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Title Analytical Dynamics of a particle and of Rigid Bodies

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ELEMENTARY
ANALYTICAL DYNAMICS OF A PARTICLE
AND OF RIGID BODIES

(INCLUDING MOMENT OF INERTIA, COMPOUND PENDULUM AND MOTION OF A RIGID BODY IN TWO DIMENSIONS UNDER FINITE AND IMPULSIVE FORCES)

FOR

B. A. Pass & Honours Students of Indian Universities

BY

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1963

S. CHAND & CO.

DELHI — NEW DELHI — BOMBAY

JULLUNDUR — LUCKNOW

S. CHAND & CO.

Fountain — DELHI
Ram Nagar — NEW DELHI
Mai Hiran Gate — JULLUNDUR
Hazratganj — LUCKNOW
Lamington Road — BOMBAY

THIRTEENTH EDITION
(*Revised & Enlarged*)

Price : **Rs. 7.00**

Published by S. Chand & Co., Ram Nagar, New Delhi-1 and Offset by Eurasia Offset Printers, Ram Nagar, New Delhi-1.

PREFACE TO THE THIRTEENTH EDITION

The present edition of this book has given me an opportunity to make a number of minor corrections and to add Chapter XVI containing suitable solved examples illustrating the motion of a rigid body under finite end impulsive forces.

The large number of examples—always a popular feature of the book—has been supplemented by some additional problems taken from recent University Papers.

It is hoped that in its present form the book will be still more useful and will continue to serve the cause of Mathematical teaching in Indian Universities.

4, Daryaganj, Delhi-6 ; S. R. GUPTA
September, 1963

PREFACE TO THE FIRST EDITION

This book is meant to be an introduction to the study of Dynamics of a Particle, and is primarily intended for the use of students preparing for the B.A. degree examination of Indian Universities, but may also be used by students preparing for the Engineering and other competitive examinations. It includes a short account of the Cycloidal Pendulum and the Planetary Motion for the benefit of students preparing for the Honours course.*

This work differs from others of its kind in the method of treatment of the subject. Elementary works already existing in the market treat the subject geometrically, whereas I have tried to popularize the analytical method, the employment of which has hitherto been restricted to advanced works only. This preference in treatment is in accordance with my conviction (which, in turn, is the result both of my personal experience of teaching the subject for the last twelve years and the various discussions which I have had with students and teachers of Indian and Foreign Universities) that no other method but the analytical can enable the beginners to grasp the subject clearly. Thus the present volume is designed to supply a real need of the students.

---

*The Treatment of Moment of Inertia, Compound Pendulum, Linear motion in a resisting medium, Motion of a rigid body in two dimensions under finite and impulsive forces, centre of Percussion, Torsional vibrations and Ballistic Pendulum have been added in subsequent editions.

To afford a fuller idea of the scope of this work, I may mention that Newton's laws of motion, motion of a particle attached to elastic strings, motion of trains and cars on curved paths, the principle of work and energy, the principle of relative velocity and the graphical method of solving Dynamical problems—all these are treated in detail and profusely illustrated by carefully selected solved examples. Besides, other examples, which are scattered throughout the book are also particularly designed to assist the student to understand the principles of Dynamics. The ready made examples are taken mostly from questions set at the Indian and Foreign Universities. Certain articles and examples which may present some difficulty to a beginner are marked with an asterisk and may conveniently be omitted by him during the first reading.

My acknowledgment is generally due to modern writers on the subject whose works I have freely used. But I am particularly indebted to Professor A.T. Banerjee of the Hindu College Delhi, for his kindness in going through the manuscript and making valuable suggestions. I shall be thankful to others who may suggest improvements for future editions, or even point out errors in this volume which may have escaped my scrutiny.

Lahore ; S. R. GUPTA
April 1931.

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	ANSWERS	

The following approximate results may be referred to for numerical calculations :—

$\sqrt{2} = 1{\cdot}414$	$\sqrt{3} = 1{\cdot}732$
$\sqrt{5} = 2{\cdot}236$	$\sqrt{6} = 2{\cdot}449$
$\sqrt{7} = 2{\cdot}646$	$\sqrt{10} = 3{\cdot}162$
$\sqrt{11} = 3{\cdot}317$	$\sqrt{13} = 3{\cdot}606$
$\pi = 3{\cdot}1416$	$\frac{1}{\pi} = 0{\cdot}31831$
$\log_{10} e = 0{\cdot}4343$	$\pi^2 = 9{\cdot}870$

Weight of one gallon of water = 10 lbs.

Weight of one cubic foot of water = 62·5 lbs.

Elementary Analytical Dynamics

CHAPTER I

VELOCITY, ACCELERATION

1·1. Dynamics is the science which treats of bodies in motion.

When a body moves all parts of the body have not necessarily the same motion but if the body is a very small one, the differences between the motions of its different parts are unimportant. In fact, we may regard the body as coinciding with a geometrical point and then consider its motion. Such a body, the position of which is defined by that of a geometrical point at each instant, is called a **particle.** Our main object in this book is to deal with motion of a particle.

1·11. If a body moves in such a way that all its points move through equal and parallel distances in all equal intervals of time, however small, the body is said to have a Motion of Translation. In such cases the body moves without rotation and between any two of its points there is no relative motion in space. Its motion is completely determined when the motion of any one of its points is known. The body is then taken as replaced by a material particle coincident with this point and having its mass equal to that of the whole body. It is usual to take the centre of gravity of the body as the point whose motion determines the motion of the whole body.

1·2. When a point changes its position relative to any object and occupies different positions at different times, it is said to be in motion relative to that object. The curve drawn through the successive positions occupied by the point, is called its *path.*

1·3. The path of a moving point may be a straight line or a curve.

In earlier chapters of this book we shall deal with the motion of a particle along a straight line and we now discuss some concepts necessary to the investigation of such a motion.

1·31. Displacement in a straight line. In order to determine the position of a point P in a straight line, a fixed point O must be marked on the line. The position of P is then defined by the length OP. If at any instant the position of a moving particle be P, and at any subsequent instant it be Q, then PQ is the change in position or the displacement of the particle in the intervening time.

Whenever a relation between the distance $OP=x$ and time t is known, the displacement or the distance travelled during a given

interval of time can be obtained by substituting the given values of t in the relation.

If, for example, in the relation $x=16t^2$, we put $t=2$, we get $x=64$ and for $t=3$ we get $x=144$, so that the displacement in the third second is $144-64=80$ ft.

1·32. Average Velocity during an Interval. If a particle moves a distance x along a straight line in time t, then $\frac{x}{t}$ is said to be the average velocity during the interval.

For example, if a train travels from Amritsar to Delhi, a distance of 300 miles in 12 hours, then $\frac{300}{12}=25$ miles per hour is its average velocity. It means that if the train were supposed to have been moving with the same velocity of 25 miles per hour during the whole interval, the train would have covered the distance of 300 miles in 12 hours. But as a matter of fact this is not so. We know from experience that the train starts from rest, gathers speed and stops at the next station and continues doing so between every two stopping stations.

It is evident, therefore, that the average velocity is related to the intervals as a whole and is simply a matter of calculation. It is distinct from the actual velocity which a moving body may have at any particular time.

With a view to understand the motion in all details, we introduce the notion of *velocity at a point.*

1·33. Velocity at a Point. Suppose that a particle begins to move from a point O along the straight line OX and comes in the position P after describing a distance x in time t. Further let it move a little distance δx in a little time δt and come in the position Q. Then $\frac{\delta x}{\delta t}$ is the *average velocity* of the particle during the interval δt. As δt and consequently δx becomes smaller and smaller, the point Q is taken nearer and nearer to the point P so that $\frac{\delta x}{\delta t}$ characterizes more and more accurately the nature of the motion at P.

$O \quad x \quad P \quad \delta x \; Q \quad X$

Thus it is natural to take $\underset{\delta t \to 0}{\text{Lt}} \frac{\delta x}{\delta t}=\frac{dx}{dt}$ as the velocity at P. Thus the derivative of x with respect to t expresses the rate of change of displacement or the velocity v at the time t,

i.e., $$v=\frac{dx}{dt}.$$

1·34. Speed. The velocity may be positive or negative according as the particle is moving in one or the other direction along a straight line. The magnitude of the velocity without any consideration of direction is usually called *speed.* A man walking north at the rate of 4 miles per hour and another walking east at 4 miles per hour

have the same speed but different velocities because the directions of motion are different.

1·35. Uniform Velocity. In case the velocity of a particle moving along a straight line is the same at every instant during an interval, *i.e.*, $\frac{dx}{dt}$ remains constant throughout the interval, the motion is said to be *uniform*.

For measuring velocity one second is usually taken as the unit of time and either one foot or one centimetre as the unit of length.

A train which travels uniformly 60 miles in one hour will travel 88 feet in one second and is said to have a velocity of 88 feet per second or 60 miles per hour.

1·36. Relative Velocity of two bodies moving in straight line.*

Let the velocities of two bodies A and B at any instant be u, and u' and let x, x' be their distances from a fixed point O in the line.

The velocity of B relative to A

$$= \text{rate of change of the distance } AB$$

$$= \frac{d}{dt}(x'-x) = \frac{dx'}{dt} - \frac{dx}{dt} = u'-u.$$

Thus the velocity of B relative to A is obtained by adding to the velocity of B, the reversed velocity of A.

If, for example, two trains A and B are moving on parallel tracks in the same direction with equal velocities, their relative velocity is zero, so that to a person in A, the train B appears to be stationary. If the trains are moving with unequal velocities of 20 and 25 miles per hour respectively in the same direction, the motion of B relative to A is $25-20$, *i.e.*, 5 miles per hour. Again, if the trains are moving in opposite directions the velocity of B relative to A is $25-(-20)$, *i.e.*, 45 miles per hour. In this case a person in B is separating from a person in A at the rate of 45 miles per hour.

1·37. Acceleration at a Point. We have seen that the rate of change of displacement at the particular instant is the velocity at that instant, similarly the rate of change of the velocity with respect to time is called *acceleration* and is generally denoted by f.

Let v be the velocity at any instant, of a particle moving with variable velocity and δv, the change in the velocity in a little time δt.

$\frac{\delta v}{\delta t}$ is then the average rate of change of the velocity in the short interval δt.

*For relative motion in a plane, see Chapter XI.

The rate of change of velocity, *i.e.*, acceleration at any instant t is therefore $= \underset{\delta t \to 0}{\text{Lt}} \frac{\delta v}{\delta t} = \frac{dv}{dt}$.

$$\therefore \quad f = \frac{dv}{dt} = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}.$$

Also $$f = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \frac{dv}{dx}.$$

Thus general expressions for acceleration of a moving particle are

$$\frac{dv}{dt} \text{ or } \frac{d^2x}{dt^2} \text{ or } v \frac{dv}{dx}.$$

The student should clearly bear in mind these *three* forms for f, and as to which one is best to employ in a given case will become clear from the examples given in the next article.

The word acceleration is used to denote any change in the velocity, whether that change be an increase, or a decrease. In other words the acceleration may be in the direction of motion, or in the contrary direction, *i.e.*, the acceleration f may have the same sign as that of the velocity v or may have the opposite sign.

In case the sign of f is opposite to that of v, the acceleration is sometimes known as **retardation.** So that *retardation implies decrease in the magnitude of velocity.*

Just like velocity, acceleration may be uniform or variable.

1·38. As acceleration is the change of velocity per unit of time, a point is said to be moving with n units of acceleration when a change of n units of velocity takes place every unit of time.

Thus a point is moving with 5 foot-second units of acceleration when a change of velocity of 5 ft. per second takes place every second; it is usual to express it by saying that the point possesses an acceleration of 5 feet per second per second or 5 ft./sec.^{2} Similarly 5 centimetre-second units of acceleration are written as 5 cm./sec.^{2} and 10 miles-hour units of acceleration as 10 mi./hr.^{2}.

Example 1. *A particle moves in a straight line in such a way that the distance from a fixed point O in the line at the end of t secs., is* $11+5t+t^3$ *ft. Find (i) its distance from O, (ii) its speed, (iii) its acceleration at the end of 4 secs.*

Here $$s = t^3+5t+11 = 95 \text{ ft. when } t=4,$$

$$v = \frac{ds}{dt} = 3t^2+5 = 53 \text{ ft. per sec. when } t=4,$$

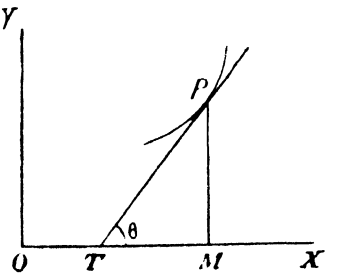
and $$f = \frac{d^2s}{dt^2} = 6t = 24 \text{ ft. per sec. per sec. when } t=4.$$

Example 2. *A body moves so that the distance x feet which it travels is given by the relation* $x = t^2+6t+11$ *where t is the time in seconds. Find the velocity and acceleration 4 secs. after the body begins to move.*

GRAPHICAL REPRESENTATION

1·4. Distance-Time Graph. If a particle moves along a straight line, the distance moved is a function of time.

If a graph is drawn such that the time is represented on some scale by the abscissa, *i.e.*, by a length drawn along OX, and the corresponding distance, by the ordinate on the same scale, the graph is called the distance-time graph, and the velocity which is $\frac{ds}{dt}$ is evidently given by the gradient of the tangent to the curve at a particular point determined by an assigned value of t.



1·5. Velocity-Time Graph. In exactly the same way, we can draw a graph representing the velocity in terms of time. The acceleration $\frac{dv}{dt}$ at a particular instant is then represented by the gradient of the tangent at the corresponding point on the graph.

When the acceleration is uniform the velocity-time graph will be evidently a straight line.

Examples

1. For the velocity of a tram car it is given that

t	0	1	3	4	5	6	7
s	0	2	9	14	20	27	35

Plot the space-time curve and find the velocity at the end of each second, hence find the acceleration at the end of the 4th second. (*Roorkee*)

2. Show that the above figures are connected by the relation $2s=t^2+3t$ and hence verify the results obtained in Example 1.

3. Show that in the velocity space curve (in which abscissae denote spaces covered and ordinates represent velocities) the sub-normal represents acceleration. (*B.U.*)

[**Hint.** Subnormal $=v\times\frac{dv}{ds}$.]

CHAPTER II

MOTION IN A STRAIGHT LINE—I

2·1. In the preceding chapter we have shown how by differentiation, we deduce the velocity from a knowledge of the relation between the distance travelled and the time taken and how we deduce the acceleration from the relation between the velocity and the time.

We now consider the more important problem of deducing the velocity from a knowledge of the acceleration, and the distance travelled from a knowledge of the velocity. This is the inverse problem and hence can be solved by integration.

2·2. Motion with constant acceleration. *A particle moves along a straight line from a fixed point in it with initial velocity u and constant acceleration f, in its direction of motion. To find the velocity v and the distance s travelled after any time t.*

The acceleration of the particle is given constant and equal to f, and s is its distance at any time t from the point at which it starts, then its equation of motion is

$$\frac{d^2s}{dt^2}=f.$$

Integrating with respect to t, we have

$$\frac{ds}{dt}=ft+c, \text{ where } c \text{ is a constant.}$$

It is given that initially when $t=0$, $\frac{ds}{dt}=u$, *i.e.*, $u=0+c$

or $$c=u.$$

$$\therefore \quad \frac{ds}{dt}=u+ft, \textit{ i.e.,} \quad \mathbf{v=u+ft.} \qquad \ldots (1)$$

which gives v, the velocity at any time.

Integrating again with respect to t, we get

$$s=ut+\tfrac{1}{2}ft^2+k \text{ (a constant)},$$

when $t=0$, $s=0$, so that $k=0$

$$\therefore \quad \mathbf{s=ut+\tfrac{1}{2}ft^2} \qquad \ldots (2)$$

Also $v\frac{dv}{ds}$ being another expression for acceleration, the equation of motion can be written as $v\frac{dv}{ds}=f$.

Integrating with respect to s, we have $\frac{v^2}{2}=fs+$

It is given that initially when $s=0$, $v=u$, so that

$$\frac{u^2}{2}=0+c_1. \qquad i.e.,\ c_1=\frac{u^2}{2}. \quad \therefore\ \tfrac{1}{2}v^2=\tfrac{1}{2}u^2+fs,$$

$$i.e.,\ \mathbf{v^2=u^2+2fs} \qquad ..(3)$$

The last equation can also be obtained by eliminating t from (1) and (2) as follows :—

From (1) $\quad v^2=u^2+2uft+f^2t^2$

$$=u^2+2f(ut+\tfrac{1}{2}ft^2)=u^2+2fs.$$

If the particle starts from rest, so that $u=0$, these equations take the simpler forms

$$v=ft, \qquad s=\tfrac{1}{2}ft^2 \qquad \text{and} \qquad v^2=2fs.$$

The three equations obtained above are of fundamental importance. They are the equations of motion of a particle moving along a straight line with constant acceleration.

Each of these equations involves four quantities and any one of them can be obtained whenever the other three are known.

In solving problems we select the equations which contain the quantities that are given and the one which is required.

Example 1. *A particle starts with a velocity of 20 ft. per sec. and travels 100 yards in 30 secs. along a straight line. Find (i) the acceleration supposed uniform, (ii) the times at which it is 166⅔ ft. from the starting point, and (iii) the time when it comes to rest.*

(*i*) Here $u=20$ ft. per sec., $s=100$ yds. $=300$ ft. and $t=30$ secs.

Substituting these in $s=ut+\frac{1}{2}ft^2$, we get

$$300=20\times30+\tfrac{1}{2}f\times30\times30$$

i.e., $\quad f=-\frac{2}{3}$ ft. per sec. per sec.

(*ii*), $\quad 166\frac{2}{3}=20t+\frac{1}{2}(-\frac{2}{3})t^2$

or $\quad 500=60t-t^2$ or $(t-10)(t-50)=0.$

$\therefore \quad t=10$ or 50 secs.

(*iii*) In the formula $v=u+ft$, we have $v=0$, $u=20$ and $f=-\frac{2}{3}$

$\therefore \quad 0=20-\frac{2}{3}t$ or $t=30$ secs.

Thus the particle moves with a retardation of $\frac{2}{3}$ ft. per sec. per sec. and it comes to rest after 30 secs., then it returns back and is again at a distance of $166\frac{2}{3}$ ft. at $t=50$ secs., the second value of t obtained above.

Example 2. *A body moving with uniform acceleration in a straight line has a velocity 12 ft. per sec. at a distance of 10 ft. from a point O and 16 ft. per sec. at 24 ft. from O. Find the acceleration and velocity at O.*

Let v be the velocity at O, then using the formula

$$v^2=u^2+2fs,$$

we have

$$144=u^2+2f\times10 \text{ and } 256=u^2+2f\times24$$

which give $f=4$ ft. per sec. per sec. and $u=8$ ft. per sec.

Example 3. *Two cars start off to race with velocities u and v and travel in a straight line with uniform accelerations α and β; if the race ends in a dead heat, prove that the length of the course is*

$$\frac{2(u-v)(u\beta-v\alpha)}{(\alpha-\beta)^2}.$$

Let the length of the course be s units. As the race ends in dead heat, the cars reach the destination at the same time, *i.e.*, each of them describes the distance s in the same time, say t.

Then by the question, we have

$$s=ut+\tfrac{1}{2}\alpha t^2=vt+\tfrac{1}{2}\beta t^2,$$

which gives

$$(u-v)\,t=\tfrac{1}{2}(\beta-\alpha)t^2 \quad \text{or} \quad t=\frac{2(u-v)}{\beta-\alpha},$$

rejecting the value $t=0$, which corresponds to the initial position.

Substituting the value of t in $s=ut+\frac{1}{2}\alpha t^2$, we now get

$$s=\frac{2u(u-v)}{\beta-\alpha}+\tfrac{1}{2}\alpha\cdot\frac{4(u-v)^2}{(\beta-\alpha)^2}=\frac{2(u-v)}{(\beta-\alpha)^2}[u(\beta-\alpha)+\alpha(u-v)]$$
$$=2(u-v)(u\beta-v\alpha)/(\alpha-\beta)^2.$$

2·21. Other results. For a body moving along a straight line with *uniform acceleration*, we have

(1) *the average velocity is the mean of the initial and final velocities and is equal to the velocity at the middle of the interval.*

Let u be the initial velocity and v, the final velocity, then

$$v=u+ft.$$

The average velocity $\dfrac{s}{t}=\dfrac{ut+\frac{1}{2}ft^2}{t}=u+\tfrac{1}{2}ft.$

The mean velocity $=\dfrac{u+v}{2}=\dfrac{u+u+ft}{2}=u+\tfrac{1}{2}ft.$

The velocity at the middle of the interval $=u+f\cdot\dfrac{t}{2}$

$$=u+\tfrac{1}{2}ft.$$

Thus all the three are equal.

(2) *The distance travelled in t seconds is equal to the product of t and the mean velocity.*

For, $$s=ut+\tfrac{1}{2}ft^2=(u+\tfrac{1}{2}ft)\,t=\frac{u+v}{2}\times t$$
$$=\textit{mean velocity}\times\textit{time}.$$

If the particle starts from rest and acquires a velocity v in time t, $u=0$ and $s=\frac{1}{2}vt$.

(3) *Distance travelled in the nth second*

$=$Distance described in n secs.$-$that described in $(n-1)$ secs.

$=un+\frac{1}{2}fn^2-[u(n-1)+\frac{1}{2}f(n-1)^2]$

$=u+\frac{1}{2}f(2n-1).$

Thus the distance described in the first, second, third..... seconds of the motion are $u+\frac{1}{2}f$, $u+\frac{3}{2}f$, $u+\frac{5}{2}f$.....which form an *A.P.* with common difference f.

Examples II

[**N.B.**—In the following questions, bodies are supposed to be moving along a straight line and their acceleration is taken to be uniform.]

1. The speed of a train increases from 30 m.p.h. to 45 m.p.h. in 11 seconds. What is the acceleration of the train and what distance does it describe during this time ?

Find also the velocity of the train when it has travelled a further distance of 847 ft.

2. If a body, moving with uniform acceleration passes over 300 feet while its velocity increases from 50 to 70 ft. per second, find the acceleration and the time of motion.

3. The velocity of a particle, which is moving in a straight line with constant retardation, decreases 10 ft./sec. while the particle travels 10 feet, and 15 ft./sec. while it travels $12\frac{1}{2}$ ft. from the starting point. Find the distance the particle travels from the starting point until it comes to rest.

4. A car is moving at 30 m.p.h. when passing one lamp-post, and at 15 m.p.h. when passing the other. If the lamp-posts are 110 yards apart, how far will the car travel before it comes to rest if this retardation is maintained ? How long after passing the first lamp-post will the car be moving ?

5. A train is moving with a speed of 45 m.p.h. and the brakes produce a retardation of 4 ft. sec.^{2} At what distance from a station should the brakes be applied so that the train may stop at the station ?

If the brakes are put on at half this distance, with what speed will the train pass the station ?

6. An engine driver whose train is travelling at the rate of 30 miles an hour sees a danger signal at a distance of 220 yards and does his best to stop the train. Supposing that he can stop the train when travelling 30 miles an hour in 440 yards, show that his train will reach the signal with a velocity of $21\frac{1}{5}$ miles per hour nearly. (*P.U.*)

7. A body starts with velocity u and moves in a straight line with constant acceleration f. When the velocity has increased to $5u$ the acceleration is reversed, its magnitude being unaltered. Find the velocity with which the body again reaches its starting point.

8. A train moving with constant acceleration takes t secs. to pass a certain point and the ends of the train pass the point with velocities u, v respectively. Find (*i*) the acceleration, (*ii*) the length of the train, (*iii*) the velocity with which the centre of the train passes the point. (*L.U.*)

9. A body starting with initial velocity and moving with uniform acceleration acquires a velocity of 20 ft. per sec. after moving through 10 ft. and a velocity of 30 ft. per sec. after moving through a further 15 ft. When and where will its velocity be 40 ft. per sec.

[**Hint.** Write down one equation for the first distance and the second equation for the sum of the two distances.]

10. In a certain interval of 10 seconds, a point passes over 220 ft., in the next interval of 5 seconds it passes over 830 ft. ; if the point is moving with uniform acceleration, find its velocity at the beginning of each of the two intervals. (*P. U.*)

11. A particle moving in a straight line with constant acceleration passes in succession through three points A, B, C, the distances AB and BC each being equal to 12 ft. The particle takes 1 sec. to travel from A to B, and 2 secs. to travel from B to C. Determine the point at which the particle comes to rest and the point at which its velocity is 8 ft. per sec. (*Bom. U.*)

12. A particle moving with uniform acceleration in a straight line

passes points P, Q, R. If $PQ=QR=b$, and if the time from P to Q is t_1, and Q to R is t_2, prove that the acceleration is $\frac{2b(t_1-t_2)}{t_1t_2(t_1+t_2)}$.

13. A body moving with uniform acceleration, moves through 530 and 770 cms. respectively in the ninth and the thirteenth seconds of its motion. Find its initial velocity and the acceleration with which it moves.

14. If the distances gone over by a particle (moving with uniform acceleration) in the pth, qth and rth seconds are respectively x, y and z; show that

$$(q-r)x+(r-p)y+(p-q)z=0. \qquad (P.U.)$$

15. Two points A and B move in the same straight line being initially at rest and B being 60 feet in front of A, B starts from rest with an acceleration of 12 ft./sec.^{2} and A starts with a velocity of 23 ft./sec. and an acceleration of 8 ft./sec^2. Prove that A will overtake and pass B after an interval of 4 seconds and that B will in turn overtake A after a further interval of $3\frac{1}{2}$ seconds.

16. A particle is projected in a straight line with a certain velocity and a constant acceleration. One second later another particle is projected after it with half the velocity and double the acceleration; when it overtakes the first the velocities are 22 and 31 ft. per second respectively. Prove that the distance traversed is 48 feet. (*D. U.*)

17. A bus is beginning to move with an acceleration of 1 ft./sec.^{2} Show that (*i*) a man who can run at 9 ft./sec. will catch the bus in 8 secs. if he is 40 ft. behind when it starts; (*ii*) if the man is $40\frac{1}{2}$ feet behind he will only just catch the bus; (*iii*) if he is 50 ft. behind he will never catch the bus the shortest distance between them being $9\frac{1}{2}$ feet.

18. A cyclist passes a car that is just starting with an acceleration of 2 ft./sec.^{2} The speed for the cyclist is 12 m.p.h., and the car maintains its acceleration for 15 seconds and then moves uniformly. How far will the car have run before overtaking the cyclist?

19. A train moving with uniform acceleration, has a velocity of 20 m.p.h. at A and 40 m.p.h. at B. Find the velocity at the middle of the interval and also the velocity midway between A and B.

20. Two particles P, Q move along a straight line AB, starting from A, P moves with velocity u and acceleration f, Q with velocity u' and acceleration f'. If they both have the same velocity at the middle point of AB, prove that

$$AB=\frac{u'^2-u^2}{f-f'} \qquad (L.U.)$$

21. A particle moves in a straight line with uniform acceleration. In motion from one point to another, show that the average velocity is equal to the velocity at the middle of the interval of time and that it is less than the velocity at the point midway in distance.

If the particle moves according to the law $s=k\sqrt{t}$, show that the average velocity is now equal to the velocity at the point midway in distance.

22. A train travels a distance S starting from rest and finishing at rest. The velocity increases uniformly till it reaches a maximum value V, and then decreases uniformly. Show that the whole time of the motion is $\frac{2S}{V}$.

23. A 100 yards sprinter starts with a speed of 20 ft. per second, accelerates uniformly to 30 ft. per sec. and finishes the race at this speed. If his total time is $10\frac{2}{5}$ secs., show that his uniform acceleration is $4\frac{1}{3}$ ft. per sec. per sec.

24. Two particles P and Q move in a straight line AB towards each other. P starts from A with velocity u and acceleration f, while Q starts from B with velocity u' and acceleration f'. They pass one another at the mid-point of AB and arrive at the other ends of AB with equal velocities. Prove that

$$(u+u')(f-f')=8(fu'-uf').$$

25. A point moves with uniform acceleration and v_1, v_2, v_3, denote the average velocities in three successive intervals of time t_1, t_2, t_3. Prove that

$$\frac{v_1-v_2}{v_2-v_3}=\frac{t_1+t_2}{t_2+t_3}. \qquad (P.U.)$$

2·3. Bodies falling vertically. It has been proved by numerous simple experiments (such as Newton's guinea and feather experiment) that all bodies let fall in a vacuum, at a particular place on the earth (not far away from the earth), move vertically downwards with constant acceleration. This acceleration is called the acceleration due to gravity and its value is about 32·2 or 32 in foot-second units and about 981 in centimetre-second units. It is generally denoted by the letter g and in solving problems on falling bodies we make use of the three equations obtained in Art. **2·2** by changing f into g.

When bodies fall freely from rest, the equations are of the form :—

$$v=gt, \quad s=\tfrac{1}{2}gt^2, \quad v^2=2gs.$$

For example the velocity acquired by a particle in falling from rest through a distance h is given by $v=\sqrt{2gh}$. This velocity is usually known as a *velocity due to a fall from a height h*.

2·4. Bodies projected vertically upwards. When a body is projected vertically upwards, it is under the effect of the downward acceleration due to gravity, *i.e.*, it moves with a retardation. Its velocity, therefore, gradually diminishes until it is totally exhausted ; the body is then for an instant at rest and immediately begins to fall with a velocity which increases numerically but is negative.

The upward direction being taken positive, the acceleration is negative and is denoted by $-g$, the equations of motion are, therefore, as follows :

$$v=u-gt \qquad ..(1)$$

$$s=ut-\tfrac{1}{2}gt^2 \qquad ..(2)$$

and $$v^2=u^2-2gs \qquad ..(3)$$

where s, the distance measured from the starting point is positive above that point and negative below it.

We now deduce the following facts from these :

(*a*) $v=0$ when from (1) $t=\dfrac{u}{g}$ and from (3) $s=\dfrac{u^2}{2g}$

which give the time to reach the highest point and the maximum height reached.

(*b*) $s=0$ when from (2) $t=0$ or $t=\dfrac{2u}{g}$.

$t=0$ refers to the instant when the body starts and $t=\dfrac{2u}{g}$ refers to the instant when the body returns to the starting point.

This is called the time of flight, which is twice the time to the

greatest height. Since the time of ascent is $\frac{u}{g}$, therefore the time of descent is also $\frac{u}{g}$.

(*c*) The equation (2) when written in the form

$$t^2-\frac{2v}{g}t+\frac{2s}{g}=0$$

gives the time taken to reach a given height s (less than the greatest).

This, in general, gives two values of t one giving the time to reach the point in the upward motion and the other the whole time to reach the same point in the descent after the body has reached the highest point.

(*d*) The equation $v^2=u^2-2gs$ gives the two equal and opposite values of v for a particular height s, one representing the velocity in the upward motion and the other in the downward motion. This shows that the motion in the downward direction is exactly the reverse of that in the upward direction.

Example 1. *A body is projected vertically upwards with a velocity of 80 f.p.s. Find*

(*i*) *how high it will go,*

(*ii*) *at what time it will be 96 ft.* (*a*) *above,* (*b*) *below the point of projection.*

(*i*) To find the total height we use $v^2=u^2-2gs$, where $v=0$, $u=80$ and $g=32$.

$$0=80\times 80-64s \text{ i.e., } s=100 \text{ ft.}$$

(*ii*) (*a*) To find the times for a height of 96 ft., we use

$$s=ut-\tfrac{1}{2}gt^2,$$

where $s=96$, $u=80$, and $g=32$.

$$\therefore \quad 96=80t-16t^2$$

$$\text{or} \quad t^2-5t+6=0 \text{ i.e., } (t-3)(t-2)=0.$$

$$\therefore \quad t=2 \text{ secs. or } t=3 \text{ secs.}$$

(*b*) In this case we put $s=-96$ and we get

$$-96=80t-16t^2$$

$$\text{or} \quad t^2-5t-6=0, \text{ i.e., } (t-6)(t+1)=0,$$

$$\text{i.e.,} \quad t=6 \text{ or } t=-1.$$

$\therefore$ the required time is 6 seconds, the second value being impossible.

Note 1. $t=1$ is the time that would be taken by the body to reach the point 96 ft. below the point of projection if it were projected *downwards*. Also when projected vertically upwards it takes $2u/g=160/32=5$ seconds to return to the point of projection. The whole time taken to reach the given point is thus 6 seconds, the same as obtained above.

Note 2. When a body is projected *upwards* and we require the time to reach a point *below* the point of projection, we need not find the whole time up and down to the point of projection and then the time taken to reach the point. We simply give a negative sign to the distance measured below the point of projection as we have done in the above example.

Example 2. *A particle is dropped from the top of a tower 200 ft. high and another particle is projected at the same time vertically upwards from the foot of the tower so as to meet the first particle at a height of 56 ft. Find the velocity of projection of the second particle.*

Let the particles meet after t secs. The first particle falls from rest a distance $200-56=144$ ft.

$$\therefore \qquad 144=\tfrac{1}{2}gt^2=16t^2 \qquad \ldots(1),$$

which gives $t=3$ secs.

If u is the required velocity of the second particle, we have

$$56=ut-\tfrac{1}{2}gt^2 \qquad \ldots(2).$$

Adding (1) and (2), we get

$$200=ut \qquad \ldots(3)$$

which gives $u=\dfrac{200}{t}=\dfrac{200}{3}=66\tfrac{2}{3}$ ft. per sec.

Or thus :—The velocity of the second particle relative to the first at time t is $u-gt-(-gt)=u$, the velocity in the beginning. Since both the particles move with the same acceleration, their relative velocity remains constant and equal to u. In order that the particles may meet, the second particle must move a distance of 200 ft. relative to the first, hence we have $200=ut$ which is the same as equation (3) above. But from (1) $t=3$, consequently $u=\frac{200}{3}$ as before.

Examples III

1. A stone is projected upwards with a velocity of 112 ft. per sec. (*i*) How high will it rise ? (*ii*) How long will it take to return to the point of projection ? (*iii*) When will its velocity be 16 f.p.s. ? (*iv*) At what time will it be 96 ft. above (*v*) 128 ft. below the point of projection ?

2. A ball is thrown vertically upwards with a speed of 128 ft./sec. Find where it is after 5 secs. and the total distance it has actually travelled in this time. If it falls past the point of projection into a well of depth 120 ft. find when it strikes the bottom.

3. A stone is projected vertically upwards with a certain velocity and it is found that when it is 384 ft. from the ground it takes 10 seconds to return to the same point again. Find the velocity of projection and the whole time of flight.

4. A stone drops from the roof of a house and falls past a window 6 ft. high in $\frac{1}{4}$ of a second, find the height of the house above the window.

5. A lift moves upwards from the bottom of a mine with acceleration of 4 ft./sec.^{2} After 12 secs. a ball is dropped from the lift. When will the ball strike the ground and where will the lift be at the instant ?

6. A body is dropped from a balloon at a height of 300 ft. from the

ground. Find its velocity on reaching the ground if the balloon is (*a*) falling (*b*) rising with a velocity of 80 ft. per sec.

What time does the body take in reaching the ground in the two cases ?

7. A balloon has been ascending vertically at a uniform rate for 4·5 seconds and a stone let fall from it reaches the ground in 7 seconds. Find the velocity of the balloon and its height when the stone is let fall. (*P.U. 1950*)

8. A stone is let fall from a height of 240 ft. and at the same time another stone is thrown upwards to meet it. With what velocity must the second be projected in order that the stones may meet at a height of 96 ft. ?

9. A particle is dropped from the top of a tower h ft. high and at the same moment another particle is projected upwards from the bottom. They meet when the upper one has described $1/n$ of the distance. Show that the velocities when they meet are in the ratio $2 : (n-2)$ and that the initial velocity of the lower is $\sqrt{\frac{1}{2}n.g.h}$.

10. A stone is thrown vertically upwards with such a velocity as will just take it to the level of the top of a tower 100 ft. high. Two seconds later another stone is thrown up from the same place with the same velocity. Find when and where the stones will meet.

11. A particle is projected vertically upwards with a velocity of u ft./sec. and after t seconds another particle is projected upwards from the same point and with the same velocity. Prove that the particles will meet at a height $\frac{4u^2-g^2t^2}{8g}$ feet, after a time $\left(\frac{t}{2}+\frac{u}{g}\right)$ secs.

12. A particle falls freely from A and a second particle is allowed to fall from B, at a distance s ft. vertically below A, at the instant when the first particle is half way between A and B. Find when and where they will be together and show that their velocities at that time will be as 3 : 1.

13. The space described by a body in the fifth second of its fall from rest was to the space described in the last second but three as 9 : 11. For how many seconds did the body fall ?

14. A stone is let fall from the top of the Qutab Minar and during the last second of its flight it is observed to fall $\frac{1}{2}\frac{0}{2}\frac{4}{5}$ of the whole of distance. Find the height of the Minar.

15. A stone dropped into a mine is heard to strike the bottom in $7\frac{9}{5}$ secs, Find the depth of the mine, the velocity of the sound being 1100 ft. per sec.

16. Two balls are projected simultaneously with the same velocity from the top of a tower, one vertically upwards and the other vertically downwards. If they reach the ground in times t_1, t_2 respectively, show that $\sqrt{t_1t_2}$ is the time which each will take to reach the ground if simply let drop from the top of the tower.

17. A particle, projected vertically upwards, takes t seconds to reach a height h ft. If t' secs. is the time from this point to the ground again prove $h=\frac{1}{2}gtt'$ and that the maximum height is $\frac{g(t+t')^2}{8}$.

Show also that the velocity of the particle at a height $\frac{1}{2}h$ is $\frac{1}{2}g\sqrt{t^2+t'^2}$.

18. A, B, C, D are points in a vertical line, the lengths $AB=BC=CI$ If a body falls from rest at A, prove that the times of describing AB, BC, CL are as

$$1 : \sqrt{2}-1 : \sqrt{3}-\sqrt{2}.$$

19. A particle takes t seconds less and acquires a velocity of s ft. pe sec. more at one place than at another in falling from rest through the same distance. Show that gravities g_1 and g_2 at the two places are related by

$$g_1g_2=s^2/t^2.$$

2·5 Space average and time-average of velocity.

Since $dx/dt=v$, therefore the distance x described by a body during the interval of time t_1 to $t_2 = \int_{t_1}^{t_2} v\,dt$.

And the *time-average* of velocity during the interval

$$= \frac{\text{total distance}}{\text{total time}} = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} v\,dt$$

Similarly the *space-average* of velocity in an interval of space from x_1 to x_2

$$= \frac{1}{x_2-x_1}\int_{x_1}^{x_2} v\,dx,$$

where v is now expressed in terms of x.

In this case the space is divided into an infinite number of equal portions assuming the velocity to remain constant in each portion and to vary from portion to portion.

Example 1. *A particle moves along a straight line with uniform acceleration. Show that the space-average of velocity when its initial and final velocities are u_1 and u_2 is* $\frac{2}{3}\dfrac{u_2^2+u_1u_2+u_1^2}{u_2+u_1}$

Let s be the total distance described and let v be the velocity and x the distance described in time t measured from the initial position as origin, then,

$$v^2 = u_1^2 + 2fx \qquad \ldots (1)$$

and

$$u_2^2 = u_1^2 + 2fs \qquad \ldots (2)$$

and the space average of velocity

$$= \frac{1}{s}\int_0^s v\,dx = \frac{1}{s}\int_0^s \sqrt{u_1^2+2fx}\,dx$$

$$= \frac{1}{s}\left[\frac{1}{2f}\cdot\frac{2}{3}\cdot(u_1^2+2fx)^{\frac{3}{2}}\right]_0^s = \frac{1}{3fs}\left[(u_1^2+2fs)^{\frac{3}{2}}-u_1^3\right]$$

$$= \frac{1}{3}\cdot\frac{2}{u_2^2-u_1^2}\left[u_2^3-u_1^3\right] \text{ by (2)} = \frac{2}{3}\cdot\frac{u_2^2+u_1u_2+u_1^2}{u_2+u_1}$$

Note 1. The time-average of velocity

$$= \frac{1}{t}\int_0^t v\,dt = \frac{1}{t}\int_0^t (u_1+ft)\,dt = u_1+\tfrac{1}{2}ft = \tfrac{1}{2}(u_1+u_2) = \text{mean velocity},$$

(compare **Art. 2·21**)

Note 2. The results of Art. 2·5 above are applicable even to the case where the acceleration is not uniform.

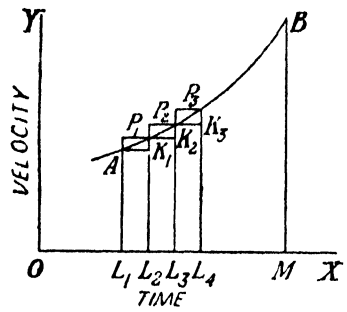
Ex. 2. A particle moves from rest with a uniform acceleration. Prove that the average velocity is half or two-thirds of the final velocity, according as the time or the space is divided into an infinite number of equal portions and the averages taken with regard to these.

Show also that in any interval the space-average of the velocity is $\frac{4}{5}$ of the time-average.

2·6. Graphical Methods. *The number of units of the distance described by a particle in a given time is represented by the number of units of area under the velocity-time graph for the same time.*

Suppose a particle is moving with a continually changing velocity, the rate of change of velocity, i.e., acceleration being uniform or variable.

Let $AP_1P_2P_3\ldots\ldots B$ be the velocity-time graph drawn so as to have the same scale for both the axes and let



$$OL_1 = t_1$$

and $$OM = t_2.$$

Let us divide the whole interval $t_2 - t_1$ represented by L_1M into n small intervals each equal to δt. If $v_1, v_2, v_3 \ldots$ be the velocities in the beginning of these intervals, and if these velocities be supposed to have remained constant during the short intervals, then the total distance described is

$$v_1\delta t + v_2\delta t + v_3\delta t \ldots\ldots + v_n\delta t$$

or $$(v_1 + v_2 + v_3 + \ldots\ldots + v_n)\delta t, \quad \ldots(1)$$

which is equal to the sum of the areas of rectangles L_1K_1, L_2K_2, L_3K_3 etc.

If now we take the constant velocity during each interval to be the velocity at the end of that interval, the total distance moved is

$$v_2\delta t + v_3\delta t + \ldots\ldots + v_{n+1}\delta t$$

or $$(v_2 + v_3 + \ldots\ldots + v_{n+1})\delta t \qquad \ldots(2)$$

which is equal to the sum of the areas of rectangles

$$L_1P_1,\ L_2P_2,\ L_3P_3, \ldots\ldots \text{etc.}$$

Evidently the actual distance moved lies between expressions (1) and (2).

Now when the short intervals of time into which $t_2 - t_1$ is divided become indefinitely small, the expressions (1) and (2) get nearer to each other, the difference between the two sets of rectangles becoming smaller and smaller, and in the limit each approaches the area AL_1MB, *i.e.*, the area included between the graph $AP_1P_2\ldots\ldots B$, the axis of x, and the two ordinates L_1A, MB corresponding to times t_1, t_2. Thus the *number* of the units of the distance travelled between the times t_1 and t_2 is graphically represented by the *number* of the units of area AL_1MB under the velocity-time graph and is given by

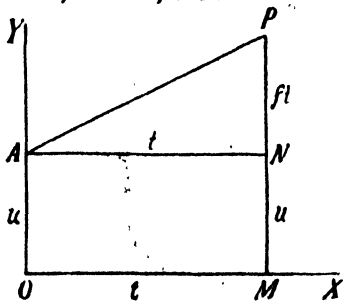
$\Sigma v\delta t$ or the integral $\int_{t_1}^{t_2} v\, dt$.

It can be shown in exactly the same way that *the number of units of velocity generated is represented by the number of units of the area under the acceleration-time graph.*

2·61. As a particular case let us consider the motion of a particle moving with constant acceleration f and initial velocity u.

The acceleration being constant, the gradient of the velocity-time graph is the same everywhere and its graph must, therefore, be a straight line.

Let AP be the graph in which $OM=t$, $OA=u$, the initial velocity and $MP=v$, the final velocity.



$$\frac{NP}{AN}=\tan \angle NAP=f$$

$$\therefore \quad NP=f.AN$$

$$=f.OM=ft.$$

Also $$NP=MP-MN$$

$$=MP-OA=v-u.$$

$$\therefore \quad v-u=ft$$

i.e., $$v=u+ft \qquad \ldots (1)$$

The number of units of the distance described

$$=\text{area } OAPM=\text{area } OANM+ANP$$

$$=OA.OM+\tfrac{1}{2}AN.NP=OA.OM+\tfrac{1}{2}\frac{NP}{AN}AN^2$$

$$=ut+\tfrac{1}{2}ft^2.$$

$$\therefore \quad s=ut+\tfrac{1}{2}ft^2 \qquad (2)$$

Also area

$$OAPM=\tfrac{1}{2}(OA+MP)OM=\tfrac{1}{2}(OA+MP)AN$$

$$=\tfrac{1}{2}(OA+MP)NP.\frac{AN}{NP}$$

$$=\tfrac{1}{2}(OA+MP)(MP-OA)\frac{AN}{NP}.$$

$$\therefore \quad s=\tfrac{1}{2}(u+v)(v-u)\frac{1}{f}=\tfrac{1}{2}(v^2-u^2)\frac{1}{f}$$

i.e., $$v^2-u^2=2fs. \qquad \ldots (3)$$

These results are the same as obtained already in Art. 2·2.

Since the problem of finding the displacement of a point in any interval of time is the same as that of finding the area of the velocity-time graph, some problems in Mechanics can easily be solved by graphical methods.

2·7. Motion of a train between two stations. *A train starts from one station and reaches another station at a distance d from the first, being uniformly accelerated for some time t_1, and uniformly retarded during the last time t_2, the velocity remaining constant for the rest of the journey for a given time t. To find the motion.*

Let A and B be the two stations and let v be the maximum speed of the train.

$A \xrightarrow{S_1,\ f_1} C \quad D \xleftarrow{S_2,\ f_2} B$

Let $AC=S_1$ be the distance and t_1, the time for which the acceleration is f_1,

then $$v=f_1t_1 \text{ and } S_1=\tfrac{1}{2}f_1t_1^2=\tfrac{1}{2}f_1t_1.t_1=\tfrac{1}{2}vt_1.$$

Let $DB=S_2$ be the distance and t_2 the time for which the retardation is f_2, then

$$0=v-f_2t_2 \text{ or } v=f_2t_2$$

and $$S_2=vt_2-\tfrac{1}{2}f_2t_2^2=vt_2-\tfrac{1}{2}vt_2=\tfrac{1}{2}vt_2.$$

Let $CD=S$, be the distance and t, the time for which the train moves with constant velocity v, then

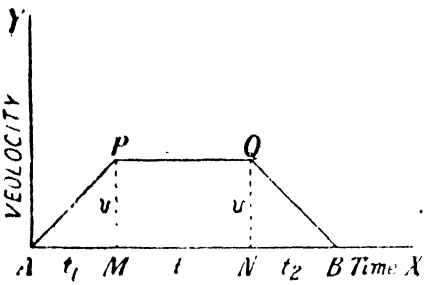
$$S=vt.$$

Hence $$d=S_1+S+S_2=\tfrac{1}{2}v(t_1+2t+t_2) \quad .. \quad (1)$$

and $$v=f_1t_1=f_2t_2 \quad .. \quad (2)$$

2·71. These two equations are important in all such cases and can be very easily obtained from velocity-time graph as follows :—

$APQB$ is the velocity-time graph for the motion of the train. The distance d=area $APQBA$



$=\triangle APM+$rect. $MPQN$ $+\triangle NQB$

$=\tfrac{1}{2}vt_1+vt+\tfrac{1}{2}vt_2$

$=\tfrac{1}{2}v(t_1+2t+t_2)$ .. (1)

The acceleration f_1 = the slope of $AP=\dfrac{MP}{AM}=\dfrac{v}{t_1}$.

The retardation f_2 = the slope of $QB=\dfrac{NQ}{NB}=\dfrac{v}{t_2}$.

$\therefore$ $$v=f_1t_1=f_2t_2 \quad .. \quad (2)$$

In the particular case when $t=0$, *i.e.*, when the train does not move any distance with constant speed, the points P and Q coincide and the first equation becomes $d=\tfrac{1}{2}v(t_1+t_2)$.

Example 1. *A train travels in 3 minutes a distance of one mile from rest at one station to rest at another station. It is uniformly accelerated for the first 30 secs and uniformly retarded for the last 15 secs., the speed being constant for the remaining time. Find the acceleration, the retardation and the maximum velocity.*

$OPQA$ is the velocity-time graph. The maximum velocity v is represented by LP or MQ.

One mile=5280 ft.

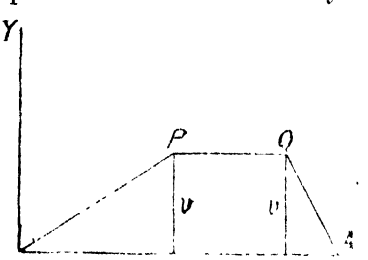
$=$Area $OPQAO$

$=\tfrac{1}{2}.30.v+135v+\tfrac{1}{2}.15.v.$

i.e., $5280=157\tfrac{1}{2}v.$

or $v=33\tfrac{1}{11}$ *f.p.s.*

The acceleration f is given by



$$f=\frac{LP}{OL}=\frac{v}{30}=1\tfrac{1}{10}\tfrac{1}{5}\text{ ft./sec}^2.$$

The retardation $f_1=\dfrac{v}{15}=2\tfrac{2}{11}$ ft./sec^2.

Example 2. *The Calcutta mail leaves the Goraya station, 16 miles from Ludhiana with a velocity of 30 m.p.h. and maintaining this speed for 15½ miles stops at Ludhiana with uniform retardation. Another*

train starts from Goraya at the instant the first passes, and being uniformly accelerated for part of the journey and uniformly retarded for the rest, stops at Ludhiana at the same time as the first. Find the greatest speed.

Let the first station Goraya be A, and the next Ludhiana be L.

PQL is the velocity-time graph for the motion of the mail train.

AP = the constant speed in the first part of the journey = 30 *m.p.h.*

Area $APQM$ = the distance moved with constant speed

$= 15\frac{1}{2} = \frac{31}{2}$ miles.

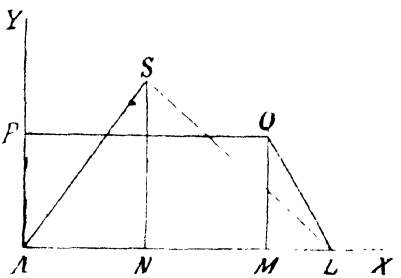
$\therefore$ time AM

$$= \frac{\text{area } APQM}{AP} = \frac{31}{2\times 30}$$

$= \frac{31}{60}$ hr $= 31$ min.

Area MQL = the distance moved under retardation $= \frac{1}{2}$ mile.

$\therefore$ time taken to describe $\frac{1}{2}$ mile



$$= ML = \frac{2MQL}{MQ} = \frac{2\times\frac{1}{2}}{30} = \frac{1}{30} \text{ hr.} = 2 \text{ min.}$$

Total time $AL = \frac{31}{60} + \frac{1}{30} = \frac{11}{20}$ hr.

Let ASL be the graph of the motion of the second train NS, representing the max. velocity, say V.

Area ASL = the whole distance = 16 miles.

$\therefore$ $\quad \frac{1}{2}V.AL = 16$ or $\frac{1}{2}V.\frac{11}{20} = 16.$

$\therefore$ $\quad V = \frac{640}{11} = 58\frac{2}{11}$ *m.p.h.*

Examples IV

1. A train travels between two stopping stations 7 miles apart in 14 minutes. Assuming that its motion is one of uniform acceleration for part of the journey and of uniform retardation for the rest, prove that the greatest speed on the journey is 60 m.p.h. *(P.U. 1954)*

2. A train starting from rest, is uniformly accelerated during the first $\frac{1}{4}$ mile of its run, then runs $\frac{3}{4}$ mile at the uniform speed acquired, and is afterwards brought to rest in $\frac{1}{4}$ mile under uniform retardation. If the time for the whole journey is 5 minutes, find the uniform acceleration and the uniform retardation and show that the latter is double of the former.

3. An electric train covers a distance of 2·76 km. between two stops in $2\frac{1}{2}$ mins. The train is first constantly accelerated for 40 secs. to a maximum speed at which it continues to run. Finally it is brought to rest in 30 secs. by a constant retardation. Find the acceleration, the retardation and the constant speed.

4. A lift ascending from a point 640 ft. deep rises during the first part of its ascent with uniform acceleration. On nearing the top the upward force is cut off and the velocity of the lift is just sufficient to carry it to the top. If the whole process occupies one minute, find the acceleration during the first part of the ascent and the maximum velocity attained.

5. A lift performs the first part of its ascent with uniform acceleration f and the remainder with uniform retardation $2f$. Prove that if h is the depth of the shaft, t the time of ascent, $h=\frac{1}{3}ft^2$.

6. Two stopping points of an electric tramcar are 440 yards apart. The maximum speed of the car is 20 miles per hour, and it covers the distance between stops in 75 seconds. If both acceleration and retardation are uniform and the latter is twice as great as the former, find the value of each of them, and also calculate how far the car runs at its maximum speed. (*D.U., P.U.*)

7. The greatest possible acceleration of a train is 3 *f.s.s.* and the greatest permissible retardation is 4 *f.s.s.* Find the least time taken to run between two stations 8 miles apart if the maximum speed is 45 miles per hour.

8. A train stopping at two stations two miles apart takes 4 minutes on the journey. If the motion is first that of uniform acceleration f, and then of uniform retardation r, prove that

$$\frac{1}{f}+\frac{1}{r}=4$$

taking one mile as the unit of distance and one minute as the unit of time.

9. A body travels a distance s in t seconds. It starts from rest and ends at rest. In the first part of the journey it moves with constant acceleration f and in the second part with constant retardation r. Show that

$$t=\sqrt{2s\left(\frac{1}{f}+\frac{1}{r}\right)}.$$

10. A lift ascends with constant acceleration f, then with constant velocity and finally stops under constant retardation f. If the total distance ascended is s and the total time occupied is t, show that the time during which the lift is ascending with constant velocity is

$$\left(t^2-\frac{4s}{f}\right)^{\frac{1}{2}}.$$

11. A train goes from one station to another, being uniformly accelerated from rest in the first quarter of the distance, and being brought to rest by uniform retardation during the last quarter. The middle half of the journey is performed at a uniform speed. Show that the average speed of the train is two-thirds of the full speed.

12. Distance between Barara and Ambala is 15 miles. A train passes Barara without stopping at 25 miles an hour and maintaining this speed for 14 miles stops at Ambala with uniform retardation. Another train starts from Barara at the instant the first train passes and being uniformly accelerated for part of the journey and uniformly retarded for the rest stops at Ambala at the same time as the first. Show that its maximum speed is $46\frac{2}{3}$ m.p.h.

REVISION QUESTIONS—I

1. A particle moves along a straight line starting with velocity u and having a constant acceleration f. Prove that the distance from the starting point after time t is $ut+\frac{1}{2}ft^2$.

A point moving in a straight line with uniform acceleration describes distances a, b feet in successive intervals t_1, t_2 seconds. Prove that the acceleration is

$$\frac{2(bt_1-at_2)}{t_1t_2(t_1+t_2)}$$

also that if the point describes successive equal distances in times t_1, t_2, t_3 then

$$\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}=\frac{3}{t_1+t_2+t_3}.$$

2. A bullet travelling horizontally pierces successively three screens placed at equal distances a apart. If the time from the first screen to the second be t_1 and from the second to the third be t_2, prove that the retardation (assumed to be constant) is $2a(t_2-t_1)/t_1t_2(t_1+t_2)$ and that the velocity at the middle screen is

$$\frac{a(t_1^2+t_2^2)}{t_1t_2(t_1+t_2)}.$$ (*P.U. 1953*)

[**Hint.** Thickness of the screens is to be neglected.]

3. A particle is moving in a straight line with constant acceleration. Show that its acceleration is

(*i*) $(S-S')/(m-n)$, where S and S' are the spaces described during the mth and nth seconds respectively.

(*ii*) $2\left(\frac{S'}{t'}-\frac{S}{t}\right)\Big/(t+t')$, where S is the space described during any t seconds and S' during next t' seconds. (*P.U. 1955*)

4. A particle moves in a straight line with constant acceleration, and its distances from an origin O on the line (not necessarily the position at time $t=0$) at times t_1, t_2, t_3 are x_1, x_2, x_3 respectively. Show that if t_1, t_2, t_3 form an A.P. whose common difference is d and x_1, x_2, x_3 are in G.P. then the acceleration is $\frac{(\sqrt{x_1}-\sqrt{x_3})^2}{d^2}$ assuming x_1, x_2, x_3 to be positive.

5. An express train is run in two parts, the first part is given 5 mins. start and attains its maximum speed of 48 m.p.h. in one mile. Prove that the two parts will run 4 miles apart but the first part would have gone 3 miles before the start of the second.

6. An engine driver suddenly puts on his brake and shuts off steam when he is running at full speed; in the first second afterwards the train travels 87 feet, and in the next 85 feet. Find the original speed of the train, the time that elapses before it comes to rest and the distance it will travel in this interval, assuming the brakesto cause a constant retardation.

Find also the time the train will take, if it be 96 yards long, to pass a spectator standing at a point 484 yards ahead of the train at the instant when the brake was applied.

*7. If the velocity of a bullet is supposed to be a uniform horizontal velocity equal to n times that of sound, show that the points at which sounds of the firing and of the bullet striking the target are heard simultaneously lie on a hyperbola of eccentricity n. (*Pub. Ser. Ex.*)

*8. A stone is dropped from an aeroplane which is rising with acceleration f, and t seconds after this another stone is dropped. Prove that the distance between the stones at a time T after the second stone is dropped is

$$\tfrac{1}{2}(g+f)(t+2T)t.$$

*9. A juggler keeps three balls going with one hand, so that at any instant two are in the air and one in his hand. Find the time during which a ball stays in the hand if each ball rises to a height (*i*) 16 ft., (*ii*) S ft.

*10. It is estimated that g is increased by the resistance of the air to $g(1+f)$ in upward motion and reduced to $g(1-f)$ in downward motion; find the time a stone is in the air when projected upwards with velocity u.

Find the value of f which will make the time the same as in unresisted motion.

*11. A train travelling at 60 m.p.h. has to slow down on account of work being done on the line. Instead of continuing at a constant speed, it, therefore, moves with a constant retardation of 1 mile per hour per second until its speed is reduced to 15 m.p.h. It then travels at a constant speed for $\frac{1}{4}$ mile, and then accelerates at $\frac{1}{4}$ mile per hour per second until its speed is once more 60 m.p.h. Find the delay caused. (*Bom., Univ.*)

12. The speed of a train increases at a constant rate α from 0 to v, then remains constant for an interval and finally decreases to 0 at a constant rate β. If l be the total distance described, prove that the total time occupied is

$$\frac{l}{v}+\tfrac{1}{2}v\left[\frac{1}{\alpha}+\frac{1}{\beta}\right].$$

Also find the least value of time when $\alpha=\beta$. (*E.A.C.*)

13. A train takes time T to perform a journey, it travels for time $\frac{T}{n}$ with uniform acceleration, then for time $(n-3)\frac{T}{n}$ with uniform speed v, and

finally for time $\frac{2T}{n}$ with constant retardation. Prove that its average speed is

$$(2n-3)\, v/2n.$$

14. For $\frac{1}{m}$ of the distance between two stations a train is uniformly accelerated and for $\frac{1}{n}$ of the distance it is uniformly retarded; it starts from rest at one station and comes to rest at the other.

Prove that the ratio of its greatest velocity to its average velocity is

$$1+\frac{1}{m}+\frac{1}{n} : 1. \qquad (A.U., D.U.)$$

***15.** A steamer is timed to run from Bombay to Aden at an average rate of 8 knots per hour, but owning to heavy sea loses 12 hours in the first part of the voyage, it then accelerates at the rate of half a knot per hour per hour uniformly till the speed reaches 10 knots which speed is maintained till the end of the voyage. If the steamer now reaches Aden in time, prove that it started accelerating when at a distance of 496 sea miles from Aden.

(*P.U.*)

16. A stone is dropped from the top of a tower into a well at its foot, and it reaches the water with a velocity of 128 ft./sec. The sound of the splash is heard $4\frac{3}{5}$ seconds after the stone was dropped. Find the speed of sound.

CHAPTER III

MOTION IN A STRAIGHT LINE—II

Nature and Nature's Laws hid in night
God said "Let Newton be" and all was light.

3.1. Newton's Laws of Motion. In the preceding chapters motion of a material body has been studied without reference to its *mass*, the quantity of matter in the body, and *force*, the cause of motion. In the present chapter we shall consider motion under the application of forces to a particle.

The treatment of the subject is based on experimental laws, known as Newton's laws of motion, which we now discuss.

3·11. Law 1. *Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by impressed forces to change that state.*

According to this law, force is necessary in the following cases :—

(1) To change the state of rest of a body or to bring the body into motion.

(2) To increase or decrease the speed of a body moving in a straight line.

(3) To change the direction of motion of a body without even changing its speed.

So that a force may be defined as follows :—

Force *is something which changes or tends to change the state of rest or uniform motion in a straight line, of a body.*

3·12. Impressed Force. An external force is usually called an *impressed force.* The mutual action and reaction between the molecules or parts of the same body are *internal forces.*

The law states that the internal forces of a body do not alter uniform motion in a straight line and that the cause of the change in motion must be *external.*

3·13. The first law of motion implies what is sometimes called the *Principle of Inertia*, that is, a body has no tendency of itself to change its state of rest or motion and if it is kept free from the action of external forces, it will remain in its state of rest or of uniform motion in a straight line. Thus the state of rest or of uniform motion in a straight line is the natural state of a body. (The Greek philosophers wrongly believed motion in a circle to be inherently natural and perfect). It is only the presence of an external force which can alter this natural state and the law asserts that a body upon which no force acts must be either at rest or else it must be

moving in a straight line with a uniform velocity and that *if a body is at rest or it moves with uniform velocity in a straight line, the external forces, if any, which act on it, balance and are in equilibrium.*

Suppose, for example, a railway carriage is at rest. Obviously some force is necessary to set it in motion. If it is pushed by an engine, a force has acted, and the carriage begins to move and acquires a certain velocity. Now, on separation from the engine, unless some forces act on the carriage it must go on moving indefinitely with the same constant velocity in the same straight line in which it is now moving. But we know from experience that its speed goes on decreasing till it stops. Hence some forces must have acted against the motion. These forces are the resistance of air and the force of friction. The law states that in the absence of friction and of any other interfering agent such as the resistance of the air, the carriage would go on for ever.

Again, if the track is curved instead of being straight, the carriage no longer moves in a straight line, there is a change in direction which must be due to the action of some force. The force in this case is the pressure of the rails on the flanges of the wheels which tends to turn the carriage in the direction of the rails. Had the flanges been absent, this force could not be exerted and motion would have taken place along a straight line causing the carriage to go off the rails.

Thus if a carriage in a railway train moves with constant velocity in a straight line, the external forces which act on it—such as the friction of the rails, the resistance of the air acting backwards, the weight of the carriage acting downwards, the pressure of the rails acting upwards, the traction of the carriage in front of it and the drag of that behind it—must all exactly balance each other.

3·2. Having studied the necessity of a force to change the natural state of a body, we now consider the law governing the measurement of force.

3·21. Law II. *The rate of change of momentum is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

Momentum is the product of the mass of a body and the velocity with which it is moving.

If m denotes the mass and v the velocity, mv is the momentum. If P denotes the force acting on the mass m, then by the first part of this law, we have $P=k\frac{d}{dt}(mv)$.

In most of the problems with which we are concerned at present, m is constant and so the equation becomes

$$P=km\frac{dv}{dt}=kmf,$$

where k is the constant of proportionality and its value depends upon

the units chosen. If the unit of force be so chosen that unit force acting on unit mass produces unit acceleration, *i.e.*, $P=1$, when $m=1$ and $f=1$, then $k=1$, and thus

$$\mathbf{P=mf.}$$

This is the fundamental equation of Dynamics and is sometimes written in the form

$$\text{Acceleration } f=\frac{P}{m}=\frac{\text{Force causing motion}}{\text{Mass moved}}.$$

Thus if a body is accelerating some force must be acting on it and in case it is not accelerating the resultant of the external forces acting on it must be nil, so that the effect of a force on a body does not in the least depend on the velocity that the body already has.

When the impressed force $P=0$ then $\frac{d}{dt}(mv)=0.$

$$\therefore \qquad mv=\text{a constant.}$$

Hence a body continues to move with constant momentum when not acted upon by an impressed force. This shows that Law I is a particular case of Law II.

3·22. Units of Force. The unit of force as implied in the fundamental equation $P=mf$ can be defined as follows :—

The unit of force is that force which acting on unit mass generates in it unit acceleration.

In the English (Foot Poudnd-Second or F.P.S.) system the unit of mass is one pound, the unit of acceleration is one foot per second per second and the unit of force is the **Poundal.**

Hence one poundal is that force which will create in a mass of one pound an acceleration of one foot per second per second.

In the French (Centimetre-Gramme-Second or C.G.S.) system the unit of mass is one gramme, the unit of acceleration is one centimetre per second per second and the unit of force is the **Dyne.**

Hence one dyne is that force which will give to a mass of one gramme an acceleration of one centimetre per second per second.

These units of force are called *Absolute Units* because their values are the same everywhere and do not depend on the earth's attraction.

3·23. Weight. Any body let fall in vacuum at a particular place (near the surface of the earth) drops downwards with constant acceleration g, showing that it is acted on by some constant force. This force is the one with which the earth attracts the body and is called the *weight* of the body. Weight is thus a force and is, therefore, measured by the formula $P=mf$.

Writing W for P and g for f, we get $W=mg$,
which gives in absolute units the weight of a body of mass m.

When m is one pound, the weight is g poundals.

Hence the weight of one pound is equal to a force of g (32) poundals.

Similarly when m is one gramme, the weight is g dynes.

Hence the weight of one gramme is equal to a force of g (981) dynes.

It is often convenient to compare force with the weight of a body, and to speak of the force of so many pound-weight or gramme-weight.

One pound-weight or simply one pound and similarly one gramme are taken as units of force in engineering practice. They are called *gravitational units* as they depend on acceleration due to gravity which is slightly different at different places of the earth's surface and this is the reason why a given mass extends the spring of a spring balance to slightly different extents at different places on the earth. Here the mass, *i.e.*, the quantity of matter in the body, remains the same, but the weight changes.

The student must distinguish carefully between the two meanings of the words *pound*, namely (*a*) mass, and (*b*) a force, two totally different physical concepts. Thus a pound of lead is a certain quantity of *matter*. If it is hung by a string, the tension in the string is a pound of *force*.

In working problems in Mechanics it is best to express force in absolute units, *i.e.*, units independent of gravity, *viz.*, the poundal and the dyne and to change them afterwards, if required in pound-weight and gramme-weight by dividing them by 32 and 981 respectively.

Example. *A body of mass 16 lbs. is pulled along a smooth horizontal table by a constant force. It describes 18 feet from rest in 3 secs. Find the magnitude of the force.*

Since $s=ut+\frac{1}{2}ft^2$ $(u=0,\ s=18,\ t=3)$,

$\therefore$ $18=\frac{1}{2}f.9$, *i.e.*, $f=4$ ft./sec.^{2}

Force is given by the formula

$$P=mf=16\times 4=64 \text{ poundals}$$
$$=\tfrac{64}{32} \text{ or } 2 \text{ lbs. wt.}$$

3·24. Physical Independence of Forces. We have learnt from the first part of the second law of motion that a force is the cause of acceleration and is proportional to the acceleration produced by it. The latter part states that the change of momentum produced by a force is in the direction in which the force acts, that is, if two or more forces act on a body at the same time each produces an acceleration proportional to it in its own line of action independent of the motion of the body. Hence we arrive at the following principle, usually called the principle of the **Physical Independence of Forces** :

When any number of forces act on a body, the acceleration due to each force is the same in direction and magnitude as if the other has not been in action.

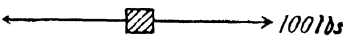
Illustration. A cricket ball thrown horizontally in air is under the effect of two forces : (*i*) resistance of the air, and (*ii*) its weight.

The resistance of the air opposes motion and produces an acceleration in a direction opposite to that of the motion of ball, while the weight of the ball moves it towards the earth and produces an acceleration vertically downwards. The two forces act independently of each other, each producing its own acceleration in the direction of its own line of action.

The effect of a number of forces acting on a particle along one line in either direction is the same as the effect of one force equal to the algebraic sum of all the forces.

So that the effect of the two forces P and Q acting on a particle along a line in one direction will be the same as that of a force $P+Q$ in the same direction and the effect of P and Q acting along a line in opposite directions will be that of a force $P-Q$ in the direction of the greater force P.

Example 1. *A stone weighing 200 lbs. is being dragged from rest by a man who can exert a uniform force of 100 lbs. wt. What distance will the stone move in two secs. when the resistance to motion is 20 lbs. wt.?*



P, the effective force on the stone $= 100-20 = 80$ lbs. wt.
$= 80 \times 32$ poundals.

The acceleration produced $= \frac{P}{m} = \frac{80 \times 32}{200} = \frac{64}{5}$ ft./sec^2.

The distance moved from rest in 2 secs.
$= \frac{1}{2}ft^2 = \frac{1}{2} \times \frac{64}{5} \times 4 = 25{\cdot}6$ ft.

Example 2. *An engine and train weigh 210 tons, and the engine exerts a pull of $3\frac{1}{2}$ tons. The resistance to the motion of the train is 14 lbs. wt. per ton. Find the time the train will take to acquire a velocity of 30 m.p.h. starting from rest.*

F, the tractive force of the engine, is $3\frac{1}{2}$ tons $= \frac{7}{2} \times 2240g$
$= 7840g$ poundals,

and R, the resistance, is $210 \times 14g = 2940g$ poundals.

$\therefore$ P, the resultant accelerating force $= F-R$
$= (7840-2940)g = 4900g$ poundals.

$\therefore$ the acceleration $= \frac{P}{m} = \frac{4900 \times 32}{210 \times 2240} = \frac{1}{3}$ ft./sec.^{2}

To find the time to acquire a velocity of 20 m.p.h. $= 44$ ft./sec., we use $v = u+ft$ $(u=0)$

$\therefore$ $44 = \frac{1}{3}t$, *i.e.*, $t = 132$ secs. $= 2\frac{1}{5}$ min.

Example 3. *In the above example, it is given that the resistance due to the brakes is 126 lbs. wt. per ton; find the distance which the train subsequently runs when (i) the steam is shut off, (ii) the steam is shut off and brakes are put on.*

(*i*) The retarding force $R=14\times210g$ poundals,

$\therefore$ the retardation $=\frac{R}{m}=\frac{210\times14\times32}{210\times2240}=\frac{1}{5}$ ft./sec^2.

To find the distance travelled in losing the velocity of 44 ft. per sec., after the steam is shut off, we use $v^2=u^2+2fs$.

$\therefore$ $0=44\times44-2\times\frac{1}{5}S$.

i.e., $S=22\times44\times5$ ft.$=1\frac{1}{2}$ mile.

(*ii*) The retarding force due to the brakes

$=126\times210g$ poundals.

$\therefore$ the additional retardation due to brakes

$=\frac{210\times126\times32}{210\times2240}=\frac{2}{5}$ ft./sec^2.

$\therefore$ the total retardation $=\frac{1}{5}+\frac{2}{5}=2$ ft./sec^2.

The distance travelled by the train in this case is given by

$0=44\times44-2\times2S$, *i.e.*, $S=484$ ft.

Examples V

1. A body of 25 lbs. mass is acted upon by a constant force which in 10 secs. gives it a velocity of 75 ft. per sec. What is the magnitude of the force ?

2. A train weighing 110 tons is running at 15 miles per hour when steam is shut off and a resistance of 10 lbs. per ton acts continuously until it is brought to rest. Find the distance it travels after steam is shut off, and the time taken to come to rest.

3. A body whose mass is 5 pounds moving with a speed of 110 ft. per sec. suddenly encounters a constant resistance equal to the weight of $\frac{1}{4}$ pound which lasts until the speed is reduced to 90 ft. per sec. For what time and through what distance has the resistance acted ?

4. A railway truck weighing half a ton is pulled by a constant force and travels 50 ft. from rest in 10 secs. Find the magnitude of the force. (*P.U.*)

5. A particle of mass $\frac{1}{4}$ lb. moves in a straight line so that its distance s from the origin at the end of time t is given by the equation

$$s=5+4\sin 3t.$$

Find the maximum force upon it during the motion.

6. A bullet weighing 81 grammes and moving at the rate of 200 cms. per sec. is fired into a log of wood into which it penetrates 10 cms. If the bullet moving with the same velocity were fired into a similar piece of wood 5 cms. thick, with what velocity would it emerge ? Find also the force of resistance, supposing it to be uniform.

7. A train weighing W tons is moving with an acceleration f, when a carriage of weight w tons is suddenly detached from it. Find the change in the acceleration of the train (resistance neglected).

8. A mass m is acted on by a constant force of P lbs. wt. under which in t secs. it moves a distance x feet, and acquires a velocity v ft. per sec. Show that

$$x=\frac{gt^2P}{2m}=\frac{v^2m}{2gP}. \qquad (M.T.)$$

9. A bucket of water weighing 160 lbs. is being raised from a well 100 ft. deep with a uniform force of 200 lbs. wt. With what acceleration will it move and what time does it take to reach the top of the well ?

10. A small balloon of total mass 10 lbs., at a height of 100 ft. above the ground, is moving upwards with a velocity of 60 ft./sec. and acceleration

4 ft./sec^2., when a mass of 1 lb. becomes detached. Find the new acceleration of the balloon assuming that the upward force acting upon it remains unaltered.

Also find how many seconds it takes the 1 lb. mass to reach the ground, neglecting air resistance. (*Lucknow U.*)

11. A ball of mass 8 oz. falls freely under gravity through a distance of 25 ft. In the process of catching a man allows his hands to drop a distance of 1 ft. Find the average pressure on his hands during the catch.

12. An engine of mass 30 tons pulls after it a train of 130 tons ; supposing the friction to be $\frac{1}{50}$th of the weight of the whole train, calculate the force exerted by the engine, if at the end of the first mile from the start the speed be raised to 45 m.p.h. (*P.U.*)

13. An engine exerts a uniform force of 2 tons weight on a train of 100 tons when there is a resistance of 14 lbs. *wt. per.* ton. Find the time required to get up a velocity of 30 miles per hour from rest.

14. A train of mass 160 tons starts from a station, the engine exerting a tractive force of 2¼ tons in excess of the resistances until a speed of 37½ m.p.h. is attained. This speed continues constant until the brakes causing retardation of 2½ ft./ sec.^{2} bring the train to rest 5 miles away. Find the time taken (*i*) during acceleration ; (*ii*) during retardation ; (*iii*) altogether.

15. A train of mass 120 tons is travelling with a uniform speed, the resistance due to friction etc. being 14 lbs. wt. per ton. If a portion of mass 20 tons is slipped, how much will the other portion have gained on it in 12 seconds assuming the pull of the engine and the resistance per ton to be the same as before ?

3·3. Law III. *To every action there corresponds an equal and opposite reaction.*

Whenever two bodies are moving or are at rest, the one cannot exert a force on the second, without the second exerting a force on first at the same time.

It is impossible for any force to exist in the universe without the existence at the same time of another force equal and opposite to it in the same straight line. If two masses act on each other in any way either by actual contact, or by connection with a string or a rod, or by mutual attraction or repulsion, the force exerted by one mass on the other has the *same magnitude* as, but is *opposite in direction* to, that which the other mass exerts on the first.

The whole phenomena of the mutual action between two bodies or between portions of a body Is called a *Stress*.

If we consider one portion of a material body and the effect on it of the other portion we call *Action*, the opposite aspect of the stress on the other portion is called *Reaction*.

The stress is measured numerically by the force exerted *on* either of the two portions of the body by the other.

Illustrations. (1) A body supported on hand, presses the hand with a force equal to the weight of the body. The hand reacts and exerts pressure on the body equal to its weight.

(2) In the case of water contained in a cistern, the water exerts some pressure at every point on the wall of the cistern and an equal pressure is exerted at the point by the wall of the cistern on the water.

(3) The earth attracts a body of mass m with a force equal to mg. The body of mass m also attracts the earth with an equa

amount of force, although it is not so obvious because the acceleration created by the body in the earth (obtained by dividing mg by the mass of the earth) is so small that it is negligible.

(4) In a tug-of-war between two parties the force exerted at any point of the rope in one direction is exactly equal to that exerted in the other direction.

Then how does one party win over the other? This must be due to some external cause. When members of one party pull at the rope they push the ground backwards with their feet, and an equal amount of the force of friction between the feet and the ground acts forwards. If the total amount of friction in one direction is, say F, and if it is F' in the other direction, it is only when F and F' are unequal that one party wins over the other.

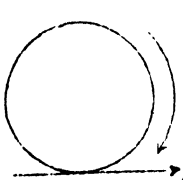
(5) **Horse and Carriage.** In the case of a carriage driven by a horse the force of tension with which the horse pulls the carriage forward is exactly the same as that with which the carriage pulls the horse backwards.

The force of tension, in this case, is not an external force, but a mere internal action (mutual action and reaction) between the two parts of the system (carriage and horse together) and so it can no more effect motion of the system as a whole than any of the other forces which hold the parts together.

The system begins to move on account of the external force, *viz.*, the friction between the horse's feet and the ground. The horse in trying to go forward, pushes the ground backward with the feet; the ground being rough, an equal amount of the force of friction, say F, acts forwards. Friction also acts on the carriage. Due to the presence of the wheels the total amount of the friction, say F', checking the motion of the carriage is much smaller than F. It is the presence of the external force $F-F'$, the difference between the two frictions, that makes the system move.

(6) **Engine and Carriage.** In the case of a carriage pulled by an engine, the tensions in the two directions at any point of the coupling chain are equal and opposite and hence the force with which the engine pulls the carriage is the same as the force with which the carriage pulls the engine. How the system begins to move may be explained thus :—

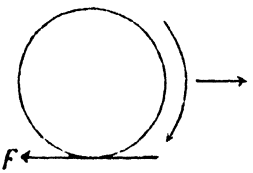
The driving wheels of the engine are made to rotate by means of a crank attached to the axle, the crank being moved by the mechanism. If there were no friction between the wheels and the rails, the wheels would slip round and rotate while the engine would remain stationary. The slipping of the wheels is opposed by the force of friction, say F, between the rails and the wheels. *This frictional force acts in the forward direction and causes the forward motion of the engine.* It is usually spoken of as the force exerted by the engine or the propulsive force.



It is important to note that F *cannot exceed* the limiting amount of friction which is μ (the co-efficient of friction) *times the pressure*

between the driving wheels and rails. This pressure being equal to the weight, to avoid slipping, as much weight as possible is brought on the driving wheels.

Again, if there were no friction between the wheels and the rails, the other wheels of the engine and the wheels of the carriage would move forward and slide without rotation. The friction in this case acts in the opposite direction to the motion of the engine and makes the wheels rotate. The resultant magnitude of this force, say F', is small as compared to F, the frictional force on the driving wheels. $F-F'$ is the resultant effective force which makes the system move.



The motion of a bicycle or a motor car is also similar—the hind wheels correspond to the driving wheels of the engine.

Note. The excess of the tractive force over all resistances is the accelerating force. If there is no excess *i.e., if the tractive force just balances the resistances, the train or the vehicle moves at a constant speed.*

Example 1. *An engine weighing 120 tons is coupled to a carriage of mass 80 tons. The system starts from rest with constant acceleration and acquires a velocity of 45 m.p.h. on the level in two minutes. Find (i) the force exerted by the engine and (ii) the tension in the coupling chain, neglecting resistances.*

The velocity acquired $=45$ m.p.h. $=66$ ft./sec.

The time to acquire this velocity $=2$ min. $=120$ sec.

$\therefore$ the acceleration f is given by

$$66=120f, \text{ or } f=\tfrac{11}{20} \text{ ft./sec.}^2$$

M, the total mass $=120+80=200$ tons $=200\times2240$ lbs.

Neglecting resistances, the force exerted by the engine is given by

$$P=Mf=200\times2240\times\tfrac{11}{20} \text{ poundals.}$$

$$=\frac{200\times2240\times11}{32\times2240\times20} \text{ or } \frac{55}{16} \text{ or } 3\tfrac{7}{16} \text{ tons wt.}$$

Now if T poundals be the tension in the coupling chain, the equation of motion for the engine is

$$mf=P-T, \textit{ i.e.,}\ 120\times2240\times\tfrac{11}{20}=P-T.$$

$$\therefore \quad T=P-\frac{120\times2240\times11}{32\times2240\times20}=3\tfrac{7}{16}-2\tfrac{1}{16} \text{ or } 1\tfrac{3}{8} \text{ tons wt.}$$

Note. The same result is obtained by writing the equation of motion for the carriage, *viz.* $80\times2240f=T$,

$$\text{or} \quad T=\frac{80\times2240\times11}{32\times2240\times20}=1\tfrac{3}{8} \text{ tons wt.}$$

Example 2. *In the above example, show that W, the part of the*

weight carried by the driving wheels of the engine cannot be less than 15⅝ tons, if μ, the co-efficient of friction is 0·22.

The maximum amount of friction available $=\mu W$.

In order that the wheels may roll and not slide, this frictional force should not be less than the effective force

i.e., $\mu W > P$ or $W > P/\mu$.

$\therefore$ the least value of $W = P/\mu = \frac{55}{16} \div \frac{22}{100} = \frac{55 \times 100}{16 \times 22}$

$= 15\frac{5}{8}$ tons.

Examples VA

1. A train of 300 tons drawn by an engine weighing 100 tons starts from rest with constant acceleration and acquires a velocity of 30 m.p.h. on the level in one minute. Find the acceleration and the force exerted by the engine.

Find also the force of tension in the coupling between the engine and the rest of the train. [*Ans.* $\frac{1}{6}\frac{1}{0}$ ft./sec.^{2} ; $9\frac{1}{2}$ tons ; $6\frac{1}{2}$ tons]

2. An engine of mass 105 tons is coupled to and pulls a carriage of mass 30 tons ; the resistance to the motion of the engine is $\frac{1}{1}\frac{1}{0}\frac{1}{0}$ of its weight and that of the carriage $\frac{1}{1}\frac{1}{5}\frac{1}{0}$ of its weight. Find the tension in the couplings if the whole tractive force exerted by the engine is equal to 6,000 lb. wt.

[*Ans.* 1159½ lbs.]

3. A train of 200 tons, uniformly accelerated acquires in two minutes from rest a velocity of 30 m.p.h. Show that, if the co-efficient of friction be 0·18, the part of the load carried by the driving wheels of the engine cannot be less than 12·7 tons.

Show also that if the part of weight of the engine supported by the driving wheels is 25 tons, the velocity at the end of a minute after starting will be less than $29\frac{5}{11}$ miles per hour.

[**Hint.** The limiting amount of friction between the driving wheels and the rails should at least be equal to the force exerted by the engine.]

MOTION OF CONNECTED PARTICLES

3·4. Field of Force. A region in which force exists is known as a *field of force*. A field of force is said to be determined when the acceleration imparted to a material particle at any point in the field is known. When the acceleration is the same at every point, the field is said to be constant or uniform. We know that all bodies are attracted towards the earth's centre and if we consider only a small region on the earth's surface, the acceleration due to the attraction of the earth may be regarded as parallel and constant at every point. This region is then a uniform field and we call it the *field of gravity*.

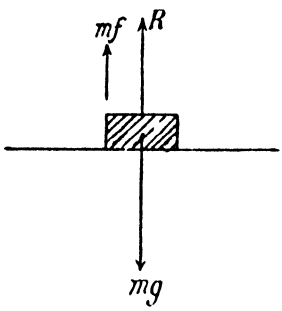
In the second chapter we considered the motion of a particle moving freely in such a field, we now consider the motion of particles which are not free to move but are connected with other particles or are in contact with other bodies.

3·5. Weight carried by a lift. *A body of mass m lbs., is carried by a lift moving with upward acceleration f, to find the pressure on the plane of the lift.*

The mass carried, being in contact with the lift, also moves with acceleration f upwards. The forces acting on the mass are—

(1) the reaction R of the plane of the lift upwards (which is equal and opposite) to the pressure which the mass exerts on the plane, and

(2) its weight mg acting downwards.



The total force upwards acting on the mass is $R-mg$.

$\therefore$ by Newton's second law, the equation of motion for the mass is

$$mf=R-mg$$

i.e.,
$$R=mg+mf \quad \ldots (1)$$

which shows that the pressure is greater than the weight of the mass by amount mf.

If the lift is moving with a downward acceleration f and if R' is the reaction of the lift on the mass, $mg-R'$ is then the total force acting *downwards* on it and its equation of motion is

$$mf=mg-R' \quad \text{or} \quad R'=mg-mf \quad \ldots (2)$$

In this case the pressure R' is less than the weight of the body.

When $f=g$, $R'=0$, *i.e.*, the body moves freely ; and when $f>g$, R' is to be negative in order that there might be contact of the mass with the lift, *i.e.*, the lift should pull the mass instead of pushing it, which is impossible. In this case the body is left behind and moves down freely.

It is important to notice that here we are concerned with only the acceleration of the lift and not with its direction of motion. The pressure will be the same when either the lift is ascending with increasing speed and acceleration f or when it is descending, its speed decreasing at the same rate, the acceleration in both cases being f in the upward direction. Thus it is only the acceleration that affects pressure and not the velocity.

When $f=0$, $R=mg$, *i.e.*, when the lift is moving upwards or downwards with constant velocity or is at rest, the pressure is equal to the weight of the body.

Example 1. *A weight of 10 lbs. is placed on a horizontal shelf which is made to ascend with (1) a uniform acceleration 2 ft./sec.² (2) a uniform velocity of 2 ft. per sec. Find the pressure on the shelf in each case.*

(1) The pressure $R=m(g+f)$

$=10(32+2)$ poundals

$=\frac{340}{32}$ lbs. wt. $=10\frac{5}{8}$ lbs. wt.

(2) $f=0$, $\therefore$ $R'=mg=$ the weight $=10$ lbs. wt.

Example 2. *A load W is to be raised by a rope, from rest to*

rest through a height h ; the greatest tension which the rope can bear is nW. Show that the least time in which the ascent can be made is

$$\left[\frac{2nh}{(n-1)g}\right]^{\frac{1}{2}}.$$

In all such cases the load W, in the first stage, is raised by a certain constant tension T $(>W)$, so that it moves with a constant acceleration and acquires speed. In the next stage, in order to bring it again to rest, the string is made slack and the load moves up freely under retardation g due to gravity till it stops.

Let f be the acceleration in the first stage in this case, then

$$Wf=Tg-Wg, \quad \text{or} \quad f=(T-W)g/W \qquad ..(1)$$

Let s_1, s_2 be the distances described in times t_1 and t_2 in the two stages and let V be the maximum velocity at the end of the first stage, then

$$V=ft_1=gt_2, \text{ so that} \qquad t_1+t_2=V\left(\frac{1}{f}+\frac{1}{g}\right) \qquad ..(2)$$

Also $$h=s_1+s_2=\tfrac{1}{2}V(t_1+t_2) \quad \text{or} \quad t_1+t_2=2h/V \qquad ..(3)$$

$\therefore$ from (2) and (3)

$$(t_1+t_2)^2=2h\left[\frac{1}{f}+\frac{1}{g}\right]=\frac{2h}{g}\left[\frac{W}{T-W}+1\right], \text{ by (1)}$$

$$=\frac{2h}{g}\cdot\frac{T}{T-W}=\frac{2h}{g}\cdot\frac{1}{1-W/T}.$$

Now this is least when T is greatest *i.e.*, when $T=nW$ as given in the question.

$\therefore$ the required least time

$$=\left[\frac{2h}{g}\cdot\frac{1}{1-W/nW}\right]^{\frac{1}{2}}=\left[\frac{2nh}{(n-1)g}\right]^{\frac{1}{2}}.$$

Examples VI

1. A man weighing 110 lbs. stands on an elevator moving downwards. The elevator moves at first with an acceleration of 6 ft. sec.^{2}, then with constant velocity, and finally with retardation of 6 ft./sec.^{2}. Find the pressure on the elevator exerted by the man in the three cases.

What happens when the acceleration of the elevator is more than the acceleration due to gravity ?

2. A diver jumps into a well with a life-belt of weight 10 lbs. on his head. What pressure does the belt exert on him before he reaches the surface of water.

3. A mass of 12 lbs. is weighed by an ordinary spring balance in a lift moving unwards with an acceleration of 2 ft./sec.^{2}. What is its apparent weight ?

4. A 4 lb. weight carried on a spring balance in a balloon has an apparent weight of 4·8 lbs. when the balloon is ascending. What is the acceleration of that balloon? What should the body weigh if the balloon is descending (*i*) with an acceleration of 10 ft./sec.^{2}, (*ii*) with uniform velocity of 20 miles per hour, (*iii*) freely ?

5. A body whose true weight is 13 oz. appeared to weigh 12 oz. when weighed by means of a spring balance in a moving lift. What was the acceleration of the lift at the time of weighing ? (*D.U.*)

6. A string can support a weight of 41 grammes. What is the greatest acceleration with which it can raise a weight of 40 gms. ?

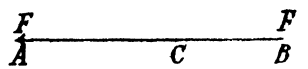
7. A load of 300 lbs. is being raised from rest with uniform acceleration by a cable. The cable is allowed to become slack, and the load comes to rest 160 ft. above its original position. The total time from rest to rest is 5 secs. Find the tension at the lower end of the cable when it is tight.

8. A lift whose weight is 19 cwt. is raised from rest by a force of one ton, the force being removed at a time such that the lift comes to rest again after moving through a total height of 20 feet. Prove that the total time from rest to rest is 5 seconds. *(P.U.)*

9. Prove that the shortest time from rest to rest in which a load of P tons can lift a weight W tons through a vertical distance h feet is

$$\sqrt{\frac{2h}{g}\cdot\frac{P}{P-W}} \text{ secs.} \qquad (D.U.)$$

3·6. Motion of two particles connected by a string. Suppose a piece of string AB is acted on by forces F and F' at its ends A and B.



Considering the string as a body of mass m, its equation of motion is

$$mf = F' - F.$$

When f is finite and the weight of the string or its mass is negligible, *i.e.*, $m=0$, then

$$F' - F = 0$$

or

$$F = F'$$

i.e., the forces at the two ends of the string are equal.

Again by considering the portions AC and CB of the string, it can be shown that the pull at C on the portion AC along CB $(=F')$ is equal to the pull on the portion BC at C along CA $(=F)$. Thus at any point of a string the two portions of the string on the two sides may be regarded as pulling each other with a force which is the same throughout the string. This force is called the tension of the string.

Hence when the weight of a string is neglected, the tension of the string is the same throughout its length and it acts in one sense or in the opposite depending on the portion of the string whose equilibrium or motion we are considering.

3·7. *Two masses m_1 and m_2, $(m_1 > m_2)$ are suspended by a light inextensible and flexible string over a smooth, fixed, small, light pulley (or a peg), to find the motion of the system and the tension of the string.*

Let l be the length of the whole string and at any instant let x, y be the depth of the masses m_1 and m_2 respectively below the top of the pulley, which is small.

Since the string is inextensible,

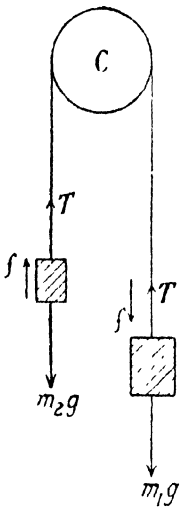
$$x + y = l.$$

$$\therefore \quad \frac{dx}{dt} + \frac{dy}{dt} = 0, \quad i.e., \quad \frac{dx}{dt} = -\frac{dy}{dt},$$

and

$$\frac{d^2x}{dt^2} = -\frac{d^2y}{dt^2}.$$

Thus the velocity v and the acceleration f of the mass m_2 upwards must throughout the motion be the same as those of m_1 downwards.



Also since the pulley is smooth, T, the tension of the string, is the same throughout the string.

Equation of motion for m_1 is

$$m_1 f = m_1 g - T \qquad ..(1)$$

and that for m_2 is

$$m_2 f = T - m_2 g \qquad ..(2)$$

Adding (1) and (2) we get

$$(m_1+m_2) f = (m_1-m_2) g,$$

or

$$f = \frac{m_1-m_2}{m_1+m_2} g \qquad ..(3)$$

Substituting this value of f in (1), we get

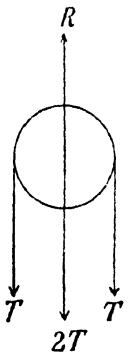
$$T = \frac{2m_1m_2}{m_1+m_2} g \qquad ..(4)$$

3·71. If $T=Mg$, then $\frac{2}{M} = \frac{m_1+m_2}{m_1m_2} = \frac{1}{m_1} + \frac{1}{m_2}$, *i.e., the tension in the string is equal to a weight which is the harmonic mean between the given weights.*

3·72. Pressure on the Pulley. The pulley is being pressed downwards by the tension along the string on both its sides.

$\therefore$ R, the pressure on the pulley

$$= 2T = \frac{4m_1m_2}{m_1+m_2} g.$$



Since $(m_1+m_2)g - \frac{4m_1m_2}{m_1+m_2} g = \frac{(m_1-m_2)^2}{m_1+m_2} g$, *the pressure on the pulley is always less than the sum of the weights of the masses.*

Example 1. *Masses 260, 230 gms. are connected by a string passing over a smooth peg. Find the distance they will describe in 3 secs. and the velocity acquired in that time.*

$$f = \frac{260-230}{260+230} g = \frac{30}{490} \times 980 = 60 \text{ cm./secs.}^2,$$

$$v = ft = 60 \times 3 = 180 \text{ cms. per sec.}$$

$$s = \tfrac{1}{2} ft^2 = \tfrac{1}{2} \times 60 \times 9 = 270 \text{ cms.}$$

Example 2. *If the string of an Atwood's machine can bear a strain of only $\frac{1}{n}$ of the sum of the two weights, show that the least possible acceleration is $\frac{\sqrt{3}}{2} g$.*

Let m_1, m_2 $(m_1 > m_2)$ be the masses of the two weights and let $m_1+m_2=m$, then

the acceleration $f=\frac{m_1-m_2}{m_1+m_2}g$, and the tension $T=\frac{2m_1m_2}{m_1+m_2}g$.

$$\therefore \quad f^2=\frac{(m_1-m_2)^2}{(m_1+m_2)^2}g^2=\frac{(m_1+m_2)^2-4m_1m_2}{(m_1+m_2)^2}g^2=g^2-\frac{2g}{m}T,$$

so that f is least when T has its greatest value $\frac{1}{8}(m_1+m_2)g=\frac{1}{8}mg$.

Hence the least acceleration is given by

$$f^2=g^2-\frac{2g}{m}\times\tfrac{1}{8}mg=\tfrac{3}{4}g^2, \quad \text{or} \quad f=\frac{\sqrt{3}}{2}g.$$

Example 3. *Two scale-pans, each of mass m, are suspended by a light string over a light pulley and in them are placed masses m_1 and m_2 ($m_1>m_2$). Find the reactions of the pans during the motion.*

f, the common acceleration of the system

$$=\frac{m_1-m_2}{2m+m_1+m_2}g.$$

If R is the reaction on m_1 of the scale pan on which it rests, then $m_1f=m_1g-R$

i.e., $$R=m_1(g-f)=\frac{2m_1(m+m_2)}{2m+m_1+m_2}g \text{ poundals.}$$

For the mass m_2 which is moving upwards, let the reaction be R_1 then $m_2f=R_1-m_2g$

i.e., $$R_1=m_2(g+f)=\frac{2m_2(m+m_1)}{2m+m_1+m_2}g \text{ poundals}$$

3·8. Atwood's Machine. In Atwood's machine two equal weights each of mass, say M, are attached to the ends of a string and suspended over a light pulley running with as little friction as possible. Over one of the weights resting on the platform A, an additional weight, called the rider, of mass m, is put. The platform A is dropped instantaneously and the system begins to move with acceleration

$$f=\frac{(m+M)-M}{(m+M)+M}g=\frac{m}{m+2M}g.$$

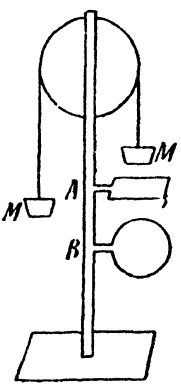
After it has moved a distance say s ($=AB$), it acquires a certain velocity, say v, given by $v^2=2fs$.

The weight with the rider is then made to pass through a ring B by which m is caught off. Weights on the two sides become equal and the system moves with constant velocity v. Further, the distance h moved with this velocity and t the time taken, are measured. Then we have

$$h=vt,$$

or $$h^2=t^2v^2=t^2.2fs=2st^2.\frac{m}{m+2M}g,$$

in which all quantities except g are known. This relation gives a rough value of g. This is one of the chief uses of Atwood's machine.



Example. The two masses in an Atwood's machine are each 240 grammes and an additional mass of 10 gms. is placed on one of them. It is found that they move a distance of 250 cms. in 5 secs. Show that $g=980$ cm./sec.^{2}

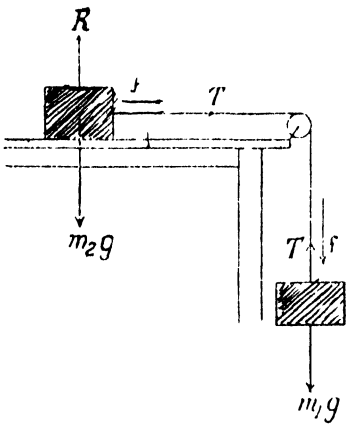
3·9. *Two particles of masses m_1 and m_2 are connected by a light inextensible string; m_2 is placed on a smooth horizontal table, the string passes over a light pulley at the edge of the table and m_1 is hanging freely. To find the motion, and the tension of the string.*

Let l be the length of the string and at any instant and let x and y be the distances of masses m_2 and m_1 respectively from the edge of the table, then

$$x+y=l,$$

$$\frac{dx}{dt}+\frac{dy}{dt}=0. \quad \therefore \quad \frac{dy}{dt}=-\frac{dx}{dt} \text{ and } \frac{d^2y}{dt^2}=-\frac{d^2x}{dt^2}.$$

Thus the velocity and acceleration of m_1 vertically downwards are at all times equal to those of m_2 along the table towards the edge.



