Gamma(k+1)}{\Gamma(k-h-1)\Gamma(h+2)} \times$$
$$\times\, \omega^{-k} \int_1^{\omega} A_l^{h+1}(t)\,(t^{-s} - \omega^{-s})\,(\omega - t)^{k-h-2} dt + \frac{\omega^{-k}}{\Gamma(h+2)} \times$$
$$\times \sum_{r=1}^{h+2} \binom{h+2}{r} \frac{\Gamma(s+r)\,\Gamma(k+1)}{\Gamma(s)\,\Gamma(k-h+r-1)} \int_1^{\omega} A_l^{h+1}(t)\, t^{-s-r} (\omega - t)^{k-h+r-2} dt.$$

PROOF. As in Lemma 3.31.

We can also deduce results analogous to Theorems 3.31-3.37. Without going through the details of the proofs, we state the results as a single theorem.

THEOREM 3.38. *If* Σa_n *is summable* (l, k), $k \geqslant 0$, *then* $\Sigma a_n l_n^{-s}$ *is summable* (l, k) *for* $\sigma > 0$, *and in fact uniformly in the angle* $|\operatorname{am} s| \leqslant a < \pi/2$, *and the sum is given by*

$$\frac{\Gamma(s+k+1)}{\Gamma(k+1)\,\Gamma(s)} \int_1^\infty A_l^k(t)\, t^{-s-k-1} dt.$$

Further, if σ_k *is the abscissa of summability* (l, k), *then* σ_k, *if positive, is given by*

$$\sigma_k = \limsup_{\omega \to \infty} \frac{\log |A_l^k(\omega)|}{\log \omega} - k.$$

It can also be shown that the sum-function is regular in the half-plane of summability.

We shall now prove a theorem which is a complement to Theorem 3.36, and whose proof depends on the use of Theorem 3.38.

THEOREM 3.39. *Let* σ_k *denote the abscissa of summability* (λ, k) *of the Dirichlet series* $\Sigma a_n e^{-\lambda_n s}$. *Then* $\Sigma a_n \lambda_n^\rho e^{-\lambda_n s}$, *where* ρ *is any complex number, and* λ_n^ρ *has its principal value, is summable* (λ, k) *for* $\sigma > \sigma_k$. *Further the summability is uniform in any finite region contained in* $\sigma \geqslant \sigma_k + \delta > \sigma_k$.

PROOF. The theorem holds if ρ equals a non-negative integer r because of Theorem 3.36. In the general case, we may write $\rho = r - s$, where $\operatorname{Re}(s) > 0$, and apply Theorem 3.38.

3.4. Abelian theorems on absolute summability

In this section we prove the analogues of the results of the foregoing section for absolute summability.

LEMMA 3.41. *Let* $k > 0$. *Then*

$$\frac{d}{d\omega}\Big[\omega^{-k} \; \Sigma a_n e^{-\lambda_n s} \, (\omega - \lambda_n)^k \Big] = k\, \omega^{-k-1} e^{-\omega s} \bar{A}_\lambda^k(\omega) +$$

$$+ \frac{\Gamma(k+1)}{\Gamma(h+1)\,\Gamma(k-h-1)}\, \omega^{-k-1} \int_0^\omega \bar{A}_\lambda^{h+1}(t)\,(e^{-ts} - e^{-\omega s})\,(\omega - t)^{k-h-2} dt +$$

$$+ \frac{\omega^{-k-1}}{\Gamma(h+2)} \sum_{r=1}^{h+1} s^r \binom{h+1}{r} \frac{\Gamma(k+1)}{\Gamma(k-h+r-1)} \int_0^\omega \bar{A}_\lambda^{h+1}(t) e^{-ts} (\omega - t)^{k-h+r-2} dt, \tag{3.41}$$

for all ω *if* $k > 1$, *and for* $\omega \neq \lambda_n$ *if* $0 < k \leqslant 1$.

PROOF. The left side of (3.41) can be written as

$$\frac{d}{d\omega}\left[\omega^{-k}\int_0^{\omega}(\omega-t)^k e^{-ts}dA_\lambda(t)\right]=k\,\omega^{-k-1}\int_0^{\omega}(\omega-t)^{k-1}e^{-ts}\,t\,dA_\lambda(t)$$

$$=k\,\omega^{-k-1}e^{-\omega s}\bar{A}_\lambda^k(\omega)+k\,\omega^{-k-1}\int_0^{\omega}(e^{-ts}-e^{-\omega s})(\omega-t)^{k-1}\,t\,dA_\lambda(t)$$

$$=k\,\omega^{-k-1}e^{-\omega s}\bar{A}_\lambda^k(\omega)+\frac{(-1)^{h+1}k\,\omega^{-k-1}}{\Gamma(h+1)}\times$$

$$\times\int_0^{\omega}\bar{A}_\lambda^{h+1}(t)D^{h+1}\left[(e^{-ts}-e^{-\omega s})(\omega-t)^{k-1}\right]dt,$$

by partial integration $(h+1)$ times. Now

$$D^{h+1}\left[(e^{-ts}-e^{-\omega s})(\omega-t)^{k-1}\right]$$

$$=(-)^{h+1}\left[\frac{\Gamma(k)}{\Gamma(k-h-1)}(e^{-ts}-e^{-\omega s})(\omega-t)^{k-h-2}+\right.$$

$$\left.+e^{-ts}\sum_{r=1}^{h+1}s^r\binom{h+1}{r}\frac{\Gamma(k)}{\Gamma(k-h+r-1)}(\omega-t)^{k-h+r-2}\right].$$

The lemma now follows upon substituting this expression for D^{h+1} in the integrand.

THEOREM 3.41. *If $\Sigma a_n\, e^{-\lambda_n s}$ is summable (λ, k), $k \geqslant 0$, for $s = s^*$, then it is summable $|\lambda, k+1|$ for every s such that $\sigma > \sigma^*$.*

PROOF. We may assume without loss of generality that $s^* = 0$. Replacing k by $k-1$, $k \geqslant 1$, the summability $(\lambda, k-1)$ of Σa_n implies, on account of (3.21), that

$$\bar{A}_\lambda^k(\omega)=o(\omega^k). \tag{3.42}$$

To prove the theorem, let us write the expression on the right of (3.41) as $\sum_{p=1}^{h+3} I_p$, and observe that it is sufficient to prove that for each $p = 1, 2,\ldots, h+3$, and $\sigma > 0$,

$$\int^{\infty}|I_p|\,d\omega<\infty.$$

First,

$$\int^{\infty} |I_1|\, d\omega \leqslant k \int^{\infty} |\bar{A}_\lambda^k(\omega)| \,.\, \omega^{-k-1} e^{-\sigma\omega}\, d\omega$$
$$< \infty,$$

by (3.42). Secondly,

$$\int^{\infty} |I_2|\, d\omega = c \int^{\infty} d\omega \left| \omega^{-k-1} \int_0^{\omega} \bar{A}_\lambda^{h+1}(t)\, (e^{-ts} - e^{-\omega s})\, (\omega - t)^{k-h-2} dt \right|$$

$$\leqslant c \int^{\infty} \omega^{-k-1}\, d\omega \int_0^{\omega} |\bar{A}_\lambda^{h+1}(t)| \,.\, |e^{-ts} - e^{-\omega s}| \,.\, (\omega - t)^{k-h-2}\, dt$$

$$\leqslant c_1 \int^{\infty} \omega^{-k-1}\, d\omega \int_0^{\omega} |\bar{A}_\lambda^{h+1}(t)| \,.\, e^{-t\sigma}\, (\omega - t)^{k-h-1}\, dt$$

$$= c_1 \int^{\infty} |\bar{A}_\lambda^{h+1}(t)| \,.\, e^{-t\sigma}\, dt \int_t^{\infty} \frac{(\omega - t)^{k-h-1}}{\omega^{k+1}}\, d\omega$$

$$\leqslant c_2 \int^{\infty} |\bar{A}_\lambda^{h+1}(t)| \,.\, e^{-t\sigma}\, t^{-h-1}\, dt$$

$$< \infty,$$

since $\bar{A}_\lambda^{h+1}(t) = o(t^{h+1})$. Finally, for $p > 0$,

$$\int^{\infty} |I_{p+2}|\, d\omega \leqslant c \int^{\infty} \omega^{-k-1}\, d\omega \left| \int_0^{\omega} \bar{A}_\lambda^{h+1}(t)\, e^{-ts}\, (\omega - t)^{k-h+p-2}\, dt \right|$$

$$\leqslant c \int^{\infty} |\bar{A}_\lambda^{h+1}(t)| \,.\, e^{-t\sigma} dt \int_t^{\infty} \frac{(\omega - t)^{k-h+p-2}}{\omega^{k+1}}\, d\omega$$

$$< c_1 \int^{\infty} t^{-h-2+p} |\bar{A}_\lambda^{h+1}(t)| \,.\, e^{-t\sigma}\, dt$$

$$< \infty,$$

since $\bar{A}_\lambda^{h+1}(t) = o\,(t^{h+1})$, and hence the theorem.

THEOREM 3.42. *If $\Sigma a_n\, e^{-\lambda_n s}$ is summable $|\lambda, k|$ for $s = s^*$, then it is summable $|\lambda, k|$ for all s such that $\sigma > \sigma^*$.*

PROOF. As usual, we may assume that $s^* = 0$, and observe that summability $|\lambda, k|$ of Σa_n is equivalent to saying that

$$\int^{\infty} \frac{|\bar{A}_{\lambda}^{k}(\omega)|}{\omega^{k+1}}\, d\omega < \infty,$$

and, *a fortiori*

$$\int^{\infty} \frac{|\bar{A}_{\lambda}^{k'}(\omega)|}{\omega^{k'+1}}\, d\omega < \infty, k' \geqslant k.$$

Using this instead of (3.42), we may argue as in Theorem 3.41, and show that

$$\int^{\infty} |I_p|\, d\omega < \infty, \text{ for } \sigma > 0, p = 1, \ldots, h + 3,$$

which will prove the theorem for $k > 0$. If $k = 0$, proof is obvious.

From Theorems 3.41 and 3.42, and from the fact that if Σa_n is summable $|\lambda, k|$ it is also summable (λ, k), we have the following

THEOREM 3.43. *There is a number $\bar{\sigma}_k$, called the abscissa of summability $|\lambda, k|$, such that $\Sigma a_n\, e^{-\lambda_n s}$ is summable $|\lambda, k|$ for $\sigma > \bar{\sigma}_k$, and not summable $|\lambda, k|$ for $\sigma < \bar{\sigma}_k$. Further*

$$\sigma_k \geqslant \bar{\sigma}_{k+1} \geqslant \sigma_{k+1}\,.$$

The next theorem gives a formula for $\bar{\sigma}_k$.

THEOREM 3.44. *Let $\bar{\sigma}_k$ be the abscissa of summability $|\lambda, k|$ of $\Sigma a_n e^{-\lambda_n s}$. If $\bar{\sigma}_k > 0$, then*

$$\bar{\sigma}_k = \limsup_{\omega \to \infty} \left[\frac{1}{\omega} \log \int_1^{\omega} t^{-k-1} |\bar{A}_{\lambda}^{k}(t)|\, dt \right], k > 0,$$

and

$$\bar{\sigma}_0 = \limsup_{n \to \infty} \left[\frac{1}{\lambda_n} \log \sum_0^n |a_r| \right].$$

PROOF. If $k > 0$, then by following the analogy of Theorem 3.34, and using Lemma 3.41, we prove: (i) if the series is summable $|\lambda, k|$ for $s = s^*, \sigma^* > 0$, then $\int_1^{\omega} t^{-k-1} |\bar{A}_{\lambda}^{k}(t)|\, dt = o(e^{\omega\sigma})$, $\sigma > \sigma^*$, and (ii) if the conclusion in (i) holds for $\sigma = \sigma^*$, then the series is summable $|\lambda, k|$, $\sigma > \sigma^*$. If $k = 0$, the result is easily verified.

We conclude this section with two theorems on the absolute summability of $\Sigma a_n l_n^{-s}$ without going through the proofs which run along the lines followed in the case of $\Sigma a_n e^{-\lambda_n s}$.

THEOREM 3.45. *If $\Sigma a_n l_n^{-s}$ is summable $|l, k|$ for $s = s^*$, then it is summable $|l, k|$ for any s such that $\sigma \geqslant \sigma^*$.*

THEOREM 3.46. *If σ_k and $\bar{\sigma}_k$ are the abscissae of summability (l, k) and $|l, k|$ respectively for the series $\Sigma a_n l_n^{-s}$, then*

$$\sigma_k \geqslant \bar{\sigma}_{k+1} \geqslant \sigma_{k+1}.$$

3.5. Relations between the abscissae of summability (λ, k) and summability (l, k)

We now mention the relation between the abscissae of summability (λ, k) and (l, k) of the Dirichlet series $\Sigma a_n e^{-\lambda_n s} = \Sigma a_n l_n^{-s}$, $l_n = e^{\lambda_n}$. It rests on the following auxiliary theorem.

THEOREM 3.51. *If Σa_n is summable (λ, k), $k \geqslant 0$, then $\Sigma a_n e^{-\lambda_n \delta}$, $\delta > 0$, is summable (l, k).*

PROOF. Set

$$d_n = a_n e^{-\lambda_n \delta}, \; T = e^t, \; W = e^\omega .$$

Then

$$D_l^k(W) \equiv \sum_{l_n \leqslant W} (W - l_n)^k d_n$$

$$= k \int_0^W (W - T)^{k-1} D_l(T)\, dT, \; D_l^0\,(T) \equiv D_l\,(T)$$

$$= k \int_0^\omega (e^\omega - e^t)^{k-1} e^t D_\lambda\,(t)\, dt.$$

Further

$$D_\lambda(\omega) = \sum_{\lambda_n \leqslant \omega} a_n e^{-\lambda_n \delta} = \int_0^\omega e^{-t\delta}\, dA_\lambda(t),$$

and hence we observe that

$$D_l^k(W) = k \int_0^\omega (e^\omega - e^t)^{k-1} e^t \, dt \int_0^t e^{-u\delta} \, dA_\lambda(u)$$

$$= k \int_0^\omega e^{-u\delta} dA_\lambda(u) \int_u^\omega (e^\omega - e^t)^{k-1} e^t \, dt$$

$$= \int_0^\omega (e^\omega - e^t)^k \, e^{-t\delta} \, dA_\lambda(t)$$

$$= \frac{(-1)^h}{\Gamma(h+1)} \int_0^\omega A_\lambda^h(t).\ D^{h+1} \left\{ (e^\omega - e^t)^k e^{-t\delta} \right\} dt,$$

by partial integrations. Now we argue as in § 2.5, and prove the result.

THEOREM 3.52. *The lines of summability of the Dirichlet series* $\Sigma a_n e^{-\lambda_n s} = \Sigma a_n \, l_n^{-s}$, $l_n = e^{\lambda_n}$, *are the same for the Riesz means of type* λ *or* l.

PROOF. If σ_k is the abscissa of summability (λ, k), and σ_k' that for (l, k), then $\sigma_k \leqslant \sigma_k'$, since a series summable (l, k) is also summable (λ, k). [cf. Th. 2.41 and the Remark in the first paragraph of § 2.2]. That $\sigma_k' \leqslant \sigma_k$ follows from Theorem 3.51.

THEOREM 3.53. *If* σ_k *is the abscissa of summability* (λ, k) *of* $\Sigma a_n e^{-\lambda_n s}$ *or of summability* (l, k) *of* $\Sigma a_n \, l_n^{-s}$, *then* σ_k *is a convex function of* k, *provided that* $\sigma_k > -\infty$. *That is, for* $0 \leqslant k \leqslant p \leqslant r$, *we have*

$$\sigma_p \leqslant \frac{(p-k)\,\sigma_r + (r-p)\,\sigma_k}{r-k}.$$

PROOF. In view of Theorem 3.52, it is enough to prove this theorem for $\Sigma a_n \, l_n^{-s}$. Let us assume that $\sigma_r > 0$. Then, by Theorem 3.38, we have

$$\sigma_r = \limsup_{\omega \to \infty} \frac{\log |A_l^r(\omega)|}{\log \omega} - r\,, \tag{3.51}$$

which implies that

$$A_l^r(\omega) = O\,(\omega^{r+\sigma_r+\varepsilon}),\ \varepsilon > 0,$$

and

$$A_l^k(\omega) = O\,(\omega^{k+\sigma_k+\varepsilon}).$$

By an appeal to Theorem 1.71, these two estimates together imply that

$$A_l^p(\omega) = O\left\{\omega^{p+[\sigma_r(p-k)+\sigma_k(r-p)]/(r-k)+\varepsilon}\right\},$$

which, by (3.51), again implies that

$$\sigma_p \leqslant \frac{\sigma_r(p-k)+\sigma_k(r-p)}{r-k} + \varepsilon.$$

Letting $\varepsilon \to 0$, we get the required result for $\sigma_r > 0$. We then extend this result to the case $\sigma_r > -M$, where M is an arbitrarily large positive number, by a change of origin (or, by considering the series $\Sigma b_n l_n^{-s}$ where $b_n = a_n l_n^M$).

3.6. Dirichlet series on the line of summability

In this section we are interested in two types of problem : (i) summability of order k on the line of summability $\sigma = \sigma_k$, and (ii) a converse of Theorem 3.35. The first is essentially a generalization of Fatou's theorem on power series, which states that if $a_n \to 0$, then $\Sigma a_n z^n$ converges at every point of regularity on the unit circle. We observe that with restrictions on the Riesz sums of order k of Σa_n and on the behaviour, on the line $\sigma = \sigma_k$, of the function $f(s)$ represented by the Dirichlet series, it is possible to deduce summability of order k of the series, at some points on the line $\sigma = \sigma_k$. The second type is concerned with the question : if a Dirichlet series, known to be summable by Riesz means of sufficiently high order in a half-plane, represents a regular analytic function which can be continued beyond that half-plane, can we say that the series is also summable in the extended region?

LEMMA 3.61. *Let $b(t)$ be integrable in the Lebesgue sense over every finite interval $0 \leqslant t \leqslant t_0$, and satisfy the condition*

$$b(t) = o(t^k),\ k \geqslant 0, \tag{3.61}$$

as $t \to \infty$, so that the integral

$$\int_0^\infty e^{-ts}\, b(t)\, dt$$

converges for $\mathrm{Re}(s) > 0$, *and*

$$F(s) = \int_0^\infty e^{-ts}\, b(t)\, dt \tag{3.62}$$

is regular in that half-plane. If $F(s)$ is assumed to be regular also at $s = i\,\tau_0$, then, for any non-negative integer r, we have

$$H(x,\omega) = e^{-\omega x} \frac{d^r}{dx^r}\left[F(x)\, e^{\omega x}\right] - \int_0^{\omega} (\omega - t)^r e^{-tx} b(t)\, dt$$

$$= o\,(\omega^k), \tag{3.63}$$

where $x = i\,\tau_0$, and (3.63) is valid uniformly in any closed interval on the imaginary axis in which $F(s)$ is regular.

PROOF. Let the function $F(s)$ be regular on the closed interval $(x_1 = i\,\tau_1, x_2 = i\,\tau_2)$, $\tau_1 < \tau_2$, of the imaginary axis. Then we can choose a number $a < 0$ such that, if b is any fixed positive number, $F(s)$ is regular inside and on the rectangle R formed by $(a + x_2,\ a + x_1,\ b + x_1,\ b + x_2)$. We shall show that

$$g_\omega(s) = \omega^{-k} e^{\omega s} (s - x_1)^{r+k+1} (s - x_2)^{r+k+1} H(s,\, \omega)$$

can be made as small as we like, uniformly on the boundary of R, by choosing ω large enough, so that the same will be true in the interior, and then the Lemma will follow easily.

On the boundary of R, we have the following inequality :

(i) $$|s - x_1|^{r+k+1} \,.\, |s - x_2|^{r+k+1} \leqslant c\,|\sigma|^{r+k+1},$$

where c is a constant.

If $b \geqslant \text{Re}(s) > 0$ and $\eta > 0$, there exists a number $\omega_0 = \omega_0(\eta)$, such that for $\omega > \omega_0$,

(ii) $$|H(s,\, \omega)| < c\,\eta\, \omega^k\, e^{-\omega\sigma}\, \sigma^{-r-k-1}.$$

For

$$|H(s,\, \omega)| = \left| \int_\omega^\infty (t - \omega)^r\, e^{-ts}\, b\,(t)\, dt \right|$$

$$< \eta \int_\omega^\infty e^{-t\sigma}\, t^k\, (t - \omega)^r\, dt,$$

for $\omega > \omega_0$, ω_0 being chosen such that $|b(t)| < \eta t^k$ for $t > \omega_0$, on account of hypothesis (3.61).

Hence

$$|H(s, \omega)| < \eta\, e^{-\omega\sigma} \int_0^\infty e^{-u\sigma} (\omega + u)^k u^r\, du$$

$$< 2^k \eta\, e^{-\omega\sigma} \int_0^\infty (\omega^k + u^k)\, e^{-u\sigma} u^r\, du$$

$$< c\, \eta\, \omega^k e^{-\omega\sigma} \sigma^{-r-k-1},$$

for $0 < \sigma \leqslant b$, and (ii) is therefore proved.

For that portion of R which is to the left of the imaginary axis, we have the inequalities :

$$\left| e^{-\omega s} \frac{d^r}{ds^r} \{F(s)\, e^{\omega s}\} \right| < c\, \omega^r,$$

and

$$\left| \int_0^\omega e^{-ts} b(t)\, (\omega - t)^r dt \right| \leqslant \omega^r \int_0^{\omega_0} |b(t)|\, e^{-t\sigma} dt + \int_{\omega_0}^{\omega} |b(t) e^{-ts} (\omega - t)^r|\, dt$$

$$\equiv I_1 + I_2,$$

say, where

$$I_1 < c\, \omega^r,$$

and

$$I_2 < \eta\, e^{-\omega\sigma} \int_{\omega_0}^{\omega} e^{(\omega - t)\sigma}\, t^k\, (\omega - t)^r dt$$

$$< \eta\, \omega^k e^{-\omega\sigma} \int_{\omega_0}^{\omega} e^{(\omega - t)\sigma} (\omega - t)^r dt$$

$$< \eta\, \omega^k e^{-\omega\sigma}\, |\sigma|^{-r-1},$$

since $\sigma < 0$, so that

(iii) $$|H(s, \omega)| < c_1\, \omega^r + c_2\, \eta\, \omega^k e^{-\omega\sigma}\, |\sigma|^{-r-1}.$$

Combining (i) and (ii), we have, on that part of the boundary of R which is to the right of the imaginary axis,

$$|g_\omega(s)| \leqslant c\, \eta.$$

On the boundary of R which is to the left of the imaginary axis, we combine (i) and (iii), and obtain

$$|g_\omega(s)| < \left[c_1 (\omega |\sigma|)^{r+k+1} \omega^{-2k-1} e^{-\omega|\sigma|} + c_2 \frac{\eta |\sigma|^{r+k+1}}{|\sigma|^{r+1}} \right]$$
$$< c_1 \omega^{-2k-1} + c_2 \eta |\sigma|^k.$$

Thus $g_\omega(s) = o(1)$, uniformly on the boundary of R, since η is arbitrary, and hence also uniformly in the interior of R, and in particular at $s = i\tau_0$, if $\tau_1 < \tau_0 < \tau_2$.

