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MECHANICS OF MACHINERY

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PREFACE TO SECOND EDITION

The first edition of this book was published in September, 1927. In preparing the second edition, the authors not only have profited from their own classroom experience, but have been aided by suggestions from others who have used the book in the past ten years. The second edition is offered as an improvement in content and arrangement of material. The same general sequence that was found so desirable in the first edition has been retained; that is, Mechanism is treated in a separate division under Part I, followed by Kinematics and Dynamics of Machinery under Part II.

The book has been completely reset. Many changes in wording, in arrangement, and in content have been made throughout. Some material has been eliminated, new material has been included, and a large number of problems have been added. The major changes embody the omission of two chapters, Fundamental Concepts and Governors, the addition of one chapter, Gear Tooth Systems and Production Methods, and the grouping of all problems into a separate division at the end of the book. Such material in the chapters omitted as it seemed wise to retain has been included at appropriate places in the other chapters.

A new feature of this edition is the large number of up-to-date problems for drafting-room work, with dimensions, scale, and location on sheet given. These problems have been selected with a view to illustrating, as far as possible, practical applications of the fundamental theory.

This work is not offered in any sense as a complete treatise. The material generally has been limited to that necessary to bring out fundamental principles and methods. In bringing this material up to date, other works on the subject, publications of manufacturers, etc., have been freely consulted. Every attempt has been made to give suitable acknowledgment, where this is in order.

The book has been used by the authors in a five-hour course (three one-hour recitation periods and two three-hour drafting periods per week) during a half year. However, Part I may be given as a two-hour or a three-hour course in Mechanism followed by Part II as a three-hour course in Kinematics and Dynamics of Machinery. It has been the purpose to cover in one textbook sufficient material to give the student an understanding of the fundamentals of both subjects in the time that is ordinarily available in an engineering curriculum. It is believed that the conventional course which includes merely the subject of Mechanism, or Kinematics, with little or no attention given to Dynamics, is inadequate for present-day requirements because of the great variety of new problems that are confronting the engineer as a result of the increased speeds in industrial machinery.

The book presupposes a knowledge of Analytical Mechanics, but Part I may be given without this requirement. The authors have kept in mind the varying requirements of other instructors who may desire to use the book, and the material is so arranged in Part II that, if time is limited to a brief course, the more complex examples and special methods in the latter part of each chapter may be omitted without interfering with the sequence of the work. On the other hand, the material may be adapted to advanced work by application to problems of the type outlined in the Appendix.

The authors wish to express their appreciation to those who have offered comments and helpful criticisms during the preparation of the second edition of the book.

C. W. HAM,
E. J. CRANE.

URBANA, ILLINOIS,
CHICAGO, ILLINOIS,
March, 1938.

CONTENTS

PAGE

PREFACE TO SECOND EDITION v

PART I

Mechanism

CHAPTER I

INTRODUCTORY CONSIDERATIONS 1

Scope of the Subject—Definitions—Machine Motion—Cycle, Period, and Phase of Motion—Continuous, Intermittent, and Reciprocating Motion—Pairing Elements, Pairs—Links—Kinematic Chains—Mechanism, Machine, Structure—Machine Representation, Skeleton Outlines—Expansion of Pairs—Inversions—Point Paths, Displacement Diagrams—General Remarks.

CHAPTER II

LINKWORK 17

The Four-link Mechanism—The Four-bar Mechanism—Dead Points—Four-bar Mechanisms in Combination—The Drag-link Mechanism—The Slider-crank Mechanism—Variations of the Slider-crank Mechanism—Connecting Rod of Infinite Length—Connecting Rod of Finite Length—The Scotch Yoke—Toggle Mechanisms—The Isosceles Linkage—Inversions of the Slider-crank Mechanism—Quick-return Mechanisms—The Oldham Coupling—Reducing Motions—Chamber Trains—Straight-line Mechanisms—Copying Devices, The Pantograph—Applications of Straight-line Mechanisms—Indicator Mechanisms—Parallel Linkages—The Universal Joint—Valve Gears—Steam-engine Valve Gears—Locomotive Valve Gears—Types of Governors—The Gravity-loaded Flyball Governor—The Spring-loaded Flyball Governor—The Shaft Governor—Principle of the Centrifugal Shaft Governor—Principle of the Inertia Shaft Governor—Action of the Two Types of Shaft Governor Compared.

CHAPTER III

CAMS 53

Cam Motions—Fundamental Procedure in a Cam Layout—Displacement Diagrams—Base Curves—Disk Cam with Radial Roller Follower—Disk Cam with Offset Roller Follower—Disk Cam with Reciprocating Flat-faced Follower—Disk Cam with

Page

Pivoted Roller Follower—Disk Cam with Pivoted Flat-faced Follower—Other Types of Cams.

CHAPTER IV

TOOTHED GEARING—SPUR GEARS 78

Nature of Motion Transmitted by Curves in Direct Contact—Angular-velocity Ratio of a Pair of Curves in Direct Contact—Curves That Give Pure Rolling Action—Motion Transmitted by Members in Pure Rolling Contact—Friction Gearing—Toothed Gearing—Definitions—Fundamental Law of Gear-tooth Action—Usual Forms of Gear-tooth Profiles—Construction of the Involute Curve—Involute Teeth Satisfy the Fundamental Law of Gear-tooth Action—Characteristics of Involute Tooth Action—Interference in Involute Gears—Limits of Addendum—Layout of a Pair of Involute Gears—Interchangeability of Involute Gears—Involute Rack and Pinion—Involute Internal Gears—Construction of Cycloidal Curves—Cycloidal Teeth Satisfy the Fundamental Law of Gear-tooth Action—Characteristics of Cycloidal Tooth Action—Layout of a Pair of Cycloidal Gears—Size of Describing Circles—Interchangeable Cycloidal Gears—Cycloidal Rack and Pinion—Internal Cycloidal Gears—Pin Gearing—Comparison of Cycloidal and Involute Tooth Forms.

CHAPTER V

BEVEL, HELICAL, AND WORM GEARING 115

Classification of Bevel Gears—Form of Bevel-gear Teeth—Bevel-gear Nomenclature—Layout of a Pair of Bevel Gears—Spiral Bevel Gears—Skew Bevel Gears—Hypoid Gears—Advantages of Helical Gears—Helical Gears for Parallel Shafts—Helical Gears for Nonparallel, Nonintersecting Shafts—Straight Worm Gearing—Hollow Worm Gearing.

CHAPTER VI

GEAR-TOOTH SYSTEMS AND PRODUCTION METHODS 131

The Brown & Sharpe 14½-deg. Involute System—The Stub-tooth Systems—The Unequal-addendum Systems—American Standard Spur-gear Tooth Forms—Standard Tooth Proportions for Bevel Gears—Standard Proportions for Worm Gearing—Producing Gear Teeth by the Forming Process—Methods of Generating Spur-gear Teeth—Methods of Generating Bevel-gear Teeth—The Grinding of Gear Teeth.

CHAPTER VII

BELTS, ROPES, AND CHAINS 153

Velocity Ratio of Belt Drives—Belt Tensions—Length of Belt—Cone Pulleys—Belt Connections Between Nonparallel Shafts—

PAGE

Crowning of Pulleys—V-belt Drives—Rope Systems—Rope Sheaves—Hoisting Chains—Conveying and Elevating Chains—Power-transmission Chains—Variable-speed Transmissions—The Block and Tackle—The Differential Chain Block.

CHAPTER VIII

INTERMITTENT-MOTION MECHANICS 173

Intermittent Movements—Ratchet Mechanisms—Escapements—Intermittent Gearing—Indexing Devices—Reversing Mechanisms—Clutches and Clutch Mechanisms—Brakes.

CHAPTER IX

TRAINS OF MECHANISM 191

Typical Examples—Gear Trains—Velocity Ratio in a Gear Train—Value of a Gear Train—Idler Gear and Direction of Rotation—Gear Trains with Nonparallel Axes—Selection of Gears for a Train—Speed Changing Gear Trains—Change Gears—Epicyclic-gear Trains—Speed Reducers.

PART II

Kinematics and Dynamics of Machinery

CHAPTER X

VELOCITIES IN MACHINES 216

Vector Quantities—Motion—Position—Motion of a Point—Combined Motions—Relative Motion—Angular Motion—Relative Angular Velocity of Two Rigid Bodies—Displacement of a Rigid Body—Fundamentals of the Method of Relative Velocities—Applications of the Method of Relative Velocities—Points on the Same Link—Application to a Simple Mechanism—The Velocity Polygon—Application to Cams and Gears—Application to a Complex Mechanism—Methods of Attack—Practical Considerations—Velocity Analysis of a Crank Shaper—Instantaneous Center of Rotation—Instantaneous Center of a Pair of Links—Location of the Instantaneous Centers in a Complete Mechanism—The Law of Three Centers—Application of the Law of Three Centers—Application of the Instantaneous Center to the Solution of Velocity Problems—Application to Slider-crank Mechanism—Application Involving Cams or Gears—Application to a Complex Mechanism—Application to Epicyclic-gear Trains—Centrodes—Special Methods—Analytical Methods.

CHAPTER XI

ACCELERATIONS IN MACHINES 268

Acceleration of a Point—Relative Acceleration of Independent Points—Relative Acceleration of Two Points on the Same Rigid

PAGE

Body—The Acceleration Polygon—Accelerations in the Four-link Mechanism—Application to Slider-crank Mechanism—Scale Relations—Four-link Mechanism—Complete Graphical Method—Slider-crank Mechanism—The Ritterhaus Construction—Relative Acceleration of a Pair of Coincident Points—Coriolis' Law—Applications—The Governor Mechanism—The Slider-crank Mechanism—Shaper Mechanism—Analytical Methods.

CHAPTER XII

STATIC FORCES IN MACHINES 306

Reaction Between Members with Friction Disregarded—Analysis of a Bell Crank—Steam-engine Mechanism—Drag-link Mechanism—Friction—Sliding Friction—Friction Analysis of a Bell Crank—Efficiency—Effect of Friction in the Engine Mechanism—Stone Crusher—Cam Friction—Spur Gears—Screw Threads—Rolling Friction—Friction of Wrapping Connectors—Forces in Parallel Planes.

CHAPTER XIII

INERTIA FORCES 329

Inertia Forces of a Rigid Body Having Plane Motion—Inertia Forces of a Floating Link—Inertia Forces of a Link Rotating about a Fixed Center—Transverse and Radial Components—Four-link Mechanism—Combined Static- and Inertia-force Analysis—Kinetically Equivalent System—Application to Engine Mechanism—Engine Mechanism—Special Methods—Inertia Forces in a Cam Mechanism—Analytical Methods.

CHAPTER XIV

BALANCING OF MACHINERY 361

Balancing of Rotating Masses—Effect of a Single Rotating Mass—Effect of Two Rotating Masses Not in the Same Plane of Rotation—Several Rotating Masses in a Single Transverse Plane—Several Rotating Masses in Different Transverse Planes—General Graphical Method of Balancing Any Number of Rotating Masses—Reduction of Masses to a Common Radius—Analytical Method of Balancing a System of Rotating Masses—The Balancing of Reciprocating Masses—Inertia Effects of the Reciprocating Masses in the Engine Mechanism—Inertia Effects of Crank and Connecting Rod—The Single-cylinder Engine—Counterbalancing—Conditions of Balance in a Multicylinder Engine—The Eight Equations of Balance—The Two-cylinder Engine—Three-cylinder Engine—The Four-cylinder Automobile-type Engine—The Six-cylinder Automobile-type Engine—The Eight-cylinder Automobile-type Engine—The Twelve-cylinder Automobile-type Engine—Balancing of the Crankshaft—Balancing the Connecting Rod—Vibrations Due to Impulse and to Fluctuating Pressure of the Working

PAGE

Fluid—Vibrations of Foundations and Supports—Critical Speeds—Balancing Machines—Description of a Balancing Machine—Balancing Rotors by Means of Electrical Networks.

APPENDIX

FORCE ANALYSIS OF A GASOLINE ENGINE 413

Indicator Diagram—Inertia Forces—Combined or Resultant Forces—Indicator Diagram—Diagram of Piston Forces—Side Pressure of the Piston—Turning-effort Diagrams for One Cylinder—Combined Turning-effort Diagrams—Pin Pressures—Diagram of Shaking Forces.

PROBLEMS 429

INDEX 471

MECHANICS OF MACHINERY

PART I
MECHANISM

CHAPTER I

INTRODUCTORY CONSIDERATIONS

Engineering is based on the sciences of *mathematics*, *physics*, and *chemistry*. These three sciences are more or less interdependent, this fact being particularly true of mathematics and physics. *Mechanics* is one of the several branches of physics. It deals with the relations that exist between masses, motions, and forces and is usually considered under three main divisions, namely, *kinematics*, *statics*, and *kinetics*. The first division deals with motion alone, the second with forces considered apart from motion, and the third with forces in conjunction with motion. Since both statics and kinetics deal with the action of forces on bodies, that part of mechanics embraced in these two subdivisions of the subject is sometimes called *dynamics*.

1. Scope of the Subject.—*Mechanics of machinery* deals with the study of masses, motions, and forces in machines. The subject is restricted so as not to include the study of elastic forces and deformations in machine parts, these lying in the province of *mechanics of materials;* neither does the subject include the determination of the sizes and shapes of machine parts necessary for carrying the loads and transmitting the forces imposed upon them, this lying in the province of *machine design*.

The subject of mechanics of machinery as presented in this book will be considered under two main divisions, as follows: Part I, Mechanism; and Part II, Kinematics and Dynamics of Machinery.

2. Definitions.—A *machine* is a combination of rigid or resistant bodies, so formed and connected that they move upon each

other with definite, relative motions and transmit force from the source of power to the resistance to be overcome. Within itself the machine thus has the two functions of transmitting definite relative motion and of transmitting force. When attention is concentrated upon the first of these—strength and rigidity needed for transmitting force being assumed—the term *mechanism* is applied to the combination of geometrical bodies constituting the machine or part of the machine. A *mechanism* may therefore be defined as a combination of rigid or resistant bodies, so formed and connected that they move upon each other with definite relative motions. The term *mechanism*, used in a broad sense as in the title of Part I of this book, is concerned primarily with a study of the functions, geometrical properties, and relative motions in the various mechanisms in common use. The study of motions in a broader sense—to include velocities and accelerations—is reserved for Part II of the text. Although the general scheme is *mechanism* first, as defined above, and then *kinematics and dynamics of machinery*, the treatment of the separate divisions is not exclusive, elementary principles of mechanics being introduced in Part I as the occasion demands.

Kinematics of machinery treats of the motions of machine parts without considering the manner in which the influencing factors (force and mass) affect the motion. It deals with the fundamental concepts of space and time, and the quantities, velocity, and acceleration derived therefrom. *Dynamics of machinery*, embracing the two subdivisions *statics of machinery* and *kinetics of machinery*, treats of machine parts that are acted upon by both balanced and unbalanced forces, taking into account the masses and accelerations of the parts as well as the external forces.

3. Machine Motion.—The motion of a machine member is usually such that all its points move in parallel planes. This type of motion is called plane motion. The motion of the driving member of most machines is one of two simple kinds of plane motion, namely, rotation or translation. The same statement is true for the motion of the final driven member which overcomes the useful resistance. The motions of intermediate members may, on the other hand, be fairly complex.

Although the majority of motions which have to be dealt with in machines are plane motions, there are two other types, helical

motion and spherical motion, which are of importance. A screw advancing through a threaded hole is an example of helical motion. It is of interest to note that helical motion is a combination of rotation about an axis and a translation along that axis. When a body has spherical motion, all points in the body move in the surfaces of spheres having a fixed point for a common center, as in the case of the flyball governor, the universal joint, ball-and-socket joint, etc.

A body that has no material connection with other bodies, so that the path of its motion varies with the external forces acting upon it, is said to have *free motion.* The planets, for example, have free motion. A body that has a material connection with another body, so that the path of its motion relative to that body must be of a definite character, regardless of the external forces that may act upon it, is said to have *constrained motion.* The crankpin of an engine, for example, has constrained motion. All machine members are dependent upon constraint for their proper action.

4. Cycle, Period, and Phase of Motion.—When, after starting from some simultaneous set of relative positions, the parts of a machine have passed through all the possible positions they could assume and have returned to their original relative positions, they are said to have completed a *cycle* of motion. The time required for a cycle is called a *period.* The simultaneous relative positions occupied by the parts of a machine at any instant during the cycle constitute a *phase.*

In the ordinary steam engine (Fig. 1) the energy cycle and also the motion cycle correspond to one revolution of the crankshaft. In the ordinary gasoline engine, however, each energy cycle requires two revolutions of the crankshaft or four strokes of the piston: the intake, compression, working, and exhaust strokes. Hence, for each energy cycle of the engine, the main parts, namely, crank, connecting rod, and piston, complete two motion cycles. The camshaft and valves, however, complete only one motion cycle during this period, so that considering the engine as a whole, the moving parts return to their initial relative positions only after two revolutions of the crankshaft or four strokes of the piston. For this reason gasoline engines of this type are referred to as four-stroke-cycle engines, or more simply as four-cycle engines.

5. Continuous, Intermittent, and Reciprocating Motion.—The motion of a machine member is *continuous* if during each successive cycle it neither stops nor reverses; *intermittent* if during each cycle it stops for a finite interval of time; and *reciprocating* if during each cycle its motion is reversed. For example, the crankshaft of the gasoline engine has continuous motion, the valve tappet rods an intermittent reciprocating motion, and the piston a purely reciprocating motion.

The rotation of a member may be continuous, intermittent, or reciprocating, the last being more commonly designated as oscillating. The translation of a body, which may be either rectilinear or curvilinear, also may be continuous, intermittent, or reciprocating, with the exception that, obviously, rectilinear translation cannot be continuous. The motion of the side rods of a locomotive is a good example of curvilinear translation. When the term translation is used without qualification it is understood to mean rectilinear translation.

6. Pairing Elements. Pairs.—The resistant parts of a machine must be so placed and so connected to adjacent members

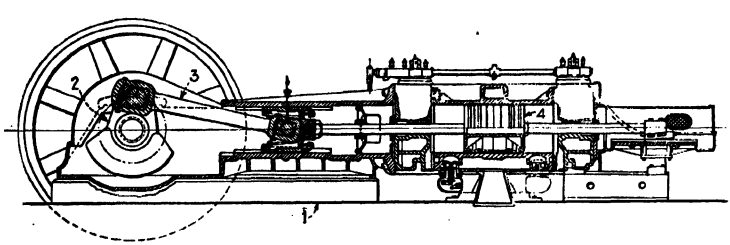


FIG. 1.

that the relative motion between any two is of a definite character. The geometrical forms placed upon two bodies so that they may be connected in this manner are called *pairing elements*. Two elements thus joined constitute what is called a *pair*, and the process of connecting them is called *pairing*. The cylindrical surface of the crosshead pin of an engine (Fig. 1) and the inside cylindrical surface of the connecting-rod end are examples of pairing elements. When joined together, they form a pair or what is commonly known as a joint. Pairs which permit surface contact are called *lower* pairs; those which permit only point or line contact are called *higher* pairs. The most common

examples of lower pairing are the turning pair and the sliding pair. The turning pair may be either a journal and its bearing or the ordinary pin-and-eye connection between two links. The sliding pair is exemplified by the crosshead and guides of the ordinary steam engine, as shown in Fig. 1. Examples of higher pairs are ball and roller bearings, the contact surfaces of a cam and its follower, and the contact surfaces of a pair of mating gear teeth. These examples are all so familiar as to require no explanatory illustrations. In general, a lower pair completely constrains the motion of the pair of links connected, whereas in the case of a higher pair the constraint is generally incomplete.

It frequently occurs that a single lower pair may have several contact surfaces. An example of a turning pair with multiple contact is the engine crankshaft and its bearings. An example of a sliding pair with multiple contact is the piston link of a steam engine, consisting of piston, piston rod, and crosshead, with sliding contact between piston and cylinder, piston rod and stuffing box, and crosshead and guides. This example is clearly illustrated in Fig. 1. Thus a single sliding pair may have any number of contact surfaces, provided these surfaces are parallel, and a single turning pair may have any number of journals and bearings, provided that they all have the same axis.

7. Links.—A *link* may be defined as a rigid body having two or more pairing elements by means of which it may be connected to other bodies for the purpose of transmitting force or motion. A link is sometimes defined in a broader sense to include bands and fluid connectors. In ordinary practice, however, the name is applied to a rigid connector, and it is in this sense that the term will be used in this work. A link with but two pairing elements is called a *simple* link. If it has more than two pairing elements, it is called a *compound* link.

In any machine there is one link that either occupies a fixed position relative to the earth or carries the machine as a whole along with it during motion. This link is called the *fixed* link. As as example showing the use of the term *link*, let the steam engine shown in Fig. 1 be considered. Taking into account only the main members of the machine, it is seen that the number of links is four. Link 1 is the fixed link and includes the frame and all other stationary parts such as the cylinder, crosshead guides, crankshaft bearings, etc. Link 2 includes the crank, crank-

shaft, flywheel, etc., all of which have a motion of rotation about the axis of the main bearing. Link 3 is the connecting rod and, being an intermediate link, has, as might be expected, a fairly complex motion. Link 4 includes the crosshead, piston and piston rod, etc., and has a motion of reciprocating translation. It is evident, from the examples discussed above, that a link includes all the auxiliary parts that are rigidly fastened to the main part and move with it as a unit.

8. Kinematic Chains.—When a number of links are connected by means of pairs, the resulting system is called a *kinematic chain*, or simply a chain. A chain may be *locked*, *constrained*, or *unconstrained*. If a chain is locked, no relative motion is possible between the links, and any motion that takes place is that of the chain as a whole. A locked chain is shown at *a* in Fig. 2. If a chain is constrained, definite relative motion between the links

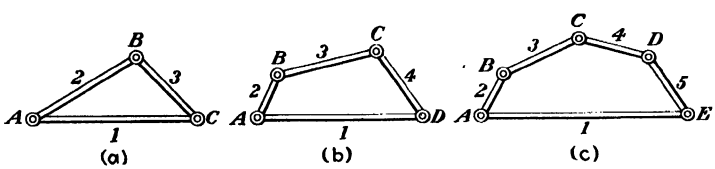


FIG. 2.

is possible; *i.e.*, if one of the links is held fixed and another put in motion, all points of the remaining links will move in certain definite paths and will always move in these same paths no matter how many times the motion is repeated. A constrained chain is shown at *b* in Fig. 2. If a chain is unconstrained, the relative motions of the links are not definite; *i.e.*, if one link is held fixed and the motion of another link is repeated, the points of the remaining links will not, in general, follow the same paths. An unconstrained chain is shown at *c* in Fig. 2.

The locked chain has its application in structures. The constrained chain is the one with which kinematics is chiefly concerned and is the basis of all machines. The unconstrained chain is of little use practically.

9. Mechanism. Machine. Structure.—If one of the links of a constrained kinematic chain is made a fixed link, the result is a *mechanism*. If a different link of the same mechanism is made the fixed link, the result is a different mechanism. Evidently

there are as many different mechanisms for the same chain as there are links in the chain. Some confusion may arise as to the distinction between the term *mechanism* and the term *machine*, both of which are fundamentally a constrained chain with one link fixed. As stated in Art. 2, the primary function of a mechanism is to transmit or modify motion, whereas that of a machine is to modify energy and do work. If, therefore, the chain is considered solely from the viewpoint of motion modified or transmitted, it should be referred to as a mechanism. If, on the other hand, the chain is considered as an agent for modifying energy or doing useful work, it should be thought of and referred to as a machine. There is a large class of instruments or apparatus, such as clockwork, adding machines, typewriters, engineers' instruments, etc., that may seem to be on the border line between mechanism and machine, according to the above definitions. But they are more properly classified as mechanisms, since the force is no greater than necessary to produce relative movement. In other words, change of position of members is the useful effect, and there is no performance of external work. When a kinematic chain is analyzed as a mechanism, no special consideration need be given to the forms or proportions of the links except in so far as concerns the location of the pairing elements. As a machine, however, the requirements of strength, stiffness, clearances, etc., make it imperative to consider the links in all their details.

The leading distinction between a *machine* and a *structure* is that the former serves to modify and transmit energy, or force and motion, while the latter modifies and transmits force only.

10. Machine Representation. Skeleton Outlines.—For the purpose of kinematic analysis, a mechanism may be represented in an abbreviated or skeleton form called the *skeleton outline* of the mechanism. The skeleton outline should give all the geometrical information necessary for determining the relative motions of the links. In Fig. 3 the skeleton outline has been drawn for the steam engine shown in Fig. 1. It will be observed that in this skeleton outline everything necessary for determining the relative motions of the main links has been shown, namely, the length AB of the crank; the length BC of the connecting rod; A, the location of the axis of the main bearing; and the right line path AC of point C, which represents the wrist-pin axis.

When drawing the skeleton outline, a link that has only two pairing elements may be represented by a straight line joining the axes of the two elements, as shown by link 2 in Fig. 3 for example. The turning pairs themselves may be represented by small circles with dots at their centers, as shown at *A*, *B*, and *C*. The fixed link need not be shown except in the vicinity of its pairing elements, where crosshatching may be used, as shown in Fig. 3, as a conventional way of indicating that it is the fixed link. A sliding link may be represented in the conventional manner shown at *C* in the figure, *i.e.*, by a rectangle pivoted on the axis of the pairing element of the link with which it has a turning connection. A simple method of representing a turning connection with the fixed link is that shown at *A* in the figure. That shown at O_2 or O_4 in Fig. 15 may be used, if preferred. A link

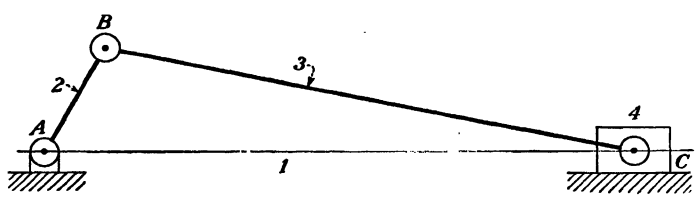


FIG. 3.

with three or more pairing elements is represented by the geometric figure formed by connecting the axes of the pairing elements, as shown by link 3 in Fig. 15, for example. An example of the skeleton outline of a more complex mechanism is illustrated by the shaper mechanism in Fig. 4.

In this mechanism, gear 2 is driven by a pinion on the shaft of the drive pulley (not shown) and carries a pin *A* on which is pivoted the block 3. Through the sliding action of the block in the slot in the vibrator (link 4) and the rotation of gear 2, link 4 is caused to oscillate. Since the ram, link 6, is so constrained as to have only reciprocating motion in a horizontal line, whereas the end of link 4 moves in the arc of a circle as it oscillates, it becomes necessary to have link 5 interposed between links 4 and 6. It is seen that link 2, the main driving link, is in effect a crank, and is so represented in the skeleton outline shown at *b*. It is also evident that link 6 is completely represented in skeleton outline by a rectangle pivoted on the end of link 5 and guided in a horizontal direction by link 1. The simplest representation of

the action between links 3 and 4 is a rectangle sliding on a line as shown.

In the case of links containing higher pairs, it is usually impossible to make much reduction from the actual form of the link

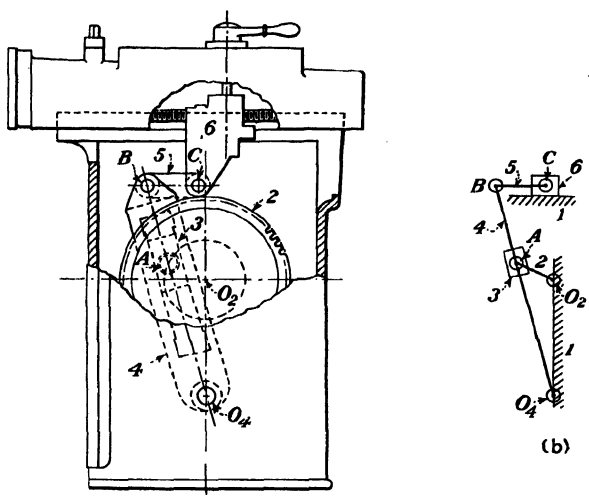


FIG. 4.

when drawing the skeleton outline; this is particularly true in the case of certain types of cams. Two spur gears in mesh may, nevertheless, be represented by their pitch circles as shown in Fig. 5. Sometimes the existence of higher pairing in a mechanism may be disregarded, as for example, in a mechanism in which a ball bearing is used in place of the ordinary pin-and-eye joint. The ball bearing is distinctly a case of higher pairing in so far as concerns its elements, but it acts as a whole exactly like an ordinary turning joint.

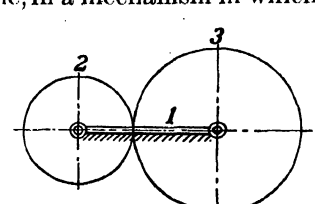


FIG. 5.

11. Expansion of Pairs.—The size of the pairing elements has no effect on the relative motion of the links connected by the pair. Sometimes by simply expanding pairing elements the appearance of a machine may be changed beyond recognition without altering in the slightest degree the character of the motion.

Thus it is often the case that mechanisms differing widely in appearance have the same or similar skeleton outlines, as will be made clear by consideration of the mechanisms shown in Figs. 6 to 11.

In each of these figures the skeleton outline has been superimposed on the drawing of the actual mechanism at the place where the expansion has been made. An inspection of the

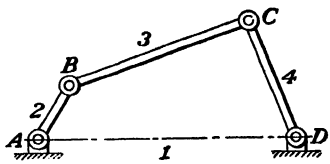


FIG. 6.

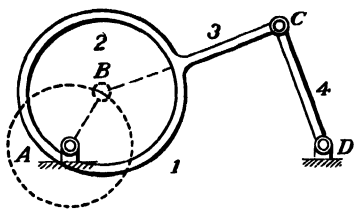


FIG. 7.

several skeleton outlines will make it evident that they are identical, thus showing that the mechanisms are all the same from a kinematic point of view. The differences in appearance are chiefly due to modifications in the detail (but not in the character) of the pairs.

With the fundamental mechanism in Fig. 6 as a starting point, link AB was replaced in Fig. 7 by an eccentric disk by the simple

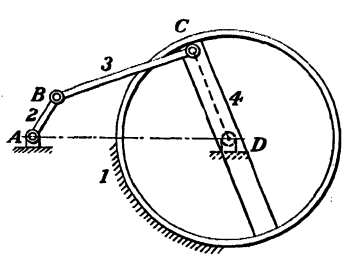


FIG. 8.

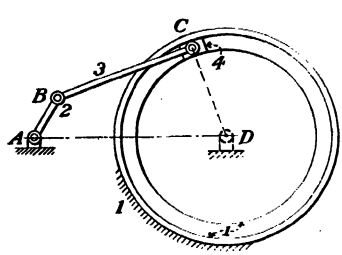


FIG. 9.

process of enlarging or expanding the pairing elements at B to include pair A. In Fig. 6 the pairing elements were represented by a small circle with the center at B. In Fig. 7 they are represented by the enlarged circle (the circumference of the disk) with the position of center B unchanged. The relative positions or motions of the two links affected, 2 and 3, have, therefore, suffered no change whatever. It is evident from a comparison of

Figs. 6 and 7 that an eccentric is equivalent to a crank whose length is equal to the eccentricity.

In Fig. 8 the pair D has been enlarged to include C. Evidently the motion of DC will be equally constrained if the sides of the circular disk are cut away, leaving the zone indicated by the solid lines. A further modification is shown in Fig. 9, where the link CD is given the form of an annular ring instead of a disk. The next step in simplification is to cut away all the ring except a small block containing the pair C. Finally, since by the rotation of crank AB the pair C is forced to reciprocate in a limited circular arc, the slot in which the block moves may as well be shortened to the length actually required as shown in Fig. 10. By comparing Fig. 10 with Fig. 6, it is seen that the

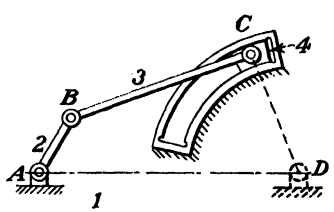


FIG. 10.

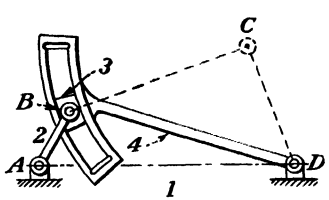


FIG. 11.

sliding pair C in Fig. 10 is equivalent to a turning pair whose center of curvature is the center of curvature D of the slot. It is also evident that the block containing the point C is equivalent to a bar of length CD equal to the radius of curvature of the slot and pivoted at the center of curvature D. By increasing the radius of curvature of the slot, the equivalent of very long links may be obtained without increasing the dimensions of the machine. Thus the ordinary sliding pair, consisting of a block moving in a straight slot, is equivalent to a turning pair whose axis is at an infinite distance.

By expanding pair C in a series of modifications similar to those described above, the mechanism shown in Fig. 11 has been derived from Fig. 6, the relative motions of the links remaining unchanged. The solution is left to the student.

12. Inversions.—*Inversion* is a term used in kinematics for a sort of reversal or interchange of form or function, as applied to pairs, kinematic chains, and mechanisms. The process of interchanging the hollow and solid elements of a pair is called

inversion of the pair. For example, instead of having a pin in the crosshead of an engine fitting into an eye in the connecting rod, the rod might contain the pin, the latter fitting into a hole in the crosshead. It should be noted that this interchange of elements in no way changes the character of the relative or absolute motions.

It was shown in Art. 9 that from a given kinematic chain as many mechanisms may be derived as the chain has links, by

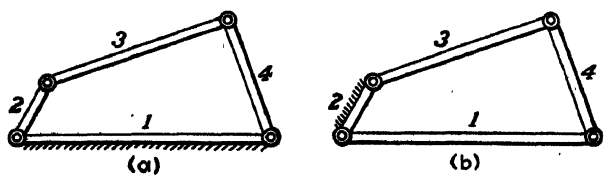


FIG. 12.

fixing each link in turn. These mechanisms are called *inversions* of the original mechanism, and, like the case of the inversion of pairs, the exchange of one fixed link for another is known as *inversion of the mechanism.*

If, for example, link 2 of the four-link mechanism shown at *a* in Fig. 12 is fixed and link 1 made free to move, the resulting mechanism, shown at *b*, is said to be an inversion of the original

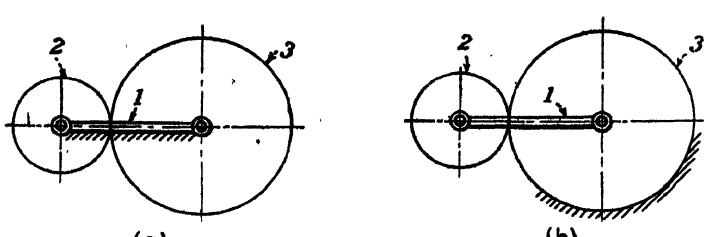


FIG. 13.

mechanism. It is important to keep in mind that the inversion of a mechanism does not change the motions of its links relative to each other, but does change their absolute motions. As another example of the inversion of a mechanism, consider the gear train at *a* in Fig. 13. The epicyclic gear train at *b* is an inversion of the original train at *a*. In making an inversion of a mechanism it may often be necessary to make an inversion of a pair or change proportions of links to suit requirements.

Inversion of function may be noted in the case of such machine tools as the shaper and planer, lathe and boring mill, etc. Whereas in the shaper the work is held stationary while the cutting tool moves, in the planer the work moves and the cutting tool is stationary. Similarly, in drilling and boring on a lathe the tool is stationary, while in a drilling machine the work is stationary. Functional inversion occurs in a machine, also, when reversal of motion causes the original driving member to become the driven member.

13. Point Paths—Displacement Diagrams.—In the kinematic study of mechanisms, it is frequently desirable to plot the paths of the motion of certain points and also to construct a diagram of the displacement of some point corresponding to the motion or displacement of some other point. In most cases the points considered are those that represent the axes of the moving pairs. In constructing a displacement diagram, the usual scheme is to plot displacements of some point on the last driven link against displacements of some point on the first driving link, employing any system of coordinates that may be found convenient. Usually rectangular coordinates are the most convenient to use. Taking first a simple illustration, in the engine mechanism (Fig. 14*a*), it is known that one end A of the connecting rod describes a circle and that the other end B describes a straight line. By assuming, say, 12 equally spaced positions of the crankpin A in the complete cycle of its motion, a corresponding number of positions of any other point such as C on the rod may be plotted, giving the elliptical curve shown in the figure. In a similar manner a diagram of the displacements of the piston B corresponding to the displacements of crankpin A may be plotted. In Fig. 14*b* displacements of the crankpin A, or angular displacements of the crank, have been plotted as abscissas, and piston displacements as ordinates. It is important to note that the length of the base line, representing abscissas, has been taken as an arbitrary length and not equal to the circumference of the crank circle, as might have been expected. Since the information sought is piston displacements corresponding to equally spaced crank positions, it is evident that in such a diagram any convenient length may be chosen for the base line. The use that is made of this particular diagram will become evident after studying Chap. II.

As a further example of problems in plotted movements, let it be required to plot the paths of the motion of B and C and the displacement diagram of piston D for the complete cycle of the Atkinson gas-engine mechanism, shown in skeleton outline in Fig. 15*a*. This type of engine is now obsolete, but the mechanism is ingenious, and therefore of kinematic interest. This was a four-cycle engine and was designed to get the four strokes of the cycle into one revolution of the crank O_2A, with longer strokes for

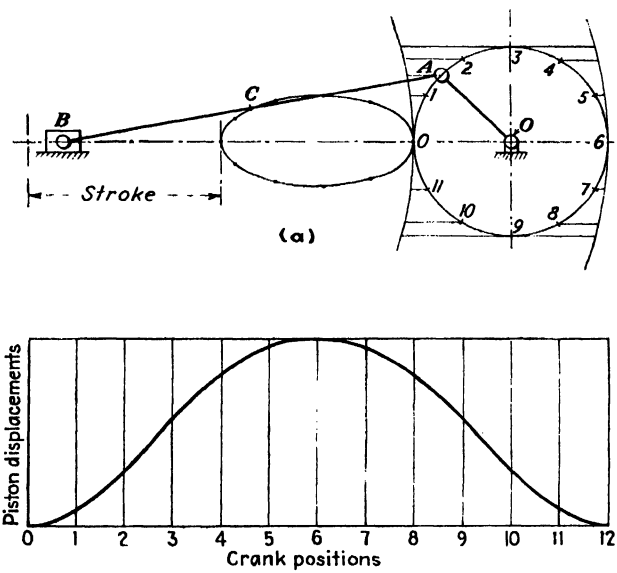
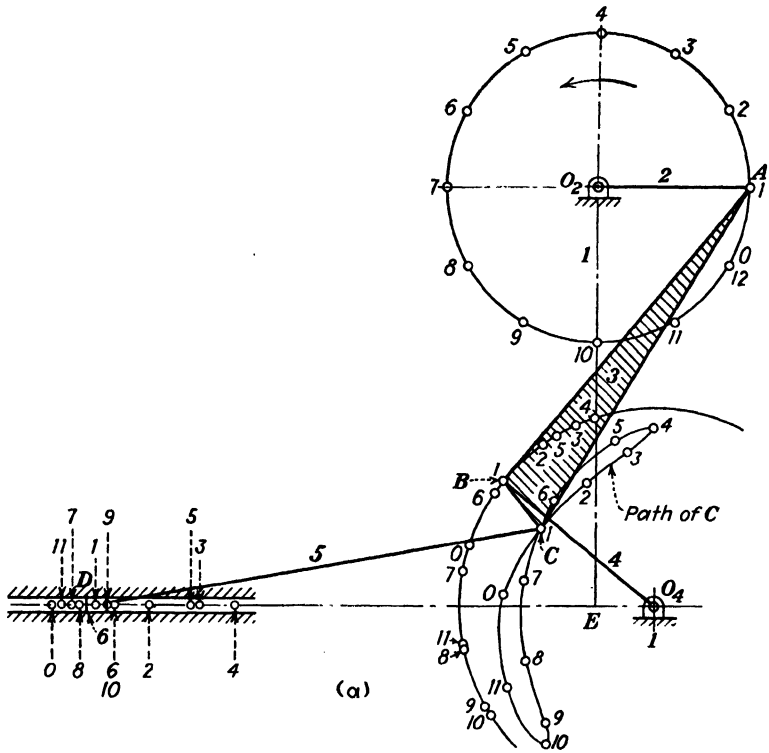
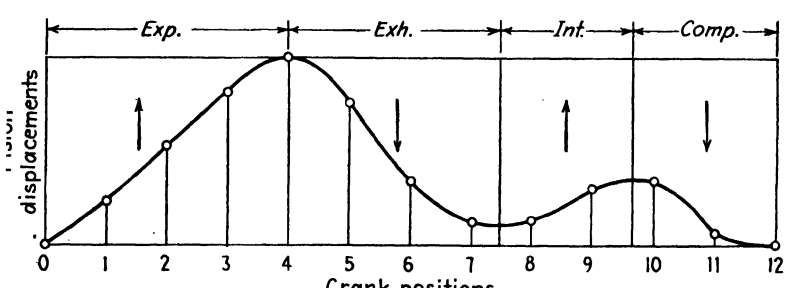


FIG. 14.

expansion and exhaust than for intake and compression. Just how well this end was attained may be seen by examining the piston-displacement diagram at Fig. 15*b*. This diagram was obtained by determining the piston displacements corresponding to the 12 crank positions indicated and plotting these displacements as ordinates on the arbitrarily chosen length of base line 0–12. The plotting of point paths of B and C is an incidental part of the work of obtaining the piston displacements. This mechanism affords an excellent example of the value of such a





DISPLACEMENT DIAGRAM FOR *D*

(b)

FIG. 15.

diagram as shown at Fig. 15*b*, as an aid in studying machine motions.

14. General Remarks.—All mechanisms derived from one kinematic chain have been shown to be inversions of that chain. These inversions frequently differ very widely in general appearance and in purpose. For example, the ordinary steam-engine mechanism and the Whitworth quick-return mechanism (Fig. 37*b* and Fig. 39*b*, Chap. II) are kinematically identical, though their general appearance would not make this fact apparent.

The *motions* in a machine depend entirely on the relative positions of the pairs and not at all upon the size and shape of the links. The *appearance* of a machine depends largely on the size and shape of its component links. Links are of diverse shapes; a crank and eccentric, for example, which are kinematically identical but quite different in appearance, are shown in Figs. 6 and 7.

In the study of mechanics of machinery it is necessary to eliminate from consideration problems of the design and manufacture of machine parts and confine attention largely to the skeleton outlines of the moving parts, since the motions of the parts are of chief concern. Once the kinematic relations between the links of a kinematic chain are known, those for all mechanisms obtained by inversions or expansions are also known. This fact is of great importance, for no matter which link is fixed, the relative angular motions of the constituent links of a kinematic chain are always the same. The study of kinematics of machinery is, therefore, greatly simplified and needless repetition avoided if the fundamental kinematic chains are studied in place of the many mechanisms which are kinematically identical.

Furthermore, the study of the subject may be approached from two different points of view, namely, the analysis of forces and motions in existing machines, and the creation of machines to produce desired motions and forces. The usual procedure is to study and analyze existing machines, and that method will be followed in this book.

CHAPTER II

LINKWORK

In Art. 7 of Chap. I a link was defined as a rigid body having two or more pairing elements by which it could be connected with other bodies for the purpose of transmitting force or motion. The term *linkage* refers to a mechanism made up of such links as cranks, levers, rods, etc., with turning and sliding pairs, thus excluding from the classification cams, gears, belts, ropes, and chains, etc.

The function of a link mechanism is to produce rotating, oscillating, or reciprocating motion from the rotation of a crank, or the reverse. Stated more specifically, linkwork may be used to convert:

1. Continuous rotation into continuous rotation, with a constant or variable angular velocity ratio.
2. Continuous rotation into oscillation or reciprocation (or the reverse), with a constant or variable velocity ratio.
3. Oscillation into oscillation, or reciprocation into reciprocation with a constant or variable velocity ratio.

15. The Four-link Mechanism.—One of the simplest examples of constrained linkwork is the *four-link* mechanism. Many of the commoner mechanisms may be formed from the four-link mechanism by means of slight modifications such as changing the character of the pairs, proportions of the links, etc. Furthermore, it may be said that any link mechanism may be analyzed as a four-link mechanism or a combination of two or more such mechanisms. A great majority of four-link mechanisms fall into the two following classes: (1) the four-bar mechanism, and (2) the slider-crank mechanism.

THE FOUR-BAR MECHANISM

16. The Four-bar Mechanism.—This linkage, in its simplest form, has four bar-shaped links and four turning pairs, as shown in Figs. 16 and 17, which represent *open* and *crossed* linkages

respectively. The fixed link may actually be bar-shaped, but more frequently it represents the frame of a machine and in that case is usually a massive casting of irregular shape.

In order to facilitate discussion, one of the rotating members of the four-bar mechanism, usually the driver, will be called the crank, or driver, and the other will be called the rocker or follower. The floating link that connects the crank and the rocker will be called the connecting rod, and the fixed link will be called the frame.

In some of the applications of the four-bar mechanism, the crank has a motion of oscillation, whereas in other applications it must be capable of making a complete rotation. The same statement may be applied to the follower. In cases of less

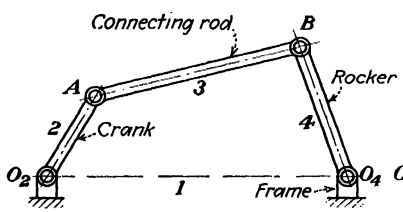


FIG. 16.

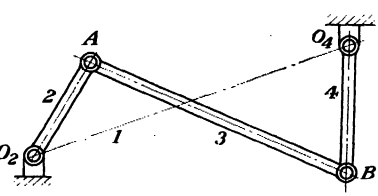


FIG. 17.

frequent occurrence, both the driver and follower must make complete rotation. In order to meet the requirements of certain types of motion, it is necessary to hold the relative lengths of the links within certain well-defined limits. In proportioning the links, consideration must also be given to the avoidance or the neutralization of dead points in the motion of the driven link.

17. Dead Points.—Referring to Fig. 18, let 2 be the driving link and 4 the driven link. The lengths are such that neither of the links may make a complete rotation. Link 2 may oscillate between the extreme positions 2‴ and 2⁗, and link 4 oscillates between the extreme positions 4′ and 4″. The position 4‴ of the follower is called a dead-point position, for it is obvious that any driving force transmitted from the driver by the connecting rod 3 will be radial and hence can have no effect on the turning of the link 4. In practical applications of such a linkage, the range of action required is usually much less than the maximum. This is a desirable condition, since the action is not smooth when the follower is near a dead point.

In Fig. 19 is shown a four-link mechanism in which the driver may make a complete rotation, and the follower 4 oscillates from 4′ to 4″. It can be seen that if link 4 should become the driver there would be two dead points in the cycle. When link 4 is the driver and complete rotation of the crank 2 is required, it becomes necessary to provide external means, as for example, a flywheel to carry the crank 2 past the dead points.

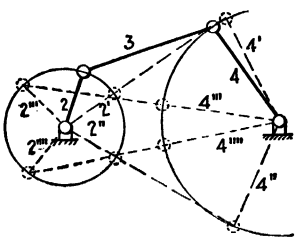


Fig. 18.

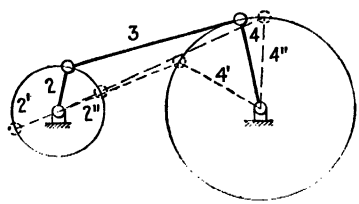


Fig. 19.

18. Four-bar Mechanisms in Combination.—With suitable proportioning and proper choice of relative positions of the links the four-bar mechanism may be made to produce a great variety of relative motions of driver and follower. The arrangement shown in Fig. 20 is just one example of the many possibilities. If the driver (link 2) is turned clockwise with uniform motion through the angle θ_2, the follower (link 4) will turn in the same sense through the much smaller angle θ_4 and with decreasing speed, which will become zero when link 2 reaches the position 2′. Reversing the motion of link 2 will cause link 4 to return toward its initial position, its motion being slow at first and then gradually increasing. This type of motion is used in the Corliss valve gear on steam engines to secure quick opening and closing of the valves. In this particular application neither of the links 2 or 4 makes a complete rotation but oscillate through comparatively small angles, generally not exceeding these shown as θ_2 and θ_4 in Fig. 20.

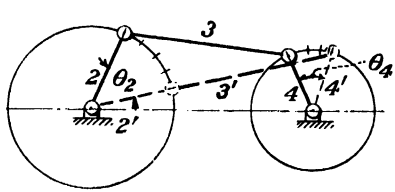


Fig. 20.

Many interesting arrangements of the four-bar mechanism that have numerous practical applications other than where

used in combination could be discussed, but since space does not permit their illustration here only a few that appear elsewhere in the text will be pointed out.

When links 2 and 4 and 1 and 3, Fig. 20, are equal, forming a parallelogram, the driver and follower have simultaneous dead points, and if the driver makes a complete rotation special means must be resorted to for complete constraint of the follower. The parallel rule, Fig. 51, represents one application in which the links 2 and 4 oscillate through a limited angle.

The four-bar linkage is the basis for a number of very simple straight-line mechanisms, such as for, example, the Watt mechanism illustrated in Fig. 45, in which the point P is constrained to move in a straight line. A somewhat different arrangement is that shown in Fig. 357 where by choosing suitable proportions and expanding link 3 into a triangular form, the point C is constrained to move in a straight line. The Roberts straight-line mechanism, Fig. 377, in which the point C is constrained to move in a straight line is an interesting example showing how a few simple changes converted the foregoing arrangement into an entirely different mechanism.

The main driving links of a Corliss valve gear are shown in skeleton outline in Fig. 21. The wrist plate 6 is given an oscillatory motion by means of the complete rotation of driving link 2, which, in the actual mechanism, is in the form of an eccentric. The linkage 1, 6, 13, 14, actuating one of the exhaust valves, will give to the link 14 a very slow motion when M is near M_1, *i.e.*, when the valve is closed, but between M and M_2, when the valve is opening or closing, the motion is much faster. The same statement applies in the case of the admission valves, as shown by the linkage 1, 6, 7, 8.

It is interesting to note that the main part of the valve gear represented in Fig. 21 is made up wholly of four-bar mechanisms, six in number, and also that it furnishes examples of four distinct types of these mechanisms. Unit 1, 2, 3, 4 has in eccentric 2 a short crank arm (in skeleton outline) that makes a complete rotation and through rod 3 gives to the rocker arm 4 a symmetrical oscillation about O_4C. In unit 1, 4, 5, 6 this oscillation is transmitted to wrist plate 6, which, so far as its function in this section of the mechanism is concerned, is sufficiently represented

by radius O_6D. Unit 1, 6, 7, 8 is an example of an *open* four-bar linkage, and unit 1, 6, 13, 14 is an example of a *crossed* four-bar linkage. No attempt is made here to show the remainder of the valve mechanism, namely, the releasing gear whereby arms 7 and 9 alternately take hold and let go of their valves.

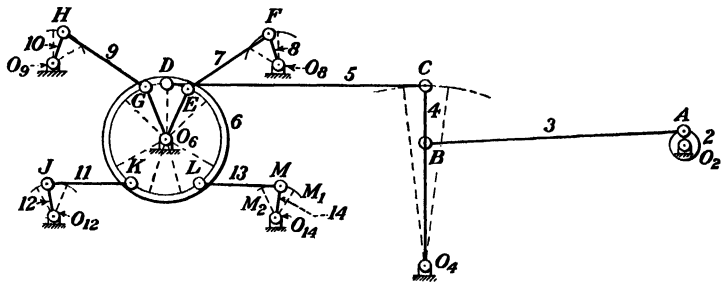


FIG. 21.

19. The Drag-link Mechanism.—A common application of the four-bar linkage, in which there is continuous rotation of both the driver and follower, is found in the *drag-link* mechanism shown in Fig. 22*a* where it takes the form of a quick-return mechanism for a slotting machine. In Fig. 22*b*, only the skeleton outline of the main rotating links is shown. This mechanism has for its object the transformation of a constant angular velocity of the crank 2 into a varying angular velocity of the follower 4, the latter making one revolution in the same time as the crank but at a varying angular velocity during the revolution. For the successful operation of the four-bar linkage in the drag-link mechanism, the requirements are that both the crank and the follower must be able to make a complete rotation and that no dead points shall occur. These requirements are met when the following relations, referring to the skeleton outline of the mechanism shown in Fig. 22*b*, are true:

$$c > d + b - a$$

and

$$c < a + b - d$$

Hence

$$d + b - a < a + b - d$$

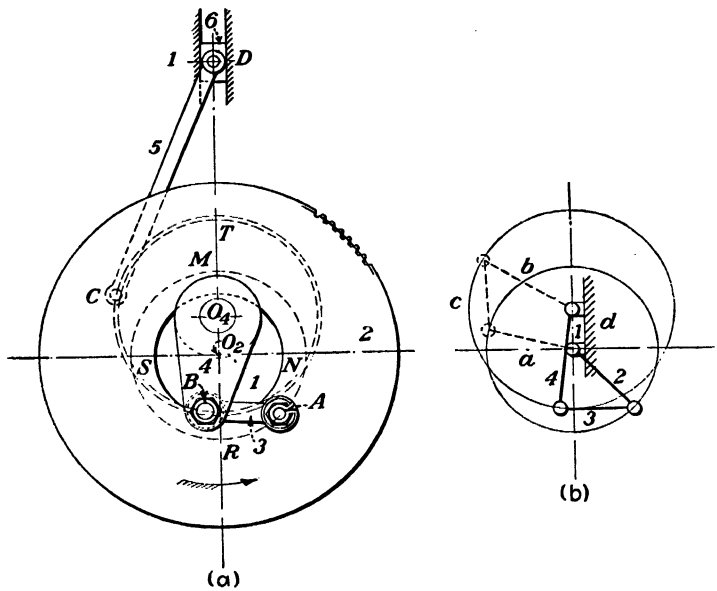

6
1
D
5
T
M
C
O_4
O_2
2
S
4
N
B
1
A
R
3
(a)
b
c
d
1
a
4
2
3
(b)

FIG. 22.

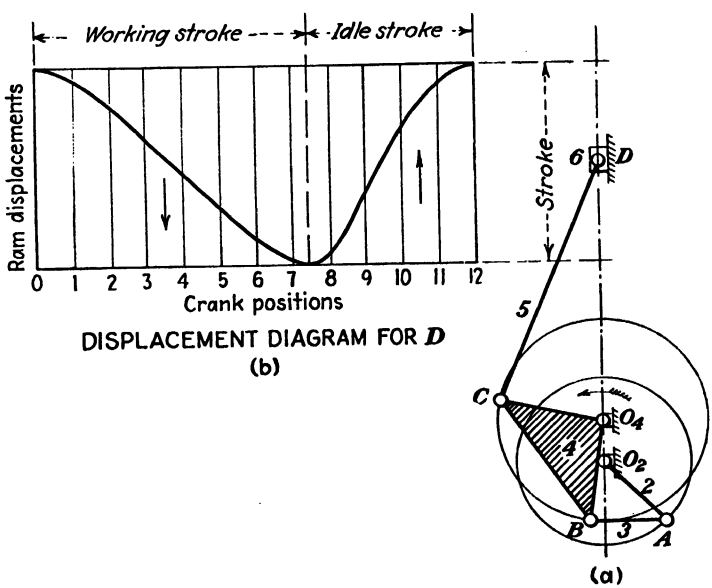

Working stroke
Idle stroke
Ram displacements
0 1 2 3 4 5 6 7 8 9 10 11 12
Crank positions
DISPLACEMENT DIAGRAM FOR D
(b)
Stroke
6
D
5
C
O_4
4
O_2
2
B
3
A
(a)

FIG. 23.

Therefore

$$d < a$$

The action of the drag-link mechanism can best be understood by plotting a displacement diagram, as shown in Fig. 23. The complete skeleton outline at *a* shows the proportions of the various links of this mechanism as used in a slotting machine of the type represented in Fig. 22*a*. Link 2 is a driving crank, which rotates uniformly, 3 is the drag link, and 4 is a follower link that drives the tool head 6 by means of connecting rod 5. At *b*, uniform angular displacements of the crank (link 2) are represented as abscissas, and corresponding displacements of the slide (link 6) are represented as ordinates. Since angular displacements are directly proportional to time, Fig. 23*b* shows that the return stroke of the slide (link 6) is made in about one-half the time required for the working stroke.

The four-bar linkage with four turning pairs has various other forms, and it has many other applications, but space does not permit further discussion here.

THE SLIDER-CRANK MECHANISM

20. The Slider-crank Mechanism.—This mechanism, which has a well-known application in the steam and the gasoline engine,

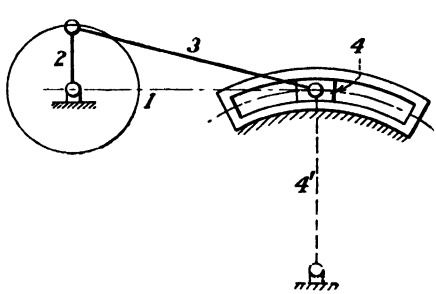


FIG. 24.

is simply a special case of the crank-and-rocker mechanism. In order to make this clear, let it first be noted that if the rocker arm 4 is of considerable length, as shown in Fig. 24, it may be desirable to replace it by a block sliding in a curved slot or guide as shown. If the length of the rocker arm is infinite, the guide

and block are no longer curved, but are straight, as shown in Fig. 25, and the linkage takes the form of the ordinary slider-crank mechanism.

21. Variations of the Slider-crank Mechanism.—There are numerous variations of the slider-crank linkage, only a few of which will be discussed here. In Fig. 26 the connecting rod has been replaced by a sliding block by the process of expanding pair C to include pair B. The skeleton outline of the mechanism (in dotted lines) shows, however, that although

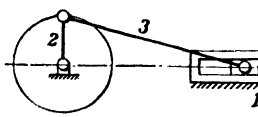


FIG. 25.

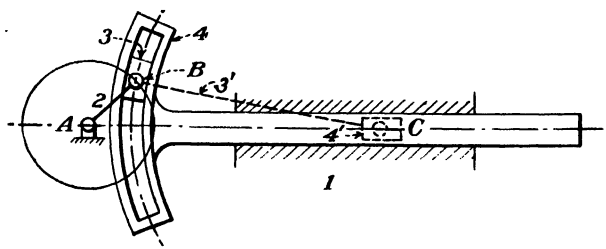


FIG. 26.

the forms of the links have been greatly modified, the kinematic essentials of the mechanism have not been changed. The relative motions of the links will, therefore, be unchanged. When the mechanism shown in Fig. 26 is only slightly modified, it has an application in punching and shearing machines. In Fig. 27 part of the head of such a machine is shown. Uniform rotation is imparted to the eccentric 2 by the main shaft, which turns about the axis O. The eccentric transmits its motion through the connecting rod 3, called the pendulum in this type of mechanism, to the ram 4, giving to the latter a motion of vertical reciprocation, the magnitude of which is equal to twice the length of the crank 2. The punch is fastened to the ram as shown at P and is

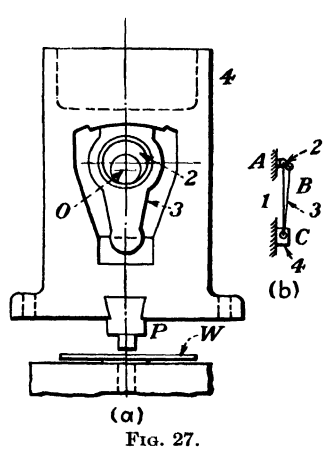


FIG. 27.

forced down through the work W by the motion of the ram. The skeleton outline of the mechanism is shown at b, bringing out very clearly the fact that kinematically the linkage is nothing more or less than the ordinary slider-crank mechanism with the pair C expanded to include both B and A.

22. Connecting Rod of Infinite Length.—When the slot in the mechanism shown in Fig. 26 is made straight, as shown in Fig. 28, the linkage is equivalent to a slider-crank mechanism having a connecting rod of infinite length. This mechanism is known as the *Scotch yoke*. Uniform rotation of the crank 2 gives simple harmonic motion of the slotted crosshead 4. This can be shown as follows: Referring to Fig. 28, let the displacement of the crosshead be measured by the displacement of the center line of the slot. When the crank angle θ is zero, the center line of the slot passes through the point B_0 which lies at a distance R from the center O_2, R being the length of the crank. As θ increases, the center line of the slot will move to the left. Then, letting the distance of the center line of the slot to the left of B_0 be represented by x, the displacement of B is

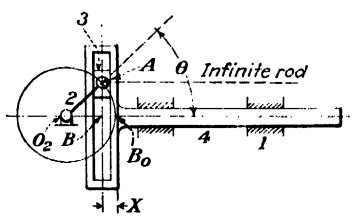


FIG. 28.

$$x = R - R \cos \theta = R(1 - \cos \theta) \tag{1}$$

The velocity of B is

$$V_b = \frac{dx}{dt} = R\omega \sin \theta \tag{2}$$

where $\omega = d\theta / dt$.

The acceleration of B is

$$A_b = \frac{d^2x}{dt^2} = R\omega^2 \cos \theta \tag{3}$$

A point is said to have simple harmonic motion if it moves in a straight line with an acceleration that is proportional to its distance from an origin and is directed toward the origin. The point B therefore has simple harmonic motion, since its acceleration is proportional to its distance $R \cos \theta$ from O_2 and is directed

toward O_2. A useful conception of simple harmonic motion is the following:

If a point moves along the circumference of a circle with uniform velocity, the projection of the point on a diameter of the circle moves with simple harmonic motion.

23. Connecting Rod of Finite Length.—In the case of the ordinary slider-crank mechanism, where the connecting rod must be of finite length, the motion of the crosshead is obviously not a simple harmonic motion. The deviation from simple harmonic motion, owing to what is known as the *angularity* of the connecting rod, may be indicated by comparing displacements. Referring to Fig. 29, let R be the length of the crank, L the length of the connecting rod, and x the displacement of the crosshead to the left of its extreme right-hand position. Then

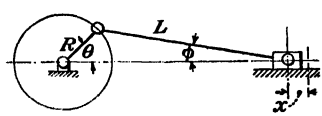


FIG. 29.

$$\begin{aligned} x &= R + L - R\cos\theta - L\cos\phi \\ &= R(1 - \cos\theta) + L(1 - \cos\phi) \\ &= R(1 - \cos\theta) + L(1 - \sqrt{1 - \sin^2\phi}) \\ &= R(1 - \cos\theta) + L\left(1 - \sqrt{1 - \left(\frac{R}{L}\right)^2 \sin^2\theta}\right) \end{aligned} \tag{4}$$

This is the exact formula. The usual scheme for simplification is to expand the quantity under the radical in accordance with the binomial theorem and discard the higher powers of the series thus obtained. Then

$$x = R(1 - \cos\theta) + \frac{R^2}{2L}\sin^2\theta \text{ (approximately)} \tag{5}$$

Comparison of this expression with that of Eq. (1) shows that in the case of the slider-crank mechanism the deviation from the simple harmonic motion is indicated by the second term of the expression.

In Fig. 30*b* displacements of the crosshead have been plotted against crank angles, both for the Scotch yoke (infinite rod) and for a value of $R/L = \frac{1}{4}$ for the slider-crank mechanism. Attention is called to the fact that, for the same length of crank, as the

length of the connecting rod becomes smaller the deviation of the crosshead from simple harmonic motion becomes greater.

A convenient graphical method for obtaining the displacements of the slide corresponding to given crank positions, in the case

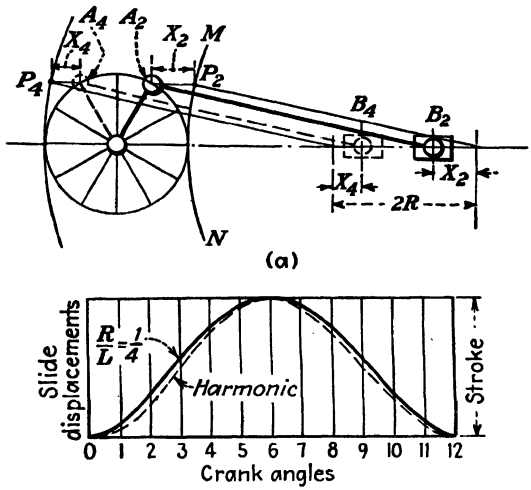


FIG. 30.

of the ordinary slider-crank mechanism, is shown in Fig. 30*a*. For any crank position, such as A_2, the displacement X_2 of the slide B from the right end of the stroke is equal to the horizontal intercept A_2P_2 between the crank circle and the arc MN, whose radius is the length of the connecting rod. In like manner the displacement X_4 of the slide from the left end of the stroke is A_4P_4.