Let f be the common acceleration. Equation of motion for m_1 is

$$m_1f=m_1g-T \qquad ..(1)$$

Equation of motion for m_2 in the horizontal direction is

$$m_2f=T \qquad ..(2)$$

The mass m_2 does not move in the vertical direction, so that $m_2g=R$, the pressure on the table, *i.e.*, the weight is balanced by the reaction of the table.

Solving (1) and (2) we have

$$f=\frac{m_1}{m_1+m_2}g \text{ and } T=\frac{m_1m_2}{m_1+m_2}g.$$

In this case the pulley is being pressed by the two equal forces T at right angles.

$\therefore$ the pressure on the pulley

$$=\sqrt{T^2+T^2}=\sqrt{2}\,T=\sqrt{2}\frac{m_1m_2}{m_1+m_2}g.$$

3·91. If the table is rough instead of smooth, the force of friction of amount μR acts on the mass m_2 in the direction of the string opposite to that of the tension, so that equation (2) becomes

$$m_2f=T-\mu R,$$

or

$$m_2f=T-\mu m_2g.$$

Solving this equation with (1) *i.e.*, $m_1f=m_1g-T$, we have

$$f=\frac{m_1-\mu m_2}{m_1+m_2}g, \text{ and } T=\frac{m_1m_2(1+\mu)}{m_1+m_2}g.$$

The body moves only when $m_1>\mu m_2$.

Examples VII

[**N.B.** The rotational inertia and friction of a pulley are to be neglected.]

1 The two masses of an Atwood's machine are 10 and 12 lbs. Find the acceleration, the tension of the string and the pressure on the pulley.

2. Two masses of 48 and 50 grammes are attached to the string of an Atwood's machine and start from rest, the greater mass falls 10 cms. in one second. Find the acceleration due to gravity.

3. The weight of 10 lbs. is attached to an end of a string, find the weight which must be attached to the other end in order that when the system is suspended from a fixed pulley, the acceleration may be half that of gravity.

4. The sum of the two weights of an Atwood's machine is 16 lbs. The heavier weight descends through 64 ft. in 4 secs. What is the value of each weight ? (*P.U.*)

5. Two scale pans of mass 2 lbs. each are connected by a string passing over a smooth pulley. Show how to divide a mass of 16 lbs. between the two scale pans so that the heavier mass may descend through a distance of 40 ft. in 4 secs.

Find in this case, the pressure on the pans.

6. Two weights are connected by a string passing over a light frictionless pulley. One is 12 lbs. and the other is 11 lbs. They are released from rest and after 2 seconds 2 lbs. are removed from the heavier weight. How soon will they be at rest again, and how far will they have moved between the instant of release and that of coming to rest again ?

7. Two particles of masses m_1 and m_2 are connected by a light inextensible string which passes over a smooth fixed pulley. If $m_1 > m_2$, and m_1 descends with an acceleration f, show that the mass which must be taken from it so that it can ascend with the same acceleration is $4\ m_1\ fg/(f+g)^2$.

8. Two heavy particles are attached together by a string passing over a pulley. If the string can only support a tension equal to one quarter of the sum of the weights at its two ends, show that the least acceleration possible is $\frac{1}{2}g\sqrt{2}$, and that the greater weight cannot be much less than six times the smaller.

9. A mass of 12 lbs. rests on a smooth horizontal platform at a distance 28 ft. from its edge. A second mass of 2 lbs. is attached to the first by means of a cord passing over the edge. Find

(*i*) the acceleration ; (*ii*) the space described in 3 secs. ; (*iii*) the time in which the 12 lbs. weight reaches the edge ; (*iv*) the tension of the string ; (*v*) the pressure on the pulley.

10. Obtain the results in example 9 when the platform is rough, the coefficient of friction μ being 0·15.

11. In example 9, if $\mu = 5/16$, and the mass on the table is 4 lbs., find the acceleration of the masses.

12. Two particles P and Q are attached to the ends of a light string, the particle P is placed on a smooth horizontal table, while Q hangs over the edge; and the system moves freely under gravity. The mass at Q is 1 lb. If P starts from rest and moves 8 ft. in the first second, determine the mass of P. (*Bom. U.*)

13. A mass of 20 lbs. on a rough horizontal table is connected by a light string passing over a smooth pulley at the edge of the table and supporting a mass of 4 lbs. which hangs freely. The frictional force on the 20 lbs. mass is 3 lbs. wt. and the 4 lbs. mass is originally 16 ins. above the floor. The system is released and after one second the string breaks.

With what velocity will the 4 lbs. mass reach the floor ? How far will the 20 lbs. mass move after the string breaks, assuming it does not reach the edge of the table ?

14. If in Art. 3·9, m_1 be replaced by m_2 and m_2 be replaced by m_3, show that the acceleration of the system remains unaltered provided m_1, m_2, m_3 are in G.P.

15. A string passing across a smooth table at right angles to two opposite edges has attached to it at the ends two masses m_1, m_2, which hang vertically. A particle of mass M is also attached to the portion of the string which is on the table. Show that the acceleration of the system when released is

$$\frac{m_2 - m_1}{m_1 + m_2 + M} g.$$

Find the tensions of the string on either side of M.

16. An Atwood's machine with masses m_1 and m_2 and string clamped, is placed on a weighing machine. Show that when the string is unclamped and the masses are allowed to move, the apparent weight of the Atwood machine is diminished by $(m_1-m_2)^2\, g/(m_1+m_2)$.

Also prove that the tension of the string is equal to the weight of a mass which is the harmonic mean between the two masses.

17. Masses P and Q in an Atwood's machine are allowed to move from rest any distance x. If P is greater than Q, show that the mass which must suddenly be removed from P at the end of distance x, so that motion in the same sense may continue a further distance nx, is

$$\frac{(n+1)\,(P^2-Q^2)}{(n+1)\,P+(n-1)Q}.$$

18. An Atwood's machine is set up in a lift accelerating upwards with 2 ft./sec.^{2}. If the masses are 3 and 5 ozs., calculate the tension and the length of time that elapses before 9 ft. of thread passes over the pulley.

[**Hint.** Reduce the lift to rest and replace g by $g+2$ in results (3) and (4), Art. 3·7.]

19. A light string passes over a light smooth pulley, fixed to the ceiling of a lift, and carries masses P of 6 lbs. and Q of 7 lbs. at its ends. The mass Q is held in contact with the pulley and the mass P is hanging with the string vertical. If Q is now released, find the tension in the string while motion occurs, and the time that elapses before Q is 3 ft. below the pulley, (*i*) when the lift remains at rest, (*ii*) when the lift is rising vertically with constant acceleration $\frac{1}{3}g$.

20. A string passes over a smooth fixed pulley and to one end there is attached a mass m_1, and to the other a smooth light pulley over which passes another string with masses m_2 and m_3 at the ends. If the system is released from rest, show that m_1 will not move if

$$\frac{4}{m_1}=\frac{1}{m_2}+\frac{1}{m_3}. \qquad (P.U.)$$

REVISION QUESTIONS—II

1. State and explain Newton's second law of motion and show that if a particle is moving in one plane, the rate of change of its moment of momentum about any fixed point in the plane is equal to the moment about that point of the forces acting on the particle.

If h_1, h_2 are the moments of momentum of a moving particle (m) about two fixed points A, B in the plane of motion, prove that its velocity perpendicular to AB is equal to

$$\frac{h_1-h_2}{m.AB}. \qquad (Pub.\ Ser.\ Ex.)$$

2. Twenty men whose total weight is 30 cwt. are being drawn up a mine shaft in a cage by a rope whose tension is constant throughout the operation. If the weight of the cage is 10 cwt. and its speed after rising 30 ft. is 12 ft. per sec., calculate the tension in the rope.

Find also in tons weight, the pressure on the floor of the cage due to the men. (*L.U.*)

3. The engine of a train weighing 400 tons is capable of exerting a tractive effort of 15 tons wt. The road resistance and the brake resistance are respectively 30 lbs. wt. per ton. and 150 lbs. wt. per ton. Find the time taken to travel between two stations $1\frac{1}{4}$ mile apart if the train starts from rest, accelerates till its speed is 45 *m.p.h.*, runs at this speed for a certain distance and then the steam is shut off and the brakes applied simultaneously, so that the train comes to stand still at the second station. (*Bom.U.*)

4. A chain which can safely bear a tension of 10 tons is to lift 8 tons as quickly as possible through a height of 200 ft. so that the load comes to rest at the top.

Find the time required to the nearest second (weight of the chain to be neglected).

5. A train of 100 tons is to be taken from rest to rest in $2\frac{1}{2}$ min. over a distance of 3610 ft. by a uniform pull against a constant resistance of 20 lbs. per ton; the brakes press the wheels with a total force of $\frac{1}{5}$ of the weight of the train and the coefficient of friction between them and the wheels is 0·18; find the magnitude of the pull and the time during which it must be applied.

6. A train of weight W lb. moving with v ft. per sec. on the level is pulled with a force F lbs. against a train resistance of R lbs. Show that in accelerating from v_0 to v_1 ft. per sec. the distance in feet described by the train is

$$\frac{W}{g}\int_{v_0}^{v_1}\frac{v\,dv}{F-R}.$$

7. Two particles, P and Q, of unknown weights are attached to the ends of a light string; the particle P is placed on a smooth horizontal table, while Q hangs over the edge, and the system moves freely under gravity. The particle P passes in succession through the points A, B, C, on the table, it passes through B and C at instants 3 secs. and 5 secs. respectively after passing through A, the distance AB and BC being each 3 ft. From these data determine the acceleration of the system and deduce that the weights P and Q are in the ratio of 159 : 1. (Take $g=32$ ft./sec.^{2}).

8. Two weights W, W' are connected by a light string passing over a light pulley. If the pulley moves with an upward acceleration equal to that of gravity, show that the tension of the string is

$$4WW'/(W+W').$$

9. A man weighing 10 stone and a weight of 8 stone are suspended by means of a light rope over a smooth pulley. Find the pressure on the pulley and the acceleration of the man.

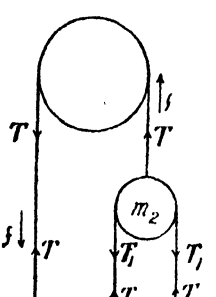
If the man pulls himself up the rope so that his downward acceleration is $g/18$, find the upward acceleration of the weight and the upward acceleration of the man relative to the rope. (*P.U.*)

10. *A mass m_1 is attached to one end of a light inextensible string which passes over a smooth fixed pulley and to the other end is attached a smooth pulley of mass m_2 over which passes another string with masses m_3 and m_4 at its ends. Find the motion.*

Let T, T_1 be the tensions in the strings passing over the fixed pulley and the movable pully (m_2).

Let the acceleration of the mass m_1 be f, measured downwards, then m_2 must have an acceleration f upwards.

If f_1 be the downward acceleration of m_3 relative to m_2 its actual acceleration will be f_1-f downwards, and that of m_4 will be f_1+f upwards.



Equations of motion for m_1, m_2, m_3 and m_4 are :—

$$m_1f=m_1g-T$$
$$m_2f=T-m_2g-2T_1$$
$$m_3(f_1-f)=m_3g-T_1$$
$$m_4(f_1+f)=T_1-m_4g.$$

Any of the unknown quantities f, f_1, T, T_1 can be obtained from these equations.

(*a*) Taking $m_1=4$ lbs., $m_2=1$ lb., $m_3=2$ lbs., $m_4=1$ lb., show that $f=\frac{1}{23}g$.

11. A string which carries at one end A a particle of mass m_3, passes over a fixed pulley at C and carries at the other end B a similar pulley. A second string carries at one end D, a particle of mass m_1, passes over the pulley at B and carries at the other end E, a particle of mass m_2. Show that when the masses of the strings and pulleys are negligible the tension in the first string is twice that in the second string.

Find also the acceleration of the particle at A when $m_1=1$ lb., $m_2=2$ lbs., $m_3=3$ lbs.

12. A light string, one end of which is fixed, has slung on it a mass $3m$ and then passes over a fixed pulley having a mass m attached to it at the other end. The portions of the string not in contact with the pulley are vertical. Find the acceleration of the two masses and the tension of the string when the system is released.

13. A mass m lying on a smooth horizontal table is attached to a string which after passing over the edge of the table hangs in a loop on which a heavy smooth ring of mass M is threaded and then passes over a smooth fixed pulley and supports a mass m'. If the free portions of the string are vertical and the whole system lies in a vertical plane, determine the tension of the string and show that the mass M will remain at rest provided that

$$\frac{2}{M} = \frac{1}{m} + \frac{1}{m'}.$$

14. Two masses m_1, m_2 lie on a smooth horizontal table. They are joined to a string on which slides a smooth ring of mass m_3. If the two portions of the string lie close together in parallel straight lines and the ring hangs down over the edge, prove that when released the tension in the thread is

$$\frac{2g}{\frac{4}{m_3} + \frac{1}{m_2} + \frac{1}{m_1}},$$

and that the acceleration of the ring downwards is $\frac{1}{4}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$ times the tension.

15. Two pulleys each of mass m, are connected by a string hanging over a smooth fixed pulley ; a string with masses $2m$ and $3m$ at its ends is hung over one pulley and another with masses m and $4m$ over the other. If the system is free to move, show that the acceleration of either pulley is $\frac{4g}{25}$. *(D.U.)*

16. A mass attached to one end of a light inextensible string which passes over a smooth pulley is descending with uniform velocity ; to the other end of the string is attached a ring without mass, through which passes a second string supporting at the extremities masses m and m'. If one of these masses on arriving at the ring becomes entangled in it, show that it will afterwards move with acceleration $\frac{(m-m')^2 g}{m^2+m'^2+6mm'}$, and the tension in the string above the ring is

$$\frac{8mm'g(m+m')}{m^2+m'^2+6mm'}. \qquad (P.U.)$$

17. A train starts from rest and covers a journey of 29·9 miles in 40 min. as follows ; it first accelerates uniformly for 3 min., reaches a speed which is unchanged for the next 10 min., it is then accelerated uniformly for a further 2 min., reaching a speed of 12 m.p.h. higher than previously ; it proceeds at this new fixed speed until the last 5 min. of journey during which it is uniformly retarded to rest. Find the two fixed speeds.

18. A motor car is driven from the hind wheels. The centre of gravity G of the car is at a height 3 ft. above the road, and the distances of the vertical through G from the front and rear axles are 8 ft. and 4 ft. respectively. Neglecting the rotatory inertia of the wheels, show that, if the co-efficient of friction between the wheels and the road is $\frac{1}{4}$, the maximum acceleration of the car is 5·7 ft./sec.^{2} *(I.R.S.E.)*

19. If P be the tractive force in tons on a train weighing W tons, and R be the resistance, prove that the least time of travelling a distance s from rest to rest is

$$\sqrt{\left[\frac{2s}{g} \cdot \frac{WP}{R(P-R)}\right]} \qquad \text{(D. U. 1961)}$$

CHAPTER IV

MOTION IN A STRAIGHT LINE—III

4·1. Motion with variable acceleration. Expressions for acceleration of a particle moving along a straight line are $\frac{d^2s}{dt^2}$, $\frac{dv}{dt}$ and $v\frac{dv}{ds}$ where s is the distance measured along the line from a fixed point in it. If m denotes the mass of the moving particle and F, the force acting on it at any instant along the line of its motion and measured in absolute units, the equation of motion of the particle as deduced from the second law of motion is,

$$m\frac{d^2s}{dt^2}=F \qquad ..(1)$$

or
$$m\frac{dv}{dt}=F \qquad ..(2)$$

or
$$mv\frac{dv}{ds}=F \qquad ..(3)$$

When the force F or the acceleration $\frac{F}{m}$ is constant, any of these equations may be used as has been done in Chapter II. When F is a function of t (1) or (2) is used, when F is a function of velocity v either (2) or (3) is used and when F is a function of the distance s, only (3) is used.

4·11. The following examples illustrate the last three cases :

Example 1. *A particle of mass m moves along a straight line starting from rest from a given point in the line. If the force acting at any instant is mk cos t, find the motion of the particle.*

Let x be the distance of the particle from the fixed point after time t, then

$$m\frac{d^2x}{dt^2}=mk\cos t.$$

Integrating we have, $\frac{dx}{dt}=k\sin t+c$,

when $t=0$, $\frac{dx}{dt}=0$, so that $c=0$ $\therefore v=k\sin t$ or $\frac{dx}{dt}=k\sin t$.

Integrating again, we get,

$$x=-k\cos t+c_1$$

when $t=0$, $x=0$. $\therefore c_1=k$.

Hence $$x=k(1-\cos t)$$

Example 2. *A steam boat is moving with velocity V when steam is shut off. Given that the retardation at any subsequent time is equal to the magnitude of the velocity at that time. Find the velocity and distance travelled in time t after the steam is shut off.*

Here $$\frac{dv}{dt}=-v \quad \text{or} \quad \frac{1}{v}\frac{dv}{dt}=-1.$$

Integrating we have, $\log v=-t+c.$

When $t=0$, $v=V$, so that $c=\log V$.

$$\therefore \quad \log \frac{v}{V}=-t \quad \text{or} \quad \frac{v}{V}=e^{-t}.$$

$$\therefore \quad v=Ve^{-t} \quad \text{or} \quad \frac{ds}{dt}=Ve^{-t}.$$

Integrating again, we have, $s=-Ve^{-t}+c_1.$

When $t=0$, $s=0$, so that $c_1=V$

$$\therefore \quad s=V-Ve^{-t}=V(1-e^{-t}).$$

Example 3. *A particle of unit mass begins to move from a distance a towards a fixed centre which repels according to the law μx. If its initial velocity is $\sqrt{\mu}\, a$, show that it will continually approach the fixed centre, but will never reach it.*

Since the particle is repelled from the centre the acceleration is in the direction in which x increases and is, therefore, positive.

Hence $$v\frac{dv}{dx}=\mu.x,$$

where the distance x is measured from the fixd centre.

Integrating, we have, $\frac{1}{2}v^2=\frac{1}{2}\mu\,.\,x^2+c.$

Initially when $x=a$, $x^2=a^2$, $v=-\sqrt{\mu a}$, $v^2=\mu a^2$, so that $c=0$.

$$\therefore \quad v^2=\mu.x^2 \quad \text{or} \quad (dx/dt)^2=\mu x^2.$$

Since the particle is moving towards the centre, x decreases as t increases and hence $\frac{dx}{dt}$ is negative.

$$\therefore \quad \frac{dx}{dt}=-\sqrt{\mu}x.$$

Integrating we have, $$\int\frac{dx}{x}=-\int\sqrt{\mu}.dt$$

i.e., $$\log x=-\sqrt{\mu}.t+c_1$$

When $t=0$, $x=a$, so that $\log a=0+c_1$.

$$\therefore \quad \log x=-\sqrt{\mu}.t+\log a \quad \text{or} \quad \log\frac{x}{a}=-\sqrt{\mu}.t.$$

$$\therefore \quad x=ae^{-\sqrt{\mu}t}$$

Now as t increases, x decreases, *i.e.*, the particle gradually approaches the centre, but since $x=0$ only when $t\to\infty$, the particle does not reach the centre for any finite value of t.

Note 2. Velocity in terms of time t can be obtained by differentiating any of these equations involving x and t.

Note 3. When the particle is on the left hand side of O, the equation of motion is $\frac{d^2x}{dt^2}$ = acceleration in the direction of $P_1A = \mu OP_1 = \mu(-x) = -\mu x$

Hence the same equation that holds on the right hand side of O, holds also on the left hand side.

4·21. The nature of Motion. The equation

$$v = \pm\sqrt{\mu}\sqrt{a^2 - x^2}$$

gives the velocity of P in terms of its distance from O. Initially, when $x=0$ at the point O, the velocity is maximum and equal to $\sqrt{\mu}\, a$. As the particle proceeds towards A, the acceleration being towards O, the velocity goes on decreasing as x increases. At A where $x=a$, it vanishes and the particle is, for an instant, at rest. Then owing to the acceleration towards O, the particle moves in the negative direction with a velocity which increases numerically as x decreases and is the greatest at O where it is $-\sqrt{\mu}\, a$. Due to this velocity, the particle proceeds further to the negative side of O, the velocity remaining negative and decreasing gradually in magnitude till the particle comes to rest at A_1 where $x=-a$. The acceleration being towards O, the particle then starts and moves towards O with a positive velocity which increases gradually till it is again maximum at O. The same motion is repeated again and again and the particle goes on *oscillating* indefinitely between A and A_1, the two positions of momentary rest.

The motion of the particle is oscillatory. All oscillatory motions are, however, not necessarily simple harmonic. In fact simple harmonic motion is the simplest and most important case of oscillatory motion which occurs in nature and it is always dominated by the differential equation

$$\frac{d^2x}{dt^2} = -\mu x.$$

The distance OA or OA_1, *i.e.*, the distance of the centre from one of the positions of rest is called the **Amplitude.**

4·22. The Period of Motion. The equation $x = a \sin \sqrt{\mu}t$ gives the time from O in terms of x, the distance of the particle measured from O.

Since $x = a \sin \sqrt{\mu}.t = a \sin (\sqrt{\mu}.t + 2\pi)$

$$= a \sin \sqrt{\mu}\left(t + \frac{2\pi}{\sqrt{\mu}}\right),$$

and $$\frac{dx}{dt} = a\sqrt{\mu} \cos \sqrt{\mu}.t = a\sqrt{\mu} \cos \sqrt{\mu}\left(t + \frac{2\pi}{\sqrt{\mu}}\right),$$

the particle has the same position, velocity and direction after time $t + \frac{2\pi}{\sqrt{\mu}}$, $t + \frac{4\pi}{\sqrt{\mu}}$, etc., as it had at the time t, *i.e.*, the particle has a periodic motion, its period of motion being $\frac{2\pi}{\sqrt{\mu}}$.

4·23. Periodic Motion. *A point is said to have a periodic motion when it moves in such a manner that after a certain fixed interval of time, it occupies the same position and moves, in the same direction with the same velocity.*

In the case of a particle moving with simple harmonic motion, the periodic time, as shown above, is $2\pi/\sqrt{\mu}$, *which is independent of the amplitude.*

4·24. The Frequency is the number of complete oscillations in one second, so that if n denotes the frequency and T the periodic time,

$$nT=1, \qquad \text{or} \qquad n=\frac{1}{T}=\frac{\sqrt{\mu}}{2\pi}.$$

Example 1. *The acceleration of a point, moving with S.H.M. is 2 f.s.s. whe. its displacement from the mid-position is 6 inches. Find the amplitude and the time of an oscillation if the maximum velocity is 4 f.s. Find also the velocity of the point when it is at a distance of half the amplitude.*

The acceleration $=2$ f.s.s. $=\mu x=\mu\times\frac{1}{2}$.

$$\therefore \qquad \mu=4.$$

The time of an oscillation $=\dfrac{2\pi}{\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{4}}=\pi=3{\cdot}14$ secs.

The maximum velocity $=4$ f.s. $=\sqrt{\mu}.a=2a$.

$$\therefore \qquad a=\text{amplitude}=2 \text{ ft.}$$

To find the velocity at half amplitude where $x=1$, we have

$$v=\sqrt{\mu}.\sqrt{a^2-x^2}=2\sqrt{4-1}=2\sqrt{3}=3{\cdot}46 \text{ f.s.}$$

Example 2. *A particle moving with S.H.M. of period 12 secs., travels 8 inches from the position of rest in 2 secs. Find the amplitude, the maximum velocity and the velocity at the end of 2 secs.*

Since $\dfrac{2\pi}{\sqrt{\mu}}=12, \qquad \therefore \sqrt{\mu}=\dfrac{\pi}{6}.$

If a ft. be the amplitude then, x, the displacement from the centre $=a-\frac{8}{12}=a-\frac{2}{3}$ ft.

t, the time from an extreme position $=2$ secs.

Since the time is given from the extreme position, we use

$$x=a\cos\sqrt{\mu}t.$$

$$\therefore \qquad a-\tfrac{2}{3}=a\cos 2\sqrt{\mu}=a\cos\frac{\pi}{3}=\frac{a}{2}.$$

$$\therefore \qquad a=\tfrac{4}{3} \text{ ft.}=16 \text{ inches,}$$

and x at the end of 2 secs. $=\frac{1}{3}-\frac{2}{3}=\frac{2}{3}$ ft.

$$\therefore v=\sqrt{\mu}.\sqrt{a^2-x^2}=\frac{\pi}{6}\sqrt{\frac{16}{9}-\frac{1}{9}}=\frac{\pi}{6}\cdot\frac{2\sqrt{3}}{3}=0{\cdot}604 \text{ ft. per sec.}$$

and max. vel. $=\sqrt{\mu}.a=\dfrac{\pi}{6}\cdot\dfrac{4}{3}=\dfrac{2\pi}{9}=0{\cdot}69$ ft. per sec.

Example 3. *A particle moves with S.H.M. in a straight line. In the first second after starting from rest, it travels a distance a and in the next second it travels a distance b in the same direction. Prove that the amplitude of the motion is*

$$\frac{2a^2}{3a-b}.$$

Measuring time t from the starting point and the distance x of the particle from the centre of motion and denoting the amplitude by A, we have

$$x=A\cos\sqrt{\mu}t.$$

Now by the question when $t=1$, $x=A-a$

and when $t=2$, $x=A-a-b$.

$$\therefore \quad A-a=A\cos\sqrt{\mu} \qquad ..(1),$$

and $$A-a-b=A\cos 2\sqrt{\mu}=A(2\cos^2\sqrt{\mu}-1) \qquad ..(2).$$

$\therefore$ from (1) and (2), we get

$$A-a-b=A\left[2\frac{(A-a)^2}{A^2}-1\right]=\frac{1}{A}[A^2-4aA+2a^2],$$

or $$A^2-aA-bA=A^2-4aA+2a^2,$$

or $$(3a-b)A=2a^2, \text{ i.e., } A=2a^2/(3a-b).$$

Example 4. *A horizontal shelf moves vertically with S.H.M. If the complete period is one second; find the greatest amplitude in centimetres that it can have so that a book, weighing 100 gms., resting on the shelf may always be in contact with it. Find also the greatest and least pressures exerted on the shelf.*

Here $$\frac{2\pi}{\sqrt{\mu}}=1 \quad \therefore \quad \sqrt{\mu}=2\pi, \quad i.e.\ \mu=4\pi^2.$$

When the shelf is moving above the centre of motion, its acceleration f is downwards which should not exceed g in order that the book may rest on the shelf. [See (2) Art. 3·5].

$\therefore$ if a is the greatest amplitude, then $f=\mu a=g$,

$$i.e., \quad a=\frac{g}{\mu}=\frac{980}{4\pi^2}=\frac{245}{\pi^2}=\frac{245}{9\cdot87}=25 \text{ cms. nearly.}$$

When the shelf is in its highest position, the acceleration g is downwards.

$\therefore$ the least pressure $R'=m(g-f)=m(g-g)=0$.

The greatest pressure R occurs when the shelf is in its lowest position where the acceleration, being directed towards the centre O, is upwards and equal to g.

$$\therefore \quad R=m(g+f)=m(g+g)=2mg=200 \text{ grammes.}$$

Example 5. *A particle is projected with a velocity V from a point A along the straight line OA. Find the motion when the acceleration is proportional to the distance of the particle from O and is directed towards O. Find also the amplitude of motion if OA = a.*

The equation of motion is $v\frac{dv}{dx}=-\mu x$.

$$\therefore \quad \tfrac{1}{2}v^2=-\tfrac{1}{2}\mu x^2+c.$$

Initially $v=V$ when $x=a$, $\therefore c=\frac{1}{2}(V^2+\mu a^2)$.

Hence $v^2=V^2+\mu(a^2-x^2)$.

The value of x when $v=0$ gives the amplitude, so that when

$$v=0, \qquad x=\sqrt{\frac{V^2}{\mu}+a^2}=\text{amplitude.}$$

***4·3. Composition of two simple harmonic motions of the same period, in the same straight line.** By equation (3) Art 4·2, the general displacement of a particle moving in S H.M. is $x=a\sin(\sqrt{\mu}t+e)$, where a is the amplitude of the motion. The quantity $\sqrt{\mu}t+e$ is called the phase of the motion and e the initial phase in angle, (given in radians), and $t+\frac{e}{\sqrt{\mu}}$ is defined to be the phase in time.

Now let $x=a\sin(\sqrt{\mu}t+e)$ and $x=b\sin(\sqrt{\mu}t+e')$ be the displacements in the two motions having the same period $\frac{2\pi}{\sqrt{\mu}}$.

Compounding these, we get

$$\begin{aligned} x&=a\sin(\sqrt{\mu}t+e)+b\sin(\sqrt{\mu}t+e') \\ &=(a\cos e+b\cos e')\sin\sqrt{\mu}t+(a\sin e+b\sin e')\cos\sqrt{\mu}t \\ &=A\sin(\sqrt{\mu}.t+E) \qquad \ldots(1) \end{aligned}$$

where and
$$\left.\begin{aligned} A\cos E&=a\cos e+b\cos e' \\ A\sin E&=a\sin e+b\sin e' \end{aligned}\right\} \qquad \ldots(2)$$

Equation (1) shows that composition of two S.H. motions of the same period, in the same st. line, gives a similar motion of the same period, whose amplitude and phase are known from equations (2).

***Example.** Prove that if two S.H. vibrations of the same period in the same line whose amplitudes are a and b and whose phases differ by α be compounded, the amplitude of the resultant vibration is $\sqrt{a^2+2ab\cos\alpha+b^2}$.

If $a=2b$, and the phases of the two motions differ by one-quarter of the period show that the amplitude of the resultant $=\sqrt{5}.b$, and its phase is in advance of the second by $\frac{1}{2\pi}\tan^{-1}2$ of the period.

Examples IX

1. A point moving with S.H.M. has a period of oscillation of π sec. and its greatest acceleration is 5 ft./sec.^{2} Find the amplitude, and the velocity when the particle is at a distance 1 ft. from the centre of oscillation.

2. A particle describing simple harmonic motion executes 100 complete vibrations per minute and its speed at its mean position is 15 ft. per sec. What is the length of its path? What is its velocity when it is half way between its mean position and an extremity of its path?

3. A particle moving in S.H.M. of period 8 secs. oscillates through a distance of 4 ft. on each side of its central position. Find the maximum speed and the speed when the particle is 2 ft. from the centre.

What time does it take in moving from this position to the nearest extreme position?

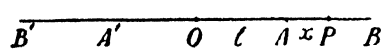
i.e., tension of an elastic string or a spring is proportional to the extension of the spring beyond its natural length. This is Hooke's law of elastic string and λ is called the **Modulus of Elasticity.**

When $x=l$, $T=\lambda$, so that λ for a string of unit cross-section is equal to the amount of force which would stretch it to twice its natural length.

Example. If an elastic string is stretched 3 ins. by a force of 2 lbs. show that the tension in the string when it is stretched 8 ins. is $5\frac{1}{3}$ lbs. wt.

HORIZONTAL ELASTIC STRING

4·5. Let one end of an elastic string be fixed to a point O on a *smooth* horizontal table and let $OA=l$ be its natural length.



If a particle of mass m is attached to the other end and if the particle is displaced along the line OA, a distance $AB=b$ and P be position of the particle at any subsequent time so that $AP=x$, then the tension in the string is $\frac{\lambda}{l}x$ which acts in the direction PA and is directed towards A.

The tension of the string being the only force which tends to move the particle, its equation of motion is

$$m\frac{d^2x}{dt^2}=-T=-\frac{\lambda}{l}x, \quad \text{or} \quad \frac{d^2x}{dt^2}=-\frac{\lambda}{lm}x. \qquad ..(1)$$

which shows that the motion about A is simple harmonic, the constant μ being replaced by $\frac{\lambda}{lm}$. The period of oscillation is

$$2\pi\sqrt{\frac{ml}{\lambda}}.$$

The particle will further move through O to a point B' at an equal distance on the other side of O and then back again and so on.

The distance from A to A' ($OA=OA'$) and back to A is moved with the velocity which the particle acquires at A. The string being slack this velocity remains the same throughout this part.

The periodic time obtained above refers to the time which the particle takes in moving from B to A, from A' to B' and then from B' to A' and from A to B. This is the only part where motion is simple harmonic.

4·51. To find the velocity we write equation (1) as

$$v\frac{dv}{dx}=-\frac{\lambda}{lm}x,$$

which on integration gives $\frac{1}{2}v^2=-\frac{\lambda}{2lm}x^2+C$

At B, $x=AB=b$ and $v=0$, $\therefore\ C=\frac{\lambda}{2lm}b^2$

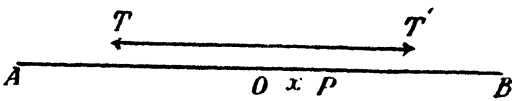
$$\therefore\ v^2=\frac{\lambda}{lm}\left(b^2-x^2\right),\quad \text{or}\quad v=-\frac{\lambda}{lm}\sqrt{b^2-x^2} \qquad ..(2)$$

the sign is taken negative because the particle is moving towards O.

Example. *An elastic string, to the middle point of which a particle of mass m is attached, is stretched to twice its natural length and placed on a smooth horizontal table and its ends are then fixed.*

(*a*) *Find the period of oscillation when the particle is displaced in the direction of the string.*

(*b*) *Find also the period of a small transverse oscillation.*



Let $2l$ be the natural length of the string, then the extended length is $4l$. Let A, B be the fixed ends and O the middle point, then $AO=OB=2l$.

(*a*) Let P be any position of the particle when displaced such that $OP=x$ (x being small).

The extended length of the portion AP of the string$=2l+x$.

$\therefore$ extension of $AP=(2l+x)-l=l+x$.

The extension of the portion $BP=(2l-x)-l=l-x$.

T, the tension in $PA=\lambda\frac{l+x}{l}$; T', the tension in $BP=\lambda\frac{l-x}{l}$.

Equation of motion for P is

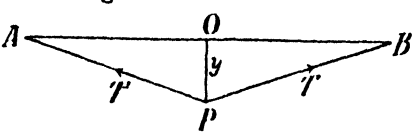
$$m\frac{d^2x}{dt^2}=T'-T=\lambda\left[\frac{l-x}{l}-\frac{l+x}{l}\right]=-\lambda\frac{2x}{l},$$

or
$$\frac{d^2x}{dt^2}=-\frac{2\lambda}{lm}x$$

which shows that the particle moves with S.H.M., the centre of motion being O.

$$\text{The period}=\frac{2\pi}{\sqrt{2\lambda/lm}}=\pi\sqrt{\frac{2lm}{\lambda}}.$$

(*b*) Suppose the particle P is displaced a small distance y, perpendicular to the string.



Let T be the tension along PA or PB, then the resultant tension in the direction $PO=2T\cos\angle OPB$.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2y}{dt^2}=-2T\cos\angle OPB.$$

Now $PB=\sqrt{OB^2+OP^2}=\sqrt{4l^2+y^2}=2l\left[1+\frac{y^2}{4l^2}\right]^{\frac{1}{2}}=2l,$

neglecting powers of $\frac{y}{l}$ higher than the first,

$$\cos\angle OPB=\frac{OP}{PB}=\frac{y}{2l}\text{ (approx.).}$$

Also T along BP or $PA=\lambda\frac{PB-l}{l}=\lambda\frac{2l-l}{l}=\lambda.$

$$\therefore\quad 2T\cos\angle OPB=2\lambda.\frac{y}{2l}=\frac{\lambda}{l}y.$$

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2y}{dt^2}=-\frac{\lambda y}{l}\quad\text{or}\quad\frac{d^2y}{dt^2}=-\frac{\lambda}{lm}y,$$

and the period of motion $=\frac{2\pi}{\sqrt{\lambda/lm}}=2\pi\sqrt{\frac{lm}{\lambda}}.$

VERTICAL ELASTIC STRING

4·6. Suppose that a particle of mass m is suspended from a fixed point O by a string (or spring) OA of a natural length l $(OA=l)$.

Let OB be the length of the string when the mass hangs in equilibrium, then $AB(=e)$, the extension of the string is given by

$$mg=\lambda\frac{AB}{l}=\lambda\frac{e}{l},\qquad\ldots(1)$$

Now if the particle is displaced vertically from B, it will oscillate in a vertical line about B and it will execute S.H.M. which can be proved as follows :—

Let P be the displaced position of the particle during its motion and let $PB=x$, then the tension, T, of the string in this position is given by

$$T=\frac{\lambda}{l}(BA+x)=mg+\frac{\lambda}{l}x.\qquad\ldots(2)$$

[by (1)].

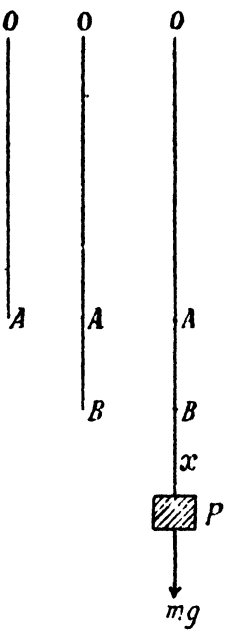
$\therefore$ the resultant force acting on the particle in the direction BP

$$=mg-T=mg-\left(mg+\frac{\lambda}{l}x\right)=-\frac{\lambda}{l}x.$$

Hence the equation of motion of the particle is

$$m\frac{d^2x}{dt^2}=-\frac{\lambda}{l}x\quad\text{or}\quad\frac{d^2x}{dt^2}=-\frac{\lambda}{lm}x\qquad\ldots(3)$$

which shows that the *particle moves with simple harmonic motion having B, the position of equilibrium, as the centre of oscillation.*



The period of motion is $2\pi\sqrt{\frac{ml}{\lambda}}=2\pi\sqrt{\frac{e}{g}}$ ..(4)

e being the extension of the string in the equilibrium position of the particle. By equation (1), e being proportional to $\frac{m}{\lambda}$, the period depends on the weight which is hung on, and on the stiffness of the string or spring to which the particle is attached. The stiffer the spring the shorter the period and the more massive the weight the longer the period.

Note 1. At B, the ultimate position of equilibrium of the particle, the forces acting on it, *viz.*, its weight and the tension of the string, balance. In all problems of this type the position of this point must be obtained first.

Note 2. The particle moves with Simple Harmonic motion only so long as the particle is *below* A, *i.e.*, so long as the string remains stretched.

If the particle rises above A (it will do so, for example when it is pulled down below, B, a distance greater than AB) the string will become slack and the part of the motion above A will be simply free vertical motion under gravity.

Example 1. *A mass of 4 lbs. suspended from a light elastic string of natural length 3 ft. extends it a distance 2 ft. One end of the string is fixed and a mass of 2 lbs. is attached to the other. The mass is held so that the string is just unstretched and is then let go.*

Find the amplitude, the period and the maximum velocity.

OA, the natural length of the string$=3$ ft.

Since the extension is 2 ft. when the force is 4 lbs. wt., we have by Hooke's law

$$4g=\lambda\tfrac{2}{3}.$$

i.e., $$\lambda=6g. \quad .. (1)$$

Let OB be the length of the string when the mass of 2 lbs. is hanging in equilibrium, (Fig. Art. 4·6) then

$$2g=\lambda\frac{AB}{3}=6g\,\frac{AB}{3} \quad \text{[by (1)]},$$

i.e., $$AB=1 \text{ ft.} \quad .. (2)$$

Now let P be any position of the mass during the motion and let it be at a distance x from the equilibrium position B after time t secs. from B, the equation of motion is

$$2\frac{d^2x}{dt^2}=2g-T.$$

But $$T=\lambda\frac{AP}{3}=6g\,\frac{1+x}{3}=2g+2gx.$$

$\therefore$ $$2\,\frac{d^2x}{dt^2}=2g-(2g+2gx)=-2gx,$$

or $$\frac{d^2x}{dt^2}=-gx \quad ..(3)$$

which shows that the motion is simple harmonic having B, the equilibrium position, as its centre of oscillation and period

$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{g}}.$$

As given in the question the particle was initially at rest at A

$\therefore$ amplitude $= AB = 1$ ft.

and max. velocity $= \sqrt{\mu a} = \sqrt{g \times 1} = 4\sqrt{2}$ ft. per sec.

Example 2. *In the last example, the particle, instead of being held at A and then let go, is pulled down below its equilibrium position B, a distance of 6 inches and then let go. What is the period of motion?*

Initially the particle is at rest at a point $\frac{1}{2}$ ft. below B; $\frac{1}{2}$ ft., being less than AB which is 1 ft., the string is never slack and the motion is entirely simple harmonic of amplitude $\frac{1}{2}$ ft.

The equation of motion of the particle is the same as equation (3) in Ex. 1.

$\therefore$ the period of motion is $2\pi/\sqrt{g}$.

Example 3. *In Ex. 1, the particle is*

(a) pulled down below B, a distance of 18 inches and then let go,

(b) it is dropped from the point O.

Discuss the motion of the particle in each case, and find the periods of motion.

$OA = l = 3$ ft., $AB = 1$ ft., $BP = x$.

Equation of motion of the particle when the string is stretched is, as obtained before

$$\frac{d^2x}{dt^2} = -gx \qquad \text{or} \qquad v\frac{dv}{dx} = -gx \qquad \ldots(1),$$

where x is measured from B, the equilibrium position.

Integrating, we have

$$\frac{v^2}{2} = -g\frac{x^2}{2} + C \qquad \ldots(2).$$

(a) The particle is initially at rest at D, $\frac{3}{2}$ ft. below B.

In this case $\frac{3}{2}$ ft. $> AB$, the extension in the equilibrium position, the motion from D to A is simple harmonic of amplitude $\frac{3}{2}$ ft. At A, the string becomes slack and the particle moves freely under gravity with a certain initial velocity at A.

At D where $x = \frac{3}{2}$, $v = 0$, so that from equation (2), $C = \frac{9}{8}g$.

$\therefore$ $v^2 = -gx^2 + \frac{9}{4}g$.

v_1, the velocity at A where $x = -1$ is given by

$$v_1^2 = -g + \frac{9}{4}g, \text{ i.e., } v_1 = \tfrac{1}{2}\sqrt{5g}.$$

The particle comes to the rest again at E at a height above A

$$= \frac{v_1^2}{2g} = \frac{5g}{8g} = \frac{5}{8} \text{ ft.}$$

t_1, the time from D to $B = \frac{1}{4}$th period of S.H.M. $= \dfrac{\pi}{2\sqrt{g}}$.

t_2, the time from B to A is given by $[x = a \sin \sqrt{\mu}\, t]$,

$$1 = \tfrac{3}{2} \sin \sqrt{g}.t_2 \quad i.e., \quad t_2 = \frac{1}{\sqrt{g}} \sin^{-1} \tfrac{2}{3},$$

t_3, the time from A to $E = \dfrac{v_1}{g} = \dfrac{\sqrt{5g}}{2g} = \dfrac{1}{2}\sqrt{\dfrac{5}{g}}$.

$\therefore$ the whole period of oscillation from D to E and back

$$=2(t_3+t_1+t_2)$$

$$=\frac{2}{\sqrt{g}}\left[\frac{\sqrt{5}}{2}+\frac{\pi}{2}+\sin^{-1}\tfrac{2}{3}\right]$$

(*b*) The particle falls freely from O acquiring a velocity $=\sqrt{2gl}=\sqrt{6g}$ at A. The string is then taut and the particle moves with S.H.M. given by equations (1) and (2).

At A where $x=-1$, $v=\sqrt{6g}$, so that from (2), $C=\frac{7}{2}g$.

$\therefore$ $$v^2=-gx^2+7g,$$

$v=0$ when $x=\sqrt{7}=2{\cdot}646$ ft.=amplitude. So that the particle comes to rest at C, a point $\sqrt{7}$ ft. below B and then moves back with S.H.M. to the point A where the string again becomes slack. The particle then moves up freely with initial velocity$=\sqrt{6g}$ at A and is brought to rest again at O.

t_1, the time from C to $B=\frac{1}{4}$ period of S.H.M.$=\dfrac{\pi}{2\sqrt{g}}$,

t_2, the time from B to A is given by

$$1=\sqrt{7}\sin\sqrt{g}.\,t_2 \qquad [\because x=a\sin\sqrt{\mu}\,t]$$

i.e., $$t_2=\frac{1}{\sqrt{g}}\sin^{-1}\frac{1}{\sqrt{7}},$$

t_3, the time from A to O or from O to $A=\sqrt{\dfrac{6}{g}}$ $[\because s=\frac{1}{2}gt^2]$.

$\therefore$ the whole period of oscillation from O to C and back.

$$=2(t_3+t_1+t_2)$$

$$=2\left[\sqrt{\frac{6}{g}}+\frac{\pi}{2\sqrt{g}}+\frac{1}{\sqrt{g}}\sin^{-1}\frac{1}{\sqrt{7}}\right]$$

$$=\frac{2}{\sqrt{g}}\left[\sqrt{6}+\frac{\pi}{2}+\sin^{-1}\frac{1}{\sqrt{7}}\right].$$

Note. If we use $x=a\cos\sqrt{\mu}\,t=\sqrt{7}\cos\sqrt{g}\,t$, in which time t is measured from the extreme position C, then by putting $x=-1$ for A, we get

$$\sqrt{g}\,t=\pi-\cos^{-1}\tfrac{1}{7}.$$

This gives time t from C to A *i.e.*, t_1+t_2.

Examples X

1. An elastic string of natural length $2l$ can just support a certain weight when it is stretched till its whole length is $3l$. One end of the string is now attached to a point on a smooth horizontal table and the same weight is attached to the other end. Prove that if the weight is pulled to any distance and then let go, the string will become slack again after a time $\frac{1}{2}\pi\sqrt{l/g}$.

2. A body of mass 5 lbs. is hung on a light spring and is found to stretch it 6 inches. The mass is then pulled down a further distance of two inches and released. Find the period of oscillation and the maximum velocity.

3. A light elastic string whose natural length is 4 ft. is such that a force of 12 lbs. wt. will stretch it to twice its natural length. One end of the string is fixed and a mass of 4 lbs. is attached to the other. Initially the string is vertical and the mass is held so that the string is just unstretched. The mass is then let go, prove that its subsequent oscillation is simple harmonic and find the periodic time. (*P.U.*)

4. A vertical spring extends a distance l when a given weight is attached to its lowest point. The weight is pulled down a further distance $a(<l)$ and let go. Find the period of the S.H.M. that ensues and show that the maximum velocity of the weight is $a\sqrt{g/l}$. *(P.U.)*

5. If t_1 and t_2 be the periods corresponding to two different attached weights and c_1 and c_2 the statical extensions due to these weights, prove that

$$g=4\pi^2(c_1-c_2)/(t_1^2-t_2^2). \qquad (P.U.)$$

6. A weight is attached to the lower end of a light spiral spring whose upper end is fixed and is released. If it oscillates in a vertical line through a space of 6 inches ; what is the period of oscillation ? *(P.U.)*

7. In Art. 4·5, $m=3$ lbs. and it is projected from A in the direction of OA with a velocity of 8 ft./sec. How far will it go and what time will it take before it begins to return ; given that the string stretches 6 ins. by a force of 12 lbs. ? What is the time from rest to rest if the length of the string is one foot ?

8. A light elastic string of modulus λ is stretched to double its length and is tied to two fixed points distant $2a$ apart. A particle of mass m, tied to its middle point, is displaced in the line of the string through a distance equal to half its distance from the fixed point and released. Find the time of a complete oscillation and the maximum velocity acquired in the subsequent motion.

9. Find the extension produced in a given spring when the weight it supports is increased by one gramme. If e be the extension and m the mass supported ; show that the time of a small oscillation for the mass m is $2\pi\sqrt{me/g}$.

10. An elastic cord AB, 6 inches long when unstretched and which would be stretched to twice its length by a steady pull of 5 pounds, is hung up at A ; and a 2 lbs. weight is attached at B. The weight is then allowed to fall from A. Find how far it will fall, and find its maximum velocity and maximum acceleration.

11. An elastic string supporting a heavy particle hangs in equilibrium, the elongation produced by the weight of the particle being equal to e. The particle is now pulled down below the equilibrium position through a distance d greater than e, and then released. Show that the height to which the particle will rise after the string ceases to be taut is $\dfrac{d^2-e^2}{2e}$.

***12.** One end of an elastic string is fixed and to the other end is fastened a particle heavy enough to stretch the string to double its natural length a. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle returns to this point in the time

$$\left(2\sqrt{3}+\frac{4\pi}{3}\right)\sqrt{\frac{a}{g}}.$$

***13.** A string of length 18 inches would be elongated 6 inches by a tension of 1 lb. weight. A weight of ½ lb. is attached to it and the string is pulled to a total length of 3 feet and released. Find the time of oscillation and the maximum velocity.

14. A heavy particle of mass m is attached to one end of an elastic string of natural length a and modulus $\frac{1}{2}mg\tan^2\theta$. The other end of the string is fixed at O and the particle is let fall from rest at O. Show that part of the motion is simple harmonic, and that the greatest depth of the particle below O is $a\cot^2\frac{1}{2}\theta$, and that the particle attains this depth in time

$$\sqrt{\frac{2a}{g}}\,[1+(\pi-\theta)\cot\theta],$$

where θ is a positive acute angle.

15. One end of an elastic string whose modulus of elasticity is λ and whose natural length is a, is tied to a fixed point on a smooth horizontal table and the other end tied to a mass m lying on the table. The particle is pulled to a distance where the extension of the string is b, and then let go ; describe the character of motion and show that the period of one complete oscillation is

$$2\left(\pi+\frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}.$$

4·7. Further Examples of Motion with Variable Acceleration.

Example. *A particle moves in a straight line, starting from rest from a distance a to a centre of attraction towards which the force per unit mass is $-\frac{\mu}{x^3}$. Show that the time required to reach the centre is $a^2/\sqrt{\mu}$.*

Let v be the velocity of the particle at a distance x from the centre.

Equation of motion of the particle is $v\frac{dv}{dx}=-\frac{\mu}{x^3}$.

Integrating, we have $\frac{1}{2}v^2=\frac{1}{2}\frac{\mu}{x^2}+c$.

Initially when $x=a$, $v=0$, so that $c=-\frac{\mu}{2a^2}$.

$$\therefore \qquad v^2=\frac{\mu}{x^2}-\frac{\mu}{a^2}=\mu\frac{a^2-x^2}{a^2x^2}.$$

Since the particle is moving towards the centre, its velocity is negative.

$$\therefore \qquad v=\frac{dx}{dt}=-\frac{\sqrt{\mu}\sqrt{a^2-x^2}}{ax}.$$

Whence $\sqrt{\mu}.t+c'=-\int\frac{ax\,dx}{\sqrt{a^2-x^2}}=-\int ax\,(a^2-x^2)^{-\frac{1}{2}}\,dx$

$$=a(a^2-x^2)^{\frac{1}{2}}.$$

Initially when $t=0$, $x=a$, so that $c'=0$

$$\therefore \qquad \sqrt{\mu}\,t=a\sqrt{a^2-x^2}.$$

Now when the particle reaches the centre, $x=0$ and $t=\frac{a^2}{\sqrt{\mu}}$.

Examples XI

1. A particle is attracted by a force varying inversely as the nth ($n>1$) power of the distance. Find the value of n when the velocity acquired from an infinite distance to a distance a from the centre is equal to the velocity which would be acquired by it in falling from rest from a distance a to a distance $a/4$.

2. A particle leaves the origin with velocity u. If it moves with an acceleration $\frac{k}{v^2}$, v being the velocity at any instant, show that the distance x described in time t is given by

$$4kx=(3kt+u^3)^{\frac{4}{3}}-u^4.$$

3. A particle is let fall from rest from a point outside the earth at a distance b from its centre. Prove that the square of the velocity of the particle on reaching the centre is $ga\left(3-\frac{2a}{b}\right)$, where a is the radius of the earth and g the value of gravity at its surface.

(It is assumed that for motion outside the earth the value of gravity varies inversely as the square of the distance from the earth's centre and that for motion inside the earth, it varies directly as the distance.)

4. A meteorite falls from rest towards the earth from infinity ; show that its velocity on reaching the earth is the same as it would have acquired in falling with constant acceleration g through a distance equal to the earth's radius. *(P.U. 1950)*

5. If the earth were still, show that if a shot were projected vertically upwards with a velocity exceeding 7 miles per second, it will never return. Assume the earth's radius to be 3,960 miles.

6. If the radius of the moon is $\frac{4}{15}$ of the earth's radius and the mass of the moon is $\frac{1}{81}$ of the earth's mass, show that the least velocity with which bodies must be projected from the surface of the moon so that they may never return to it, is about 1·5 miles per second.

7. A particle starts from A and moves in a straight line $AO(=a)$ with an acceleration which is directed towards O and varies, inversely as the square of its distance from O ; show that the particle arrives at O with an infinite velocity after a time $\frac{\pi}{2} \cdot \frac{a^{\frac{3}{2}}}{\sqrt{2\mu}}$, μ being the constant of variation.

8. Show that the time of descent to a centre of force the force varying inversely as the square of the distance from the centre, through the first half of its initial distance is to that through the last half as $\pi+2 : \pi-2$. *(P.U., D.U.)*

9. A particle moves in a straight line from a distance a, to a centre of force. Show that the time required to reach the centre is $\frac{a^2}{\sqrt{\mu}}$ or $\frac{2a^{\frac{4}{3}}}{\sqrt{3\mu}}$ according as the force of attraction is μr^{-3} or $\mu r^{-\frac{5}{3}}$.

10. A particle whose mass is m is acted upon by a force $m\mu\left(x+\frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance a, show that it will arrive at the origin in time $\pi/4\sqrt{\mu}$. *(Agra U.)*

11. A particle oscillates with simple harmonic motion of amplitude a and periodic time T. Find an expression for the velocity v (1) in terms of a, T, x ; (2) in terms of a, T, t, where x is the distance from the centre and t is the time measured from an instant when $x=0$.

Prove that $$\int_0^T v^2 dt = \frac{2\pi^2 a^2}{T}$$ *(Bm. U.)*

12. Prove that if a straight tunnel were bored between any two points on the earth's surface, and if the resistance of the air is neglected, a train running on smooth rails would traverse the tunnel under gravity alone in about 42 minutes.

What will be the maximum velocity of the train if the length of the tunnel is nearly 300 miles.

Given that the attraction of the earth at points inside its surface varies as the distance from its centre and that the radius of the earth is about 4,000 miles.

13. A thread of natural length 30 cms. extends to a length of 50 cms. when supporting 5 gms. One end of the thread is fastened 30 cms. below a small hole in a table while the other end is passed through the hole and attached to a particle of weight 1 gm. The particle is placed at rest on the table at a distance of 10 cms. from the hole and is let go. Find the time in which it performs a complete oscillation and its velocity as it passes over the hole. *(Bm. U.)*

REVISION QUESTIONS—III

1. A load of mass M lbs. is being pulled up with a uniform velocity u by a long steel cable when the upper end of the cable is suddenly fixed. Having given that a weight of m lbs. would extend the cable 1 ft., show that the amplitude of oscillation of the load is $u\sqrt{M/mg}$.

2. A light elastic string of natural length l has one extremity fixed at a

point A and the other attached to a stone the weight of which, in equilibrium, would extend the string to a length l_1, show that if the stone be dropped from rest at A, it will come to instantaneous rest at a depth $\sqrt{l_1^2-l^2}$ below the equilibrium position.

3. If u, v, w be the velocities of a particle at distances a, b, c from a fixed point O (not the centre of motion) on the straight line along which the particle is moving with S. H. M., show that the period T is given by

$$(b-c)u^2+(c-a)v^2+(a-b)w^2=4\pi^2(b-c)(c-a)(a-b)/T^2.$$

[**Hint.** Let O', $(OO'=d)$ be the centre of motion and k the amplitude, then $O'A=a-d$, $O'B=b-d$, $O'C=c-d$,

and $u^2=\mu[k^2-(a-d)^2]=\mu(k^2-a^2-d^2+2ad)$ etc.]

4. A smooth light pulley is suspended from a fixed point by a spring of natural length l and modulus of elasticity λ. If masses m_1 and m_2 hang over the pulley, show that the pulley executes a simple harmonic motion about a point whose depth below the point of suspension is

$$l\left[1+\frac{4m_1m_2g}{(m_1+m_2)\lambda}\right].$$

† 5. Two bodies M and M' $(<M)$ are attached to the lower end of an elastic string whose upper end is fixed, and are hung at rest; M' falls off; show that the distance of M from the upper end of the string at time t is

$$a+b+c\cos\sqrt{\frac{g}{b}}\,t,$$

where a is the unstretched length of the string, and b and c the distances by which it would be stretched when supporting M and M' respectively.

6. A heavy particle of mass m is attached to one end of an elastic string of natural length l ft. whose modulus of elasticity is equal to the weight of the particle and whose other end is fixed at O. The particle is let fall from rest at O. Show that part of the motion is simple harmonic and that the greatest depth of the particle below O is $(2+\sqrt{3})\,l$ ft.

Show also that this depth is attained in time

$$\left(\sqrt{2}+\pi-\cos^{-1}\frac{1}{\sqrt{3}}\right)\sqrt{\frac{l}{g}}\text{ secs.} \qquad (D.U.)$$

*7. A light elastic string AB of length l is fixed at A and is such that if a weight W be attached to B, the string will be stretched to double its length. If a weight $W/4$ be fastend to B and let fall from A prove that

(a) the subsequent motion is harmonic.

(b) the distance through which it will fall is $2l$

(c) the period of oscillation is $\sqrt{\frac{l}{4g}}\,(4\sqrt{2}+\pi+2\sin^{-1}\frac{1}{3})$.

(d) the amplitude is $\frac{3}{2}l$. $(P.U.)$

*8 A particle is suspended from two fixed points in the same horizontal line by equal elastic threads and hangs in equilibrium with each thread of length l, making an angle of 60° with the vertical. Having given that the modulus of elasticity is equal to the weight of the particle, prove that the period of small oscillation is $2\pi\sqrt{2l/5g}$.

† *9. A particle m is attached to a light wire which is stretched tightly between two fixed points under a tension P. If a, b, be the distances of the particle from the two ends, prove that the period of a transverse oscillation is

$$2\pi\sqrt{\frac{m}{P}\cdot\frac{ab}{a+b}}. \qquad (P.U.\ Hons.)$$

*10. A particle moves in a straight line in such a manner that its velocity, t seconds after it is projected with velocity u from the point from which the distance s is measured, is $ue^{\alpha(t+s)}$, where α is a positive constant.

Prove that the time taken to attain the velocity $2u$ is $\frac{1}{\alpha}\log\frac{2u+2}{2u+1}$, and the distance travelled during this interval is $\frac{1}{\alpha}\log\frac{2u+1}{u+1}$.

*11. Particles of masses m_1, m_2, are attached to the ends of a light string of modulus λ and natural length l. They are attached also to the ends of a light inextensible string which passes over a smooth horizontal peg. The system is released from rest with the string unstretched, the lighter particle m_1 vertically below the peg, and the particle m_2, vertically below the particle m_1. Prove that the subsequent motion of each particle is simple harmonic and find the period and amplitude.

*12. A particle moves in a straight line starting from rest from a distance a to a centre of attraction towards which the force per unit mass is μ/x. Show that the time required to reach the centre is $a\sqrt{\pi}/\sqrt{2\mu}$.

[**Hint.** In the final integral put $\log\frac{a}{x}=z$ and use $\int_0^\infty z^{\frac{1}{2}}e^{-z}\,dz=\sqrt{\pi}$.]

CHAPTER V

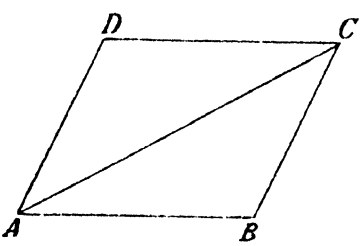
COMPOSITION AND RESOLUTION OF VELOCITIES, ACCELERATIONS AND FORCES

5·1. A particle may have several velocities and accelerations simultaneously. In the present chapter, we discuss the methods of finding their resultant.

As the velocity or the acceleration of a particle at any instant will have a certain magnitude and a certain direction, they may, therefore, be represented by a straight line of certain magnitude drawn in a certain direction.

5·11. Co-existence of displacements. A body may receive two or more displacements simultaneously in different directions, *e.g.*, if a ball is rolled on the deck of a vessel in a direction other than that of the vessel in a river, the ball receives simultaneously two displacements one on the deck of the vessel and the other that of the vessel on the stream. If moreover, the river is supposed to have a motion relative to its banks, the ball will also undergo the displacement of the stream when the earth is considered to be the reference body relative to which we consider the motion.

5·12. Composition of displacements. *If a particle receives two displacements represented by the two sides AB and AD of the parallelogram ABCD, the resultant displacement will be represented by the diagonal AC.*

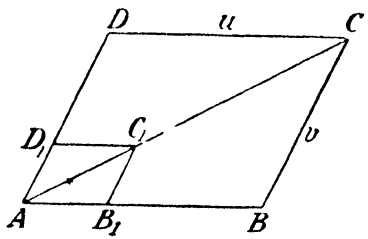


We may suppose that a tube coincides with AB, in which a particle is moving and that the tube itself is moving parallel to itself such that its end A undergoes the displacement AD. Suppose that in the same time, the displacement of the particle relative to the tube is AB. It is clear that the position of the particle due to the two displacements will be C. Hence the resultant displacement is represented by AC.

5·2. Composition of velocities. As the velocity is the rate of displacement, the co-existence of displacements implies the co-existence of velocities. Displacements and velocities are compounded by the same law which, in the case of velocities, is known as *the Parallelogram of Velocities* and may be stated thus :

5·21. Parallelogram of velocities. *If two velocities are represented in magnitude and direction by the two sides of a parallelogram drawn from a point, their resultant is represented in magnitude and direction, on the same scale, by the diagonal of the parallelogram passing through the point.*

Suppose that a particle possesses simultaneously two velocities u, v, represented in magnitude and direction by the sides AB, and AD respectively of the parallelogram $ABCD$. The resultant will be represented in magnitude and direction by the diagonal AC.



The particle may be supposed to move along AB with speed u, while the line AB moves parallel to itself such that its end A describes AD with speed v.

In unit time the particle moves the distance AB while the line AB moves to the position DC, so that the particle occupies the position C.

The displacement of the particle in the time t will be ut, say AB_1, along AB, while the line AB will move parallel to itself, its end A moving a distance vt, say AD_1, along AD.

Let AC_1 be the diagonal of the parallelogram of which AB_1 and AD_1, are the sides. The particle will thus be at C_1 at the end of the interval t. Join CC_1. In the triangles AB_1C_1 and ABC.

$$\frac{AB_1}{B_1C_1}=\frac{AB_1}{AD_1}=\frac{ut}{vt}=\frac{u}{v}=\frac{AB}{AD}=\frac{AB}{BC}.$$

$\therefore$ A, C_1, C are on the same straight line.

Hence the particle moves along AC, so that AC is the *direction* of the resultant velocity of the particle.

Also if V is the resultant velocity, $AC_1=Vt$.

But since $\frac{AC_1}{AC}=\frac{AB_1}{AB}=\frac{ut}{u}=t$, we have $AC=\frac{AC_1}{t}=\frac{Vt}{V}=V$,

i.e., AC represents the *magnitude* of the resultant velocity on the same scale as AB and AD represent u and v.

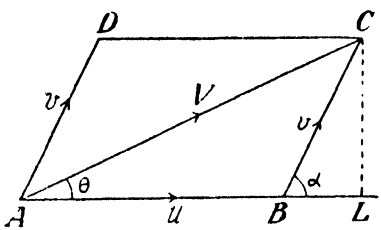
Hence AC represents both in magnitude and in direction the resultant velocity.

5·22. Now if we denote the $\angle BAD=\angle CBL$ by α.

we have
$$\begin{aligned}AC^2&=AB^2+BC^2-2AB.BC\cos\angle ABC\\&=AB^2+BC^2-2AB.BC\cos(180°-\alpha)\\&=AB^2+AD^2+2AB.AD\cos\alpha.\end{aligned}$$

Hence $V=\sqrt{u^2+v^2+2uv\cos\alpha}$. ..(1)

Also if the angle $BAC=\theta$ and CL is drawn perpendicular to AB we have



$$\begin{aligned}\tan\theta&=\frac{CL}{AB+BL}\\&=\frac{BC\sin\alpha}{AB+BC\cos\alpha}\\&=\frac{v\sin\alpha}{u+v\cos\alpha}\end{aligned}\quad\ldots(2)$$

V is called the resultant of u and v, and u and v are called the components of V in directions AB and AD.

5·23. In the particular case when the two velocities u and v are equal, we put $v=u$ in (1) and (2) and we get

$$V=\sqrt{u^2+u^2+2u^2\cos\alpha}=u\sqrt{2(1+\cos\alpha)}=2u\cos\frac{\alpha}{2},$$

and $$\tan\theta=\frac{u\sin\alpha}{u+u\cos\alpha}=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}=\tan\frac{\alpha}{2},$$

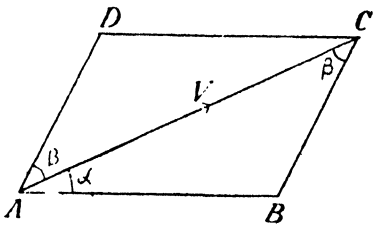
i.e., the line of action of the resultant velocity bisects the angle between its equal components.

In case u and v are at right angles, $\alpha=90°$

$$\therefore \qquad V=\sqrt{u^2+v^2} \text{ and } \tan\theta=\frac{v}{u}.$$

5·24. Resolution of velocities. A velocity may be resolved into component velocities in an infinite number of ways, for the straight line representing the velocity may be the diagonal of an infinite number of parallelograms.

The components of the velocity V in directions making angles α and β with it can be found as follows :—



Let AC represent V. Draw AB, AD making angles α and β with AC and complete the parallelogram $ABCD$.

Then AB and AD or AB and BC represent the required components.

$\therefore$ from the triangle ABC we have

$$\frac{AB}{\sin\beta}=\frac{BC}{\sin\alpha}=\frac{AC}{\sin B}=\frac{V}{\sin(\alpha+\beta)}.$$

Hence $$AB=\frac{V\sin\beta}{\sin(\alpha+\beta)}, \text{ and } BC=\frac{V\sin\alpha}{\sin(\alpha+\beta)}.$$

5·25. In the particular case when the components are to be at right angles, let $\alpha=\theta$, then

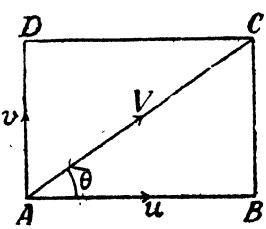
$$\beta=90°-\theta \text{ and } \alpha+\beta=90°.$$

So that the components make angles θ and $90°-\theta$ with AC and are known as the *resolved parts* of the velocity V. If we denote them by u and v respectively, we have

$u=V\cos\theta$ and $v=V\sin\theta$.

Since $\sin\theta=\cos(90°-\theta)$,

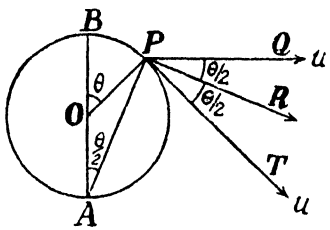
it follows that *the resolved part of a velocity in a certain direction is obtained by multiplying the magnitude of the velocity by the cosine of the angle between the direction of velocity and the given direction.*



To find the components of a velocity we must be told the two directions into which it is to be decomposed but to find the resolved parts, we need only know one direction as the second direction must be at right angles to the known direction.

5·26. Example. *A wheel is rolling, without slipping, in a vertical plane, along a horizontal straight line, its centre having the speed u, to find the velocity of any point on its rim.*

Let O be the centre, A the lowest point in contact with the ground and B, the highest point of the wheel at a particular instant. Take P any point on the rim such that $\angle POB=\theta$.



The centre moves in a straight line with velocity u and the wheel turns uniformly round the centre. Since the wheel does not slip and since each point of the rim, in succession, touches the ground, it follows that the time which any point on the rim takes to describe the perimeter of the wheel is the same in which the centre of the wheel moves a distance equal to the perimeter. Hence any point P of the rim possesses two velocities, each equal to u, one along the tangent PT relative to the centre, and the other along the horizontal line PQ, equal to the velocity of O.

Since $\angle QPT=\theta$, $\therefore$ the resultant of these equal velocities

$$=2u\cos\frac{\theta}{2} \qquad \ldots(1)$$

$$=2u\cos\angle PAB$$

$$=2u\frac{AP}{AB}.$$

It takes place along PR which bisects $\angle QPT$, is perpendicular to AP and passes through B (when produced backwards).

The velocity of B where $\theta=0$ is $2u$, *by (1), i.e., the highest point of the wheel moves with double the velocity of its centre.*

The velocity of A where $\theta=\pi$ is zero, *i.e., the lowest point is, for an instant, at rest.*

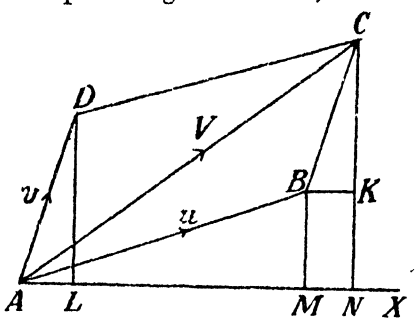
5·27. Triangle of velocities. In the figure of parallelogram of velocities, Art. 5·21, since $BC=AD$, AC represents in magnitude and direction, the resultant of velocities represented in magnitude and direction, by AB and BC or in other words the resultant of two *velocities represented in magnitude and direction by the two sides of a triangle in the same sense, is represented in magnitude and direction by the third side of the triangle taken in the opposite sense.*

Thus theorem is called the *triangle of velocity.*

Since the resultant of velocities represented by AB and BC is represented by AC, therefore, the resultant of velocities represented by AB, BC and CA will be nil and thus if a particle has three simultaneous velocities which can be represented in magnitude and direction by the three sides of a triangle, taken in order, in the same sense, the particle is reduced to rest.

5·3. *The sum of the resolved parts of two velocities in a given direction is equal to the resolved part of their resultant in the same direction.*

Let AB and AD represent the two velocities u and v. Complete the parallelogram $ABCD$, then the diagonal AC represents the resultant velocity V



Through A, draw AX parallel to the given direction. Draw BM, CN, DL, perpendiculars to AX and BK perpendicular to CN.

AM is the resolved part of u in the direction AX, AL is the resolved part of v while AN is the resolved part of V in the same direction.

It is easy to prove that

$$MN = BK = AL$$

$$\therefore \quad AN = AM + MN = AM + AL$$

which proves the theorem.

By its repeated application the theorem can be easily extended to the more general theorem which may be stated as :

The sum of the resolved parts of a number of velocities in a given direction is equal to the resolved part of their resultant in the same direction.

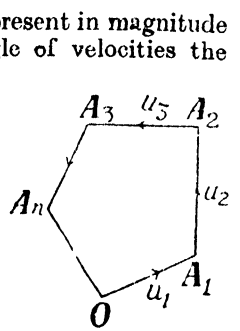
5·4. Composition of a number of velocities. The resultant of a number of velocities, $u_1, u_2, \ldots\ldots u_n$, can be obtained in several ways.

(1) *By the repeated application of the parallelogram law of velocities.*

Find the resulant of u_1 and u_2 let it to be v_1 ; find the resultant of v_1 and u_3, let it be v_2 and so on. Finally find the resultant of v_{n-2} and u_n which gives V, the resultant sought.

(2) *By means of the Polygon of Velocities.*

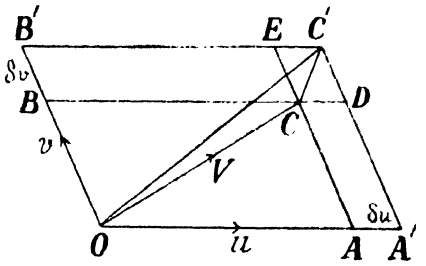
Draw OA_1, A_1A_2, A_2A_3, etc., in order to represent in magnitude and direction u_1, u_2, u_3 etc. Then by the triangle of velocities the resultant of u_1 and u_2 will be ropresented in magnitude and direction by OA_2; call it v_1. The resultant of v_1 and u_3, will be v_2 represented by OA_3, and so on. Finally the resultant of v_{n-2} and u_n will be represented by OA_n, the line which completes the polygon, and taken in the sense opposite to that in which the other sides of the polygon are taken.



(3) *By resolution along two axes at right angles* (*i.e.*, by Art. 5·3).

Take any two axes OX and OY at right angles and let the velocities u_1, u_2 etc., make angles α, β, etc., with OX, then the

In the position O at time t let a particle be moving with two simultaneous velocities u and v represented in magnitude and direction by OA and OB. Let these velocities be variable and let δu, δv, the small changes in the velocities in the short interval δt, be represented by AA' and BB' respectively.



Then OA', and OB' represent the velocities at time $t+\delta t$;

$$CD = AA' = \delta u,$$

and $$CE = BB' = \delta v.$$

The resultant velocity V at time t is represented by OC and $V+\delta V$, the resultant at time $t+\delta t$ is represented by OC', therefore, by the theorem of the triangle of velocities, δV, the change in the resultant velocity in time δt must be represented by CC'.

Thus, if δu and δv, the small change (in the short interval δt) in the two velocities posssessed by a particle, are represented by the two sides CD and CE of the parallelogram $CDC'E$, δV the resultant change in the velocity is represented by the diagonal CC'.

Now $\frac{\delta u}{\delta t}$, $\frac{\delta v}{\delta t}$ and $\frac{\delta V}{\delta t}$ are respectively proportional to δu, δv and δV.

$\therefore$ if $\frac{\delta u}{\delta t}$, $\frac{\delta v}{\delta t}$ are represented by the two sides of a parallelogram drawn through a point, $\frac{\delta V}{\delta t}$, can be represented, on the same scale, by the diagonal of the parallelogram passing through the point, and this remains true, however small δt may be.

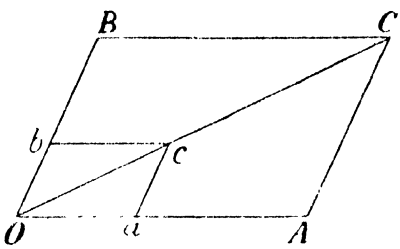
Hence in the limit when $\delta t \to 0$, if $\frac{du}{dt}$ and $\frac{dv}{dt}$, the accelerations in two directions are represented by the two sides of a parallelogram drawn from a point, then $\frac{dV}{dt}$, the resultant acceleration, must be represented by the diagonal of the parallelogram passing through the point.

From the parallelogram law of accelerations, it follows that accelerations are resolved and compounded in exactly the same way as velocities.

5·6. Parallelogram of forces. *If two forces are represented in magnitude and direction by the two sides of a parallelogram drawn from a point, then the resultant force is represented in magnitude and direction by the diagonal of a parallelogram passing through that point.*

Let F_1 and F_2 be the two forces represented by OA and OB and let f_1 and f_2 be the accelerations produced by them on a particle of mass m. Let f_1 and f_2 be represented by Oa and Ob. Complete the parallelograms $Oacb$ and $OACB$.

Then by the second law of motion, $F_1 = mf_1$ and $F_2 = mf_2$.



i.e., $\dfrac{OA}{Oa} = \dfrac{F_1}{f_1} = m$

and $\dfrac{OB}{Ob} = \dfrac{F_2}{f_2} = m$

$\therefore \quad \dfrac{OA}{Oa} = \dfrac{OB}{Ob} = \dfrac{AC}{ac}$

i.e., Oc produced passes through C and triangles Oac and OAC are similar.

Hence $\dfrac{OC}{Oc} = \dfrac{OA}{Oa} = m$ *i.e.*, $OC = m.Oc$.

Now by the parallelogram law of accelerations Oc represents the resultant acceleration, say f.

$\therefore \quad OC\ (= mOc = mf)$ must represent the resultant force, say F.

Thus if F_1, F_2 are represented by the sides OA and OB, the resultant force F is represented by the diagonal OC.

Cor. Momenta being proportional to velocities, also follow the parallelogram law.

Forces and momenta can be resolved and compounded in exactly the same way as velocities.

5·7. Vector quantities. Velocities, accelerations, momenta and forces are examples of quantities, called *vector quantities*, as distinguished from those known as *scalar quantities*.

Vector quantities involve both magnitude and direction and they can, therefore, be represented by straight lines of specified lengths drawn in specified directions. All such quantities are compounded by the parallelogram law.

Scalar quantities such as lengths, masses, and times are compounded by the ordinary algebraic rule. Thus if a body weighs 2 lbs. and another 3 lbs., the two together will weigh $2+3=5$ lbs., but a velocity of 2 ft. per sec., and a velocity of 3 ft. per sec. are never equal to a velocity of 5 ft. per sec. unless, in the special case, when they are taken in the same direction along a straight line.

MOTION ON AN INCLINED PLANE

5·8. Smooth inclined plane. *A body moves along a smooth inclined plane under the action of gravity alone, to find its motion.*

Let m be the mass of the body and α the inclination of the plane.

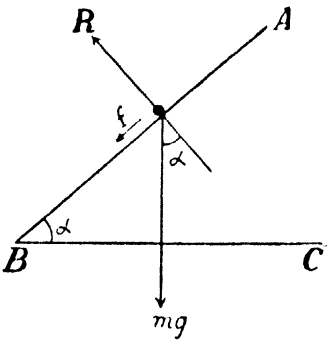
The only forces acting on the body are its weight mg acting vertically and the pressure R of the plane acting perpendicular to the plane.

The vertical force mg is equivalent to (*i*) a force $mg\cos\alpha$ perpendicular to the plane and (*ii*) a force $mg\sin\alpha$ down the plane.

Since there is no motion and, therefore, no acceleration perpendicular to the plane, we have

$$R-mg\cos\alpha=0$$

or $$R=mg\cos\alpha \quad ..(1)$$



If the acceleration down the plane is f then

$$mf=mg\sin\alpha$$

or $$f=g\sin\alpha \quad ..(2)$$

Hence the body moves down the plane with a constant acceleration $g\sin\alpha$ and it is, therefore, evident that the motion of a body, on a smooth inclined plane is similar to that of a body moving vertically under gravity, except that instead of g we have to take $g\sin\alpha$ for acceleration.

Thus, if a body is projected up a smooth inclined plane along a line of greatest slope with an initial velocity u, then v, the velocity, and s, the distance described in time t are given by the equations

$$v=u-g\sin\alpha.t,$$

$$s=ut-\tfrac{1}{2}g\sin\alpha.t^2,$$

and $$v^2=u^2-2g\sin\alpha.s.$$

Note 1. The *line of greatest slope* through any point on an inclined plane is the line on the plane drawn through that point perpendicular to the line of intersection of the inclined plane with a horizontal plane.

Note 2. If a particle is released from rest or projected with an initial velocity from a point on a smooth inclined plane, up or down the plane, and moves under the action of gravity alone, the particle moves along the line of greatest slope with a constant downward acceleration equal to $g\sin\alpha$.

Note 3. If the body is made to slide down the plane in a direction inclined to a line of greatest slope (*e.g.*, in a smooth groove) the acceleration is not $g\sin\alpha$, but $g\sin\alpha\cos\beta$ where β is the inclination of the direction of motion with the line of the greatest slope.

Example. *The resistance to the motion of a train, due to friction etc., is 14 lbs. per ton. If the train moving at the rate of 30 miles an hour, comes to the foot of an incline of 1 in 200, the steam being turned off, find how far the train will go up the inclined plane before it comes to rest.*

30 m.p.h.=44 f.p.s.

The inclination α of the plane is given by $\sin\alpha=\frac{1}{200}$

Retardation due to the weight $=g\sin\alpha=g\times\frac{1}{200}=\frac{32}{200}=\frac{4}{25}$.

If the mass of the train is m tons, the retardation due to friction

$$=\frac{\text{friction}}{\text{mass}}=\frac{14mg}{m.2240}=\frac{14g}{2240}=\frac{14\times32}{2240}=\frac{1}{5}.$$

$\therefore$ total retardation $= \frac{4}{25} + \frac{1}{5} = \frac{9}{25}$.

The required distance s is given by the equation

$$0 = u^2 - 2fs$$

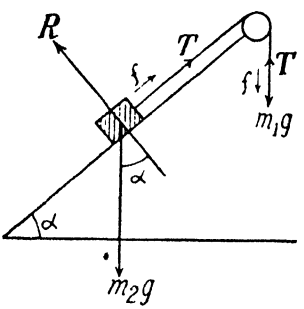
$$\therefore \quad 44 \times 44 = 2 \times \tfrac{9}{25} s$$

i.e.,

$$s = \frac{44 \times 44 \times 25}{2 \times 9} = 2688{\cdot}89 \text{ ft.}$$

5·81. *A mass m_1 hanginy vertically pulls a mass m_2 up a smooth inclined plane of inclination α, by means of an inextensible string passing over a pulley at the top of the plane (the part of the string between m_2 and the pulley being parallel to a line of greatest slope), to find the motion.*

The velocity and acceleration of m_2 up the plane are equal to the velocity and acceleration of m_1 vertically downwards.



Let f be the common acceleration, then the equation of motion for m_1 is

$$m_1 f = m_1 g - T \qquad \ldots (i)$$

The equation of motion for m_2 along the plane is

$$m_2 f = T - m_2 g \sin \alpha \qquad \ldots (ii)$$

$\therefore$ from (i) and (ii), we have

$$f = \frac{m_1 - m_2 \sin \alpha}{m_1 + m_2} g$$

and

$$T = \frac{m_1 m_2 (1 + \sin \alpha)}{m_1 + m_2} g.$$

$\alpha = 0$ gives the same results as obtained when m_2 is placed on a horizontal table.

And $\alpha = 90°$ gives results obtained when both the particles are suspended vertically as in an Atwood's machine.

Examples XIII

1. A body moves up a smooth inclined plane starting with a velocity of 48 ft. per sec. If the plane is inclined at an angle of 30°; find the velocity of the body when it has moved a distance of 64 ft. from the starting point. *(P.U.)*

2. Prove that the velocity acquired by a heavy particle sliding freely down a smooth inclined plane of given height, is independent of the inclination of the plane.

Prove also that the time of descent varies as the cosecant of the inclination.

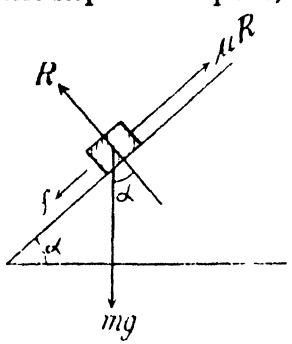
3. Show that the time of fall down an inclined plane of given base is least when the inclination is 45°. *(P.U. 1950)*

4. A right-angled triangle is placed with its sides horizontal and vertical. Find their ratio so that the time of falling down the vertical and describing the base with the velocity acquired, may be equal to the time of descent down the hypotenuse. *(P.U. 1952)*

5. *(a)* Prove that the time of descent along all smooth chords of a vertical circle drawn from its highest point is the same.

5·9. Rough inclined plane. *A body slides along a rough inclined plane of inclination α and coefficient of friction μ, to find the motion.*

Let the motion be down a line of greatest slope of the plane, then the force of friction acts up the plane.



If f is the acceleration down the plane, the equation of motion along the plane is

$$mf = mg \sin\alpha - \mu R \quad ..(i),$$

and since, there is no motion perpendicular to the plane, we have

$$R - mg\cos\alpha = 0,$$

i.e., $$R = mg\cos\alpha \quad ..(ii),$$

$\therefore$ $$mf = mg\sin\alpha - \mu\, mg\cos\alpha$$

or $$f = g(\sin\alpha - \mu\cos\alpha) \quad ..(iii).$$

If $\sin\alpha < \mu\cos\alpha$ or $\tan\alpha < \mu$, then there will be no acceleration down the plane and as the body obviously cannot move up the plane by itself, this means that it will remain at rest.

If the body is projected up the plane with a given velocity u, the force of friction will act down the plane and the resultant retardation f' down the plane (obtained by changing the sign of μ) is given by

$$f' = g(\sin\alpha + \mu\cos\alpha) \quad ..(iv).$$

Due to this retardation, the body will come to rest after describing a length l of the plane given by

$$0 = u^2 - 2f'l,$$

i.e., $$u^2 = 2gl(\sin\alpha + \mu\cos\alpha).$$

Now if $\tan\alpha > \mu$, it will move down the plane with an acceleration f given by (*iii*) and thus it will return back to the starting point with a velocity v, given by

$$v^2 = 2fl = 2gl(\sin\alpha - \mu\cos\alpha).$$

Obviously $v < u$. This is due to the fact that the plane is rough.

5·91. *A mass m_1 hanging vertically pulls a mass m_2 up a* **rough** *inclined plane of inclination α, by means of an inextensible string passing over a pulley at the top of the plane, to find the motion.*

(Compare Art. 5·81, P. 76 and see the figure given there.)

Resolving forces perpendicular to the plane for m_2, we have

$$R - m_2 g\cos\alpha = 0$$

i.e., $$R = m_2 g\cos\alpha \quad ..(1).$$

Let f be the resulting acceleration of the system, then the equation of motion for m_1 which moves vertically, is

$$m_1 f = m_1 g - T \quad ..(2)$$

The plane being rough and the motion of m_2 being along the plane upwards, the force of friction μR will act down the plane, μ being the coefficient of friction.

$\therefore$ the equation of motion for m_2 is

$$\begin{aligned} m_2 f &= T - m_2 g \sin\alpha - \mu R \\ &= T - m_2 g \sin\alpha - \mu\, m_2 g \cos\alpha \qquad \text{[by (1)]} \\ &= T - m_2 g\,(\sin\alpha + \mu\cos\alpha) \qquad \ldots(3). \end{aligned}$$

Adding (2) and (3) we get

$$(m_1+m_2)\,f = m_1 g - m_2 g\,(\sin\alpha + \mu\cos\alpha) \qquad \ldots(4).$$

Cor. In case the motion takes place in the opposite direction, the force of friction will act up the plane and the corresponding equations of motion will be

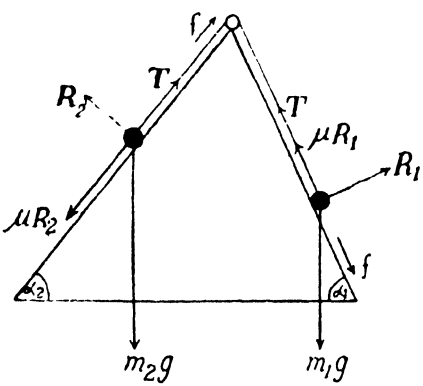
$$m_1 f = T - m_1 g.$$

$$\begin{aligned} m_2 f &= m_2 g \sin\alpha - T - \mu R \\ &= m_2 g(\sin\alpha - \mu\cos\alpha) - T \end{aligned}$$

and $$(m_1+m_2)\,f = m_2 g(\sin\alpha - \mu\cos\alpha) - m_1 g.$$

Example. *Two equally rough planes, coefficient of friction μ, of equal height and inclinations α_1, α_2 are placed back to back, masses m_1 and m_2 are placed on them and are connected by a light inextensible string passing over a pulley at the common vertex of the two planes (the string being parallel to the lines of greatest slopes of the planes through the common vertex). If m_1 descends, find the motion.*

Let T be the tension of the string and f, the common acceleration.



Equation of motion for m_1 moving down the plane of inclination α_1, is

$$\begin{aligned} m_1 f &= m_1 g \sin\alpha_1 - \mu R_1 - T \\ &= m_1 g\,(\sin\alpha_1 - \mu\cos\alpha_1) - T \qquad (i) \end{aligned}$$

and the equation of motion for m_2, moving up the second plane is

$$\begin{aligned} m_2 f &= T - m_2 g \sin\alpha_2 - \mu R_2 \\ &= T - m_2 g(\sin\alpha_2 + \mu\cos\alpha_2) \qquad \ldots(ii) \end{aligned}$$

Adding (i) and (ii), we have

$$(m_1+m_2)\,f = g[m_1\,(\sin\alpha_1 - \mu\cos\alpha_1) - m_2(\sin\alpha_2 + \mu\cos\alpha_2)].$$

Examples XIV

1. A rough plane is 100 ft. long and is inclined to the horizon at an angle $\sin^{-1}\frac{3}{5}$, the coefficient of friction being $\frac{1}{2}$, and a body slides down it from rest at the highest point ; find its velocity on reaching the bottom.

If the body were projected up the plane from the bottom so as just to reach the top, find its initial velocity.

2. A body is projected up a rough inclined plane with the velocity which would be acquired in falling freely through 20 feet and just reaches the top of the plane. If the inclination of the plane is 30° and the angle of friction is 60°, find the height of the plane.

3. A body slides down a rough incline of slope 30° to the horizon. If the coefficient of friction between the plane and the body be $\sqrt{3}/4$, find the

CHAPTER VI

MOTION OF A PARTICLE IN A PLANE—I

PROJECTILES

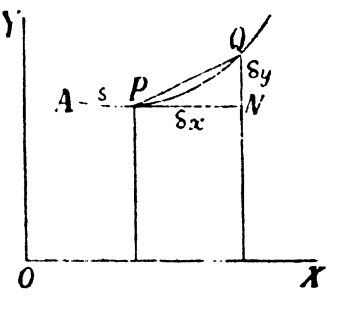
6·1. Position in a plane. So far we have been dealing with motion of a point along a straight line, the position of the point being completely determined by a single measurement along the line from a given point in it. Now we consider the motion of a point in a plane where the position of the point P is determined either by two distances x, y measured parallel to Ox, Oy in the plane or by r, the distance of the point P from a fixed point O and the angle θ which OP makes with some fixed line OX in the plane.

In the case of motion of a particle along a straight line, the direction is constant and we consider only the magnitude of the velocities and accelerations with positive or negative signs, their resultant being obtained simply by their algebraic sum.

In the case of motion of a particle in a plane, we find the components of velocities and accelerations in two directions, generally at right angles, their resultant being obtained by the parallelogram law as discussed in the previous chapter.

6·2. Expressions for velocity and acceleration of a particle moving in a plane.

Components of Velocity along OX, OY. Let the particle be moving along the curve APQ and let it move from the position P at time t to the neighbouring point Q at the time $t+\delta t$.



Let the co-ordinates of P and Q be (x, y) and $(x+\delta x, y+\delta y)$ referred to rectangular axes, OX, OY.

The displacement PQ may be represented by its components $(\delta x, \delta y)$ parallel to the axes.

The displacement of the particle parallel to OX in time δt is δx.

$\therefore$ the resolved part of the velocity parallel to OX

$$= \operatorname{Lt}_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}.$$

Similarly the resolved part of the velocity parallel to OY

$$= \operatorname{Lt} \frac{\delta y}{\delta t} = \frac{dy}{dt}.$$

Thus if x, y are the co-ordinates of the point P at any instant

referred to axes OX, OY, the component velocities parallel to the axes are the rates of change of x and y, *i.e.*, $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

6·21. Resultant Velocity. The resultant velocity V is given by,

$$V=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \quad \ldots (i)$$

and if θ be the angle which the direction of motion makes with the axis of x, we have

$$\tan\theta=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{dy}{dx} \quad \ldots (ii)$$

i.e., the direction of motion is the same as the direction of the tangent to the path.

6·211. Another expression for the resultant velocity is obtained as follows :

Let A be a fixed point on the curve, s the arc AP and δs, the arc PQ of the curve, then

$$V=\mathrm{Lt}\frac{\text{chord } PQ}{\delta t}=\mathrm{Lt}\frac{\text{chord } PQ}{\text{arc } PQ}\cdot\frac{\text{arc } PQ}{\delta t}=1\times\mathrm{Lt}\frac{\delta s}{\delta t}=\frac{ds}{dt}.$$

Thus the actual velocity at P is along the tangent at P and is equal to $\frac{ds}{dt}$ which is equivalent to expression (*i*), a fact which is otherwise obvious, [$\because$ $(\delta s)^2=(\delta x)^2+(\delta y)^2$].

6·22. Speed. $\frac{ds}{dt}$ *i.e.*, the rate of change of *arcual* distance without any consideration of the direction of motion is known as speed.*

So that speed is another name for magnitude of velocity. To specify velocity we require both the speed and the direction of motion.

6·23. Components of Acceleration along OX, OY. If u, v denote the resolved parts of the velocity of the particle parallel to the axes at time t and $u+\delta u$, $v+\delta v$ denote the resolved parts at time $t+\delta t$, then the resolved parts of the acceleration are

$$\mathrm{Lt}_{\delta t\to 0}\frac{\delta u}{\delta t}=\frac{du}{dt}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{d^2x}{dt^2},$$

and

$$\mathrm{Lt}_{\delta t\to 0}\frac{\delta v}{\delta t}=\frac{dv}{dt}=\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{d^2y}{dt^2}.$$

The consideration of component velocities and accelerations is of great importance when we have to deal with cases of motion where the path is not a straight line.

6·24. Equations of motion of a particle moving in a plane. The position of a point in a straight line being determined by one co-ordinate, only one equation of motion is sufficient to determine the motion completely. In the case of a particle moving in a plane two equations of motion are required in order to obtain the

two co-ordinates which define the position of a point in a plane. The two equations of motion are obtained by resolving the forces in any two convenient directions at right angles to one another.

If the two directions are taken parallel to the co-ordinate axes the equations of motion, as deduced from the second law of motion, will be of the form

$$m\frac{d^2x}{dt^2}=F_x \quad \text{and} \quad m\frac{d^2y}{dt^2}=F_y$$

where F_x, F_y are the sums of the resolved parts of the forces parallel to the axes of x and y.

Example 1. A particle of mass 5 lbs. is moving at any instant in a north-east direction with a speed of 10 ft. per sec. It is then acted on by a force of 20 pounds towards the north. Find the path it describes and its position after one second.

Take the axis of x towards the east, the axis of y towards the north and the initial position of the particle as the origin. Let (x, y) be its position at any time t.

Equations of motion for the particle are :

$$5\frac{d^2y}{dt^2}=20g \text{ and } 5\frac{d^2x}{dt^2}=0.$$

$$\therefore \frac{dy}{dt}=4gt+c \text{ and } \frac{dx}{dt}=c_1.$$

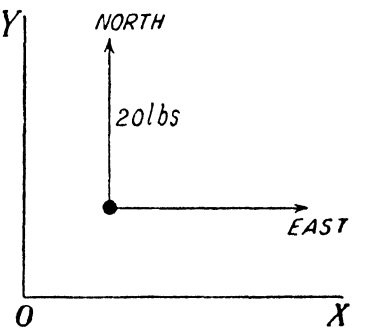
When $t=0$,

$$\frac{dx}{dt}=\frac{10}{\sqrt{2}}=5\sqrt{2}, \quad \frac{dy}{dt}=\frac{10}{\sqrt{2}}=5\sqrt{2}.$$

$$\therefore \frac{dy}{dt}=4gt+5\sqrt{2} \text{ and } \frac{dx}{dt}=5\sqrt{2}.$$

$$\therefore \quad y=2gt^2+5\sqrt{2}t+k$$

and $x=5\sqrt{2}.t+k_1$



when $t=0$, $x=y=0$, $\therefore k=k_1=0$.

Hence $y=2gt^2+5\sqrt{2}.t$ and $x=5\sqrt{2}.t$.

Eliminating t, we get

$$y=\frac{2x^2g}{50}+x \quad \text{or} \quad gx^2=25(y-x)$$

which represents a parabola.

After 1 second $x=5\sqrt{2}$ ft., $y=2\times 32+5\sqrt{2}$ ft. $=71{\cdot}07$ ft.

Example 2. A particle of 10 lbs. is moving towards the north with a speed of 6 ft. per second. It is acted upon by two forces, one of 30 poundals acting towards the north-east, the other of 10 poundals acting towards the north. Find the position of the particle after t seconds.

Example 3. Show that the path of a particle moving under a constant force is necessarily a straight line or a parabola.

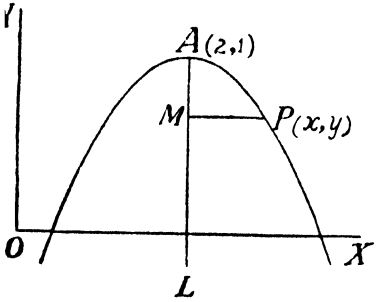
6·3. Equation to a parabola. The equation to a parabola is the simplest when its axis is taken as the axis of x and the tangent at the vertex as the axis of y. The equation then is of the form

$$y^2=4ax, \textit{ i.e.}, \text{ (Ordinate)}^2=\text{latus rectum}\times\text{abscissa},$$

the ordinate being the distance of any point on the curve from the axis and the abscissa being the distance of the ordinate from the vertex.

Suppose now we take a parabola having its vertex upwards at a point whose co-ordinates are (2, 1) with reference to two perpendicular axes OX and OY and having its axis AL parallel to the axis of y.

P is any point taken on the parabola having co-ordinates (x, y) with reference to the same axes.



Then the ordinate $MP=x-2$ and the abscissa $MA=1-y$.

If l denotes the length of the latus-rectum, equation of the parabola will be $MP^2=l\times MA$

or $(x-2)^2=l(1-y)$

$=-l(y-1)$.

Hence it follows that any equation of the form $(x-h)^2=-l(y-k)$ represents a parabola having its vertex upwards at the point (h, k) and having its latus-rectum equal to l, the numerical value of the co-efficient of y.