Hence

$$|H(x, \omega)| = \frac{|g_\omega(x)| \omega^k}{|x - x_1|^{r+k+1} . |x - x_2|^{r+k+1}} = o(\omega^k),$$

for $x = i\tau_0$, which proves the lemma.

THEOREM 3.61. *Let $b(t)$ and $F(s)$ be defined as in Lemma* 3.61. *Then*

(a) *the integral*

$$\int_0^\omega e^{-ts} b(t) dt$$

is 'summable by typical means of order k' to the value $F(s)$ at every point on the imaginary axis at which it is regular, and uniformly in every interval in which it is regular, that is

$$\lim_{\omega \to \infty} \omega^{-k} \int_0^\omega (\omega - t)^k e^{-xt} b(t)\, dt = F(x),$$

if $F(s)$ is regular at $s = x = i\tau_0$;

(b) *for $k > k' > -1$, we have*

$$\int_0^\omega (\omega - t)^{k'} e^{-tx} b(t)\, dt = o(\omega^k),$$

$$\int_0^\omega (\omega - t)^{k'-1} (e^{-tx} - e^{-\omega x})\, b(t)\, dt = o(\omega^k).$$

PROOF. If k is an integer, the theorem follows easily from Lemma 3.61, if we observe that for $s = x$ we have

$$e^{-\omega x}\frac{d^k}{dx^k}\{F(x)e^{\omega x}\} = \omega^k F(x) + \sum_1^k \binom{k}{p}\omega^{k-p}F^{(p)}(x)$$
$$= \omega^k F(x) + o\,(\omega^k),$$

and use this in (3.63).

If k is not an integer, we choose $r = [k]$ in Lemma 3.61, and set

$$B^k(x, \omega) = \int_0^\omega (\omega - t)^k e^{-tx} b(t)\, dt.$$

Then

$$B^k(x, \omega) = \frac{\Gamma(k+1)}{\Gamma(r+1)\,\Gamma(k-r)}\int_0^\omega B^r(x, t)\,(\omega - t)^{k-r-1}dt,$$

and

$$B^k(x, \omega) - F(x)\omega^k$$
$$= \frac{\Gamma(k+1)}{\Gamma(r+1)\,\Gamma(k-r)}\int_0^\omega \{B^r(x, t) - t^r F(x)\}\,(\omega - t)^{k-r-1}dt$$
$$= \left[\int_0^{\omega-1} + \int_{\omega-1}^{\omega}\right] = I_1 + I_2,$$

say. Now

$$I_2 = o(\omega^k),$$

by Lemma 3.61, and

$$I_1 = c_1\{B^{r+1}(x, \omega) - (\omega - 1)^{r+1}F(x)\} +$$
$$+ c_2\int_0^{\omega-1}\{B^{r+1}(x, t) - t^{r+1}\;F(x)\}\,(\omega - t)^{k-r-2}\,dt$$
$$= o\,(\omega^k) + \int_0^{\omega-1} o\,(t^k)\;.\;(\omega - t)^{k-r-2}\,dt, \qquad \text{by Lemma 3.61 ;}$$
$$= o\,(\omega^k) + o\left[\omega^k\int_0^{\omega-1}(\omega - t)^{k-r-2}\,dt\right], k - r - 2 < -1,$$
$$= o\,(\omega^k).$$

Combining the estimates for I_1 and I_2, we prove part (a) of the theorem. The proof of (b) runs along the same lines.

From Lemma 3.61 and Theorem 3.61 we have the following general result.

THEOREM 3.62. *Let σ_k be the abscissa of summability (λ, k) of the Dirichlet series $\Sigma a_n e^{-\lambda_n s}$. If*

$$A_\lambda^k(\omega) = o(\omega^k e^{\omega\sigma_k}),\ k \geqslant 0,\ \sigma_k \geqslant 0, \tag{3.64}$$

then $\Sigma a_n e^{-\lambda_n s}$ is summable (λ, k) at every point on the line $\sigma = \sigma_k$ at which the function represented by the Dirichlet series is regular.

PROOF. If $f(s)$ is the function represented by the Dirichlet series, then we know that

$$f(s) = \frac{s^{h+2}}{\Gamma(h+2)} \int_0^\infty A_\lambda^{h+1}(t)\, e^{-ts}\, dt, \quad \sigma > \sigma_k.$$

If we set

$$F(s) = \Gamma(h+2) f(s)\, s^{-h-2}, \quad b(t) = A_\lambda^{h+1}(t)\, e^{-t\sigma_k},$$

we have

$$F(s) = \int_0^\infty b(t)\, e^{-t(s-\sigma_k)}\, dt.$$

Since it follows from (3.64) that $b(t) = o(t^{h+1})$, we can apply Theorem 3.61. We assume that $s \neq 0$, since when $s = 0$, the result follows trivially from (3.64) alone. Since $F(s)$ is regular at all points of $\sigma = \sigma_k$ at which $f(s)$ is regular($s \neq 0$), it follows from Theorem 3.61 that

$$\omega^{-h-1} \int_0^\omega (\omega - t)^{h+1}\, b(t)\, e^{-it\tau} dt - F(\sigma_k + i\,\tau) = o(1),$$

or

$$\omega^{-h-1} \int_0^\omega (\omega - t)^{h+1} e^{-ts}\, A_\lambda^{h+1}(\omega)\, dt - F(s) = o(1), \tag{3.65}$$

where $s = \sigma_k + i\tau \neq 0$ is a point of regularity of $F(s)$. Now to prove that

$$\omega^{-k} \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^k a_n\, e^{-\lambda_n s} - f(s) = o(1),$$

we have only to use formula (3.31) in which the first term can be estimated by using (3.64), the last by using (3.65), and the others by using Theorem 3.61 (b).

The assumption that $f(s)$ is regular at a point on the line $\sigma = \sigma_k$, may be replaced by the hypothesis that $f(s)$ has boundary-values on the line $\sigma = \sigma_k$, and is bounded to the right of it. In as much as the proof of Theorem 3.62 depends on that of Theorem 3.61, we shall prove a generalization of Theorem 3.61 only.

THEOREM 3.63. *If in Theorem 3.61 the hypothesis that $F(s)$ is regular on the imaginary axis is replaced by the hypothesis that $F(s)$ has boundary-values on an interval $(i\,\tau_1, i\,\tau_2)$, $\tau_1 < \tau_2$, of the imaginary axis, and is bounded to the right of that interval, then*

(i) *for* $k > 0$, $$\lim_{\omega\to\infty} \omega^{-k} \int_0^{\omega} (\omega - t)^k e^{-it\tau} b(t)\,dt = F(i\,\tau),$$

for almost all τ in (τ_1, τ_2), in particular at all points of continuity;

(ii) *and for* $k = 0$, $\tau_1 < \tau < \tau_2$,

$$\int_0^{\infty} e^{-it\tau}\, b(t)\, dt$$

converges at the point $i\tau$, if $F(i\tau)$ satisfies a Lipschitz condition or any of the sufficient conditions for the convergence of a Fourier series.

PROOF. Let $x_1 = i\,\tau_1$, $x_2 = i\,\tau_2$, and let C denote a smooth curve starting from x_1 and ending with x_2 and lying to the right of the imaginary axis. Also, for $\delta > 0$, let C_δ denote a similar smooth curve starting from $\delta + x_1$ and ending with $\delta + x_2$, which tends to C as $\delta \to 0$. Then by Cauchy's theorem,

$$F(s) = \frac{1}{2\pi i} \int_{\delta+x_2}^{\delta+x_1} \frac{F(z)}{z-s}\,dz + \frac{1}{2\pi i} \int_{C_\delta} \frac{F(z)}{z-s}\,dz,$$

for any s lying to the right of the line $\sigma = \delta$ and to the left of C_δ. Now if $\delta \to 0$, then, by Lebesgue's theorem on dominated convergence, we have

$$F(s) = \frac{1}{2\pi i} \int_{x_2}^{x_1} \frac{F(z)}{z-s}\,dz + \frac{1}{2\pi i} \int_{C} \frac{F(z)}{z-s}\,dz$$

$$\equiv F_1(s) + F_2(s), \text{ say,}$$

for any s lying to the right of the imaginary axis and to the left of C.

$$F_1(s) = -\frac{1}{2\pi i}\int_{x_2}^{x_1} F(z)\int_0^\infty e^{t(z-s)}\,dt = \int_0^\infty e^{-ts}\,dt\int_{C_2}^{x_1}\left[-\frac{1}{2\pi i}F(z)\,e^{tz}\right]dz$$

$$= \int_0^\infty \varphi(t)\,e^{-ts}\,dt,$$

say, where

$$\varphi(t) = -\frac{1}{2\pi}\int_{i\tau_2}^{i\tau_1} F(i\,t)\,e^{i\tau t}\,dt,$$

so that $\varphi(t) = o(1)$ by the well-known Riemann-Lebesgue lemma.

If we write

$$B(t) = \int_0^t b(u)\,du,$$

$$\Phi(t) = \int_0^t \varphi(u)\,du,$$

$$A(t) = B(t) - \Phi(t),$$

we observe that

$$F_2(s) = \int_0^\infty e^{-ts}\,dA(t).$$

Since $F_2(s)$ is regular on the imaginary axis, we can now apply Theorem 3.61 with $b(t) = A'(t)$, and conclude that

$$\lim_{\omega\to\infty}\omega^{-k}\int_0^\omega e^{-it\tau}(\omega-t)^k\,dA(t) = F_2(i\,\tau),\quad \tau_1<\tau<\tau_2.$$

The left side is, however, equal to

$$\omega^{-k}\int_0^\omega e^{-it\tau}(\omega-t)^k\,b(t)\,dt - \omega^{-k}\int_0^\omega e^{-it\tau}(\omega-t)^k\,\varphi(t)\,dt.$$

Since $\varphi(t)$ is in the form of a Fourier integral, the second term can be tackled as in the theory of Fourier series, and we conclude that it tends to $F_1(i\tau)$ at a point of continuity of $F(i\tau)$ if $k > 0$, and if $k = 0$ at a point where $F(i\tau)$ satisfies a Lipschitz condition. Hence

$$\lim_{\omega\to\infty} \omega^{-k} \int_0^\omega e^{-it\tau} (\omega - t)^k b(t)\, dt$$

$$= \lim_{\omega\to\infty} \omega^{-k} \int_0^\omega e^{-it\tau} (\omega - t)^k \varphi(t)\, dt + F_2(i\tau)$$

$$= F_1(i\tau) + F_2(i\tau) = F(i\tau).$$

We shall now state a theorem on convergence, which cannot be deduced directly either from Lemma 3.61 or from Theorem 3.61.

THEOREM 3.64. *Let* $\Sigma a_n e^{-\lambda_n s}$ *converge for* $\sigma > 0$, *where it represents a regular function* $f(s)$, *and let*

$$a_n = o(\lambda_n - \lambda_{n-1}),\ \lambda_n - \lambda_{n-1} = O(1).$$

Then $\Sigma a_n e^{-\lambda_n s}$ *converges at every point on the imaginary axis at which* $f(s)$ *is regular; and more generally, at any interior point of an interval* $(i\tau_1, i\tau_2)$ *on which* $f(s)$ *has boundary-values, and to the right of which* $f(s)$ *is bounded, provided that the boundary-function satisfies at that point any of the sufficient conditions for the convergence of a Fourier series.*

PROOF. We indicate the proof in the case where $f(s)$ is regular; the other cases can be treated as in Theorem 3.63. Let $(i\tau_1, i\tau_2)$, $\tau_1 < \tau_2$, be a closed interval where $f(s)$ is regular, and let $x = i\tau$, $\tau_1 < \tau < \tau_2$.

Set

$$H(x, \omega) = [\sum_{\lambda_r \leqslant \omega} a_r e^{-\lambda_r x} - f(x)],$$

$$g_s(\omega) = e^{\omega s} (s - i\tau_1).(s - i\tau_2).\ H(s, \omega),$$

and argue as in Lemma 3.61. We easily derive all the inequalities proved there with $k = 0$, using the hypothesis on a_n. For example, corresponding to the inequality (ii) in Lemma 3.61, we have

$$|H(s, \omega)| = |\sum_{n+1}^{\infty} a_r e^{-\lambda_r s}| = o\{\sum_{n+1}^{\infty} (\lambda_r - \lambda_{r-1}) e^{-\lambda_r \sigma}\}$$

$$= o\left(\int_{\lambda_n}^{\infty} e^{-t\sigma} dt\right) = o(\sigma^{-1} e^{-\lambda_n \sigma}),\ \lambda_n \leqslant \omega < \lambda_{n+1}.$$

so that $|g_s(\omega)| = o\{e^{(\omega-\lambda_n)\sigma}\} = o(1)$, since $\lambda_{n+1} - \lambda_n = O(1)$. (iii) can be treated similarly.

The above theorems have their analogues, with one exception, in the case of the Dirichlet series $\Sigma a_n l_n^{-s}$. The exception is Theorem 3.63 in which there was a distinction between the two cases $k = 0$ and $k > 0$, with different conditions on the boundary-behaviour of the function. Here, however, in both cases, we require the function to satisfy a Lipschitz condition or any of the conditions sufficient for the convergence of Fourier series. We have closely to follow the foregoing arguments substituting (not changing the variable!) t^{-s} for e^{-ts} and $\omega^{\sigma k}$ for $e^{\omega\sigma_k}$. We simply state the theorem in its general form.

THEOREM 3.65. *Let $\sigma_k \geqslant 0$ be the abscissa of summability (l, k) of the Dirichlet series $\Sigma a_n l_n^{-s}$, and let*

$$A_l^k(\omega) = o\,(\omega^{k+\sigma_k}),\ k \geqslant 0.$$

Then $\Sigma a_n l_n^{-s}$ is summable (l, k) at any point of the line $\sigma = \sigma_k$ at which $f(s)$, the sum of the Dirichlet series, is regular; and more generally, at an interior point of any interval of the line $\sigma = \sigma_k$ on which $f(s)$ has boundary-values, and to the right of which $f(s)$ is bounded, provided that the boundary-function satisfies at that point any of the sufficient conditions for the convergence of Fourier series, such as a Lipschitz condition.

We shall now proceed to prove a class of theorems of the same nature as the previous one, with this difference: that the hypothesis on $A_\lambda^k(x)$ is replaced by a hypothesis on the behaviour of $f(s)$ as $s \to \infty$ along lines parallel to the imaginary axis. These results are, in a sense, converses of Theorem 3.35. The proofs depend on the explicit expression of the Riesz sums in terms of the function represented by the Dirichlet series, which is called Perron's formula.

LEMMA 3.62. *If* $\sigma > 0,\ k > 0,$

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} s^{-k-1}\, e^{us}\, ds = \begin{cases} u^k/\Gamma(k+1), & u \geqslant 0, \\ 0, & u \leqslant 0. \end{cases}$$

If $\sigma > 0$,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} s^{-1} e^{us}\, ds = \begin{cases} 1, & u > 0, \\ \tfrac{1}{2}, & u = 0, \\ 0, & u < 0, \end{cases}$$

it being understood that the principal value of the integral is taken if $u = 0$.

LEMMA 3.63. *If* $\sigma > 0$, $k > 0$,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(k+1)\,\Gamma(s)}{\Gamma(k+1+s)}\, v^s\, ds = \begin{cases} (1-1/v)^k, & v \geqslant 1, \\ 0, & 0 < v \leqslant 1. \end{cases}$$

PROOF. If we write

$$(1-x)^k = \sum_0^\infty B_r^k x^r,$$

we have

$$\frac{\Gamma(k+1)\,\Gamma(s)}{\Gamma(k+1+s)} = \int_0^1 x^{s-1}(1-x)^k\, dx = \sum_0^\infty \frac{B_r^k}{s+r},$$

and if we observe that

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{v^s\, ds}{s+r} = \begin{cases} v^{-r}, & v > 1, \\ \tfrac{1}{2}, & v = 1, \\ 0, & v < 1, \end{cases}$$

where the principal value of the integral is taken in case $v = 1$, we see that the lemma follows by substituting the series $\sum_0^\infty \frac{B_r^k}{s+r}$ in the integrand of the Lemma, and integrating termwise.

Lemma 3.62 will be used for the study of the Dirichlet series $\Sigma a_n e^{-\lambda_n s}$ and Lemma 3.63 for $\Sigma a_n l_n^{-s}$. From Lemma 3.62, we deduce

LEMMA 3.64. *Let* $\Sigma a_n e^{-\lambda_n s}$ *be summable* (λ, k), $k \geqslant 0$, *for* $\sigma > \sigma_k$. *Then for* $\sigma > \sigma_k$ *and* $\sigma > \sigma^*$, *we have*

$$\frac{1}{\Gamma(k+1)} \sum_{\lambda_n \leqslant \omega}' a_n e^{-\lambda_n s^*} (\omega - \lambda_n)^k$$

$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{f(s)}{(s-s^*)^{k+1}}\, e^{\omega(s-s^*)}\, ds,$$

where $f(s)$ is the sum of the series. The dash indicates that if $k = 0$, $\omega = \lambda_n$, then the last term of the sum has to be multiplied by $\frac{1}{2}$.

PROOF. Let $\lambda_m \leqslant \omega < \lambda_{m+1}$. Set

$$g(s) = e^{\omega s} \{f(s) - \sum_0^m a_n e^{-\lambda_n s}\}$$
$$= e^{\omega s} h(s),$$

where $h(s)$ is the sum-function of the summable series

$$a_{m+1} e^{-\lambda_{m+1} s} + a_{m+2} e^{-\lambda_{m+2} s} + \ldots \quad .$$

This series is summable (μ, k), where $\mu_n = \lambda_{m+n+1}$ for $\sigma > \sigma_k$.

Now

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{h(s)\, e^{\omega(s-s^*)}\, ds}{(s-s^*)^{k+1}} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{g(s)\, e^{-\omega s^*}\, ds}{(s-s^*)^{k+1}}$$
$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{f(s)\, e^{\omega(s-s^*)}\, ds}{(s-s^*)^{k+1}} - \frac{1}{\Gamma(k+1)} \sum_{\lambda_n \leqslant \omega}{}' \; a_n e^{-\lambda_n s^*} (\omega - \lambda_n)^k,$$

by Lemma 3.62, and the result will be proved if we can show that this is zero.

Let us consider a rectangle R formed by the points

$$\sigma - i\,T_1,\; \sigma + i\,T_2,\; \Omega + i\,T_2,\; \Omega - i\,T_1,$$

where T_1, T_2, Ω are large positive numbers. By Cauchy's theorem,

$$\frac{1}{2\pi i}\int_R \frac{g(s)\, e^{-\omega s^*}}{(s-s^*)^{k+1}}\, ds = 0,$$

that is

$$\frac{1}{2\pi i}\int_{\sigma-iT_1}^{\sigma+iT_2} = \frac{1}{2\pi i}\left[\int_{\sigma-iT_1}^{\Omega-iT_1} + \int_{\Omega-iT_1}^{\Omega+iT_2} + \int_{\Omega+iT_2}^{\sigma+iT_2}\right]$$
$$\equiv I_1 + I_2 + I_3,$$

say. If we fix T_1 and T_2, and let $\Omega \to \infty$, we observe that $I_2 \to 0$, since the numerator of the integrand is bounded, by Theorem 3.37. As $\Omega \to \infty$,

$$I_1 \to \frac{1}{2\pi i}\int_{\sigma-iT_1}^{\infty-iT_1} = o(1), \text{ as } T_1 \to \infty,$$

since $h(s) = o\{|\tau|^{k+1}\}$, by Theorem 3.35. Similarly

$$\lim_{T_2\to\infty} \lim_{\Omega\to\infty} I_3 = 0.$$

Hence

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{g(s)\, e^{-\omega s^*}}{(s-s^*)^{k+1}}\, ds = 0.$$

Corresponding to Lemma 3.64 we have

LEMMA 3.65. *Let $\Sigma a_n l_n^{-s}$ be summable (l, k), $k \geqslant 0$, for $\sigma > \sigma_k$, and let $f(s)$ denote its sum. Then for $\sigma > \sigma_k$ and $\sigma > \sigma^*$, we have*

$$\omega^{-k} \sum_{l_n\leqslant\omega}{}' a_n\, l_n{}^{-s^*} (\omega - l_n)^k = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} f(s)\frac{\Gamma(k+1)\,\Gamma(s-s^*)}{\Gamma(k+1+s-s^*)}\omega^{s-s^*}\, ds.$$

The dash indicates that the last term of the sum has to be multiplied by $\frac{1}{2}$, if $k = 0$ and $\omega = l_n$.

PROOF. The reasoning runs on the same lines as in Lemma 3.64. We observe that if $\Sigma a_n l_n^{-s}$ is summable (l, k), then $\sum\limits_{m+1}^{\infty} a_n l_n^{-s}$ is summable (μ, k), where $\mu_n = l_{m+n+1}$.

From Lemma 3.65 we can deduce theorems on the summability of the Dirichlet series. We assume that for sufficiently large values of σ, say $\sigma > d$, the Dirichlet series is summable by Riesz means of sufficiently high order, and that the sum-function is regular in a larger half-plane, say $\sigma > \eta$, $\eta < d$, and satisfies a condition like $f(s) = O(|s|^k)$. We then observe that the Dirichlet series is summable for $\sigma > \eta$ as well. More precisely we have the following

THEOREM 3.66. *Let the Dirichlet series $\Sigma a_n l_n^{-s}$ be summable (l, p), where p is sufficiently large, for $\sigma > d$, and let the function $f(s)$ defined by this series be regular for $\sigma > \eta$, where $\eta < d$. Further, let $f(s)$ satisfy the condition*

$$f(s) = O(|\tau|^{k'}),\ k' \geqslant 0,$$

uniformly for $\sigma \geqslant \eta + \varepsilon$, $\varepsilon > 0$. Then the Dirichlet series is summable (l, k) for $k > k'$ and $\sigma > \eta$.