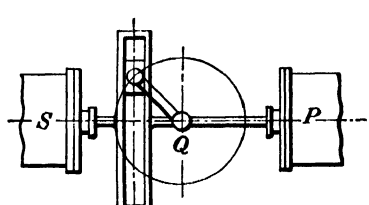


FIG. 31.

24. The Scotch Yoke.—Figure 31 shows an application of the Scotch yoke as it has occasionally been used on small engines and steam pumps. P indicates the pump cylinder and S the steam cylinder. The flywheel is carried on the crankshaft Q. The practical value of this modification of the slider-crank linkage is to provide a more compact mechanism than could be obtained with

a connecting rod of finite length. Arranged as in Fig. 31, it gives a very compact means of introducing flywheel control into a steam pump or compressor, but it is an awkward arrangement, and the rubbing surfaces between block and slide are difficult to keep in proper lubrication.

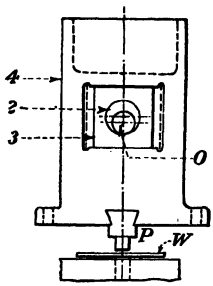


FIG. 32.

Proportioned as in Fig. 32 with a large block surrounding an eccentric of small throw, the mechanism thus formed is of common occurrence in punching and shearing machines, its function being the same as that of the similar arrangement shown in Fig. 27. It should be understood, however, that it is only the form shown in Fig. 31 that may be designated as the Scotch yoke.

25. Toggle Mechanisms.—When the slider-crank mechanism approaches dead center, as indicated by the positions of links 2, 3, and 4 in Fig. 33, there is a rapid rise in the ratio of the resistance Q to vertical driving force P. Let Q be the resistance at C, P the vertical force applied at B, and α the

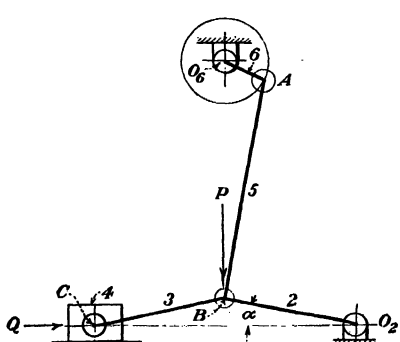


FIG. 33.

angle that each link makes with the line of motion of C. Then, if 2 and 3 are equal in length,

$$\frac{P}{Q} = \frac{2 \sin \alpha}{\cos \alpha} = 2 \tan \alpha$$

This means of overcoming a very great resistance with a small force by straightening out a flat angle between two links is called a toggle effect. The toggle mechanism is applied in a

variety of forms in stone crushers, presses, pneumatic riveters, friction clutches, etc. Figure 34 is a diagram showing an example of a toggle mechanism as applied in one type of stone crusher.

26. The Isosceles Linkage.—Another modification of the slider-crank mechanism that is of interest is that obtained by

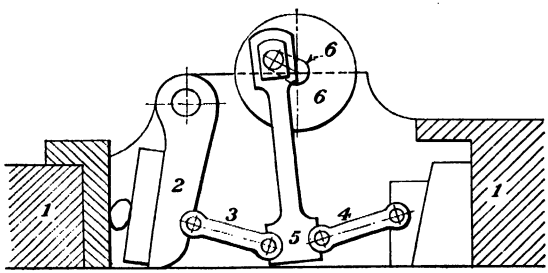


FIG. 34.

making the crank and connecting rod of equal length, as in Fig. 35. In this arrangement, these two members always form an isosceles triangle ABC with the line of travel DE of the sliding member. For this reason the arrangement is sometimes called

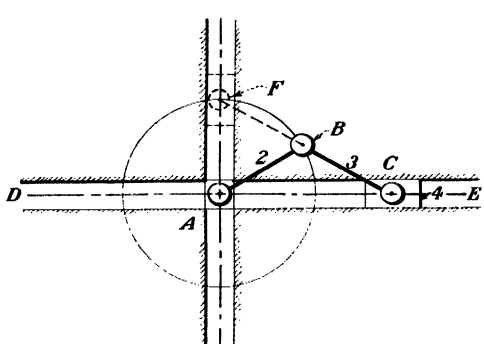


FIG. 35.

the *isosceles linkage*. It is evident that for a complete rotation of crank 2, the length of stroke of the sliding member C is $DE = 4AB$. With the arrangement ABC, the slide C cannot have positive motion at the mid-point A of its stroke. In practical applications of this linkage, positive action is secured by the simple expedient of extending link CB to F where another sliding

member is attached to it and guided in exactly the same manner as at C. With this addition, the motions of the several links become independent of AB and hence the link AB may be dispensed with as far as constraint is concerned.

The isosceles linkage is the basis for several useful devices, among these being the *ellipsograph*, an instrument for drawing ellipses. The principle of the ellipsograph is illustrated in Fig. 36. If a bar AC has the two points A and C guided in perpendicular straight lines, any other point on the bar such as D will trace an ellipse.

Let

$$DC = b$$

and

$$DA = a$$

then

$$DE = y = b \sin \theta$$

and

$$DF = x = a \cos \theta ,$$

whence

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \theta + \sin^2 \theta = 1 \qquad (6)$$

the equation of an ellipse. For the particular point B midway between A and C, the ellipse becomes a circle. This can be easily seen by referring back to Fig. 35.

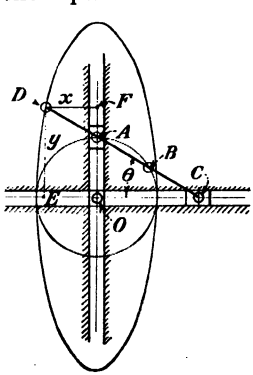


FIG. 36.

The isosceles linkage is also the basis for straight-line motions as applied in certain types of indicator mechanisms.

27. Inversions of the Slider-crank Mechanism.—The meaning of the term *inversion* as applied to *mechanisms* was explained in Chap. I. The inversions of the slider-crank linkage give rise to several well-known mechanisms. The most important, of course, is the engine mechanism that has been considered in Chap. I.

Figures 37, 38, and 39 have been selected and arranged with the purpose of showing not only common examples of inversions

of the slider-crank linkage, but also typical examples of the kinematic identity of machines that, at first sight, appear to be very different.

In each figure the skeleton outline is shown at *a*, and *b* and *c* represent two kinematically identical mechanisms that have been derived from it. The links in the actual mechanisms are numbered similarly to those in the skeleton outlines, so that the figures are almost self-explanatory.

a. The First Inversion.—In the first inversion (Fig. 37) with link 1 fixed, *b* represents the ordinary engine mechanism in which the piston 4 is the driver, and *c* represents a pump mecha-

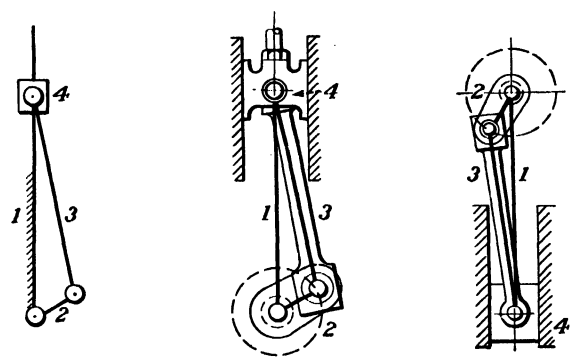


FIG. 37.

nism where the crank 2 is the driver. It should be noted, in this connection, that in machines of this type the crank is commonly constrained by the action of a flywheel to a practically uniform rotation; therefore, regardless of whether the drive is actually from piston to crank, or from crank to piston, it is usual and logical to take the crank as kinematically the driving member.

b. The Second Inversion.—The second inversion (Fig. 38), results from fixing connecting rod 3 of the original mechanism. At *b* is shown a form of oscillating engine in which sliding block 4 has been elaborated to form a cylinder on trunnions and the original frame 1 changed into a piston and rod. This type of engine has had limited application as a hoisting engine. Its chief advantage is cheapness of construction, since it permits of a very simple scheme for supplying steam to the cylinder. This is also the usual form of toy engine. A more useful inversion is the

crank-shaper mechanism shown at *c*. Suitable extension of the linkage is required beyond link 4 for connection to the ram of the shaper. The object of this mechanism as applied to the shaper is to secure a slow cutting stroke and a quick return stroke of the tool.

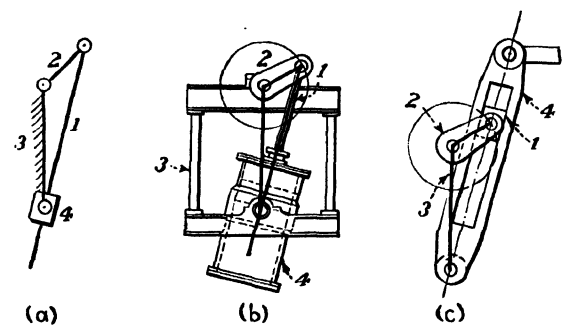


FIG. 38.

c. The Third Inversion.—In the third inversion (Fig. 39), crank 2 of the original mechanism becomes the fixed link, or frame. The Whitworth quick-return mechanism shown at *b* is derived from this inversion and is used in shapers and other

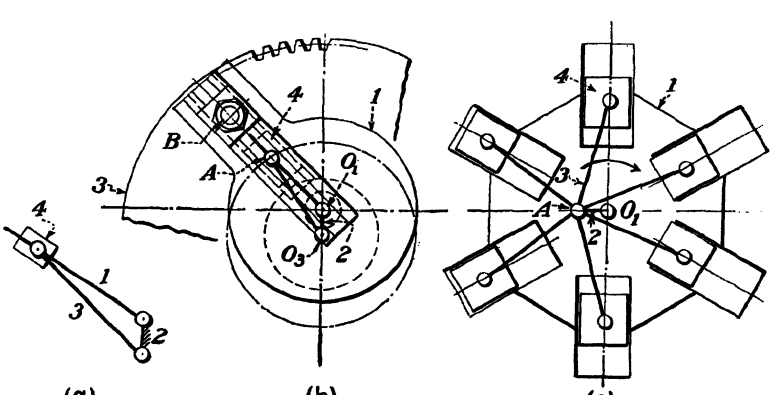


FIG. 39.

machine tools. Gear 3, rotating on axis O_3 and carrying pin A, on which slide 4 is pivoted, corresponds with the connecting rod 3 of the original mechanism and acts as the driver. Link 1 rotating on axis O_1 is driven with a variable angular velocity and carries

crankpin B, which can be adjusted to various radial distances from O_1, to change length of stroke of slide to which it is connected by an additional link. The rotary type of engine shown at c, at one time used in air service, is another interesting example of this inversion in which the crank is the fixed member. The cylinders are mounted upon a rotating frame that turns on axis O_1, and the connecting rods are pivoted on fixed crankpin A.

d. The Fourth Inversion.—The fourth inversion, obtained by fixing the sliding block (link 4) of the original mechanism, does not give a mechanism of sufficient practical value to warrant discussion. Certain mechanisms, of kinematic interest only, have been devised, but space does not permit of their discussion here.

28. Quick-return Mechanisms.—It was shown above that two very important quick-return mechanisms were derived from

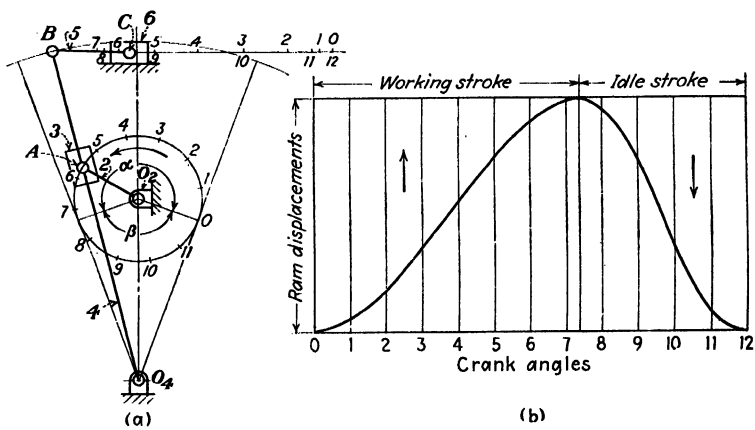


FIG. 40.

inversions of the slider-crank mechanism, namely, the ordinary crank-shaper mechanism (Fig. 38*c*) and the Whitworth quick-return motion (Fig. 39*b*), which is used in shapers, slotting machines, etc.

A quick-return mechanism may be defined as one in which a slow working stroke of the tool and a fast idle or return stroke is produced from uniform rotation of a crank. Just how well this end is attained in case of the crank shaper may be determined by plotting the displacement diagram shown in Fig. 40. At *a* is shown the skeleton outline of the mechanism of Fig. 38*c*, with

suitable extension of the linkage beyond link 4 for connection to the ram of the shaper, which is represented by link 6. In the displacement diagram at *b*, angular displacements of crank 2 have been plotted as abscissas, and corresponding displacements of ram 6 as ordinates.

In the study of quick-return mechanisms, considerable use is made of what is called the time ratio. This is simply the ratio of the time required for the cutting stroke to the time required for the return stroke, both on the assumption that the driving link has a motion of uniform rotation. Referring to Fig. 40, it is evident that, in the case of the crank-shaper mechanism, the ram is at the end of its stroke when the crank 2 is perpendicular to the rocker arm 4. The cutting stroke takes place while the crank is turning through the angle α, and the return stroke while the crank is returning through the angle β. Since the crank rotates uniformly, it is evident that the time required for the cutting and return strokes of the tool must be proportional to those angles; that is, the time ratio is equal to α/β.

Another type of quick-return mechanism, in which there are no sliding pairs between the first driving link and the last driven link, is the drag-link mechanism shown in Figs. 22 and 23.

Elliptical gears were formerly employed to some extent in shaper mechanisms to secure the quick-return motion. Although abandoned for this application, they are occasionally used in other mechanisms where a quick-return effect is desired.

29. The Oldham Coupling.—A striking example of inversion will be found in the Oldham coupling illustrated in Fig. 41*a*. This is a well-known form of flexible coupling used to transmit uniform angular velocity between parallel shafts whose axes may be out of line. As shown in the skeleton outline in Fig. 41*b* it is seen to be an inversion of the mechanism of Fig. 35 (in its practical form) with link 3 instead of 1 as the fixed member. With link 1 fixed (Fig. 41*b*) the locus O of the center of the cross is a circle of diameter AB. Let O and O' be two positions of the center of the cross. Then angles OAO' and OBO', as may be seen from the geometry of the figure, are equal. Hence, if link 2 turns through an angle OBO', 3 will turn through an equal angle OAO'. If, therefore, 2 and 3 are mounted on shafts as shown in the actual form of the coupling in Fig. 41*a* rotary motion may be transmitted from one to the other with an angular

velocity ratio equal to unity. Evidently the center O of the cross will travel around a circle whose diameter AB is the distance between the axes of the shafts. Disk 4, instead of being shown with two slots at right angles as in the diagram (Fig. 41*b*), is in its actual form made with a slot on one side and with a guide strip on the other side at right angles to the slot, as shown in Fig. 41*a*.

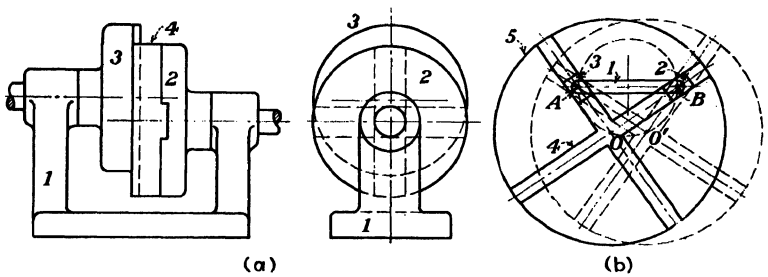


FIG. 41.

30. Reducing Motions.—In Fig. 42 the three mechanisms that may be used for the same purpose show interesting kinematic variation in the slider-crank mechanism. The crank 2 is arranged here for oscillating motion only, and the purpose is to give at cord C, which actuates the drum of a steam-engine indicator, a miniature copy of the motion of the crosshead 4.

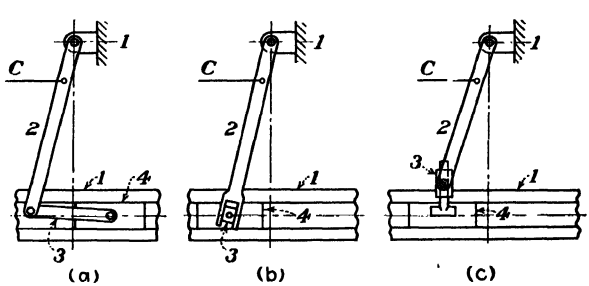


FIG. 42.

31. Chamber Trains.—The term "chamber train" covers mechanisms of the type shown in Fig. 43, used as pumps for comparatively low pressures and as blowers delivering against moderate pressures. Although chamber trains are made in a great variety of constructions, they really fall into two general divisions: (*a*) those which have a single rotor or impeller placed

eccentrically within a cylindrical casing, as in Fig. 43 (it will be found that this type is usually built around some form of the slider-crank mechanism) and (*b*) those which have two rotors, each concentric with its part of the casing (Fig. 44), the two working together like a pair of gears, though actuated by a positive drive from outside the casing.

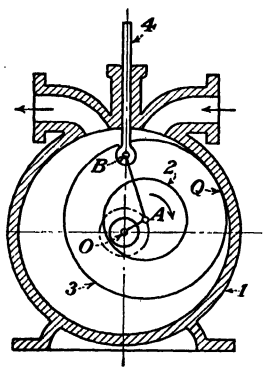


FIG. 43.

In Fig. 43 the fact that the mechanism is essentially of the slider-crank form is clearly brought out by the skeleton form outlined on the figure. The crank and connecting rod are embodied in the inner eccentric 2 and the outer ring 3 respectively, and the slide 4 in the reciprocating partition. It is interesting to observe that eccentric 2 may be considered as formed by expansion of the pairing elements at A (Art. 11), thus replacing crank OA; and that likewise ring 3 may be considered as formed by a further expansion of the outer, or hollow, element at A, thus replacing the connecting rod AB. The variable displacement is contained between contact point Q of rotor with casing 1 and the sliding partition 4.

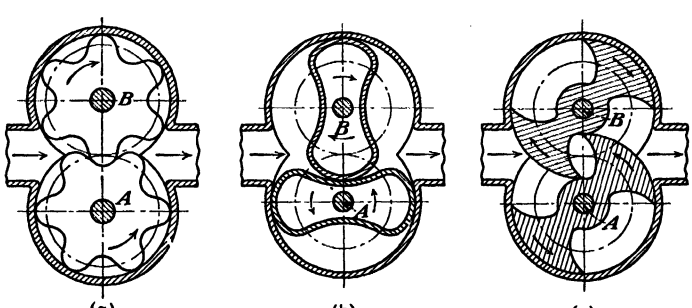


FIG. 44.

The arrangements in Fig. 44 are regularly used in large positive blowers. Correct relative motion between the two rotors in each case is maintained by the pair of equal gears (indicated by the dotted circles) outside the casing. Since these chamber-wheel mechanisms are toothed wheels, discussion of their kine-

matic properties should be taken up under the subject of toothed gears, rather than under the subject of linkwork.

VARIOUS LINK MECHANISMS

32. Straight-line Mechanisms.—A straight-line mechanism is a linkage designed to make a point on one of the links move exactly or approximately in a straight line. In order to avoid the friction and also the structural difficulties arising from the use of straight guides, links with turning pairs are almost invariably employed in a straight-line mechanism. A large number of linkages have been devised for the purpose, only a few of which will be considered here.

a. Approximate Straight-line Mechanisms.—The Watt straight-line mechanism, one of the earliest and best known of the approxi-

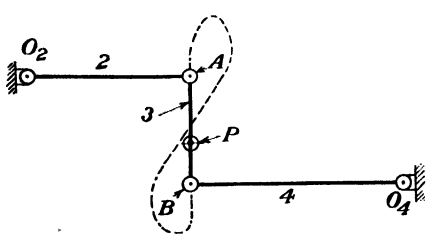


FIG. 45.

mate straight-line mechanisms, is shown in Fig. 45 in its simplest form. Links 2 and 4 are pivoted at O_2 and O_4, and their free ends are connected by link 3. When in mid-position, links 2 and 4 are parallel and connecting link 3 is shown perpendicular to both. It is not absolutely necessary, however, for link 3 to be perpendicular to links 2 and 4. P is a point on link 3 that traces an approximate straight line, within certain limits of the motion. The complete path of P is the *figure* 8 shape of curve shown. The path of P is a function not only of arms 2 and 4, but also its location on the intermediate link 3. It can be seen that the principle of action is based on the fact that as the tracer bar is deflected by the arm at one end, it receives a compensating deflection at the other end. If 2 and 4 are of the same length, P is located at the mid-point of link 3. In general, the segments AP and BP are inversely as the lengths of the adjacent arms.

b. Exact Straight-line Mechanisms.—The best-known of the exact straight-line mechanisms is the Peaucellier mechanism,

shown in Fig. 46. Two equal links 2 and 3 have a fixed center O. The links 4, 5, 6, and 7 are equal, and link 8 with a fixed center at Q equals the distance OQ. P is constrained to move in an exact straight line mn, within the working limits of the construction.

The above statement may be proved as follows:

$$\overline{OA}^2 - \overline{OD}^2 = \overline{AP}^2 - \overline{PD}^2$$

or

$$\begin{aligned}\overline{OA}^2 - \overline{AP}^2 &= \overline{OD}^2 - \overline{PD}^2 \\ &= (OD + PD)(OD - PD) \\ &= OP \times OC\end{aligned}$$

Therefore, the product $OP \times OC$ is a constant. Now suppose that the linkage is moved until C falls at C', and let P' be the new position of P. Then $OC' \times OP' = OC \times OP$, or

$$\frac{OC'}{OC} = \frac{OP}{OP'}$$

and $OQ = QC = QC'$, so that O, C, and C' lie on a semicircle, which makes the angle OCC' a right angle.

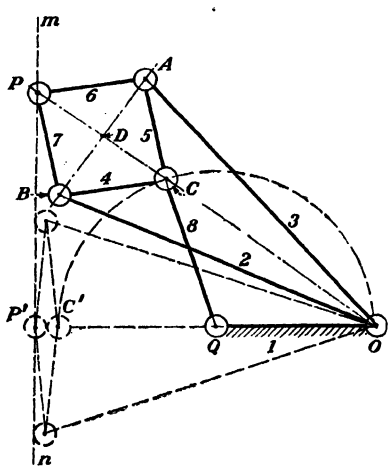


Fig. 46.

In the triangles COC' and POP', the angle POP' is common, and since the sides are proportional, the triangles are similar.

Thus, since the angle OCC' is a right angle, the angle $PP'O$ is also a right angle.

In the same way it can be shown for any other position of the linkage that the line PP' is perpendicular to OP', and therefore P moves in a straight line.

33. Copying Devices. The Pantograph.—This class of mechanisms is composed mostly of variations of the pantograph. The pantograph is a four-bar linkage so arranged as to form a parallelogram in which, if one point is made to travel in any path some

other point will be constrained to describe a similar path, enlarged or reduced.

Figure 47 shows one such arrangement in which the links form a parallelogram with one corner fixed at O. A point p on ab that lies on the line connecting P and O will move in a path similar to that traced by P. Assume that P moves to P' along

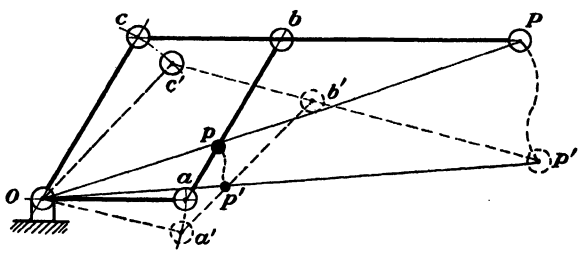


FIG. 47.

the path indicated by the dotted line; then p moves to p' along a similar path. From similar triangles

$$\frac{OP}{Op} = \frac{cP}{cb}$$

also

$$\frac{OP'}{Op'} = \frac{c'P'}{c'b'}$$

But

$$cP = c'P' \qquad \text{and} \qquad cb = c'b'$$

Therefore

$$\frac{OP}{Op} = \frac{OP'}{Op'}$$

Hence, the distance of p from O is proportional to the distance of P from O. As p always lies on the line OP, since OcP and pbP are similar triangles, the angular motion of p about O is equal to the angular motion of P about O. Any path of P is determined by its angular motion about O and its radius vector to O as a pole. As the angular motions of P and of p about O are seen to be equal for any motion of either of these points, and as the radius vector of p bears a constant ratio to that of P, the path of p is similar to that of P.

The pantograph is the standard instrument for copying and reducing or enlarging drawings, maps, engravings, etc. It is also frequently used to increase or reduce motion in some definite proportion, as in the indicator rig on an engine where the motion of the crosshead is reduced proportionally to the desired length of the indicator diagram. The practical objection to its use as an indicator rig is the great number of joints and consequent likelihood of lost motion from wear.

34. Applications of Straight-line Mechanisms.—In the early days of the steam engine, the straight-line and parallel mechanisms were of greater importance than now. These mechanisms represented the method whereby the extremity of the piston rods of the beam engine of that time (shown in Fig. 48) were

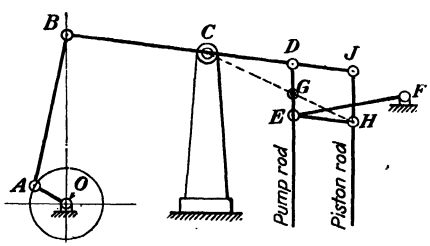


FIG. 48.

constrained to move in a straight-line path. The modern device, that of crosshead and guides, was not feasible because of structural and machining difficulties. The straight-line link motion was, therefore, an ingenious solution providing suitable constraint of the piston rod of a beam engine. Watt combined the pantograph with his straight-line mechanism so that the steam piston rod, and pump rod were guided in straight lines by one combination of links as shown diagrammatically in Fig. 48. The links *CD*, *DE*, and *EF* form the Watt straight-line mechanism, and *DE* is divided at *G* so that $DG/GE = EF/DC$. *CD* is one arm of the oscillating beam of the engine, and it is extended to the point *J*. On this link *CDJ*, the pantograph *CDJHE* is constructed, the lengths of *DJ* and *EH* being such that the point *H* traces a path similar to that of *G*. From previous discussions of the pantograph, it will be recognized that the condition for this motion is that *C*, *G*, and *H* lie on a straight line. The main piston rod is then connected at *H* and the pump rod at *G*. Although straight-line mechanisms are no longer required for the

original purpose for which they were designed, they still have an important use in connection with multiplying gear for recording instruments, of which the steam-engine indicator rig is typical.

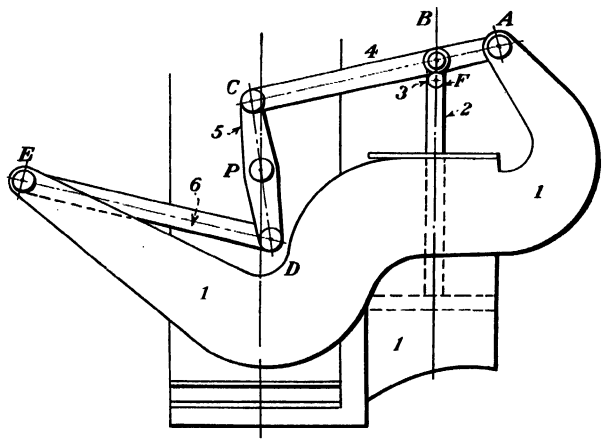


FIG. 49.

35. Indicator Mechanisms.—The function of the indicator is to give a magnified copy of the movement of the indicator piston, as variable pressure (against resistance of spring) causes that piston to move up and down. To the kinematic requirements of straight-line path and constant-travel ratio for the pencil is added the necessity for minimum mass, in order that the instrument may be quick acting without serious disturbance by inertia.

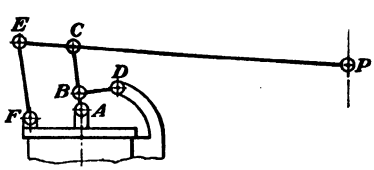


FIG. 50.

The Richards indicator (Fig. 49), one of the earliest of the modern types, used the Watt linkage *ACDE* to guide the pencil point at *P* in a straight line.

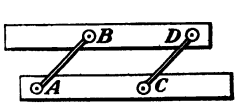


FIG. 51.

The Crosby indicator, a modern type, is shown in Fig. 50. This mechanism is a modification of the pantograph. The guiding is all concentrated on member *CA* and dependent upon radius rod *BD*.

36. Parallel Linkages.—These are commonly applications of the four-bar linkage in which the links form a parallelogram.

The *parallel rule* (Fig. 51), an old-fashioned piece of drafting equipment, furnishes an example of its simplest application. The *universal drafting machine* (Fig. 52), a modern substitute for the T-square and triangle, is an interesting example of its present-day use. This instrument consists primarily of a combination of parallelograms *ABCD* and *EFGH* in which the coupling ring *CDEF* is an important feature.

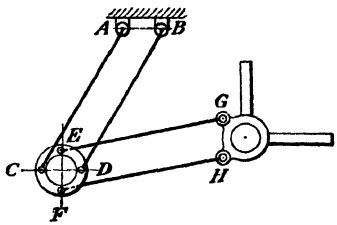
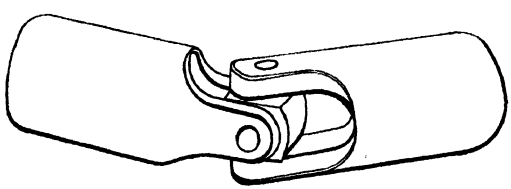
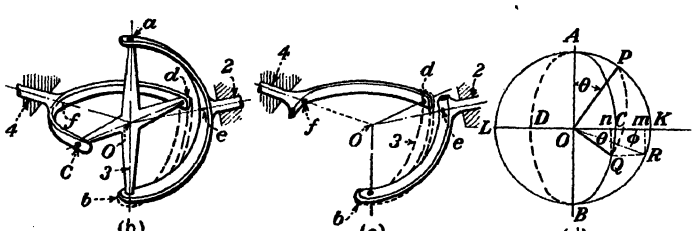


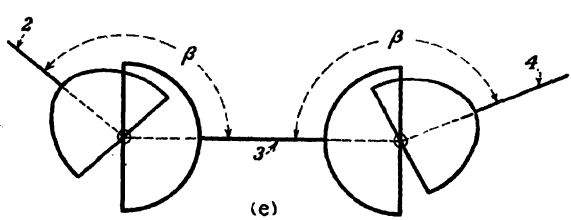
FIG. 52.

37. The Universal Joint.—The universal joint (Fig. 53*a*), also known in the older literature as *Hooke's coupling*, is a spherical



(a)





(e)

FIG. 53.

linkage in common use for connecting two shafts that intersect. Regardless of how constructed or proportioned for practical

use, it has essentially the form in Fig. 53*b*, consisting of two semi-circular forks 2 and 4, pin-jointed to a right-angle cross 3. If only one-half of each fork is considered, as *eb* of 2 and *fd* of 4, and these are assumed to be connected by a spherical link *db* equal to the fixed distance between the two adjacent points of the cross 3, a four-bar conic linkage is produced in which the axes of all the turning pairs intersect in *O*. With this arrangement, the fork could be omitted and there would be, as the kinematic equivalent of the original mechanism, the arrangement shown in Fig. 53*c*.

The driver 2 and the follower 4 make complete revolutions in the same time, but the velocity ratio is not constant throughout the revolution. The following analysis will show how complete information as to the relative motions of driver and follower may be obtained for any phase of the motion.

a. Analysis of a Universal Joint.—If a plane of projection is taken perpendicular to the axis of 2, the path of *a* and *b* will be a circle *AKBL* (Fig. 53*d*). If the angle between the shafts is β, the path of *c* and *d* will be a circle that is projected as the ellipse *ACBD*, in which $OC = OD = OK \cos \beta = OA \cos \beta$.

If one of the arms of the driver is at *A*, an arm of the follower will be at *C*; and if the driver arm moves through the angle θ to *P*, the follower arm will move to *Q*; *OQ* will be perpendicular to *OP*; hence $COQ = \theta$. But *COQ* is the projection of the real angle described by the follower. *Qn* is the real component of the motion of the follower in a direction parallel to *AB*, which line is the intersection of the planes of the driver's and the follower's paths. The true angle ϕ described by the follower, while the driver describes the angle θ, can be found by revolving *OQ* about *AB* as an axis into the plane of the circle *AKBL*. Then *OR* = the true length of *OQ*, and $ROK = \phi$ = the true angle that is projected as $COQ = \theta$.

Now

$$\tan \phi = \frac{Rm}{Om}$$

and

$$\tan \theta = \frac{Qn}{On}$$

But

$$Qn = Rm$$

Hence

$$\frac{\tan \theta}{\tan \phi} = \frac{Om}{On} = \frac{OK}{OC} = \frac{1}{\cos \beta}$$

Therefore

$$\tan \phi = \cos \beta \tan \theta \tag{7}$$

The ratio of the angular motion of the follower to that of the driver is found as follows, by differentiating Eq. (7), remembering that β is constant

$$\frac{\omega_4}{\omega_2} = \frac{d\phi}{d\theta} = \frac{\cos \beta \sec^2 \theta}{\sec^2 \phi} = \frac{\cos \beta \sec^2 \theta}{1 + \tan^2 \phi} \tag{8}$$

Eliminating ϕ by means of Eq. (7)

$$\frac{\omega_4}{\omega_2} = \frac{\cos \beta \sec^2 \theta}{1 + \cos^2 \beta \tan^2 \theta} = \frac{\dfrac{\cos \beta}{\cos^2 \theta}}{\dfrac{\cos^2 \theta + \sin^2 \theta \cos^2 \beta}{\cos^2 \theta}}$$

$$= \frac{\cos \beta}{\cos^2 \theta + \sin^2 \theta(1 - \sin^2 \beta)} = \frac{\cos \beta}{1 - \sin^2 \theta \sin^2 \beta} \tag{9}$$

By a similar process θ could be eliminated, giving

$$\frac{\omega_4}{\omega_2} = \frac{1 - \cos^2 \phi \sin^2 \beta}{\cos \beta} \tag{10}$$

It is seen from Eq. (9) that $\omega_4 \div \omega_2$ is a minimum where $\sin \theta = 0$, or when $\theta = 0, \pi$, etc., which corresponds to a value of $\phi = 0, \pi$, etc. The same thing is seen from Eq. (10), which gives a minimum value of $\omega_4 \div \omega_2$ when $\cos \phi = 1$, or $\phi = 0, \pi$, etc. Also $\omega_4 \div \omega_2$ is a maximum when $\sin \theta = 1$, or $\cos \phi = 0$, corresponding to $\theta = 90$ deg., etc.; $\phi = 90$ deg., etc. To summarize the above, when the driver has a uniform angular motion, the follower has a minimum angular motion when the driving arm is at A or B and the follower arm is at C or D. The follower has a maximum angular motion when the driving arm is at K or L and the following arm is at A or B.

b. The Effect of a Double Universal Joint.—By using a double joint (Fig. 53*e*), the variation of angular motion between driver and follower can be entirely avoided. This compensating arrangement is to place an intermediate shaft 3 between the two main shafts 2 and 4, making the same angle β with each. The two forks of this intermediate shaft must be parallel. If the first shaft rotates uniformly, the angular motion of the intermediate shaft will vary according to the law deduced above. This variation is exactly the same as if the last shaft rotated uniformly, driving the intermediate shaft. Since, therefore, the variable motion transmitted to the intermediate shaft by the uniform rotation of the first shaft is exactly compensated for by the motion transmitted from the intermediate to the last shaft, the uniform motion of either of these shafts will impart, through the intermediate shaft, uniform motion to the other. This is the combination frequently seen in machine-tool drives, and elsewhere.

VALVE GEARS

38. Valve Gears.—The linkages so far considered have, in general, been made up of not more than four to six links. Linkwork of a much greater degree of complexity is found in great variety in the mechanisms that actuate the valves of steam, gasoline, and locomotive engines. Linkages of this class are called *valve gears.* Although in some cases, particularly in gasoline engines, the actuating motion for the valve is derived from or mostly applied through cams, the great majority of valve gears are link motions that appropriately come under the heading of this chapter and merit extended discussion. It is not practicable, however, to consider more than a few examples here.

Kinematic study of valve-gear linkages usually involves problems relating to the determination of point paths, displacement of valve corresponding to displacement of crank or piston, and relative motions of various links.

39. Steam-engine Valve Gears.—The simplest form of valve gear, used on the ordinary steam engine, adds to the main slider-crank linkage a secondary linkage of the same kind. A skeleton outline of the arrangement is shown in Fig. 54. Since the crankshaft, crank, and eccentric form one rigid link, they are represented by the compound link OAC, OA representing the crank

and OC the eccentric. Whereas in the steam engine the length of the main connecting rod 3 is usually from five to seven times that of the crank arm OA, the length of eccentric rod 5 varies from twenty to forty times the throw OC of the eccentric. For this reason, in analysis of the motions, it is usual to neglect the effect on the valve movement of the finite length of rod, except in case of close analysis.

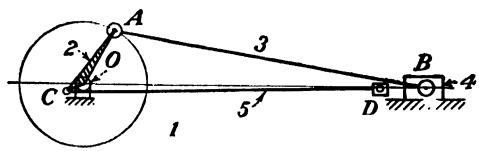


FIG. 54.

The simple form of the regular Corliss type of valve gear used on steam engines (Fig. 21) has been discussed briefly in Art. 18. Because of the time element in the dashpot releasing gear, the full Corliss type of valve gear represented in Fig. 21 is suitable only for engines running at comparatively low speeds. For high-speed steam engines there have been developed modifications of the Corliss gear in which the dashpot control has been eliminated.

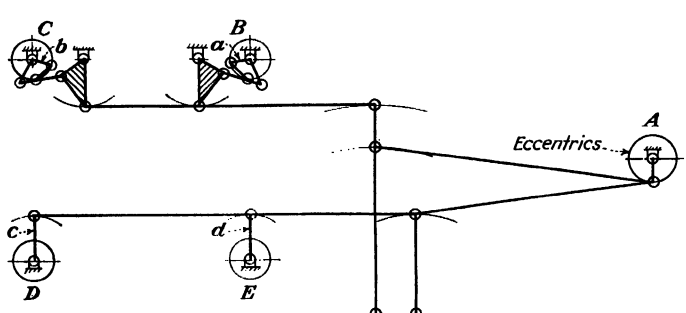


FIG. 55.

A valve gear of this type, known as the Corliss nonreleasing gear, is outlined in Fig. 55. Through the linkwork shown, the eccentrics at A actuate the steam-valve mechanisms at B and C and the exhaust-valve mechanisms at D and E. The steam valves rotate through the same angles as links a and b, to which they are rigidly connected. In like manner the exhaust valves rotate through the same angles as their driving links c and d.

40. Locomotive Valve Gears.—Of the many varieties of locomotive valve gears, two typical examples will now be considered: the *Stephenson valve gear*, which for a long time held predominant place in locomotive design, and the *Walschaert valve gear*, the predominant modern type.

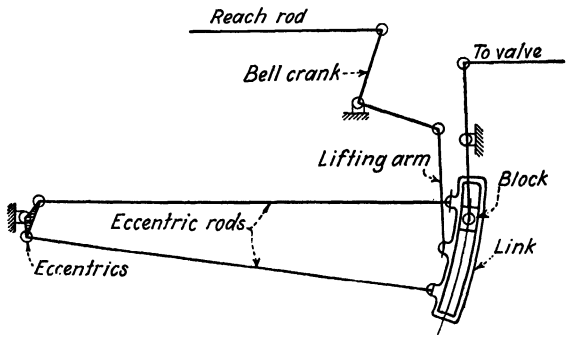


FIG. 56.

The Stephenson gear is shown in Fig. 56. Located between the wheels, this gear was successful up to the time when modern developments in locomotive practice began uncomfortably to crowd the space between the wheels. When this condition became serious, the Stephenson gear was abandoned in favor of the Walschaert type (Fig. 57), which, in various modifications, has been adopted on practically all modern locomotives.

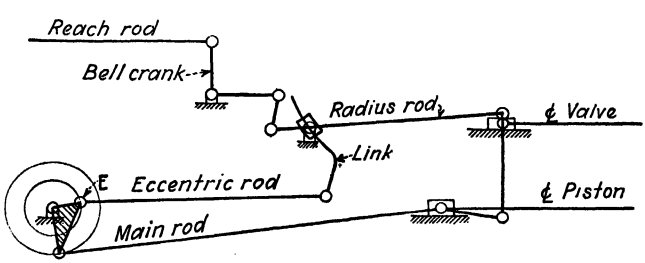


FIG. 57.

The principal advantage of the Walschaert gear lies in the accessibility of its parts, which are placed entirely outside the driving wheels. This facilitates inspection, oiling, and cleaning.

In heavy locomotives employing the Stephenson valve gear, the eccentrics must be of large diameter to secure the required

throw. This increases the velocity of the rubbing surfaces and the tendency to heat. Also, the inertia effects of the heavy slotted links are objectionable.

The various parts of the Walschaert gear are pin-connected and are easily lubricated; hence troubles due to overheating are reduced to the minimum. The Walschaert gear also transmits the moving force of the valve in practically a straight line; consequently there are less springing and yielding of the parts than in the Stephenson gear. By removal of the valve gear from between the wheels, a better opportunity is afforded to introduce frame bracing, thus reducing the possibility of frame breakage. The Walschaert type of valve gear also possesses advantages as regards steam distribution and operation.

The general adoption of the Walschaert valve gear has been accompanied by a number of variations which, however, do not eliminate its essential elements. The Baker, the Young, and the Southern valve gears, other well-known modern locomotive valve gears, are simply variations of the Walschaert gear.

GOVERNORS

41. A governor may be defined as a device for regulating automatically the output of a machine. The governor of a steam engine, steam turbine, gas engine, or other prime mover is a linkage that automatically regulates the supply of steam, gas, or other fluid so as to keep the driving force exerted by the working fluid constantly adjusted to the resistance to be overcome.

The output of a prime mover is continually varying. If, for example, a steam turbine is driving a generator used for lighting purposes, the number of lights in use varies from time to time. The same condition exists in the case of an engine that furnishes power for a machine shop, since the power demands from the various machine tools in the shop are continually changing.

The governor partakes of the motion of the machine to which it is attached so that a change in the speed of the machine due to a change in the load causes a corresponding change in the moving parts of the governor, which, in turn, cause, by means of a suitable mechanism, a change in the pressure of the fluid or a change in the quantity of the fluid delivered to the machine.

The regulating action of a governor should not be confused with that of the flywheel of an engine. The flywheel that acts

as a reservoir of energy is useful in regulating speed throughout the very short period of a revolution or engine cycle, whereas the function of the governor is to regulate speed over greater periods, maintaining a balance between the energy supplied to the engine and the resistance to be overcome.

42. Types of Governors.—Classified according to disposition of the revolving masses and the method of attaching them to the prime mover, governors are of two general types: flyball governors and shaft governors.

Flyball governors are of two kinds:

1. Gravity loaded (Fig. 58), in which the centrifugal force due to the revolving masses is largely balanced by gravity.

2. Spring loaded (Fig. 59), in which the centrifugal force is largely balanced by springs.

Shaft governors are of two kinds:

1. Centrifugal governors (Fig. 60), in which centrifugal force plays the major part in the regulating action.

2. Inertia governors (Fig. 61), in which the inertia effect predominates.

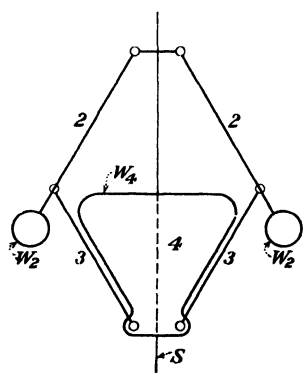


FIG. 58.

43. The Gravity-loaded Flyball Governor.—This type of governor is older than the shaft governor. It is the original type used by Watt on his engines and has been much used since on low-speed steam engines. Modifications of the earlier designs have been adapted to modern gas-engine and steam-turbine practice.

The older forms of steam-engine flyball governors are gravity loaded and run at moderate speeds. A typical gravity-loaded flyball governor of the conic-pendulum form used on a Corliss engine is illustrated in Fig. 58.

In its simplest form it has two or more weights W_2 carried by arms 2 which in turn are pivoted on a rotating spindle S. The spindle is driven at a speed proportional to the speed of the engine. As the weights revolve with the spindle, they tend to move outward under the action of centrifugal force. If this force is great enough, the weights will move outward, and by means of links 3 will move the weighted slide 4 along the spindle S. Through a suitable linkage this slide actuates the valve

mechanism, and thus the supply of working fluid is controlled by the position of 4, decreasing as 4 rises.

44. The Spring-loaded Flyball Governor.—More recent designs of the flyball governor such as are used for steam turbines are spring loaded, as illustrated in Fig. 59. A spring-controlled governor may be considered a special case of the gravity-loaded flyball governor, in which a spring is substituted for the dead weight for the purpose of rendering the governor more effective. In large turbines the flyball governor may act indirectly. That is, it may take only the light service of moving a small pilot valve of an auxiliary device that operates the main valve gear or valve.

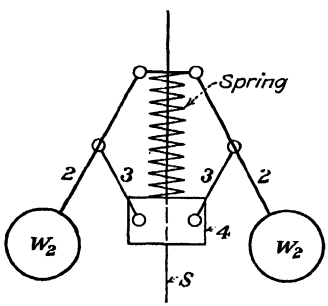


FIG. 59.

45. The Shaft Governor.—This governor is a later development than the flyball type, having come into use in this country mainly as an adjunct to the high-speed steam engine, which requires a governor of greater effort and power than is usually possible with the flyball type. Builders of high-speed steam engines have governed them almost entirely by varying the point of cutoff of the steam. This has been accomplished by changing the angle of advance and also the throw of the eccentric by means of a governor that causes the center of the eccentric to vary in position relative to the crank, according to the load, resulting in a change in the events of the stroke. Since the governor usually acts directly upon the eccentric, it is necessary to have a powerful governor in order properly to overcome the resistance of the valve. Again, since the governor acts directly on the eccentric, it is convenient to have the governor on the crankshaft and hence running at the engine speed. The term *shaft governor* is therefore appropriate for this type of governor.

46. Principle of the Centrifugal Shaft Governor.—The elements of a governor of this type are shown in Fig. 60. The governor parts are usually mounted inside the flywheel and rotate with it on the engine power shaft O_2. The weight W is pivoted at O_3 to one of the arms of the flywheel so that it can assume various positions with respect to the flywheel. As the flywheel

rotates, the weight W tends to move outward under the action of centrifugal force. This motion is opposed by the tension of spring S. If the speed increases beyond a certain point, weight W will assume a new position, and by means of suitable linkage (not shown) the eccentric will be shifted relative to the shaft O_2, thus regulating the supply of working fluid.

In addition to the centrifugal force, there is another force tending to move weight W relative to the flywheel. Let it be assumed that the flywheel is rotating about shaft center O_2 with angular velocity ω. If the load is decreased, the engine tends to speed up and the flywheel receives an angular acceleration α. Owing to its inertia the governor weight W resists this acceleration and thus tends to rotate about the center O_3 in a sense opposite that of ω. This action is known as the inertia effect. In this case, however, the inertia effect is relatively small and the centrifugal effect very powerful.

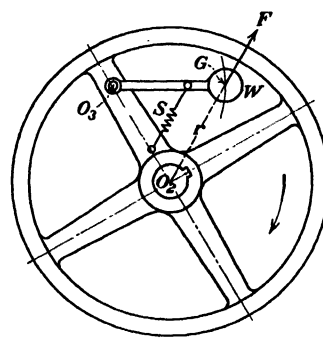


FIG. 60.

When a shaft governor is so proportioned that the centrifugal action predominates as in Fig. 60, the governor is known as a *centrifugal shaft governor*. It is not practicable to construct a shaft governor that operates by inertia alone, but when the inertia effect is relied upon to furnish the greater part of the regulating force, the governor is called an *inertia governor*.

47. Principle of the Inertia Shaft Governor.—One of the simplest forms of inertia governor is outlined in Fig. 61. It consists of a single piece of cast iron in the general form of a bar pivoted to the flywheel at P, with the center of mass very close to the shaft center O. The eccentric that operates the valve is attached to this bar, or weight, by means of a suitable linkage. Neglecting the small centrifugal force that will be set up, it is evident that, as long as the flywheel rotates with a uniform angular velocity, there will be no relative motion between the weight and the flywheel. Now, if the flywheel is given an angular acceleration, the weight, because of its inertia, will resist this acceleration. As a result, there will be a relative motion between the flywheel and weight that will cause the eccentric to shift its position relative

to the shaft. Although the centrifugal force has a certain desired effect in regulation, owing to the fact that the mass center is placed a short distance from the shaft center O, it is evident that in the action of this governor inertia force plays decidedly the major part.

48. Action of the Two Types of Shaft Governor Compared.—It has been shown that shaft governors operate as a result of combined centrifugal force and inertia force and are classified according to the force that predominates. It is important to have a clear understanding of the principle of action of each type. These principles will be discussed further in Part II.

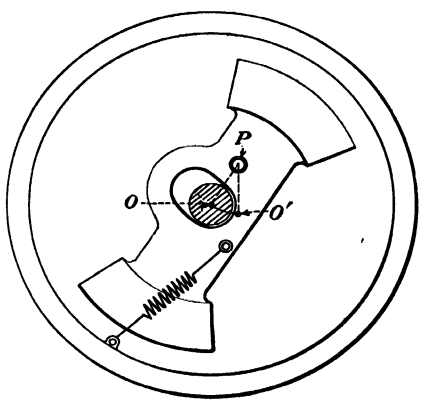


FIG. 61.

In the centrifugal type the governing force depends upon the *amount* of change in the angular velocity of the flywheel. Hence there must be a definite amount of change in the speed before the change in the centrifugal force is great enough to move the valve gear against the frictional forces.

In the inertia governor, on the other hand, the governing force depends on the *rate* of change of angular velocity, which is usually greatest just at the beginning of the change of speed. Hence it begins to act just as soon as the flywheel begins to change its speed. It follows, therefore, that the inertia governor is more sensitive than the centrifugal governor.

CHAPTER III

CAMS

The transformation of one of the simple motions, such as uniform rotation, into any other arbitrarily determined irregular motion is most conveniently accomplished by means of a cam mechanism, consisting usually of two moving elements, the cam and the follower, mounted in a fixed frame. A *cam* may be defined as a machine element having a curved outline or a curved groove, which, by its oscillating or rotating motion, gives a predetermined motion to another element called the *follower*. The cam has a very important function in the operation of many classes of machines, especially those of the automatic type, such as printing presses, shoe machinery, textile machinery, gear-cutting machines, screw machines, etc. In any class of machinery where automatic control and accurate timing are paramount, the cam is an indispensable part of the mechanism. The possible applications of cams are unlimited, and their shapes occur in great variety. Some of the most common forms will be considered in this chapter.

49. Cam Motions.—Two simple examples of a disk cam and follower are shown in Figs. 62 and 63, one with a roller follower and one with a flat-faced follower. If the cam (Fig. 62 or 63) is given a motion of rotation, it will first lift the follower by direct contact to its extreme position, will next permit its return to its initial position, and will then let it remain at rest in its initial position during a given interval of time, after which the cycle of motion will be repeated. In cams of this type the return motion of the follower must be brought about by some force external to the cam, by means of a spring, for example. The roller, by means of which contact is made between cam and follower (Fig. 62), serves to reduce friction to a minimum, but it is sometimes desirable or necessary to use a flat-faced follower, such as represented in Fig. 63. That part of the follower face which comes in contact with the cam is usually provided with a hardened surface to prevent excessive wear.

50. Fundamental Procedure in a Cam Layout.—In the layout of a cam, the initial position, length of stroke, direction, and character of the motion of the follower are, as a rule, completely known. The angular motion of the cam and the location of the axis of the cam relative to the follower are usually known also. The problem, therefore, is to determine the shape of the cam profile that will impart the required motion to the follower.

The fundamental principles to be observed in the layout of a cam will be brought out by means of the examples given in Figs. 62 and 63. In Fig. 62 is shown a disk cam A whose clockwise

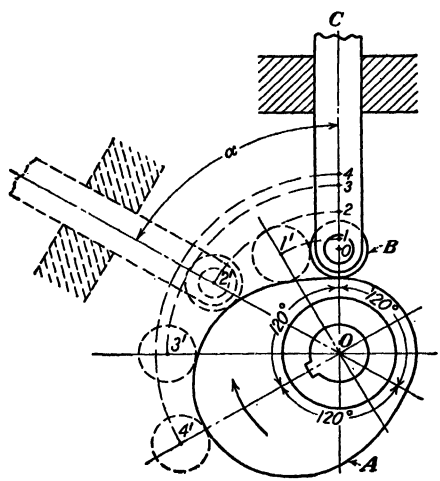


FIG. 62.

rotation is to drive the roller follower B radially outward along the line OC. The cam is assumed to rotate clockwise at a uniform rate.

The outline of the cam is to be such that for each 30 deg. of uniform rotation of the cam for the first 120 deg. the axis of the roller follower is to move progressively from its initial position at 0 to 1, 2, 3, and 4 and then return progressively from its extreme position at 4 to 3, 2, 1, and 0 during each 30-deg. rotation of the cam for the second 120 deg., remaining at rest in its initial position during the remaining 120 deg. of the cycle.

The first step in the solution is to draw the four radial lines dividing the first 120-deg. rotation of the cam into the required 30-deg. intervals. The next step is to locate the centers 1′, 2′,

3′, and 4′ of the roller follower along these radial lines at distances from the center O of the cam successively equal to the distances $O1$, $O2$, $O3$ and $O4$ measured on the initial radial line OC, and draw in the outline of the roller follower in these positions. The final step in the solution is to determine the required outline of this part of the cam by drawing a smooth curve tangent to the initial position and the four constructed positions of the roller follower.

In determining the constructed positions 1′, 2′, 3′, and 4′ of the roller follower, as described above, it is convenient to apply the principle of *inversion* (see Art. 27). That is, to assume that the cam is the fixed link and that the frame that guides the follower and carries the bearing supporting the cam shaft is the moving link. Obviously, fixing the cam and rotating the frame in a sense opposite that assumed for the cam will give the same relative motion between the follower and frame as would be given by the actual setup, *i.e.*, rotation of the cam with the frame fixed. In the present instance, *e.g.*, when the cam has rotated clockwise through an angle α, which in this case is equal to 60 deg., the axis of the roller follower is to have moved from 0 to 2; or, if the cam is fixed and the roller and frame are rotated counterclockwise through the angle α to the dotted position shown in Fig. 62, the roller will move a distance equal to that of 0 to 2 relative to the frame. Hence point 2′ is where an arc of radius $O2$ intersects the radial line $O2'$. Obviously, if a circle of radius equal to that of the roller follower is drawn from point 2′ and a part of the cam outline drawn tangent to this curve and the initial position 0 of the roller, and the cam is then rotated clockwise through 60 deg., the center of the roller follower will be positively driven from 0 to 2. Having determined the remaining points 1′, 3′, and 4′ in a similar manner, the outline of the working surface of the cam for the first 120-deg. rotation was determined as shown in the figure. Since the method of determining the working surface of the cam for the second 120 deg. of its motion is exactly the same as for the first 120 deg., the outline has been drawn in without showing the construction. Since the follower is to remain stationary while the cam rotates through the last 120 deg., this part of the cam outline will be a circular arc drawn from O tangent to the initial position of the roller follower.

In order to avoid confusion in a cam layout, the constructed positions of the follower, such as 0, 1′, 2′, 3′, and 4′ in Fig. 62 should always be numbered to correspond to the numbered positions 0, 1, 2, 3, and 4, which define the motion of the follower along its path.

Figure 63 illustrates the layout when the roller follower of Fig. 62 has been replaced by a flat-faced follower. The specifications are otherwise exactly the same as for the layout of Fig. 62, and the solution is made in the same manner. The working

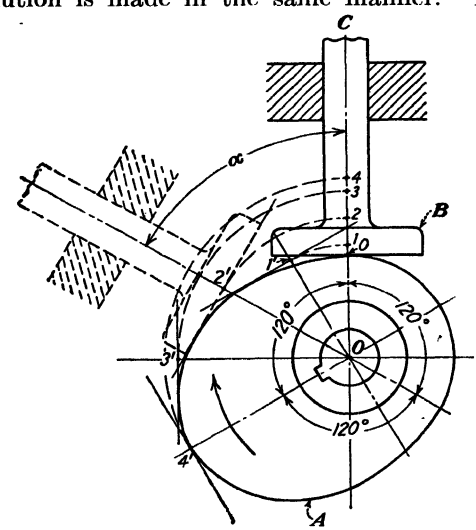


FIG. 63.

surface of the cam is a smooth curve tangent to the several constructed positions of the face of the follower as in the previous case.

It should be noted that in this case the face of the follower makes an angle of 90 deg. with the initial radial line *OC*, and, therefore, in its constructed positions the face should make the same angle with the corresponding radial lines, as illustrated by the dotted position of the follower at position 2′.

51. Displacement Diagrams.—In the layout of a cam mechanism it is frequently desirable to make use of a *displacement diagram.* Such a diagram (Fig. 64) shows graphically the relation between the motions of the driver and the follower. The length of the diagram represents the time of one revolution of the

cam, and the height of the diagram the total displacement of the follower. If the cam rotates with uniform angular motion, which is usually the case, the time of a revolution is divided into any convenient number of equal parts, called time periods, which appear as equal spaces on the base line. The ordinates corresponding to the several time periods represent displacements of the follower from its initial position.

Figure 64 represents the displacement diagram for the cam mechanism in Fig. 62 (or Fig. 63) as it would be plotted on an arbitrary length of base line representing 12 time intervals for one rotation of the cam. The total number of intervals in the present instance was determined from the fact that four were

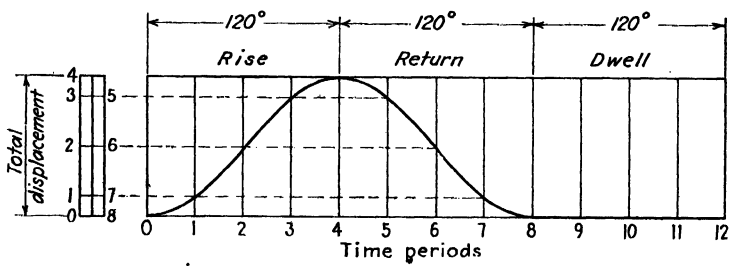


FIG. 64.

used in the first one-third rotation. The ordinates in the diagram represent the correspondingly numbered displacements shown along the line of stroke of the follower in Fig. 62 (or Fig. 63). The law of variation of displacement with time during the rise and return of the follower is clearly indicated by the curves in the diagram.

In Figs. 62 and 63 the construction for the return motion of the follower during the second 120 deg. of motion of the cam was omitted for sake of clearness. But since the return motion is exactly the same as the rise, except in reverse order, the return motion is represented in the displacement diagram (Fig. 64) by the reversed curve shown from positions 4 to 8. The remaining 120-deg. period of the cam motion, where the follower is at rest in its initial position, is represented in the diagram between positions 8 to 12. Hence, positions 0 to 4 in the displacement diagram represent the rise of the follower, 4 to 8 the return, and 8 to 12 a rest, or dwell, in the initial position until the beginning of the next cycle.

In addition to providing a very convenient means for studying the character of the motion of the follower, a displacement diagram may also be of value in showing the proper sequence of events in a mechanism. In general the diagram may be drawn on any suitable length of base line, since this dimension is arbitrary.

Cam outlines should be designed so as to avoid shock at the beginning and end of the follower stroke. In order to secure smooth action no abrupt changes should appear anywhere in the displacement diagram. One method sometimes used in the design of a cam is to lay out a displacement diagram such as shown in Fig. 64, drawing in with the compass or irregular curve an outline that will represent a smooth motion for the follower. The intersection of this outline with the ordinates of the diagram will give a scale for the motion of the follower similar to that shown at the left of the diagram in Fig. 64. In general, however, the travel of the follower conforms to some definite law of motion, such as is discussed in the next article, in which case the variation of the motion between its extreme positions (scale of motion) can be laid out without the aid of a displacement diagram. The desirability or necessity of making a displacement diagram in any given case is thus governed by the requirements of the problem.

52. Base Curves.—The curves in the displacement diagram that represent the motion of the follower during its rise or fall are called *base curves*. The motion of the follower may be made to conform to a great variety of base curves, but only the simple forms that are more commonly used will be considered here, namely, the modified straight-line base curve, the harmonic base curve, and the parabolic base curve.

For purposes of comparison the base curves shown at *a* in Figs. 65 to 68 have been drawn on base lines of equal length, and for equal displacements. Each of these base curves, therefore, may be considered as applying to a cam which imparts to the follower the same total displacement during the same angular motion of the cam. The difference lies in the law of motion of the follower. The velocity and acceleration diagrams shown directly beneath the base curves, at *b* and *c* respectively in Figs. 65 to 68, give a comparison of the characteristics of the follower motion for the base curves shown at *a* in these figures.

a. Straight-line Base Curve.—If the motion of the follower were represented by the straight line, Fig. 65*a*, it would have equal displacements in equal units of time, *i.e.*, uniform velocity from the beginning to the end of its stroke, as shown at *b*. The acceleration except at the ends of the stroke would be zero, as shown at *c*. The diagrams indicate very clearly the abrupt changes of motion, with the consequent large forces, at the beginning and end of the stroke. This condition is undesirable in any case and especially so when the cam rotates at high velocity. The straight-line base curve is, therefore, only of theoretical interest.

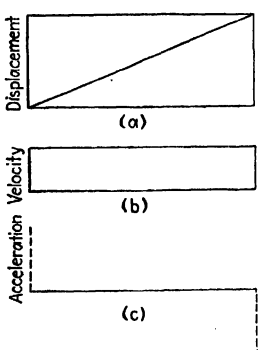


FIG. 65.

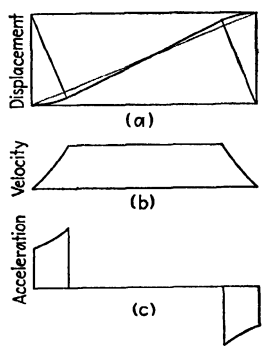


FIG. 66.

b. Modified Straight-line Base Curve.—To avoid the very undesirable condition described in *a* above, the law of motion of the follower should be such as to give gradually increasing velocity at the beginning of the stroke and then gradually decreasing velocity at the end of the stroke. This can be accomplished in the case of the straight-line base curve of Fig. 65*a* by easing off the corners of the diagram as shown in Fig. 66*a*, giving what is known as the modified straight-line base curve. The lengths of the periods for acceleration and retardation at the ends of the stroke, and for uniform motion between, depend entirely on the "easing-off" radius. This radius may be taken as any convenient length, within reasonable limits. The shorter the radius the nearer the approach to the undesirable conditions of the straight-line base curve. The longer the radius the more gradual the action at the ends of the stroke, but the more abrupt at the middle of the stroke. A radius equal to the follower displace-

ment is commonly used in practice. The diagrams of Fig. 66 are based on this radius. Although the velocity and acceleration diagrams at *b* and *c* in this figure show a decided improvement in the character of the follower motion as compared with the similar diagrams at *b* and *c* in Fig. 65, the motion is not so smooth as that of Figs. 67 and 68, which will be discussed presently.

c. Harmonic Base Curve.—A cam with the base curve at *a* in Fig. 67 will impart simple harmonic motion to the follower. This curve is sometimes called the *crank curve*, owing to the fact that when a crankpin moves with uniform speed in a circular path the motion of the projection of its center on the diameter of the circle

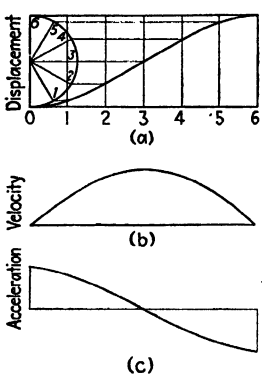


FIG. 67.