MOTION OF A PROJECTILE

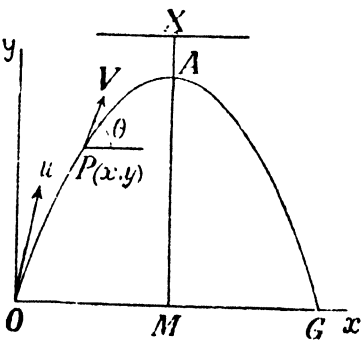
We discussed in the second chapter, the motion of a particle projected vertically upwards in the field of gravity. We now consider the motion of a *projectile*, that is, the motion of a body which is small enough to be regarded as a particle and which is projected in a direction *oblique* to the direction of gravity. We shall suppose the body to be projected in *vacuum* near the surface of the earth or, in other words, we shall suppose the resistance due to air and the slight variation in the force of gravity to be negligible.

6·4. *A particle of mass m is projected into the air with velocity u, in a direction making an angle α with the horizontal, to find its motion and the path described.*

Let O, the point of projection, be taken as the origin and let the *horizontal* and the *vertical* lines through O be taken as the axes of x and y.

Let $P(x, y)$ be the position of the moving point, after time t.

During the motion of the projectile, the only force acting on it is its weight acting downwards.



The equations of motion, therefore, are

$$m\frac{d^2x}{dt^2}=0 \quad \text{or} \quad \frac{d^2x}{dt^2}=0,$$

and

$$m\frac{d^2y}{dt^2}=-mg \quad \text{or} \quad \frac{d^2y}{dt^2}=-g.$$

Integrating these equations, we get

$$\frac{dx}{dt}=c \text{ and } \frac{dy}{dt}=-gt+c_1, \text{ where } c, c_1 \text{ are constants.}$$

Initially at O, when $t=0$, $\frac{dx}{dt}=u\cos\alpha$, $\frac{dy}{dt}=u\sin\alpha$.

$$\therefore \quad \frac{dx}{dt} = u \cos \alpha, \text{ constant throughout the motion} \quad ..(1)$$

and $$\frac{dy}{dt} = u \sin \alpha - gt \quad ..(2)$$

Integrating these equations again and applying initial conditions, *viz.*, when $t=0$, $x=y=0$, we get,

$$x = u \cos \alpha \, t \quad ..(3)$$

and $$y = u \sin \alpha \, t - \tfrac{1}{2}gt^2 \quad ..(4)$$

Equations (1) and (2) give the components of the velocity and (3) and (4) the displacements of the particle in the horizontal and vertical directions at any time t.

These equations could also be written down at once by regarding the particle to be projected with a constant velocity $u \cos \alpha$ in the horizontal direction and with an initial velocity $u \sin \alpha$ under a retardation g in the vertical direction.

Eliminating t from (3) and (4) we get,

$$y = x \tan \alpha - \tfrac{1}{2}g \cdot \frac{x^2}{u^2 \cos^2 \alpha} \quad ..(5)$$

which can be written as

$$x^2 - \frac{2u^2 \sin \alpha \cos \alpha}{g} x = -\frac{2u^2 \cos^2 \alpha}{g} y$$

or $$\left(x - \frac{u^2 \sin \alpha \cos \alpha}{g}\right)^2 = -\frac{2u^2 \cos^2 \alpha}{g}\left(y - \frac{u^2 \sin^2 \alpha}{2g}\right) \quad ..(6)$$

Cor. In the particular case when the particle is projected horizontally, $\alpha=0$, and the equation (5) of the path reduces to

$$y = -\frac{gx^2}{2u^2} \quad \text{or} \quad x^2 = -\frac{2u^2}{g} y$$

which is obviously a parabola the length of whose latus-rectum is $2u^2/g$.

If the y-axis is taken downwards this equation can be written as

$$y = gx^2/2u^2 \quad ..(i)$$

u being the velocity at the highest point of the path in this case.

This result can be obtained independently.

We now deduce the following facts from the six equations just obtained :—

(*a*) **The Path.** Equation (5) is of the second degree and the second degree term (x^2) is a perfect square. It follows, therefore, that the *path of the particle is a parabola.*

Equation (5) when written in the form (6) shows that *the latus-rectum of the parabolic path* $= \dfrac{2u^2 \cos^2 \alpha}{g}$.

[This can be remembered as the reciprocal of the coefficient of x^2 in (5)].

Equation (6) also shows that the vertex of the parabola is the point $x = \dfrac{u^2 \sin \alpha \cos \alpha}{g}$, $y = \dfrac{u^2 \sin^2 \alpha}{2g}$.

The path of a projectile is called its *trajectory.*

(*b*) **The time of flight, T**, is the time which the particle takes in reaching the horizontal plane through the point of projection.

Putting $y=0$ in (4), we get either $t=0$ (at O),

or $$t=\frac{2u\sin\alpha}{g}$$

$$\therefore \quad T=\frac{2u\sin\alpha}{g}.$$

(*c*) **Horizontal Range.** OG=the range R, on the horizontal plane through the point of projection=the horizontal distance described by the particle in the time of flight T.

i.e., $$R=u\cos\alpha.\,T=u\cos\alpha.\frac{2u\sin\alpha}{g}=\frac{u^2\sin 2\alpha}{g}$$

=twice the abscissa of the vertex.

R can also be obtained by putting $y=0$ in equation (5).

The abscissa of the focus=the abscissa of the vertex =half the range=$u^2\sin 2\alpha/2g$ which is the same as given in (*a*).

(*d*) **Maximum Horizontal Range.** The range R is maximum when $\sin 2\alpha=1$, *i.e.*, when $\alpha=\frac{\pi}{4}$ or 45°.

$$\therefore \quad \text{the maximum range}=\frac{u^2}{g}.$$

For a given velocity of projection, *the horizontal range is the greatest when the angle of projection is 45°.*

(*e*) **Directions of projection for a given Horizontal Range.** Since $\frac{u^2}{g}\sin 2(\tfrac{1}{2}\pi-\alpha)=\frac{u^2}{g}\sin(\pi-2\alpha)=\frac{u^2\sin 2\alpha}{g}$, it follows that the angles $\frac{1}{2}\pi-\alpha$ and α give the same range.

Hence with a given velocity of projection, to have a particular range or to hit a given point on the horizontal plane passing through the point of projection, *there are two directions of projection which are such that the inclination of the one with the horizontal is the same as that of the other with the vertical*, *i.e.*, they are equally inclined to the direction of the maximum range.

If u is enormous, the range $\frac{u^2\sin 2\alpha}{g}$ being fixed, α is small, so that one path will be very flat and the other nearly vertical up and down.

As u is diminished the two paths will approach each other till they coalesce when u reaches its minimum possible value for the range.

(*f*) **Greatest height** MA=the ordinate of the vertex.

$$=\frac{u^2\sin^2\alpha}{2g} \qquad \text{from (6)}$$

This is also obtained either by finding, by differentiation, the maximum value of y from (4) or by the fact that at the greatest

height the vertical component of the velocity must vanish, *i.e.*, from (2) $\frac{dy}{dt}=0$ whence $t=\frac{u\sin\alpha}{g}$.

Substituting this in (4) and simplifying we get $y=\frac{u^2\sin^2\alpha}{2g}$.

(*g*) **The height of the directrix**$=MX=MA+AX$

$=$The height of the vertex$+\frac{1}{4}$ latus-rectum

$=\frac{u^2\sin^2\alpha}{2g}+\frac{1}{4}.\frac{2u^2\cos^2\alpha}{g}=\frac{u^2}{2g}$,

which shows that the height of the directrix above the point of projection is the height to which the particle would rise if projected *vertically upwards* or that *the velocity at O is the same as acquired by a particle in falling freely from the level of the directrix.*

It also shows that all the trajectories which have the same point of projection and the same initial velocity have a common directrix.

(*h*) **The focus.** The height of the focus

$=$the height of the vertex$-\frac{1}{4}$ latus-rectum

$=\frac{u^2\sin^2\alpha}{2g}-\frac{1}{4}.\frac{2u^2\cos^2\alpha}{g}$

$=-\frac{u^2}{2g}(\cos^2\alpha-\sin^2\alpha)=-\frac{u^2\cos 2\alpha}{2g}$

and the co-ordinates of the focus are

$$\left(\frac{u^2\sin 2\alpha}{2g},-\frac{u^2\cos 2\alpha}{2g}\right).$$

Hence the focus lies above, on, or below the horizontal line OX, according as $-\frac{u^2\cos 2\alpha}{2g}$ is positive, zero, or negative, *i.e.*, according as α is greater than, equal to, less than $\frac{\pi}{4}$.

In the case of the maximum range the focus lies in the range.

Example 1. *A particle is projected with a velocity of 24 ft. per sec. at an angle of elevation 60°. Find (a) the equation to its path, (b) the length of the latus-rectum of the path, (c) the greatest height attained, (d) the height of the directrix of the path, (e) the time for the range, (f) the length of the range.*

Here $u=24$ ft./sec., $\alpha=60°$, $\sin\alpha=\frac{\sqrt{3}}{2}$, $\cos\alpha=\frac{1}{2}$, $\tan\alpha=\sqrt{3}$

$\therefore$ (*a*) the equation to the path is $y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}$

i.e., $y=x\sqrt{3}-\frac{32x^2}{2\times24\times24\times\frac{1}{4}}=x\sqrt{3}-\frac{1}{9}x^2$.

(*b*) Latus-rectum$=\frac{2u^2\cos^2\alpha}{g}=\frac{2\times24\times24}{32}\times\frac{1}{4}=9$ ft.

(*c*) Greatest height$=\frac{u^2\sin^2\alpha}{2g}=\frac{24\times24}{2\times32}\times\frac{3}{4}=6\frac{3}{4}$ ft.

(*d*) Height of the directrix

$=\frac{u^2\sin^2\alpha}{2g}+\frac{1}{4}.\frac{2u^2\cos^2\alpha}{g}=\frac{u^2}{2g}=\frac{24\times24}{2\times32}=9$ ft.

(e) T, the time for the range

$$=\frac{2u\sin\alpha}{g}=\frac{2\times24}{32}\times\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{4}\text{ secs.}$$

(f) The hor. range $=u\cos\alpha\times T=24\times\frac{1}{2}\times\frac{3\sqrt{3}}{4}=9\sqrt{3}$ ft.

Example 2. *Find the maximum horizontal range of cricket ball projected with a velocity of 48 ft./sec.*

If the ball is to have a range of $36\sqrt{3}$ ft., find the least angle of projection and the least time taken.

Hor. range $=u\cos\alpha\times\frac{2u\times\sin\alpha}{g}=\frac{u^2\sin2\alpha}{g}$.

Here $u=48$ and for max. hor. range $\alpha=45°$.

$\therefore$ the max. hor. range $=\frac{48\times48\sin90°}{32}=72$ ft.

If $\frac{u^2\sin2\alpha}{g}=36\sqrt{3}$ ft., then $\sin2\alpha=\frac{36\sqrt{3}\times32}{48\times48}=\frac{\sqrt{3}}{2}$.

$\therefore$ $2\alpha=60°$ or $120°$ *i.e.*, $\alpha=30°$ or $60°$.

Thus the least angle of projection $=30°$

and the least time taken $=\frac{2u\sin\alpha}{g}=\frac{2\times48}{32}\times\frac{1}{2}=1\frac{1}{2}$ secs.

Example 3. *A ball is thrown from the top of the Qutab Minar 200 ft. high with a velocity of 80 ft. per sec. at an elevation of 30° above the horizon; find the horizontal distance from the foot of the Minar to the point where it hits the ground.*

We take the top of the Minar as the origin.

$\therefore$ by the equation $y=u\sin\alpha\,t-\frac{1}{2}gt^2$,

we have $-200=80\sin30°t-\frac{1}{2}gt^2=40t-16t^2$

i.e., $2t^2-5t-25=0$

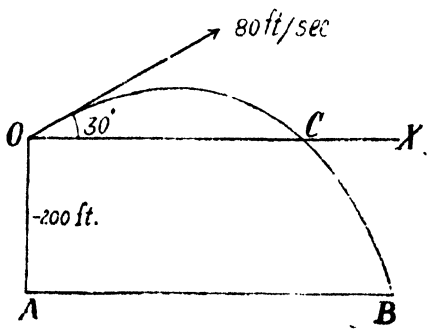
or $(2t+5)(t-5)=0$.

Neglecting the negative value of t, we get $t=5$.

$\therefore$ the horizontal distance moved in 5 secs.

$=80\cos30°\times5=200\sqrt{3}=346{\cdot}4$ ft.

Aliter. Putting $\alpha=30°$ and $u=80$ in equation (5), we get



$$y=x.\frac{1}{\sqrt{3}}-\frac{gx^2}{2\times80\times80\times\frac{3}{4}}$$

or $y=\frac{x}{\sqrt{3}}-\frac{x^2}{300}$.

For the horizontal plane through the foot of the Qutab Minar, $y=-200$, which gives

$x^2-100\sqrt{3}\,x-200\times300=0$

i.e., $(x-200\sqrt{3})(x+100\sqrt{3})=0$

$\therefore$ rejecting the negative value, we have

$x=200\sqrt{3}=346{\cdot}4$ ft

$$V\cos\theta = u\cos\alpha,$$

and $$V\sin\theta = u\sin\alpha - gt.$$

By division, we get $\tan\theta = \dfrac{u\sin\alpha - gt}{u\cos\alpha}$ ..(1)

which gives the direction of motion.

Squaring and adding, $V^2 = u^2 - 2u\sin\alpha\, gt + g^2t^2$ ..(2)

$$= u^2 - 2g(u\sin\alpha\, t - \tfrac{1}{2}gt^2) = u^2 - 2g.y$$

$$= 2g\left(\frac{u^2}{2g} - y\right) \quad ..(3)$$

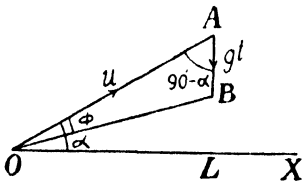
$=2g$ (height of the directrix—ordinate of P)

$=2g$ (height of the directrix above P).

It follows that the magnitude of the velocity at any point of the path is the same as would be acquired in falling freely a vertical distance from the directrix to the point.

(*b*) **Geometrical Method of finding the velocity.**

The velocity of a particle, t secs. after it has been projected from O with velocity u at an angle α above the horizontal, can also be obtained geometrically as follows :—



Draw a straight line OA to represent u in magnitude and direction ; on the same scale draw AB vertically downwards to represent gt. Then OB represents V in magnitude and direction.

Since $\angle XOA = \alpha$,

$\therefore$ $\angle OAB = 90° - \alpha$.

From the triangle OAB, we have

$$OB^2 = OA^2 + AB^2 - 2OA.AB\cos\angle OAB,$$

i.e., $$V^2 = u^2 + g^2t^2 - 2ugt\sin\alpha.$$

(*c*) **Change in the direction of motion.**

If the direction of motion of the particle has turned through an angle ϕ from the original direction,

$\angle AOB = \phi$ and since $\angle OAB = 90° - \alpha$, $\therefore$ $\angle OBA = 90° - \phi + \alpha$

From the triangle OAB, we have

$$\frac{AB}{\sin\phi} = \frac{OA}{\sin(90° - \phi + \alpha)},$$

i.e., $$\frac{gt}{\sin\phi} = \frac{u}{\cos(\phi - \alpha)},$$ so that $t = \dfrac{u\sin\phi}{g\cos(\phi - \alpha)}$.

In particular the velocity is perpendicular to its original direction if $\phi = 90°$, which gives $t = \dfrac{u}{g\sin\alpha}$.

Example. *A body is projected at an elevation α with a velocity of 32 ft./sec. In what time will the direction of motion be $\frac{1}{2}\alpha$?*

Putting $u = 32$ and $\theta = \frac{1}{2}\alpha$ in (1) of Art. 6·41 we have

$$\tan\frac{\alpha}{2} = \frac{32\sin\alpha - gt}{32\cos\alpha} = \tan\alpha - \frac{t}{\cos\alpha} \qquad [\because\ g = 32].$$

$$\therefore \quad \frac{t}{\cos\alpha} = \tan\alpha - \tan\frac{\alpha}{2} = \frac{\sin\alpha\cos\frac{1}{2}\alpha - \cos\alpha\sin\frac{1}{2}\alpha}{\cos\alpha\cos\frac{1}{2}\alpha}$$

$$= \frac{\sin\frac{1}{2}\alpha}{\cos\alpha\cos\frac{1}{2}\alpha}.$$

$$\therefore \quad t = \sin\tfrac{1}{2}\alpha/\cos\tfrac{1}{2}\alpha = \tan\tfrac{1}{2}\alpha.$$

Examples XV (B)

1. A stone is projected with a speed of 64 ft./sec. at an angle of elevation 30°. Find its speed

(a) after half a second ; (b) at the highest point of its path ;
(c) at a height of 15 ft.

Find also the interval of time between two points at which the stone has the same speed of 56 ft./sec.

2. A particle is projected under gravity with a velocity v at an angle α to the horizontal. Find its velocity after a time t, and the interval before it has again that velocity.

If $v=240$ feet per second, and the time between the two points in the path of the projectile at which the velocity is 208 feet per second is 5 seconds, show that $\tan\alpha=\frac{5}{12}$. (*P. U.*)

3. At any instant a projectile is moving with a velocity u in a direction making an angle α to the horizon. After a certain interval of time t the direction of its path makes an angle β with the horizontal. Prove that

$$u\cos\alpha = \frac{gt}{\tan\alpha - \tan\beta}.$$

Prove also that the direction of motion turns through an angle θ after a time $\frac{u\sin\theta}{g\cos(\theta-\alpha)}$ and that the direction of motion will be at right angles to the former direction after a time $u/g\sin\alpha$.

4. If v_1 and v_2 be the velocities at the ends of a focal chord of a projectile's path and u the horizontal component of the velocity, show that

$$\frac{1}{v_1^2} + \frac{1}{v_2^2} = \frac{1}{u^2}. \qquad (P.U.)$$

5. If α be the angle between the tangents at the extremities of any arc of a parabolic path, v, v_1 the velocities at these extremities and u their horizontal component, show that the time of describing the arc is

$$\frac{vv_1\sin\alpha}{gu}.$$

6. A particle is projected with velocity V and elevation α ($<\pi/4$) from a point O situated at a height h above a horizontal plane. If the particle is moving at right angles to its direction of projection when it hits the plane after time t, show that

$$(i)\ t = \frac{V}{g\sin\alpha}. \qquad (ii)\ h = \frac{V^2\cos 2\alpha}{2g\sin^2\alpha}$$

7. A ball thrown at a vertical wall 30 ft. away, strikes it perpendicularly at a point 25 ft. above the point of projection. Find the height to which the ball would rise if projected vertically with the same speed.

8. Prove that if a particle is projected from O at an elevation α, and if after time t the particle is at P, then

$$\tan\beta = \tfrac{1}{2}(\tan\alpha + \tan\theta),$$

where β and θ are respectively the inclinations to the horizontal of OP and of the direction of motion of the particle when at P.

9. Two particles are projected simultaneously, one with velocity u up a smooth plane inclined at an angle of 30° to the horizon and the other with a certain velocity v in a direction making an angle of 30° with the plane. The two particles are relatively at rest at a time t. Find t and v in terms of u.

6·5. Directions of projection to hit a given point

The equation of the path *viz.*, equation (5) of Art. 6·4 is

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha} \qquad ..(i)$$

If the projectile has to pass through the point $P(a, b)$, then

$$b=a\tan\alpha-\frac{ga^2}{2u^2}(1+\tan^2\alpha) \qquad ..(ii)$$

or
$$\tan^2\alpha-\frac{2u^2}{ga}\tan\alpha+1+\frac{2u^2b}{ga^2}=0 \qquad ..(1)$$

which, being a quadratic in tan α, shows that *there are, in general two directions in which a particle projected with a given velocity from a given point O may pass through another point P.*

The roots will be real if $\frac{4u^4}{g^2a^2}>4\left(1+\frac{2u^2b}{ga^2}\right)$

i.e., if $u^4>g^2a^2+2u^2gb$

i.e., if $(u^2-gb)^2>g^2(a^2+b^2)$

i.e., if $u^2>g(b+\sqrt{a^2+b^2})$.

Hence the *least velocity of projection V, to hit a given point* (a, b) *is given by*

$$V^2=g(b+\sqrt{a^2+b^2}) \qquad ..(2)$$

Cor. If $OP=r$ and it makes an angle α with the horizon, then $\sqrt{a^2+b^2}=r$, and $b=r\sin\alpha$

$$\therefore \qquad V^2=gr(1+\sin\alpha) \qquad ..(3)$$

6·6. Times to pass through a given point.

If the projectile has to pass through a given point (a, b) then

$$a=u\cos\alpha\, t$$

and
$$b=u\sin\alpha\, t-\tfrac{1}{2}gt^2.$$

Eliminating α, we have, $(b+\frac{1}{2}gt^2)^2+a^2=u^2t^2$ ..(1)

or
$$t^4+\frac{4}{g^2}(bg-u^2)t^2+\frac{4(a^2+b^2)}{g^2}=0 \qquad ..(2)$$

If t_1, t_2, are the roots of this equation,

$$t_1^2t_2^2=\frac{4(a^2+b^2)}{g^2}=\frac{4r^2}{g^2},$$

where r is the distance of the point hit from the point of projection

$$\therefore \qquad t_1t_2=2r/g.$$

Hence the product of the two times of transits in the two different paths is independent of the initial velocity.

Also since $r=\frac{1}{2}gt_1t_2$, it follows that *the product of the two times of transits of a projectile through a given point is equal to the square of the time taken by a particle in falling from rest vertically through a distance equal to r (the distance of the given point from the point of projection).* [∵ in the latter case $r=\frac{1}{2}gt^2$]

Cor. Replacing a by x and b by y in equation (1), we get

$$(y+\tfrac{1}{2}gt^2)^2+x^2=u^2t^2. \qquad ..(3)$$

This shows that particles projected simultaneously from the same point in different directions with the same velocity in a vertical plane will all lie on a circle after a time t.

Example 1. *A particle projected with a speed of 120 ft./sec., just clears a vertical wall 25 ft. high at a distance of 300 ft. Find the possible angles of projection.*

Here $u=120$, $a=300$, $b=25$,

$\therefore$ from (*ii*) Art. 6·5, we have

$$25=300\tan\alpha-\frac{32(300)^2}{2(120)^2}(1+\tan^2\alpha),$$

or $$4\tan^2\alpha-12\tan\alpha+5=0.$$

$\therefore$ $$(2\tan\alpha-5)(2\tan\alpha-1)=0,$$

i.e., $$\alpha=\tan^{-1}(\tfrac{5}{2}) \quad \text{or} \quad \tan^{-1}\tfrac{1}{2}.$$

Example 2. *The line joining C to D is inclined at an angle α to the horizon. Show that the least velocity V required to shoot from C to D is tan $(\frac{1}{4}\pi+\frac{1}{2}\alpha)$ times the least velocity V' required to shoot from D to C.*

From (3), Art. 6·5, we have

$$V^2=gr(1+\sin\alpha),$$

and replacing α by $-\alpha$, we have

$$V'^2=gr(1-\sin\alpha)$$

$$\therefore \quad \frac{V}{V'}=\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}=\frac{\cos\frac{1}{2}\alpha+\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha-\sin\frac{1}{2}\alpha}=\frac{1+\tan\frac{1}{2}\alpha}{1-\tan\frac{1}{2}\alpha}$$

$$=\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

Example 3. *A ball is projected from a point on the ground distant a from the foot of a vertical wall of height b, the angle of projection being α to the horizontal. If the ball just clears the wall prove that the greatest height reached is $a^2\tan^2\alpha/4(a\tan\alpha-b)$,*

Let u be the velocity of projection, then since the ball passes through the top of the wall, a point (a, b), we have

$$b=a\tan\alpha-\frac{ga^2}{2u^2\cos^2\alpha}, \quad \text{or} \quad \frac{ga^2}{2u^2\cos^2\alpha}=a\tan\alpha-b.$$

Now the greatest height reached by the ball

$$=\frac{u^2\sin^2\alpha}{2g}=\frac{\sin^2\alpha}{2g}\cdot\frac{ga^2}{2(a\tan\alpha-b)\cos^2\alpha}=\frac{a^2\tan^2\alpha}{4(a\tan\alpha-b)}.$$

Example 4. *A particle is projected with a velocity u so as just to pass over the highest possible post at a horizontal distance d from the point of projection O. Prove that the greatest height above O attained by the particle in its flight is*

$$\frac{u^6}{2g(u^4+g^2d^2)}.$$

Taking θ as the angle of projection and substituting d for x in the equation to the path, we have

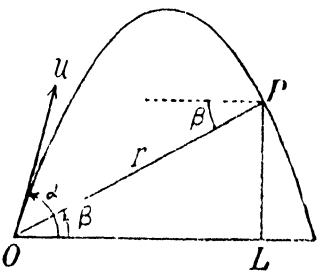
$$y=d\tan\theta-\frac{gd^2}{2u^2\cos^2\theta}=d\tan\theta-\frac{gd^2}{2u^2}(1+\tan^2\theta).$$

Also [redacted] the angle of projection be 45° and the greatest height above the p[redacted]on be $3h$, then the horizontal distance travelled is $6h(1+\sqrt{2})$.

29. [redacted] projected so as to have a range R on the horizontal plane through [redacted] of projection. If α, β are the possible angles of projection and t_1, t_2, the corresponding times of flight, show that

$$\frac{t_1^2-t_2^2}{t_1^2+t_2^2}=\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}.$$

6·7 Range on an inclined plane. Let a particle be projected from a point O on a plane of inclination β, in the vertical plane through OP, the line of greatest slope of the inclined plane.



Let the velocity of projection be u at an elevation α to the horizontal.

The equation to the path of the particle is

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha} \qquad ..(1).$$

If the particle strikes the inclined plane at the point P, the distance OP is called the range on the inclined plane.

If $OP=r$ then the co-ordinates of $P(r\cos\beta, r\sin\beta)$ must satisfy equation (1).

$$\therefore r\sin\beta=r\cos\beta\tan\alpha-\frac{gr^2\cos^2\beta}{2u^2\cos^2\alpha}$$

$$(\alpha)\ \therefore \text{the range } r=\frac{2u^2}{g}\cdot\frac{(\cos\beta\tan\alpha-\sin\beta)\cos^2\alpha}{\cos^2\beta}$$

$$=\frac{2u^2}{g}\cdot\frac{\sin(\alpha-\beta)\cos\alpha}{\cos^2\beta} \qquad ..(2)$$

$$=\frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta] \qquad ..(3).$$

The range *down* the plane may be obtained by putting $-\beta$ for β, for in this case the slope of the inclined plane is downwards.

(*b*) **Maximum range.** u and β being known, the range varies with α, and it will be max. when $\sin(2\alpha-\beta)$ is max.

i.e., when $2\alpha-\beta=\frac{\pi}{2}$ or $\alpha-\beta=\frac{\pi}{2}-\alpha$.

Hence *for maximum range, the direction of projection must bisect the angle between the vertical and the inclined plane.*

If OT be the direction of projection, then OT is tangent to the path at O, and the vertical through O is perpendicular to the directrix. OT being equally inclined to OP and the vertical, the focus to the path must, therefore, lie on the line OP of the inclined plane, *i.e., in the case of maximum range the focus lies in the range.*

The value of the maximum range is

$$\frac{u^2(1-\sin\beta)}{g\cos^2\beta}=\frac{u^2}{g(1+\sin\beta)} \qquad ..(4)$$

(*c*) **Directions of projection for a give[redacted]** For a given velocity of projection when the value of the [redacted] than the maximum, is given, sin $(2\alpha-\beta)$ in equation [redacted] own and thus we get two values for $2\alpha-\beta$, each less th[redacted]

If α_1 and α_2 be the corresponding values o[redacted]

$$2\alpha_1-\beta=\pi-(2\alpha_2-\beta)$$

i.e., $$\alpha_1-\tfrac{1}{2}\left(\frac{\pi}{2}+\beta\right)=\tfrac{1}{2}\left(\frac{\pi}{2}+\beta\right)-\alpha_2$$

$\frac{1}{2}\left(\frac{\pi}{2}+\beta\right)$ being the direction for the maximum range, it follows that *for a given range, there are two directions of projection which are equally inclined to the direction for maximum range.*

(*d*) **Time of flight.** If the particle strikes the plane after a time T, then

$$OL=u\cos\alpha.\,T \text{ where } OL=r\cos\beta.$$

$$\therefore \quad T=\frac{r\cos\beta}{u\cos\alpha}=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}. \qquad \text{[From (2)]}$$

(*e*) **Velocity at P.** If v is the velocity at P,

$$v^2=u^2-2gy=u^2-2gr\sin\beta,$$

and if θ is the angle which the direction at P makes with the horizontal, then

$$\tan\theta=\frac{u\sin\alpha-gT}{u\cos\alpha}=\tan\alpha-\frac{g}{u\cos\alpha}\times\frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

$$=\tan\alpha-2(\tan\alpha-\tan\beta)=2\tan\beta-\tan\alpha.$$

*(*f*) **The condition that the particle may strike the plane at right angles.**

We put $\theta=-(\frac{1}{2}\pi-\beta)$, for in this case the direction of motion of the particle at P is downwards making an angle $\frac{1}{2}\pi-\beta$ with the horizontal.

Hence the required condition becomes

$$\tan(\beta-\tfrac{1}{2}\pi)=2\tan\beta-\tan\alpha$$

or $$\cot\beta=\tan\alpha-2\tan\beta. \qquad ..(5).$$

By adding $\tan\alpha$ to both sides and simplifying it may be put in the following form :

$$\cot\beta=2\tan(\alpha-\beta). \qquad ..(6).$$

(For another and direct proof see the next article.)

Note. In the above article the direction of projection is expressed in terms of the angle with the horizontal. It can also be expressed in terms of the angle with the inclined plane. In working problem care must be taken to see which of these angles is given.

6·8. Another Method. When a particle is projected from a point O (Fig. Art. 6·7) on an inclined plane, in the vertical plane through OP, (the line of greatest slope through O), its range on the plane and the time of flight can also be obtained by resolving its velocity u and the acceleration g along and perpendicular to OP (the perpendicular being drawn upwards in the vertical plane through OP).

CHAPTER VII

MOTION OF A PARTICLE IN A PLANE—II

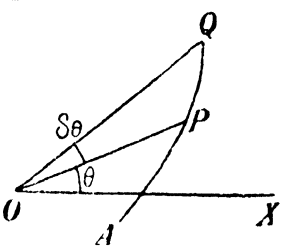
MOTION ALONG A CIRCLE

In the following two chapters we shall consider the motion of a particle moving in a circle with uniform or variable speed.

7·1. Angular Velocity and Acceleration. Let OX be a fixed line and let P be any point moving along the curve APQ.

At any time t, let the position of the moving point P be defined by the angle XOP.

In an additional time δt, let the point move to Q, OP describing an additional $\angle POQ = \delta\theta$.



$\frac{\delta\theta}{\delta t}$ is then the average rate of change of the angle θ.

$\therefore$ the angular velocity of P about $O = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta\theta}{dt} = \frac{d\theta}{dt}$.

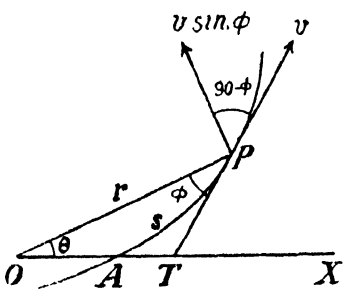
Thus if a point P moves in a plane, and if O is a fixed point in the plane and OX a fixed straight line through O, *the angular velocity of P about O (or of the line OP) is the rate of change of the angle XOP.*

If the angular velocity is denoted by ω, the angular acceleration

$$= \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}.$$

θ is always measured in radians, angular velocity in radians per second and angular acceleration in radians per second per second.

7·11. Relation between Angular Velocity and Linear Velocity. Let the velocity of the point P be v and let the tangent at P make an angle ϕ with OP, then



$$\omega = \frac{d\theta}{dt} = \frac{d\theta}{ds} \cdot \frac{ds}{dt}$$

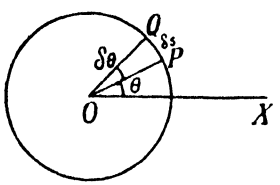
$$= \frac{ds}{dt} \cdot \frac{rd\theta}{ds} \cdot \frac{1}{r} = \frac{v \sin\phi}{r}$$

i.e., the angular velocity of P about O

$$= \frac{\textit{Vel. of } P \textit{ resolved perp. to } OP}{OP}.$$

In particular, if the curve is a circle and the angular velocity is measured about the centre then ϕ being a right angle, $\omega = \frac{v}{r}$ so that *the*

angular velocity of a particle moving in a circle about the centre at any instant is obtained by dividing its linear velocity at that instant by the radius of the circle.



This can be proved independently as follows :—

Since $\delta s = \text{arc } PQ = r\,\delta\theta$

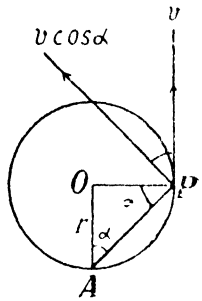
$$\therefore \quad v = \operatorname*{Lt}_{\delta t \to 0} \frac{\delta s}{\delta t} = \operatorname*{Lt} \frac{r\delta\theta}{\delta t} = r\frac{d\theta}{dt} = r\omega$$

or $$\omega = \frac{v}{r}.$$

Example 1. *If a point P is moving along a circle with constant speed v, find its angular velocity and acceleration about any point A on the circle.*

If the chord AP makes an angle α with the radius $OA = r$, then

$$AP = 2r\cos\alpha.$$



The point P has a velocity v in the direction of the tangent at P. Its component in a direction perpendicular to AP is $v\cos\alpha$.

$\therefore$ the angular velocity of P about A

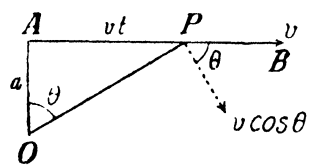
$$= \frac{v\cos\alpha}{AP} = \frac{v\cos\alpha}{2r\cos\alpha} = \frac{v}{2r}$$

= half the angular velocity about the centre.

This being constant, the angular acceleration is zero.

Example 2. *A point moves uniformly along a given straight line ; show that its angular velocity about a fixed point varies inversely as the square of its distance from the fixed point.*

Let AB be the fixed straight line along which the particle moves and let O be the fixed point. Draw OA perpendicular to AB and let $OA = a$.



Let P be the position of the particle after a time t, t being measured from the instant when the particle was at A. then $AP = vt$.

$$\text{Now } \tan\theta = \frac{AP}{OA} = \frac{vt}{a}$$

so that $$\sec^2\theta\,\frac{d\theta}{dt} = \frac{v}{a}.$$

$$\therefore \quad \omega = \frac{d\theta}{dt} = \frac{av}{a^2\sec^2\theta} = \frac{av}{OP^2} \text{ which varies as } \frac{1}{OP^2}.$$

Aliter. $$\omega = \frac{\text{Vel. perp. to } OP}{OP} = \frac{v\cos\theta}{OP} = \frac{v}{OP}\cdot\frac{a}{OP} = \frac{av}{OP^2}.$$

Example 3. *A flywheel making 180 revolutions per minute is to a constant angular retardation of two radians per sec. per sec.*

How many revolutions does it make before coming to rest and what is its angular velocity after π secs.?

180 *r.p.m.* $=3$ *r.p.s.* $=6\pi$ rad. per sec.

$\therefore$ the initial angular velocity $=6\pi$ rad. per sec.

Angular acceleration $=\dfrac{d\omega}{dt}=-2$. $\therefore\ \omega=-2t+c$.

When $t=0$, $\omega=6\pi$, $\therefore\ c=6\pi$, $\therefore\ \omega=-2t+6\pi$ ..(1).

When $t=\pi$, $\omega=6\pi-2\pi=4\pi$ rad. per sec.

When the wheel stops $\omega=0$ and $t=3\pi$ sec. [from (1)].

From (1) $\dfrac{d\theta}{dt}=6\pi-2t$.

$\therefore\ \theta=6\pi t-t^2+c_1$,

but $\theta=0$ when $t=0$, $\therefore\ c_1=0$ $\therefore\ \theta=6\pi t-t^2$ ..(2)

When $t=3\pi$, $\theta=18\ \pi^2-9\pi^2=9\pi^2$

which is the angle turned in 3π secs.

$\therefore$ the number of revolutions $=\dfrac{\theta}{2\pi}=\dfrac{9\pi^2}{2\pi}=\dfrac{9}{2}\pi$.

Note 1. Equations (1) and (2) are particular cases easily deducible from the general formulae,

$$\omega=\omega_0+ft \quad \text{and} \quad \theta=\omega_0 t+\tfrac{1}{2}ft^2.$$

where ω_0 is the initial angular velocity and f the constant angular acceleration.

Note 2. The abbreviations *r.p.m.* and *r.p.s.* stand for 'revolutions per minute' and 'revolutions per second' respectively.

Examples XVIII

1. A flywheel is making 180 revolutions per minute and after 20 secs. it is turning at 140 *r.p.m.* How many revolutions will it make and what time will elapse before stopping, if the retardation is uniform? (*P.U.*)

2. A wheel starting from rest receives a uniform angular acceleration of $1\frac{1}{2}$ radians per second per second. How many revolutions per second will it be making at the end of one minute?

If the effort is taken off and if the frictional resistance of the bearings is equivalent to a uniform negative acceleration of $\frac{1}{2}$ radian per second per second, in how many minutes will the wheel come to rest?

3. A wheel rolls uniformly on the ground, without sliding, its centre describing a straight line; find the velocity of any point on its rim and show that its angular velocity about the point of contact of the wheel with the ground is equal to the angular velocity of the wheel about its centre.

4. A particle falls vertically down a straight line $x=a$ starting from rest from the axis of x. Find the angular velocity and acceleration of the line joining the particle to the origin. Show that the particle has fallen a vertical distance $a/\sqrt{3}$ when the angular acceleration is zero.

5. Two points are moving with uniform velocities u and v along perpendicular lines OX, OY the motion being towards O. When $t=0$ they are at distances a, b respectively from O. Calculate the angular velocity of the line joining them at any time t, and show that it is greatest when

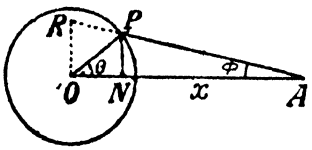
$$t=\frac{au+bv}{u^2+v^2}.$$

6. The ends of a rod which rotates with constant angular velocity move on two perpendicular straight lines OX, OY. Prove that the projection of any other point on the rod on OX or OY executes simple harmonic motion.

7. Prove that the angular velocity of a projectile P about the focus S of its path varies inversely as its distance from the focus.

[**Hint.** Tangent at P makes equal angles with SP and the vertical.]

8. A crank OP rotates about a fixed centre O while the end A (the piston) of the connecting rod PA is constrained to move along OA. If angles POA and PAO are θ and φ, prove that when the angular velocity of P is ω, the linear velocity of A is



$$-OP\frac{\sin(\theta+\varphi)}{\cos\varphi}\omega.$$

[**Hint.** $x=OA=ON+NA=OP\cos\theta+PA\cos\varphi$
also $OP\sin\theta=PN=PA\sin\varphi$]

9. In *Q.* 8, if the perp. through O to the line OA meets AP produced in R; show th .t the velocity of A varies as OR.

[**Hint.** Show that $OR/OP=\sin(\theta+\phi)/\cos\phi$.]

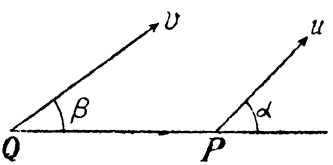
10. A circular disc of radius a is free to turn round its centre in its own plane and a string whose thickness may be neglected, is wound round the rim of the disc. If the free end of the string is withdrawn along a radius with speed v, find the angular velocity of the disc when the distance of this end from the centre is x.

7·12. Angular Velocity of a line joining two points moving in a plane. The angular velocity about a fixed point Q of the point P moving with a velocity u at an angle α with PQ

$$=\frac{\text{Vel. of } P \text{ perp. to } PQ}{PQ}=\frac{u\sin\alpha}{PQ},$$

the component velocity of P along PQ having no effect on the direction PQ.

But when P and Q both are moving in a plane with velocities u and v, then either possesses an angular velocity relative to the other. Let the directions of motion make angles α and β with PQ, the line joining the moving points at any instant, then the angular velocity of P relative to Q, *i.e.*, the angular velocity of the line PQ



$$=\frac{\text{The component of velocity of } P \text{ relative to } Q \text{ perp. to } PQ}{PQ}$$

$$=\frac{\text{Diff. of Vel. components perp. to } PQ}{PQ}$$

$$=\frac{u\sin\alpha-v\sin\beta}{PQ}$$

the component velocities of P and Q along PQ having no effect on the direction of PQ.

Whenever $u\sin\alpha=v\sin\beta$, the point P has no angular velocity relative to Q, *i.e.*, the line PQ moves parallel to itself and the linear velocity of P relative to Q is in the direction of the line PQ.

If, however, PQ remains constant, P must have the same velocity as Q in the direction of PQ, *i.e.*, $u\cos\alpha=v\cos\beta$.

Ex. 1. The line joining two points P and Q, is of constant length a and the velocities of P and Q are in directions which make angles α and β respectively with PQ. Prove that the angular velocity of PQ is $\frac{u\sin(\alpha-\beta)}{a\cos\beta}$, where u is the velocity of P.

Ex. 2. If two particles describe the circle of radius r in the same sense and with the same speed v, show that the relative angular velocity of each with respect to the other is v/r.

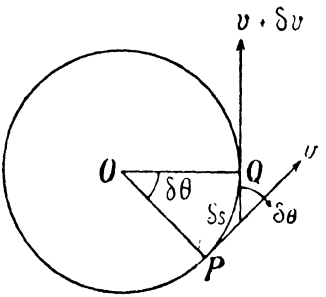
Ex. 3. Show how to measure the angular velocity—

(*i*) of a point about another point, and

(*ii*) of the straight line joining two points moving in a given manner in a plane.

$ABCD$ is a square of side 100 ft., a person P starts from A and walks towards B with a speed of 4 *f.p.s.*, at the same instant a second person Q starts from B and walks towards C with a speed of 3 *f.p.s.*, show that the angular velocity of the straight line PQ after t seconds from the instant of starting is $\frac{12}{t^2-32t+400}$ radians per sec.; and that it is the greatest when PQ is 60 ft. (*P.U.*)

7·2. Expressions for tangential and normal accelerations for motion along a circle. Suppose that a particle moves along a circle of radius r. Let P be the position and v the velocity of the particle at any instant and Q its position and $v+\delta v$ its velocity after a short interval of time δt. Let the arc PQ be δs and the angle POQ be $\delta\theta$, so that $\delta s = r\delta\theta$.



Now the change in velocity in time δt along the tangent at P is

$$(v+\delta v)\cos\delta\theta - v.$$

$\therefore$ the tangential acceleration at P

$$= \operatorname*{Lt}_{\delta t\to 0} \frac{(v+\delta v)\cos\delta\theta - v}{\delta t}$$

$$= \operatorname*{Lt}_{\delta t\to 0} \frac{(v+\delta v)[1-\frac{1}{2}(\delta\theta)^2+\ldots\ldots]-v}{\delta t}$$

$$= \operatorname{Lt}\frac{\delta v}{\delta t} - \operatorname{Lt}\tfrac{1}{2}\delta\theta.\,(v+\delta v)\frac{\delta\theta}{\delta t} = \frac{dv}{dt} + 0 = \frac{dv}{dt},$$

and since $v = \frac{ds}{dt}$, therefore $\frac{dv}{dt} = \frac{d^2s}{dt^2}$ or $= \frac{dv}{ds}\cdot\frac{ds}{dt}$, *i.e.*, $v\frac{dv}{ds}$.

Again, the change in the velocity parallel to PO, the normal at P is

$$(v+\delta v)\sin\delta\theta - 0,$$

$\therefore$ the normal acceleration at P

$$= \operatorname*{Lt}_{\delta t\to 0} \frac{(v+\delta v)\sin\delta\theta - 0}{\delta t} = \operatorname*{Lt}_{\delta t\to 0} (v+\delta v)\frac{\delta\theta}{\delta t}$$

$$= \operatorname*{Lt}_{\delta t\to 0} (v+\delta v)\frac{\delta s}{\delta t}\cdot\frac{1}{r} = v\times\frac{ds}{dt}\times\frac{1}{r} = \frac{v^2}{r}$$

And since $v = r\omega$, ω being the angular velocity of the particle,

$$\therefore \qquad \frac{v^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2.$$

Thus, if a particle moves along a circle of radius r and has the speed v at any instant, the resolved part of its acceleration respectively along the tangent and normal (towards the centre) are

$$\frac{dv}{dt}\left(\text{or } v\frac{dv}{ds} \text{ or } \frac{d^2s}{dt^2}\right) \text{ and } \frac{v^2}{r} \text{ (or } r\omega^2\text{)}.$$

In particular, if the speed v is constant, $\frac{dv}{dt}=0$, *i.e.*, the acceleration has no resolved part along the tangent and the resultant acceleration is, therefore, *completely directed towards* the centre and is of a constant magnitude $\frac{v^2}{r}$.

Note 1. Whenever a particle moves along a circular path, the tangential component of the acceleration changes the velocity in *magnitude* while the normal component causes change in the direction of the velocity. If the former is zero, the particle moves with a constant speed but if the latter is zero, the motion must take place along a straight line only.

Note 2. So far as the expression for the tangential acceleration is concerned, the proof given above does not depend upon any property peculiar to the circle and is, therefore, applicable to the motion of a particle along *any* curve and so the expressions $\frac{d^2s}{dt^2}$ or $\frac{dv}{dt}$ or $v\frac{dv}{ds}$ can be used whenever motion along a curve is to be dealt with.

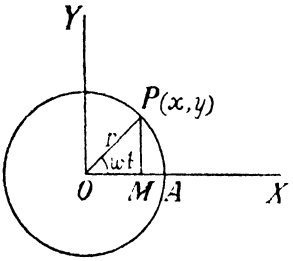
Example. *A particle is moving along a circle of radius a such that the arcual distance of the particle at any time t from a fixed point of the circle is s. Show that the magnitude of the acceleration of the particle at time t is*

$$\left[\left(\frac{d^2s}{dt^2}\right)^2+a^{-2}\left(\frac{ds}{dt}\right)^4\right]^{\frac{1}{2}}.$$

7·21. We give below another method of proving the particular case of the previous particle *viz.*, *if a particle moves along a circle of radius r with* **uniform** *speed v, its acceleration is* $\frac{v^2}{r}$ *towards the centre of the circle.*

(One other proof based on *hodograph* will be given in Chap. XI.)

As the particle moves along the circle with *uniform* speed, its angular velocity about the centre must also be uniform.



Let ω be the angular velocity and AOP be the angle described by the particle in time t, then

$$\angle AOP=\omega t,$$

If (x, y) are the co-ordinates of the particle with reference to the centre as origin, we have

$$x=r\cos\omega t, \qquad y=r\sin\omega t,$$

$$\frac{dx}{dt}=-r\omega\sin\omega t, \qquad \frac{dy}{dt}=r\omega\cos\omega t,$$

$$\frac{d^2x}{dt^2}=-r\omega^2\cos\omega t, \qquad \frac{d^2y}{dt^2}=-r\omega^2\sin\omega t.$$

$\therefore$ the resultant acceleration

$$\sqrt{\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2}=r\omega^2=r\frac{v^2}{r^2}=\frac{v^2}{r}.$$

Also since $\frac{d^2y}{dt^2}\Big/\frac{d^2x}{dt^2}=\tan\omega t$, and since both components of the acceleration are negative, it follows that the acceleration acts along PO.

7·22. *When a particle describes a circle with uniform speed, its projection on a diameter of the circle moves with simple harmonic motion.*

As shown above, for the point M, we have

$$x=r\cos\omega t, \text{ and } \frac{d^2x}{dt^2}=-\omega^2 r\cos\omega t=-\omega^2 x.$$

It shows that the acceleration of the point M is proportional to OM and is directed towards O, *i.e.*, the point M moves with *S.H.M.*

This is a very important property of simple harmonic motion and is frequently used as its definition.

7·23. Whenever a particle of mass m describes a circle of radius r, with uniform speed v, the acceleration $\frac{v^2}{r}$ towards the centre must be supplied by some force. If F is the required force, then

$$F=\frac{mv^2}{r}=mr\omega^2.$$

If the particle makes n revolutions per second, then ω, the angle described in one second, is $2\pi n$

$\therefore$ $$F=mr.4\pi^2n^2.$$

The force given by $\frac{mv^2}{r}$ or $mr\omega^2$ is always in *absolute units* and may be produced in various ways.

In the case of a particle attached to one end of a string and whirled round, the other end of the string being held in the hand, the required force is supplied by the tension of the string. The action and reaction being equal and opposite an equal amount of force is exerted through the string by the particle at the hand in the opposite direction. As the string is in a state of tension it is sometimes wrongly believed that there is a tendency for the particle to move *away* from the centre along the radius of the circle. In fact, if the string breaks the particle continues to move in its direction of motion along the *tangent* of the circle.

In the case of a railway carriage moving on a curved track, the force necessary to compel the carriage to follow the curved path is supplied by the pressure of the outer rails on the flanges of the wheels, the wheels exerting an equal amount of pressure on the rails which, if not fixed, will leave their place causing derailment.

In the case of motor car the force is provided by the force of friction between the wheels and the ground.

In case of the earth moving round the sun, the required force which makes the earth follow a curved path, is supplied by the attraction of the sun.

Example. *A mass of 12 oz. revolves horizontally at the end of the cord 3 ft. long. If the string makes 20 revolutions per minute, find the tension in the string.*

Here $m=12$ oz$=\frac{3}{4}$ lb., $r=3$ ft.,

and $\omega=20$ r.p.m.$=\dfrac{20}{60}$ r.p.s. $=\dfrac{2\pi}{3}$ radians per sec.

$$\therefore \quad T=mr\omega^2=\tfrac{3}{4}.3.\ \frac{4\pi^2}{9}=\pi^2=9{\cdot}87 \text{ poundals.}$$

Examples XIX

1. A particle of mass 10 lbs. is attached to a fixed point on a smooth table by means of an inelastic string 20 ft. long. The particle is set in circular motion on the table with a velocity of 12 f.p.s., find the tension of the string. (*P.U.*)

2. A mass of 20 gms. tied by a thread to a point on a smooth table is describing a circle of 60 cms. radius in every 4 seconds. Find the tension of the thread.

3. Four masses, each of 10 lbs., are fastened together with four strings each 3 ft. long so as to form a square. This square rotates in a horizontal plane on a smooth table at a speed of one revolution per second ; find in pounds weight the tension in the strings.

4. A cyclist travels on a level track of radius 220 ft., and the coefficient of friction between the tyres and the ground is 0·32. Find the greatest speed at which he may travel. (*P.U.*)

5. An engine of mass 20 tons is travelling round a level curved track at the rate of 30 m.p.h. If the curve is an arc of a circle of radius 1200 ft., find the horizontal thrust exerted by the rails on the engine.

6. A motor car weighing one ton is travelling on a curved level road of radius one quarter of a mile at the rate of 45 m.p.h. Find the necessary force of friction between the wheels and the ground.

7. A string can just bear a weight of 18 lbs. without breaking. If a weight $\frac{1}{4}$ lb. be attached to it and whirled round in a horizontal circle of radius 3 ft. find the number of revolutions that the string must make in a second in order that the string may just escape breaking.

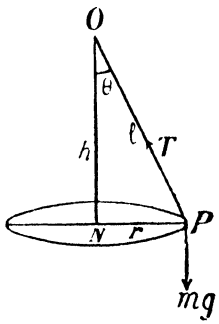
8. Two particles each of mass m, are attached to the ends of a light string of length a and placed on a horizontal turn-table at distances a, $2a$ from the centre of rotation. Prove that when the table rotates with angular velocity ω the particles can remain upon it without slipping provided that the coefficient of friction is not less than $3a\omega^2/2g$.

9. A heavy particle of mass m is connected by a string passing through a hole in a smooth table with a particle of mass $4m$ which hangs vertically. Find the number of revolutions per minute which the particle on the table must perform in a circle of radius 6 inches in order to keep the other particle at rest.

10. A particle of mass m on a smooth table is fastened to one end of a fine string which passes through a small hole in the table and supports at its other end a particle of mass $2m$, the first particle being held at a distance c from the hole. Find the velocity with which m must be projected so that it may describe a circle of radius c.

11. A mass of 10 lbs. is constrained to move in a circular path and makes one revolution in one second. It is attached to the centre of a circle by a light elastic string whose unstretched length is 1 ft. and which extends 1/12 foot under a pull of 4 pounds weight. Find the radius of the circle and the tension of the string.

7·3. Conical Pendulum. When a particle is tied to one end of a light string, the other end of which is fixed at a point O, and the particle describes a circle in a horizontal plane, so that the string describes a cone whose axis is the vertical line through O, the system is called a *Conical pendulum.*



Let N be the centre and r the radius of the horizontal circle described by P, a particle of mass m, with uniform angular velocity ω and let l be the length of the string and θ the angle which it makes with the vertical.

The forces acting on the particle are its weight mg downwards and T, the tension of the string.

As there is no motion of the particle in the vertical direction, the vertical component of T must balance the weight.

$$\therefore \quad T\cos\theta = mg \qquad ..(1)$$

The central force $= \frac{mv^2}{r}$ or $mr\omega^2$ along PN which is necessary to keep the particle moving in the circle, is supplied by the component of the tension in the horizontal direction.

$$\therefore \quad T\sin\theta = \frac{mv^2}{r} = mr\omega^2 = ml\sin\theta\,\omega^2 \qquad ..(2)$$

If θ is zero, the string hangs vertically and from (1) $T=mg$, which is otherwise obvious.

When θ is not zero, we have from (2)

$$T = ml\omega^2 \text{ poundals} \qquad ..(3)$$

and from (1)

$$\cos\theta = \frac{mg}{T} = \frac{mg}{ml\omega^2} = \frac{g}{l\omega^2} \qquad ..(4)$$

Thus, in order that the particle may move in a conical pendulum, ($\cos\theta$ being <1) $l\omega^2 > g$

or $$\omega^2 > \frac{g}{l}.$$

7·31. Now $$ON = h = l\cos\theta = \frac{g}{\omega^2} \qquad ..(5)$$

i.e., the vertical depth of P below O varies inversely as the square of the angular velocity and is independent of the length of the string.

Also from (5), $$\omega^2 = \frac{g}{h},$$

and t, the time of one revolution $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}$.

If n is the number of revolutions made per second, $\omega = 2\pi n$ and from (3)

$$T = 4\pi^2 n^2 ml \text{ poundals.}$$

7·32. When a particle moving in a conical pendulum is also

constrained to move on a smooth horizontal table, equation (1) becomes

$$T\cos\theta+R=mg$$

or $$R=mg-T\cos\theta=mg-4\pi^2n^2mh \quad [\because \cos\theta=h/l]$$

which gives R, the pressure on the table. In order that the particle may not leave the table, $mg>4\pi^2n^2mh$

i.e., $$n<\frac{1}{2\pi}\sqrt{\frac{g}{h}}; \qquad \text{or} \quad \omega<\sqrt{\frac{g}{h}}.$$

Example 1. *A mass of 2 lbs. suspended by a string 1 ft. long is revolving in a conical pendulum. Find the greatest number of complete revolutions per second of the pendulum when the greatest tension that the string can bear is 10 lbs. weight.*

Let n be the required number of revolutions per sec., then

$$\omega=2\pi n$$

$\therefore$ $$T\sin\theta=mr\,\omega^2=ml\sin\theta.4\pi^2n^2$$

i.e., $$T=ml.4\pi^2n^2=2.1.4\pi^2n^2.$$

But the greatest value of $T=10$ lb. wt. $=320$ poundals.

$\therefore$ $$320=8\pi^2n^2 \qquad \text{or} \quad n^2=\frac{40}{\pi^2}=4 \text{ approximately.}$$

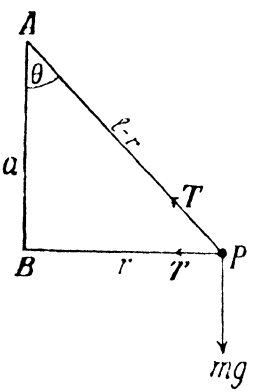
Hence the greatest number of revolutions per sec. is 2.

Example 2. *A string of length l has its ends fastened to two points A and B in the same vertical line at a distance a apart. If a smooth bead P on the string rotates uniformly about B so that BP is always horizontal, prove that the angular velocity is*

$$l\left[\frac{2g}{a(l^2-a^2)}\right]^{\frac{1}{2}}.$$

Show also that P is nearer to B than A by $\frac{a^2}{l}$.

Let the required angular velocity be ω, and let $BP=r$, so that $AP=l-r$.



The bead being smooth, the tension along the string is the same throughout its length, so that tension along PB = tension along $PA=T$, say

$\therefore$ for motion of the bead, we have vertically, $T\cos\theta=mg$,

or $$T=mg\sec\theta$$

and horizontally,

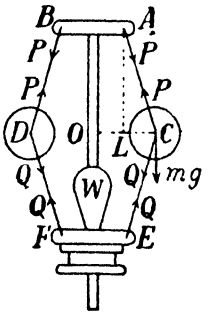
$$mr\omega^2=T+T\sin\theta=mg\sec\theta\,(1+\sin\theta)$$

$\therefore$ $$r\omega^2=g\frac{l-r}{a}\left(1+\frac{r}{l-r}\right)=\frac{gl}{a},$$

or $$\omega^2=\frac{gl}{ar}.$$

*17. *The Governor of a steam engine, making 180 revolutions per minute, takes up the position shown in the sketch, the rods AC, BD, CE and DF being inclined at the same angle of 30° with the vertical. It is required to find W, the weight of the loaded sleeve, the mass of each ball being given.*

The piece AB is fixed to the vertical spindle and the rods AC, BD, CE and DF are freely hinged at both ends, so that when the spindle rotates the sleeve EF can slide up and down, the balls C and D moving in a horizontal circle.



Take $AB=EF=2''$; $AC=BD=CE=DF=6''$, m, the mass of each ball$=2$ lbs. and the mass of the rods negligible.

Now, r the radius of the circle described by each ball$=OC=OL+LC=\frac{1}{2}AB+AC\sin 30°=1+6\times\frac{1}{2}=4''$
$=\frac{1}{3}$ ft.

ω, the angular velocity of each ball$=180$ r.p.m.

$$=\frac{180}{60}\times 2\pi=6\pi \text{ rad./sec.}$$

Let P be the pull in each of the rods AC and BD
and Q ,, ,, ,, CE and DF
For motion of one of the balls, say C, we have :—

vertically, $P\cos 30°=Q\cos 30°+2g$, *i.e.*, $P-Q=\frac{4}{\sqrt{3}}g$ ..(1)

horizontally, $(P+Q)\sin 30°=mr\omega^2=2\times\frac{1}{3}\times 36\times\pi^2$, *i.e.*, $P+Q=48\pi^2$ ..(2).

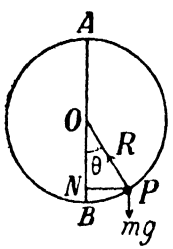
$\therefore$ $2Q=48\pi^2-\frac{4}{\sqrt{3}}g=\left(\frac{3}{2}\pi^2-\frac{4}{\sqrt{3}}\right)g=\frac{3}{2}\pi^2-\frac{4}{\sqrt{3}}$ lbs. wt.

Now resolving vertically for the sleeve EF, we have

$$W=2Q\cos 30°=\left(\frac{3}{2}\pi^2-\frac{4}{\sqrt{3}}\right)\frac{\sqrt{3}}{2}=\frac{3\times 1\cdot 73\times 9\cdot 9}{4}-2=10\cdot 84 \text{ lbs.}$$

18. In Ex. 17, find W when
(*i*) AB, AC etc., are as before, $m=1$ lb., $\omega=210$ *r.p.m.*
(*ii*) $AB=3''$, $AC=10''$, $m=3$ lbs., $\omega=120$ *r.p.m.*

7·4. Rotating Sphere. *A hollow smooth sphere is rotating with uniform angular velocity ω about a vertical diameter, to find the position of relative rest of a particle placed in it.*



Let P be the position of the particle and let
$OP=a$, the radius of the sphere,
$PN=r$, the radius of the horizontal circle described by P,
$\angle PON=\theta$
and R, the pressure of the sphere on the particle, then

$$R\cos\theta=mg \quad ..(1).$$

and $$R\sin\theta=m\omega^2 r=m\omega^2 a\sin\theta \quad ..(2).$$

$\therefore$ from (1) and (2), we get

$$\tan\theta=\frac{\omega^2 a}{g}\sin\theta \quad ..(3).$$

or $$\sin\theta\cos\theta=\frac{g}{a\omega^2}\sin\theta.$$

Either $\sin\theta=0$ *i.e.*, $\theta=0$ and from (1) $R=mg$, which gives the lowest position as the position of rest,

or $$\cos\theta=\frac{g}{a\omega^2}.$$

Since, $\cos\theta<1$, $\frac{g}{a\omega^2}<1$, $\therefore \quad \omega^2>\frac{g}{a}$.

Thus if the angular velocity fall short of $\sqrt{\frac{g}{a}}$, the particle will rest only at the lowest point.

Examples

1. A particle moving on the inside of a smooth sphere of radius r, describes a horizontal circle at a distance $\frac{1}{2}r$ below the centre of the sphere. Show that its spead is $\frac{1}{2}\sqrt{6gr}$.

2. A smooth cup in the form of a paraboloid is set whirling with angular velocity ω about its axis which is vertical. Show that a particle will remain in equilibrium on any part of its surface if $\omega=\sqrt{g/2a}$, where $4a$ is the latus-rectum of parabolic section of the cup.

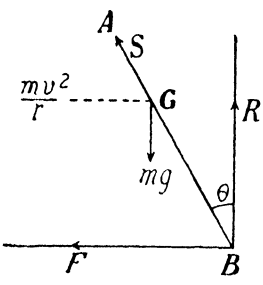
3. A body is moving in a circle making n revolutions per sec. on the surface of a smooth hollow cone of vertical angle 2α. Show that the distance of the body from the axis of the cone is

$$\frac{g\cot\alpha}{4\pi^2n^2}.$$

4. A cyclist's circular racing track is part of a cone with axis vertical, vertex downwards, semi-vertical angle 60°, and radius 120 ft. If the coefficient of friction is 0·3, find the maximum and minimum speeds possible on the track. (*Punjab Public Service Commission*)

[*Ans.* 64 ; 30 ft. per sec. approx.]

7·5. Motion of a bicyclist on a curved path. When a man runs or rides a bicycle on a curved path (supposed to be nearly circular), he always leans towards the centre of the path. By this the reaction of the ground becomes inclined to the vertical.



Let the machine and the rider be represented by the line AB inclined at an angle θ to the vertical. The forces acting on the machine are :—

(i) the total weight mg acting at G, the centre of gravity of the machine and the rider, and (ii) reaction S, passing through B, the point of contact with the ground.

Now for motion in a horizontal circle, there must be a force mv^2/r acting towards the centre of motion. In this case this necessary force which has to act at G is the resultant of S and mg.

Since mg acts at G, therefore, S must also pass through G.

Resolving vertically and horizontally, we have

$$S\cos\theta=mg \qquad ..(1)$$

and

$$S\sin\theta=\frac{mv^2}{r} \qquad ..(2)$$

$\therefore$

$$\tan\theta=\frac{v^2}{rg}, \qquad ..(3)$$

which gives the inclination of the machine to the vertical in terms of the velocity and the radius of the path described.

The result shows that whenever v is large and r is small, θ is large, *i.e.*, the cycle is in danger of falling to the ground if the cyclist either takes a sharp turn or takes a turn while moving at great speed.

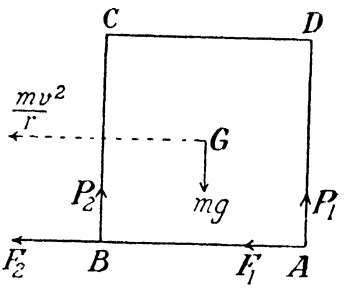
F, the horizontal component of S, is the force of friction and

$$= S \sin\theta = \frac{mv^2}{r},$$

R, the vertical component of $S = S\cos\theta = mg$.

Note. It is important to note that $\frac{mv^2}{r}$ is not an external force but is really the resultant effect of the forces mg and S.

7·6. Motion of a railway carriage or a motor car on a curved level track. When a railway carriage moves along a curved horizontal path (supposed to be circular), the force $\frac{mv^2}{r}$ towards the centre of the path is supplied by the horizontal component of the pressure of the rails on the flanges of the wheels.



Let $ABCD$ be the section of a railway carriage by the vertical plane perpendicular to the direction of motion through G, its centre of gravity (supposed to lie midway between A and B), A and B being the points where the wheels touch the lines.

Let AB, the lateral distance between the wheels be $2a$, and h the height of G above AB.

Let P_1, P_2 be the resultant vertical components and F_1, F_2, $(F_1+F_2=F)$ the resultant horizontal components of the pressures of the rails on the wheels on the two sides.

Also, let r be the radius of the curved path described, and v the velocity of the train.

The effective force $\frac{mv^2}{r}$ acting at G is the resultant effect of all the forces (F_1, F_2, P_1, P_2, mg) acting on the carriage.

Resolving in the horizontal and vertical directions, we have

$$P_1+P_2=mg \qquad ..(1)$$

and

$$F=F_1+F_2=\frac{mv^2}{r} \qquad ..(2).$$

The last equation shows that the force $\frac{mv^2}{r}$ is equivalent to a force acting along the line AB, not passing through G, the centre of mass of the carriage.

7·61. Upsetting of a carriage on a curved level track.

As all the forces on the carriage are equivalent to $\frac{mv^2}{r}$ acting

at G, therefore the sum of the moments of the forces about any point is the same as the moment of $\frac{mv^2}{r}$.

Taking moments about A, we have

$$\frac{mv^2}{r} \cdot h = mg \cdot a - P_2 \cdot 2a$$

i.e.,
$$P_2 = \tfrac{1}{2}m\left(g - \frac{v^2 h}{ra}\right) \qquad ..(3).$$

$\therefore$ from (1) *i.e.*, $P_1 + P_2 = mg$ and (3), we have

$$P_1 = mg - P_2 = \tfrac{1}{2}m\left(g + \frac{v^2 h}{ra}\right) \qquad ..(4).$$

This shows that P_1 is always positive and increases with v, while P_2 decreases as v increases and vanishes when $v^2 = \frac{gra}{h}$ so that the wheels on the inside of the curve are no longer in contact with the ground and the carriage begins to tilt about A *towards the outside of the curve* if

$$v > \sqrt{\frac{gra}{h}} \qquad ..(5).$$

It shows that v is large when a is large and h is small so that the danger of upsetting is minimized by keeping the centre of gravity of the carriage as low as possible and distance between the wheels as large as possible.

These being fixed for any particular carriage, the danger is further reduced by keeping v small and r large, *i.e.*, by reducing the velocity of the carriage and avoiding a sharp turn.

Equations (3) and (4) give the vertical pressures on the inner and outer wheels.

The motion of a motor car is exactly similar to the motion of a railway carriage. The forces F_1, F_2 are, in this case, the forces of friction between the tyres of the wheels and the ground.

7·62. The result (5) given above can also be arrived at, and the phenomenon of upsetting can be explained as follows :—

The central force $\frac{mv^2}{r}$ necessary to cause the circular motion, should really act at G, the centre of gravity of the carriage but, actually, being produced by F_1 and F_2 or their resultant, F, it can act only at the points of contact with the rails or ground. Now the force F acting along AB is equivalent to an equal horizontal force F acting at G together with a couple which tends to make the carriage rotate in direction $BCDA$, *i.e.*, to lift the inner wheels off the ground and to tilt the carriage about A.

This fact is responsible for the upsetting of a carriage rounding a curve at high speed. The weight mg acting vertically at G is the only force which prevents this rotation.

The moment of the couple tending to upset the carriage is $F.h=\frac{mv^2}{r}h$ while the moment of the weight about A is $mg.\ a$,

$\therefore$ the carriage will upset,

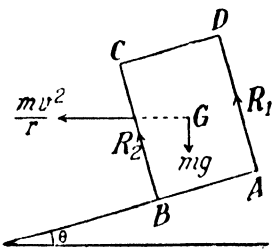
$$\text{if } \frac{mv^2h}{r} > mga \quad \text{or} \quad \text{if } v > \sqrt{\frac{agr}{h}}.$$

7·63. We have seen that in the case of a railway carriage the force F acts on the wheels in the horizontal direction and action and reaction being equal and opposite the wheels also exert an equal amount of pressure on the rails in the opposite direction. These lateral pressures wear away the rails and are undesirable. In the case of the motion of a motor car or a carriage moving on a curved road, the force F appears as the force of friction between the wheels and the ground.

We have also seen that the force $F=\frac{mv^2}{r}$ acting along the line AB and not acting at G, the centre of mass, tends to upset the carriage.

In order to avoid these objectionable lateral pressures and the danger of upsetting, the plane of the track is banked up, *i.e.*, it is made inclined to the horizontal. In a railway track this is done by raising the outer rail, so that the floor of a railway carriage moving on a curved track is not horizontal.

7·7. Motion on a banked up track. How the banking up of the track prevents the lateral pressure and upsetting of a carriage can be shown as follows :—



Let θ be the inclination of the floor to the horizontal and R_1, R_2 the normal reactions perpendicular to the floor.

In order that there may be no lateral thrust, the necessary force $\frac{mv^2}{r}$ towards the centre of the horizontal path must be the resultant of the external forces R_1, R_2, and mg acting at G.

Equating the sum of the resolved parts of the forces, and their resultant in the horizontal and vertical directions, we have

$$(R_1+R_2)\ \sin\theta=\frac{mv^2}{r} \quad \text{and} \quad (R_1+R_2)\cos\theta=mg.$$

$$\therefore \qquad \tan\theta=\frac{v^2}{rg} \qquad \ldots(1)$$

which gives the required inclination of the track.

Thus, when the plane of the track is made inclined to the horizontal, at an angle given by this equation, there is no side thrust on the track.

Also as the resultant of the weight and the normal reactions is $\frac{mv^2}{r}$ acting at G, there will be no tendency to upset.

Note. Equation (1) can also be obtained at once by equating the resolved part of the resultant and the forces in the direction AB.

Thus we have $\frac{mv^2}{r}\cos\theta = mg\sin\theta$ or $\tan\theta = \frac{v^2}{rg}$.

7·71. When banking up the road, the engineers adjust the angle θ with reference to some moderate velocity with which the trains (or cars on roads) are generally expected to move. If the actual velocity, say V, of a train is different from v, which the engineers have taken into account in banking up the road, some lateral thrust, (friction in the case of a car) say F, comes into play.

Suppose that the force F acts along AB when the velocity of the carriage is V, then $\frac{mV^2}{r}\cos\theta$, the component of the central force down the slope is supplied by F, and $mg\sin\theta$ the component of the weight.

Hence $$\frac{mV^2}{r}\cos\theta = F + mg\sin\theta$$

i.e., $$F = \frac{mV^2}{r}\cos\theta - mg\sin\theta$$

$$= m\cos\theta\left(\frac{V^2}{r} - g\tan\theta\right)$$

$$= m\cos\theta\left(\frac{V^2 - v^2}{r}\right) \text{ [by (1) Art. 7·7.]}.$$

Thus, if V (the actual velocity of the train) is greater than v (the velocity for which there is no side thrust for the inclination θ), F is positive, *i.e.*, it acts in the direction AB and is caused by the pressure of the outer rail at A while if V is less than v, F is negative, it acts in the direction BA and is caused by the inner rail at B.

Exactly the same results apply to a *banked up* motor track.

Examples XXI

1. A bicyclist is revolving round a circular garden at the rate of 15 *m.p.h.* the diameter of the garden being 100 yards. Find at what angle to the vertical he should keep himself inclined in order to maintain the speed.

2. Find the horizontal thrust exerted on the rails by an engine of mass 25 tons going round a curve of radius 440 yards at 36 miles per hour.

If there is to be no horizontal thrust on the rails, show that the track must be banked at $\tan^{-1} 0{\cdot}066$.

3. An aeroplane moves at 90 *m.p.h.* in a curve of 110 yards radius. Show that the plane must be banked at nearly 59° to the horizontal.

4. A mass of one pound is suspended by a string 2 ft. long in a railway carriage. Show that when the train is moving round a curve whose radius is 1,452 ft. at the rate of 30 *m.n.h.* the tension of the string is increased by about $\frac{1}{\pi^2}$ of an ounce and the horizontal displacement of the mass is nearly one inch.

5. If a particle of weight W is suspended from the roof of a railway carriage which is moving with a constant speed v, in a curve, prove that, in the position in which the particle is at rest relatively to the carriage, the tension of the cord is $W\left(1 + \frac{v^4}{\rho^2 g^2}\right)^{\frac{1}{2}}$, where ρ is the radius of curvature of the path.

6. A merry-go-round is revolving in a horizontal circle of radius, 3 ft. at the rate of 7 revolutions in 22 secs. A child of weight 20 lbs. rides a wooden horse suspended by a string $\sqrt{5}$ ft. long.

Find the tension in the string and show that its inclination to the vertical is $\tan^{-1}\frac{1}{2}$.

At what point will the child fall on the ground if he loses hold of the horse whose height above the ground is 4 ft.

7. If on a railway curve of radius 400 yards the distance between the rails is 5 ft. and the super-elevation of the outer rail is 3 inches, and if there is to be no lateral thrust on the rails, show that the velocity of a train must be about 30 miles per hour.

What is the lateral thrust on the track produced by an engine of weight 10 tons (1) at rest, (2) moving at a speed of 25 *m.p.h.*, (3) at a speed of 35 *m.p.h.* ?

8. Find the velocity with which a train can travel without side pressure, on a curved portion of the Kalka-Simla railway track of gauge 30″ : it is given that

$$S=1\tfrac{1}{4}''\text{ and }R=261\text{ ft.}$$

where S stands for the super-elevation and R for the radius of the track.

Also find the same when $S=2\frac{1}{4}''$, and $R=140$ ft.

9. A railway of gauge 5 ft. is taken round a curve of $\frac{1}{4}$ mile radius. What super-elevation must be given to the outer rail in order that a train travelling round that curve at 30 and 60 *m.p.h.* may impose the same side pressure on the inner and outer rail respectively ?

10. A motor racing track of radius a is banked at an angle α. Obtain an equation which will give the speed for which the track is designed. Show that if the speed of a car is one-half this speed, there will be a total transverse frictional force of $\frac{1}{2}W\sin\alpha$ between the car and the ground, W being the weight of the car.

11. An engine of weight W is at rest on a railway curve of radius r, which is banked up to suit a speed v, show that there is an internal lateral thrust on the rail of magnitude.

$$W\frac{v^2}{\sqrt{v^4+r^2g^2}}.$$

12. The axles of a motor car are 4 ft. 6 in. in length and the centre of gravity of the car is 2 ft. above the ground. Prove that the car will overturn if it is driven round an arc of radius 30 ft. at a speed greater than 22·5 *m.p.h.* It is assumed that the car does not skid.

13. The velocity of a motor-car is v, the distance between its wheels is a and the height of its centre of gravity above the ground is h, show that the radius of the smallest circle which the centre of gravity can describe, without the inner wheels leaving the ground is $\frac{2v^2h}{ag}$.

14. A tram with wheels 5 ft. apart weighs 2 tons and has its *C.G.* 5 ft. above the ground. Find the pressure on the sides of the track and the vertical reactions on the wheels when it moves round a curve of 100 yards radius at 24 ft. per second.

What is the smallest radius of the curve in which the tram can move at this speed without overturning ?

REVISION QUESTIONS VI

1. Show that the angular velocity of a moving point P about a fixed origin O is vp/r^2, where $OP=r$; v is the velocity of P and p is the perpendicular from O on the direction of motion. (*Roorkee*)

2. If a particle describes a circle of radius a uniformly making one revolution in n seconds, prove that its acceleration is directed towards the centre and is

$$\frac{4\pi^2a}{n^2}.$$

Show that the effect of the rotation of the earth is to lessen the apparent weight of a body at the equator by about $\frac{1}{289}$ of itself.

[Take the circumference of the earth as 40,000 kms. and g as 980 C.G.S. units.]

3. A belt, the mass of which is m lbs. per foot length, passes round half the circumference of a pulley wheel, the speed of belt being v ft., per sec. If T_1 and T_2 be the tensions in the two sides of the belt, show that, neglecting the weight of the pulley, the total pull on the pulley is equal to

$$\left[T_1 + T_2 - \frac{2mv^2}{g}\right] \text{lbs.}$$

4. A particle of mass m is attached to a point O by an inextensible string of length l. Prove that it can describe a horizontal circle about a point vertically below O with uniform angular velocity ω, provided $l\omega^2 > g$.

Prove that if the string is elastic this condition is replaced by

$$l_0\lambda \frac{\omega^2}{g} > \lambda - ml_0\omega^2 > 0,$$

where λ is the modulus of elasticity and l_0 the natural length of the string.

5. A string of length l has its ends fastened to two points A and B in the same vertical line, A being above B. A bead C on the string rotates about AB with uniform angular velocity ω. Show that

(*i*) if the string is not vertical, $\cos A - \cos B = \frac{gl}{ab\omega^2}$.

(*ii*) BC is horizontal, if $\omega^2 = \frac{2gl^2}{c(l^2-c^2)}$.

6. The point A is vertically above B and $AB = l$. The ends of a string ABC of length $2l$ are fixed at A and B. A bead of mass m, which can slide freely on the string describes a horizontal circle with angular velocity ω about AB. The plane in which C moves is at depth y below A. Show that

$$y = \tfrac{1}{2}\, l + 4g/3\omega^2.$$

7. A railway carriage is travelling on a curve of radius r with velocity v; if h be the height of the centre of inertia of the carriage above the rails and $2a$ the distance between them, show that the weight of the train is divided between the inner and the outer rails in the ratio of $gra - v^2h$ to $gra + v^2h$ and hence show that the carriage will upset if

$$v > \sqrt{\frac{gra}{h}}. \qquad (P.U.\ ;\ D.U.\ Hons.\ 1927)$$

8. If the maximum and minimum speeds of trains on a certain curve of radius r are v and u respectively, show that, if the track is banked up so that the sleepers are inclined at an angle β to the horizontal such that

$$\tan\beta = \frac{v^2+u^2}{2gr},$$

then the outer lateral thrust of the fastest train is equal to the inner lateral thrust of the slowest train, assuming their weights to be equal.

9. A train starts from rest and moves on a level uniform curve with acceleration f. The outer rail is raised so that the floor of the carriage is inclined at an angle α to the horizontal. Show that a body cannot rest on the floor of the carriage unless the co-efficient of friction between the body and the floor is greater than $\dfrac{\sqrt{f^2+g^2\sin^2\alpha}}{g\cos\alpha}$.

10. A cyclist rides with velocity v on a circular track of radius a on level ground. Determine the angle of friction, λ, between the wheels and the ground necessary to prevent slipping outwards. ($Ans.$ $\tan^{-1} v^2/ag$)

If the track were banked up to an elevation β, prove that the necessary angle of friction would be $\lambda - \beta$, where λ is the angle determined in the first part of the question. ($Bm.\ U.$)

CHAPTER VIII

MOTION ALONG A SMOOTH VERTICAL CIRCLE

SIMPLE PENDULUM

8·1. When a particle is constrained to move along a circle we generally find its motion by resolving the forces in the tangential and normal directions and writing the equations of motion in the form

$$mv\frac{dv}{ds}=F_t \quad \text{and} \quad \frac{mv^2}{r}=F_n.$$

In the particular case of a particle moving along a horizontal circle, if F_t is zero, the first equation shows that v is constant. This case was discussed in the previous chapter.

Now we consider another particular case, namely, the motion of a particle on a smooth circle in a vertical plane when the force acting on it is the force of gravity only.

(We may conceive of a smooth curve by the concave or convex side of a circular strip of smooth metal.)

In this case, the first equation, gives rise to a simple relation between the velocities at any two points of the circle and the vertical distance between them. This relation is obtained in the next article and will be frequently used in later discussions of circular motion.

8·2. *A particle is projected along the inside of the arc of a smooth vertical circle, from any point in it with velocity* v_1. *To prove that* v_2, *the velocity of the particle, after it has moved through a height* h, *is given by*

$$v_2^2=v_1^2-2gh.$$

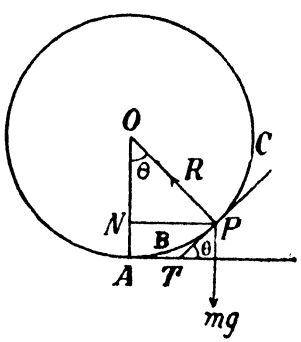
Let O be the centre, A the lowest point and r, the radius of the vertical circle.

Let a particle of mass m move along the arc BPC, B being its initial position where its velocity is v_1 and C the final position where its velocity is v_2.

Let v be the velocity of the particle at any point P and let $\angle AOP=\theta$ and arc $AP=s$.

The only external forces acting on the particle are its weight mg, acting vertically downwards, and the pressure R of the curve in the direction of the normal.

Equation of motion of the particle along TP, the tangent at P, is



$$mv\frac{dv}{ds} = \text{sum of the components of the forces along } TP$$

$$= -mg\sin\theta. \qquad \ldots(1).$$

But $\dfrac{dv}{ds} = \dfrac{dv}{d\theta}\cdot\dfrac{d\theta}{ds} = \dfrac{1}{r}\cdot\dfrac{dv}{d\theta}$ [since $s = r\theta$].

$\therefore$ the equation (1) can be written as: $v\dfrac{dv}{d\theta} = -rg\sin\theta$.

Integrating, we have $\frac{1}{2}v^2 = rg\cos\theta + c = g.NO + c$

$$= g(AO - AN) + c = g(r - y) + c,$$

where y is the height of P above A.

Now at B where $v = v_1$, let $y = y_1$, so that

$$\tfrac{1}{2}v_1^2 = g(r - y_1) + c.$$

Also at C where $v = v_2$, let $y = y_2$, then

$$\tfrac{1}{2}v_2^2 = g(r - y_2) + c.$$

Subtracting, we get $\frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 = -g(y_2 - y_1)$

or $$v_2^2 = v_1^2 - 2g(y_2 - y_1) = v_1^2 - 2gh,$$

where h is the height of the point C above the point B.

If, instead of moving upwards, the particle slides down the arc of the circle with an initial velocity u, the velocity v after moving a vertical distance h is given by $v^2 = u^2 + 2gh$.

If the particle slides from rest, $v^2 = 2gh$.

Thus the tangential equation of motion of a particle, moving along a circular arc, gives us its velocity at any point on the arc.

The normal reaction R on the particle, is then obtained from the equation of motion along the normal, *viz.*,

$$\frac{mv^2}{r} = \text{sum of the components of the forces along the normal}$$

$$= R - mg\cos\theta.$$

Similar results for the general case of motion of a particle along *any* curve will be given in Chapter XIII.

Example. A heavy particle is free to move in a vertical circle of radius l, the particle is projected with velocity u from the lowest point A of the circle. If v be its velocity when it has risen a vertical distance h, prove that $v^2 = u^2 - 2gh$. If it just reaches a point B, show that

$$u = \sqrt{\frac{g}{l}}\,AB. \qquad (P.U.)$$

We now consider various cases of motion of a particle moving along a smooth vertical circle, *i.e.*, along the concave or convex side of a smooth strip of metal bent into the form of a vertical circle.

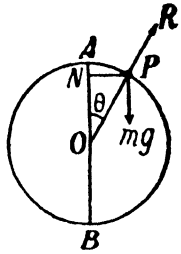
8·3. Motion on the outside of a smooth vertical circle. *A particle starts from rest at the highest point and moves down along the outside of the arc of a smooth vertical circle, to discuss the motion.*

AOB is the vertical diameter of the circle of radius r, P is any position of the particle and PN is drawn perpendicular to AB.

Let $AN=h$ and $\angle AOP=\theta$.

If v is the velocity at P, then by Art. 8·2 we have

$$v^2=2g.AN=2gh \qquad ..(1).$$



Equation of motion along the normal PO, is

$$\frac{mv^2}{r}=mg\cos\theta-R$$

(the particle being outside the arc, the pressure on it is outwards.)

$$\therefore\quad R=m\left(g\cos\theta-\frac{v^2}{r}\right)=m\left(g\frac{r-h}{r}-\frac{2gh}{r}\right)=mg\frac{r-3h}{r},$$

which gives R, the pressure of the curve on the particle.

R vanishes when $h=\frac{r}{3}$, the particle no longer presses the arc and leaves it. It afterwards moves in a free parabolic path.

8·31. *To find the equation to the subsequent parabolic path.*

Take the point where it leaves the curve as origin, the horizontal line as the axis of x and the *downward* vertical line as the axis of y. If v is the velocity at that point, then

$$v^2=2gh=2g.\frac{r}{3}=\frac{2}{3}gr.$$

The direction of motion at the point makes an angle θ with the horizontal given by

$$\cos\theta=\frac{ON}{OP}=\frac{\frac{2}{3}r}{r}=\frac{2}{3}\quad\text{or}\quad\tan\theta=\frac{\sqrt{5}}{2}.$$

The equation to the path is $y=x\tan\theta+\frac{gx^2}{2v^2\cos^2\theta}$.

$$\therefore\qquad y=\frac{\sqrt{5}}{2}x+\frac{gx^2}{2.\frac{2}{3}gr.\frac{4}{9}}=\frac{\sqrt{5}}{2}x+\frac{27}{16r}x^2.$$

This is the required equation to the parabolic path whose latus-rectum$=\frac{8}{27}r$.

8·4. Motion on the inside of a smooth vertical circle. *A particle is projected from the lowest point with velocity u and moves along the inside of the arc of a smooth vertical circle, to discuss the motion.*

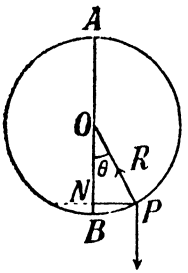
AOB is the vertical diameter of the circle. P is any position of the particle and PN is drawn perpendicular to AB.

Let $\angle BOP=\theta$, $OP=r$, and $BN=h$

then $h=r-r\cos\theta$, a relation between h and θ.

If v is the velocity at P, then by Art. 8·2, we have

$$v^2=u^2-2gh \qquad ..(1)$$



Equation of motion along the normal PO is

$$\frac{mv^2}{r}=R-mg\cos\theta.$$

$$\therefore\ R=\frac{mv^2}{r}+mg\cos\theta=m\left(\frac{u^2-2gh}{r}+g\frac{r-h}{r}\right)$$

$$=\frac{m}{r}[u^2-3gh+gr] \quad \text{in terms of } h, \qquad ..(2)$$

or $$=\frac{m}{r}(u^2-2gr+3gr\cos\theta) \quad \text{in terms of } \theta.$$

Equation (1) gives the velocity of the particle and equation (2) gives R, the pressure of the curve on the particle.

(*a*) At the lowest point B, $h=0$, ($\theta=0$) and $R=\frac{mu^2}{r}+mg$ (3)

(*b*) At highest point A, where $h=2r$ ($\theta=\pi$),

$$R=\frac{m}{r}[u^2-5gr] \text{ and by (1), } v^2=u^2-4gr.$$

In order that the particle may not leave contact with the arc of the circle, R should not vanish till the particle reaches the highest point, *i.e.*, $u^2>5gr$ for which value of u, the velocity v also does not vanish at any point ; thus the particle makes a complete revolution.

It must also go on making complete revolutions subsequently provided the resistance of the air is altogether avoided and the arc is ideally smooth which in practice is impossible.

When $u^2=5gr$, then from (3) $R=6mg$ so that *if the particle is projected with a velocity just sufficient to take it to the highest point, pressure at the lowest point=6mg, six times the weight of the particle.*

To consider other cases, suppose that the velocity v vanishes at a height h_1 and the pressure R vanishes at a height h_2, then from (1) and (2), we have

$$h_1=\frac{u^2}{2g} \text{ and } h_2=\frac{u^2+gr}{3g}$$

(*c*) $$h_1<h_2 \text{ if } \frac{u^2}{2g}<\frac{u^2+gr}{3g} \text{ *i.e.*, if } u^2<2gr.$$

Thus if the velocity of projection, u is less than the velocity sufficient to take the particle to the level of the horizontal diameter, the velocity vanishes before the pressure vanishes. The particle, therefore, stops and moves down, acquires some velocity at the lowest point B, moves on the other side of B through an equal height and goes on moving to and fro just like the bob of a clock pendulum.

(*d*) $$h_2<h_1 \text{ if } \frac{u^2+gr}{3g}<\frac{u^2}{2g} \text{ *i.e.*, } u^2>2gr.$$

So that if $u^2>2gr$, and $u^2<5gr$ *i.e.*, when the velocity of projection is sufficient to take the particle higher than the level of the

horizontal diameter and is insufficient to allow the particle to make complete revolutions, the pressure vanishes before the velocity vanishes. The particle, therefore, leaves the arc and on account of the velocity it possesses, it moves freely along a parabolic path whose equation can be obtained.

Hence we obtain the following results :

(*i*) *When* $u^2 < 2gr$, *the particle oscillates on each side of the lowest point.*

(*ii*) *When* $u^2 > 2gr$ *but* $< 5gr$, *the particle leaves the curve somewhere above the level of O, and describes a parabola.*

(*iii*) *When* $u^2 > 5gr$, *it makes complete revolutions.*

8·41. Bead on a smooth vertical circular wire.

In the case of motion of a bead on a smooth vertical circular wire or that of a particle moving inside a smooth vertical circular tube, the bead or the particle necessarily keeps to the circular path and the question of its leaving the circle does not arise.

In the case when $u^2 > 2gr$, and $< 5gr$, the pressure vanishes for an instant somewhere in the upper half of the circle and then it changes sign, so that instead of the bead pressing the wire outwards, it begins to press it inwards above that point.

In this case the particle can reach the highest point if $u^2 > 4gr$.

8·42. Motion of a particle attached to the end of a string. If a particle is hanging from a fixed point by a light inextensible string and is projected with a certain horizontal velocity, the motion is exactly the same as that of a particle moving inside a smooth vertical circle. We have only to substitute the tension T for the pressure R, and the length of the string l, for the radius r of the circle in the discussion of Art. 8·4.

In the case when $u^2 > 2gl$ and $< 5gl$, the tension vanishes somewhere above the point of suspension, the string becomes slack, and the particle describes a parabola freely so long as the string does not become tight again.

In order that the particle may make complete revolutions, the string must be strong enough to bear a tension of at least six times the weight of the particle.

[This can be deduced from (3) Art. 8·4, by replacing R by T, r by l and taking the least value of u given by $u^2 = 5gl$.]

Note. A heavy particle attached to a fixed point by a light string and swinging in a vertical plane is usually known as a *simple pendulum.*

Example. One end of a string OA, l feet long is fixed and a body of weight w lbs. is attached to A. The body is projected horizontally from its lowest position with velocity u ft./sec. Show that as long as the string remains taut, the tension in the string is

$$\frac{w}{gl}(u^2 - 2gl + 3gl\cos\theta) \text{ lbs.},$$

where θ is the angle OA makes with the downward vertical.

(*i*) If $u^2 = 11gl$, show that the maximum tension is twice the minimum tension. (*P.U.*)

(*ii*) If $u^2 = (2+\sqrt{3})\,gl$, show that the string becomes slack after it has described an angle $\cos^{-1}(-1/\sqrt{3})$. (D. U.)

Examples XXII-A

1. The roadway of a bridge over a canal is in the form of a circular arc of radius 60½ ft. What is the greatest velocity with which a motor cycle can cross the bridge without leaving the ground at the highest point.

2. A glass containing some water is suspended by a string of length r and made to turn along a vertical circle of that radius. Show that in order to prevent water from spilling outside, the velocity at the highest point must be at least $\sqrt{gr}$.

3. An aeroplane describes a vertical circle of 150 yards radius. Find the least velocity it must have when upside down at the highest point in order that no force may be necessary to keep the pilot in his seat.

What force is required to keep the pilot in his seat when the aeroplane is moving upside down at the lowest point of the circle with a velocity which is just sufficient to take it to the highest point ?

4. A trick cyclist rides inside a sphere; riding head downwards past the highest point. If the greatest velocity with which he can reach half-way up the sphere is 30 m.p.h. and if he does not pedal in the upper hemisphere, what is the diameter of the largest sphere in which he can perform his feat without leaving contact with the sphere ?

Also find the least speed the cyclist must have when upside down at the highest point of a sphere of radius 21 ft., the mass of the machine and rider being supposed to be concentrated at a point 3 ft. from the track.

5. In a centrifugal railway a car runs down a long slope and then the rails take a turn round a vertical circle of 30 feet diameter, so that the car and passengers are upside down at the top. What is the least vertical height through which the car must descend before entering the circle in order that it may not leave the rails.

6. A particle attached by a light string of length $3a$ to a fixed point describes a horizontal circle of radius a with uniform angular velocity ω, prove that $2\sqrt{2}a\omega^2 = g$.

If when the particle is moving in this manner, it is suddenly stopped and then let go, find its velocity when the string is vertical in the subsequent motion.

7. A particle slides from rest down the rim of a smooth vertical circular disc, starting from the position in which the radius through the particle makes an angle α with the vertical. Prove that the particle leaves the disc when the radius through it makes an angle of $\cos^{-1}(\frac{2}{3}\cos\alpha)$ with the vertical.

(*a*) If the particle starts from rest at a depth $\frac{1}{3}r$ below the highest point show that it leaves the disc at a height $\frac{1}{3}r$ above the centre.

(*b*) If it starts from the highest point with an initial velocity due to a fall from a height h above the centre, show that it will fly off the disc at a height $\frac{2}{3}h$ above the centre.

8. A particle is projected horizontally with speed $\sqrt{\frac{1}{4}ag}$ from the highest point of the outside of a fixed smooth sphere of radius a. Show that it will leave the sphere at the point whose vertical distance below the point of projection is $\frac{1}{4}a$. Find the magnitude and direction of the velocity of the particle when it strikes the plane upon which the sphere stands.

9. The bob of a freely hanging simple pendulum of length l is projected from its lowest position with speed $\sqrt{6gl}$. Find the velocity of the bob and tension in the string when the string makes 120° with the downward vertical. Weight of the bob is W. (*P.U. 1951*)

10. Find the greatest angle through which a person can oscillate on a swing, ropes of which can support twice the person's weight when at rest.

If the ropes are strong enough and he swings through 180° and if v is his speed at any point, prove that the tension in the rope at the point is $\frac{3mv^2}{2l}$ where m is the mass of the person and l the length of the string.

11. A stone of weight w is tied to one end of a string and is describing a circle in a vertical plane, the other end of the string being fixed. If the max. speed of the stone is twice the min. speed, prove that when the string is horizontal its tension is $10\,w/3$.

Also find the tension when the string makes an angle of 45° with the downward vertical.

12. With what velocity must a particle be projected from the lowest point of a smooth sphere of radius a in order that moving inside the sphere it may just reach (*i*) the end of the horizontal diameter, (*ii*) the highest point ?

(*a*) Show that the reaction at any point is proportional to the depth below the horizontal diameter in the first case and below the highest point in the second case.

(*b*) If the velocity of projection $=\sqrt{2gh}$, and if $\frac{5}{2}a > h > a$, prove that the circular motion ceases when the particle has reached the height $\frac{1}{3}(a+2h)$.

13. A heavy particle hanging from a fixed point by a light inextensible string of length a is projected horizontally with velocity u. Show that during circular motion the tension of the string at any time is proportional to the depth of the particle at that moment below a certain horizontal line.

(*P.U. 1954*)

14. A weight of 2 lb. is whirled round in a vertical circle with a constant speed of 20 ft. per sec., at the end of an elastic cord. The natural length of the cord is 6 ft. and it extends 1 foot for every 7 lbs. of tension. Find the length of the cord at the top and bottom of the path of the weight.

15. A heavy particle is suspended by a string from a fixed point and rotates in a vertical circle. Show that the sum of the tensions of the string when the particle is at the opposite ends of a diameter is the same for all diameters.

16. A mass of 10 lbs. is attached to one end of a string the other end of which is fixed at O. The mass is held such that the string is tight and horizontal and it is released so that it describes a vertical circle. Find the tension in the string when it is vertical.

In this position the string presses against a peg vertically below O, the tension in the string is thereby trebled. Find the depth of the peg below O.

17. A heavy particle is projected horizontally along the interior of a smooth fixed sphere of radius a with a velocity $2\sqrt{ag}$ from a point at a depth $\frac{1}{2}a$ below the centre of the sphere. Prove that the particle will just rise to the level of the centre.

18. A heavy bead is constrained to move on a smooth vertical circular wire of radius a. Show that the bead will describe the whole circle if projected from the lowest point with a velocity just greater than $2\sqrt{ag}$.

What happens when instead of the bead, a particle is projected from the lowest point with the same velocity.

19. A particle slides outside a fixed smooth circle in a vertical plane, starting from rest at the top, and another is projected from the lowest point, inside the circle with just sufficient velocity to carry it to the top ; prove that both leave the circle at the same point, and proceed to describe parts of the same parabola.

20. A small ball is tied to one end of an inelastic string, the other end of which is attached to a fixed point O. It is held with the string tight at a point which is $1\frac{1}{2}$ feet above O and then let fall. If the length of the string be 3 feet, find its velocity immediately after the string becomes tight and the height above O to which it subsequently rises. (*D.U.*)

21. A particle of mass m, attached to a fixed peg O by a string of length l, is lifted up with the string horizontal and then let go. Prove that when the string makes angle θ with the horizontal, the resultant acceleration is $g\sqrt{1+3\sin^2\theta}$.