PROOF. Let us suppose that the Dirichlet series is summable $(l, k+m)$, for $\sigma > d$, where m is a sufficiently large positive integer. Then by Lemma 3.65, we have, for $\sigma > d$, $\sigma > \sigma^*$,

$$\omega^{-k-m} \sum_{l_n \leqslant \omega} (\omega - l_n)^{k+m} a_n l_n^{-s^*}$$
$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(k+m+1)\Gamma(s-s^*)}{\Gamma(k+m+1+s-s^*)} f(s)\,\omega^{s-s^*}\,ds.$$

We first observe that in the above formula $\sigma > d$ can be replaced by $\sigma > \sigma^* > \eta$. For we have only to apply Cauchy's theorem to the rectangle whose sides are $\sigma = c$, $(c > \sigma^* > \eta)$, $\sigma = \nu$, $(\nu > d)$, $\tau = -T_1$ and $\tau = T_2$, and observe that the integrals on the sides parallel to the real axis tend to zero as $T_1 \to \infty$ and $T_2 \to \infty$, on account of the hypothesis on f. Next we observe that we can take $m = 0$. For we can multiply both sides by ω^{k+m}, differentiate with respect to ω, and divide by $(k+m)\,\omega^{k+m-1}$ on both sides. This will lead to a formula with $m - 1$ in the place of m. We may perform this process successively m times, observing each time that we get a convergent integral on the right side. We thus obtain the formula

$$\omega^{-k} \sum_{l_n \leqslant \omega} (\omega - l_n)^k a_n l_n^{-s^*} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(k+1)\Gamma(s-s^*)}{\Gamma(k+1+s-s^*)} f(s)\,\omega^{s-s^*}\,ds,$$

$\sigma > \sigma^* > \eta$. The integral on the right converges absolutely if $k > k'$, for

$$H(s - s^*) \equiv \frac{\Gamma(k+1)\,\Gamma(s-s^*)}{\Gamma(k+1+s-s^*)} = O(|\tau|^{-k-1}),$$

and $f(s) = O(|\tau|^{k'})$, by hypothesis. Finally we extend the formula to $\sigma < \sigma^*$. The function $H(s - s^*)$ has a pole at $s = s^*$ with residue 1, so that by another application of Cauchy's theorem, we will have, for $\eta < \sigma < \sigma^* < \eta + 1$,

$$\omega^{-k} \sum_{l_n \leqslant \omega} (\omega - l_n)^k a_n l_n^{-s^*} - f(s^*) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} H(s-s^*) f(s)\,\omega^{s-s^*}\,ds.$$

Using the hypothesis on $f(s)$ and the known order of $H(s - s^*)$, we observe that the right side is $O(\omega^{\sigma - \sigma^*}) = o(1)$ since $\sigma < \sigma^*$, which proves the theorem.

A corresponding theorem for $\Sigma a_n e^{-\lambda_n s}$ is also true, which we proceed to prove.

THEOREM 3.67. *Let the Dirichlet series* $\Sigma a_n e^{-\lambda_n s}$ *be summable* (λ, p), *where* p *is sufficiently large, for* $\sigma > d$, *and let the function* $f(s)$ *defined by the series be regular for* $\sigma > \eta$ *where* $\eta < d$. *Further, let* $f(s)$ *satisfy the condition*

$$f(s) = O(|\tau|^{k'}), \; k' \geqslant 0,$$

uniformly for $\sigma \geqslant \eta + \varepsilon$, $\varepsilon > 0$. *Then the Dirichlet series is summable* (λ, k) *for* $k > k'$ *and* $\sigma > \eta$.

PROOF. Let us put $l_n = e^{\lambda_n}$. Since $\Sigma a_n e^{-\lambda_n s}$ is summable (λ, p) for $\sigma > d$, it is also summable (l, p) for $\sigma > d$, by Theorem 3.51. So by an application of Theorem 3.66, $\Sigma a_n e^{-\lambda_n s}$ $(\equiv \Sigma a_n l_n^{-s})$ is summable (l, k) for $\sigma > \eta$, and therefore also summable (λ, k) for $\sigma > \eta$, by the second consistency theorem.

We now raise the question: under what further conditions can one assert summability on the line $\sigma = \eta$? We state two theorems similar to Theorem 3.63.

THEOREM 3.68. *If in Theorem* 3.67 *we further assume that* $f(s)$ *has boundary-values on* $\sigma = \eta$, *and that*

$$f(s) = O\{(1 + |\tau|)^{k'}\}, \; k' \geqslant 0, \; |\tau| \geqslant 0,$$

uniformly for $\sigma > \eta$, *then* $\Sigma a_n e^{-\lambda_n s}$ *is summable* (λ, k), $k > k'$, *at every point on the line* $\sigma = \eta$ *at which* $f(s)$ *is continuous in the complex sense (when the neighbourhood of approach to the point is on the right side of the line* $\sigma = \eta$).

THEOREM 3.69. *If in Theorem* 3.66 *we further assume that* $f(s)$ *has boundary-values on* $\sigma = \eta$, *and that*

$$f(s) = O\{(1 + |\tau|)^{k'}\}, \; k' \geqslant 0, \; |\tau| \geqslant 0,$$

uniformly for $\sigma > \eta$, *then* $\Sigma a_n l_n^{-s}$ *is summable* (l, k), $k > k'$, *at every point on the line* $\sigma = \eta$ *at which* $f(s)$ *satisfies a Lipschitz condition or any of the conditions sufficient for the convergence of a Fourier series.*

PROOF OF THEOREM 3.68. Let $x = \eta + i\tau$ be any point on the line $\sigma = \eta$, at which $f(s)$ is continuous in the given sense. Then, from Lemmas 3.62 and 3.64, we have, for $\sigma > \eta$ and $k > k'$,

$$\frac{1}{\Gamma(k+1)}\Sigma a_n e^{-\lambda_n x}(\omega-\lambda_n)^k - \frac{\omega^k}{\Gamma(k+1)} f(x)$$
$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{f(s)-f(x)}{(s-x)^{k+1}} e^{\omega(s-x)}\,ds.$$

By Lebesgue's theorem on dominated convergence, we observe that the above formula is true with $\sigma = \eta$, provided we go round the point $s = x$ along a semi-circle to the right of the line $\sigma = \eta$ with centre x and radius r. Thus

$$\frac{1}{\Gamma(k+1)}\Sigma a_n e^{-\lambda_n x}(\omega-\lambda_n)^k - \frac{\omega^k}{\Gamma(k+1)} f(x) = \frac{1}{2\pi i}\left[\int_{\eta-i\infty}^{\eta-ir} + \int_C + \int_{\eta+ir}^{\eta+i\infty}\right]$$

where C denotes the semi-circle with centre x and radius r. Given $\varepsilon > 0$, we determine δ such that

$$|f(s) - f(x)| < \varepsilon, \text{ for } |s-x| \leqslant \delta.$$

We further choose $r = 1/\omega$. Then for $1/\omega < \delta$, we write

$$\int_{\eta+i\omega^{-1}}^{\eta+i\infty} = \int_{\eta+i\omega^{-1}}^{\eta+i\delta} + \int_{\eta+i\delta}^{\eta+i\infty}.$$

Now

$$\int_{\eta+i\delta}^{\eta+i\infty} = o(1) = o(\omega^k),$$

as $\omega \to \infty$, by the Riemann-Lebesgue lemma, and

$$\int_{\eta+i\omega^{-1}}^{\eta+i\delta} = O(\varepsilon\,\omega^k),$$

so that

$$\int_{\eta+i\omega^{-1}}^{\eta+i\infty} = O(\varepsilon\,\omega^k).$$

Similarly

$$\int_{\eta-i\infty}^{\eta-i\omega^{-1}} = O(\varepsilon\,\omega^k).$$

Finally

$$\left|\int_C\right| = O(\varepsilon\,\omega^k).$$

Since ε is arbitrary, the theorem follows.

REMARK. It is clear that the above method can also be used for the proof of Theorem 3.67.

Theorem 3.69 is similarly proved, by using Lemma 3.65. We have to observe that

$$H(s - s^*) = \frac{\Gamma(k+1)\,\Gamma(s - s^*)}{\Gamma(k + 1 + s - s^*)}$$

has a simple pole at $s = s^*$ with residue 1.

3.7. Some converse theorems on the abscissae of summability

In this section we are concerned with a class of theorems on the abscissae of summability, which are Tauberian in nature, though not explicitly recognizable as such. Corresponding results for the ordinary Dirichlet series $\Sigma a_n n^{-s}$ were first proved by G. H. Hardy and J. E. Littlewood, and later generalized by K. Ananda-Rau to cover a larger class of series.

Using the results of § 1.8, we prove inequalities for the abscissae of summability (l, k) of the Dirichlet series $\Sigma a_n l_n^{-s}$, where $A_l(t)$ satisfies Tauberian hypotheses, and combining them with Theorem 3.66, we prove Theorem 3.72 which is a complement to Theorem 3.66. Finally we apply these results to $\Sigma a_n l_n^{-s}$, where the growth of $\{l_n\}$ is restricted, and obtain a generalization of the Schnee-Landau theorem.

LEMMA 3.71. *If*

$$A_l^k(\omega) - c\,\omega^k = o(\omega^{k+\beta}),\ \beta \geqslant 0, \tag{3.71}$$

then $\Sigma a_n l_n^{-\beta}$ *is either summable* (l, k), *or never summable* (l, r) *for any* r.

PROOF. We assume without loss of generality that $c = 0$. It is obviously enough to show that if $\Sigma a_n l_n^{-\beta}$ is summable $(l, k+m)$, m a positive integer, then it is summable (l, k). Now set

$$b_n = a_n l_n^{-\beta},\ c_n = b_n l_n^p = a_n l_n^{p-\beta},$$

where p is an integer greater than β. Then

$$C_l^k(\omega) = \sum_{l_n \leqslant \omega} (\omega - l_n)^k a_n l_n^{p-\beta}$$
$$= o\,(\omega^{k+p}),$$

on using (3.71) in Lemma 3.32 with $p - \beta$ for $-s$. *A fortiori*

$$C_l^r(\omega) = o\,(\omega^{r+p}), \tag{3.72}$$

for every $r > k$. But

$$C_l^r(\omega) = \sum_{l_n \leqslant \omega} (\omega - l_n)^r a_n l_n^{p-\beta} = \sum_{l_n \leqslant \omega} (\omega - l_n)^r l_n^p b_n$$
$$= \sum_{l_n \leqslant \omega} (\omega - l_n)^r \{\omega - (\omega - l_n)\}^p b_n$$
$$= \omega^p B_l^r(\omega) - p\omega^{p-1} B_l^{r-1}(\omega) + \ldots + (-)^p B_l^{r+p}(\omega),$$

or

$$B_l^r(\omega) = \omega^{-p} C_l^r(\omega) + p\omega^{-1} B_l^{r+1}(\omega) - \ldots + (-)^{p+1}\omega^{-p} B_l^{r+p}(\omega). \tag{3.73}$$

From (3.72) and (3.73) we observe that

$$B_l^{r+1}(\omega) = o\,(\omega^{r+1}) \tag{3.74}$$

implies

$$B_l^r(\omega) = o\,(\omega^r).$$

That is to say, under the hypothesis of the Lemma, if $\Sigma b_n = \Sigma a_n l_n^{-\beta}$ is summable $(l, r+1)$, then it is summable (l, r). Setting $r = k+m-1, k+m-2, \ldots, k$, we observe that, if $\Sigma a_n l_n^{-\beta}$ is summable $(l, k+m)$, then it is summable (l, k). Hence the Lemma.

REMARK. The Lemma is valid even if β is complex with a non-negative real part.

THEOREM 3.71. *If σ_r denotes the abscissa of summability (l, r) of the Dirichlet series $\Sigma a_n l_n^{-s}$, $\sigma_r < +\infty$, and if*

$$A_l(l_n + t) - A_l(l_n) = O\,(t^\mu l_n^\alpha),\ \mu > 0,\ 0 < t = O(l_n), \tag{3.75}$$

then, either $\sigma_r < \alpha + \mu$, in which case

$$\sigma_k \leqslant \frac{(\alpha+\mu)(r-k) + (k+\mu)\sigma_r}{r+\mu},\ 0 \leqslant k \leqslant r,$$

or $\sigma_r \geqslant \alpha + \mu$, in which case $\sigma_k = \sigma_r$, $0 \leqslant k \leqslant r$.

Proof. Let γ be real and $\gamma > \sigma_r$. Set $b_n = a_n l_n^{-\gamma}$. Since Σb_n is summable (l, r), we may write

$$B_l^r(\omega) = c\omega^r + o(\omega^r), \tag{3.76}$$

where c is the sum of the series. Further, (3.75) implies

$$B_l(l_n + t) - B_l(l_n) = O(t^\mu \, l_n^{a-\gamma}) \tag{3.77}$$

if $t = O(l_n)$, for

$$B_l(l_n + t) - B_l(l_n) = \sum_{l_n < l_p \le l_n + t} a_p \, l_p^{-\gamma} = \int_{l_n}^{l_n+t} u^{-\gamma} \, dA_l(u)$$

$$= (l_n + t)^{-\gamma} \left\{A_l(l_n + t) - A_l(l_n)\right\} + $$

$$+ \gamma \int_{l_n}^{l_n+t} u^{-\gamma-1} \left\{A_l(u) - A_l(l_n)\right\} du$$

$$= O(t^\mu \, l_n^{a-\gamma}),$$

if $t = O(l_n)$. Now (3.76) and (3.77) imply, by Theorem 1.82,

$$B_l^k(\omega) - c\omega^k = o\left\{\omega^{k+(r-k)(a+\mu-\gamma)/(r+\mu)}\right\}, \tag{3.78}$$

if $a + \mu - \gamma > 0$, $0 \leqslant k < r$. Now (3.78) implies, by Lemma 3.71, that

$$\Sigma b_n \, l_n^{-\beta}, \qquad \beta = \frac{(a + \mu - \gamma)(r - k)}{r + \mu} \geqslant 0,$$

is summable (l, k) since it is summable (l, r). Since $\Sigma b_n \, l_n^{-\beta} = \Sigma a_n \, l_n^{-\gamma-\beta}$, we observe that

$$\sigma_k \leqslant \gamma + \beta = \frac{(a + \mu)(r - k) + \gamma(k + \mu)}{r + \mu}.$$

As γ can be any number $> \sigma_r$, the first part of the theorem follows.

To prove the second part, we observe that if $\gamma > a + \mu$, then $\Sigma a_n \, l_n^{-\gamma}$ is convergent if it is summable (l, r), for (3.77) implies

$$B_l\,(l_n + t) - B_l\,(l_n) = O(t^\mu \, l_n^{-\mu}), \; t = O(l_n),$$

which, together with (3.76), implies, by Theorem 1.82,

$$B_l\,(l_n) - c = o(1).$$

Corollary 3.71. *If* $a_n = O\{l_n^a\,(l_n - l_{n-1})\}$, *and if* σ_r *denotes the abscissa of summability* (l, r) *of the Dirichlet series* $\Sigma a_n \, l_n^{-s}$, *then*

$$\sigma_k \leqslant \frac{(a + 1)(r - k) + \sigma_r(k + 1)}{r + 1}, \; 0 \leqslant k \leqslant r.$$

We have only to put $\mu = 1$ in Theorem 3.71, and observe that $\sigma_r \leqslant \sigma_0 \leqslant a + 1$.

COROLLARY 3.72. *If* $\sum_{\nu=0}^{n} |a_\nu|^p\, l_\nu^p\, (l_\nu - l_{\nu-1})^{1-p} = O\{l_n^{p(1+a)+1}\}$, *where* $p > 1$, $a + 1 + 1/p \geqslant 0$, *then for* $0 \leqslant k \leqslant r$,

$$\sigma_k \leqslant \frac{(a+1)(r-k) + \sigma_r (k + 1 - 1/p)}{r + 1 - 1/p}.$$

The hypothesis in this corollary implies (3.75) with $1 - 1/p$ for μ, $a + 1/p$ for a, and $\sigma_0 \leqslant a + 1$.

We can combine Theorem 3.66 and Theorem 3.71, and obtain

THEOREM 3.72. *If*

$$A_l(l_n + t) - A_l(l_n) = O(t^\mu l_n^a),\ \mu > 0,\ 0 < t = O(l_n), \qquad (3.75)$$

if the Dirichlet series $\Sigma a_n l_n^{-s}$ *is summable* (l, p), *where* p *is sufficiently large, for sufficiently large values of* σ, *and if the function* $f(s)$ *represented by the Dirichlet series is regular for* $\sigma > \eta$, $\eta < a + \mu$, *satisfying the condition*

$$f(s) = O(|\tau|^r),\ r \geqslant 0,$$

uniformly for $\sigma \geqslant \eta + \varepsilon$, $\varepsilon > 0$, *then* $\Sigma a_n l_n^{-s}$ *is summable* (l, k) *for*

$$\sigma > \frac{(a+\mu)(r-k) + \eta(k+\mu)}{r+\mu},\quad 0 < k \leqslant r.$$

PROOF. If $r' > r$, then by Theorem 3.66, $\Sigma a_n l_n^{-s}$ is summable (l, r') for $\sigma > \eta$, and hence $\sigma_{r'} \leqslant \eta$, where $\sigma_{r'}$ is the abscissa of summability (l, r') of $\Sigma a_n l_n^{-s}$. We can now apply Theorem 3.71, and deduce that $\Sigma a_n l_n^{-s}$ is summable (l, k) for

$$\sigma > \frac{(a+\mu)(r'-k) + \eta(k+\mu)}{r'+\mu}.$$

If we let $r' \to r$, we obtain the theorem.

COROLLARY 3.73. *If, instead of* (3.75), *we have*

$$a_n = O\{l_n^a (l_n - l_{n-1})\},$$

in Theorem 3.72, *with* $\eta < a + 1$, *then* $\Sigma a_n l_n^{-s}$ *is summable* (l, k) *for*

$$\sigma > \frac{(a+1)(r-k) + \eta(k+1)}{r+1},\ 0 \leqslant k \leqslant r.$$

COROLLARY 3.74. *If, instead of* (3.75), *we have*

$$\sum_{\nu=0}^{n} |a_\nu|^p \, l_\nu^p \, (l_\nu - l_{\nu-1})^{1-p} = O\{l_n^{p(a+1)+1}\}, \; p > 1, \; a + 1 + 1/p \geqslant 0,$$

in Theorem 3.72, *with* $\eta < a + 1$, *then* $\Sigma a_n l_n^{-s}$ *is summable* (l, k) *for*

$$\sigma > \frac{(a+1)(r-k) + \eta(k+1-1/p)}{r+1-1/p}, \quad 0 < k \leqslant r.$$

We conclude this section with a few results on the summability of the Dirichlet series $\Sigma a_n l_n^{-s}$, where the increase of l_n is restricted.

THEOREM 3.73. *Let*

$$\frac{l_n}{l_n - l_{n-1}} = O(l_n^h), \; h \geqslant 0,$$

and let $a_n = O(l_n^\delta)$ *for every* $\delta > 0$, *so that* $\Sigma a_n l_n^{-s}$ *is absolutely convergent for* $\sigma > h$. *Further let the function represented by the Dirichlet series* $\Sigma a_n l_n^{-s}$ *be regular in the region* $\sigma > \eta$ $(\eta < h)$, *satisfying the condition*

$$f(s) = O(|\tau|^r), \; r \geqslant 0,$$

uniformly in $\sigma > \eta$. *Then* $\Sigma a_n l_n^{-s}$ *is summable* (l, k) *for*

$$\sigma > \frac{hr + \eta - k(h-\eta)}{r+1}, \quad 0 \leqslant k < r.$$

PROOF. This follows easily from Corollary 3.73 on putting $a + 1 = \delta + h$, and then letting $\delta \to 0$.

It is clear that we can have more general conditions on a_n, as in Theorem 3.72 or Theorem 3.73.

THEOREM 3.74. *If* σ_r *denotes the abscissa of summability* (l, r) *of the Dirichlet series* $\Sigma a_n l_n^{-s}$, *and*

$$\frac{l_n}{l_n - l_{n-1}} = O(l_{n-1}^h), \quad h > 0,$$

then

$$\sigma_k - \sigma_r \leqslant h(r-k), \quad 0 \leqslant k \leqslant r.$$

PROOF. Let $\gamma > \sigma_r$ and set $b_n = a_n l_n^{-\gamma}$. Since Σb_n is summable (l, r),

$$B_l(\omega) = o[\{l_{n+1}/(l_{n+1} - l_n)\}^r], \quad \text{(by Theorem 1.62)}$$
$$= o(l_n^{hr}) = o(\omega^{hr}),$$

where $l_n \leqslant \omega < l_{n+1}$. Hence, by Theorem 1.71, since $B_l^r(\omega) = O(\omega^r)$,

$$B_l^k(\omega) = o\,(\omega^{k+h\,(r-k)}),$$

which implies by Lemma 3.71 that $\Sigma a_n\, l_n^{-\gamma-h\,(r-k)}$ is summable (l, k) if it is summable. But it is summable (l, r), since $\gamma + h\,(r-k) \geqslant \gamma > \sigma_r$. Hence $\sigma_k \leqslant \gamma + h\,(r-k)$, or $\sigma_k - \sigma_r \leqslant h\,(r-k)$, if we let $\gamma \to \sigma_r$.

COROLLARY 3.75. *If* $l_n/(l_n - l_{n-1}) = O(l_{n-1}^{\delta})$ *for every* $\delta > 0$, *then* $\sigma_0 = \sigma_k$ *for every* $k > 0$.

3.8. Tauberian theorems

In this section we are concerned with a class of problems extensively studied by G. H. Hardy and J. E. Littlewood, generalizing the classical theorem of Tauber on power series. Tauber's theorem states that if a power series $\Sigma a_n x^n$ converges for $|x| < 1$, and the sum-function tends to a limit c, as $x \to 1 - 0$, and if $na_n = o(1)$, then Σa_n converges to c. We are concerned with generalizing this result to the Dirichlet series $\Sigma a_n e^{-\lambda_n s}$. We assume that this series converges for $\sigma > 0$, and that its sum $f(s)$ tends to a finite limit as $s \to 0$ along the positive real axis. We observe that Σa_n then converges, if a_n satisfies any of the Tauberian conditions in § 1.8.