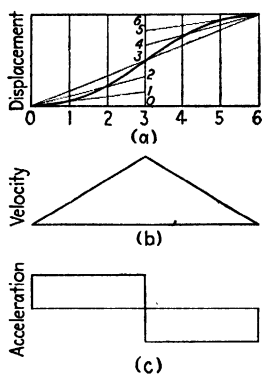


FIG. 68.

is a simple harmonic motion. The velocity diagram at *b* indicates smooth action. The acceleration, as shown at *c*, is a maximum at the initial position, zero at the mid-position, and a negative-maximum at the final position. The method of constructing the harmonic base curve is shown in the figure.

d. Parabolic Base Curve.—The parabolic base curve is shown in Fig. 68*a*. As indicated at *b*, the velocity increases at a uniform rate during the first half of the motion and decreases at a uniform rate during the second half of the motion. The acceleration is constant and positive throughout the first half of the motion, as shown at *c*, and is constant and negative throughout the second half. Because of the constant acceleration this base curve is sometimes called the *gravity curve*. A method of constructing this curve is shown in the figure.

53. Disk Cam with Radial Roller Follower. *a. Layout of the Cam.*—A disk cam rotating clockwise at a uniform rate is to impart harmonic motion to a roller moving in a radial path as follows:

Up ¾ in. while the cam turns through 120 deg.

Rest while the cam turns through 60 deg.

Down ¾ in. while the cam turns through 90 deg.

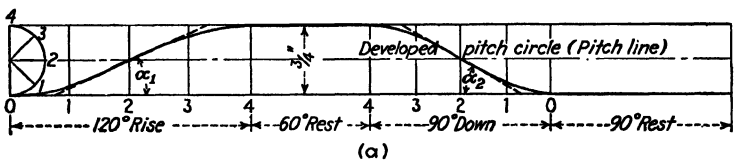
Rest while the cam turns through 90 deg.

The motion of the follower is fully represented by the displacement diagram shown in Fig. 69*a*, and its lowest position with respect to the axis of the camshaft is shown in Fig. 69*b*. The first step in the solution is to lay out the scale of motion of the follower along the path of the axis of the roller between its initial and final positions, 0 to 4 respectively. This may be accomplished either by transferring points from the displacement diagram or by laying out a semicircle of ¾ in. diameter, dividing it into a number of equal parts (four were assumed in this case), and projecting to the diameter to obtain the divisions 0 to 4 shown along the line of the follower stroke.

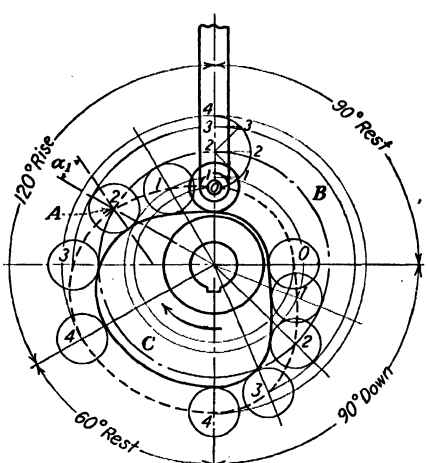
Since the follower is to complete its cycle of motion during one revolution of the cam, the next step is to lay out around the axis of the cam the four angular divisions, 120, 60, 90, and 90 deg., required for the motion of the follower. In the solution of this problem, the same expedient is employed as in Art. 50, the cam being assumed as fixed and the frame and follower rotated in a sense the reverse of that specified for the cam. Since the cam motion is clockwise, the angles referred to above will therefore be laid out in a counterclockwise sense from the path of motion of the follower, as shown in the figure. Now, since the displacement scale has been constructed on the basis of four equal time intervals for both the 120-deg. up and the 90-deg. down motion, these angular spaces are divided into four equal divisions and numbered 0 to 4 as shown. Arcs with radii equal to distances from the axis of the cam to the numbered divisions on the displacement scale will intersect correspondingly numbered radial lines at points that represent constructed positions of the axis of the roller follower. Arcs equal to the radius of the roller follower are drawn about these centers, and the working surface of the cam obtained by drawing a smooth curve tangent to these arcs. The working surfaces of the cam for the periods of rest of the

follower are obtained by connecting the working surfaces for the up and down motion of the follower with circular arcs drawn from the axis of the camshaft.

b. Pressure Angle.—In general, the length of the displacement diagram is immaterial. If, however, in cases like Fig. 69, where a roller follower moves in a radial path, the length of the displacement diagram is made equal to the circumference of the



(a)



(b)

FIG. 69.

circle *ABC*, the diagram will represent a true development of the pitch surface of the cam. This circle, whose radius is equal to the distance from the axis of the cam to the mid-point of the travel of the follower, is called the *pitch circle.* In Fig. 69*a* the length of the displacement diagram was made equal to the circumference of the pitch circle. The developed pitch circle in the displacement diagram is called the *pitch line.*

The angle that the tangent to the base curve in the displacement diagram makes with the base line is known as the *pressure*

angle. In Fig. 69*a* the maximum pressure angle is obviously at the mid-point of the stroke, and it is indicated as α_1, on the up stroke and α_2 on the down stroke. It may be noted that the velocity of the follower is proportional to the tangent of the pressure angle. Referred to the actual cam outline, the pressure angle is the angle between a normal at the point of contact of the cam and follower and the axis of the follower stem. In Fig. 69*b*, for example, the maximum pressure angle α_1 on the working stroke is located at the point *A*. For this type of cam mechanism the maximum pressure angle is at the point where the pitch circle intersects the pitch surface of the cam.

It has been commonly assumed that 30 deg. should be the maximum limit for the pressure angle on the working stroke, but this limit must be governed by the character of the design, and in many cases a much greater angle may be used with satisfactory results.

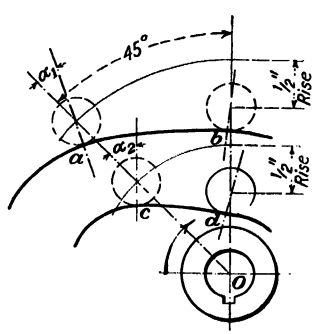


FIG. 70.

c. Size of Cam.—For a given stroke, or travel, of the follower, the maximum size of the cam is governed only by the space available. The minimum size may be determined by constructive details such as size of camshaft, or by the consideration of pressure angle. The size chosen should be large enough to insure an easy motion of the follower. The larger the cam the easier will be the motion of the follower for any given travel in any given angular rotation of the cam, as shown in Fig. 70. Let *O* be the center of the camshaft and let 45 deg. be the angular motion of the cam required to raise the roller follower ½ inch. It can be seen that the part of the larger cam outline *ab*, which produces the rise, imparts a more gradual motion to the follower than does curve *cd* on the smaller cam. It should be noted also that in the case of the larger cam the pressure angle is smaller. This fact may be observed by comparing angles α_1 and α_2.

Although the parabolic base curve combines the two most important theoretical considerations, namely, smoothest motion and least power for operation, the harmonic curve permits of a smaller cam for a given pressure angle than does the parabola,

provided the follower is a roller moving in a radial direction. In other words, for the same size of cam, the pressure angle is smaller for the harmonic curve, and therefore the action of the harmonic curve is easier in this respect than the action of the parabolic curve.

This fact is evident from Figs. 71 and 72. For the same maximum pressure angle α and the same rise t, the length l of the base curve will bear a definite relation to the rise t. Since the displacement diagram, when drawn to the proper scale, represents the development of the pitch surface of the cam, the line CD (equal to l) in Fig. 71 is a development of a part of the pitch circle of the cam. Such a pitch circle cde in its relation to the cam and follower may be seen in Fig. 72. Its circumference is equal to

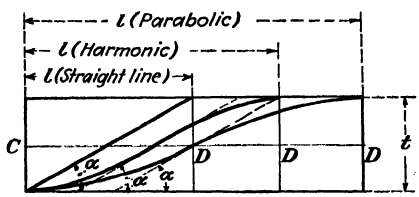


FIG. 71.

the length of the complete displacement diagram, and it passes through the point A of the pitch surface of the cam where the cam has its greatest side pressure (pressure angle) against the follower. This discussion applies, however, only to cams in which the center of the roller follower moves in a radial line.

It will now be shown that the radius r of the pitch circle cde is less for the harmonic base curve than for the parabolic base curve, when the maximum pressure angles α are the same. Let it be assumed that the rise t of the follower is to take place while the cam turns through an angle β and that cd in Fig. 72 is equal to the length l of the displacement diagram (Fig. 71).

$$\text{Arc } cd = ft = \frac{2\pi r\beta}{360} \tag{11}$$

where

$$f = \frac{l}{t} \text{ (called the cam factor, Fig. 71)}$$

therefore

$$r = \frac{360ft}{2\pi\beta} = kf \tag{12}$$

where

$$k = \text{a constant} = \frac{360l}{2\pi\beta}$$

For a maximum pressure angle of 30 deg., $f = 2.72$ for the harmonic curve, and 3.06 for the parabolic curve. It is evident, therefore, that the cam with the harmonic base curve is the smaller and that, in general, for the same maximum pressure angle, the size of the cam is directly proportional to the length l of the base curve.

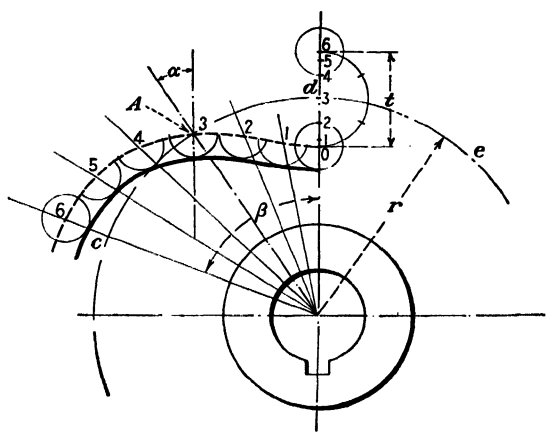


FIG. 72.

The pitch circle serves no purpose other than that of obtaining a cam of minimum size for a given pressure angle in cases like the preceding where the *follower is a roller moving in a radial path.* For other motions of the roller follower, and for flat-faced followers, the cam is generally independent of a pitch circle based on pressure angle, and hence no reference will be made to these terms in the discussion of other types of cams.

54. Disk Cam with Offset Roller Follower.—In Fig. 73 the roller follower is required to move in the straight-line path from 0 to 4 with harmonic motion while the cam turns clockwise with uniform motion through 120 deg. Since the straight-line path of the roller (extended) does not pass through the axis of the camshaft, this is known as an offset roller follower. Consideration of only the upward motion of the follower as indicated in

the figure will be sufficient for the present purpose. The working surface of the cam is determined by the general method outlined in Art. 50. One important difference, however, should be carefully noted when comparing this solution with that of Fig. 62. Assuming the vertical line through the axis of the cam as the zero or initial line for the layout, the angle of 120 deg. is laid off in a counterclockwise sense from this line and subdivided, as

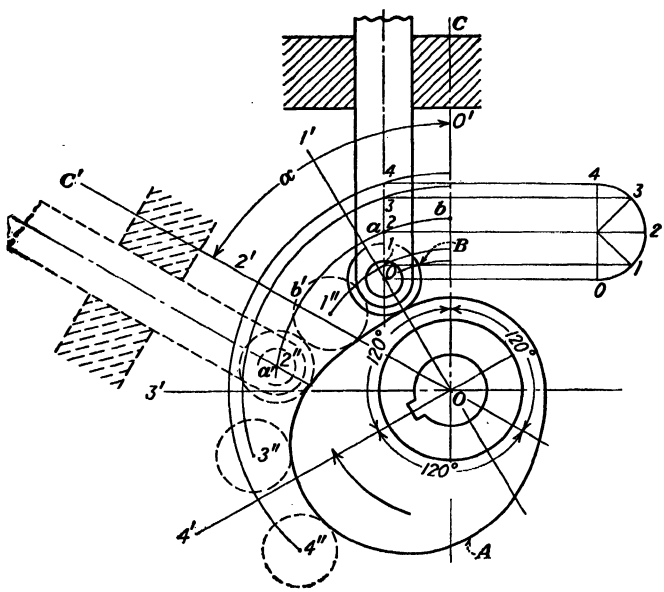


FIG. 73.

required, by the equally spaced radial lines 0′, 1′, 2′, 3′, and 4′. The constructed positions of the roller follower are not located along the radial lines, but are offset from corresponding radial lines the same distance as the actual positions of the roller from the initial radial line *OC*. For example, in position 2″, arc *a′b′* must equal arc *ab* so that when line *OC′* on the cam rotates into initial position *OC* the axis of the roller will be in its required position at 2 along its line of travel.

It should be noted here that the construction shown in Fig. 73 is only one of several possible methods that could be used in determining the constructed positions of the roller follower such

as shown at 2″. The initial or zero line *OC*, for example, could have been assumed as a line drawn from *O* through the initial, or zero, position of the roller, in which case the offset such as *a′b′* would vary. Another method is to let the path of the roller represent the initial or zero line. In this case the four division

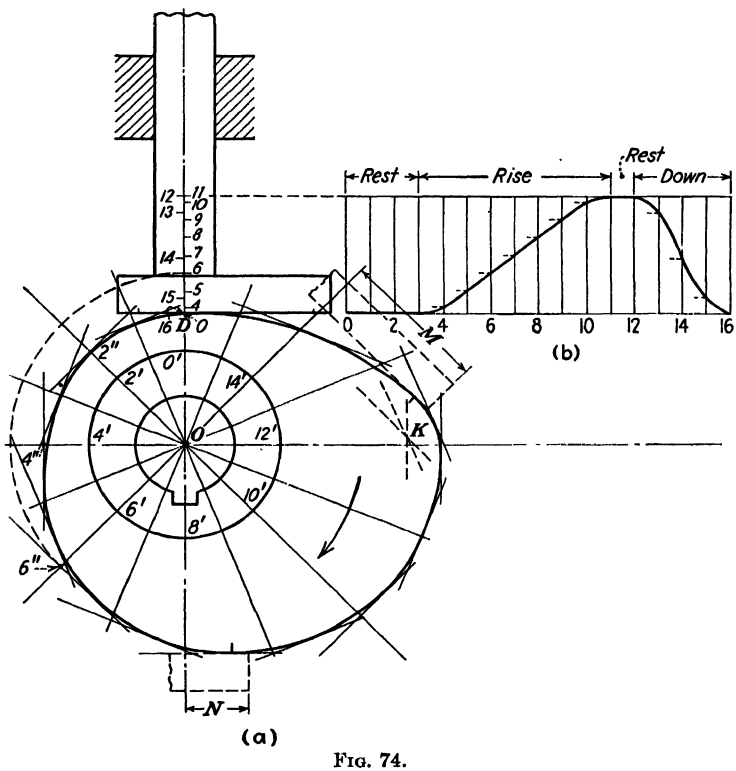


FIG. 74.

lines such as 0′, 1′, 2′, 3′, 4′ can be drawn tangent to a circle through *O*, in which case no offsets will be required.

55. Disk Cam with Reciprocating Flat-faced Follower.—In this problem it is required to determine the working surface of a disk cam (Fig. 74*a*) to impart motion to a reciprocating flat-faced follower in accordance with the displacement diagram shown in Fig. 74*b*. The cam is to displace the follower a total of 1¼ in. during one revolution while turning clockwise at a uniform rate.

The time for one revolution of the cam has been divided into 16 equal time intervals. The follower is to dwell in its initial position during three time intervals, rise during eight time intervals, dwell in its extreme position during one time interval, and return to its initial position in four time intervals. The displacement diagram given does not represent any definite law of motion for the follower. However, it was drawn so as to give a gradual motion at the beginning and end of the stroke.

The initial, or lowest, position of the follower was taken as $1\frac{3}{8}$ in. from the axis of the cam. If the displacement diagram is placed in line with the displacement of the follower, the displacement scale shown in the figure is easily constructed by projection. The most convenient point from which to start the displacement scale is the point D where a perpendicular through O, the axis of the cam, intersects the contact surface of the follower. As the cam rotates clockwise, point D in the contact surface of the follower advances to positions 1, 2, 3, etc.; or, following the usual scheme of assuming an inversion of the mechanism while the construction is in progress, as the follower is rotated counterclockwise with the cam stationary, point D moves to the radial positions 1″, 2″, 3″, etc. Hence, for the sixth position, *e.g.*, the intersection of an arc of radius $O6$ with radial line $O6'$ of the cam determines the position 6″ of the point D of the contact surface of the follower. Since this contact surface is perpendicular to the center line of the stem of the follower, a line through 6″ perpendicular to $O6'$ determines one line to which the working surface of the cam must be tangent. Lines representing the contact surface of the follower having been similarly located for the remaining positions, the working surface of the cam is obtained by drawing a smooth curve tangent to them as shown.

In the layout of cams with flat-faced followers, limitations may be found as to minimum size, because below a certain size it may not be possible to draw the outline of the cam tangent to all the constructed positions of the follower. For example, had the minimum radius OD of the cam been taken as $1\frac{1}{16}$ in., the lines for positions 12, 13, and 14 would have intersected in a point as shown at K, thus requiring the outline of the cam to come to a sharp point. Had the minimum radius OD been taken still smaller, the line for position 13 would fall outside the

intersection of the lines for positions 12 and 14, in which case a solution in accordance with the specifications given would be impossible.

The length of face of the follower is determined by the maximum distances of the line of contact to the right or left of *D*. It is a maximum to the right for position 14 and a maximum to the left for position 8 as shown by the dotted outlines in the figure. For obvious practical reasons the actual length of contact surface should extend somewhat beyond the theoretical points of contact as indicated by dimensions *M* and *N*.

If the stem of the follower is cylindrical and its axis located as shown in Fig. 74 but offset from the mid-plane of the cam, the friction between the cam and follower face will cause the follower to rotate while reciprocating, thus requiring the follower face to be in the form of a circular disk of radius *M*. Such a follower is known as a *mushroom follower*. This arrangement has the advantage of assisting lubrication and distributing wear.

For the reciprocating flat-faced follower, the pressure between the cam and follower, friction neglected, is at right angles to the face of the follower. The pressure angle is constant, therefore, and equal to 90 deg. minus the angle between the face of the follower and the stem. In the present instance the angle between the face and the stem is 90 deg., and hence the pressure angle is zero.

56. Disk Cam with Pivoted Roller Follower.—Another case of the roller follower is that in which the center of the roller is constrained to move in a circular arc. Let it be required to lay out a disk cam which will give harmonic motion to a pivoted roller follower in the following manner:

Up 15 deg. while the cam turns through 120 deg.

Rest while the cam turns through 60 deg.

Down 15 deg. while the cam turns through 90 deg.

Rest while the cam turns through 90 deg.

The cam is to have a uniform clockwise rotation. The location of the follower pivot relative to the axis of the cam, and the lowest position of the roller, are shown in Fig. 75.

The first step is to subdivide the motion of the follower harmonically. If the chord of the arc through which the roller travels is taken as the diameter of the circle that is used in laying out the harmonic motion, the construction will, in general,

be sufficiently accurate. The numbered positions of the follower correspond to the equal subdivisions (similarly numbered) of the angles through which the cam turns while the total rise or fall of the follower is taking place. The next step is to consider the cam mechanism "inverted" and to locate the follower (its working profile is sufficient) in positions corresponding to the

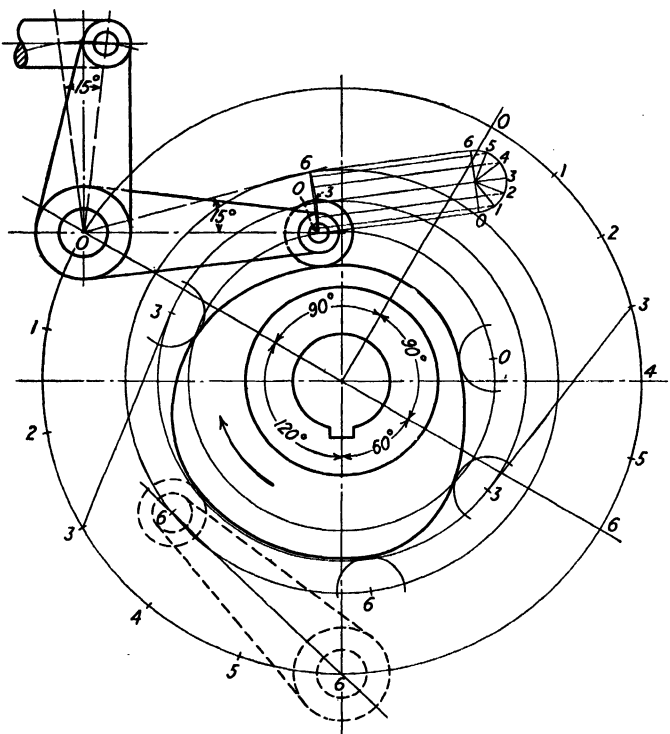


FIG. 75.

numbered positions in the original mechanism. This has been accomplished by a simple geometrical construction as indicated at position 6. Other geometrical constructions that may occur to the student may be used if desired. The working profile of the cam during the periods of rise and fall may now be drawn tangent to the various positions of the working profile of the follower. For the sake of clarity, only positions 0, 3 and 6 have been shown in the figure. In the actual layout all the positions

from 0 to 6 would be used. The parts of the profile of the cam that make contact during the periods of rest are arcs of circles having their centers at the axis of the camshaft.

57. Disk Cam with Pivoted Flat-faced Follower.—As a further example of a cam layout, let it be required to lay out a disk

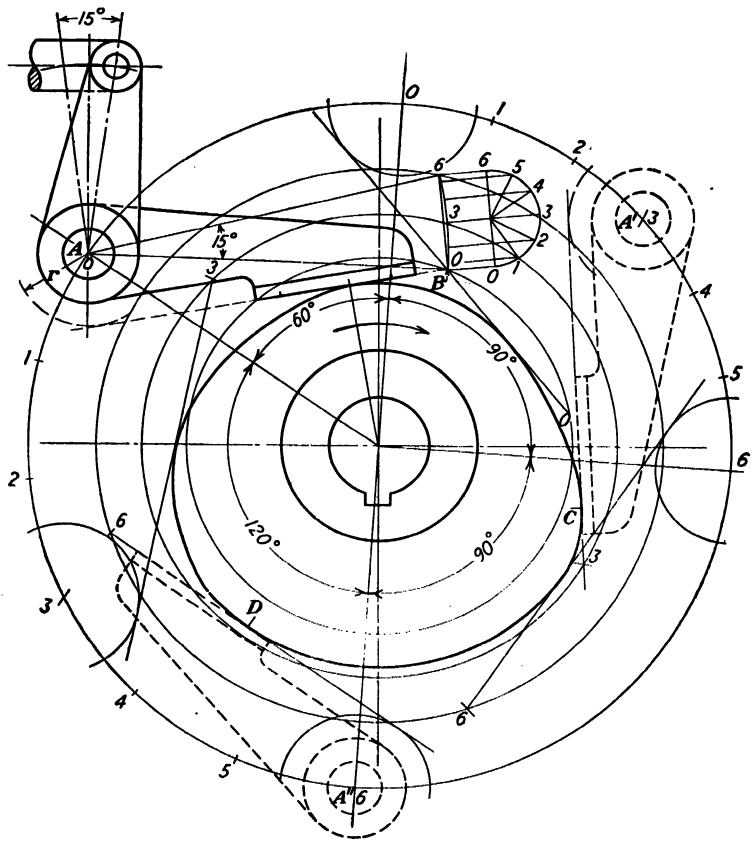


FIG. 76.

cam which will give harmonic motion to a pivoted flat-faced follower in the following manner:

Up 15 deg. while the cam turns through 120 deg.

Rest while the cam turns through 90 deg.

Down 15 deg. while the cam turns through 90 deg.

Rest while the cam turns through 60 deg.

The cam is to have a uniform clockwise rotation. The location of the follower pivot relative to the axis of the cam and the lowest position of the face of the follower are shown in Fig. 76.

The first step in the layout is to subdivide the angular displacement of the follower in accordance with harmonic motion. A little forethought here will materially reduce the labor of laying out the cam later on. The working profile of the follower is a straight line, and it will eventually be necessary to locate this straight line for the various positions of the follower in the "inverted" mechanism. The one point of the follower that has no motion relative to the fixed link is the center A about which the follower swings. The straight-line working profile of the follower will always be tangent to a circle of radius r drawn from the successive positions of this center. The position of one other point B on the straight line will be sufficient to locate it. If now an arc with A as a center is drawn through any point B and 15 deg. laid off on this arc, the total angular motion of the follower will be completely represented. This arc may now be subdivided in accordance with the required motion as shown in the figure. Using the chord instead of the arc for this construction is an approximation, but is sufficiently accurate for almost any condition. The location of the follower in the various positions in the "inverted" mechanism is now accomplished without difficulty, and the working profile of the cam is drawn in the usual manner. For the sake of clarity, only the construction for positions 0, 3, and 6 has been shown in Fig. 76.

The length and location of that part of the face of the follower that comes into contact with the cam during a complete revolution may be determined by measuring the longest and shortest distances from the follower pivot A to the points of tangency of cam and follower. In the figure these distances are $A'C$ and $A''D$ respectively, and, when laid off on the follower in the original mechanism, they completely determine the working face of the follower. For obvious practical reasons a small amount should be added to the theoretical length as determined above.

58. Other Types of Cams.—As previously indicated, the return motion of the follower in the types of cams that have just been discussed must be brought about by some force external to the cam. This is usually accomplished by the use of weights or springs. Where a positive action is desired, the usual method is

to add a second working profile of the cam on the outside of the roller in the manner shown in Fig. 77.

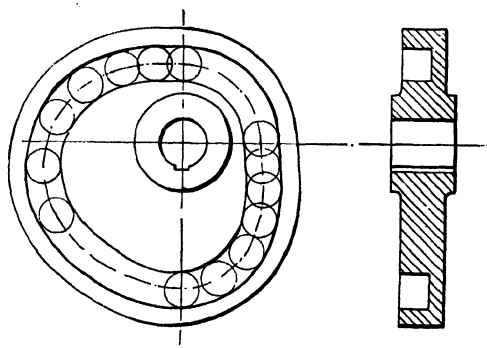


FIG. 77.

In designing positive-motion cams, the radius of the follower roller must be considered. This radius may be equal to, but in general should be less than, the shortest radius of curvature of

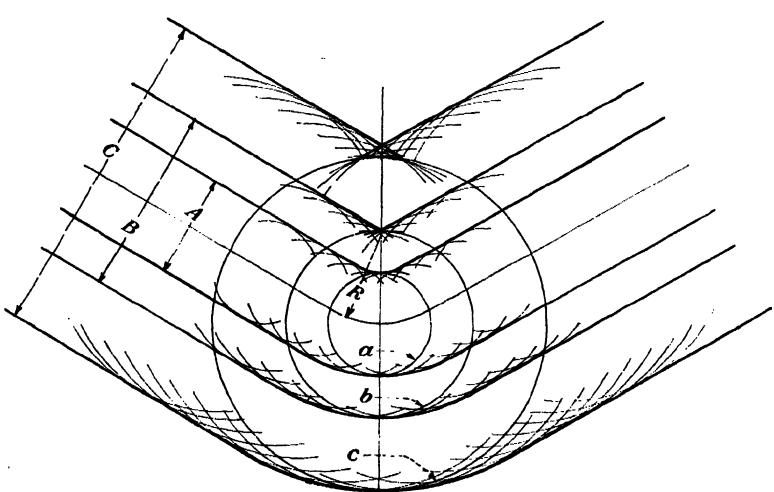


FIG. 78.

the pitch surface, when measured on the working-surface side. If the radius of the roller is not so taken, the follower will not have the motion for which it was designed. Thus, in Fig. 78,

let *a*, *b*, and *c* represent three rollers moving in grooves of widths *A*, *B*, and *C*, respectively. The radius of curvature of the pitch surface common to the three grooves at the point under consideration is indicated by *R*. Roller *a* has a radius smaller than *R*, roller *b* a radius equal to *R*, and roller *c* a radius greater than *R*. It is evident from the figure that when the radius of the roller is greater than *R*, as in the case of roller *c*, the width of groove at that point becomes greater than the diameter of the roller, and positive motion cannot be assured.

A *plate cam*, or *sliding cam*, such as shown in Fig. 79, has a reciprocating motion between fixed guides. It carries a curved face or slot that may impart reciprocating motion to a roller follower or oscillating motion to a pivoted-roller-follower arm.

An inversion of the plate cam in which the roller becomes the driver and the plate

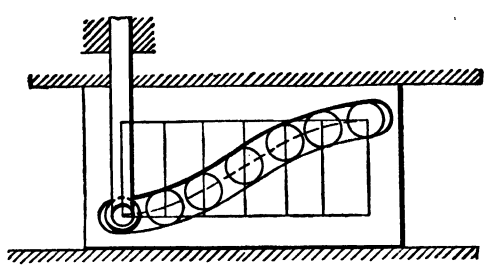


FIG. 79.

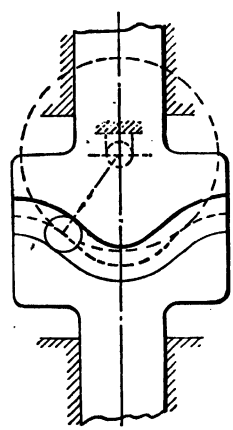


FIG. 80.

the follower is sometimes used in light mechanisms such as sewing machines. An example of this type of cam, known as an *inverse cam*, is given in Fig. 80.

A *cylindrical cam* may be used to give the same motion as a sliding cam. A cam of this form may be looked upon as a sliding cam bent to a cylindrical shape. As the cylinder is rotated on its axis, the follower is given precisely the same motion as when the sliding cam is given a motion of translation. The general method of laying out a cylindrical cam with a pivoted roller follower is shown in Fig. 81. The first step is to lay out a displacement diagram in which the height of the diagram is equal to the stroke of the follower and in which the length of the diagram is equal to the circumference of the cylinder. The diagram is next wrapped around the cylinder and the required cam outline

transferred to the surface of the cylinder. The groove in the cylinder may now be milled by a cutter of the same size and shape

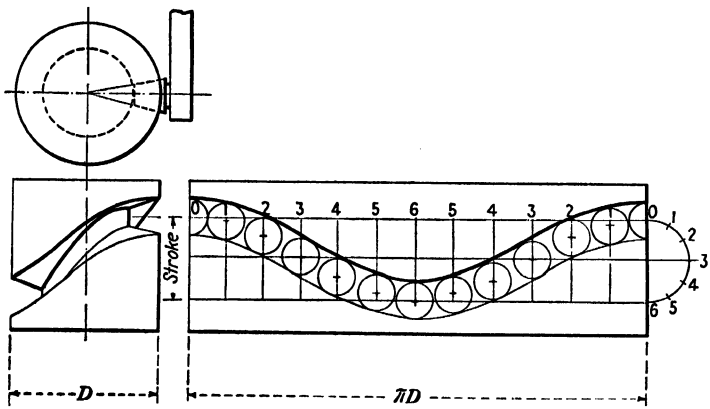


FIG. 81.

as the roller, the cutter being guided so as to conform to the curve marked on the cylindrical surface. It should be noted that the roller follower is usually the frustum of a cone whose apex is on the axis of the cam.

Another form of cylindrical cam is obtained by fastening adjustable strips on the surface of a cylinder. This form is very much used on automatic screw machines.

Disk cams with roller or mushroom followers, such as illustrated in Fig. 82, are usually employed to operate the valves of automobile engines. In order that such cams may be accurately made on a production basis their outlines are usually made up of circular arcs, or straight lines and circular arcs, as indicated in the figure.

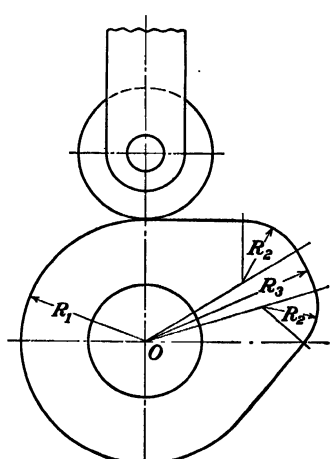


FIG. 82.

In Fig. 83 is shown a positive-return disk cam whose outline is made up of circular arcs of two different radii. In order that the cam may be of constant breadth, the centers O, B, and C must be at the apexes of an

equilateral triangle. It can be seen that there is a dwell at each end of the stroke while the cam turns through 60 deg. The

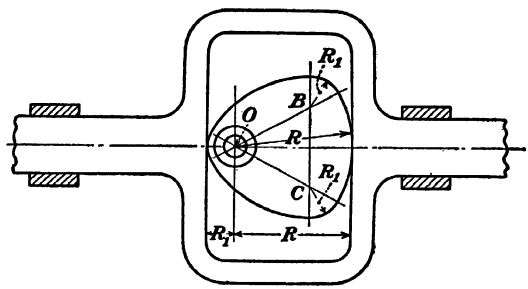


FIG. 83.

total displacement $R - R_1$ takes place while the cam turns through 120 deg. Such cams are designed for a given total displacement of the follower. The law of displacement, therefore, must depend upon the assigned outline of the cam.

In Fig. 84 is shown an oscillating disk cam that imparts reciprocating motion to a flat-faced follower. Because of its form and action this combination is often spoken of as a

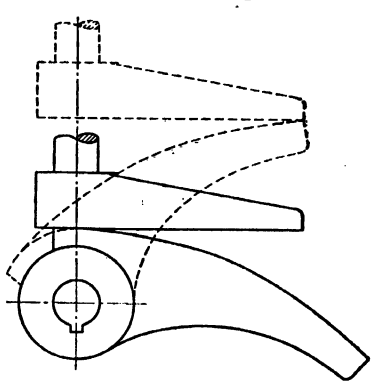


FIG. 84.

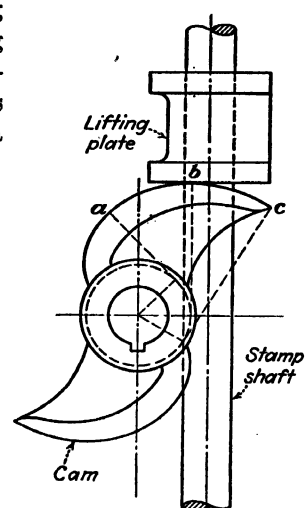


FIG. 85.

toe-and-wiper cam. It is frequently seen in the valve mechanism of marine engines.

A well-known form of cam used in stamp mills for pulverizing crushed ore, is the *stamp-mill cam* shown in Fig. 85. It consists

of a reciprocating flat-faced follower actuated by a rotating disk in the form of a double involute cam. When the cam rotates at a uniform rate, the involute outline of the cam imparts uniform velocity to the follower. The construction of the involute outline is indicated in the figure by the dotted construc-

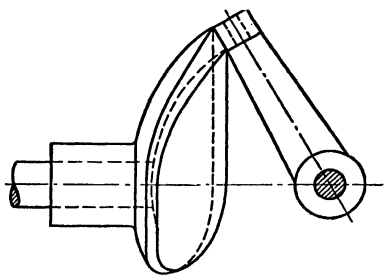


FIG. 86

tion at the three points *a*, *b*, and *c*. The stamp is raised and dropped twice during each revolution of the cam.

A spherical form of cam is shown in Fig. 86. There are many other interesting types of cams that enter into the construction of modern machinery in a great variety of forms, but space does not permit their discussion here. The fundamental principles involved in laying out any of these cams are the same as those that have been applied in the examples given in this chapter.

CHAPTER IV

TOOTHED GEARING—SPUR GEARS

The study of gear-tooth action is fundamentally a study of the motion transmitted by a pair of members having curved surfaces in direct contact. Preliminary to the study of gearing, therefore, the general case of a pair of curved members in driving contact will be considered. Figure 87*a* shows two such members, with the driving member 2 pivoted at O_2 and the driven member 3 pivoted at O_3. Except for obvious practical limitations the contact surfaces may be of any form, and they may move upon each other with a pure rolling action, a pure sliding action, or with a combined rolling and sliding action. The last condition is the most general.

59. Nature of the Motion Transmitted by Curves in Direct Contact.—In *pure rolling* action no one point of either member comes in contact with two successive points of the other, for example, two circles rolling upon each other without slipping. If a point on one member comes in contact with all successive points of the acting surface of the other, within the limits of the motion, the action is *pure sliding*. This action occurs in the case of one circle (in the above example) being held stationary and the other made to rotate while in contact with it. *Combined rolling and sliding* is exemplified in the case of a pair of circles rolling upon each other with some slipping permitted. The precise nature of these actions will now be considered.

Referring to Fig. 87*a*, it is evident that all points in 2 must rotate about O_2, and that all points in 3 must rotate about O_3. Consequently, the motion of any point in either 2 or 3 is represented for the instant by a line through that point perpendicular to the radius connecting it with its center O_2 or O_3. The point of contact P between the members may be considered as a pair of coincident points. Considered as a point on 2, P will be designated as P_2, and as a point on 3 it will be designated as P_3

Then P_2M_2 and P_3M_3 will represent the motions of P_2 and P_3, respectively.

Motion by direct contact can be transmitted only by normal pressure between the surfaces. The pressure between 2 and 3

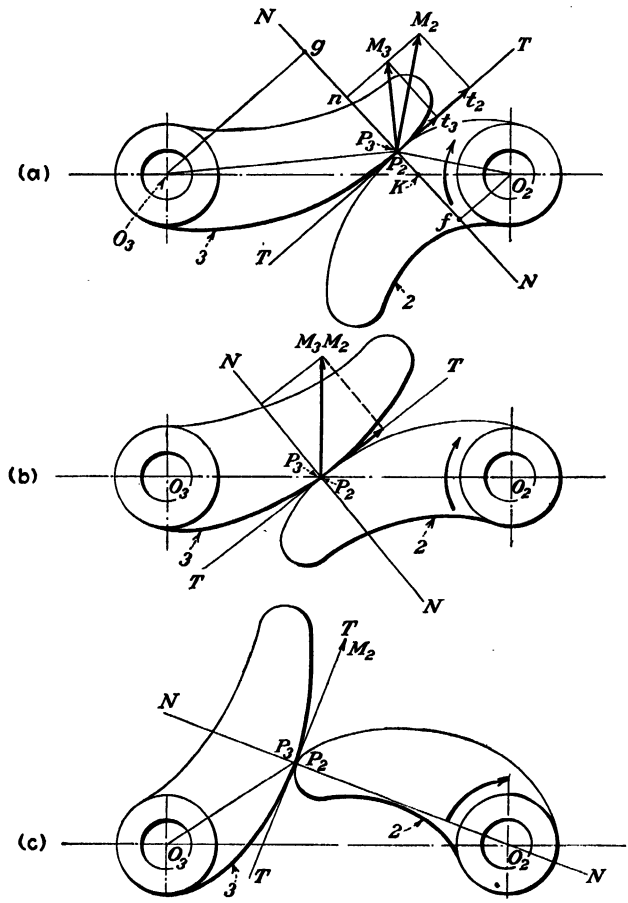


FIG. 87

is, therefore, transmitted in the direction NN, the common normal to the two surfaces at the point of contact, and whatever the actual motions of P_2 and P_3 the components Pn of these two motions along the line of the normal NN must be equal. If this were not the case, the surfaces would have either to separate or to cut into each other. The tangential components of the

motion of the coincident contact points P_2 and P_3 must be in the direction of the common tangent TT, but they may have different magnitudes, in either the same or opposite directions. In Fig. 87*a* it is seen that these tangential components P_2t_2 and P_3t_3 are in the same direction, but of different magnitudes. It is evident that this difference of the tangential components represents the rate of the sliding action for this particular phase of the motion. When the tangential components are in the opposite directions, the rate of sliding is evidently their sum. Thus *the rate of sliding is the algebraic difference of the tangential components.*

When the tangential components are equal, the nature of the action is pure rolling. In Fig. 87*b* the members of Fig. 87*a* have been rotated into positions to illustrate this condition, and the motions P_2M_2 and P_3M_3 are identical. But P_2, as a point on 2, moves for the instant at right angles to O_2P_2, and P_3 as a coincident point on 3 moves at right angles to O_3P_3. Therefore, when P_2M_2 and P_3M_3 are coincident, O_2P_2 and O_3P_3 are both perpendiculars to the same line at the same point, and they must, therefore, lie in one straight line. In order that this may occur, the contact point P must lie on the line of centers. Thus *the condition of pure rolling is that the point of contact shall always lie on the line of centers.*

When the common normal NN passes through the center about which the driver rotates, as in Fig. 87*c*, the radius O_2P_2 coincides with it and the motion P_2M_2 of the contact point of the driver lies in the direction of the common tangent TT; hence the normal component of this motion is zero, and P_3 has no motion. It is seen, therefore, that for this particular phase of the motion, the action is for the instant pure sliding.

60. Angular-velocity Ratio for a Pair of Curves in Direct Contact.—In the general case of a pair of members with curved surfaces in contact, as in Fig. 87*a*, the angular-velocity ratio of driver to follower is continually varying. This ratio for any instant may be determined by the most elementary principles of mechanics. Since the ratio of the angular velocities is independent of the actual angular velocities, the angular velocity of one member may be assumed, and this affords a means of determining the corresponding angular velocity of the other member at that instant.

If in Fig. 87*a* the angular velocity of link 2 is ω_2, the linear velocity P_2M_2 of P as a point on 2 can be found from the relation

$$P_2M_2 = O_2P_2 \times \omega_2 \tag{13}$$

The direction of the velocity of P_3 as a point on 3 (perpendicular to O_3P_3) is known, and it has been shown that its normal component must be equal to that of P_2, or Pn. The magnitude of the velocity of P_3 (P_3M_3) can now be found, and hence the angular velocity of 3 is

$$\omega_3 = \frac{P_3M_3}{O_3P_3} \tag{14}$$

The ratio of the angular velocities is therefore

$$\frac{\omega_3}{\omega_2} = \frac{P_3M_3}{O_3P_3} \times \frac{O_2P_2}{P_2M_2} \tag{15}$$

The above expression may be simplified in the following manner: Drop the perpendiculars O_2f and O_3g from O_2 and O_3 upon the common normal NN. P_2M_2n and O_2P_2f are similar triangles, and hence

$$\frac{P_2M_2}{O_2P_2} = \frac{P_2n}{O_2f}$$

Similarly, P_3M_3n and O_3P_3g are similar triangles and

$$\frac{P_3M_3}{O_3P_3} = \frac{P_3n}{O_3g}$$

Therefore

$$\frac{\omega_3}{\omega_2} = \frac{P_3n}{O_3g} \times \frac{O_2f}{P_2n} = \frac{O_2f}{O_3g} \tag{16}$$

The common normal NN intersects the line of centers O_2O_3 at K, O_2Kf and O_3Kg are similar triangles, and

$$\frac{\omega_3}{\omega_2} = \frac{O_2f}{O_3g} = \frac{O_2K}{O_3K} \tag{17}$$

It follows, therefore, that, for a pair of links with curved surfaces in direct contact, *the angular velocities are inversely as the segments into which the line of centers is divided by the common normal.*

From the above analysis it follows, also, that *for constant angular-velocity ratio the common normal must intersect the line of centers in a fixed point.* The two propositions just stated are of fundamental importance in the theory of toothed gearing, as will appear later in this chapter.

61. Curves That Give Pure Rolling Action.—There are many pairs of curves that will satisfy the condition of either constant-velocity ratio or pure rolling. Several cases of the former will be taken up later in connection with the study of gear teeth. Among those curves that will give a pure rolling action, the following may be mentioned: a pair of circles rotating about their centers; a pair of equal ellipses each rotating about one of its foci, with a distance between centers equal to the common major axis; a pair

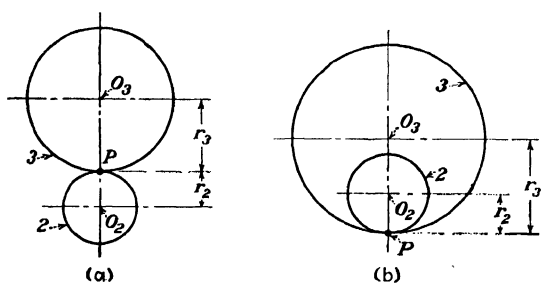


FIG. 88.

of similar logarithmic spirals rotating about their foci; a pair of equal parabolas; and a pair of equal hyperbolas.

There is but one class of curves, however, *i.e.*, circular arcs rotating about their centers, that can have *both constant angular-velocity ratio and pure rolling action.* It has been shown that for constant-velocity ratio, the common normal must intersect the line of centers in a fixed point; and that for pure rolling the point of contact must lie on the line of centers. If both these conditions are satisfied, the point of contact must be at a fixed place in the line of centers. As a result, the contact radii must be constant and, therefore, the curves are circular arcs.

62. Motion Transmitted by Members in Pure Rolling Contact.—It has been stated that there are several classes of curves that may be so paired as to give pure rolling action. In the present article, however, only those cases that have common application in the study of the theory of toothed gearing will be

considered, namely, rolling cylinders, rolling cones, and rolling hyperboloids.

a. Rolling Cylinders.—If two cylinders in contact rotate on each other about fixed parallel axes without slipping, the ratio of the angular velocities of the two cylinders will at all times be constant. Referring to *a* and *b* in Fig. 88, if there is no slipping at the point of contact P, P_2 and P_3 will have the same velocity at any instant. The angular velocities of the cylinders may be expressed in terms of the linear velocity of the point of contact, as follows:

$$\omega_2 = \frac{V_p}{O_2P} \qquad \text{and} \qquad \omega_3 = \frac{V_p}{O_3P}$$

Therefore

$$\frac{\omega_2}{\omega_3} = \frac{V_p}{O_2P} \times \frac{O_3P}{V_p} = \frac{O_3P}{O_2P} \tag{18}$$

That is, the angular velocities of the cylinders are inversely proportional to their radii. If one cylinder drives the other, the ratio of their angular velocities is designated in a definite manner, being the ratio obtained by dividing the angular velocity of the driver by the angular velocity of the follower or driven member. This ratio is called the *velocity ratio* of the pair of cylinders. This method of designating velocity ratio is in accordance with universal practice in discussions on gearing.

It should be noted that at *a* in Fig. 88 the cylinders rotate in opposite directions, and that at *b* they rotate in the same direction.

b. Rolling Cones.—The derivation of rolling cones from rolling cylinders is illustrated by the use of two right cylinders combined with two right cones as shown in Fig. 89. Each cylinder has one base in common with that of one of the cones; hence the axes of this cylinder and cone must coincide. Although the bases of the cones need not be equal, the slant heights must be the same. The bases of the two cones have a common tangent in their plane that passes through B (Fig. 89*a*), and that is perpendicular to the plane of the paper. If the axis O_3A_3 is caused to rotate about this tangent, the apex O_3 will coincide with the apex O_2, the bases of the cones will still have the common tangent at B and will be in contact along element O_2B (Fig. 89*b*). Therefore,

the cones will roll upon each other in their new positions, the contact points at their bases having equal velocities along the common tangent, as in the original positions where these bases were in common with those of the two rolling cylinders. Any other corresponding transverse sections of the cones equidistant

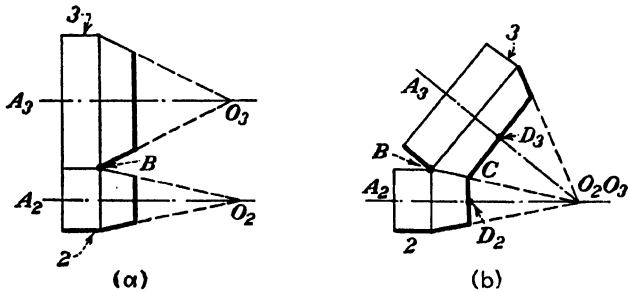


FIG. 89.

from the common apex O_2, as at C, will roll together with the same velocity ratio as their bases.
Therefore,

$$\frac{\omega_2}{\omega_3} = \frac{CD_3}{CD_2} = \frac{r_3}{r_2} \tag{19}$$

where r_2 and r_3 are the radii of the bases of the cones 2 and 3.

A pair of rolling cones may be constructed to transmit a given velocity ratio between two given shafts that intersect at some angle α, in the following manner:

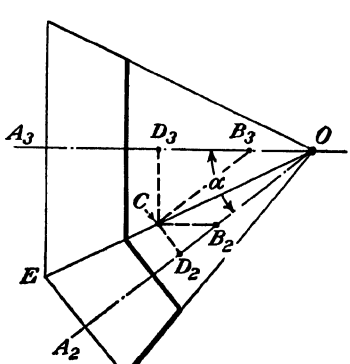


FIG. 90.

If, in Fig. 90, OA_2 and OA_3 are the given axes, the position of the contact element may be found by laying off from O on these shafts the distances OB_2 and OB_3 directly proportional to the given angular velocities of these shafts, thus

$$\frac{\omega_2}{\omega_3} = \frac{OB_2}{OB_3}$$

If on OB_2 and OB_3 a parallelogram is constructed, the diagonal of this parallelogram OC is the required common contact element.

The following proof will show the above construction to be correct:

From C drop the perpendiculars CD_2 and CD_3 to the axes OA_2 and OA_3. From similar triangles it can be seen that

$$\frac{CB_3}{CB_2} = \frac{CD_3}{CD_2} = \frac{A_3E}{A_2E} \tag{20}$$

But

$$\frac{CB_3}{CB_2} = \frac{OB_2}{OB_3}$$

Therefore

$$\frac{\omega_2}{\omega_3} = \frac{A_3E}{A_2E} \tag{21}$$

and the cones (or frustums of the cones) with any selected radii

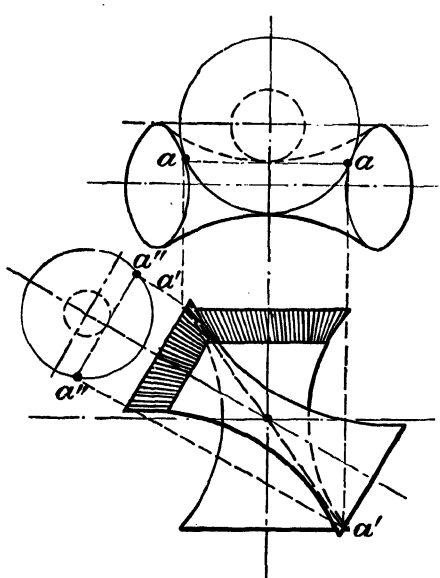


FIG. 91.

of bases as A_2E and A_3E will roll together with the required angular-velocity ratio.

c. Rolling Hyperboloids.—Figure 91 represents a pair of rolling hyperboloids of revolution tangent to each other along a common element AA. Since the elements of the surfaces are rectilinear,

any pair of elements can be placed in a tangent position. If the axes are fixed in positions corresponding to such tangency, the two surfaces will continue to be tangent along a common element as the two figures rotate about these axes. It is characteristic of the action of these two members as they roll together that there is a certain amount of sliding along the common contact element. In a sense, therefore, there is a departure from pure rolling action. This does not, however, prohibit the use of these forms as pitch surfaces for toothed gears. For a more complete discussion of rolling hyperboloids, see Barr's "Kinematics of Machinery."

63. Friction Gearing.—When two plain wheels of circular cross section are held in contact with each other by properly constructed bearings, the frictional force between their surfaces will be sufficient to transmit rotary motion from one axis to the other. Such an arrangement constitutes what is known as *friction gearing*, and it is chiefly limited to rolling cylinders, rolling cones, and disk wheels, though other rolling surfaces are possible. Some slipping between the contact surfaces is inevitable; therefore, a constant-velocity ratio cannot be assured. In many instances it is not essential to maintain this definite relation between driver and driven member, and in certain applications the likelihood of slipping when subjected to the excessive loads is desirable, since it serves to protect the driven mechanism against excessive stresses.

Figure 92*a* illustrates the principle of a friction drive between parallel shafts. The action is purely frictional as the pair rotate upon each other, and the velocity ratio remains constant unless the resistance of motion exceeds the frictional resistance. Friction gearing of the bevel type for transmitting power between intersecting shafts is illustrated in Fig. 92*b*. Figure 92*c* shows friction gearing of the disk-wheel form, which has been used extensively for varying feed movements in machine tools. The small *brush wheel*, A in this case, bears against the flat face of a cast-iron disk B, and the relative speeds are varied by changing the radial position of wheel A. The power that can be transmitted through friction gearing depends upon the physical characteristics of the materials forming the surfaces in contact and also upon the normal pressure between the surfaces. The driving-wheel face is usually made of a material that has a high coefficient of friction, as wood, compressed paper, or leather; while the

driven wheel is usually made of cast iron. In order to secure greater frictional resistance between the wheels than is possible with the plain surfaces in Fig. 92*a*, without at the same time increasing the pressure on the bearings, the wheels are sometimes made with V-shaped grooves and ridges, which mate with a wedging action, as shown in Fig. 92*d*.

Although limited in use to the transmission of small powers, friction gearing finds considerable application in speed-controlling devices, light power hoists, coal screens, friction-board drop hammers, etc.

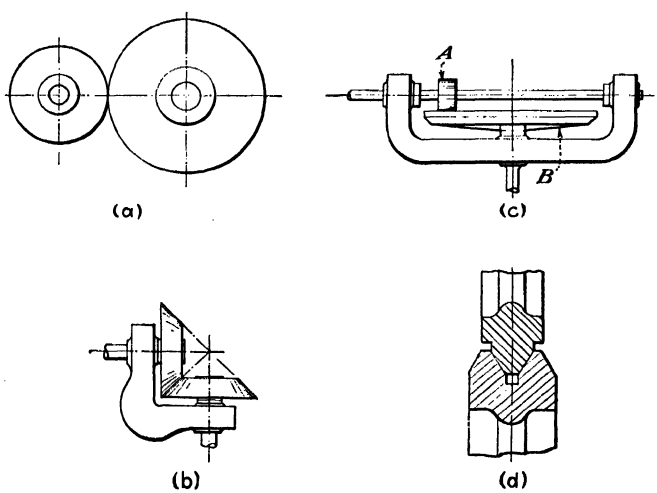


FIG. 92.

64. Toothed Gearing.—In the design of toothed gearing the contact surfaces of mating teeth are so shaped that the motion transmitted is the same as that of friction gearing when no slipping occurs. Kinematically, therefore, friction gearing running without slip and toothed gearing are identical, but as pointed out, friction wheels are of limited service in practice, and toothed wheels are employed where considerable power must be transmitted and where a constant-velocity ratio must be maintained at every phase in the cycle of motions. The surfaces of the rolling bodies which are kinematically identical with the toothed wheels which replace them are called *pitch surfaces* of the gears; and right sections of these surfaces are called *pitch lines* or *pitch curves*.

Toothed gearing may be classified according to relation of the axes and pitch surfaces, as in Table 1.

TABLE 1.—CLASSIFICATION OF TOOTHED GEARING

Name	Kind	Relation of axes	Pitch surfaces
Spur gears		Parallel	Cylinders
Bevel gears	Straight	Intersecting	Cones
	Spiral	Intersecting	Cones
	Skew	Not in one plane	Hyperboloids
	Hypoid	Not in one plane	Cones
Helical Gears		Parallel	Cylinders
		Not in one plane	Cylinders
Worm and Wheel		Not in one plane	Cylinders[1]

[1] Except in case of the Hindley worm (see Art. 97).

The characteristics of the various types of gears in the above classification will be discussed in some detail in the articles that

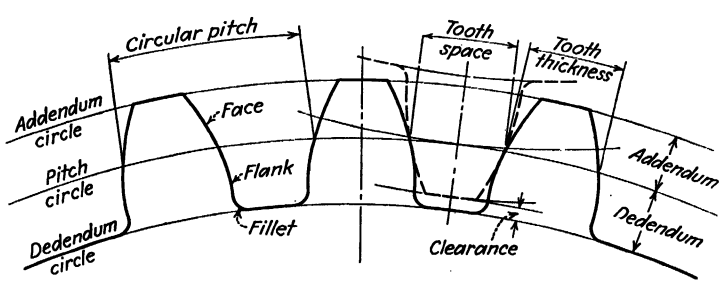


FIG. 93.

follow, after the fundamental principles that apply to toothed gearing in general have been considered.

65. Definitions.—In order to facilitate the discussion of toothed gearing, it is necessary to define certain terms that are in common use. Those terms that are most frequently mentioned are defined below. It will be noted that the definitions pertaining to gear-tooth parts may, in general, be represented on a right section of the gear. Figures 93 and 94 will, therefore, be of assistance in obtaining a clear understanding of these definitions.

1. The *pitch surface* is the surface of the imaginary rolling cylinder (cone, etc.) that the toothed gear may be considered to replace.

2. The *pitch circle* (also called pitch line) is a right section of the pitch surface.

3. The *addendum circle* is the circle bounding the ends of the teeth, in a right section of the gear.

4. The *dedendum*, or *root circle*, is the circle bounding the spaces between the teeth, in a right section of the gear.

5. The *addendum* is the radial distance between the pitch circle and the addendum circle.

6. The *dedendum* is the radial distance between the pitch circle and the dedendum circle.

7. The *clearance* is the difference between the dedendum of one gear and the addendum of the mating gear.

8. The *face* of a tooth is that part of the tooth surface lying outside the pitch surface.

9. The *flank* of a tooth is that part of the tooth surface lying inside the pitch surface.

10. The *tooth thickness* (also called the *circular thickness*) is the thickness of the tooth measured on the pitch circle. It is the length of an arc and not the length of a straight line.

11. The *tooth space* is the distance between adjacent teeth measured on the pitch circle.

12. The *backlash* is the difference between the tooth thickness of one gear and the tooth space of the mating gear.

13. The *circular pitch* is the width of a tooth and a space, measured on the pitch circle.

14. The *diametral pitch* is the number of teeth of a gear per inch of its pitch diameter.

A definite relation exists between the circular pitch and the diametral pitch, which may be determined as follows:

A toothed gear must have an integral number of teeth. The circular pitch, therefore, is equal to the pitch circumference divided by the number of teeth. The diametral pitch is, by definition, the number of teeth divided by the pitch diameter. That is,

$$p' = \frac{\pi D}{T} \quad \text{and} \quad p = \frac{T}{D}$$

Hence

$$pp' = \pi \tag{22}$$

where p' = circular pitch.
p = diametral pitch.
T = number of teeth.
D = pitch diameter.

That is, the product of the diametral pitch and the circular pitch $= \pi$.

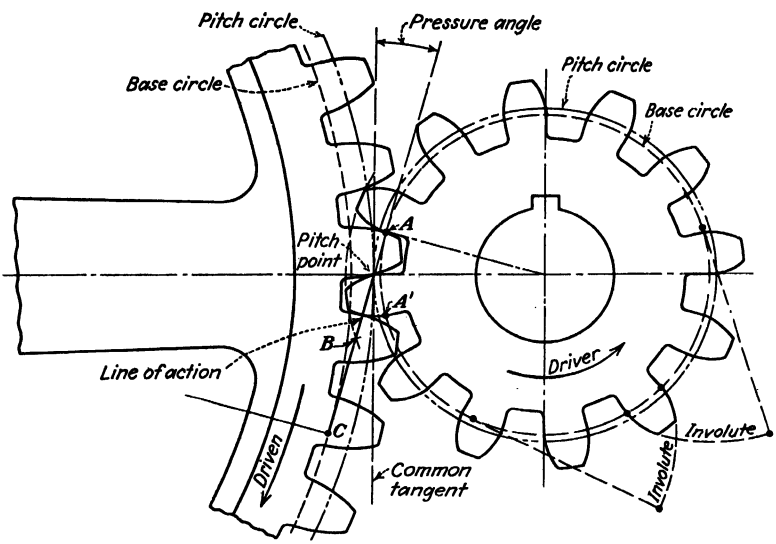


FIG. 94.

15. The *fillet* is the small radius that connects the profile of a tooth to the dedendum circle.

16. A *pinion* is always designated as the smaller of any pair of mating gears.

17. The *velocity ratio* in a pair of mating gears is always the ratio of the number of revolutions of the driving gear to the number of revolutions of the driven gear, in a unit of time.

18. The *pitch point* is the point of tangency of the pitch circles of a pair of mating gears.

19. The *common tangent* is the line tangent to the pitch circles at the pitch point.

20. The *line of action* is a line normal to a pair of mating tooth profiles at their point of contact.

21. The *path of contact* is the path traced by the contact point of a pair of tooth profiles.

22. The *pressure angle* (also called the angle of obliquity) is the angle between the line of action and the common tangent.

23. The *base circle* is an imaginary circle used in involute gearing to generate the involutes that form the tooth profiles. It is the circle drawn from the center of each of a pair of mating gears tangent to the line of action.

66. Fundamental Law of Gear-tooth Action.—In order to have a pair of toothed gears transmit the same motion as the rolling members (usually cylinders or cones) that they replace, it is necessary for the tooth profiles to have definite forms. There is a fundamental kinematic requirement to which the tooth profiles of any pair of gears must conform if the gears are to transmit the correct motion referred to above. This requirement will now be discussed.

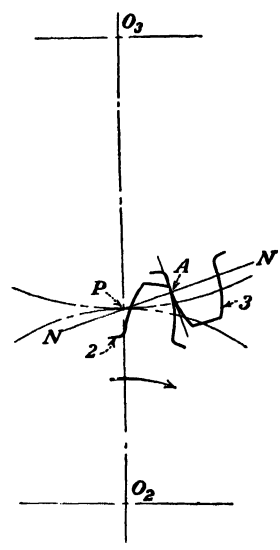


FIG. 95.

Figure 95 shows a pair of gear teeth in contact. All of each gear has been removed except one tooth, the resulting mechanism being merely a pair of members with curved surfaces in contact (see Art. 59). Now let it be assumed that the pitch circles of the original gears are tangent to each other at the point P on the line of centers. If the pair of tooth profiles in contact are such as to satisfy the requirement of constant-velocity ratio, the common normal NN at the point of contact A must, at any instant, intersect the line of centers in a fixed point P (see Art. 60); and this fixed point must be at the point of tangency of the pitch circles (called the *pitch point*), if the velocity ratio is to be the same as that of the rolling pitch circles. The fundamental law of gear-tooth action may now be stated as follows: *The common normal to the tooth profiles at the point of contact must always pass through the pitch point.*

67. Usual Forms of Gear-tooth Profiles.—A number of tooth forms will satisfy the fundamental requirement of correct gear-

tooth action, as outlined in the foregoing articles, but only two have been widely used, namely, the cycloidal form and the involute form, so named because of the curves on which they are based. Up to about 60 years ago the cycloidal tooth form predominated, but now the involute form has replaced the cycloidal almost entirely. Cycloidal outlines are still used in some special cases for both cast teeth and cut teeth. The cycloidal principle is still applied also in the design of pin gearing for watches, clocks, and instruments.

The cycloidal tooth form is discussed at some length in the following pages largely because of its historical and kinematic interest, and also because it is believed that the student will gain a broader viewpoint of the fundamental theory of gear-tooth action than is possible with his attention confined merely to the one commonly used tooth form, namely, the involute.

INVOLUTE TOOTH PROFILES

The curve most commonly used for gear-tooth profiles is the involute of a circle. This curve is the path traced by a point on a line as the line rolls without slipping on the circumference of a circle. It may also be defined as the path traced by the end of a string as the string is unwound from a circle. The circle from which an involute is derived is called its *base circle.*

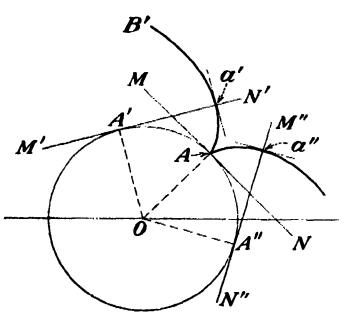


FIG. 96.

Referring to Fig. 96, let line MN roll without slipping in a counterclockwise sense on the circumference of the circle whose radius is OA. When the line has reached the position $M'N'$, its original point of tangency A has reached the position a', having traced the involute curve Aa' during the motion. As the motion continues, the point A will trace the curve $Aa'B'$. The involute possesses several interesting and useful properties. In the first place let it be noted that, since the line MN rolls without slipping on the circle, the distance $A'a'$ is equal to the arc $A'A$. For any instant the instantaneous center of the motion of the line is its point of tangency with the circle. Therefore, the motion of the

point that is tracing the involute is perpendicular to the line at any instant, and hence the curve traced will also be perpendicular to the line; *i.e.*, the normal at any point of an involute is tangent to the base circle.

Figure 96 also shows that by rolling the line MN in the opposite sense, the involute Aa'' would be generated. This curve is the same as Aa' except for position.