If the particle is let fall from a point in the horizontal through O at a

distance $l\cos\theta$ from O, show that when it is vertically below O, its velocity is

$$\sqrt{2gl(1-\sin^3\theta)},$$

and the tension of the string is $mg(3-2\sin^3\theta)$.

Examples XXII-B

*1. Find the velocity with which a particle must be projected along the interior of a smooth vertical hoop of radius r, from the lowest point in order that it may leave the hoop at an angular distance of 30° from the vertical.

Show that it will strike the hoop again at an extremity of the horizontal diameter.

Show also that if the velocity of projection be $\sqrt{\frac{1}{2}gr}$, the particle will leave the hoop and return to the lowest point. [*Ans.* $\sqrt{\frac{1}{2}gr(4+3\sqrt{3})}$]

2. A simple pendulum is set swinging in a vertical plane so as to make complete revolutions. If l be its length, show that the velocity of projection must not be less than $\sqrt{5gl}$.

In the subsequent motion ω is the angular velocity at any point of the particle's path. If the maximum angular velocity is greater than ω by an amount α and the minimum angular velocity is less by the same amount, prove that $l\omega\alpha=g$.

3. A heavy particle of mass m makes complete revolutions in a smooth circular tube fixed in a vertical plane. Its greatest speed is n times its least speed. Prove that the pressure in the tube when the particle is moving vertically is $\frac{2(n^2+1)mg}{n^2-1}$, that its speed then is $\sqrt{\frac{n^2+1}{2}}$ times its speed at the highest point. (*P.U.*(

*4. A hollow cylinder of internal radius a rotates round its horizontal axis with constant angular velocity ω. Show that a small particle inside the cylinder will start to slide when its radius makes an angle θ with the downward vertical given by

$$\sin(\theta-\lambda)=\frac{a\omega^2}{g}\sin\lambda,$$

λ being the angle of friction.

*5. A wheel whose spokes are of length a is made to revolve about a horizontal axis with uniform angular velocity ω, on one spoke a small ring fits loosely, and initially the ring is at rest in contact with the lowest point of the rim of the wheel. Prove that the angle α through which the wheel turns before the ring slides is given by

$$g\cos(\alpha-\lambda)+a\omega^2\cos\lambda=0,$$

where λ is the angle of friction between the ring and the spoke.

*6. A particle hangs freely from a string of length l, it is projected horizontally with a velocity $2\sqrt{gl}$. Show that it will rise to a height of $\frac{5}{3}l$ before the string becomes slack.

Also show that subsequently it will pass at a height $\frac{9}{16}l$ directly over the point of suspension.

*7. A particle at the end of a string of length l, the upper end of which is fixed, is projected horizontally with the velocity $\sqrt{ngl}$. If the string becomes slack before the particle reaches the top of the circle, show that it does so at a height $l(1+n)/3$ above the lowest point. Prove also that the vertex of the parabola that it afterwards describes is higher than the point of projection by $\frac{l}{54}(1+n)^2(8-n)$.

*8. A particle is projected along the inner side of a smooth vertical circle, of radius a the velocity at the lowest point being u. Show that if $u^2<5ga$ ($>2ga$) the particle will leave the circle before arriving at the highest point and will describe a parabola whose latus rectum is

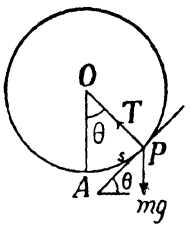
$$\frac{2(u^2-2ga)^3}{27g^3a^2}.$$ (*D.U.*)

SIMPLE PENDULUM

8·5. If a heavy particle is tied to one end of a light inextensible string the other end of which is fixed, and oscillates in a vertical circle, we have what is called a Simple Pendulum.

We now obtain the time of oscillation of such a pendulum when it is allowed to oscillate through a small angle only.

Let O be the fixed point, A the lowest position of the particle, and P any position such that the arc $AP=s$ and $\angle AOP=\theta$.



The forces acting on the particle are its weight mg, and T the tension of the string.

The equation of motion in the direction of the tangent at P is

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad ..(1).$$

If the amplitude of oscillation is supposed to be so small that $\sin\theta$ may be replaced by θ (radians), we have

$$\frac{d^2s}{dt^2}=-g\theta.$$

If l is the length of the string, $\theta=\frac{AP}{l}=\frac{s}{l}$

so that

$$\frac{d^2s}{dt^2}=-\frac{g}{l}s \qquad ..(2).$$

The equation is of the same type as the equation for a simple harmonic motion and it shows that when a simple pendulum swings through a small amplitude, its motion is approximately harmonic. On integration, this equation will give the velocity and time as obtained in Art. 4·2.

The period of a complete oscillation $=2\pi\Big/\sqrt{\frac{g}{l}}=2\pi\sqrt{\frac{l}{g}}$.

Note. If the equation of motion of a particle in a particular problem happens to be of the form $\frac{d^2x}{dt^2}=-\frac{g}{k}x$, then on comparing it with equation (2) we find that the period of motion of the particle is the same as that of a simple pendulum of length k. k is, therefore, known as the *length of an equivalent simple pendulum, e.g.*, in the equation $\frac{d^2x}{dt^2}=-\frac{\lambda}{am}x=-\frac{\lambda g}{amg}\cdot x$, the length of the equivalent simple pendulum is $\frac{amg}{\lambda}$.

The pendulum of a clock beats at both the extreme positions of momentary rest.

$\therefore$ time of one beat or a swing $=\frac{1}{2}$ the period $=\pi\sqrt{\frac{l}{g}}$.

A clock is correct when its pendulum beats seconds,

i.e., when $\pi\sqrt{\frac{l}{g}}=1$, which gives $l=\frac{g}{\pi^2}$. ..(3)

Hence the length of a correct simple pendulum is,

$$l=\frac{g}{\pi^2}=\frac{32{\cdot}2}{9{\cdot}87}\text{ft.}=\frac{32{\cdot}2\times 12}{9{\cdot}87}\text{ or }39{\cdot}14\text{ inches.}$$

Taking $g=981$ cms./sec^2., $l=99{\cdot}4$ cms.

8·6. Loss or gain of Oscillations. The time of an oscillation of a simple pendulum depends upon its length and the value of g at a place. When owing to a change in either of them, the clock does not beat seconds, it loses or gains a number of vibrations in a certain interval.

If n is the number of beats in a certain interval of time T seconds, and t, the time of one beat, then

$$T=nt=n\pi\sqrt{\frac{l}{g}},$$

or
$$n=\frac{T}{\pi}\sqrt{\frac{g}{l}}. \qquad ..(1).$$

If now the number of beats in the same interval becomes $n+dn$ when l becomes $l+dl$ and g becomes $g+dg$ *i.e.*, if dn is the gain in the number of beats due to an increase of dl in l and dg in g, then

$$n+dn=\frac{T}{\pi}\sqrt{\frac{g+dg}{l+dl}}$$

$$=\frac{T}{\pi}\sqrt{\frac{g}{l}}\left(1+\frac{dg}{g}\right)^{\frac{1}{2}}\left(1+\frac{dl}{l}\right)^{-\frac{1}{2}}.$$

$$\therefore \quad \frac{n+dn}{n}=\left(1+\frac{dg}{g}\right)^{\frac{1}{2}}\left(1+\frac{dl}{l}\right)^{-\frac{1}{2}} \qquad \text{[from (1)]}$$

$$\therefore\ 1+\frac{dn}{n}=\left(1+\tfrac{1}{2}\frac{dg}{g}\right)\left(1-\tfrac{1}{2}\frac{dl}{l}\right)=1+\tfrac{1}{2}\frac{dg}{g}-\tfrac{1}{2}\frac{dl}{l},$$

dg and dl being taken so small that their *second and higher powers* are neglected.

Hence
$$\frac{dn}{n}=\tfrac{1}{2}\frac{dg}{g}-\tfrac{1}{2}\frac{dl}{l}.$$

The same result may also be obtained thus:—

From (1), by taking logarithms, we have

$$\log n=\log\frac{T}{\pi}+\tfrac{1}{2}\log g-\tfrac{1}{2}\log l.$$

Differentiating and noticing that $\log\dfrac{T}{\pi}$ is a constant whose differential is zero, we have

$$\frac{dn}{n}=\tfrac{1}{2}\frac{dg}{g}-\tfrac{1}{2}\frac{dl}{l} \qquad ..(2),$$

where the differentials dn, dg, dl represent the changes in n, g and l respectively.

8·61. If only the length l changes to $l+dl$, $dg=0$.

$$\therefore \qquad dn=-\frac{n}{2}\frac{dl}{l} \qquad ..(3).$$

The negative sign shows that there is a loss in the number of vibrations, so that *an increase in the length of the pendulum makes the clock go slow.*

8·62. If only g changes to $g+dg$, $dl=0$.

$$\therefore \qquad dn=\frac{n}{2}\frac{dg}{g} \qquad \ldots(4).$$

The result being positive, the nubmer of vibrations is gained *i.e., an increase in the value of g (acceleration due to gravity) makes the clock go fast.*

Example 1. *Find how many seconds a clock would lose per day if the length of its pendulum were increased in the ratio 900 : 901.*

Here $$\frac{l+dl}{l}=\frac{901}{900} \text{ or } \frac{dl}{l}=\frac{1}{900},$$

n=the number of seconds in a day

$=24\times60\times60=86400.$

Since $\frac{dn}{n}=-\frac{1}{2}\frac{dl}{l}$ $\therefore$ $dn=-\frac{n}{2}\frac{dl}{l}=-\frac{86400}{2\times900}=-48$ secs.

Hence the number of seconds lost per day is 48.

Example 2. *A clock keeps correct time at a place where $g=32{\cdot}1$ ft./sec.^{2} how many seconds would it gain or lose per day if taken to a place where $g=32{\cdot}2$ ft./sec.^{2}.*

Here $$\frac{g+dg}{g}=\frac{32{\cdot}2}{32{\cdot}1} \text{ or } \frac{dg}{g}=\frac{0{\cdot}1}{32{\cdot}1}=\frac{1}{321}.$$

Since $\frac{dn}{n}=\frac{1}{2}\frac{dg}{g}$, $\therefore$ $dn=\frac{n}{2}\frac{dg}{g}=\frac{86400}{2\times321}=134{\cdot}6.$

The result being positive the clock gains 134·6 secs. per day.

Example 3. *A clock with a seconds pendulum loses 20 seconds per day at the place where acceleration due to gravity is 32 ft./sec^2. Find what change (i) in length, (ii) in gravity is necessary to make it accurate.*

(i) Suppose the length of this pendulum is $l+dl$, then

$$\frac{dn}{n}=-\frac{1}{2}\frac{dl}{l},$$

where n=number of seconds in one day

$=24\times60\times60=86400$

dn=number of seconds gained$=-20$

l=correct length of the pendulum$=\frac{g}{\pi^2}$ [Art. 8·5, (3)]

$$\therefore \qquad \frac{-20}{86400}=-\frac{1}{2}\frac{dl}{l},$$

or $$dl=\frac{40\,l}{86400}=\frac{40\times g}{86400\pi^2}\text{ ft.}=\frac{40\times32\times12}{86400\times9{\cdot}87} \text{ or } 0{\cdot}018 \text{ in.}$$

Thus the length of the pendulum was originally too long by 0·018 in. and must, therefore, be shortened by 0·018 in.

(*ii*) Let g be the value of gravity for correct oscillations, and at this place let it be $g+dg=32$, then

Since $\frac{dn}{n}=\frac{1}{2}\frac{dg}{g}$, $n=86400$ and $dn=-20$.

$$\therefore \quad \frac{g+dg}{g}=\frac{n+2dn}{n}=\frac{86400-2\times 20}{86400}=\frac{86360}{86400}$$

i.e., $$\frac{32}{g}=\frac{8636}{8640} \text{ or } g=\frac{32\times 8640}{8636}=32\cdot 015$$

Thus, g must be increased by 0·015.

8·63. Change in g due to position. (*a*) At points above the surface of the earth g varies as $\frac{1}{r^2}$.

i.e., $$g=\frac{\mu}{r^2},$$

where r is the distance from the earth's centre and μ is a constant.

$\therefore$ $\log g=\log \mu-2\log r$ and hence $\frac{dg}{g}=-\frac{2dr}{r}$.

Now if h is the height above the earth's surface, then

$$dr=h \quad \text{and} \quad \frac{dg}{g}=-\frac{2h}{r}.$$

Hence from equation (4) of Art. 8·62, we get

$$dn=\frac{n}{2}\frac{dg}{g}=-\frac{nh}{r} \qquad \ldots(1).$$

So that *the number of oscillations is lost and the clock becomes slow when taken to a high mountain.*

(*b*) *At points inside the surface of the earth g varies as r,*

i.e., $$g=\mu r.$$

$$\therefore \quad dg=\mu\, dr=\frac{g}{r}dr \text{ or } \frac{dg}{g}=\frac{dr}{r}.$$

If D is the depth below the earth's surface, dr is negative $=-D$,

and, therefore, $$\frac{dg}{g}=\frac{dr}{r}=-\frac{D}{r}$$

$$\therefore \quad dn=\frac{n}{2}\cdot\frac{dg}{g}=-\frac{nD}{2r} \qquad \ldots(2).$$

So that *the number of oscillations is decreased i.e., the clock becomes slow also when taken to a deep mine.*

Example 4. *A clock which keeps correct time at the surface of the earth is carried to the top of a mountain 4 miles high; find the number of seconds it will lose in a day.*

Here $h=4$ miles,

$r=$ radius of the earth $=4000$ miles,

$n=24\times 60\times 60=86400$.

CHAPTER IX

WORK, ENERGY AND IMPULSE

1. WORK AND ENERGY

9·1. Work. When a body moves due to the action of a force, the force is said to do work In order to distinguish it from various other kinds of work, such work is called Mechanical work.

The work done by a constant force is equal to the product of the force and the displacement of its point of application in the direction of the force.

Let F be the force, s the displacement, and W the work done.

If the displacement takes place in the direction and line of action of the force, then

$$W = Fs.$$

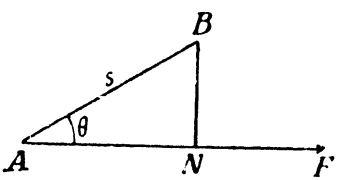
If the displacement is in the same line as the force but in the opposite direction, the work done by the force is negative

i.e., $$W = -Fs.$$

If the displacement takes place along a line at an angle θ to the line of action of the force, then

$$W = F.AN = F.s \cos \theta.$$

the result being negative when θ is an obtuse angle.



Since $F.s \cos \theta = s.F \cos \theta$, the work done by the force is equal to the product of displacement and the resolved part of the force in the direction of the displacement.

If $\theta = 90°$, $W = 0$, that is when the displacement takes place at right angles to the force, the force does no work.

As an illustration let W denote the work done by the force of gravity on a body of mass m :

When the body drops vertically a distance, h, $W = mgh$.

When the body rises vertically a distance, h, $W = -mgh$.

i.e., mgh units of work is required in raising the body through the height h.

When the body slides down a length l of an inclined plane of inclination α, $W = mg.l \sin \alpha$ or mgh,

where h is the vertical distance in the direction of the force.

When $\alpha = 0$, $W = 0$, *i.e.*, if the body moves along a horizontal plane the force of gravity does no work.

9·11. Units of Work. *The absolute unit of work* is the work done by an absolute unit of force in moving its point of application through a distance of unit length in the direction of the force.

In the English (F.P.S.) system, the absolute unit of work is the work done by a force of one poundal in moving its point of appli-

cation through one foot in the direction of the force and is called a **Foot Poundal.**

In the French (C.G.S.) system, the absolute unit of work is the work done by a force of one dyne in moving its point of application through one centimetre in the direction of the force and is called an **Erg.**

The *gravitational* or *practical unit of work* is the work done in lifting the weight of a unit mass through a height of unit length. Owing to the slight difference in the weight of a unit mass at different places of the earth's surface the gravitational unit of work differs slightly at different places.

The *F.P.S.* practical unit of work is the **Foot-Pound** *i.e.*, the **work done in raising a mass of one pound vertically through a height of one foot.**

Since weight of one pound $=g$ poundals,

$\therefore$ *one foot-pound* $=g$ *foot poundals.*

The *C.G.S.* practical unit of work is the *gramme-centimetre*, *i.e.*, work done in raising one gramme vertically through a height of one centimetre.

Since the weight of one gramme $=g$ dynes,

$\therefore$ one gramme centimetre $=g\,(981)$ ergs.

Example. *A motor car weighing 20 cwt. and travelling at 15 m.p.h. is brought to rest in 50 feet by the application of its brake. Find the work done by the resistance.*

Speed of 15 *m.p.h.* $=22$ *f.p.s.*, *mass* $m=20$ cwt. $=20\times112$ lb.

Retardation $f=\dfrac{v^2}{2s}=\dfrac{22\times22}{2\times50}=\dfrac{121}{25}$.

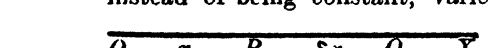
F, the force of resistance $=mf=20\times112\times\dfrac{121}{25}$ poundals

$\therefore$ the work done $=F.s=\dfrac{20\times112\times121}{25}\times50$ ft. poundals

$=\dfrac{20\times112\times121\times2}{32}$ ft. lb.

$=\dfrac{121\times2}{32}$ or $\dfrac{121}{16}$ ft. tons $=7{\cdot}56$ ft. tons.

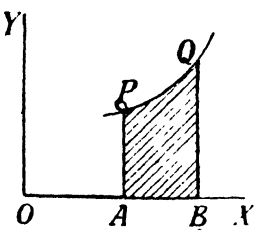
9·2. Work done by a variable force. Suppose the force, instead of being constant, varies and acts along the line OX and suppose that the particle also moves along the same line. If F is the magnitude of the force at P at a distance x from O, the small amount of work δW, done by the force F, when the particle moves a small distance δx from P, is approximately equal to $F.\delta x$,



i.e., $\delta W=F.\delta x$

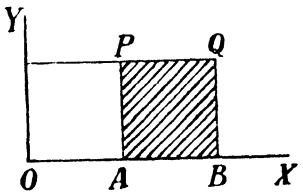
$\therefore$ the total amount of work done by the variable force F when the particle moves from the point $x=a$ to the point $x=b$, is given by

$$W=\int_a^b F\,dx$$



9·21. Graphically it is evidently represented by the area $ABQP$ under the force-distance graph between the ordinates $x=a$ and $x=b$.

In the particular case when the force is constant the graph is a straight line parallel to the X-axis and the work is represented by the area of the rectangle $ABQP$ and is equal to $F(b-a)$.



Example. *A particle is attracted towards a fixed point with a force which varies inversely as the square of the distance from the point. Find the work done by the attraction for any displacement of the particle along the line of action of the force.*

Let the attraction be $\frac{\mu}{x^2}$ per unit mass at a distance x from the fixed point and let the particle (mass m) move from the point $x=a$ to the point $x=b$ $(a<b)$, then for the small displacement δx, the work done by the attraction is $-\frac{\mu m}{x^2}\delta x$, the sign being negative as the displacement takes place in the direction opposite to that of the force.

$$\therefore \qquad W=\int_a^b \left(\frac{-m\mu}{x^2}\right)dx=m\mu\left(\frac{1}{b}-\frac{1}{a}\right),$$

which is negative as $b>a$.

9·22. Work done in stretching an elastic string.

Let one end of the string be fixed and let it be stretched by pulling it at the other end.

Let l be the natural length of the string and let it be stretched to a length $l+x$. The extension is x and the tension T of the string is $\lambda\frac{x}{l}$ which varies as x, the extension of the string beyond its natural length.

Now as the string is being stretched the force applied by an agent at any instant is always equal to the tension of the string at that instant.

$\therefore$ δW, the small amount of work done in stretching the string a further distance δx against the tension $T=T'.\delta x$.

i.e., $$\delta W=\lambda\frac{x}{l}\delta x.$$

$\therefore$ W, the total work done in stretching the string from a length l to a length $l+x$ is given by

$$W=\int_0^x \lambda\frac{x}{l}dx=\tfrac{1}{2}\frac{\lambda}{l}x^2 \qquad ..(1)$$

So that the work done in stretching an elastic string varies as the square of the extension.

Now the work done for an extension $x_1=\frac{1}{2}\frac{\lambda}{l}x_1^2$;

and the work done for an extension $x_2=\frac{1}{2}\frac{\lambda}{l}x_2^2$.

Hence the work done in increasing the extension from x_1 to x_2

$$=\tfrac{1}{2}\frac{\lambda}{l}x_2^2-\tfrac{1}{2}\frac{\lambda}{l}x_1^2=\frac{\lambda}{l}\frac{x_1+x_2}{2}(x_2-x_1).$$

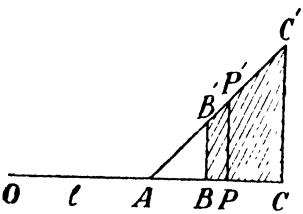
$\frac{\lambda}{l}x_1$ is the initial tension and $\frac{\lambda}{l}x_2$ is the final tension,

$\therefore$ $\frac{\lambda}{l}\cdot\frac{x_1+x_2}{2}$ is the mean of the initial and final tensions and x_2-x_1 is the extension produced.

Hence *the work done in stretching an elastic string is equal to the product of the mean of the initial and final tensions and the extension.*

Since $\frac{\lambda}{l}\cdot\frac{x_1+x_2}{2}$ is also the tension for an extension $\frac{x_1+x_2}{2}$, therefore, the work done is also equal to the product of the tension at $\frac{x_1+x_2}{2}$ and the extension.

9·23. The same can be easily proved graphically as follows :



Let $OA=l$ be the natural length of the string and let

$OB=l+x_1$, and $OC=l+x_2$.

Tension when the length of the string is OP

$$=\lambda\frac{OP-OA}{OA}=\lambda\frac{AP}{l}.$$

Denoting the tension by y, and the extension AP by x, we have $y=\frac{\lambda}{l}x$, which shows that the force-distance graph is the straight line $AB'C'$, having its slope $=\frac{\lambda}{l}$.

$\therefore$ the required work is equal to the area $BB'C'C$

$=BC.\frac{1}{2}(BB'+CC')$

$=(x_2-x_1)\times$(mean of the initial and final tensions).

Also the work done in stretching the string from the natural length OA to a length OP = area $APP'=\frac{1}{2}AP\times PP'$

$$=\tfrac{1}{2}x.\frac{\lambda}{l}x=\tfrac{1}{2}\frac{\lambda}{l}x^2.$$

Example 1. *An elastic string of natural length 3 ft. is stretched 2 inches by a weight of 1 lb. hanging on it. Find the work done in stretching it from a length of $3\frac{1}{2}$ ft. to $4\frac{1}{2}$ ft.*

Since 1 lb. wt. stretches the string 2 inches, and $l=3$ ft,

$\therefore$ by Hooke's law $1=\frac{\lambda}{3}\cdot\frac{2}{12}$, *i.e.*, $\lambda=18$ lb. wt.

The tension for an extension of $3\frac{1}{2}-3$ or $\frac{1}{2}$ ft.

$$=\frac{\lambda}{3}\cdot\frac{1}{2}=\frac{18}{6}=3 \text{ lb. wt.}$$

The tension for an extension of $4\frac{1}{2}-3$ or $\frac{3}{2}$ ft.

$$=\frac{\lambda}{3}\cdot\frac{3}{2}=\frac{18}{2}=9 \text{ lb. wt.}$$

$\therefore$ the mean of the initial and final tensions

$$=\frac{3+9}{2}=6 \text{ lb. wt.}$$

The extension $=4\frac{1}{2}-3\frac{1}{2}=1$ ft.

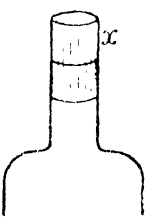
$\therefore$ the work done $=6\times 1=6$ ft. lb.

An example of a different type is given below to illustrate how we calculate work done by a variable force.

Example 2. *A cork 3 inches long is drawn slowly from the neck of a bottle, the force exerted at any instant being proportional to the area of the cork in contact with the bottle. Calculate the work done in drawing the cork, assuming that initially the whole cork is in the bottle and that the pull at starting is 48 lbs.*

If a ft. is the radius of the neck of the bottle, the area of the cork in contact with the bottle, initially

$$=2\pi a.1\tfrac{3}{2}=\frac{\pi a}{2} \text{ sq. ft.}$$



$\therefore$ the initial pull $=k\frac{\pi a}{2}=48$ lbs.

i.e., $$k=\frac{96}{\pi a}.$$

The pull when a length x of the cork has been drawn out

$$=k.2\pi a(\tfrac{1}{4}-x)$$
$$=2\times 96(\tfrac{1}{4}-x) \text{ lb. wt.}$$

The work done in drawing out a further length δx of the cork

$$=2\times 96(\tfrac{1}{4}-x)\delta x.$$

$\therefore$ the required work

$$=\int_0^{\frac{1}{4}} 2\times 96(\tfrac{1}{4}-x)dx=2\times 96[\tfrac{1}{16}-\tfrac{1}{2}.\tfrac{1}{16}]$$
$$=6 \text{ ft. lb.}$$

Examples XXIV

1. Find the uniform force that will move a 1 lb. mass from rest through 1 foot in one second. If this force is exerted through 100 yards from rest, find the number of ft. lb. of work done by the force.

2. A truck weighing 400 lbs. requires a force of 10 lbs. to keep it moving at a constant speed. What work must be done to move the truck 20 ft. with an acceleration of 2 ft. per sec. per sec. ?

3. Find the greatest uniform velocity with which an engine capable of producing 140,000 foot-pounds of work per sec. can draw a train of 100 tons along a horizontal plane, the resistance due to friction etc., being $\frac{1}{80}$th of the weight. (*P.U.*)

4. Show that the work done in stretching an elastic string
$=\frac{1}{2}$ Tension $\times$ Extension.

A spiral spring of length 1 foot is stretched to 13 inches when supporting a four ounce weight. Find the work done in stretching it to a length of 18 inches.

Find also the period of a vertical oscillation of a three ounce weight supported by the same spring.

5. Prove that the work done in stretching an elastic string AB, of natural length l and modulus λ, from tension T_1 to tension T_2 is

$$\frac{l}{2\lambda}(T_2^2 - T_1^2).$$

6. A spider hangs from the ceiling by a thread of modulus of elasticity equal to its weight. Show that it can climb to the ceiling with an expenditure of work equal to only three quarters of what would be required if the thread were inelastic.

7. A man of weight W hangs at the end of a light extensible rope whose modulus is nW, the other end being fastened to a fixed point. He proceeds to climb up the rope. Prove that when he reaches the fixed point he has done $\frac{2n+1}{2n+2}$ times the work he would have to do in climbing the same distance up an inextensible rope. (*Indian Audit Service*)

8. A man of 12 stone ascends a mountain 11,000 ft. high in 7 hours and the difficulties in the way are equivalent to carrying a weight of 3 stone: one of Watt's horse could pull him up the same height without impediments in 56 minutes. Show that the horse can do as much work as 6 such men in the same time.

9. A body whose mass is m falls vertically to the earth's surface from a height a equal to the radius of the earth, show that $\frac{1}{2}mga$ units of work is done by the earth's attraction which varies inversely as the square of the distance from the earth's centre.

10. A man is drawing water from a well with a light bucket which leaks uniformly; the pail when full weighs 50 lbs. and when it arrives at the top half of the water remains. Find the work done if the depth of the well is 100 ft. (*E.A.C.*)

11. A cylindrical cork of length l and radius r is slowly extracted from the neck of a bottle. If the normal pressure per unit of area between the bottle and the unextracted part at any instant be constant and equal to P, show that the work done in extracting it is $\pi r\mu Pl^2$, where μ is the co-efficient of friction.

12. Prove that the work done in raising a system of bodies is measured by the product of the total weight and the vertical height through which the centre of gravity of the system has been raised.

13. A load of 2 tons is suspended by a vertical rope 300 feet long, the rope itself weighing 6 lb. per foot. In winding up the load to the top, how many foot-pounds of work are done?

9·3. Power *is the rate of doing work.* It is the amount of work that an agent is capable of performing per unit time.

The power may be measured in foot pound per second or gramme centimetre per second.

The power of an engine is usually measured in **Horse-Power** (*H.P.*). One Horse-power is the equivalent of 550 foot-pounds per second or 33,000 foot-pounds per minute.

If δW is the small amount of work done in a small displacement δs under the action of a force F at any time t, then

$$\delta W = F\delta s, \quad \text{and} \quad \frac{\delta W}{\delta t} = F\frac{\delta s}{\delta t},$$

$$\therefore \quad \frac{dW}{dt} = F\frac{ds}{dt} = Fv,$$

i.e., the power or the rate of doing work is equal to the product of the force exerted and the velocity of the point of application of the force.

Thus, if at any instant, an engine moving with a velocity of v ft. per second, is exerting a force of F pounds in the direction of its motion, the rate of work, *i.e.*, the work done in one second is Fv ft. lb.

The Horse-power of the engine, $H=\frac{Fv}{550}$,

or the force $F=\frac{550H}{v}$ lb. wt.

If the speed is uniform then F, the pull of the engine, is equal to R, the resistance due to friction, etc., and the horse-power $H=\frac{RV}{550}$, V being the uniform speed.

If the train is accelerating, the work done per second is more than RV as work is also being done in accelerating the train.

Note. The *C.G.S.* units of power is the Watt which is 10^7 ergs (=1 Joule per sec.)

One horse-power is equivalent to about 746 watts.

Example 1. *A train of 100 tons is pulled by an engine on the level at a constant speed of 30 m.p.h. the resistance due to friction, etc., is 10 lbs. per ton. Find the minimum horse power of the engine.*

The resistance $R=100\times 10 =1000$ lb. wt.

Speed $V=30$ m.p.h.$=44$ f.p.s.

The speed being constant, the force exerted by the engine is equal to the resistance.

$\therefore$ the engine's rate of work$=RV=1000\times 44$ ft. lb. per sec.

$\therefore$ the horse-power $H=\frac{RV}{550}=\frac{1000\times 44}{550}=80.$

Example 2. *In Ex. 1, find the acceleration of the train at the instant when it is moving with a velocity of 10 miles per hour, the engine working at its minimum rate.*

In this case the engine, working at its original rate, will accelerate the train so long as the train does not acquire its original velocity of 30 m.p.h.

H the power of the engine$=80$ *H.P.* ;

m, the mass$=100$ tons

v, the speed of the train$=10$ m.p.h.$=\frac{44}{3}$ f.p.s.

F, the total force exerted by the engine

$$=\frac{550H}{v}=\frac{550\times 80}{\frac{44}{3}}=3000 \text{ lb. wt.}$$

R, the resistance$=1000$ lb. wt.

If the required acceleration is f, then by Newton's second law of motion, we have

$mf=(F-R)$ in poundals$=(3000-1000)\times 32$ poundals.

$\therefore$ $$f=\frac{2000\times 32}{m}=\frac{2000\times 32}{100\times 2240}=\frac{2}{7} \text{ ft./sec.}^2$$

Example 3. *Find the H.P., required for a car weighing 10 tons if it has to attain a speed of 30 m.p.h. in 30 secs. from rest up an incline of 1 in 100, the frictional resistance being 20 lbs. per ton.*

m, the mass of the car $= 10$ tons $= 10 \times 2240$ lbs.

R, the resistance $= 20 \times 10 = 200$ lb. wt.

G, the component of the weight down the slope

$$= 10 \times 2240 \times \tfrac{1}{100} = 224 \text{ lb. wt.}$$

f, the acceleration to attain a speed of 30 m.p.h. or 44 f.p.s. in 30 secs.

$$= \tfrac{44}{30} = \tfrac{22}{15} \text{ ft./sec.}^2$$

the accelerating force $= mf = 10 \times 2240 \times \tfrac{22}{15}$ poundals or 1026·6 lb. wt.

If F lb. wt. be the total force exerted by the engine of the car, then

$$mf = F - R - G \text{ or } F = R + G + mf$$

$$\therefore \quad F = 200 + 224 + 1026{\cdot}6 = 1450{\cdot}6 \text{ lb. wt.}$$

The work per sec. required by the car

$$= Fv = 1450{\cdot}6 \times 44 \text{ ft. lb.}$$

$$\therefore \text{ the required } H.P. = \frac{1450{\cdot}6 \times 44}{550} = 116 \text{ approximately.}$$

Example 4. *A train of mass 200 tons is ascending an incline of 1 in 100, the resistance to motion being 15 pounds weight per ton. When its speed has reached 12 m.p.h., what is its acceleration if the horse power developed is 600?*

m, the mass of the train $= 200 \times 2240$ lbs.

v, the velocity $= 12$ m.p.h. $= \tfrac{88}{5}$ ft. per sec.

F, the force exerted by the engine $= \dfrac{550H}{v}$

$$= \frac{550 \times 600}{\frac{88}{5}} = 18750 \text{ lb. wt.}$$

R, the resistance $= 15 \times 200 = 3000$ lb. wt.

G, the component of the weight along the plane

$$= 200 \times 2240 \times \tfrac{1}{100} = 4480 \text{ lbs.}$$

If f is the required acceleration, then

$$mf = (F - R - G) \text{ in poundals}$$
$$= (18750 - 3000 - 4480) \times 32$$
$$= 11270 \times 32.$$

$$\therefore \quad f = \frac{11270 \times 32}{200 \times 2240} = \frac{161}{200} = 0{\cdot}805 \text{ ft./sec.}^2$$

Example 5. *An engine of mass M tons working at H horse power draws n carriages, each of mass m tons, at the uniform rate of v m.p.h. supposing the resistance to be proportional to the weight, prove that the tension in the coupling between the engine and the nearest carriage is*

$$\frac{75\,H\,mn}{448(M+mn)v} \text{ tons wt.} \qquad (Madras\ U.)$$

Let the resistance be k lb. wt. per ton, then R, the total resistance on the train $= (k \times$ total mass in tons$)$ lbs. $= k(M+mn)$ lbs.

run before stopping? If the engine continues to work at the same horse power, what will be the value of the speed when it again becomes steady? (*P.U.*)

23. Find the horse-power of an engine which can just pull a train of mass m tons with velocity v m.p.h., up an incline of 1 in n, the resistance to motion being x lb. per ton on level.

If $v=40$ m.p.h., $n=500$ and $x=15$, show that the engine will be able to pull the train down the same incline at 74 m.p.h. approximately. (*P.U.*)

24. An engine of weight 50 tons hauls a train of weight 150 tons up an incline of 1 in 32. Starting from rest, and moving with constant acceleration against resistances amounting to 12 lb. wt. per ton, the train attains a speed of 30 m.p.h. in 1 mile. Find the tension in the coupling between the engine and the train.

Find also the maximum horse-power exerted.

Examples XXV—B

***1.** *A horse pulls a wagon of 10 tons from rest against a constant resistance of 50 lb. wt. The pull exerted is initially 200 lb. wt., and decreases uniformly with the distance covered until it falls to 50 lb. wt. at a distance of 167 ft. from the start. Show that the resulting velocity of the wagon is very nearly 6 ft. per. sec.*

Let the pull be P lb. wt. at a distance of x ft. from the initial position,

then $$P=200-kx,$$

when $x=167$ ft., $P=50$ lb. wt., so that $50=200-k.167$.

$$\therefore \quad k=\frac{200-50}{167}=\frac{150}{167}.$$

The resistance R is constant and equal to 50 lb. wt.

and the mass of the wagon $=10\times 2240$ lbs.

$\therefore$ the equation of motion of the wagon is

$$mf=Pg-Rg=\left(200-\frac{150}{167}x\right)g-50g$$

or $$10\times 2240\, v\frac{dv}{dx}=150g\left(1-\frac{1}{167}x\right)$$

$$\therefore \quad \tfrac{1}{2}v^2=\frac{15\times 32}{2240}\left(x-\frac{1}{167}\cdot\frac{x^2}{2}\right)+c$$

or $$v^2=\frac{3}{7}\left(x-\frac{1}{334}x^2\right), \quad \left[\because\ v=0,\ \text{when } x=0,\ \therefore\ c=0\right]$$

Now when $x=167$, $$v^2=\frac{3}{7}\left[167-\frac{167\times 167}{334}\right]=\frac{3\times 167}{14}$$

$$=3\times 12 \text{ nearly.}$$

Hence $$v=\sqrt{36} \text{ or } 6 \text{ ft./sec. nearly.}$$

***2.** A body of mass M lb. starts with a speed u ft./sec. The body moves under the action of a force which does work at the constant rate of H horse-power. Prove that the equation which determines the speed v at a time t is

$$Mv\frac{dv}{dt}=550gH.$$

Write down the initial acceleration of the body in terms of M, H and u and prove that the time which the acceleration takes to fall to one-quarter of its initial value is $$\frac{3Mu^2}{220gH}.$$

***3.** In starting a train the pull of the engine on the rails is at first constant and equals P; after the speed attains a certain value u the engine works at a constant rate $R=Pu$. Prove that when the engine has attained a speed $v>u$, the time t and distance x from the start are given by

$$t=\tfrac{1}{2}\frac{M}{Rg}(v^2+u^2), \quad x=\tfrac{1}{3}\frac{M}{Rg}(v^3+\tfrac{1}{2}u^3),$$

where M is the mass of the engine and the train. (*Pub. Ser. Com., D.U.*)

*4. Two trains of equal mass are drawn along smooth level lines by engines one of which, X, exerts a constant pull while the other, Y, works at a constant rate. Both start with the same initial velocity u and after a time t both trains again have the same velocity v. Show that Y describes a greater distance during the interval.

*5. A body moves in a straight line a total distance of a ft. from A to B. It accelerates uniformly from rest at A until it attains a speed of V ft. per sec. with which speed it travels a distance of b ft. It slows down to rest at B in the last c ft. under the action of a retarding force which does work at a constant rate. If the whole journey takes t sec., show that

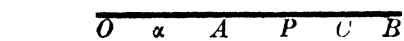
$$V=\frac{4a-2b-c}{2t}.$$

*6. A train of mass 300 tons is originally at rest on a level track. It is acted on by a horizontal force F which uniformly increases with the time, in such a way that $F=0$ when $t=0$, $F=5$, when $t=15$; F being measured in tons weight, t in seconds. When in motion the train may be assumed to be acted on by a frictional force of 3 tons, independent of the speed of the train.

Find the instant of starting, and show that at $t=15$, the speed of the train is 0·64 ft. per sec., whilst the horse-power required at the instant is about 13. [**Ans.** 9 sec.] (*M.T.*)

9·4. Kinetic Energy. If a particle is moving with a velocity v and no force acts on it, it will continue to move uniformly in a straight line with the same velocity v. But if a force acts on the particle in a direction opposite to that of its motion, the velocity of the particle, according to the second law of motion, decreases so long as the force acts, the particle being ultimately reduced to rest if the force continues to act for a sufficiently long time. Although the force does work upon the particle, in reducing it to rest, the particle is said to possess the capacity to do work. A particle in motion is, therefore, said to have the capacity to do work or to possess energy called the *Kinetic energy.*

Thus, the **Kinetic Energy** *of a particle is the capacity to do work in virtue of its motion and is measured by the amount of work that the impressed force will do in bringing the particle to rest.*



Suppose that a particle of mass m moves along the straight line OAB. Let the velocity of the particle be V at A ($OA=a$) and let it be reduced to zero at B ($OB=b$) by a constant or a variable force acting in the line of motion of the particle.

Now suppose that at any point P at a distance s from O, the magnitude of the force is F and the velocity of the particle is v, then

$$F=-mv\frac{dv}{ds}.$$

The work done by the force when the particle moves an additional distance δs is $F\delta s$.

$\therefore$ the whole work done by the force till the particle is reduced to rest at B, is

$$\int_a^b F\,ds=-\int_a^b mv\frac{dv}{ds}\,ds=-\int_V^0 mv\,dv=\tfrac{1}{2}m\,V^2.$$

Note. When the force is constant and equal to P and if f is the constant retardation produced by it, then,

$$P=mf \qquad \ldots(1)$$

and

$$0=V^2-2f.AB \qquad \ldots(2)$$

$\therefore$ $\tfrac{1}{2}mV^2=mf.AB=P.AB=$ work done by the force.

Further suppose that the forces do not act in the direction of the original motion of the particle, so that the particle moves along a plane curve. Let a, b, s be the lengths of the arcs measured from a fixed point O on the curve. If F_t is the sum of the components of the forces along the tangent at P, then

$$F_t = -mv\frac{dv}{ds}.$$

The small amount of work done by the force F_t when the particle moves a little distance δs along the arc of the curve is $F_t.\delta s$.

$\therefore$ the work done by the external forces till the particle is reduced to rest

$$= \int_a^b F_t\, ds = -\int_a^b mv\frac{dv}{ds}\, ds = -\int_V^0 mv\,dv = \tfrac{1}{2}mV^2.$$

Thus the **Kinetic Energy** *of a particle of mass m moving with velocity V at any instant, (or the amount of work done by the impressed forces to reduce the particle to rest), is given by the expression* $\frac{1}{2}mV^2$ (measured in absolute units). ..(I)

It may be noted that this result depends only on the mass and the velocity of the particle and is quite independent of the nature and direction of the impressed forces.

Note. The change in the Kinetic energy per unit of space

$$= \frac{d}{ds}(\tfrac{1}{2}mV^2) = mV\frac{dV}{ds} = \text{the force.}$$

9·41. If during the motion of the particle moving along a straight line (or along a curved path), its velocity changes from U at A to V at any other point $C(OC=c)$, *the total work done by the impressed forces while the particle moves from A to C* in the direction of the resultant force F

$$= \int_a^c F ds = \int_a^c mv\frac{dv}{ds}ds = \int_U^V mv\,dv$$

$$= \tfrac{1}{2}mV^2 - \tfrac{1}{2}mU^2 = \textit{Change in Kinetic Energy.}$$

Thus, *in any displacement of a particle the change in the Kinetic Energy is equal to the Work done by the forces acting on the particle.* ..(II)

This theorem expresses the important principle known as the *principle of Conservation of Work and Energy.*

Note. If the particle moves along a straight line under a constant force P which produces a constant acceleration f, then

$$P = mf \quad \text{and} \quad V^2 = U^2 + 2f.AC$$

$\therefore$ $\frac{1}{2}mV^2 - \frac{1}{2}mU^2 = mf.AC = P.AC =$ work done by the force.

Example 1. *A man raises a bucket of water weighing 10 lbs. from a well. When the bucket has risen 18 ft. its velocity is 16 ft. per sec. How much work has the man done up to this time ?*

What is the horse power of an agent who can do ten times as much work every second ?

Work done against gravity $= mgh$

$= 10 \times 18g$ ft. poundals

$= 180$ ft. lb.

Work done in imparting $K.E. = \frac{1}{2}mv^2$

$= \frac{1}{2} \times 10 \times 16 \times 16$ ft. poundal

$= 40$ ft. lb.

$\therefore$ the total work done $= 180 + 40 = 220$ ft. lb.

Work done by the agent $= 10 \times 220$ ft. lb. per second.

1 H.P. $= 550$ ft. lb. per sec.

$\therefore$ H.P. of the agent $= \dfrac{220 \times 10}{550} = 4$.

Example 2. *A train travelling at 45 m.p.h. picks up 240 cu. ft. of water from a trough $\frac{1}{4}$ mile long. Find the extra power that must be developed to maintain the speed.*

(The height to which the water is raised is to be neglected).

The velocity $v = 45$ m.p.h. $= 66$ ft. per sec.

Time to travel $\frac{1}{4}$ mile $= \dfrac{\textit{distance}}{\textit{velocity}} = \dfrac{1760 \times 3}{4 \times 66} = 20$ secs.

Weight of 1 cu. ft. of water $= 62{\cdot}5$ lbs.

$\therefore$ m, weight of water picked up every sec.

$$= \frac{240 \times 62{\cdot}5}{20} = 750 \text{ lbs.}$$

K.E. imparted to picked up water

$$= \tfrac{1}{2}mv^2 = \tfrac{1}{2} \times 750 \times 66 \times 66 \text{ ft. poundals.}$$

$$= \frac{750 \times 66 \times 66}{2 \times 32} \text{ ft. lb.}$$

$\therefore$ the extra horse power

$$= \frac{\frac{1}{2}mv^2}{550} = \frac{750 \times 66 \times 66}{2 \times 32 \times 550} = \frac{1485}{16}$$

$= 92{\cdot}8$ nearly.

Example 3. *A bullet moving at the rate of 200 ft. per sec. is fired into a thick target which it penetrates to the extent of 6 inches; if fired into a target 3 inches thick with equal velocity, with what velocity would it emerge, supposing the resistance to be uniform and the same in both cases?*

Let m be the mass of the bullet and R poundals the resistance, then since the total Kinetic energy of the bullet is equal to the work done by R when the bullet moves a distance of 6 inches, we have

$$\tfrac{1}{2}m \times 200 \times 200 = R.\tfrac{1}{2} \qquad \ldots(1)$$

In the second case when the target is 3″ thick, let v be the velocity of emergence, then since the decrease in the $K.E.$ is equal to the work done by R,

we have $\tfrac{1}{2}m \times 200 \times 200 - \tfrac{1}{2}mv^2 = R.\tfrac{1}{4} = \tfrac{1}{4} \times m \times 200 \times 200$, by (1)

$\therefore$ $80{,}000m - 2mv^2 = 40000m$

i.e., $v^2 = 20000$

$\therefore$ $v = 100\sqrt{2} = 141$ ft. per sec. nearly.

Example 4. *A mass of 12 lbs. falls 36 feet and is then brought to rest by penetrating $\frac{1}{2}$ ft. into a soft bed of mud. Find the average resistance of the mud.*

If v is the velocity of the mass on reaching the ground, then

$$v^2 = 2gh = 2 \times 32 \times 36 = 64 \times 36.$$

Let R poundals be the average resistance of the mud. Since the weight of the mass is acting downwards, the total force upwards acting on the mass is $R-mg=R-12g$ poundals.

Hence $\frac{1}{2}mv^2=(R-12g)s$

i.e., $\frac{1}{2}\times 12\times 64\times 36=(R-12g)\frac{1}{2}$

$\therefore$ $R=12\times 64\times 36+12g$ poundals

$\doteq 24\times 36+12$ or 876 lb. wt.

9·5. Potential Energy. *Potential energy of a body is its capacity to do work in virtue of its position and is represented by the amount of work that the forces acting on the body will do when the body moves from its given position to some other position, taken as the standard position.*

For example, a body of weight w lb. placed at a height h feet from the ground, possesses potential energy of amount wh ft lb. which is equal to the work that the weight of the body will do when the body is allowed to fall to the earth's surface, the standard position.

In winding a clock spring one has to do work. In this case the particles on the inner side of the spring have been forced together and those on the outer side drawn apart against the elastic forces of cohesion that hold them together. The spring is said to have potential energy, *i.e*, the forces acting on the spring will do some amount of work when it is allowed to unwind itself to its natural shape (the standard position). The motion of the hands of a clock is thus the result of the potential energy of the compressed spring.

Compressed air is another example of potential energy, the standard position in this case being the volume at atmospheric pressure.

Examples XXVI-A

1. What is the potential energy of 8,000 gallons of water which has been raised 50 ft. above the ground ?

If 8,000 ft. lb. of work is required to drive a machine for one hour, how long will the machine go before the whole water has flowed down to the ground?

(1 gallon of water weighs 10 lbs.)

2. An engine is raising water from a depth of 55 feet and discharging 960 gallons a minute with a velocity of 44 ft./sec. Find the H.P. at which the engine is working.

3. A 3 lbs. weight extends a spiral spring 1 inch. How much work is done on the spring when it is extended 8 inches ?

If this amount of work is done in winding the spring of a clock and if 0·25 ft. lbs. of work is expended to drive the clock for one hour, for how many hours will the clock go ?

4. What is the Kinetic energy of a bullet weighing 4 lbs. when moving with a velocity of 20 ft. per sec. ?

The bullet loses a velocity of 4 ft. per sec. in passing through a plank, find how many such uniform planks will it pass through completely before coming to rest, assuming the resistance of the planks to be uniform.

5. A bullet moving horizontally with velocity u strikes and penetrates a fixed vertical target of thickness a and emerges with horizontal velocity v. Prove that the same bullet would have just failed to penetrate the target had the thickness been $\frac{au^2}{u^2-v^2}$, the average resistance to penetration being the same in the two cases.

6. A bullet fired with a velocity of 1600 ft./sec. passes through three planks in succession. If it loses a velocity of 400 ft./sec. in passing through each plank, find the ratio of their thicknesses to which the resistance is supposed to be proportional. (*I.I.T. Kharagpur*)

7. A mass of 10 lbs. falls 100 feet and is brought to rest by penetrating 1 foot into sand. Find the resistance of the sand.

8. A particle is projected downwards with a velocity of 80 ft./sec. and just penetrates a uniform horizontal board of thickness 1 ft. whose upper surface is 16 ft. below the point of projection. If the same particle were projected vertically upwards with a velocity of 80 ft./sec. from a point 16 ft. below the lower surface of the board, find how far it would penetrate assuming that the resistance offered by the board is constant. (*L.U.*)

9. Determine the horse-power which could be obtained from the Kinetic energy of a river at a place where the width is 100 ft., the mean depth 20 ft. and the mean velocity $4\frac{1}{2}$ miles per hour.

10. If work is done on a particle at a constant rate prove that the velocity acquired in describing a space x from rest varies as $x^{\frac{1}{3}}$.

11. A mass of 18 gms. moving in a straight line is brought to rest in such a way that it loses kinetic energy at a uniform rate of one erg per. sec. If the initial kinetic energy is 900 ergs, prove that the distance travelled before coming to rest is 6,000 cm.

Prove also that the velocity at any point is proportional to the cube root of the distance from the stopping point.

12. Prove that a train of W tons going up an incline of 1 in m will acquire a velocity

$$\left(\frac{P}{W}-\frac{1}{m}-\frac{R}{2240}\right)gt$$

and Kinetic energy $\frac{1}{2}W\left(\frac{P}{W}-\frac{1}{m}-\frac{R}{2240}\right)^2 gt^2$

foot tons after t seconds from rest. P being the pull of the engine in tons and R the resistance on the level in pounds per ton.

9·6. Conservative Forces. In examples given in Art. 9·5, the forces acting on the bodies are such that the algebraic total work done in performing a series of displacements which bring the bodies to their original position is *nil*; *e.g.*, the work done by gravity when a particle of mass m ascends a height h and then comes back to the surface of the earth is $-mgh+mgh$ or zero.

Such is not the case with all forces. If, for example, a body is dragged through a distance s against a constant frictional force F, the work done is Fs. To bring the body to its former position on the same path an equal amount of work Fs is to be done again. Thus the total work performed to bring the body to its original position is $2Fs$ (or more depending on the path followed by the body) and not zero. This is the essential difference between the two types of forces against which work is performed. Forces of the first type are called *Conservative forces.*

A *Conservative system of forces* is, therefore, of such a nature that if a particle, after describing any path in the field of the forces, returns to its original position the total work performed during the displacement is *nil.* It follows from this that in the case of such forces, the work done in bringing a particle from one position to another depends only on the initial and final positions and is independent of the path followed by the particle.

CONSERVATION OF ENERGY

9·61. *A particle of mass m falls from rest at a height h above the ground. The sum of its Potential and Kinetic energies remains constant throughout the motion.*

Let us consider a particle of mass m at a height h ft. from the earth's surface. It possesses potential energy of amount mgh foot poundals. When allowed to fall freely, it acquires a certain velocity. If the velocity is v after falling a distance s, we have $v^2=2gs$.

The kinetic energy of the particle $=\frac{1}{2}mv^2=\frac{1}{2}m.2gs=mgs$, and its potential energy in this position $=mg(h-s)$.

$\therefore$ the sum of kinetic energy and potential energy

$=mgs+mg(h-s)=mgh=$ original potential energy.

On reaching the ground, its potential energy is zero, the ground being the standard position, and the velocity v is given by $v^2=2gh$.

$\therefore$ the kinetic energy $=\frac{1}{2}mv^2=\frac{1}{2}m.2gh=mgh$

$=$ original potential energy.

Thus, for all positions of the particle the sum of its kinetic and potential energies remains always constant.

This is a very simple illustration of an important principle in Mechanics known as the principle of *Conservation of energy.*

9·62. To take another example, let a particle of mass m slide the length l of a smooth inclined plane of inclination, α, and height h.

At the top of the plane, the particle is at rest and possesses potential energy equal to $mg.l \sin \alpha=mgh$.

On arriving at the base of the plane, the potential energy is zero (the standard position being taken at the level of the base) and the particle acquires a velocity v, given by

$$v^2=2g \sin \alpha.l=2gh.$$

The kinetic energy of the particle $=\frac{1}{2}mv^2=\frac{1}{2}m \times 2gh=mgh$.

Thus, the sum of the energies remains the same in the two positions, the whole of the potential energy at the top being converted into kinetic energy at the bottom.

9·63. If the plane is rough instead of smooth the co-efficient of friction being μ, the acceleration down the plane is

$g(\sin \alpha-\mu \cos \alpha)$ and for the velocity v, $v^2=2g(\sin \alpha-\mu \cos \alpha)l$

$\therefore$ $\frac{1}{2}mv^2=mg(\sin \alpha-\mu \cos \alpha)l=mgh-\mu mgb$,

where $l \cos \alpha=b$ is the length of the base of the inclined plane.

It shows that there is a loss in the energy of amount μmgb which is equal to the work done against the force of friction in moving the particle a distance equal to the base of the plane. This loss is only apparent, for, in reality, the energy lost has been transformed into other kinds of energy, *viz.*, sound energy or heat energy. If it is possible to measure all kinds of energy, it will be found that there is no loss of energy, in nature.

In Dynamics, we are mainly concerned with the kinetic and potential energies and we may, therefore, state the principle of conservation of energy as follows :—

If a material particle describes any path under the action of a conservative system of forces, the sum of the kinetic and potential energies remains always constant.

In other words, the total amount of potential and kinetic energies in a mechanic system does not change unless some work is done on it from outside or unless some of the energy of the system is removed from it.

In the last case, the sum of the kinetic and potential energies is not constant because one of the forces acting on the particle *viz.*, the force of friction is not a conservative force. The principle of energy should, therefore, never be applied to problems where there is any friction or any sudden jerk or impact. In such cases energy is always converted to various other forms, *e.g.*, heat, light and sound.

9·64. In general, when a particle moves along a straight line (or along a curve) under a force F, its velocity changing from U at point A to V at any other point C, the work done

$$= \int_a^c F ds = \int_a^c mv \frac{dv}{ds} ds = \int_U^V mv dv$$

$$= \tfrac{1}{2}mV^2 - \tfrac{1}{2}mU^2 \qquad \text{[Art. 9·411.}$$

But work done by the force is equal to the P.E. lost by the particle $= P_1 - P_2$, P_1 being the P.E. at A and P_2 that at C.

Thus we have $P_1 - P_2 = \tfrac{1}{2}mV^2 - \tfrac{1}{2}mU^2$

i.e., $P_1 + \tfrac{1}{2}mU^2 = P_2 + \tfrac{1}{2}mV^2$

i.e., the sum of the *P.E.* and *K.E.* remains constant during the motion.

A more general proof is as follows :

9·65. Principle of Energy. *When a particle undergoes displacement under the action of a conservative system of co-planar forces, to prove that*

(i) the change in the kinetic energy is equal to the work done by the forces.

(ii) the sum of its kinetic and potential energies remains constant.

At any point $P(x, y)$ occupied by the particle at any instant, let ds be the displacement and F be the sum of the components of the forces in the direction of ds, then the small work done by the forces is $F.ds$.

If X, Y be the sums of the components of the forces in the direction of the co-ordinate axes and dx, dy the displacements in these directions, then the work done is also equal to $Xdx + Ydy$.

∴ the total work done in a finite displacement when the particle moves from a point $A(x_1, y_1)$ to another point $B(x_2, y_2)$ is

$$\int_A^B F ds = \int_A^B (Xdx + Xdy) \qquad ..(1).$$

If v be the velocity of the particle in the position P and m be its mass, then $F = mv \frac{dv}{ds}$.

$$\therefore \quad \int_A^B F\,ds = \int_A^B mv\frac{dv}{ds}ds = \left|\tfrac{1}{2}mv^2\right|_{u_1}^{u_2} = \tfrac{1}{2}mu_2^2 - \tfrac{1}{2}mu_1^2$$

$$= K_B - K_A \qquad ..(2),$$

where u_1, u_2 are the velocities of the particle at A and B respectively.

Thus the left hand side of the equation (1) *i.e., the work done by the forces, is equivalent to the change in the kinetic energies* in the two positions.

We denote it by $K_B - K_A$ where K stands for the kinetic energy.

Now imagine that there is some function of x and y denoted by V such that, $\quad X = -\dfrac{\partial V}{\partial x}$ and $Y = -\dfrac{\partial V}{\partial y}$.

$$\therefore \quad \int_A^B (X\,dx + Y\,dy) = -\int_A^B \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy\right) = -\int_A^B dV$$

$$= V_A - V_B \qquad ..(3).$$

So that the nature of the forces is such that the work done by them depends only on the initial and final positions A and B and does not depend on the path followed by the particle. Such forces, as already defined, are known as *Conservative Forces and V is known as the* **Potential Function.**

Thus the right hand side of equation (1) *i.e., the work done by the forces is equivalent to the change in the Potential Energy of the particle.*

From (2) and (3), we have

$$K_B - K_A = V_A - V_B \quad \text{or} \quad K_B + V_B = K_A + V_A,$$

i.e., the sum of the kinetic and potential energies remains constant.

The theorem can be easily extended for a system of particles.

In its general form, the principle of conservation of energy states that :

The total amount of energy in the universe is constant, energy cannot be created or destroyed although it may be converted into various forms.

9·66. The principle of conservation of mechanical energy is of great value in simplifying a certain type of problems in Dynamics.

For example, suppose we have to find the velocity v_2 of a particle projected upwards with velocity v_1 along the arc of a smooth vertical circle from any point on it, when the particle has described a vertical distance h. [A problem discussed in Art. 8·2].

Taking the horizontal level through the point of projection as the level of zero potential energy, we have

Initial P.E. $= 0$ and K.E. $= \tfrac{1}{2}mv_1^2$.

Final P.E. $= mgh$, and K.E. $= \tfrac{1}{2}mv_2^2$.

$\therefore$ by the principle of energy, we get

$$\tfrac{1}{2}mv_1^2+0=\tfrac{1}{2}mv_2^2+mgh,$$

or $$v_2^2=v_1^2-2gh,$$

which is the same as obtained in Art. 8·2.

We give below a few more examples on the application of the principle of energy :—

Example 1. *Apply the principle of energy in finding the acceleration of two particles connected by a string placed over a pulley. (Art. 3·7).*

Let m, m' $(m>m')$ be the particles and let x be the distance moved by each.

The kinetic energy of the system $=\tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2+\tfrac{1}{2}m'\left(\frac{dx}{dt}\right)^2$

Work done by gravity $=mgx-m'gx.$

The work done by the tension of the string on the two particles being equal and opposite, its effect on the system as a whole is nil.

$$\therefore \qquad \tfrac{1}{2}(m+m')\left[\frac{dx}{dt}\right]^2=(m-m')gx.$$

Differentiating both sides *w. r.* to t, we have

$$(m+m')\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}=(m-m')g\frac{dx}{dt}.$$

$$\therefore \qquad \frac{d^2x}{dt^2}=f=\frac{m-m'}{m+m'}g.$$

Example 2. *Find, by the principle of energy, the velocity of a projectile* [*Art. 6·41 (a)*].

Suppose that a projectile of mass m is projected with a velocity u and that its velocity at a height y above the point of projection is v.

Regarding the point of projection as the level of zero potential energy, initial P.E. is zero and initial K.E. is $\frac{1}{2}mu^2$. K.E. in the second position is $\frac{1}{2}mv^2$ and P.E. in that position is mgy.

$$\therefore \qquad \tfrac{1}{2}mv^2+mgy=\tfrac{1}{2}mu^2.$$

i.e., $$v^2=u^2-2gy.$$

Example 3. *A particle moves with simple harmonic motion, its position of rest being at a distance a from the centre. Find, by the principle of energy, the velocity at the centre.* [*Art. 4·2*].

Let v be the velocity at the centre, then

K.E. $=\tfrac{1}{2}mv^2=$ work done by the force of attraction as the particle moves from the position of rest to the centre.

The magnitude of the force F at a distance x from the centre being equal to $m\mu x$ and dx being measured in a direction opposite to that of the force, the work done by the force

$$=-\int_a^0 F\,dx=-\int_a^0 m\mu x\,dx=\tfrac{1}{2}m\mu a^2$$

$$\therefore \qquad \tfrac{1}{2}mv^2=\tfrac{1}{2}m\mu a^2, \quad \text{or} \quad v=\sqrt{\mu}\,a.$$

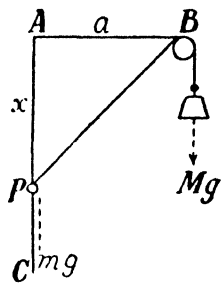
Example 4. *A ring of mass m slides on a smooth vertical rod ; attached to the ring is a light string passing over a smooth peg distant a*

from the rod, and at the other end of this string is a mass M ($>m$). The ring is held on a level with the peg and released. Show that it first comes to rest after falling a distance

$$\frac{2mMa}{M^2-m^2}.$$

Let B be the peg and AC the vertical rod, A being the initial position of the ring so that $AB=a$.

The ring will move down under its weight and will subsequently come to rest in some position, say at P. Let $AP=x$.



Now, P.E. lost by the ring $=mgx$.

The distance through which M moves up by this time

$$=PB-AB=\sqrt{a^2+x^2}-a.$$

$\therefore$ P.E. gained by $M=Mg[\sqrt{a^2+x^2}-a]$.

Hence as the system starts from rest and comes to rest, by the principle of energy, we have

$$mgx=Mg[\sqrt{a^2+x^2}-a],$$

or $$(mx+Ma)^2=M^2(a^2+x^2)$$

i.e., $$m^2x^2+2mMax=M^2x^2 \text{ or } (M^2-m^2)x=2mMa.$$

$\therefore$ $$x=\frac{2mMa}{M^2-m^2}.$$

Examples XXVI-B

1. A heavy particle is free to move in a vertical circle of radius l, the particle is projected with a velocity u from the lowest point A of the circle, and just reaches the point B ; show by applying the principle of energy that

$$u=\sqrt{\frac{g}{l}}\cdot AB.$$

2. Two particles of masses $3m$ and m are attached to the ends, A and B respectively of a light rod 3 ft. long which is freely hinged at a fixed point O in the rod where $BO=2$ ft. and the rod is constrained to rotate in a vertical plane. Initially A is above O, show that, if the rod is just disturbed A will pass through the lowest position with velocity $\frac{1}{2}\sqrt{28g}$.

If the mass at A were $2m$, show that the rod can rest in any position.

3. An elastic cord, modulus λ, natural length l, hangs vertically, one end being attached to a fixed point and the other to a body of mass M. The mass is raised till the string has its natural length and is then released from rest. Show that its greatest fall in the subsequent motion is $\frac{2Mgl}{\lambda}$.

4. A mass m is attached to one end of an elastic string of length a, the other end being fixed to a peg. Initially m is held near the peg and projected with a velocity v vertically downwards. If the particle moves all along in a straight line, show that the depth h below the peg, of m when it is first at rest is given by the equation

$$v^2+2hg=\frac{\lambda}{ma}(h-a)^2,$$

where λ is the modulus of elasticity of the string. (*Bm. U.*)

5. An elastic string of natural length 6 ft. stretches to 6 ft. 6 in. when supporting a weight of 4 lbs. One end of the string is fixed at a point A and the mass is held at A and allowed to fall vertically.

Find by the principle of energy or otherwise

(*i*) the extension of the string when the mass reaches the lowest point, and (*ii*) the velocity of the mass when the string is stretched 1 ft. beyond its natural length.

6. Two equal weights P and P are supported by a string passing over smooth pegs A and B in the same horizontal line, a weight $Q=\frac{2}{\sqrt{3}}P$ is attached to the middle point of the string between A and B. Prove that Q will descend until QAB forms an equilateral triangle.

7. A light string $ABCDE$ whose middle point is C passes through smooth rings, B, D which are fixed in a horizontal plane at a distance $2a$ apart. To each of the points A, C, E is attached a mass m. Initially C is held at rest at O, the middle point of BD, and is then set free. Show that C will come to instantaneous rest when $OC=\frac{4}{3}a$.

Show also that when C has fallen through $\frac{4}{3}a$ from O, its velocity is

$$\sqrt{\frac{25ag}{86}}. \qquad (D.U.)$$

8. A heavy ring of mass m slides on a fixed smooth vertical rod and is attached to a fine string which passes over a smooth peg distant a from the rod and then after passing through a smooth ring of mass $3m$, is tied to the peg. Show that, if m is dropped from the point in the rod at the same horizontal level as the peg, then it will oscillate through a distance $12a/5$. Find the velocity of m when it has fallen through a distance a.

9. A uniform string, of mass M and length $2a$, is placed symmetrically over a smooth peg and has particles of masses m and m' attached to its ends, m being $>m'$; show that the string runs off the peg when its velocity is

$$\sqrt{\frac{[M+2(m-m')]ag.}{M+m+m'}}. \qquad (D.U.\ 1959)$$

10. ACB is a horizontal straight line of length $a+b$. A heavy particle at C is attached by an inextensible string of length b to A, and by an unstretched elastic string of length a to B. If, when the particle starts from rest, the string AC just oscillates through a right angle, and the modulus of elasticity of the elastic string is equal to twice the weight of the particle, prove that

$$4a^3=b(a+2b)^2.$$

11. Two equal particles each of mass m are connected by a light elastic string of natural length a and modulus $\frac{1}{2}mg$ and placed together on the upper edge of a smooth fixed triangular prism. The sides of the prism are each inclined at an angle 30° to the horizontal, and the particles are released from rest, so that they slide down opposite sides. Show that they next come to rest when the length of the string is $a\,(2+\sqrt{3})$. *(L.U.)*

II—IMPULSE AND ENERGY

9·7. Impulse of a constant force. *When a force is constant in magnitude and direction, the impulse is the product of the force and the time during which it acts.*

If a constant force P, acting on a particle of mass m, produces an acceleration f and changes the velocity of the particle from U to V in time t, we have

$$\text{Impulse}=Pt=mft=m(V-U)=mV-mU$$

i.e., the impulse = change in momentum.

If $U=0$, i.e., if the particle starts from rest

$$Pt=mV$$

i.e., *Impulse = momentum generated in time t,*

or *Force P = momentum generated per unit time.*

9·71. Impulsive Forces. When a moving body strikes a fixed object or when moving bodies impinge on one another there is a sudden change of motion. The forces acting on the bodies are great and act on them for such a short time that it is difficult to estimate either their intensity or the time during which they act. Such forces are known as *Impulsive Forces* or *Blows*.

In these cases instantaneous changes of velocities take place and it is comparatively easy to measure the effect of the forces by their *impulses* or the changes of momentum produced.

It is important to remember that *an impulse is not a force.*

If, for example, a hammer strikes a ball of m lb. and sends it off with a velocity of u ft./sec., then mu is the magnitude of the *impulse* which is given to the ball and nothing can be known about the *force* exerted by the hammer unless the time during which the force acts is also known.

It may be recalled that momentum is a *vector* quantity and so care must always be taken to equip it with the proper sign.

Example. *A ball 2 oz. in weight falls from a height of 36 feet, and rebounds to a height of 25 ft., find the impulse and the average force between the ball and floor, if the duration of impact is* $\frac{1}{100}$ *sec.*

Velocity of the ball just before impact

$$= \sqrt{2 \times 32 \times 36} = 48 \text{ f. p. s. (downwards).}$$

Velocity of the ball just after impact

$$= \sqrt{2 \times 32 \times 25} = 40 \text{ f. p. s. (upwards)}$$

$\therefore$ impulse = change of momentum

$$= \frac{2}{16} \times 40 - \left(-\frac{2}{16} \times 48\right) = 11 \text{ lb. ft. sec. units.}$$

If P is the upward force exerted by the floor, then since the weight of the ball acts downwards, the total force acting on the ball upwards is $P - mg$.

$\therefore$ $(P - mg)\, t$ = Impulse

or
$$P = \frac{\text{Impulse}}{\text{time}} + mg = \frac{11}{\frac{1}{100}} + \frac{2}{16} \times 32$$
$$= 1104 \text{ poundals} = 34{\cdot}5 \text{ lb. wt.}$$

9·72. Impact of water on a surface. When a jet of water strikes against a fixed surface in a direction normal to it and when there is no splash, *i.e.*, the water flows along the surface after impact, the pressure exerted by the surface continues destroying the momentum of the water. By putting $t = 1$ in the equation $Pt = mV$, we find that P, the average pressure exerted by the surface is obtained by calculating the amount of momentum destroyed per unit time.

If the water strikes the surface with velocity v after passing through a pipe, the area of whose cross-section is a and if m is the mass of a unit volume of water then the mass of water which strikes the wall each second is mav and the momentum destroyed each second is $mav.v$ or mav^2 which must be equal to P, the pressure exerted on the wall.

The pressure per unit area of the wall is mv^2 in absolute units.

Note. 1 cu. ft. of water weighs 62·5 lbs.

Example. *Water issuing from a nozzle, of 2 inches diameter, with a velocity 60 ft. per sec., impinges on a vertical wall, the jet being at right angles to the wall. If there is no splash, find the pressure exerted on the wall.*

Since there is no splash the water will flow along the surface of the wall after impact and the whole momentum of the water is destroyed by P, the pressure exerted on the wall.

Here $a=\pi(\frac{1}{12})^2$ and $P=mav^2=62{\cdot}5\times\dfrac{\pi}{144}\times60\times60$ poundals

$=153{\cdot}4$ lbs. wt. nearly.

Examples XXVII-A

1. A hammer of mass 4 cwt. falls through 4 ft. and comes to rest after striking a mass of iron, the duration of the blow being 1/50 of a second; find the average force exerted by the hammer on the iron.

2. A ball whose mass is 4 lb. is moving *horizontally* at the rate of 80 ft. per sec. What force expressed in lb. wt. will stop it (*i*) in 2 secs., (*ii*) in 2 feet?

3. A mass of 4 lb. falls *vertically* through 100 feet from rest and is then brought to rest by penetrating into sand. Find the mean force exerted by the sand if it stops the mass (*i*) in 2 secs., (*ii*) in 2 feet.

4. A stream of fluid of density m, meets directly a plane obstacle with velocity v. If the fluid be assumed to flow outward at the rim parallel to the plane, show that the pressure on the obstacle is mv^2 per unit area.

Calculate the pressure in lb. wt. per square foot due to a wind of 50 miles an hour, if a cubic foot of air weighs $1\frac{1}{4}$ oz.

5. A jet of water of 1 sq. in. cross-section, impinges normally on a fixed vertical plate so that the water after impact leaves the plate parallel to it. The force on the plate is measured and is found to be 5·5 pounds weight. Prove that the velocity of the jet is 20 ft. per sec. approximately.

6. A jet of water leaves a nozzle of 1 inch diameter at a speed of 50 ft. per sec. and impinges on a plate fixed at right angles to its direction. What pressure is exerted on the plate?

If the nozzle is drawn backwards with a velocity of 20 ft. per sec. in the line of the jet, what pressure is then exerted on the plate? (*P.U.*)

7. Find the pressure in lb. wt. per acre due to the impact of a fall of rain three inches in 24 hours supposing the rain to have a velocity due to falling freely through 400 feet.

8. A ball of 2 oz. in weight falls from a height of 25 ft. Find the impulse and the average force between the ball and the floor, if the duration of the impulse is $\frac{1}{100}$ sec.

9. A bucket weighing 5 lbs. is suspended from a tap which is delivering 6 gallons of water per min. with a downward velocity of 16 ft./sec. Find in lb. wt. the vertical force the bucket exerts on the tap when the bucket contains a gallon of water.

9·73. Impulse of a variable force. Now suppose that the force is variable and that P is its value at a particular instant t, then the impulse of the force is defined as the integral

$$\int_0^T P\,dt,$$

where T is the time during which the force acts

If the velocity of the particle at the instant t is v, then

$$P=m\frac{dv}{dt}$$

$\therefore$ the whole impulse in time T during which the velocity of the particle changes from U to V

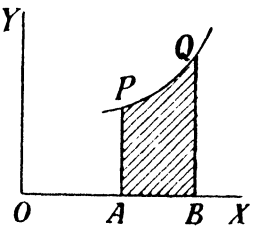
$$=\int_0^T P\,dt=\int_0^T m\frac{dv}{dt}\,dt=\int_U^V m\,dv=m(V-U).$$

In particular when P is constant,

$$\int_0^T P\,dt=P\int_0^T dt=P.T=m(V-U) \quad \textbf{(Art. 9·7).}$$

Thus, *whether a force is constant or variable, its impulse is measured by the change of momentum produced by it.*

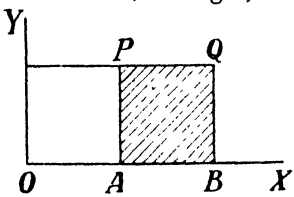
9·74. Graphically the impulse of a force is equal to the area under the force-time graph included between the two ordinates corresponding to the two times, *i.e.*, if $OA=t_1$ and $OB=t_2$ the impulse



$$\int_{t_1}^{t_2} P\,dt=\text{area } ABQPA$$

In particular when P is constant, the impulse $=P(t_2-t_1)=$area of rectangle $ABQP$.

9·75. If a constant force P acting for time t, changes the velocity of a body of mass m from u to v, in a distance s, we have



$$Pt=mft=m(v-u)$$

i.e., force × time = change of momentum,

and $Ps=mfs=\frac{1}{2}m.2fs=\frac{1}{2}m(v^2-u^2)$,

i.e., force × space = change in Kinetic energy.

These equations may also be written as

$$P=\frac{mv-mu}{t}, \quad P=\frac{\frac{1}{2}mv^2-\frac{1}{2}mu^2}{s}$$

i.e., when the force is variable, we calculate the time-average of the force from the change of momentum and the space-average of the force from the change of Kinetic energy.

Also since $Ps=\frac{1}{2}m(v^2-u^2)=m(v-u)\frac{u+v}{2}$,

it shows that *the work done by an impulsive force is equal to the product of its impulse and the mean of the velocities before and after the blow.*

Example. *It is estimated that the actual work done in accelerating a steamer of 1,000 tons, from rest, for the first 210 seconds, is 4,000 foot-tons, and the distance travelled is 2,000 feet. Find (1) the space-average, (2) the time-average of the accelerating force.*

$$\text{Space average of force}=\frac{K.E.}{\text{distance}}=\frac{\text{Work}}{\text{distance}}=\frac{4{,}000}{2{,}000}$$
$$=2 \text{ tons wt.}$$

The velocity v acquired by the steamer is given by

$\frac{1}{2}mv^2$ = work in foot poundals.

$\therefore$ $\frac{1}{2} \times 1000 \times 2240\ v^2 = 4000 \times 2240 \times 32$

or $v = 16$ ft. per sec.

Time average of the force $= \frac{\text{momentum}}{\text{time}} = \frac{mv}{t}$

$= \frac{1000 \times 2240 \times 16}{210}$ poundals $= \frac{1000 \times 16}{210 \times 32}$ or 2·38 tons-wt.

9·8. Motion of two bodies. Suppose that a body of mass m moving with a velocity v is acted upon by a force F in the line of motion, then by the second law of motion, we have $\frac{d}{dt}(mv) = F$. *i.e.*, the rate of change of momentum = the impressed force.

If no external force acts on the body $\frac{d}{dt}(mv) = 0$, or, mv = constant, *i.e.*, the body continues to move with a constant momentum.

Similarly, if there are *two* bodies of masses m, m', moving in the same straight line with velocities u, u', then on the assumption, in accordance with Newton's Third Law, that the mutual action between the bodies are equal and opposite, we have

$$\frac{d}{dt}(mu) = F_1 + R,$$

and

$$\frac{d}{dt}(m'u') = F_2 - R,$$

where F_1, F_2 are the external forces acting on the bodies respectively and forces $+R$, $-R$ are the mutual action and reaction.

By addition, we have $\frac{d}{dt}(mu + m'u') = F_1 + F_2$,

which shows that the rate of change of momentum is equal to the total external force and is unaffected by the mutual action.

If no external force acts on the system, then

$$\frac{d}{dt}(mu + m'u') = 0 \quad \text{or} \quad mu + m'u' = \text{a constant},$$

i.e., the total linear momentum of the system remains constant.

It is evident that if the bodies are not moving in the same straight line and if no external force acts on them, the sum of the resolved parts of their momenta in any particular direction will remain constant.

Thus, if we consider two bodies as a single system and if no external force acts, we may say that the two bodies together will continue to move always with uniform momentum. If at any time during the motion they come in contact with each other, the only forces acting on them are the forces of action and reaction which being equal and opposite cancel each other. Whenever two bodies collide, the change in momentum of the one must be equal to the change in momentum of the second, but in the opposite direction, their total momentum remaining unchanged in any particular direction. This is called the principle of *conservation of linear momentum* and may be stated as follows :

9·81. Conservation of Linear Momentum. *If the sum of the external forces acting on any system of mutually attracting and impinging particles resolved in any direction is always zero, the total momentum of the system in that direction remains constant during the motion.*

Illustration. (1) When a gun is fired the powder is converted into a gas at a very high pressure which drives the shot out. During the short time when the shot moves into the barrel of the gun, the force exerted on the shot forwards is equal and opposite to the force exerted by the shot on the gun backwards, *i.e.*, the impulse of the gun on the shot is equal and opposite to the impulse of the shot on the gun. Hence the momentum generated in the shot is equal and opposite to that generated in the gun. If free to move the gun will recoil in the direction opposite to the motion of the shot.

Suppose m is the mass and u the velocity of the shot and M the mass and V the velocity of the gun, then

$MV + mu$ = momentum before the shot was fired = 0

$$\therefore \qquad V = -\frac{mu}{M}$$

the negative sign shows that the gun recoils, *i. e.*, moves in the direction opposite to that of the shot.

(2) When a shell moving through the air, is caused to explode by means of a time fuse, the motion of the shell cannot change whatever may happen inside it so long as the pressure of the gas is not sufficient to burst the shell. It is only when the shell bursts that the fragments become independent objects themselves and are thrown in different directions on account of the difference of pressure of the air surrounding them.

9·82. Suppose a body of mass m, moving with velocity v impinges on another body of mass M which is at rest and is free to move in the direction of m's motion.

Now if the bodies are *inelastic, i.e., the bodies adhere together* and move forwards as a single body of mass $m+M$ with some velocity, say V, then, as there is no loss of momentum

$$\therefore \qquad (m+M)V = mv \quad \text{or} \quad V = \frac{m}{m+M}\, v,$$

which gives the common velocity of the bodies after impact.

9·83. It is important to note that although there is no loss of momentum due to the impact, there is a loss of Kinetic energy, for, the Kinetic energy before impact $= \frac{1}{2}mv^2$,

and the Kinetic energy after impact

$$= \tfrac{1}{2}(m+M)V^2 = \tfrac{1}{2}(m+M).\frac{m^2v^2}{(m+M)^2} = \frac{m}{m+M}.\tfrac{1}{2}mv^2.$$

The latter is obviously less than the former since $\frac{m}{m+M}$ is less than unity.

In problems where impulsive forces are involved energy is always lost and the principle of energy is never used.

9·84. The principle of conservation of momentum can be applied only in a direction in which there is no *external* impulsive force. If, for example, a shot hits a smooth target obliquely, there is no change in its momentum parallel to the surface of the target, but the whole of its momentum perpendicular to the surface may be destroyed on account of the impulsive reaction of the target.

Note. The word *inelastic* denotes the properties of bodies by which they adhere at the time of impact and move forward as one mass.

The collision of *elastic* bodies where they separate after impact will be dealt with in the next Chapter.

Example 1. *A body of mass 12 lbs. moving with a velocity of 9 ft. per sec., meets a body of mass 8 lbs. moving (i) in the same direction (ii) in opposite direction, with a velocity of 4 ft. per sec. If they coalesce into one body, find the velocity of the compound body. Find also the loss of Kinetic energy.*

(*i*) Let V be the required velocity. Since the sum of the momenta of the two bodies remains unaltered by the impact, we have

$$(8+12)V = 8\times 4 + 12\times 9 = 140$$

$$\therefore \quad V = 7 \text{ ft per sec.}$$

$$\text{Initial } K.E. = \tfrac{1}{2}\times 8\times (4)^2 + \tfrac{1}{2}\times 12\times (9)^2 \text{ ft. poundals}$$

$$= \tfrac{5}{3}\tfrac{5}{2}\tfrac{0}{} \text{ or } 17{\cdot}19 \text{ ft. lb.}$$

$$\text{Final } K.E. = \tfrac{1}{2}(8+12)\times (7)^2 \text{ ft. poundals}$$

$$= \tfrac{4}{3}\tfrac{9}{2}\tfrac{0}{} \text{ or } 15{\cdot}31 \text{ ft. lb.}$$

$$\text{Loss of } K.E. = 17{\cdot}19 - 15{\cdot}31 = 1{\cdot}88 \text{ ft. lb.}$$

(*ii*) Let V_1 be the required velocity. By conservation of momentum, we have

$$(8+12)V_1 = 12\times 9 - 8\times 4 = 76$$

$$\therefore \quad V_1 = 3{\cdot}8 \text{ ft. per sec.}$$

$$\text{Final } K.E. = \tfrac{1}{2}(8+12)V_1^2 = 144{\cdot}4 \text{ ft. poundals}$$

$$= 4{\cdot}51 \text{ ft. lb.}$$

$$\text{Loss of } K.E = 17{\cdot}19 - 4{\cdot}51 = 12{\cdot}68 \text{ ft. lb.}$$

Observe that the loss is much greater in the second case than in the first.

Observe also that when *velocities in one direction are taken as positive those in the opposite direction must be considered as negative.*

Example 2. *A shot of mass 1 ounce, is projected with a velocity of 800 ft. per sec. from a gun of mass 10 lbs., find the velocity with which the latter begins to recoil. Also compare the Kinetic energies of the shot and the gun.*

Let v be the required velocity, and let the direction of motion of the shot be regarded as positive.

Since the total momentum of the shot and the gun in any direction must remain unaltered, we have

$$\tfrac{1}{16}\times 800 - 10v = \text{initial total momentum} = 0$$

$$\therefore \quad v = 5 \text{ ft. per sec.}$$

$K.E.$ of the gun $= \tfrac{1}{2}\times 10\times 5\times 5 = 125$ ft. poundals.

$K.E.$ of the shot $= \tfrac{1}{2}\times \tfrac{1}{16}\times 800\times 800 = 125\times 160$ ft. poundals.

The Kinetic energy of the shot is thus 160 times that of the gun, and it is to this great superiority in Kinetic energy of the shot that its destructive power is due.

Examples XXVII—B

1. A truck weighing 6 tons standing at rest on smooth rails is struck by another truck weighing 4 tons moving at 10 ft. per sec. and both trucks move together. What is their common velocity ?

2. A bullet of mass 2 ounces is fired into a target with a velocity of 1280 ft. per sec. The mass of the target is 10 lbs. and it is free to move : find the loss of Kinetic energy by the impact in foot-pounds. (*P.U.*)

3. A car weighing 2 tons and moving at 60 m.p.h. along a main road collides with a lorry of weight 10 tons which emerges at 30 m.p.h. from a cross-road at right angles to the main road. If the two vehicles lock, what will be their common velocity after the collision ?

4. A gun of mass 5 tons discharges a shot of mass 112 lbs. horizontally. The recoil of the gun is opposed by a constant force of 2 tons weight which brings it to rest in $1\frac{1}{4}$ sec. Find the velocity of the shot, and also the energy of the explosion in foot-tons.

5. A bullet weighing $\frac{1}{4}$ oz. is fired with a velocity of 1,000 f.p.s. from a rifle weighing 10 lbs. The rifle kicks back through 4 inches. Find the average pressure applied by the hand in bringing it to rest.

6. A railway truck of mass m when moving with a velocity $3v$, strikes a truck of equal mass at rest, and moves after impact with velocity v in the same direction as before. Show that the velocity of the second truck after impact is $2v$, and that the kinetic energy lost is $2mv^2$. (*P.U.*)

7. A shot leaves the barrel of a gun of mass 150 tons with a velocity of 600 ft. per sec., if the gun recoils with a velocity of 5 ft. per sec., find the mass of the shot.

8. Two ice yachts stand at rest on perfectly smooth ice with their keels in the same direction. A man jumps from the first to the second and then immediately back again on to the first. Show that the final velocities of the yachts are in the ratio $M : M+m$, where M is the mass of each ice yacht and m the mass of the man.

FURTHER EXAMPLES ON IMPULSE AND ENERGY

9·85. When two masses connected by a string passing over a pulley are in motion or when a moving body penetrates in or coalesce with another which is free to move, *the principle of conservation of momentum must be applied whenever any sudden jerk or impact takes place.*

It is important to note that in writing down equations *for the instantaneous change of momentum produced by impulsive forces, all impulses due to finite forces, such as weight, are neglected.*

Example 1. *Two masses of 5 oz. and 3 oz. are connected by a light inextensible string passing over a smooth fixed pulley. The system starts from rest and after it has moved for 2 seconds, a weight of 4 oz at rest is suddenly attached to the 3 oz. wt. Find the time which elapses before the system comes to instantaneous rest.*

f, the acceleration of the system $= \frac{5-3}{5+3} g = \frac{1}{4}g = 8$ ft./sec.^{2}

v, the velocity after 2 secs. $= 8 \times 2 = 16$ ft./sec. ..(1)

The momentum of the system $= (5+3)16 = 128$ oz. ft./sec. units.

When the 4 oz. mass is suddenly picked up by the 3 oz. mass, an impulsive action takes place and the new mass also partakes of the motion of the system.

If V is common velocity of the system after picking up the 4 oz. mass, then

$$(5+3+4)V=128, \text{ i.e., } V=\tfrac{32}{3} \text{ ft./sec.} \quad ..(2).$$

The masses are now 5 and 7 oz. and if f' is the retardation,

$$f'=\frac{7-5}{7+5}g=\frac{1}{6}g=\frac{16}{3} \text{ ft./sec.}^2$$

$\therefore$ the time t, taken by the system to come to instantaneous rest after the 4 oz. wt. is attached, is given by

$$0=\tfrac{32}{3}-\tfrac{16}{3}t, \text{ i.e., } t=2 \text{ sec.}$$

Note. Equation (2) may be explained as follows—

Suppose I is the impulse due to tension of the string (compared to which the impulses of finite weights are neglected) at the instant when the 4 oz. wt. is suddenly picked up. Since a string cannot push but can pull, I acts upwards on the weights on either side of the string.

Taking the upward direction as positive and considering the 3 oz. wt. which picks up the 4 oz. wt., we get

I = change of momentum = New momentum − Old momentum

$$=(3+4)V-3\times 16 \text{ oz. ft. sec.} \quad ..(1).$$

Again, considering the 5 oz., we have

I = change of momentum in the upward direction,

$$=5(-V)-5(-16)=5\times 16-5V \quad ..(2).$$

From (1) and (2), we get

$$(3+4)V-3\times 16=5\times 16-5V$$

or

$$(3+4+5)V=(3+5)16=128.$$

Example 2. *A shot of mass m penetrates a thickness s of a fixed plate of mass M; prove that if M is free to move, the thickness penetrated is*

$$s\Big/\left(1+\frac{m}{M}\right).$$

Let u be the velocity of the shot and R poundals the average resistance to penetration.

In the first case the shot is reduced to rest after penetrating a distance s.

$\therefore$ by the principle of work and energy, we have

$$Rs=\tfrac{1}{2}mu^2 \quad ..(1).$$

Let s' be the thickness penetrated in the second case and v the final common velocity of the shot and the plate when their relative velocity ceases after impact, then the principle of conservation of momentum gives

$$mu=(m+M)v \quad ..(2),$$

and the principle of work and energy gives

$$Rs'=\tfrac{1}{2}mu^2-\tfrac{1}{2}(m+M)v^2$$

$$=\tfrac{1}{2}mu^2-\tfrac{1}{2}\frac{m^2u^2}{m+M} \quad \text{[from (2)].}$$

$$=\tfrac{1}{2}mu^2.\frac{M}{m+M} \quad ..(3).$$

$\therefore$ from (1) and (3),

$$s' = \frac{M}{m+M}\, s = s \Big/ \left(1 + \frac{m}{M}\right).$$

Example 3. *An inelastic pile, of mass m pounds, is driven vertically into the ground a distance of a feet at each blow of a hammer of mass M pounds, which falls vertically through h feet. Show that*

(i) the weight which would have to be placed on the top of the pile to drive it slowly into the ground would be

$$M + \frac{M^2 h}{(m+M)a} \text{ pounds,}$$

and (ii) the time during which the pile is in motion is

$$\frac{(m+M)a}{M}\sqrt{\frac{2}{gh}}.$$

Let u be the velocity of the hammer just before impact, then

$$u^2 = 2gh \qquad \ldots(1).$$

Let v be the velocity of the hammer and pile just after impact, then by conservation of momentum, we have

$$(M+m)v = Mu \qquad \ldots(2).$$

If P poundals be the average resistance of the ground, then the total force acting on the pile in the upward direction is $P-(M+m)g$, and this force brings the pile and hammer to rest in a space of a feet

$\therefore$ by the principle of energy and work, we have

$$\tfrac{1}{2}(M+m)v^2 = [P-(M+m)g]a \qquad \ldots(3).$$

$\therefore$
$$P = (M+m)g + (M+m)\frac{v^2}{2a}$$
$$= (M+m)g + \frac{M^2}{M+m}\cdot\frac{u^2}{2a}, \qquad \text{[from (2)]}$$
$$= (M+m)g + \frac{M^2}{M+m}\cdot g\frac{h}{a} \qquad \text{[from (1)]}$$

so that a weight $M + \dfrac{M^2}{M+m}\cdot\dfrac{h}{a}$ lb. placed on the pile would slowly overcome the resistance and drive the pile into the ground.

Also the K.E. transformed by the blow

$$= \tfrac{1}{2}Mu^2 - \tfrac{1}{2}(M+m)v^2$$
$$= \tfrac{1}{2}u^2\left(M - \frac{M^2}{M+m}\right) = \frac{mM}{M+m}\, gh.$$

(ii) If t is the required time, then since the impulse of the force acting on the pile is equal to the momentum lost by it, we have

$$[P-(M+m)g]t = (m+M)v \qquad \ldots(4)$$

$\therefore$ From (3) and (4) we get, $\dfrac{t}{a} = \dfrac{2}{v}$

or $$t=\frac{2a}{v}=\frac{2a(M+m)}{Mu}=\frac{(M+m)a}{M}\sqrt{\frac{2}{gh}}.$$

Note. Equation (2) may be explained as follows :—During the time that elapses between the instant the hammer comes in contact with the pile and the instant at which they begin to move together with a common velocity (this infinitesimally small time is of course different from the finite time taken by the hammer and pile to penetrate the ground), an equal and opposite impulse, say I (compared to which the finite impulses due to weights and resistance of the ground are neglected) acts upwards on the hammer and downwards on the pile. Taking the downward direction to be positive, we have

For the pile I=change of momentum$=mv-0=mv$.

For the hammer, $-I$=change of momentum$=Mv-Mu$.

Hence $$Mv-Mu+mv=0$$

i.e., $$(M+m)v=Mu.$$

Example 4. *A bird of mass m is flying horizontally at a height h with velocity v when it is struck by a bullet of mass M moving vertically with a velocity V. If the bullet kills the bird and remains embedded in it, prove that the bird will fall to the ground at a distance d from the point of projection of the bullet where*

$$d=\frac{mv}{g(M+m)^2}\left[MV+\sqrt{M^2V^2+2gh\,(M+m)^2}\right].$$

(I.I.T. Kharagpur)

Let the velocity of the dead bird with the bullet embedded in it be u at an angle α to the horizontal, then by the principle of conservation of momentum, we have

horizontally $(m+M)\,u\cos\alpha=mv+M\times o=mv$

and vertically $(m+M)\,u\sin\alpha=m\times o+MV=MV$

$\therefore$ by division, $$\tan\alpha=MV/mv \qquad ..(1),$$

and by squaring and adding, we get

$$(m+M)^2u^2=m^2v^2+M^2V^2 \qquad ..(2).$$

Now the equation of motion for the dead bird, is

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}=x\tan\alpha-\frac{gx^2\sec^2\alpha}{2u^2}.$$

For a point on the ground, from where the bullet was projected $y=-h$, so that the required distance d is given by

$$-h=d\tan\alpha-\frac{gd^2}{2u^2}(1+\tan^2\alpha).$$

Using (1) and (2) for $\tan\alpha$ and u^2 and simplifying, we get the quadratic equation in d in the form

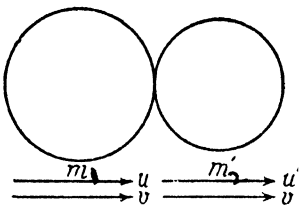
$$g(m+M)^2d^2-2mMvVd-2m^2v^2h=0,$$

which gives the required result at once.

Examples XXVIII

1. The masses of 10 and 8 oz. are connected by a light string which passes over a smooth fixed pulley. The system starts from rest and the 8 oz. mass, after it has risen 3 inches, passes through a fixed ring on which rests a bar of mass 4 oz. and so carries the 4 oz. mass on in its motion. Show that the 4 oz. mass will be carried nearly 2¼ inches above the ring.

But we know, as a matter of ordinary experience, that when two bodies of any hard material impinge on each other, they separate almost immediately and a finite change of velocity is generated in each by their mutual action depending on the material of the bodies. Hence the spheres, if free to move, will have after impact, different velocities say v and v'.



The equation of momentum now becomes

$$mv+m'v'=mu+m'u'. \qquad ..(1)$$

10·21. This single equation is not sufficient to determine the two unknown quantities v and v'. Another relation between the velocities is supplied by **Newton's Experimental Law** which states that *when two bodies impinge directly, their relative velocity after impact, is in a constant ratio to their relative velocity before impact, and is in the opposite direction.*

If bodies impinge obliquely, the same fact holds for their component velocities along the common normal at the point of contact.

The equation derived from this law for the above spheres is,

$$\frac{v-v'}{u-u'}=-e$$

or
$$v-v'=-e(u-u'), \qquad ..(2)$$

v, v', u and u' being *all measured in the same direction.*

10·22. The constant ratio, e is called the **co-efficient of elasticity or restitution.** It depends on the substances of which the bodies are made and is independent of the masses of the bodies and their velocities before impact. The value of e differs considerably for different bodies and varies from 0 to 1.

When $e=0$, the bodies are said to be *inelastic.* In this case we have from (2) $v=v'$, *i.e., if two inelastic spheres impinge they move with the same velocity after impact.*

When $e=1$ the bodies are said to be *perfectly elastic.*

Both these are ideal cases never actually realised in nature.

10·23. To find the velocities of the spheres after direct impact we solve equations (1) and (2) and get

$$v=\frac{mu+m'u'-em'(u-u')}{m+m'} \text{ and } v'=\frac{mu+m'u'+em(u-u')}{m+m'}.$$

10·24 When $m=m'$ and $e=1$, we have $v=u'$ and $v'=u$.

i.e., if two equal perfectly elastic spheres impinge directly they interchange their velocities after impact.

Example 1. *A ball of mass 8 lb. moving with a velocity of 4 ft. per sec. is overtaken by a ball, of mass 12 lb., moving with a velocity of 9 ft. per sec., (i) in the same direction as the first, (ii) in the opposite direction. If $e=\frac{1}{2}$, find the velocities of the balls after impact. Find also the loss of Kinetic energy in the first case.*

Let the direction of motion of the first ball be taken as positive and let v, v' be the velocities after impact, then

(*i*) $8v+12v'=8\times4+12\times9=140$

and $v-v'=-\frac{1}{5}(4-9)=1,$

which give $v=7{\cdot}6$ *f.p.s.* and $v'=6{\cdot}6$ *f.p.s.*

Loss of K.E. $=\frac{1}{2}\times8\times(4)^2+\frac{1}{2}\times12\times(9)^2-\frac{1}{2}\times8\times(7{\cdot}6)^2-\frac{1}{2}\times12\times(6{\cdot}6)^2$

$=64+486-231{\cdot}04-261{\cdot}36$

$=57{\cdot}6$ ft. poundals $=1{\cdot}8$ ft. lb.

(*ii*) $8v+12v'=8\times4-12\times9=-76$

and $v-v'=-\frac{1}{5}[4-(-9)]=-\frac{13}{5}=-2{\cdot}6$

which give $v=-5{\cdot}36$ *f.p.s.*, $v'=-2{\cdot}76$ *f.p.s.*

In this case the first ball turns back after impact.

It should be noted that the velocities are measured algebraically, that is, all velocities in one direction are taken as positive while those in the opposite direction as negative.

Example 2. *A ball A, moving with velocity u, impinges directly on an equal ball B moving with velocity v in the opposite direction. If A be brought to rest by the impact, show that*

$$u : v = 1+e : 1-e$$

where e is the co-efficient of restitution.

Let V be the velocity of B after impact and let m be the mass of each, then since A is reduced to rest after the impact, we have

$$m\times0+mV=mu+m(-v), \quad \text{or} \quad V=u-v,$$

and $$0-V=-e[u-(-v)], \quad \text{or} \quad V=e(u+v)$$

$\therefore$ $$u-v=V=e\,(u+v) \text{ or } (1-e)u=(1+e)\,v.$$

Hence $$u : v=(1+e) : (1-e).$$

Examples XXX

1. A truck weighing 2 tons and moving with a velocity of 5 f.p.s. overtakes a truck weighing 8 tons which has a velocity of 2 f.p.s. Find the velocities of the trucks after impact if the co-efficient of restitution of buffers is $\frac{2}{3}$.