THEOREM 3.81. *Let* $\Sigma a_n e^{-\lambda_n \sigma}$ *converge for* $\sigma > 0$ *to the sum* $f(\sigma)$, *and let* $f(+0)$ *exist. Then a necessary and sufficient condition that* $\lim_{\omega\to\infty} A(\omega)$ *should exist is that*

$$\bar{A}_\lambda^1(\omega) = \int_0^\omega t\, dA_\lambda(t) = o(\omega). \tag{3.81}$$

PROOF. The necessity of the condition follows from the identity

$$\omega^{-1}\bar{A}_\lambda^1(\omega) = A_\lambda(\omega) - \omega^{-1}\int_0^\omega A_\lambda(t)\,dt = A_\lambda(\omega) - \omega^{-1} A_\lambda^1(\omega). \tag{3.82}$$

For if $\lim_{\omega\to\infty} A_\lambda(\omega)$ exists, then $A_\lambda(\omega)$ and $\omega^{-1} A_\lambda^1(\omega)$ converge to the same limit, and their difference tends to zero.

To prove the sufficiency, we set

$$C_\lambda^1(\omega) = \omega^{-1} A_\lambda^1(\omega) = \int_0^\omega \frac{\bar{A}_\lambda^1(t)}{t^2}\, dt; \tag{3.83}$$

$$f_1(\sigma) = \sigma \int_0^\infty C_\lambda^1(t)\, e^{-\sigma t}\, dt = \sigma \int_\sigma^\infty u^{-2} f(u)\, du, \qquad (3.84)1$$

where

$$f(u) = \Sigma a_n\, e^{-\lambda_n u} = u \int_0^\infty A_\lambda(t)\, e^{-ut}\, dt = u^2 \int_0^\sigma A_\lambda^1(t)\, e^{-ut}\, dt. \qquad (3.84)2$$

Now since $f(+0)$ exists, it follows from (3.84) that $f_1(+0)$ also exists and

$$f_1(+0) = f(+0).$$

Further

$$\begin{aligned} f_1(\sigma) &= \sigma \int_0^\infty C_\lambda^1(t)\, e^{-\sigma t}\, dt \\ &= \sigma \int_0^\infty e^{-\sigma t}\, dt \int_0^t u^{-2} \bar{A}_\lambda^1(u)\, du \\ &= \int_0^\infty u^{-2} \bar{A}_\lambda^1(u)\, e^{-\sigma u}\, du. \end{aligned} \tag{3.85}$$

Therefore, from (3.83) and (3.85),

$$\begin{aligned} C_\lambda^1(\omega) - f_1(\sigma) &= \int_0^\omega t^{-2} \bar{A}_\lambda^1(t)\, dt - \int_0^\infty t^{-2} \bar{A}_\lambda^1(t)\, e^{-\sigma t}\, dt \\ &= \int_0^\omega t^{-2} \bar{A}_\lambda^1(t)\, (1 - e^{-\sigma t})\, dt - \int_\omega^\infty t^{-2} \bar{A}_\lambda^1(t)\, e^{-\sigma t}\, dt \\ &\equiv I_1 - I_2, \end{aligned}$$

say. Now

$$I_1 = O\left[\sigma \int_0^\omega |\bar{A}_\lambda^1(t)|\, t^{-1}\, dt\right] = o(\sigma\omega),$$

$$I_2 = o\left[\int_\omega^\infty t^{-1} e^{-\sigma t}\, dt\right] = o\left\{(\sigma\omega)^{-1} e^{-\sigma\omega}\right\}.$$

Choosing $\omega\,\sigma = 1$, we observe that

$$C_\lambda^1(\omega) - f_1(\sigma) = o(1),$$

which proves that Σa_n is summable $(\lambda, 1)$. The convergence of Σa_n now follows from (3.81) and (3.82).

THEOREM 3.82. *If $f(+0)$ exists, and if Σa_n has non-negative partial sums, i.e. $A_\lambda(t) \geqslant 0$, then Σa_n is summable $(\lambda, 1)$ to the sum $f(+0)$, i.e. $\omega^{-1} A_\lambda^1(\omega) \to f(+0)$ as $\omega \to \infty$.*

PROOF. We prove the theorem in two stages: (α). If $f(+0)$ exists, and if $A_\lambda(t) \geqslant 0$, then Σa_n is summable (λ, k) for sufficiently large values of k. (β). If $A_\lambda(t) \geqslant 0$, and if Σa_n is summable (λ, k), $k > 1$, then Σa_n is summable $(\lambda, 1)$.

PROOF OF (α). Now

$$f(\sigma) = \Sigma a_n\, e^{-\lambda_n \sigma} = \sigma \int_0^\infty e^{-\sigma t} A_\lambda(t)\, dt,$$

and since

$$\lim_{\sigma\to 0} f(\sigma) = f(+0)$$

exists, we observe that

$$\lim_{\sigma\to 0} f(m\,\sigma) = f(+0),\ m > 0,$$

that is,

$$\lim_{\sigma\to 0} \sigma \int_0^\infty A_\lambda(t)\, e^{-m\sigma t}\, dt = \frac{1}{m} f(+0) = f(+0) \int_0^\infty e^{-mt}\, dt. \qquad (3.86)$$

From (3.86) we note that if $P_n(x)$ is a polynominal in x of degree n, then

$$\lim_{\sigma\to 0} \sigma \int_0^\infty P_n(e^{-\sigma t})\, A_\lambda(t)\, e^{-\sigma t} dt = f(+0) \int_0^\infty P_n\,(e^{-t})\, e^{-t}\, dt. \qquad (3.87)$$

We next note that by an appeal to Weierstrass's theorem on the approximation of continuous functions by polynomials, we can replace the polynomial $P_n(x)$ in (3.87) by a function $\varphi(x)$ continuous in $0 \leqslant x \leqslant 1$. Thus given any positive number ε, we can find polynomials $P_n(x)$ and $Q_m(x)$ such that

$$Q_m(x) \leqslant \varphi(x) \leqslant P_n(x)$$

and

$$\varphi(x) - Q_m(x) \leqslant \varepsilon,\ P_n(x) - \varphi(x) \leqslant \varepsilon.$$

So, $A_\lambda(t)$ being non-negative,

$$\sigma \int_0^\infty Q_m(e^{-\sigma t})\, A_\lambda(t)\, e^{-\sigma t}\, dt \leqslant \sigma \int_0^\infty \varphi(e^{-\sigma t})\, e^{-\sigma t}\, A_\lambda(t)\, dt$$

$$\leqslant \sigma \int_0^\infty P_n(e^{-\sigma t})\, e^{-\sigma t}\, A_\lambda(t)\, dt.$$

If we let $\sigma \to 0$, we observe that the last integral tends to

$$f(+0) \int_0^\infty P_n(e^{-t})\, e^{-t}\, dt \leqslant f(+0) \int_0^\infty \varphi(e^{-t})\, e^{-t}\, dt + \varepsilon\, f(+0),$$

while the first tends to

$$f(+0) \int_0^\infty Q_m(e^{-t})\, e^{-t}\, dt \geqslant f(+0) \int_0^\infty \varphi(e^{-t})\, e^{-t}\, dt - \varepsilon\, f(+0).$$

In other words,

$$f(+0) \left[\int_0^\infty \varphi(e^{-t})\, e^{-t}\, dt - \varepsilon \right] \leqslant \varliminf_{\sigma \to 0} \sigma \int_0^\infty A_\lambda(t)\, \varphi(e^{-\sigma t})\, e^{-\sigma t}\, dt$$

$$\leqslant f(+0) \left[\int_0^\infty \varphi(e^{-t})\, e^{-t}\, dt + \varepsilon \right]$$

Since ε is arbitrary, we obtain

$$\lim_{\sigma \to 0} \sigma \int_0^\infty A_\lambda(t)\, \varphi(e^{-\sigma t})\, e^{-\sigma t}\, dt = f(+0) \int_0^\infty \varphi(e^{-t})\, e^{-t}\, dt. \tag{3.88}$$

On putting $\sigma = 1/\omega$, where $\omega \to \infty$, and

$$\varphi(x) = \begin{cases} (1 - \log x^{-1})^{k-1}, & \text{for } e^{-1} < x \leqslant 1, \\ 0, & \text{for } 0 \leqslant x \leqslant e^{-1}. \end{cases}$$

(3.88) leads to summability (λ, k), $k > 1$, which completes the proof of (α).

Proof of (β). This is Corollary 1.83.

COROLLARY 3.81. *If $f(+0)$ exists, and if $A_\lambda(t) = O_L(1)$, then Σa_n is summable $(\lambda, 1)$. If, however, $A_\lambda(t) = O(1)$, then Σa_n is summable (λ, k) for every $k > 0$.*

THEOREM 3.83. *If $f(+0)$ exists, and if*

$$\sum_0^n \lambda_\nu a_\nu = O_L(\lambda_n), \text{ i.e. } \bar{A}_\lambda^1(\omega) = O_L(\omega),$$

then Σa_n is summable $(\lambda, 1)$, i.e.

$$\omega^{-1} A_\lambda^1(\omega) \to f(+0),$$

as $\omega \to \infty$.

PROOF. From (3.82) and (3.84) we deduce that

$$f(\sigma) - f_1(\sigma) = \sigma \int_0^\infty \left(A_\lambda(t) - \frac{A_\lambda^1(t)}{t} \right) e^{-\sigma t}\, dt = \sigma \int_0^\infty \frac{\bar{A}_\lambda^1(t)}{t} e^{-\sigma t}\, dt$$

$$= o(1), \tag{3.89)1}$$

as $\sigma \to 0$. Hence, if

$$\bar{A}_\lambda^1(t) \geqslant -Kt,$$

then

$$\sigma \int_0^\infty \left(\frac{\bar{A}_\lambda^1(t)}{t} + K \right) e^{-\sigma t}\, dt \to K,$$

as $\sigma \to 0$, and by Theorem 3.82,

$$\omega^{-1} \int_0^\omega t^{-1} \bar{A}_\lambda^1(t)\, dt \to 0.$$

But the left side is, by (3.82),

$$\omega^{-1} \int_0^\omega \{ A_\lambda(t) - t^{-1} A_\lambda^1(t) \}\, dt = \omega^{-1} \left[A_\lambda^1(\omega) - \int_0^\omega t^{-1} A_\lambda^1(t)\, dt \right]$$

$$= \omega^{-1} \int_0^\omega t\, d\, C_\lambda^1(t),$$

where $C_\lambda^1(t) = t^{-1} A_\lambda^1(t)$. Thus

$$\omega^{-1} \int_0^\omega t\, d\, C_\lambda^1(t) = o(1). \tag{3.89)2}$$

Now (3.89)2 and the fact, implied by (3.89)1, that

$$f_1(\sigma) = \sigma \int_0^\infty C_\lambda^1(t)\, e^{-\sigma t}\, dt \to f(+0)$$

as $\sigma \to 0$, imply, by Theorem 3.81, that

$$\lim_{\omega\to\infty} C_\lambda^1(\omega) = \lim_{\omega\to\infty} \{\omega^{-1} A_\lambda^1(\omega)\} = f(+0),$$

which proves the theorem.

THEOREM 3.84. *If $f(+0)$ exists, and if*

$$\liminf_{\omega\to\infty} \min_{\omega<t<(1+\delta)\omega} \{A_\lambda(t) - A_\lambda(\omega)\} \geqslant -\varphi(\delta) \to 0, \text{ as } \delta \to 0,$$

then Σa_n converges, i.e. $\lim_{\omega\to\infty} A_\lambda(\omega)$ exists.

PROOF. The theorem is proved in two stages: (α). The hypothesis on a_n or $A_\lambda(t)$ implies $\bar{A}_\lambda^1(\omega) = O_L(\omega)$, and hence by Theorem 3.83, $\omega^{-1}A_\lambda^1(\omega)$ tends to a limit as $\omega \to \infty$. (β). The hypothesis on $A_\lambda(t)$ also implies that if $\omega^{-1} A_\lambda^1(\omega)$ tends to a limit, then $A_\lambda(\omega)$ tends to the same limit as $\omega \to \infty$.

PROOF OF (α). See Theorem 1.88(α).

PROOF OF (β). See Theorem 1.88(β).

COROLLARY 3.82. *If $\lambda_n a_n = O_L(\lambda_n - \lambda_{n-1})$, $\dfrac{\lambda_n}{\lambda_{n-1}} \to 1$, and $f(+0)$ exists, then Σa_n converges.*

PROOF. The hypothesis of this Corollary implies the hypothesis of Theorem 3.84, as we have already remarked in the proof of Theorem 1.89.

THEOREM 3.85. *If $\lambda_n a_n = O(\lambda_n - \lambda_{n-1})$, and if $f(+0)$ exists, then Σa_n converges.*

The proof of this theorem is similar to that of Theorem 3.84, except that we use Corollary 1.81 (with $\alpha = -1$, $\beta = k$) instead of Theorem 1.88 (β).

3.9. Dirichlet product of summable series

If we associate summability by Riesz means of type λ with the series Σa_n, and of type μ with Σb_n, we may form the sequence of numbers ν_n, which are numbers $\lambda_p + \mu_q$ arranged in increasing order of magnitude, and associate summability by Riesz means of type ν with the series Σc_n where

$$c_n = \sum_{\lambda_p + \mu_q = \nu_n} a_p b_q.$$

We define Σc_n as the *Dirichlet product* of Σa_n and Σb_n. This definition suggests itself naturally because the formal product of $\Sigma a_n e^{-\lambda_n s}$ and $\Sigma b_n e^{-\mu_n s}$ may be written as $\Sigma c_n e^{-\nu_n s}$. If $\lambda_n = \mu_n = n$, the Dirichlet product is precisely the *Cauchy product*.

It is the aim of this section to discuss the relationship between the orders of summability, ordinary or absolute, of the series Σa_n, Σb_n and their Dirichlet product Σc_n. We first prove a few formulae for the Riesz means of the Dirichlet product, which we state as lemmas, then a few theorems generalizing known theorems on the convergence of the Cauchy product, and finally add a few results of a Tauberian character.

LEMMA 3.91. *If* $k \geqslant 0$, $l \geqslant 0$, then

$$C_\nu^{k+l+1}(\omega) = \frac{\Gamma(k+l+2)}{\Gamma(k+1)\,\Gamma(l+1)} \int_0^\omega B_\mu^l(\omega - t)\, A_\lambda^k(t)\, dt. \quad (3.91)$$

PROOF. If we consider the expression on the right of (3.91), we observe that the term a_p occurs in $A_\lambda^k(t)$ if $\lambda_p \leqslant t$, with the coefficient $(t - \lambda_p)^k$, while the term b_q occurs in $B_\mu^l(\omega - t)$ with the coefficient $(\omega - t - \mu_q)^l$, if $\mu_q \leqslant \omega - t$, so that the term $a_p b_q$ will occur on the right side, if $\lambda_p + \mu_q \leqslant \omega$, with the coefficient

$$\frac{\Gamma(k+l+2)}{\Gamma(k+1)\,\Gamma(l+1)} \int_{\lambda_p}^{\omega - \mu_q} (t - \lambda_p)^k (\omega - t - \mu_q)^l\, dt$$
$$= (\omega - \lambda_p - \mu_q)^{k+l+1}.$$

This is the coefficient of $a_p b_q$ in the expression on the left of (3.91) and hence the lemma.

LEMMA 3.92. *If* $k \geqslant 0$, $l \geqslant 0$, *then*

(i) *for* $k = 0$, $l \geqslant 0$,

$$C_\nu^l(\omega) = \sum_{\lambda_p \leqslant \omega} a_p B_\mu^l(\omega - \lambda_p), \quad (3.92)1$$

and

(ii) *for* $k > 0,\ l \geqslant 0$,

$$C_\nu^{k+l}(\omega) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l+1)} \int_0^\omega B_\mu^l(\omega - t)\, d A_\lambda^k(t)$$

$$= \frac{k\,\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l+1)} \left[\int_0^\omega B_\mu^l(\omega - t)\, t^{-1} \bar{A}_\lambda^k(t)\, dt + \right.$$

$$\left. + \int_0^\omega B_\mu^l(\omega - t)\, t^{-1} A_\lambda^k(t)\, dt \right]. \qquad (3.92)2$$

PROOF. Case (i) is easily seen as follows.

$$C_\nu^l(\omega) = \sum_{\lambda_p + \mu_q \leqslant \omega} (\omega - \lambda_p - \mu_q)^l\, a_p\, b_q = \sum_{\lambda_p \leqslant \omega} a_p \cdot \sum_{\mu_q \leqslant \omega - \lambda_p} (\omega - \lambda_p - \mu_q)^l\, b_q$$

$$= \sum_{\lambda_p \leqslant \omega} a_p\, B_\mu^l(\omega - \lambda_p).$$

Case (ii) is proved as follows : by Case (i),

$$C_\nu^{k+l}(\omega) = \sum_{\lambda_p \leqslant \omega} a_p\, B_\mu^{k+l}(\omega - \lambda_p)$$

$$= \int_0^\omega B_\mu^{k+l}(\omega - t)\, dA_\lambda(t)$$

$$= \frac{\Gamma(k+l+1)}{\Gamma(k)\,\Gamma(l+1)} \int_0^\omega d A_\lambda(t) \int_0^{\omega - t} (\omega - t - u)^{k-1} B_\mu^l(u)\, du,$$

by (1.21);

$$= \frac{\Gamma(k+l+1)}{\Gamma(k)\,\Gamma(l+1)} \int_0^\omega dA_\lambda(t) \int_t^\omega (x - t)^{k-1}\, B_\mu^l(\omega - x)\, dx$$

$$= \frac{\Gamma(k+l+1)}{\Gamma(k)\,\Gamma(l+1)} \int_0^\omega B_\mu^l(\omega - x)\, dx \int_0^x (x - t)^{k-1}\, dA_\lambda(t)$$

$$= \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l+1)} \int_0^\omega B_\mu^l(\omega - x)\, dA_\lambda^k(x).$$

The second part of (3.92)2 follows upon substituting

$$(x - t)^{k-1} = x^{-1} \left\{ t(x - t)^{k-1} + (x - t)^k \right\}$$

in the formula immediately preceding the last.

LEMMA 3.93. *If* $k > 0,\ l > 0$, *then*

$$\bar{C}_{\nu}^{k+l}(\omega) = \frac{\Gamma(k+l)}{\Gamma(k+1)\,\Gamma(l+1)} \left[k \int_0^\omega \bar{A}_\lambda^k(\omega - t)\, dB_\mu^l(t) + \right.$$
$$\left. + \, l \int_0^\omega \bar{B}_\mu^l(\omega - t)\, dA_\lambda^k(t) \right], \tag{3.93}$$

for all ω *if* $k + l \geqslant 1$, *and for* $\omega \neq \lambda_p + \mu_q$ *if* $k + l < 1$.

PROOF.

$$\bar{C}_{\nu}^{k+l}(\omega) = \sum_{\lambda_p + \mu_q \leqslant \omega} (\omega - \lambda_p - \mu_q)^{k+l-1} (\lambda_p + \mu_q)\, a_p\, b_q$$
$$= \sum_{\lambda_p + \mu_q \leqslant \omega} (\omega - \lambda_p - \mu_q)^{k+l-1} \lambda_p a_p \,.\, b_q + \sum_{\lambda_p + \mu_q \leqslant \omega} (\omega - \lambda_p - \mu_q)^{k+l-1} a_p \,.\, \mu_q b_q.$$

The first term in this formula is

$$= \sum_{\mu_q \leqslant \omega} b_q \,.\, \sum_{\lambda_p \leqslant \omega - \mu_q} (\omega - \lambda_p - \mu_q)^{k+l-1} \lambda_p\, a_p$$
$$= \sum_{\mu_q \leqslant \omega} b_q\, \bar{A}_\lambda^{k+l}(\omega - \mu_q) = \int_0^\omega \bar{A}_\lambda^{k+l}(\omega - t)\, d\, B_\mu(t).$$

Now, if $\omega - t \neq \lambda_p$,

$$\bar{A}_\lambda^{k+l}(\omega - t) = \sum_{\lambda_p \leqslant \omega - t} (\omega - t - \lambda_p)^{k+l-1} \lambda_p\, a_p$$
$$= \frac{\Gamma(k+l)}{\Gamma(k)\,\Gamma(l)} \sum_{\lambda_p \leqslant \omega - t} \lambda_p\, a_p \int_{\lambda_p}^{\omega - t} (\omega - t - x)^{l-1} (x - \lambda_p)^{k-1}\, dx$$
$$= \frac{\Gamma(k+l)}{\Gamma(k)\,\Gamma(l)} \int_0^{\omega - t} (\omega - t - x)^{l-1} \left[\sum_{\lambda_p \leqslant x} (x - \lambda_p)^{k-1} \lambda_p\, a_p \right] dx$$
$$= \frac{\Gamma(k+l)}{\Gamma(k)\,\Gamma(l)} \int_0^{\omega - t} (\omega - t - x)^{l-1}\, \bar{A}_\lambda^k(x)\, dx.$$

Hence

$$\int_0^\omega \bar{A}_\lambda^{k+l}(\omega - t)\, dB_\mu(t)$$

$$= \frac{\Gamma(k+l)}{\Gamma(k)\,\Gamma(l)} \int_0^\omega dB_\mu(t) \int_0^{\omega-t} (\omega - t - x)^{l-1} \bar{A}_\lambda^k(x)\, dx$$

$$= \frac{\Gamma(k+l)}{\Gamma(k)\,\Gamma(l)} \int_0^\omega dB_\mu(t) \int_t^\omega (y - t)^{l-1} \bar{A}_\lambda^k(\omega - y)\, dy$$

$$= \frac{\Gamma(k+l)}{\Gamma(k)\,\Gamma(l)} \int_0^\omega \bar{A}_\lambda^k(\omega - y)\, dy \int_0^y (y - t)^{l-1}\, dB_\mu(t)$$

$$= \frac{\Gamma(k+l)}{\Gamma(k)\,\Gamma(l+1)} \int_0^\omega \bar{A}_\lambda^k(\omega - y)\, dB_\mu^l(y).$$

A similar argument applies to the second term of the formula.

Using the above formulae, we shall deduce a number of theorems on the summability of the Dirichlet product of two summable series.