68. Construction of the Involute Curve.—A method of constructing the involute curve is shown in Fig. 97. The arc of the circle to the left of 0 is divided into equal arcs of convenient

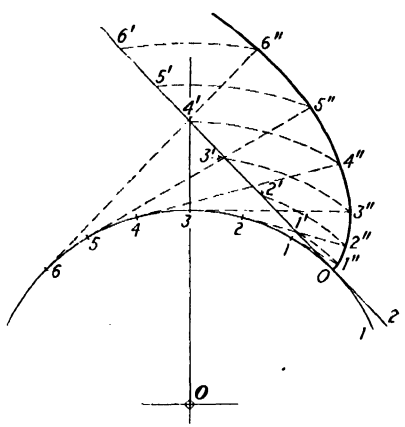


FIG. 97.

length, 01, 12, 23, etc.; and the straight line tangent to the circle at 0 is divided, to the left of 0, into equal parts 01′, 1′2′, 2′3′, etc., of the same length as the arcs. Hence when the straight line is rolled on the circle, the point 1′ will come into coincidence with 1, 2′ with 2, 3′ with 3, etc., and the describing point 0 on the line will have moved to 1″, 2″, 3″, etc., corresponding to the dotted positions of the generating line. It is obvious from the method employed in plotting the involute curve that points 1′ and 1″, 2′ and 2″, 3′ and 3″, etc., must be on concentric arcs from O as a center.

A construction that is sufficiently accurate for ordinary purposes is shown in Fig. 98. Step off a number of divisions 01, 12, 23, etc. on the circumference of the circle and draw tangents to the circle at these points. From point 1 with a radius equal to

chord 01 strike an arc extending from 0 about halfway to tangent 22′. From 2, in turn, with a radius great enough to connect with the preceding arc, strike a second arc extending about halfway to tangent 33′, and so on as indicated in the figure.

Where, for illustrative purposes, it is desired to draw the teeth of a gear, a templet for the purpose may be made from the involute curve that has been constructed in accordance with one of the above methods. A more convenient method, and one that is sufficiently accurate for ordinary purposes, is to find, by trial, an arc that will closely approximate the involute and draw in the

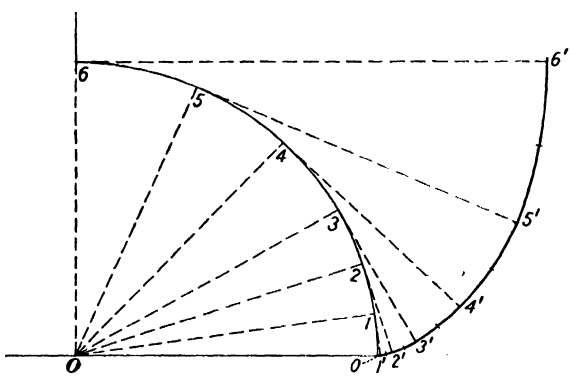


FIG. 98.

tooth outlines with the compass. In most cases it will be found that the locus for the center of this arc can be taken on the base circle.

69. Involute Teeth Satisfy the Fundamental Law of Gear-tooth Action.—The construction of a pair of mating involute gear-tooth outlines, and their action, is illustrated in Fig. 99. From centers O_2 and O_3 the pitch circles 2 and 3 are drawn in accordance with the required velocity ratio. Through the pitch point P the line of action NN is drawn at an angle to the common tangent MM equal to the given pressure angle θ. The base circles 2′ and 3′ can now be drawn tangent to the line of action at points B_2 and B_3. Now, if the common tangent B_2B_3 is assumed to be cut at any point b and the segment bB_2 rolled on base circle 2′, the point b will describe the involute curve C_2D_2, forming the outline of a tooth on gear 2. If next the segment bB_3 is rolled on the base circle 3′, the point b will describe the involute curve

C_3D_3, forming the outline of the mating tooth on gear 3. In the actual operation all the motions take place simultaneously as the gears rotate about their axes. Hence, the curves are traced simultaneously by the point b on the common tangent B_2B_3 and thus are in contact for the instant at that point. After the mating profiles have thus been determined the complete tooth forms can be drawn as shown in the figure.

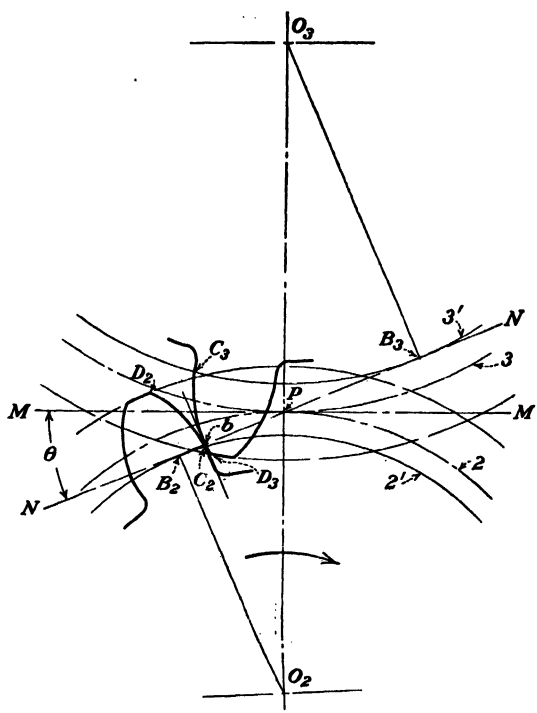


FIG. 99.

It is evident that, since the common tangent B_2B_3 cannot change its position, it will always intersect the line of centers O_2O_3 at the same point P. Since any normal to an involute is tangent to the base circle of the involute, it follows that the segments bB_2 and bB_3 must form a common normal to the two involute curves at their point of contact b. Since this common normal is coincident with the common tangent to the two base circles, it will always intersect the line of centers at the pitch point P. The above discussion can be shown to hold true for any

position of the curves. It is evident, therefore, that gears with involute tooth profiles will satisfy the fundamental law of gear-tooth action.

70. Characteristics of Involute Tooth Action.—Since the common normal to a pair of involute gear teeth in contact is always a line tangent to the two base circles, this line is called the *line of action.* It follows that the point of contact is always situated

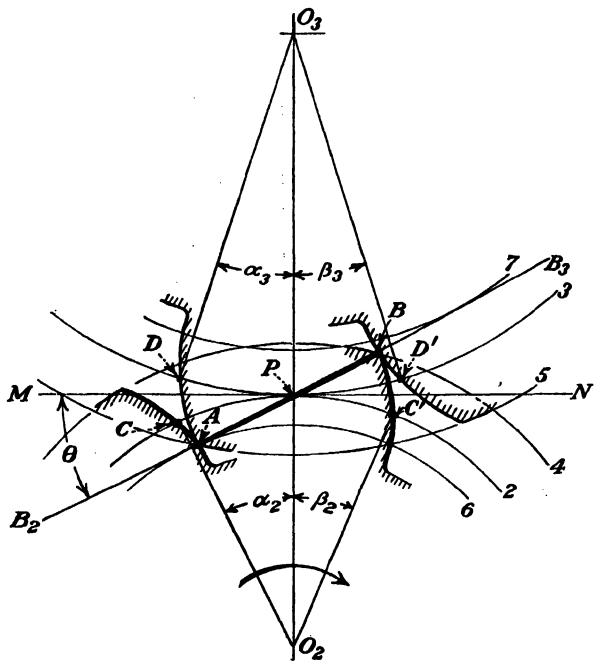


FIG. 100.

on this line. The part of the line of action along which contact takes place is determined as follows: Referring to Fig. 99, it is evident that contact cannot begin until point D_3 (end of the tooth) crosses the locus of contact, which is along the line of action B_2B_3. It is also evident that contact cannot continue after the point D_2 crosses the locus of contact. Therefore, contact along line B_2B_3 is limited by the intersection of the two addendum circles and the tangent to the two base circles (line of action).

In Fig. 100 let 2 and 3 represent the pitch circles of a pair of mating gears, 4 and 5 the corresponding addendum circles, and

6 and 7 the base circles. Let gear 2 be the driver, and assume that it rotates in the direction indicated by the arrow. The shaded lines at A and B represent a pair of mating tooth profiles drawn in at their first and last points of contact. It is evident that these profiles must begin contact at A where the addendum circle of gear 3 cuts the line B_2B_3 tangent to the base circles, and that as the gears rotate, the contact point will move along B_2B_3 to the point B where this line is cut by the addendum circle of gear 2. With any further motion the profiles will separate, and thus B marks the end of contact. The line APB (drawn in heavy) is called the *path of contact.*

Referring again to Fig. 100 and considering the driving gear 2, it can be seen that the point C where the tooth profile at the beginning of contact cuts the pitch circle will move through the arc CP of the pitch circle as the tooth profile approaches the line of centers, and will move through the arc PC' of the pitch circle as the tooth profile recedes from the line of centers to the point where contact ceases. Arc CP is called the *arc of approach,* and arc PC'_1 the *arc of recess.* The angle α_2 subtended by the arc of approach is called the *angle of approach* and is evidently the angle through which the tooth moves from the position where contact begins to the position where contact is on the line of centers. The angle β_2 subtended by the arc of recess is called the *angle of recess* and is evidently the angle through which the tooth moves from its contact position on the line of centers to the position where contact ceases.

The *arc of action* is the sum of the arcs of approach and recess and is, therefore, the arc measured on the pitch circle from the position of the tooth at which contact begins to the position of the tooth at which contact ends. The *angle of action* is the sum of the angles of approach and recess and is, therefore, the angle through which a tooth moves from the position where contact begins to the position where contact ceases.

The methods of determining the arcs of approach and recess DP and PD' and the angles of approach and recess α_3 and β_3 of the driven gear 3 are the same as those just described for the driving gear 2.

The direction of the common normal in the case of involute gears is the same for all positions of the teeth. The line MN drawn through the pitch point P perpendicular to the line of

centers O_2O_3 is tangent to both pitch circles at that point. The angle θ which the common normal AB makes with the line MN is called the *pressure angle* (or angle of obliquity) and is evidently a constant for all positions of the mating teeth.

71. Interference in Involute Gears. Limits of Addendum.—Figure 101 represents the layout of a pair of involute gears in which the line of action is tangent to the base circles at A_2 and A_3. It will now be shown that for correct tooth action the path of contact must lie entirely within these tangent points, A_2 and A_3. If pinion 2 is the driver, and its addendum circle is the circle

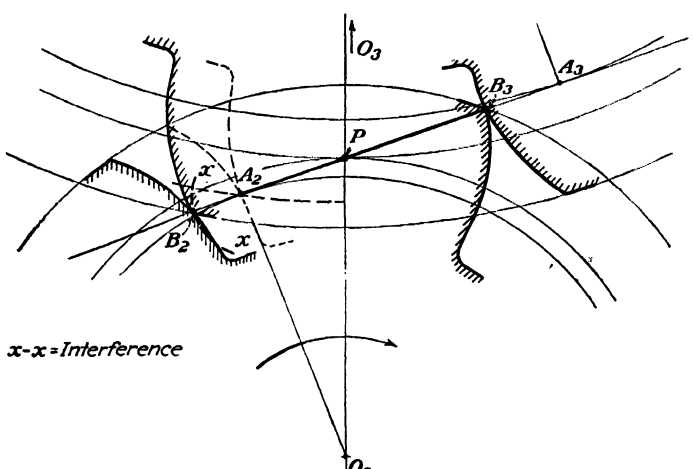


FIG. 101.

passing through the point B_3, and the addendum circle of gear 3 is the dotted circle passing through the tangent point A_2, contact will begin at A_2 and end at B_3, and the teeth will work together properly. But the actual addendum circle of the gear 3 intersects the line of action A_2A_3 at B_2 outside of its point of tangency with the base circle. Considering first the dotted addendum circle that passes through A_2, the contact on the pinion profile begins at A_2 on the base circle where the involute part of the profile begins and advances along the profile to the end of the tooth while the profile is rotating to position B_3, where the teeth pass out of contact.

Consider next the addendum circle that passes through B_2 and the increased addendum of the gear tooth that this implies.

If the teeth begin contact at B_2, it is evident that the addendum of the gear tooth must act upon that part of the pinion profile that lies inside its base circle. But this part of the pinion profile was arbitrarily drawn in as a radial line, since it was not possible to extend the involute curve inside the base circle. Correct tooth action can result only from the action of involute upon involute. The tooth action from position B_2 to A_2 is involute upon a radial line, and the face of the gear tooth must cut into the radial flank of the pinion tooth as shown in order to maintain the constant angular-velocity ratio desired. This is called *interference* of involute teeth, and to avoid it the flanks must be hollowed out or the points of the teeth *eased off*. The latter is the usual remedy.

In a full depth 14½-deg. involute system the smallest number of teeth in a gear that will mesh with a rack without interference is 32; but in the 14½-deg. composite system the cutters have been so designed that the teeth are slightly *eased off* on the end, so that practically, the minimum number of teeth is 12 (refer to Art. 74 and Fig. 103).

72. Layout of a Pair of Involute Gears.—The general method of procedure in laying out a pair of involute gears will be illustrated by a numerical example. Let it be required to lay out a pair of involute spur gears in accordance with the following specifications:

The tooth proportions are to conform to the 14½-deg. composite system (see Art. 98).
Distance between shaft centers = 12 in.
Driving shaft turns 400 r.p.m., clockwise
Driven shaft turns 200 r.p.m.
Diametral pitch = 4
Backlash = 0

The following computations may now be made:

$$\text{Pitch radius of the pinion } R_2 = \frac{200}{200 + 400} \times 12 = 4 \text{ in.}$$

$$\text{Pitch radius of the gear } R_3 = \frac{400}{200 + 400} \times 12 = 8 \text{ in.}$$

$$\text{Number of teeth in pinion} = pD_2 = 4 \times 8 = 32$$
$$\text{Number of teeth in gear} = p \times D_3 = 4 \times 16 = 64$$

$$\text{Circular pitch } p' = \frac{\pi}{4} = 0.7854 \text{ in.}$$

From Art. 98:

$$\text{Addendum} = \frac{1}{p} = \frac{1}{4}\text{ in.}$$

$$\text{Dedendum} = \frac{1}{p} + 0.05\,p' = \frac{1}{4}\text{ in.} + 0.05 \times 0.7854\text{ in.} = 0.289\text{ in.}$$

Radius of addendum circle (pinion) $= 4 + 0.25 = 4.25$ in.
Radius of addendum circle (gear) $= 8 + 0.25 = 8.25$ in.
Radius of dedendum circle (pinion) $= 4 - 0.289 = 3.711$ in.
Radius of dedendum circle (gear) $= 8 - 0.289 = 7.711$ in.

Lay out the pitch circles, addendum circles, and dedendum circles as shown in Fig. 102. Through the pitch point P, draw

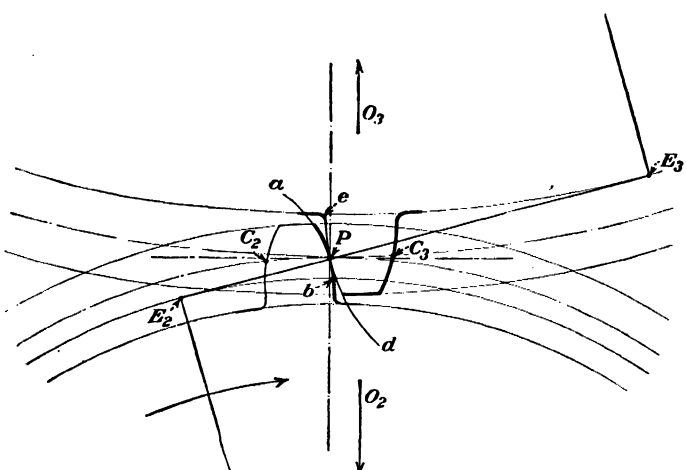


FIG. 102.

the line of action E_2E_3 at an angle of 14½ deg. with the line tangent to the pitch circles at the pitch point. The base circles are tangent to this line at points E_2 and E_3. The direction of the line of action is governed by the direction of rotation of the driver, which, in this case, is clockwise as indicated by the arrow.

Construct the involute profile bPa of the pinion tooth by rolling the line PE_2 on the base circle of the pinion and construct the involute profile ePd of the gear tooth by rolling the line PE_3 on the base circle of the gear. Draw radial lines to complete the parts of the tooth flanks that lie inside the base circles and

join the flanks to the dedendum circles by small fillets (see Art. 98). Space off the tooth thicknesses PC_2 and PC_3 on the pitch circles and through C_2 and C_3 draw the tooth profiles (reversed) to form the opposite sides of the teeth. The tops of the teeth and bottoms of the spaces between adjacent teeth are completed by the bounding addendum and dedendum circles. The complete gears, or as many teeth as desired, can be represented by stepping off the tooth thickness on the pitch circles and drawing the profiles through the division points. These profiles may be drawn with the compass, in the manner suggested in Art. 68.

The arcs of approach and recess may now be found, if desired, by the methods described in the preceding articles.

There is an important relation between the circular pitch and the arc of action that must be considered in the design of any pair of gears. In properly designed gears there must be continuous tooth action; that is, one pair of teeth must not cease contact before the adjacent pair which follows begins contact. It is evident that when the arc of action and the circular pitch are equal, one pair of teeth is ceasing contact just as the adjacent pair is beginning contact. It follows, therefore, that the circular pitch must not be greater than the arc of action, for in that case continuous contact becomes impossible.

73. Interchangeable Involute Gears.—A set of gears is said to be interchangeable when any two of the set will run together properly. The conditions of interchangeability for involute gears are that all gears of the set have the *same pitch* and the *same pressure angle.*

74. Involute Rack and Pinion.—When the radius of the pitch circle becomes infinite, as in the case of a rack, the radius of the base circle also becomes infinite, and the involute will become a straight line. Hence in an involute rack the tooth profiles become straight lines, and these lines are perpendicular to the line of action. The general layout of an involute rack and pinion is shown in Fig. 103. No interference can ever occur between the face of the pinion tooth and the flank of the rack tooth, but between the face of the rack tooth and the flank of the pinion tooth, conditions are such as to give the maximum of interference. Figure 103 shows the interference of an uncorrected rack in mesh with a 12-tooth gear having a 14½-deg. pressure angle. A slight interference is also indicated in the case of two 12-tooth

gears meshing with each other. It is evident that the straight-sided rack teeth will mesh properly with any involute gear of the same pitch and the same pressure angle. This fact is of great

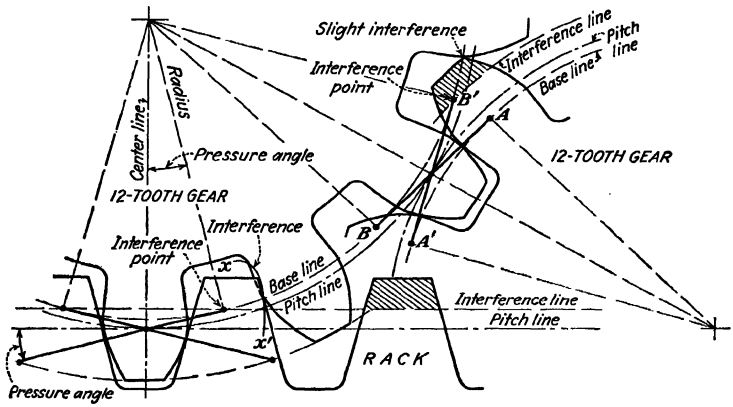


FIG. 103.

importance in its bearing on the manufacture of gears, for it makes possible the generating of the teeth of any gear of an interchangeable set with a cutting tool of the simplest form, namely,

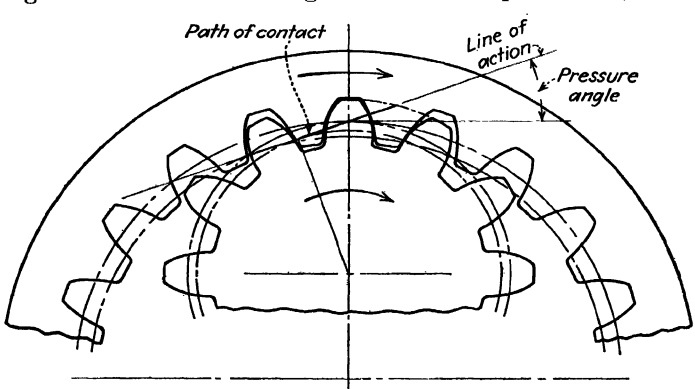


FIG. 104.

one having a straight side. This principle will be discussed more fully in Art. 105.

75. Involute Internal (or Annular) Gears.—Figure 104 shows the general layout of an involute internal (or annular) gear and

pinion. It will be observed that in the internal gear the tooth profiles are concave instead of convex, as in the case of the external gear (spur gear). Because of this shape an interference condition exists that is not found in the case of either the external gear or the rack; that is, if the ratio of the number of teeth in the internal gear exceeds a certain limit, a *secondary* interference will occur between the teeth after they have ceased contact along the path of contact. Obviously, the rack with its straight-sided teeth is the limiting case of both the external and the internal gear.

CYCLOIDAL TOOTH PROFILES

A *cycloid* is the curve described by a point on the circumference of a circle as the circle rolls on a straight line. If the circle rolls on the outside of another circle, the curve described is called an *epicycloid;* when it rolls on the inside of another circle, the curve described is called a *hypocycloid.*

76. Construction of Cycloidal Curves.—A convenient method of drawing an epicycloid is as follows: Let R be the radius of the fixed circle and r the radius of the rolling or describing circle (Fig. 105). Draw through C_0 a circle CD concentric with the fixed circle. Evidently the center of the describing circle will always be on CD as it rolls on the fixed circle. Lay off on the describing circle convenient divisions such as 0-1, 1-2, 2-3, etc.; then lay off divisions of the same length on the circumference of the fixed circle, as 0-1′, 1′-2′, etc. If the divisions are taken small enough, the differences between the chords and arcs may be assumed to be negligible, and the work will be sufficiently accurate if the chords are used in the construction. Through the points of division on the fixed circle, draw radii and produce them to intersect CD, the locus of centers for the describing circle, in points C_0, C_1, C_2, etc. The points thus located will be the successive positions of the center C of the describing circle. If the describing circle is drawn in any one of these positions, as C_3, its intersection d with the circular arc through 3, about center O, will be a point on the required epicycloid om. (It is usually more accurate to lay off $gd = f3$.) Other points on the curve are determined in the same manner.

The construction for the cycloid is the same as that for the epicycloid except that the center of the fixed circle is moved to infinity, and the circumference becomes a straight line.

The construction for the hypocycloid *on* is indicated in Fig. 105. The method is the same as for the epicycloid, the locus for center C in this case coming inside the fixed circle, as shown. The figure shows how the epicycloid and hypocycloid form the face and flank, respectively, of a gear tooth.

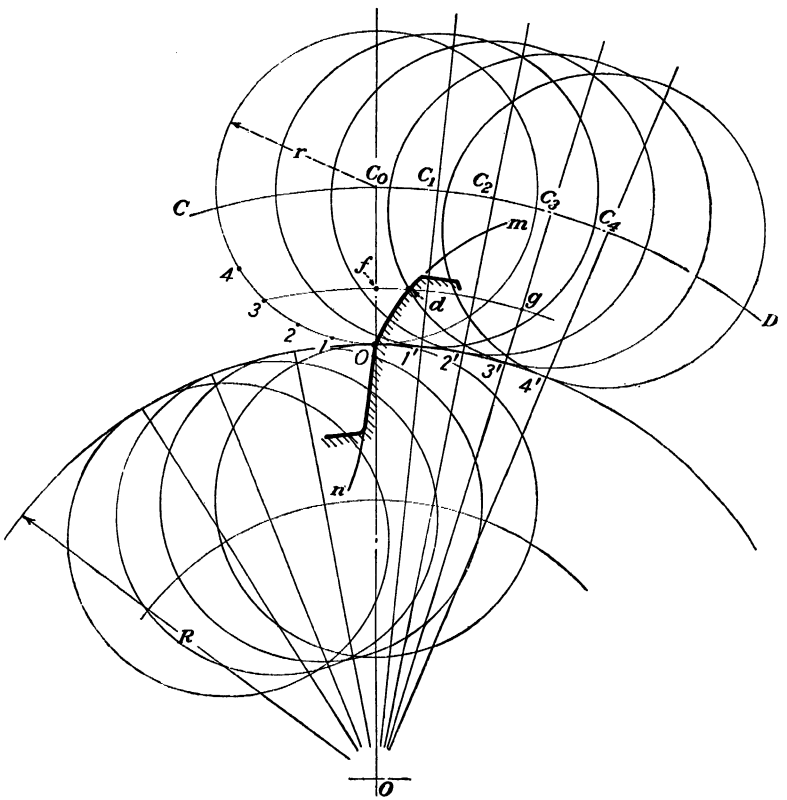


FIG. 105.

An important case of the hypocycloid is that in which the diameter of the describing circle is equal to the radius of the fixed circle in which it rolls. In this case any point on the describing circle has its path along a diameter of the fixed circle, and the hypocycloid thus becomes a straight line.

77. Cycloidal Teeth Satisfy the Fundamental Law of Gear-tooth Action.—In Fig. 106, 2 and 3 are two pitch circles whose centers are O_2 and O_3, respectively, and 4 and 5 are describing

circles. The epicycloid *cd*, forming the face of a tooth on gear 2, is described by point *c* as the describing circle rolls on the outside of pitch circle 2. The hypocycloid *ce*, forming the flank of a tooth on gear 3, is traced by point *c* as the describing circle rolls on the inside of pitch circle 3. By construction, therefore, arcs *Pc*, *Pe*, and *Pd* are equal. In the actual operation, all the motions take place simultaneously, the circles all remaining in contact at the pitch point *P*. Hence, the curves *cd* and *ce* are traced simultaneously by the point *c* on the describing circle, and thus are in contact for the instant at that point. It follows, also, that the curves must have a common normal and a common tangent at that point. When a circle rolls on a line, straight or curved, the point of contact is the instantaneous center for the motion of all points of the circle. It is evident, therefore, that *P* is the instantaneous center for the motion of point *c* on the describing circle, which moves in the direction of the common tangent *TT*, and that *Pc* is for that instant the common normal at the contact point of the curves *cd* and *ce*. The above proof can be shown to hold true for any other position of these curves. It can be seen that the common normal will always cut the line of centers at the pitch point *P*. It is evident, therefore, that *gears with cycloidal tooth outlines will satisfy the fundamental law of gear-tooth action.*

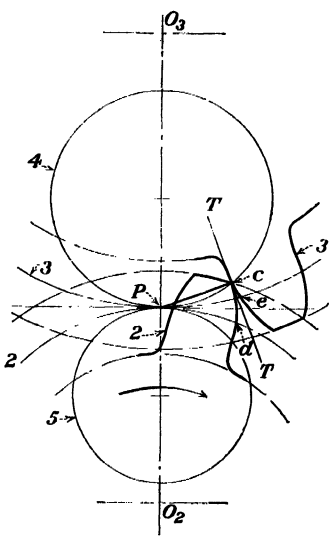


FIG. 106.

In the above proof it was only necessary to consider the face *cd* of the tooth on gear 2 mating with the flank *ce* of the tooth on gear 3. The other face and the other flank necessary for completing the pair of tooth profiles are described by the rolling of describing circle 5 on the two pitch circles.

78. Characteristics of Cycloidal Tooth Action.—It can be seen that from the very nature of the method of describing a pair of mating cycloidal tooth outlines, the point of contact must always lie on the circumferences of the describing circles; and

further, that the limits of this locus of contact must be at the points where the addendum circles intersect the describing circles. This is clearly shown in Fig. 107. Let 2 and 3 represent the pitch circles of a pair of mating gears, 4 and 5 the corresponding addendum circles, and 6 and 7 the describing circles. Let gear 2 (pinion) be the driver and assume that it rotates in the

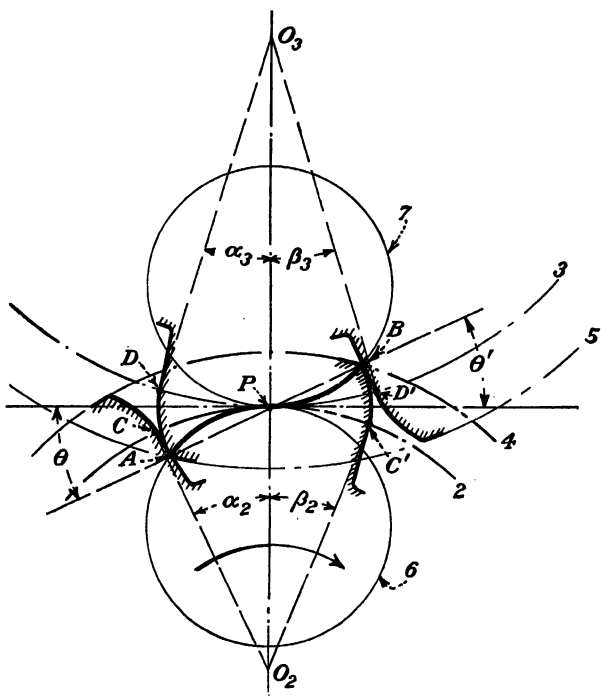


FIG. 107.

direction indicated by the arrow. The shaded lines at A and B represent a pair of mating tooth profiles drawn in at their first and last points of contact. It is evident that these profiles must begin contact at A where the addenum circle 5 of gear 3 cuts the describing circle 6, and that as the gears rotate, the contact point will move along the circumference of this describing circle to the pitch point P. From this point, contact proceeds along the circumference of describing circle 7 to the point B where this circumference is cut by the addendum circle 4 of gear 2. With

any further motion the profiles will separate, and thus B marks the end of contact. The curve APB (shown in heavy lines) is called the *path of contact.*

Referring again to Fig. 107 and considering the driving gear 2, it can be seen that the point C where the pinion tooth profile at the beginning of contact cuts the pitch circle will move through the arc CP of the pitch circle as the tooth profile approaches the line of centers, and will move through the arc PC' of the pitch circle as the tooth profile recedes from the line of center to the point where contact ceases. Arc CP is called the *arc of approach,* and arc PC' the *arc of recess.* The angle α_2 subtended by the arc of approach is called the *angle of approach* and is evidently the angle through which the tooth moves from the position where contact begins to the position where contact is on the line of centers. The angle β_2 subtended by the arc of recess is called the *angle of recess* and is evidently the angle through which the tooth moves from its contact position on the line of centers to the position where contact ceases.

The *arc of action* is the sum of the arcs of approach and recess and is, therefore, the arc measured on the pitch circle from the position of the tooth at which contact begins to the position of the tooth at which contact ends. The *angle of action* is the sum of the angles of approach and recess and is, therefore, the angle through which a tooth moves from the position where contact begins to the position where contact ceases.

The methods of determining the arcs of approach and recess DP and PD' and the angles of approach and recess α_3 and β_3 of the driven gear 3 are the same as those just described for the driving gear 2.

The *pressure angle* (angle of obliquity) is the angle formed by the common normal as PA or PB at the point of tooth contact and the common tangent to the pitch circles at the pitch point P. Evidently this angle varies from zero where tooth contact is on the line of centers to a maximum when the tooth contact is just beginning or ending. In Fig. 107, θ is greater than θ', and therefore θ is the maximum pressure angle for this pair of gears.

79. Layout of a Pair of Cycloidal Gears.—The general method of procedure in laying out a pair of cycloidal gears can best be illustrated by a numerical example. Let it be required to lay

out a pair of cycloidal spur gears in accordance with the following specifications:

The tooth proportions are to conform to those of the 14½-deg. composite system for involute-cut teeth (see Art. 98)
Distance between shaft centers = 12 in.
Driving shaft turns 400 r.p.m., clockwise
Driven shaft turns 200 r.p.m.
Diametral pitch = 4
Backlash = 0
Diameter of describing circle for pinion = 3 in.
Diameter of describing circle for gear = 6 in.

The following computations may now be made:

$$\text{Pitch radius of pinion } R_2 = \frac{200}{200 + 400} \times 12 = 4 \text{ in.}$$

$$\text{Pitch radius of gear } R_3 = \frac{400}{200 + 400} \times 12 = 8 \text{ in.}$$

Number of teeth in pinion $= pD_2 = 4 \times 8 = 32$
Number of teeth in gear $= p \times D_3 = 4 \times 16 = 64$
Circular pitch $= \pi/4 = 0.7854$

From Art. 98:

Addendum $= 1/p = 1/4$

$$\text{Dedendum} = \frac{1}{p} + 0.05p' = \tfrac{1}{4} + 0.05 \times 0.7854 = 0.289 \text{ in.}$$

Radius of addendum circle (pinion) $= 4 + 0.25 = 4.25$ in.
Radius of addendum circle (gear) $= 8 + 0.25 = 8.25$ in.
Radius of dedendum circle (pinion) $= 4 - 0.289 = 3.711$ in.
Radius of dedendum circle (gear) $= 8 - 0.289 = 7.711$ in.

Lay out the pitch circles describing circles, addendum circles, and dedendum circles, as shown in Fig. 108.

Construct the face *Pa* of the pinion tooth by rolling the describing circle *A* on the pitch circle of the pinion and construct the flank *Pb* of this tooth by rolling the describing circle *B* inside the pitch circle of the pinion. On the pitch circle of the pinion lay off *Pc* equal to the thickness of the tooth (one-half the circular pitch) and through *c* draw the curves *Pa* and *Pb* (reversed) to form the opposite side of the pinion tooth. Join the flanks of the tooth to the dedendum circle by a small fillet (see Art. 98). The top of the tooth and the bottom of the space between adjacent teeth are completed by the bounding addendum and dedendum circles.

A tooth of the gear may be constructed in the same manner. The face of the gear tooth is determined by rolling describing circle B on the outside of the pitch circle of the gear, and the flank by rolling describing circle A inside the pitch circle of the gear.

The complete gears, or as many teeth as desired, can be laid out by dividing both pitch circles into arcs equal to one-half the circular pitch and by then drawing the curves already found, in their proper order, through these points. These outlines may be drawn in with the compass after finding, by trial, arcs that closely approximate the true cycloidal outlines.

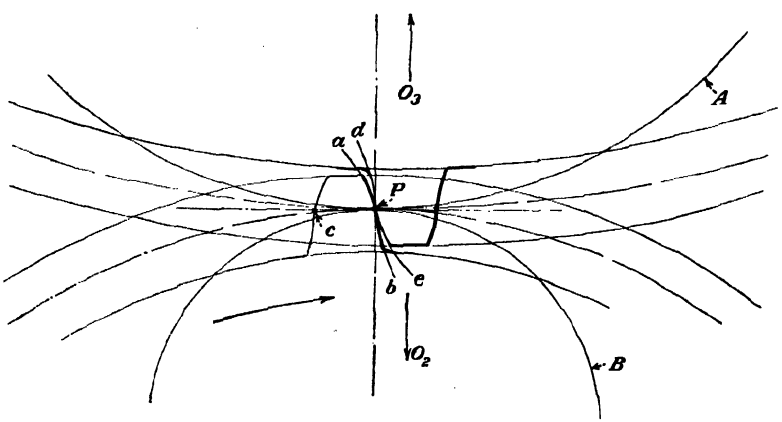


FIG. 108.

The arcs of approach and recess, maximum angle of obliquity, etc., may now be found by the methods described in the preceding article.

There is an important relation between the circular pitch and arc of action that must be considered in the design of any pair of gears. In properly designed gears there must be continuous tooth action; *i.e.*, one pair of teeth must not cease contact before the adjacent pair that follows begins contact. It is evident that when the arc of action and the circular pitch are equal, one pair of teeth is ceasing contact just as the adjacent pair is beginning contact. It follows, therefore, that the circular pitch must not be greater than the arc of action, for in that case continuous contact becomes impossible.

80. Size of the Describing Circles.—In Fig. 109 are shown three gear teeth of the same proportions constructed on the three

equal pitch circles indicated. At *a* the diameter of the describing circle that forms the flank of the tooth is *less* than the radius of the pitch circle. This describing circle describes the hypocycloid *Pa*, which gives a *spreading* flank and, therefore, a strong tooth shape. At *b* the diameter of the describing circle is *equal* to the radius of the pitch circle. The hypocycloid in this case becomes a straight line coincident with the diameter of the pitch circle. The result is a *radial* tooth flank, which results in a tooth narrower at the base than at the pitch circle and, therefore, of weak form.

At *c* the diameter of the describing circle is *greater* than the radius of the pitch circle, and the flank of the tooth is *undercut* as shown; *i.e.*, the hypocycloid *Pa* falls inside the radial line,

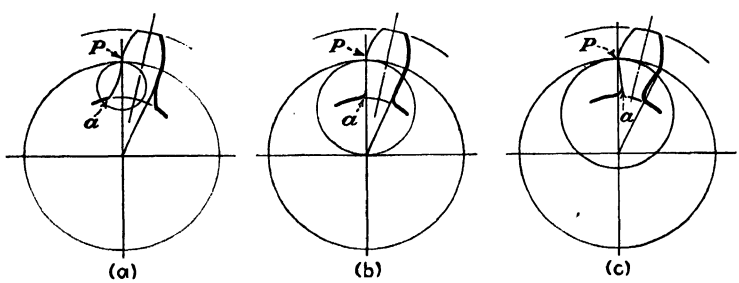


FIG. 109.

and a very weak form of tooth results. It follows, therefore, from the standpoint of strength of the teeth, that it is desirable to have the diameter of the describing circle less than the radius of the pitch circle.

Referring again to Fig. 108, it can be seen that the faces of the teeth of one gear act only on the flanks of the teeth of the mating gear. Hence, the form of the tooth face on one gear is independent of that of its own flank. There is no fixed relation, therefore, as to the size of the describing circles used for a pair of gears, as far as proper tooth action is concerned. There are however, practical conditions that govern the size of describing circles to be used in any given case.

81. Interchangeable Cycloidal Gears.—In Fig. 108 it was shown that, in the teeth of two gears 2 and 3, the faces of the former and the flanks of the latter were described by the circle *A*, and that the faces of 3 and the flanks of 2 were described by another circle *B*. It was also shown that these two gears would

work together properly. A third gear, 4, of the same pitch, however, cannot work properly with *both* 2 and 3; for if the faces of 4 are described by A and its flanks are described by B, it may engage properly with 3, but it cannot act correctly with 2, for the faces of 2 and the flanks of 4 are not described by the same circle; neither are the flanks of 2 and the faces of 4, and the conditions of constant-velocity ratio are not satisfied by this construction.

Practical considerations make it desirable to provide for the working of any gear of a set with any other gear of the same set. When this is possible, the set is said to be *interchangeable.* The conditions necessary for interchangeability in a set of cycloidal gears are that *all the gears of the set have the same pitch and that all the teeth have outlines generated by the same describing circle.* The size of the describing circle usually chosen is such that the smallest gear of the set shall have radial flanks. This gear is commonly taken as having from 12 to 15 teeth.

82. Cycloidal Rack and Pinion.—If gear 3 of Fig. 108 had the center of its pitch circle moved to infinity, the resulting com-

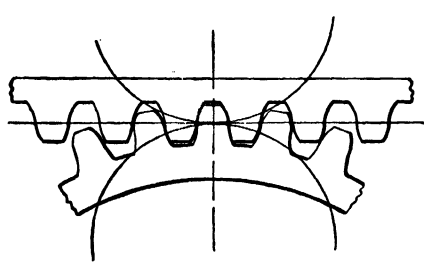


FIG. 110.

bination would be a rack and pinion (Fig. 110). Thus a rack is simply a gear with a pitch circle of infinite radius, and it is evident that the theory discussed in the preceding articles applies equally well in this case. The faces and flanks of the rack are both cycloids, and in an interchangeable set of gears they are alike. In this case any gear of the set will work properly with the rack. The rack is necessarily limited in length, and its motion must be reciprocating.

83. Cycloidal Internal (or Annular) Gears.—Figure 111 shows an internal (also called annular) gear combination. The larger of the pair is called the internal or annular gear. The theory of the tooth action applies here in the same manner as for the ordi-

nary spur-gear combination. Here, however, the faces of the pinion teeth and the flanks of the gear teeth are both epicycloids, and the flanks of the pinion teeth and the faces of the gear teeth are both hypocycloids.

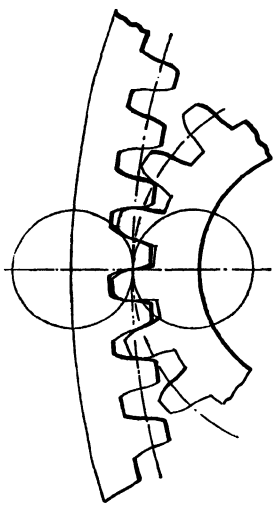


FIG. 111.

84. Pin Gearing.—An interesting form of cycloidal gearing called *pin gearing* is illustrated in Fig. 112. Although now confined mainly to clockwork, this is an old type of gearing much used in early times when teeth were commonly made of wood. The kinematic requirement in this class of gearing is that the describing circle shall be equal to one of the pitch circles. The hypocycloid in this case becomes a point, and this point acted upon by an epicycloid described on the other gear by this same describing circle will transmit motion identical with the rolling of the two pitch circles.

The primary tooth forms are, thus, a point P on the pitch circle of the pinion 2 (Fig. 112) and the dotted epicycloid which this point describes on the mating gear 3. In actual use point P serves as the center

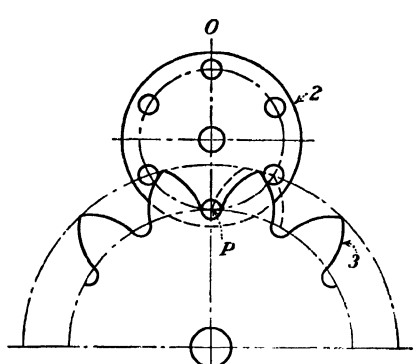


FIG. 112.

of a pin of sensible diameter, the actual tooth profiles being *parallel* to the primary curves, and at distances from these curves equal to the radius of the pin.

With the primary curves (the point and epicycloid) the angle of action is entirely on one side of the line of centers. In toothed gearing the tooth action is always smoother during recess than during approach. In order to have the action take place during recess and thus secure smooth action in pin gearing, the pin wheel should be the follower. It can be seen, however, that with a pin of sensible diameter there must always be a small angle of approach.

85. Comparison of Cycloidal and Involute Tooth Forms.—The fact that the involute form of tooth has almost entirely superseded the cycloidal form is in itself an indication that the former must possess decided advantages as compared with the latter. The situation may be summed up as follows:

Advantages of involute gears:

1. In involute gears the distance between centers may vary without affecting the velocity ratio. In cycloidal gears the tooth action is not correct unless the exact center distance is maintained.

2. Involute rack teeth have straight sides, and therefore cutting tools of simple design may be used for manufacturing involute gears. Furthermore, involute teeth are of single curvature and this simple form is conducive to economy in manufacture.

3. In the involute tooth form the angle of obliquity is constant. This is generally considered more favorable to smooth running and durability than the varying obliquity in the cycloidal form.

Advantages of cycloidal gears:

1. There is no interference in cycloidal gears.

2. The cycloidal tooth has, in general, spreading flanks, whereas, the standard involute tooth has radial flanks. Therefore, the cycloidal tooth is of somewhat stronger shape. For this reason the cycloidal form has been favored for cast teeth.

3. In cycloidal gears a convex surface is always in contact with one that is concave. This condition tends to result in less wear than in the case of involute teeth where convex surfaces are in contact. The difference in wear probably is negligible.

In the above comparison the advantages are more in the favor of the involute tooth than a superficial examination would indicate. For example, interference in the involute form of tooth is obviated in practice by slight modification of the involute profile, or by modifications in tooth proportions. The stronger shape of the cycloidal tooth is of little importance in modern industrial

gears, because the allowable tooth loads are generally based upon wear rather than upon strength.

Furthermore, when it comes to the matter of production, with the simplicity of the single curve and the great importance of the straight-line rack profile as a fundamental form, and in securing and maintaining conditions for correct operation, the geometrical advantages of the cycloidal curves are greatly outweighed by the greater simplicity and flexibility of the involute.

CHAPTER V

BEVEL, HELICAL, AND WORM GEARING

The form of gearing used to connect shafts that intersect[1] is called bevel gearing. In Art. 62 the contact surfaces of a pair of bevel friction wheels were shown to be surfaces of the frustums of a pair of cones having a common apex. If teeth are formed on these surfaces in a manner analogous to the methods used in spur gearing, the cones become the pitch surfaces of a pair of bevel gears.

86. Classification of Bevel Gears.—In the great majority of bevel-gear drives the shafts are at right angles, but the angle between the shafts may be either greater or less than 90 deg. In such cases the gears are called *angular bevel gears.* When the angle between the shafts is 90 deg. and the two gears of a pair are equal, *i.e.*, the angular-velocity ratio is 1:1, the gears are called *miter gears.*

Bevel gears are nearly always made to work together in pairs and are not interchangeable. The involute form of tooth is used, and the Brown and Sharpe 14½-deg. involute tooth proportions (see Art. 98) have been widely used in the past for ordinary work; but for the better class of bevel gearing and especially for high-speed gearing, the teeth conform to standards that have been developed for bevel gears (see Art. 102). Bevel gears may be divided into four classes:

1. Straight-tooth bevel gears, in which the elements of the tooth surfaces are straight lines, converging at the apex of the pitch cone. The pitch surfaces are cones.

2. Spiral bevel gears, in which the teeth are curved. The pitch surfaces are cones.

3. Skew bevel gears, which may be used to connect nonparallel, nonintersecting shafts. The pitch surfaces are hyperboloids (see Art. 62) and the teeth are straight.

4. Hypoid gears, which connect nonparallel nonintersecting shafts. The pitch surfaces are cones and the teeth are curved.

[1] The shafts are nonintersecting in the case of skew bevel gears.

87. Form of Bevel-gear Teeth.—The motion of a pair of spur gears comes under the head of plane motion, and the relative motions can be represented by a right section. Thus, an involute tooth profile is described by a point in a line that rolls on a base circle. In reality this is simply a representation, in a plane, of a tooth surface that is swept up by a line in a plane as the plane containing the line rolls on a base cylinder. This simple treatment cannot be applied to bevel gears. Although each bevel

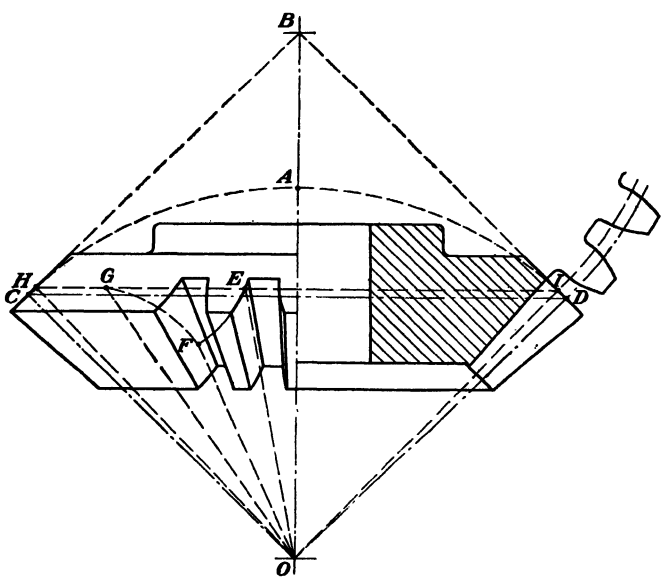


FIG. 113.

gear considered separately has plane motion (rotation) about its own axis, the relative motion, considering the two gears rolling together, is spherical motion.

a. Theoretically Correct Form.—A simple illustration will show that the method of forming the surface of an involute bevel-gear tooth is fundamentally the same as that for forming the surface of an involute spur-gear tooth. In Fig. 113 let the cone *OHI* represent the base cone of the bevel gear shown, on which the involute tooth surfaces are to be formed. Imagine the base cone to be enclosed in a thin flexible covering that is cut along the line *OE*. Now take hold of the edge *OE* and, keeping the cover-

ing taut, unwrap it. The surface *OEF* swept up by the edge or element *OE* is the desired involute surface. The point *E* is constrained to remain a constant distance from the cone center *O*, equal to *OE*, and consequently moves in the surface of a sphere *HAI*. For this reason the curve *EF* is called a *spherical involute*.

b. Tredgold's Approximation.—The difficulty of laying out a tooth profile that is a curve lying in the surface of a sphere is apparent, since the surface of a sphere cannot be developed. No appreciable error will be introduced if the conical surface *CBD* is substituted for the spherical surface *CAD*. The cone *CBD*, called the back cone, is tangent to the sphere at the circle *CD*, this circle representing the common base of the pitch cone and back cone. The back cone is thus *normal* to the pitch cone; *i.e.*, the elements of the back cone are perpendicular to corresponding elements of the pitch cone. It can be seen in the figure that the comparatively short distance necessary to include the entire tooth profile practically coincides with the spherical surface. Within the limits of practice, therefore, such profiles agree quite closely with the theoretically correct forms.

The above approximation was first published by Tredgold and is known as *Tredgold's approximation*. In the application of the method to the layout of bevel-gear teeth, the teeth are laid out on the developed surface of the back cone as shown in Fig. 113, and the back-cone surface is then wrapped back to its original position. The profiles thus constructed on the back cone will form the ends of the bevel-gear teeth, and straight lines drawn from the pitch-cone center *O* and following the profiles will sweep up the tooth surfaces, all elements of which are straight lines converging at *O*.

The teeth are laid out on the developed back cone by exactly the same methods as used for spur gears. In Fig. 113 the slant height *BD* of the back cone is the pitch radius of the equivalent spur gear and *BI* is the radius of the base circle.

88. Bevel-gear Nomenclature.—As far as the layout of the teeth and their action is concerned, the definitions and nomenclature for bevel gears are in every respect the same as those given for spur gears in Arts. 65 and 70. Bevel gears, however, require the additional nomenclature shown in Fig. 114, which is self-explanatory.

In bevel gearing *formative number of teeth* is the number of teeth of the given pitch that would be contained in a complete spur gear having a pitch radius equal to the back-cone radius of the bevel gear. This number of teeth is used in laying out the teeth, in the kinematic analysis of the tooth action, in calculating

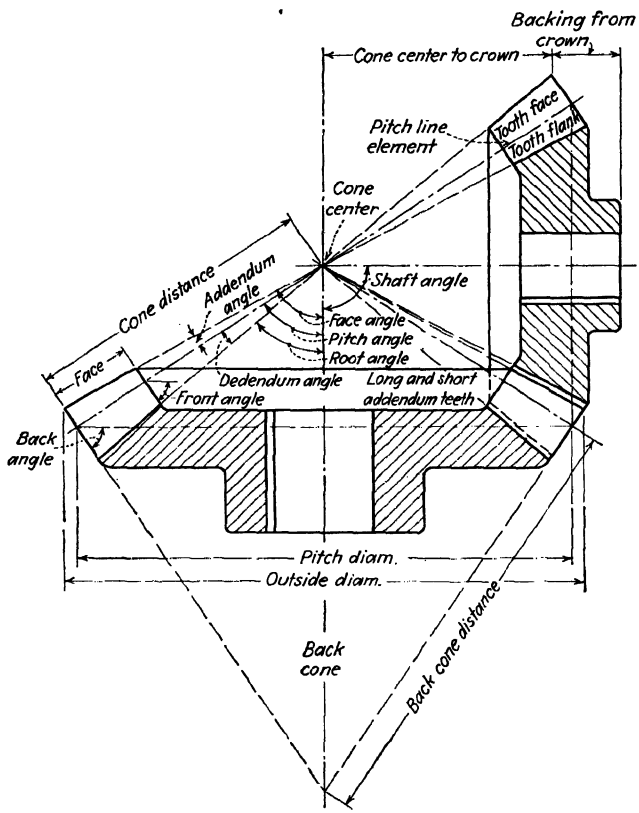


FIG. 114.

the strength of the teeth, in designing and selecting the proper cutting tools, etc.

89. Layout of a Pair of Bevel Gears.—Figure 115 shows a pair of straight-tooth bevel gears laid out in accordance with the following specifications:

14½-deg. composite system (Arts. 98 and 101)
Diametral pitch = 2½

Velocity ratio = 2 to 1
Angle between shafts = 69 deg.
Number of teeth in pinion = 15
Number of teeth in gear = 30.
Length of face of the gears = $2\frac{3}{4}$ in.

The following items may be calculated before starting the layout (all dimensions given in inches).

	Pinion	Gear
Pitch diameter	6.000	12.000
Back-cone radius	3.228	8.853
Addendum	0.400	0.400
Dedendum	0.463	0.463
Formative number of teeth	16.14	44.26

Lay out the pitch cones in accordance with the method described in Art. 62, and then draw the back cones *HDB* and *HEC*. The gears will be shown in sectional view, the usual practice for working drawings. At *H* lay off on the back cones the addendum and dedendum distances for the pair of teeth in contact. Next lay off the face length *HK* on the contact element of the pitch cones and draw the lines that converge at *O*, representing the tops and bottoms of the pair of teeth. Repeat the above tooth layout at *E* and *D* and complete the sectional view of the gears by drawing in suitable hub, rim, and web proportions, as shown in the figure.

Although the above layout is sufficiently complete for ordinary shop purposes, it is frequently necessary to show the true shapes of the tooth profiles. To do this, develop parts of the back cones as shown. The teeth will then be laid out as the teeth of a pair of involute spur gears having pitch circles whose radii are *BH* and *CH* and whose pitch is the same as that of the bevel gears. The formative number of teeth for the pinion = $2BH \times p = 2 \times 3.228 \times 2.5 = 16.14$, and the formative number of teeth for the gear = $2 \times CH \times p = 2 \times 8.853 \times 2.5 = 44.26$. That is, the teeth are laid out as the teeth of a pair of equivalent spur gears having 16.14 teeth in the pinion and 44.26 teeth in the gear, whose center distance is *BC* and whose pitch is the same as that of the bevel gears.

Before generating machines were developed, the teeth of bevel gears were proportioned as in Fig. 115 and cut on the milling machine by the use of formed cutters of the Brown and Sharpe type. To a very limited extent bevel gears are still cut by this method, which is inherently incorrect. When the requirements are at all exacting, the teeth of bevel gears are proportioned in accordance with the Gleason System for Bevel Gears (Art. 102) and cut to a high degree of accuracy on Gleason machines. It

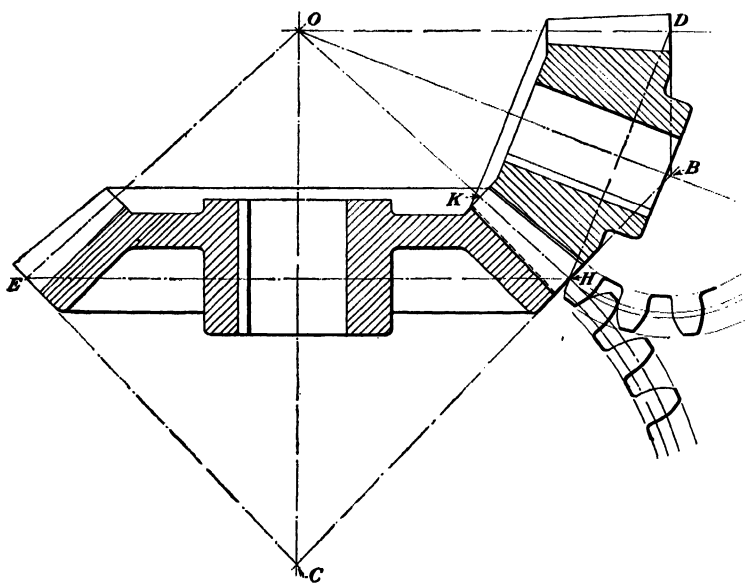


FIG. 115.

should be observed that if the teeth in Fig. 115 had been laid out in accordance with the Gleason System they would appear as in Fig. 114.

90. Spiral Bevel Gears.—Spiral bevel gears have been almost universally used for the rear-axle drive of the automobile, but at the present time hypoid gears are replacing them to a considerable extent for this purpose. They are also used in machine tools, motion-picture machinery, sewing machines, and many other classes of machinery where quiet and smooth operation at high speeds is essential, or where the quantity is such that the cost is less than for straight-tooth bevel gears.

The spiral type of bevel gear is illustrated in Fig. 116. Theoretically, the teeth are curved on a spiral, but in order to facilitate manufacture they are actually curved on the arc of a circle that very closely approximates the true spiral within the limits of the length of the tooth. The use of a circular arc instead of a spiral not only makes rapid production possible but results in an adjustability valuable from the standpoint of assembly and operation.

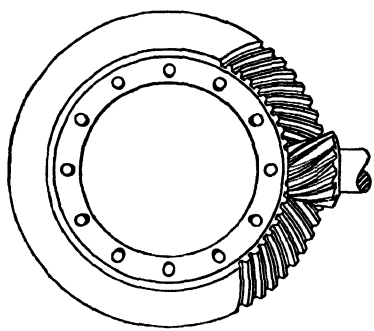


FIG. 116.

The outstanding advantage of spiral bevel gears as compared with straight-tooth bevel gears is the greater quietness of operation of spiral bevel gears, particularly at high speeds.

The reasons for the smooth and quiet action of spiral bevel gears are:

1. More teeth are in contact than in the corresponding straight-tooth bevel gears.

2. The teeth of spiral bevel gears engage with one another gradually, the contact beginning at one end and gradually working over to the other end, whereas in straight-tooth bevel gears the contact takes place along the entire face of the tooth at the same instant.

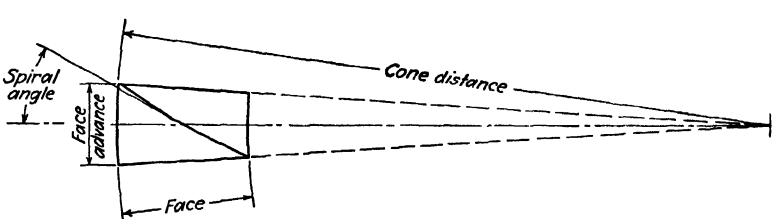


FIG. 117.

The points of superiority of spiral bevel gears over straight-tooth bevel gears are identical in almost every respect with the points of superiority of helical gears over spur gears (see Art. 94).

The fundamental theory of toothed gearing as applied to straight-tooth bevel gearing applies also to spiral bevel gearing.

The tooth proportions of spiral bevel gearing conform to what is called the *Gleason system for spiral bevel gears* (see Art. 102).

The diagram in Fig. 117 shows how the angle of spiral (analogous to helix angle in helical gears) is designated.

91. Skew Bevel Gears.—Skew bevel gears may be used to connect shafts whose axes do not intersect, as illustrated in Fig. 118. The teeth are constructed on rolling hyperboloids as pitch surfaces (see Art. 62). The tooth elements are straight lines which, however, do not converge to a common point as in the case of ordinary straight-tooth bevel gears. The teeth are in contact along straight lines, and there is sliding action along the tooth elements as well as at right angles to the tooth elements. Skew

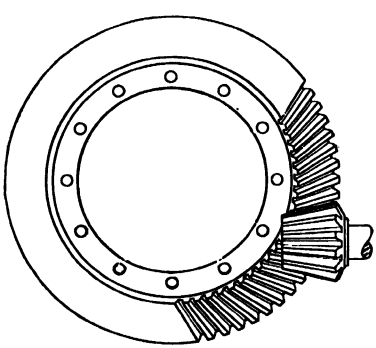


FIG. 118.

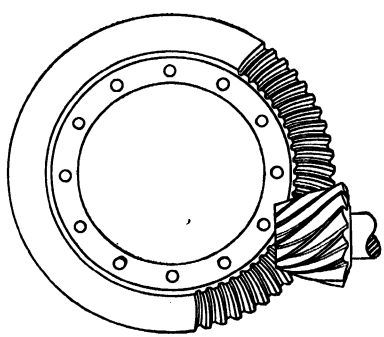


FIG. 119.

bevel gears are rarely used in machine construction because of the difficulties involved in producing correct tooth forms. This type of gearing, however, has certain well-recognized advantages, and numerous efforts have been made to overcome the difficulties in its design and production.

92. Hypoid Gears.—The outcome of a recent development in the attempt to secure the advantages possessed by skew bevel gears has been a new type of bevel gear called the *hypoid gear* (illustrated in Fig. 119), which was developed by the Gleason Works, Rochester, N. Y. The term *hypoid* is a contraction of the term hyperboloid and is believed to have been first used by Geo. B. Grant in his "Treatise on Gearing." Hypoid gears possess the main characteristics of skew bevel gears, namely, offset axes and a certain amount of sliding action in the direction of the tooth elements. But hypoid gears have curved teeth,

whereas true skew bevel gears have straight teeth constructed on rolling hyperboloids of revolution as pitch surfaces.

The hypoid gear was developed in 1925 and was first used in the rear-axle drive of the Packard automobile. Its chief application has been in the automobile, where its use is rapidly increasing. It is also having gradually increasing use in the industrial field. The use of hypoid gears in the automobile permits lowering of the drive shaft and is thus advantageous in the design of cars with low bodies.

Some important advantages result from the application of hypoid gears to industrial purposes. The pinion may be mounted on a continuous shaft, bearings may be placed on both sides of the gear and pinion, and in some cases, the raised or lowered position of the pinion allows arrangements of machinery that would not otherwise be possible. Owing to the offset axes, the pinion of a pair of hypoid gears is larger and stronger than that of the equivalent pair of spiral bevel gears. Also, the teeth of the hypoid gears have a certain amount of endwise sliding. Although this sliding action results in quieter operation, the sliding, when in combination with the high unit pressures, makes it advisable to give special attention to the matter of lubrication.

HELICAL GEARING

Helical gearing is a term applied to all classes of gears in which the teeth are of helical form. Helical gearing is used to connect parallel shafts as well as nonparallel, nonintersecting shafts. The pitch surfaces are cylindrical as in spur gearing, but the teeth, instead of being parallel to the axes, wind around the cylinders helically like screw threads.

The distinction between helical gearing and worm gearing may be stated as follows: If the number of threads, or teeth, on the pitch cylinder is such that no one thread makes a complete turn, the gear is called a helical gear. If, on the other hand, a thread (from one to four is the usual number) makes a complete turn, the result is a *worm* and the mating gear is called a *worm wheel.*

Helical gearing is used mainly between parallel shafts, but it is also employed to some extent to connect nonparallel, nonintersecting shafts, usually at 90 deg.

93. Helical Gears for Parallel Shafts.—Helical gears connecting parallel shafts are illustrated in Fig. 120*a*. The conception

of a helical gear is simplified by considering it as a spur gear with the teeth twisted. Assume a spur gear to be cut by a series of equidistant planes perpendicular to the axis and let the resulting slices or laminations be twisted relative to each other as shown in Fig. 121. The result is a *stepped gear* in which there is a series of narrow teeth derived from the original gear coming into contact successively with similarly constructed mating teeth, instead of a pair of wide teeth coming into contact at the same instant along the entire length, as in the original pair of gears. If the slices

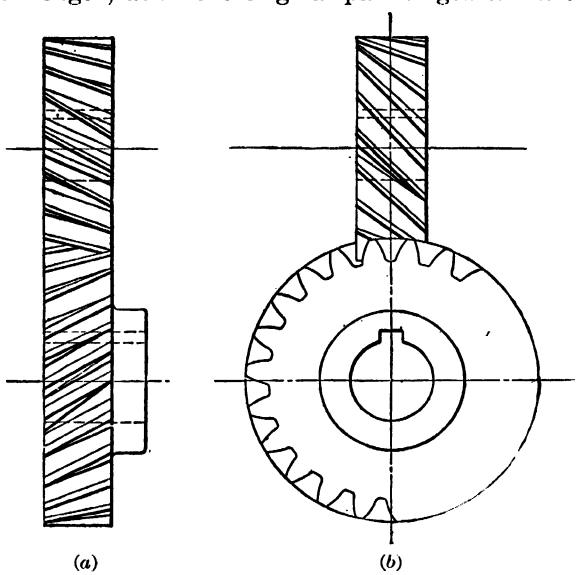


FIG. 120.—(*Fellows Gear Shaper Co., Springfield, Vt.*)

become thin enough, lines joining corresponding points of the stepped teeth become helices, and the gear is, therefore, called a helical gear. The teeth of helical gears with parallel axes have line contact, just as do spur gears. An instantaneous line of contact, however, is not parallel to the axes but runs diagonally across the tooth faces, so that in successive sections all conjugate points of a pair of profiles are in contact at the same time. The teeth of helical gears are of the involute form, usually of a stub type, with a pressure angle greater than that of the standard 14½-deg. systems (see Art. 101).