2. A particle A of mass 4 lb. and moving with a velocity of 10 f.p.s. impinges on a particle B of mass 12 lb. at rest. How many foot pounds of energy disappear in the collision and what is the co-efficient of elasticity if A is brought to rest.

3. Two sphere of masses 2 and 3 oz. are moving in the line of centres towards each other with velocities of 24 ft. per sec. and 30 ft. per sec., and their co-efficient of restitution is $\frac{1}{2}$. Find their velocities after impact, and the amount of kinetic energy transformed in the collision.

4. If a ball overtakes a ball of twice its own mass moving with one-seventh of its velocity and if the co-efficient of restitution between them be $\frac{3}{4}$, show that the first ball after striking the second ball will remain at rest. (*D.U.*)

5. Two elastic spheres impinge directly with equal and opposite velocities ; find the ratio of their masses so that one of them may be reduced to rest by the impact, the co-efficient of elasticity being e.

6. If the masses of two balls be as 2 : 1 and their respective velocities before impact be as 1 : 2 in opposite directions. Show that if the co-efficient of restitution is $\frac{5}{6}$, each ball moves back, after impact, with $\frac{5}{9}$th of its original velocity.

7. Two spheres of masses M, m impinge directly when moving in opposite directions with speeds u, v respectively, and the sphere of mass m is brought to rest by the collision; prove that $v(m-eM)=M(1+e)u$.

After the collision the sphere of mass M is acted on by a constant

retarding force which brings it to rest after travelling a distance a. Prove that magnitude of this force is $Me^2(u+v)^2/2a$.

8. A mass of 5 lb. impinges directly on a mass of 10 lb. which is at rest, with a velocity of 12 ft. per sec. and is observed to recoil with a velocity of 1 f.p.s. Find the co-efficient of elasticity and the energy lost in the impact.

9. Two spheres A and B of masses 4 and 8 lb. moving with velocities 9 and 3 ft./sec., in opposite directions, collide. If A rebounds with velocity of 1 ft./sec., find the velocity of B after impact, the co-efficient of elasticity and the loss of Kinetic energy. (*D.U.*)

10. A ball is projected vertically with a velocity of 80 ft./sec. The moment it reaches the highest point, a second equal ball is thrown after it from the same point with the same velocity. How high will they collide ?

If the co-efficient of restitution be 3/5, find the time the second ball takes to reach the ground. (*P.U.*)

11. A sphere impinges directly on an equal sphere at rest ; if the co-efficient of restitution is e show that their velocities after the impact are as

$$(1-e)/(1+e).$$

If the mass of the first sphere be m and that of the second m' show that the first cannot have its velocity reversed if $m>em'$.

12. Two bodies A and B whose elasticity is e, moving in opposite direcions with velocities a and b, impinge directly upon each other ; show that their distance at time t after impact is $te\,(a+b)$.

13. Two bodies whose elasticity is $\frac{2}{3}$, moving in opposite directions with velocities 25 ft./sec. and 16 ft./sec., impinge directly. Find the distance between them $4\frac{1}{2}$ sec. after the moment of impact.

14. Three perfectly elastic balls of masses m, $2m$ and $3m$ are placed in a straight line. The first impinges directly on the second with a velocity u and then the second impinges on the third. Find the velocity of the third ball after impact.

If the masses of the second and third balls are m_1 and m_2, show that the third ball will move after impact with velocity u if

$$(m+m_1)(m_1+m_2)=4mm_1.$$

15. If A, B, C be the masses of three perfectly elastic balls having velocities a, b, c respectively, and A impinges upon B and B upon C so that their velocities after the impact are u, v and w respectively, v being the velocity of B after impact on C ; prove that

$$Aa^2+Bb^2+Cc^2=Au^2+Bv^2+Cw^2.$$

16. A ball A weighing 2 lb. and moving with a velocity of 5 ft./sec., communicates a velocity of 2 ft./sec., to another ball B of weight 8 lb. by means of an intermediate ball C. If the balls are perfectly elastic show that the mass of C is equal to either that of A or that of B. Show also that C again collides with A in the first case.

17. The masses of three spheres, A,B, C are $7m$, $7m$, m ; their co-efficient of restitution is unity, their centres are in a straight line and C lies between A and B. Initially A and B are at rest and C is given a velocity along the line of centres towards A. Show that it strikes A twice and B once and the final velocities of A, B, C are proportional to 21, 12, 1. (*D.U.*)

18. Three smooth spheres A, B, C of equal volume have masses $3m$, m, $4m$ respectively and are initially at rest in a straight line on a horizontal plane. A and B are projected along this line towards C with speeds of 10 ft./sec., and 5 ft./sec., respectively. A overtakes B before B reaches C. Find the speed of C after B collides with it. The co-efficient of restitution is $\frac{1}{4}$ at each impact.

19. A series of n elastic spheres whose masses are 1, e, e^2 etc. are at rest separated by intervals, with their centres on a straight line. The first is made to impinge directly on the second with velocity u. Show that finally the first $n-1$ spheres will be moving with the same velocity $(1-e)u$ and the last with velocity u.

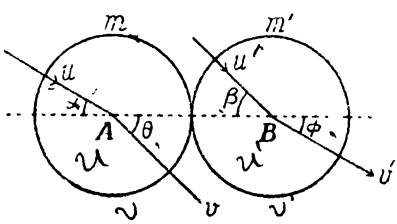
Prove that the final *K.E.* of the system is $\frac{1}{2}(1-e+e^n)\,u^2$.

20. Two equal spheres, A, B, lie on a smooth horizontal circular groove at opposite ends of a diameter. A is projected along the groove and at the end of time T impinges on B. Show that the second impact will occur after a further time $2T/e$, where e is the co-efficient of elasticity. (*P.U.*)

21. A small ball of mass m is just disturbed from the highest point of a smooth circular tube in a vertical plane and impinges on a smooth ball of mass $2m$ resting inside the tube at the lowest point of the circle. Find the height to which each ball rises in the tube after (*i*) first impact (*ii*) second impact, when the co-efficient of elasticity is $\frac{1}{2}$.

10·3. Oblique Impact of two smooth spheres. Suppose that at the moment of impact the direction of motion of the spheres is not along the line joining their centres.

Let m, m' be the masses of the two spheres with centres A and B at the time of impact,



u, u' the velocities just before impact,

α, β the angles the directions of motion make with AB before impact,

v, v' the velocities after impact,

and θ, φ angles the directions of motion make with AB after impact.

Since the spheres are smooth, there is no impulse perpendicular to the line of centres and hence the resolved parts of velocities of the two spheres in the direction perpendicular to AB remain unaltered.

$$\therefore \qquad v \sin \theta = u \sin \alpha \qquad \ldots(1)$$

$$\text{and} \qquad v' \sin \varphi = u' \sin \beta. \qquad \ldots(2)$$

Since the impulsive forces acting during the collision on the two spheres along their line of centres are equal and opposite, the total momentum along AB remains unchanged.

$$\therefore mv \cos \theta + m'v' \cos \varphi = mu \cos \alpha + m'u' \cos \beta. \qquad \ldots(3)$$

By Newton's experimental law for relative velocities resolved along the common normal AB, we have

$$v \cos \theta - v' \cos \varphi = -e(u \cos \alpha - u' \cos \beta). \qquad \ldots(4)$$

From (3) and (4), we obtain

$$v \cos \theta = \frac{(mu \cos \alpha + m'u' \cos \beta) - em'(u \cos \alpha - u' \cos \beta)}{m + m'}, \qquad \ldots(5)$$

$$v' \cos \varphi = \frac{(mu \cos \alpha + m'u' \cos \beta) + em(u \cos \alpha - u' \cos \beta)}{m + m'}. \qquad \ldots(6)$$

Squaring and adding (1) and (5) we obtain v^2 and dividing (1) by (5), we obtain $\tan \theta$. Similarly we get v'^2 and $\tan \varphi$ from (2) and (6).

10·31. We deduce the following particular cases from the above equations :—

(*i*) If $u' = 0$, from (2) $\varphi = 0$, ($\because v' \neq 0$), *i.e., if the sphere of mass m' were at rest, it will move along the line of centres after impact.*

(*ii*) If $u'=0$ and $m=em'$ from (2) $\varphi=0$ and then from (5) $\theta=90°$, *so that if a sphere of mass m impinges obliquely on a sphere of mass m' at rest, the directions of motion of the spheres after impact will be at right angles if* $m=em'$.

This evidently holds true when the spheres are equal and perfectly elastic *i.e.*, when $u'=0$, $e=1$ and $m=m'$.

(*iii*) If $m=m'$, $e=1$, then, from (5) and (6), we have

$$v \cos \theta = u' \cos \beta \text{ and } v' \cos \varphi = u \cos \alpha$$

i.e., if two equal and perfectly elastic spheres impinge they interchange their velocities in the direction of their line of centres.

Also in this case, by using (1) and (2), we get :

$$\tan \theta \tan \varphi = \tan \alpha \tan \beta.$$

It follows that *if two equal and perfectly elastic spheres impinge at right angles, their directions after impact will still be at right angles.*

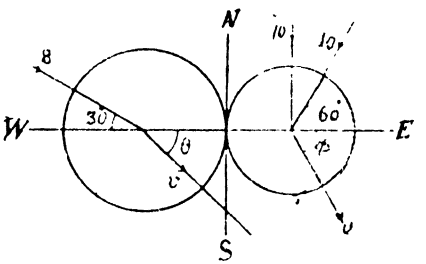
The student is advised to prove these particular cases independently.

Example 1. *A smooth sphere of mass 5 lb. moving with a velocity of 8 ft. per sec. from a direction 30° N of W collides with a sphere of 3 lb. moving with a velocity of 10 ft. per sec. The impact takes place in such a way that the line joining the centres of the spheres is E. and W. at the time of impact. Determine the subsequent motion of the second sphere and the deviation in its direction of motion, if its direction of motion before impact is*

(*i*) *from N. to S.* (*ii*) *from 30° E. of N. and* $e=0{\cdot}6$.

Let the velocities after impact be v and v' at angles θ, φ to the line of centres.

Since the spheres are smooth, the velocities perpendicular to the line of centres are unaltered.



(*i*) In the first case

$\therefore \quad v \sin \theta = 8 \sin 30 = 4 \quad ..(1)$

and $v' \sin \varphi = 10 \sin 90° = 10 \quad ..(2)$

Since the momentum after impact along the line of centres

= momentum before impact,

$$\therefore \quad 5v \cos \theta + 3v' \cos \varphi = 5 \times 8 \cos 30° + 3 \times 10 \cos 90$$
$$= 20\sqrt{3}. \qquad ..(3)$$

By Newton's law,

$$v \cos \theta - v' \cos \varphi = -\tfrac{3}{5} (8 \cos 30° - 10 \cos 90°)$$
$$= -\tfrac{12}{5}\sqrt{3} \qquad ..(4)$$

From (3) and (4), we have

$$3v' \cos \varphi + 5 v' \cos \varphi = 20\sqrt{3} + 12\sqrt{3}, \text{ or } v' \cos \varphi = 4\sqrt{3} \quad ..(5)$$

From (2) and (5), we have

$$v' = \sqrt{148} = 12{\cdot}17 \text{ f.p.s.}$$

and $$\tan\varphi=\frac{5}{2\sqrt{3}}=\tfrac{5}{6}\sqrt{3}=1\cdot4433 \text{ or } \varphi=55^\circ\ 17'.$$

$\therefore$ the required deviation $=90^\circ-55^\circ\ 17'=34^\circ\ 43'$ E. of S.

(*ii*) In the second case, the direction of motion of the second sphere before impact is from 30° E. of N. *i.e.*, it is from 60° N. of E. and therefore the corresponding five equations of motion are as follows :—

$$v\sin\theta=8\sin 30^\circ=4, \qquad ..(1)$$

$$v'\sin\varphi=10\sin 60^\circ=5\sqrt{3}, \qquad ..(2)$$

$$5v\cos\theta+3v'\cos\varphi=5\times8\cos 30^\circ-3\times10\cos 60^\circ$$
$$=20\sqrt{3}-15 \qquad ..(3)$$

$$v\cos\theta-v'\cos\varphi=-\tfrac{3}{5}[8\cos 30^\circ-(-10\cos 60^\circ)]$$
$$=-\tfrac{12}{5}\sqrt{3}-3. \qquad ..(4)$$

From (3) and (4),

$$3v'\cos\varphi+5v'\cos\varphi=32\sqrt{3} \text{ or } v'\cos\varphi=4\sqrt{3}. \qquad ..(5)$$

$\therefore$ from (2) and (5), we have

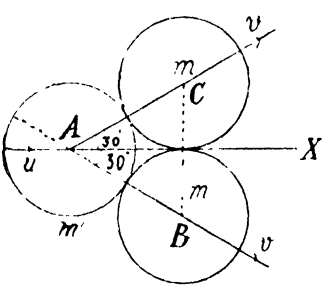
$$v'=\sqrt{123}=11\cdot1 \text{ f.p.s.}$$

and $$\tan\varphi=\tfrac{5}{4}=1\cdot25 \text{ or } \varphi=51^\circ\ 20'$$

$\therefore$ the required deviation $=30^\circ+90^\circ-51^\circ\ 20'$
$=30^\circ+38^\circ\ 40'=68^\circ\ 40'$.

Example 2. *Two equal spheres, each of mass m are in contact on a smooth horizontal table, a third equal sphere of mass m′ impinges symmetrically on them, prove that this sphere is reduced to rest by the impact if* $2m'=3me$.

Before impact, let u be the velocity of m', which is reduced to rest after the impact.



The spheres being equal in size, $\triangle ABC$ is equilateral and AX, the common tangent to B and C makes equal angles, each 30°, with AB and AC.

By symmetry the velocities of B and C will be equal, say each equal to v; also the total momentum of the spheres along AX will remain constant.

$\therefore$ $mv\cos 30^\circ+mv\cos 30^\circ=m'u$

or $$\sqrt{3}\,mv=m'u \qquad ..(1)$$

By Newton's law of relative velocity along AB for the spheres A and B, we have

$$v-0=-e(0-u\cos 30^\circ),$$

or $$v=\tfrac{1}{2}eu\sqrt{3} \qquad ..(2)$$

$\therefore$ from (1) and (2), we get

$$\sqrt{3}\,m\times\tfrac{1}{2}e\sqrt{3}=m', \text{ i.e., } 2m'=3me.$$

Cor. If $m'=m$, then $e=\frac{2}{3}$.

Example 3. *A smooth sphere A, of diameter a, is projected in a certain direction so as to have impact with another equal sphere B at rest. If the direction of motion of A before and after impact makes angles θ and ϕ respectively with the line of centres at the time of impact, prove that*

(i) $\tan\phi = 2\tan\theta/(1-e)$ *and*

(ii) in order to produce the greatest deviation in the direction of A, it must be projected in a direction making an angle

$$\sin^{-1}\left[a\sqrt{1-e}/c\sqrt{3-e}\right]$$

with the line (of length c) joining the centres of A and B.

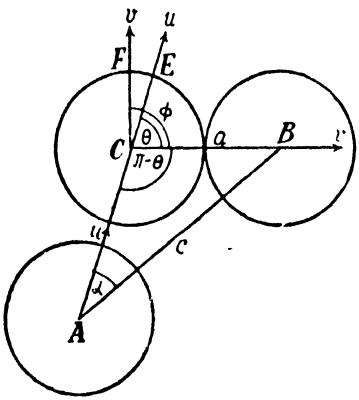
Let the sphere A be projected with velocity u along the line ACE.

Let B be the centre of the sphere B at rest and $A(AB=c)$ the centre of the moving sphere A in its original position.

Let BC' $(=a)$ be the line of centres at the time of impact of the two spheres, C being the centre of the sphere A at that time.

Let v be the velocity of A and v' that of B after the impact, so that $\angle BCE=\theta$ and $\angle BCF=\phi$.

The sphere B, having no velocity before impact, will obviously move along CB after the impact.



(i) By conservation of momentum along BC, and by Newton's law of relative velocity, we have

$$mv\cos\phi + mv' = mu\cos\theta,$$

or
$$v\cos\phi + v' = u\cos\theta, \qquad ..(1)$$

and
$$v\cos\phi - v' = -e(u\cos\theta - 0). \qquad ..(2)$$

From these, we get

$$v\cos\phi = \tfrac{1}{2}(1-e)u\cos\theta. \qquad ..(3)$$

Also since the velocity of the sphere A perp. to CB remains the same, we have

$$v\sin\phi = u\sin\theta. \qquad ..(4)$$

$\therefore$ from (3) and (4), we get

$$\tan\phi = 2\tan\theta/(1-e). \qquad ..(5)$$

(ii) Now let $\angle ECF$, the deviation in the direction of motion of the sphere A, be D, so that $\phi = \theta + D$, putting this in (5) and differentiating w.r. to θ, we have

$$\tan(\theta+D) = 2\tan\theta/(1-e),$$

and
$$\sec^2\left(\theta+D\right)\left[1+\frac{dD}{d\theta}\right] = \frac{2}{1-e}\sec^2\theta = \frac{2}{1-e}\left(1+\tan^2\theta\right).$$

In order that D may be the greatest $dD/d\theta = 0$ and thus we get,

$$\frac{2}{1-e}\left(1+\tan^2\theta\right)=\sec^2(\theta+D)=\sec^2\phi$$

$$=1+\frac{4\tan^2\theta}{(1-e)^2} \text{ by (5)}$$

or $$\frac{2}{1-e}-1=\left[\frac{4}{(1-e)^2}-\frac{2}{1-e}\right]\tan^2\theta$$

or $$[4-2(1-e)]\tan^2\theta=2(1-e)-(1-e)^2$$

or $$2(1+e)\tan^2\theta=1-e^2.$$

$\therefore$ $$\tan^2\theta=\tfrac{1}{2}(1-e), \quad i.e., \tan\theta=\sqrt{1-e}/\sqrt{2},$$

$\therefore$ $$\sin\theta=\sqrt{1-e}/\sqrt{3-e}.$$

Now to find $\angle CAB=\alpha$, from $\triangle ABC$, we have

$$\frac{\sin\alpha}{CB}=\frac{\sin(\pi-\theta)}{AB} \text{ or } \frac{\sin\alpha}{a}=\frac{\sin\theta}{c}$$

Hence $$\sin\alpha=a\sin\theta/c=a\sqrt{1-e}/c\sqrt{3-e}.$$

Examples XXXI

1. A ball of mass 5 lb. moving with a velocity of 15 ft. per sec., impinges on a second ball of double its mass and one-third its velocity; if their velocities before impact be parallel and inclined at an angle of 30° to the line of centres at the time of impact, find the resulting motion of the first sphere if $e=\frac{1}{2}$.

2. Two smooth spheres of mass 1 lb. each moving with velocity 20 f.p.s. in directions at right angles to one another, impinge in such a way that the line joining their centres is the direction of motion of one of them. Find the velocities of the spheres after impact if $e=\frac{1}{2}$.

3. Two equal balls moving with equal speeds impinge, their directions being inclined at 30° and 60° to the line of centres at the time of impact; show that if $e=1$, the balls move in parallel directions after the impact, inclined at 45° to the line of centres.

4. A sphere of mass 4 lb. moving with a velocity of 5 f.p.s. impinges on a sphere of mass 2 lb. moving with a velocity of 3 f.p.s. If spheres are moving (*i*) in like parallel directions (*ii*) in opposite parallel directions inclined at an angle of 30° to the line of centres at the time of impact, show that the velocities of the second sphere after impact are the same in the two cases, the coefficient of restitution being $\frac{1}{2}$.

What are the velocities of the first sphere after impact?

5. An inelastic sphere impinges obliquely on a second sphere at rest, whose mass is twice its own, in a direction making an angle of 30° with the line joining the centres of the spheres at the instant of impact; show that its direction of motion is turned through an angle of 30°.

6. A smooth sphere approaches a stationary sphere of equal size and mass its centre moving along a tangent to the stationary sphere. On impact the path of the moving sphere is caused to deviate through 30°. Find the coefficient of restitution between the spheres, and show that their speeds after impact are equal.

7. A ball impinges on another at rest prove that if the co-efficient of restitution be equal to the ratio of their masses, the balls will leave in directions at right angles to each other.

What will happen in the case of direct collision? (*P.U.*, *D.U.*)

8. Two equal smooth spheres whose co-efficient of restitution is unity, moving in directions at right angles to each other, impinge obliquely. Show

and $v \sin \theta = u \sin \alpha = 8 \sin 30° = 4.$

$\therefore$ $v^2 = 12 + 16 = 28$

i.e., $v = 5{\cdot}29$ ft. per sec.,

and $\cot \theta = \frac{\sqrt{3}}{2}$, *i.e.*, $\theta = 49° 6'$.

Loss of $K.E. = \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}m(8 \times 8 - 28)$ ft. poundals
$= \frac{1}{3}\frac{8}{2}$ ft. lb. $= 9$ ft. oz. $(\because\ m = 1)$

$$\text{Impulse} = m(1+e)u \cos \alpha = 1 \times \frac{3}{2} \times 8 \times \frac{\sqrt{3}}{2}$$
$$= 6\sqrt{3} = 10{\cdot}4 \text{ lb. ft. sec.}$$

Example 2. *A particle falls from a height h in time t upon a fixed horizontal plane. It rebounds and reaches the maximum height h' in time t', show that*

$$t' = et \quad \text{and} \quad h' = e^2h.$$

Also prove that the whole distance (up and down) described by the particle before it has finished rebounding is $h\frac{1+e^2}{1-e^2}$ and that the time that elapses is

$$\frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$$

The time t taken and velocity u acquired on reaching the horizontal plane are given by

$$u^2 = 2gh, \qquad \ldots(1)$$

and $$u = gt \qquad \ldots(2)$$

or $$h = \tfrac{1}{2}gt^2, \text{ i.e., } t = \sqrt{2h/g} \qquad \ldots(3).$$

The velocity of rebound being eu, the time t' and the max. height h' attained by the particle to reach the highest point where its velocity is reduced to zero, are given by

$$0 = eu - gt' \text{ i.e., } eu = gt' \qquad \ldots(4)$$

and $$0 = e^2u^2 - 2gh', \text{ i.e., } e^2u^2 = 2gh' \qquad \ldots(5).$$

From (2) and (4), we get $e \cdot gt = gt'$, *i.e.*, $t' = et$,

and from (1) and (5), we get $e^2\ 2gh = 2gh'$, *i.e.*, $h' = e^2h$.

It will reach the horizontal plane again with the same velocity eu, in the same time $t' = et$, after moving the same distance $h' = e^2h$.

Thus after the first rebound and before the second rebound takes place, the particle takes time $2et$ and describes the total distance up and down equal to $2e^2h$.

The same process will be repeated subsequently till the particle comes to rest.

Thus the whole time taken by the particle during the motion

$$= t + 2et + 2e^2t + \ldots\ldots = t + 2t(e + e^2 + \ldots\ldots)$$
$$= t + \frac{2et}{1-e} = \left(1 + \frac{2e}{1-e}\right)t = \frac{1+e}{1-e}t = \frac{1+e}{1-e}\sqrt{\frac{2h}{g}} \quad \text{[by (3)]},$$

and the whole distance described by it

$$= h + 2e^2h + 2e^4h + \ldots\ldots = h + \frac{2e^2h}{1-e^2} = \frac{1+e^2}{1-e^2}h.$$

Examples XXXII-A

1. A ball weighing 10 lb. and moving with a velocity of 30 ft. per sec. impinges on a smooth fixed plane in a direction making 60° with the plane. If the co-efficient of restitution is $\frac{1}{2}$ find

(*i*) the impulse on the plane due to the impact.

(*ii*) the direction of motion of the ball after impact.

2. At what angle must a body whose elasticity is $\frac{1}{2}$ be incident on a perfectly hard plane so that the angle between the directions before and after impact may be a right angle ?

3. An imperfectly elastic sphere whose elasticity is equal to tan 30° impinges upon a plane with a velocity such that

the velocity after impact = the velocity before impact × sin 45°,

find the angles of incidence and reflection.

4. A marble dropped on a stone floor from a height of 12 ft. is found to rebound to a height of 10 ft. Find the co-efficient of restitution.

5. A steel ball is released from rest and falls upon a fixed steel anvil and rebounds, the co-efficient of restitution being 0·9. The lowest point of the ball is initia y at a distance of one foot above the anvil. Show that the ball finally comes to rest on the anvil 4·75 secs. after its release and that the total distance travelled by the ball is $19\frac{1}{19}$ ft.

6. A tennis ball is dropped from a point P at a height H above the surface of the court and rebounds to a height h. Prove that the velocity v with which it must be projected vertically downwards from P in order that it may just return to P after rebound is given by the equation

$$hv^2 = 2gH(H-h).$$

Find the impulse at the point of impact with the court in the latter case.

7. A ball, whose co-efficient of restitution with the ground is $\frac{2}{3}$, is projected upwards from the ground with a velocity of 16 f.p.s. Find the total distance described by the ball before it comes to rest, and the time occupied in describing this distance.

8. An elastic ball of mass m is projected vertically upwards from a point on a horizontal plane with a velocity u. If e be the co-efficient of restitution, find the total space described by it and the time that elapses up to the instant of its nth rebound. What is the kinetic energy after the nth rebound ?

(*P.U.* 1955)

9. Two smooth balls, one of mass double that of the other, are moving with equal velocities in opposite parallel directions and impinge, their directions of motion at the instant of impact making angles of 30° with the line of centres. If $e = \frac{1}{2}$, show that the smaller ball behaves after impact as if it were a perfectly elastic ball impinging on a fixed plane.

10. Two perfectly elastic balls fall from rest from heights of 16 ft. and 9 ft. above a horizontal plane descending along the same vertical line. They impinge upon a plane inclined at an angle of 45° and move along the horizontal plane. Prove that they will collide after moving a distance 24 ft. on the horizontal plane.

10·42. A projectile impinging on a plane. A particle projected with a certain velocity in a certain direction, other than the horizontal and vertical, may strike a horizontal, a vertical or an inclined plane, and may continue its motion after rebounding from the plane.

Example 1. *From a point in a smooth horizontal plane a ball is projected with velocity u at an angle α to the horizon. Show that it will keep rebounding from the plane for a time $\frac{2u \sin \alpha}{g(1-e)}$ and will have a range $\frac{u^2 \sin 2\alpha}{g(1-e)}$, e being the co-efficient of elasticity.*

The ball at first describes a parabola and on meeting the horizontal plane, it rebounds and so describes another parabola, and again rebounds and thus describes a series of parabolas.

The initial vertical component of the velocity is $u \sin \alpha$ and vertical velocities of the ball at the successive rebounds are $eu \sin \alpha$, $e^2u \sin \alpha$, etc.

So the times of describing the various parabolic paths are

$$\frac{2u \sin \alpha}{g}, \frac{2eu \sin \alpha}{g}, \frac{2e^2u \sin \alpha}{g}, \text{ etc.}$$

Hence the total time that elapses before the ball ceases to rebound is

$$\frac{2u \sin \alpha}{g} + \frac{2eu \sin \alpha}{g} + \frac{2e^2u \sin \alpha}{g} + \ldots$$
$$= \frac{2u \sin \alpha}{g}(1 + e + e^2 + \ldots) = \frac{2u \sin \alpha}{g} \cdot \frac{1}{1-e}.$$

The plane being *smooth*, the horizontal velocity $u \cos \alpha$ of the ball remains unaltered during the motion.

$\therefore$ the horizontal distance described in this time

$$= \frac{2u \sin \alpha}{g(1-e)} \cdot u \cos \alpha = \frac{u^2 \sin 2\alpha}{g(1-e)}.$$

After the ball has ceased to rebound, it will move along the plane with constant velocity $u \cos \alpha$.

Example 2. *A particle after falling from rest through a distance h strikes a smooth plane inclined at an angle α to the horizon. Show that the distance between the first two points at which it strikes the plane is*

$$4he(1+e) \sin \alpha.$$

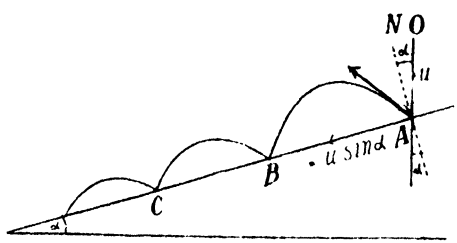
(i) Find this value when $e = \frac{1}{2}$, $\alpha = 60°$ and $h = 2$ ft.

(ii) Prove also that the whole range on the inclined plane when the particle ceases to rebound is $\dfrac{4he \sin \alpha}{(1-e)^2}$.

Let the particle start from O and strike the plane at A with velocity u, so that

$$OA = h,$$

and $$u^2 = 2gh \quad \ldots (1).$$



Just before impact at A, the velocity components of the particle are :—

$u \cos \alpha$ along the normal AN in the direction NA, and $u \sin \alpha$ along the plane downwards.

Just after the impact, they are :—

$eu \cos \alpha$ along the normal in the direction AN,

and $eu \sin \alpha$ along the plane, the plane being smooth this remains constant throughout for all impacts.

Considering motion in a direction perp. to the plane, the time t which the particle takes to strike the plane again at B is given by

$0 = eu \cos\alpha\, t - \frac{1}{2} g \cos\alpha\, t^2$, so that $t = 2eu/g$ ..(2)

Now considering motion in the direction of the plane, the range AB is given by

$$AB = u \sin\alpha .\, t + \tfrac{1}{2} g \sin\alpha .\, t^2 = u \sin\alpha . \frac{2eu}{g} + \tfrac{1}{2} g \sin\alpha . \frac{4e^2u^2}{g^2}$$

$$= \frac{2e \sin\alpha}{g}(1+e)u^2 = \frac{2e \sin\alpha}{g}(1+e) \times 2gh$$

$$= 4he(1+e) \sin\alpha \quad ..(3)$$

(*i*) Substituting $e = \frac{1}{2}$, $\alpha = 60°$ and $h = 2$ ft., we have

$$AB = 4 \times 2 \times \tfrac{1}{2} \times \frac{3}{2} \times \frac{\sqrt{3}}{2} = 3\sqrt{3} \text{ or } 5{\cdot}2 \text{ ft. nearly.}$$

(*ii*) Just after the impact at B, the velocity component perp. to the plane is $e^2u \cos\alpha$.

$\therefore$ as in (2), time for the second range $BC = \dfrac{2e^2u}{g}$,

and so on for other ranges.

$\therefore$ the whole time T till the particle ceases to rebound

$$= \frac{2eu}{g} + \frac{2e^2u}{g} + \ldots\ldots = \frac{2eu}{g}(1 + e + e^2 + \ldots\ldots) = \frac{2eu}{g(1-e)} .$$

$\therefore$ the whole range on the inclined plane in this time

$$= u \sin\alpha .\, T + \tfrac{1}{2} g \sin\alpha .\, T^2$$

$$= u \sin\alpha \times \frac{2eu}{g(1-e)} + \frac{g \sin\alpha}{2} \cdot \frac{4e^2u^2}{g^2(1-e)^2}$$

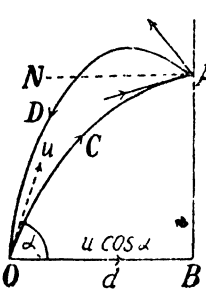
$$= \frac{2e \sin\alpha}{g(1-e)} \left[1 + \frac{e}{1-e} \right] u^2 = \frac{2e \sin\alpha}{g(1-e)} \times \frac{1}{1-e} \times 2gh$$

$$= \frac{4he \sin\alpha}{(1-e)^2}$$

Example 3. *A ball is projected with a velocity u at an elevation α, from a point distant d from a smooth vertical wall in a plane perpendicular to it. After rebounding from the wall it returns to the point of projection. Prove that*

$$u^2 \sin 2\alpha = gd\left(1 + \frac{1}{e}\right).$$

Let O, be the point of projection, AB, the wall and AN perpendicular to AB.



Let t_1 be the time for motion from O to A along the path OCA, and t_2 the time for the return journey from A to O along the path ADO, then

$d = t_1 \times$ horizontal component of the velocity before striking the wall

$= t_1 .\, u \cos\alpha$,

also $d = t_2 \times$ horizontal velocity after impact with the wall

$= t_2 .\, eu \cos\alpha$.

$$\therefore \quad (t_1+t_2)\ u\cos\alpha = d+\frac{d}{e} = d\left(1+\frac{1}{e}\right) \qquad ..(1)$$

The wall being smooth, the vertical component $u \sin \alpha$ of the velocity is unaffected by the impact and the final vertical displacement of the ball being zero, we have

$$0 = u\sin\alpha.\ (t_1+t_2) - \tfrac{1}{2}g(t_1+t_2)^2, \quad \text{or} \quad t_1+t_2 = \frac{2u\sin\alpha}{g} \qquad ..(2).$$

Substituting this in (1), wet get

$$\frac{2u\sin\alpha}{g}.\ u\cos\alpha = d\left(1+\frac{1}{e}\right) \quad \text{or} \quad u^2\sin 2\alpha = gd\left(1+\frac{1}{e}\right).$$

Note. In case u and α are known, the co-efficient of restitution between the ball and the wall, as obtained from this equation, is given by

$$e = gd/(u^2\sin 2\alpha - gd).$$

Examples XXXII-B

1. A particle of elasticity e is projected in a direction inclined to the vertical and bounces on the smooth horizontal plane. The range of one rebound is r; find the range of the next. [Ans. er.] *(P.U. 1957)*

2. A particle is projected from a point in a horizontal plane with a velocity of 32 ft. per sec. at an angle of 30° with the horizon. If the co-efficient of restitution is $\frac{3}{4}$, find the distance described by it horizontally before it ceases to rebound. [Ans. 111 ft. nearly.] *(D.U.)*

3. A ball is projected from a point in a horizontal plane and makes one rebound, show that if the second range is equal to the greatest height which the ball attains, the angle of projection is $\tan^{-1} 4e$, e being the co-efficient of elasticity.

4. A smooth elastic ball is projected horizontally from the top of a tower 100 ft. high with a velocity of 100 f.p.s.; show that after one rebound it will describe a horizontal range of 40 ft. if the co-efficient of elasticity is $\frac{2}{5}\sqrt{5}$.

5. A ball falls vertically for 2 seconds and hits a plane inclined at 30° to the horizon. If the co-efficient of restitution be 3/4, show that the time that elapses before it again hits the plane is 3 seconds.

6. A ball is projected in a plane perpendicular to a smooth vertical wall so that after rebounding, it passes through the point of projection. If the point is at a maximum distance d from the wall, prove that V the velocity of projection is given by $V^2 = gd\left(1+\frac{1}{e}\right)$, where e is the co-efficient of restitution.

7. A ball projected from a point O hits a vertical wall, rebounds and passes through the point 16 ft. above O two seconds after projection. If the distance of the wall from O is $30\sqrt{3}$ ft. and the co-efficient of restitution is $\frac{3}{5}$, find the magnitude and the direction of the initial velocity of the ball.

Find also the height above the level of O, of the point at which the ball hits the wall. [Ans. 80 ft./sec. at 30° with horizon ; 21 ft.]

8. A particle is projected from a point in a smooth horizontal plane so as to strike a smooth vertical wall at right angles, and after rebounding from the wall, and once from the horizontal plane, returns to the point of projection; prove that the co-efficient of elasticity is $\frac{1}{2}$. *(P.U.)*

9. A perfectly elastic ball is projected from a point in a plane of inclination α; show that θ, the angle of projection with the horizon in order that after striking the plane, the ball may be reflected vertically upwards is given by

$$2\tan\theta = 3\tan\alpha + \cot\alpha.$$

10. A smooth elliptical tray is surrounded by a smooth vertical rim, prove that a perfectly elastic particle projected from a focus along the tray in any direction will after two impacts return to the same focus.

11. A small sphere, moving with velocity V on a smooth horizontal plane towards a fixed vertical plane in a line at right angles to it, impinges directly upon an equal sphere at rest at a distance a from the vertical plane. The co-efficient of restitution between the spheres and also between a sphere and the vertical plane is $\frac{1}{2}$.

Prove that there are only two impacts between the spheres, the interval between them being $12a/5V$. Also find the final velocity of the first sphere.

[Ans. $7V/32$.]

10·5. Kinetic energy lost by impact. In general there is always a loss of kinetic energy whenever two bodies impinge.

We have, by Algebra

$$(m+m')(mu^2+m'u'^2)=(mu+m'u')^2+mm'(u-u')^2, \quad ..(1)$$

and $$(m+m')(mv^2+m'v'^2)=(mv+m'v')^2+mm'(v-v')^2 \quad ..(2)$$

(1) and (2) are mere identities.

I. **Direct Impact.** In the case of direct impact of two spheres as given in Art. 10·2, we have

$$mv+m'v'=mu+m'u' \quad ..(3)$$

and $$v-v'=-e(u-u') \quad ..(4)$$

Subtracting (2) from (1), dividing by $2(m+m')$ and using (3) and (4), we have

$$\text{Loss in K.E.}=(\tfrac{1}{2}mu^2+\tfrac{1}{2}m'u'^2)-(\tfrac{1}{2}mv^2+\tfrac{1}{2}m'v'^2)$$

$$=\tfrac{1}{2}\frac{mm'}{m+m'}(u-u')^2(1-e^2). \quad ..(5)$$

Since $e<1$, this is positive and there is always a loss of Kinetic energy. This vanishes when $e=1$, *i.e.*, when the spheres are perfectly elastic, there is no loss of energy.

II. **Oblique Impact.** When the collision is oblique, we have

$$\text{Loss in K.E.}=(\tfrac{1}{2}mu^2+\tfrac{1}{2}m'u'^2)-(\tfrac{1}{2}mv^2+\tfrac{1}{2}m'v'^2)$$

$$=\tfrac{1}{2}m(u^2\cos^2\alpha+u^2\sin^2\alpha)+\tfrac{1}{2}m'(u'^2\cos^2\beta+u'^2\sin^2\beta)$$

$$-\tfrac{1}{2}m(v^2\sin^2\theta+v^2\cos^2\theta)-\tfrac{1}{2}m'(v'^2\sin^2\varphi+v'^2\cos^2\varphi).$$

But the components of the velocities at right angles to the line of centres are unaltered by the impact and thus, as in § 10·3, we have

$$v\sin\theta=u\sin\alpha \quad \text{and} \quad v'\sin\varphi=u'\sin\beta.$$

Hence loss in K.E.

$$=\tfrac{1}{2}mu^2\cos^2\alpha+\tfrac{1}{2}m'u'^2\cos^2\beta-\tfrac{1}{2}mv^2\cos^2\theta-\tfrac{1}{2}m'v'^2\cos^2\varphi.$$

It follows that it is only velocity components in the line of centres that can affect a change in the Kinetic energy of the spheres

Hence proceeding as before in the case of the direct impact and writing respectively, $u\cos\alpha$, $u'\cos\beta$, $v\cos\theta$ and $v'\cos\varphi$ for u, u', v and v' in the four equations, we obtain a result similar to (5).

$$\therefore \quad \text{Loss in K.E.}=\tfrac{1}{2}\frac{mm'}{m+m'}(u\cos\alpha-u'\cos\beta)^2(1-e^2).$$

Note. If $u'=0$ and $m'\to\infty$, the result is the same as (5) in Art. 10·41 for the loss of energy of a sphere striking a fixed plane.

10·6. Compression and Restitution. When a wet tennis ball falls on to a hard floor or strikes a wall, it leaves a mark which is not merely a point, but a circle of considerable area. It shows that at the time of impact, the ball must have undergone a deformation near the point of contact. Similarly when two elastic spheres of

any hard material impinge on each other, each is slightly compressed by the other, so that they are no longer perfect spheres. Finally each resumes its original shape and separates from the other. The period of impact may thus be divided into two parts :

(*i*) **The Period of Compression,** during which the distance between the centres of the spheres diminishes until the distance is a minimum at the *moment of the greatest compression.*

(*ii*) **The Period of Restitution,** during which the spheres resume their original shape and separate completely.

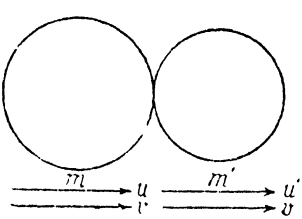
At the instant of greatest compression the distance between the centres of the spheres is minimum and, therefore, its rate of change is then zero and consequently the relative velocity of the spheres along their common normal vanishes. Thus *at the time of greatest compression, the components of the velocities of the spheres along the line of the centres must be the same.*

***10·7. Impulses of Compression and Restitution.** Whenever two spheres impinge, the whole time of action between the spheres is so exceedingly short that it is not possible to follow the process in detail. Yet during the momentary contact, forces are called into play rising from zero to a high value and dying away to zero again, so that a great and instantaneous change of velocities takes place. We now find the relation between these forces and the changes in the velocities of the two spheres.

Let m, m' be the masses of the spheres, u and u' their velocities before impact, and v, v' their velocities after impact, the velocities being all measured in the same direction.

Let U be the common velocity of the spheres at the time of the greatest compression.

The time from the instant at which the spheres touch to the instant of the greatest compression is so small that the forces producing the changes in the velocities of the spheres may be regarded as impulsive. In whatsoever manner these forces may change during the impact, they, being of the nature of the action and reaction, must be equal and opposite at a particular instant.



Thus, if I is impluse of compression on the sphere of mass m, the impulse on the sphere of mass m' will be $-I$ and we have

$$\left.\begin{aligned} I &= \text{change of momentum in } m = m(U-u) \\ -I &= \text{change of momentum in } m' = m'(U-u') \end{aligned}\right] \quad \ldots(1)$$

$$\therefore \quad \frac{I}{m} + \frac{I}{m'} = U-u-(U-u') = u'-u \quad \ldots(2)$$

After the instant of the greatest compression, the period of restitution begins and another system of forces comes into play until the spheres recover their shape and separate from each other with velocities v and v'.

If I' be the impulse of restitution, then

I'=change in momentum in $m=m(v-U)$

and $-I'$=change in momentum in $m'=m'(v'-U)$ ..(3)

$$\therefore \quad \frac{I'}{m}+\frac{I'}{m'}=v-U-(v'-U)=v-v' \qquad ..(4)$$

From (2) and (4), we have

$$\frac{I'}{I}=\frac{v-v'}{u'-u}=-\frac{v-v'}{u-u'}=-(-e) \text{ by Newton's law.}$$

Hence $I'=eI$

i.e., *Impulse of restitution* $=e\times$ *impulse of compression.*

Conversely if $I'=eI$, then from (2) and (4), we have

$$v-v'=e(u'-u)=-e(u-u').$$

Total impulse $=I+I'=(1+e)I=(1+e)\dfrac{mm'(u'-u)}{m+m'}$, from (2).

Also from (1), $m(U-u)=-m'(U-u')$

i.e., $(m+m')U=mu+m'u'$

Similarly from (3),

$$(m+m')U=mv+m'v'$$

Hence $$U=\frac{mu+m'u'}{m+m'}=\frac{mv+m'v'}{m+m'}.$$

Note. These equations prove the fact that the total momentum remains the same throughout the motion.

10·8. Kinetic Energy lost by direct impact during the two periods.

U, the common velocity of the spheres at the time of the greatest compression is given by

$$U=\frac{mu+m'u'}{m+m'}=\frac{mv+m'v'}{m+m'}$$

$\therefore$ E_1=loss of K.E. during the period of compression

$$=\tfrac{1}{2}mu^2+\tfrac{1}{2}m'u'^2-\tfrac{1}{2}(m+m')U^2$$

$$=\tfrac{1}{2}\left[mu^2+m'u'^2-\frac{(mu+m'u')^2}{m+m'}\right]=\tfrac{1}{2}\frac{mm'}{m+m'}(u-u')^2$$

[by (1) § 10·5]

E_2=loss of K.E. during the period of restitution

$$=\tfrac{1}{2}(m+m')U^2-\tfrac{1}{2}mv^2-\tfrac{1}{2}m'v'^2$$

$$=-\tfrac{1}{2}\left[mv^2+m'v'^2-\frac{(mv+m'v')^2}{m+m'}\right]$$

$$=-\tfrac{1}{2}\frac{mm'}{m+m'}(v-v')^2=-\tfrac{1}{2}\frac{mm'}{m+m'}e^2(u-u')^2=-e^2E_1.$$

The negative sign shows that a part of the energy lost in the first period is regained in the second period.

Total loss in energy $=E_1+E_2=E_1-e^2E_1=(1-e^2)E_1$

$$=\tfrac{1}{2}\frac{mm'}{m+m'}(u-u')^2(1-e^2),$$

this result is the same as obtained in § 10·5.

Examples XXXIII

1. Two elastic spheres, each of mass m, collide directly. Show that the energy lost during the impact is $\frac{1}{4}m(u^2-v^2)$, where u and v are the relative velocities before and after impact.

If the velocity of one of the spheres is exactly reversed by the impact, and if the co-efficient of restitution is $\frac{1}{3}$, show that the energy lost is four times that of the sphere whose velocity is reversed.

2. A tram car weighing 10 tons and running at 10 m.p.h. overtakes and collides with another tram car weighing 10 tons and running at $9\frac{1}{2}$ m.p.h. After the impact the cars separate with a relative velocity of $\frac{1}{2}$ f.p.s. Find the amount of energy lost in the collision. (*Ans.* $50\frac{1}{3}\frac{5}{6}$ ft. lb.)

3. Two equal smooth billiard balls, whose co-efficient of elasticity is e, moving with equal velocites v in opposite directions, impinge obliquely, the line of centres on impact being inclined at 45° to the direction of motion.

Prove that the loss of energy by impact is half what it would have been had the impact been direct.

4. Given the masses, the co-efficient of elasticity and the magnitude but not the direction of the velocity of each of the two smooth spheres, show that the loss of kinetic energy is greatest when the impact is direct and the spheres are travelling towards each other.

5. Two smooth homogeneous spheres whose velocities are u and v and masses m, m' impinge directly. Prove that the momentum transferred from one to the other or the impulse which each receives is

$$\frac{mm'}{m+m'}(u-v)(1+e),$$

where e is the co-efficient of elasticity.

6. Explain the action that takes place between two elastic bodies during collision.

If the impulsive force during the period of restitution bears a constant ratio e to the impulsive force during the period of compression, show that

(*i*) the relative velocity after impact is $-e$ times the relative velocity before impact, and

(*ii*) the momentum of the system is the same after impact as before.
(*P.U.*)

REVISION QUESTIONS—IX

1. Prove that when two smooth spheres impinge directly, the K.E. is lost by impact, unless the elasticity is perfect. (*P.U. 1954*)

2. Describe the action that takes place when one elastic body impinges on another. Show that the ratio of the impulses of the forces of restitution and compression is equal to the co-efficient of restitution. (*P.U. 1955*)

3. Two balls, each of mass 1 lb., are placed on a smooth horizontal plane and connected by a light elastic string of natural length 6 ft. and modulus 6 lb. wt. The balls are drawn apart so that the distance between them is 8 ft. and then let go. Find the velocity of the balls when the string becomes slack and show that if the co-efficient of elasticity is $\frac{1}{2}$, the balls return, after impact, with half of this velocity, and that the subsequent maximum elongation of the string is one foot. [*Ans.* 8 ft./sec.]

4. A small ball hangs from a point in a vertical wall by a string of length l and rests in contact with the wall. The ball is pulled back in a vertical plane perpendicular to the wall, the string making an angle α with the vertical, and let go. If e be the co-efficient of restitution show that the angular amplitude of the rebound after n impacts is

$$2\sin^{-1}(e^n \sin \tfrac{1}{2}\alpha).$$

5. An imperfectly elastic sphere of mass m, moving with velocity u impinges directly on another sphere of mass m' at rest. The second sphere afterwards strikes a vertical cushion at right angles to its path : show that there will be no further impact of the spheres if $m(1+e'+ee')<em'$, where e is the co-efficient of restitution between the spheres and e' that between the second sphere and the cushion. *(Bm. U.)*

6. A ball is projected from the middle point of one side of a billiard table so as to strike first an adjacent side, and then the middle point of the side opposite to that from which it started ; show that the ball will hit the other adjacent side at a distance $\frac{be}{1+e}$ from the end nearest the opposite side, b being the length of the adjacent side.

7. A square table $ABCD$ whose side is a has raised edges. A particle of elasticity e is projected from a point P in AB and hits the sides BC, CD, DA in Q, R, S ; prove that the paths PQ and RS are parallel.

If α be the angle QPB and BP be x, prove that, if the particle returns to P,

$$x(1-e)=a(1-e\cot\alpha). \qquad (D.U.\ 1959)$$

8. A smooth circular table is surrounded by a smooth rim whose interior surface is vertical. Show that a ball projected along the table from a point on the rim in a direction making an angle θ with the radius through the point, will return to the point of projection after

(*i*) two impacts if $\tan\theta=\frac{e^{\frac{3}{2}}}{\sqrt{1+e+e^2}}$ and that the time of describing the first chord is e-times the time for the last chord.

(*ii*) three impacts if $\tan\theta=e^{\frac{3}{2}}$, e being the co-efficient of elasticity.

9. A smooth sphere, of mass m, is tied to a fixed point by an inextensible string and another sphere, of mass m', impinges directly on it with velocity v in a direction making an acute angle α with the string. Show that m begins to move with velocity

$$\frac{m'(1+e)\sin\alpha}{m+m'\sin^2\alpha}v. \qquad (P.U.\ Hons.)$$

10. A particle of elasticity e goes bounding down an endless flight of stairs each of height h and breadth b. It hits each stair at an exactly similar point at the same angle and with the same speed. Find the angle of impact and show that the normal velocity of impact is

$$\sqrt{2gh/(1-e^2)}.$$

[*Ans.* $\tan^{-1}[b(1-e)/2h]$ with the vertical.]

11. In a certain game a ball is rolled along a horizontal plane until it strikes an inclined plane of inclination θ from which it rebounds. The object of the game is to make the ball after rebounding fall into a hole in the inclined plane at a distance d from its junction with the horizontal plane. Show that in order that the ball may enter the hole, its velocity of projection V must be given by

$$gd=2e\,V^2\sin\theta\,(1-e\tan^2\theta).$$

12. A perfectly elastic ball is projected from the foot of a plane inclined at an angle θ to the horizon. If after striking the plane at a distance l from the point of projection it rebounds and retraces its former path, show that the velocity of projection is

$$\sqrt{\frac{gl(1+3\sin^2\theta)}{2\sin\theta}} \qquad (Agra\ U.\ ;\ P.U.\ Hons.)$$

13. If two balls of masses m, m' moving with velocities u, u' impinge directly prove that the condition that each loses the same amount of kinetic energy is

$$(3+e)(mu+m'u')+(1-e)(mu'+m'u)=0.$$

14. Two equal smooth spheres of radius r, move with the same speed in opposite directions along parallel lines which are at a distance d apart. Prove that the direction of motion of each deviates, on impact, through a right angle if

$$d^2(1+e)=4er^2.$$

15. An elastic ball is projected from a point at the foot of one of the two smooth parallel vertical walls so that after three impacts it may return to the point of projection. If the last impact be direct, show that

$$e^3+e^2+e=1.$$

16. A heavy perfectly elastic particle is dropped from a point P on the inside surface of a smooth sphere. Prove that the second point of impact on the surface of the sphere will be in the same horizontal plane as the first if the angular distance of P from the highest point of the sphere is

$$\cos^{-1}\{(2^{\frac{1}{2}}+1)^{\frac{1}{2}}/2\}. \qquad (D.U.\ 1961)$$

CHAPTER XI

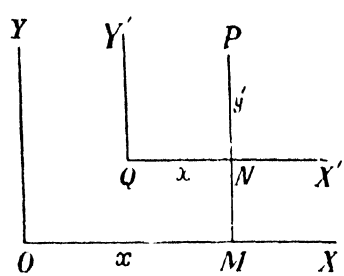
MISCELLANEOUS PROBLEMS

1.—RELATIVE MOTION

11·1. Rest and motion relative terms. When we say that a body is at rest, we simply mean that the body has no velocity with respect to another on which the body stands, as for instance, the surface of the earth. But really the body is not at rest as it is partaking in the motion of rotation of the earth about its axis and the motion of the whole earth round the sun. Again, the sun itself with the whole solar system is moving through space towards the star Vega. Vega, the sun and a large number of stars are themselves supposed to be moving with respect to another system of stars and so on. There is thus no absolutely fixed point from which we can measure our motion, so that *all motion is relative*. It is, therefore, unscientific to distinguish between rest and motion as between two different states of a body in itself, since it is impossible to speak of a body being at rest or in motion except with reference, expressed or implied, to some other body.

11·2. Relative Velocity. The velocity of a point P relative to a point Q is the rate of change of position of P with respect to Q or in other words the velocity of P relative to Q is the velocity which P appears to have when viewed from Q.

The relative velocity of two bodies moving along a straight line was discussed in the first chapter. We now consider the relative velocity of bodies moving over a plane.



Let the position of the point P at any time t be defined by the co-ordinates (x, y) relative to two perpendicular axes through a point O (supposed fixed) and let the co-ordinates of the point Q at the same time with reference to the same axes be (ξ, η).

If (x', y') are the co-ordinates of P relative to the axes through Q parallel to those through O, then

$$x' = x - \xi \quad \text{and} \quad y' = y - \eta \qquad ..(1)$$

Differentiating, we have

$$\frac{dx'}{dt} = \frac{dx}{dt} - \frac{d\xi}{dt} \quad \text{and} \quad \frac{dy'}{dt} = \frac{dy}{dt} - \frac{d\eta}{dt} \qquad ..(2)$$

which shows that when all velocities are measured relative to a given point, the velocity components of the point P relative to the moving point Q are obtained by adding to the actual velocity components of

P, the actual velocity components which are equal and opposite to those of Q.

The velocity of P relative to Q may, therefore, be regarded as the resultant of two velocities, *viz.*,

(*i*) the actual velocity of P,

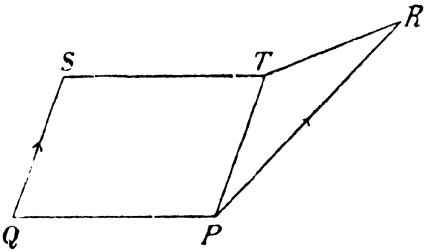
and (*ii*) the reversed velocity of Q.

Thus, we have the following :—

The velocity of P relative to Q is obtained by compounding the absolute velocity of P with the reversed velocity of Q.

11·21. This can also be proved geometrically as follows :—

Let the velocities of P and Q be represented in direction and magnitude by the lines PR and QS. Complete the parallelogram $PQST$ and join TR.



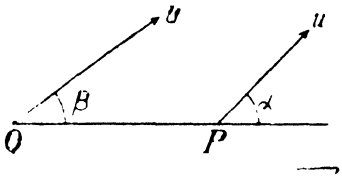
By triangle of velocities, the velocity PR is equivalent to two velocities represented by PT and TR. But PT, being parallel and equal to QS, represents the velocity of Q. Thus the velocity of P is equivalent to two velocities, one represented by PT, equal and parallel to that of Q, and the other represented by TR.

Hence TR represents the velocity of P relative to Q.

Again, TR is equivalent to PR, the velocity of P, and TP, a velocity equal and opposite to that of Q.

Hence the velocity of P relative to Q is obtained by compounding with the actual velocity of P, a velocity equal and opposite to that of Q.

11·22. Thus, if the velocities of P and Q be u and v in directions making angles α and β with the axis of x and if V be the magnitude and θ the direction of the velocity of P relative to Q, then



$$V\cos\theta = u\cos\alpha - v\cos\beta$$

and $$V\sin\theta = u\sin\alpha - v\sin\beta$$

$$\therefore \quad \tan\theta = \frac{u\sin\alpha - v\sin\beta}{u\cos\alpha - v\cos\beta},$$

and $$V^2 = (u\cos\alpha - v\cos\beta)^2 + (u\sin\alpha - v\sin\beta)^2$$

$$= u^2 + v^2 - 2uv\cos(\alpha - \beta).$$

These equations give the magnitude and direction of the relative velocity.

11·23. Equations (2) Art. 11·2, may also be written in the form

$$\frac{dx}{dt} = \frac{d\xi}{dt} + \frac{dx'}{dt} \quad \text{and} \quad \frac{dy}{dt} = \frac{d\eta}{dt} + \frac{dy'}{dt}$$

i.e., the actual velocity of P is obtained by compounding with the velocity of Q, the velocity of P relative to Q.

Example 1. *A railway train running at the rate of 60 miles per hour is struck by a stone moving at right angles to the train with a velocity of 33 ft. per sec. Find the velocity with which the stone appears to strike the train.*

Let V be the required velocity in a direction making an angle θ with the direction of the train, then since 60 m.p.h.$=88$ f.p.s.

$$\therefore \quad V\cos\theta = 0-88 = -88$$

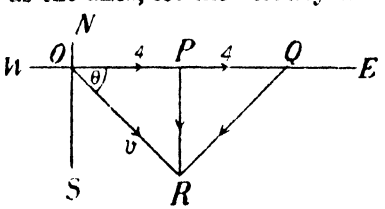
and $$V\sin\theta = 33-0 = 33$$

$$\therefore \quad V = \sqrt{88\times 88+33\times 33} = 11\sqrt{73} = 93{\cdot}9 \text{ ft. per sec.,}$$

and $$\tan\theta = \frac{33}{-88} = -\frac{3}{8}.$$

Example 2. *A person going eastward with a speed of 4 m.p.h. finds that the wind appears to blow directly from the north. He doubles his speed and the wind seems to come from N.E. In what direction and with what velocity is the wind blowing?*

Let OE and OS drawn in the east and south directions be taken as the axes, let the velocity of the wind be v, and let it be blowing in a direction making an angle θ with OE, then the components of this velocity are $v\cos\theta$, $v\sin\theta$.



Components of the velocity of the man are 4, 0.

$\therefore$ components of the relative velocity are

$$v\cos\theta - 4, \qquad v\sin\theta - 0.$$

$\therefore$ in the first case when the wind appears to come from the north, *i.e.*, the relative velocity is perp. to OE,

its component along OE must vanish.

$$\therefore \quad v\cos\theta - 4 = 0 \qquad i.e., \quad v\cos\theta = 4 \qquad \ldots(1)$$

In the second case when the man doubles his speed, the components of the relative velocity are $v\cos\theta - 8$ and $v\sin\theta - 0$.

$$\therefore \quad \frac{v\sin\theta - 0}{v\cos\theta - 8} = \tan\angle EQR = \tan 135^\circ = -1$$

i.e., $$v\sin\theta = -v\cos\theta + 8 = -4+8 = 4 \qquad \ldots(2)$$

$\therefore$ from (1) and (2), we have $v = 4\sqrt{2}$ and $\theta = \tan^{-1}1 = 45^\circ$

i.e., the wind is actually blowing from the north-west.

Or thus: the man's speeds are represented by OP and OQ and let OR represent the velocity of the wind, then

OR compounded with PO and QO *i.e.*, OP and OQ taken in the reversed directions, gives the relative velocities PR and QR respectively.

But PR is given to be from the north, so that

$$\angle OPR = 90^\circ = \angle QPR,$$

and QR is given to be from the north-east, so that

$$\angle PQR = 45°.$$

$$\therefore \angle PRQ = 45° \text{ and } PR = PQ,$$

but $PQ = OP$, $\therefore$ $OP = PR$ and $\angle POR = 45°$.

Hence $OR = OP \sec 45° = 4\sqrt{2}$.

Example 3. *Two motor cars A and B are moving uniformly on two straight roads at right angles to one another at 40 and 20 miles per hour respectively. A passes the intersection of the roads when B has still to move 50 miles to reach it. How much time will elapse before they are close together and what is the shortest distance between them?*

If z is the distance between the cars after t hours, then

$$z^2 = (40t)^2 + (50 - 20t)^2,$$

and $$z\frac{dz}{dt} = 40 \times 40t - 20(50 - 20t),$$

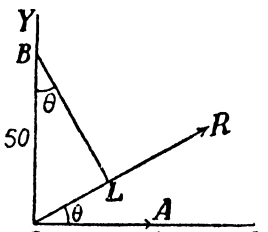
$$= 2000t - 1000$$

z is least when $\frac{dz}{dt} = 0$,

i.e., when $t = \frac{1}{2}$ hour $= 30$ mins.,

which gives $z = 20\sqrt{5} = 44{\cdot}7$ miles,

the shortest distance between the cars.



Or thus :—Let O be the position of A when B is 50 miles from O initially. A is moving along OX and B is moving along YO. In this position let a velocity of 20 m.p.h. in the direction OY be impressed upon both of them.

B is reduced to rest and A moves with a velocity of $\sqrt{40 \times 40 + 20 \times 20} = 20\sqrt{5}$ m.p.h. along OR inclined at an angle

$$\theta = \tan^{-1}\tfrac{40}{20} = \tan^{-1}\tfrac{1}{2} \text{ with } OX.$$

OR is the direction of the relative velocity of A with respect to B.

$\therefore$ the shortest distance between the cars

$= BL$, perpendicular from B on OR

$$= OB \cos\theta = 50 \times \frac{2}{\sqrt{5}} = 20\sqrt{5} = 44{\cdot}7 \text{ miles.}$$

The time t when it occurs

= time to describe OL with the relative velocity.

$$= \frac{OL}{20\sqrt{5}} = \frac{1}{20\sqrt{5}} \times OB \sin\theta = \frac{1}{20\sqrt{5}} \times \frac{50}{\sqrt{5}} = \tfrac{1}{2} \text{ hour} = 30 \text{ mins.}$$

Example 4. *If the distance between two points P and Q at any time be a, if V be their relative velocity, and if u and v be the components of V respectively in, and perpendicular to the direction of a, show that their distance when they are nearest to one another is av/V, and that the time that elapses before they arrive at their nearest distance is au/V².*

In the Fig. of Ex. 3, let O be the position of P and B that of Q, so that $OB = a$.

Let OR be the path of P relative to Q, so that when Q is

supposed to be at rest at B, P moves along OR with relative velocity V, then by the question

$V \sin\theta$ = component of V in the direction of a, *i.e.*, along $OB = u$,

$V \cos\theta$ = component of V perp. to a, *i.e.*, along $OX = v$.

Now the shortest distance between them

$$= BL = a\cos\theta = a.\frac{v}{V} = \frac{av}{V},$$

and the time to reach this position

$$= \frac{OL}{V} = \frac{a\sin\theta}{V} = \frac{a}{V}\,\frac{u}{V} = \frac{au}{V^2}.$$

***11·3. Velocity of a shot fired from a gun free to recoil horizontally.** Let a shot of mass m be fired from a gun of mass M at rest on a horizontal plane. Let the barrel be inclined at an angle α to the horizon and let u be the velocity of the shot relative to the gun. This velocity, being in the direction of the bore of the gun, will be inclined at the angle α to the horizon.

Let the gun recoil with a velocity v. Then the actual horizontal velocity of the shot = its horizontal velocity relative to the gun ($u\cos\alpha$) + the velocity of the gun ($-v$)

$$= u\cos\alpha - v.$$

Also the vertical component of the actual velocity of the shot

$$= u\sin\alpha + 0 = u\sin\alpha.$$

If θ be the inclination of the actual direction of motion of the shot at the time of firing, then

$$\tan\theta = \frac{\text{vertical vel. of the shot}}{\text{its horizontal vel.}} = \frac{u\sin\alpha}{u\cos\alpha - v} \quad ..(1).$$

Now by the principle of conservation of momentum in the horizontal direction, we have

$$m(u\cos\alpha - v) - Mv = 0$$

i.e.,
$$v = \frac{mu\cos\alpha}{m+M} \quad ..(2).$$

$$\therefore \quad u\cos\alpha - v = u\cos\alpha - \frac{mu\cos\alpha}{m+M} = \frac{Mu\cos\alpha}{m+M}.$$

Hence from (1), $\tan\theta = \dfrac{u\sin\alpha}{Mu\cos\alpha}(M+m) = \left[1+\dfrac{m}{M}\right]\tan\alpha.$

* **Ex. 1.** A cannon-ball of mass m is shot from a gun of mass M (which is free to recoil in a horizontal direction) so that its muzzle-velocity relative to the ground is V. Show that its greatest range is $\dfrac{V^2}{g}$ and is obtained by giving the gun an elevation of

$$\cot^{-1}\left[1+\frac{m}{M}\right].$$

[**Hint.** For the greatest range $\theta = 45°$.]

***Ex. 2.** A shot of mass m is fired from a gun of mass M with velocity u relative to the gun, show that the actual velocities of the shot and the gun are

$\frac{Mu}{m+M}$ and $\frac{mu}{M+m}$ respectively and show that the total K.E. generated is

$$\tfrac{1}{2}\frac{Mm}{M+m}u^2. \qquad \text{(Here take } x=0\text{)}$$

*Ex. 3. A gun of mass M free to move on smooth horizontal rails, fires a shell of mass m in the vertical plane midway between the rails. Assuming that the velocity of the shell relative to the gun is of magnitude u whatever the elevation of the gun, show that t the time taken by the shell to reach a point at horizontal and vertical distances a, b, from the initial position of the gun is given by

$$\left[\frac{(M+m)a}{M}\right]^2+(b+\tfrac{1}{2}gt^2)^2=u^2t^2.$$

Examples XXXIV

1. A train moving at the rate of 30 m.p.h. is struck by a stone moving at right angles to the train with a velocity of 33 f.p.s. Find the velocity with which the stone appears to strike the train.

2. A ball of mass 8 ounces after falling vertically for 40 ft. is caught by a man in a motor car travelling horizontally at 30 m.p.h. Find the inclination to the vertical at which it will appear to him to be moving and the magnitude of the impluse on the hand when the ball is caught.

3. To a passenger in a train moving at 45 m.p.h. the rain though falling vertically appears to make an angle of 60 degrees with the vertical. Find the velocity of the rain.

4. To a man walking at the rate of 4 miles an hour rain appears to fall vertically. If its real velocity is 8 miles an hour, find its real direction of motion. (*P.U.*)

5. A road and a railway line cross at right angles. If at a certain instant a car is approaching the crossing at 30 ft./sec. and a train at 40 ft./sec. find the magnitude and direction of the velocity of the train relative to the car at the said instant. (*P.U. 1951*)

6. Two railway lines are inclined at an angle of 60° ; two engines run, one on each line from the point of intersection of the lines, at the rate of 20 m.p.h. Find their relative velocity in magnitude and direction.

7. Two motor cars are moving uniformly along straight roads making an angle of 60° with each other, with velocities of 20 and 12 miles per hour. If the first car be moving towards and the second away from the junction of the roads, find the relative velocity of the first with respect to the second.

8. A cyclist travelling at 10 m.p.h. is overtaken by a car moving at 20 m.p.h. as they both arrive at a cross-road. The cyclist keeps straight on, but the motorist proceeds along the road that makes 120° with his previous line of motion. With what velocity and in what direction does the car appear to the cyclist to move ?

9. A steamship is travelling north at the rate of 15 m.p.h. and there is a north-east wind blowing at the rate of $10\sqrt{2}$ m.p.h. Find the direction in which the smoke from the funnel appears to move to an observer in the ship.

10. To a cyclist going at 9 m.p.h. due east, the wind seems to blow from a direction 60° south of west at 6 m.p.h. Find the true direction and velocity of the wind.

11. A ship is sailing due west and the apparent direction of the wind as shown by the fluttering of the flag on the mast, is from north. The wind is known to be blowing from a direction 30° east of north. Show that its velocity is double that of the ship.

12. Two ships are sailing, one towards the east with a velocity u and the other towards the south with a velocity v. The wind is blowing with velocity w in a direction making an angle α to the south of west. Find the angle between the streams of smoke of the ships and show that if $\alpha=45°$ and $u=v$, this angle is $\tan^{-1} u^2/w^2$.

13. To a passenger in an open car travelling at 32 m.p.h. the wind appears to come from a direction 30° to the right and from ahead at 8 m.p.h What is the true direction and velocity of the wind ?

14. To a man walking at 4 m.p.h. along a road running due west, wind appears to blow from the south, while to a cyclist travelling in the same direction at 8 m.p.h. it appears to come from the south-west. What is the true direction and velocity of the wind ?

15. To a man travelling due east, the wind appears to come from the noarth-east, but when he doubles his speed it appears to come from a direction $\tan^{-1}\frac{1}{2}$ north of east. Find the direction of the wind. (*P.U.*)

16. A person travelling towards the north-east finds that the wind appears to blow from the north but when he doubles his speed it seems to come from a direction inclined at an angle $\cot^{-1} 2$ on the east of north. Prove that the true direction of the wind is towards the east. (*P.U.*)

17. A motor car is proceeding due east and the motorist notices that the wind appears to come from a direction 30° east of north. When he drives due west at the same speed the wind appears to come from a direction 60° west of north, prove that the wind is actually blowing from a direction 30° west of the north with the same speed as that of the car.

18. ' ABC is a triangle right-angled at C, a particle P starts from A and moves along AC with uniform velocity u : a second particle Q starts from C at the same instant and moves along CB with uniform velocity v ; show that the shortest distance between P and Q will be

$$\frac{v.AC}{\sqrt{u^2+v^2}} \text{ after a time } \frac{u.AC}{u^2+v^2}.$$

19. Two scouts, cycling at 9 and 12 m.p.h. along straight roads which cross at right angles at O, are at some instant both at a distance of one mile from O. Find the shortest distance between the scouts in the subsequent motion.

(You may use a graphical or an analytical method). (*P.U.*)

[**Hint.** Both may be taken to be cycling towards O.]

20. A steamer A moving due north at 12 m.p.h. and another steamer B which moves at 20 m.p.h. is two miles due east of A. Show that the least time in which B can catch up A is $7\frac{1}{2}$ mins.

Find graphically or otherwise, the least distance apart of the vessels if B steams in a direction 30° north of west.

21. A foot-ball player A gets the ball at a point P and runs northwise with it at 24 ft./sec. At the same time an opponent B starts at a speed of 26 ft./sec. from a point Q due west of P to try to tackle. In what direction should he run ?

(*i*) Show that the velocity of B relative to A must be eastward. Find its value.

(*ii*) If $PQ=25$ ft., how long will B take to catch A ? (*Posts and Tele.*)

22. A bomber moves due east at 100 miles/hr. over a town X at a certain time. Six minutes later a pursuit plane sets off from a station Y which is 40 miles due south of X and flies north-east. If both maintain their course, find graphically or by calculation the velocity with which the plane must fly in order to overtake the bomber. If the pursuit plane flies at the rate of $150\sqrt{2}$ miles/hr. along its course, find its nearest approach to the bomber in miles.

(*Pub. Ser. Com.*)

23. Two straight roads OX and OY are inclined to each other at an acute angle α ; one car moves along XO with speed u while a second car moves along OY with speed v.

If the first car is at a distance d from O when the second is at O, show that the cars are at their least distance apart after time

$$\frac{(u+v\cos\alpha)\,d}{u^2+v^2+2uv\cos\alpha}.$$

Also show that when the cars are at the least distance apart, the ratio of their distances from O is

$$v+u\cos\alpha : u+v\cos\alpha.$$

24. Two particles are moving with velocities v and $2v$ respectively in the circumference of a circle. In what positions is their relative velocity greatest and least, and what values has it then ?

25. A train of length 880 ft. travels round a curve of radius $\frac{1}{2}$ mile at a speed of 60 m.p.h. Calculate the velocity of the front of the train relative to the rear.

26. A wheel of radius r is rolling (without slipping and in a vertical plane) in a straight line along a horizontal plane, its centre having a speed v. If P, Q are the ends of a diameter PQ, show that the velocity of P relative to Q is $2v$ in a direction perpendicular to PQ.

27. AB is the vertical diameter of a circle. Particles start simultaneously from the highest point A and any other point P and fall down along chords AP and PB separately ; show that the least distance between them is equal to the distance of P from AB.

28. One particle slides down a smooth inclined plane of inclination 30° and a second drops vertically, both starting together from the top of the plane. Find the relative velocity in direction and magnitude at the end of 4 secs.

29. The position of a point P is defined by $x=t$, $y=2t$ and that of another point Q by $x=3t$, $y=5t^2$. Show that the path of Q relative to P is a parabola and find the velocity of Q relative to P when $t=3$.

30. Two ships are steaming along straight courses with such constant velocities that they will collide unless their velocities are altered. Show that to an observer on either ship the other appears to be always moving directly towards him.

31. You are in a railway train standing at a station. Another train arrives on a parallel track from the opposite direction. As the second train is about to stop at the station, you feel as if your train has begun to move in its own direction. Why is it so ?

32. A shot of mass m is fired from a gun of mass M with a velocity u relative to the gun. Find the energy of the recoil and the velocity with which the shot would be fired if the gun was fixed. *(P.U.)*

[The barrel of the gun may be taken as horizontal.]

REVISION QUESTIONS—X

1. Define *relative* velocity, and from your definition deduce that

velocity of B=velocity of A *plus* the velocity of B relative to A,

plus denoting vector addition.

A wheel rolls along the ground with uniform angular velocity ω. Show that the velocity of any point P on its rim is $\omega.AP$ and is perpendicular to AP, A being the point of contact with the ground at any instant. *(P.U.)*

2. Three straight lines are equally inclined to one another in a plane, and three equal particles P, Q, R move along them with velocities u, v, w respectively. Show that the velocity of P relative to the centroid of the particles is

$$\tfrac{1}{3}(4u^2+v^2+w^2+2uv+2uw-vw)^{\frac{1}{2}}. \qquad (P.U.)$$

3. If particles P and Q are travelling in straight lines with uniform velocities u and v towards a point O, show that the least distance between them during the motion is $\frac{u\sim v}{v}p$, where p is the perpendicular distance of O from the direction of the velocity of P relative to Q when Q and P are equidistant from O. *(P.U.)*

4. A and B are projectiles describing parabolic paths under gravity. Prove that, relative to A, B appears to describe a straight line uniformly.

5. Two particles start simultaneously from the same point and move along two straight lines, one with uniform velocity u and the other from rest, with uniform acceleration f. Show that their relative velocity is least after a time $\frac{u\cos\alpha}{f}$ — and that the least relative velocity is $u\sin\alpha$, where α is the angle between the lines.

6. An aeroplane has a speed of u miles/hr. and a range of action x miles (*i.e.*, can carry enough petrol to go x miles out and x miles back) in calm weather. Prove that in a steady wind of v miles/hr., its range of action in a direction inclined at θ to the wind is

$$\frac{x(u^2-v^2)}{u(u^2-v^2\sin^2\theta)^{\frac{1}{2}}}$$

Find also the magnitude and direction of the maximum range.

[*Ans.* $x\sqrt{u^2-v^2}/u$, perp. to the wind].

7. An aviator flies round a triangular course each side of which is c miles long, while the wind blows at u m.p.h. parallel to a side, show that he takes

$$\frac{c(v+\sqrt{4v^2-3u^2})}{v^2-u^2}$$

hours to complete the circuit in either direction, v being his velocity relative to the air. (*P.U. Hons.*)

11·4. Relative Acceleration. Differentiating the equations given in Art. 11·23, we have

$$\frac{d^2x}{dt^2}=\frac{d^2\xi}{dt^2}+\frac{d^2x'}{dt^2},\quad \frac{d^2y}{dt^2}=\frac{d^2\eta}{dt^2}+\frac{d^2y'}{dt^2}$$

i.e., the actual acceleration of P is obtained by compounding with the actual acceleration of Q, the acceleration of P relative to Q.

***Example.** A particle of mass m slides from rest down the smooth face of inclination α, of a wedge. The wedge is of mass M and is free to move on a smooth horizontal table on which it rests. To determine the motion of the particle and of the wedge.*

The forces acting on the particle are :—

(*i*) its weight mg, vertical.

(*ii*) R, the pressure of the wedge perpendicular to the face (shown outwards in the figure).

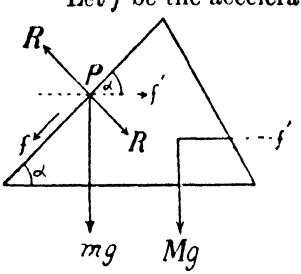
The forces acting on the wedge are :—

(*i*) its weight Mg, vertical,

(*ii*) R, the pressure of the particle on the wedge (shown inwards in the figure),

(*iii*) the pressure, say S, of the table on the wedge, vertical (the table being smooth).

Let f be the acceleration of the particle down the wedge relative to it and f', the horizontal acceleration of the wedge.



The acceleration of the particle relative to the ground is the resultant of the acceleration of the wedge and the acceleration of the particle relative to the wedge.

Therefore the components of the acceleration of the particle are—

(1) along the face of the wedge downwards

$$=f-f' \cos \alpha.$$

(2) perpendicular to the face$=f' \sin \alpha$, (inwards).

Hence equations of motion for m along and perpendicular to the face are

$$m(f-f' \cos \alpha)=mg \sin \alpha \quad ..(1),$$

and

$$mf' \sin \alpha=mg \cos \alpha-R \quad ..(2).$$

The horizontal component of R on the wedge being $R \sin \alpha$, its equation of motion is

$$Mf'=R \sin \alpha \quad ..(3).$$

Solving (2) and (3), we get

$$f'=\frac{mg \sin \alpha \cos \alpha}{M+m \sin^2 \alpha} \quad \text{and} \quad R=\frac{Mm}{M+m \sin^2 \alpha} \cdot g \cos \alpha.$$

From (1), we have $f=f' \cos \alpha+g \sin \alpha$

$$=\frac{M+m}{M+m \sin^2 \alpha} \cdot g \sin \alpha.$$

S, the reaction of the ground$=Mg+R \cos \alpha$

$$=\frac{M(M+m)}{M+m \sin^2 \alpha} g.$$

In the particular case when some external force, say F, for example the force of friction between the wedge and the table, is sufficient to stop the motion of the wedge, $f'=0$ and therefore from (1) and (2), we have

$$f=g \sin \alpha, \text{ and } R=mg \cos \alpha.$$

Also in this case $S=Mg+R \cos \alpha=Mg+mg \cos^2 \alpha=g(M+m \cos^2 \alpha)$ and the horizontal force F necessary to prevent the wedge from moving $=R \sin \alpha=mg \sin \alpha \cos \alpha$.

*Eamples XXXV

1. Two bodies start simultaneously from rest, one sliding down a smooth fixed inclined plane of inclination θ, and the other falling freely. Prove that their relative acceleration is $g \cos \theta$.

Hence show that either body as seen by a person moving with the other appears to be moving from the observer in a straight line perpendicular to the plane with uniform acceleration.

2. A wedge of mass 3 lb. and angle 30°, rests with one face on a smooth horizontal plane. If a particle of mass 1 lb., slides down another smooth face, determine the horizontal force necessary to prevent the wedge from moving.

3. The angle of a smooth wedge of mass M is α. The wedge is placed with one face on a smooth horizontal table and a particle of mass m is allowed to slide down its other face. Prove that a horizontal force $mg \sin \alpha \cos \alpha$ must be applied to the wedge to keep it from moving, and find the reaction between the wedge and the table.

4. A particle is placed on the face of a smooth wedge which can slide on a horizontal table ; find how the wedge must be moved in order that the particle may neither ascend nor descend.

Find in this case, the pressure between the particle and the wedge.

5. A particle of mass 2 lb. is placed on the smooth face of an inclined plane of mass 7 lb., and slope 30°, which is free to slide on a smooth horizontal

plane. Show that if the system starts from rest, the particle will slide down a distance of 15 feet along the face of the plane in 1·25 seconds. (*P.U.*)

6. A body of mass m with two plane smooth faces inclined at an angle α ($<\pi/2$) is placed with one face in contact with a smooth horizontal table, and a particle of mass $2m$ slides down its other face under gravity. Show that the horizontal acceleration of the body and the pressure on the table are respectively

$$\frac{g \sin 2\alpha}{2-\cos 2\alpha}, \quad \frac{3mg}{2-\cos 2\alpha}.$$

7. A mooth wedge of mass M rests on a smooth horizontal table and a particle of mass m is allowed to slide from rest down its face, which is inclined at an angle α to the horizontal. Show that when the particle has moved a distance l along the face of the wedge, the wedge will have moved a distance

$$\frac{ml \cos \alpha}{M+m}.$$

8. A smooth wedge whose section is an isosceles right angled triangle, rests on a perfectly smooth table, the hypotenuse being in contact with the table. Two masses m_1 and m_2 are attached to the ends of a string which passes over a smooth pulley at the summit of the wedge. Prove that the wedge moves on the table with acceleration

$$(m_1-m_2)g/(2M+m_1+m_2),$$

where M is the mass of the wedge and $m_1 > m_2$. (*P.U. Hons.*)

9. A wedge of mass M can slide on a smooth horizontal plane and the wedge has a face inclined at an angle α to the horizontal. Initially the wedge is at rest and a particle of mass m is projected directly up the inclined face. If the particle rises to a height h above the point of projection, show that the velocity of projection is $\sqrt{2gh\,\frac{M+m}{M+m\sin^2\alpha}}$. (*D.U. Hons.*)

Also prove that if the velocity of projection of the particle is V, it will return to the point in the wedge from which it was projected after a time

$$\frac{2V(M+m\sin^2\alpha)}{(m+M)\,g\sin\alpha}.$$

10. A straight uniform tube of mass M and length $2l$ lies at rest on a smooth horizontal table. A particle of mass m is projected with velocity v along the tube from one end. The interior of the tube is uniformly rough and the particle comes to rest relative to the tube at the middle point of the tube. Show that the tube will have moved a distance $ml/(m+M)$ before the particle comes to relative rest in it.

11. A small block of mass 1 lb. rests on the end A of a plank AB of mass 10 lb. and length 8 ft. and the plank lies on a smooth horizontal table ; a horizontal impulse is applied to the block in the direction AB, and the block slides along the plank. If the block comes to rest relatively to the plank when it reaches the other end B, and if the co-efficient of friction between the block and the plank is $\frac{1}{4}$; find the magnitude of the impulse. (*H.C.*)

12. A mass of 8 lb. is at relative rest on the rough horizontal floor of a car. Find the friction between the mass and the car when the car is moving

(*i*) with a uniform velocity of 40 f.p.s.,

(*ii*) with the same velocity and with a retardation of 10 ft./sec^2.

If the co-efficient of friction is 0·3 and the retardation continues to act until the car is brought to rest, prove that the relative motion ensues and find how far the mass moves in the carriage before the carriage is finally at rest.

13. You are in a train that is getting up speed on leaving a station. Which will be the easier to walk along the floor of a carriage in the direction of motion or against the direction of motion ?

11·5. Relative Angular Velocity of two points moving in concentric circles.

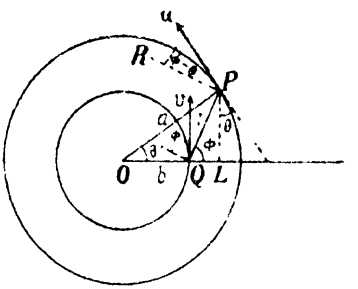
Example. *Two points P and Q are describing concentric circles of radii a and b and centre O with velocities u and v.*

(i) Find the velocity of P relative to Q when the angle POQ is θ.

(ii) Find θ when the angular velocity of the one relative to the other is zero.

(iii) Also prove that the angular velocity of PQ when its length is r, is $[(r^2+a^2-b^2)\omega+(r^2-a^2+b^2)\omega']/2r^2$,
where ω, ω′ are the angular velocities of P and Q.

Let the angle which PQ makes with OQ produced be φ.



Component of the relative velocity along $OQ = -u\sin\theta - 0$.

Component of the relative velocity perp. to $OQ = u\cos\theta - v$

(i) ∴ the resultant relative velocity

$$= \sqrt{(-u\sin\theta)^2+(u\cos\theta-v)^2}$$
$$= \sqrt{u^2+v^2-2uv\cos\theta} \quad ..(1)$$

(ii) Now when the angular velocity of P relative to Q is zero the linear velocity of P relative to Q must be in the direction of PQ.

$$\therefore \quad \frac{u\cos\theta-v}{-u\sin\theta} = \tan\varphi = \frac{LP}{QL} = \frac{LP}{OL-OQ} = \frac{a\sin\theta}{a\cos\theta-b},$$

$$\therefore \quad (u\cos\theta-v)(a\cos\theta-b)+au\sin^2\theta = 0$$

which gives $$\cos\theta = (au+bv)/(av+bu) \quad ..(2).$$

Aliter. Since $\angle PQL = \varphi$, $\angle POL = \theta$,

$$\therefore \quad \sin\varphi = \frac{LP}{PQ} = \frac{a\sin\theta}{r}, \quad \cos\varphi = \frac{QL}{PQ} = \frac{OL-OQ}{PQ} = \frac{a\cos\theta-b}{r} \quad ..(3).$$

∴ V, the velocity of P relative to Q, perp. to PQ

$$= u\cos(\varphi-\theta) - v\cos\varphi$$
$$= (u\cos\theta-v)\cos\varphi + u\sin\theta\sin\varphi$$

∴ by (3), $$rV = (u\cos\theta-v)(a\cos\theta-b)+u\sin\theta\,.\,a\sin\theta$$
$$= au(\cos^2\theta+\sin^2\theta)+bv-(av+bu)\cos\theta$$
$$= au+bv-(av+bu)\cos\theta \quad ..(4).$$

Thus $V=0$ gives the same result as (2).

(iii) Now since $u = a\omega$, $v = b\omega'$, and from $\triangle OPQ$,

$$\cos\theta = (a^2+b^2-r^2)/2ab,$$

∴ from (4), $$rV = a^2\omega+b^2\omega'-(ab\omega'+ab\omega)(a^2+b^2-r^2)/2ab$$
$$= \tfrac{1}{2}[2(a^2\omega+b^2\omega')-(\omega+\omega')(a^2+b^2)+(\omega+\omega')r^2]$$
$$= \tfrac{1}{2}[(r^2+a^2-b^2)\omega+(r^2-a^2+b^2)\omega']$$

∴ the angular velocity of the line PQ

$$= \frac{V}{r} = \frac{rV}{r^2} = \frac{1}{2r^2}\left[(r^2+a^2-b^2)\omega+(r^2-a^2+b^2)\omega'\right]$$

Note. This example finds its application in astronomy on the subject of motion of planets round the sun.

Ex. 1. Two particles A and B are describing circular paths with common centre O, of radii 1 ft. and 2 ft. respectively with constant speeds and with the same sense of rotation. A makes one complete revolution in 4 seconds and B in 6 seconds. If initially the particles are in line with A between O and B, find the velocity of A relative to B two seconds later.

Find also the angular velocity of the line joining them at that instant.

$$\left(Ans.\ \frac{\pi}{6}\sqrt{13}\ \text{f.p.s.},\ \frac{\pi}{3}\right)$$

Ex. 2. Two points describe concentric circles of radii a and b with speeds varying inversely as the radii ; show that the relative velocity is parallel to the line joining the points when the angle θ between the radii to these points is given by

$$\cos\theta=\frac{2ab}{a^2+b^2}.$$

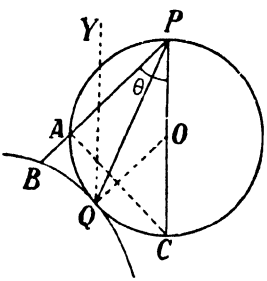
Ex. 3. Two points describe concentric circles of radii a and b with speeds varying inversely as the square roots of the radii ; show that their relative angular velocity vanishes when the angle θ between the radii to these points is given by

$$\cos\theta=\frac{\sqrt{ab}}{a-\sqrt{ab}+b}.$$

II. LINES OF QUICKEST AND SLOWEST DESCENT

11·6. *If a number of particles slide down a series of smooth chords of a vertical circle starting from rest at the highest point, the times taken are the same.*

Suppose that a particle takes time t to describe the chord PA making an angle θ with the vertical diameter PC of the circle of radius a, then



$$PA=\tfrac{1}{2}ft^2=\tfrac{1}{2}g\cos\theta\, t^2$$

$$\therefore\quad t^2=\frac{2PA}{g\cos\theta}=\frac{2PC\cos\theta}{g\cos\theta}=\frac{4a}{g}.$$

This result, being independent of θ, shows that the time taken by all the particles sliding down different chords through the highest point, is the same.

11·61. Lines of Quickest Descent. To find the line of quickest descent from a given point P to a given curve, we have to find a point Q on the curve such that the time of descent of a particle down PQ is a minimum. [See Fig. Art. 11·6]

If the vertical circle PAQ of the last article touches a curve at Q, the time taken by a particle in sliding from rest at P along the chord PQ is equal to the time from rest down the chord PA and accordingly less than the time down PB.

Thus, the time down the chord drawn from P, the highest point of the circle, to Q the point of contact of the circle with the curve, is less than the time down any other chord drawn from P to the curve.

Hence the line of quickest descent from a given point to a curve in the same vertical plane is obtained by describing a vertical circle having its highest point at the given point and touching the given curve at some point and then by joining the given point with the point of contact.

It can similarly be proved that the time taken by a particle in sliding down any chord of a vertical circle ending in the lowest point is the same and hence that the line of quickest descent from *a given curve to a given point* in the same vertical plane is obtained by describing a vertical circle having the given point as its lowest point and touching the curve at some point and then by joining the point of contact with the given point.

It is important to note that the line of quickest descent is not usually the shortest line geometrically.

11·611. *The line of quickest descent bisects the angle between the normal to the curve and the vertical at the point where it meets the curve.*

In the figure of Art. 11·6, OQ is normal to the circle as well as to the curve.

Join PQ, OQ and draw QY vertical.

Since QY and OP are parallel,

$\therefore \quad \angle PQY = \angle QPO$

Also since $\quad OP = OQ,$

$\therefore \quad \angle QPO = \angle PQO.$

Hence $\quad \angle PQY = \angle PQO$, *i.e.*, PQ bisects $\angle OQY$.

11·62. We now give geometrical constructions for the lines of quickest descent in a few special cases. The points and curves are assumed to lie in the same vertical plane.

(The student is advised to draw figures and verify the results).

(*i*) (*a*) *From a given point P to a given straight line.*

(*b*) *From a straight line to a given point P.*

Draw through P, a horizontal straight line meeting the given line in A and cut off along the given line $AQ = AP$, the point Q being below A in case (*a*) and above A in case (*b*). Then PQ is the required line.

A circle drawn to touch the given line at Q will have P as its highest point in case (*a*) and as its lowest point in case (*b*).

(*ii*) (*a*) *From a given point P without a given circle to the given circle* (*P higher than the lowest point of the circle*) *or from a given circle to a given point P within the circle.*

Join P to the lowest point of the circle meeting the circle in Q, then PQ is the required line.

(*b*) *From a given point P within a given circle to the circle or from a given circle to a given point P, without the circle* (*P is lower than the highest point of the circle*).

Join the given point P to the highest point of the circle meeting it in Q.

(*iii*) (*a*) *From a given straight line without a given circle to the circle.*

Draw a horizontal line through the lowest point B of the circle to meet the given line in K and cut off along the line $KP = BK$. Join PB meeting the circle in Q then PQ is the required line.

(*b*) ***From a given circle to a given straight line without the circle.***

In this case the horizontal line is to be drawn from the highest point of the circle.

(*iv*) (*a*) ***From a given circle to another circle without it.***

Join the highest point of the upper circle to the lowest point of the lower circle, cutting the two circles in P and Q, then PQ is the required line.

(*b*) ***From a given circle to another circle lying wholly within it.***

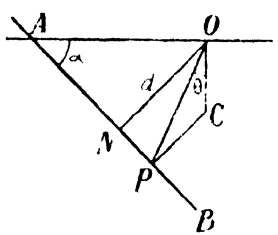
In this case join the lowest points of the circles.

(*c*) ***From a given circle to another circle within which it lies.***

In this case join the highest points of the circles.

Example. *A particle slides down a smooth wire from a point O to a line AB inclined at an angle α to the horizontal. Show that the time from O to any point on AB is least when the wire makes an angle $\frac{\alpha}{2}$ with the vertical and that the least time is $\sqrt{\frac{2d}{g}}\sec\frac{\alpha}{2}$, where d is the distance of O from AB.*

First Method. *Geometrical.* Draw through O a horizontal straight line meeting the line AB in A, cut off $AP=AO$, then OP is the required position of the wire. [See *i* (*a*), Art. 11·62].



Draw PC perp. to AB meeting the vertical through O in C.

Then since

$$\angle OAP=\alpha, \text{ and } OA=AP,$$

$$\therefore \quad \angle AOP=\angle APO=\tfrac{1}{2}(180^\circ-\alpha)$$

$$=90^\circ-\tfrac{1}{2}\alpha.$$

so that $\quad \angle COP=\angle CPO=\tfrac{1}{2}\alpha,$

and $\quad PC=CO.$

Hence if with C as centre and CP or CO as radius, a circle is drawn, it will have its highest point at O and will touch the line AB in P. This evidently satisfies the geometrical condition for the line OP to be the line of quickest descent from O to AB. [Art. 11·61.]

Now, to find the time we have

f, the acceleration down $OP=g\cos\frac{\alpha}{2}$.

Draw ON perp. to AB, then $ON=d$, $\angle AON=90^\circ-\alpha$

and $\quad \angle NOP=\angle AOP-\angle AON=90^\circ-\tfrac{1}{2}\alpha-(90^\circ-\alpha)=\tfrac{1}{2}\alpha.$

$$\therefore \qquad OP=d\sec\frac{\alpha}{2}.$$

Hence if t is the time taken by the particle in sliding down OP, then

$$OP=\tfrac{1}{2}ft^2,$$

i.e., $$d \sec \frac{\alpha}{2} = \tfrac{1}{2} g \cos \frac{\alpha}{2} . t^2,$$

or $$t = \sqrt{\frac{2d}{g}} \sec \frac{\alpha}{2}.$$

Second Method. *Analytical.* Let OP be the wire making an angle θ with the vertical and ON $(=d)$ perp. to AB, then

$$\angle PON = \angle POA - \angle NOA = (90° - \theta) - (90° - \alpha) = \alpha - \theta,$$

and $$OP = d \sec \angle PON = d \sec (\alpha - \theta).$$

If t is the time taken by the particle in sliding down OP, then

$$OP = \tfrac{1}{2} g \cos \theta \, t^2,$$

i.e., $$t^2 = \frac{2OP}{g \cos \theta} = \frac{2d \sec (\alpha - \theta)}{g \cos \theta} = \frac{2d}{g \cos \theta \cos (\alpha - \theta)}.$$

Now t will be least when $\cos \theta \cos (\alpha - \theta)$ is maximum,

i.e., when $\tfrac{1}{2}[\cos \alpha + \cos (2\theta - \alpha)]$ is maximum,

or when $\cos (2\theta - \alpha)$ is maximum *i.e.*, when $2\theta - \alpha = 0$,

which gives $\theta = \frac{\alpha}{2}$ and $t = \sqrt{\frac{2d}{g \cos^2 \alpha/2}} = \sqrt{\frac{2d}{g}} \sec \frac{\alpha}{2}$.

11·63. Lines of Slowest Descent. To find the lines of slowest descent from a given point P to a given curve, we have to find a point Q on the curve such that the time of descent of a particle down PQ is maximum.

A few special cases are given below. Curves and points are assumed to lie in the same vertical plane.

(*i*) *From a given point P to a given circle, P being without the circle and higher than the highest point of the circle.*

Joint P to the highest point of the circle and produce it to meet the circle in Q, then PQ is the required line of the slowest descent. For, a circle can be drawn to have P as its highest point and to touch the given circle at Q and it can be shown that the time of descent down PQ is greater than the time down another line joining P to any other point of the given circle.

(*ii*) *From a given circle to a given point P without it.*

In this case join P to the lowest point of the circle and produce it to meet the circle in Q then PQ is the required line.

(*iii*) *From a given circle to another given circle lying wholly without it, the lowest point of the first circle being higher than the highest point of the second.*

Join the lowest point of the first circle with the highest point of the second and produce it both ways to meet the circles in P and Q, then PQ is the required line.

Examples XXXVI

1. Find the shortest time in which a ring can be made to slide down a smooth wire to a vertical wall from a point distant 50 ft. from it.

Examples XXXVII

1. *Find the hodograph of the motion of a point whose rectangular co-ordinates at time t are given by $x=\frac{1}{3}t^3$, $y=\frac{1}{2}t^2$.*

Here $\frac{dx}{dt}=t^2$ and $\frac{dy}{dt}=t$.

$\therefore$ $X=K\frac{dx}{dt}=Kt^2$ and $Y=K\frac{dy}{dt}=Kt$.

Eliminating t, we have the parabola $Y^2=KX$, which is the equation to the hodograph.

2. A point moves on the parabola $y^2=4ax$ in such a way that its hodograph is the same parabola ; prove that, if it passes through the point (x_0, y_0) at $t=0$, its co-ordinates at any time are given by

$$x=x_0\,e^{kt}, \qquad y=y_0\,e^{2t}.$$

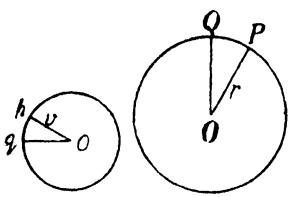
3. *The hodograph of a point projected with a velocity u in a direction making an angle α with the horizontal and thus describing a parabola is a vertical straight line.*

For $u\cos\alpha$ the horizontal component of the velocity being constant, the ends of the straight lines representing the velocities at different points of the path shall all lie on a vertical straight line at a distance $u\cos\alpha$ from the fixed point.

4. *Show that the hodograph of a point moving in a circle of radius r, with uniform speed v is another circle of radius v.*

Hence prove that the acceleration of the point is v^2/r directed towards the centre of its path.

Suppose P and Q are two positions of the moving point and p and q the corresponding positions in the hodograph.



As the particle moves with constant speed v, $o'p=o'q=v$, so that $o'p$ is constant and the hodograph of P is a circle of radius v units.

Also since $\angle po'q$=the angle between the tangents at P and $Q=\angle POQ$.

$\therefore$ $\frac{\text{arc } pq}{o'p}=\frac{\text{arc } PQ}{OP}$, *i.e.*, arc pq=arc $PQ\,\frac{v}{r}$.