THEOREM 3.91. *If Σa_n is bounded (λ, k), $k \geqslant 0$, and summable (λ, k'), $k' > k$, and Σb_n is summable (μ, l), $l \geqslant 0$, then Σc_n is summable $(\nu, k + l + 1)$. The sum of Σc_n is equal to the product of the sums of Σa_n and Σb_n.*

PROOF. Let

$$A_\lambda^k(\omega) = O(\omega^k), \tag{3.94}$$

$$A_\lambda^{k'}(\omega) = a\omega^{k'} + o(\omega^{k'}),\ k' > k \geqslant 0, \tag{3.95}$$

and

$$B_\mu^l(\omega) = b\omega^l + o(\omega^l),\ l \geqslant 0\,. \tag{3.96}$$

Then by (3.91),

$$C_\nu^{k+l+1}(\omega) = \frac{\Gamma(k+l+2)}{\Gamma(k+1)\,\Gamma(l+1)} \int_0^\omega A_\lambda^k(t)\, B_\mu^l(\omega - t)\, dt$$

$$= b\, A_\lambda^{k+l+1}(\omega) + \frac{\Gamma(k+l+2)}{\Gamma(k+1)\,\Gamma(l+1)} \times$$

$$\times \int_0^\omega A_\lambda^k(t) \left\{ B_\mu^l(\omega - t) - b(\omega - t)^l \right\} dt$$

$$= ab\ \omega^{k+l+1} + o(\omega^{k+l+1}) + \int_0^\omega O(t^k)\,.\,o(\omega - t)^l\,dt$$

$$= ab\ \omega^{k+l+1} + o(\omega^{k+l+1}),$$

by (3.94), (3.95) and (3.96), which proves the theorem.

COROLLARY 3.91. *If Σa_n is summable (λ, k), $k \geqslant 0$, and Σb_n is summable (μ, l), $l \geqslant 0$, then Σc_n is summable $(\nu, k + l + 1)$.*

COROLLARY 3.92. *If Σa_n, Σb_n and Σc_n are summable by Riesz means of sufficiently high order and of type λ, μ and ν respectively, and if their sums are a, b and c, then $c = ab$.*

In view of Corollary 3.92, we shall not explicitly state, in the following theorems, that the sum of the product series is equal to the product of the sums, but it shall remain implicit.

THEOREM 3.92. *If Σa_n is summable $|\lambda, k|$, $k \geqslant 0$, and Σb_n is summable $|\mu, l|$, $l \geqslant 0$, then Σc_n is summable $|\nu, k + l|$.*

PROOF. We will consider three cases: (i) $k > 0$, $l > 0$, (ii) $k = 0$, $l > 0$, or $k > 0$, $l = 0$, and (iii) $k = 0$, $l = 0$.

CASE (i). $k > 0$, $l > 0$. We have to show that

$$\int_0^\infty |\, d\ \{ \omega^{-k-l}\, C_\nu^{k+l}(\omega) \}\, | < \infty,$$

which is the same thing as saying

$$\int^\infty \frac{|\bar{C}_\nu^{k+l}(\omega)|}{\omega^{k+l+1}}\, d\omega < \infty. \tag{3.97}$$

Now, by (3.93), for $\omega \neq \lambda_p + \mu_q$,

$$\bar{C}_\nu^{k+l}(\omega) = c_1 \int_0^\omega \bar{A}_\lambda^k(\omega - t)\, dB_\mu^l(t) + c_2 \int_0^\omega \bar{B}_\mu^l(\omega - t)\, d A_\lambda^k(t)$$

$$= c_1 \left[\int_0^\omega \bar{A}_\lambda^k(\omega - t)\, \frac{B_\mu^l(t)}{t}\, dt + \int_0^\omega \bar{A}_\lambda^k(\omega - t)\, \frac{\bar{B}_\mu^l(t)}{t}\, dt \right] +$$

$$+ c_2 \left[\int_0^\omega \bar{B}_\mu^l(\omega - t)\, \frac{A_\lambda^k(t)}{t}\, dt + \int_0^\omega \frac{\bar{A}_\lambda^k(t)}{t}\,.\,\bar{B}_\mu^l(\omega - t)\, dt \right]$$

$$\equiv I_1 + I_2 + I_3 + I_4, \text{ say.}$$

Now

$$\int_0^\infty \omega^{-k-l-1} |I_1| \, d\omega = c_1 \int_0^\infty \omega^{-k-l-1} \left| \int_0^\omega \bar{A}_\lambda^k(\omega - t) \frac{B_\mu^l(t)}{t} dt \right| d\omega$$
$$\leqslant c_1 \int_0^\infty |\bar{A}_\lambda^k(t)| \, dt \int_t^\infty \frac{|B_\mu^l(\omega - t)|}{\omega^{k+l+1}(\omega - t)} d\omega$$
$$< c \int^\infty |\bar{A}_\lambda^k(t)| \, dt \int_t^\infty (\omega - t)^{l-1} \omega^{-k-l-1} \, d\omega$$
$$< c \int^\infty t^{-k-1} |\bar{A}_\lambda^k(t)| \, dt$$
$$< \infty.$$

Here we have used the fact that if Σa_n is summable $|\mu, l|$, it is summable (μ, l).

Again

$$\int^\infty \omega^{-k-l-1} |I_2| \, d\omega = c \int_0^\infty \omega^{-k-l-1} \, d\omega \left| \int_0^\omega \bar{A}_\lambda^k(t) \bar{B}_\mu^l(\omega - t)(\omega - t)^{-1} dt \right|$$
$$\leqslant \int^\infty |\bar{A}_\lambda^k(t)| \, dt \int_t^\infty \frac{|\bar{B}_\mu^l(\omega - t)|}{\omega^{k+l+1} \cdot (\omega - t)} d\omega$$
$$\leqslant c \int_0^\infty t^{-k-1} |\bar{A}_\lambda^k(t)| \, dt \int_t^\infty \frac{|\bar{B}_\mu^l(\omega - t)|}{(\omega - t)^{l+1}} d\omega$$
$$< \infty.$$

Similarly we show that

$$\int_0^\infty \omega^{-k-l-1} |I_3| \, d\omega$$

and

$$\int_0^\infty \omega^{-k-l-1} |I_4| \, d\omega$$

are finite, which will prove (3.97) for Case (i).

CASE (ii). $k = 0$, $l > 0$, or $l = 0, k > 0$.

Let $k = 0$. We proceed as in Case (i) and observe that expressions corresponding to I_1, I_2 will be obtained as sums and not as integrals, with a single sum corresponding to $I_3 + I_4$. Thus if $k = 0$, $l > 0$, and $\omega \neq \lambda_p + \mu_q$,

$$\bar{C}_\nu^{k+l}(\omega) = \sum_{\lambda_p + \mu_q \leqslant \omega} (\omega - \lambda_p - \mu_q)^{l-1} (\lambda_p + \mu_q)\, a_p b_q$$

$$= \sum_{\lambda_p + \mu_q \leqslant \omega} (\omega - \lambda_p - \mu_q)^{l-1} \lambda_p a_p \,.\, b_q + \sum_{\lambda_p + \mu_q \leqslant \omega} (\omega - \lambda_p - \mu_q)^{l-1} a_p \mu_q b_q.$$

The first sum on the right is

$$= \sum_{\lambda_p \leqslant \omega} \lambda_p a_p \,.\, \sum_{\mu_q < \omega} (\omega - \lambda_p - \mu_q)^{l-1} b_q$$

$$= \sum_{\lambda_p \leqslant \omega} \lambda_p a_p \,.\, \frac{1}{\omega - \lambda_p} \Big[B_\mu^l(\omega - \lambda_p) + \bar{B}_\mu^l(\omega - \lambda_p) \Big]$$

$$\equiv I_1 + I_2,$$

and the second is

$$= \sum_{\lambda_p \leqslant \omega} a_p \bar{B}_\mu^l(\omega - \lambda_p) \equiv I_3,$$

say. Now I_1, I_2 and I_3 can be treated as in Case (i). The proof is similar if $l = 0$, $k > 0$.

CASE (iii). $k = l = 0$. This case is simply the familiar result that if Σa_n and Σb_n are absolutely convergent, then Σc_n is absolutely convergent.

THEOREM 3.93. *If Σa_n is summable $|\lambda, k|$, $k \geqslant 0$, and Σb_n is summable (μ, l), $l \geqslant 0$, then Σc_n is summable $(\nu, k + l)$.*

PROOF. We consider two cases: (i) $k > 0$, $l \geqslant 0$, and (ii) $k = 0$, $l \geqslant 0$, and assume that b is the sum of Σb_n and a that of Σa_n.

CASE (i). By (3.92)2,

$$C_\nu^{k+l}(\omega) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l+1)} \int_0^\omega B_\mu^l(\omega - t)\, dA_\lambda^k(t)$$

$$= \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l+1)} \int_0^\omega [\, B_\mu^l(\omega - t) - b(\omega - t)^l \,]\, dA_\lambda^k(t) +$$

$$+ \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l+1)} \,.\, b \int_0^\omega (\omega - t)^l\, dA_\lambda^k(t).$$

The second expression on the right is $ab\omega^{k+l} + o(\omega^{k+l})$. So it is enough to show that the first is $o(\omega^{k+l})$. Now if we set $\varphi(\omega - t) = B_{\mu}^{l}(\omega - t) - b(\omega - t)^{l}$, then the first integral, without the constant factor, is

$$\int_0^{\omega} \varphi(\omega - t)\, dA_{\lambda}^{k}(t)$$

$$= \int_0^{\omega} \varphi(\omega - t) \frac{\bar{A}_{\lambda}^{k}(t)}{t}\, dt + \int_0^{\omega} \varphi(\omega - t) \frac{A_{\lambda}^{k}(t)}{t}\, dt, \text{ if } k > 0,$$

$$= I_1 + I_2,$$

say, where

$$I_2 = \int_0^{\omega} o(\omega - t)^{l} \,.\, O(t^{k-1})\, dt = o(\omega^{k+l}).$$

We write I_1 as $I_{1,1} + I_{1,2}$, where

$$I_{1,1} = \int_0^{\omega_0}, \quad I_{1,2} = \int_{\omega_0}^{\omega},$$

and ω_0 is to be fixed presently. Given $\varepsilon > 0$, we can choose ω_0 so that for $\omega > \omega_0$,

$$\int_{\omega_0}^{\omega} \frac{|A_{\lambda}^{k}(t)|}{t^{k+1}}\, dt < \varepsilon,$$

since Σa_n is summable $|\lambda, k|$, and then

$$I_{1,2} = O(\omega^{k+l}) \int_{\omega_0}^{\omega} \frac{|\bar{A}_{\lambda}^{k}(t)|}{t^{k+1}}\, dt = O(\varepsilon\, \omega^{k+l}).$$

$$\text{Also } I_{1,1} = o\{(\omega - \omega_0)^{l}\} \int_0^{\omega_0} \frac{|\bar{A}_{\lambda}^{k}(t)|}{t}\, dt = o(\omega^{l}) = o(\omega^{k+l}).$$

Combining the estimates for I_2, $I_{1,1}$ and $I_{1,2}$ we have, since ε is arbitrary,

$$C_{\nu}^{k+l}(\omega) = ab \,.\, \omega^{k+l} + o(\omega^{k+l}),$$

under the assumption $k + l > 0$, $k > 0$.

CASE (ii). If $k = 0$, we start with (3.92)1.

$$C_\nu^{k+l}(\omega) = C_\nu^l(\omega) = \sum_{\lambda_p \leqslant \omega} a_p \, B_\mu^l(\omega - \lambda_p)$$

$$= b \sum_{\lambda_p \leqslant \omega} (\omega - \lambda_p)^l a_p + \sum_{\lambda_p \leqslant \omega} a_p \left\{ B_\mu^l(\omega - \lambda_p) - b(\omega - \lambda_p)^l \right\}$$

$$= b \, . \, A_\lambda^l(\omega) + \sum_{\lambda_p \leqslant \omega} a_p \left\{ B_\mu^l(\omega - \lambda_p) - b\,(\omega - \lambda_p)^l \right\}.$$

The first expression on the right is $ab\omega^l + o(\omega^l)$, while the second is seen to be $o(\omega^l)$ by splitting it as we did for I_1 before.

The next three theorems are of a Tauberian nature.

THEOREM 3.94. *Let* Σa_n *be summable* (λ, k), $k \geqslant 0$, *and* Σb_n *summable* (μ, l), $l \geqslant 0$, *and let*

$$\int_0^\omega |d\bar{A}_\lambda^{k+1}(t)| = O(\omega^{k+1}), \quad \int_0^\omega |d\bar{B}_\mu^{l+1}(t)| = O(\omega^{l+1}).$$

Then Σc_n *is summable* $(\nu, k + l)$.

PROOF. If $k > 0$, $l > 0$, the above conditions can be written as

$$\int_0^\omega |\bar{A}_\lambda^k(t)|\, dt = O(\omega^{k+1}), \quad \int_0^\omega |\bar{B}_\mu^l(t)|\, dt = O(\omega^{l+1});$$

while if $k = 0$, $l = 0$, the same conditions will become

$$\sum_{\lambda_\nu \leqslant \omega} |\lambda_\nu a_\nu| = O(\omega), \quad \sum_{\mu_\nu \leqslant \omega} |\mu_\nu b_\nu| = O(\omega).$$

Summability of Σa_n and Σb_n implies

$$\bar{A}_\lambda^{k+1}(\omega) = o(\omega^{k+1}), \tag{3.98)1}$$

$$\bar{B}_\mu^{l+1}(\omega) = o(\omega^{l+1}), \tag{3.98)2}$$

for

$$\bar{A}_\lambda^{k+1}(\omega) = \omega\, A_\lambda^k(\omega) - A_\lambda^{k+1}(\omega),$$

or

$$\omega^{-k-1}\, \bar{A}_\lambda^{k+1}(\omega) = \omega^{-k}\, A_\lambda^k(\omega) - \omega^{-k-1}\, A_\lambda^{k+1}(\omega) = o(1)\,;$$

and similarly for $\bar{B}^{l+1}(\omega)$. Now by Corollary 3.91, Σc_n is summable $(\nu, k + l + 1)$, and hence, to prove summability $(\nu, k + l)$, it is enough to show that

$$\bar{C}_\nu^{k+l+1}(\omega) = o(\omega^{k+l+1}).$$

But, by integrating (3.93) by parts, we see that

$$\bar{C}_{\nu}^{k+l+1}(\omega) = c_1 \int_0^{\omega} B_{\mu}^{l}(\omega - t)\, d\bar{A}_{\lambda}^{k+1}(t) + c_2 \int_0^{\omega} A_{\lambda}^{k}(\omega - t)\, d\bar{B}_{\mu}^{l+1}(t)$$

$$\equiv I_1 + I_2,$$

say. Now

$$I_1 = c_1 b \int_0^{\omega} (\omega - t)^l\, d\bar{A}_{\lambda}^{k+1}(t) +$$

$$+ c_1 \int_0^{\omega} \{B_{\mu}^{l}(\omega - t) - b(\omega - t)^l\}\, d\bar{A}_{\lambda}^{k+1}(t)$$

$$= I_{1,1} + I_{1,2},$$

say, where

$$I_{1,1} = c_1 b.\ \bar{A}_{\lambda}^{k+l+1}(\omega) = o\,(\omega^{k+l+1}),$$

by (3.98) and the first consistency theorem. Also

$$I_{1,2} = \int_0^{\omega} o(\omega - t)^l.\, |d\bar{A}_{\lambda}^{k+1}(t)| = o(\omega^{k+l+1}),$$

by hypothesis. Hence $I_1 = o(\omega^{k+l+1})$. Similarly we prove $I_2 = o(\omega^{k+l+1})$, and hence the theorem.

Corollary 3.93. *If* Σa_n *and* Σb_n *are convergent, and if* $\lambda_n a_n = O\,(\lambda_n - \lambda_{n-1})$ *and* $\mu_n b_n = O\,(\mu_n - \mu_{n-1})$, *then* Σc_n *is convergent.*

This reduces to the case $k = l = 0$ of Theorem 3.94.

Theorem 3.95. *Let* Σa_n *and* Σb_n *be summable* (λ, k), $k \geqslant 0$, *and* (μ, l), $l \geqslant 0$, *respectively, and let* $\int_0^{\omega} |d\bar{A}_{\lambda}^{k+1}(t)| = O(\omega^{k+1})$, $k > 0$. *Then* Σc_n *is bounded* $(\nu, k + l)$ *and summable* $(\nu, k + l + \varepsilon)$ *for every* $\varepsilon > 0$.

Proof. By Corollary 3.91, Σc_n is summable $(\nu, k + l + 1)$, and by Corollary 1.71 it follows that Σc_n is summable $(\nu, k + l + \varepsilon)$ if we prove that Σc_n is bounded $(\nu, k + l)$.

Now we first observe that if $k > 0$, and if

$$\int_0^\omega |d\bar{A}_\lambda^{k+1}(t)| = O(\omega^{k+1}),$$

then

$$\int_0^\omega t^{-1}|d\bar{A}^{k+1}(t)| = O(\omega^k), \tag{3.99}$$

by partial integration. Next, from (3.92)2, we can write

$$C_\nu^{k+l}(\omega) = c' \int_0^\omega B_\mu^l(\omega - t)\, t^{-1} A_\lambda^k(t)dt + c'' \int_0^\omega B_\mu^l(\omega - t)\, t^{-1} d\bar{A}_\lambda^{k+1}(t).$$

Each of the expressions on the right is seen to be $O(\omega^{k+l})$ on account of the hypothesis and (3.99).

Theorem 3.96. *If Σa_n and Σb_n are summable (λ, k), $k \geq 0$, and (μ, l), $l \geq 0$, respectively, and if $\int_0^\omega |d\bar{A}_\lambda^{k+1}(t)| = o(\omega^{k+1})$, $k > 0$, then Σc_n is summable $(\nu, k+l)$.*

Proof is as in Theorem 3.95, except that we have to use the analogous formula for (3.99) with o instead of O on the right.

NOTES ON CHAPTER III

§3.1. For an introductory study of Dirichlet series of the form $\Sigma a_n n^{-s}$ see E. C. Titchmarsh, *The theory of functions*, ed. 2, Oxford (1939).

§3.2. If $0 < k < 1$, the function $\bar{A}_\lambda^k(\omega)$ exists except for $\omega = \lambda_n$, and is integrable, and the formula (3.21) may be written in the form

$$k \int_0^\omega \bar{A}_\lambda^k(t)\, dt = \omega A_\lambda^k(\omega) - A_\lambda^{k+1}(\omega)$$

for all values of $k > 0$.

§3.3. It is well to notice the essential difference between the region of convergence of a power series and that of a Dirichlet series. The circle of convergence of a power series passes through the singularity nearest to the centre. For a Dirichlet series there is not necessarily any singularity on the line of convergence. See, for instance, Titchmarsh, loc. cit., 294.

Most of the results of this section on the summability of Dirichlet series can be found in the *Tract.*

In connexion with Theorem 3.31 see G. L. Isaacs, *Jour. London Math. Soc.* 26 (1951), 285-290. Our attention was called to this at the stage of proof-correction.

§3.4. Absolute summability of Dirichlet series was discussed by N. Obrechkoff, *Math. Zeitschrift,* 30 (1929), 375-386. He has also shown that for $\sigma \geqslant \bar{\sigma}_k + \varepsilon$, $f(s) = O(|\tau|^k)$.

It should be noted that in Theorem 3.42, if $k = 0$, the conclusion is valid for $\sigma \geqslant \sigma^*$.

As Dr. Bosanquet points out, Theorem 3.42 is false with $\sigma = \sigma^*$ (but Theorem 3.45 is true !), e.g. $\Sigma n^{-1} e^{in}$ is summable $|n, 1|$ (by Theorem 3.46, for instance) but Σn^{-1} is not summable $|n, 1|$. In a paper which is soon to appear he is stressing this point.

For theorem 3.44 see Obrechkoff, loc. cit.

Theorem 3.46 was proved by L. S. Bosanquet, *Jour. London Math. Soc.* 23 (1948), 35-38. He points out that the less delicate result of Theorem 3.41 could also be obtained by replacing $u^{-p-\delta}$ by $e^{-\delta u}$ throughout his proof.

The inequality

$$\bar{\sigma}_k - \sigma_k \leqslant \limsup_{n\to\infty} \frac{\log n}{\log l_n}$$

was proved for integral k by L. S. Bosanquet, *Jour. London Math. Soc.* 22 (1947), 190-195. Dr. Bosanquet informs us that M. C. Austin has proved it for all values of $k > 0$ except possibly one, and that his proof will appear in the *Jour. London Math. Soc.* 27 (1952).

§3.5. It is possible that the results of this section have their analogues for absolute summability, but we are not aware of any literature on the subject.

§3.6. For Lemma 3.62 see E. T. Whittaker and G. N. Watson, *A course of modern analysis*, ed. 4, Cambridge (1927), 238, and

E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Leipzig (1909), 342. Theorems 3.61-3.63 were proved by M. Riesz, *Acta Szeged*, 2 (1924), 18-31. In connexion with Theorem 3.62 see A. C. Offord, *Proc. London Math. Soc.* 37 (1934), 147-160.

Theorem 3.64 is a variant of a result due to M. Riesz, *Acta Mathematica*, 40 (1916), 349-361. The conditions on a_n could take the form: $a_n = o(1)$ and $a_n = o(\lambda_n - \lambda_{n-1})$. See also A. E. Ingham, *Proc. London Math. Soc.* 38 (1935), 463.

The statement that $f(s)$ has boundary-values on $\sigma = \eta$ should be understood to mean that $f(s)$ has a unique limit as s tends to any point on the line $\sigma = \eta$ from the right. The boundary-function is the function defined on the line $\sigma = \eta$ by means of the boundary-values.

Theorems 3.66 is Theorem 41 of the *Tract*. It is due to M. Riesz, *Comptes Rendus*, 5 July 1909. Theorems 3.68 and 3.69 are the modified versions of Theorems 42 and 43 of the *Tract*.

§3.7. Historically, G. H. Hardy and J. E. Littlewood were the first to prove analogous results for the Dirichlet series $\Sigma a_n n^{-s}$, see *Proc. London Math. Soc.* 11 (1912), 411-478. They were later generalized by K. Ananda-Rau, *Proc. London Math. Soc.* 34 (1932), 414-440.

If in Theorems 3.71, 3.72, we replace the discrete variable l_n in the hypothesis (3.75) by a continuous variable ω, then it can be shown that $\sigma_0 \leqslant \alpha + \mu$. For we may assume that $\alpha + \mu \neq 0$, and consider the cases: (i) $\alpha + \mu > 0$, and (ii) $\alpha + \mu < 0$. In (i) we note, by a familiar argument (cf. p. 26, line 7), that $A_l(t) = O(t^{\alpha+\mu})$, which implies the desired conclusion. (ii) can then be proved by considering the series Σb_n, $b_n = a_n l_n^p$, $\alpha + \mu + p > 0$, and then applying (i).