In Fig. 121 helices formed by the intersection of the pitch surface and two adjacent tooth surfaces are shown. The dis-

tance between the helices measured around the pitch surface in a plane at right angles to the axis is the *circular pitch.* The distance measured in a similar manner at right angles to the helices is called the *normal pitch.* The *helix angle* is always the angle between the tangent to the helix and the axis of the gear, as shown in the figure. The velocity ratio follows exactly the same law as for spur gears.

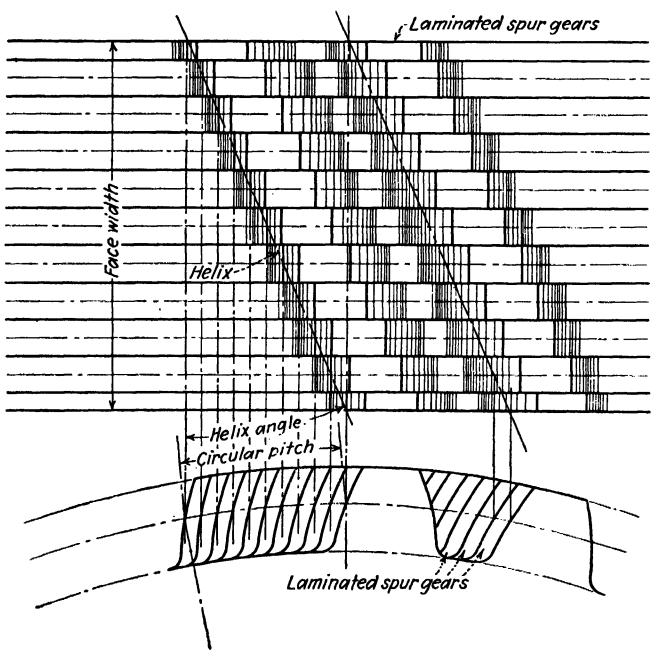


FIG. 121.

From Fig. 121 it is evident that single-helical gears produce an axial thrust on the shaft. This objection is overcome by using *double-helical* or *herringbone* gears (Fig. 122), which are simply two single-helical gears with teeth of the opposite hand combined.

Double-helical gears, as manufactured in this country, are of two general types, as follows:

1. The type in which the teeth meet at a common apex at the center of the face as shown in Fig. 122*b*. A modification of this type in which the central part has been removed with the object of providing tool clearance is shown in Fig. 122*a*.

2. The type known as the *Wuest* gear in which the teeth are staggered as shown in Fig. 122*c*. The central portion is cut away somewhat by the tools but the blank is not grooved as in Fig. 122*a*.

Figure 122*d* illustrates an interesting type of helical gear called the *Citroën gear*, manufacture by the Citroën Gear Company, Ltd., Paris, France. The teeth may be cut to any required shape, and may be either double helical or double herringbone. Figure 122*d* shows a typical case of a double-herringbone Citroën gear used for connecting parallel shafts. Since the teeth of Citroën gears are produced with an end-milling cutter, the teeth

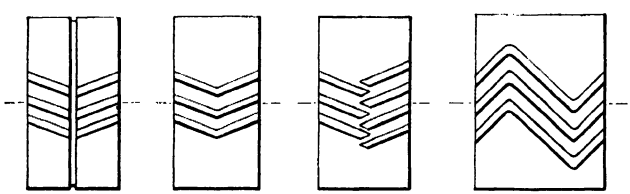


FIG. 122.

can be readily formed on bevel gears also, in either the double-helical or double-herringbone form.

94. Advantages of Helical Gears.—The outstanding advantage of helical gears, as compared with corresponding spur gears, is that helical gears run more smoothly and more quietly at high speeds and under other severe service conditions.

The reasons for the smooth and quiet action of helical gears are:

1. There are more teeth in contact than in the corresponding spur gears.

2. The teeth come into engagement gradually, the contact beginning at an end and gradually working over to the other end. In spur gears the contact takes place along the entire face of the tooth at the same instant.

95. Helical Gears for Nonparallel, Nonintersecting Shafts.—While helical gearing for nonparallel, nonintersecting shafts may be designed for any angle between the shafts, the shafts are usually at right angles as in Fig. 120*b*. The general case of helical gearing may be considered as exemplified by some angular relation of shafts between 0 and 90 deg., and shafts parallel

and at right angles might be considered to represent special cases of helical gearing.

It is important to note that although the design is essentially the same, the tooth action of helical gears for nonparallel, nonintersecting axes is quite different from that of helical gears for parallel shafts. The former have merely point contact, while the latter have line contact. In the former case, also, there is a large amount of sliding in the direction of the common tangent to the tooth elements, which is entirely absent in the latter case. Helical gears for nonparallel, nonintersecting shafts may consequently be used only for comparatively light service. Such gears are, therefore, of secondary importance.

WORM GEARING

Worm gearing is commonly employed to obtain higher velocity ratios than can conveniently be obtained from other forms of

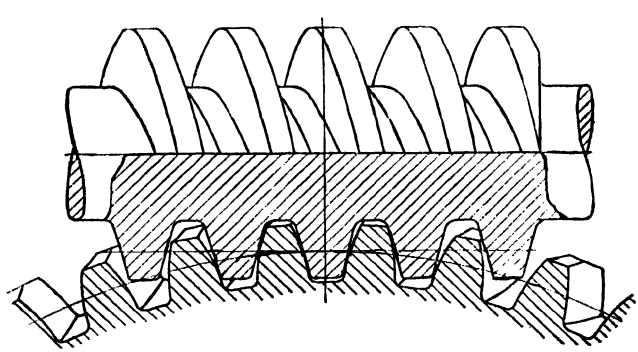


FIG. 123.

gearing. In a worm-gear combination, the shafts are nonintersecting and nearly always make an angle of 90 deg. with each other. The worm, which is usually the driver, is a special form of helical gear in which the angle of the helix is such that the tooth becomes a thread making one or more complete turns around the pitch cylinder. The worm wheel, the driven member, becomes a form of helical gear in which the helix angle is the complement of that of the worm.

There are two classes of worm gearing in use: the *straight or parallel type* illustrated in Fig. 123, and the *hollow type* (generally

known as Hindley worm gearing), shown in Fig. 124. Of these two classes the straight type is the more frequently used.

96. Straight Worm Gearing.—The pitch surfaces of straight worm gearing are cylinders, and the teeth are of the involute form. The proportions now regarded as standard are given in Art. 103. The contact between a pair of teeth is theoretically line contact.

The pitch of the worm is the distance between corresponding points of adjacent threads or teeth measured parallel to the axis of the worm, and is called the *linear pitch*. The linear pitch of the

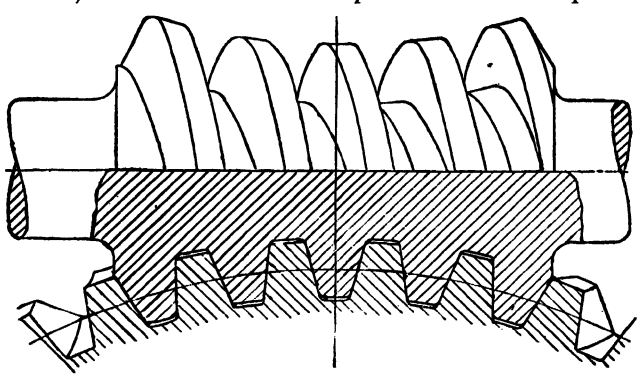


FIG. 124.

worm must, of course, be equal to the circular pitch of the worm wheel. The *lead* of a worm thread is the axial distance traversed by a point moving on the helix in one complete turn of the helix. Worms are designated as right-hand (or left-hand) single-thread worm, right-hand (or left-hand) double-thread worm, etc. In a single-thread worm the lead equals the linear pitch; in a double-thread worm the lead equals twice the linear pitch; etc. The distinction between lead and pitch is made clear by examination of Fig. 125, in which the difference between a single- and a multiple-thread worm is clearly shown.

The velocity ratio in worm gearing does not depend upon the diameters of the worm and gear but is the ratio between the number of teeth on the worm wheel and the number of threads on the worm. Thus, for a single-thread worm and a worm wheel having 36 teeth, the velocity ratio is 36:1; and for a triple-thread worm and a worm wheel with 36 teeth the velocity ratio is 36:3, or 12:1.

In general, when the pressure angle is 14½ deg., 32 teeth is the smallest number advisable for the worm wheel on account of interference, or undercutting (see Art. 71). Hence, if the velocity ratio is less than 30:1, the worm must be provided with a multiple thread in order to avoid this condition.

It can be seen in Fig. 126 that in a central plane containing the axis, the straight worm is an involute rack, and the teeth, therefore, have straight sides. In the central plane perpendicular to

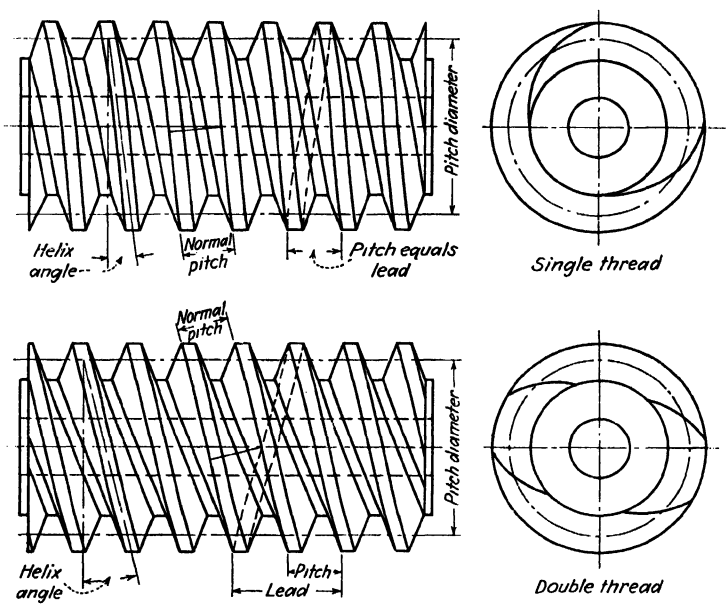


FIG. 125.

the axis of the worm wheel, the worm wheel is an ordinary involute spur gear. The worm and gear are, therefore, equivalent to an involute rack and pinion, and in making a layout of a worm and gear, the method employed is the same as that for the involute rack and pinion.

A characteristic of worm gearing that is not found in other types of gearing discussed in this chapter is its self-locking property; that is, a worm drive, in general, is not reversible. This is obviously a useful property in many applications such as elevators, hoists, automobile steering devices, etc. In rare cases, as

in the cream separator, for example, the worm wheel is the driver, in which case the helix angle is very large.

97. Hollow Worm Gearing.—In the hollow type of worm gearing the worm has a shape similar to that of an hour glass. Worms of the hollow type are generally known as Hindley worms. As

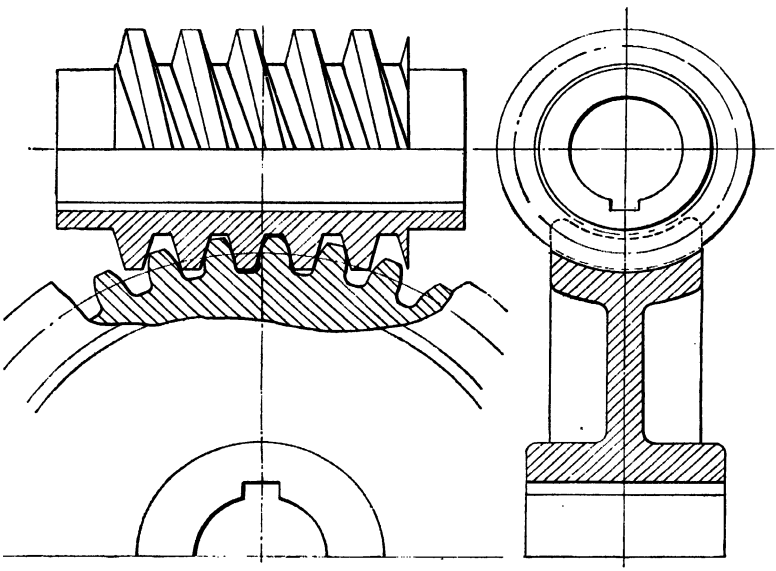


FIG. 126.

can be seen in Fig. 124, the object in making the worm hollow is to make it conform to the shape of the worm wheel. The chief advantage claimed is larger contact surface between the mating teeth than in the straight worm type. In securing this advantage, however, manufacturing difficulties are greater, and the requirements as to mounting are more exacting than in the case of straight worm gearing.

CHAPTER VI

GEAR-TOOTH SYSTEMS AND PRODUCTION METHODS

The first toothed wheels were made of wood, and the first metallic gears were cast. Cast gears are still used to some extent in certain machines and appliances where the requirements are not exacting and in places where they are exposed to the elements and operate only occasionally. The teeth of cast gears are made from forms carefully shaped by the patternmaker to secure as close an approximation of conjugate gear-tooth action as possible. The proportions for cast teeth have never been standardized. The proportions that have been used differ from those for cut teeth only in the provision for greater clearance and greater backlash.

Although the use of toothed gears extends back into ancient times, it was only about 250 years ago that the theory of correct tooth shapes was enunciated, and only about 75 years ago that a means of producing accurately cut teeth was made commercially available. At that time the formed milling cutter of the type shown in Fig. 135 was introduced by the Brown and Sharpe Manufacturing Company. This development marked the first real advance in gear manufacture. At this time the cycloidal form of tooth prevailed, but it was not long thereafter that it began to give way rapidly to the involute form.

Since the introduction of the involute form of tooth, a variety of gear-tooth proportions have been devised to meet the changing requirements of industry. The different systems that have come into use have been developed primarily around the method employed to produce them. It is only recently, however, that attempts at standardization have generally been successful.

Before entering into a discussion of these recently adopted gear standards that are now gaining wide acceptance, a brief review will be given of those systems for cut teeth that have been developed in the past and are still employed to a considerable

extent. These systems will generally fall within the following classification:

1. The Brown and Sharpe 14½-deg. system.
2. The stub-tooth systems.
3. The unequal-addendum systems.

98. The Brown and Sharpe 14½-deg. Involute System.—When this system was introduced by the Brown and Sharpe Manufacturing Company, it was given the same tooth proportions as the cycloidal system that it has almost completely superseded. This system, like its predecessor, the cycloidal system, was developed primarily for formed milling cutters, and it has long been the recognized standard for this widely used method of cutting gear teeth. This system has been adopted as one of the new American Standards and it is now known as the 14½-deg. composite system (see Art. 101).

The tooth proportions of the Brown and Sharpe 14½-deg. involute system are given as functions of either the diametral pitch p or the circular pitch p' and are as follows:

Pressure angle = 14½ deg.

$$\text{Addendum (in inches)} = \frac{1}{\text{diametral pitch}} = \frac{1}{p}$$

$$\text{Dedendum (in inches)} = \text{addendum} + \text{clearance} = \frac{1}{p} + 0.05p'$$

$$\text{Clearance} = 0.05 \times \text{circular pitch} = 0.05p'$$

$$\text{Whole depth of tooth} = 2 \times \text{addendum} + \text{clearance} = 2 \times \frac{1}{p} + 0.05p'$$

$$\text{Working depth of tooth} = 2 \times \text{addendum} = 2 \times \frac{1}{p}$$

$$\text{Thickness of tooth} = \frac{\text{circular pitch}}{2} = \frac{p'}{2}$$

$$\text{Width of tooth space} = \frac{\text{circular pitch}}{2} = \frac{p'}{2}$$

$$\text{Minimum radius of fillet} = \text{clearance} = 0.05p'$$

It is noted that in the above proportions the backlash is zero. This is sometimes desirable, but it is common practice now to provide backlash, and when standard cutters are used, this is accomplished by cutting the teeth slightly deeper than standard. In any case, the backlash is only a few thousandths of an inch

for ordinary cut gears and does not enter into the kinematic study of gear-tooth action.

The proportions of cast gear teeth have not been standardized, but the proportions that have been commonly used do not differ greatly from those given above for cut teeth. Obviously, the backlash and clearance in cast teeth must be considerably greater than for cut teeth.

The profiles in the Brown and Sharpe 14½-deg. involute system are not true involutes, the curves produced by the standard cutters deviating slightly from the true involute in order to avoid interference.

99. The Stub-tooth Systems.—While the Brown and Sharpe 14½-deg. involute system was long the recognized standard for

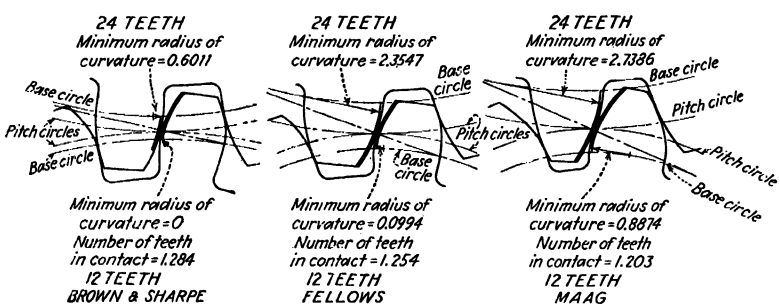


FIG. 127.

most ordinary purposes, unusual conditions arose where the standard tooth form was not suitable. A demand arose for a stronger tooth shape and one that would minimize interference. Consequently, teeth of greater obliquity of action and less depth than the preceding standard have become quite common. Gears with this special form of tooth are called *stub-tooth* gears and are being successfully used in automobile drives, machine tools, hoisting machinery, etc.

Stub teeth are of the involute form, usually having a 20-deg. angle of obliquity, and have a shorter addendum and dedendum than that of the 14½-deg. standard. Of the several stub-tooth systems that have been proposed, that adopted by the Fellows Gear Shaper Company (Fig. 127) has become the most widely used. In the *Fellows stub-tooth system* the pitch is expressed as a combination of two standard diametral pitches, as for example,

$\frac{6}{8}$ pitch which is read *six-eight* pitch and not as a fraction *six-eighths*. Thus, a $\frac{6}{8}$-pitch tooth has a thickness on the pitch line equal to that of a standard 6-pitch tooth, while the addendum is equal to that of a standard 8-pitch tooth. The clearance is made greater than that of the standard tooth, being 0.25/diametral pitch, so that the dedendum is 1.25 times the addendum. In the case of the $\frac{6}{8}$ pitch tooth the dedendum would be $\frac{1}{8} + 0.25/8 = 1.25/8$, or $1.25 \times \frac{1}{8}$.

It is thus evident that the first number of the expression for the combined pitch governs the thickness of the tooth, and the second number governs the height of the tooth.

The pitches that have been adopted and are in use are as follows: $\frac{4}{5}$, $\frac{5}{7}$, $\frac{6}{8}$, $\frac{7}{9}$, $\frac{8}{10}$, $\frac{9}{11}$, $\frac{10}{12}$, and $\frac{12}{14}$.

100. The Unequal Addendum Systems.—The exacting requirements of high-speed gearing led to the adoption by certain gear manufacturers of other special tooth forms that aimed at eliminating the defects of the standard involute system and at securing the best possible results as to quietness, strength, and durability. Important features of these tooth forms, all of which have the true involute outline, are a long addendum for the driving pinion and a short addendum for the driven gear, accompanied by a varying pressure angle; *i.e.*, for a pair of gears of a given velocity ratio the addenda and the pressure angle are chosen so as to give the best running conditions for this particular ratio.

Typical of the form of tooth coming under the above description is the system of gearing originally developed by Max Maag of Zurich, Switzerland, and known as the *Maag system* (Fig. 127).

The Maag system has been successfully applied in many classes of machinery where high-speed gearing of the spur or helical type is required.

Another tooth form of the unequal addendum type, known as the *Gleason system for bevel gears* is in common use in the rear-axle drives of automobiles, and in many other places where high-speed bevel gearing is used.

Some of the advantages claimed for the tooth forms of the unequal-addendum type described above are as follows:

1. The elimination of interference, accompanied by a tooth shape of greater strength. The most advantageous part of the involute curve is used, so that the flanks of the teeth are true involutes, and pinions with a small number of teeth can be used

successfully. One of the defects of an equal-addendum system is the interference and undercutting in pinions with small numbers of teeth.

2. The arc of recess is greater than the arc of approach, and there is a maximum of rolling contact and a minimum of sliding contact. These factors result in smoother and quieter running and less wear than in the case of equal-addendum teeth.

3. The teeth of the pinion and gear can be proportioned so as to be of equal strength, whereas with equal-addendum teeth this is not possible without resorting to the use of different materials for the pinion and gear.

In the case of the unequal-addendum tooth forms such as referred to above, the tooth proportions vary with the velocity ratio, and hence the proportions cannot be given by simple formulas as in the case of equal-addendum teeth, where the proportions are constant. For example, the addenda and pressure angle for a pair of gears of 2:1 ratio would be different from the addenda and pressure angle of a pair of gears of the same pitch having a 4:1 ratio. Elaborate tables are required to cover the data for all the ratios in common use. The tables for the Gleason system for straight-tooth bevel gears are given in Art. 102. Since the tooth profiles of all such systems are involutes, the fundamental theory of the gear-tooth action is the same for all.

Figure 127 is given for the purpose of comparing the tooth proportions in a particular case, for the Brown and Sharpe 14½-deg. involute system, a stub-tooth system, and an unequal-addendum system. In each of the three cases represented, the pitch, pitch diameters, and velocity ratio are the same.

101. American Standard Spur-gear Tooth Forms.—Since any two spur gears that will mesh properly with a rack will mesh properly with each other, rack-tooth proportions may be used as the basis for an interchangeable system of spur gears. Of the various spur-gear tooth standards that have been proposed for an interchangeable system, the best known and most widely used have been the Brown and Sharpe 14½-deg. involute system and the Fellows 20-deg. stub-tooth system. Of the unequal-addendum systems for noninterchangeable spur gears, that proposed by Maag is typical and is perhaps the best known. It was probably the first used to produce to any extent noninterchangeable spur gears on a commercial basis. The above-mentioned tooth

forms have been discussed in Arts. 98 to 100 and illustrated in Fig. 127.

In June 1921, the American Standards Association, with the American Gear Manufacturer's Association and the American Society of Mechanical Engineers as joint sponsors, began a survey of the various standards on gears in use in both the United States and Europe with the view of developing standards to meet the requirements of industry. With the findings of this survey as a background, the American Standards Association in November

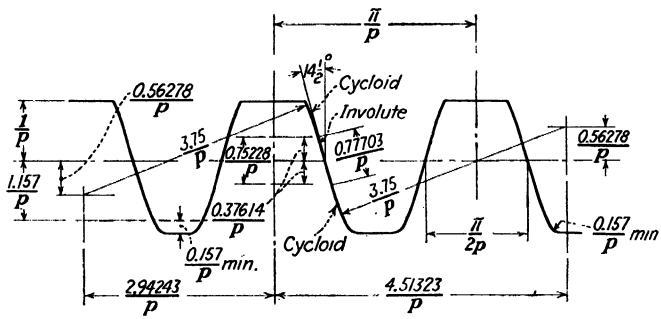


FIG. 128.

1932, approved and designated as American Standards, four spur-gear tooth forms that are listed as follows:

14½-deg. composite system.

14½-deg. full-depth involute system.

20-deg. full-depth involute system.

20-deg. stub involute system.

The proportions approved for the basic rack of the 14½-deg. composite system are shown in Fig. 128. In this system the 12-tooth pinion has been adopted as the smallest gear of the set, and the tooth profile of the basic rack is a combination of involute and cycloidal curves. Reference to Fig. 129 will show how the proportions given in Fig. 128 were established. The line *bPc* is drawn at an angle of 14½ deg. with the pitch line 3 of the rack, circle 2 being the pitch circle of a 12-tooth pinion with its center at O_2. The corresponding base circle 2′ is tangent to line *bPc* at *c*. Since there can be no involute action beyond *c*, the distance *ce* is the maximum addendum of a 14½-deg. involute rack tooth. Since PcO_2 is evidently a right triangle, point *c* lies on a circle 2″ having a diameter equal to the radius

of the 12-tooth pinion. Hence if *kl* is described by rolling 2″ on pitch line 3 and *fg* is described by rolling 2″ on the inside of 2, the cycloidal face *kl* of the rack tooth will mesh properly with the hypocycloidal flank *fg* of the pinion tooth. Beyond *c* the path of contact *cd* would therefore follow the describing circle 2″. The flank *mn* of the rack tooth will mate properly with the face *bh* of the pinion if the flank *mn* is made a cycloid described by rolling 3″ on pitch line 3, and the face *bh* is made an epicycloid by rolling 3″ on the outside of 2′. The rack-tooth profile is therefore cycloidal from *n* to *m*, involute from *m* to *k*, and

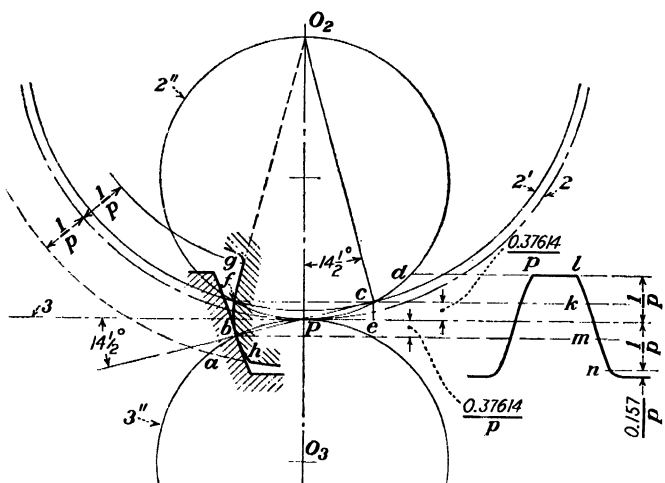


FIG. 129.

cycloidal from *k* to *l*; whereas the pinion-tooth profile is hypocycloidal from *g* to *f*, involute from *f* to *b*, and epicycloidal from *b* to *h*. With the diameter of 2″ equal to the radius of 2, the hypocycloid *fg* becomes a radial line. If the rack tooth is to be basic for an interchangeable system of gears, the generating circle 3″ must be made equal to 2″, in which case the path of the point of contact *abPcd* becomes symmetrical with respect to the center line O_2P. It was in the above manner that the basic rack of the 14½-deg. composite system as shown in Fig. 128 was derived. In this figure are given the radii and location of the centers that would very closely approximate the true cycloidal faces and flanks of the rack.

The tooth proportions of the 14½-deg. composite system are identical with those of the old Brown and Sharpe 14½-deg. involute system (see Art. 98), and the basic racks of the two systems have substantially the same tooth profiles. It is important to note, therefore, that the new system was established by practically adopting the old system as a definite standard, and that gears cut in accordance with the two systems are interchangeable.

The 14½-degree composite system finds its chief application where the teeth are produced by the form-milling process, particularly by those concerns that have been using the Brown and Sharpe cutters for many years in jobbing and repair work.

The tooth proportions of the 14½-deg. full-depth involute system (Fig. 130) and the 20-deg. full-depth involute system

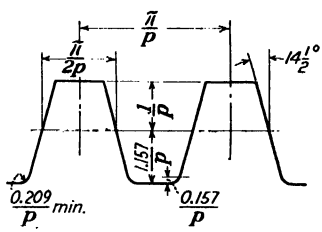


Fig. 130.

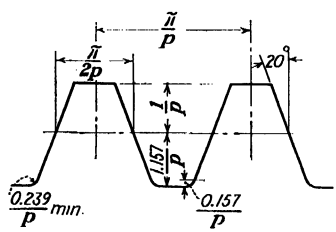


Fig. 131.

(Fig. 131) are identical with those of the 14½-deg. composite system, but the teeth of the basic racks have straight sides throughout the full working depth. These two systems are employed when the teeth are cut by the generating methods, and their purpose is to obtain full involute action. However, to obtain full involute tooth action on pinions of 31 teeth and smaller in the 14½-deg. full-depth involute system and on pinions of 17 teeth and smaller in the 20-degree full-depth involute system, the outside diameter of the pinion must be increased. To maintain standard center distance where a gear is substituted for the rack, the outside diameter of the gear must be decreased the same amount. Tabulations of the amounts of increase and decrease in diameter and the corresponding circular tooth thicknesses on the pitch circle of both gear and pinion are included in tables published by the American Standards Association.

For gears with a relatively large number of teeth, say 40 or more, the 14½-deg. full-depth involute system is customarily

employed, but for gears with fewer teeth, the 20-deg. full-depth involute system is given preference.

The proportions for the basic rack of the 20-deg. stub involute system are given in Fig. 132. This system, also, applies to gearing where the generating method is employed. A pinion having 14 teeth is the smallest that will mesh with the rack without interference. These proportions are identical with those of the American Gear Manufacturer's Association recommended practice, for herringbone gears. It may be observed,

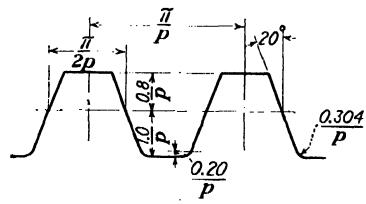


FIG. 132.

TABLE 2.—COMPARISON OF STANDARD INTERCHANGEABLE INVOLUTE SYSTEMS

	14½-deg. Brown and Sharpe system	14½-deg. composite system	14½-deg. full-depth involute system	20-deg. full-depth involute system	Fellows stub-tooth system[3]	20-deg. stub-involute system
Pressure angle, deg	14½	14½	14½	20	20	20
Addendum	$\frac{1}{p}$	$\frac{1}{p}$	$\frac{1}{p}$	$\frac{1}{p}$	$\frac{1}{p_d}$	0.80 / p
Working depth	$\frac{2}{p}$	$\frac{2}{p}$	$\frac{2}{p}$	$\frac{2}{p}$	$\frac{2}{p_d}$	1.60 / p
Clearance	$\frac{0.157}{p}$	$\frac{0.157}{p}$	$\frac{0.157}{p}$	$\frac{0.157}{p}$	$\frac{0.25}{p_d}$	0.20 / p
Dedendum	$\frac{1.157}{p}$	$\frac{1.157}{p}$	$\frac{1.157}{p}$	$\frac{1.157}{p}$	$\frac{1.25}{p_d}$	1 / p
Whole depth	$\frac{2.157}{p}$	$\frac{2.157}{p}$	$\frac{2.157}{p}$	$\frac{2.157}{p}$	$\frac{2.25}{p_d}$	1.80 / p
Tooth thickness[1]	$\frac{p'}{2}$	$\frac{p'}{2}$	$\frac{p'}{2}$	$\frac{p'}{2}$	$\frac{\pi}{2p_n}$	$\frac{p'}{2}$
Tooth space[1]	$\frac{p'}{2}$	$\frac{p'}{2}$	$\frac{p'}{2}$	$\frac{p'}{2}$	$\frac{\pi}{2p_n}$	$\frac{p'}{2}$
Fillet radius[2]	$\frac{0.157}{p}$	$\frac{0.157}{p}$	$\frac{0.209}{p}$	$\frac{0.239}{p}$	$\frac{0.25}{p_d}$	0.304 / p

The symbols p and p' denote diametral and circular pitch respectively. All dimensions are in inches.

[1] In these proportions backlash has not been taken into account.

[2] These may be assumed as minimum values. The fillet radius has not been standardized.

[3] For this fractional form of pitch, the symbol p_n represents the figure in the numerator and p_d the figure in the denominator. Example: $p_n/p_d = \frac{1}{6}$.

also, that these proportions approach very closely those of the Fellows stub-tooth system. The 20-deg. stub-involute system is particularly applicable where the tooth numbers are relatively small in both pinion and gear, as in automobile transmission assemblies. It is also used to some extent for heavy mill gears with large tooth numbers, where the increased strength of the shorter teeth is a distinct asset.

Table 2 is given for convenience in making a comparison of the standard spur-gear tooth forms that have been discussed in this and in previous articles.

102. Standard Tooth Proportions for Bevel Gears.—Standards have been developed by the Gleason Works, Rochester, N. Y., covering in a very comprehensive manner the tooth forms of straight-tooth bevel gears, spiral bevel gears, and hypoid gears. The teeth are of the unequal-addendum type and the gears are not interchangeable. This system, known as the *Gleason system for bevel gears*, has been adopted as a recommended standard by the American Gear Manufacturer's Association. The system applies to any pair of generated straight-tooth or spiral bevel gears operating at right angles where the pinion is the driver and has 10 or more teeth in the case of straight-tooth bevel gears and 5 or more teeth in the case of spiral bevel gears. The data covering that part of this system relating to straight-tooth bevel gears are given in Tables 3 and 4. Similar tables of data are available for spiral bevel gears and hypoid gears, but space does not permit their inclusion here. In these tables the term *diametral pitch* is designated by the abbreviation D.P. When this term is designated by a symbol, in this text, the letter p is used (see Art. 65).

103. Standard Proportions for Worm Gearing.—The proportions recommended by the American Gear Manufacturer's Association for worm gearing for general commercial use are given in Table 5. The range covered by these proportions may be defined as follows:

Linear pitches, inches $\frac{1}{4}$, $\frac{5}{16}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{4}$, 1, $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, 2.
Multiplicity of threads, single to quadruple.
Gear ratios.....10:1 to 100:1.
Relation of axes.....90 deg. only.

The *thread form* regarded as *standard* is the form produced by a straight-sided milling cutter having a diameter not less than the

TABLE 3.—GLEASON WORKS SYSTEM FOR GENERATED STRAIGHT-TOOTH BEVEL GEARS

(For straight-tooth bevel gears only, operating at right angles, where pinion is driver and has 10 or more teeth)

Working depth $= \frac{2.000 \text{ in.}}{\text{D.P.}}$ Full depth $= \frac{2.188 \text{ in.}}{\text{D.P.}}$

Pressure Angle:	Deg.
Ratios having 14 or more teeth in pinion	14½
Ratios 13:13 to 13:24	17½
Ratios 13:25 and higher	14½
Ratios 12:12 and higher	17½
Ratios 11:11 to 11:14	20
Ratios 11:15 and higher	17½
Ratios 10:10 and higher	20

Addendum:

Addendum of gear $= \frac{\text{Addendum for 1 D.P. (Table 4)}}{\text{D.P.}}$

Addendum of pinion $= \frac{2.000 \text{ in.}}{\text{D.P.}}$ − addendum of gear

Dedendum:

Dedendum of gear $= \frac{2.188 \text{ in.}}{\text{D.P.}}$ − addendum of gear

Dedendum of pinion $= \frac{2.188 \text{ in.}}{\text{D.P.}}$ − addendum of pinion

Circular Thickness:

Circular thickness of gear for all ratios using 14½ deg. $= \frac{1.071 \text{ in.}}{\text{D.P.}} + (0.5 \times \text{addendum of gear}) - \frac{K \text{ (Table 4)}}{\text{D.P.}}$

Circular thickness of gear for all ratios using 17½ deg. $= \frac{0.971 \text{ in.}}{\text{D.P.}} + (0.6 \times \text{addendum of gear}) - \frac{K}{\text{D.P.}}$

Circular thickness of gear for all ratios using 20 deg. $= \frac{0.871 \text{ in.}}{\text{D.P.}} + (0.7 \times \text{addendum of gear}) - \frac{K}{\text{D.P.}}$

Circular thickness of pinion for 14½ deg., 17½ deg., or 20 deg. $= \frac{3.142 \text{ in.}}{\text{D.P.}}$ − circular thickness of gear

TABLE 4.—GLEASON WORKS SYSTEM FOR STRAIGHT-TOOTH BEVEL GEARS

Addendum[1] for 1 diametral pitch

Ratios		Add., in.	Ratios		Add., in.	Ratios		Add., in.	Ratios		Add., in.
From	To		From	To		From	To		From	To	
1.00	1.00	1.000	1.15	1.17	0.880	1.42	1.45	0.760	2.06	2.16	0.640
1.00	1.02	0.990	1.17	1.19	0.870	1.45	1.48	0.750	2.16	2.27	0.630
1.02	1.03	0.980	1.19	1.21	0.860	1.48	1.52	0.740	2.27	2.41	0.620
1.03	1.04	0.970	1.21	1.23	0.850	1.52	1.56	0.730	2.41	2.58	0.610
1.04	1.05	0.960	1.23	1.25	0.840	1.56	1.60	0.720	2.58	2.78	0.600
1.05	1.06	0.950	1.25	1.27	0.830	1.60	1.65	0.710	2.78	3.05	0.590
1.06	1.08	0.940	1.27	1.29	0.820	1.65	1.70	0.700	3.05	3.41	0.580
1.08	1.09	0.930	1.29	1.31	0.810	1.70	1.76	0.690	3.41	3.94	0.570
1.09	1.11	0.920	1.31	1.33	0.800	1.76	1.82	0.680	3.94	4.82	0.560
1.11	1.12	0.910	1.33	1.36	0.790	1.82	1.89	0.670	4.82	6.81	0.550
1.12	1.14	0.900	1.36	1.39	0.780	1.89	1.97	0.660	6.81	∞	0.540
1.14	1.15	0.890	1.39	1.42	0.770	1.97	2.06	0.650			

TABLE 4.—GLEASON WORKS SYSTEM FOR STRAIGHT-TOOTH BEVEL GEARS.—(*Continued*)

Values[2] of K for circular thickness formula

Number of teeth in pinion	Ratios														
	1.00 to 1.25	1.25 to 1.50	1.50 to 1.75	1.75 to 2.00	2.00 to 2.25	2.25 to 2.50	2.50 to 2.75	2.75 to 3.00	3.00 to 3.25	3.25 to 3.50	3.50 to 3.75	3.75 to 4.00	4.00 to 4.50	4.50 to 5.00	5.00 and higher
	Values of K (in inches) for different ratios														
10	0.025	0.070	0.100	0.120	0.140	0.160	0.175	0.190	0.205	0.215	0.225	0.230	0.240	0.250	0.255
11	0.010	0.015	0.050	0.080	0.105	0.125	0.145	0.160	0.170	0.180	0.190	0.195	0.200	0.210	0.220
12	0.000	0.040	0.070	0.100	0.120	0.140	0.155	0.170	0.180	0.185	0.190	0.195	0.205	0.210	0.215
13	0.000	0.015	0.040	0.045	0.050	0.060	0.070	0.080	0.090	0.100	0.110	0.120	0.135	0.150	0.165
14	0.000	0.015	0.030	0.050	0.065	0.080	0.090	0.100	0.110	0.120	0.125	0.130	0.140	0.150	0.160
15 to 17	0.000	0.000	0.010	0.020	0.030	0.045	0.060	0.070	0.080	0.090	0.095	0.100	0.110	0.115	0.120
18 to 21	0.000	0.000	0.000	0.000	0.010	0.030	0.045	0.060	0.070	0.080	0.085	0.090	0.095	0.100	0.100
22 to 29	0.000	0.000	0.000	0.000	0.010	0.030	0.040	0.050	0.060	0.065	0.070	0.070	0.080	0.085	0.085
30 and up	0.000	0.000	0.000	0.000	0.010	0.025	0.035	0.040	0.045	0.050	0.055	0.060	0.065	0.070	0.070

[1] To obtain addendum select from table value corresponding to ratio given by the formula:

$$\text{Ratio} = \frac{\text{number of teeth in gear}}{\text{number of teeth in pinion}}.$$

[2] Select value corresponding to number of teeth in pinion and ratio given by formula.

TABLE 5.—PROPORTIONS OF STANDARD COMMERCIAL WORMS AND WORM WHEELS RECOMMENDED BY THE AMERICAN GEAR MANUFACTURER'S ASSOCIATION (GEAR RATIOS 10:1 TO 100:1)
(Refer to Figs. 133 and 134)

Terms for worm (Fig. 133):	Symbol	Single- and double-thread worms	Triple- and quadruple-thread worms
Linear pitch	p'	.....	.....
Pitch diam.	d_p	$2.4 \times p' + 1.1$	$2.4 \times p' + 1.1$
Outside diam.	d_o	$3.036 \times p' + 1.1$	$2.972 \times p' + 1.1$
Root diam.	d_r	$1.664 \times p' + 1.1$	$1.726 \times p' + 1.1$
Hub diam.	d_h	$1.664 \times p' + 1.0$	$1.726 \times p' + 1.0$
Bore (max.)	d_b	$p' + 0.625$	$p' + 0.625$
Face length	f	$p' \times \left(4.5 + \frac{N}{50}\right)$	$p' \times \left(4.5 + \frac{N}{50}\right)$
Hub extensions	h_e	p'	p'
Hub length	..	$f + 2 \times p'$	$f + 2 \times p'$
Keyway	..	A.G.M.A. Std.	A.G.M.A. Std.
Addendum	..	$0.318 \times p'$	$0.286 \times p'$
Whole depth	..	$0.686 \times p'$	$0.623 \times p'$
Lead angle	λ	$\cot \lambda = \frac{d_p \times 3.1416}{\text{lead}}$	$\cot. \lambda = \frac{d_p \times 3.1416}{\text{lead}}$
Nor. tooth thickness	..	$0.5 \times p' \times \cos. \lambda$	$0.5 \times p' \times \cos \lambda$
Top round	r_t	$0.05 \times p'$	$0.05 \times p'$
Lead	..	No. of thds. $\times p'$	No. of thds. $\times p'$
Nor. pressure angle	α	14½ deg.	20 deg.

Terms for wheel (Fig. 134):	Symbol	Single- and double-thread worms	Triple- and quadruple-thread worms
Circular pitch	p'	.....	.....
No. of teeth	N	.....	.....
Pitch diam.	D_p	$N \times 0.3183 \times p'$	$N \times 0.3183 \times p'$
Throat diam.	D_t	$D_p + (0.636 \times p')$	$D_p + (0.572 \times p')$
Outside diam.	D_o	$D_t + (0.4775 \times p')$	$D_t + (0.3183 \times p')$
Hub diam.	D_h	$1.875 \times$ bore	$1.875 \times$ Bore
Keyway	..	A.G.M.A. Std.	A.G.M.A. Std.
Face width	F	$2.38 \times p' + 0.25$	$2.15 \times p' + 0.2$
Hub extensions	..	$0.25 \times$ bore	$0.25 \times$ bore
Hub length	H	$F + 0.5 \times$ bore	$F + 0.5 \times$ bore
Radius of face	R_f	$0.882 \times p' + 0.55$	$0.914 \times p' + 0.55$
Radius of rim	R_r	$2.2 \times p' + 0.55$	$2.1 \times p' + 0.55$
Edge round	R_e	$0.25 \times p'$	$0.25 \times p'$
Center distance	C	$(D_p + d_p) \times 0.5$	$(D_p + d_p) \times 0.5$
Nor. pressure angle	α	14½ deg.	20 deg.

outside diameter of the worm nor greater than $1\frac{1}{4}$ times the outside diameter of the worm, the sides of the cutter having an angle of obliquity of $14\frac{1}{2}$ deg. in the case of single- and double-

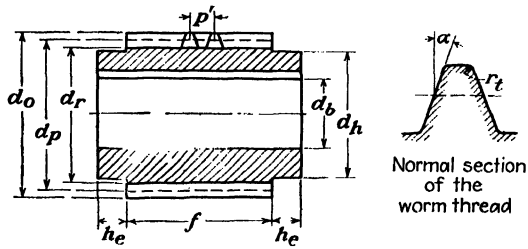


FIG. 133.

thread worms and an angle of obliquity of 20 deg. in case of triple- and quadruple-thread worms.

METHODS OF PRODUCING GEAR TEETH

Gear teeth are produced commercially by two distinct processes: molding and machining. Formerly all gear teeth were cast,[1] the molds being formed from complete patterns of the gears. At the present time, however, cast teeth are generally produced by a machine-molding process which is superior to the earlier method. Even with the improved foundry methods, however, cast teeth are not suitable for high speeds and are found only in the cheaper classes of machinery.

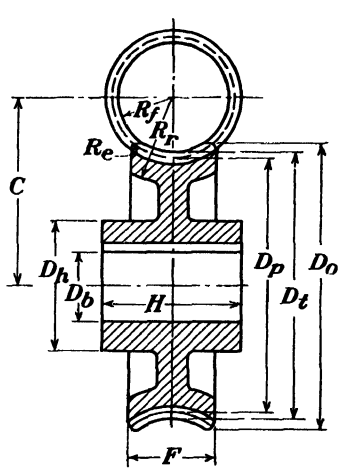


FIG. 134.

In the machining process two general methods are followed: *forming* and *generating*. In the forming method the teeth are cut either by a rotary milling cutter that has been formed to the exact shape required, or by a planing tool which forms the cutline required by following a previously shaped templet.

104. Producing Gear Teeth by the Forming Process.—The *rotary milling cutter* and its action are indicated in Fig. 135. The

[1] In ancient times wooden gears were used.

cutter is shaped so that the required tooth outlines are formed by setting the cutter to the proper depth and cutting the spaces between adjacent teeth. The standard rotary cutters are made by the Brown and Sharpe Manufacturing Company and may be obtained in sets to cover a wide range of work. This is one of the oldest and most widely used methods of producing cut teeth. It is extensively employed in the production of many of the gears that are required to meet only ordinary conditions.

The teeth of Citroën gears illustrated in Fig. 122*d* are cut with a rotating end-milling cutter formed to the normal contour of the teeth.

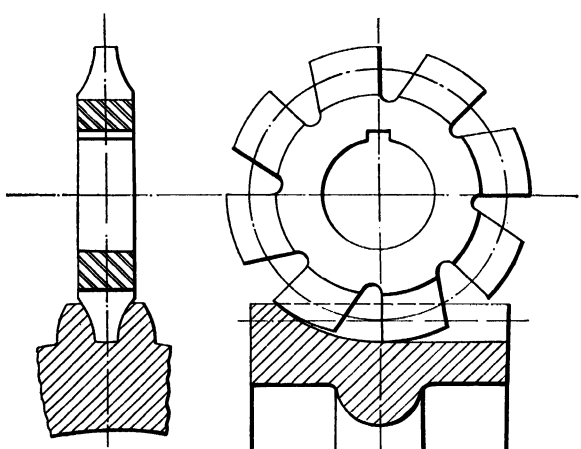


FIG. 135.

Stamping, which may be considered a process in which all the teeth of the gear are formed by a single stroke of the cutter (die), is used extensively in producing small thin gears such as are used in watches, clocks, instruments, etc.

The *templet-planing method* is shown in Fig. 136 as applied in cutting the teeth of bevel gears. The apex of the pitch cone of the bevel gear is located at *O*. *OA* is the axis of a roller in contact with a master form or templet, called a *former*, whose shape corresponds exactly to an enlarged profile of the tooth to be cut. It can be seen that as the roller swings about *O* and remains in contact with the *former*, the line *OA* will sweep up the required tooth surface. In the actual cutting process the point of the tool

reciprocates along the line OA, the cutting action taking place as the tool travels toward the cone center O. In cutting the teeth of spur gears by this method, the point of the tool is constrained to move parallel to the axis of the gear, instead of being directed toward the apex of a cone.

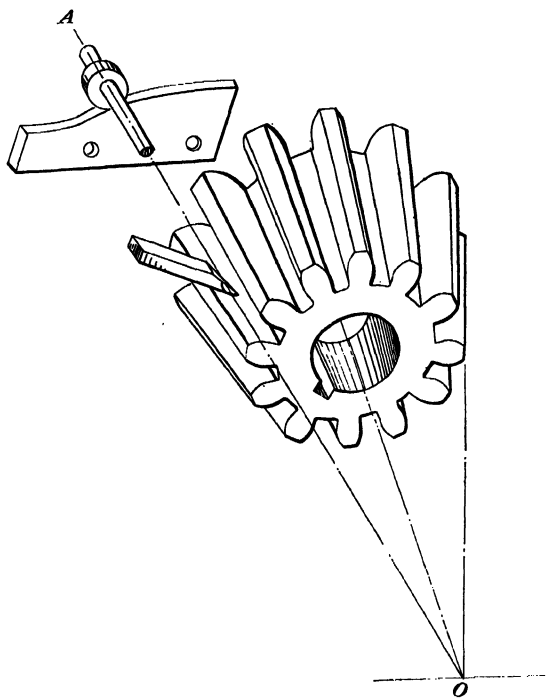


FIG. 136.

105. Methods of Generating Spur-gear Teeth.—In the *generating method* a machine generates mathematically correct tooth profiles by virtue of the motion given to the cutter and the gear blank; *i.e.*, the machine gives both the cutter and the work motion, while the cutting is in process, identical with the motion they would have if they were two similar gears in engagement.

Machines for generating gear teeth are of three distinct types: one employs a rotating worm-shaped cutter called a hob; another employs a reciprocating pinion-shaped cutter; and in the third type, a reciprocating rack-shaped cutter is used.

The *hobbing process* consists in rotating and advancing a worm-shaped cutter through a rotating blank. Figure 137 illustrates a hob in position for generating the teeth of a spur gear. The usual hob consists of a straight-sided worm-shaped cutter, its axial section representing the developed form of the generating rack. If the axis of the hob is set square with the axis of the gear

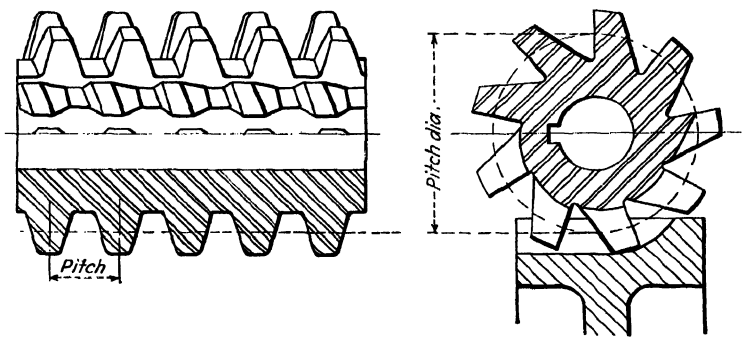


FIG. 137.

blank, its axial section represents the generating rack. As the hob is usually set at an angle to the blank, the angle of the sides of the hob teeth are altered accordingly. The hobbing process has the advantage of requiring only one hob to cut gears with any number of teeth of the same pitch. All motions are continuous

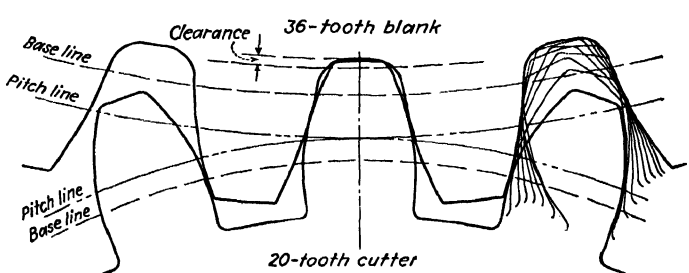


FIG. 138.

and, except for the feeding of the hob through the blank, all motions are rotary. Hobbing is one of the most rapid methods of producing gear teeth.

The Fellows gear shaper is the best-known machine employing the *pinion-shaped cutter*. The cutting process is indicated in Fig. 138. Both the cutter and the gear blank are rotated the

proper amount between the strokes of the cutter. In this machine the cutter is made exactly like a pinion such as may be in engagement with the gear about to be made. The cutter differs in one respect from the pinion in that its addendum is made long enough to provide clearance on the gear to be cut. The cutter is mounted on a vertical ram that has a reciprocating motion; and both cutter and work are made to rotate as if they were two similar gears in engagement. One cutter cuts gears with any number of teeth of the same pitch. This method is extensively used in the automotive industries in producing the well-known Fellows form of stub tooth.

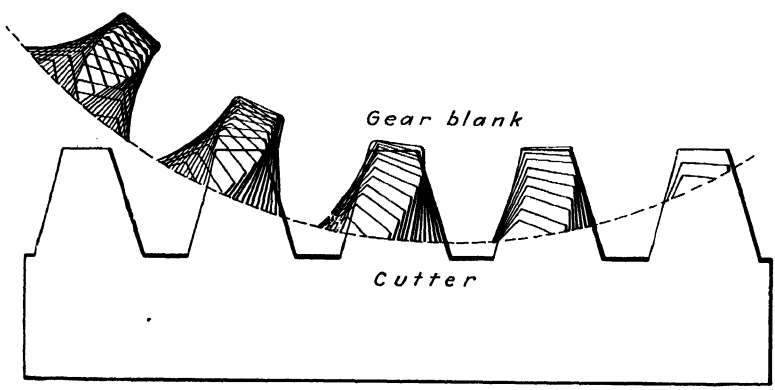


FIG. 139.

The Maag machine is typical of the generating machines using the *rack-shaped cutter*. The cutting process is shown in Fig. 139, as applied in generating the teeth of a spur gear. Between each stroke of the cutter the blank is revolved and advanced the proper distance along the face of the rack cutter. When the blank has advanced a distance equal to the circular pitch, it returns to its starting point without any rotary motion. This method of generating gears has the same advantage as hobbing and cutting with pinion-shaped cutters in that one cutter cuts gears with any number of teeth of the same pitch.

Any of the methods described above may be adapted to the cutting of either spur or helical gearing. No one method has all the advantages. Specific requirements determine which is best to use.

106. Methods of Generating Bevel-gear Teeth.—The generating of bevel-gear teeth is greatly complicated by the fact that the tooth elements converge toward the apex of the pitch cone. The reciprocating-rack principle is employed, but instead of the cutting tool consisting of several rack teeth, it is in the form of one side of a single rack tooth.

a. Straight-tooth Bevel Gears.—The cutting process as employed in the Gleason bevel-gear generator of the straight-tooth type is illustrated in Fig. 140. In the generating motion, by which the profile of the tooth is obtained, the cutting edges of the tools represent the sides of adjacent teeth of a crown gear, which is to the bevel-gear system what a rack is to the spur-gear system. As the tools rotate about the axis of this imaginary crown gear they cut in the direction of the tooth elements. The gear to be cut is set so that the apex of its pitch cone coincides with the apex of the crown gear, and so that the pitch surfaces of both are tangent to each other. Means are then provided to rotate tools and gear about their axes while maintaining the same speed of rotation that they would have if they were finished gears properly in mesh. Since the tools are taking the place of the teeth of the crown gear, they cut away the interfering metal on the gear tooth as they reciprocate, cutting alternately, one on each side of the tooth, and thus leave the correct tooth profile.

Figure 140 shows the tools and gear in three positions. *A* is the beginning of the roll, *B* the middle, and *C* the top. In gears of coarse pitch the gears are previously gashed, or roughed out. In gears of fine pitch, however, the teeth may be generated from the solid blank.

b. Spiral Bevel Gears.—The principle of action of the Gleason spiral bevel-gear generator is shown in Fig. 141. The three generating stages shown correspond to those of Fig. 140. A circular cutter of the face-mill type is used with blades whose cutting edges are straight sided to correspond to the tooth shape of a crown gear, which is to the bevel-gear system what a rack is to the spur-gear system. The teeth are generated by rolling the revolving cutter with the gear blank in the proper relation to conform to the meshing action of the crown-gear tooth, to which the cutter corresponds.

107. The Grinding of Gear Teeth.—With the increasing use of hardened gears, machines of various types have been developed

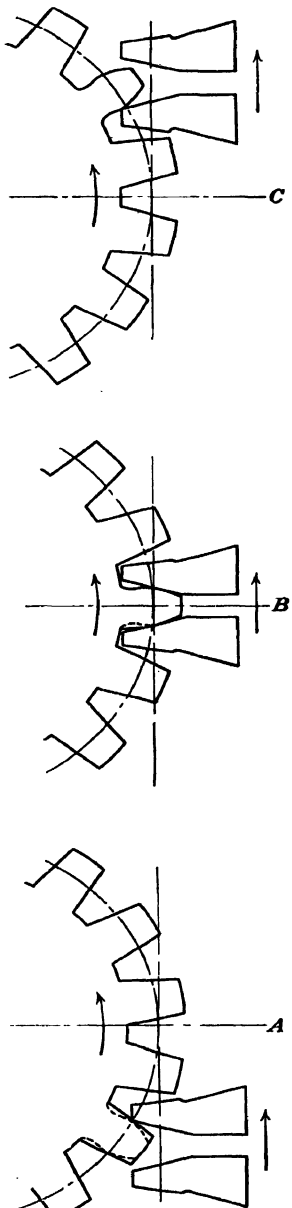


FIG. 140.

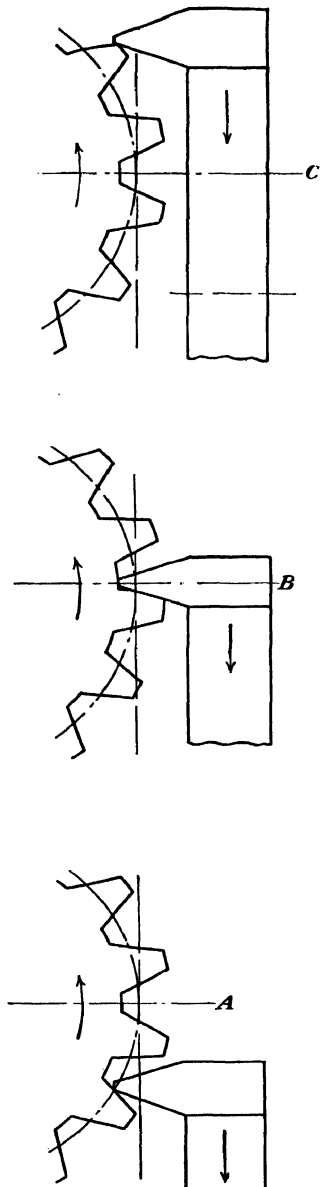


FIG. 141.

for the grinding of gear teeth. Some machines are of the form-grinding type, and others are of the generating-rack type. The machines that operate on the rack principle have the generating features common to the generating machines that employ cutting tools. The chief object in grinding gear teeth is to make correc-

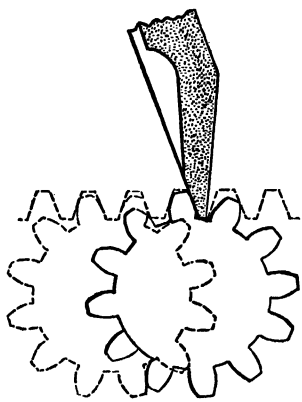


FIG. 142.

tions for the irregularities resulting from the distortion due to heat treatment. Figure 142 is a diagram showing how the flat side of a grinding wheel generates involute tooth curves as the gear rolls along an imaginary rack.

Burnishing and lapping processes are also extensively employed in producing smooth and polished tooth surfaces to meet the exacting requirements of modern industrial gears.

CHAPTER VII

BELTS, ROPES, AND CHAINS

When the distance between shafts becomes so great as to render the use of gear trains impracticable, rotary motion is transmitted by means of belts, ropes, or chains. In these cases the belt, rope, or chain is passed in a continuous band over pulleys, sheaves, or sprockets, and the motion is transmitted as though the wheels rolled directly upon each other. Owing to conditions that will be explained later in this chapter, the first two of the above methods of driving cannot be used where exact velocity ratio is required. These methods, nevertheless, have a wide range of employment in the transmission of motion and power, because of the numerous applications where exact velocity ratio is not necessary.

FLAT BELTS

When a belt is in the form of a thin flat band designed to run on a cylindrical surface it is known as a *flat belt*. The flat-belt drive has been one of the most widely used methods of transmitting power. Although very largely superseded by more modern methods, it still has applications. The use of belts is subject to certain limitations. For example, there is a limit to the distance over which power can be transmitted with economy by this method. No specific figures can be given for this limit, as each installation is a special case and must be treated as such. The distance between centers for a great number of flat-belt drives, however, varies from 15 to 30 ft. Another limitation upon the use of belts results from the fact that the driving force transmitted depends upon the friction between the belt and the faces of the pulleys it connects. There is always the possibility of some slipping between the belt and the faces of the pulleys, and hence the character of the motion transmitted is not positive. Where positive action is required, gears or chains must be used.

108. Velocity Ratio of Belt Drives.—When the rotation of the driven pulley of a belt drive must have the same sense as that of

the driving pulley, the scheme shown in Fig. 143 is employed. When on the other hand, the sense of rotation of the driven pulley must be opposite that of the driving pulley, the belt is arranged in the manner shown in Fig. 144. These two arrangements are called *open* and *crossed* belt drives, respectively.

The velocity ratio of a belt drive is defined as the ratio of the number of turns of the driving pulley to the number of turns of the driven pulley in a unit of time. This may be determined with sufficient accuracy in the following manner: Let it be assumed that the center of the belt section has a constant linear velocity,

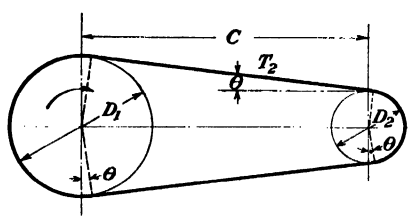


FIG. 143.

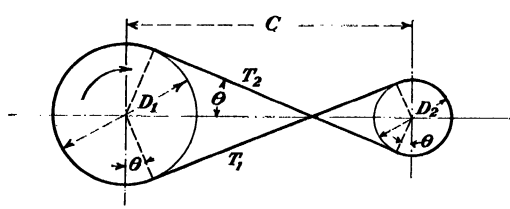


FIG. 144.

whether it is between the pulleys or passing around one of them. Let this velocity in feet per second be V; the thickness of the belt t; the diameters of the driving and driven pulleys D_1 and D_2, respectively; and the r.p.m. of the driving and driven pulleys N_1 and N_2, respectively. If there is no slip, the linear velocity of the belt, while passing around one of the pulleys, will be equal to the angular velocity of that pulley multiplied by the sum of its radius and half the thickness of the belt.

Hence,

$$V = \frac{2\pi N_1}{60}\left(\frac{D_1}{2} + \frac{t}{2}\right) = \frac{2\pi N_2}{60}\left(\frac{D_2}{2} + \frac{t}{2}\right)$$

$$N_1(D_1 + t) = N_2(D_2 + t)$$

$$\frac{N_1}{N_2} = \frac{D_2 + t}{D_1 + t} \qquad (23)$$

Since the thickness of the belt is usually small compared to the radii of the pulleys, the approximate formula

$$\frac{N_1}{N_2} = \frac{D_2}{D_1} \tag{24}$$

is sufficiently accurate, except for rare cases where a relatively thick belt runs on a comparatively small pulley.

109. Belt Tensions.—Referring to Fig. 144, it is evident that if the belt is to stay on the pulleys, both the part marked T_1, called the tight side, and the part marked T_2, called the slack or loose side, must exert a pull on the pulleys; that is, the slack side is only relatively slack. If the tensions in pounds are repre-

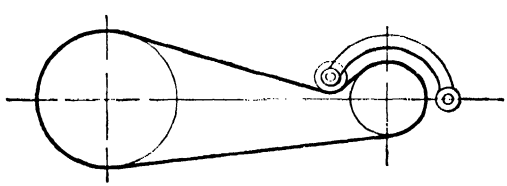


FIG. 145.

sented by T_1 and T_2 as shown, the effective pull or the force transmitted from one pulley to the other is

$$T = T_1 - T_2$$

If D_1 is the diameter of the driving pulley in feet, and N_1 is the number of revolutions it makes in a minute, the horsepower transmitted is

$$\text{hp.} = \frac{\pi D_1 N_1}{33{,}000}(T_1 - T_2) \tag{25}$$

Methods of determining the effective pull, or as it is often called, the net tension, will be found in works on machine design. The thickness of the belt is disregarded in the formula for the horsepower given above, the error resulting therefrom being negligible in view of the incompleteness of available information regarding the net tension.

Frequently an idler pulley is used to increase the arc of contact, especially on the smaller pulley, and also to produce more tension in the belt. When a swinging idler is used, as shown in Fig. 145, the belt tension can be regulated and may be released altogether, if desired.