The acceleration of the point P in its path

=the velocity of p in the hodograph

$$=\text{Lt}\,\frac{pq}{\delta t}=\text{Lt}\,\frac{v}{r}\cdot\frac{PQ}{\delta t}=\frac{v}{r}\,\text{Lt}\,\frac{PQ}{\delta t}=\frac{v}{r}\cdot v=\frac{v^2}{r}.$$

The direction of motion of p, being along the tangent at p in the hodograph, is evidently parallel to PO.

5. A bead moves under the action of gravity along a smooth vertical circle of radius a, starting from rest at the highest point, show that a polar equation of a hodograph is $r=2\sqrt{ag}\,\sin\frac{1}{2}\theta$, the centre of the circle being taken as the origin.

6. Show that the polar equation of the hodograph of a point which describes a circle of radius a with a constant angular acceleration α is

$$r^2=a^2[\alpha(2\theta-\pi)+\omega^2],$$

its initial angular position and angular velocity being zero, and ω respectively.

7. The co-ordinates of a particle at time t are $x=a\sin nt$, $y=a\cos 2nt$; show that the equation $16x^4=n^2a^2(16x^2-y^2)$ is a possible equation for the hodograph.

8. What is the hodograph of a point moving in a straight line with (*a*) constant velocity, (*b*) constant acceleration, (*c*) having $x=a+bt$, $y=c+dt$, as its co-ordinates at time t ?

[*Ans.* (*a*) A point, (*b*) a straight line described with constant velocity, (*c*) the point (b, d)].

This equation gives the velocity of the particle and it follows from this that as t increases and tends to infinity, the velocity continually approaches the limit $\frac{g}{k}$ which, as it is also evident from (1), is the velocity which the particle ultimately attains when the acceleration vanishes. This is thus the *terminal velocity* of the particle.

Equation (2) can be written as

$$\frac{dx}{dt}=\frac{g}{k}-\frac{g}{k}e^{-kt}.$$

Integrating, we get

$$x=\frac{g}{k}\cdot t+\frac{g}{k^2}e^{-kt}+c'.$$

Initially when $t=0$, $x=0$,

$$\therefore \qquad c'=-\frac{g}{k^2}.$$

Hence
$$x=\frac{gt}{k}-\frac{g}{k^2}+\frac{g}{k^2}e^{-kt} \qquad ..(3)$$

which gives the distance described in time t.

Eliminating t between (2) and (3), we get

$$x=-\frac{v}{k}-\frac{g}{k^2}\log\left(1-\frac{kv}{g}\right) \qquad ..(4),$$

a relation between x and v which can also be obtained directly by writing the equation of motion in the form

$$v\frac{dv}{dx}=g-kv.$$

12·4. *A particle falls vertically from rest, in a medium whose resistance varies as the square of the velocity, to find the motion of the particle.*

Here the equation of the motion is

$$\frac{dv}{dt}=g-kv^2 \qquad ..(1).$$

In this case the terminal velocity is $\sqrt{g/k}$. Denoting it by V, we have

$$\frac{dv}{dt}=k\left(\frac{g}{k}-v^2\right)=k(V^2-v^2).$$

Integrating, we get

$$\int\frac{dv}{V^2-v^2}=kt+c \quad \text{or} \quad \frac{1}{2V}\log\frac{V+v}{V-v}=kt+c.$$

Initially when $t=0$, $v=0$; $\therefore$ $c=0$,

Hence
$$\log\frac{V+v}{V-v}=2kVt$$

or $$v=V\frac{e^{2kvt}-1}{e^{2kvt}+1}=V\frac{1-e^{-2kvt}}{1+e^{-2kvt}} \qquad ..(2)$$

This equation gives v the velocity of the particle after a time t and it shows that as $t\to\infty$, $v\to V$.

Equation (2) can be written as

$$v=\frac{dx}{dt}=V\tanh\frac{gt}{V} \qquad ..(3) \qquad \left[\because k=\frac{g}{V^2}\right].$$

Integrating, we have

$$x=\frac{V^2}{g}\log\cosh\frac{gt}{V}+c'.$$

Initially when $t=0, x=0, \quad \therefore\ c'=0.$

Hence $$x=\frac{V^2}{g}\log\cosh\frac{gt}{V} \qquad ..(4).$$

Eliminating t between (3) and (4), we get

$$v^2=V^2\left(1-e^{-\frac{2gx}{V^2}}\right) \qquad ..(5),$$

a relation between v and x which can also be obtained directly from the equation $v\frac{dv}{dx}=g-kv^2.$

12·5. *A particle is projected upwards, with velocity U against the resistance kv^2, to find the motion.*

Taking the upward direction as positive, the equation of motion of the particle is

$$\frac{dv}{dt}=-g-kv^2 \qquad ..(1)$$

$$=-k(V^2+v^2), \qquad \left(\text{taking } \frac{g}{k}=V^2\right)$$

Integrating, we get

$$\int\frac{dv}{V^2+v^2}=-kt+c \quad \text{or} \quad \frac{1}{V}\tan^{-1}\frac{v}{V}=-kt+c.$$

Initially when $t=0, v=U,$

$$\therefore \qquad c=\frac{1}{V}\tan^{-1}\frac{U}{V}.$$

$$\therefore \qquad Vkt=\tan^{-1}\frac{U}{V}-\tan^{-1}\frac{v}{V}. \qquad ..(2).$$

Again equation (1) can be written as

$$v\frac{dv}{dx}=-g-kv^2=-k(V^2+v^2).$$

$$\therefore \qquad \int\frac{vdv}{V^2+v^2}=-kx+c', \text{ i.e., } \log(V^2+v^2)=-2kx+2c'.$$

Initially when $x=0, v=U,$

$$\therefore \qquad 2c'=\log(V^2+U^2)$$

Hence $$2kx=\log\frac{V^2+U^2}{V^2+v^2} \qquad ..(3).$$

Equation (2) gives the velocity at the end of any time t while equation (3) gives the velocity in terms of the distance described by the particle.

The greatest height h to which the particle rises is found by putting $v=0$ and $x=h$ in equation (3), thus we have

$$h=\frac{1}{2k}\log\left(1+\frac{U^2}{V^2}\right)=\frac{V^2}{2g}\log\left(1+\frac{U^2}{V^2}\right) \quad ..(4).$$

The particle then begins to descend. Taking the highest point reached as the origin and using the results of the previous article, we can further find completely the motion of the particle. The velocity of the particle when it returns to the point of projection is obtained by substituting the above value of h for x in equation (5) of the previous article, so that after simplification, we get

$$v^2=V^2\left(1-\frac{V^2}{V^2+U^2}\right)=\frac{U^2V^2}{V^2+U^2} \quad ..(5).$$

Example 1. *Take U, the velocity of projection=50 ft. per sec. and V, the terminal velocity=150 ft. per sec. in the above Article. Find the greatest height reached by the particle and its velocity when it returns to its starting point.*

Using equation (4), we have

$$h=\frac{150\times150}{2\times32}\log_e\left[1+\frac{50\times50}{150\times150}\right]=\frac{75\times75}{16}\log_{10}\tfrac{9}{8}/\log_{10}e$$

$$=\frac{75\times75}{16}\left[\frac{1-0{\cdot}9542}{0{\cdot}4343}\right] \text{ from the tables}=37 \text{ ft. (approx.)}$$

Using equation (5), we get

$$v^2=\frac{50\times50\times150\times150}{150\times150+50\times50}=15\times150.$$

$$\therefore \quad v=15\sqrt{10}=47{\cdot}43 \text{ ft. per sec.}$$

Example 2. *A motor cycle which with its rider weighs 3 cwt. is found to run at 30 m.p.h. up an incline of 1 in 20, and at 50 m.p.h. down the same incline. Assuming that the resistance is proportional to the square of the velocity, and that the engine is working at the same horse-power find the speed that would be attained on the level, and show that the horse power is $2\frac{1}{3}$ nearly.*

Let v ft. per sec. be the velocity of the cycle and F lb. wt. be the tractive force exerted by the engine at any instant, then the equations of motion (1) on the level (2) up the inclined plane (3) down the plane are:

$$m\frac{dv}{dt}=(F-kv^2)g \quad ..(1),$$

$$m\frac{dv}{dt}=\left(F-kv^2-\frac{3\times112}{20}\right)g \quad ..(2),$$

$$m\frac{dv}{dt}=\left(F-kv^2+\frac{3\times112}{20}\right)g \quad ..(3).$$

If H be the constant horse power of the engine, then

$$Fv=550H \quad \text{or} \quad F=\frac{550H}{v} \qquad ..(4)$$

The maximum speed obtained by the cycle while moving up the plane is given to be 30 m.p.h. or 44 f.p.s. and, therefore, in equation (2) $\frac{dv}{dt}=0$ when $v=44$.

Hence from (2) and (4), we have

$$\frac{550H}{v}-kv^2-\frac{3\times 112}{20}=0$$

or $$550H-kv^3-\tfrac{84}{5}v=0, \text{ where } v=44 \qquad ..(5)$$

Similarly from (3), we have

$$550H-kv'^3+\tfrac{84}{5}v'=0 \qquad ..(6),$$

where $v'=50$ m.p.h. $=\frac{220}{3}$ f.p.s.

Subtracting (6) from (5), we have

$$k=\tfrac{84}{5}\cdot\frac{v'+v}{v'^3-v^3}=\tfrac{84}{5}\cdot\frac{\frac{220}{3}+44}{(\frac{220}{3})^3-(44)^3}=\frac{27}{35\times 121}.$$

Substituting this value of k in (5) and simplifying, we get

$$H=\tfrac{81}{35}\tfrac{6}{0}=2\tfrac{11}{35}\tfrac{6}{0}=2\tfrac{1}{8} \text{ nearly.}$$

Now to find the speed attained by the cycle on the level, we equate $\frac{dv}{dt}=0$ in (1), which gives

$$F=kv^2 \quad \text{or} \quad \frac{550H}{v}=kv^2.$$

Substituting the values of H and k and simplifying, we get

$$v^3=\frac{55\times 121\times 816}{27}=\left(\frac{22}{3}\right)^3\times 510=\left(\frac{22\times 8}{3}\right)^3 \text{ (approx.)}$$

$$\therefore\ v=\frac{22\times 8}{3}\text{f.p.s.}=\frac{22\times 8\times 15}{3\times 22} \text{ or } 40 \text{ m.p.h.}$$

Examples XXXIX

1. A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is k(velocity). Show that the velocity v after a time t and the distance s described in that time are given by

$$v=Ve^{-kt}, \qquad s=\frac{V}{k}(1-e^{-kt})$$

2. A particle is projected with velocity u along a smooth horizontal plane in a medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance d it has described in time t and its velocity V at that time are given by

$$d=\frac{1}{\mu u}[\sqrt{1+2\mu u^2 t}-1]\ ; \quad V=\frac{u}{\sqrt{1+2\mu u^2 t}}.$$

Show also that $d/t=2uV/(u+V)$.

3. A particle is projected upwards with a velocity g/k in a medium whose resistance is kv per unit mass, where v is the velocity of the particle and g is the acceleration due to gravity. Prove that the height attained by the particle after a time $\frac{1}{k}$ is $\frac{g(e-2)}{k^2 e}$.

4. A particle of mass m, is projected vertically under gravity, the resistance of the air being mk times the velocity ; show that the greatest height attained by the particle is

$$\frac{V^2}{g}[\lambda - \log(1+\lambda)],$$

where V is the terminal velocity of the particle and λV is the initial vertical velocity.

5. A particle is projected vertically upwards with velocity u, and the resistance of the air produces a retardation kv^2 where v is the velocity. Show that the velocity V with which the particle will return to the point of projection given by

$$\frac{1}{V^2} = \frac{1}{u^2} + \frac{k}{g}.$$

If E is the kinetic energy of the particle in its upwards path at a given point, show that the loss of energy when it passes the same point on the way down is $\frac{E^2}{E+E'}$, where E' is the limit to which its energy approaches in its downward course.

6. A particle of mass m can move in a resisting medium in which the resistance varies as the square of the speed, the magnitude of the terminal velocity being V. If it is projected upwards with a speed $V \tan \alpha$, show that it will return to the point of projection with a speed $V \sin \alpha$.

Show that the amount of energy, kinetic and potential together, which is lost in its ascent is

$$\tfrac{1}{2}mV^2(\tan^2 \alpha - 2 \log_e \sec \alpha). \qquad (L.U.)$$

7. A particle falls from rest in a medium in which the resistance is kv^2 per unit mass. Prove that the distance fallen in time t is

$$\frac{1}{k} \log \cosh (t\sqrt{gk}).$$

If the particle were ascending, show that its distance at any instant below the highest point of its path is

$$\frac{1}{k} \log \sec (t\sqrt{gk}),$$

where t denotes the time it will now take to reach its highest point.

8. If the resistance per unit mass is kv^2 and the particle slides on a smooth straight wire inclined at angle θ to the vertical, prove that the space s described in time t from rest is given by

$$e^{ks} = \tfrac{1}{2}(e^{bt} + e^{-bt}),$$

where $b^2 = kg \cos \theta$.

9. A cyclist works at the constant rate of P horse-power. When there is no wind, he can ride at 22 ft./sec. on level ground and at 11 ft./sec. up a hill making an angle $\sin^{-1}\frac{1}{20}$ with the horizon. The total mass of the man and cycle is 180 lb. The resistance of the air is kv^2 lb. wt. where the velocity of the man relative to air is v ft./sec. ; the other frictional forces are negligible.

Find P and show that the speed of the cyclist when riding on level ground against a wind of 22 ft./sec. is between 10 and 10·5 ft./sec. $(M.T.)$

$$(Ans.\ P = \tfrac{3.6}{17.5})$$

10. A locomotive drawing a total weight of 264 tons on the level is exerting a tractive force of 20,000 lb. wt. at the speed of 15 m.p.h. It works at a constant horse-power until its speed is 60 m.p.h. when it is just able to overcome the resistance to motion, which may be taken to vary as the square of the velocity. Show that it reaches the speed of 45 m.p.h., from the speed of 15 m.p.h. in a distance of approximately 5,080 feet. $(M.T.)$

11. The resistance to an aeroplane when landing is $a+bv^2$ absolute units per unit mass ; v being the velocity, and a, b, constants. For a particular machine $b = 10^{-3}$ ft. lb. sec. units and it is found that if the landing speed is 50 m.p.h. the length of the run of the machine before coming to rest is 150 yds. Calculate the value of the constant a.

$$[Ans.\ \log_e\left(1+\frac{242}{450}\right) = 0{\cdot}9 \text{ or } a = 3{\cdot}68]$$

CHAPTER XIII

MOTION OF A PARTICLE IN A PLANE—III

ELLIPTIC HARMONIC MOTION : CYCLOIDAL MOTION : PLANETARY MOTION

13·1. A few simple cases of motion of a particle in a plane were dealt with in Chapters VI, VII and VIII. In the present chapter we will discuss similar problems of a more difficult type. As was pointed out in Art. 6·24, the motion of a particle moving in a plane is determined by two equations of motion obtained by resolving forces in two directions usually at right angles to one another. The choice of these directions is a matter of convenience depending upon the nature of the problem.

A particle may be *free* to move along any path which it may describe according to the forces acting upon it (*e.g.*, parabolic path of a projectile) or it may be *constrained* to move along any given curve or surface, (*e.g.*, may slide along a smooth vertical circle).

In the former case the motion of the particle is discussed by resolving forces in directions parallel to two axes which are usually at right angles, in particular, if the *forces are central i.e.*, they are directed towards a fixed point, the most useful directions for the axes are along the radius vector and perpendicular to the radius vector. In the case of constrained motion, since in addition to other forces, there is always present a force of normal reaction of the curved surface along which the particle is constrained to move, the motion can best be discussed by resolving forces along the tangent and the normal to the path.

1. MOTION REFERRED TO TWO FIXED AXES

13·11. In this case the position of the particle at any instant is defined by the co-ordinates x and y referred to two fixed axes OX and OY and by resolving forces parallel to these axes, the equations of motion as given in Art. 6·24, are of the form

$$m\frac{d^2x}{dt^2}=F_x \quad \text{and} \quad m\frac{d^2y}{dt^2}=F_y.$$

Example 1. *Find the law of force parallel to an asymptote under which a rectangular hyperbola can be described.*

Let the rectangular hyperbola $xy=c^2$ be described by a particle of unit mass, under a force parallel to the y-axis, which is one of its asymptotes.

Then $\dfrac{d^2x}{dt^2}=0$, and $\therefore \quad \dfrac{dx}{dt}=$ a constant $=u$, say.

Now since $y=\dfrac{c^2}{x}$, $\dfrac{dy}{dt}=-\dfrac{c^2}{x^2}\cdot\dfrac{dx}{dt}=-\dfrac{c^2u}{x^2}$.

$$\therefore \qquad \frac{d^2y}{dt^2}=\frac{2c^2u}{x^3}\cdot\frac{dx}{dt}=\frac{2c^2u^2}{x^3}.$$

It shows that the force varies inversely as cube of x, the distance of the particle from the asymptote parallel to which force acts.

Example 2. *A particle describes freely the catenary $y=c\cosh x/c$ under the action of a force parallel to the ordinate. Show that the force and the velocity at any point vary as the ordinate of the point.*

As the force is parallel to the ordinate, it has no component parallel to the axis of x.

$$\therefore \qquad m\frac{d^2x}{dt^2}=0, \text{ so that } \frac{dx}{dt}=\text{const.}=u, \text{ say} \qquad ..(1)$$

From this and given equation of the curve, we have

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dx}=c\times\frac{1}{c}\sinh\frac{x}{c}\cdot\frac{dx}{dt}=u\sinh\frac{x}{c} \qquad ..(2)$$

$\therefore$ v, the velocity of the particle is given by,

$$v^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=u^2+u^2\sinh^2\frac{x}{c}=u^2\cosh^2\frac{x}{c}=u^2.\frac{y^2}{c^2},$$

which gives $v=\frac{u}{c}y$ and shows that the velocity at any point varies as the ordinate.

Also since $\frac{d^2x}{dt^2}=0$, $\therefore$ for the resultant acceleration f, we have from (2),

$$f=\frac{d^2y}{dt^2}=\frac{u}{c}\cosh\frac{x}{c}\cdot\frac{dx}{dt}=\frac{u^2}{c^2}.c\cosh\frac{x}{c}=\frac{u^2}{c^2}y,$$

which shows that the acceleration and consequently the force varies as the ordinate.

Examples XL

1. If the co-ordinates x, y of a moving point are given by

$$x=a(\cos\theta+\theta\sin\theta),\quad y=a(\sin\theta-\theta\cos\theta)$$

and θ increases at a uniform rate ω, prove that the velocity of the point is $a\theta\omega$.

2. A point moves in a plane such that its velocities parallel to axes of x and y are $u+cy$ and $v+cx$ respectively, where u, v and c are constants, show that the path of the point is a conic section.

3. A particle describes the parabola $y^2=4ax$ whose axis is horizontal. Prove that

(*i*) if its horizontal velocity is constant, its vertical velocity varies inversely as its ordinate,

(*ii*) if its vertical velocity is constant, the acceleration of the foot of the ordinate is constant, and

(*iii*) if its velocity is constant, the acceleration of the foot of the ordinate varies inversely as the square of its distance from the directrix.

4. A particle moving in a plane is subject to a force towards the x-axis and proportional to its distance from that axis. If initially it is projected from the origin with velocity u in the direction making an angle α with the x-axis, prove that it will cross the x-axis again at this angle after a time $\frac{\pi}{\sqrt{\mu}}$, where μ is the constant of proportionality. (The mass $m=1$).

Also prove that the maximum displacement from the x-axis is proportional to $u\sin\alpha$.

5. A particle moves in a plane under a constant acceleration $a\mu$ parallel to Ox, and an acceleration $-2\mu y$, parallel to Oy where Ox, Oy are rectangular axes. If the particle starts from rest at the point $(0, a)$, show that it will describe the curve $y=a\cos(2\sqrt{x}/\sqrt{a})$.

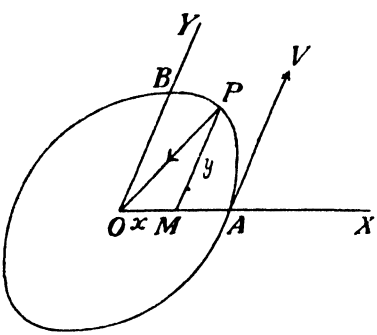
6. A particle moves on a curve $y=a\log\sec\frac{x}{a}$ in such a way that the tangent to the curve rotates uniformly, prove that the acceleration of the particle varies as the square of the radius of curvature.

7. A particle acted upon by gravity is describing a curved path APB. The resolved part of its velocity at A at right angles to the chord AB is u, show that the resolved part of the velocity at B in the same direction is $-u$.

13·12. In examples given above and for the motion of a *projectile* given in Chapter VI, the axes used are rectangular. In the problem discussed in the next Article it will be found more convenient to use oblique axes.

13·13. Elliptic Harmonic Motion. *A particle moves in a plane with an acceleration which is always directed towards a fixed point and varies directly as the distance from it; to prove that the particle describes an ellipse having the fixed point as its centre.*

Let the fixed point O towards which the acceleration is directed be taken as the origin and let the particle be projected in a certain direction from a point A with a velocity V.



Take OA as the axis of x and a line OY parallel to the direction of the initial velocity V, as the axis of y.

Let (x, y) be the co-ordinates of P in any subsequent position of the particle. The acceleration of the particle is μPO whose components parallel to the axes are μPM and μMO or $-\mu y$ and $-\mu x$.

Hence the equations of motion are

$$\frac{d^2x}{dt^2}=-\mu x \qquad \text{and} \qquad \frac{d^2y}{dt^2}=-\mu y.$$

These equations show that the moving particle has a simple harmonic motion of period $\frac{2\pi}{\sqrt{\mu}}$ parallel to each of the axes of x and y.

Solving these equations, we have

$$x=A\cos\sqrt{\mu}\,t+B\sin\sqrt{\mu}\,t \qquad ..(1),$$

and

$$y=C\cos\sqrt{\mu}\,t+D\sin\sqrt{\mu}\,t \qquad ..(2).$$

Initially when $t=0$, $x=OA=a$, $y=0$, $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=V$

Hence from equations (1) and (2), we get

$$A=a,\ B=C=0,\ D=V/\sqrt{\mu}.$$

$$\therefore \quad x=a\cos\sqrt{\mu}\,t, \text{ and } \quad y=\frac{V}{\sqrt{\mu}}\sin\sqrt{\mu}\,t. \qquad ..(3).$$

Eliminating t, we have

$$\frac{x^2}{a^2}+\frac{y^2}{V^2/\mu}=1 \qquad ..(4).$$

Thus the locus of P is an ellipse having its centre at the centre of attraction and having OX, OY, a pair of conjugate diameters, as axes.

13·131. By putting $x=0$ in (4), we get y^2 or $OB^2=V^2/\mu$ which gives $V=\sqrt{\mu}.OB.$

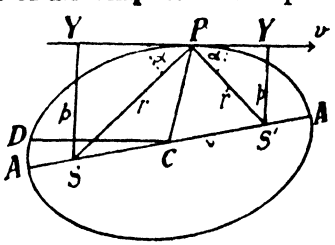
Thus the speed of the particle at any point A, is proportional to OB, the semi diameter conjugate of OA.

Velocity at any point P. Let v be the velocity at P and let $\angle AOB=\omega$, then

$$v^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+2\,\frac{dx}{dt}\,\frac{dy}{dt}\cos\omega$$
$$=a^2\mu\sin^2\sqrt{\mu}\,t+V^2\cos^2\sqrt{\mu}\,t$$
$$-2aV\sqrt{\mu}\sin\sqrt{\mu}\,t\cos\sqrt{\mu}\,t\cos\omega, \quad [by\ (3)].$$

Note. In the particular case when the velocity V is at right angles to OX, the two axes being perpendicular to each other, are the principal axes of the ellipse.

13·132. Geometrical properties of an ellipse. A few properties of an ellipse are given below for ready reference.



The letters used have their usual meanings, CD being the semi-diameter conjugate to CP,

$$SP+SP'=r+r'=2a \qquad ..(1)$$
$$SP.\,S'P=r\,r' \;=CD^2 \qquad ..(2)$$
$$SY.\,SY'=pp' \;=b^2 \qquad ..(3)$$

$$\text{Sin}\,\alpha=\frac{p}{r}=\frac{p'}{r'}=\frac{\sqrt{pp'}}{\sqrt{r\,r'}}=\frac{b}{\sqrt{r\,r'}} \text{ or } \frac{b}{CD} \qquad ..(4).$$

Also from (4)

$$\frac{b^2}{p^2}=\frac{rr'}{r^2}=\frac{r'}{r}=\frac{2a-r}{r}\,[by\ (1)]=\frac{2a}{r}-1 \qquad ..(5).$$

the pedal equation of an ellipse referred to a focus.

If, however, p denotes the perpendicular from the centre C on the tangent at P and $CP=r$, then

$$CP^2+CD^2=a^2+b^2 \qquad ..(6),$$

and

$$p.\,CD=ab \qquad ..(7).$$

From (6) and (7),

$$CP^2+\frac{a^2b^2}{p^2}=a^2+b^2,$$

or

$$\frac{a^2b^2}{p^2}=a^2+b^2-CP^2=a^2+b^2-r^2 \qquad ..(8).$$

the pedal equation of an ellipse referred to its centre.

Example 1. *A particle is describing an ellipse under a force to the centre. If V be the velocity at the end of the minor axis, and v_1, v_2, the velocities at the ends of any pair of conjugate diameters, prove that* $v_1^2+v_2^2=V^2+\mu b^2$.

By Art. 13·131, we have

$$v_1=\sqrt{\mu}\, CD, \quad \text{and} \quad v_2=\sqrt{\mu}\, CP.$$

Also since the semi-diameter conjugate to the minor-axis is the semi-major axis$=a$, $\therefore V=\sqrt{\mu}\, a$ or $V^2=\mu a^2$.

$$\therefore \quad v_1^2+v_2^2=\mu\,(CP^2+CD^2)=\mu\,(a^2+b^2) \qquad \text{[by (6)]}$$
$$=V^2+\mu b^2.$$

Example 2. *A particle describes an ellipse with an acceleration μr directed towards the centre C, show that (i) its angular velocity about a focus is inversely proportional to its distance from the focus,*
(ii) the rate of change of its direction of motion is $ab\sqrt{\mu}/CD^2$.

(*i*) Let v be the velocity of the particle at P and let ω be its angular velocity about the focus S.

By Art. 13·131, $\qquad v=\sqrt{\mu}\, CD \qquad ..(i)$

$$\text{Now} \quad \omega=\frac{\text{vel. perp. to } SP}{SP}=\frac{v\sin\alpha}{SP}=\frac{\sqrt{\mu}\, CD}{SP}\cdot\sin\alpha$$
$$=\frac{\sqrt{\mu}\, CD}{SP}\cdot\frac{b}{CD}\ \text{[by (4)]}\ =\frac{\sqrt{\mu}\, b}{SP}.$$

Hence ω varies inversely as SP.

(*ii*) Let ψ be the angle which the tangent at P makes with the major axis, then

the rate of change of the direction of motion

$$=\frac{d\psi}{dt}=\frac{d\psi}{ds}\cdot\frac{ds}{dt}=\frac{v}{\rho}=\frac{\sqrt{\mu}.CD}{\rho} \qquad \text{[by (}i\text{)]}$$

Now to find ρ we use equation (8), $a^2b^2/p^2=a^2+b^2-r^2$.

Differentiating this we get,

$$\frac{-2a^2b^2}{p^3}=-2r\frac{dr}{dp}, \quad \therefore \rho=r\frac{dr}{dp}=\frac{a^2b^2}{p^3}=\frac{CD^3}{ab} \qquad \text{[by (7)]}$$

Hence $\dfrac{d\psi}{dt}=\sqrt{\mu}.CD\times\dfrac{ab}{CD^3}=\dfrac{ab\sqrt{\mu}}{CD^2}$.

Examples XLI

1. Show that the speed of the particle at any point P of the ellipse $=\sqrt{\mu}r'$, where r' is the semi conjugate diameter of P.

Hence show that the same ellipse is a hodograph of the motion of P.

2. Show that the time taken by the particle to describe the whole ellipse is $2\pi/\sqrt{\mu}$.

3. A particle is acted on by attractions to a number of centres of force, each being proportional to the distance. If it be projected in any direction, show that it will describe an ellipse.

4. The ends of a rod which rotates with constant angular velocity move on two intersecting straight lines at right angles; prove that any other point on the rod executes an elliptic harmonic motion.

5. The components of acceleration of a point are $-4\mu x$, $-\mu y$ parallel to fixed rectangular axes Ox, Oy; initially it is at rest at the point (a, b). Show that it will oscillate to and fro on the arc of the parabola $2ay^2=b^2(x+a)$; and find its velocity at the point where its path first crosses the axis of y.

$$\left[Ans.\ \sqrt{\tfrac{1}{2}\mu(8a^2+b^2)},\ \text{inclined to } Ox \text{ at } \pi+\tan^{-1}\frac{b}{2\sqrt{2}a}.\right]$$

6. A particle moves in a plane with an acceleration which is always directed away from a fixed point and varies directly as the distance from it. Prove that the particle describes a hyperbola.

7. If a particle describes an ellipse about a centre of force in the centre, show that sum of the reciprocals of its angular velocities about the foci is constant.

8. Prove that the speed at any point P of an elliptic orbit described under a central force, varying directly as the distance to the centre O of the ellipse, is proportional to the length of the diameter conjugate to OP.

A particle is projected from a point P in a direction making an angle $\pi/4$ with the line joining P to the above centre of force O. Its speed of projection is that which it would acquire in moving directly from rest at P to the centre of force O. Prove that the eccentricity e of the orbit is given by

$$e^2=2(\sqrt{2}-1). \qquad (P.U.)$$

9. Prove that in an elliptic harmonic motion the time-average of the Kinetic Energy is equal to the arithmetic mean of the greatest and least values of the Kinetic Energy. (*D.U. 1960*)

II. CONSTRAINED MOTION

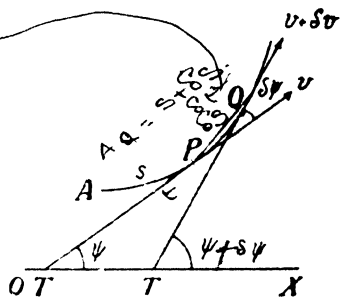
13·2. One particular case of constrained motion, *viz.*, the motion of a particle along a smooth vertical circle was discussed in Chapter VIII. We now consider similar motion in a more general manner. In this case the equations of motion are obtained by resolving forces along the tangent and the normal to the path.

13·21. Expressions for tangential and normal accelerations.

Suppose that a particle moves along the curve APQ. Let A be a fixed point and P and Q, the positions of the particle after times t and $t+\delta t$.

Let arc $AP=s$

and arc $AQ=s+\delta s$.



Chord PQ is the displacement of the particle in time δt.

$\therefore$ the velocity of the particle

$$= \operatorname{Lt}_{\delta t\to 0} \frac{\text{chord } PQ}{\delta t}$$

$$= \operatorname{Lt}_{\delta t\to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta t} = \frac{ds}{dt}$$

along the tangent at P.

It follows that if the point P is constrained to move along a given plane curve, the velocity of the point P at any instant is in the direction of the tangent to the curve at the point and its magnitude is $\frac{ds}{dt}$ where s denotes the length of the arc of the curve measured from any fixed point on it.

Now let v be the velocity at P and let it be $v+\delta v$ at Q.

If the tangents at P and Q make angles ψ and $\psi+\delta\psi$ with the fixed line OX, the change in velocity in direction of the tangent at $P=(v+\delta v)\cos\delta\psi-v$.

$$\therefore \text{ tangential acceleration}=\underset{\delta t\to 0}{\text{Lt}}\frac{(v+\delta v)\cos\delta\psi-v}{\delta t}=\frac{dv}{dt}$$

and since $v=\frac{ds}{dt}$ $\therefore$ $\frac{dv}{dt}=\frac{d^2s}{dt^2}$ or $v\frac{dv}{ds}$.

(the same as obtained in Art. 7·2)

The change in velocity in the direction of the normal at P

$$=(v+\delta v)\sin\delta\psi-0.$$

$\therefore$ the acceleration in the direction of the normal at P

$$=\underset{\delta t\to 0}{\text{Lt}}\frac{(v+\delta v)\sin\delta\psi}{\delta t}=\text{Lt}(v+\delta v)\left(\frac{\sin\delta\psi}{\delta\psi}\right)\frac{\delta\psi}{\delta t}$$

$$=v\frac{d\psi}{dt}=v\frac{d\psi}{ds}\cdot\frac{ds}{dt}=v^2\frac{d\psi}{ds}=\frac{v^2}{\rho},$$

where ρ is the radius of curvature of the curve at P and is equal to $\frac{ds}{d\psi}$ by Calculus.

In particular when a particle moves along a circle of radius r the radius of curvature is then the radius of the circle and the normal acceleration is $\frac{v^2}{r}$ as already obtained in Art. 7·2.

Whenever a particle moves along a curve the resultant tangential acceleration changes the velocity in magnitude while the resultant normal acceleration causes change in the direction of the velocity. If the former is zero, the particle moves with a constant speed, but if the latter is zero, the motion must take place along a straight line only.

Example 1. *A particle is moving in a parabola with uniform angular velocity about the focus, prove that its normal acceleration at any point is proportional to the radius of curvature of its path at that point.*

Let the radius vector SP make an angle θ with the axis.

Then by the question $d\theta/dt$=a constant=k (say).

Also by the property of a parabola, the angle ψ which the tangent at P makes with the axis=$\frac{1}{2}\theta$, so that

$$\frac{d\psi}{dt}=\tfrac{1}{2}\frac{d\theta}{dt}=\frac{k}{2} \text{ and } v=\frac{ds}{dt}=\frac{ds}{d\psi}\times\frac{d\psi}{dt}=\frac{k\rho}{2}$$

Hence the normal acceleration $\frac{v^2}{\rho}=\frac{k^2\rho^2}{4\rho}=\frac{k^2\rho}{4}$.

Example 2. *A particle describes a curve with uniform speed v. If the acceleration at any point P be $cv^2/(c^2+s^2)$ where c is a constant, prove that the curve described is a catenary.*

Since the speed v is constant, $\therefore$ the tangential acceleration $dv/dt=0$.

$\therefore$ the resultant acceleration=normal acceleration$=\frac{v^2}{\rho}=v^2\frac{d\psi}{ds}$.

$\therefore$ by the question, $v^2\frac{d\psi}{ds}=\frac{cv^2}{c^2+s^2}$

or $$\psi=\int\frac{c\,ds}{c^2+s^2}=\frac{c}{c}\tan^{-1}\frac{s}{c}+C.$$

Let $\psi=0$ when $s=0$, so that $C=0$.

$\therefore$ $\psi=\tan^{-1}\ s/c$, or $s=c\tan\psi$, a catenary.

Example 3. *A particle moves in an ellipse with constant speed, prove that the magnitude of its acceleration is proportional to the cube of the length of the perpendicular from the centre on the tangent.*

The speed being constant equal to v, say, the acceleration is only normal$=v^2/\rho$.

To find ρ, we use pedal equation (8) of Art. 13·132, so that we have

$$\frac{a^2b^2}{p^2}=a^2+b^2-r^2,\quad\text{and}\quad-\frac{2\,a^2b^2}{p^3}=-2r\frac{dr}{dp}\quad\text{or}\quad r\frac{dr}{dp}=\frac{a^2b^2}{p^3}$$

$$\therefore\quad\rho=r\frac{dr}{dp}=\frac{a^2b^2}{p^3},\quad\text{and}\quad\therefore\ \text{acceleration}=\frac{v^2}{\rho}=\frac{v^2}{a^2b^2}p^3.$$

Hence v being a constant, the acceleration varies as p^3, p being the length of the perpendicular from the centre on the tangent to the ellipse at the point.

Examples XLII-A.

1. If the speed of a particle moving in a curve varies as the radius of curvature, show that its direction of motion revolves with a constant angular velocity.

[**Hint.** $v=ds/dt=k\rho=kds/d\psi$, $\therefore$ $d\psi/dt=k$.].

2. A particle moves in a catenary $s=c\tan\psi$ the direction of its acceln. at any point makes equal angles with the tangent and the normal to the path at that point. If the speed at the vertex ($\psi=0$) be u, show that the velocity and acceln. at any other point are given by

$$ue^{\psi}\quad\text{and}\quad\frac{\sqrt{2}}{c}u^2e^{2\psi}\cos^2\psi.$$

3. A point moves in a plane in such a manner that its tangential and normal accelerations are always equal. Show that the equation to the path is

(*i*) the equiangular spiral $ks=Ae^{\psi}+B$, when the angular velocity of the tangent is constant and equal to k.

(*ii*) the catenary $s=c\tan\psi$, when the linear velocity of the point varies as e^{θ}, where $\tan\theta=\frac{s}{c}$.

4. A particle moves in a smooth tube in the form of a catenary, being attracted to the directrix by a force proportional to the distance from it. Show that the motion is simple harmonic. (*D.U. Hons.*)

[**Hint.** Resolve along the tangent and use $y=c\sec\psi$, $s=c\tan\psi$]

5. A particle starting from rest moves in a circle in such a way that its speed increases at a constant rate k. Show that when the particle has described an angular distance θ about the centre, its acceleration is

$$k\sqrt{4\theta^2+1}.$$

6. A particle is describing a circle of radius a in such a way that the tangential component of its acceleration bears to the normal component a constant ratio k, if u is the initial velocity, show that the circumference will be described in the time

$$\frac{a}{ku}\left(1-e^{-2\pi k}\right).$$

7. If throughout the motion of a particle describing a plane curve, the tangential and normal components of acceleration are constant and each equal to A, show that the angle φ through which the direction of motion turns whilst the velocity changes from V_0 to V is given by $\varphi=\log\frac{V}{V_0}$, and that if t is the time taken for the change

$$\varphi=\log\left(1+\frac{At}{V_0}\right). \qquad (P.U.)$$

8. Prove that the angular acceleration of the direction of a point moving in a plane is

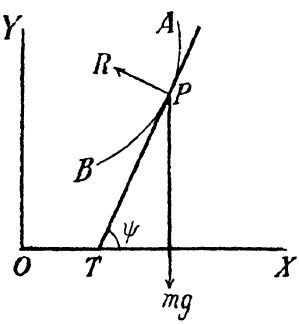
$$\frac{v}{\rho}\cdot\frac{dv}{ds}-\frac{v^2}{\rho^2}\cdot\frac{d\rho}{ds}.$$

$\left[\text{Hint. } \frac{d\psi}{dt}=\frac{d\psi}{ds}\cdot\frac{ds}{dt}=\frac{v}{\rho}\right]$.

13·22. Motion of a Heavy Particle on a smooth Curve in a Vertical Plane. *A particle is projected with a velocity v_1 along a smooth curve in a vertical plane. Its velocity v_2 after moving through a height h is given by*

$$v_2^2=v_1^2-2gh.$$

Let a particle of mass m move along the curve BPA, B being its initial position where its velocity is v_1, and A, the final position where the velocity is v_2.



Let P be any position of the particle where its velocity is v.

The only external forces acting on the particle are its weight, mg, acting vertically downwards and R, the pressure of the curve in the direction of the normal.

Equation of motion along the tangent is

$$mv\frac{dv}{ds}=\text{sum of the components of the forces along } TP$$
$$=-mg\sin\psi \qquad ..(1).$$

We know that for any curve, $\sin\psi=\frac{dy}{ds}$ and, therefore, the equation becomes,

$$v\frac{dv}{ds}=-g\frac{dy}{ds}.$$

Integrating, we get $\frac{1}{2}v^2=-gy+c$.

At B where $v=v_1$, let $y=y_1$,

then $\frac{1}{2}v_1^2=-gy_1+c$.

At A where $v=v_2$, let $y=y_2$,

then $\frac{1}{2}v_2^2=-gy_2+c$.

Subtracting, we get

$$\tfrac{1}{2}v_2^2-\tfrac{1}{2}v_1^2=-g(y_2-y_1) \qquad ..(2)$$

i.e.,
$$v_2^2=v_1^2-2g(y_2-y_1)=v_1^2-2gh.$$

If instead of moving upwards, the particle slides down the curve with an initial velocity u, the velocity v after moving a vertical distance h is given by,

$$v^2=u^2+2gh \qquad ..(3),$$

If the particle slides from rest,

$$v^2=2gh \qquad ..(4).$$

The final velocity in each case depends on the vertical distance moved and not on the form of the curve.

Thus, when a particle moves along a plane curve, the tangential equation of motion gives the velocity of the particle and the normal pressure is then obtained from the equation of motion along the normal, *viz.*,

$$\frac{mv^2}{\rho}=\text{the sum of the components of the forces along the normal}=R-mg\cos\psi \qquad ..(5)$$

13·23. Cor. If m is not cancelled from equation (1), equation (2) will assume the form,

$$\tfrac{1}{2}mv_2^2-\tfrac{1}{2}mv_1^2=-mgy_2+mgy_1$$

or
$$\tfrac{1}{2}mv_2^2+mgy_2=\tfrac{1}{2}mv_1^2+mgy_1,$$

which shows that regarding the axis of x as zero level

Kinetic Energy+Potential Energy at A=K.E.+P.E. at B
=*Constant.*

i.e., when a particle moves along a smooth vertical curve under gravity, the sum of its kinetic and potential energies remains always constant.

Thus, the principle of energy is applicable to the motion of a particle constrained to move along a smooth vertical plane curve under gravity. Its application to a few particular cases was shown in Chapter IX.

Example. *A thin wire in the form of the curve $s=c\tan\psi$ is fixed with its axis vertical and vertex downwards and a small bead of mass m slides freely on it. If the bead starts from rest at a height nc above the vertex, prove that the reaction at the vertex will be $(2n+1)mg$.*

By (4), the velocity v at the vertex is given by

$$v^2=2gh=2gnc.$$

By (5), the reaction R at the vertex where $\psi=0$ is given by

$$mv^2/\rho=R-mg\cos\psi=R-mg.$$

But $$\rho=\frac{ds}{d\psi}=c\sec^2\psi=c, \text{ at the vertex, where } \psi=0.$$

$\therefore$ $$R=mg+mv^2/\rho=mg+m.2gnc/c=(2n+1)mg.$$

Examples XLII—B

1. A particle falls from rest at the vertex of an inverted catenary $s=c\tan\psi$, prove that the particle leaves the curve at the point where $\psi=\sec^{-1}2$.

Show that at this point the length of the arc described is $\sqrt{3}$ times the vertical distance through which the particle falls.

[**Hint.** $\rho=c\sec^2\psi$, $y=c\sec\psi$.]

2. A particle of mass m moves in a smooth horizontal circular tube of centre C and radius a under the action of a force directed towards a fixed point A of the tube and of magnitude μr per unit mass, where r is the distance of the particle from A. If the particle starts from rest at B, the other end of the diameter through A, prove that when the particle is at P, where $\angle BCP=\theta$, its velocity v and the pressure R of the tube on it are given by

$$v^2=2\mu a^2(1-\cos\theta),\ R=ma\mu(1-3\cos\theta).$$

3. A bead, which slides on a wire in the form of a parabola with axis vertical and vertex upwards, is just displaced from rest at the highest point; show that in any subsequent position the velocity of the bead varies as its distance from the axis of the parabola and that the pressure on the curve varies as its curvature.

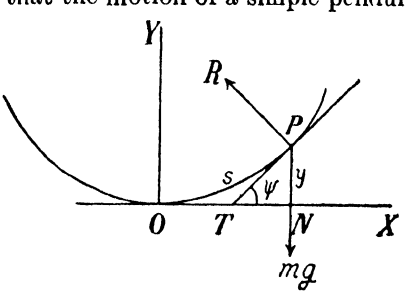
4. A smooth parabolic tube is placed, vertex downwards in a vertical plane and a particle slides down the tube from rest under gravity. Prove that in any position the reaction of the tube is

$$2w(h+a)/\rho,$$

where w is the weight of the particle, ρ the radius of curvature, $4a$ the latus rectum and h, the original vertical height of the particle above the vertex.

5. A smooth wire is bent into the form $y=\sin x$ and placed in a vertical plane with the axis of x horizontal. A bead of mass m slides down the wire starting from rest at $x=\frac{1}{2}\pi$. Show that the pressure on the wire as the bead passes through the origin is $mg/\sqrt{2}$, and find the pressure as it passes through $x=-\frac{1}{2}\pi$. [*Ans.* $mg(3+\sqrt{2})$].

13·24. · The Cycloidal Pendulum. In Art. 8·5, we have seen that the motion of a simple pendulum is simple harmonic motion only when the angle of swing is so small that $\sin\theta$ is very nearly equal to θ and the amplitude to the motion is so small that it may be treated as infinitesimal. If, however, the amplitude of motion is not small and the particle is supposed to be constrained to move under gravity, along the arc of a smooth cycloid in a vertical plane, the equation of motion of the particle along the tangent to the curve is



$$m\frac{d^2s}{dt^2}=-mg\sin\psi \qquad ..(1),$$

where ψ is the angle which the tangent to the curve makes with the horizontal and s is the length of its arc measured from the vertex, the cycloid being placed with its vertex downwards and axis vertical.

We know by the Calculus that the intrinsic equation of the cycloid is

$$s=4a\sin\psi \qquad ..(2),$$

s being measured from the vertex where $\psi=0$, and a being the radius of the generating circle.

From (1) and (2), we have

$$\frac{d^2s}{dt^2}=-\frac{g}{4a}s \qquad ..(3)$$

which shows that the motion shall be simple harmonic no matter how great the amplitude.

The time of a complete oscillation is $2\pi\sqrt{\frac{4a}{g}}$ which is constant for oscillations, small or large.

Thus, *if a particle is constrained to move along a smooth cycloidal curve, its period of motion is absolutely independent of the amplitude.*

(This is an answer to the question which interested the mathematicians of the 18th century—in what curve should the bob of a pendulum swing in order that the period of oscillation may be absolutely independent of the amplitude ?)

The oscillations on a cycloidal arc are called *isochronous* because the period is the same for large or small oscillations.

This important property of a cycloid finds its application in the formation of clocks.

A cycloidal pendulum may be constructed by causing the cord of the pendulum to wind and unwind itself on the evolute of the path.

13·241. In order to find the pressure of the curve on the particle, we write its equation of motion in the direction of the normal at the point, namely

$$\frac{mv^2}{\rho}=R-mg\cos\psi \qquad ..(4),$$

where ρ is the radius of curvature of the curve, R the normal pressure and v the velocity of the particle obtained from equation (3) by integration.

Note 1. The student acquainted with elements of differential equation will note that

$$s=A\cos\sqrt{\mu}t+B\sin\sqrt{\mu}t.$$

where $\mu=\frac{g}{4a}$ is the most general solution of equation (3).

Note 2. Since $s=4a\sin\psi=4a\frac{dy}{ds}$, $\therefore s^2=8ay$, y and s being measured from the vertex of the cycloid.

Example 1. *A particle oscillates in a cycloid under gravity the amplitude of the motion being l and the periodic time being T. Show that its velocity at a time t measured from a position of rest is*

$$\frac{2\pi l}{T}\sin\frac{2\pi t}{T}.$$

Equation of motion of the particle is

$$\frac{d^2s}{dt^2}=-\frac{g}{4a}s=-n^2s,$$

where $n^2=\frac{g}{4a}$ and $T=\frac{2\pi}{n}$ or $n=\frac{2\pi}{T}$.

The solution of the equation gives

$$s = A\cos nt + B\sin nt \qquad ..(1)$$

$$\therefore \quad v = \frac{ds}{dt} = -An\sin nt + Bn\cos nt \qquad ..(2)$$

By the question when $t=0$, $s=l$ and $v=0$.

$\therefore$ from (1) $A=l$ and from (2) $B=0$.

Hence $s = l\cos nt$

and $v = -nl\sin nt$

$$= \frac{2\pi l}{T}\sin\frac{2\pi t}{T} \text{ (in magnitude).}$$

Example 2. *A particle of mass m starts from rest at the cusp of a smooth cycloidal arc whose axis is vertical and vertex downwards. Find the pressure on the curve at any point.*

The height of a cusp above the vertex being $2a$, the velocity v, at a point P of the cycloid at a height y above the vertex is given by

$$v^2 = 2g(2a-y)$$

$$= 2g\left(2a - \frac{s^2}{8a}\right) \qquad \text{[By Note 2, Art. 13·241]}$$

$$= 2g\left(2a - \frac{16a^2\sin^2\psi}{8a}\right) = 4ag\cos^2\psi.$$

By (4), 13·241, the pressure R is given by

$$\frac{mv^2}{\rho} = R - mg\cos\psi, \text{ where } \rho = \frac{ds}{d\psi} = 4a\cos\psi.$$

Hence $R = mg\cos\psi + m\,4ag\cos^2\psi/4a\cos\psi = 2mg\cos\psi$,

where ψ is the angle which the tangent at A makes with the tangent at the vertex.

Examples XLIII

1. A particle oscillates in a complete cycloid from cusp to cusp. Prove that the velocity at any point P is equal to the resolved part of the velocity at the vertex along the tangent at P.

2. A particle starts sliding down a smooth inverted cycloid $s=4a\sin\psi$ from rest at the vertex. Show that it will leave the curve when it has descended a vertical distance equal to a.

3. A particle of mass m starts from rest at the cusp of a smooth cycloidal arc whose axis is vertical and vertex downwards. Prove that

(*a*) the time occupied in falling half the distance measured along the arc to the vertex is $\frac{1}{2}$ of the time required to reach the vertex.

(*b*) the time occupied in falling down first half of the vertical height is equal to the time of falling down the second half;

(*c*) the pressure on the curve at any point is equal to twice that due to the weight of the particle alone.

4. Two particles are let drop from the cusp of a cycloid down the curve at an interval of time t, prove that they will meet at a time

$$2\pi\sqrt{\frac{a}{g}} + \frac{t}{2}.$$

5. A particle is projected with velocity V from a cusp of a smooth

cycloid down the arc. Show that the time of reaching the vertex is

$$2\sqrt{\frac{a}{g}}\tan^{-1}\frac{\sqrt{4ag}}{V}.$$

6. Show that the time a train, if unresisted, takes to pass through a tunnel under a river in the form of an arc of a cycloid of length $2l$ and height h, cut by a horizontal line is

$$\frac{2l}{\sqrt{2gh}}\tan^{-1}\frac{\sqrt{2gh}}{v}, \text{ or } \frac{l}{\sqrt{2gh}}\cos^{-1}\left[\frac{v^2-2gh}{v^2+2gh}\right],$$

where v is the velocity with which the train enters and leaves the tunnel.

7. Two particles $2m$, m are connected by a light string ABC, the heavier rests in a smooth open tube in the form of a cycloid with axis vertical and vertex downwards, and the other particle hangs vertically from an open end B at which the tangent is vertical. Prove that in the position of equilibrium the tangent at A to the cycloid is inclined at 30° to the horizontal and that the time of a small oscillation about this position is

$$2\pi\sqrt{\frac{6a}{g}}.$$

8. A heavy bead slides on a smooth fixed circular wire of radius a. If it be projected from the lowest point with a velocity just sufficient to carry it to the highest point, prove that the radius through the bead will turn through an angle θ in time t given by the equation

$$t=\sqrt{\frac{a}{g}}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{4}\right).$$

9. A parabola of latus rectum $4a$, is placed with its axis horizontal and plane vertical, and a heavy smooth particle is projected from the vertex with a velocity due to a height h and moves on the concave side of the arc. Prove that if θ be the angle described about the focus before leaving the curve

$$h=a\left(\tan^3\frac{\theta}{2}+3\tan\frac{\theta}{2}\right).$$

Also by taking the particular case $h=14a$ prove that the vertical height attained before leaving the arc is two-thirds of the greatest height attained.

10. A smooth parabolic cylinder is fixed with its generators and the axis of each of its normal sections horizontal. A particle is placed upon it at a height above the axial plane equal to the latus rectum, prove that it will run off at the extremity of the latus rectum and will then describe a parabola of equal latus rectum.

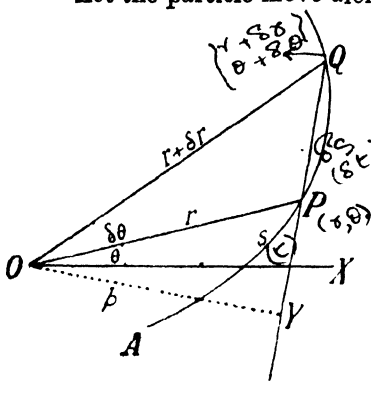
11. A particle is describing the ellipse $\frac{b^2}{p^2}=\frac{2a}{r}-1$ in such a manner that its acceleration at any point is made up of two components equal in magnitude and in directions of the two foci of the ellipse. Prove that the speed of the particle along the curve is constant, and that the magnitude of either component of the acceleration is inversely proportional to the product of the focal distances of particle. (*P.U.*)

III. MOTION UNDER THE ACTION OF A CENTRAL FORCE

13·3. In this case a particle moves under a *central force i.e.*, a force directed towards a fixed point called the *centre of force*. The path described by the particle is called a *central orbit*. The position of the particle at any instant is defined by the polar co-ordinates r and θ referred to the centre of force O as the origin and any fixed line OX through O as the initial line. The equations of motion are then obtained by resolving forces along and perpendicular to the radius vector, as given later in Art. 13·43.

13·31. Areal Velocity. *When a particle moves along a plane curve, the rate of change of the area traced out by the radius vector joining the particle to a fixed point is called the areal velocity of the particle.*

Let the particle move along the curve APQ and let it describe the arc $PQ=\delta s$ in time δt.



Let (r, θ) be the co-ordinates of P and $(r+\delta r, \theta+\delta\theta)$ be those of Q, then the areal velocity at P

$$=\text{Lt}\ \frac{\triangle OPQ}{\delta t}=\text{Lt}\ \frac{\frac{1}{2}r(r+\delta r)\sin\delta\theta}{\delta t}.$$

$$=\text{Lt}\ \frac{\frac{1}{2}r^2\delta\theta}{\delta t}=\frac{1}{2}r^2\frac{d\theta}{dt}.$$

If p is the length of the perpendicular from O on the line PQ, and v the velocity of the particle at P, then

$$\text{the areal velocity}=\text{Lt}\frac{\triangle OPQ}{\delta t}=\text{Lt}\ \frac{\frac{1}{2}p\delta s}{\delta t}=\frac{1}{2}p\frac{ds}{dt}=\frac{1}{2}pv.$$

Again, if (x, y), $(x+\delta x, y+\delta y)$ be the co-ordinates of P and Q then

$$\text{the areal velocity}=\text{Lt}\frac{\triangle OPQ}{\delta t}=\text{Lt}\ \frac{\frac{1}{2}(x\delta y-y\delta x)}{\delta t}=\frac{1}{2}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right).$$

Thus, we have :—

$$\textbf{Twice the areal velocity}=\mathbf{r^2\frac{d\theta}{dt}=vp=x\frac{dy}{dt}-y\frac{dx}{dt}}.$$

Example 1. *A point P moves in a parabola with constant speed V, show that (i) the areal velocity about the focus S varies as $\sec^2\frac{1}{2}\theta$, and (ii) the angular velocity about S varies as $\cos^3\frac{1}{2}\theta$, or inversely as $SP^{\frac{3}{2}}$, where θ is the angular distance from the vertex.*

In a parabola with focus as origin, we have

$$p^2=ar \text{ and } \frac{l}{r}=1+\cos\theta=2\cos^2\tfrac{1}{2}\theta, \textit{ i.e., } r=\tfrac{1}{2}l\sec^2\tfrac{1}{2}\theta \qquad ..(1).$$

$\therefore$ (*i*) twice the areal velocity about S at any point

$=Vp=V\sqrt{ar}=V\sqrt{\frac{1}{2}al}\sec\frac{1}{2}\theta$, which varies as $\sec\frac{1}{2}\theta$.

(*ii*) ω, the angular velocity about S is given by

$$\omega=\frac{d\theta}{dt}=\frac{Vp}{r^2}=\frac{V\sqrt{ar}}{r^2}=\frac{V\sqrt{a}}{r^{\frac{3}{2}}}=\frac{V\sqrt{a}}{(\frac{1}{2}l)^{3/2}}\cos^3\tfrac{1}{2}\theta \qquad [\textit{by } (1)].$$

It shows that the angular velocity varies either as $\cos^3\frac{1}{2}\theta$ or inversely as $r^{\frac{3}{2}}$ *i.e.*, inversely as $(SP)^{\frac{3}{2}}$.

Example 2. *A particle is moving in a parabola with uniform angular velocity about the focus, prove that its normal acceleration at any point varies as the radius of curvature ρ at that point.*

In this case $d\theta/dt$ is constant$=c$, say,

If v be the velocity at any point, we have

$$c=\frac{d\theta}{dt}=\frac{vp}{r^2}, \quad \text{or} \quad v^2=\frac{c^2r^4}{p^2}=\frac{c^2r^4}{ar}=\frac{c^2r^3}{a} \quad (\because p^2=ar).$$

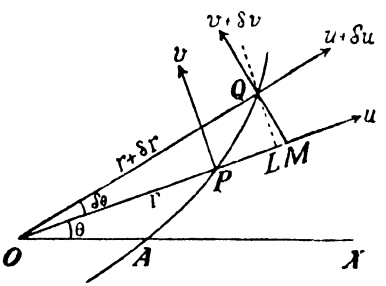
This shows that v varies as $r^{\frac{3}{2}}$ or $(SP)^{\frac{3}{2}}$.

Now since $p^2=ar$, $\quad \therefore \quad 2p=a\frac{dr}{dp}$,

$$\therefore \quad \rho=r\frac{dr}{dp}=\frac{2pr}{d}=\frac{2r^{\frac{3}{2}}}{\sqrt{a}}, \quad \text{or} \quad r^3=\frac{a\rho^2}{4}.$$

$$\therefore \quad \text{normal acceleration}=\frac{v^2}{\rho}=\frac{c^2r^3}{a\rho}=\frac{c^2}{a\rho}\cdot\frac{a\rho^2}{4}=\frac{c^2}{4}\rho.$$

13·4. Expressions for Radial and Transverse Components of Velocity and Acceleration. Let a particle move along a curve from $P(r, \theta)$ to $Q(r+\delta r, \theta+\delta\theta)$ in time δt. It is required to find expressions for the components of velocity and acceleration of the particle in the position P at the time t, in the direction of the radius vector OP and at right angles to it, the senses being those in which r and θ increase. If LQ be perp. to OP, then



u, the component of the velocity along OP

$$=\text{Lt}\ \frac{\text{displacement along } OP \text{ in time } \delta t}{\delta t}$$

$$=\text{Lt}\ \frac{PL}{\delta t}=\text{Lt}\ \frac{(r+\delta r)\cos\delta\theta-r}{\delta t}=\frac{dr}{dt} \qquad ..(1),$$

v, the component of the velocity perpendicular to OP

$$=\text{Lt}\ \frac{\text{displacement perp. to } OP \text{ in time } \delta t}{\delta t}$$

$$=\text{Lt}\ \frac{QL}{\delta t}=\text{Lt}\ \frac{(r+\delta r)\sin\delta\theta}{\delta t}=r\frac{d\theta}{dt} \qquad ..(2).$$

Hence $\frac{dr}{dt}$ and $r\frac{d\theta}{dt}$ are the radial and transverse velocities of the particle, the actual velocity being the resultant of these two.

Now to find the components of acceleration, suppose $u+\delta u$, $v+\delta v$ are the components of the velocity of the particle in the position

Q, at time $t+\delta t$, in the direction OQ and perp. to it, then f_1, the component of acceleration along OP

$$= \text{Lt}\ \frac{\text{change of vel. along } OP \text{ in time } \delta t}{\delta t}$$

$$= \text{Lt}\ \frac{[(u+\delta u)\cos\delta\theta - (v+\delta v)\sin\delta\theta] - u}{\delta t}$$

$$= \frac{du}{dt} - v\frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\right) - r\frac{d\theta}{dt}\times\frac{d\theta}{dt} \qquad \text{[by (1) and (2)]}$$

$$= \frac{\mathbf{d^2r}}{\mathbf{dt^2}} - \mathbf{r}\left(\frac{\mathbf{d\theta}}{\mathbf{dt}}\right)^2 \qquad \ldots(3),$$

and f_2, the component of acceleration perp. to OP

$$= \text{Lt}\ \frac{\text{change of vel. perp. } OP \text{ in time } \delta t}{\delta t}$$

$$= \text{Lt}\ \frac{[(u+\delta u)\sin\delta\theta + (v+\delta v)\cos\delta\theta] - v}{\delta t}$$

$$= u\frac{d\theta}{dt} + \frac{dv}{dt} = \frac{dr}{dt}\cdot\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right) \qquad \text{[by (1) and (2)]}$$

$$= 2\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = \frac{\mathbf{1}}{\mathbf{r}}\frac{\mathbf{d}}{\mathbf{dt}}\left(\mathbf{r^2}\frac{\mathbf{d\theta}}{\mathbf{dt}}\right)$$

the actual acceleration is the resultant of f_1 and f_2.

Cor. In the particular case when the path of a particle is a circle with its centre as origin, r is constant, $dr/dt=0$ and thus the radial and transverse components of acceleration reduce to

$$-r\left(\frac{d\theta}{dt}\right)^2 \text{ and } r\frac{d^2\theta}{dt^2} \text{ respectively.}$$

13·41. The above results can also be obtained as follows :—

Suppose that the Cartesian co-ordinates of P are (x, y), then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Differentiating, we have

$$\left.\begin{aligned} \frac{dx}{dt} &= \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt} \\ \text{and} \quad \frac{dy}{dt} &= \sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt} \end{aligned}\right\} \qquad \ldots(1).$$

u, the component of the velocity along OP

$$= \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta = \frac{dr}{dt} \qquad \text{by (1),}$$

and v, the component of the velocity perp. to OP

$$= \frac{dy}{dt}\cos\theta - \frac{dx}{dt}\sin\theta = r\frac{d\theta}{dt}, \qquad \text{by (1).}$$

Differentiating equations (1), we have

$$\frac{d^2x}{dt^2} = \frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2},$$

$$\frac{d^2y}{dt^2}=\frac{d^2r}{dt^2}\sin\theta+2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta-r\sin\theta\left(\frac{d\theta}{dt}\right)^2+r\cos\theta\frac{d^2\theta}{dt^2}.$$

The component of acceleration along OP

$$=\frac{d^2x}{dt^2}\cos\theta+\frac{d^2y}{dt^2}\sin\theta=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2,$$

and the component of acceleration perpendicular to OP

$$=\frac{d^2y}{dt^2}\cos\theta-\frac{d^2x}{dt^2}\sin\theta=2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

Example. *A particle describes the equianuglar spiral* $r=ae^{\theta}$, *in such a manner that its acceleration has no radial component. Prove that its angular velocity is constant and that the magnitude of the velocity and acceleration is each proportional to* r.

Here the radial component of acceln. $=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=0.$

$$\frac{d^2r}{dt^2}=r\left(\frac{d\theta}{dt}\right)^2 \qquad ..(1).$$

Also, since $r=ae^{\theta}$, $\quad\frac{dr}{dt}=ae^{\theta}\frac{d\theta}{dt}=r\frac{d\theta}{dt}$ $\quad\therefore(2),$

and from (2), $\quad\frac{d^2r}{dt^2}=\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}=r\left(\frac{d\theta}{dt}\right)^2+r\frac{d^2\theta}{dt^2}$ $\quad..(3).$

$\therefore$ by (1) and (3) $r\frac{d^2\theta}{dt^2}=0$, so that $\frac{d\theta}{dt}=$ a constant, say ω, *i.e.*, the particle moves with a constant angular velocity.

Now from (2), $\quad\frac{dr}{dt}=r\frac{d\theta}{dt}=r\omega$ $\quad..(4).$

Hence the velocity of the particle $=\sqrt{\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2}$

$=\sqrt{(r\omega)^2+(r\omega)^2}=\sqrt{2r^2\omega^2}=\omega r\sqrt{2}$ *i.e.*, varies as r.

Also the radial acceln. being zero, the resultant acceln.

$=$the transverse acceln. $=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{1}{r}\frac{d}{dt}(r^2\omega)=\frac{2r\omega}{r}\cdot\frac{dr}{dt}$

$=2r\omega^2$ by (4)

which varies as r.

Examples XLIV

1. A particle moves along $r=Ae^{k\theta}$, $\theta=Bt$, prove that its acceleration is proportional to r and makes a constant angle with the radius vector.

2. If the angular velocity of a particle about a point in its plane of motion be constant, prove that the transverse component of its acceleration is proportional to the radial component of its velocity.

3. If the velocities of a particle along and perpendicular to the radius vector from a fixed point are λr and $\mu\theta$, show that the accelerations along and perpendicular to the radius vector are

$$\lambda^2r-\frac{\mu^2\theta^2}{r} \quad\text{and}\quad \mu\theta\left(\lambda+\frac{\mu}{r}\right).$$

4. The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$. Show that the equation to the path of the particle is

$$\frac{\lambda}{\theta} = \frac{\mu}{2r^2} + C.$$

Also show that the components of acceleration are

$$2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}, \quad \lambda\mu r\theta^2 + 2\frac{\mu^2\theta^3}{r}.$$

5. If the radial and transverse velocities of a point are always proportional to each other, show that the equation to the path is an equiangular spiral.

If, in addition, the radial and transverse accelerations are also always proportional to each other, show that the velocity of the particle varies as some power of the radius vector. (*All. U.*)

6. If the path of a particle is $r = a \tan\theta$ and the acceleration is directed towards the origin, show that the acceleration in terms of r is

$$\frac{h^2}{r^3}\left[3 + \frac{2a^2}{r^2}\right] \text{ where } h = r^2\frac{d\theta}{dt}.$$

7. Prove that the path of a point P which possesses two constant velocities, one along a fixed direction and the other perpendicular to the radius vector OP drawn from a fixed point O, is a conic section.

8. A point moves in a circular path of radius a, so that its angular velocity about a fixed point in the circumference of the circle is constant and equal to ω, show that the resultant acceleration of the point at every point of the path is of constant magnitude $4a\omega^2$.

[Take $r = 2a\cos\theta$ as the equation of a circle.]

9. A particle describes the circle $r = 2a\sin\theta$ with speed $k\,\mathrm{cosec}^2\theta$. Prove that its acceleration is wholly along the line joining the particle to the origin.

10. A particle moves along the circle $r = 2a\cos\theta$ in such a way that the acceleration towards the origin is always zero. Prove that

$$\frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} = -2\omega^2\cot\theta, \ \omega \text{ being the angular velocity.}$$

Deduce that the transverse acceleration is proportional to $\mathrm{cosec}^5\theta$.

11. The ends of a straight rod are moving in two straight grooves intersecting at right angles at O, and P is the foot of the perpendicular from O on the rod. Show that the velocity of P perpendicular to OP is $\omega.OP$ and along OP is $2\omega.CP$, where C is the middle point of the rod and ω is the angular velocity of C about O.

12. A smooth straight tube OA $(=a)$ rotates in horizontal plane about one end O with uniform angular velocity ω.

(*a*) When initially a particle is at rest at A relative to the tube, show that at time t, the distance of the particle from O is $a\cosh(\omega t)$ and that the force exerted by the tube on the particle at that time is proportional to $\sinh(\omega t)$.

(*b*) When a particle is suddenly projected from A towards O with velocity $a\omega$, show that the particle will have travelled half way down the tube after a time $\frac{1}{\omega}\log 2$ and will not reach O in any finite time.

13. A particle is placed at rest in a rough tube at a distance a from one end and the tube starts rotating horizontally with a uniform angular velocity ω about this end. Show that the distance of the particle at time t is

$$ae^{-\omega t\tan\lambda}[\cosh(\omega t\sec\lambda) + \sin\lambda\sinh(\omega t\sec\lambda)].$$

where $\tan\lambda$ is co-efficient of friction. (*P. U.*)

13·42. Angular Momentum. The angular momentum of a *particle* moving along a plane curve obout a point in the plane is the moment of its momentum about that point.

The magnitude of the moment of momentum of a particle P of mass m, at any instant when its velocity is v, about any point O (in the plane of its motion) is

$$mvp \qquad ..(1),$$

where p is the length of the perpendicular from O to the tangent to the path of the particle at P.

The sign is taken positive or negative according as the rotation is counter-clockwise or clockwise.

If the velocity v is resolved along and perpendicular to OP in the direction of θ increasing, then the components of the momentum of the particle in these directions are $m\frac{dr}{dt}$, and $mr\frac{d\theta}{dt}$. Hence another expression for the moment of momentum about O is

$$mr^2\frac{d\theta}{dt} \qquad ..(2),$$

the moment of $m\frac{dr}{dt}$ about O being zero.

Again, if the velocity is resolved along two perpendicular lines OX, OY, then the components of the momentum are $m\frac{dx}{dt}, m\frac{dy}{dt}$ and the moment of momentum about O is

$$m\left[x\frac{dy}{dt}-y\frac{dx}{dt}\right] \qquad ..(3).$$

Hence the *angular momentum* of a particle about the origin

$$=mr^2\frac{d\theta}{dt}=mvp=m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right).$$

Also the expression for the kinetic energy of the particle in polar co-ordinates is

$$\tfrac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2\right].$$

13·43. Equations of Motion. We have already discussed motion of a particle in a plane by writing down its equations of motion either in the directions of two fixed co-ordinate axes or in the direction of the tangent and normal to the path described by the particle. However a large number of dynamical problems, where a particle moves under a central force, are readily solved, as already pointed out, by writing the equations of motion in the direction of the radius vector and in a direction perpendicular to it.

These equations are of the form

$$m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]=F \qquad ..(1),$$

and

$$\frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=T \qquad ..(2),$$

where m is the mass of the particle and F and T are the sums of the components of the forces in the radial and transverse directions.

13·431. Central Force. When a particle moves under a *central* force, *i.e.*, a force directed towards a fixed point with its intensity as some function of the distance from the fixed point, then $T=0$ and from (2), we have

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0, \text{ i.e., } r^2\frac{u\theta}{dt}=h \text{ (a constant).}$$

So that *in a central orbit the radius vector joining the particle to the centre of force sweeps out equal areas in equal times and the angular momentum of the particle remains constant throughout its motion.*

Also from Art. 13·31, we have :

$$r^2\frac{d\theta}{dt}=vp=\text{twice the areal velocity}=h.$$

$$\therefore \qquad \frac{d\theta}{dt}=\frac{h}{r^2} \text{ and } v=\frac{h}{p}.$$

i.e., in the motion of a particle subject to any central force,

(*i*) *the angular velocity of the particle varies inversely as the square of the distance* ; and

(*ii*) *velocity of the particle varies inversely as the perpendicular drawn from the centre upon the tangent to the path.*

These results hold true in the case of planets moving round the sun, the sun being the centre of force.

Since the moment of F about O is obviously zero and since any point may be taken as the origin, equation (2) Art. 13·43, when written in the form

$$\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right)=rT,$$

shows that *for the motion of a particle along a curve, the rate of change of its angular momentum about any point O is equal to the sum of the moments about the same point O, of the forces acting on the particle.*

Examle 1. *A particle P of unit mass describes an ellipse with centre C under the action of a force $\mu.CP$. Show that the normal acceleration at any instant varies inversely as the speed of the particle.*

Let ϕ be the angle which the tangent at P makes with CP, then the normal component of the acceleration

$$=\mu.CP\sin\phi=\mu r\sin\phi=\mu p=\frac{\mu vp}{v}=\frac{\mu h}{v},$$

where h is a constant.

For, the force being a central force, vp is constant, the constant being taken as h.

Example 2. *A particle P of mass m, moves on a smooth horizontal table. A light string attached to P passes through a small hole in the table and an equal particle Q is suspended at this end of the string.*

If initially $OP=a$, and the particle P is moving with velocity $\sqrt{\frac{1}{2}ag}$ in a direction at right angles to OP, determine the extreme values of OP during the subsequent motion and find the tension of the string at any instant.

Let T be the tension of the string and $OP=r$ at any instant t.

The table being smooth, there is no force on P perp. to OP and T is the only force in the direction PO.

Therefore, the equations of motions for P are :—

$$\frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0,$$

$$i.e.,\ r^2\frac{d\theta}{dt}=\text{constant}=\text{its initial value}=a\times\sqrt{\tfrac{1}{2}ag} \qquad ..(1),$$

and
$$m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]=-T \qquad ..(2).$$

If l be the length of the string, the equation of motion for Q is

$$m\frac{d^2r}{dt^2}\left(l-r\right)=mg-T,\ \ i.e.,\ m\frac{d^2r}{dt^2}=T-mg \qquad ..(3).$$

Subtracting (2) from (3) and using (1), we have :

$$2T-mg=mr\left(\frac{d\theta}{dt}\right)^2=mr\,\frac{\frac{1}{2}a^3g}{r^4}=\frac{mga^3}{2r^3}.$$

$$T=\tfrac{1}{4}mg(2+a^3/r^3).$$

Again, adding (2) and (3) and using (1), we obtain :

$$2m\frac{d^2r}{dt^2}-mr\left(\frac{d\theta}{dt}\right)^2=-mg \quad \text{or} \quad 2\frac{d^2r}{dt^2}-\frac{ga^3}{2r^3}=-g.$$

Multiplying this equation by $\dfrac{dr}{dt}$ and integrating, we have

$$\left(\frac{dr}{dt}\right)^2+\frac{ga^3}{4r^2}=-gr+c.$$

Initially when $r=a$, $\dfrac{dr}{dt}=0$, which gives $c=\frac{5}{4}ag$.

Thus,
$$\left(\frac{dr}{dt}\right)^2+\frac{ga^3}{4r^2}=-gr+\tfrac{5}{4}ag.$$

For the max. or min. value of r, $\dfrac{dr}{dt}=0$, so that $\dfrac{ga^3}{4r^2}+gr=\frac{5}{4}ag$,

$$i.e.,\qquad 4r^3-5ar^2+a^3=0 \quad \text{or} \quad (r-a)(4r^2-ar-a^2)=0.$$

This gives $r=a$ or $r=\dfrac{1+\sqrt{17}}{8}\,a=0{\cdot}64a$,

the negative value of r *viz.*, $\dfrac{1-\sqrt{17}}{8}\,a$, being rejected.

Examples XLV

1. A particle capable of moving on a smooth horizontal table is connected by a light inextensible string which passes without friction through a small hole, to a second equal particle hanging freely below the table. When the first particle is at a distance a from the hole, it is moving at right angles to the string with the velocity $\sqrt{\frac{2}{3}ag}$. Prove that the hanging particle will move up and down through a distance a and that the tension of the string will vary between 2/3 and 11/6 of the weight of the particle.

2. A string passes through a small hole in a smooth horizontal table, and has equal particles attached to its ends, one hanging vertically and the other lying on the table at a distance a from the hole. This latter is projected with a velocity $\sqrt{ag}$ perpendicular to the string. Show that the hanging particle will remain at rest, and that if it be slightly disturbed, the time of a small oscillation will be

$$2\pi\sqrt{2a/3g}.$$

3. Two particles of masses m, $2m$ lie on a smooth table and connected by a taut, inextensible, light string of length $2a$ which passes through a fixed smooth ring fixed at a point O on the table. The particle m is at a distance a from O and is projected along the table with velocity u perpendicular to the portion of the string attached to it. Prove that the particle $2m$ reaches O after a time $3a/u$.

4. Two particles of masses $3m$ and m are attached to the ends of an inextensible string which passes through a smooth fixed ring, the whole resting on a horizontal table. The particle m being projected at right angles to the string show that path is $a=r\cos\frac{1}{2}\theta$, a being the initial distance of m from the ring.

5. A particle of unit mass lies on a smooth horizontal table and is attached to one end of an elastic string of natural length l and modulus of elasticity λ, whose other end is fixed on the table. The particle is held so that the string is just tight, and projected at right angles to the string with velocity $2\sqrt{\lambda l/3}$, show that the greatest length of the string in the resulting motion is $2l$.

6. One end of a light inextensible string is attached to a particle of mass m which moves on a smooth horizontal table. The other end is passed through a hole in the table and supports a particle of mass M which hangs vertically below the hole. Show that the path described by the particle on the table satisfies the diffential equation

$$(m+M)\frac{d^2u}{d\theta^2}+mu=\frac{Mg}{h^2u^2},\quad \text{where } h \text{ is a constant and } u=\frac{1}{r}.$$

13·5. Central Orbits. *To find the differential equation of the path of a particle moving in a plane under a force which is directed to a fixed centre.*

Polar Equation. Let (r, θ) be the co-ordinates of the particle P referred to the fixed centre O and let mF be the attracting force acting on it, then the equations of motion are

$$\frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0 \qquad ..(1)$$

and

$$m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]=-mF \qquad ..(2)$$

Integrating (1), we have

$$r^2\frac{d\theta}{dt}=h \text{ (a constant)} \qquad ..(3)$$

i.e., $\dfrac{d\theta}{dt}=\dfrac{h}{r^2}=hu^2$, where $u=\dfrac{1}{r}$ or $r=\dfrac{1}{u}$.

$\therefore$ $\dfrac{dr}{at}=\dfrac{d}{d\theta}\left(\dfrac{1}{u}\right)\cdot\dfrac{d\theta}{dt}=-\dfrac{1}{u^2}\dfrac{du}{d\theta}\cdot\dfrac{d\theta}{dt}=-\dfrac{1}{u^2}\dfrac{du}{d\theta}hu^2=-h\dfrac{du}{d\theta}$,

and $\dfrac{d^2r}{dt^2}=\dfrac{d}{d\theta}\left(\dfrac{dr}{dt}\right)\dfrac{d\theta}{dt}=\dfrac{d}{d\theta}\left(-h\dfrac{du}{d\theta}\right)hu^2=-h^2u^2\dfrac{d^2u}{d\theta^2}$.

Substituting in equation (2), we get

$$-h^2u^2\frac{d^2u}{d\theta^2}-\frac{1}{u}h^2u^4=-F$$

or
$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2} \qquad ..(4)$$

which is the equation of the orbit in polar co-ordinates, F being the *force per unit mass or the acceleration.*

13·51. Pedal Equation. If p be the length of the perpendicular from O upon the tangent at P, then by the Calculus

$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2 \qquad ..(5).$$

Differentiating equation (5), with respect to r, we have

$$-\frac{2}{p^3}\cdot\frac{dp}{dr}=2u\frac{du}{d\theta}\frac{d\theta}{dr}+2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}\frac{d\theta}{dr}$$

$$=2\frac{du}{d\theta}\frac{d\theta}{dr}\left(u+\frac{d^2u}{d\theta^2}\right)=2\frac{du}{dr}\left(u+\frac{d^2u}{d\theta^2}\right)=-\frac{2}{r^2}\left(u+\frac{d^2u}{d\theta^2}\right)$$

$$=-\frac{2}{r^2}\frac{F}{h^2u^2}\text{[by (4)]}=-\frac{2F}{h^2}$$

$\therefore$
$$\frac{h^2}{p^3}\frac{dp}{dr}=F \qquad ..(6)$$

which is the pedal equation of the orbit.

Equations (4) and (6) are important for the solution of two problems :—

(*i*) Given the orbit, to determine the law of central force.

(*ii*) Given the law of central force, to determine the orbit.

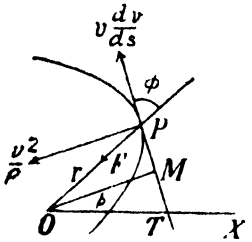
13·52. To find the time we use equation (3) which gives

$$t=\frac{1}{h}\int r^2d\theta.$$

13·53. Equation (6) can be obtained independently as follows :

Let φ be the angle between the outward-drawn radius vector and the direction of motion.

As the force F (per unit mass) acts along PO, resolving along the normal we have



$$\frac{v^2}{\rho} = F \sin\varphi \text{ or } v^2 = F\frac{p}{r}\rho.$$

$$\therefore v^2 \text{ or } \frac{h^2}{p^2} = F\frac{p}{r} \times r\frac{dr}{dp}$$

$$\therefore \quad F = \frac{h^2}{p^3}\frac{dp}{dr}.$$

13·6. Velocity.

First Method. By using equation (5).

$$v^2 = \frac{h^2}{p^2} = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] \qquad ..(7)$$

which gives the velocity when the path is known.

Second Method. By using equation (6).

Integrating equation (6), we have

$$-\tfrac{1}{2}\frac{h^2}{p^2} = \int F\,dr + c, \text{ i.e., } \tfrac{1}{2}v^2 = -\int F\,dr + C \qquad ..(8)$$

which gives the velocity when F, the force, is known.

13·61. Equation (8) can also be obtained independently as follows :

Resolving along the tangent (See Fig. given above), we have

$$v\frac{dv}{ds} = -F\cos\varphi = -F\frac{dr}{ds}.$$

Integrating with respect to s, we get

$$\tfrac{1}{2}v^2 = -\int F\frac{dr}{ds}\,ds + C = -\int F\,dr + C.$$

13·62. Velocity from Infinity at any point of a central orbit means the velocity that a particle would acquire if it is moved from infinity to that point under the action of the given attractive force associated with the orbit.

From (8), the velocity from infinity is given by

$$\tfrac{1}{2}v^2 = -\int_{\infty}^{r} F\,dr \qquad ..(9).$$

For example, if $F = \frac{\mu}{r^2}$, $v^2 = -2\int_{\infty}^{r}\frac{\mu}{r^2}\,dr = \frac{2\mu}{r}$.

13·63. Multiplying (9) by m, the mass of the particle, and supposing that the velocity changes from v_1 to v_2 as r changes from r_1 to r_2, we have

$$\tfrac{1}{2}mv_2^2-\tfrac{1}{2}mv_1^2=-\int_{r_1}^{r_2}F\,dr$$

which shows that the change in the kinetic energy is equal to the work done by the force as the particle moves from one position to another.

Example 1. *A particle describes the path $r=a\tan\theta$ under a force to the origin. Find its acceleration and velocity in terms of r.*

Here $r=a\tan\theta$ *i.e.*, $au=\cot\theta$ ..(1).

$\therefore$ $a\dfrac{du}{d\theta}=-\operatorname{cosec}^2\theta=-(1+\cot^2\theta)=-(1+a^2u^2)$ ..(2),

and $a\dfrac{d^2u}{d\theta^2}=2\operatorname{cosec}^2\theta\cot\theta=2au(1+a^2u^2)$ ..(3).

$\therefore$ by (4) Art. 13·5, we have

The acceln. $F=h^2u^2\left[u+\dfrac{d^2u}{d\theta^2}\right]=h^2u^2[u+2u(1+a^2u^2)]$

$$=h^2u^3[3+2a^2u^2]=\frac{h^2}{r^3}\left[3+\frac{2a^2}{r^2}\right].$$

Also by (7) Art. 13·6, we have

$$v^2=\frac{h^2}{p^2}=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=h^2\left[u^2+\frac{(1+a^2u^2)^2}{a^2}\right],\text{ by (2)}$$

$$=\frac{h^2}{a^2r^4}\left(a^4+3a^2r^2+r^4\right).$$

Example 2. *Find the law for a central attractive force for which the velocity in a circle at any distance is equal to the velocity acquired in falling from infinity to that distance.*

Let O be the centre of force F, v the velocity at a point P at a distance r from O and let $F=\mu/r^n$, then by (9) Art. 13·62, we have

$$v^2=-2\int_\infty^r F\,dr=-2\int_\infty^r\frac{\mu}{r^n}=-\frac{2\mu}{(1-n)r^{n-1}}=\frac{2\mu}{(n-1)r^{n-1}}\quad..(1).$$

Now velocity v in a circle at a distance r means that if the point P were to describe a circle of radius r at P, its acceleration or force F on it will be v^2/r.

Thus we have $\dfrac{\mu}{r^n}=F=\dfrac{v^2}{r}=\dfrac{2\mu}{(n-1)r^n}$ [by (1)].

Hence we get $n-1=2$ or $n=3$,

and thus we find that law of the force is inverse cube of the distance.

Examples XLVI

1. Find the law of force in the following orbits, the pole being the centre of attraction :—

(*i*) $r=a\sec^2\frac{1}{2}\theta$. (*Ans.* c/r^2).

(*ii*) $r^2=a^2\cos 2\theta$. (*Ans.* c/r^7).

(*iii*) $2a^3/r^3=1+\cos 3\theta$. (*Ans.* Const.)

(*iv*) $r=\dfrac{a}{\theta^2+b}$. $\left[\text{Ans. } \dfrac{h^2(2r+a)}{ar^3}\right]$.

2. If a particle describes the cardioid $r=a(1-\cos\theta)$ under a force to the pole, show that the force is proportional to the inverse fourth power of the distance.

If P be the force at the apse ($\theta=\pi$), and V the velocity, prove that

$$3V^2=4aP.$$

3. A particle moving under the influence of a central force, describes a circle *through* the centre of the force. Prove that the force is attractive and inversely proportional to the fifth power of the distance.

[**Hint.** Equation of the circle is $r=2a\cos\theta$].

4. Show that the curve $r^n=a^n\cos n\theta$ can be described under a force to the pole varying inversely as $(2n+3)$rd power of the distance.

Deduce the law of attraction under which a particle describes :

(*i*) a parabola ($n=-\frac{1}{2}$),

(*ii*) a cardioid ($n=\frac{1}{2}$),

(*iii*) a straight line ($n=-1$),

(*iv*) a circle about a point on the circumference ($n=1$),

(*v*) rectangular hyperbola ($n=-2$).

(*vi*) a Lemniscate ($n=2$).

5. The motion of a point in a plane at time t is defined by $x=at^2$, $y=\dfrac{b}{t}$. Prove that the areal velocity is constant and verify that the acceleration is everywhere directed from a fixed point according to the law $\dfrac{2r}{t^2}$, r being the distance of the moving point from the fixed point.

[**Hint.** Use the third result of Art. 13·31].

Force varying inversely as the square of the distance

13·7. Planetary Motion. The laws according to which planets move round the sun are as follows :

(*i*) *The orbit of a planet round the sun is an ellipse, in one focus of which the centre of the sun is situated.*

(*ii*) *The radius vector, drawn from the centre of the sun to the planet describes equal areas in equal times.*

(*iii*) *The square of the periodic times of the various planets are proportional to the cubes of the semi-major axes of their orbits.*

These three laws were discovered by Kepler and were deduced by him entirely from observations of the movements of the planets without any reference to the nature of the forces which control these movements.

Newton, however, proved them to be consequences of the law of universal gravitation which states that *every particle of matter attracts every other particle with a force varying directly as the product of their masses and inversely as the square of the distance between them.*

The law of inverse square of the distance for a particle describing an ellipse under a force to the focus is proved in the following article :—

13·71. To find the law of force to the focus when the orbit is an ellipse. *A particle moves in an ellipse under a force which is always directed towards one focus ; to prove that the attraction of the particle varies inversely as the square of the distance from the focus.*

Also to find the velocity and the periodic time.

First Method. By using the polar equation.

The polar equation of an ellipse is

$lu=1+e\cos\theta$, l being the semi-latus-rectum,

then $\dfrac{du}{d\theta}=-\dfrac{e\sin\theta}{l}$ and $\dfrac{d^2u}{d\theta^2}=-\dfrac{e\cos\theta}{l}$.

By equation (4) Art. 13·5, we have

$$\frac{F}{h^2u^2}=\frac{d^2u}{d\theta^2}+u=-\frac{e\cos\theta}{l}+\frac{1}{l}+\frac{e\cos\theta}{l}=\frac{1}{l}$$

Hence $$F=\frac{h^2}{l}u^2=\frac{\mu}{r^2} \qquad \ldots(1),$$

where $$h^2=\mu l \qquad \ldots(2).$$

This shows that the force varies inversely as the square of the distance from the focus from which r is measured.

Second Method. By using the pedal equation.

The pedal equation of an ellipse with reference to the focus is

$$\frac{b^2}{p^2}=\frac{2a}{r}-1$$

Differentiating, we get

$$-\frac{2b^2}{p^3}\frac{dp}{dr}=-\frac{2a}{r^2} \text{ or } \frac{1}{p^3}\frac{dp}{dr}=\frac{a}{b^2r^2}.$$

$\therefore$ by (6) Art. 13·51

$$F=\frac{h^2}{p^3}\frac{dp}{dr}=\frac{h^2}{b^2}\cdot\frac{a}{r^2}=\frac{\mu l}{al}\cdot\frac{a}{r^2}=\frac{\mu}{r^2}.$$

13·72. The velocity.

First Method. By using the polar equation.

By (7) Art. 13·6, $v^2=\dfrac{h^2}{p^2}=h^2\left[u^2+\left(\dfrac{du}{d\theta}\right)^2\right]$

$$=\mu l\left(\frac{(1+e\cos\theta)^2}{l^2}+\frac{e^2\sin^2\theta}{l^2}\right)$$

$$=\mu\left(\frac{1+2e\cos\theta+e^2}{l}\right)=\mu\left(2\frac{1+e\cos\theta}{l}-\frac{1-e^2}{l}\right)$$

$$=\mu\left(\frac{2}{r}-\frac{1}{a}\right) \qquad \left[\text{for } l=\frac{b^2}{a}=a(1-e^2)\right].$$

It the particle describes a parabola $e=1$ and $v^2=\frac{2\mu}{r}$, so that in this case, the kinetic energy of the particle is inversely proportional to its distance from the centre of force.

Second Method. By using the pedal equation.

$v^2=\frac{h^2}{p^2}=\frac{\mu l}{p^2}$, l being the semi-latus rectum $=\frac{b^2}{a}$,

$$=\frac{\mu}{a}\frac{b^2}{p^2}=\frac{\mu}{a}\left(\frac{2a}{r}-1\right)=\mu\left(\frac{2}{r}-\frac{1}{a}\right)$$ in the case of ellipse,

$$=\frac{\mu}{a}\left(\frac{2a}{r}+1\right)=\mu\left(\frac{2}{r}+\frac{1}{a}\right)$$ in the case of hyperbola,

$=\frac{\mu.2a}{p^2}=\frac{\mu.2a}{ar}=\frac{2\mu}{r}$ ($\because$ $l=2a$) in the case of parabola.

Thus the orbit is an ellipse, a hyperbola, or a parabola according as v^2 is less than, greater than or equal to $\frac{2\mu}{r}$, the velocity from infinity.

It may be noted that the class of a conic described depends only on the velocity of projection, and not on the direction.

Example. *A particle describes an ellipse under a force to the focus S. When the particle is at one extremity B of the minor axis, its kinetic energy is doubled without change in the direction of motion. Prove that the particle proceeds to describe a parabola.*

Let $2a$ be the length of a major axis, so that $SB+S'B=2a$ and $SB=S'B=a$, the velocity v at B is given by

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)=\mu\left(\frac{2}{a}-\frac{1}{a}\right)=\frac{\mu}{a}.$$

Now when the kinetic energy is doubled, the subsequent velocity v' is then given by

$$v'^2=2v^2=\frac{2\mu}{a}.$$

But this gives the velocity of a particle describing a parabolic path at a distance a from the centre of force and since the class of a conic described by a particle depends on the velocity of projection, the particle, therefore, proceeds to describe a parabola with S as the focus.

13·73. Periodic time. Since $\frac{1}{2}h$ is equal to the area described in a unit time it follows that if T be the time that the particle takes to describe the whole ellipse, then

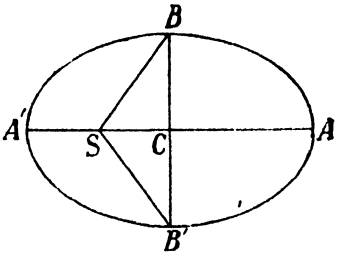
$\frac{1}{2}h\times T=$ area of the ellipse $=\pi ab$.

Hence $T=\frac{2\pi ab}{h}=\frac{2\pi}{\sqrt{\mu}}\cdot\frac{ab}{\sqrt{l}}=\frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}}$ $\left[\because\ l=\frac{b^2}{a}\right]$.

Example 1. *Prove that the time taken by the earth to travel over*

half its orbit, remote from the sun, separated by the minor axis is about 2 days more than half the year. The eccentricity of the orbit is 1/60.

Let C be the centre,
$ACA'=2a$, the major axis ; and
$BCB'=2b$, the minor axis of the earth's orbit.



If S be the sun in the focus nearer A', we have
area $SB'ABS$
$=$area $BAB'+\triangle B'SB$.
$=\frac{1}{2}$ area of the ellipse$+BC\times SC$
$=\frac{1}{2}\pi ab+bae$.

Since the time of describing an arc is proportional to the sectorial area, therefore, t the required time is given by

$$\frac{t}{\text{a year}}=\frac{\text{area } SB'ABS}{\text{area of ellipse}}=\frac{\frac{1}{2}\pi ab+abe}{\pi ab}=\frac{1}{2}+\frac{e}{\pi}.$$

So that $t=\left(\frac{1}{2}+\frac{e}{\pi}\right)$ of a year$=\frac{1}{2}$ a year$+\frac{1}{60\pi}\times 365\frac{1}{4}$ days.

$=\frac{1}{2}$ a year$+$about 2 days.

Example 2. *Prove that the time required to describe an arc of a parabola under the action of the force μ/r^2 to the focus, starting from the nearer end of the axis is given by*

$$\sqrt{\frac{2a^3}{\mu}}\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right],$$

where a is the apsidal distance, and θ is measured from the axis.

Let P be any point on the parabola whose focus is S, vertex $A(AS=a)$, and $\angle ASP=\theta$.

The velocity v at any point is given by,

$$v^2=\frac{2\mu}{r}=\frac{2\mu}{a} \text{ at } A \text{ or at } A,\ v=\sqrt{\frac{2\mu}{a}}.$$

$\therefore$ h, the moment of the velocity at A about S

$$=\sqrt{2\mu/a}\times a=\sqrt{2\mu a}.$$

[or, $\because$ $h^2=\mu l$ and $l=2a$, $\therefore$ $h^2=2\mu a$, and $h=\sqrt{2\mu a}$].

From the equation of the parabola, we have

$$lu=1+\cos\theta=2\cos^2\tfrac{1}{2}\theta, \text{ or } r=\tfrac{1}{2}l\sec^2\tfrac{1}{2}\theta=a\sec^2\tfrac{1}{2}\theta.$$

Now $$r^2\frac{d\theta}{dt}=h=\sqrt{2a\mu},$$

or $$\sqrt{2a\mu}.\ t=\int_0^\theta r^2 d\theta=a^2\int_0^\theta \sec^4\tfrac{1}{2}\theta\, d\theta$$

$$=a^2\int_0^\theta (1+\tan^2\tfrac{1}{2}\theta)\sec^2\tfrac{1}{2}\theta\, d\theta$$

$$=a^2[2\tan\tfrac{1}{2}\theta+\tfrac{2}{3}\tan^3\tfrac{1}{2}\theta]$$

Hence $$t=\sqrt{\frac{2a^3}{\mu}}\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right].$$

Examples XLVII

1. A particle describes an ellipse as a central orbit about the focus, prove that the velocity at the end of the minor axis is a geometric mean

(*i*) between the greatest and the least velocities.

(*ii*) between the velocities at the ends of any diameter.

2. A particle describes an elliptic orbit under a central force to one of the foci, the greatest and least velocities in the orbit being 110 f.p.s. and 90 f.p.s. respectively, and the periodic time being 20 min.; show that the eccentricity is $\frac{1}{10}$ and the length of the major axis is about 7·2 miles.

3. A comet describe a parabola about the sun, show that :—

(*i*) the sum of the squares of the velocities at the extremities of a focal chord is constant,

(*ii*) its velocity perpendicular to the axis of its orbit varies inversely as the radius vector from the sun.

4. Prove that in an elliptic orbit described about a centre at a focus, the radial velocities at the two ends of any focal chord are equal.

5. Show that the velocity of a planetary body at any point of its orbit is the same as it would have been if it had fallen to that point from rest at a distance from the sun equal to the length of the major axis.

6. A particle describes an ellipse about the focus. Show that the greatest and least angular velocities occur at the ends of the major axis and also that if α, β be these angular velocities, the mean angular velocity is

$$\frac{2(\alpha\beta)^{\frac{1}{4}}}{\sqrt{\alpha}+\sqrt{\beta}}$$

7. Prove that if a particle moves subject to a central acceleration μ/r^2 and has a velocity v at a distance r from the centre of force, its periodic time is

$$\frac{2\pi}{\sqrt{\mu}}\left(\frac{2}{r}-\frac{v^2}{\mu}\right)^{-\frac{3}{2}}$$

8. If ω is the angular velocity at the nearer end of the major axis of a planetary body prove that its period is

$$\frac{2\pi}{\omega}\sqrt{\frac{1+e}{(1-e)^3}}.$$

9. If the velocity of a body in a given elliptic orbit (major axis $2a$) is the same at a certain point P, whether the orbit is being described in periodic time T about one focus S or in periodic time T' about the other focus S', prove that

$$SP=\frac{2aT'}{T+T'}\quad,\quad S'P=\frac{2aT}{T+T'}.$$

10. If a particle is describing an ellipse of eccentricity $\frac{1}{2}$, under the action of a force to a focus and when it arrives at an *apse* (an end of the major axis) the velocity is doubled, show that the new orbit will be a parabola or a hyperbola according as the *apse* is the further or the nearer one.

11. The period of Halley's comet is about 76 years. Show that the ratio of the major axis of its orbit to the earth's distance from the sun is about 35·88.

12. Assuming that the eccentricity, e, of a planet's orbit is a small fraction, show that the ratio of the time taken by the planet to travel over the two halves of its orbit separated by the minor axis is nearly

$$1+\frac{4e}{\pi}.$$

Show that in the case of the earth's orbit for which $e=\frac{1}{60}$, the difference in times is about 3·87 days.

13. The eccentricity of the earth's orbit round the sun is $\frac{1}{60}$, prove that the earth's distance from the sun exceeds the semi-major axis of the orbit during about two days more than half the year.