Corollaries 3.71, 3.73, 3.75 and Theorems 3.73, 3.74 are due to Ananda-Rau, loc. cit. Theorem 3.73 is a generalization of the Schnee-Landau theorem. See W. Schnee, *Acta Mathematica*, 35 (1911), 357-398. Corollary 3.75 is obtained from Theorem 3.74 by letting $h \to 0$. In this Corollary it may further be verified that $\sigma_0 = \bar{\sigma}_0$. See K. Ananda-Rau, loc. cit., Theorem 13.

Corollaries 3.72, 3.74 are due to V. Ganapathy Iyer, *Annals of Math.* 36 (1935), 100-116. For an alternative hypothesis on a_n in these corollaries see Notes on §1.8. Here again we assume that $l_{-1} = 0$.

§ 3.8. For Tauberian theorems on power series see G. H. Hardy, *Divergent series*, Oxford (1949), 148-175. For an account of Tauberian theorems on Dirichlet series see O. Szász, *Trans. American*

Math. Soc. 39 (1936), 117-130. For generalizations of theorems of this section, where $f(\sigma)$ is assumed to tend to infinity like a logarithmico-exponential function, see J. Karamata, *Journal für reine und ang. Math.* 164 (1931), 27-39.

§3.9. For a discussion of various kinds of multiplication of series see G. H. Hardy, loc. cit., 227-246.

There should be no confusion between the $C_\nu^k(\omega)$ defined here as the Riesz sum of Σc_n, and the $C_\lambda^k(x)$ defined in § 1.1 as the Riesz mean of Σa_n.

It should be borne in mind that when 'inner integrals' arise from the inversion of a repeated Stieltjes integral, they are in general Lebesgue-Stieltjes integrals existing almost everywhere.

Theorems 3.91, 3.92 and 3.93 are generalizations of the theorems of Abel, Cauchy and Mertens for the Cauchy product of convergent series. As an application of Theorem 3.91 we may observe the following: Let a_n be a positive monotone null sequence and let $A_n = a_1 + \dots + a_n$. Then $\Sigma(-1)^{n-1} A_n$ is summable $(n, 1)$ to the sum $\frac{1}{2}\Sigma(-1)^{n-1} a_n$, for the former series is the Cauchy product of $\Sigma(-1)^{n-1}$ and $\Sigma(-1)^{n-1} a_n$.

Corollary 3.93 is Theorem 58 of the *Tract*. The conclusions of Theorems 3.95 and 3.96 hold if Σb_n is bounded (μ, l) and summable (μ, l') for some $l' > l$. These two theorems may fail for $k = 0$. The Cauchy product of $\Sigma a_n \equiv \Sigma(-)^{n-1} 1/n$ and $\Sigma b_n \equiv \Sigma(-)^{n-1} (\log\log n)^{-1}$ is $\Sigma(-)^{n-1} c_n$ where $c_n = \Sigma \frac{1}{(n-\nu)\log\log\nu} \sim \frac{\log n}{\log\log n} \to \infty$, and hence cannot have bounded partial sums. This shows that Theorem 3.95 may fail for $k = 0$. Similarly, to show that Theorem 3.96 need not be true for $k = 0$, we observe that the Cauchy product of $\Sigma(-1)^{n-1}(n\log n)^{-1}$ and $\Sigma(-1)^{n-1}(\log\log n)^{-1}$ cannot converge. In Theorem 3.95, if $k = 0$, a part of the conclusion, namely summability $(\nu, k + l + \varepsilon)$ for every positive ε, can be proved, since the hypothesis on $\bar{A}_\lambda^k$ is true for every $k > 0$, if it is true for $k = 0$.

It is an interesting and important problem to determine the abscissae of summability, ordinary or absolute, of the Dirichlet series $\Sigma c_n e^{-\nu_n s}$ knowing those of $\Sigma a_n e^{-\lambda_n s}$ and $\Sigma b_n e^{-\mu_n s}$. Except for one result, namely Theorem 59 of the *Tract*, no progress has been made in this direction.

IV

APPLICATIONS TO FOURIER SERIES

4.1. Introduction

IN this chapter we shall be concerned with the application of Riesz means to the study of the summability of Fourier series. We shall deal with multiple series summed over spheres, and demonstrate that such a method of summation makes it possible for us to utilize results on Fourier series for proving a classical identity in the theory of lattice points, and several of its generalized versions.

We first prove a formula, due to S. Bochner, for the Riesz mean of a Fourier series in k variables ($k \geqslant 1$), and use the formula to prove that if a function is continuous at a point, its Fourier series is summable at that point by Riesz means of a certain order δ which depends on the number k. We then impose a further condition on the order of magnitude of the Fourier coefficients, and deduce summability of order less than δ, including convergence. We next consider the summability of series derived from Fourier series by repeated application of Laplace's differential operator. For the summability of such series at a given point, we impose on the function a hypothesis in the nature of differentiability in a neighbourhood of the point in question. Here again we obtain further results of a Tauberian nature by restricting the order of magnitude of the Fourier coefficients, one of which is applicable to the study of summations over lattice points. We then prove analogous results on absolute summability. We conclude the chapter by stating conditions which are necessary and sufficient for the summability, ordinary or absolute, of a Fourier series at a given point.

4.2. Spherical means

Let $f(x) \equiv f(x_1,\ldots,x_k)$ be a function of the Lebesgue class L_1, periodic with period 2π in each of the k variables. We write the Fourier series of $f(x)$ as follows :

$$f(x) \sim \sum_{-\infty}^{\infty} \cdots \sum a_{n_1\ldots n_k} e^{i(n_1x_1+\ldots+n_kx_k)},$$

where

$$a_{n_1 \dots n_k} = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} f(x)\, e^{-i(n_1 x_1 + \dots + n_k x_k)} dx_1 \dots dx_k.$$

We write

$$A_n(x) = \sum_{n_1^2 + \dots + n_k^2 = n} a_{n_1 \dots n_k} e^{i(n_1 x_1 + \dots + n_k x_k)}, \tag{4.21}$$

with the convention that $A_n(x) \equiv 0$ if n cannot be represented as the sum of k squares. We now define the n-th *spherical partial sum* of the Fourier series as

$$S_n(x) = \sum_{r=0}^{n} A_r(x).$$

Our object will be to study the Riesz means of the series $\Sigma A_r(x)$. Accordingly we define, for $\delta > 0$,

$$S_R^\delta(x) = \sum_{n_1^2 + \dots + n_k^2 \leqslant R^2} \left(1 - \frac{n_1^2 + \dots + n_k^2}{R^2}\right)^\delta a_{n_1 \dots n_k} e^{i(n_1 x_1 + \dots + n_k x_k)}$$

$$= \sum_{r=0}^{n} \left(1 - \frac{r}{R^2}\right)^\delta A_r(x) \qquad n < R^2 < n+1$$

$$= \frac{2\delta}{R^2} \int_0^R \left(1 - \frac{u^2}{R^2}\right)^{\delta - 1} S(u)\, u\, du, \tag{4.22}$$

where

$$S(R) \equiv S_R^0(x) \equiv S_n(x).$$

We next define, for any fixed x and for $0 < t < \infty$, the *spherical mean* $f_x(t)$ of the function $f(x)$ as follows :

$$f_x(t) = \frac{\Gamma(k/2)}{2(\pi)^{k/2}} \int_\sigma f(x_1 + t\,\xi_1, \dots, x_k + t\,\xi_k)\, d\sigma_\xi, \tag{4.23}$$

where σ is the sphere $\xi_1^2 + \dots + \xi_k^2 = 1$, and $d\sigma_\xi$ is its $(k-1)$-dimensional volume-element. If $k = 1$, we define $f_x(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$.

Considered as a function of the single variable t, $f_x(t)$ exists for almost all $t > 0$, and is integrable in every finite t-interval.

We shall now state two lemmas governing the behaviour of $f_x(t)$, which follow directly from the behaviour of $f(x)$.

LEMMA 4.21. *If $f(x) \in L_1$, and is periodic with period 2π in each variable, we have*

$$\int_0^\varepsilon |f_x(s)|\, s^{k-1}ds = o(1) \text{ as } \varepsilon \to 0, \tag{4.24}$$

and

$$\int_0^t |f_x(s)|\, s^{k-1}ds = O(t^k) \text{ as } t \to \infty. \tag{4.25}$$

PROOF. The conclusions will result from the inequality

$$\int_a^b |f_x(s)|\, s^{k-1}ds \leqslant \int \underset{\Omega}{\cdots} \int |f(x)|\, dx_1 \dots dx_k,$$

where Ω is the spherical shell with radii a and b, and centre x.

LEMMA 4.22. *Let $f(x)$, $f^1(x)$, $f^2(x)$,...be a sequence of periodic functions, with period 2π, belonging to the Lebesgue class L_1, and let*

$$\lim_{r\to\infty} \int \underset{E}{\cdots} \int |f(x) - f^r(x)|\, dx_1 \dots dx_k = 0$$

uniformly in all unit spheres E. Then, uniformly in x,

$$\lim_{r\to\infty} \int_0^1 |f_x(s) - f_x^r(s)|\, s^{k-1}ds = 0, \tag{4.26}$$

and, given $\varepsilon > 0$, and $t_0 > 0$, there exists r_0 such that, for $t \geqslant t_0$ and $r > r_0$, we have

$$\int_0^t |f_x(s) - f_x^r(s)|\, s^{k-1}ds < \varepsilon t^k. \tag{4.27}$$

PROOF. (4.26) is an immediate consequence of our hypothesis. (4.27) is proved as follows:

$$\int_0^t |f_x(s) - f_x^r(s)|\, s^{k-1}ds \leqslant \int \underset{\Omega_t}{\cdots} \int |f(x) - f^r(x)|\, dx_1 \dots dx_k, \tag{4.28}$$

where Ω_t is a sphere with centre x and radius t. Since this sphere can be covered by a finite number of unit spheres whose total volume is ct^k, where c is a numerical constant independent of t, and since $f^r(x)$ approximates to $f(x)$ as $r \to \infty$ uniformly in all such spheres, the right side of (4.28) is less than εt^k.

4.3. Bessel functions

If $J_\mu(t)$ denotes the Bessel function of the first kind and of order μ, we have, for $\mu > -1$,

$$J_\mu(t) = \left(\frac{t}{2}\right)^\mu \sum_{n=0}^{\infty} \frac{(-)^n t^{2n}}{4^n\, n!\, \Gamma(\mu+n+1)}. \tag{4.31}$$

We define

$$V_\mu(t) \equiv \frac{J_\mu(t)}{t^\mu}. \tag{4.32}$$

It is known that

$$J_\mu(t) = \begin{cases} O(t^\mu), \text{ as } t \to 0, \\ O(t^{-1/2}), \text{ as } t \to \infty. \end{cases} \tag{4.33}$$

The following formulae, which are well known, will be required in the sequel.

$$V_\nu(xR) = \frac{2^{\mu+1-\nu}}{\Gamma(\nu-\mu)\, x^{2\nu}} \int_0^x (x^2-y^2)^{\nu-\mu-1} y^{2\mu+1} V_\mu(yR)\,dy, \tag{4.34}$$

for $\nu > \mu > -1$. If $\nu > \mu + 1 > 0$, then

$$\int_0^\infty \frac{J_\nu(\omega t)\, J_\mu(nt)}{t^{\nu-\mu-1}}\,dt$$

$$= \begin{cases} \dfrac{n^\mu}{2^{\nu-\mu-1}\Gamma(\nu-\mu)\,\omega^{2\mu-\nu+2}}\left(1-\dfrac{n^2}{\omega^2}\right)^{\nu-\mu-1}, \ 0 < n < \omega, \\ 0, \quad \text{if } n \geqslant \omega. \end{cases} \tag{4.35}$$

$$\frac{d}{dx}\left[V_\mu(x)\right] = -x\, V_{\mu+1}(x). \tag{4.36}$$

$$\int_0^\infty \frac{J_\nu(t)}{t^{\nu-\mu+1}}\,dt = \frac{\Gamma(\mu/2)}{2^{\nu-\mu+1}\Gamma(\nu-\mu/2+1)}, \ 0 < \mu < \nu + 3/2. \tag{4.37}$$

4.4. A formula of S. Bochner

Our object is to establish a formula which expresses the Riesz mean $S_R^\delta(x)$ defined in (4.22) in terms of the spherical mean $f_x(t)$ defined in (4.23), for any fixed x. This formula of Bochner, which may be viewed as a generalization of the classical integral of Fejér, is as follows :

$$S_R^\delta(x) = c_1 R^k \int_0^\infty f_x(t)\, t^{k-1} V_{\delta+k/2}(tR)\, dt, \ \delta > \tfrac{1}{2}(k-1), \qquad (4.41)$$

where $c_1 = 2^{\delta-k/2+1}\, \Gamma(\delta+1) \{\Gamma(k/2)\}^{-1}$.

We shall first prove the formula, as a lemma, in the special case where $f(x)$ is an exponential polynomial. In the general case where $f(x) \in L_1$, we shall make use of the fact that there exists a sequence of exponential polynomials $\{f^n(x)\}$ approximating to $f(x)$ in L_1-norm, in the space of (x), and show that this approximation allows us to extend the formula to arbitrary $f(x) \in L_1$.

LEMMA 4.41. *Formula* (4.41) *holds when $f(x)$ is an exponential polynomial, for every $\delta \geqslant 0$.*

PROOF. If

$$g(x) = e^{i(n_1 x_1 + \ldots + n_k x_k)},$$

its spherical mean $g_x(t)$ is given by

$$g_x(t) = 2^{(k-2)/2}\, \Gamma(k/2)\, V_{(k-2)/2}(n^{1/2} t)\, e^{i(n_1 x_1 + \ldots + n_k x_k)},$$

where, as before, we have $n = n_1^2 + \ldots + n_k^2$.

Hence the spherical mean of the exponential polynomial

$$f^r(x) \equiv \sum b^r_{n_1 \ldots n_k}\, e^{i(n_1 x_1 + \ldots + n_k x_k)} \qquad (4.42)$$

is given by

$$f^r_x(t) = \sum b^r_{n_1 \ldots n_k}\, e^{i(n_1 x_1 + \ldots + n_k x_k)}\, V_{(k-2)/2}(n^{1/2} t).\ 2^{(k-2)/2}\, \Gamma(k/2).$$

Hence we have

$$c_1 R^k \int_0^\infty f^r_x(t)\, t^{k-1}\, V_{\delta+k/2}(tR)\, dt$$

$$= c_2 \sum b^r_{n_1 \ldots n_k}\, e^{i\Sigma n_j x_j} \int_0^\infty \frac{J_{(k-2)/2}(n^{1/2} t)\, J_{(\delta+k/2)}(Rt)}{t^\delta} \cdot \frac{R^{k/2-\delta}}{n^{(k-2)/4}}\, dt$$

$$= \sum_{n<R^2} b^r_{n_1 \ldots n_k} e^{i\Sigma n_j x_j} \left(1 - \frac{n}{R^2}\right)^\delta,$$

by (4.35), which proves the lemma.

Theorem 4.41. *For any $f(x)$ defined as in § 4.2, we have*

$$S_R^\delta(x) = c_1 \int_0^\infty R^k t^{k-1} f_x(t)\, V_{\delta+k/2}(Rt)\, dt, \tag{4.43}$$

provided that $\delta > (k-1)/2$.

Proof. There exists a sequence of exponential polynomials $\{f^r(x)\}$, $r = 1, 2, \ldots$, such that

$$\lim_{r\to\infty} \int_E |f^r(x) - f(x)|\, dx_1 \ldots dx_k = 0$$

uniformly in all unit spheres E. This implies that

$$\lim_{r\to\infty} b^r_{n_1 \ldots n_k} = a_{n_1 \ldots n_k}, \tag{4.44}$$

where $b^r_{n_1 \ldots n_k}$ is defined as in (4.42). Let the Riesz mean corresponding to $f^r(x)$ be denoted by

$$S_R^\delta(x; f^r) = \sum_{n<R^2} \left(1 - \frac{n}{R^2}\right)^\delta b^r_{n_1 \ldots n_k}\, e^{i\Sigma n_j x_j}; \tag{4.45}$$

then we have already seen, in Lemma 4.41, that

$$S_R^\delta(x\,;\, f^r) = c\, R^k \int_0^\infty t^{k-1} f_x^r(t)\, V_{\delta+k/2}(tR)\, dt.$$

Keeping R fixed, we let $r \to \infty$, and show that

$$\lim_{r\to\infty} S_R^\delta(x; f^r) = \lim_{r\to\infty} c\, R^k \int_0^\infty t^{k-1} f_x^r(t)\, V_{\delta+k/2}(tR)\, dt$$

$$= c\, R^k \int_0^\infty t^{k-1} f_x(t)\, V_{\delta+k/2}(tR)\, dt \tag{4.46}$$

uniformly in x. This is seen as follows. Let

$$R^k \int_0^\infty t^{k-1} [f_x^r(t) - f_x(t)]\, V_{\delta+k/2}(tR)\, dt = \int_0^1 + \int_1^\infty = I_1 + I_2,$$

say. Using relations (4.26) and (4.33), we obtain

$$I_1 = O\left(\int_0^1 |f_x(t) - f_x^r(t)|\, t^{k-1}\, dt\right) = o(1),$$

and
$$I_2 = O\left(\int_1^\infty |f_x(t) - f_x^r(t)|\, t^{-\delta+k/2-3/2}\, dt\right)$$
$$= O\left(\int_1^\infty t^{-\delta-k/2-1/2}\, d\varphi\right) = o(1),$$

as $r \to \infty$, since, for an arbitrary $\varepsilon > 0$, we have, for $t > t_0$ and $r > r_0$,

$$\varphi(t) \equiv \int_0^t |f_x(s) - f_x^r(s)|\, s^{k-1}\, ds < \varepsilon\, t^k,$$

by (4.27). This establishes (4.46). Thus, for each r, the function $S_R^\delta\ (x\ ;\ f^r)$ is periodic, and the sequence converges uniformly in x as $r \to \infty$; therefore the limit-function is again periodic, and its Fourier series is the formal limit of the right side of (4.45). Using (4.44), we see that the limit-function has the Fourier series

$$\sum_{n<R^2} \left(1 - \frac{n}{R^2}\right)^\delta a_{n_1\ldots n_k}\, e^{i\Sigma n_j x_j}\ ;$$

therefore

$$\lim_{n\to\infty} S_R^\delta(x\,;f^n) = S_R^\delta(x),$$

and this, in conjunction with (4.46), yields the formula (4.43).

4.5. Summability theorems

We shall now prove a few results on the summability and convergence of multiple Fourier series by making use of Bochner's formula. Before doing so, we wish to observe that if the order of summability is sufficiently high, then the summability of the series at a point x depends only on the behaviour of the function in a neighbourhood of that point.

LEMMA 4.51. *If* $\eta > 0$ *and* $\delta > (k-1)/2$, *then*

$$R^k \int_\eta^\infty t^{k-1} f_x(t)\, V_{\delta+k/2}(tR)\, dt = o(1),$$

as $R \to \infty$, *uniformly for all* x.

PROOF. We have, on using (4.33),

$$\left| R^k \int_\eta^\infty t^{k-1} f_x(t) \; V_{\delta+k/2}(tR)\, dt \right| = O\left[\frac{1}{R^{\delta-k/2+1/2}} \int_\eta^\infty \frac{t^{k-1}\,|f_x(t)|}{t^{\delta+(k+1)/2}}\, dt \right]$$

$$= O\left[\frac{1}{R^{\delta-(k-1)/2}} \int_\eta^\infty \frac{dF}{t^{\delta+(k+1)/2}} \right], \quad F(t) = \int_0^t s^{k-1}\,|f_x(s)|\, ds,$$

$$= O\left(\frac{1}{R^{\delta-(k-1)/2}} \right) = o(1),$$

by (4.25), which proves the lemma.

Combining this lemma with formula (4.41), we obtain

THEOREM 4.51. *Riesz summability (n, δ) for $\delta > (k-1)/2$ is a 'local property' for multiple Fourier series summed spherically.*

Because of Theorem 4.51, we can state several simple conditions for summability of order δ when δ exceeds $\frac{1}{2}(k-1)$; one such condition is embodied in the following

THEOREM 4.52. *If $f_x(t) \to l$ as $t \to 0$, for a fixed x, or more generally, if*

$$\lim_{t\to 0} \; t^{-k} \int_0^t s^{k-1}\,|f_x(s) - l|\, ds = 0,$$

then

$$\lim_{R\to\infty} \; S_R^\delta(x) = l, \;\; \delta > (k-1)/2.$$

PROOF. On account of (4.37), we may assume that $l = 0$ without loss of generality. Choose $\eta > 0$ arbitrarily, and define $F(t)$ as in Lemma 4.51. In view of Lemma 4.51, we have only to prove that

$$I_1 = R^k \int_0^\eta t^{k-1} f_x(t) \; V_{\delta+k/2}(tR)\, dt = o(1),$$

as $R \to \infty$. This follows from the fact that

$$I_{1,} = \left[\int_0^{1/R} \right] = O\left[R^k \int_0^{1/R} t^{k-1}\,|f_x(t)|\, dt \right], \text{ by (4.33);}$$

$$= o(1), \text{ as } R \to \infty,$$

and

$$I_{1,2} = \left[\int_{1/R}^{\eta}\right] = O\left[R^k \int_{1/R}^{\eta} t^{k-1}\,|f_x(t)|\,.\,(tR)^{-\delta-k/2-1/2}\,dt\right]$$

$$= O\left[R^{-\delta+k/2-1/2}\int_{1/R}^{\eta}\frac{dF}{t^{\delta+k/2+1/2}}\right]$$

$$= O\left(R^{-\delta+k/2-1/2}\right)\left[\left(\frac{F(t)}{t^{\delta+k/2+1/2}}\right)_{1/R}^{\eta} + \int_{1/R}^{\eta}\frac{F(t)\,dt}{t^{\delta+k/2+3/2}}\right]$$

$$= o(1),$$

as $R \to \infty$, if $\delta > (k-1)/2$.