110. Length of Belt.—The determination of the correct length of belt for a given installation is a somewhat complicated problem, even when the effect of sag is not taken into account. Formulas for this length will be deduced, first for the crossed and then for the open type of belt drive.

a. Crossed Belt.—Referring to Fig. 144, let D_1, D_2, and C be known. The total length of the belt is made up of the part around the smaller pulley, the part around the larger pulley, and the part between the pulleys.

$$\text{Part around the smaller pulley} = (\pi + 2\theta)\frac{D}{2}$$

$$\text{Part around the larger pulley} = (\pi + 2\theta)\frac{D_1}{2}$$

$$\text{Part between the pulleys} = 2C \cos \theta$$

The total length is then

$$L = \left(\frac{\pi}{2} + \theta\right)\left(D_1 + D_2\right) + 2C \cos \theta \tag{26}$$

$$\sin \theta = \frac{D_1 + D_2}{2C} \quad \text{and} \quad \cos \theta = \sqrt{1 - \left(\frac{D_1 + D_2}{2C}\right)^2}$$

$$L = \frac{\pi}{2}(D_1 + D_2) + (D_1 + D_2) \arcsin \frac{D_1 + D_2}{2C} + 2C\sqrt{1 - \left(\frac{D_1 + D_2}{2C}\right)^2} \tag{27}$$

Equation (27) may be simplified by making use of the series obtained by expanding the arc sine and the quantity under the radical as follows:

Let

$$\frac{D_1 + D_2}{2C} = x$$

Then, $\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$

and

$$\sqrt{1 - x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \cdots$$

Substituting the above quantities in Eq. (27), an expression is obtained for the length of the belt that involves only the diameters and center distance of the pulleys, thus

$$L = 2C + \frac{\pi}{2}(D_1 + D_2) + \frac{(D_1 + D_2)^2}{4C} + \frac{(D_1 + D_2)^4}{192C^3} + \frac{(D_1 + D_2)^6}{2{,}560C^5} + \cdots \quad (28)$$

The number of terms to be taken in calculating the length of a crossed belt by means of Eq. (28) is dependent upon the degree of accuracy desired. Not more than three terms are necessary, all powers above the square being negligible. The final expression for the length of a crossed belt then becomes

$$L = 2C + \frac{\pi}{2}(D_1 + D_2) + \frac{(D_1 + D_2)^2}{4C} + \cdots \quad (29)$$

For rough calculations, as in making a cost estimate, the first two terms will give sufficiently accurate results. The values of the quantities D_1 and D_2 in the above equation are usually taken as the diameters of the pulleys under consideration. To be strictly accurate the term D_1 (or D_2) should be taken as the diameter of the pulley plus twice the distance from the face of the pulley to the neutral plane of the belt. The latter distance is approximately equal to one-half the thickness of the belt, thus making the value of D_1 equal to the diameter of the pulley plus the thickness of the belt. Such accuracy is seldom required.

b. Open Belt.—The case for open belts is similar. Referring to Fig. 143 it may be seen that the component parts of the total length are as follows:

$$\text{The part around the smaller pulley} = (\pi - 2\theta)\frac{D_2}{2}$$

$$\text{The part around the larger pulley} = (\pi + 2\theta)\frac{D_1}{2}$$

$$\text{The part between the pulleys} = 2C \cos \theta$$

By a process of reasoning similar to that employed for crossed belts, it may be shown that the length of an open belt is

$$L = 2C + \frac{\pi}{2}(D_1 + D_2) + \frac{(D_1 - D_2)^2}{4C} + \cdots \quad (30)$$

From a comparison of Eqs. (29) and (30), it is evident that for relatively large center distances the same belt may be used for either crossed or open drives.

111. Cone Pulleys.—It is sometimes desirable so to construct a belt drive that its velocity ratio may be changed. This may be accomplished by means of a single belt and a pair of stepped cone pulleys as shown in Fig. 146. This change may be accomplished in a similar manner by the use of a pair of *speed cones* which are merely cones tapered down smoothly instead of by steps. It is customary to construct cone-pulley drives in such a manner that the speeds of the driven cone for a constant speed of the driving cone increases in geometric ratio for each step as the belt is shifted from one side of the pair to the other. Referring to the figure, let ω_1 be the constant angular velocity of the driving cone, and let ω_2 be the lowest angular velocity of the driven cone. Then, if the speeds are to increase in geometric ratio, the next pair of pulleys on the cones should give the driven cone an angular velocity of $k\omega_2$; the succeeding pair should give it a velocity of $k^2\omega_2$; the next $k^3\omega_2$; etc. The corresponding velocity ratios are

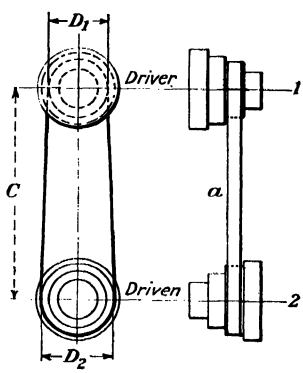


FIG. 146.

$$\frac{\omega_1}{\omega_2}, \frac{\omega_1}{k\omega_2}, \frac{\omega_1}{k^2\omega_2}, \text{ etc.}$$

After the velocity ratios for the various steps of a cone-pulley drive have been determined, the diameters of the pulleys for one step may be chosen at will, subject, of course, to the requirements of space and satisfactory operation. The choice of the diameters of the other pairs of pulleys is complicated by the fact that the same belt is used for all the pairs. The problem will first be taken up for the crossed belts and then for the open belts.

a. Cone Pulleys with Crossed Belt.—Referring to Fig. 146, let N_1/N_{2a} be the velocity ratio for the position of the belt shown. The length of the belt for the diameters D_{1a} and D_{2a} is

$$L = 2C + \frac{\pi}{2}(D_{1a} + D_{2a}) + \frac{(D_{1a} + D_{2a})^2}{4C} + \cdots \quad (31)$$

For any other pair of diameters x, the length of the belt is

$$L = 2C + \frac{\pi}{2}(D_{1x} + D_{2x}) + \frac{(D_{1x} + D_{2x})^2}{4C} + \cdots \quad (32)$$

Since this length is a constant, the two expressions for the length may be equated as follows:

$$\frac{\pi}{2}(D_{1a} + D_{2a}) + \frac{(D_{1a} + D_{2a})^2}{4C} + \cdots = \frac{\pi}{2}(D_{1x} + D_{2x}) + \frac{(D_{1x} + D_{2x})^2}{4C} + \cdots \quad (33)$$

This equation can be true only if

$$D_{1a} + D_{2a} = D_{1x} + D_{2x} \quad (34)$$

For the velocity ratio N_1/N_{2x}, the diameters D_{1x} and D_{2x} must also satisfy the relation

$$\frac{D_{2x}}{D_{1x}} = \frac{N_1}{N_{2x}} \quad (35)$$

These two relations may be solved simultaneously for D_{1x} and D_{2x}, and the diameters of any pair of pulleys of a crossed-belt cone-pulley drive thus determined.

b. Cone Pulleys with Open Belt. The case for open belts is as follows. Referring again to Fig. 146, the length of belt required for pulleys a is given by

$$L = 2C + \frac{\pi}{2}(D_{1a} + D_{2a}) + \frac{(D_{1a} - D_{2a})^2}{4C} + \cdots \quad (36)$$

This value may be calculated and substituted in the equation

$$L = 2C + \frac{\pi}{2}(D_{1x} + D_{2x}) + \frac{(D_{1x} - D_{2x})^2}{4C} + \cdots \quad (37)$$

From the equation expressing the velocity ratio,

$$D_{2x} = \frac{N_1}{N_{2x}} D_{1x} \quad (38)$$

Substituting,

$$L = 2C + \frac{\pi}{2}\left(1 + \frac{N_1}{N_{2x}}\right)D_{1x} + \frac{\left(1 - \frac{N_1}{N_{2x}}\right)^2}{4C} D_{1x}^2 + \cdots \quad (39)$$

This equation may easily be solved for D_{1x}, and from the preceding equation D_{2x} may then be found.

Convenient graphical methods of obtaining the results expressed by the above analytical expressions may be found in several well-known treatises on mechanism.

Of these, that known as the Burmester method, illustrated in Fig. 147 is the most convenient. The line $AM = C$ = the distance between the centers of the shafts is drawn at an angle of 45 deg. with a horizontal line AB. Next $MN = C/2$ is drawn perpendicular to AM at M. With A as a center, a circular arc is drawn passing through N. On this arc the point E is located so that the vertical distance between E and a point F on AM is $EF = D_{1a}/2 - D_{2a}/2$, the difference between the radii of the given pair of steps. The line EF is extended to G, making $FG = D_{2a}/2$. Through G the horizontal line OG is drawn intersecting AM at O. Then $EG = D_{1a}/2$, and $OG = D_{2a}/2$. Now draw OE and let the angle $EOG = \theta$. Then $\tan \theta = EG/OG$, and $D_{2a}/D_{1a} = N_1/N_{2a}$, the given velocity ratio.

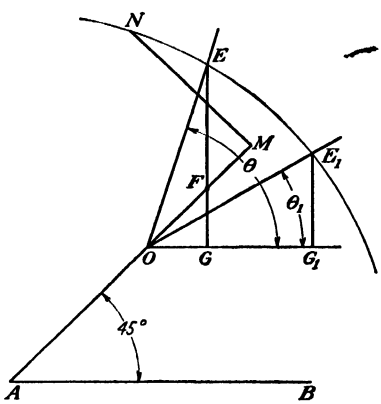


FIG. 147.

The radii $D_{1x}/2$ and $D_{2x}/2$ for any other velocity ratio N_1/N_{2x} are found as follows: Through O draw OE_1, intersecting the circular arc at E_1 and making an angle θ_1 with OG such that $\tan \theta_1 = N_{2x}/N_1$. Finally draw the vertical line E_1G_1 intersecting OG produced at G_1. Then $E_1G_1 = D_{1x}/2$ and $OG_1 = D_{2x}/2$.

To secure satisfactory results, the above construction should be accurately drawn to as large a scale as possible.

112. Belt Connections between Nonparallel Shafts.—Belts may be used for drives between shafts that are not parallel, provided the pulleys are so located as to conform to a fundamental

principle that governs the operation of all belt drives, namely: *The center line of that part of the belt approaching a pulley must lie in the central plane of that pulley.* The angle at which the belt leaves the pulley is immaterial. This will be made clear by reference to Fig. 148. In this case the shafts 1 and 2 are turning in the directions indicated by the arrows. The center line of the belt leaves the driving pulley *A* at the point *m*. The driven pulley *B* is in such a position on shaft 2 that a plane through the middle of its face and perpendicular to its axis contains the center line *mn*. The belt will, therefore, run properly on to pulley *B*. It is evident that if the direction of motion were reversed, the belt would immediately leave the pulleys. Changes in direction of rotation thus necessitate corresponding changes in relative positions of the pulleys. In Fig. 148 the axes of the pulleys are at 90 deg. with each other. The belt would run equally well if the pulleys were turned at any angle about *xx* as an axis. It can be seen that the extreme practical limits of motion about this axis would in one direction result in an open belt drive between parallel shafts, and in the other direction a crossed belt drive between parallel shafts. Belts arranged as in Fig. 148 are called *quarter-turn* belts.

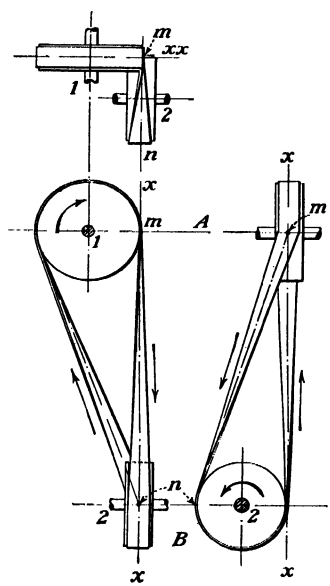


FIG. 148.

If the drive between two nonparallel shafts is to be such that the pulleys may run in either direction and still deliver the belt properly, in accordance with the fundamental principle already explained, it is necessary to make use of intermediate pulleys to guide the belt into the proper plane. Such pulleys are called *guide* pulleys. It is impossible to arrange a reversible belt drive between nonparallel shafts using only two pulleys. An example of a reversible belt drive is shown in Fig. 149.

113. Crowning of Pulleys.—Pulleys are crowned to prevent the belt from running off their sides. By crowning is meant making the pulley slightly conical or spherical as shown in

Fig. 150*a* and Fig. 150*b*. Just how the crowned pulley tends to center the belt may be seen from the following considerations: Referring to Fig. 150*c*, if a belt is led on to a double conical pulley as shown, it will be immediately subjected to two influences, the first of which is to lie flat, thus causing the belt to be bent laterally

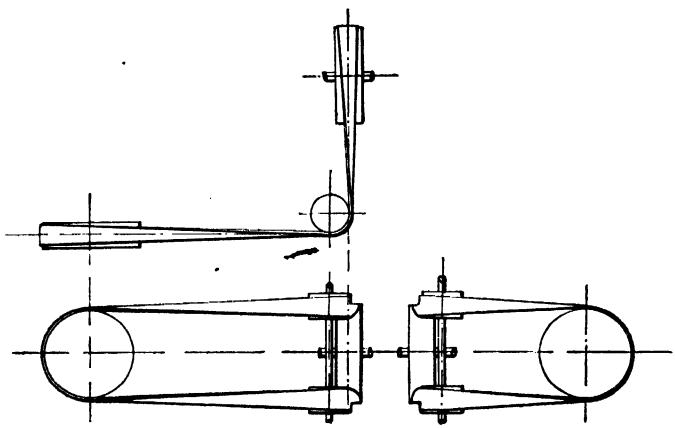


FIG. 149.

as shown. The second is to resist this lateral bending and, as a result, that part of the belt that is approaching the pulley will be thrown over to the right. Once in contact there is not much tendency for the belt to slip sidewise, and, since the first point of

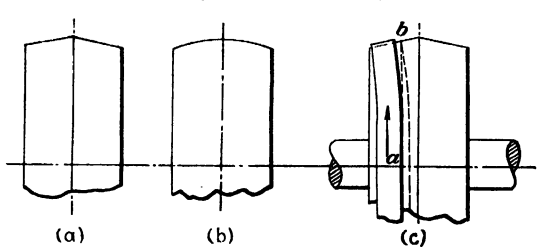


FIG. 150.

contact tends to be thrown towards the center of the pulley, the belt as a whole will run centrally.

V-BELTS

When a belt is trapezoidal in section, designed to run in a V-shaped groove, it is known as a *V-belt*. The usual form of V-belt and pulley is shown in section at *a* in Fig. 151, where *D*

indicates the nominal pitch diameter of the pulley. This form of belt has long been in use for special purposes, the ordinary automobile fan belt being perhaps the most familiar form of this type of drive. The modern V-belts, made of fabric and vulcanized rubber with a cotton-cord tension element, run on sheaves having a varying number of grooves to suit the horsepower ratings. They are in certain cases used with a grooved driving pulley and a flat driven pulley. Also, these belts may be adapted to quarter-turn drives and, with modifications in design, as shown at *b* in the figure, to reversed bending about sheaves. Where the distance between centers of shafts is small the V-belt is rapidly replacing the older flat belt drive, since it transmits a larger amount of power from a pulley of a given width of face. Also, being almost positive and slipless in action, the V-belt drive is displacing chain and gear drives for many short-center-distance machine drives.

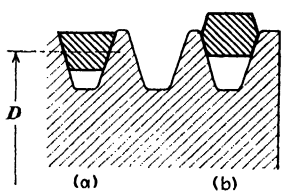


FIG. 101.

114. V-Belt Drives.—The discussion in the preceding articles relating to belts applies specifically to flat-belt drives. It should be noted, however, that the part pertaining to velocity ratio and length of belt may be applied to V-belt drives by using the proper diameters.

ROPES

Rope drives are used where power must be transmitted over greater distances than those for which it is economical to use belts. Ropes are also used in cases where the conditions of service are such that the life of a belt would be very short. One of the objections to the use of a rope drive from the standpoint of economy is that the sheave must be grooved to prevent the rope from running off.

115. Rope Systems.—Ropes used in engineering operations are made of a fibrous material such as manila, hemp, and cotton, or of iron and steel. As to the kind of service, rope may be classified as follows: (1) for hoisting and transporting of loads, and (2) For the transmission of power.

Fibrous rope is used for both hoisting tackle and transmission of power, whereas the present-day application of wire rope is

chiefly to hoisting, haulage, and transporting service, and but little to the actual transmission of power.

There are two distinct systems of driving with fibrous rope, each of which has its advantages. One is the *multiple-rope* or *English system.* This is the simpler of the two and consists of independent ropes running side by side in multiple-grooved

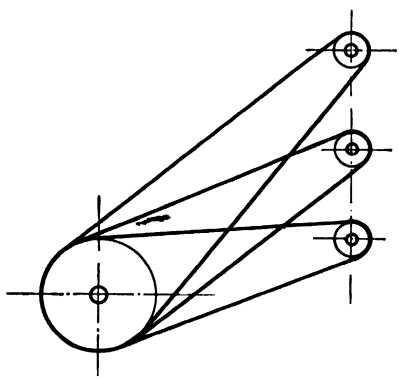


FIG. 152.

sheaves. An illustration of this system is shown in Fig. 152. The other system, illustrated in Fig. 153, is called the *continuous* or *American system.* One rope is wound around the driving and driven sheaves several times, and conducted back from the last groove of each sheave to the first groove of each sheave by means

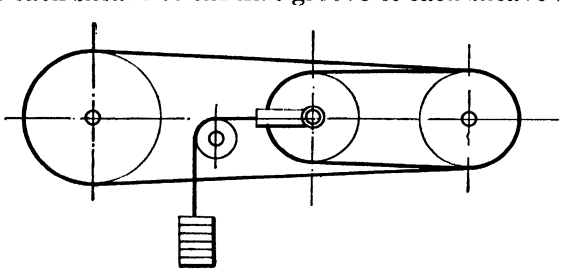


FIG. 153.

of one or more intermediate sheaves which also serve the purpose of maintaining a uniform tension throughout the entire rope. The slack should be taken up on the loose side just off the driving sheave.

The transmission of power by means of manila rope gives satisfactory results for distances between shafts as great as 175 ft.

without the use of carrying pulleys, and with carriers the distance may be increased almost indefinitely. Manila rope is also well adapted to short distances.

Wire rope as a medium for the transmission of power is used where the distances are too great for the use of manila ropes. The development of electrical transmission has gradually crowded out the wire rope, though for distances of from 300 to 1,500 ft. it is considered a cheap and simple method of transmitting power. Two systems are used, namely, the *continuous* or *endless rope*, used in operating cableways, haulage systems, and tramways, and the *single loop*, the latter being simply a modification of belt driving.

116. Rope Sheaves.—The forms of groove required on rope sheaves in the different kinds of service for which the sheaves

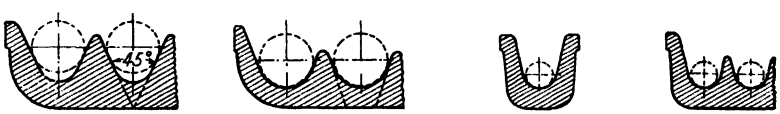


FIG. 154.

are used are shown in Fig. 154. For fibrous rope the form of groove at *a* is used on power-transmission sheaves, and that at *b* for idler sheaves. For wire rope the type of groove shown at *c* is employed on power-transmission sheaves, and that at *d* for idler sheaves. The transmission sheaves for wire rope are made with a space (not shown) in the bottom of the groove that is filled with leather, rubber, or hardwood blocks. This provision is for the purpose of increasing the friction and of preventing injury to the rope, which cannot be wedged in a groove as in the case of fibrous rope.

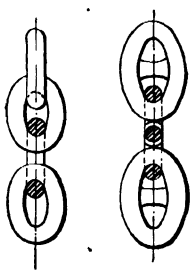
CHAINS

Chain drives are used where positive action is required and where it is desired to obtain a more compact arrangement than would be possible by the use of a flat belt. In general, the power is transmitted over a shorter distance in a chain drive than is the case for a flat-belt drive. The wheels over which chains run are called sprockets and have their surfaces shaped to conform to the type of chain used. Chains and sprockets are made in a large variety of forms, according to the service to be performed.

The various types of chains found in engineering practice may be classified as follows:

Hoisting chains	Coil
	Stud-link
Conveying and elevating chains	Detachable, or hook joint
	Closed joint
Power-transmission chains	Block
	Roller
	Inverted-tooth

117. Hoisting Chains.—The coil chain (Fig. 155*a*) is used on hoists, cranes, dredges, etc., and the stud-link chain (Fig. 155*b*) is used mainly in marine work in connection with anchors and moorings. The chief advantage of the stud-link chain is that it will not kink or entangle so readily as a coil chain.



(*a*) (*b*)

FIG. 155.

118. Conveying and Elevating Chains.—Conveying chains are used for conveying and elevating all kinds of materials under a wide range of conditions. The detachable or hook-joint chain (Fig. 156) is used extensively for conveying and elevating, but is not well adapted for conveying gritty bulk material. It is also used to some extent for power transmission where the speeds are low. The closed-joint chain, one type of which is shown in Fig. 157, is well adapted to the elevating and conveying of gritty bulk material, as well as the transmission of power at moderate speeds.

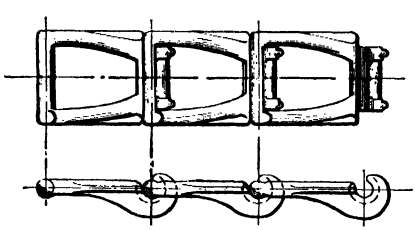


FIG. 156.

119. Power-transmission Chains.—The types of chains discussed in the preceding paragraphs are not well adapted to any service requiring speeds above 600 f.p.m., and consequently are not suitable for the transmission of power where the speed exceeds this limit. For this class of service the special forms of power-

transmission chains, all of which are machined fairly accurately, have been devised.

Figure 158 illustrates a block chain. As the name indicates, the block chain consists of solid steel blocks, shaped like the letter B or the figure 8, to which the side links are fastened by hard-

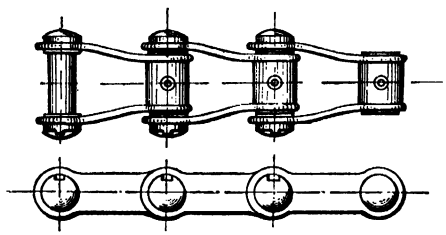


FIG. 157.

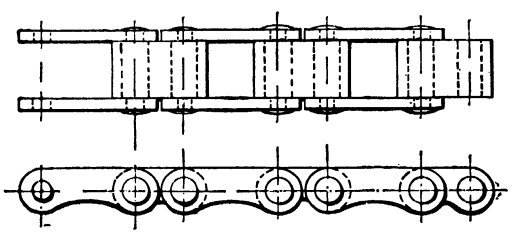


FIG. 158.

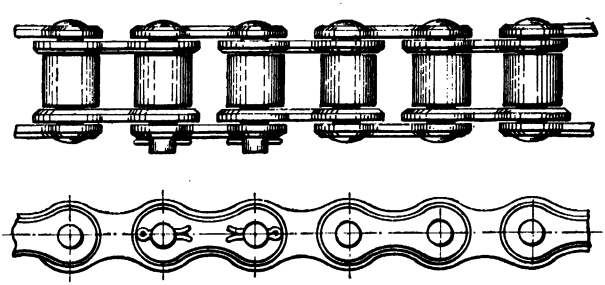


FIG. 159.

ened steel rivets. Chains of the block type are less expensive to make than roller or inverted-tooth chains, and have proved satisfactory for light power transmission where the speeds do not exceed 800 to 900 f.p.m.

A typical roller chain is shown in Fig. 159. This type of chain is used to some extent in motor-vehicle service, especially on trucks, as well as for general power transmission. A chain

speed of 1,500 f.p.m. with proper lubrication is about the limit for general use.

The most widely used type of chain for transmitting power at relatively high linear speeds is the inverted-tooth chain. There are now in use several well-known makes of inverted-tooth chain that run satisfactorily at high speeds. One example (Fig. 160) will serve to illustrate this type of chain, since the several designs have, in general, the same form of link, differing only in the form of joint used. If properly designed and constructed, an installation of an inverted-tooth chain will be just as efficient, it is claimed, as a gear drive for the same conditions of operation.

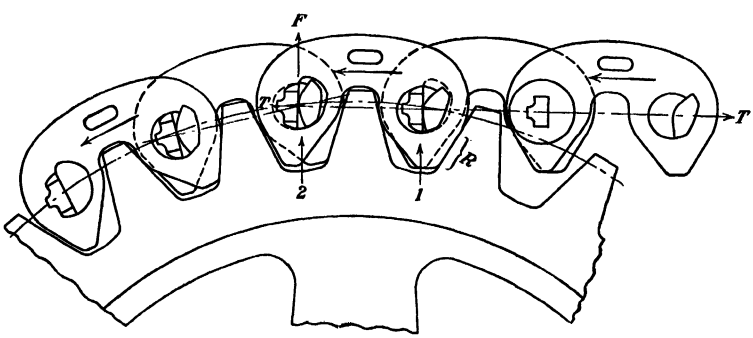


FIG. 160.

Where a positive drive is essential, and where the shafts are too far apart for gearing, inverted-tooth chains are used extensively. Inverted-tooth chains are well adapted for transmitting power economically at speeds of 1,200 to 1,500 f.p.m., and the smaller sizes of chain may be run as high as 2,000 f.p.m. The higher limit of speed, however, is attained at a cost of reduced life of the chain.

VARIABLE-SPEED TRANSMISSIONS

It has been observed (Art. 111) that with the flat belt, change of speed is accomplished by moving the belt along the axes of the cone pulleys or speed cones to give a different pair of working diameters on the driving and driven pulleys. These may be regarded as the elementary mechanisms through which variable-speed transmission is accomplished with flexible connectors.

One of the best-known modern devices for accomplishing variable-speed transmission through the use of a belt is the *Reeves*

variable-speed transmission. It consists essentially of a pair of pulleys connected by a V-shaped belt in the manner indicated in Fig. 161. Each of the pulleys consists of a pair of beveled disks (cones) keyed to the shaft so that disks and shaft must turn together but the disks are free to be moved along the shaft. By means of suitable adjusting devices the two conical disks on one shaft are caused to approach each other at the same time that the two disks on the other shaft separate. Bringing the two disks nearer together causes the V-shaped belt to be in contact with them farther out from the axis, and separating them causes the belt to be in contact nearer the axis. Thus, the effective

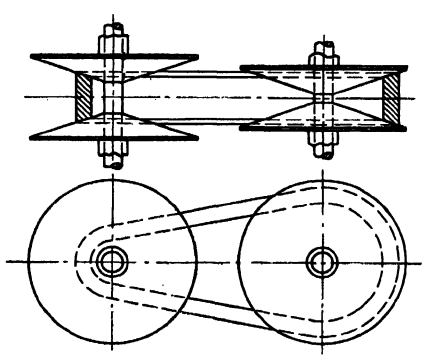


FIG. 161.

radius of one pulley is increased at the same time that the effective radius of the other pulley is decreased. In this manner, the ratio of driving diameter to driven diameter is readily and quickly changed, thus securing any desired speed, between the maximum and minimum, without the necessity of stopping the machine. The device is in the form of a compact unit that can readily be interposed between the source of power and the driven machine.

Variable-speed transmission in the case of the chain drive may be secured by a device similar to that described above for the V-shaped belt. One such device, known as the *P.I.V.* (*positive, infinitely variable*) *gear* consists essentially of a V-shaped link chain with teeth on the sides that engage with radial teeth in a pair of axially adjustable conical disks on the driving and driven shafts. Each pair of disks thus becomes an adjustable-diameter

wheel in the transmission of power, so that an infinite number of speeds between the minimum and maximum are available.

HOISTING TACKLE

Under the title of *hoisting tackle* may be grouped all those combinations of ropes and chains with pulleys and pulley blocks, the purpose of which is to overcome a considerable resistance that acts through a relatively small distance by means of a relatively small force that acts through a considerable distance.

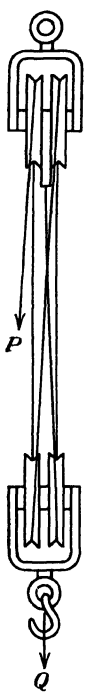


FIG. 162.

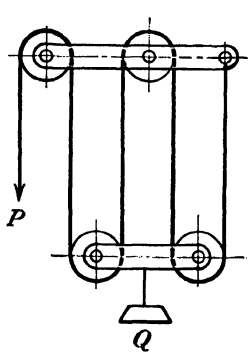


FIG. 163.

120. The Block and Tackle.—Probably the simplest example of hoisting tackle is the ordinary block and tackle illustrated in Fig. 162. For purposes of analysis it is more convenient to use the diagrammatic representation of the block and tackle, as shown in Fig. 163. For the case shown, a force P acting through a distance p will, if friction is disregarded, lift the load Q, which is equal to $4P$, through the distance $\frac{1}{4}p$. The mechanical advantage of a block and tackle, as in the case of any other machine, is the ratio of the resistance overcome to the force that would have to be applied if friction were disregarded. There are a number of methods for determining the mechanical advantage of a block and tackle, but probably the most convenient is simply to count the ropes supporting the load. This number is equal to the mechanical advantage.

In some cases block-and-tackle systems are used to lift a small load through a considerable distance by means of a large force acting through a small distance. An example of such an application is the hydraulic elevator system shown in Fig. 164. In this system the sheaves, instead of being placed side by side as in the ordinary pulley block, are placed one above the other in a line, interference between the ropes being avoided by making the diameters of the outer sheaves somewhat larger than those of the inner sheaves. The hydraulic cylinder is the stationary

element, the plunger being forced down by means of high-pressure water in the cylinder. The *mechanical advantage* of this system is one-sixth. Attention is called to the fact that if the plunger were stationary and the cylinder the moving element, the mechanical advantage would be one-fifth.

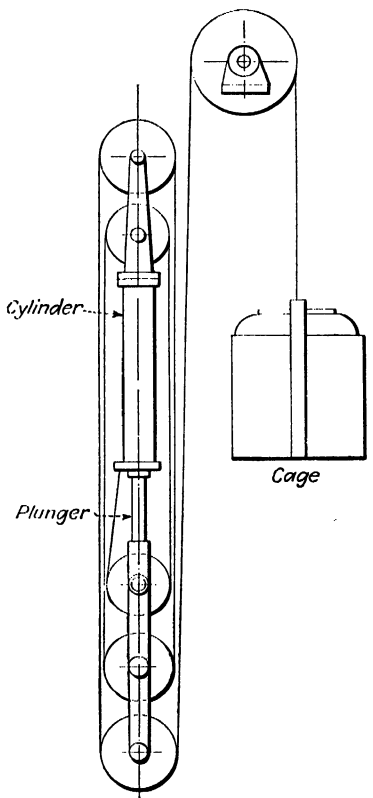


Fig. 164.

Combinations of tackle like that shown in Fig. 165 may be employed in cases where a considerable mechanical advantage is required. Referring to the figure, it is evident that P_2 is equal to $5P_1$, neglecting friction, and that Q is equal to $4P_2$; that is, the total mechanical advantage is 20.

121. The Differential Chain Block.—Although the hoisting

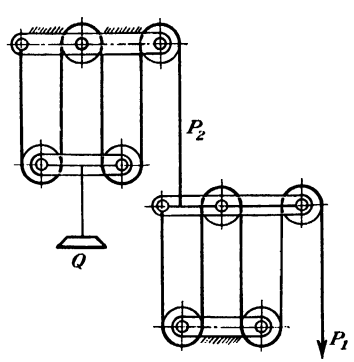


Fig. 165.

mechanism shown in Fig. 166 differs somewhat in principle from the ordinary block and tackle, it is used for the same purpose, namely, the raising of a heavy load by means of a small applied force. The upper block contains two sheaves of slightly different diameters that are fastened together so as to move as a unit. The lower block has but one sheave and is attached to the upper block by means of the endless chain, as shown. The force is applied at P, and the load Q is fastened to

the hook on the lower block. The relation between P and Q may be determined as follows: Let the diameter of the larger sheave in the upper block be D_a, and let the diameter of the smaller sheave be D_b. If these sheaves are rotated in a counterclockwise sense, the chain that is run off the smaller sheave will increase the total length of chain between the upper and lower blocks, and that which is run on to the larger sheave will decrease this length. The net change of the total length of chain between the blocks will be the difference of the lengths run on and off the upper sheaves. This change of length is equally divided between the two parts of the chain that connect the upper and lower blocks, and hence the change of center distance of the two blocks is equal to one-half the difference of the lengths of chain run on and off the sheaves in the upper block. Let the sheaves in the upper block be turned through one revolution under the action of the force P. The load Q will be raised through a distance,

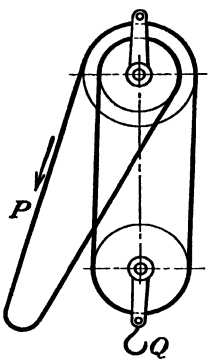


FIG. 166.

$$h = \frac{\pi}{2}(D_a - D_b) \tag{40}$$

Assuming no friction, the amount of work delivered to the hoist by P must equal the amount done in raising Q. Therefore,

$$P\pi D_a = Qh = Q\frac{\pi}{2}[D_a - D_b]$$

$$P = \frac{Q}{2}\left[\frac{D_a - D_b}{D_a}\right] \tag{41}$$

CHAPTER VIII

INTERMITTENT-MOTION MECHANISMS

In addition to the standard agencies for transmitting motion, namely, linkwork, cams, gears, belts, ropes, and chains, there are numerous auxiliary devices for controlling and regulating the motions of machine parts and adapting such motions to a great variety of purposes. By far the greater number of these supplementary mechanisms consist of combinations such as ratchets used in feed and stop mechanisms, escapements for controlling clockwork, clutches, and brakes for controlling motions between a pair of shafts; or the combinations may be of a more complex nature as in indexing and reversing mechanisms for automatic machines. Such combinations are usually classified as *intermittent-motion mechanisms*, and they will be discussed in this chapter under the following subdivisions:

1. Ratchets, escapements, intermittent gearing, and indexing devices; for producing rotary or rectilinear motion by steps.
2. Reversing mechanisms; for producing reversible running of trains of mechanisms.
3. Clutches and brakes; for stopping and starting parts having rotary motion of the uniform type.

122. Intermittent Movements.—The action of all automatic machinery is based upon intermittent movements so synchronized that one or more members will automatically perform a certain function and then remain at rest while the other members do their part by continuous motion. From various combinations of the two main movements possible in a machine, namely, rotation and reciprocation, all the complicated movements in machinery are composed. For example, in the case of rotary motion, any number of revolutions or any fractional part of a revolution may be followed by a period of rest equivalent to any number of revolutions or fractional part of a revolution. Combinations of these movements may be varied in any way, and a series of different motion periods followed by required rest periods.

123. Ratchet Mechanisms.—A wheel provided with suitably shaped teeth, receiving an intermittent circular motion from an oscillating or reciprocating member, is called a ratchet wheel. A simple form of ratchet mechanism is shown in Fig. 167. A is the ratchet wheel, and B is an oscillating lever carrying the driving pawl C. A supplementary pawl at D prevents backward motion of the wheel. When arm B moves counterclockwise, the pawl C will force the wheel through a fractional part of a revolution dependent upon the motion of B. When the arm moves back, the pawl will slide over the points of the teeth while the wheel remains at rest because of the fixed pawl, and will be ready to push the wheel on its forward motion as before. The amount of backward motion possible varies with the pitch of the teeth. This motion could be reduced by using small teeth, but this procedure would result in weak teeth, and the expedient is sometimes adopted of placing several pawls side by side on the same axis, the pawls being of different lengths. The contact surfaces of wheel and pawl should be so inclined that they will not tend to disengage under pressure. As shown in Fig. 167, this means that the common normal at N should pass between the pawl and ratchet-wheel centers. If this common normal should pass outside these limits, the pawl would be forced out of contact under load unless held by friction. In many ratchet mechanisms the pawl is held against the wheel during motion by the action of a spring.

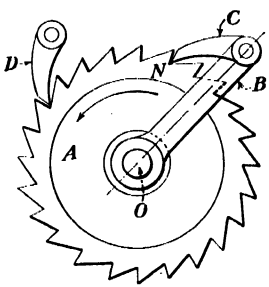


FIG. 167.

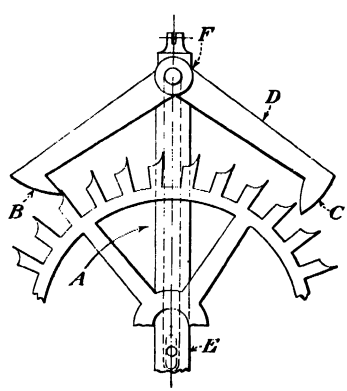


FIG. 168.

The usual form of the teeth of a ratchet wheel is that shown in Fig. 167, but in feed mechanisms such as used on many machine tools it is necessary to modify the tooth shape for a reversible pawl so that the drive may be in either direction.

Ratchet mechanisms occur in a great variety of forms and find a wide range of application. They may be seen in ratchet drills, lifting jacks, feed and stop mechanisms in machine tools, capstans, windlasses, etc.

124. Escapements.—Under the head of escapements may be grouped a number of self-acting stopping and releasing ratchet mechanisms in which the driven link is alternately released and stopped. The most familiar example is found in clockwork where the driving weight or spring is permitted to move the parts and, therefore, the hands by a definite amount at regular intervals.

One common form of escapement such as used in clocks is shown in Fig. 168. The escape wheel A turns in the direction indicated by the arrow and is supplied with long pointed teeth acting upon the pawl ends B and C, which are known as pallets. The double pawl D is oscillated by the pendulum E, which is attached to it at F. The pendulum has a normal period of vibration corresponding to its length, and if the pendulum is so heavy that the rotative effort of the escape wheel cannot alter this period, the pendulum in swinging will control the motion of the wheel. The tendency of the wheel to produce vibration of the pendulum may be made sufficient to overcome the frictional resistance which acts to stop the pendulum, and thus the amplitude of the vibrations is maintained. Many modifications of the escapement have been devised to meet special requirements. The problem of design is an intricate one requiring the skill of specialists. In watches, chronometers, etc., a balance wheel is used instead of a pendulum to regulate the period of the vibrating member, but all these escapements operate on the same general principle.

125. Intermittent Gearing.—A pair of rotating members may be so designed that for continuous rotation of the driver, the follower will alternately roll with the driver and remain stationary. Such an arrangement is known by the general term *intermittent gearing*. This type of gearing occurs in some variety in counting mechanisms, motion-picture machines, feed mechanisms, etc.

a. The Simplest Case.—The simplest form of intermittent gearing, illustrated in Fig. 169, has the same kind of teeth as for ordinary gears designed for continuous motion. This example is

merely a modification of a pair of 20-tooth gears to meet the requirement that the follower shall advance one-tenth of a turn for each turn of the driver. The interval of action is the two-pitch angle α (indicated on both gears), the single tooth on the driver engaging with each space on the follower to produce the required motion of one-tenth turn of the follower. During the remainder of a driver turn, the follower is locked against rotation in the manner shown in the figure. In order to vary the relative movements of the driver and follower, the meshing teeth may be arranged in various ways, to suit requirements. For example, the driver may have more than one tooth, and the periods of rest of the follower

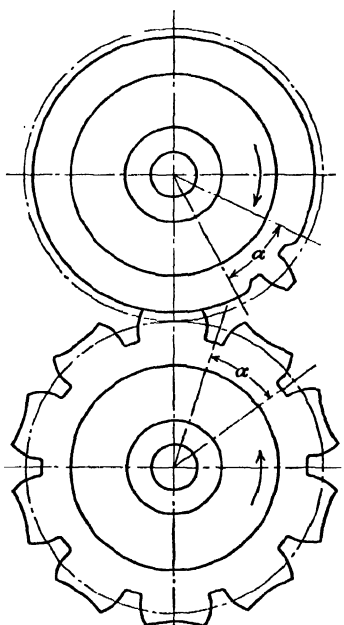


FIG. 169.

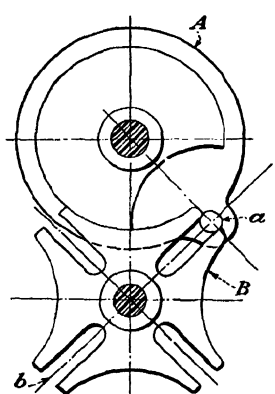


FIG. 170.

may be uniform or may vary considerably. Counting mechanisms are often equipped with gearing of this general type.

b. The Geneva Wheel.—An interesting example of intermittent gearing is the *Geneva wheel* combination shown in Fig. 170. In this particular case the driven wheel B makes one-fourth of a turn for one turn of the driver A, the pin a working in the slots b causing the motion of B. The circular portion of the driver, coming in contact with corresponding hollow circular parts of the driven wheel, retains it in position when the pin or tooth a is out of action. The wheel A is cut away near the pin a as

shown, in order to provide clearance for wheel B in its motion. If one of the slots is closed, A can make only a little more than $3\frac{1}{2}$ revolutions in either direction before pin a strikes the closed slot and thus stops the motion. The device in this modified form was early used in watches, music boxes, etc., to prevent overwinding, and from this application received the name *Geneva stop*. In the arrangement as a stop, wheel A is secured to the spring shaft, and B turns on the axis of the spring barrel. The number of slots or interval units in B depends upon the number of times it is desired to turn the spring shaft.

As applied in some types of motion-picture machines, the wheel B has four slots, so that it is given a quarter turn for every revolution of wheel A. Wheel B is secured to the sprocket that meshes with the openings in the edge of the film and jerks it down before the lens, picture by picture, as the pin in wheel A pulls B around one-quarter of a turn.

The Geneva wheel is sometimes used on machine tools for indexing or rotating some part through a fractional part of a revolution.

126. Indexing Devices.—By indexing, as a shop term, is meant the operation of exactly dividing the periphery of a circular piece into a number of equal parts, or what is the equivalent, one turn of a wheel, or 360 deg., into a number of equal angular divisions. Mechanisms in considerable variety have been devised to produce this result, most of them concerned with the production of toothed gearing. Indexing is mostly done on milling machines and gear-cutting machines. The indexing mechanisms on gear-cutting machines are usually more or less complex. Indexing in its simplest form is found on the milling machine where, for general service, the well-known type of index head, also called dividing head, which forms a part of the standard equipment of the machine, is used.

a. The Index Head.—Index heads vary considerably in design but all embody the essential features of the mechanism of the plain index head outlined in Fig. 171. Spindle A is carried in a headstock with which comes a companion tailstock. Work can then be held either between centers or in a chuck attached to the spindle. On the spindle is a 40-tooth worm wheel B driven by a single-threaded right-hand worm C. On the end of the worm shaft and located at the side of the head is the adjustable index

crank D with handle E and plunger pin F which can be let into any hole in the index plate G. This plate, which, with sector H, is shown in more detail in Fig. 172, carries the essential dividing element of the apparatus in the form of a number of circles of equally spaced holes.

Each index head is provided with several interchangeable index plates in order to cover a wide range of work. The plate shown in Fig. 172 has circles of 15, 16, 17, 18, 19, and 20 equally spaced holes, but for the sake of clearness in illustrating the use of the plate, only the circle of 17 holes is shown.

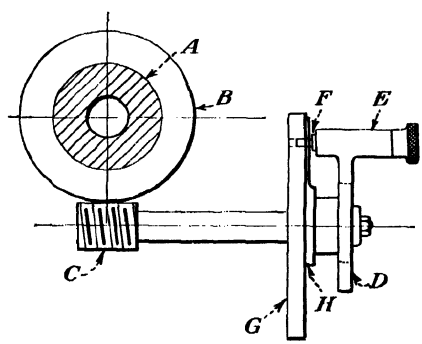


FIG. 171.

As 40 turns of the crank are required to make one turn of the work, since the work turns with the spindle, in order to find how many turns of the crank are necessary for a certain division of the work, 40 is divided by the number of divisions desired. The rule then may be stated as follows: *Divide* 40 *by the number of divisions desired, and the quotient will be the number of turns, or part of a turn, of the crank which will give each desired division.* Applying the rule, to make 40 divisions of the work the crank would be given one complete turn to obtain each division. To obtain 20 divisions the crank would be given two complete turns for each division.

b. Function of the Index Plate.—The index plate becomes necessary when the crank has to be turned only a part of the way around to obtain the necessary divisions. For example, if the work is to be given 120 divisions the crank must be turned one-third the way around, and an index plate should be selected that has a circle with a number of holes divisible by three, as 15 or 18 (Fig. 172).

c. Function of the Index Sector.—The index sector is of service in obviating the necessity of counting the holes at each partial turn of the crank. The sector has two arms that can be set and clamped together at any angle. To illustrate the use of the sector it will be assumed that it is desired to divide the work into 136 parts. Applying the rule, 40 divided by 136 gives $\frac{5}{17}$, showing that the crank must be rotated five-seventeenths of a turn to obtain each of the 136 divisions on the work. An index plate with a circle of 17 holes is selected (Fig. 172), and the sector is set to measure off 5 spaces. Starting with the crankpin in

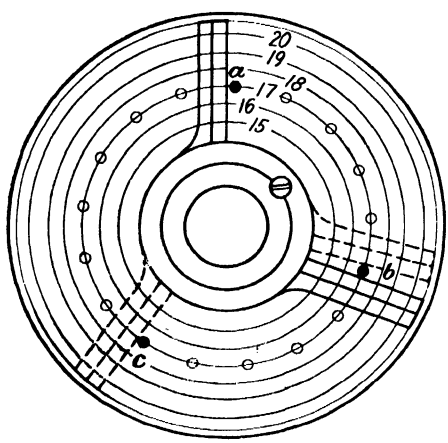


FIG. 172.

hole *a*, for example, a cut would be made in the work and the crank would then be turned to bring the pin into the hole *b* and a second cut made in the work. The sector would then be turned to the position shown by dotted lines. After the second cut had been made, the crank would again be turned to bring the pin into hole *c*, and so on until the operation had been repeated 136 times. The operator of the machine is provided with index tables so that for any given case the number of turns of the crank and number of holes in the circle may be readily ascertained.

For rapid indexing in simple numbers, as in milling four- or six-fluted reamers, provision is usually made for indexing directly through a plate on the spindle *A* itself. This plate has circles commonly of 24, 30, or 36 holes and indexes any number dividing directly into these numbers. To change from one indexing

method to the other, a cam or eccentric throw-out is provided for disengaging the worm. With the worm out of mesh with the worm wheel, the spindle is then indexed by hand.

Index heads may be used not only on milling machines, for which they are primarily designed, but also on shapers, planers, and any other machine tool where there is work that must be indexed.

d. The Geneva Wheel as an Indexing Device.—The process of indexing involves the starting of a mass from rest, moving it through a fixed distance, bringing it back to rest, and accurately locating it. Regardless of the magnitude of the mass the starting and stopping are accompanied by a shock. In the design of certain kinds of automatic machinery where provision must be made for indexing the work or the tools it is necessary to reduce this shock as much as possible, and good results have been achieved by the use of the Geneva wheel.

The mechanism consists of a driving and a driven member arranged so that the driving member, rotating uniformly about its axis, carries a roller that engages a slot in the driven member and causes the latter to rotate with a variable angular motion until the roller leaves the slot. The driven member then remains stationary until the driving member has rotated to a position where engagement is made with the next slot.

In Fig. 173, the roller shown entering the slot at a moves with uniform angular motion about the center O_2 of the driver in the direction indicated, leaves the slot at l, and continues its motion until point a is again reached where it is then ready to engage the next slot. In the meantime the first slot of the Geneva wheel has moved about its center of rotation O_3 through the angle aO_3l, when it again comes to rest.

It will be seen from the figure that, since the driving arm O_2a is of constant length, it forces the roller moving in the slot of the driven member toward the center O_3 of the driven member as the latter rotates about that center. The effective arm O_3a will therefore decrease to O_3m as its minimum length. At this point the driven member 3 is rotating at its maximum velocity. The driven member 3 has a gradual motion at the start as shown by the travel of the roller from a to b while the point a of the wheel moves from a to b'. Acceleration of the wheel 3 is positive until the roller is on the center line O_2O_3 at m, after which the

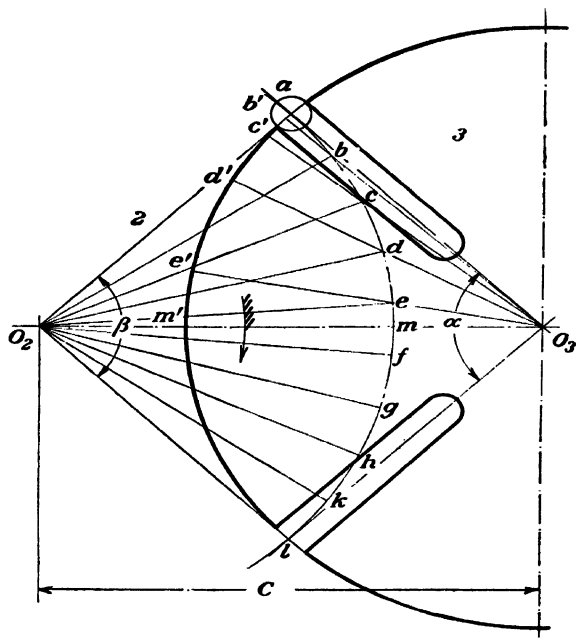
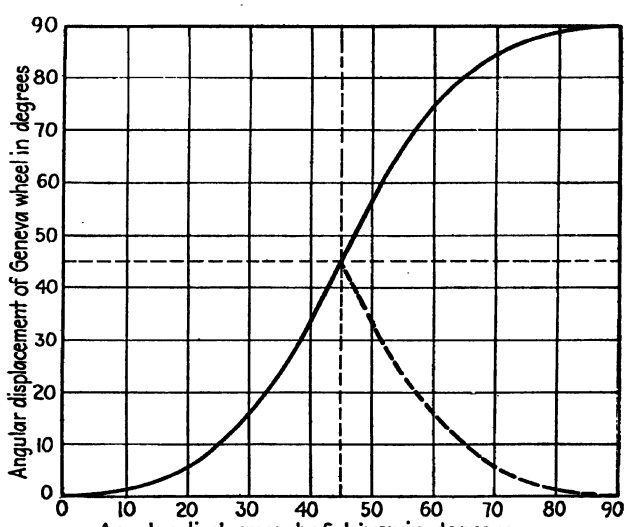


FIG. 173.



DISPLACEMENT DIAGRAM FOR GENEVA WHEEL

FIG. 174.

acceleration is negative until the roller leaves the slot at l, where the wheel comes to rest.

The relative angular motions of the driving and the driven member are very clearly shown by the angular-displacement diagram in Fig. 174. If the second half of the motion of the driven member is plotted as shown by the dotted line in the figure, then the ordinates of the curves in the lower half of the figure will represent, to a suitable scale, the angular velocities

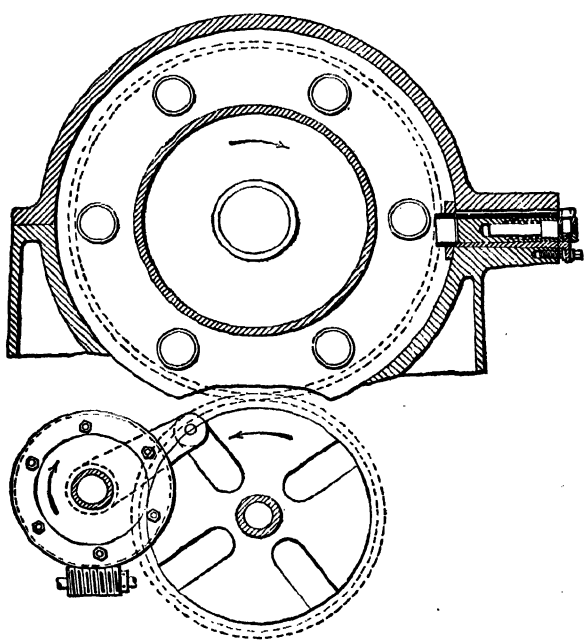


FIG. 175.

of the driven member corresponding to the constant angular velocity of the driving member at any given positions indicated by the abscissas.

With the mass of the driven member known, and means for finding its velocity in any phase of its motion, the accelerations and consequently the forces acting may be determined by the methods outlined in Part II of this book.

In laying out a Geneva wheel it is necessary to know the diameter and number of slots. The procedure then is to draw the circle of radius O_3a (Fig. 173) and lay off the angle $\alpha = 360/N$.

where N is the number of slots. Next the angle α is bisected by the line O_3O_2, and through point a the line aO_2 is drawn tangent to the circle, making an angle $\beta/2$ with the center line O_3O_2. The length of driving arm O_2a and center distance C between driver and follower are thus determined. If greater accuracy is required than can be obtained by the graphical method, these distances can easily be calculated.

In Fig. 175 is shown an indexing mechanism in which it was found necessary to introduce a geared connection between the Geneva wheel and the member to be indexed. A locking device, to prevent any rotation of the last driven member while the driving member is moving around to the position of engagement with the next slot, is shown in the figure.

127. Reversing Mechanisms.—A reversing mechanism is an arrangement of machine parts whose function is to produce reversible motion of sliding or rotating members or entire trains of mechanism. The types of reversing mechanisms in common use vary greatly, both as to principle of operation and as to form. Some are hand controlled, but by far the greater number are operated automatically. Some must be so designed that the reversal of motion occurs at a fixed point in a cycle of carefully timed movements, and with others the point of reversal may be changed by means of adjustable dogs or trips that are attached to a moving part and control the action of the reversing mechanism.

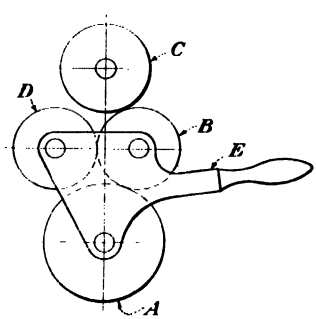


FIG. 176.

a. The Idler-gear Arrangement.—One of the simplest methods for obtaining reversal of motion in a gear train provides for the insertion of an additional gear in the train, as in the old type of lathe feed train, illustrated in Fig. 185 (Chap. IX). In Fig. 176 is shown diagrammatically the part of the train essential to this discussion. The two intermediate gears B and D are carried on a swinging arm E pivoted on the axis of gear A, which can be adjusted for engaging either of the intermediate gears with gear C. In the position shown, the drive is from C through B to A. When the arm is shifted to the other extreme position, the drive

is through both intermediate gears, or from C through D and B to A, thus reversing the direction of rotation of A.

b. The Epicyclic-gear Arrangement.—Reversal of motion may be obtained by means of an epicyclic gear train, as illustrated diagrammatically in Fig. 177. This figure shows the principle of operation of the transmission gearing used in some of the early designs of automobiles.

The gear a that is keyed to the crankshaft is the driver in each case. At a is shown the combination for obtaining the slow forward speed. The internal gear c is held stationary by the application of a brake band to its periphery. The pinions b carried by the driven member are then forced by the driving gear a to roll around inside the internal gear c, thus transmit-

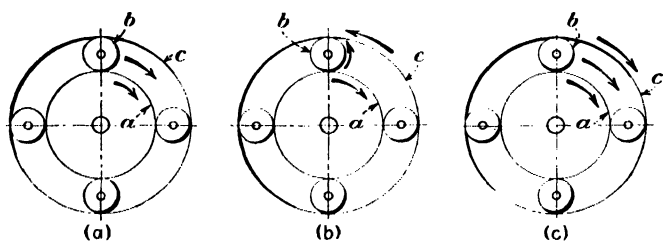


FIG. 177.

ting a slow rotating motion to the driven member attached to the pinions.

The reversal of motion is shown by the combination at b. The member carrying the pinions is prevented from rotating by the application of another brake band, so that the pinions are forced to rotate on their stud and thus produce rotation of the internal gear c. In this case the internal gear that is attached to a driving sprocket becomes the driven member.

For obtaining the direct high-speed drive the gears are all locked together and revolve as a unit with the crankshaft, as shown by the combination at c. The relative motions of the gears of each combination are clearly indicated by the arrows.

c. The Bevel-gear Device.—A bevel-gear type of reversing mechanism is illustrated diagrammatically in Fig. 178. This arrangement is found in many classes of mechanisms in which the reversing action is automatic. Gear a, which is usually the driver, is constantly in mesh with the pinions b and c. The pinions are loose on the driven shaft and have a toothed clutch d

interposed between them. The clutch is free to slide axially on a feather key that forces it to revolve with the shaft. Each pinion is provided with a clutch end so that its engagement with the clutch compels it to revolve with the shaft. Since the pinions

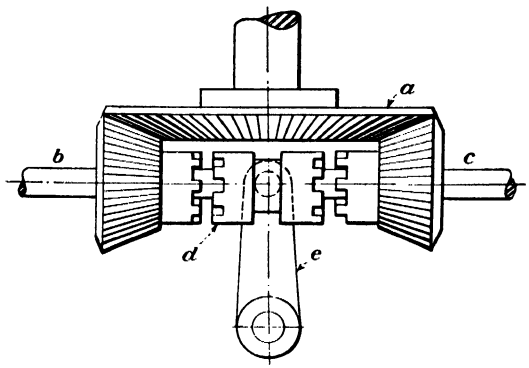


FIG. 178.

rotate in opposite directions, the rotation of the driven shaft is reversed as the clutch is shifted from one pinion to the other. No motion is transmitted to the driven shaft when the clutch is in the neutral position shown in the figure. The clutch is shifted by means of the yoke *e*, which may receive its motion in various ways.

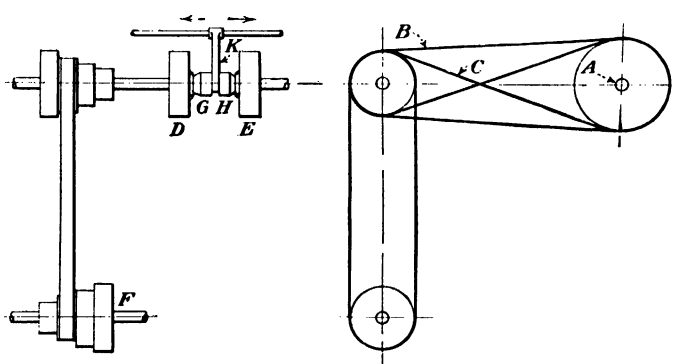


FIG. 179.

d. The Use of Open and Crossed Belts.—Reversal of rotation may be accomplished by the use of open and crossed belts. One familiar device for reversing the motion of the drive pulley of a machine is outlined in Fig. 179. This is the older form of drive

that has been much used for an ordinary engine lathe. Power is taken from a continually running line shaft A by means of open and crossed belts B and C which drive countershaft pulleys D and E in opposite directions, thus reversing the rotation of the machine drive pulley F. Provision is made for starting and stopping and for reversing the machine by means of friction clutches G and H connected with each pulley. The shifting yoke K is usually operated by a hand-controlled lever. The older countershafts for many other machine tools requiring a reversal of motion have been commonly arranged in this manner.

c. The Automatic Belt Shifter.—A belt-shifting arrangement in which the reversing action is automatic is exemplified in the standard belt-shifting mechanism of the ordinary planer of the older type. Two adjustable dogs clamped to the bed of the planer acting near the ends of the forward and return stroke trip a lever arm that shifts the open and crossed belts in such a manner as to reverse the motion of the planer table.

CLUTCHES AND BRAKES

128. Clutches and Clutch Mechanisms.—A clutch is a form of connection between a driving and a driven member on the same axis, so designed that the two members may be engaged or disengaged at will either by a hand-operated device or automatically by the action of some power-driven device. Several distinct types of clutches are in common use, and they are made in a great variety of designs. The common types of clutches may be divided into two general classes, namely, the positive or jaw clutch and the friction clutch. The jaw clutch is simpler and more powerful and is used when it is not objectionable to start the driven member suddenly and where the resistance to motion is not so great as to cause injurious shock when the clutch teeth come into engagement. Since a friction clutch may be engaged gradually and will slip while the driven member is being accelerated, it can be used successfully with great differences in speed.

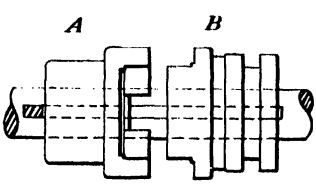


FIG. 180.

a. Jaw Clutches.—A simple form of jaw clutch is shown in Fig. 180. Part *A* is keyed to its shaft, and part *B* is splined so that it can be moved axially by a shifting device that engages it by means of a groove in the flange as shown. Several forms of this clutch differ mainly in regard to the shapes of teeth or angle of the engaging surfaces.

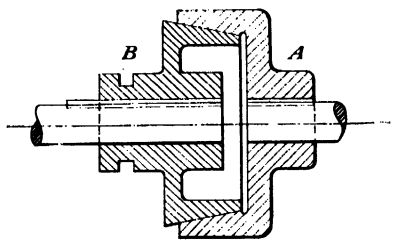


FIG. 181.

b. Friction Clutches.—The different types of friction clutches vary mainly in regard to the form of the friction surfaces and with respect to the kinds of material used to obtain sufficient frictional resistance. The frictional surfaces may be *conical* or *cylindrical*, or in the form of one or more *flat rings* or *disks*. A simple form of conical clutch is shown in Fig. 181.

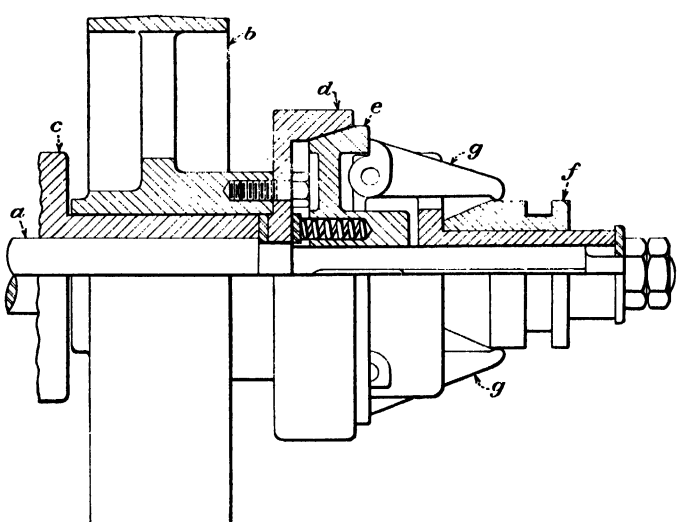


FIG. 182.

Motion is transmitted from the driving member *A* to the driven member *B* by the frictional resistance of the conical surfaces.

c. Clutch-operating Mechanisms.—Since friction clutches require the development of heavy pressure just as they close up, the toggle principle (see Art. 25, Chap. II) is very generally employed

in the operating mechanism of the clutch. In some types of friction clutch, the members are maintained in engagement by means of springs which are compressed in order to release the clutch. Where toggle mechanisms are employed, springs are frequently used to separate the friction members and thus eliminate any tendency to drag when the clutch is open.

An example of a conical clutch and its operating mechanism as applied to a machine tool is shown in Fig. 182. The driving pulley runs loose on the hub of the bearing *c*, and fastened to the pulley is the cast-iron cone *d*. The sliding cone *e*, which engages with cone *d*, is keyed to the driving shaft *a* by means of a feather

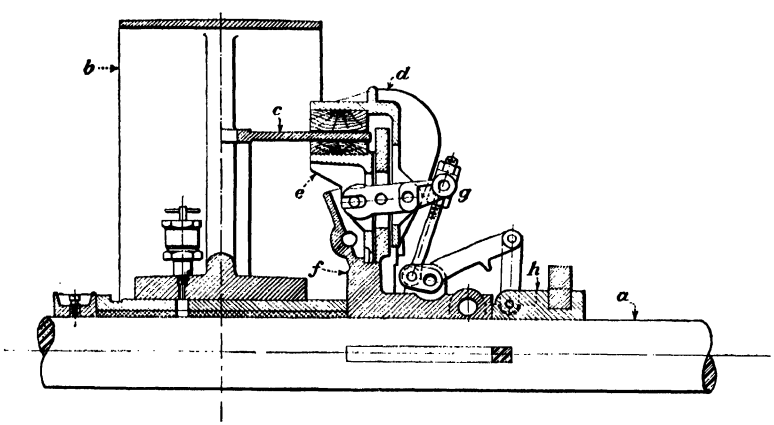


FIG. 183.

key. The cones are engaged by means of the sliding member *f* and the levers *g*. Several helical springs, one of which is shown in the figure, are placed in the hub of the cone *e* for the purpose of disengaging the two cones when the sliding member *f* is shifted so as to relieve the pressure between the cones.

In the type of clutch shown in Fig. 183, the operating mechanism consists of an assemblage of several vise arms, *d* and *e*, each double gripping and thereby driving an annular ring *c* integral with the pulley *b*, which runs loose when not in engagement with the clutch. By means of the toggle mechanism at *g*, the vise arms *d* and *e*, which are lined with shoes made of hard maple, are forced to grip the annular ring from both sides. The shaft *a* is driven by means of the member *f*, which is keyed to it. The

clutch is thrown in and out of engagement by means of the sliding member *h*.