14. Prove that the two parts into which the earth's orbit is divided by the latus-rectum are described in 178·7 and 186·5 days respectively ($e=\frac{1}{60}$).

15. Prove that in an elliptic orbit the time average of the kinetic energy is the same as that of a particle revolving in a circle of radius equal to the mean distance a with the mean angular velocity.

The Converse Problem

13·8. Force μ/r^2, to find the orbit and the velocity. *A particle moves under the action of a force to a fixed point varying inversely as the square of the distance; to prove that the orbit is a conic section with one focus at the centre of force.*

First Method. By using the polar equation of the orbit.

$$F=\frac{\mu}{r^2}=\mu u^2.$$

Hence equation (4) of Art. 13·5 gives

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}=\frac{\mu u^2}{h^2u^2}=\frac{\mu}{h^2},$$

or $$\frac{d^2}{d\theta^2}\left(u-\frac{\mu}{h^2}\right)+u-\frac{\mu}{h^2}=0 \qquad \left[\because \ \frac{\mu}{h^2} \text{ is const.}\right]$$

Solving this equation, we have

$$u-\frac{\mu}{h^2}=A\cos(\theta-\alpha),$$

or $$\frac{h^2}{\mu r}=1+\frac{h^2A}{\mu}\cos(\theta-\alpha)$$

i.e., $$\frac{l}{r}=1+e\cos(\theta-\alpha),$$

where $$\frac{h^2A}{\mu}=e \text{ and } h^2=\mu l.$$

This is the polar equation of a conic section referred to the focus as the origin.

Second Method. By using the pedal equation of the orbit.

We have $$\frac{h^2}{p^3}\frac{dp}{dr}=F=\frac{\mu}{r^2}$$

Integrating, we have

$$\frac{h^2}{p^2}=\frac{2\mu}{r}+C \qquad \ldots(1).$$

Suppose $h^2=\mu l=\mu\times$ semi-latus rectum $=\mu\dfrac{b^2}{a}$, then

$$\frac{b^2}{p^2}=\frac{2a}{r}+\frac{aC}{\mu} \qquad \ldots(2).$$

Equation (2) is the pedal equation of a conic referred to the focus as pole. It will represent an ellipse, a hyperbola or a parabola according as $\dfrac{aC}{\mu}$ is -1, 1 or 0,

$$\left(i.e., \ C \text{ is } -\frac{\mu}{a}, \ \frac{\mu}{a} \text{ or } 0\right).$$

13·81. To find the velocity. From equation (1), we have

$$v^2=\frac{h^2}{p^2}=\frac{2\mu}{r}+C$$

$$=\mu\left(\frac{2}{r}-\frac{1}{a}\right) \quad \text{in the case of ellipse,}$$

$$=\mu\left(\frac{2}{r}+\frac{1}{a}\right) \quad \text{in the case of hyperbola,}$$

$$=\frac{2\mu}{r} \quad \text{in the case of parabola,}$$

the results being the same as obtained in Art. 13·72.

Force varying directly as the distance.

13·9. To find the law of force to the centre when the orbit is an ellipse.

The pedal equation to an ellipse referred to its centre as origin is

$$\frac{a^2b^2}{p^2}=a^2+b^2-r^2. \qquad ..(1)$$

Differentiating with respect to r, we have

$$-\frac{2a^2b^2}{p^3}\frac{dp}{dr}=-2r.$$

Hence $F=\frac{h^2}{p^3}\frac{dp}{dr}=\frac{h^2}{p^3}\times\frac{p^3r}{a^2b^2}=\frac{h^2}{a^2b^2}r=\mu r$; where $\mu=\frac{h^2}{a^2b^2}$.

It shows that the force varies directly as the distance.

[Compare Art. 13·13].

Conversely, *if the force varies directly as the distance, the orbit is an ellipse with its centre at the centre of force.*

This can be proved as follows :—

$$\frac{h^2}{p^3}\frac{dp}{dr}=F=\mu r.$$

$\therefore$ integrating, we have $-\frac{h^2}{2p^2}=\frac{1}{2}\mu r^2+C$ or $\frac{a^2b^2\mu}{p^2}=-\mu r^2-2C$,

so that $$\frac{a^2b^2}{p^2}=C'-r^2=a^2+b^2-r^2,$$

the same as (1) the constant C' being taken equal to a^2+b^2.

13·91. To find the velocity v. In this case we have

$$v^2=\frac{h^2}{p^2}=\frac{\mu a^2b^2}{p^2}=\mu(a^2+b^2-r^2)=\mu r'^2, \qquad \text{or} \qquad v=\sqrt{\mu}.r'$$

which shows that the velocity is proportional to r', the semi-diameter conjugate to r. [Compare Art. 13·131]

13·92. Apse and Apsidal Distance. An *apse* is a point on central orbit at which the radius vector drawn from the centre of a force is a maximum or minimum.

The length of the radius vector at such a point is known as the apsidal distance.

The analytical condition for a maximum or minimum value of the length of the radius vector is that $\frac{du}{d\theta}$ shall vanish and that the first differential co-efficient which does not vanish shall be of an even order.

Now if φ be the angle between the radius vector and the tangent to the curve, then by the Calculus,

$\tan\varphi = -u\frac{d\theta}{du}$, or $\frac{du}{d\theta} = -u\cot\varphi$, so that when $\frac{du}{d\theta}=0$, $\varphi=\frac{\pi}{2}$.

Hence *the tangent at an apse is perpendicular to the radius vector.*

In the case of a planet moving round the sun in an ellipse, the ends of the major axis are the two *apses*, the one nearer to the sun is called, *perihelion* and the further one is called *aphelion.*

13·93. Example 1. *If a particle is projected from an apse at a distance a with the velocity from infinity under the action of a central force* μr^{-2n-3}, *prove that the path is* $r^n = a^n \cos n\theta$.

Here $$F = \mu r^{-2n-3} = \mu u^{2n+3}.$$

$$\therefore \quad u + \frac{d^2u}{d\theta^2} = \frac{F}{h^2u^2} = \frac{\mu}{h^2}u^{2n+1}.$$

Multiplying by $2du/d\theta$ and integrating, we get

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{\mu u^{2n+2}}{h^2(n+1)} + C \qquad \ldots(1).$$

V, the velocity from infinity at the apse $r=a$, is given by

$$V^2 = -2\int_\infty^a F dr = -2\int_\infty^a \mu r^{-2n-3} dr = \frac{\mu}{(n+1)a^{2n+2}}.$$

$\therefore$ h^2, the square of the moment of the velocity at the apse

$$= a^2V^2 = \mu/(n+1)a^{2n}.$$

Now to find the constant C in (1), we substitute the initial values at the apse *viz.*, $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$ and $h^2=\frac{\mu}{(n+1)a^{2n}}$, and we get $C=0$.

$$\therefore \quad u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{\mu u^{2n+2}}{(n+1)h^2} = a^{2n}u^{2n+2}.$$

This gives $$\int \frac{du}{u\sqrt{a^{2n}u^{2n}-1}} = \theta + \alpha.$$

$$-\int \frac{r^{n-1}\,dr}{\sqrt{a^{2n}-r^{2n}}} = \theta+\alpha, \qquad \because \ u=\frac{1}{r}.$$

Therefore, $\cos^{-1}\frac{r^n}{a^n} = n\theta + n\alpha$, *i.e.*, $r^n = a^n\cos(n\theta+n\alpha)$.

If we measure θ from the apse line, then $r=a$ when $\theta=0$,

$\therefore$ $\cos n\alpha = 1$, *i.e.*, $\alpha = 0$.

Hence $$r^n = a^n\cos n\theta.$$

Example 2. *Find the velocity necessary for the description of a circular orbit of radius a under a central force* $2\mu a^2u^5-\mu u^3$. *Show that the orbit is unstable.* *(D.U. 1960)*

For a circular orbit of radius a,

$$u=\frac{1}{a} \text{ and } F=2\mu a^2.\frac{1}{a^5}-\frac{\mu}{a^3}=\frac{2\mu}{a^3}-\frac{\mu}{a^3}=\frac{\mu}{a^3}.$$

But force or acceleration F for the velocity v for a circular orbit of radius a is v^2/a.

$$\therefore \quad \frac{v^2}{a}=F=\frac{\mu}{a^3}, \text{ or } v^2=\frac{\mu}{a^2}, \text{ i.e. } v=\frac{\sqrt{\mu}}{a}.$$

This gives the required velocity.

Also $\quad h^2=v^2p^2=v^2a^2=\frac{\mu}{a^2}\times a^2=\mu.$

Now $$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}=\frac{2\mu a^2u^5-\mu u^3}{\mu u^2}=2a^2u^3-u \qquad ..(1).$$

Suppose the particle is slightly displaced from the circular path in such a way that h remains unaltered.

In (1), put $u=x+1/a$, where x is small, then

$$\frac{d^2x}{d\theta^2}+\frac{1}{a}+x=2a^2\left(\frac{1}{a}+x\right)^3-\left(\frac{1}{a}+x\right)$$

$$=2a^2\left[\frac{1}{a^3}+\frac{3x}{a^2}\right]-\frac{1}{a}-x=\frac{1}{a}+3x,$$

neglecting square and higher powers of x.

$$\therefore \quad \frac{d^2x}{d\theta^2}=4x.$$

$$\therefore \quad x=Ae^{2\theta}+Be^{-2\theta}.$$

It shows that as θ increases, x also increases, so that x is not always small and the orbit does not continue to be nearly circular and hence the orbit is unstable.

Ex. Integrating equation (1), show that if a slight disturbance takes place inwards, the path may be represented by

$$r=a\tanh\theta.$$

Examples XLVIII-A

1. The law of force is μu^5 and a particle is projected from an apse at a distance a. Find the orbit when the velocity of projection is $\sqrt{\mu}/a^2\sqrt{2}$.

[*Ans.* $r=a\cos\theta$.]

2. A particle acted on by a central attractive force μ/r^3 is projected with velocity $\sqrt{\mu}/a$ at the angle $\pi/4$ with its initial distance, a, from the centre of force ; prove that its orbit is the equiangular spiral $r=ae^{\theta}$.

3. A particle of mass m moves under a central force $m\mu/r^3$, and is projected at an angle α to the radius at a distance $r=a$ from the centre of force with the velocity which would be acquired by a fall from rest at infinity to the point of projection. Prove that the orbit is an equiangular spiral.

Show also that the time elapsed till the particle reaches a distance r from the centre is $\frac{1}{2}(r^2-a^2)\sec\alpha/\sqrt{\mu}$, α being a constant.

4. (*a*) If in a central orbit under the force $\mu u^3(3+2a^2u^2)$, a particle be projected at a distance a with a velocity $\sqrt{5\mu}/a$ in a direction making $\tan^{-1}\frac{1}{2}$ with the radius, show that the equation to the path is

$$r=a\cot\left(\frac{\pi}{4}+\theta\right).$$

(*b*) If the law of force be $5\mu u^3+8\mu c^2u^5$, and the particle is projected from an apse at the distance c with velocity $3\sqrt{\mu}/c$, prove that the orbit is

$$r=c\cos\tfrac{2}{3}\theta.$$

5. A particle is projected at right angles at a distance a under a central force μ/r^n with the velocity from infinity ; show that the orbit is

$$r^{\frac{n-3}{2}}=a^{\frac{n-3}{2}}\cos\frac{n-3}{2}\theta.$$

6. A particle under an attraction μ/r^4 per unit mass, is projected from an apse at a distance b from the centre of attraction with velocity $(2\mu)^{\frac{1}{2}}b^{-\frac{3}{2}}$ show that there is another apsidal distance equal to $\frac{1}{2}b(\sqrt{3}-1)$.

7. A particle of mass m moves under a central force

$$m\mu[3au^4-2(a^2-b^2)u^5],$$

a being $>b$ and is projected from an apse at the distance $a+b$, with velocity $\sqrt{\mu/(a+b)}$; show that its orbit is

$$r=a+b\cos\theta.$$

Examples XLVIII-B

1. A particle of mass, m describes a parabola under an attraction to the focus ; prove that the attraction at a distance, r, is $m\mu/r^2$ and that the velocity is $(2\mu/r)^{\frac{1}{2}}$.

Two particles describe in equal times the arc of a parabola bounded by the latus-rectum, one under an attraction to the focus, and the other with constant acceleration, g, parallel to the axis. Show that the acceleration of the first particle at the vertex of the parabola is $\frac{16}{9}g$. (*D.U. Hons., 1924*)

[*See* solved Ex. 2, Art. 13·73]

2. A comet describes a parabola about the sun, its nearest distance from the sun being one-half of the radius of the earth's orbit supposed circular. Show that the comet will remain within the earth's orbit for about 77 days.

3. Show that the velocity of a particle P moving in an ellipse about a centre of force in the focus S can be resolved into components, of constant magnitude, one μ/h perpendicular to the radius SP and the other $\mu e/h$ perpendicular to the major axis AS.

4. Deduce from Ex. 3, that the velocity of P can be resolved into

$$\mu(e+\cos\theta)/h \quad \text{and} \quad \mu\sin\theta/h,$$

respectively perpendicular and parallel to the major axis, where θ is the angle ASP.

Hence show that the hodograph of the motion is a circle.

5. A particle of mass m is projected under the central force $m\mu/r^2$. If it be projected from the point $r=c$, $\theta=0$ at right angles to the radius, with a velocity $\sqrt{n\mu/c}$, prove that

$$\left(r\frac{dr}{dt}\right)^2=\frac{\mu}{c}(r-c)[nc-(2-n)r].$$

6. Show that, in elliptic motion about a focus under attraction μ/r^2, the radial velocity is given by the equation

$$r^2\left(\frac{dr}{dt}\right)^2=\frac{\mu}{a}[a(1+e)-r][r-a(1-e)] \qquad (P.U.)$$

or

$$r^2\left(\frac{dr}{dt}\right)^2=\frac{2\mu(r_1-r)(r-r_2)}{r_1+r_2},$$

where r_1 and r_2 are the greatest and least radial distances respectively.

7. A planet is describing an ellipse about the sun as focus ; show that its velocity away from the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it is then

$$2\pi ae/T\sqrt{1-e^2},$$

where $2a$ is the major axis, e the eccentricity and T the periodic time. (*P.U.*)

8. A particle describes an ellipse about a centre of force at the focus, show that at any point of its path the angular velocity about the other focus (*i*) varies inversely as the square of the normal at the point.

(*ii*) varies inversely as the square of conjugate diameter and is equal to $n\,ab/CD^2$, where CD is the semi-diameter conjugate to CP and n is the mean angular velocity.

9. If the velocity of the earth at any point of its orbit (supposed circular) were increased by about one-half, prove that it would describe a parabola about the sun as focus.

10. Obtain the velocity at any point of an elliptic orbit about a focus. Find, when possible, the point at which, if the centre of force were transferred to the empty focus, the orbit would be a parabola. Prove that there is no such point unless the eccentricity is greater than $\sqrt{5}-2$. (*D.U. 1953*)

11. Show that if a parabolic orbit and a rectangular hyperbolic orbit have the same latus-rectum under forces towards the foci, the velocity at the end of the latus-rectum is $\sqrt{\frac{3}{2}}$ times as great in the latter case as in the former.

12. A body of mass m is describing a parabola under an attraction varying inversely as the square of the distance. It collides with another mass of magnitude $(\sqrt{2}-1)m$ which is at rest, at the vertex, and the two masses coalesce into one. Prove that the collected mass will describe a circle having its centre at the centre of force.

13. A body is describing a circle in a central orbit, the force varying inversely as the square of the distance. If the attraction is suddenly decreased by one-half, show that the subsequent path is a parabola.

14. If, when a planet is at the end of the minor axis of an ellipse, the mass of the sun is suddenly increased by one-half, show that the planet will continue to describe an elliptic orbit of axes $\frac{3}{2}a$, $b\sqrt{2}$, where $2a$, $2b$ are the axes of the former orbit.

15. A particle is moving in an ellipse of eccentricity e, under an acceleration μ/r^2 to a focus. When the particle is nearest to the focus, this acceleration is suddenly replaced by an acceleration $\mu' r$ towards the centre of the ellipse. If the particle continues to move in the same ellipse, prove that

$$\mu=\mu' a^3\,(1-e)^2.$$

16. A particle is describing an ellipse under an attraction to the focus. When it is at any point of its path, the centre of attraction is transferred to the other focus, and the attraction continues to vary as the inverse square of the distance, but the absolute force is changed so that the magnitude of the attraction is unaltered, show that the particle continues to describe the same ellipse.

REVISION QUESTIONS—XI

1. A particle slides down the catenary, $s=c\tan\psi$, with vertex upwards, the velocity at any point being that due to a fall from the directrix. Prove that the particle moves inside the curve and that the pressure at any point is inversely proportional to the distance of that point from the directrix.

2. If a particle be acted upon by a force always parallel to the axis of y and proportional to the square of the radius of curvature at the point, prove that it will describe the curve

$$\frac{y-b}{a}=\log\sec\frac{x}{a},$$

the particle moving parallel to the axis of x at the point (o, b) and a being a constant.

3. A particle moves on the outside of a smooth elliptic cylinder whose axis is horizontal, the major axis to the principal elliptic section being vertical and the eccentricity being e. The particle starts from rest from the highest

generator, and moves in a vertical plane. Prove that it will leave the cylinder at a point where the eccentric angle φ is given by

$$e^2 \cos^3 \varphi = 3 \cos \varphi - 2.$$

4. A smooth elliptic ring, semi-axes a and b, is placed with its major axis vertical, and a particle is projected from the lowest point, along the concave side of the arc, with velocity due to height h above the centre. Show that the particle will leave the arc at the point whose eccentric angle is given by

$$(a^2 - b^2) \cos^3 \varphi = 3a^2 \cos \varphi + 2ah.$$

If the ring is placed with its minor axis vertical, show that the least velocity of projection u, that will carry the particle completely round the ring, without leaving contact, is given by $u^2 = g(a^2 + 4b^2)/b$.

5. If a particle slides on a curve under a constant force, show that it leaves it at the point where its velocity is due to a fall under the force through one quarter of the chord of curvature in the direction of the force.

Verify this in the case of a particle starting from rest from the highest point of a vertical circle.

6. If the resistance of the air to a particle's motion be n times its weight, and the particle be projected horizontally with velocity V, show that the velocity of the particle, when it is moving at an inclination ψ to the horizontal, is

$$V(1-\sin\psi)^{\frac{n-1}{2}}(1+\sin\psi)^{-\frac{n+1}{2}}. \qquad (D.U.\ 1959)$$

7. A particle describes an ellipse with uniform areal velocity about one focus. Show that the attraction of the particle is directed towards that focus and varies inversely as the square of the distance from the focus.

Show also that the hodograph of the motion of the particle is a circle. (*P.U.*)

8. Find an expression for the velocity of a particle moving under a central force.

If the law of force is inverse square of the distance, find the velocity at any point and prove that the particle will describe an ellipse, a parabola or a hyperbola according as the velocity at any point of the path is less than, equal to, or greater than, that acquired in falling from infinity.

9. In the motion of a particle in an ellipse under a force to a focus, prove that

$$h^2 = \mu l, \quad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right).$$

If the path described is $l = r(1 + e\cos\theta)$ and U and V are radial and transverse components of the velocity, prove that

$$\frac{dU}{dt} = \frac{\mu e \cos\theta}{r^2} \quad \text{and} \quad \frac{dV}{dt} = -\frac{\mu e \sin\theta}{r^2}. \qquad (L.U.)$$

10. A particle moves in a circular groove, of radius a under an attraction μ/r^2 to a point O which is in the plane of the circle and distant b from its centre. The particle is projected with velocity u from the point of the circle nearest to O. Show that the particle will perform complete revolution if

$$u^2 > 4\mu b/(a^2 - b^2).$$

11. A particle is constrained to move in a circle of radius a under an attraction μr per unit mass to a point O inside the circle distant c from its centre. If the particle be placed at its greatest distance from O and started with an infinitesimal velocity, prove that it will pass over the second quadrant of the circle in the time

$$\sqrt{\frac{a}{\mu c}} \log(\sqrt{2}+1).$$

12. A particle moves with a central acceleration $\mu\left(r + \frac{a^4}{r^3}\right)$ being projected from an apse at a distance a with a velocity $2a\sqrt{\mu}$. Show that it describes the curve

$$r^2(2 + \cos\sqrt{3}\theta) = 3a^2.$$

13. If the orbit of a particle, moving under a central attraction μ/r^2, is a rectangular hyperbola, then the velocity V at a point where the radius vector makes an angle of 45° with the tangent to the orbit is given by

$$rV^2 = \mu(1 + \sqrt{3}). \qquad (P.U.\ Hons.)$$

CHAPTER XIV

MOMENT OF INERTIA. COMPOUND PENDULUM

14·1. Moment of Inertia. If m be the mass of a particle of a given body and r be its distance from a given line, then mr^2 is called the moment of inertia of the particle about the given line and the sum of similar expressions taken for all the particles of the body, *i.e.*, Σmr^2 is called the *moment of inertia of the body* about the given line.

If the body consists of a finite number of isolated particles, the value Σmr^2 is calculated by multiplying the mass of each particle by the sq are of its distance from the given line and then adding these results for all the particles.

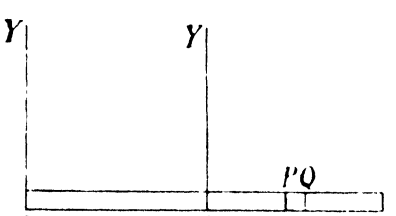
If, however, the body consists of a continuous mass where the number of particles is infinitely great, the value of Σmr^2 is calculated by integration.

If we write $\frac{\Sigma mr^2}{M} = K^2$, where M is the whole mass of the body then the length K is called the **Radius of Gyration** of the body about the given line.

If M is given in pounds and K in feet, the moment of inertia is measured in pound-foot squared or briefly lb./ft.^{2} units, the same quantity in C.G.S. units being measured in gm./cm.^{2} units.

The moment of inertia (briefly denoted by M. I.) for a few symmetrical bodies about their lines of symmetry is calculated in the following articles :

14·2. Moment of Inertia of a rod. Let $2a$ be its length, α the area of its cross-section, O its middle point and ρ its density.



When the rod is uniform the mass M of the rod AB is $2a\alpha\rho$ and the mass of the element PQ of length δx at P is $\rho\alpha\delta x$ where $OP=x$, then

(*a*) M. I. of the rod about OY

$$= \Sigma\rho\alpha\delta x.x^2 = \int_{-a}^{a} \rho\alpha x^2 dx$$

$$= \rho\alpha\frac{2a^3}{3} = 2a\alpha\rho.\frac{a^2}{3} = M\frac{a^2}{3}$$

(*b*) M. I. about an axis through O inclined at an angle θ to the rod

$$= \int_{-a}^{a} \rho\alpha x^2 \sin^2\theta\, dx = M\frac{a^2}{3}\sin^2\theta,$$

$x \sin\theta$ being the distance of the element PQ from the axis in this case.

(*c*) M. I. about $AY' = \int_0^{2a} \rho\alpha x^2 dx = \rho\alpha \frac{8a^3}{3} = 2a\alpha\rho . \frac{4a^2}{3} = M . \frac{4a^2}{3}$,

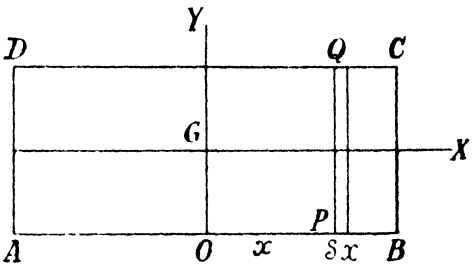
x being the distance of the element PQ from AY' in this case.

(*d*) When the rod is *not uniform* and its density varies as the distance from one end A, *i.e.*, when $\rho = kx$ at P.

Its M. I. about $AY' = \int_0^{2a} kx\, \alpha x^2 dx = k\alpha . 4a^4 = 2k\alpha a^3 . 2a^2 = M . 2a^2$.

the mass M being $= \int_0^{2a} kx\alpha dx = 2k\alpha a^3$.

14·21. Moment of inertia of a uniform rectangular lamina. Let $ABCD$ be the rectangular lamina where $AB = 2a$ and $AD = 2b$; let G be the centre of gravity of the lamina and GX, GY be the axes parallel to AB and AD.



To obtain the moment of inertia about GY, we divide the rectangle into strips such as $PQ(OP = x)$ drawn parallel to GY. If ρ is the mass per unit area, the mass of strip PQ is $2b\delta x.\rho$ and *since the distance of every point on PQ from GY can be taken as x*, its M.I. about GY is $2b\rho\delta x.x^2$. Hence

(*a*) M.I. of the lamina about GY

$$= \Sigma mx^2 = \int_{-a}^{a} 2b\rho x^2 dx = \frac{4\rho ba^3}{3} = 4ab\rho . \frac{a^2}{3} = M . \frac{a^2}{3}.$$

(*b*) M.I. about $AD = \int_0^{2a} 2b\rho x^2 dx = \frac{16\rho ba^3}{3} = 4ab\rho . \frac{4a^2}{3} = M . \frac{4a^2}{3}$,

x being measured from AD.

Similarly by taking strips parallel to GX, we have

(*c*) M.I. about $GX = \Sigma my^2 = \int_{-b}^{b} 2a\rho y^2 dy = M . \frac{b^2}{3}$.

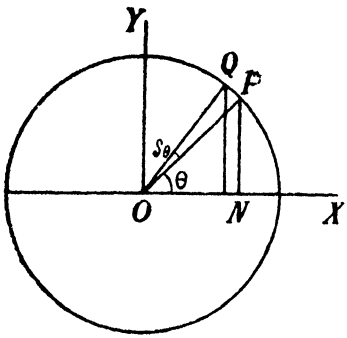
(*d*) M.I. about AB $= \int_0^{2b} 2a\rho y^2 dy = M . \frac{4b^2}{3}$.

(*e*) M.I. about GZ, an axis through G perpendicular to the plane of the rectangle

$$= \Sigma mr^2 = \Sigma m(x^2 + y^2) = \Sigma mx^2 + \Sigma my^2 = M . \frac{a^2}{3} + M . \frac{b^2}{3} = M \frac{a^2 + b^2}{3},$$

r, being the distance from the point G (being also the distance from GZ) of any element of mass m at the point $P(x, y)$ of the rectangle.

14·3. Moment of Inertia of a uniform circular ring. Let OX, OY be two perpendicular diameters of the ring of radius a, through its centre O and let P be any point on the ring, XOP being equal to θ, then $NP = a \sin\theta$. If m be the mass per unit length of the arc, then the mass of the element PQ is $ma\delta\theta$. Hence



(*a*) M.I. about OX

$$= \Sigma ma\delta\theta . a^2 \sin^2\theta = ma^3 \int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= \tfrac{1}{2}ma^3 \int_0^{2\pi} (1 - \cos 2\theta) d\theta = m\pi a^3$$

$$= 2\pi am \cdot \frac{a^2}{2} = M \cdot \frac{a^2}{2}.$$

Similarly (*b*) M.I. about $OY = ma^3 \int_0^{2\pi} \cos^2\theta \, d\theta = M \cdot \frac{a^2}{2}$.

Also (*c*) M.I. about OZ, an axis through O perp. to the plane of the ring

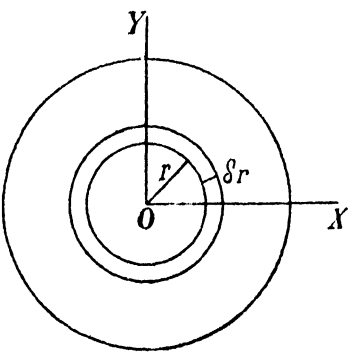
$$= \Sigma ma\delta\theta \cdot a^2 = ma^3 \int_0^{2\pi} d\theta = 2\pi ma^3 = M \cdot a^2.$$

14·31. Moment of inertia of a uniform circular disc. Let O be the centre of the disc, OX, OY its two perpendicular diameters and let m be the mass per unit area.

Divide the disc into elements in the form of concentric rings.

The mass of a ring of radius r and breadth δr is $m\, 2\pi r.\delta r$ and since by Art. 14·3 M.I. of the ring about OX is

$$2m\pi r\delta r . \frac{r^2}{2} = m\pi r^3 \delta r.$$



$\therefore$ (*a*) M.I. of the disc about OX

$$= \int_0^a m\pi r^3 dr = m\pi a^2 \cdot \frac{a^2}{4} = M \cdot \frac{a^2}{4}.$$

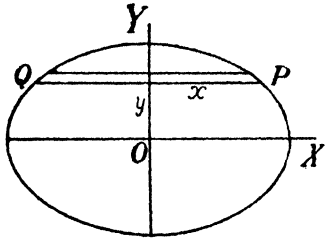
Similarly (*b*) M.I. about $OY = M \cdot \frac{a^2}{4}$.

Also (*c*) M.I. about OZ an axis through O perpendicular to the plane of the disc.

$$= \int_0^a 2m\pi r dr . r^2 = \frac{m\pi a^4}{2} = m\pi a^2 . \frac{a^2}{2} = M . \frac{a^2}{2}.$$

Cor. The moment of inertia of a circular cylinder of radius a, about its axis is $M \frac{a^2}{2}$.

14·32. Moment of inertia of a uniform elliptic disc. Let OX, OY be its principal axes and let m be its mass per unit area.



Divide the disc into strips such as PQ perp. to OY.

Let the co-ordinates of P be

(x, y) or $(a\cos\theta, b\sin\theta)$,

θ being the eccentric angle.

M.I. of the strip about $OY = 2x\delta y m.\frac{x^2}{3}$.

Hence (a) M.I. of the disc about OY

$$= \int_{-b}^{b} \tfrac{2}{3} m x^3 dy = \tfrac{2}{3} m \int a^3 \cos^3\theta d(b\sin\theta)$$

$$= \tfrac{2}{3} m a^3 b \times 2 \int_0^{\pi/2} \cos^4\theta \; d\theta = \tfrac{1}{3} m a^3 b . \frac{3\pi}{16} = m\pi ab . \frac{a^2}{4} = M.\frac{a^2}{4}.$$

Similarly (b) M.I. about $OX = M.\frac{b^2}{4}$.

14·4. Moment of inertia of a rectangular parallelopiped (or rectangular prism).

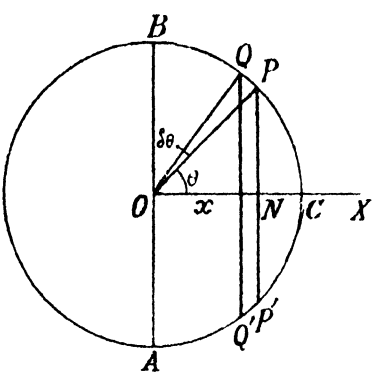
Let the lengths of the sides be $2a$, $2b$, $2c$ and let OX, OY, OZ be the principal axes, *i.e.*, axes through the centre O, perp. to the faces of the solid body.

To find the moment of inertia about OX, divide the body into parallel rectangular slices all perp. to OX. Each of the slices has sides $2b$ and $2c$. M.I. of each about OX is the product of its mass and $\frac{b^2+c^2}{3}$. [Art. 14·21 (e)].

Hence M.I. of the whole body about $OX = M.\frac{b^2+c^2}{3}$.

Similarly M.I. about $OY = M.\frac{a^2+c^2}{3}$ and about $OZ = M.\frac{a^2+b^2}{3}$.

14·5. Moment of inertia of a solid sphere about a diameter. Let O be the centre of the sphere, a its radius and m the mass per unit volume of the sphere. Let OX be a diameter of the sphere.



Divide the sphere into circular slices such as $PP'Q'Q$ (distant x from O), all perpendicular to the diameter OX, then

$$NP^2 = y^2 = a^2 - x^2.$$

$\therefore$ the mass of the circular slice $PP'Q'Q$

$$= m\pi y^2 \delta x = m\pi(a^2 - x^2)\delta x,$$

and its M.I. about OX

$$= m\pi y^2 \delta x . \frac{y^2}{2} \quad \text{[Art. 14·31(c)]}$$

$$=m\pi(a^2-x^2)\delta x\cdot\frac{a^2-x^2}{2}.$$

Hence M.I. of the whole sphere about OX

$$=\int_{-a}^{a}\tfrac{1}{2}m\pi(a^2-x^2)^2dx=m\pi\int_0^a(a^2-x^2)^2dx=m\pi\frac{8}{15}a^5$$
$$=\frac{4}{3}m\,\pi a^3\times\frac{2a^2}{5}=M\cdot\frac{2a^2}{5}.$$

Cor. For the solid hemisphere ACB, the limits of integration will be from o to a, and the result will be $m\pi\frac{4}{15}a^5=\frac{2}{3}m\pi a^3\times\frac{2}{5}a^2=M.\frac{2}{5}a^2$, the same as for the whole sphere.

14·51. Moment of inertia of a hollow sphere about a diameter. Let m be the mass per unit area of the surface of the sphere. Divide the surface into elements in the form of circular bands such as $PP'Q'Q$, perp. to OX (Fig. Art. 14·5).

Let $\angle XOP=\theta$ and $\angle XOQ=\theta+\delta\theta$, so that the breadth of the band=arc $PQ=a\delta\theta$, its radius$=NP=a\sin\theta$,
its mass$=m2\pi a\sin\theta.a\delta\theta$, and its M.I. about OX is

$$2m\pi a\sin\theta\,.\,a\delta\theta\,.\,a^2\sin^2\theta \qquad \text{[Art. 14·3 (c)]}.$$

Hence M.I. for the hollow sphere about OX

$$=\int_0^{\pi}2m\pi a^4\sin^3\theta d\theta=\tfrac{4}{3}m\pi a^4=4m\pi a^2\times\tfrac{2}{3}a^2=M.\tfrac{2}{3}a^2.$$

Cor. For the hemis erical shell ACB, limits of integration are from o to $\frac{1}{2}\pi$, and the result is $\frac{1}{3}m\pi a^4=2m\pi a^2\times\frac{2}{3}a^2=M.\frac{2}{3}a^2$, the same as for the whole spherical shell.

14·6. Routh's Rule. The following rule given by Dr. Routh in his Treatise on Rigid Dynamics, is useful for remembering the results obtained in the previous articles and other similar results for moments of inertia of certain symmetrical *solid* bodies about their principal axes.

M.I. about an axis of symmetry

$$=\text{Mass}\times\frac{\text{the sum of squares of perp. semi-axes}}{3,\ 4,\ 5},$$

the denominator being 3 or 4 or 5 according as the body is rectangular (including the rod and a square) or elliptical (including a circle) or ellipsoidal (including a sphere).

14·7. Sometimes we require the moment of inertia of a body about axes other than the principal axes. The three theorems given below will be found useful for the purpose.

14·71. Moment of inertia about perpendicular axes. *If the moments of inertia of a lamina about OX and OY, two perdendicular axes in its plane, meeting in O are A and B, then the moment of inertia of the lamina about OZ, an axis through O perpendicular to the plane of the lamina, is $A+B$.*

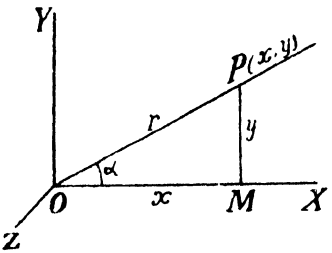
Let m be the mass of an element of the lamina at $P(x, y)$ and let $OP=r$; then M.I. about OZ, a line perp. to both OX and OY

$=\Sigma mr^2=\Sigma m(x^2+y^2)$

$=\Sigma my^2+\Sigma mx^2$

=M.I. about OX+M.I. about OY

$=\quad A \quad + \quad B$



For example :—

(*i*) M.I. of a rectangular lamina, (Art. 14·21) about, GZ

$$=M\left(\frac{a^2+b^2}{3}\right)=M\frac{a^2}{3}+M\frac{b^2}{3}=\text{M.I. about } GY+\text{M.I. about } GX.$$

(*ii*) M.I. of a circular ring (Art. 14·3) about OZ

$$=Ma^2=M\cdot\frac{a^2}{2}+M\frac{a^2}{2}=\text{sum of the values about } OX \text{ and } OY.$$

(*iii*) M.I. of a circular disc (Art. 14·31) about OZ

$$=M\frac{a^2}{2}=M\frac{a^2}{4}+M\frac{a^2}{4}=\text{sum of the values about } OX \text{ and } OY.$$

(*iv*) M.I. of an elliptic lamina about OZ

$$=\text{M.I. about } OX+\text{M.I. about } OY=M\frac{b^2}{4}+M\frac{a^2}{4}=M\left(\frac{a^2+b^2}{4}\right).$$

14·72. Moment of inertia about parallel axes. *If the moment of inertia of a body of mass M, about an axis through its centre of gravity is I, then I', moment of inertia about a parallel axis at a distance d from the first axis, is given by*

$$I'=I+Md^2.$$

Let the co-ordinates of G, the centre of gravity of the body referred to rectangular axes OX, OY, OZ be (f, g, h) and let the co-ordinates of any point P of the body referred to the same axes be (x, y, z). Also let (x', y', z') be the co-ordinates of P referred to GX', GY', GZ', axes through G parallel to the original, then

$$x=x'+f, \qquad y=y'+g, \qquad z=z'+h$$

and the distance between the parallel axes OZ and $GZ'=\sqrt{f^2+g^2}$.

If m is the mass of a particle of the body at P then I', M.I. of the body about OZ is given by

$$I'=\Sigma m(x^2+y^2)=\Sigma m[(x'+f)^2+(y'+g)^2]$$

$$=\Sigma m(x'^2+y'^2)+(f^2+g^2)\Sigma m+2f\Sigma mx'+2g\Sigma my' \qquad ..(1).$$

Now $\Sigma m(x'^2+y'^2)=I$, the M.I. of the body about GZ', an axis through G parallel to the axis OZ.

$f^2+g^2=d^2$, the distance between the parallel axes.

Also $\dfrac{\Sigma mx'}{\Sigma m}$ and $\dfrac{\Sigma my'}{\Sigma m}$ are the distances of the centre of gravity of the body from the planes $Y'GZ'$ and $Z'GX'$ respectively, but as G

itself is the centre of gravity of the body, these expressions and consequently $\Sigma mx'$ and $\Sigma my'$ vanish. Also $\Sigma m = M$.

Hence from equation (1), we get,

$$I' = I + Md^2. \qquad ..(2).$$

It can also be written as

$$I = I' - Md^2 \qquad ..(3),$$

which gives I when I' is known.

It follows from this theorem that if a number of axes are drawn in a given direction, the moment of inertia of a body is a minimum about the one that passes through the centre of gravity of the body.

14·721. The following examples illustrate the result (2) :—

(*i*) M.I. of a rod about AY' (Art. 14·2)

$$= M\frac{4a^2}{3} = M\frac{a^2}{3} + Ma^2 = \text{M.I. about } GY + Ma^2.$$

(*ii*) M.I. of a rectangular lamina about AB (Art. 14·21)

$$= M\frac{4b^2}{3} = M.\frac{b^2}{3} + Mb^2 = \text{M.I. about } GX + Mb^2.$$

(*iii*) M.I. of a circular disc of radius a about a diameter

$$= M \cdot \frac{a^2}{4}; \qquad \text{(Art. 14·31)}$$

Hence M.I. about a tangent parallel to the diameter

$$= M\frac{a^2}{4} + Ma^2 = M \cdot \frac{5}{4}a^2.$$

Note. It is important to note that the theorem of parallel axes is to be applied only when the moment of inertia of the body about the axis through the *C.G.* of the body has first been obtained.

14·722. Moment of inertia of a hemisphere about a diameter in its base.

(*i*) **Solid Hemisphere.** In Art. 14·5, M.I. of the elementary disc $PP'Q'Q$ of radius y and thickness dx about its diameter PNP'

$$= m\pi y^2 dx . \frac{y^2}{4}.$$

$\therefore$ by (2), Art. 14·72, its M.I. about AOB, a diameter of the base parallel to PNP'

$$= m\pi y^2 dx(\tfrac{1}{4}y^2 + x^2) = m\pi y^2 dx . \tfrac{1}{4}(y^2 + 4x^2)$$
$$= m\pi(a^2 - x^2)dx . \tfrac{1}{4}(a^2 - x^2 + 4x^2).$$

Hence M.I. of the hemisphere about AOB is given by

$$I = \tfrac{1}{4}m\pi \int_0^a (a^2 - x^2)(a^2 + 3x^2)dx = \tfrac{1}{4}m\pi \times \tfrac{8}{15}a^5$$
$$= \tfrac{2}{3}m\pi a^2 \cdot \tfrac{2}{5}a^2 = \tfrac{2}{5}Ma^2.$$

Cor. Also by (3), M.I. about an axis through G ($OG = \frac{3}{8}a$), the *C.G.* of the hemisphere, parallel to AOB

$$= I - M(\tfrac{3}{8}a)^2 = \tfrac{2}{5}Ma^2 - \tfrac{9}{64}Ma^2 = \tfrac{83}{320}Ma^2.$$

(*ii*) **Hollow Hemisphere.** In Art. 14·51, M.I. of the elementary

band (ring) $PP'Q'Q$ of radius $NP=a\sin\theta$ and breadth $a\delta\theta$ about its diameter PNP'

$$=m.2\pi a\sin\theta\, a\delta\theta.\tfrac{1}{2}a^2\sin^2\theta.$$

$\therefore$ by (2), Art. 14·72. its M.I. about AOB, a diameter of the base parallel to PNP'

$$=2m\pi a^2\sin\theta\delta\theta(\tfrac{1}{2}a^2\sin^2\theta+x^2) \text{ where } x=a\cos\theta$$
$$=\tfrac{1}{2}.2m\pi a^2(a^2\sin^2\theta+2a^2\cos^2\theta)\sin\theta\delta\theta$$
$$=m\pi a^4(1+\cos^2\theta)\sin\theta\delta\theta.$$

Hence M.I. of the hemispherical cap about AOB is given by

$$I=m\pi a^4\int_0^{\frac{1}{2}\pi}(1+\cos^2\theta)\sin\theta d\theta=\tfrac{5}{3}m\pi a^4=2m\pi a^2\times\tfrac{2}{3}a^2$$
$$=\tfrac{2}{3}Ma^2.$$

Cor. Also by (3), M.I. about an axis through G ($OG=\frac{1}{2}a$), the $C.G.$ of the hemispherical cap, parallel to AOB

$$=I-M(\tfrac{1}{2}a)^2=\tfrac{2}{3}Ma^2-\tfrac{1}{4}Ma^2=\tfrac{5}{12}Ma^2.$$

Example 1. *Find the moment of inertia of a solid right circular cylinder of length l and radius a about OY, a diameter of one of its end faces.*

Divide the cylinder into circular discs all parallel to OY and let one of them be at a distance x from it. If m is the mass per unit volume and δx the thickness of this disc, its mass$=m\pi a^2.\delta x$. The moment of inertia of the disc about its own diameter parallel to OY is

$$m\pi a^2.\delta x.\frac{a^2}{4}=\frac{m\pi a^4}{4}\delta x.$$

$\therefore$ its moment of inertia about OY, by Art. 14·72

$$=\frac{m\pi a^4}{4}\delta x+m\pi a^2\delta x.x^2 \qquad ..(1).$$

Hence M.I. of the cylinder about OY

$$=\int_0^l\frac{m\pi a^4}{4}dx+\int_0^l m\pi a^2x^2dx$$
$$=\tfrac{1}{4}m\pi a^4l+\tfrac{1}{3}m\pi a^2l^3=m\pi a^2l\left(\frac{a^2}{4}+\frac{l^2}{3}\right)=M\left(\frac{a^2}{4}+\frac{l^2}{3}\right).$$

Also from equation (3) Art. 14·72, M.I. of the cylinder about an axis through its centre of gravity perpendicular to its axis *i.e.*, parallel to OY

$$=M\left(\frac{a^2}{4}+\frac{l^2}{3}\right)-M\left(\frac{l}{2}\right)^2=M\left(\frac{a^2}{4}+\frac{l^2}{12}\right).$$

Note. The distance from OY of every particle on the disc at a distance x from it is not the same and this is the reason why its M.I. about OY consists of two terms as given in equation (1) and is not simply $m\pi a^2\delta x.x^2$.

Example 2. *Find the moment of inertia of a solid right circular cone of height h and vertical angle 2α, about a line through the vertex parallel to the base.*

Let the cone be generated by a line OP revolving about OX as axis, making a constant angle α with it. [*See* Fig. Art. 14·71].

Let m be the mass per unit volume of the cone. Take an element of the cone in the form of a disc of radius y and thickness dx, at a distance x from the vertex O.

m', the mass of this element $= m\pi y^2 dx = m\pi x^2 \tan^2\alpha\, dx$.

M.I. of this disc about a diameter $= m'y^2/4$.

$\therefore$ its M.I. about OY, a line drawn through the vertex O, parallel to the diameter

$$= m'(\tfrac{1}{4}y^2 + x^2) = \tfrac{1}{4}m'(x^2\tan^2\alpha + 4x^2) = \tfrac{1}{4}m'x^2(4+\tan^2\alpha)$$
$$= \tfrac{1}{4}m\pi x^4 \tan^2\alpha(4+\tan^2\alpha)dx.$$

Hence M.I. of the whole cone about OY

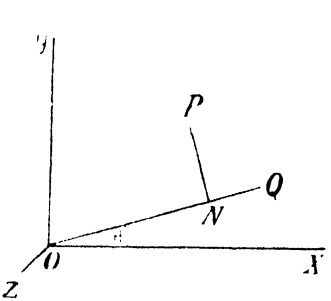
$$= \tfrac{1}{4}m\pi \tan^2\alpha(4+\tan^2\alpha)\int_0^h x^4 dx = \tfrac{1}{4}m\pi \tan^2\alpha(4+\tan^2\alpha).\tfrac{1}{5}h^5$$
$$= \tfrac{1}{3}m\pi h^3 \tan^2\alpha \times \tfrac{3}{10}h^2(4+\tan^2\alpha) = \tfrac{3}{20}Mh^2(4+\tan^2\alpha).$$

Ex. 3. Find the moment of inertia of a triangular lamina about a line through one of its vertices.

[*See* (2), Art. 14·81 with full process.]

14·73. Moment of inertia of a lamina about a line in its plane through the origin.

Let ON be the line making an angle θ with OX.



Let $P(x, y)$ be the position of an element of mass m of the lamina and PN be the perp. to ON, then

$$PN = y\cos\theta - x\sin\theta$$

and I the *moment of inertia about* ON, is given by

$$I = \Sigma m PN^2 = \Sigma m(y\cos\theta - x\sin\theta)^2$$
$$= \cos^2\theta\,\Sigma my^2 + \sin^2\theta\,\Sigma mx^2 - 2\sin\theta\cos\theta\,\Sigma mxy$$

i.e., $I = A\cos^2\theta + B\sin^2\theta - 2F\sin\theta\cos\theta$ ..(1),

where $\Sigma my^2 =$ M.I. about $OX = A$, $\Sigma mx^2 =$ M.I. about $OY = B$ and Σmxy, known as the product of inertia about OX, $OY = F$.

So that when A, B and F are known, equation (1) gives I, the moment of inertia of the lamina about ON, an axis through O.

Also the *product of inertia* about ON and a line perpendicular to it through O

$$= \Sigma m\, NP.ON = \Sigma m(y\cos\theta - x\sin\theta)(x\cos\theta + y\sin\theta)$$
$$= \Sigma m[(y^2 - x^2)\sin\theta\cos\theta + xy(\cos^2\theta - \sin^2\theta)]$$
$$= (\Sigma my^2 - \Sigma mx^2)\sin\theta\cos\theta + \cos 2\theta\,\Sigma mxy$$
$$= (A-B)\sin\theta\cos\theta + F\cos 2\theta.$$ ..(2).

14·731. Momental Ellipse.

If a point $Q(x, y)$ is taken on the line ON such that $OQ = r$ and *the moment of inertia of the lamina about* ON *is inversely proportional to* OQ^2, *i.e.*, if $I = \dfrac{Me^4}{OQ^2} = \dfrac{Me^4}{r^2}$, equation (1) gives

$$A\cos^2\theta + B\sin^2\theta - 2F\sin\theta\cos\theta = \frac{Me^4}{r^2}$$ ..(1),

which gives $Ax^2+By^2-2Fxy=Me^4$, ..(2)

where M denotes the mass and e, a length which is introduced to satisfy the requirements of physical dimensions.

This shows that the locus of Q is a conic, and since, by construction every radius is real and finite, therefore the conic is an ellipse. known as the *momental ellipse at O.*

The principal axes of this ellipse are called the *principal axes of inertia* and the corresponding moments are called the *principal moments of inertia at O.*

In the particular case when $F=0$, the co-ordinate axes OX, OY are the principal axes of inertia at O and equation (2) reduces to

$$Ax^2+By^2=Me^4.$$

In case F is not zero, the co-ordinate axes are not the principal axes of the lamina. But at any point O there is always a set of principal axes, their position being obtained by equating to zero the product of inertia of the lamina about these axes.

Thus if ON and a line perpendicular to it through O be the principal axes at O, the product of inertia $\Sigma(m\ NP.ON)$ must vanish.

∴ the angle θ which ON, one of the principal axes makes with OX, obtained by equating to zero the product of intertia as given in (2) Art. 14·73, is

$$\tfrac{1}{2}\tan^{-1}\frac{2F}{B-A}.$$

Note. The product of inertia of a lamina about its principal axes at a point O always vanishes and conversely if the product of inertia about two perpendicular lines through a given point of a lamina vanishes, the lines are the principal axes of the lamina.

Example. *Show that the principal axes at a corner of a rectangular lamina of sides 2a, 2b make with a side angles* θ, $\frac{1}{2}\pi+\theta$, *where*

$$\tan 2\theta=\tfrac{3}{2}\,ab/(a^2-b^2).$$

Let m be the mass per unit area and M the total mass of the lamina, so that $M=4abm$.

Let the sides $OA=2a$, $OB=2b$, be the co-ordinate axes, then

$$A,\ \text{M.I. about } OA=M.\frac{4b^2}{3} \qquad \text{(Art. 14·21)}$$

$$B,\ \text{M.I. about } OB=M.\frac{4a^2}{3},$$

and F, the product of inertia about OA, OB

$$=\int_0^{2a}\int_0^{2b} m\,xy\,dx\,dy$$

$$=\frac{m\,4b^2\,.\,4a^2}{2\,.\,2}$$

$$=4ab\,m\,.\,ab=Mab$$

$$\therefore\quad \tan 2\theta=\frac{2F}{B-A}=\frac{2Mab}{\frac{1}{3}M(a^2-b^2)}=\frac{3ab}{2(a^2-b^2)}.$$

Examples XLIX

1. Find the moment of inertia of the following :—

(*a*) An isosceles triangle ABC about an axis through the vertex A perpendicular to (*i*) the opposite side, (*ii*) the plane of the triangle.

[*Ans.* (*i*) $Ma^2/24$; (*ii*) $\frac{1}{8}Ma^2(\frac{1}{3}+\cot^2\frac{1}{2}A)$, where $BC=a$].

(*b*) An arc of a circle about an axis passing through its middle point and (*i*) through the centre, (*ii*) perpendicular to its plane.

$$[Ans.\ (i)\ M\frac{a^2}{2}\left(1-\frac{\sin 2\alpha}{2\alpha}\right);\ (ii)\ 2Ma^2\left(1-\frac{\sin\alpha}{\alpha}\right),$$

where a is the radius and 2α the angle subtended by the arc at the centre.]

(*c*) A solid right circular cone whose height is h and the radius of whose base is a about (*i*) its axis, (*ii*) a line perpendicular to the axis through its centre of gravity, (*iii*) about a diameter of the base.

[*Ans.* (*i*) $3Ma^2/10$, (*ii*) $3M(h^2+4a^2)/80$, (*iii*) $M(3a^2+2h^2)/20$.]

2. A uniform triangular lamina ABC is of mass M ; prove that its moment of inertia about BC is $Mh^2/6$, where h is the distance of A from BC.

Deduce that the moment of inertia of a uniform rectangular lamina about a diagonal is $Ma^2b^2/6(a^2+b^2)$, where a and b are the sides of the rectangle and M is its mass.

3. Prove that the moment of inertia of a uniform triangular lamina ABC of mass m about an axis through A perpendicular to the plane of the lamina is $\frac{1}{12}m(3b^2+3c^2-a^2)$. (*P.U.*)

4. Prove that for a regular polygon of n sides, the square of the radius of gyration about an axis through the centre perpendicular to the plane is

$$\frac{a^2}{12}\cdot\frac{2+\cos 2\pi/n}{1-\cos 2\pi/n},$$

a being a side.

5. Particles of equal mass are placed at the corners of a regular polygon. Prove that the squares of the principal radii of gyration at any point O in the plane of the polygon are $a^2/2$ and $(a^2+2h^2)/2$, where a is the radius of the circumscribing circle of the polygon and h is the distance of O from the centre of the circle. (*D.U. 1957*)

6. Prove that the moment of inertia of a right circular solid cone of semi-vertical angle α, height h and mass M, about a generator is

$$\tfrac{3}{4}Mh^2\sin^2\alpha(1+\tfrac{1}{5}\sec^2\alpha).$$

7. Show that the radius of gyration K of the frustum of a solid cone about its axis of symmetry is given by $K^2=\frac{3}{10}\cdot\frac{a^5-b^5}{a^3-b^3}$, where a and b are the radii of the flat ends.

8. Show that the moment of inertia of a hollow sphere, whose external and internal radii are a and b about a diameter is

$$\frac{2M}{5}\cdot\frac{a^5-b^5}{a^3-b^3}.$$

9. Show that the radius of gyration K of (*i*) a hollow cylinder and (*ii*) a solid cylinder about a tangent line which lies in a plane passing through the centre of mass and perpendicular to the axis is given by

$$(i)\ K^2=\frac{3a^2}{2}+\frac{h^2}{3};\quad (ii)\ K^2=\frac{5a^2}{4}+\frac{h^2}{3},$$

where a is the radius and $2h$ the length of the cylinder.

10. Show that the square of the radius of gyration of a parabolic area of latus rectum $4a$, cut off by an ordinate at a distance h from the vertex is $\frac{2}{7}h^2$ about the tangent at the vertex and $\frac{4}{5}ah$ about the axis.

11. Show that the radius of gyration of a paraboloid of revolution about its axis is $1/\sqrt{3}$ of the radius of its base.

12. Show that the moment of inertia of a uniform elliptic plate of semi-axes a, b and of mass m (*i*) about a line perpendicular to its plane through an end of the major axis is $\frac{1}{4}m(b^2+5a^2)$; (*ii*) about a diameter inclined at an angle θ to the major axis is $\frac{m}{4}(a^2\sin^2\theta+b^2\cos^2\theta)$.

13. If k_1, k_2 be the radii of gyration of an elliptic lamina about two conjugate diameters, show that

$$\frac{1}{k_1^2}+\frac{1}{k_2^2}=4\left(\frac{1}{a^2}+\frac{1}{b^2}\right). \qquad (D.U.\ 1953)$$

14. ABC is a uniform triangular plate, C being a right angle, $CA=a$, $CB=b$. Show that the principal axes of inertia at C makes with the sides an angle

$$\tfrac{1}{2}\tan^{-1}\frac{ab}{a^2-b^2}.$$

15. A wire is in the form of a semi-circle of radius a ; show that at an end of a diameter the principal axes are inclined to the diameter at angles

$$\tfrac{1}{2}\tan^{-1}\frac{4}{\pi} \text{ and } \frac{\pi}{2}+\tfrac{1}{2}\tan^{-1}\frac{4}{\pi}.$$

16. Show that at the centre of a quadrant of an ellipse of axes $2a$, $2b$ the principal axes are inclined at an angle

$$\tfrac{1}{2}\tan^{-1}\left(\frac{4}{\pi}\cdot\frac{ab}{a^2-b^2}\right) \text{ to the axes.}$$

17. Find the M.I. of the lamina given by $y=\sin x$, and the x-axis, about the axis of y, x changing from 0 to π. [*Ans.* $\frac{1}{8}(\pi^2-4)m$.]

18. For the area bounded by the curve $r^2=a^2\cos 2\theta$, prove that

(*i*) the square of the radius of gyration about an axis through the origin perp. to its plane is $\frac{1}{4}\pi a^2$,

(*ii*) the moment of inertia about its axis is $\frac{1}{16}Ma^2(\pi-\frac{8}{3})$;

(*iii*) the principal axes at the node of a half loop are inclined to the initial line at angles

$$\tfrac{1}{2}\tan^{-1}\tfrac{1}{2} \text{ and } \tfrac{1}{2}\pi+\tfrac{1}{2}\tan^{-1}\tfrac{1}{2}.$$

19. Show that for a thin hemispherical shell of radius a and mass M, the principal moments of inertia at the centre of gravity are $\frac{5}{12}Ma^2$, $\frac{5}{12}Ma^2$, $\frac{2}{3}Ma^2$. (*D.U.*)

20. A uniform rectangular plate whose sides are of lengths $2a$, $2b$ has a portion cut out in the form of a square whose centre is the centre of the rectangle and whose mass is half the mass of the plate. Show that the axes of greatest and least moment of inertia at a corner of the rectangle make angles θ, $\frac{1}{2}\pi+\theta$ with a side, where

$$\tan 2\theta=6ab/5(a^2-b^2). \qquad (D.U.\ 1958)$$

14·74. Moment of Inertia of a body. *To find the moment of inertia of a body about a line OQ through the origin O, whose direction cosines with reference to three rectangular axes OX, OY, OZ are l, m, n.*

Let A, B, C, be the moments of inertia of a body about OX, OY, OZ, and D, E, F, its products of inertia with respect to OY, OZ ; OZ, OX ; and OX, OY respectively.

Let $P(x, y, z)$ be the position of an element of mass m' of the body and let PN be drawn perpendicular to the line OQ.

In the figure of Art. 14·73, let OZ be perp. to the plane of the paper and suppose OQ to have direction cosines l, m, n.

Then by Solid Geometry,

$$PN^2=OP^2-ON^2=(x^2+y^2+z^2)-(lx+my+nz)^2$$
$$=x^2(m^2+n^2)+y^2(n^2+l^2)+z^2(l^2+m^2)-2mnyz-2nlzx-2lmxy,$$
since $l^2+m^2+n^2=1$
$$=l^2(y^2+z^2)+m^2(z^2+x^2)+n^2(x^2+y^2)-2mnyz-2nlzx-2lmxy$$

$\therefore$ I, the *M.I.* of the body about OQ
$$=l^2\Sigma m'(y^2+z^2)+m^2\Sigma m'(z^2+x^2)+n^2\Sigma m'(x^2+y^2)$$
$$-2mn\Sigma m'\,yz-2nl\Sigma m'zx-2lm\Sigma m'xy$$

Now since y^2+z^2, z^2+x^2, x^2+y^2 are respectively squares of the distances of the point P from OX, OY, OZ, we have

$$\mathrm{I}=Al^2+Bm^2+Cn^2-2Dmn-2Enl-2Flm \qquad ..(1).$$

14·741. Momental Ellipsoid. Now along a line OQ drawn through O, take $OQ=r$, and the point $Q(x, y, z)$, such that the moment of inertia of the body about OQ may be *inversely proportional to* OQ^2, *i.e., let* $\mathrm{I}=\dfrac{Me^4}{OQ^2}=\dfrac{Me^4}{r^2}$, where M denotes the mass of the body and e, a length to satify the requirements of dimensions, then

$$\frac{Me^4}{r^2}=A\,l^2+B\,m^2+Cn^2-2\,Dmn-2Enl-2Flm$$

$$or \quad Ax^2+By^2+Cz^2-2\,Dyz-2\,Ezx-2Fxy=Me^4 \qquad ..(2).$$

Hence the locus of Q is an ellipsoid called the momental ellipsoid.

If, with reference to a set of rectangular axes, $D=E=F=0$, these axes are then the *principal axes* and ellipsoid referred to them is reduced to the form

$$A'x^2+B'y^2+C'z^2=Me^4.$$

Further if the three principal moments at any point O are equal to each other, the ellipsoid becomes a sphere.

For example, the perpendiculars from the *C.G.* of a cube to three faces are principal axes and the three moments of inertia about them are, by symmetry, equal. Hence the momental ellipsoid is a sphere.

14·8. Equimomental Bodies or Particles. Two bodies or two systems of particles are said to be equimomental when their moments of inertia about all straight lines are equal each to each.

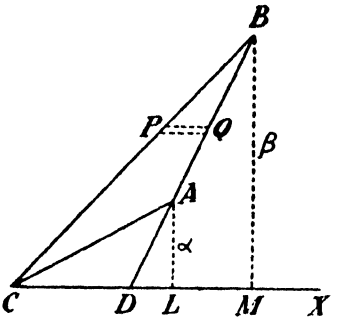
If this be so, the two systems will necessarily have the same mass and the same centre of gravity.

For example a straight uniform rod of mass m is equimomental with particles of mass $\frac{1}{3}m$ at each of its ends and a particle of mass $\frac{2}{3}m$ at its middle point.

14·81. *A uniform triangular lamina of mass M is equimomental with three particles, each of mass $\frac{1}{3}M$, placed at the middle points of its sides.*

As a first step, we find *M.I.* of the triangle ABC about CX, a line drawn in its plane through one of its vertices C.

Let α β be the distances of A and B from CX and let BA meet it in D. If m be the mass per unit area, then



M, the mass of the triangle ABC
$= m(\triangle BCD - \triangle ACD)$
$= \frac{1}{2}mCD(\beta - \alpha)$ ..(1).

Let PQ be a strip of the $\triangle BCD$ drawn parallel to its base CD at a distance y from CX and let dy be its breadth, then since PQ is at a distance $\beta - y$ from B, the area of the strip $= \frac{CD}{\beta}(\beta - y)\,dy$. $\left[\because \frac{PQ}{CD} = \frac{\beta - y}{\beta}\right]$.

M.I. of $\triangle BCD$ about CX

$$= \int_0^\beta \frac{m\,CD}{\beta}(\beta - y)y^2\,dy = \frac{m\,CD}{\beta}\left(\frac{1}{3}\beta^4 - \frac{1}{4}\beta^4\right) = \frac{1}{12}m\,CD\,\beta^3.$$

Similarly M.I. of $\triangle ACD$ about $CX = \frac{1}{12}mCD\,\alpha^3$.

Hence I, M.I. of $\triangle ABC$ about CX

$$= \frac{m}{12}CD(\beta^3 - \alpha^3) = \frac{1}{8}m\,CD(\beta - \alpha) \times \frac{1}{3}(\beta^2 + \alpha\beta + \alpha^2)$$
$$= \frac{1}{8}M(\beta^2 + \alpha\beta + \alpha^2) \quad ..(2).$$

Now consider a system of three particles, each of mass $\frac{1}{3}M$. at the middle points of the sides of the $\triangle ABC$.

Their C.G. coincides with that of the triangle and their M.I. about $CX = \frac{M}{3}\left[\left(\frac{\alpha}{2}\right)^2 + \left(\frac{\beta}{2}\right)^2 + \left(\frac{\alpha+\beta}{2}\right)^2\right] = \frac{1}{6}M\,(\alpha^2 + \alpha\beta + \beta^2)$

which is the same as I given in (2).

This establishes the equality of the moments of inertia about CX, a line through an angular point.

Now suppose d is the distance of G, the common $C.G.$ of the particles and the triangle, from CX.

Then by (3) Art. 14·72, $M\,I.$ of the triangle, or of the particles about a parallel line through G is $I - Md^2$.

Again, by (2) Art. 14·72, $M.I.$ of either system about a parallel line at a distance h from G is obtained by adding Mh^2 to $I - Md^2$.

Hence M.I. of the particles about any line parallel to CX is the same as that of the triangle.

But the direction of CX is arbitrary and so the theorem is proved.

Note. In the particular case when AB does not intersect CX and is parallel to it, the proof is very much simplified and is left to the student as an exercise.

If A and B are on opposite sides of CX, the term $\alpha\beta$ is negative, α, β being of opposite signs.

Ex. 1. Show that a uniform triangular lamina of mass m is equimomental with three equal particles of mass $\frac{1}{3}m$ placed at the angular points and a particle of mass $\frac{1}{2}m$ placed at the centre of inertia ($C.G.$) of the triangle. (*D. U. 1959*)

Ex. 2. Show that a lamina of mass M in the form of a parallelogram is equimomental to the system of five particles four of $\frac{1}{4}M$ each at the middle points of the sides and one of $\frac{1}{2}M$ at the point of intersection of the diagonals.

Ex. 3. Obtain the result of Ex. 3 P. 286 by the application of this article.

Ex. 4. Show that the moment of inertia of a regular hexagon of mass M and side a about one of its sides is $\frac{23}{24} Ma^2$.

Ex. 5. Show that, if α, β, γ are the distances of the vertices of a triangle from a straight line in its plane, the moment of inertia of the triangle about this line is

$$\tfrac{1}{3}M(\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta),$$

where M denotes its mass.

14·9. Motion of a rigid body about a fixed axis. *A rigid body* is supposed to be one whose shape and size are invariable, so that the distance between any two points of it remain the same during any motion.

A *rigid body* is said to be moving in two dimensions when all points of the body move in parallel planes.

A rigid body moving in two dimensions has three degrees of freedom and, therefore, requires three co-ordinates to specify its position and consequently three equations are necessary to determine its motion.

In the particular case when a rigid body is free only to rotate about a fixed axis, its position is specified completely by a *single co-ordinate*, which is taken to be the angle θ which some plane through the axis of rotation, fixed in the body, makes with some other plane, fixed in space, through the same axis.

In such cases, the angular velocity and acceleration, at any instant, is the same for *every* point of the body.

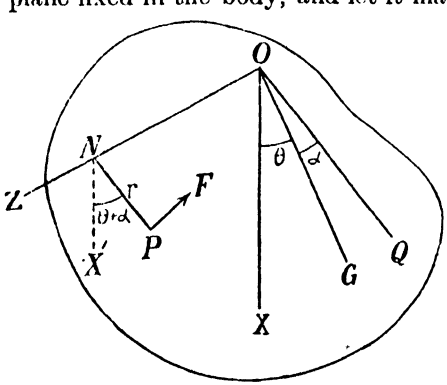
This is proved as follows :—

***14·91.** Suppose a rigid body of mass M rotates about a fixed horizontal axis OZ drawn perpendicular to the plane of the paper.

Let the plane through the centre of gravity G of the body perpendicular to the axis be the plane of paper ; let it meet the axis in O and let us call it the plane p.

Let XOZ, the vertical plane through the axis OZ be a plane fixed in space, meeting the plane p along OX.

Let GOZ, the plane through the axis OZ and through G be a plane fixed in the body, and let it make, at any instant, an angle θ, with the plane XOZ, so that $\theta = \angle XOG$ is the angle which a plane through the axis fixed in the body, makes with a plane through the axis fixed in space.



This angle specifies the position of the body at any instant.

Let a plane ZOQ through OZ and any point P of the body make an angle α with the plane GOZ and cut the plane p in the line OQ so that $\angle GOQ = \alpha$ and $\angle XOQ = \theta + \alpha$.

As the body rotates about the axis OZ, $\angle GOQ$ remains always the same and, therefore, we have

$$\frac{d}{dt}(\theta+\alpha)=\frac{d\theta}{dt} \quad \text{and} \quad \frac{d^2}{dt^2}(\theta+\alpha)=\frac{d^2\theta}{dt^2}.$$

The results being independent of α, show that the angular velocity and acceleration, at any instant, is the same for *every* point of the body.

14·92. The equation of motion of a rigid body rotating about a fixed axis. From P draw $PN(=r)$ perpendicular to the axis OZ and then from N draw NX' parallel to OX, then P being in the plane ZOQ, $\angle X'NP=\theta+\alpha$.

Let a particle of mass m be situated at P, then as the body rotates about OZ, PN or r remains constant and the particle at P moves in a circle of radius r about N as centre.

$\therefore$ its acceleration perpendicular to PN is

$$r\frac{d^2}{dt^2}(\theta+\alpha)=r\frac{d^2\theta}{dt^2}. \qquad \text{(Cor. Art. 13·4)}$$

If F be the sum of the resolved parts of the forces acting on P, perpendicular to PN, we have

$$mr\frac{d^2\theta}{dt^2}=F$$

$$\therefore \quad mr^2\frac{d^2\theta}{dt^2}=Fr \quad \text{or} \quad \frac{d}{dt}\left(r.mr\frac{d\theta}{dt}\right)=Fr \qquad ..(1),$$

r being a constant $dr/dt=0$, and, therefore, $rd\theta/dt$ is the velocity, $mrd\theta/dt$, the momentum and $r.mrd\theta/dt$, is the moment of the momentum of the particle of mass m at P.

Equation (1), therefore, shows that the rate of change of the moment of momentum or of the angular momentum of the particle P about the axis OZ is equal to the sum of the moments about the same axis of *all* the forces acting at P.

Summing up for all the particles of the body, we have

$$\Sigma\left(mr^2\frac{d^2\theta}{dt^2}\right)=\Sigma Fr.$$

$\frac{d^2\theta}{dt^2}$ being the same for all the particles, we have

$$\frac{d^2\theta}{dt^2}\Sigma(mr^2) \quad \text{or} \quad MK^2\frac{d^2\theta}{dt^2}=\Sigma(Fr), \qquad ..(2),$$

where K is the radius of gyration of the body about OZ. (Also *see* Art. 15·4).

$\Sigma(Fr)$ is equal to the sum of the moments about the axis, of all the forces on all the particles of the body. These forces are partly external to the system and partly of the nature of internal action and reaction among the particles.

On the *assumption* that the sum of the moments about the axis of the mutual action among the different particles of the body vanishes, $\Sigma(Fr)$ represents the sum of the moments about the axis of all the *external forces* acting on the body.

Equation (2) is then the required equation of motion and it shows that *the rate of change of the angular momentum of the body about the axis of rotation is equal to the sum of the moments about the same axis* of all the forces acting on the body.

Kinetic Energy. Since the velocity of P is $r\frac{d\theta}{dt}$, the kinetic energy of the particle at P is $\frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2$ or $\frac{1}{2}mr^2\omega^2$,

where $\omega=\frac{d\theta}{dt}$ is the same for every point of the body.

Hence the total kinetic energy of the body

$$\Sigma\tfrac{1}{2}mr^2\omega^2=\tfrac{1}{2}\omega^2\Sigma mr^2=\tfrac{1}{2}MK^2\omega^2. \qquad ..(3).$$

Example. *Find the kinetic energy, of a circular disc of radius 2 ft. and weight 10 lbs., rolling, without sliding, on a horizontal plane such that its centre moves forward with a speed of 20 ft. per sec.*

Here $m=10$ lbs., $r=2$ ft., $K^2=\frac{1}{2}r^2=\frac{1}{2}\times4=2$.

Also since the speed $v=20$ ft./sec.,

$\therefore\quad \omega=v/r=20/2=10$ radians per sec.

The kinetic energy due to translation of the disc

$$=\tfrac{1}{2}mv^2=\tfrac{1}{2}\times10\times(20)^2=2000 \text{ poundals}$$

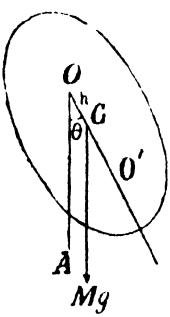
The kinetic energy due to rotation

$$=\tfrac{1}{2}mK^2\omega^2=\tfrac{1}{2}\times10\times2\times(10)^2=1000 \text{ poundals.}$$

Hence the total kinetic energy

$$=\frac{2000+1000}{32}=\frac{375}{4}=93\tfrac{3}{4} \text{ ft. lb.}$$

14·93. The Compound Pendulum. Any rigid body free to turn about a smooth fixed horizontal axis is known as a *compound pendulum*.



Let AOG be the vertical plane, perp. to the horizontal axis about which the rotation takes place, through G, the centre of gravity of the body; and let O, called the *centre of suspension*, be the point of intersection of this plane and the axis of rotation. Let OG make an angle θ with the vertical line OA, (the initial position of OG) and let $OG=h$.

Let M be the mass, and K, the radius of gyration of the body about the axis.

The equation of motion is

$$MK^2\frac{d^2\theta}{dt^2}=\text{sum of the moments of the forces acting on the body about the axis of rotation}$$

$$=-Mgh\sin\theta.$$

Now if θ is so small that $\sin\theta$ can be replaced by θ, the equation becomes

$$\frac{d^2\theta}{dt^2}=-\frac{g}{K^2/h}\theta,$$

which shows that the motion is simple harmonic, the length l of the equivalent simple pendulum being equal to

$$\frac{K^2}{h} = \frac{K_1^2+h^2}{h},$$

where K_1 is the radius of gyration of the body about the axis through G parallel to the axis through O.

If $$OO'=l=\frac{K_1^2+h^2}{h}.$$

Then $$K_1^2=hl-h^2=h(l-h)=OG.GO'.$$

O' is called the *centre of oscillation* and the symmetry of the result shows that if the body were suspended from a parallel axis through O', O would be the centre of oscillation, so that the *centres of oscillation and suspension are interchangeable.*

Example 1. *A pendulum is supported at O, and O' is the centre of oscillation. Show that, if an additional weight is rigidly attached at O', the period of oscillation is unaltered.*

By Art. 14·93, $OO'=l=K^2/h$, where $OG=h$.

Let M be the mass of the pendulum and m that of the additional weight attached at O'.

Let K' be the radius of gyration about the axis through O, and h' the distance from O of the $C.G.$ for the system M and m, then

$$(m+M)h'=Mh+ml=Mh+m\frac{K^2}{h} \quad ..(1),$$

and $$(m+M)K'^2=MK^2+ml^2$$

$$=MK^2+m\frac{K^4}{h^2}=\frac{K^2}{h}\left[Mh+m\frac{K^2}{h}\right] \quad ..(2).$$

Dividing (2) by (1), we get $$\frac{K'^2}{h'}=\frac{K^2}{h}.$$

Hence the period of oscillation remains unaltered.

Example 2. *A solid homogeneous cone of height h and vertical angle 2α, oscillates about a horizontal axis through its vertex. Show that the length of the equivalent simple pendulum is*

$$\tfrac{1}{5}h(4+\tan^2\alpha).$$

From solved Ex. 2 Art. 14·72, we have

$$K^2=\tfrac{3}{20}h^2(4+\tan^2\alpha),$$

and h', the distance of the $C.G.$ from the vertex $=\frac{3}{4}h$.

Hence the required length of the simple equivalent pendulum

$$=K^2/h'=\tfrac{1}{5}h(4+\tan^2\alpha).$$

Ex. 3. If the cone of Ex. 2 makes small oscillations about a diameter of the base as axis, show that the period of a complete oscillation is

$$2\pi\left[\frac{(2+3\tan^2\alpha)h}{5g}\right]^{\frac{1}{2}}$$

Examples L

1. A light rod AB of length $2l$ has two particles of equal mass, one attached to the end B and the other to a point distant $\frac{3}{4}l$ from B. Find the period of small oscillation about a horizontal axis through A.

[*Ans.* $2\pi\sqrt{26l}/\sqrt{15g}$.]

2. A uniform rod of length l is suspended from one end and swings in a vertical plane. Prove that it oscillates like a simple pendulum of length $\frac{2}{3}l$.

Prove that this rod will oscillate most quickly if its point of suspension is at a distance from one end equal approximately to $0{\cdot}21\ l$.

3. A weightless straight rod ABC, of length $2a$, is movable about the end A which is fixed and carries two particles of the same mass, one fastened to the middle point B and the other to the end C of the rod. If the rod be held in a horizontal position and then let go, show that its angular velocity when vertical is $\sqrt{6g/5a}$, and that $\frac{5}{3}a$ is the length of the simple equivalent pendulum. (*D.U.*)

4. A compound pendulum consists of a thin uniform rigid rod of length $2a$ and mass m, with a heavy particle of mass $2m$ at its middle point. Show that there are two points of suspension on each side of the mid-point of the rod, about each of which the small oscillation will have the same period as those of a simple pendulum of length a, and find the distance between these points.

[*Ans.* $0{\cdot}745a$.]

5. A uniform circular disc of radius a oscillates in its own plane about a horizontal axis perpendicular to it passing through a point (*i*) $\frac{1}{2}a$ from the centre, (*ii*) on its rim. Show that the length of the equivalent simple pendulum is the same in both the cases.

6. Compare the times in which a circular plate will vibrate about a horizontal tangent and about the horizontal axis through the point of contact at right angles to the tangent. [*Ans.* $\sqrt{5/6}$.]

7. Show that the length of equivalent simple pendulum for a circular wire of angle 60° and radius a with its mid-point as the point of suspension is $2a$.

8. Three particles of the same mass m are fixed to a uniform circular hoop of mass M and radius a, at the corners of an equilateral triangle. The hoop is free to move in a vertical plane about the point on the circumference opposite to one of the masses m. Find the length of the equivalent simple pendulum.

[*Ans.* $2a$.]

9. A uniform circular hoop, whose centre is C, radius a, and mass m, can move freely about an axis which is perpendicular to its plane and is horizontal and passes through a point O of the hoop. A particle of mass m is fixed to the hoop at a point A such that the angle OCA is a right angle. Find the length of the equivalent simple pendulum. [*Ans.* $4a/\sqrt{5}$.]

10. A triangular lamina ABC oscillates in its own plane (which is vertical) about the vertex A, prove that the length of the equivalent simple pendulum is

$$(3b^2+3c^2-a^2)/4\sqrt{2b^2+2c^2-a^2}. \qquad (D.U.\ 1954)$$

11. A uniform cube is free to turn about one edge which is horizontal. If l is the length of a seconds pendulum, find in terms of l, the length of the edge of the cube so that it may execute a complete oscillation in 2 seconds.

[*Ans.* $3l/2\sqrt{2}$.]

12. Find the centre of oscillation of a cone whose axis of oscillation is a diameter of the base.

Prove that a simple pendulum of length $\frac{7}{5}h$ has the same period as a solid cone of height h and semi-vertical angle 60° if the axis of rotation is a line perpendicular to the axis of the cone through the vertex.

[*Ans.* $\frac{1}{5}(2+3\tan^2\alpha)h$ from the base.]

13. Find the length of a simple pendulum equivalent to a compound pendulum, consisting of a rod of mass m and length a carrying at one end a

sphere of mass m_1 and diameter $2b$, the other end of the rod being fixed. (*P.U.*)

Ans. $\left[m\frac{a^2}{3}+m_1\left((a+b)^2+\frac{2b^2}{5}\right)\right]\ \left[\frac{ma}{2}+m_1(a+b)\right]$.

14. An ellipse of semi-axes a, b and a circle of radius b are cut out from the same sheet of thin uniform metal and are superimposed and fixed together with their centres coincident. The figure is free to move in its own vertical plane about the end of the major axis; show the length of the equivalent simple pendulum is

$$(5a^2-ab+2b^2)/4a. \qquad (P.U.;\ D.U.)$$

15. A rigid body of mass M can turn freely round a horizontal axis about which its moment of inertia is I. Prove that oscillations under gravity are of period

$$2\pi\left(\frac{I}{Mgh}\right)^{\frac{1}{2}},$$

where h is the length of the perpendicular GA drawn from G, the centre of gravity, to the axis of rotation.

This period is to be halved by attaching a heavy particle to the body at some point P, in the straight line AG, show that the mass of the particle required is a minimum if $AP=\dfrac{I}{8Mh}$. (*D.U.*)

16. A sphere of radius a, is suspended by a fine wire from a fixed point at a distance l from the centre; show that the time of a small oscillation is given by

$$2\pi\sqrt{\frac{5l^2+2a^2}{5lg}}\left[1+\frac{\alpha^2}{16}\right],$$

where α represents the amplitude of vibration and is taken so small that its cube and higher powers are neglected.

Find also the corresponding result for a simple pendulum of length l.

17. A uniform rod AB of length $2a$ and Mass M, is freely attached to a fixed support at A and swings in a vertical plane through A. Find the frequency of the small oscillations.

If a particle of mass kM is attached to the rod at B, prove that the frequency of the small oscillations is reduced in the ratio

$$\sqrt{(1+2k)\,/\,(1+3k)} \qquad (D.U.\ 1961)$$

18. A uniform lamina is bounded by a parabolic arc, of latus-rectum $4a$, and a double ordinate at a distance b from the vertex. If $b=\frac{a}{3}(7+4\sqrt{7})$, show that two of the principal axes at the end of a latus-rectum are the tangent and normal there.

[**Hint.** Equate to zero the product of inertia about the tangent and normal.]

19. Show that the equation of the momental ellipsoid at the corner of a cube of side $2a$, referred to its principal axes is

$$2x^2+11(y^2+z^2)=\text{const.} \qquad (D.U.\ 1963)$$

CHAPTER XV

MOTION OF A RIGID BODY IN TWO DIMENSIONS

15·1. A rigid body is one in which the distance between every pair of particles remains constant and the body retains its shape and size even when it is in motion.

A rigid body is said to *move in two dimensions* or *to move in one plane* when every particle of the body moves parallel to a fixed plane.

As pointed out in Art. 14·9, a rigid body moving in two dimensions has three degrees of freedom and requires three co-ordinates to specify its position and consequently three equations are necessary to determine its motion.

The three co-ordinates usually taken are the co-ordinates $(\bar{x}, \bar{y})$ of the centre of gravity G of the body with reference to two rectangular axes in the plane of motion and θ, the angle which a line fixed in the body makes with a line fixed in space.

15·2. Equations of Motion. Let us consider a particle of mass m of the body situated at a point (x, y). Its equations of motion parallel to the co-ordinate axes are,

$$m\frac{d^2x}{dt^2}=X+X' \quad ..(1) \quad \text{and} \quad m\frac{d^2y}{dt^2}=Y+Y' \quad ..(2),$$

where X, Y are the sums of the resolved parts of the external forces acting on the particle and X', Y' the sums of the resolved parts of the internal forces *i.e.*, forces due to actions of the other particles on it.

Multiplying (1) by y and (2) by x and subtracting, we get

$$m\left[x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right]=(xY-yX)+(xY'-yX') \quad (3).$$

Summing up for all the particles of the body, we have

$$\Sigma m\frac{d^2x}{dt^2}=\Sigma X+\Sigma X', \qquad \Sigma m\frac{d^2y}{dt^2}=\Sigma Y+\Sigma Y' \quad ..(4),$$

and

$$\Sigma m\left[x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right]=\Sigma(xY-yX)+\Sigma(xY'-yX') \quad ..(5).$$

The internal forces are of the nature of mutual actions and reactions between the various particles, and, therefore, by Newton's Third Law, cancel each other in pairs and hence

$$\Sigma X'=0, \ \Sigma Y'=0 \quad \text{and} \quad \Sigma(xY'-yX')=0.$$

Thus equations (1), (2) and (3) reduce to the form

$$\Sigma m\frac{d^2x}{dt^2}=X, \qquad \Sigma m\frac{d^2y}{dt^2}=Y \quad ..(A),$$

and

$$\Sigma m\left[x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right]=\Sigma(xY-yX) \quad ..(B).$$

15·21. D'Alembert's Principle. $m\frac{d^2x}{dt^2}$ is called the effective force parallel to the axis of X on a particle of mass m and thus equations (A) can be stated as,

the sums of the components, parallel to the co-ordinate axes, of the effective forces are respectively equal to the sums of the components of the external impressed forces parallel to the same axes.

And equation (B) can be stated as,

the sum of the moments about the origin O (or about OZ, a line through O perp. to the XY plane) of the effective forces is equal to the sum of the moments of the external forces.

Equations (A) can be written in the form

$$\Sigma X - \Sigma m\frac{d^2x}{dt^2}=0, \qquad \Sigma Y - \Sigma m\frac{d^2y}{dt^2}=0,$$

i.e., the reversed effective forces and the external forces of the system are in equilibrium.

This is D'Alembert's principle.

15·22. Equations (A) of Art. 15·2 can also be written in the form,

$$\frac{d}{dt}\left(\Sigma m\frac{dx}{dt}\right)=\Sigma X, \qquad \frac{d}{dt}\left(\Sigma m\frac{dy}{dt}\right)=\Sigma Y,$$

i.e., the rate of change of the linear momentum of the system in any direction is equal to the sum of the resolved parts of the external forces in that direction.

It follows that if there is a direction, say that of the axis of x, in which the sum of the resolved parts of the external forces is zero, *i.e.*, $\Sigma X=0$, then

$$\Sigma m\frac{d^2x}{dt^2}=0, \quad i.e., \quad \frac{d}{dt}\left(\Sigma m\frac{dx}{dt}\right)=0 \quad \therefore \quad \Sigma m\frac{dx}{dt}=\text{const.}$$

i.e., the linear momentum in that direction is constant.

This is **the principle of Conservation of linear momentum.**

Again since $$\frac{d}{dt}\left[x\frac{dy}{dt}-y\frac{dx}{dt}\right]=\frac{dx}{dt}\frac{dy}{dt}+x\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{dx}{dt}-y\frac{d^2x}{dt^2}$$
$$=x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2},$$

equation (B) can be written as,

$$\frac{d}{dt}\left[\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)\right]=\Sigma(xY-yX),$$

i.e., the rate of change of the moment of momentum or of the angular momentum of the system about any fixed origin (or axis) is equal to the sum of the moments of the external forces about that origin (or axis).

From this it obviously follows that:

If the sum of the moments of the external forces about any fixed point is zero, the moment of momentum or the angular momentum of the system about that point is constant.

This is **the principle of Conservation of angular momentum.**

15·3. Motion of the centre of mass. If $(\bar{x}, \bar{y})$ be the co-ordinates of G, the centre of mass of the body of mass M, then

$$M\bar{x}=\Sigma mx \qquad \text{and} \qquad M\bar{y}=\Sigma my.$$

Differentiating these equations, we get

$$M\frac{d^2\bar{x}}{dt^2}=\Sigma m\frac{d^2x}{dt^2}, \qquad M\frac{d^2\bar{y}}{dt^2}=\Sigma m\frac{d^2y}{dt^2}.$$

Hence equations (A) become,

$$\Sigma m\frac{d^2x}{dt^2} \text{ or } M\frac{d^2\bar{x}}{dt^2}=\Sigma X, \qquad \Sigma m\frac{d^2y}{dt^2} \text{ or } M\frac{d^2\bar{y}}{dt^2}=\Sigma Y \quad ..(C).$$

Further if $\Sigma X=0=\Sigma Y$, then

$$M\frac{d\bar{x}}{dt}=\text{const., and } M\frac{d\bar{y}}{dt}=\text{const.}$$

So that if *the body is not acted upon by external forces, its C.G. is either at rest or moves with a constant velocity.*

The equations (C) indicate that :

The motion of the centre of mass G is the same as if the whole mass of the body were collected into a single particle at G and all the forces were to act at G.

15·31. Motion relative to the centre of mass. Now let (x', y') be the co-ordinates of the particle P of mass m relative to parallel axes through G $(\bar{x}, \bar{y})$, the co-ordinates of P relative to the original axes being (x, y), then

$$x=\bar{x}+x', \qquad y=y+y' \quad ..(1),$$

$$\therefore \quad \Sigma mx=\Sigma m\bar{x}+\Sigma mx'=M\bar{x}+\Sigma mx'$$

But since $\Sigma mx=M\bar{x}$,

$$\therefore \quad \Sigma mx'=0 \quad \text{and so also} \quad \Sigma my'=0 \quad ..(2),$$

and $$\therefore \quad \Sigma m\frac{d^2x'}{dt^2}=0, \quad \text{and also} \quad \Sigma m\frac{d^2y'}{dt^2}=0 \quad ..(3).$$

Substituting (1) in equation (B) of Art. 15·2, *viz.*,

$$\Sigma m\left[x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right]=\Sigma(xY-yX) \quad ..(B),$$

we get,

$$\Sigma m\left[(\bar{x}+x')\frac{d^2}{dt^2}(\bar{y}+y')-(\bar{y}+y')\frac{d^2}{dt^2}(\bar{x}+x')\right]$$
$$=\Sigma[(\bar{x}+x')Y-(\bar{y}+y')X].$$

Taking $\bar{x}$, $\bar{y}$ and their derivatives outside Σ and using results (2) and (3), this equation reduces to

$$M\left[\bar{x}\frac{d^2\bar{y}}{dt^2}-\bar{y}\frac{d^2\bar{x}}{dt^2}\right]+\Sigma m\left[x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right]$$
$$=(\bar{x}\Sigma Y-\bar{y}\Sigma X)+\Sigma(x'Y-y'X).$$

In view of equations (C) Art. 15·3, this further reduces to

$$\Sigma m\left[x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right]=\Sigma(x'Y-y'X) \quad ..(D).$$

This equation is of the same form as equation (B) given above and could have been obtained if the C.G. of the body were regarded as a fixed point.

Now equations (C) Art. 15·3 give the motion of translation of the body concentrated as a particle of mass M at the C.G. ; while

equation (D) gives the motion of rotation of the body about its C.G. independent of any linear motion. Thus the motion of translation and rotation are independent of each other.

This is known as the **principle of the independence of translation and rotation.**

15·4. Equation (D) can also be written as

$$\frac{d}{dt}\left[\Sigma m\left(x'\frac{dy'}{dt}-y'\frac{dx'}{dt}\right)\right]=\Sigma(x'Y-y'X) \qquad ..(E),$$

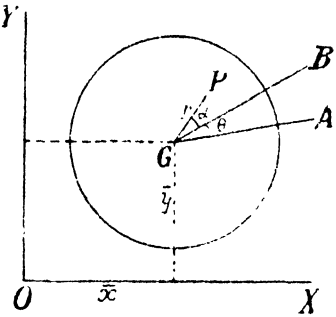
i.e., the rate of change of the moment of momentum about G is equal to the sum of the moments of the external forces.

Now $x'\frac{dy'}{dt}-y'\frac{dx'}{dt}$ is equal to the moment about G of the velocity relative to G of the particle m at P (x', y'), when the velocity components are expressed in cartesian co-ordinates.

In case the point P is expressed in polar co-ordinates with reference to G, equation (E) takes the form

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\mathbf{MK^2}\frac{\mathbf{d\theta}}{\mathbf{dt}}\right)=\Sigma(\mathbf{x'Y-y'X}) \qquad ..(F),$$

where K is the radius of gyration of the body about G, and θ is the angle which a line fixed in the body makes with a line fixed in space.



To obtain this we proceed as follows :—

Let GA be a line fixed in space and GB, a line fixed in the body, so that $\angle AGB=\theta$ changes as the body rotates about G.

[$\because$ GA remains fixed and GB changes its position.]

Let $GP=r$, $\angle BGP=\alpha$, which remain constant during the motion.

The components of the velocity of P relative to G along and perpendicular to GP are

$$\frac{dr}{dt}=0, \text{ and } r\frac{d}{dt}(\theta+\alpha)=r\frac{d\theta}{dt},$$

r and α being constants.

$\therefore$ the moment of the velocity of P about $G=r\,.\,r\frac{d\theta}{dt}$, so that the moment of momentum of the particle of mass m at P about G

$$=mr^2\frac{d\theta}{dt}.$$

Summing up for the whole body, its moment of momentum about G

$$=\Sigma mr^2\frac{d\theta}{dt}=\frac{d\theta}{dt}\Sigma mr^2=MK^2\frac{d\theta}{dt},$$

θ being independent of the position of P, $d\theta/dt$ can be taken outside Σ.

Hence equation (E) takes the form

$$\frac{d}{dt}\left(MK^2\frac{d\theta}{dt}\right)=\Sigma(x'Y-y'X)$$

or $MK^2\dfrac{d^2\theta}{dt^2}$=sum of the moments of the external forces about G,

which is written briefly as

$$I\frac{d^2\theta}{dt^2}=N,$$

where I stands for the moment of inertia of the body about G, and N for the sum of the moments about G of the external forces.

15·41. Kinetic Energy. As seen above the velocity of P relative to G is $r\,d\theta/dt$.

Hence the kinetic energy of the body relative to G

$$=\tfrac{1}{2}\Sigma mr^2\left(\frac{d\theta}{dt}\right)^2=\tfrac{1}{2}\left(\frac{d\theta}{dt}\right)^2\Sigma mr^2=\tfrac{1}{2}MK^2\left(\frac{d\theta}{dt}\right)^2=\tfrac{1}{2}I\omega^2,$$

where ω is the angular velocity about G.

If the body has also a velocity of translation equal to v, its total kinetic energy is equal to $\tfrac{1}{2}Mv^2+\tfrac{1}{2}MK^2\left(\dfrac{d\theta}{dt}\right)^2=\tfrac{1}{2}Mv^2+\tfrac{1}{2}I\omega^2$.

15·42. If we now denote the co-ordinates of the centre of mass by $(\bar{x}, \bar{y})$ and the angle through which the body rotates by θ, the three equations of motion (C) and (D) of Art. 15·3, 15·31 take the form :—

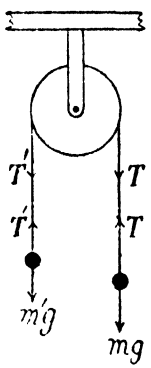
$$M\frac{d^2\bar{x}}{dt^2}=X,\quad M\frac{d^2\bar{y}}{dt^2}=Y,\quad ..(1),\quad MK^2\frac{d^2\theta}{dt^2}=N.\quad ..(2).$$

Example 1. *A fine string has two masses m and m' $(m>m')$ tied to its ends and passes over a rough pulley, of mass M, whose centre is fixed, if the string does not slip over the pulley and the pulley can rotate about the horizontal axis through the centre, show that m will descend with acceleration*

$$\frac{(m-m')g}{m+m'+M\bar{K}^2/a^2},$$

where a is the radius and K the radius of gyration of the pulley about the axis.

In Art. 3·7, the pulley was supposed to be light and fixed, the tensions in the string on its two sides were, therefore, taken to be equal.



In the present case since the rotation of the pulley of mass M is to be taken into account, the tensions on its sides must be different say T, T'.

When the pulley has turned through an angle θ, let the depths of masses m, m' below its centre be x, y, then the equations of motion are :

$MK^2\ddot{\theta}$=sum of the moments of the tensions about the centre of the pulley

$=Ta-T'a$, for the pulley by (2),

and $m\ddot{x}=mg-T$, and

$m'\ddot{y}=m'g-T'$, for the masses.

But the string being inextensible $x+y$ remains constant throughout the motion, so that $\ddot{y}=-\ddot{x}$ and hence $m'\ddot{x}=T'-m'g$.

Also since the string does not slip over the pulley

$$\dot{x}=a\dot{\theta} \text{ and } \therefore \quad \ddot{x}=a\ddot{\theta} \text{ or } \ddot{\theta}=\ddot{x}/a.$$

So the first equation can be written as

$$MK^2\ddot{x}/a^2=T-T'.$$

Adding this to the other equations, we have

$$(m+m'+MK^2/a^2)\ddot{x}=mg-m'g=(m-m')g,$$

which gives the required acceleration.

Aliter. By Art. 15·41 Kinetic energy of the pulley

$$=\tfrac{1}{2}MK^2\dot{\theta}^2=\tfrac{1}{2}(MK^2/a^2)\dot{x}^2,$$

and Kinetic energies of the masses are $\frac{1}{2}m\dot{x}^2$, $\frac{1}{2}m'\dot{x}^2$.

Hence total K.E. $=\frac{1}{2}(m+m'+MK^2/a^2)\dot{x}^2$

$$=\text{work done}=mgx-m'gx=(m-m')gx$$

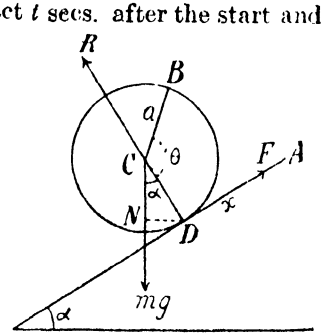
Differentiating this with respect to time t, we have

$$(m+m'+MK^2/a^2)\dot{x}\ddot{x}=(m-m')g\dot{x}$$

Hence $(m+m'+MK^2/a^2)\ddot{x}=(m-m')g$, as before.

Example 2. *A uniform circular disc of radius a and mass m rolls down a rough inclined plane without sliding ; to find the motion.*

Suppose D is the point of contact t secs. after the start and that B was in contact with A in the beginning.



Let $AD=x$, $\angle DCB=\theta$ and α the inclination of the plane.

The external forces acting on the disc are :—

R, the normal pressure of the plane,

F, the force of friction up the plane,

mg, the weight of the disc.

For the motion of C, the C.G. of the disc parallel to the plane, we have

$$m\frac{d^2x}{dt^2}=mg\sin\alpha-F \qquad ..(1).$$

And since there is no motion perp. to the plane

$$0=mg\cos\alpha-R \qquad ..(2).$$

For rotational motion obtained by taking moments about C, we have

$$mk^2\frac{d^2\theta}{dt^2}=F\,.\,a. \qquad ..(3).$$

Since there is no sliding, arc $BD=AD$, *i.e.*, $x=a\theta$ and

$$\therefore \quad \frac{d^2x}{dt^2}=a\frac{d^2\theta}{dt^2} \qquad ..(4).$$

Using (4) and eliminating F between (1) and (3), we have

$$m(a^2+k^2)\frac{d^2x}{dt^2}=mga^2\sin\alpha \quad \text{or} \quad \frac{d^2x}{dt^2}=\frac{a^2}{a^2+k^2}g\sin\alpha \qquad ..(5).$$

For the disc $k^2=\frac{1}{2}a^2 \quad \therefore \quad d^2x/dt^2=\frac{2}{3}g\sin\alpha.$

Thus the centre of the disc moves with constant acceleration

$$f=\tfrac{2}{3}g\sin\alpha,$$

then its velocity v and the distance x at time t are given by

$$v=ft=\tfrac{2}{3}g\sin\alpha\,.\,t, \quad \text{and} \quad x=\tfrac{1}{2}ft^2=\tfrac{1}{2}\times\tfrac{2}{3}g\sin\alpha\,.\,t^2$$

Now from (1), the friction $F=mg\sin\alpha-mf$

$$=mg\sin\alpha-\tfrac{2}{3}mg\sin\alpha$$
$$=\tfrac{1}{3}mg\sin\alpha.$$

Also from (2), the pressure $R=mg\cos\alpha$.