The hypothesis on f in the foregoing theorem is in the nature of a restriction on its continuity, in the neighbourhood of a fixed point, and the conclusion is the summability of the Fourier series for $\delta > (k-1)/2$. If we impose another hypothesis on the order of the Fourier coefficients (or, what is the same, on the partial sums), it should be possible to reduce the order of summability by the use of Theorem 1.81, and, in special cases, to derive ordinary convergence. Actually we shall see that it is possible to tie up the two hypotheses in such a way that when the continuity-condition on the function is strengthened, the order-condition on the Fourier coefficients is correspondingly weakened, the two together yielding summability of order γ where $0 \leqslant \gamma \leqslant (k-1)/2$. We need two preliminary lemmas.

LEMMA 4.52. *If $r_k(n)$ denotes the number of lattice points on the sphere $x_1^2 + \ldots + x_k^2 = n$, then*

$$r_k(n) = O\,(n^{k/2-1+\varepsilon}),\ \varepsilon > 0.$$

Further

$$R_k(x) \equiv \sum_{n \leqslant x} r_k(n) = \pi^{k/2}\left\{\Gamma(k/2+1)\right\}^{-1} x^{k/2} + O(x^{k/2-k/(k+1)}).$$

LEMMA 4.53. *If at a point x,*

$$f_x(t) - l = O\,(t^{\theta}),\ \theta > 0,$$

then

$$S_R^{\delta}(x) - l = O\,(R^{-\theta}),\ \delta > \theta + (k-1)/2.$$

PROOF. As before, we may assume that $l = 0$. We have

$$S_R^\delta(x) = c_1 \int_0^\infty R^k t^{k-1} f_x(t) V_{\delta+k/2}(tR)\, dt$$

$$= \left[\int_0^{1/R} + \int_{1/R}^{\eta} + \int_{\eta}^{\infty} \right] = I_1 + I_2 + I_3,$$

say. As in Lemma 4.51, we obtain

$$I_3 = O\left(\frac{1}{R^{\delta-(k-1)/2}}\right) = O(R^{-\theta}),\ \delta > (k-1)/2 + \theta.$$

Again

$$I_1 = O(R^{-\theta}),$$

using the hypothesis on $f_x(t)$, while

$$I_2 = O\left[\int_{1/R}^{\eta} t^{-k/2-\delta-1/2} dF(t) \right] = O(R^{-\theta}),$$

for $\delta > \theta + (k-1)/2$.

THEOREM 4.53. *If at a point x, we have*

$$f_x(t) - l = o(t^\theta),\ \theta > 0, \tag{4.51}$$

and

$$a_{n_1 \dots n_k} = O\{(n_1^2 + \dots + n_k^2)^{-\alpha}\}, \tag{4.52}$$

where

$$\alpha = (k/2) - \tfrac{1}{2}\{\theta(1+\beta)/(\delta-\beta)\},\ \beta \geqslant 0,\ \delta > \tfrac{1}{2}(k-1) + \theta,$$

then

$$S_R^\beta(x) - l = o(1).$$

PROOF. Hypothesis (4.51) implies, in virtue of Lemma 4.53, that

$$S_R^\delta(x) - l = o(R^{-\theta}), \tag{4.53}$$

for $\theta > 0$ and $\delta > (k-1)/2 + \theta$. Again, for $t = O(R)$,

$$|S\{(R+t)^{1/2}\} - S(R^{1/2})| \leqslant \sum_{R<n\leqslant R+t} |A_n(x)|$$

$$= \sum_{R<n\leqslant R+t} |a_{n_1\dots n_k}|$$

$$= O\left[\sum\left\{(n_1^2 + \dots + n_k^2)^{-a}\right\}\right]$$

$$= O\left(\sum_{R<n<R+t} r_k(n)\, n^{-a}\right)$$

$$= O\left(\int_R^{R+t} x^{-a}\, d\, R_k(x)\right)$$

$$= O\left[t\, R^{k/2-a-1}\right], \qquad (4.54)$$

by partial integration and an application of the second part of Lemma 4.52.

Now by applying Theorem 1.81 we deduce that

$$S_R^\beta(x) - l = o\left[R^{\frac{2}{\delta+1}\left\{\delta\beta + (\delta-\beta)\left(\frac{k}{2} - a\right) + \beta\left(1 - \frac{\theta}{2}\right) - \frac{\theta}{2}\right\} - 2\beta}\right]$$

for $0 \leq \beta < \delta$. The exponent of R will be zero if

$$(\delta - \beta)\,(k/2 - a) = \tfrac{1}{2}\,\theta\,(1 + \beta),$$

which will certainly hold if

$$a = \frac{k}{2} - \frac{\theta(1+\beta)}{2(\delta - \beta)}.$$

REMARKS. (i). In (4.51) and (4.52) the o and O can be interchanged.

(ii). The result holds for $\theta = 0$ because, in that case, Theorem 4.52 will take the place of Lemma 4.53.

(iii). Given any $\theta > 0$, for $a = k/2 - \theta/2\gamma$, where γ is some number exceeding $(k-1)/2 + \theta$, we have ordinary convergence at x.

(iv). If $\theta = 0$ and $a = k/2$, then too we have convergence at x.

4.6. Derived Fourier series

In this section we shall consider the summability of series obtained by successively applying the Laplacian to a given Fourier series. We shall actually prove a generalization of Theorem 4.52. The proof is based on the idea that we can differentiate Bochner's formula.

Conforming to the notation $x = (x_1, \ldots, x_k)$, we set

$$|x| = [x_1^2 + \ldots + x_k^2]^{1/2}.$$

We shall say that $f(x) = f(x_1, \ldots, x_k)$ *has a zero at the point* x if

$$\lim_{|\xi| \to 0} |\xi|^{-k} \int \ldots \int_{E_\xi} |f(x_1 + \xi_1, \ldots, x_k + \xi_k)| \, d\xi_1 \ldots d\xi_k = 0,$$

where E_ξ denotes the k-dimensional sphere of radius $|\xi|$ with the origin as centre. If q is a non-negative integer, we shall say that $f(x)$ *has a* q*-fold zero at* x, if

$$\lim_{|\xi| \to 0} |\xi|^{-k} \int \ldots \int_{E_\xi} |f(x + \xi)| \, . \, |\xi|^{-q} \, d\xi_1 \ldots d\xi_k = 0.$$

If we write the spherical mean of $|f(x)|$ as $\bar{f}_x(t)$ then this condition amounts to saying that

$$\lim_{\varepsilon \to 0} \varepsilon^{-k} \int_0^\varepsilon \bar{f}_x(t) \, t^{k-1-q} \, dt = 0.$$

THEOREM 4.61. *For every point x at which*

$$\lim_{\varepsilon \to 0} \varepsilon^{-k} \int_0^\varepsilon \bar{f}_x(t) \, t^{k-1-2q} \, dt = 0 ,$$

we have

$$\lim_{R \to \infty} \Delta_x^q \, S_R^\delta(x) = 0, \quad \delta > 2q + (k-1)/2,$$

where q is a non-negative integer, and

$$\Delta_x^q = \left(- \frac{\partial^2}{\partial x_1^{\,2}} - \ldots - \frac{\partial^2}{\partial x_k^{\,2}} \right)^q .$$

PROOF. We have

$$S_R^\delta(x) = c_1 \, R^k \int_0^\infty f_x(t) \, t^{k-1} \, V_{\delta+k/2} \, (tR) \, dt.$$

Set

$$c_1 \, R^{k-1} \, t^{k-1} \, V_{\delta+k/2} \, (tR) = H_\delta \, (tR).$$

Then

$$S_R^\delta(x) = R \int_{-\infty}^{\infty} \cdots \int f(x+\xi) \frac{H_\delta(|\xi|R)}{|\xi|^{k-1}} \, d\xi_1 \ldots d\xi_k$$

$$= R \int_{-\infty}^{\infty} \cdots \int f(\xi) \frac{H_\delta(|x-\xi|R)}{|x-\xi|^{k-1}} \, d\xi_1 \ldots d\xi_k . \qquad (4.61)$$

Now, by a change of variable, we see that

$$R \int_{-\infty}^{\infty} \cdots \int f(\xi) \, \Delta_x^q \frac{H_\delta(|x-\xi|R)}{|x-\xi|^{k-1}} \, d\xi_1 \ldots d\xi_k \qquad (4.62)$$

$$= R \int_{-\infty}^{\infty} \cdots \int f(x+\xi) \, \Delta_\xi^q \left[\frac{H_\delta(|\xi|R)}{|\xi|^{k-1}} \right] d\xi_1 \ldots d\xi_k . \qquad (4.63)$$

Further

$$\Delta_\xi^q \frac{H_\delta(|\xi|R)}{|\xi|^{k-1}} = \begin{cases} O(R^{k-1+2q}) \ , \ \text{as} \ |\xi| \to 0, \\ O(|\xi|^{-k-2q-\varepsilon} R^{-1-\varepsilon}) \ , \ \text{as} \ |\xi| \to \infty, \end{cases} \qquad (4.64)$$

for some $\varepsilon > 0$, since for a function $f(x_1, \ldots, x_k)$ whose value depends only on the radius $|x| = r$, we have

$$\Delta_x^q f = \left[\frac{d^2}{dr^2} + \frac{(k-1)}{r} \frac{d}{dr} \right]^q f,$$

and we have then only to use (4.36) and (4.33). Using (4.64) in (4.63), we easily obtain

$$\left| R \int \cdots \int f(x+\xi) \, \Delta_\xi^q \frac{H_\delta(|\xi|R)}{|\xi|^{k-1}} \, d\xi_1 \ldots d\xi_k \right| \qquad (4.65)$$

$$\leqslant c_3 \left[R^{k+2q} \int_0^{1/R} \bar{f}_x(t) \, t^{k-1} \, dt + R^{-\varepsilon} \int_{1/R}^{\infty} \frac{\bar{f}_x(t)}{t^{1+2q+\varepsilon}} \, dt \right]$$

$$= I_1 + I_2,$$

say. We shall show that (4.65) is finite for each fixed R, and tends to zero as $R \to \infty$, provided that $\delta > (k-1)/2 + 2q$. For

$$|I_1| = O\left[R^k \int_0^{1/R} \bar{f}_x(t) \, t^{k-1-2q} \, dt \right]$$

$$= o(1), \text{ as } R \to \infty,$$

and

$$I_2 = \left[\int_{1/R}^{\eta} + \int_{\eta}^{\infty} \right] = I_{2,1} + I_{2,2},$$

say, where η is chosen so small that in the interval $(0, \eta)$ the hypothesis on $\bar{f}_x(t)$ operates. Now

$$I_{2,1} = R^{-\varepsilon} \left[\left(\frac{\bar{F}(t)}{t^{k+\varepsilon}} \right)_{1/R}^{\eta} + c \int_{1/R}^{\eta} \frac{\bar{F}(t)}{t^{k+\varepsilon+1}} dt \right],$$

where

$$\bar{F}(t) \equiv \int_0^t \bar{f}_x(s)\, s^{k-1-2q}\, ds < \varepsilon\, t^k,$$

for $1/R \leqslant t \leqslant \eta$. Hence

$$I_{2,1} = o(1),$$

as $R \to \infty$. In $I_{2,2}$ we use (4.25) with $\bar{f}_x(t)$ in place of $f_x(t)$, and by an argument similar to the one employed in Lemma 4.51, prove that

$$I_{2,2} = o(1)\ ,\ \text{as } R \to \infty.$$

Hence (4.62) is finite for each R, and from (4.61) we see that it is equal to $\Delta_x^q\, S_R^\delta(x)$. We have further proved that this tends to zero as $R \to \infty$ for $\delta > (k-1)/2 + 2q$, which completes the proof of the theorem.

We shall now show that given any function $f(x) \in L_1$ which is periodic, if it is differentiable in a neighbourhood of a given point, we can then subtract an exponential polynomial from it such that the difference has all its derivatives equal to zero at the point near which differentiability is assumed. We need the following

LEMMA 4.61. *If $k \geqslant 1$, $0 \leqslant n < \infty$, and if the numbers $a_{n_1 \ldots n}$ are arbitrarily given for $0 \leqslant n_1 \leqslant n$, $0 \leqslant n_2 \leqslant n, \ldots, 0 \leqslant n_k \leqslant n$, then there exists an exponential polynomial*

$$P(x_1, \ldots, x_k) = \sum_{r_1=0}^{n} \cdots \sum_{r_k=0}^{n} c_{r_1 \cdots r_k}\, e^{i \Sigma r_j x_j}$$

such that

$$\left(\frac{\partial^{n_1+\dots+n_k} P}{\partial x_1^{n_1}\dots\partial x_k^{n_k}}\right)_{x=(0)} = a_{n_1\dots n_k}.$$

PROOF. If arbitrary numbers $b_{n_1,\dots,n_k}$, $0 \leqslant n_j \leqslant n, j = 1,\dots, k$, are given, then there exists an ordinary polynomial

$$Q(y_1, \dots, y_k) = \sum_{s_1=0}^{n} \dots \sum_{s_k=0}^{n} d_{s_1\dots s_k} y_1^{s_1}\dots y_k^{s_k}$$

such that

$$\left(\frac{\partial^{n_1+\dots+n_k} Q}{\partial y_1^{n_1}\dots\partial y_k^{n_k}}\right)_{y=(0)} = b_{n_1,\dots,n_k},$$

namely the one with $d_{n_1\dots n_k}\cdot n_1!\dots n_k! = b_{n_1\dots n_k}$. Now the transformation

$$y_1 = e^{ix_1} - 1, \dots, y_k = e^{ix_k} - 1$$

transforms a $P(x)$ into a $Q(y)$ and conversely, under preservation of n, and for assigned values of $a_{n_1\dots n_k}$ this leads to values $b_{n_1\dots n_k}$, and inversely from the b's to the a's, and hence the lemma.

REMARK. By a suitable change of co-ordinates, it can be seen that the lemma holds at any point x, not necessarily the origin.

THEOREM 4.62. *If $f(x)$ (defined as in §4.2) has continuous derivatives of total order $\leqslant 2q$ in a neighbourhood of the point $x = x_0$, then at that point we have*

$$\lim_{R\to\infty} [\Delta_x^q S_R^\delta(x) - \Delta_x^q f(x)] = 0$$

for $\delta > (k-1)/2 + 2q$.

PROOF. The conclusion is trivial when $f(x)$ is an exponential polynomial $P(x)$. In the general case, on account of Lemma 4.61, we may write $f(x) = P(x) + \varphi(x)$, where for $\varphi(x)$ all partial derivatives of total order $\leqslant 2q$ are zero at the point $x = x_0$. But $\varphi(x)$ has also continuous derivatives of order $2q$ in a neighbourhood of x, therefore it satisfies the condition of Theorem 4.61, from which the conclusion follows.

We shall next prove a theorem where the hypotheses are of a composite nature, consisting of 'local' differentiability of the function as in Theorem 4.62, and a restriction on the order of magnitude of the Fourier coefficients.

THEOREM 4.63. *If $A_n = O(n^a)$, then at a point x in a neighbourhood of which $f(x)$ possesses partial derivatives of all orders, the series $\Sigma A_n n^h$ is summable (n, δ), for $\delta \geqslant 0$ and $\delta > 2a + 1 + 2h$.*

PROOF. By Theorem 4.62, we find that $\Sigma A_n n^q$ is summable (n, δ) for $\delta > (k-1)/2 + 2q$, where q is a non-negative integer. Since $A_n n^q = O(n^{a+q})$, it follows from Corollary 3.71 that $\Sigma A_n n^h$ is summable (n, η) for $\eta \geqslant 0$, and

$$q - h > \frac{(a+q+1)(\delta-\eta)}{\delta+1},$$

or

$$\eta > \delta - \frac{(q-h)(\delta+1)}{a+q+1}.$$

Since δ may be any number greater than $(k-1)/2 + 2q$, this implies that any

$$\eta > \frac{\{(k-1)/2\}(a+1+h) + h + 2q(a+\frac{1}{2}+h)}{a+q+1}$$

is admissible. Given k, a, h, since q may be chosen as large as we please, the theorem is true for $\eta > 2a + 1 + 2h$.

4.7. Summations over lattice points

Let

$$r_k(n) = \sum_{n_1^2+\ldots+n_k^2=n} 1$$

for integral values of n_k; representations of n which differ only in sign or order being counted as distinct. Let

$$\bar{R}_k(x) = \sum_{n\leqslant x}' r_k(n),$$

the dash denoting that the last term should be replaced by $\frac{1}{2} r_k(x)$ if x is an integer. It is well known that $\bar{R}_k(x)$ can be represented as a series of Bessel functions. In the case $k = 2$, the following identity is classical:

$$\bar{R}_2(x) = \pi x - x^{1/2} \sum_{n=1}^{\infty} \frac{r_2(n) J_1[2\pi\sqrt{(nx)}]}{n^{1/2}}. \tag{4.71}$$

For general k, the series representing $\bar{R}_k(x)$ is no longer convergent but summable. Thus it is known that the series

$$\Sigma\, r_k(n) J_{k/2}[2\pi\sqrt{(nx)}]\, n^{-k/4} \tag{4.72}$$

is summable (n, δ) for $\delta > (k-3)/2$, and *not* summable for $\delta = (k-3)/2$. We shall show that the above expansion is a (spherical) multiple Fourier Series of a simple function at the origin, and then apply the foregoing theorems to study its summability. For this, we need the following lemmas.

LEMMA 4.71. *For $\beta > -1$, let*

$$g(x_1, \ldots, x_k) = \begin{cases} \left[\xi^2 - \left(\sum_1^k x_j^2\right)\right]^\beta, \text{ if } \sum x_j^2 < \xi^2, \\ 0, \text{ if } \sum x_j^2 \geqslant \xi^2. \end{cases}$$

Let

$$f(x_1, \ldots, x_k) = \sum \underset{p_k=-\infty}{\overset{\infty}{\cdots}} \sum g(x_1 + p_1, \ldots, x_k + p_k) \tag{4.73}$$

where $\{p_k\}$ are integers. Then $f(x_1, \ldots, x_k)$ is a periodic function with period 1 in each variable, which belongs to L_1, and its Fourier coefficients are given by

$$a_{n_1 \ldots n_k} = \frac{\xi^{\beta+k/2}\, \Gamma(\beta+1)\, \pi^{-\beta} J_{\beta+k/2}(2\pi\xi\sqrt{n})}{(n_1^2 + \ldots + n_k^2)^{\beta/2+k/4}} \tag{4.74}$$

and

$$a_{0,\ldots,0} = \frac{\xi^{2\beta+k}\, \Gamma(\beta+1)\, \pi^{k/2}}{\Gamma(\beta+1+k/2)}.$$

PROOF. We have

$$a_{n_1\ldots n_k} = \int_{-1/2}^{1/2} \cdots \int f(x_1, \ldots, x_k)\, e^{-2\pi i \Sigma n_j x_j} dx_1 \ldots dx_k$$

$$= \int_{-\infty}^{\infty} \cdots \int g(x_1, \ldots, x_k)\, e^{-2\pi i \Sigma n_j x_j} dx_1 \ldots dx_k$$

$$= (2\pi)^{k/2} \int_0^\xi (\xi^2 - t^2)^\beta t^{k-1} V_{k/2-1}(2\pi t\sqrt{n})\,dt.$$

We now obtain the required result if we use (4.34).

We observe that the function in (4.73) is differentiable in a neighbourhood of the origin, and its Fourier series is therefore summable (n, δ) for $\delta > (k-1)/2$ at the origin. We can write down the series by using (4.74), and by an appeal to Theorem 4.52, obtain

THEOREM 4.71. *If ξ^2 is non-integral,*

$$\sum_{n\leqslant\xi} (\xi^2 - n^2)^\beta \, r_k(n)$$

$$= \lim_{R\to\infty} \left[c_1 + c_2 \sum_{n\leqslant R^2} \left(1 - \frac{n}{R^2}\right)^{(k-1)/2+\varepsilon} \frac{r_k(n)\, J_{\beta+k/2}(2\pi\xi\sqrt{n})}{n^{\beta/2+k/4}} \right].$$

This theorem does not yield the identity (4.71) for $k = 2$, because the series on the right has been proved to be only summable (n, δ) for $\delta > \frac{1}{2}$. We can, however, obtain a result which does yield the identity as a special case, if we appeal to Theorem 4.63, by noting that

$$r_k(n) = O\{n^{(k-2)/2+\varepsilon}\},$$

as in Lemma 4.52.

THEOREM 4.72. *If ξ^2 is non-integral, then*

$$\sum r_k(n)\, J_{k/2+\beta}\, (2\,\pi\,\xi\,\sqrt{n}) n^l$$

is summable (n, η) for $\eta \geqslant 0$ and $l < 3/4 - k/2 + \eta/2$, $\beta > -1$.

PROOF. If ξ^2 is non-integral, the function $f(x_1,\dots,x_k)$ defined in (4.73) is infinitely differentiable in a neighbourhood of the origin, and its Fourier coefficients satisfy the condition

$$A_n = O\{n^{(k-2)/2+\varepsilon-k/4-\beta/2-1/4}\}$$
$$= O\{n^{(k-5)/4-\beta/2+\varepsilon}\},$$

in the notation of (4.21). Applying Theorem 4.63 now, we find that

$$\sum \frac{r_k(n)\, J_{k/2+\beta}\, (2\,\pi\,\xi\,\sqrt{n}) n^p}{n^{k/4+\beta/2}}$$

is summable (n, η) for $\eta > (k - 3)/2 - \beta + 2\,p$. Setting $l = p - k/4 - \beta/2$, we observe that $\sum r_k(n) J_{k/2+\beta}(2\,\pi\,\xi\,\sqrt{n})\, n^l$ is summable (n, η) for $\eta > 2\,l + k - 3/2$.

4.8. Absolute summability

In this section we shall consider the analogues, for absolute summability, of some of the foregoing theorems. If we aim at summability $|n, \delta|$ for $\delta > (k - 1)/2$ at a point x, we have to impose conditions on $f_x(t)$ in the entire interval $0 < t < \infty$; if we consider the case $\delta > (k + 1)/2$, however, we can prove that summability $|n, \delta|$ is a *local* property.