A clutch of the disk type is represented in Fig. 184, as applied to a pulley *b* running loose on a shaft *a*. The disk *c* is faced with hard-maple segments, and the construction is such that the segments may easily be renewed. The flange *d* slides on the flanged hub *e*, which is keyed to the shaft *a*. By means of the toggle mechanism consisting of levers *f*, *g*, and *h* and the sliding member

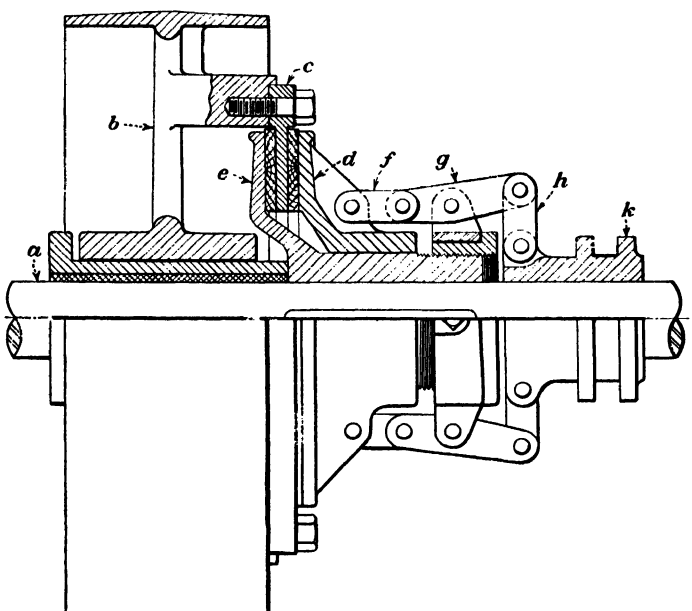


FIG. 184.

k, the flanges *d* and *e* are pressed against the disk *c*, so that the power is transmitted from the pulley to the shaft or vice versa.

129. Brakes.—In their action, brakes are very closely related to friction clutches, and in some starting and stopping mechanisms the clutch and brake principles are used in combination. The study of clutches and brakes more properly belongs in books on machine design, and, therefore, no attempt will be made to go farther than the few illustrations already shown, which have been given merely for introductory purposes, as examples of intermittent-motion mechanisms.

A vast number of ingenious devices of kinematic interest may properly be included under the heading of this chapter, but the object has been merely to call attention to this important class of mechanisms by a few simple examples rather than to attempt a thorough treatment. An extended discussion of such mechanisms may be found in several well-known treatises on mechanism.

CHAPTER IX

TRAINS OF MECHANISM

Since the primary purpose of a machine is to modify energy, all machines must have as constituent parts a mechanism or a series of mechanisms between the point where the power is received and the point where the useful work is delivered. Similarly, mechanisms, whether they form constituent parts of machines or are employed merely in their primary function of modifying motion, are made up of combinations of units arranged in trains. Such combinations of units are called *trains of mechanism*. These trains may consist of a great variety of constituents: linkwork, cams, gears, chains, belts, ropes, etc., any or all.

When the distance between shafts is comparatively great, rope, belt, or chain connections may be used. When the distance between shafts is comparatively small and positive transmission is essential, gears are used. When this last condition is not a requisite, and the distance between shafts is too small to use belts or chains advantageously, frictional gears are occasionally employed. It is possible with a train of mechanism to secure any desired result as to plane and direction of the ultimate motion, and as to variety of motion, whether rotary, reciprocating, constant, intermittent or irregular, etc. Trains of mechanism are common to all classes of machinery, and in connecting the source of power to the useful work, they must provide separately or in combination mechanical advantage, a definite velocity ratio, flexibility as to speed ratio, and compactness.

130. Typical Examples.—A typical example of a comparatively simple train of mechanism, and one with which students are ordinarily familiar, is found in the ordinary metal-cutting lathe of the older style. The mechanism of this lathe is shown in diagrammatic form in Fig. 185. The main train at A drives the spindle S, which carries the work piece and rotates it against the cutting resistance of the tool. The first secondary train B, an end view of which is also shown, drives the lead screw; this

screw engages a split nut on the tool carriage or saddle, so as to give the latter a smooth and steady motion along the *ways* for screw cutting. The belt at C connects L and M, and the gears at D drive the feed rod, which by means of gear trains E and F can give a longitudinal feed to the whole saddle or a cross-feed to the slide resting on the saddle.

The cone pulley a, with pinion b fastened to it, is the first member of the main train. This combined pulley and gear is driven by a belt from an overhead countershaft and turns freely on the spindle to which the gear e is fastened. Through the back-gear train b-c-d-e, a slow and powerful movement is given to the

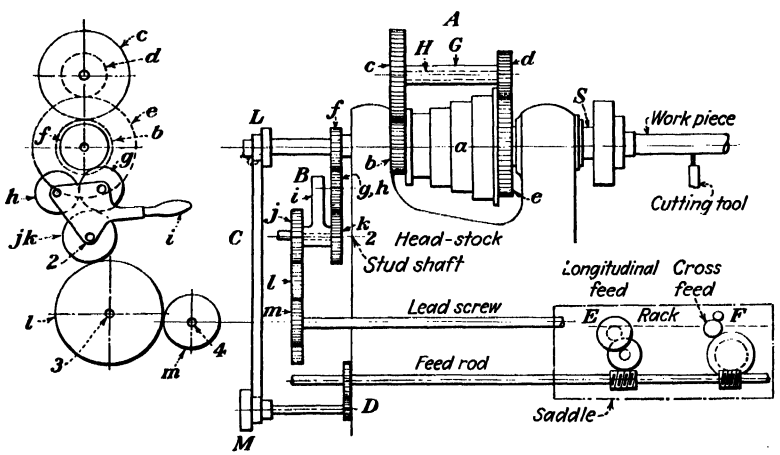


FIG. 185.

spindle. For high-speed work the back gears c and d, which are fastened to a hollow shaft G, are thrown out by means of the shaft H, which has eccentric bearings in the headstock. Gear e is then locked to cone a, so that the belt drives the spindle directly. As regards the reversing gears, it is only necessary to state that the plate carrying these intermediate gears is pivoted on axis 2 and that arm i can be latched in position for forward, neutral, and reverse. Different gears can be slipped on studs 2, 3, and 4 so as to give different speed ratios of lead screw to spindle, stud 3 being adjustable to any position required.

The belt drive for the feed rod is an old-fashioned arrangement, modern lathes having some form of all-gear drive. In the old type of lathe the lead screw could be used for ordinary longitu-

dinal feeding, but its general use would cause unevenly distributed wear, so that it is generally reserved exclusively for the operation of screw cutting.

The tool-carriage feed trains *E* and *F* are carried in the *apron* that extends down from the saddle, and they connect with the

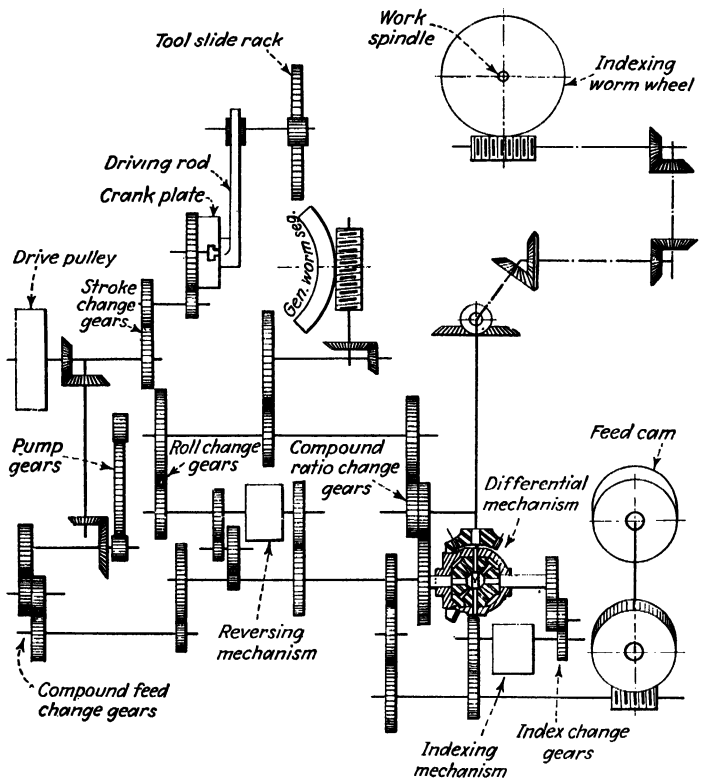


FIG. 186.

fixed rack on the lathe bed for the longitudinal feed and with the screw for the cross-feed, respectively.

A typical example of a more complex train of mechanism is shown in Fig. 186. This figure is a diagram of the driving mechanism found in a certain type of automatic gear-cutting machine. It may be noted that the constituents common to trains of mechanism occur here in considerable variety, namely, pulleys; spur, bevel, and worm gearing; cams; linkwork; etc.

The mechanism as a whole is composed of a number of auxiliary trains branching off from the main drives to the cutting tools and the work, to actuate the indexing, reversing, and feeding mechanisms, etc. In a machine of this type careful timing is necessary, so that the velocity ratios are generally exact.

131. Gear Trains.—If motion is transmitted entirely through gearing, the combination of gears is called a gear train. Gear trains may be divided into two classes, *simple* and *compound*. In a simple gear train each shaft of the mechanism carries one gear only, as in Fig. 187. In a compound train of gears each shaft except the first and last carries two gears as in Fig. 188. It is kinematically possible to transmit motion between any two shafts

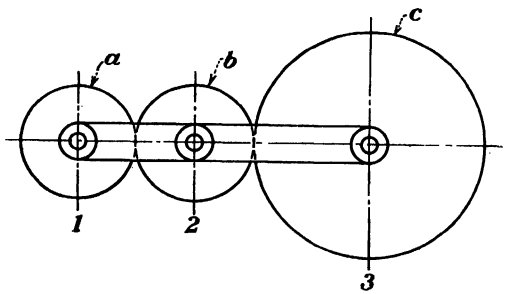


FIG. 187.

by a single pair of gears, but there are practical considerations that often make this arrangement impracticable.

A numerical example will show the advantage of a compound train over a simple train. In Fig. 188 let a, b, c, and d have 12, 60, 16, and 64 teeth, respectively. The velocity ratio between a and d is therefore $^{60}/_{12} \times {}^{64}/_{16} = {}^{20}/_{1}$. To get the same velocity ratio by employing a simple train, a 12-tooth pinion meshing with a 240-tooth gear would be required, assuming 12 to be the smallest number of teeth practicable for the pinion. If the pitch of the gears is taken as 4, the diameter of the 240-tooth gear would be 60 in., whereas in the compound train the diameter of the largest gear would be 16 in. Moreover, in case of the single pair, the pinion would be subjected to twenty times the tooth wear of the gear, a very undesirable condition. In addition, the faster pair of gears in the compound train may have a smaller pitch and narrower face than the slower pair, a condition that results in further compactness.

Still further compactness in a compound gear train may be obtained by placing the first and last gears on the same axis, as shown in Fig. 189, one of the gears being attached to a sleeve or hollow shaft. This arrangement is sometimes referred to as a *reverted* gear train. A familiar example of this arrangement is

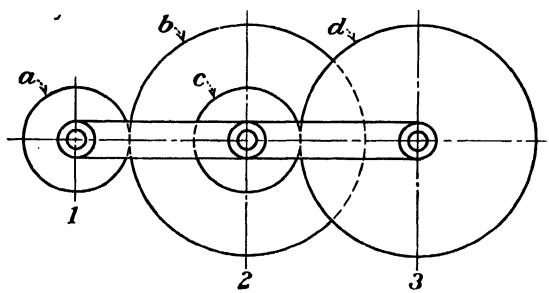


FIG. 188.

observed in clockwork, where the hour and minute hands are attached to gears on the same axis.

Compound trains of the type shown in Figs. 188 and 189 find wide application in speed-reducing mechanisms used in connection with high-speed prime movers such as the steam turbine and the high-speed electric motor. Such a train is generally enclosed in a housing, forming a unit that connects the prime mover to the driven machine, and is called a *speed reducer*. Descriptions

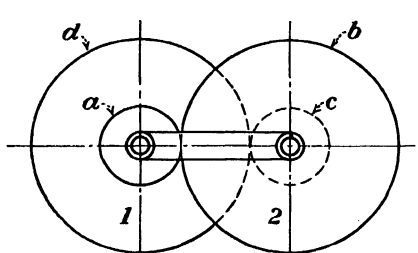


FIG. 189.

of the several types of speed reducers in common use are given in Art. 139.

132. Velocity Ratio in a Gear Train.—In Chap. IV velocity ratio in the case of a pair of mating gears was defined as the ratio of the angular velocity of the driving gear to that of the

driven gear. It can be shown that a similar relation exists for the velocity ratio of a gear train.

Assume four axes, 1, 2, 3 and 4 (Fig. 190) to be arranged as shown and connected by toothed gears represented by their pitch circles a, b, c, etc. Gear a meshes with b, c meshes with d, and e meshes with f. Both the gears b and c are secured to the shaft 2; hence they must rotate as one piece, having the same angular velocity at any instant. Likewise d and e are both secured to shaft 3 and thus have the same angular velocity. Let the angular velocities of the shafts 1, 2, 3, and 4 be represented by ω_1, ω_2, ω_3, and ω_4, respectively. Two gears that mesh together must have the same pitch; hence the numbers of teeth are proportional

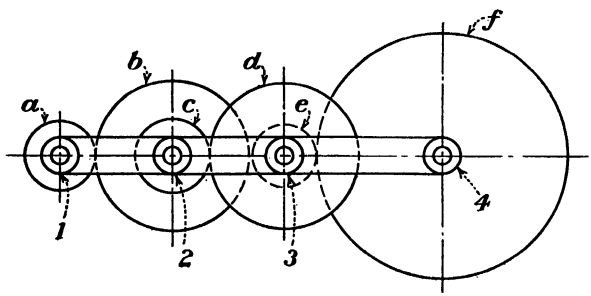


FIG. 190.

to the circumferences, to the diameters, or to the radii. But their angular velocities are inversely as the radii, and therefore inversely as the numbers of teeth in the gears. If a, b, c, etc. are the numbers of teeth in the gears designated by these letters, it follows that

$$\frac{\omega_1}{\omega_2} = \frac{b}{a}; \frac{\omega_2}{\omega_3} = \frac{d}{c}; \frac{\omega_3}{\omega_4} = \frac{f}{e}$$

Therefore

$$\frac{\omega_1}{\omega_4} = \frac{\omega_1}{\omega_2} \times \frac{\omega_2}{\omega_3} \times \frac{\omega_3}{\omega_4} = \frac{b}{a} \times \frac{d}{c} \times \frac{f}{e} = \frac{bdf}{ace} \qquad (42)$$

In this train a is the driver and b is the follower in the first pair; c is the driver and d the follower in the second pair; and e is the driver and f the follower in the third pair. The important conclusion expressed by ω_1/ω_4 in the above equation may be put into the following general statement, good for all such trains:

The angular-velocity ratio of the first driving gear to the last driven gear equals the continued product of the numbers of teeth in the driven gears divided by the continued product of the numbers of teeth in the driving gears.

In finding the velocity ratio of a train, any of the factors ω_1/ω_2, ω_2/ω_3, etc., may be expressed in terms of the numbers of teeth, radii, diameters, or revolutions per unit of time of the pair of gears involved. If the last relation is used, however, the ratio is direct, and not an inverse ratio as in the case of the other terms. Nor is it necessary that these different factors all be given in the same terms. Thus in Fig. 190, if a has 15 teeth and b has 60 teeth; if c is 3 in. in diameter and d is 15 in. in diameter; if e makes 75 r.p.m. and f makes 25 r.p.m., then

$$\frac{\omega_1}{\omega_4} = \frac{60}{15} \times \frac{15}{3} \times \frac{75}{25} = \frac{60}{1}$$

Hence, if shaft 1 makes 120 r.p.m., shaft 4 will make 2 r.p.m.

133. Value of a Gear Train.—In most problems involving gear trains the speed of the first gear is known and that of the last required. It is convenient, therefore, in making calculations to have a factor by which to multiply the first in order to get the last speed. This factor is the reciprocal of the velocity ratio of a train. The relation may be expressed by writing the reciprocal of Eq. (42) as follows:

$$\frac{\omega_4}{\omega_1} = \frac{ace}{bdf} \qquad \text{or} \qquad \omega_4 = \omega_1 \times \frac{ace}{bdf} \tag{43}$$

The reciprocal of the velocity ratio, *i.e.*, the continued product of the numbers of teeth in the driving members divided by the continued product of the numbers of teeth in the driven members, is known as the *value of the train.* Referring to Fig. 188, if a, b, c, and d have 12, 60, 16, and 64 teeth, respectively, the train value is $a/b \times c/d = 12/60 \times 16/64 = 1/20$; and if $\omega_1 = 100$ r.p.m., $\omega_4 = \omega_1 \times ac/bd = 100 \times 1/20 = 5$.

In making gear-train calculations, it is a matter of choice as to whether the results are obtained by consideration of the overall velocity ratio or train value. A little practice, however, will show that the use of the train value is generally more convenient.

134. Idler Gear and Direction of Rotation.—If two gears are placed in direct contact, their sense of rotation, except in the

case of an internal gear, will be opposite; but if one intermediate gear is placed between them, their sense of rotation will be the same. In Fig. 187 let the letters that represent the gears also designate their respective numbers of teeth. Gear a drives b, which in turn drives c. Gear b is, therefore, both a driven and a driving gear. Such a gear is called an *idler*. The velocity ratio is

$$\frac{\omega_a}{\omega_c} = \frac{b}{a} \times \frac{c}{b} = \frac{c}{a}$$

Thus the introduction of an idler in a train serves to change the sense of rotation, but not the velocity ratio or the train value. The chief function of an idler is to change the sense of rotation. Sometimes it is used to bridge the space between driver and follower and so enable a given velocity ratio to be transmitted by use of smaller gears than would otherwise be possible. If a train is made up entirely of spur gears, the adjacent shafts rotate in opposite sense and the alternate shafts in the same sense. If there is an odd number of shafts, the first and last axes will rotate in the same sense; and if there is an even number of shafts, the first and last shafts will rotate in the opposite sense. In general the safest procedure is to determine the directions of rotation in a train of mechanism by inspection and not attempt to go by rule.

A familiar example of the case in which an idler provides a simple and convenient reversing device is embodied in the *tumbler gears* common to many mechanisms. A common application of the tumbler-gear arrangement occurs in the headstock of the screw-cutting lathe (Fig. 185), where it provides a convenient means of changing the direction of feed, or for cutting either a right- or left-hand thread. Two idlers, g and h, are mounted on a swinging arm i so that they are always in mesh, but at different center distances from axis 2 of the pivot. In the configuration shown, the train runs f-g-k, and f and k rotate in the same direction. By swinging the arm through a small angle, h is brought into mesh with f. The train now runs f-h-g-k, and f and k rotate in opposite directions.

135. Gear Trains with Nonparallel Axes.—In the preceding articles only gear trains with parallel axes have been described. The methods deduced in Arts. 132 and 133 for the velocity ratio and value of a train are general and apply to all cases when the

proper substitutions are made. A train is shown in Fig. 191 in which the axes are not all parallel. A pinion a on shaft 1 drives the spur gear b on shaft 2; a pair of bevel gears c and d connect shafts 2 and 3, and a worm e on shaft 3 drives the worm wheel f on shaft 4. If the numbers of teeth on a, b, c, d, e, and f are 16,

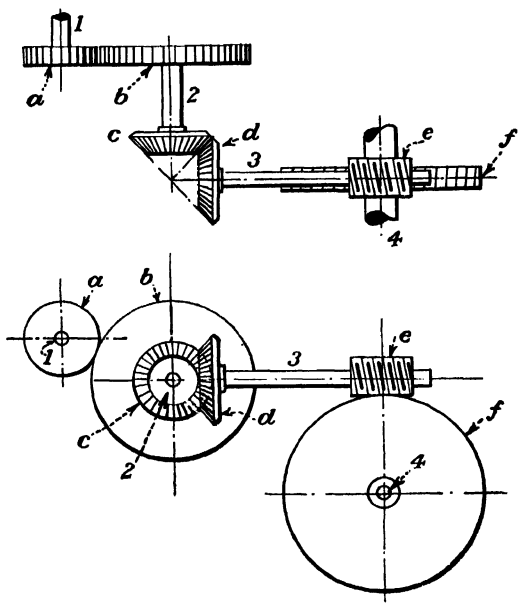


FIG. 191.

32, 15, 30, 1, and 40, respectively, and if shaft 1 runs at a speed of 320 r.p.m.,

$$\omega_4 = \omega_1 \times \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = 320 \times \frac{16}{32} \times \frac{15}{30} \times \frac{1}{40} = 2$$

136. Selection of Gears for a Train.—When no restrictions or limitations are imposed as to the number, size, pitch, etc. of the gears, the variations in the make-up of gear trains that would transmit a given velocity ratio are almost unlimited. No definite rules or formulas can be followed, the process being mainly one of *cut and try* until the desired result is obtained. There are, however, certain limiting conditions in practice that confine the selection of combinations within comparatively narrow limits. Almost every drive is subject to certain special restrictions as to

space and arrangement of the gears. The power transmitted determines the pitch of the gears in any given case, and since it is common practice to limit the number of teeth in a pinion to 12, the minimum size of the pinion is fixed. When the function of the train as a whole is to reduce the speed, smoothness of action requires that there should be no *gearing up* at any point in the train; *i.e.*, the driving gear should be smaller than the driven gear in every case.

a. The Hunting Tooth.—Another consideration that often must be taken into account in the selection of gears is the frequency of contact between a pair of teeth of mating gears. Although it is a debated question, it is considered by many that greater evenness of wear on the teeth will result when a given pair of teeth on two gears comes into contact the least number of times. To illustrate this point, assume a pair of mating gears to have 30 teeth each, the velocity ratio thus being unity; then a given tooth of one gear will come into contact with a given tooth of the other gear at each revolution of each gear. Now if the number of teeth in one gear is increased to 31, the velocity ratio is nearly the same as before, and yet a given pair of teeth will come into contact only after 30 revolutions of one of the gears and 31 revolutions of the other. The tooth inserted is called a *hunting tooth*, because a pair of teeth, after once being in contact, gradually separate and then approach by one tooth in each turn, and thus appear to hunt each other as they go around. The result of the use of a hunting tooth is most completely obtained when the numbers are prime to each other, as in the case just cited. A hunting-tooth arrangement is frequently found in automobile rear-axle drives, where, for example, a 13-tooth pinion driving a 53-tooth ring gear is just as satisfactory, as far as velocity ratio is concerned, as the exact-multiple ratio of 13 to 52. But there are a great many cases where an exact-multiple ratio is required, thereby rendering the use of a hunting tooth impossible.

b. Illustrative Examples.—It is evident from the above discussion that in selecting gears for a train there are some general lines of attack which may be followed, and which may best be understood by studying certain typical problems.

1. As an example let it be required to select the gears for a train arranged as shown in Fig. 188. The train value is to be $\frac{1}{20}$ and the gears selected are to be subject to the limitation that no

gear is to have fewer than 12 teeth. The first step in the solution of the problem is to write the equation,

$$\frac{a}{b} \times \frac{c}{d} = \frac{1}{20}$$

It is desirable in general to have the two factors a/b and c/d as nearly as may be of the same value. The square root of $\frac{1}{20}$ is between $\frac{1}{4}$ and $\frac{1}{5}$; so a trial pair of factors may be taken as $\frac{1}{4}$ and $\frac{1}{5}$. Then,

$$\frac{a}{b} \times \frac{c}{d} = \frac{1}{4} \times \frac{1}{5} = \frac{13}{52} \times \frac{12}{60} = \frac{1}{20}$$

and the equation is satisfied if a, b, c, and d, have 13, 52, 12, and 60 teeth, respectively. By factoring differently or by using other numbers of teeth for the pinion, other successful combinations of gears may be selected.

The above arrangement has what is considered by many the disadvantage of having the same tooth of a pinion always making contact with the same tooth in a gear. This condition may be avoided and the hunting-tooth condition obtained by the following arrangement:

$$\frac{a}{b} \times \frac{c}{d} = \frac{12}{52} \times \frac{13}{60} = \frac{1}{20}$$

2. As a further example let it be required to select the gears for a train with the axis of the last gear coincident with the axis of the first gear, as in Fig. 189. The train value is to be $\frac{1}{20}$, no gear is to have fewer than 12 teeth, and all gears are to be of the same pitch. As in the preceding example, a trial pair of factors may be taken as $\frac{1}{4}$ and $\frac{1}{5}$ so that

$$\frac{a}{b} \times \frac{c}{d} = \frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$$

Now since the pitches are all the same,

$$a + b = c + d$$

Let x represent this sum. Then a value must be chosen for x such that it may be broken up into two parts whose ratio is 1 to 4 and also two parts whose ratio is 1 to 5. If x is made equal to the

least common multiple of (1 + 4) and (1 + 5), the conditions will be satisfied. This least common multiple is 30. Then $a/b = 6/24$ and $c/d = 5/25$. But the values in the numerators are too small for the numbers of teeth required in the pinions, so that numerator and denominator of each fraction must be multiplied by some number such that no number will be less than the number of teeth allowed in the smallest gear. In this case multiplying $\frac{6}{24}$ by $\frac{3}{3}$ and $\frac{5}{25}$ by $\frac{3}{3}$ gives

$$\frac{a}{b} \times \frac{c}{d} = \frac{18}{72} \times \frac{15}{75}$$

Therefore the conditions of the problem are satisfied when a, b, c, and d have 18, 72, 15, and 75 teeth, respectively. When the pitches of the two pairs of gears are not the same, a slight modification of the method is necessary. The solution in such a case is left to the student.

137. Speed-changing Gear Trains. Change Gears.—The gears of a train may all be interposed permanently between the driving and driven members, or certain gears in the train may be changed from time to time to vary the velocity ratio of the train. Gears thus subject to change are called *change gears*. Speed-changing gear trains are mostly used in machine tools and motor cars. The oldest and simplest arrangement of change gears is that in which certain gears may be removed from their shafts or studs and others substituted in their place.

The most convenient and desirable arrangement, and the one that represents the modern tendency in design, is that in which the gears are not removed from the machine, but are enclosed in a box or housing called a *speed box* or *gear box*, so designed that various combinations of gears may be selected automatically and the speed changes quickly obtained by manipulating a hand lever that slides or shifts the gears into any desired positions.

The mechanisms of a great variety of machines might be chosen to illustrate the two general schemes outlined above for effecting speed changes. But since the principles are fundamentally the same for all, it is desirable to select for illustration a type of machine with which students are commonly familiar.

The ordinary engine lathe serves this purpose remarkably well. The mechanism of the lathe, particularly in respect to its screw-cutting provision, illustrates a typical application of a

gear train in which speed changes are required. In lathes of the older type the change gears are removable, whereas most modern lathes are of the quick-change type with an elaborate train of gears in which the speed changes are made in a manner similar to the gear shift of automobiles.

a. Speed-changing Mechanism of the Engine Lathe.—The general arrangement of gearing in a simple lathe of the older type, equipped for screw cutting, is shown in Fig. 185 and was briefly described in Art. 130. On the spindle S are the gears b, e, and f and the stepped cone a. The cone is connected by belt to the countershaft, which supplies the power. The gear e is keyed to the spindle, and the stepped cone is free to rotate on the spindle except when it is locked to gear e by means of a pin. The gear b is fastened to the stepped cone and rotates with it; c and d are the *back gears* and are keyed to a shaft that is parallel to the spindle. This shaft has eccentric bearings so that the back gears may be thrown out of mesh with b and e.

When the back gears are thrown out, e and the stepped cone are locked together, and as many speeds can be obtained as there are steps on the cone, whereas with the back gears thrown in, e and the stepped cone are not locked, and the number of speeds of the spindle is therefore doubled.

The purpose of the spindle gear f is to drive the thread-cutting train. The tumbler gears g and h are for the purpose of changing the direction of rotation of the lead screw. Gear g meshes with k, which is the inside stud gear, and the arrangement is such that f and g may be thrown out of mesh and h be made to mesh directly with f. f and k are usually made the same size, but, when they are not the same, k is usually made twice the size of f, so that k makes one-half as many revolutions as the spindle. On the same shaft with k is the outside stud gear j. Meshing with j is an intermediate gear l, which meshes with the lead-screw gear m.

In thread cutting it is necessary to change the stud gear j and the lead-screw gear m. The intermediate gear l is held on a slotted bracket called a *quadrant*, not shown in the figure. This allows l to be adjusted to accommodate different-sized gears j and m.

In order to cut a greater range of threads without adding more change gears, the intermediate gear l may be *compounded*, that is, replaced by two gears of different sizes, and fastened together,

one of them meshing with the stud gear and the other with the lead-screw gear. When compounded, the intermediate gears are subject to change and therefore become change gears along with the stud gear and lead-screw gear.

One of the inherent advantages of the involute system, namely, the possibility of varying the center distance without affecting the velocity ratio, is manifest in change-gear sets, because no provision is made for adjusting these gears to the theoretical center distance. To do so would greatly complicate the design.

The cutting tool is held in a tool holder mounted on a carriage that is moved back and forth along the work by means of a split nut engaging the lead screw. If there were eight threads per inch on the lead screw, then eight revolutions of the lead screw would advance the cutting tool 1 in., and if the work made four revolutions to eight of the lead screw, four threads per inch would be cut on the work.

b. Selecting Change Gears for the Screw-cutting Train.- -An illustrative example will show the method of selecting the change gears to do a given piece of work. A certain lathe of 16-in. swing has a lead screw of four threads per inch, and a set of change gears of the following numbers of teeth: 24, 30, 36, 42, 48, 54, 60, 66, 69, 72, 78, 84. Let it be required to find the numbers of teeth in the stud and lead-screw gears j and m, when f and k are the same size.

Since the gears g, h, and l are idlers, they need not enter into the calculations, and since the spindle and stud shafts make the same number of revolutions, it is only necessary to find the number of revolutions of the stud shaft.

Let n = number of threads per inch to be cut on the work.
N = number of threads per inch on lead screw.
t = number of teeth on lead-screw gear.
T = number of teeth on stud gear.

Then

$$\frac{N}{n} = \frac{T}{t} \tag{44}$$

Let it be required to select change gears for cutting six threads per inch on the work.

$$\frac{T}{t} = \frac{N}{n} = \frac{4}{6} = \frac{2}{3}$$

Any two gears of the set may be taken that have numbers of teeth in the ratio of T to t. In this particular case a 24-tooth gear would be selected for the stud and a 36-tooth gear for the lead screw.

If the change gears are compounded, the pair of intermediate gears, unlike the idler l, which they replace, affect the velocity ratio between the spindle and the lead screw because of their difference in diameters. The velocity ratio as found by the preceding method must be multiplied by the ratio of the two intermediate gears. Thus if 35 threads are to be cut on the same lathe, it will be necessary to use the compound intermediate gears. Take the ratio $TT/tt = 4/35$ and split it into the obvious

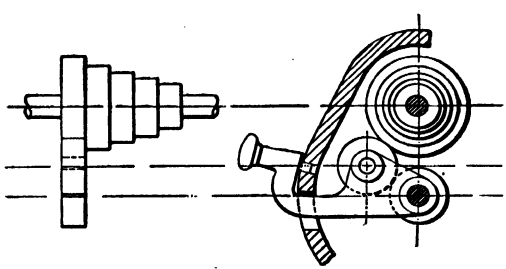


FIG. 192.

factors $\frac{2}{5}$ and $\frac{2}{7}$, which can be realized with gears of 24, 60, 20, and 70 teeth.

The primary unit of nearly all quick-change lead-screw trains is a cone-and-tumbler pair (Fig. 192) based on the arrangement shown by gears j, l, and m, (Fig. 185). The secondary train, which will not be discussed here, multiplies and divides the cone ratio by 2. Of the two gears j and m in Fig. 185, one will serve as a pinion (constant), and the other will become a cone (in effect, a gear of varying diameter). Whether f or j is expanded into a cone depends upon the system of screw threads.

The above screw-cutting trains are given as examples of the old and the modern arrangements; but among lathes of various makes there are many modifications in detail to be found. All ordinary screw-cutting lathes, however, have mechanisms that are fundamentally those given above.

c. The Automobile Transmission.—Figure 193 is a drawing of a variable-speed transmission showing how the speed-changing principle is applied in the automobile drive. The standard

arrangement gives three forward speeds to the drive shaft (one without using the gears) and one reverse. The power from the engine is delivered through a friction clutch (not shown) to the shaft E and to this shaft is secured a gear A having also a part of a jaw clutch B on its right-hand side. Gear A meshes with another gear G on the countershaft M, which carries also the gears H, J, and K keyed to it, and, whenever the engine shaft E operates, the gears G, H, J, and K are running. On the right is shown the power shaft P, which connects with the rear or driving axle; this shaft is in line with E and carries the gears D and F and also the inner part C of the jaw clutch, which engages with the outer part B. The gears D and F are forced to rotate

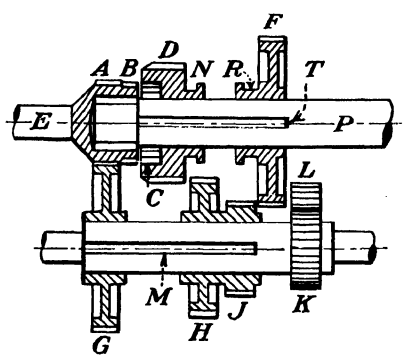


FIG. 193.

with P by means of keys at T, but both are sliding gears and may be moved separately along the splined shaft by means of a shifting lever that connects with the collars at N and R, respectively. In addition to the gears already mentioned there is another, an idler gear L, which meshes with K and runs on a bearing behind the gear K.

When it is desired to operate the car at maximum speed (third or high), the operator throws F into the position shown and pushes D to the left so that the clutch piece C engages with B, in which case P runs at the same speed as the engine shaft E. The second highest speed (second or intermediate) is obtained by slipping D to the right until it comes into contact with H, the ratio of the gears (train value) then being $A/G \times H/D$; F remains as shown. For the lowest speed (first or low) D is placed as shown in the figure and F moved into contact with J, the ratio

of the gears in this case being $A/G \times J/F$. The shaft P and the car are reversed by moving F to the right until it meshes with L, the gear ratio being $A/G \times K/L \times L/F$. The shifting arrangement is such that it is not possible to have more than one set of gears in operation at one time.

138. Epicyclic-gear Trains.—If one of the gears in a train is fixed or stationary and another gear (or gears) in mesh with it revolves about the stationary gear in addition to rotating relative to its own axis, the mechanism is known as an epicyclic gear train, because points on the revolving gears describe epicycloidal curves.

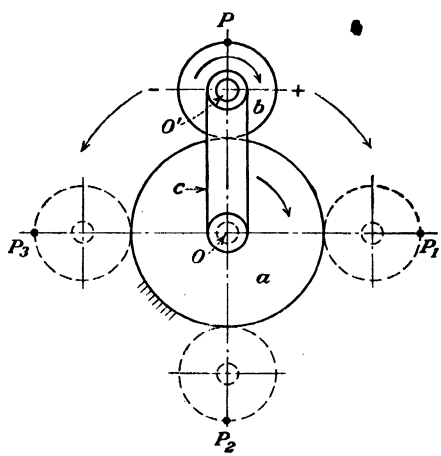


FIG. 194.

a. Simple Epicyclic-gear Train.—In Fig. 194 let a and b be two gears mounted on an arm c, so that if c were fixed, a and b would form a simple gear train. Now let it be assumed that a is the fixed member of the train. Then c can rotate about O, carrying b with it, b itself rotating relative to c around its axis at O'. It is required to find the number of revolutions that b will make *around its own axis* relative to the fixed member a for every revolution of c around O.

First, let a be disconnected from the fixed frame, so that a, b, and c (locked together) can make one revolution as one piece, in the direction indicated, around O. Then b will make one revolution around O' solely because of its motion around O. This can be seen by noting the positions of any point, as P, relative to

O' during different phases of the revolution, as shown. Now if c is held stationary and a is rotated backward one revolution, a is brought back to its original position, and thus its resultant motion is zero. At the same time b must receive a/b turns forward. Then the total number of revolutions that b will make for one revolution of c around O is $1 + a/b$, and its direction of rotation will be the same as that of c. It is evident that c can be rotated in either direction, and the result obtained will still hold.

The total motion of each link may be tabulated in the following manner: Assume clockwise rotation as +, and counterclockwise

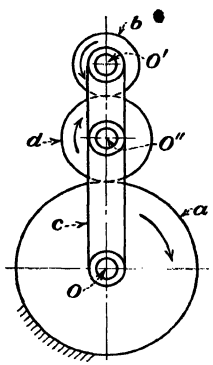


FIG. 195.

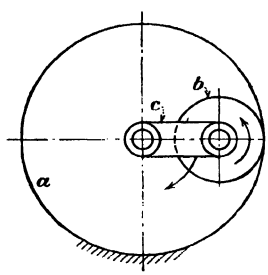


FIG. 196.

rotation as −. Then considering the whole mechanism locked and given one revolution about O in the positive direction, each member will rotate +1 revolution. Next consider that the arm c is fixed and that a is given one revolution in the negative direction. The total motions in this case are: −1 revolution for a; $+ a/b$ revolutions for b; and 0 revolutions for c. The total motion of each link is then given by the following table:

	a	b	c
Motions with c	+1	+1	+1
Motions relative to c	−1	$+\frac{a}{b}$	+0
Total motions	0	$1+\frac{a}{b}$	+1

Then the angular-velocity ratio between the gear b and the arm c may be expressed as follows: $\frac{\omega_b}{\omega_c} = \frac{1 + \frac{a}{b}}{1}$, and the signs indicate that both rotate in the same direction.

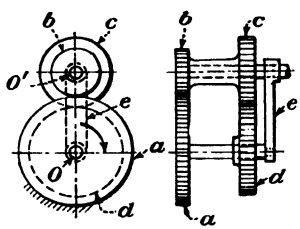


Fig. 197.

If an idler is placed between a and b, as in Fig. 195, or if a is an internal gear (Fig. 196), the direction of motion of b is reversed.

b. Compound Epicyclic-gear Train.—The tabular scheme may be followed if other gears such as c and d are added as in Fig. 197, forming a compound reverted train. If b is fastened to c, and d is free to revolve about O, the motions are shown in the following table.

	a	b	c	d	e
Motions with e.....	$+1$	$+1$	$+1$	$+1$	$+1$
Motions relative to e.....	-1	$+\frac{a}{b}$	$+\frac{a}{b}$	$-\frac{a}{b} \times \frac{c}{d}$	0
Total motions.....	0	$1 + \frac{a}{b}$	$1 + \frac{a}{b}$	$1 - \frac{ac}{bd}$	$+1$

It follows that $\frac{\omega_d}{\omega_e} = \frac{1 - \frac{ac}{bd}}{1}$

This form of epicyclic train is used extensively for obtaining great velocity ratios between the arm e and the last gear d.

As an extreme example:

Let a have 99 teeth.

b have 100 teeth.

c have 101 teeth.

d have 100 teeth.

Then

$$\frac{\omega_d}{\omega_e} = \frac{1 - \frac{ac}{bd}}{1} = \frac{\left(1 - \frac{99 \times 101}{100 \times 100}\right)}{1}$$

$$= \frac{\frac{1}{10{,}000}}{1}$$

The application of epicyclic gearing to hoisting devices will be obvious from the above. Epicyclic gears are also used for feed mechanisms on large boring bars, in machines for making wire rope, in speed reducers, etc.

c. Epicyclic Spur-gear Trains. The Triplex Hoist.—In the triplex hoist, shown in diagrammatic form in Fig. 198*a*, is found an interesting practical application of the epicyclic-gear train.

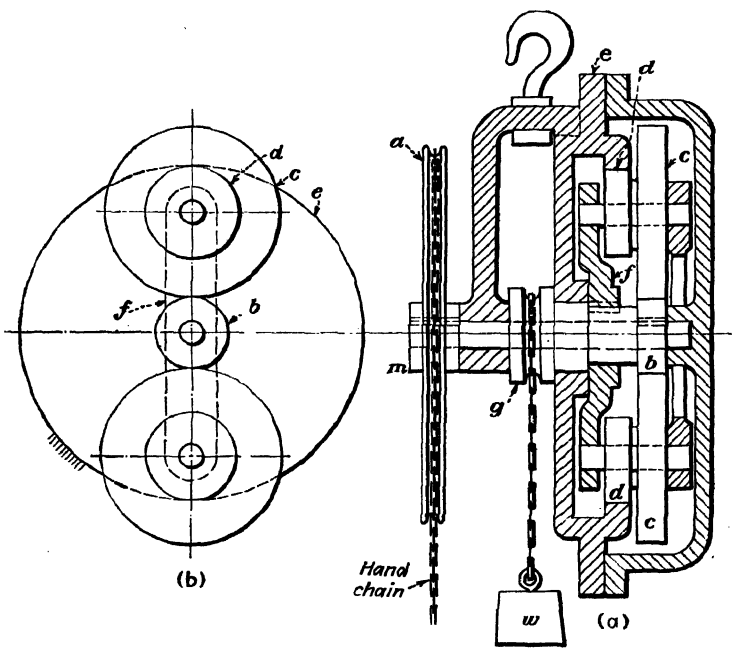


FIG. 198.

The frame *e* is provided with bearings that carry the hoisting sprocket *g*, and in the casting *f*, which is keyed to the hoisting sprocket, are studs, each carrying a pair of compound gears *cd*, the smaller one (*d*) meshing with an internal gear integral with the frame *e*, and the larger one (*c*) of the pair meshing with a pinion *b* on the end of the shaft *m* to which the hand sprocket wheel *a* is attached. When an operator pulls on the hand sprocket chain, the hand sprocket wheel *a* is rotated, and with it the pinion *b*, which in turn sets the compound gears *cd* in motion. As one of these gears meshes with the fixed annular gear on the frame *e*,

the only motion possible is for the spider and the studs carrying the compound gears to revolve, and thus carry with them the hoisting sprocket wheel *g*.

The velocity ratio between hand sprocket wheel *a* and hoisting sprocket wheel *g* is evidently the same as that between gear *b* and rotating casting *f*. In finding the velocity ratio for any specific case, it is convenient to draw the skeleton outline of the gear train as shown in Fig. 198*b*. The two pairs of gears *c* and *d* are for the purpose of distributing the load. In finding the velocity ratio it is only necessary to consider one-half of the symmetrical arrangement, *i.e.*, the gears that are lettered in Fig. 198*b*.

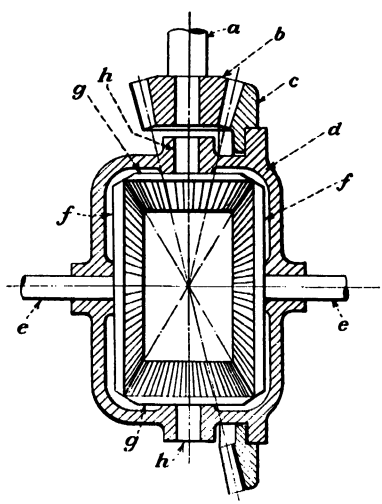


FIG. 199.

d. Epicyclic Bevel-gear Trains. The Automobile Rear-axle Drive.—Figure 199 shows the arrangement of bevel gears in the automobile rear axle. Shaft *a* is driven from the engine and has keyed to it the bevel pinion *b* meshing with ring gear *c*, which is fastened to the frame *d*. The frame turns freely on the axles *e*, to which gears *f* and the rear wheels are keyed. The frame *d*, to which *c* is fastened, carries the pinions *g*, which are free to turn on the studs *h*. There are usually three of these pinions, in order to distribute the load. When the automobile is going straight ahead, *a* drives *b*, and all the other gears revolve as a unit with *c*, without any relative motion. As soon, however, as the car starts to turn a corner, say, toward the right, the left-hand wheel will have to travel faster than the right-hand wheel, and therefore the gears begin to move relative to each other, the action being that of an epicyclic train.

Let it be assumed that the left-hand wheel is jacked up so that it may turn freely while the right-hand wheel is left on the floor, thus holding right-hand gear *f* from turning. Consider fixed gear *f* as the first gear of the train and the frame *d* as the arm

carrying gears g. The velocity ratio of free gear f relative to fixed gear f may be found in the same manner as for epicyclic spur-gear trains. Thus using the tabular method:

	f(fixed)	d	g	f(free)
Motion with d	+1	+1	...	+1
Motion relative to d	−1	0	...	$+\frac{f}{g} \times \frac{g}{f}$
Total motion	0	+1	...	$1 + \frac{fg}{gf} = 2$

Whence it is shown that the left-hand wheel turns twice as fast as gear c. It is evident that in the case of the epicyclic bevel-gear train, direction of rotation cannot be definitely defined as clockwise or counterclockwise, as is done with spur gears, so that direction of rotation in any given case must be determined by inspection.

139. Speed Reducers.—The turbine with its high speed is more efficient than the low-speed engine; and the small high-speed motor is more efficient, just as powerful, and less costly than the larger one of slow speed. The use of the high-speed prime mover, however, calls for a compact and efficient device that will decrease or step down the speed to a value suitable for practical conditions, *i.e.*, to a speed or range of speed at which the driven member operates economically and efficiently. Belt, chain, and rope drives, open gearing, or any combination of these methods of speed reduction may be found in practically every industry. Although these methods of speed reduction possess certain individual merits, they are also accompanied by many objectionable features. Since the maximum practical ratio for a belt drive is about 6 to 1, and that of a chain or spur-gear drive 8 to 1, it is evident that great reduction is possible only by the use of a series of reducing units. Satisfactory service on double-belt transmission requires a minimum pulley diameter of 12 in., triple and quadruple belts requiring pulleys of larger diameter. It is evident then that transmission that requires considerable speed reduction would necessitate the use of a great deal of space.

Open drives such as the above are also difficult to lubricate properly and are subject to wear and deterioration, particularly

where exposed to dirt, dust, grit, moisture, and acid and alkaline fumes. Another very important consideration is that of safety. If not carefully guarded, open drives are a constant source of danger on account of rapidly moving exposed parts.

The advent of the efficient high-speed prime movers, and the disadvantages of the prevalent methods of speed reduction resulted in the development of units known generally as *speed reducers*. These units vary in design, but they all consist essentially of a train of gears totally enclosed in a cast-iron housing. Speed reductions in ratios as great as 300 to 1 are accomplished in standard constructions, and special construction makes ratios much larger than this possible.

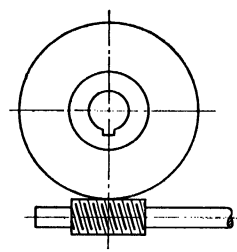


FIG. 200.

Speed reducers fall into four general classifications: (*a*) worm gear, (*b*) spur gear, (*c*) internal gear, and (*d*) planetary gear.

a. The Worm-gear Reducer.—The worm type of reducer (Fig. 200) is commonly used to secure high-speed reductions between shafts that are at right angles to each other. Combined bevel- and spur-gear reducers are also used for angle drives.

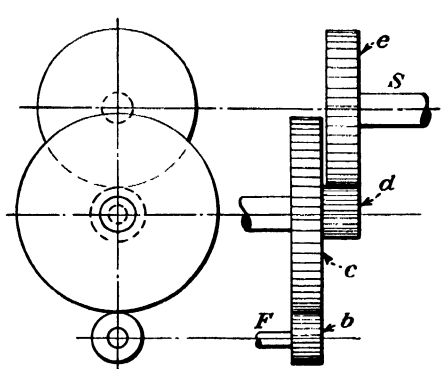


FIG. 201.

b. The Spur-gear Reducer.—The ordinary spur-gear type of speed reducer consists of a train of spur gears arranged as in Fig. 201 or 202, each having its peculiar advantages. The arrangement in Fig. 201 is commonly found in large steam-turbine drives in which double-helical gears are used. The most used

arrangement is that of Fig. 202, the outstanding advantages being compactness and reduction in a straight line.

The arrangement shown in Fig. 202 reduces in a straight line, the low-speed shaft S being to all outward appearances a con-

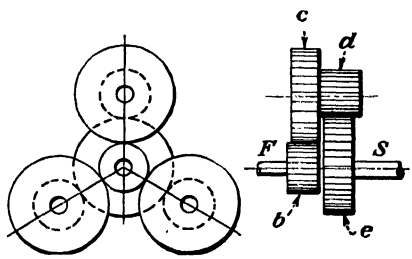


FIG. 202.

tinuation of the high-speed shaft F. The high-speed pinion b meshes usually with three idler gears c making contact at 120-deg. points of the pinion. This three-point application of the load is for the purpose of keeping down the size of a single-tooth load. Keyed to the same shaft of each of these gears is another smaller gear d, which in turn drives a larger gear e making contact at 120-deg. points of the larger gear. This large gear is keyed to the slow-speed shaft S. Any reduction required may be attained by proper gear ratios, or by making series combinations of the units just described.

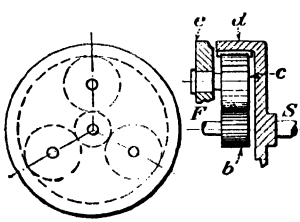


FIG. 203.

c. The Internal-gear Reducer.—In the internal-gear type of reducer (Fig. 203), the small high-speed pinion b makes three-point contact with three intermediate gears c, which in turn make three-point contact with an internal gear d. In this case, however, the intermediate gears c are *journaled* in the frame e, and the internal gear d, which is keyed to the low-speed shaft S, is free to rotate. Greater speed reduction is obtained by making series combinations of this single stage.

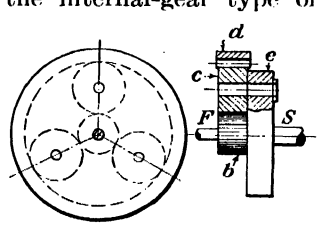


FIG. 204.

d. The Planetary-gear Reducer.—In the planetary type of speed reducer, the gear train contains an epicyclic pair arranged as shown in Fig. 204. Here, as in the previous cases, the high-speed pinion *b* meshes with the three intermediate gears *c*. The gears *c* are *journaled* on studlike shafts that protrude from a plate *e*, which is keyed to the low-speed shaft *S*. As the three intermediate gears *c* rotate about their centers, they mesh with a stationary internal gear *d*. This causes the entire unit, consisting of the three intermediate gears, the plate in which they are journaled, and the low-speed shaft, to rotate. Greater speed reduction is obtained by making series combinations of this single stage.

PART II

KINEMATICS AND DYNAMICS OF MACHINERY

CHAPTER X

VELOCITIES IN MACHINES

In this chapter two general methods will be developed for determining the velocity of any point in a machine having constrained motion, when the velocity of any other point is known. These two methods will be called the *method of relative velocities* and the *method of instantaneous centers;* they may be applied either graphically or analytically, but are more often applied graphically. In the case of many problems, however, a solution is more easily effected either analytically or by a combination of graphical and analytical work.

The reasons for making a velocity analysis vary somewhat with the type of machine analyzed. In the case of high-speed machinery, it is important to have a knowledge of the inertia forces set up in a given machine. In order to determine these inertia forces, the accelerations of certain points of the machine must be determined, and this usually requires that a complete velocity analysis be made first. In many cases some idea of the accelerations may be obtained from a velocity analysis alone. In the case of the quick-return mechanisms used on shapers, slotters, and other machine tools, a velocity analysis will show the cutting and return speeds of the tool. It is frequently more convenient to determine the mechanical advantage of a given mechanism by means of a velocity analysis than by means of a force analysis.

140. Vector Quantities.—The fundamental quantities of mechanics may be divided into two groups, since they are either vector or scalar quantities. A vector quantity has both magnitude and direction, whereas a scalar quantity has magnitude only. A vector is a straight line, the length of which is proportional to the magnitude of the quantity it represents, and the direction of which (indicated by means of an arrowhead on the

line, or by some other similar means) is the same as that of the quantity. The initial point on the straight line is called the origin of the vector, and the final point is called its extremity. These are sometimes called the tail and head of the vector respectively.

Some of the fundamental quantities of mechanics, such as force and velocity, are obviously vector quantities. Many of the others that are not so obviously vector quantities may, nevertheless, be represented by means of vectors, and actually are thus represented in the study of vectorial mechanics. The latter subject, however, is not particularly well suited to the needs of the engineer, and for this reason vector representation

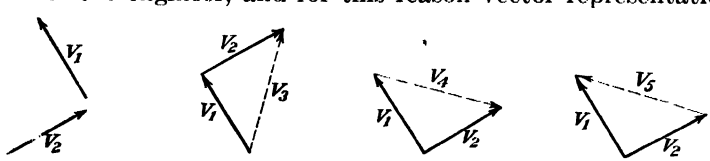


FIG. 205.

will be used in what follows only for those quantities where it is a distinct aid to visualization.

It is shown in textbooks on vector analysis that it is possible to manipulate vectors in much the same manner as algebraic quantities. That is, a set of operating rules may be formulated by means of which operations very similar to those commonly performed upon algebraic quantities (such for example as addition, multiplication, differentiation, etc.) may also be performed upon vectors. The only operations suited to the purposes of this text, however, are those known as vector addition and vector subtraction.

The addition and subtraction of two vectors may be understood by referring to Fig. 205. The vectors shown at *a* are added at *b*, the vector V_3 being their sum. It should be noted that the origin of the vector representing the sum is coincident with the origin of the first vector of the sum, and that its extremity is coincident with the extremity of the last vector. The result of vectorially subtracting V_1 from V_2 is shown at *c*, and the result of subtracting V_2 from V_1 is shown at *d*. The vectors representing the results of the last two operations are equal in magnitude and opposite in direction.

The operations described above may be indicated as follows:

$$\begin{aligned} V_1 \nleftrightarrow V_2 &= V_3 \quad \text{(Fig. 205}b\text{)} \\ V_2 \rightarrow V_1 &= V_4 \quad \text{(Fig. 205}c\text{)} \\ V_1 \rightarrow V_2 &= V_5 \quad \text{(Fig. 205}d\text{)} \end{aligned} \tag{45}$$

The signs $\nleftrightarrow$ and $\rightarrow$ indicate that the addition and subtraction are vectorial and not algebraic.

The fact that the order in which two vectors are added has no effect upon their sum is brought out in Fig. 206. It is of course evident that the same statement applies to any number of vectors.

It should not be inferred from the foregoing discussion that in order to obtain the sum of a number of vectors it is necessary for all of them to lie in the same plane. The operations of addition and subtraction may be performed upon vectors in space in exactly the same manner as upon vectors lying in the same plane. In order to perform these operations graphically, however, considerable labor is involved in the way of projection, and it is therefore better when dealing with vectors in space to employ analytical methods.

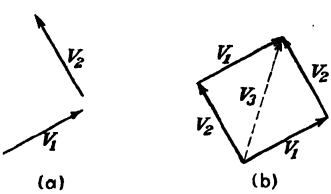


FIG. 206.

MOTION CONCEPTS

141. Motion.—As stated in Chap. I most machine members are constrained to have plane motion; *i.e.*, their points move in parallel planes. In such cases it is possible to study the motion of a machine member by means of its projection in a plane parallel to the plane of motion; that is, by means of a working drawing of the machine of which the member is a part. This is due to the fact that all the points of the member situated on any one line perpendicular to the plane of motion have exactly the same motion.

142. Position.—Motion is change of position, and hence the first requirement in the study of motion is a clear and definite means of defining the position of a rigid body at any instant.

The position of a rigid body in space may be defined by defining the positions of any three of its points. Two points are not sufficient, for in this case the body can rotate about the axis formed by the straight line joining the two points without in any

way disturbing the position of the latter. If a rigid body is constrained to have plane motion, two points are sufficient to determine its position, provided, however, that the points are not situated on a line perpendicular to the plane of motion.

The position of a point in space may be defined:

1. In terms of its distance from three fixed rectangular axes.

2. In terms of the length and direction of a line drawn to it from a single fixed point, the direction of the line being measured by means of the angles that it makes with three fixed rectangular axes passing through the fixed point.

The position of a point in a plane may be defined:

1. In terms of its distance from a pair of fixed rectangular axes.

2. In terms of the length and direction of a line drawn to it from a single fixed point in the plane, the direction of the line being measured by means of the positive angle that it makes with some fixed line in the plane.

In graphical representation the position of a point in a plane is usually referred to the paper on which the representation is made and is indicated by means of a letter or some other suitable character.

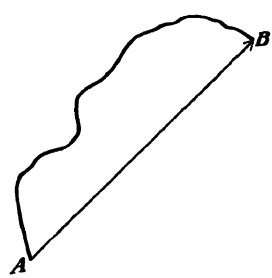


FIG. 207.

143. Motion of a Point.—The motion of a rigid body is defined in terms of the motions of its points. In order, therefore, to study machine motions it is first necessary to consider the motion of a single point. Because of the fact that machine members usually have plane motion, the discussion that follows will deal largely with the motion of a point in a plane, and it will be assumed that the motions dealt with hereafter are *plane motions* unless otherwise stated.

a. Displacement of a Point.—Referring to Fig. 207, if a point moves from A to B along any path whatsoever (as indicated by the curve), its change of position is measured by the straight line AB; *i.e.*, it is the distance AB measured along the straight line in the direction AB. This change of position is called the *displacement* of the point. Having both magnitude and direction, the displacement is obviously a vector quantity. It should be noted that, although the total distance traveled by the point

is the length of the curve AB, its change of position is the straight line AB.

Instead of representing the two positions of the point merely by the letters A and B, let these positions be defined in terms of the lengths and directions of the lines drawn to them from some fixed point O as shown in Fig. 208. The position of A relative to O is given by the straight line OA, and the position of B relative to O by the straight line OB. These two position lines are vector quantities, having as they do both magnitude and direction. The displacement AB therefore may be considered the vector change of the moving point's position. It is evident from the figure that

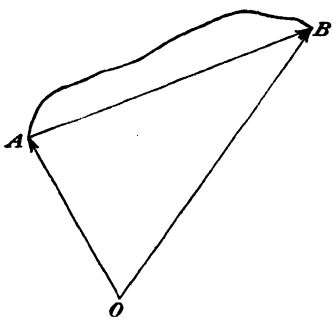


FIG. 208.

$$\overline{OB} = \overline{OA} \leftrightarrow \overline{AB}$$

b. Velocity of a Point.—The concept of velocity is introduced by considering the time required for a given displacement. Referring to Fig. 209 let a point be moving along the plane curve AB, and let it be at position 1 at time T_1 and at position 2 at time $(T_1 + \Delta T)$. The displacement of the point during the interval is ΔL. Its *average velocity* during the interval is a vector quantity having the same direction as ΔL and a magnitude as given by the ratio $\Delta L/\Delta T$. The average *speed* of the point during the same interval is $\Delta S/\Delta T$.

The average velocity of a moving point during an infinitesimal time interval is called its *instantaneous velocity*. In this case the quantities ΔS and ΔL become the infinitesimals dS and dL, respectively. Since the latter are practically equal in length, the instantaneous speed ds/dt is the same as the magnitude of the instantaneous velocity, and, since the direction of dL is practically the same as that of the tangent, the direction of the instantaneous velocity is also the same as that of the tangent.

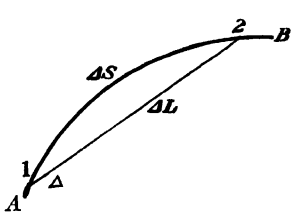


FIG. 209.

The quantity dS is equal to the product $R \cdot d\theta$, where R is the radius of curvature and $d\theta$ is the angle between the two radii drawn from the center of curvature to the ends of the infinitesimal segment dS. The magnitude of the instantaneous velocity is therefore equal to $R\frac{d\theta}{dt}$. If the radius of curvature be thought of as moving with the point, the quantity $d\theta/dt$ may be considered its angular velocity.

The term *velocity* is usually intended to mean *instantaneous velocity* and will be so understood hereafter.

c. General Relations.—It has been brought out that the position, displacement, and velocity of a moving point are all vector quantities or may be defined in terms of vector quantities. The position of the moving point relative to any fixed point is, at any instant, defined by a vector, the origin of which is at the fixed point, and the extremity of which is at the moving point. The displacement of a moving point is the vector change of its posi-

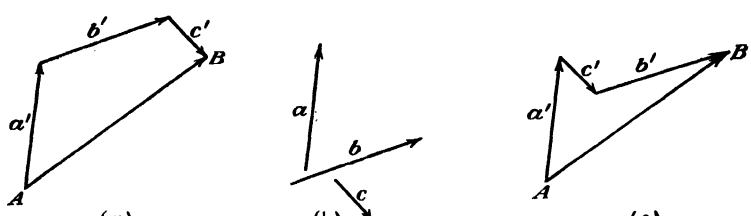


FIG. 210.

tion. The velocity of a moving point is the time rate of the vector change of its position. The acceleration of a moving point will be shown later to be the time rate of the vector change of its velocity.

144. Combined Motions. *a. Displacements.*—If a point is given a number of successive displacements, the total displacement of the point will be equal to the vector sum of the several displacements. For example, let a point situated at A in Fig. 210*a* be given displacements that are equal to the vectors a, b, and c shown at Fig. 210*b*. The path of the point is made up of the displacements a', b', and c', equal to the vectors a, b, and c. The total displacement is evidently the vector AB, which is the vector sum of a, b, and c. As shown at Fig. 210*c* the total displacement will be the same regardless of the order in which the intermediate displacements are taken.

The next step is to consider displacements that occur simultaneously. In order to illustrate the case of simultaneous displacements, let the example shown in Fig. 211 be considered. A man rowing from A to B straight across a river in which a strong current is flowing has not only to cover the distance between the two banks, but also to avoid being carried downstream by the motion of the water. If he were to make no attempt to resist the stream, *i.e.*, if he were to row at right angles to the direction of the current, the actual motion of the boat would be in some diagonal direction such as AC. The total displacement of the boat AC will be the result or sum of the displacements AB, given it by the man, and BC, given it by the current. The two component displacements occur simultaneously, and hence the path of the boat will be along the diagonal indicated, but the total displacement AC will be exactly the same as though it were given the displacements AB and BC separately.

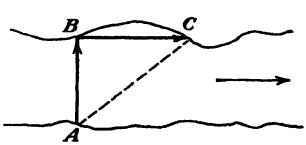


FIG. 211.

It is evident from the foregoing example that if a point is given several displacements simultaneously, the magnitude and direction of the total displacement are exactly the same as though they were given the several displacements separately. Similarly a given displacement may, for the purposes of analysis, be considered the combined effect of several displacements which occurred simultaneously, provided, of course, that the vector sum of the component displacements is equal to the given displacement.

b. Velocities.—Now let it be assumed that a point P is given several simultaneous infinitesimal displacements during an interval of time dt. The total displacement dL of the point during this interval is, of course, the same as though the several simultaneous displacements occurred separately, and it is given by their vector sum. The velocity of the point may be represented by a vector having the same direction as the displacement dL, and a magnitude as given by the ratio dL/dt. It is also possible to think of the point as having component velocities corresponding to each of the several simultaneous infinitesimal displacements. In each case the component velocity may be represented by a vector having the same direction as its corresponding displacement, and a magnitude as defined by the ratio of the magnitude of its corresponding displacement to the

quantity dt. Since the time required for each of these displacements is the same, the total velocity and all the component velocities will be proportional to their corresponding displacements in the same ratio. It follows that, if the several component velocities are added vectorially, a figure will be obtained that is similar to that obtained by adding the simultaneous infinitesimal displacements vectorially. The total velocity dL/dt, therefore, is equal to the vector sum of the component velocities. The velocity polygon may be thought of as a large scale drawing of the infinitesimally small displacement polygon. This conception is a valuable aid to visualization. Since a given displacement may be considered the combined effect of several simultaneous displacements, so also may the velocity of a point be considered the combined effect of several component velocities.

c. Acceleration.—The method of dealing with accelerations in connection with combined motions will be shown later to be similar in every respect to that employed for displacements and velocities; *i.e.*, the total acceleration of a point may be considered as made up of a number of component accelerations the vector sum of which gives the total acceleration.