In order that there may be no sliding of disc, $F<\mu R$,

i.e., $\quad \frac{1}{3}mg\sin\alpha<\mu\,.\,mg\cos\alpha. \quad \therefore \quad \mu>\frac{1}{3}\tan\alpha.$

Note. It may be noted that we can avoid F by taking moment about D, the point of contact. In this case

$$k^2 \text{ about } D=k^2 \text{ about } C+a^2=\tfrac{3}{2}a^2.$$

$$\therefore \quad m.\tfrac{3}{2}a^2\frac{d^2\theta}{dt^2}=\text{moment of the weight } mg \text{ about D}=mg\,a\sin\alpha.$$

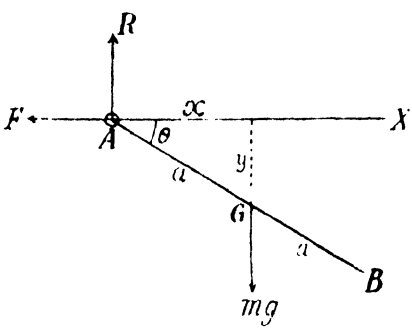
Hence using (4), we get

$$\frac{d^2x}{dt^2}=\tfrac{2}{3}g\sin\alpha.$$

Example 3. *A small light ring is threaded on a fixed thin horizontal wire. One end of a uniform rod, of mass m and length 2a, is freely attached to the ring. The co-efficient of friction between the ring and the wire is μ. The system is released from rest when the rod is horizontal and in the vertical plane containing the wire. Show that if the ring slips on the wire when the rod has turned through an angle α, then*

$$\mu(10\tan^2\alpha+1)=9\tan\alpha. \qquad (M.T.)$$

When the ring does not slip, the forces acting on the rod are :—



Its weight mg ;

R, the pressure of the wire at A;

F, the force of friction at A.

Let (x, y) be the co-ordinates of G, the $C.G.$ of the rod as shown in the figure and let θ be the angle through which the rod turns in time t, then

$$x=a\cos\theta \text{ and } y=a\sin\theta.$$

$\therefore$ denoting differentiations by dots, we have

$$\ddot{x}=-a\cos\theta\,.\,\dot{\theta}^2-a\sin\theta\,.\,\ddot{\theta},$$

and $$\ddot{y}=-a\sin\theta\,.\,\dot{\theta}^2+a\cos\theta\,.\,\ddot{\theta}.$$

Now the equations of motion for the rod are,

$m\ddot{x}=-F$ *i.e.*, $F=-m\ddot{x}=ma(\cos\theta\,.\,\dot{\theta}^2+\sin\theta\,.\,\ddot{\theta})$ ..(1),

and $m\ddot{y}=mg-R$ *i.e.*, $R=mg-m\ddot{y}=mg+ma(\sin\theta\,.\,\dot{\theta}^2-\cos\theta.\ddot{\theta}$ (2),

and in order to avoid action at the ring A, we take moments about A, K^2 about which for the rod $=\frac{4}{3}a^2$

$$\therefore\quad mK^2\ddot{\theta},\ i.e.,\ m\,.\,\frac{4a^2}{3}\,\ddot{\theta}=mg\,a\cos\theta\ i.e.,\ a\ddot{\theta}=\tfrac{3}{4}g\cos\theta \qquad ..(3)$$

Integrating this, we get

$$a\dot{\theta}^2=\tfrac{3}{2}g\sin\theta+C=\tfrac{3}{2}g\sin\theta,$$

as $\dot{\theta}=0$ when $\theta=0$.

Substituting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ in equations (1) and (2), and simplifying, we have,

$$F=\tfrac{9}{4}mg\sin\theta\cos\theta \quad\text{and}\quad R=\tfrac{1}{4}mg(10-9\cos^2\theta).$$

If the rod is about to slip when $\theta=\alpha$, then the force of friction F attains its limiting value μR, *i.e.*, $F=\mu R$.

Hence $\tfrac{9}{4}mg\sin\alpha\cos\alpha=\mu\,.\,\tfrac{1}{4}mg(10-9\cos^2\alpha)$

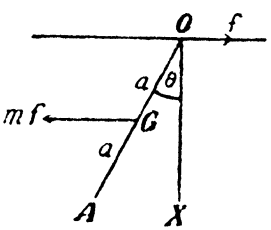
or $$9\tan\alpha=\mu(10\sec^2\alpha-9)$$
$$=\mu(10\tan^2\alpha+1).$$

Example 4. *The door of a railway carriage, has its hinges (supposed smooth) towards the engine, which starts with an acceleration f. Prove that the door closes in time*

$$\left(\frac{a^2+k^2}{2af}\right)^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{\sin\theta}}$$ *with an angular velocity* $\left(\frac{2af}{a^2+k^2}\right)^{\frac{1}{2}}$,

where $2a$ is the breadth of the door and k its radius of gyration about a vertical axis through G, the centre of mass.

Let θ be the angle through which the door turns in time t. Let O represent the line of hinges and $OA(=2a)$ be the horizontal section of the door through G.



When the train starts with accln. f, say, towards the right, the door will have a relative accln. backwards towards the left, so that the resultant force mf acts on it at G which tends to close the door.

To avoid forces acting on the line of hinges, we take moments about O.

$\therefore$ the equation of motion for the door is

$$m(a^2+k^2)\ddot{\theta}=mfa\cos\theta \qquad ..(1).$$

Multiplying by $2\dot{\theta}$ and integrating, we have

$$(a^2+k^2)\dot{\theta}^2=2af\sin\theta+C.$$

But as the door is initially at rest $\theta=0$ when $\theta=0$, $\therefore\ C=0$.

Hence $$(a^2+k^2)\dot{\theta}^2=2af\sin\theta,$$

or $$\frac{d\theta}{dt}=\sqrt{\frac{2af}{a^2+k^2}}\sin\theta. \qquad ..(2).$$

This gives the angular velocity of the door when it has turned through an angle θ.

The time when the door closes, *i.e.*, when $\theta=\frac{1}{2}\pi$, is given by

$$t=\sqrt{\frac{a^2+k^2}{2af}}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{\sin\theta}}.$$

when by (2), the angular velocity is $\sqrt{\dfrac{2af}{a^2+k^2}}$.

Note. If the door is supposed to be a uniform thin rectangular plate, then $k^2=\frac{1}{3}a^2$, $a^2+k^2=\frac{4}{3}a^2$ and by (2), the angular velocity of the door when it has turned through an angle θ is $\sqrt{\dfrac{3f}{2a}\sin\theta}$.

Examples LI

1. If in worked example 2, the body is a sphere instead of a disc, show that the least co-efficient of friction is $\frac{2}{7}\tan\alpha$.

2. A uniform solid cylinder rolls without sliding on a rough plane of inclination 45°, show that the least co-efficient of friction μ is $\frac{1}{3}$.

If the cylinder be hollow, show that the least value of μ is $\frac{1}{2}$.

3. A circular ring rolls down a perfectly rough inclined plane in one minute, show that a circular disc will roll down the same distance in 52 secs. nearly.

4. Show that the kinetic energy of a body revolving about a fixed axis is

$$\tfrac{1}{2}I\omega^2,$$

where I is its moment of inertia about that axis and ω its angular velocity.

A thin hollow cylinder, open at both ends and weighing 96 lbs.

(*a*) slides with a speed of 10 ft. per sec. without rotating,

(*b*) rolls with the same speed, without slipping.

Compare the kinetic energies of the cylinder in the two cases. [*Ans.* 1 : 2]

5. Which will roll more quickly down an inclined plane, a solid sphere or a hollow cylinder ?

A solid disc rolls down a plane inclined at 30° to the horizontal. How far will it move down the plane in 20 secs. from rest ? What is the velocity of its centre and if it weighs 10 lbs. how much energy has it ? (*Roorkee 1953*)
(*Ans.* Sphere ; 2133⅓ ft., 213⅓ ft. per sec., 10666⅔ ft. lbs.)

6. The weight of a wagon is 2 tons, of which the wheels weigh ¼ ton. The diameter of the wheels is 2 ft. and the radius of gyration is 0·9 ft. Find the total kinetic energy of the wagon when travelling at 40 *m. p. h.* (*Roorkee 1953*)
[*Ans.* 118·44 ft. tons.]

7. A circular disc partly rolls and partly slides down a rough inclined plane, show that the acceleration of its C. G. is the same as for a body which simply slides and does not roll.

Show that the result is the same if the disc is replaced by a cylinder or a sphere.

[**Hint.** In worked Ex. 2, take $F=\mu R$. Equation (4) does not hold in this case.]

8. A wheel and axle has a moment of inertia I about the axis, which is fixed horizontally and about which it can revolve freely. Masses m and M hang by light threads from the wheel (radius a) and axle (radius b) respectively. Show that the angular acceleration under gravity will be

$$\frac{(ma-Mb)g}{ma^2+Mb^2+I}.$$

9. The end of a weightless thread wound round a reel, (a solid cylinder of mass m), is held fixed and the reel is allowed to fall in a vertical plane, its axis remaining horizontal. Show that the acceleration of the reel is $\frac{2}{3}g$ and the tension of the thread is $\frac{1}{3}mg$.

10. A uniform rod of length $2a$ is held in a vertical position with one end resting upon a perfectly rough table, and when released rotates about the end in contact with the table.

Show that when it is inclined at an angle θ to the vertical its angular acceleration is $\frac{3}{4a}g \sin\theta$ and that the pressure on the table vanishes but the rod does not leave the table, when $\cos\theta = \frac{1}{3}$.

11. A uniform rod of mass m and length $2a$ can turn freely about a fixed end. Show that the least angular velocity with which it must be started from the lowest position so as to make complete revolution is

$$\sqrt{3g/a}.$$

Show also that with this as the initial angular velocity, time taken by the rod to turn through an angle θ is given by

$$t = \sqrt{\frac{4a}{3g}} \log \tan \tfrac{1}{4}[\pi + \theta].$$

12. A uniform hoop can rotate in its own plane which is vertical about a smooth hinge at one end of a diameter. It is allowed to fall from the position in which the diameter is horizontal. Prove that when the horizontal component of the reaction at the hinge is a maximum, the vertical component is 1·25 of the weight of the hoop. (*Indian Railway Service*).

[Proceed as in worked Ex. 3.]

13. AB and CD are two rods of lengths $2a$ and b respectively, the mass of each rod being m per unit length. The rods are rigidly joined together at right angles at C, the middle point of AB, and the system is free to rotate in a vertical plane about D. If the system is held with AB vertical and then let go, calculate the angular velocity when AB is horizontal. (*D.U.*)

[Ans. $[3g(4a+b)b/(2a^3+6ab^2+b^3)]^{\frac{1}{2}}$.

15·5. Torsional Vibrations. When a body such as a bar, a circular disc, or a lamina is supported in a horizontal position by a vertical wire attached to its C.G. and is rotated from the position of equilibrium through an angle θ, the wire is twisted and a restoring couple is brought into action due to which the body executes vibrations called *torsional vibrations*.

Let the body be a uniform circular disc of mass m and radius a suspended by a wire attached to its centre.

Suppose the restoring couple is proportional to the angle θ through which the disc is rotated, then its equation of motion is

$$mk^2\frac{d^2\theta}{dt^2} = -\lambda\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{\lambda}{mk^2}\theta = -\frac{2\lambda}{ma^2}\theta, \left(\because\ k^2 = \frac{a^2}{2}\right).$$

The motion is, therefore, simple harmonic, the period of oscillation being equal to

$$2\pi \div \sqrt{\frac{\lambda}{mk^2}} = 2\pi\sqrt{\frac{mk^2}{\lambda}} = 2\pi\sqrt{\frac{ma^2}{2\lambda}}.$$

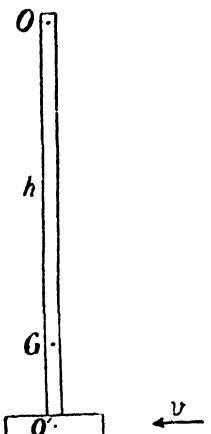
The length of the equivalent simple pendulum in this case is $\frac{1}{2}ma^2g/\lambda$.

*** 15·6. Ballistic Pendulum.** An important application of the compound pendulum dealt with in Art. 14·93 is made in determining the velocity of a rifle bullet.

A pendulum with a massive bob and considerable time of oscillation is so formed that the bob is in the form of a hollow iron cylinder open at one end, stuffed with a block of wood inside it.

Let a bullet of mass m be fired in a horizontal line through the centre of oscillation O' of the pendulum.

Let v be the velocity of the bullet which is to be determined, M the mass and I the moment of inertia (MK^2) of the pendulum including the embedded bullet. Also let h be the distance of G, the $C.G.$ of the pendulum, from O, the centre of suspension.



Then the equation of motion as given in Art. 14·93 is

$$MK^2\ddot{\theta} = -Mgh \sin\theta \text{ or } I\ddot{\theta} = -Mgh \sin\theta \quad ..(1).$$

If θ is taken to be a small angle and the time T of oscillation is determined,

then $$I\ddot{\theta} = -Mgh\theta$$

and $$T = 2\pi\sqrt{\frac{I}{Mgh}} \text{ or } I = \frac{MghT^2}{4\pi^2} \quad ..(2).$$

Again multiplying (1) by $2\dot{\theta}$ and integrating, we get, $$I\dot{\theta}^2 = 2Mgh\cos\theta + C \quad ..(3).$$

When $\theta = 0$, let $\dot{\theta} = \omega$, the initial angular velocity of the pendulum, then

$$I\omega^2 = 2Mgh + C.$$

If the pendulum swings through an angle α, so that $\dot{\theta} = 0$ when $\theta = \alpha$, then $0 = 2Mgh\cos\alpha + C$.

Hence $$I\omega^2 = 2Mgh(1-\cos\alpha) = 4Mgh\sin^2\frac{\alpha}{2},$$

or $$\omega = 2\sqrt{\frac{Mgh}{I}}\sin\frac{\alpha}{2} = \frac{4\pi}{T}\sin\frac{\alpha}{2} \quad ..(4), \text{ [from (2)]}.$$

Now there being no external force, the moment of momentum or the angular momentum of the system remains constant (Art. 15·22).

If $l = OO'$, the distance of the centre of oscillation from the centre of suspension, the angular momentum just before the bullet is embedded $= mv.l = mlv$.

And the angular momentum just after the bullet is embedded is

$$I\dot{\theta} = I\omega.$$

$$\therefore \quad mlv = I\omega \quad \text{or} \quad v = I\omega/ml.$$

Substituting the values of I and ω from (2) and (4), the velocity v of the bullet is given by,

$$v = \frac{1}{ml} \times \frac{MghT^2}{4\pi^2} \times \frac{4\pi}{T}\sin\frac{\alpha}{2} = \frac{TMgh}{\pi ml}\sin\frac{\alpha}{2}.$$

In the above, $M.I.$ and T may be taken as the quantities relating to the pendulum before the bullet is embedded in it, m being small, the errors so made are small.

The angle α is usually measured by a piece of tape attached to the bottom of the pendulum and wound on a reel, the amount of the tape unwound is measured and the amplitude of oscillation is determined.

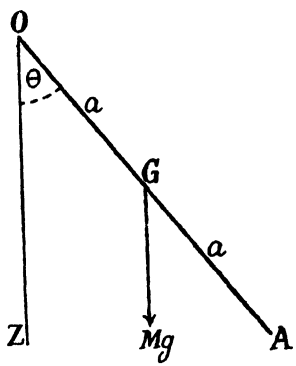
CHAPTER XVI

FURTHER EXAMPLES OF MOTION OF A RIGID BODY IN TWO DIMENSIONS

16·1. Rotation about an axis. *A uniform rod OA, of mass M and length $2a$ can turn freely about one end O which is fixed. It is started with an angular velocity ω from the position in which it hangs vertically. To find the least value of ω in order that the rod may make complete revolutions.*

Let the rod make an angle θ with the vertical OZ, then taking moments about O, we have

$$MK^2 \frac{d^2\theta}{dt^2} = -Mg.a \sin\theta \qquad \ldots (1).$$



Since $K^2 = \frac{a^2}{3} + a^2 = \frac{4}{3}a^2$,

$$\therefore \quad \ddot{\theta} = -\frac{3g}{4a}\sin\theta.$$

Multiplying by $2\dot{\theta}$ and integrating, we get

$$\therefore \quad \dot{\theta}^2 = \frac{3g}{2a}\cos\theta + C.$$

Since $\dot{\theta} = \omega$ when $\theta = 0$,

$$\therefore \quad C = \omega^2 - 3g/2a.$$

Hence $$\dot{\theta}^2 = \omega^2 - \frac{3g}{2a}(1-\cos\theta) \qquad \ldots (2).$$

Now when $\theta = \pi$, $\dot{\theta}^2 = \omega^2 - 3g/a$.

In this position $\dot{\theta}$ is positive if $\omega^2 > 3g/a$.

Hence the least value of ω for the rod to make complete revolutions is $\sqrt{3g/a}$.

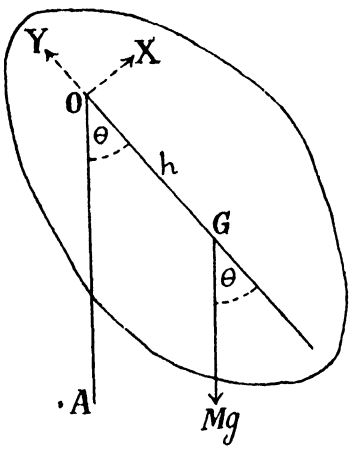
Example 1. *Show that with this particular value of ω, the time t to describe any angle θ is given by* $t = \sqrt{\frac{4a}{3g}} \log \tan\left(\frac{\pi+\theta}{4}\right)$.

Example 2. *A body makes complete revolutions about a fixed horizontal axis, about which its radius of gyration is K and from which the distance of its C. G. is h. If the greatest and least angular velocities are one per cent greater and one per cent less than ω, prove that* $K^2\omega^2 = 100\,gh$.

16·2. Reaction of the axis of Rotation. *A body moves about a fixed horizontal axis under the force of gravity alone, to find the reaction of the axis.*

It is assumed that the body is symmetrical about the plane through, G, the centre of gravity of the body, perpendicular to the axis.

Let AOG be the vertical plane perpendicular to the axis about which rotation takes place, and let it intersect the axis in O.



Let OG make an angle θ with the vertical line OA (the initial position of OG), let K be the radius of gyration of the body about a line through G parallel to the axis and let $OG=h$.

Then by taking moments about the axis, we have

$$M(K^2+h^2)\ddot{\theta}=-Mg\,h\sin\theta \quad \ldots (1).$$

Multiplying this by $2\dot{\theta}$ and integrating, we get

$$(K^2+h^2)\,\dot{\theta}^2=2gh\cos\theta+C$$
$$=2gh\,(\cos\theta-\cos\alpha) \quad \ldots(2),$$

taking $\dot{\theta}=0$, when $\theta=\alpha$.

Now let X, Y be the components of the reaction at right angles to and along GO, then resolving in these directions, we have

$$M\,h\,\ddot{\theta}=X-Mg\sin\theta \quad \ldots (3),$$

and
$$Mh\,\dot{\theta}^2=Y-Mg\cos\theta \quad \ldots (4).$$

From (1) and (3), we get

$$X-Mg\sin\theta-\frac{Mg\,h^2}{K^2+h^2}\sin\theta=\frac{Mg\,K^2}{K^2+h^2}\sin\theta \quad \ldots (5).$$

From (2) and (4), we get

$$Y=Mg\cos\theta+\frac{2Mg\,h^2}{K^2+h^2}(\cos\theta-\cos\alpha)$$

The resultant of X and Y gives the required reaction.

Note. For small oscillations equation (1) becomes $\ddot{\theta}=-\frac{gh}{K^2+h^2}\theta$.

The period of a small oscillation T is $2\pi\sqrt{(K^2+h^2)/gh}$ and its frequency $=1/T=\sqrt{gh}/2\pi\sqrt{K^2+h^2}$.

Examples LII

1. A thin uniform rod has one end attached to a smooth hinge and is allowed to fall from a horizontal position. Show that the horizontal strain on the hinge is greatest when the rod is inclined at an angle of 45° to the vertical, and that the vertical strain is then $\frac{11}{8}$ times the weight of the rod.

2. A circular area of weight W can turn freely about a horizontal axis perpendicular to its plane which passes through a point C on its circumference. If it starts from rest with the diameter through C vertically above C, show that, when the diameter has turned through an angle θ, the components of the strain at C along and perpendicular to this diameter are respectively

$$\tfrac{1}{4}\,W\,(7\cos\theta-4) \quad \text{and} \quad \tfrac{1}{4}\,W\sin\theta$$

Show also that when this diameter is horizontal the resultant pressure on the axis is $\frac{1}{8}\sqrt{17}$ W and that when it is vertically below C, the resultant pressure is $\frac{1}{3}\frac{1}{2}$ W

(*b*) Show that in the case of a uniform rod which just makes complete revolutions, these components of the strain are $\frac{1}{2}$ W (3+5 cos 0) and $\frac{1}{4}$ W sin 0,

[**Hint.** See Art. 16·1 and use (3) and (4) of Art. 16·2.]

3. A uniform hoop can rotate in its own plane which is vertical, about a smooth hinge at one end of a diameter. It is allowed to fall from the position in which this diameter is horizontal. Prove that when the horizontal component of the reaction at the hinge is maximum, the vertical component is $\frac{5}{4}$ of the weight of the hoop.

4. A right circular cone, of weight W and of semi-vertical angle α, can turn freely about an axis passing through the centre of its base and perpendicular to its axis. If the cone starts from rest with its axis horizontal, show that, when the axis is vertical, the thrust on the fixed axis is

$$W(6+3\cos^2\alpha)/(6-2\cos^2\alpha).$$

5. A uniform rod AB of length $2a$ and mass M, is freely attached to a fixed support at A and swings in a vertical plane through A. Find the reaction of the support and the frequency of the small oscillations.

Prove that if a particle of mass kM is attached to the rod at B, the frequency of the small oscillations is reduced in the ratio

$$\sqrt{(1+2k)/(1+3k)}. \qquad (D.\ U.\ 1961)$$

16·3. *A uniform rod AB held at an inclination α to the vertical with one end A in contact with a rough horizontal table. To show that if released, the rod will commence to slide at once if the co-efficient of the friction μ is less than 3 sin α cos α / (1 + 3 cos^2 α).*

Let m be the mass and $2a$ the length of the rod.

Let (x, y) be the co-ordinates of the middle point of the rod when it is inclined at an angle θ to the vertical and let R be the normal reaction and F, the friction in this position, then

Since $x = a \sin\theta$,

$$\dot{x} = a\cos\theta.\dot{\theta} - a\sin\theta.\dot{\theta}^2,$$

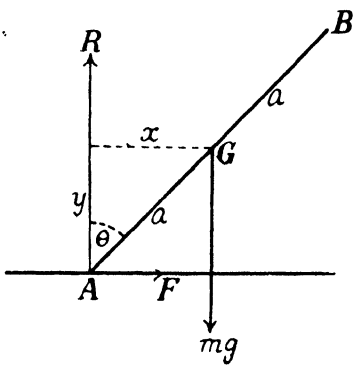
and since $y = a\cos\theta$,

$$\dot{y} = -a\sin\theta.\ddot{\theta} - a\cos\theta.\dot{\theta}^2$$

The equations of motion are :—

$$m\ddot{x} = m(a\cos\theta.\ddot{\theta} - a\sin\theta.\dot{\theta}^2)$$
$$= F \qquad \ldots (1),$$

and $m\ddot{y} = m(-a\sin\theta.\dot{\theta} - a\cos\theta.\dot{\theta}^2)$
$$= R - mg \qquad \ldots (2).$$



Taking moments about A, we have

$$m\frac{4a^2}{3}\ddot{\theta} = mg.a\sin\theta$$

or $\quad 4a\ddot{\theta} = 3g\sin\theta \qquad \ldots (3).$

Multiplying (3) by $\dot{\theta}$ and integrating, we have

$$2a\dot{\theta}^2 = -3g\cos\theta + C = 3g(\cos\alpha - \cos\theta) \qquad \ldots (4),$$

since $\dot{\theta} = 0$ when $\theta = \alpha$.

Substituting $\ddot{\theta}$ and $\dot{\theta}$ from (3) and (4) in (1) and (2), we get

$$F = \tfrac{3}{4}mg(3\cos\theta - 2\cos\alpha)\sin\theta,$$

and $\quad R = \frac{1}{4}mg(1 + 9\cos^2\theta - 6\cos\theta\cos\alpha).$

When $\theta=\alpha$, $F=\frac{3}{4}\,mg\sin\alpha\cos\alpha$, and $R=\frac{1}{4}\,mg\,(1+3\cos^2\alpha)$.

The rod will slide at once *i.e.*, when $\theta=\alpha$,

if $F>\mu R$, *i.e.* if $\frac{3}{4}\,mg\sin\alpha\cos\alpha>\frac{1}{4}\,\mu\,mg\,(1+3\cos^2\alpha)$

i.e., if $\mu<3\sin\alpha\cos\alpha/(1+3\cos^2\alpha)$.

Cor. (*i*) If the rod starts from the vertical position then $\alpha=0$, and $F=\frac{3}{4}\,mg\sin\theta\,(3\cos\theta-2)$, and $R=\frac{1}{4}\,mg\,(1-3\cos\theta)^2$.

It shows that when $\cos\theta=\frac{1}{3}$, although R vanishes but it does not become negative, so that the end A of the rod does not leave the table even when the pressure vanishes.

(*ii*) In this case, F changes its sign and its direction is reversed as θ passes through the value $\cos^{-1}\frac{2}{3}$.

Note. Equation (4) can be written down at once from the principle of energy.

$$K.E.=\tfrac{1}{2}mI\,\omega^2=\tfrac{1}{2}m\,\tfrac{1}{3}\,a^2\,\dot\theta^2$$

and work done by the weight of the rod $mg\,(a\cos\alpha-a\cos\theta)$, equating these we get equation (4).

Examples LIII

1. A uniform straight rod of length $2a$ is held at an inclination α to the vertical with its lower end in contact with a horizontal table and let go. Show that the angular velocity of the rod when it becomes horizontal is $\sqrt{3\,g\cos\alpha/2a}$ whether the table is perfectly smooth or perfectly rough.

2. A uniform rod AB slides down in a vertical plane with its lower end B resting on a smooth horizontal plane and the upper end A resting aganist a smooth vertical wall. It starts from rest when it is inclined at an angle α to the horizontal. Show that the end A leaves the wall when the inclination of the rod with the horizontal is $\sin^{-1}(\frac{2}{3}\sin\alpha)$ and that its angular velocity is then equal to $\sqrt{g\sin\alpha/2a}$.

3. In solved Ex. 3, P. 302, if the horizontal wire is smooth instead of being rough show that the centre G of the rod moves in a vertical line through its initial position and that the velocity of either end of the rod is $\sqrt{6\,ag}$ when the rod is vertical.

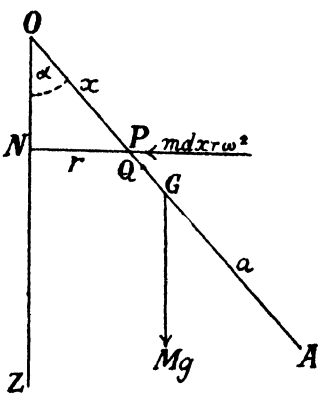
4. A thin uniform rod, of mass m and length $2a$, is held horizontally, and is in contact with a fixed rough peg at a point distant d from the middle point of the rod. The rod is then released, show that slipping commences when the rod makes with the horizontal the angle $\tan^{-1}\frac{\mu\,a^2}{9d^2+a^2}$ and that the initial reaction of the peg is $\frac{mg\,a^2}{3d^2+a^2}$.

16·4. *A uniform rod OA, of mass M and length $2a$, is free to turn about a fixed hinge at one end O and revolves about a vertical line OZ so as to describe a cone of semi-vertical angle α, to find its angular velocity ω.*

Let m be the mass per unit length of the rod, so that $2am=M$, and consider an element $PQ=dx$ at a distance x from the fixed point O.

Draw PN perpendicular to OZ. Since the element PQ describes a circle of radius r about N, its acceleration towards N, is $r\omega^2$, where $r=PN=x\sin\alpha$.

$\therefore$ the force towards N is $mdx.\ r\omega^2$ and its moment about O is $mdx.\ r\omega^2.\ ON=mdx\,\omega^2.\ x\sin\alpha.\ x\cos\alpha$
$=m\omega^2\sin\alpha\cos\alpha.\ x^2dx.$



Taking moments about O for the whole rod, we have

$$\int_0^{2a} m\omega^2\sin\alpha\cos\alpha.\ x^2dx$$

$=$moment of the weight
$=Mg.\ a\sin\alpha$,
$=2\,amg.\ a\sin\alpha.$

$\therefore\ m\omega^2\sin\alpha\cos\alpha.\ \frac{8}{3}a^3=2\,a^2\,mg\sin\alpha$

Hence either $\alpha=0$, *i.e.* the rod hangs vertically,

Or $$4a\omega^2\cos\alpha=3g,\ \textit{i.e.}\ \cos\alpha=\frac{3g}{4a\omega^2}. \qquad ...(1).$$

This shows that $\cos\alpha$ being <1, $4a\omega^2>3g$, *i.e.* $\omega^2>3g/4a$.

Example 1. *Prove that the direction of the reaction at the hinge makes with the vertical the angle* $\tan^{-1}(\frac{3}{4}\tan\alpha)$.

[**Hint.** Vertically $Y=Mg$. Horizontally $X=\int_0^{2a} m\omega^2x\sin\alpha\,dx$

$=m\omega^2\sin\alpha.\ 2a^2=M\omega^2a\sin\alpha$. Now use (1) for ω^2].

Example 2. *A rod, of length* $2a$, *is suspended by a string, of length* l, *attached to one end; if the string and rod revolve about the vertical with uniform angular velocity, and their inclinations to the vertical be* θ *and* ϕ *respectively, show that*

$$\frac{3l}{a}=\frac{(4\tan\theta-3\tan\phi)\sin\phi}{(\tan\phi-\tan\theta)\sin\theta}. \qquad (D.U.\ 1963)$$

[**Hint.** If T be the tension of the string and m the mass per unit length of the rod, then resolving vertically, $T\cos\theta=mg$.

The distance of an element at P from $OZ=x\sin\phi+l\sin\theta$.

Resolving horizontally, $T\sin\theta=\int_0^{2a} m\omega^2(x\sin\phi+l\sin\theta)\,dx$

Taking of moments about O gives

$$2a\,mg.\ a\sin\phi=\int_0^{2a} m\omega^2(x\sin\phi+l\sin\theta)\,x\cos\phi\,dx].$$

Example 3. *A uniform rod AB is freely movable on a rough inclined plane about a smooth pin fixed through the end A. The rod is held in the horizontal position in the plane and allowed to fall from this position. The inclination of the plane to the horizon is α and the co-efficient of friction is μ. Show that if θ be the angle through which the rod falls from rest, then*

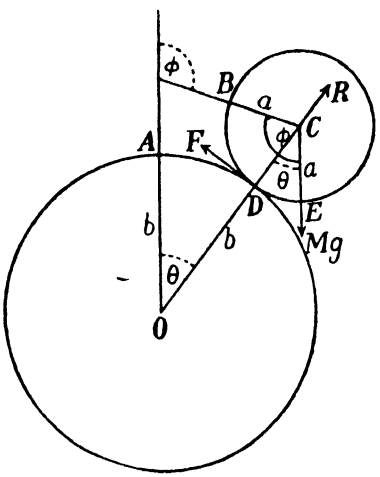
$$\sin\theta/\theta=\mu\cot\alpha.$$

16·5. *A solid homogeneous sphere is resting on the top of another rough fixed sphere, and rolls down it. To find the motion.*

Let a and b be the radii and C and O the centres of the moving and fixed spheres respectively.

Let D be the point of contact, F the friction and R the normal reaction on the moving sphere at any time t.

Let B be the point of the moving sphere which was initially in contact with A, the highest point of the fixed sphere.



Let $\angle AOC=\angle OCE=\theta$, and $\angle BCE=\phi$.

The geometrical condition is arc BD=arc AD, or $a(\phi-\theta)=b\theta$,

i.e. $$a\phi=(a+b)\theta \qquad \ldots (1).$$

Since the centre C of the moving sphere describes a circle of radius $a+b$,

$\therefore$ its equations of motion are :—

$$M(a+b)\ddot{\theta}=Mg\sin\theta-F \qquad \ldots (2),$$

and

$$M(a+b)\dot{\theta}^2=Mg\cos\theta-R \qquad \ldots (3).$$

Also since CB, a line fixed in the moving sphere makes with the vertical, a direction fixed in space, an angle ϕ, by taking moments about C and using (1), we have

$$MK^2\ddot{\phi}=Fa, \text{ or by (1), } MK^2\frac{a+b}{a}\ddot{\theta}=Fa \qquad \ldots (4).$$

Taking $K^2=\frac{2}{5}a^2$, it becomes $\frac{2}{5}M(a+b)\ddot{\theta}=F$, and added to (2) it gives

$$(a+b)\ddot{\theta}=\tfrac{5}{7}g\sin\theta \qquad \ldots (5).$$

Multiplying this by $2\dot{\theta}$ and integrating and applying the initial condition that $\dot{\theta}=0$ when $\theta=0$, we have

$$(a+b)\dot{\theta}^2=-\tfrac{10}{7}g\cos\theta+C=\tfrac{10}{7}g(1-\cos\theta) \qquad \ldots (6).$$

Now from (3), $R = Mg\cos\theta - \frac{10}{7} Mg(1-\cos\theta)$

$$= \tfrac{1}{7} Mg(17\cos\theta - 10) \qquad \ldots (7)$$

and from (2) and (5), $F = Mg\sin\theta - \frac{5}{7} Mg\sin\theta = \frac{2}{7} Mg\sin\theta \qquad \ldots (8).$

Notes. 1. The sphere will slip if $F = \mu R$,

i.e., if $\frac{2}{7} Mg\sin\theta = \mu . \frac{1}{7} Mg\ (17\cos\theta - 10)$ or $2\sin\theta = \mu\ (17\cos\theta - 10)$,

Taking $\mu = \tan\lambda$, this can be written as

$$2\sin(\theta-\lambda) = 5\sin\lambda\ (3\cos\theta - 2).$$

2. If the sphere is rough enough to prevent slipping, then by (7) R will be zero and change sign, *i.e.*, the upper sphere will leave the lower when $\cos\theta = 10/17$, a result independent of magnitudes of the spheres.

3. Equation (6) can be obtained directly from the principle of energy as follows :—

Kinetic energy of the sphere $= \frac{1}{2} M v^2 + \frac{1}{2} I\omega^2$

$$= \frac{1}{2} M(a+b)^2\ \dot\theta^2 + \frac{1}{2} M.\frac{2}{5} a^2 \dot\Phi^2 = \frac{7}{10} M(a+b)^2\ \theta^2 \quad [\text{by (1)}]$$

The work done by the weight $= Mgy = Mg[a+b-(a+b)\cos\theta]$
$= Mg(a+b)(1-\cos\theta).$

$\therefore\ \frac{7}{10} M(a+b)^2\theta^2 = Mg(a+b)(1-\cos\theta)$ or $(a+b)\dot\theta^2 = \frac{10}{7}\ g(1-\cos\theta).$

Example 1. (*a*) *Show that if the spheres were both smooth, the upper sphere will leave the lower when* $\cos\theta = \frac{2}{3}$.

(*b*) *If the upper sphere starts moving from the position of rest at* $\theta = \alpha$, *show that it will slide when*

$$2\sin(\theta-\lambda) = 5\sin\lambda\ (3\cos\theta - 2\cos\alpha),$$

and that if it is sufficiently rough so as not to slip, it will leave the fixed sphere when $\cos\theta = \frac{10}{17}\cos\alpha$.

Example 2. *A solid homogeneous sphere of radius a and mass m, rolls in the interior of a fixed hollow sphere of radius b, the centres of the two being always in the same vertical plane. Show that*

(*i*) *the equation of motion of the sphere is the same as that of a simple pendulum of length* $\frac{7}{5}(b-a)$.

(*ii*) *That the pressure on the sphere is* $\frac{1}{7} mg\ (17\cos\theta - 10\cos\alpha)$, *where* θ *is the angle between the common radius and the vertical and* α *the extreme value of* θ.

(*iii*) *That the sphere will make complete revolutions when in its lowest position, the pressure on it is greater than* $\frac{31}{7}$ *times its own weight.*

Example. 3. *A circular plate of mass m, rolls down the inner circumference of a rough circle under the action of gravity, the planes of both the plate and the circle being vertical. When the line joining their centres is inclined at an angle* θ *to the vertical, show that the friction between the bodies is* $\frac{1}{3} mg\sin\theta$.

16·6. Impulsive Forces. Equations of motion as given in Art. 15·42 of the previous chapter are :—

$$M\frac{d^2\bar{x}}{dt^2}=X,\ M\frac{d^2\bar{y}}{dt^2}=Y \ \ldots\ (1);\ \text{and}\ MK^2\frac{d^2\theta}{dt^2}=N \ \ldots\ (2),$$

where $(\bar{x}, \bar{y})$ are the co-ordinates of the point G, the centre of gravity of the body.

Equations (1) show that G, the centre of gravity of the body moves as if M, the whole mass of the body, were collected there and all the external forces act on it.

Equation (2) gives the motion of rotation of the body where N is the sum of the moments of the external forces about G as if it were a fixed point. Integrating these equations, we have

$$M\frac{d\bar{x}}{dt}=\int Xdt,\ M\frac{d\bar{y}}{dt}=\int Ydt,\ \text{and}\ MK^2\frac{d\theta}{dt}=\int Ndt.$$

Suppose X, Y, N are large and act for a very small time t, their effect is then measured by their impulses, viz.

$$\int_0^t Xdt=X_1,\ \int_0^t Ydt=Y_1,\ \text{and}\ \int_0^t Ndt=N_1,$$

where X_1, Y_1 are the impulsive forces and N_1 is the impulsive couple.

Thus $$X_1=\int_0^t Xdt=\left[M\frac{d\bar{x}}{dt}\right]_0^t,\ Y_1=\int_0^t Ydt=\left[M\frac{d\bar{y}}{dt}\right]_0^t$$

$$\text{and}\ N_1=\int_0^t Ndt=MK^2\left[\frac{d\theta}{dt}\right]_0^t$$

Now suppose that just before the impulses act (*i.e.* when $t=0$), the velocities of G, the centre of mass of the rigid body, are u_1, v_1 parallel to the axes and ω_1 is the angular velocity and that after the action of the impulses (*i.e.* at time t) these velocities are u_2, v_2 and ω_2 respectively.

Then these equations become

$$X_1=M(u_2-u_1),\ Y_1=M(v_2-v_1),\ \text{and}\ N_1=MK^2(w_2-w_1).$$

The first two equations show that change in the momentum of mass M, supposed to be collected at G, in any direction is equal to the sum of the impulses in that direction and the last equation shows that change in the moment of momentum (or in the angular momentum) about G is equal to the moment about G of the impulses of the forces.

Note. It may be remembered that an *impulse is not a force*. The former being measured by the change of momentum is of physical dimensions MLT^{-1} while the latter is of dimensions MLT^{-2}.

16·61. These principles are illustrated in the following examples :—

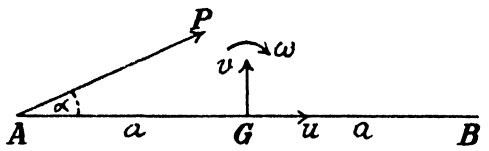
Example 1. *A uniform rod AB of mass m and length 2a, is set in motion by a blow P acting at the end A and inclined at an angle α to the rod. Find the velocity of A immediately after the impact, and prove that the energy communicated to the rod is $P^2(1+3\sin^2\alpha)/2m$.*

(The rod may be supposed to be placed on a smooth horizontal table.)

Let G be the $C.G.$ of the rod, u, v velocities of G along and perp. to AB, and let ω be the angular velocity, then the equations of motion are :—

$$mu = P\cos\alpha,$$
$$mv = P\sin\alpha.$$

and $m\frac{a^2}{3}\omega$ = moment of P about $G = P\,a\sin\alpha$.



The component velocities of A, just after the action of P are

$$u = \frac{P\cos\alpha}{m}, \text{ and } v + a\omega = \frac{P\sin\alpha}{m} + \frac{3\,P\sin\alpha}{m} = \frac{4\,P\sin\alpha}{m}.$$

The kinetic energy of the rod $= \frac{1}{2}m\,(u^2+v^2+K^2\omega^2)$

$$= \frac{1}{2}m\left[\frac{P^2\cos^2\alpha}{m^2} + \frac{P^2\sin^2\alpha}{m^2} + \frac{1}{3}\cdot\frac{9\,P^2\sin^2\alpha}{m^2}\right] = \frac{P^2}{2m}(1+3\sin^2\alpha).$$

Example 2. *Two equal uniform rods AB and BC, each of mass m and length 2a are smoothly jointed at B and placed on a smooth table so that ABC is a straight line. A horizontal blow of impulse P acts at A at right angles to AB. Show that*

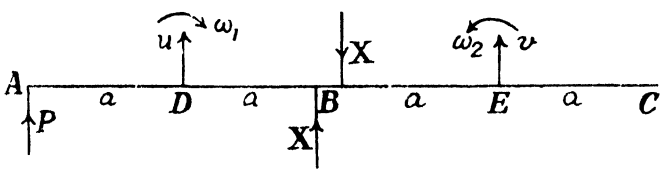
(i) the resulting velocity U of A is $3\frac{1}{2}$ times that of B,

(ii) the kinetic energy generated is $\frac{1}{7}\,m\,U^2$,

show also that the kinetic energy generated is $\frac{7}{8}$ times what it would be if the rods were rigidly fastened together at B.

Let u, v be the velocities of centres D and E of the rods and let ω_1 and ω_2 be their angular velocities after the action of the blow.

Let X be the impulsive action at B in opposite directions on the two rods, then for the motion of the rod AB, we have



$$mu = P + X \qquad \ldots(1),$$

and
$$m\frac{a^2}{3}\omega_1 = Pa - Xa \qquad \ldots(2),$$

and for the rod BC,

$$mv = -X \quad \ldots(3), \text{ and } m\frac{a^2}{3}\omega_2 = Xa \qquad \ldots(4).$$

The velocity of A being given U, $\therefore\ u+a\omega_1=U$ ...(5).

The velocity of the end B of the rod $AB=u-a\omega_1$, also the velocity of the end B of the rod $BC=v-a\omega_2$

Equating them, we get $u-a\omega_1=v-a\omega_2$ ...(6).

Substituting from the first four equations, we get

$$\frac{P+X}{m}-\frac{3(P-X)}{m}=-\frac{X}{m}-\frac{3X}{m} \text{ or } 8X=2P, \text{ i.e. } X=\tfrac{1}{4}P.$$

$$\therefore v=-\frac{P}{4m},\quad u=\frac{5P}{4m},\quad a\omega_1=\frac{9P}{4m},\quad \text{and } a\omega_2=\frac{3P}{4m}.$$

(*i*) Now velocity of $A=u+a\omega_1=\dfrac{14P}{4m}=\dfrac{7P}{2m}=U$ ...(7),

and velocity of $B=v-a\omega_2=-\dfrac{4P}{4m}=\dfrac{P}{m}$ in magnitude.

Hence the velocity of $A=3\frac{1}{2}$ times that of B.

(*ii*) The kinetic energy $T=\frac{1}{2}m\left[u^2+v^2+\frac{1}{3}a^2\omega_1^2+\frac{1}{3}a^2\omega_2^2\right]$

$$=\tfrac{1}{2}m\left[\frac{25P^2}{16m^2}+\frac{P^2}{16m^2}+\frac{81P^2}{48m^2}+\frac{9P^2}{48m^2}\right]=\frac{7P^2}{4m} \quad ...(8)$$

But $$\frac{7P^2}{4m}=\frac{m}{7}\times\frac{49P^2}{4m^2}=\tfrac{1}{7}mU^2. \text{ [by (7)]}.$$

(*iii*) When the rods are rigidly fastened together at B, it becomes one rod of length $4a$ and mass $2m$. If V be the velocity of its middle point B and ω be its angular velocity, then

$$2mV=P, \text{ and } 2m\times\frac{(2a)^2}{3}\omega=P.\ 2a \text{ or } ma\omega=\frac{3}{4}P.$$

The kinetic energy T' in this case $=\frac{1}{2}.\ 2mV^2+\frac{1}{2}.\ 2m\ \frac{(2a)^2}{3}\omega^2$

$$=m\left[\frac{P^2}{4m^2}+\frac{4}{3}\cdot\frac{9P^2}{16m^2}\right]=\frac{P^2}{m} \quad ...(9).$$

$\therefore$ from (8) and (9) $\qquad T=\dfrac{7}{4}T'.$

Example 3. *In the above example if the blow P strikes the rod AB not at A but at another point Q, show that A and B will begin to move in the same direction if AQ is greater than $\frac{1}{3}AB$ and less than $\frac{7}{9}AB$.*

To use the same figure as given in the last example, let $AQ<AD$ and let $AQ=x$, so that $QD=a-x$.

Taking moments about the middle point D for the rod AB, we have

$$m\frac{a^2}{3}\omega_1=P(a-x)-Xa \quad ...(2).$$

Substituting from this equation and from (1), (3), and (4) in (6), which remain the same as given in the last example, we get

$$\frac{P+X}{m}-\frac{3P(a-x)}{ma}=-\frac{X}{m}-\frac{3X}{m}$$

or $$8X = 2P - \frac{3Px}{a}, \text{ i.e. } X = \frac{P}{4} - \frac{3Px}{8a}.$$

$$\therefore u = \frac{P+X}{m} = \frac{P}{m} + \frac{P}{4m} - \frac{3Px}{8am} = \frac{(10a-3x)P}{8am},$$

and $$a\omega_1 = \frac{3P(a-x) - 3Xa}{ma} = \frac{3P(a-x)}{ma} - \frac{3}{m}\left[\frac{P}{4} - \frac{3Px}{8a}\right]$$

$$= \frac{3(6a-5x)P}{8am}.$$

$\therefore$ Velocity of $A = u + a\omega_1 = (28a - 18x)\,P/8am$,

and Velocity of $B = u - a\omega_1 = (12x - 8a)\,P/8am$.

The velocity of A will be positive if $18x < 28a$

i.e. if $x < \frac{7 \times 2a}{9}$ or $\frac{7AB}{9}$.

The velocity of B will also be positive if $12x > 8a$,

i.e. $x > \frac{2a}{3}$ or $\frac{AB}{3}$.

Thus A and B will move in the same direction

if $\frac{1}{3}\,AB < x < \frac{7}{9}AB$.

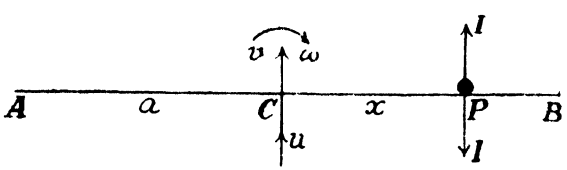
Note. The velocities of both A and B will be negative if $x > \frac{7}{9}\,AB$ but $< \frac{1}{3}\,AB$, which is obviously impossible.

Example 4. *A uniform rod of length $2a$ and mass m is in motion in a horizontal plane p, with speed u at right angles to its length when it collides with a stationary small uniform elastic sphere of equal mass whose centre lies in the plane p. If the sphere is free to move, prove that the angular velocity acquired by the rod cannot exceed*

$$(1+e)\,u\sqrt{6}/4a,$$

where e is the co-efficient of restitution.

Let the rod AB strike the sphere at a distance x from its centre C, and let v be linear and ω the angular velocity of the rod and v_1 the velocity of the sphere just after the impact. Then the equations of motion are :—



For the rod $$m(v-u) = -I \qquad \ldots(1)$$

and $$m\,\frac{a^2}{3}\,\omega = I\,x \qquad \ldots(2)$$

For the sphere $$mv_1 = I, \qquad \ldots(3)$$

where I is the equal and opposite impulse on the sphere and the rod.

From these equations we get

$$(v_1-v)+u=2I/m=\tfrac{2}{3}a^2\omega/x \qquad \ldots(4).$$

The velocity of the point P of the rod just after the impact is $v-x\omega$.

$\therefore$ by the law of relative velocity for impact, we have

$$v_1-(v-x\omega)=-e\,(0-u)=eu, \quad \text{or} \quad v_1-v+x\omega=eu \qquad \ldots(5).$$

Now from (4) and (5), we get

$$u-x\omega+eu=\tfrac{2}{3}\,a^2\omega/x \quad \text{or} \quad (3x^2+2a^2)\omega=3xu\,(1+e)$$

$$\therefore \quad \omega=\frac{3u\,(1+e)x}{3x^2+2a^2} \text{ and } \frac{d\omega}{dx}=\frac{3u\,(1+e)\,[3x^2+2a^2-6x^2]}{(3x^2+2a^2)^2}$$

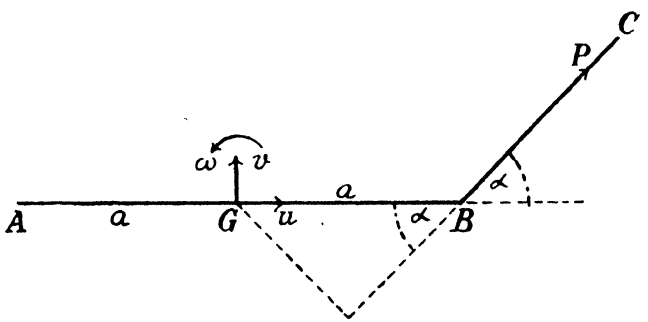
For max. value of ω, $d\omega/dx=0$, which gives $2a^2-3x^2=0$

$$\text{or} \qquad x=\sqrt{\tfrac{2}{3}}a.$$

For this value of x, $\omega=\dfrac{3u\,(1+e)\sqrt{2a}}{\sqrt{3}\,(2a^2+2a^2)}=\dfrac{(1+e)\,u\sqrt{6}}{4a}$.

Example 5. *A uniform rod AB, of mass m and length $2a$, is struck by an instantaneous impulse P acting at B in a direction BC which makes an angle α with AB produced. Show that AB will become parallel to BC after a time t given by $t=ma\alpha/3P\sin\alpha$.*

Let u, v be the components of the linear velocity of the middle point G, and ω the angular velocity of the rod, then



$$mu=P\cos\alpha, \qquad mv=P\sin\alpha,$$

and $m\dfrac{a^2}{3}\omega=$ moment about G of the impulse $P=P.a\sin\alpha$,

or $$\omega=3P\sin\alpha/ma.$$

After the action of the impulse the motion becomes steady and the rod will become parallel to BC after it has rotated through the angle α in time t given by

$$t=\frac{\alpha}{\omega}=\frac{ma\alpha}{3P\sin\alpha}.$$

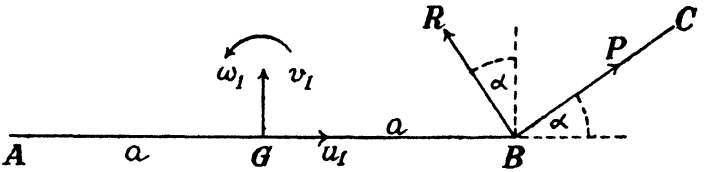
Example 6. *In the above example if the end B, instead of being free, is compelled, by a frictionless constraint, to move along the line*

BC, show that the impulse P will be accompanied by a simultaneous impulsive reaction on the constraint of magnitude.

$$3P \sin\alpha \cos\alpha/(1+3\cos^2\alpha).$$

Let u_1, v_1 be the components of the velocity of G and ω_1, the angular velocity of the rod in this case.

The constraint being frictionless, the impulsive reaction will be perpendicular to BC.



Resolving along and perpendicular to AB, we have

$$mu_1 = P\cos\alpha - R\sin\alpha,$$
$$mv_1 = P\sin\alpha + R\cos\alpha,$$

and taking moments about G, we have

$$m\frac{a^2}{3}\omega_1 = P.a\sin\alpha + R\ a\cos\alpha.$$

The velocity of B is u_1 along AB and $v_1+a\omega_1 \perp AB$.

$\therefore$ the point B will move along BC if

$$\tan\alpha = \frac{v_1+a\omega_1}{u_1} = \frac{P\sin\alpha + R\cos\alpha + 3(P\sin\alpha + R\cos\alpha)}{P\cos\alpha - R\sin\alpha}$$

or $(P\cos\alpha - R\sin\alpha)\sin\alpha = 4(P\sin\alpha + R\cos\alpha)\cos\alpha$,

which gives in magnitude $R = \dfrac{3P\sin\alpha\cos\alpha}{\sin^2\alpha + 4\cos^2\alpha} = \dfrac{3P\sin\alpha\cos\alpha}{1+3\cos^2\alpha}$.

Example 7. *Two equal uniform rods, each of mass m and length 2a, rests on a smooth table, one extremity of one rod being connected with one extremity of the other by a light inextensible string, the rods and the string form the sides of a rectangle and the string is taut. If an impulsive couple of moment G suddenly acts upon one of the rods so as to produce an impulsive tension in the string, prove that the energy communicated to the system is $15G^2/16a^2m$.*

Let u, u_1 be the velocities of the middle points and ω, ω_1 be the angular velocities of the rods, and let T be the impulsive tension in the string, just after the application of the impulsive couple G. Then the equations of motion are :

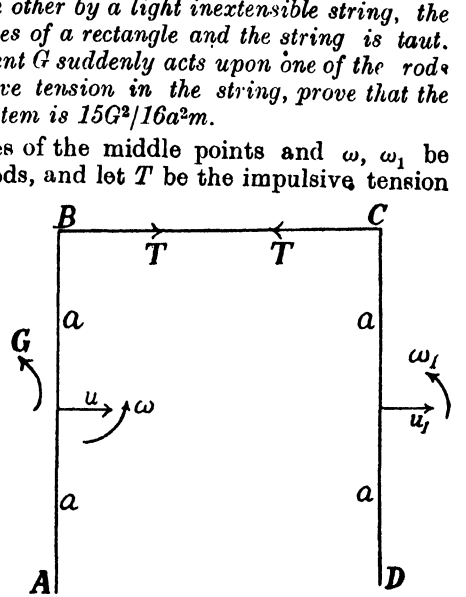
$$mu = T \quad \ldots(1),$$

$$m\frac{a^2}{3}\omega = G - Ta \ldots(2),$$

and $$mu_1 = -T \quad \ldots(3),$$

$$m\frac{a^2}{3}\omega_1 = Ta \quad \ldots(4).$$

Since the string is inextensible, the velocity of B = the velocity of C.



$$\therefore \quad u-a\omega=u_1-a\omega_1$$
$$\text{or} \quad u-u_1=a\omega-a\omega_1 \qquad \ldots(5).$$

Substituting the values of u, u_1, ω, ω_1 in (5), we get

$$2T=3G/a-3T-3T, \quad \text{or} \quad aT=\tfrac{3}{8}G.$$

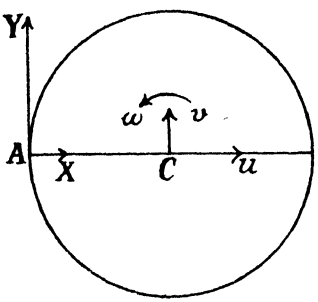
$$\therefore \quad a\omega=\frac{3G}{am}-\frac{3T}{m}=\frac{8T}{m}-\frac{3T}{m}=\frac{5T}{m} \text{ and } a\omega_1=\frac{3T}{m}$$

$$\therefore \text{ Kinetic energy}=\tfrac{1}{2}m[u^2+u_1^2+\tfrac{1}{3}a^2\omega^2+\tfrac{1}{3}a^2\omega_1^2]$$
$$=\tfrac{1}{2}m\left[\frac{T^2}{m^2}+\frac{T^2}{m^2}+\frac{25T^2}{3m^2}+\frac{9T^2}{3m^2}\right]$$
$$=\frac{40T^2}{6m}=\frac{40}{6m}\cdot\frac{9G^2}{64a^2}=\frac{15G^2}{16a^2m}.$$

Example 8. *A uniform circular disc of mass m, initially at rest but free to move in any manner, is suddenly set in motion by an impulse, in its plane, applied to a point A of its circumference. The initial velocity of A has components U along the radius through A, and V along the tangent at A. Prove that the initial kinetic energy of the disc is $\frac{1}{2}m\,(U^2+\frac{1}{3}V^2)$.*

Let the diameter through and the tangent at A be taken as the axes.

Let ω be the angular velocity of the disc and u, v the components of the velocity of its centre C, and let X, Y be the components of the impulse at A.



Then we have

$$mu=X, \quad mv=Y, \text{ and } \quad mK^2\omega=-Ya$$
$$\text{or} \quad m\frac{a^2}{2}\omega=-aY \text{ i.e. } ma\omega=-2Y$$
$$\therefore \quad 2mv+ma\omega=2Y-2Y=0$$
$$\text{or} \quad a\omega=-2v$$

For the motion of A.

$$U=u, \text{ and } V=v-a\omega=3v,$$
$$\text{i.e., } \quad v=\tfrac{1}{3}V \text{ and } \quad a\omega=-2v=-\tfrac{2}{3}V$$

$$\text{The Kinetic energy}=\tfrac{1}{2}m(u^2+v^2+K^2\omega^2)$$
$$=\tfrac{1}{2}m\left[U^2+\frac{1}{9}V^2+\frac{a^2}{2}\times\frac{4V^2}{9a^2}\right]$$
$$=\tfrac{1}{2}m[U^2+\tfrac{1}{3}V^2].$$

Examples LIV

1. A uniform rod AB of length $2a$, is lying on a smooth horizontal plane and is struck by a horizontal blow, of impulse P, perpendicular to the rod at a distance x from the centre. Find the motion and show that the point about which the rod begins to turn is at a distance $(a^2+3x^2)/3x$ from the line of action of P.

If P acts at the end A of the rod, show that the velocity of A is twice that of the end B.

2. A *light* rigid rod has particles, each of mass m, attached at A, B and C, where $AB=a$, $BC=b$ $(a>b)$. A blow P perpendicular to the rod is applied at the middle point of AC, show that the angular velocity acquired is

$$\frac{P}{4m}\cdot\frac{a-b}{a^2+ab+b^2}.$$

If the blow P acts at a point distant from A equal to BC, show that the kinetic energy of the system is

$$\frac{P^2}{2m} \cdot \frac{a^2-ab+b^2}{a^2+ab+b^2}$$

3. A circular disc is revolving in its plane about its centre with angular velocity ω. Show that its new angular velocity if a point on the rim is suddenly fixed is $\frac{1}{2}\omega$.

4. An impulse is applied at a point P in the rim of a uniform circular disc lying on a smooth table, in such a manner that P starts to move along the tangent to the rim at P. Prove that the initial velocity of the centre of the disc is $\frac{1}{2}$ times that of P.

5. A uniform circular hoop, lying on a smooth table, receives a blow, in its plane, at a point P, inclined at an angle α with PO, the radius through P. Show that P begins to move in a direction inclined to PO at an angle

$$\tan^{-1}(2 \tan \alpha).$$

6. A square lamina $ABCD$, of side $2a$, rests on a smooth horizontal plane. If the corner A is made to move with velocity u along the line BA produced, prove that the initial angular velocity of the lamina is $3u/8a$.

7. Two equal uniform rods, each of mass m, are freely jointed at A and laid on a smooth horizontal table so that angle BAC is a right angle. The rod AB is struck by a blow P at B in a direction perpendicular to AB. Show that

(*i*) the resulting velocities of the middle points of AC and AB are in the ratio 2 : 7

and (*ii*) the initial velocity of A is $2P/5m$.

8. A uniform lamina of mass m in the form of an equilateral triangle of side $2a$, moves along with velocity u in a direction parallel to one side AB. Suddenly the foremost end A of the side is fixed. Show that the subsequent angular velocity is $\sqrt{3}\,u/5a$ and that the impulse of the force at the fixed point is $\frac{1}{5}mu\sqrt{19}$ in a direction inclined at an angle $\tan^{-1}(\sqrt{3}/4)$ to the side AB of the triangle.

9. Three equal rods AB, BC, CD are hinged freely at their ends, B and C, so as to form three sides of a square and are laid on a smooth table; the end A is struck by a horizontal blow J at right angles to AB. Show that initial velocity of A is 19 times that of D, and that the impulsive actions at B and C are respectively $\frac{5}{12}J$ and $\frac{1}{12}J$.

10. A uniform rod AB, of mass M and length $2a$, lies on a smooth table and a particle of mass m is attached to B by a light inextensible string. The particle is projected from B along the table with velocity u at right angles to the rod. Find the velocity of the centre of the rod and its angular velocity immediately after the string tightens, and show that a fraction $M/(M+4m)$ of the original energy is lost. [*Ans.* $mu/(M+4m)$, $3mu/a(M+4m)$].

11. A uniform rod falling without rotation, strikes a smooth horizontal plane. Prove that for all values of the co-efficient of restitution, the angular velocity of the rod immediately after the impact is a maximum if the rod before impact makes with the horizontal an angle $\cos^{-1}(1/\sqrt{3})$.

12. The end B of a uniform rod, of mass $2m$ and length $2a$, is smoothly hinged to a point on the rim of a uniform circular disc of mass m and radius r. The rod and disc are laid on a smooth horizontal table so that the direction of AB passes through the centre of the disc.

An impulse P is applied at A at right angles to AB. Prove that the kinetic energy produced is $9P^2/10m$, and that the impulsive reaction at the hinge is $\frac{1}{5}P$.

13. A uniform rod, of length 2 feet and weight 5 lb., is pivoted freely at one end about a horizontal axis. The rod is slightly displaced from the position of unstable equilibrium. With what angular velocity will it reach the horizontal position?

What will be the impulsive blow on a stop which catches the end and prevents the rod from moving past that position?

[*Ans.* $4\sqrt{3}$; $40/\sqrt{3}$ lb. ft. sec. units.]

14. A circular hoop of radius a rolling on a rough horizontal plane, impinges on a rough peg of height $\frac{1}{5}a$ fixed in the plane. Find the angular velocity with which the hoop begins to turn about the peg. If V be the velocity of the centre before impact, prove that if $81V^2 > 80ga$, the hoop leaves the peg immediately. [*Ans.* $9V/10a$.]

15. Compare the kinetic energies imparted to an elliptic disc of semi-axes a and b by a blow J in its plane through a focus at right angles to the major axis

(*i*) when the disc is free (*ii*) when the centre of the disc is fixed.

[*Ans.* $(5a^2-3b^2)/4(a^2-b^2)$].

16·7. Centre of Percussion. *If an impulse is applied to a rigid body which is free to turn about a fixed axis so as not to produce any impulsive reaction on the axis, the line of action of the impulse is called the line of percussion and the point in which this line meets the plane through the centre of gravity G of the body and the axis of rotation is called the* **Centre of Percussion.**

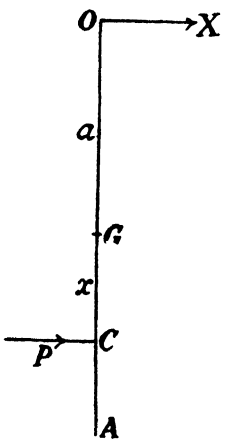
In the case of a thin rod the centre of percussion is the point in the rod where the line of action of the impulse meets it.

When the line of action of the impulse is given, the axis about which the body begins to turn freely is called the *axis of spontaneous rotation*.

16.71. As a simple example let us consider the following :—

A uniform rod OA of mass m and length 2a suspended freely from one end and struck by a horizontal blow P at a point C at a distance x from G, the middle point of the rod.

Let ω be the angular velocity acquired by the rod just after the application of P and let X be the impulsive reaction of the fixed axis on the rod.



Then the velocity of G is $a\omega$.

$$\therefore \quad ma\omega = P + X \qquad \ldots (1).$$

Taking moments about O, we have

$$m(K^2+a^2)\omega = \text{moment of the impulses about } O = P(a+x) \qquad \ldots (2),$$

where K is the radius of gyration of the rod about G, *i.e.*, $K^2 = \frac{1}{3}a^2$.

From these by eliminating P, we get

$$m[K^2+a^2-a(a+x)]\omega = -(a+x)X,$$

or $$X = m(ax-K^2)\omega/(a+x).$$

Now $X=0$, *i.e.*, there is no impulsive action at O, if $ax=K^2$, *i.e.*, if $OG.GC=K^2$ ... (3).

If this condition is satisfied the point C is the centre of percussion and equation (3) shows that the centre of percussion coincides with the centre of oscillation in this case.

It also shows that if the axis of rotation passes through C, O is then the centre of percussion *i.e.*, the axis of rotation and the line of impulse are interchangeable.

Note 1. In the example given above $x=K^2/a$. If x is not qqual to K^2/a, X exists and is not zero. It is positive or negative according as $x>K^2/a$, or $<K^2/a$ *i.e.*, the impulsive reaction on the rod is in the same or opposite direction as the blow according as the blow is applied at a point below or above the centre of percussion.

Note 2. If in a body (symmetrical about a plane through G perp. to the axis of rotation), rotating about a fixed axis; $OG=h$, then the position of C is given by $x=GC=K^2/h$, where K is the radius of gyration about a parallel axis through G.

If may be compared with the Compound Pendulum discussed in Art. 14.93, P. 292.

16·72. The principle of the centre of percussion has important practical application in the design of instruments.

For example, the handle of a hammer is made of wood and not of the metal of which the hammer-head is made. Wood being lighter, the main part of the mass is collected in the hammer-head, so that the blow acts at a point very near to the centre of percussion and thus action on the axis of rotation *i.e.*, on the hand of the workman is very slight.

If the handle were made of the same material as the head, the workman is bound to feel the jar at his hand.

Similar is the case with a cricket bat, the hands of the batsman occupying a small portion of its handle is an approximation to a fixed axis. A very little jar on the batsman's hands is felt if the bat hits the ball at the proper place. Any one who has used a bat knows what difference it makes to strike the ball at the correct point on the blade of the bat. The point is, of course, the centre of percussion.

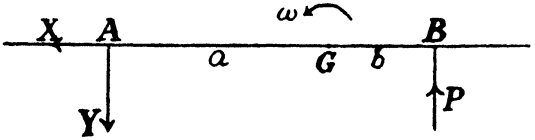
The height above the table at which a billiard ball should be struck by a horizontal blow, in order that it may not slide, is

$$\frac{a^2+K^2}{a}=\frac{a^2+\frac{2}{5}a^2}{a}=\frac{7}{5}a,$$

where a is the radius of the ball.

Example 1. *A uniform rod of mass m and length $2l$ lies on a smooth horizontal plane and is freely hinged at a point A distant a from the centre G of the rod. It is struck by a blow of impulse P in a direction at right angles to its length at a point B at a distance b from G and $a+b$ from A. Find the angular velocity ω imparted to the rod and the impulsive reaction at the hinge. Find also the position of B in order that there may be no reaction at the hinge.*

Let X and Y be the components of the impulse at the hinge A, along and at right angles to the rod.



Then $X=0$ ...(1),

$ma\omega=P-Y$...(2),

and $m\frac{l^2}{3}\omega=Pb+Ya$ ... (3).

From (2) and (3), we get $P(a+b)=m(a^2+\frac{1}{3}l^2)\omega$.

$$\therefore \quad \omega=\frac{3P(a+b)}{m(3a^2+l^2)} \quad \text{and} \quad Y=P-ma\omega=\frac{l^2-3ab}{3a^2+l^2}P.$$

In order that the impulse at the hinge may be zero,

$$h=l^2/3a=l^2/3\ AG,$$

so that the centre of percussion in this case is distant $l^2/3AG$ from G.

Cor. If the rod were not hinged and is free to move and if the point A were at rest immediately after the blow, equations (2) and (3) then become,

$ma\omega=P$, and $m\omega\,\frac{1}{3}l^2=Pb$, from which $a=l^2/3b$.

Thus the rod begins to turn about the point distant $l^2/3b$ from the centre.

Subsequently G moves with uniform velocity $a\omega=P/m$, at right angles to the initial direction of the rod and at the same time the rod rotates with uniform angular velocity $3Pb/ml^2$.

Example 2. *A pendulum is constructed of a solid sphere, of mass M and radius a, which is attached to the end of a rod AB, of mass m and length b. Where should it be struck at each oscillation that there may be no impulsive pressure to wear the point of support?*

Let G be the $C.\ G.$ and K the radius of gyration of the whole body and let the point of support be at A, then

$(M+m)\,h$ or $(M+m)\,AG=M(a+b)+m.\ \frac{1}{2}\ b$,

and $$(M+m)\,K^2=M[\tfrac{2}{5}a^2+(a+b)^2]+m[\tfrac{1}{3}(\tfrac{1}{2}b)^2+(\tfrac{1}{2}b)^2].$$

$\therefore$ the required point C is given by

$$AC=\frac{K^2}{h}=\frac{M[\frac{2}{5}a^2+(a+b)^2]+\frac{1}{3}mb^2}{M(a+b)+\frac{1}{2}mb}.$$

Examples LV

1. Find the distance of the centre of percussion from the axis of rotation in the following cases :—

(*i*) A uniform bar of length $2a$, free to turn about one end.

(*ii*) A uniform circular plate of radius a with axis perpendicular to its plane through a point on its rim.

(*iii*) A uniform circular plate of radius b with axis a horizontal tangent.

(*iv*) An equiangular triangular lamina of side $2a$, rotating about a side which is horizontal.

[*Ans.* (*i*) $\frac{4}{3}a$ (*ii*) $\frac{3}{2}a$ (*iii*) $\frac{5}{4}b$ (*iv*) $\frac{1}{2}\sqrt{3}\ a$.]

2. A uniform circular lamina whose centre is O, rests on a smooth horizontal plane, show that it will commence to turn about a point A on its circumference if it is struck a blow acting perpendicular to AO at a distance from A equal to $\frac{1}{5}$ of a diameter of the lamina.

3. A pendulum of length 1 ft. 4 ins. consists of a uniform circular disc of diameter 4 ins. at one end of a uniform rod of length, 1 ft., which is suspended from the other end. Find the ratio of the masses of the rod and the disc if a horizontal blow applied in the plane of the disc $\frac{5}{8}$ in. above the centre produces no impulsive reaction at the point of support. [*Ans.* 1 : 3]

ANSWERS

Examples I. Page 5.

1. (*i*) $18\frac{1}{3}$ f. p. s. (*ii*) 21 ft./sec.^{2} (*iii*) 39 f.p.s.
2. (*i*) zero, -2ft./sec.^{2} ; 1 ft./sec., 4 ft./sec.^{2} (*ii*) zero, $-\frac{4}{27}$ ft. (*iii*) $-\frac{2}{27}$ ft.
7. From $x=0$ to $x=2$ or from -2 to $-\infty$.
9. $10\frac{40}{121}$m.p.h. 10. $1\frac{4}{11}$. 11. 7 : 8. 14. 2000 ft./sec.
15. 6 m.p.h.

Examples II. Page 11.

1. 2 ft./sec.^{2} ; 605 ft. ; 60 m.p.h. 2. 4 ft./sec.^{2} ; 5 secs.
3. $13\frac{1}{3}$ ft. 4. 110 ft. further ; 20 secs.
5. $544\frac{1}{2}$ ft. ; $33\sqrt{2}=46{\cdot}7$ f.p.s. 7. $7u$.
8. (*i*) $(v-u)/t$. (*ii*) $\frac{1}{2}(u+v)t$. (*iii*) $\sqrt{(u^2+v^2)/2}$.
9. 21 ft. from the last position ; after $\frac{3}{5}$ sec.
10. $-7\frac{1}{3}$ f.p.s. ; $51\frac{1}{3}$ f.p.s.
11. $\frac{1}{2}$ ft. beyond C ; $4\frac{1}{2}$ ft. beyond B.
13. 20 cm./sec., 60 cm./sec.^{2} 18. $319\frac{1}{11}$ ft.
19. 30 m.p.h. ; 31·6 m.p.h.

Examples III. Page 15.

1. (*i*) 196 ft. (*ii*) 7 secs. (*iii*) After 3 secs.
   (*iv*) After 1 sec. and 6 secs. (*v*) After 8 secs.
2. 240 ft. ; 272 ft. ; 8·8 secs. 3. 224 ft./sec. ; 14 secs.
4. $17\frac{13}{16}$ft. 5. After 6 secs. ; 648 ft. above bottom.
6. 160 ft./sec. ; $2\frac{1}{2}$ secs. ; $7\frac{1}{2}$ secs. 7. $68\frac{4}{13}$ f.p.s. ; $306\frac{12}{13}$ ft.
8. 80 ft./sec.
10. $3\frac{1}{7}$ secs. from the instant of throw of the first stone ; 84 ft. above the point of projection.
12. At a distance $s/8$ below B and $\frac{1}{2}\sqrt{s/g}$ sec. after the start of the second particle.
13. 9 secs. 14. 225 ft. 15. $756\frac{1}{4}$ ft.

Examples IV. Page 21.

2. $\frac{9}{80}$ mi./min.^{2} ; $\frac{9}{25}$ mi./min.^{2}
3. 60 cm./sec.^{2} ; 80 cm./sec.^{2} ; 2400 cm./sec.^{2}
4. $\frac{32}{55}$ ft./sec.^{2} ; $21\frac{1}{3}$ f.p.s. 6. $1\frac{1}{8}$ ft./sec.^{2}, $\frac{7}{8}$ ft./sec.^{2} ; 440 ft.
7. 10 min., 59·25 secs.

REVISION QUESTIONS—I. Page 22

6. 60 m.p.h. ; 44 secs. ; 1936 ft. ; 8 secs.
9. 1 sec. ; $\frac{1}{4}\sqrt{S}$ sec. 10. $\frac{n}{g}\left(\frac{1}{1+f}+\frac{1}{\sqrt{1-f^2}}\right)$; $\frac{1}{\sqrt{2}}$.
11. $1\frac{19}{32}$ min. 12. $2\sqrt{\frac{l}{\alpha}}$. 16. 1120 ft./sec.

Examples V. Page 30.

1. 5·86 lb. wt. 2. 1694 ft. ; 154 secs. 3. 12·5 secs. ; 1250 ft.
4. 35 lb. wt. 5. 9 poundals.
6. 141·4 cms. per sec. ; 165·1 gm. wt.
7. $\frac{wf}{W-w}$. 9. 8 ft./sec.^{2} ; 5 secs. 10. 8 ft./sec.^{2} ; 5 secs.
11. 13 lb. wt. 12. $5\frac{3}{10}$ tons wt. 13. 100 secs.
14. (*i*) 110 secs. (*ii*) 22 secs. (*iii*) 546 secs.
15. 17·28 ft.

Examples VI. Page 36.

1. $89\frac{3}{8}$ lbs. ; 110 lbs. ; $130\frac{5}{8}$ lbs. 2. Zero. 3. $12\frac{3}{4}$ lbs. wt.
4. 6·4 ft./sec.^{2} ; $2\frac{3}{4}$ lb. ; 4 lb. ; zero. 5. $2\frac{6}{13}$ ft./sec.^{2}
6. 24·53 cm./sec^2. 7. 500 lb. wt.

Examples VII. Page 40.

1. $\frac{1}{11}g$, $10\frac{10}{11}$ lb. wt., $21\frac{9}{11}$ lb. wt. 2. 980 cm./sec^2.
3. 30 lb. or $3\frac{1}{2}$ lb. 4. 10 lb., 6 lb.
5. 9 lb. 9 oz., 6 lb. 7 oz. ; 8·1 lb. wt., 7·44 lb. wt.
6. $1\frac{19}{24}$ sec. ; $5\frac{17}{25}\frac{1}{25}$ ft.
9. $\frac{32}{7}$ ft./sec.^{2}, $20\frac{4}{7}$ ft., $3\frac{1}{2}$ secs., $1\frac{5}{7}$ lb. wt., 2·42 lb. wt.
10. $\frac{16}{35}$ ft./sec.^{2}., 2·06 ft., 11·06 sec., 1·97 lb. wt., 2·79 lb. wt.
11. 4 ft./sec.^{2} 12. 1 lb. 13. $6\frac{2}{3}$ ft. per sec. ; $2\frac{2}{3}$ in.
15. $\frac{M+2m_1}{M+m_1+m_2}m_2g$, $\frac{M+2m_2}{M+m_1+m_2}m_1g$.
18. $3\frac{3}{8}\frac{3}{4}$ oz. wt. ; 1·46 sec.
19. (*i*) $6\frac{67}{73}$ lb. wt. ; $\frac{1}{4}\sqrt{39}$ sec. (*ii*) $7\frac{7}{73}$ lb. wt. ; $\frac{1}{8}\sqrt{78}$ sec.

REVISION QUESTIONS—II. Page 42.

2. 43 cwt. ; 1·61 tons wt.
3. 2 mins. $55\frac{1}{11}\frac{1}{8}$ secs. 4. 8 secs. nearly.
5. 1·98 tons ; $138\frac{1}{3}$ secs. 7. $\frac{1}{5}$ ft./sec.^{2}
9. $17\frac{1}{3}$ stone wt. ; $g/9$; $13g/72$; $g/8$. 11. $g/17$.
12. $\frac{1}{7}g$, $\frac{2}{7}g$ ft./sec.^{2} ; $\frac{2}{7}mg$ poundals.
13. $3g \div \left(\frac{1}{m}+\frac{1}{m'}+\frac{4}{M}\right)$. 17. 42 m.p.h. ; 54 m.p.h.

Examples VIII. Page 47.

1. $-2v^3/c^2$. 11. $\log_e 4 = 1{\cdot}4$ sec. nearly.
12. 350 $\log_e 2$ sec. ; $2\frac{1}{11}$ miles.
13. $\frac{1}{\sqrt{\mu}}\log\frac{\sqrt{\mu}x+\sqrt{\mu x^2+V^2}}{V}$; $\frac{V}{\sqrt{\mu}}\sinh\sqrt{\mu}\,t$; $\frac{1}{\sqrt{\mu}}\cosh^{-1}2$.

Examples IX. Page 52.

1. 1. ft. 3 in. ; $1\frac{1}{2}$ ft./sec. 2. 2·86. ft. ; 13 ft./sec. nearly.
3. 3·14 f.p.s. ; 2·7 f.p.s., $1\frac{1}{3}$ sec. 4. 2π sec. ; $3\sqrt{2}$ in. per sec.
5. 16 sec. ; 16·97 inches. 8. $3\sqrt{2} = 4{\cdot}24$ ft./sec. ; 0·17.
11. $6\pi = 18{\cdot}85$ f.p.s. ; 666, 333 lb. wt. nearly. 12. $\frac{7}{3}a$.
17. $-\frac{1}{2}$, $-\frac{1}{2}$, 0. 18. $5\pi^2/16g$. 19. 35·08, 28·92 poundals.
20. $30\sqrt{g}/\pi = 54$ nearly.. 22. $\pi/3n$.
23. 3·14 sec. ; 5ft. ; 10 f.p.s., 20 ft./sec^2.

Examples X. Page 60.

2. $\frac{\pi}{4}$ sec., $1\frac{1}{2}$ f.p.s. 3. $\pi/\sqrt{6}$ sec. 4. $2\pi\sqrt{\frac{l}{g}}$.

6. $\frac{\pi}{\sqrt{g}}$ sec. 7. 6 in.; $\frac{\pi}{32}$ sec.; $\frac{\pi}{16}+\frac{1}{4}$ sec.

8. $\pi\sqrt{am/\lambda}$, $\sqrt{\lambda a/m}$.

10. 1·19 ft.; $\frac{3}{5}\sqrt{15}=6{\cdot}19$ f.p.s., 78·4 ft./sec^2.

13. $\frac{\pi}{8\sqrt{2}}+\frac{1}{4\sqrt{2}}\sin^{-1}\frac{1}{5}+\frac{\sqrt{3}}{2}$ sec., 14·14 f.p.s.

Examples XI. Page 62.

I. $n=\frac{3}{2}$. 11. $\frac{2\pi}{T}\sqrt{a^2-x^2}$; $\frac{2\pi a}{T}\cos\frac{2\pi t}{T}$.

12. 664 m.p.h. nearly. 13. $\frac{4\pi}{\sqrt{g}}$ sec., $5\sqrt{g}$ cm./sec.

REVISION QUESTIONS—III. Page 63.

11. $\pi\left[\frac{(m_1+m_2)l}{\lambda}\right]^{\frac{1}{2}}$; $\frac{(m_2-m_1)g}{4\lambda}$.

Examples XII. Page 71.

1. 28·35 f.p.s. at $\tan^{-1}\frac{5\sqrt{3}}{27}$ with the car's direction of motion.

3. 1·73; 3 m.p.h. 4. 7 m.p.h.

5. At $90°+\sin^{-1}\frac{4}{5}=143°\ 8'$ with his own direction of motion.

7. 30° North of West; $10\sqrt{3}$ m.p.h.

8. 28·57 min. nearly, ; nearly 30·5° East of South.

9. $(a^2+s^2t^2+2st\sqrt{a^2-b^2})^{\frac{1}{2}}/t$.

12. $v>u$; $\frac{d\sqrt{u^2-v^2}}{v}$, $\frac{ud}{v\sqrt{u^2-v^2}}$.

14. At 30° with QP; 6 min.; at right angles.

15. $\sqrt{35}\,PQ$. 17. 20, 17·32 f.p.s. at $\pm30°$ to the horizon.

18. $60\sqrt{3}$ m.p.h.; at $\pm30°$ to the horizon.

Examples XIII. Page 76.

1. 16 ft./sec. 4. 4 : 3.

6. $1:(\sqrt{2}-1)$; 1 : 3; 1 : 3 : 5. 7. 1 in 48.

8. $\sqrt{gh}$; 0; $\sqrt{gh}$.

10. At a distance of $\frac{1}{4}$th length of the plane from the bottom; 32 ft.

12. $2182\frac{2}{3}$ yds., 43·25 f.p.s.

13. 3220 lb. wt.; $1861\frac{7}{17}$ ft. 14. $\frac{g}{3}\sqrt{6}$ ft./sec.

15. 13 ft. 4 in. 16. 26 lb. 10 oz. 17. 4 : 5; 11 : 19.

18. (*a*) 5, $2\frac{1}{2}$, $3\frac{3}{4}$ lb. wt. (*b*) $2\frac{1}{2}$ sec., 50 ft. 20. 30°. 21. 2 ft.

22. The radius through the point makes 60° with the vertical.

Examples XIV. Page 80.

1. $16\sqrt{5}$ ft./sec.; 80 ft./sec. 2. 5 ft. 3. 4 ft./sec^2.; 2 secs.

5. $\frac{1}{2}\sqrt{5}$ sec.; $\frac{1}{2}\sqrt{5}$ ft. per sec. 7. $\frac{1}{14}$ sec.

8. The larger mass descends with acceln. $\frac{2\sqrt{3}-3}{9}g$; the masses do not move. 9. $\frac{2761\sqrt{3}}{36}$ yds.

REVISION QUESTIONS—IV. Page 81.

4. $\sin^{-1}\frac{V^2}{g^2T^2}$. 8. $\mu=(m\sin\theta-M)/m\cos\theta$.

Examples XV-A. Page 91.

1. (*i*) 72 ft. (*ii*) 200 ft. (*iii*) 98 ft.; 15°, 75°.
2. $80\sqrt{110}$, $48\sqrt{110}$ ft./sec. 3. 1 : $\sqrt{3}$; equal.
5. (*b*) 5 sec., 692·8 ft., 100 ft., 400 ft. 6. $60\sqrt{3}$ ft.
11. 4·33 secs.; 25 yards. 12. (*a*) 2400·0 ft./sec. (*b*) 192 ft.
13. 2·12 ft. 14. $27\frac{3}{11}$ m.p.h.
17. With a velocity $10\sqrt{10}$ ft./sec. at $\tan^{-1}3$ with the horizontal.

Examples XV-B. Page 94.

1. 57·7, 55·4, 56 ft./sec.; $\frac{1}{2}$ sec.
2. $(v^2-2v\sin\alpha.\,gt+g^2t^2)^{\frac{1}{2}}$; $2(v\sin\alpha-gt)/g$. 7. 34 ft.
9. $v=\frac{2u}{\sqrt{3}}$, $t=\frac{2u}{3g}$.

Examples XVI. Page 98.

1. 45°, $\tan^{-1}\frac{3}{2}=56°\,18'$; 450 ft., 415·4 ft. 2. 45°, $\tan^{-1}3$.
4. $\tan^{-1}\frac{3}{5}$. 8. 32 ft./sec., after $\frac{5}{4}$ sec.; 24 ft./sec.
9. $\sqrt{8ag}$. 12. 20 ft. 16. 13·4 ft.
19. $\frac{v}{g}\sqrt{v^2+2gh}$. 21. 14·6 ft. from the nearest wall; 50·6 f.p.s.
26. 86·5 f.p.s.; 170·64 f.p.s.

Examples XVII. Page 104.

1. 4·3 sec.; 300 ft. 2. 64 yds.; 192 yds.
4. 3333·33 ft. 5. 121·42 yds., $52\frac{1}{2}°$ with the vertical.
10. $\tan^{-1}\frac{1}{2}$.

Examples XVIII. Page 111.

1. 135; 90 secs. 2. $\frac{45}{\pi}$ r.p.s.; 3 min. 10. $\frac{v}{ax}\sqrt{x^2-a^2}$.

Examples XIX. Page 116.

1. $2\frac{1}{4}$ lb. wt. 2. 3·02 gm. wt. 3. 18·5 lb. wt.
4. 47·4 ft./sec. 5. $1\frac{1}{120}$ ton. wt. 6. 231 lb. wt.
7. 3. 9. $480/\pi$ r.p.m. 10. $\sqrt{2cg}$.
11. 1·346 ft.; 16·61 lb. wt.

Examples XX. Page 119.

1. $13\frac{1}{2}$ lb. wt. 2. 3 lb. 6 oz. wt. nearly. 3. 48.

5. 41 r.p.m. nearly; 0·115 lb. wt.
8. (*i*) 12 cm.; 81 cm. per sec. nearly. (*ii*) 32 gm. wt. nearly. (*iii*) 2 r.p.s.
9. $m(g-2v^2/3l)$. 18. (*i*) 7·7 lb. (*ii*) 10·85 lb.

Examples XXI. Page 126.

1. $\tan^{-1}\frac{11}{20}$. 2. 1·65 tons wt.
6. 22·4 lb. wt. nearly; $4\sqrt{2}$ ft. from projection of the centre on the ground.
7. 1120, 336, 117 lb. wt. nearly.
8. 12·7, 12·5 m.p.h. 9. 3·44 inches.
14. 268·8, 1971·2, 2508·8 lb. wt.; 36 ft.

Examples XXII-A. Page 134.

1. 30 m.p.h. 3. 81·82 m.p.h., 6 times the weight of the pilot.
4. 40·3 ft., 24 f.p.s. 5. $37\frac{1}{2}$ ft. 6. $[2ag(3-2\sqrt{2})]^{\frac{1}{2}}$.
8. With a velocity $\frac{3}{2}\sqrt{2ag}$ at $\tan^{-1}\frac{11}{5}\sqrt{\frac{7}{5}}$ with the horizontal.
9. $\sqrt{3gl}$; $\frac{5}{2}$ W. 10. 120°. 11. $\frac{1}{6}(20+9\sqrt{2})w$.
12. $\sqrt{2ag}$, $\sqrt{5ag}$. 14. 6·3 ft., 6·8 ft.
16. 30 lb. wt.; $\frac{3}{4}$th of the length of the string.
18. The particle leaves the circle at a height $\frac{5}{3}a$ above the lowest point.
20. 12 ft./sec., $\frac{1}{2}$ ft.

Examples XXIII. Page 141.

1. 39·03 inches, 99·3 cm. 2. 324 ft. nearly; 6·36 ft.
3. 4·32. 6. Shortened by 1·77 mm.
9. 32·2 ft./sec^2. 10. Gains 270 sec. 11. $\frac{21}{25}\frac{600}{601}$. 12. $\frac{43}{43}\frac{21}{19}$.
13. Loses 16·36 sec. 14. 108 secs.
15. 1630 yds. nearly, 5 secs. per day.
16. $\frac{25}{27}$ mile. 17. 1%. 20. $2\pi\sqrt{mb/F}$.

Examples XXIV. Page 149.

1. 2 poundals, $18\frac{3}{4}$ ft. lb. 2. 700 ft. lb.
3. 50 f.p.s. 4. 12 ft. poundals; $\frac{\pi}{8\sqrt{2}}$ sec.
10. 3750 ft. lb. 13. 1,614,000 ft. lb.

Examples XXV-A. Page 153.

1. 125 H.P. 2. $\frac{61}{250}$ ft./sec^2.; $24\frac{53}{128}$ m.p.h.
3. $8\frac{8}{15}$ H.P., 95·57 H.P. 4. 3413·664 H.P.
5. 672; 90 m.p.h. 6. $11\frac{5}{11}$ H.P.
7. 28·64 H.P. 8. 53 nearly.
9. 13·4 m.p.h. nearly 10. $18\frac{3}{4}$ m.p.h.
11. 1·168 ft./sec^2. 12. 12 m.p.h.; $1\frac{11}{14}$ ft./sec^2.
13. $21\frac{2}{3}$ m.p.h.
14. 12450 lb. wt., $905\frac{5}{11}$ H.P.; 0·189 ft./sec.^{2}
19. 50 m.p.h. 20. $435\frac{1}{5}$ H.P., $\frac{78}{175}$ ft./sec^2.
21. 500; 490 H.P. 22. $14\frac{2}{3}$, $2032\frac{4}{5}$ ft., $34\frac{8}{13}$ m.p.h.
23. $\frac{v}{375}\left(\frac{2240n}{n}+mx\right)$H.P. 24. 14225 lb. wt.; $1517\frac{1}{3}$ H.P.

Examples XXVI-A. Page 159.

1. 40,00,000 ft. lb. ; 500 hours. 2. 24·8. 3. 8 ft. lb. ; 32 hours.
4. 25 ft. lb. ; 2. 6. $\sqrt{7} : \sqrt{5} : \sqrt{3}$. 7. 1010 lb. wt.
8. $8\frac{3}{8}\frac{2}{9}$ inches. 9. 1020·94 H.P.

Examples XXVI-B. Page 165.

5. (*i*) 3 ft., (*ii*) $8\sqrt{6}$ ft./sec. 8. $\sqrt{\frac{8}{11}(5-3\sqrt{2})ga}$.

Examples XXVII-A. Page 168.

1. 104 cwt. 2. (*i*) 5 lb. wt. (*ii*) 200 lb. wt.
3. (*i*) 9 lb. wt. (*ii*) 204 lb. wt. 4. 13·13 lb. wt.
6. 26·64 lb. wt. : 9·59 lb. wt. 7. 39·39 lb. wt.
8. 5 units ; $15\frac{3}{4}$ lb. wt. 9. $15\frac{1}{2}$ lb. wt.

Examples XXVII-B. Page 173.

1. 4 f.p.s. 2. $3160\frac{1}{3}\frac{0}{2}$ ft. lb.
3. 26·9 m.p.h. in a direction making 21° 48′ with the cross road.
4. 1600 ft./sec., 2020 ft. tons.
5. 4·58 lb. wt. nearly. 7. $1\frac{1}{4}$ ton.

Examples XXVIII. Page 176.

2. 2 ft. $9\frac{3}{4}$ in., 1·68. sec. 4. 4 ft., 4 oz. wt., $\frac{1}{6}\sqrt{15}$ sec.
5. $2v^2/g$, $2mv/g$, $2v/\sqrt{3}$. 7. 58·8 f.p.s., $2\frac{1}{8}$ in.
10. $3971\frac{3}{7}$ lb. wt., $1714\frac{2}{7}$ ft. lb. 12. $772\sqrt{10}$ f.p.s.
16. $\frac{n(n-1)S}{2V}$ sec. ; $\frac{V}{n}$; $\frac{n-1}{n}$.
17. 2035·42 f.p.s. ; 964·58 f.p.s.

REVISION QUESTIONS—VIII. Page 181.

3. 105 sec., $73\frac{1}{3}$ f.p.s., $7\frac{1}{3}$ H.P. 4. 0·21.
12. $\frac{1}{3}\sqrt{V^2-2\mu ga}$.
16. The integer just $> \frac{MV}{mv}$, $\frac{Mm(V+v)^2}{2(M+m)}$; 4, 2·48 ft. lb.

Examples XXX. Page 186.

1. 1 f.p.s., 3 f.p.s. 2. $4\frac{1}{3}$ ft. lb., $\frac{1}{3}$.
3. 32·7 f.p.s., 7·8 f.p.s. ; both reversed ; 47·8 ft. poundals.
5. $(1+2e) : 1$. 8. $\frac{5}{3}$; 4·57 ft. lb.
9. 2 f.p.s. reversed ; $\frac{1}{4}$; $5\frac{5}{8}$ ft. lb. 10. 75 ft. ; 2·79 secs.
13. 123 ft. 14. $\frac{8}{15}/4$. 18. 4 ft./sec. (approx.).
21. $0, \frac{1}{4}d$; $\frac{1}{4}d$, $\frac{1}{16}d$, d being the diameter.

Examples XXXI. Page 192.

1. $5\sqrt{3}$ ft. per sec. at 60° to the line of centres.
2. 5 f.p.s. along the line of centres, 25 f.p.s. in a direction making $\tan^{-1}\frac{4}{3}$ with the line of centres.
4. 4·27 f.p.s. at $\tan^{-1}\frac{5}{4\sqrt{3}}$ to the line of centres ;
2·65 f.p.s. at $\tan^{-1}\frac{5}{\sqrt{3}}$ to the line of centres. 6. $\frac{1}{3}$.

7. The first ball comes to rest, the second moves with e times the original velocity of the first.

Examples XXXII-A. Page 196.

1. 433 lb. ft. sec. ; $\tan^{-1}(2/\sqrt{3})$ with the plane.
2. 30° 3. 30°, 45°. 4. 0·91.
6. $m\sqrt{2gH}(1+\sqrt{H/h})$. 7. 14·4 ft., 3 secs.
8. $\frac{u^2}{g(1-e^2)}$; $\frac{2u(1-e^n)}{g(1-e)}$; $\frac{1}{2}mu^2e^{2n}$.

Examples XXXIV. Page 211

1. 55 f.p.s. at $\tan^{-1}\frac{3}{4}$ with the rear of the train.
2. $\tan^{-1}\frac{11}{4\sqrt{10}}$, 33·5 units nearly. 3. 38 f.p.s. nearly.
4. At 60° with the man's direction of motion.
5. 50 ft./sec. at $\tan^{-1}\frac{3}{4}$ with the railway line.
6. 20 m.p.h. at 120° with the direction of motion of one of them.
7. 28 m.p.h. at $\tan^{-1}\frac{3\sqrt{3}}{13}$ with the direction of motion of the first car.
8. 26·46 m.p.h., at 139° 6′ with the direction of the cyclist.
9. At $\tan^{-1}\frac{5}{2}$ S of W.
10. 13 m.p.h. nearly at $\tan^{-1}\frac{\sqrt{3}}{4}$ north of east.
12. $\tan^{-1}\frac{uv-uw\sin\alpha+vw\cos\alpha}{w^2+uw\cos\alpha-vw\sin\alpha}$.
13. 25·4 m.p.h. from the right rear at 9° 3′ with the direction of motion of the car. 14. 5·66 m.p.h. from $S.E.$
15. From the direction $\tan^{-1}\frac{1}{2}$ E of N.
19. $\frac{1}{8}$ mile. 20. 404 yards (approx.).
21. $\sin^{-1}\frac{2}{3}$ nroth of east ; (*i*) 10 ft./sec. ; (*ii*) $2\frac{1}{2}$ secs.
22. 188·56 m.p.h. ; $\sqrt{10}$ miles.
24. $3v$ or v according as the points are moving in the opposite or in the same direction when they are at the ends of a diameter.
25. 29 ft./sec. nearly. 28. 110·85 f.p.s., perp. to the incline.
29. 28·08. 32. $\frac{1}{2}M\left(\frac{mu}{m+M}\right)^2$, $\frac{Mu}{m+M}$

Examples XXXV. Page. 215.

2. $\sqrt{3}/4$ lb.wt. 3. $g(M+m\cos^2\alpha)$.
4. The wedge must be moved with acceleration $g\tan\alpha$; $mg\sec\alpha$.
11. 11·9 lb. ft. sec. 12. Zero. 2·5 lb. wt., 3·2 ft.
13. Against the direction of motion

Examples XXXVI.. Page 221.

1. 2·5 secs.
3. From the hypotenuse BC, cut off BP equal to the horizontal side BA, and join AP.
4. Measure from the lowest point along the hypotenuse a length equal to the horizontal side.
10. Inclined at 60° with the axis in each case.
11. Inclined at $\cos^{-1}\frac{1}{e\sqrt{2}}$ with major axis ; $e=\frac{1}{\sqrt{2}}$.
12. A circle passing through the lowest point of the given circle and touching it there.

Examples XXXVIII. Page 230.

1. $21\frac{9}{11}$. 2. $1{\cdot}348\times10^7$ ergs. 3. $\frac{7}{75}$. 4. $\frac{7}{1375}$
5. $\frac{11}{40}$ sec., 9·68 ft. 6. $4\frac{4}{9}$ ft., $1\frac{1}{3}$ sec.
7. $\frac{3}{4}$ lb., 10 ft., $\frac{1}{2}$ sec. 8. 6 gm., $1\frac{2}{3}$ cm.
9. 50 lb., 6·4 f.p.s. 10. 600 ft. $7\frac{1}{2}$ sec., 1200 lb.
14. Power.