THEOREM 4.81. *For a fixed x, if we have*

$$\int_0^\infty |df_x(t)| = O(1),$$

then

$$\int_0^\infty |dS_R^\delta(x)| = O(1), \textit{ for } \delta > (k-1)/2.$$

PROOF. Setting
$$\psi(u) = u^{k-1} V_{k/2+\delta}(u),$$
and
$$\psi_1(u) = \int_0^u \psi(v)\, dv,$$
we find, for $\delta > (k-1)/2$, that
$$\psi(u) = O(u^{-1-\varepsilon}), \qquad \varepsilon > 0, \tag{4.81}$$
as $u \to \infty$. Now
$$\begin{aligned} c_1 S_R^\delta(x) &= R \int_0^\infty \psi(tR) f_x(t)\, dt = \int_0^\infty f_x(t)\, d\psi_1(tR) \\ &= \Big[f_x(t)\, \psi_1(tR) \Big]_0^\infty - \int_0^\infty \psi_1(tR)\, df_x(t) \end{aligned}$$
so that, by (4.37),
$$\begin{aligned} c_1[S_R^\delta(x) - f_x(+\infty)] &= -\int_0^\infty df_x(t) \int_0^{tR} \psi(u)\, du \\ &= -\int_0^\infty df_x(t) \int_0^R t\, \psi(ut)\, du \\ &= -\int_0^R du \int_0^\infty t\, \psi(ut)\, df_x(t), \end{aligned} \tag{4.82}$$
the last step being justified by (4.81) and the hypothesis on $f_x(t)$. Substituting for ψ, we get
$$c \int_0^\infty |dS_R^\delta(x)| < \int_0^\infty u^{k-1}\, du \left| \int_0^\infty t^k V_{\delta+k/2}(ut)\, df_x(t) \right|$$

$$\leqslant \int_0^\infty u^{k-1}\,du \int_0^\infty t^k\,|V_{k/2+\delta}(ut)|\,.\,|df_x(t)|$$

$$= \int_0^\infty t^k\,|df_x(t)|\,.\,\int_0^\infty u^{k-1}\,|V_{k/2+\delta}(ut)|\,du,$$

which is justified since $\delta > (k-1)/2$ (in view of (4.33)).

If

$$I = \int_0^\infty u^{k-1}|V_{k/2+\delta}(ut)|\,du$$

$$= \left[\int_0^{1/t} + \int_{1/t}^\infty\right] \equiv I_1 + I_2,$$

say, we have

$$I_1 = O\left[\int_0^{1/t} u^{k-1}\,du\right] = O(t^{-k}),$$

and

$$I_2 = O\left[\frac{1}{t^{k/2+\delta+1/2}}\int_{1/t}^\infty u^{k/2-\delta-3/2}\,du\right] = O(t^{-k}),$$

so that

$$I = O\,(t^{-k}),$$

and hence

$$\int_0^\infty |dS_R^\delta(x)| = O\left[\int_0^\infty |df_x(t)|\right] = O(1).$$

Remark. The integral $\int_0^\infty |df_x(t)|$ should be interpreted as

$$\lim_{\varepsilon\to+0}\int_\varepsilon^\infty |df_x(t)|.$$

We next prove the analogue for the q-th Laplacian of $S_R^\delta(x)$, where q is a non-negative integer, as before.

THEOREM 4.82. *If*

$$\int_0^\infty |df_x(t)|\, t^{-2q} = O(1),$$

then

$$\int_0^{\sigma} |d_R\, \Delta_x^q\, S_R^\delta(x)| = O(1),$$

for $\delta > (k-1)/2 + 2q$.

PROOF. As in the proof of Theorem 4.61, we use the relation

$$\Delta_x^q S_R^\delta(x) = R \int_{-\infty}^{\infty} \cdots \int f(x+\xi)\, \Delta_\xi^q \frac{H_\delta(|\xi| R)}{|\xi|^{k-1}}\, d\xi_1 \dots d\xi_k .$$

We write

$$\Delta_\xi^q \frac{H_\delta(|\xi| R)}{(|\xi| R)^{k-1}} = P(|\xi| R),$$

and

$$s^{k-1} P(s) = Q(s),$$

so that

$$\Delta_x^q\, S_R^\delta(x) = R \int_0^\infty f_x(t)\, Q(tR)\, dt.$$

Using the estimate

$$Q(s) = O(s^{-2q-\varepsilon-1}),\ \varepsilon > 0,$$

as $s \to \infty$, which is obtained from (4.64), and proceeding as in (4.82), we obtain

$$\int_0^\infty |d_R\, \Delta_x^q\, S_R^\delta(x)| \leqslant \int_0^\infty s\,|df_x(s)| \int_0^\infty |Q(sR)|\, dR$$

$$= O\left[\int_0^\infty s^{-2q} |df_x(s)|\right] = O(1),$$

since

$$\int_0^\infty |Q(sR)|\, dR = O(s^{-2q-1}),$$

which can be seen as follows :

$$\int_0^{1/s} |Q(sR)|\, dR = O\left[\int_0^{1/s} s^{k-1} R^{k-1+2q}\, dR\right] = O(s^{-2q-1}),$$

and

$$\int_{1/s}^{\infty} |Q(sR)|\, dR = O\left[\int_{1/s}^{\infty} s^{-2q-\varepsilon-1} R^{-\varepsilon-1}\, dR\right] = O(s^{-2q-1}),$$

on account of the estimate (4.64).

A remark similar to the one made at the end of the previous theorem applies here as well. We shall next prove that summability $|n, \delta|$ of a sufficiently high order is a *local* property.

THEOREM 4.83. *Riesz summability* $|n, \delta|$ *for* $\delta > (k+1)/2$ *is a 'local' property for Fourier series.*

PROOF. Let

$$f^1(r) = \frac{2\,\Gamma(k/2+1)}{\Gamma(k/2)r^k} \int_0^r t^{k-1} f_x(t)\, dt.$$

Integrating Bochner's formula, given in (4.43), by parts once, and using the fact that $f^1(r) = O(r^k)$, as $r \to \infty$, we obtain

$$\begin{aligned} S_R^\delta(x) &= c_1 R^k \int_0^\infty t^{k-1} f_x(t)\, V_{\delta+k/2}(tR)\, dt \\ &= c_2 R^{k+2} \int_0^\infty t^{k+1} f_x^1(t)\, V_{\delta+k/2+1}(tR)\, dt, \end{aligned}$$

provided that $\delta > (k-1)/2$. In the same manner as in (4.82), we show that

$$\begin{aligned} \int |dS_R^\delta(x)| &= \int dR \left| R^{k+1} \int_0^\infty t^{k+2} V_{k/2+\delta+1}(tR)\, df_x^1(t) \right| \\ &= \int dR \left| R^{k+1} \left[\int_0^\eta + \int_\eta^\infty\right] \right| \\ &= \int dR\, |I_1 + I_2|, \text{ say.} \end{aligned} \tag{4.83}$$

We show that

$$\int |I_2|\, dR < \infty, \text{ for } \delta > (k+1)/2. \tag{4.84}$$

We first observe that

$$\frac{d}{dt}\{f_x^1(t)\} = \frac{1}{t}\{c_3 f_x(t) + c_4 f_x^1(t)\};$$

using this in I_2, we have

$$I_2 = c_1 R^{k+1} \int_\eta^\infty t^{k+1} f_x(t)\, V_{k/2+\delta+1}(tR)\, dt$$

$$+ c_2 R^{k+1} \int_\eta^\infty t^{k+1} f_x^1(t)\, V_{k/2+\delta+1}(tR)\, dt$$

$$\equiv I_{2,1} + I_{2,2},$$

say. Now

$$|I_{2,2}| = O\left(\frac{1}{R^{\delta-1-k/2+3/2}}\right) \int_\eta^\infty \frac{dt}{t^{\delta-1-k/2+3/2}}$$

$$= O\left(\frac{1}{R^{\delta-1-k/2+3/2}}\right), \tag{4.85}$$

if $\delta > (k+1)/2$, since $f_x^1(t) = O(1)$ as $t \to +\infty$. And

$$|I_{2,1}| = O\left(\frac{1}{R^{\delta-1-k/2+3/2}}\right), \tag{4.86}$$

if $\delta > (k+1)/2$, as in the proof of Lemma 4.51. (4.85) and (4.86) prove (4.84); if the latter is used in (4.83), we see that a necessary and sufficient condition for the validity of $\int |dS_R^\delta| = O(1)$ is that $\int |I_1|\, dR = O(1)$, which proves the theorem.

4.9. Necessary and sufficient conditions for summability

In the foregoing sections we have considered various conditions which are *sufficient* for the summability (ordinary or absolute) of a Fourier series at a given point. It is possible similarly to prove that certain conditions are *necessary* for the summability of a Fourier series at a given point. To demonstrate this, we consider the spherical mean $f_x(t)$ as the mean of order zero, and define spherical means of higher order in analogy with Riesz means of series. We then prove that if the Fourier series is summable to a sum l at a point, then the spherical mean, of some order, of the function tends to l as $t \to 0$. We prove the converse proposition by a slight extension of Theorem 4.52. Combining the two, we state a necessary and sufficient condition for the summability of the series at a point. We next prove the analogue for absolute summability.

We define

$$\varphi_{x,p}(r) = \frac{1}{2^{p-1}\Gamma(p)} \int_0^r (r^2 - t^2)^{p-1} t^{k-1} f_x(t)\, dt,\ p > 0,$$

$$f_{x,p}(r) = \frac{2^p\,\Gamma(p)}{\mathrm{B}(p, \frac{1}{2}k)} \cdot \frac{\varphi_{x,p}(r)}{r^{k+2p-2}}, \quad p > 0,$$

$$f_{x,0}(r) = f_x(r), \quad \text{as in } §4.2.$$

It is easily verified that

$$\varphi_{x,p+q}(r) = \frac{1}{2^{q-1}\Gamma(q)} \int_0^r (r^2 - t^2)^{q-1} t\varphi_{x,p}(t)\, dt, \tag{4.91}$$

if $p + q \geqslant 1$, and hence it follows that

$$f_{x,p+q}(r) = \frac{2\Gamma(p + k/2 + q)}{\Gamma(p+k/2)\,\Gamma(q)\, r^{2(p+q)+k-2}} \int_0^r (r^2-t^2)^{q-1} t^{2p+k-1} f_{x,p}(t)\, dt. \tag{4.92}$$

It is easy to see that

$$f_{x,p}(t) = O(1) \tag{4.93}$$

for $p \geqslant 1$, as $t \to \infty$. For

$$|f_{x,1}(t)| = O\left[\frac{1}{t^k}\int_0^t u^{k-1}|f_{x,0}(u)|\, du\right]$$
$$= O(1),$$

by (4.2). Using this in formula (4.92), we obtain (4.93).

We now generalize Bochner's formula (4.43), so as to admit spherical means of order greater than zero.

THEOREM 4.91. *If p is a positive integer, then*

$$S_R^\delta(x) = c_2 R^{k+2p} \int_0^\infty t^{k+2p-1} f_{x,p}(t)\, V_{\delta+p+k/2}(tR)\, dt, \tag{4.94}$$

provided that $\delta > p + (k-3)/2$. If $p = 0$, we require $\delta > (k-1)/2$.

PROOF. We have only to integrate by parts, $(p-1)$ times, the right side of the formula

$$S_R^\delta(x) = c_1 R^k \int_0^\infty t^{k-1} f_{x,0}(t)\, V_{\delta+k/2}(tR)\, dt,$$

each time using

(i) $$\varphi_{x,p+1}(t) = \int_0^t s\varphi_{x,p}(s)\, ds,$$

(ii) $f_{x,p}(t) = O(1)$, as in (4.93), and

(iii) $\dfrac{d}{dx}[\,V_\mu(x)\,] = -\,x V_{\mu+1}(x)$, as in (4.36).

From Theorem 4.91 we deduce the following

THEOREM 4.92. *If, at a point x, we have*

$$\lim_{t\to 0} f_{x,p}(t) = l,$$

for a non-negative integer p, then at that point

$$\lim_{R\to\infty} S_R^\delta(x) = l,\ \delta > p + (k-1)/2.$$

The proof of this theorem follows from Theorem 4.91, in the same manner as the proof of Theorem 4.52 follows from that of Theorem 4.41.

Again we have a result on absolute summability.

THEOREM 4.93. *If*

$$\int_0^\infty |df_{x,p}(t)| = O(1),$$

for a non-negative integer p, then

$$\int_0^\infty |dS_R^\delta(x)| = O(1),\ \delta > p + (k-1)/2.$$

The proof of this theorem follows from that of Theorem 4.91, in the same manner as the proof of Theorem 4.81 follows from that of Theorem 4.41.

We now proceed to state some converse theorems. For this purpose we first need a formula which is a 'converse' of (4.94).

Theorem 4.94. *If $p > 1$, and $\delta > (k-1)/2$, then*

$$f_{x,p}(y) = c_3 y^{2\delta+2} \int_0^\infty S_R^\delta(x)\; R^{2\delta+1}\; V_{p+\delta+k/2}(yR)\, dR. \qquad (4.95)$$

Proof. From the definition of S_R^δ as

$$S_R^\delta(x) = \sum_{r=0}^{n} \left(1 - \frac{r}{R^2}\right)^\delta A_r(x),\; n \leqslant R^2 < n+1,$$

and the estimate for $r_k(n)$ in Lemma 4.52, it follows that

$$S_R^\delta(x) = o(R^k), \text{ as } R \to \infty,$$

if we note that the Fourier coefficients of a function in L_1 tend to zero. Hence the integral

$$I = y^{2\delta+2} \int_0^\infty R^{2\delta+1}\, S_R^\delta(x)\; V_\gamma(Ry)\, dR \qquad (4.96)$$

converges for $\gamma > 2\,\delta + 3/2$. Hence, using formula (4.94) in the case $p = 0$, we get

$$I = c_4\, y^{2\delta+2} \int_0^\infty R^{2\delta+k+1}\, V_\gamma(Ry)\, dR \int_0^\infty t^{k-1} f_{x,0}(t)\;\; V_{\delta+k/2}(tR)\, dt$$

$$= \frac{c_4}{y^{\gamma-2\delta-2}} \int_0^\infty t^{k/2-\delta-1} f_{x,0}(t)\, dt \int_0^\infty R^{k/2+\delta+1-\gamma} J_{\delta+k/2}(tR)\, J_\gamma(yR)\, dR,$$

as the double integral is absolutely convergent. Hence, by (4.35),

$$I = \frac{c_5}{y^{2\gamma-2\delta-2}} \int_0^y t^{k-1} f_{x,0}(t)\, (y^2 - t^2)^{\gamma-\delta-k/2-1}\; dt$$

$$= c_5 f_{x,p}\,(y),$$

where $p = \gamma - \delta - k/2$, $\gamma > 2\,\delta + 3/2$, $\delta > (k-1)/2$, which proves the theorem.

Remark. Here p need not be an integer.

Using formula (4.95), we now prove some converse theorems.

Theorem 4.95. *If $S_R^\gamma(x) \to l$ as $R \to \infty$, then $f_{x,p}(y) \to l$ as $y \to 0$, provided that*

$$p > \max\left\{1,\; \gamma - (k-3)/2\right\}.$$

PROOF. We may assume that $l = 0$, without loss of generality, on account of (4.37). If $\gamma > (k - 1)/2$, we choose $\delta = \gamma$; if $\gamma \leqslant (k - 1)/2$, we choose $\delta = (k - 1)/2 + \theta$, $\theta > 0$. With this choice of δ, we make use of formula (4.95). We may write the integral I from the previous theorem as

$$I = cy^{2\delta+2}\left[\int_0^\omega + \int_\omega^\infty\right] = I_1 + I_2,$$

say, and obtain

$$|I_1| = O\left[\int_0^{\omega y} z^{2\delta+1}\frac{|J_{p+\delta+k/2}(z)|}{z^{p+\delta+k/2}}\,dz\right]$$
$$= o(1),$$

as $y \to 0$, and

$$|I_2| = o\left[\,y^{2\delta+2}\int_\omega^\infty R^{2\delta+1}|\,V_{p+\delta+k/2}(yR)|\,dR\,\right]$$
$$= o\left[\int_{y\omega}^\infty z^{2\delta+1}\frac{|J_{p+\delta+k/2}(z)|}{z^{p+\delta+k/2}}\,dz\right]$$
$$= o(1),$$

if $p > \delta - (k-3)/2$. Thus $I = o(1)$, for $p > \max\{1, \gamma - (k-3)/2\}$, and the theorem is proved.

Analogously we obtain

THEOREM 4.96. *If*

$$\int_0^\infty |dS_R^\delta(x)| = O(1),$$

then

$$\int_0^\infty |df_{x,p}(t)| = O(1),$$

for $p > \max\{1, \delta - (k-3)/2\}$.

PROOF. We have only to use the formula in Theorem 4.95, and adopt the same argument as in Theorem 4.81.

Combining Theorems 4.92 and 4.95, we obtain

THEOREM 4.97. *A necessary and sufficient condition that a multiple Fourier series of a function $f(x)$ should be summable at a point x is that the spherical mean $f_{x,p}(t)$, of some order p, of the function should have a finite limit as $t \to 0$.*

Combining Theorems 4.93 and 4.96, we obtain

THEOREM 4.98. *A necessary and sufficient condition that the multiple Fourier series of a function $f(x)$ should be absolutely summable at a point x is that the spherical mean $f_{x,p}(t)$ should satisfy the condition*

$$\int |df_{x,p}(t)| = O(1)$$

for some p.

NOTES ON CHAPTER IV

§4.1. Throughout this chapter we confine attention to the *spherical* summation of multiple Fourier series, since it affords a good illustration of the use of typical means, and facilitates a unified treatment.

The literature that exists in the case $k = 1$ is so vast that, for the sake of convenience, we give only the most obvious references. Fuller information can be obtained from the references contained in the papers cited hereunder, and from A. Zygmund, *Trigonometrical series*, Warsaw (1935).

§4.2. For the spherical summation of multiple Fourier series and integrals see S. Bochner, *Trans. American Math. Soc.* 40 (1936), 175-207.

§4.3. For the properties of Bessel functions see G. N. Watson, *A treatise on Bessel functions*, Cambridge (1922). In particular, for (4.34) see p. 372, and for (4.37) see p. 392. For (4.35) see E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford (1937), 183.

§4.4. For formula (4.41) see S. Bochner, loc. cit. For an alternative approach see S. Minakshisundaram, *American Jour. Math.* 71 (1949), 60-66. For further generalizations see K. Chandrasekharan, *Proc. London Math. Soc.* 50 (1948), 210-229. Also see S. Bochner and K. Chandrasekharan, *American Jour. Math.* 71 (1949), 50-59.

§4.5. Theorem 4.52 ceases to hold if $\delta = (k-1)/2$, and this value of δ is therefore called the critical exponent. If $k > 1$, S. Bochner has shown (loc. cit., 193) that there exists a periodic and Lebesgue-integrable function $f(x)$ which vanishes in a neighbourhood of the origin, such that lim sup $S_R^{(k-1)/2}(0) = \infty$. For further information on the critical case see S. Bochner and K. Chandrasekharan, *Annals of Math.* 49 (1948), 966-978.

For Lemma 4.52 see A. Walfisz, *Math. Zeitschrift*, 19 (1924), 300-307.

For Lemma 4.53 see K. Chandrasekharan, *Proc. London Math. Soc.* 50 (1948), 219. For the original version of Theorem 4.53 with $\beta = 0$, $k = 2$, see K. Chandrasekharan and S. Minakshisundaram, *Duke Math. Journal* 14 (1947), 731-753.

If $k = 1$, Theorem 4.51 is true for $\delta = (k-1)/2$ since convergence is known to be a 'local' property for ordinary Fourier series. See B. Riemann, *Ges. Werke*, Aufl. 2, Leipzig (1892), 227-271. Theorem 4.52 yields, in the case $k = 1$, a generalization of Fejér's theorem, see G. H. Hardy, *Proc. London Math. Soc.* 12 (1913), 365-372. For the case $k = 1$ of Lemma 4.53 see N. Obrechkoff, *Bull. Soc. Math.* 62 (1934), 84-109.

§4.6. For the summability of derived (multiple) Fourier Series see S. Bochner, *Annals of Math.* 37 (1936), 345-356. For Lemma 4.61 and Theorems 4.62 and 4.63 see S. Bochner and K. Chandrasekharan, *Acta Szeged*, XII B (1950), 1-15.

For the formula for the Laplacian of a radial function, used in the proof of Theorem 4.61, see R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, II, Berlin (1937), 227.

§4.7. For literature pertaining to summations over lattice points see S. Bochner and K. Chandrasekharan, *Quarterly Jour. Math.* (Oxford) 19 (1948), 238-248, (2) (1950), 80 and *Acta Szeged*, loc. cit.

§4.8. For Theorems 4.81 and 4.83 on the absolute summability of Fourier series see K. Chandrasekharan, *Proc. London Math. Soc.* 50 (1948), 223-229. Theorem 4.82 is due to appear in a paper by S. Bochner and K. Chandrasekharan.

The case $k = 1$ of Theorem 4.81 is due to L. S. Bosanquet, *Jour. London Math. Soc.* 11 (1936), 11-15. Theorem 4.83 has been proved to be 'best-possible' if $k = 1$ by L. S. Bosanquet and H. Kestelman, *Proc. London Math. Soc.* 45 (1939), 88-97.

§4.9. For a definition of the higher spherical means, and for Theorems 4.91-4.98, see K. Chandrasekharan, *Proc. London Math. Soc.* 50 (1948), 210-229. Theorems 4.91-4.93 are valid without the restriction that p is integral.

It is clear that if $f_{x,p}(t) \to 0$, as $t \to 0$, then $f_{x,q}(t) \to 0$, as $t \to 0$, for $q > p$ (cf. first theorem of consistency). In the proof of Theorem 4.94, we use the Riemann-Lebesgue lemma for several variables. See S. Bochner and K. Chandrasekharan, *Fourier transforms*, Princeton (1949), 57.

For Theorem 4.92 in the case $k = 1$, $0 < p < 1$, see G. H. Hardy and J. E. Littlewood, *Proc. Cambridge Phil. Soc.* 23 (1927), 681-684. For the case $k = 1, p > 0$, of Theorems 4.92 and 4.95 see L. S. Bosanquet, *Proc. London Math. Soc.* 31 (1930), 144-164; R. E. A. C. Paley, *Proc. Cambridge Phil. Soc.* 26 (1930), 173-203; S. Verblunsky, ibid., 152-157. For Theorems 4.93 and 4.96 in the case $k = 1$ see L. S. Bosanquet, *Proc. London Math. Soc.* 41 (1936), 517-528. For the case $k = 1$ of Theorem 4.97, which effects a solution of the 'summability problem' for Fourier series, see G. H. Hardy and J. E. Littlewood, *Math. Zeitschrift*, 19 (1924), 67-96, and for the corresponding case of Theorem 4.98 see L. S. Bosanquet and J. M. Hyslop, *Math. Zeitschrift*, 42 (1937), 489-512.

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