145. Relative Motion. *a. Relative Displacement.*—It was shown in Art. 143 that the displacement of a moving point is the vector change of its position relative to some fixed point. Instead of referring the position of a moving point to some fixed point, it may, at any given instant, be referred to the position occupied by some other moving point at the same instant. The position of one moving point relative to another is defined in exactly the same terms as those given in Art. 143 for the position of a moving point relative to some fixed point, and it is, therefore, a vector quantity of the same kind. Referring to Fig. 212*a*, let the two moving points A and B be considered. The position of the point B relative to the point A is given by the line AB measured in the direction AB. If the points are given the displacements S_a and S_b to the positions A' and B', the position of B relative to A will then be given by the line $A'B'$. Now let BB'' be drawn parallel and equal to S_a. Since by construction $A'B''$ is equal to the relative position vector AB, it follows that $B''B'$ $(= S_{ba})$ is the vector change of relative position. By analogy this vector is called the displacement of point B relative to the point A. The displacement of one point relative to another, therefore, is defined as the vector change of the relative

position of the two points. It can be seen from the figure that

$$S_b = S_a \leftrightarrow S_{ba}$$

The construction at Fig. 212*b* shows that the displacement S_{ab} of A relative to B is equal and opposite the displacement of B relative to A.

b. Relative Velocity.—Now let the displacement of two points A and B, during an infinitesimal interval of time dt, be considered. The infinitesimal displacements themselves are too small to be shown in a figure, but if the velocities of A and B are known, they

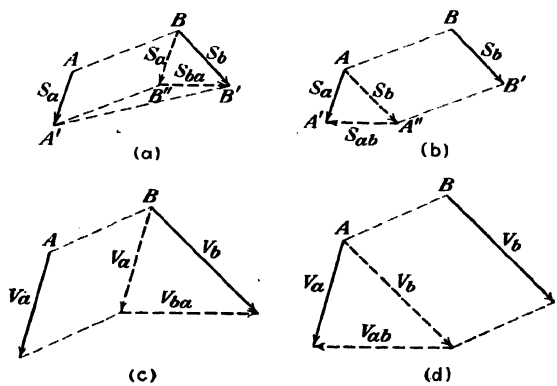


FIG. 212.

may be represented by vectors as shown at *c* and *d* in Fig. 212, and these vectors will be proportional to the actual infinitesimal displacements. It follows that the vector V_{ba} is proportional to the infinitesimal displacement of B relative to A. By analogy this vector is said to represent the velocity of B relative to A. It is evident from the figure that

$$V_b = V_a \leftrightarrow V_{ba} \tag{46}$$

The foregoing is a fundamental principle of extreme importance and should be thoroughly understood before proceeding any further. The expedient of considering the velocity vectors as large-scale representations of the actual infinitesimal displacements is of assistance in visualizing relative motion. Thus the "displacement" V_b may be considered the net effect of two simultaneous "displacements" V_a and V_{ba}. The final displacement would be the same if the two simultaneous displacements were

taken separately, and the point B may be thought of as moving first with A, *i.e.*, parallel to V_a, to the end of "displacement" V_a, and thence on alone to its final position at the ends of "displacements" V_b and V_{ba}. During the first part of the journey, its position relative to A remains unchanged, and therefore all the change of relative position must take place during the second part. It follows that V_{ba} must be the "displacement" of B relative to A, and if the vectors are again taken to represent velocities, the vector V_{ba} must represent the relative velocity.

c. Relative Acceleration.—The relative displacement of two points has been defined as the vector change of their relative position, and their relative velocity as the time rate of the vector change of their relative position. By analogy the relative acceleration of two points may be defined as the time rate of the vector change of their relative velocity. This will be discussed in a later chapter.

d. General Considerations.—It has been brought out in the foregoing discussion that the relative motion of any two points is, in all its phases, merely the vector difference of the absolute motions of the points. It will be shown later that relative angular motion may be defined as the difference of two absolute angular motions, only in this case the difference is algebraic, angular motion being considered a scalar quantity. This concept of relative motion will be used almost exclusively in the discussions that follow. Other concepts are possible, but they are likely to lead to confusion in dealing with certain types of problems.

146. Angular Motion.—The angular position of a line in a plane is defined in terms of the angle that the line makes with some arbitrarily chosen reference line in the plane. In order to secure definiteness it is customary to choose positive directions for both the line and the reference line as indicated by the arrows in Fig. 213. The angular position of the line is then defined as the angle measured to it in a positive sense (usually counterclockwise) from the reference line. The angular position of the line AB, for example, is the angle θ_1, and that of the line CD is the angle θ_2.

The angular motion of a line is the change of its angular position. If, for example, the line AB in Fig. 214 moves to a new position as indicated by the letters $A'B'$ it is said to have moved or turned through an angle $\Delta\theta$. This angle is called the angular

displacement of the line. In defining angular displacement it is unnecessary to choose a positive direction along the line displaced, definiteness being secured by the order in which the line takes up its two positions.

The angular displacement of a line does not completely define its change of position. For example, the line AB might have

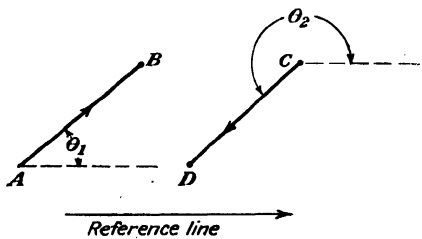


FIG. 213.

been moved to some other position as indicated by the letters $A''B''$ and still have been given the same angular displacement $\Delta\theta$. Its final angular position, however, would be the same regardless of whether it were moved to $A'B'$ or $A''B''$.

If the angular displacement of a line takes place during the time interval ΔT, then the *average angular velocity* of the line during the interval is given by the ratio $\Delta\theta/\Delta T$. If an infinitesimal time interval be considered, the ratio $d\theta/dt$ gives the *instantaneous angular velocity*. The term *angular velocity* usually means *instantaneous angular velocity* and will be so understood hereafter. Angular velocity is usually represented by the Greek letter ω.

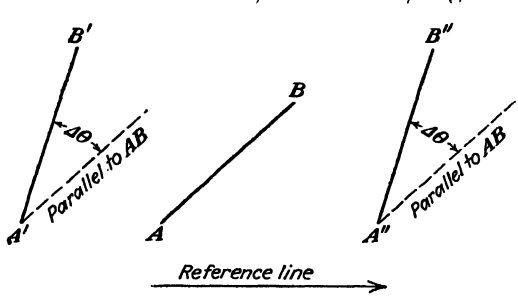


FIG. 214.

If, during the time interval ΔT, the angular velocity of a line changes by an amount $\Delta\omega$, then the quantity $\Delta\omega/\Delta T$ is called the average angular acceleration of the line during the specified time

interval. For an infinitesimal interval $d\omega/dt$ is the instantaneous angular acceleration; $d\omega/dt$ is equal to the second derivative $d^2\theta/dt^2$. As before, the term *instantaneous* is usually dropped. Angular acceleration is usually represented by the Greek letter α.

147. Relative Angular Velocity of Two Rigid Bodies.—Referring to Fig. 215, let the reference lines AB and CD be drawn on the projections of the rigid bodies 2 and 3, respectively, and let the fixed reference line OX be drawn. The angular position of the rigid body 2 may be defined by the angle ϕ_2; that of the rigid body 3 by the angle ϕ_3. Now let it be assumed that during the interval of time ΔT the two bodies are displaced as indicated by the dotted lines. The angular displacements of bodies 2 and 3 are $\Delta\phi_2$ and $\Delta\phi_3$, respectively.

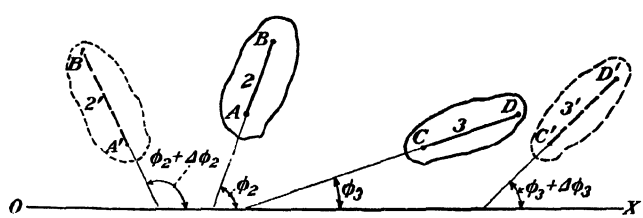


FIG. 215.

The angular positions of 2 relative to 3 at the beginning and at the end of the interval ΔT are given by the angles ϕ_{23} and $\phi_{2'3'}$ (not shown) respectively, where

$$\phi_{23} = \phi_2 - \phi_3$$
$$\phi_{2'3'} = (\phi_2 + \Delta\phi_2) - (\phi_3 + \Delta\phi_3)$$

The *change* $\Delta\phi_{23}$ of the angular position of 2 relative to 3 is

$$\begin{aligned}\Delta\phi_{23} &= \phi_{2'3'} - \phi_{23} \\ &= [(\phi_2 + \Delta\phi_2) - (\phi_3 + \Delta\phi_3)] - [\phi_2 - \phi_3] \\ &= \Delta\phi_2 - \Delta\phi_3\end{aligned}$$

and, dividing both sides of the equation by ΔT,

$$\frac{\Delta\phi_{23}}{\Delta T} = \frac{\Delta\phi_2}{\Delta T} - \frac{\Delta\phi_3}{\Delta T}$$

That is, the average angular velocity of 2 relative to 3 is given by the difference of the average absolute angular velocities. Passing to the case of an infinitesimal interval of time dt,

$$\frac{d\phi_{23}}{dt} = \frac{d\phi_2}{dt} - \frac{d\phi_3}{dt} \text{ or } \omega_{23} = \omega_2 - \omega_3 \tag{47}$$

where ω is a general term indicating angular velocity. In a similar manner it can be shown that

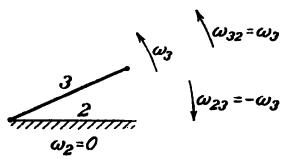


FIG. 216.

$$\omega_{32} = \omega_3 - \omega_2 = -\omega_{23} \tag{48}$$

That is, the angular velocity of 2 *relative to* 3 *is equal and opposite the angular velocity of* 3 *relative to* 2.

It is to be noted that the absolute angular velocity of a rigid body is merely a special case of relative angular velocity. Referring to Fig. 216, let the rigid body 3 have an angular velocity ω_3 in the direction shown, and let the rigid body 2 be assumed as fixed. The angular velocity

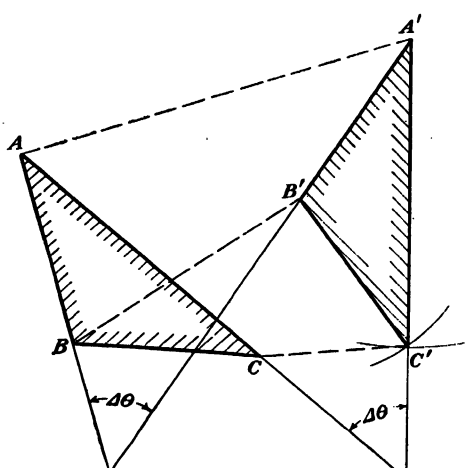


FIG. 217.

of 3 relative to 2 is

$$\omega_{32} = \omega_3 - \omega_2 = \omega_3 - 0 = \omega_3$$

also,

$$\omega_{23} = \omega_2 - \omega_3 = 0 - \omega_3 = -\omega_3$$

That is, the fixed body 2 may be thought of as having an angular velocity relative to the moving body 3 that is equal and opposite to the absolute angular velocity of the latter.

148. Displacement of a Rigid Body.—Referring to Fig. 217, let the rigid body represented by the outline ABC be given a displacement to the position $A'B'C'$ during the interval of time ΔT. It is evident from the figure that in general the different points of the body have different displacements. The displacement of the point A is, for example, much greater than that of the point B. If the displacements of two of the points (*e.g.*, A and B) are known, the displacement of any third point C can be determined by drawing arcs from A' and B' as shown.

Referring again to the figure, it can be seen that not only have all the points of the body been displaced, but also that the body as a whole has been turned through an angle; *i.e.*, it has been given an angular displacement. For example, after the side AB has been displaced, it makes an angle $\Delta\theta$ with its original position; and it can be seen that the two positions of any other line on the body make the same angle $\Delta\theta$ with each other. The ratio $\Delta\theta/\Delta T$ is the average angular velocity of the body during the interval of time ΔT.

METHOD OF RELATIVE VELOCITIES

149. Fundamentals of the Method of Relative Velocities.—In order to bring out the fundamental principles underlying the method of relative velocities, let it first be noted that the displacements of the points of the body shown in Fig. 217 all took place during the time interval ΔT and were, therefore, simultaneous. It was shown in Art. 144 that any displacement of a point may, for the purpose of analysis, be assumed to be made up of two or more simultaneous displacements, the vector sum of which is equal to the actual displacement. Referring to Fig. 218, let it be assumed that the body AB has been displaced to the position $A'B'$ during the time interval ΔT, and let it be further assumed that the displacement of any point of the body such as B is made up of the two simultaneous displacements, BB'', equal to the displacement AA' of some other point A, and $B''B'$, the displacement of B relative to A.

If the displacements BB'' and $B''B'$ are taken separately, the net result will be the same; *i.e.*, if A and B move to A' and B'', and then if B moves from B'' to B' along the arc as shown, the final positions of the two points will be the same as though the displacements occurred simultaneously. Since two points deter-

mine the position of a body having plane motion, it is evidently quite possible for the body to move as a whole along with the two points A and B in the manner just described, and it follows that if the body *is* moved in this manner the displacement of any third point will be made up of two displacements similar to those of B.

The vector equation for the displacement of the point B is

$$\overline{BB'} = \overline{BB''} \leftrightarrow \overline{B''B'} = \overline{AA'} \leftrightarrow \overline{B''B'}$$

Dividing both sides of the equation by ΔT,

$$\frac{\overline{BB'}}{\Delta T} = \frac{\overline{AA'}}{\Delta T} \leftrightarrow \frac{\overline{B''B'}}{\Delta T} \text{ or } V_{b_{avg}} = V_{a_{avg}} \leftrightarrow V_{ba_{avg}}$$

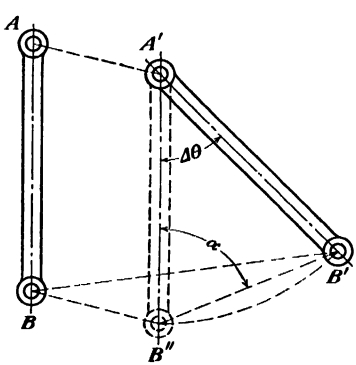


FIG. 218.

The magnitude of $V_{ba_{avg}}$ is given by

$$V_{ba_{avg}} = \frac{\overline{B''B'}}{\Delta T} = \overline{AB} \cdot \frac{2 \sin \frac{\Delta\theta}{2}}{\Delta T}$$

where $\Delta\theta$ is the angular displacement of the body. The direction $V_{ba_{avg}}$ makes an angle α with both the initial and final positions of the line AB. This angle is obviously the same for any other point of the body.

Now let it be assumed that the displacements all take place during the infinitesimal interval of time dt. The average velocities become instantaneous velocities; the angle α becomes, except for negligible quantities, equal to 90 deg.; and the quantity $\frac{2 \sin \frac{\Delta\theta}{2}}{\Delta T}$ becomes $\frac{d\theta}{dt}$, the angular velocity of the body.

Therefore

$$V_b = V_a \leftrightarrow V_{ba} \tag{49}$$

$$V_{ba} = \overline{AB} \cdot \frac{d\theta}{dt} = \overline{AB} \cdot \omega \tag{50}$$

That is,

1. The velocity of any point of a rigid body is given by the vector sum of the velocity of some other point and the velocity of the first point relative to the second.

2. The velocity of any point of a rigid body relative to any other point of the body is a vector quantity having a magnitude equal to the product of the angular velocity of the body and the distance between the points, and a direction at right angles to the line connecting the points.

Let it be noted that the first statement is merely a special case of the statement of the relation that exists between the velocities of two independent points and their relative velocity.

150. Applications of the Method of Relative Velocities. Points on the Same Link.—The simplest and most fundamental applica-

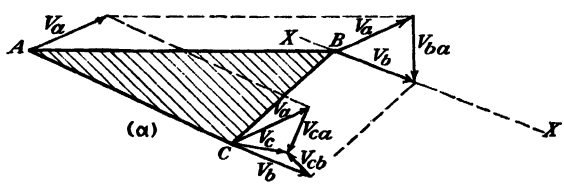


FIG. 219.

tion of the method of relative velocities is that shown in Fig. 219. The velocity of point A of the rigid body is assumed to be completely known, and it is also assumed that the direction of the velocity of the point B is along the line XX. The velocity of the point B is

$$V_b = V_a + V_{ba}$$

The direction of V_{ba} is perpendicular to the line AB. If, therefore, the vector V_a is laid off from B and a line drawn through its extremity at right angles to AB, the intersection of this line with the line XX will determine the velocity vectors V_b and V_{ba} as shown.

Now let it be required to determine the velocity of any third point C. First, let the two following vector equations be written:

$$V_c = V_a + V_{ca}$$
$$V_c = V_b + V_{cb}$$

The simultaneous solution of these two vector equations may be obtained as follows: Let the vectors V_a and V_b be laid off from the

point C, and let lines be drawn through their extremities perpendicular to AC and BC respectively. The intersection of these two lines locates the extremity of the vector V_c. If now the construction at C is compared with the preceding equations, it will be found to satisfy both of them, and the vector V_c must, therefore, represent the velocity of the point C.

The angular velocity of the rigid body is

$$\omega = \frac{V_{ba}}{AB} = \frac{V_{ca}}{AC} = \frac{V_{cb}}{BC}$$

151. Application to a Simple Mechanism. The Velocity Polygon.—In Fig. 220*a* is shown a four-link mechanism in which V_a, the velocity of point A on link 2, is assumed to be known and

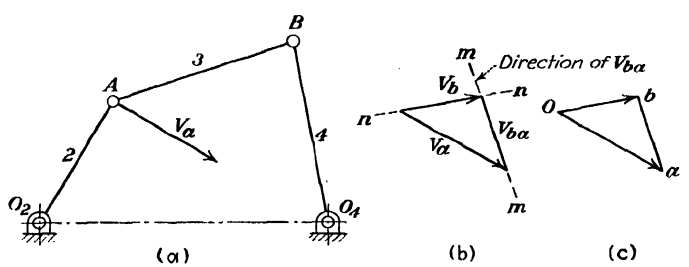


FIG. 220.

V_b, the velocity of a point on link 4, is required. The velocity of A has a magnitude determined from the angular velocity and length of link 2, ($V_a = \overline{O_2A} \cdot \omega_2$). It has a direction perpendicular to O_2A. The point A, being the center of a turning joint, is actually the position of a pair of coincident points, one on link 2 and one on link 3. These points must have the same velocity, as otherwise links 2 and 3 would separate at A. The velocity of one point on link 3 is therefore known as soon as V_a is known. Likewise B represents a pair of coincident points on links 3 and 4. Hence, since A and B may be considered as points on link 3, the analysis of the preceding article applies. Therefore $V_b = V_a + V_{ba}$. In the vector addition representing this equation, the sides of the triangle are:

V_a, known in magnitude and direction.

V_b, known in direction (perpendicular to OB).

V_{ba}, known in direction (perpendicular to BA).

The construction is shown in Fig. 220*b*. First the known vector V_a is laid off, to some scale. Next a line *mm* is drawn from the extremity of V_a in the known direction of V_{ba}. From the origin of vector V_a a line *nn* is drawn in the known direction of V_b. The intersection of these lines determines the magnitudes of V_b and V_{ba}. Making the vector addition on the drawing of the mechanism at the point whose velocity is required, as shown in Fig. 219, has certain advantages but results in confusion of lines in complex mechanisms. Hence, in applying the method of relative velocities, such constructions will generally be made off the figure, as in Fig. 220*b*.

In Fig. 220*c* is shown the same diagram with a simplified notation. In this diagram it is observed that the absolute-velocity

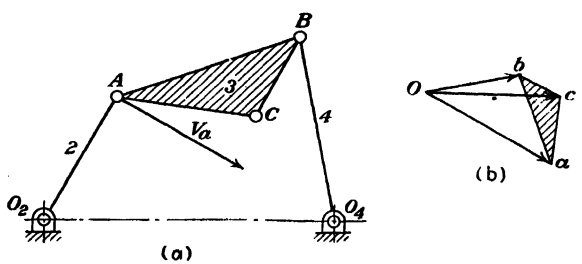


FIG. 221.

vectors radiate from a pole *o*, and the relative-velocity vector is represented by the line joining the extremities of the absolute-velocity vectors. The small letters at the extremities of the vectors correspond with the capital letters on the mechanism. Thus *oa* represents V_a, *ob* represents V_b, and *ab* and *ba*, in the order given, represent V_{ba} and V_{ab}, respectively. This diagram, Fig. 220*c*, is known as the *velocity polygon*. In drawing velocity polygons it is important to keep in mind that all absolute-velocity vectors are lines radiating from the pole, whereas all relative-velocity vectors are lines joining the ends of absolute-velocity vectors. In what follows arrows representing the sense will be placed on absolute-velocity vectors but will be omitted from relative-velocity vectors for convenience in reading these vectors in either sense as desired.

In Fig. 221*a* the four-link mechanism of Fig. 220*a* has been redrawn with link 3 extended to include a third point *C*, the velocity of which is to be determined. First, a pole *o* is chosen

(Fig. 221*b*), and, with *oa* (the velocity of *A*) known, *ob* (the velocity of *B*) is determined, as in Fig. 220*c*.

To determine the velocity of point *C*, consider the following equation:

$$V_c = V_a + V_{ca}$$

In this equation, V_a is known in both magnitude and direction, and V_{ca} is known in direction only (perpendicular to the line *CA*). If from the extremity of vector *oa* in the velocity polygon (Fig. 221*b*) a line is drawn perpendicular to *CA*, it is known from the equation that the extremity of vector *ac* will lie on this line. The complete determination of V_c, however, requires the use of another equation

$$V_c = V_b + V_{cb}$$

In this equation, V_b is known in both magnitude and direction, and V_{cb} is known in direction only (perpendicular to the line *CB*). If, therefore, from the extremity of vector *ob* in the velocity polygon, a line is drawn perpendicular to *CB*, it is known from the equation that the extremity of the vector *bc* will lie on this line. The intersection *c* of the two lines perpendicular respectively to *AC* and *BC* determines the magnitudes of the relative velocity vectors *ac* and *bc* and at the same time the direction and magnitude of vector *oc*, which represents V_c, the velocity of *C*. This method of determining the velocity of *C* is essentially the same as that employed in connection with Fig. 219; it has been described in detail to bring out the manner in which the velocity polygon may be used to reduce graphical work.

In the velocity polygon (Fig. 221*b*), the line *ab* representing V_{bn} was drawn perpendicular to the line *AB*. The lines *ac* and *bc* were drawn perpendicular respectively to *AC* and *BC*. Thus, the three sides of the shaded triangle *abc* of the velocity polygon are perpendicular respectively to the sides of the link *ABC*, and it follows that the triangle *abc* in the polygon is similar to the triangle *ABC* in the mechanism. For this reason, the triangle *abc* in the velocity polygon is known as the *velocity image* of the link *ABC*. Each line or link in the mechanism has its image in the velocity polygon. Lines *ab*, *bc*, and *ac* are images of lines *AB*, *BC*, and *AC* respectively. Line *oa* is the image of O_2A (link 2), *ob* is the image of O_4B (link 4), etc. The image of link 1

(the fixed link) is the pole o, the point of zero velocity, since link 1 has no motion.

152. Application to Cams and Gears.—In the cam mechanism (Fig. 222), which can also be taken for the purposes of analysis as representing a pair of gear teeth, link 3 is driven by link 2. At the instant under consideration, point A_2 of the driver is in contact with point A_3 of the follower. From the angular velocity of the driver (link 2), the linear velocity V_{a_2} of point A_2 can be found. That is, $V_{a_2} = \overline{O_2A_2} \cdot \omega_2$. The problem is to find the velocity V_{a_3} of point A_3 on the follower. It is known that

$$V_{a_3} = V_{a_2} \nleftrightarrow V_{a_3a_2}$$

Velocities V_{a_2} and V_{a_3} are known to be perpendicular respectively to O_2A_2 and O_3A_3. To draw the velocity polygon, the direction

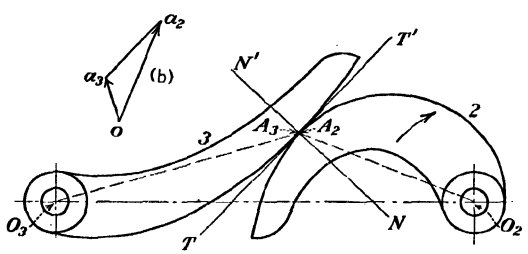


FIG. 222.

of $V_{a_3a_2}$ must be known. Since 2 and 3 are rigid links, there can be no motion in the direction of the normal NN'. Therefore A_2 and A_3 can move relative to each other only along the common tangent TT'. The polygon is shown in Fig. 222*b*. Vector oa_2 is laid off equal to the known velocity V_{a_2}. A line is drawn from o in the known direction of V_{a_3} and a line from a_2 in the known direction of $V_{a_3a_2}$ (parallel to the tangent TT'). The intersection of these two lines at a_3 gives the magnitudes of V_{a_3} and $V_{a_3a_2}$. Now with the velocity of one point on 3 known, the velocity of any other point on the link, or the angular velocity of the link, can easily be determined.

153. Application to a Complex Mechanism.—In Fig. 223*a* is shown the skeleton outline of a toggle press. Link 2, the driving link, has a constant angular velocity. Link 8 is the last driven link. If crank O_2A has a length of 10 in. and an angular velocity

of 1 radian/sec., the point A will have a velocity of 50 f.p.m. Knowing V_a, the problem is to determine V_f [the velocity of the ram (link 8)], by the method of relative velocities. The mechanism is shown at about the beginning of its stroke, where V_f is large compared with its value during the pressure portion of the stroke.

Figure 223*b* is the velocity polygon, which was drawn beginning with V_a (equal to *oa*) and progressing through the mechanism to V_f (equal to *of*). A velocity scale of 1 in. = 50 f.p.m. was used, and V_f on being scaled off was found to be equal to 73 f.p.m. for the configuration of the mechanism shown. Note that V_{fa} (= *af*) has no practical significance here, but was drawn simply to show how the relative velocity of two such points (A and F) could be determined from the polygon.

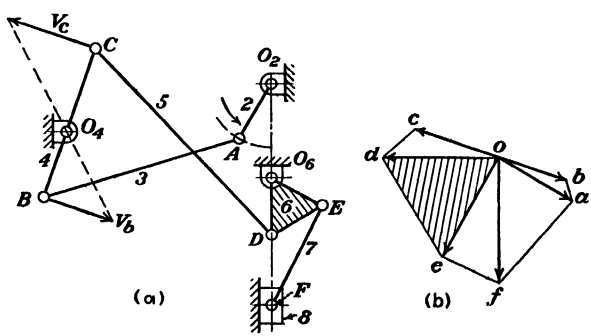


FIG. 223.

154. Methods of Attack.—The solution of a velocity problem requires in general the exercise of a certain amount of ingenuity, as almost every problem is more or less a special case. In general the best method of obtaining a solution depends largely upon the conditions of the particular problem at hand. By way of illustration let the mechanism shown in Fig. 224 be considered. The velocity of the point A is known, and the velocities of the points B and C are required. The velocities V_b and V_{ba} are determined by the methods used in the preceding articles. In the determination for point C, however, these methods break down because of the impossibility of securing intersections. In order to determine V_c, let it first be noted that

$$V_{ba} = \overline{AB} \cdot \omega_3 \text{ and } V_{ca} = \overline{AC} \cdot \omega_3$$

Therefore,

$$\frac{V_{ca}}{V_{ba}} = \frac{AC}{AB}$$

The vector V_{ca}, being proportional to V_{ba} in the ratio AC/AB, may be determined by means of the ordinary similar triangle construction shown at b in the figure. This construction may, if desired, be made on the drawing of the mechanism itself as shown. The relative velocity V_{ca} having been determined, the remainder

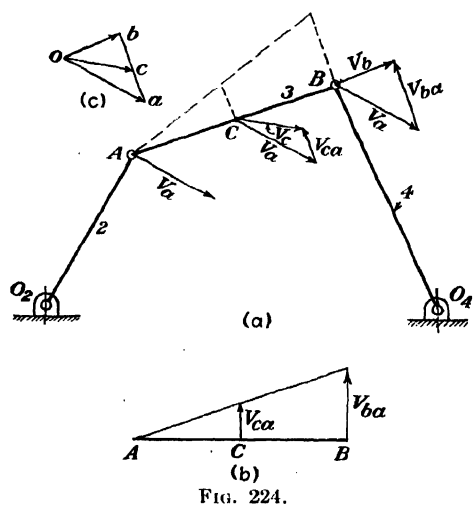


FIG. 224.

of the solution for V_c follows without difficulty. The velocity polygon is shown at c.

155. Practical Considerations.—In the velocity analysis of a machine, it is seldom the case that the velocities are required for a single position only of the mechanism. Usually the velocities must be determined for a sufficient number of positions to show just how the motions of the machine members vary during one complete cycle. In such cases, much time and trouble can be saved by giving a little consideration to the method of attack before actually starting the problem; and, as previously stated, practically every problem may profitably be considered as a special case. The problem given in the following article is intended as an illustration of the foregoing statements; it is not

in any sense intended to illustrate a specific method of procedure applicable to any large number of different problems, because in general no such method exists.

156. Velocity Analysis of a Crank Shaper.—Let it be required to make a velocity analysis of the crank shaper shown in Fig. 225. In order to vary the stroke of the ram the length of the crank arm may be varied by the bevel gears and screw shown in the figure. In order to avoid extended discussion, the analysis will be made here for one length of crank arm only. This involves the determination of the velocities of the several members of the

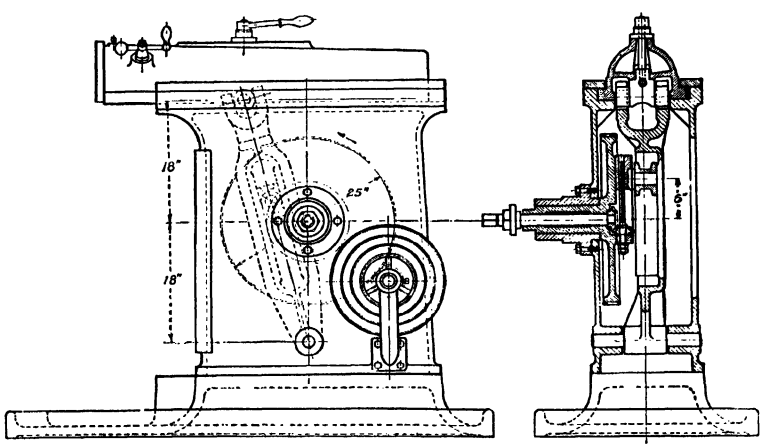


FIG. 225.

machine for all positions during one complete cycle; *i.e.*, during one revolution of the crank. The general method of procedure is to make velocity determinations for a number of positions during the cycle and to approximate the velocities corresponding to the intermediate positions from the data thus obtained.

The crank gear of the shaper under consideration is assumed to turn at a constant angular velocity of $10\frac{1}{2}$ r.p.m. The length of the crank arm is 6 in. It is required to determine the velocity of the ram for all positions of the crank, the mechanical advantage of the mechanism, and any other items of interest that may appear as the analysis progresses.

a. Preliminary Analysis.—The first step in the analysis is to choose a suitable scale (a scale of $1\frac{1}{2}$ in. per ft. was chosen for the original drawing) with which to draw the skeleton outline

of the mechanism in any convenient position, as shown in Fig. 226*a*, and to calculate the velocity of the crankpin as follows:

$$V_{a_2} = \frac{6 \times 2\pi}{12} \times 10\frac{1}{2} = 33 \text{ f.p.m.}$$

The requirement now is to determine V_{b_5}, the velocity of the cutting tool. After choosing a suitable scale (a scale of 1 in. = 30 f.p.m. was chosen for the original drawing), the velocity V_{a_2}

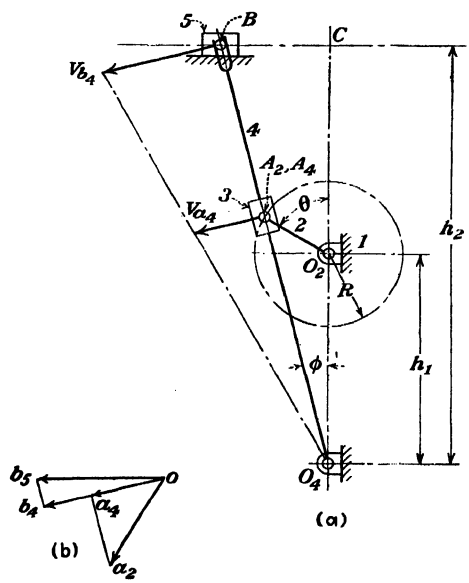


FIG. 226.

of the center of the crankpin may be laid off as the vector oa_2 in the polygon at b in the figure. Now let the velocity of the point A_4 be considered. A_4 is a point on link 4 coincident with the point A_2 on link 2 and may be thought of as being situated on link 4 directly beneath the point A_2. Since link 4 rotates about the fixed center O_4, the velocity V_{a_4} will have a direction perpendicular to the line O_4A_4, and this direction may therefore be laid off as oa_4 in the polygon. The relative motion of A_2 and A_4 is along the center line of the slide, which in this case is the line O_4A_4. If, therefore, a line is drawn from the extremity of vector oa_2 and parallel to the line O_4A_4, the intersection of this

line with the line oa_4 will determine the point a_4 and therefore the magnitudes of the velocities V_{a_4} and $V_{a_4a_2}$. The situation at B is similar. The point on link 4 situated at B has a velocity of which the direction is perpendicular to the line O_4B. Since link 4 rotates about a fixed axis, the velocity V_{b_4} is proportional to V_{a_4} in the ratio O_4B/O_4A, and may be determined by the similar-triangle method shown on the figure and drawn as ob_4 in the polygon. The ram, link 5, is constrained to move in a horizontal direction, *i.e.*, it has a motion of horizontal translation, and the point B_5 will, therefore, also move in a horizontal direction. The relative motion of B_4 and B_5 is along the center line of the fork, *i.e.*, along the line O_4B, from which it follows that the velocities $V_{b_5b_4}$ and V_{b_5} may be determined by drawing a line through the extremity of ob_4, in the polygon parallel to the line O_4B, to intersect the horizontal line ob_5 drawn from the pole o. The vector ob_5 represents the velocity of the cutting tool.

b. Analysis for All Positions of the Crank.—The velocity determination having been made for a single position of the crank, ways and means may now be considered for reducing the labor involved in dealing with the other positions. As the next step in the solution, let the skeleton outline be redrawn as shown in Fig. 227 and the crankpin circle divided into 24 equal parts, starting at the extreme right where the ram is at the beginning of the cutting stroke. Now let the velocity polygon of Fig. 226 (which is the polygon for crank position 11) be redrawn for the present purpose, as shown at *b* in Fig. 227. If this polygon is turned through 90 deg., as shown at *c* in Fig. 227, it will be observed that oa_2 is parallel to the crank, that ob_4 is parallel to the rocker arm, and that ob_5 is in the direction of the vertical center line of the mechanism. This suggests the possibility of using a modified drawing of the mechanism itself as the velocity polygon. (A *revolved velocity polygon* such as shown in Fig. 227*c* could be drawn for any mechanism, and its use is recommended by some authors because of the convenience in drawing the vectors in the polygon parallel instead of perpendicular to the center lines of the links of the mechanism.)

The next step in the solution is to take the revolved polygon at *c* in Fig. 227 and superimpose it on the mechanism as shown by the heavy lines, with the pole o placed at the center O_4. The parallel dotted lines drawn from A and B show how the propor-

tion $oa_4/ob_4 = O_4A_4/O_4B_4$, necessary to locate point b_4, can be conveniently obtained in the scheme employed here. Now let the circle be drawn with O_4a_2 (the velocity vector V_{a_2}) as a radius, with numbered divisions corresponding to those of the crankpin circle. The velocity polygon for any other position of the crank, and hence the desired velocity V_{b_5} of the ram, can now be easily determined by lightly drawing in the new configuration of the

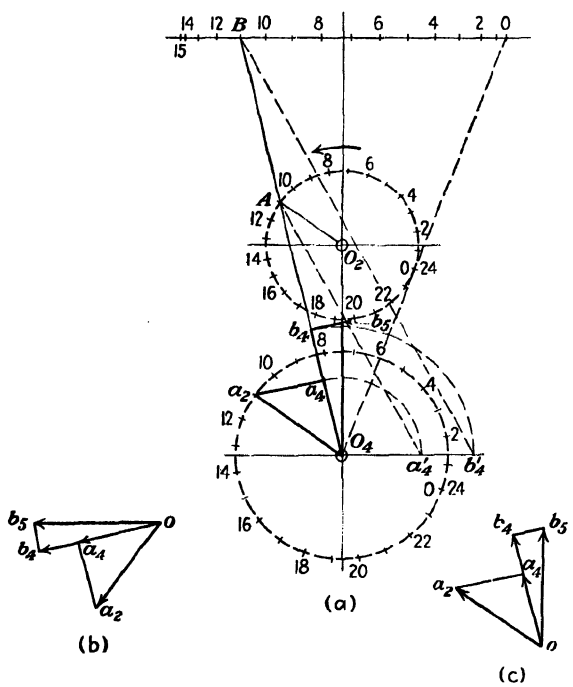


FIG. 227.

mechanism and drawing the lines of the revolved polygon, from O_4 as the pole, parallel to or coincident with the links of the mechanism as illustrated for crank position 11 in the foregoing discussion.

Since it is the velocity vector ob_5 (which represents the velocity of the ram) that is required for each of the 24 crank positions, it will be observed as the work progresses that merely the location of intermediate points will suffice, and many of the lines in the constructions radiating from the pole O_4 can be omitted. Since

the motion of the ram on each side of the midposition is symmetrical, the velocity determinations need be made for one-half the crank circle only.

c. The Velocity-space Diagram.—Starting with the ram in the zero position, at the extreme right (Fig. 228) and continuing through the complete cycle, the velocity of the ram V_{b_5} (vector ob_5 from Fig. 227) has been plotted on the stroke of the ram as a

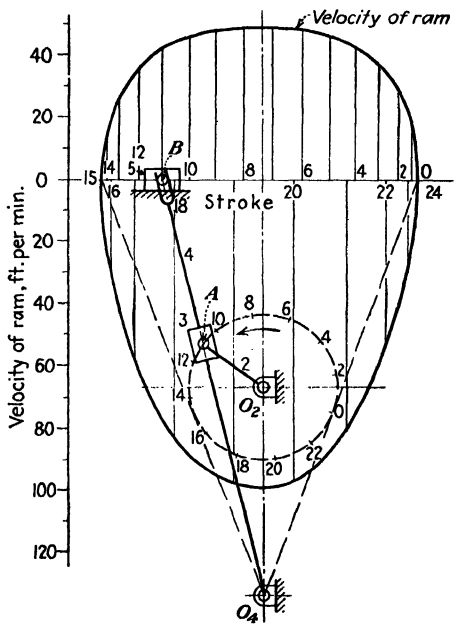


FIG. 228.

base line. The ordinates of the curve above the base line represent the speed of the cutting tool during the working stroke, and the ordinates below the base line represent the speed of the tool during the idle, or return, stroke. Not only is this a useful diagram in the study of an existing mechanism of this kind, but it is also useful in the original design, where various arrangements and proportions of links may be tried out and the diagrams compared in order to secure the most desirable arrangement. In some designs modifications of the standard or conventional linkage are made in order to secure a flatter curve on the cutting stroke, *i.e.*, a more uniform cutting speed.

d. The Velocity-time Diagram.—Since the motion of the crank is uniform, the 24 equal divisions on the crank circle represent equal intervals of time. If, therefore, 24 equal spaces of any convenient length are laid off on a horizontal line, as shown in Fig. 229, and if the velocities of the ram are laid off as ordinates perpendicular to this line, the velocity-time diagram for the motion of the ram will be obtained. This diagram also brings out very clearly the relation between the cutting speed and the return speed of the tool.

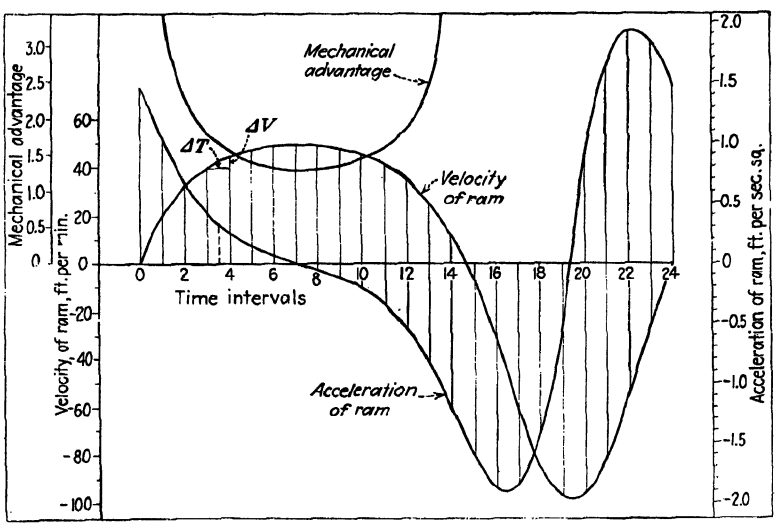


FIG. 229.

The mechanical advantage during the cutting stroke may be determined as follows: For this case the mechanical advantage may be defined as the ratio of the resistance encountered by the tool to the tangential force exerted upon the crank gear at the pitch line, on the assumption of 100 per cent efficiency. It may be shown that this ratio is equal to the ratio of the pitch-line velocity of the crank gear to the velocity of the cutting tool. The pitch-line velocity of the crank gear, whose pitch diameter is 25 in., may be calculated as follows:

$$V = \pi DN = \pi \times 25\tfrac{1}{12} \times 10.5 = 68.71 \text{ f.p.m.}$$

If, therefore, 68.71 is divided by the velocity of the ram for any position in the cutting stroke, the mechanical advantage for that

position will be obtained. These values have been calculated and plotted in Fig. 229. The curve shows very clearly that near the end of the stroke the mechanical advantage becomes very large, so that if the tool meets any unusual obstruction when the ram is in this position, very large forces are set up.

e. Acceleration and Inertia Force of Ram.—The velocity-time diagram furnishes data for an approximate determination of the inertia forces of the ram. The method employed is as follows: Referring to Fig. 229, let it be noted that a length equal to 24 position spaces represents the time required for one revolution of the crank. One space, therefore, represents an increment of time

$$\Delta t = \frac{60}{10.5 \times 24} = 0.238 \text{ sec.}$$

Between numbered positions 3 and 4, for example, the increment of velocity V_{b_3} is

$$\Delta V_{b_3} = 0.740 - 0.660 = 0.080 \text{ f.p.s.}$$

The average acceleration is

$$A_{b_3} = \frac{\Delta V_{b_3}}{\Delta t} = \frac{0.080}{0.238} = 0.336 \text{ f.p.s.}^2$$

After a suitable scale has been chosen (in this case 1 in. = 0.6 f.p.s.^{2}) the average acceleration is laid off midway between positions 3 and 4. This having been done for all such positions, and a curve drawn through the points, the approximate accelerations of the ram corresponding to the numbered positions may be obtained by scaling the ordinates at these points. The weight of the ram of this machine is estimated as 300 lb. The inertia force of the ram at position 4, for example, where the scaled value of the acceleration is 0.258 f.p.s.^{2}, is $F = MA_{b_4} = 300/32.2 \times 0.258 = 2.42$ lb. The maximum inertia force (near positions 16 and 22) is about 18 lb.

METHOD OF INSTANTANEOUS CENTERS

157. Instantaneous Center of Rotation.—It was shown in Art. 149 that the displacement of a body having plane motion may, for the purpose of analysis, be considered as made up of a

translation *with* one of its points and a rotation *about* that point. Upon this fundamental concept the method of relative velocities was based. The method of instantaneous centers is based upon a similar concept, namely, that any displacement of a body having plane motion may be considered as a pure rotation about some center.

Referring to Fig. 230, let the rigid body AB be displaced so that it takes up any position such as $A'B'$. Now let the perpendicular bisectors of the lines representing the displacements AA' and BB' be erected, and let the intersection of these bisectors be designated by O. It can be seen that AO and $A'O$ are equal;

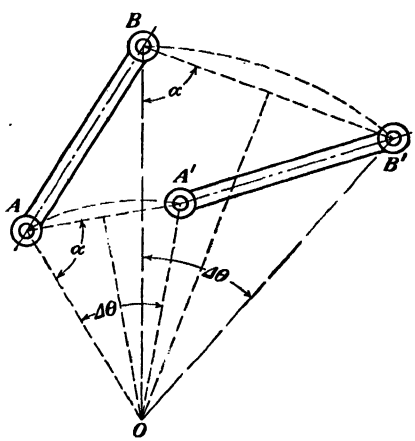


FIG. 230.

also that BO and $B'O$ are equal. Hence it is evident that the line AB of the rigid body, without being lengthened, shortened, or otherwise deformed, can move to $A'B'$ as part of the triangle ABO turning about its vertex O. A similar proof could be given for any other line on the rigid body. The indicated displacement of the rigid body AB could therefore have been effected by means of a rotation about the fixed point O, and it follows that any required plane displacement of a rigid body can be effected by a rotation of the body about some fixed point.

Referring again to Fig. 230, let the line that joints any point of the body to the center O be called the radius of that point. The radius of the point A, for example, is the line OA. Any radius will, during the displacement, sweep through an angle $\Delta\theta$

equal to the angular displacement of the body. The displacement of any point, therefore, may be determined in terms of its radius and the angle $\Delta\theta$ as follows:

$$\text{Displacement of } A = \overline{OA} \cdot 2 \sin \frac{\Delta\theta}{2}$$

$$\text{Displacement of } B = \overline{OB} \cdot 2 \sin \frac{\Delta\theta}{2}$$

It is seen from the above equations that the displacement of any point is proportional to its radius. Reference to the figure will bring out the fact that the displacements of the various points all make the same angle α with their corresponding radii.

If, instead of the finite displacement shown in the figure, the body is given an infinitesimal displacement during the time dt, the directions of the infinitesimal displacements of its points will all be at right angles to their corresponding radii, and the magnitudes of these displacements will be equal to the product of the infinitesimal angular displacement of the body and the lengths of the corresponding radii (the second statement follows from the fact that when $\Delta\theta$ becomes small, $2 \sin \frac{\Delta\theta}{2}$ is equal, except for negligible quantities, to the angle $\Delta\theta$). That is,

$$ds = R \cdot d\theta$$

where ds is the infinitesimal displacement of any point of the body, R the radius of that point, and $d\theta$ the infinitesimal angular displacement of the body. Dividing both sides by dt,

$$\frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \tag{51}$$

where ω is the angular velocity of the body, the quantity ds/dt being obviously the magnitude of the velocity of the point. The direction of this velocity is perpendicular to the radius R (since α becomes a right angle). For an infinitesimal displacement of the body such as that just discussed, the center O is called the *instantaneous center* of rotation, and the distance of any point of the body from that center (*i.e.*, the radius R) is called the *instantaneous radius* of the point.

The practical significance of the instantaneous-center concept can best be brought out by means of a simple example. Refer-

ring to Fig. 231, let it be assumed that the link AB has plane motion, that V_a is the known velocity of point A at a given instant, and that the direction of V_b at the same instant is also known. The instantaneous radii of the points A and B are, as seen above, perpendicular to the directions of the velocities V_a and V_b respectively, and hence may be drawn from A and B as indicated by the dashed lines. Their intersection locates the instantaneous center O. The magnitude of V_b can then be determined from the relations:

$$V_a = \overline{OA} \cdot \omega \qquad V_b = \overline{OB} \cdot \omega$$

i.e.,

$$\frac{V_b}{V_a} = \frac{\overline{OB}}{\overline{OA}} \qquad (52)$$

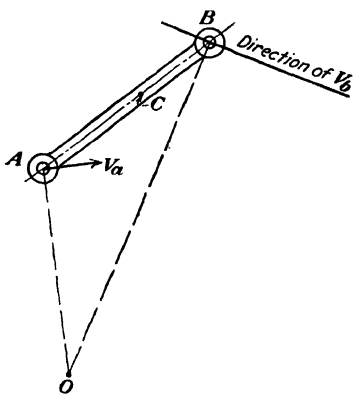


FIG. 231.

the velocities being, therefore, proportional to their instantaneous radii. Similarly, any third point C has a velocity perpendicular and proportional to its instantaneous radius $\overline{OC}$ (not shown).

It is important to note that the link AB is not *necessarily* constrained to *continue* to rotate about the point O and will not in general do so. As a matter of fact, all that has been done in the foregoing analysis is to draw lines from A and B perpendicular to the known directions of the velocities of these points, and then to assume that the body as a whole is turning about the intersection of these two lines, the justification for this assumption being simply the analogy that this case bears to the case of the finite displacement illustrated in Fig. 230. It can be shown, however, by other methods of analysis (the method of relative velocities, for example), that the *velocities* of the various points of link AB are, at the instant under consideration, exactly the same as they would be if the link were constrained to rotate about the point O. Actually, the link is *not* "rotating" about O; and at any subsequent instant the velocities of its points would no longer be related to O in the manner just stated. In velocity analyses, however, we are concerned only with the conditions that exist at definite specified instants, and it is convenient, therefore, to

think of the motion of the link AB as being a rotation about the instantaneous center O at the instant under consideration.

A special case of the instantaneous center is illustrated in Fig. 232. Here the link has a motion of straight-line translation, and hence the instantaneous radii of any two of its points are parallel and intersect at infinity.

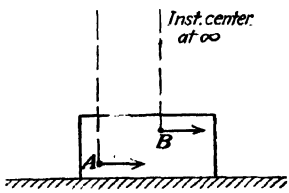


FIG. 232.

158. Instantaneous Center of a Pair of Links.—The instantaneous center of a pair of links is the point about which one of the links may be considered to be rotating at a given instant relative to the other link, which may be either fixed or moving. Referring to Fig. 233, let the vectors V_a and V_b indicate the relative velocities at some instant (and hence the directions of relative motion at that instant) of two pairs of coincident points at A and B respectively, one point of each pair being on link 2 and the other on link 1, which may be either fixed or moving. Link 1 in this case is represented by an irregular shaped body on which link 2 is outlined. Point A on 2 may be considered as moving

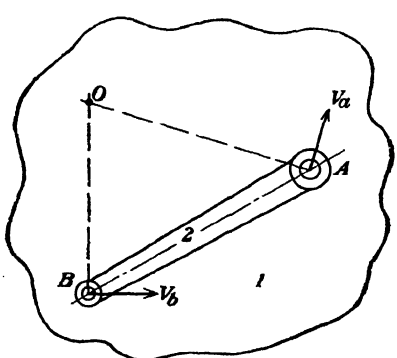


FIG. 233.

relative to A on 1 at the given instant about any point on a line through A perpendicular to the vector V_a. Likewise, point B on 2 may be considered as moving relative to B on 1 at the given instant about any point on the line through B perpendicular to the vector V_b. The intersection of these two lines at O is obviously the only point about which the two points A and B can move (and therefore about which link 2 can rotate) at the given

instant relative to link 1. Therefore, the point O must be the instantaneous center of the rotation of link 2 relative to link 1. Obviously this point also represents the instantaneous center for the motion of link 1 relative to link 2, since the relative rotation between a pair of links is the same, regardless of which link is considered as fixed, or whether both are moving relative to some other link. Ordinarily, a center of rotation such as found above would be designated by the symbol O, or more specifically, as O_{12}, in this case indicating center of the relative rotation of link 2 and link 1. For brevity the symbol O will be dropped, and from this point throughout the text instantaneous centers will be designated, for example, as 12, 23, 34, etc. These symbols should be read one-two, two-three, three-four, etc., and not twelve, twenty-three, and thirty-four, etc.

Further examination of Fig. 233 will show that, in addition to being the point about which one body may be considered to rotate relative to another, the instantaneous center is *the point at which the two bodies have zero relative velocity*, and hence the only point at which they may have the *same* velocity relative to points on other bodies coincident with it. The instantaneous center of two links may be thought of as the position of a pair of coincident points, one on each link, having zero relative velocity and identical velocities relative to a coincident point on some other link. For this reason the instantaneous center is sometimes referred to as the point of common velocity.

It is important to note, and clearly to appreciate, that the instantaneous center of two moving links is related, in the manner described above, to the relative motion of *coincident* points *only* of the two links, and *not* to the relative motion of noncoincident points. This can be seen by considering two cases, both illustrated by Fig. 233, it being assumed in the first case that link 1 is fixed, and in the second that link 1 is moving, the relative angular velocity ω_{21} and the relative position of the links and their instantaneous center O being assumed the same in both cases.

In the first case (link 1 fixed), the velocity V_a of the point at A on 2, relative to the coincident point at A on 1 (called A_2 and A_1 respectively hereafter), is obviously the same as the *absolute* velocity of A_2 and is equal to $\overline{OA} \cdot \omega_{21}$, the second factor being equal to the absolute angular velocity, ω_2 of link 2 in this case

(since link 1 is fixed). Since the relative velocity of two points in general is given by the vector difference of their absolute velocities, and since when link 1 is fixed the velocities of *all* of its points are zero, it is evident that V_a in this case is also the velocity of A_2 relative to *any* point on link 1.

With link 1 moving, however, an entirely different situation is found. First let it be noted that the motion of link 2 as a whole relative to link 1 is, at the instant under consideration (and as far as velocities are concerned), exactly the same as the motion it would have relative to link 1 if the two links were permanently connected by a turning joint at the instantaneous center O. It follows that the point A_2 is moving away from point A_1 at a rate given by

$$V_{a_2a_1} = \overline{OA} \cdot \omega_{21}$$

which shows that the relative velocity of the pair of coincident points at A is exactly the same as in the previous case. This is not true for noncoincident points. The velocity of A_2 relative to any other point on link 1 is, by definition, the vector difference of the absolute velocities of the two points. Since the absolute velocity of this other point on link 1 obviously differs from that of A_1, it follows that the velocity of A_2 relative to a noncoincident point on link 1 differs from $V_{a_2a_1}$, which was unchanged when link 1 was set in motion under the conditions stated. However, with link 1 fixed, the velocity of A_2 relative to *any* point on link 1 is $V_a (= V_{a_2a_1})$ as seen above. It follows that the velocity of A_2 relative to a noncoincident point on link 1 is changed when the latter is set in motion.

It is evident from the foregoing discussion that, regardless of how complex the motion of a body may be, its motion at any instant may be considered as a rotation about an instantaneous center. This is equivalent to saying that its motion through a complete cycle consists of a series of rotations about successive instantaneous centers. The connecting rod in an engine, for example, has a motion of pure rotation at one end and pure translation at the other end, thus producing a combined motion of considerable complexity. Since, however, in any phase of its motion it may be considered as rotating about such a center as O in Fig. 233, its velocity relations may be very easily determined by this method.

159. Location of the Instantaneous Centers in a Complete Mechanism.—In the preceding article the method of locating the instantaneous center of the motion of one link relative to another was discussed. The present article will deal with the location of instantaneous centers in a complete mechanism. In Fig. 234 is shown a four-link mechanism in which link 1 is fixed. Evidently the mechanism will have as many instantaneous centers as there are possible combinations of the links, taken two at a time. In this case the number will be six, as follows: 12, 13, 14, 23, 24, and 34. Links 2 and 4 rotate relative to 1 about their respective centers 12 and 14. Also the motions of 2 relative to 3 and of 3 relative to 4 are about the respective centers 23 and 34. It is to be noted that the four centers thus far obtained have been determined by inspection. The remaining centers, 13 and 24, may be found by the method outlined in the preceding article. First, with link 1 fixed, the center 13 can be found as follows: Since link 2 rotates about the fixed center 12, the direction of motion of 23 (23 is the point of common velocity for links 2 and 3), must be perpendicular to line 12-23. Hence the direction of motion of 23 as one point on link 3 is known. Again, since link 4 rotates about center 14, the direction of motion of 34 (34 is the point of common velocity for links 3 and 4), must be perpendicular to line 14-34. Hence the direction of motion of 34, a second point on link 3 is known. Thus, the directions of motion of two points on link 3 (23 and 34) have now been determined, and 13, the instantaneous center of the rotation of 3 relative to 1, is found at the intersection of the lines drawn through these points perpendicular to their directions of motion. It can be seen that these lines are extensions of lines 12-23 and 14-34.

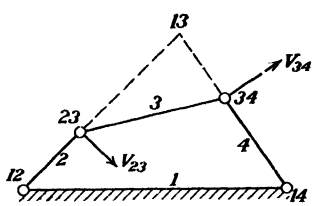


FIG. 234.

The determination of the position of the instantaneous center 24 depends upon the application of a principle which as yet has been brought out by inference only, and which may be stated specifically as follows: *The center, instantaneous or otherwise, of the relative rotation of two bodies having plane motion is the point about which one of them would for the instant rotate if the other were*

fixed, provided, of course, that the relative rotation of the bodies remains unchanged when one of them is fixed. The latter condition is automatically fulfilled in a constrained mechanism.

With link 4 assumed to be temporarily fixed, therefore, as shown in Fig. 235, the same reasoning as that applied in the preceding paragraph will locate 24, the instantaneous center of the relative rotation of links 2 and 4, at the intersection of the lines 23-34 and 12-14.

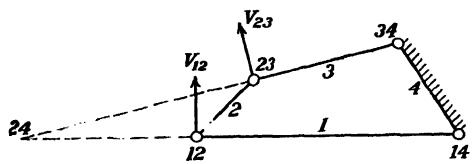


FIG. 235.

160. The Law of Three Centers.—The method of locating the instantaneous centers of a four-link mechanism, based on the application of simple fundamental principles, was demonstrated in the preceding article. It is obvious that this method would be quite complicated and laborious for mechanisms of six, seven, eight, or more links. This article suggests a simpler and more practical method of locating instantaneous centers, namely, the use of what is known as the *law of three centers.* This law may be stated as follows: *When any three bodies have plane motion, their centers of relative rotation (whether they be fixed, permanent, or instantaneous) lie on the same straight line.* In Fig. 236, link 1 is any link, fixed or moving; links 2 and 3 are moving links. The three possible relative rotations are between 1 and 3, between 1 and 2, and between 2 and 3. Hence there will be three centers, 12, 13, and 23. Point 12 is the center of the rotation of 2 relative to 1 and point 13 is the center of the rotation of 3 relative to 1. From the preceding discussion, it is known that the third center 23 must be situated where a pair of coincident points of links 2 and 3 have zero relative motion. Let it be assumed that center 23 might have a position such as that of A in Fig. 236. As a point

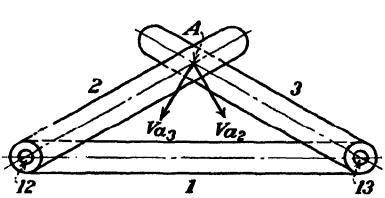


FIG. 236.

on 2 the direction of motion of A is indicated by the vector V_{a_2} at A perpendicular to radius 12-A. As a point on 3 the direction of motion of A is indicated by the vector V_{a_3} at A perpendicular to radius 13-A. It is evident that, even though these vectors may be of equal length, they do not coincide as is necessary if the two coincident points are to have zero relative motion. This condition is possible only when A, that is 23, lies on the line joining 12 and 13. Therefore, the three centers 12, 23, and 13 must lie in the same straight line, as stated in the law of three centers.

161. Application of the Law of Three Centers.—The procedure in locating all the instantaneous centers of a mechanism by

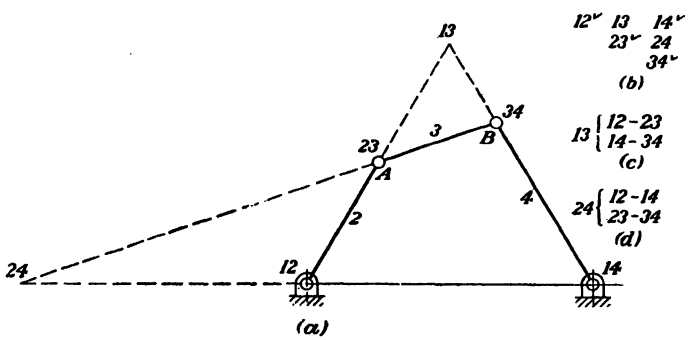


FIG. 237.

applying the law of three centers may be outlined as follows: First, determine the number of centers. The number of instantaneous centers N for any mechanism composed of n links is the number of possible combinations of the links taken two at a time. Hence

$$N = \frac{n(n-1)}{2} \tag{53}$$

Second, write down a list of all the centers. Third, locate by inspection the known centers and check them on the list. Finally, locate the remaining centers, using the law of three centers.

As an example of the above procedure, let the four-link mechanism shown in Fig. 237 be considered. The number of centers is found to be six, and these are set down in the form of

a half square as shown at *b*. Centers 12, 23, 34, and 14 are found by inspection and checked off in *b* as shown. To locate center 13 write down the centers as shown at *c*. Numbers 1 and 3 outside the braces, designating the center 13, are combined first with 2 to form centers 12 and 23, which with center 13 must lie in a straight line, according to the law of three centers. Next, 1 and 3 are combined with 4 to form centers 14 and 34, which with center 13 must also lie in a straight line. All the possible combinations of the remaining links have now been made with 1 and 3. The intersection of the line through centers 12 and 23 and the line through centers 14 and 34 determines the location of center 13. Since the remaining center 24 is located in exactly the same manner, the procedure should be obvious from the figure.

162. Application of Instantaneous Centers to the Solution of Velocity Problems.—This article will show how instantaneous

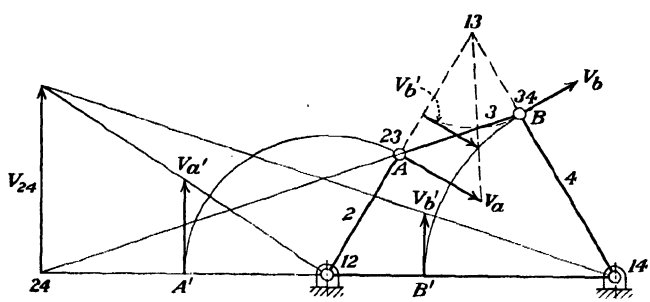


FIG. 238.

centers are used in the solution of a velocity problem. Figure 238 is Fig. 237*a* redrawn showing all the instantaneous centers. Assuming that V_a, the velocity of A is known, let it be required to find V_b, the velocity of B, using the method of instantaneous centers. According to the instantaneous-center concept, link 3 may be considered to be turning about center 13 with respect to link 1. Point A, which is a point on both link 2 and link 3, is moving with a velocity as shown by vector V_a. Since A and B are two points on link 3, the velocities V_a and V_b are directly proportional to the distances 13-23 and 13-34, respectively, of these points from center 13. This method of determining V_b is shown graphically by the dotted-line construction in Fig. 238. When a vector is placed in a position other than its true position,

for construction purposes, its symbol will be primed as shown in the case of vectors V_a and V_b in Fig. 238.

The velocity V_b may also be found by the use of center 24. To find V_b, assuming V_a known as before and using center 24 instead of center 13, consider 24 as the two coincident points on links 2 and 4 that have the same velocity. Since center 24 is a point on link 2 (extended) that turns about center 12, it is moving with an absolute velocity $V_{24} = V_a \cdot \frac{12 - 24}{12 - 23}$, *i.e.*, with a velocity represented by the vector V_{24} shown in Fig. 238. Now, center 24 is also a point on link 4 (extended) that turns about center 14. Therefore, point B is moving with a velocity $V_b = V_{24} \cdot \frac{14 - 34}{14 - 24}$. This method of determining V_b is shown graphically by the solid-line construction in Fig. 238.

It is sometimes necessary to determine the ratio of the angular velocities of two members of a mechanism with respect to a third member. Such determinations by the use of instantaneous centers may be shown as follows:

In Fig. 238 let it be recalled that center 23 is the point of common velocity for links 2 and 3. From the fundamental relation $\omega = V/R$, the angular velocity of link 3 relative to link 1 is equal to the velocity of point 23 divided by the distance $\overline{13\text{-}23}$, *i.e.*, expressed as an equation;

$$\omega_{31} = \frac{V_{23}}{13 - 23} \quad \text{Likewise } \omega_{21} = \frac{V_{23}}{12 - 23}$$

$$\text{Therefore, } \frac{\omega_{31}}{\omega_{21}} = \frac{12 - 23}{13 - 23}. \quad \text{Similarly, } \frac{\omega_{41}}{\omega_{21}} = \frac{12 - 24}{14 - 24} \tag{54}$$

Thus, it is observed that angular-velocity relations between links of a mechanism may be expressed in terms of distances between instantaneous centers without involving linear velocities in any manner whatever.

163. Application to Slider-crank Mechanism.—In Fig. 239*a* is shown the skeleton outline of a slider-crank mechanism. Link 4 is the piston, which has a motion of translation, *i.e.*, its center of rotation 14 is at infinity in a direction perpendicular to its path. Centers 12, 23, and 34 are also located by inspection and checked off at *b*. To locate the remaining centers 13 and 24, use the combinations shown at *c* and *d* in the figure. Knowing

the velocity of the crankpin center (point A), the velocity of the piston (V_b) can be found using center 13 (dotted-line construction) or center 24 (solid-line construction), as shown in the figure.

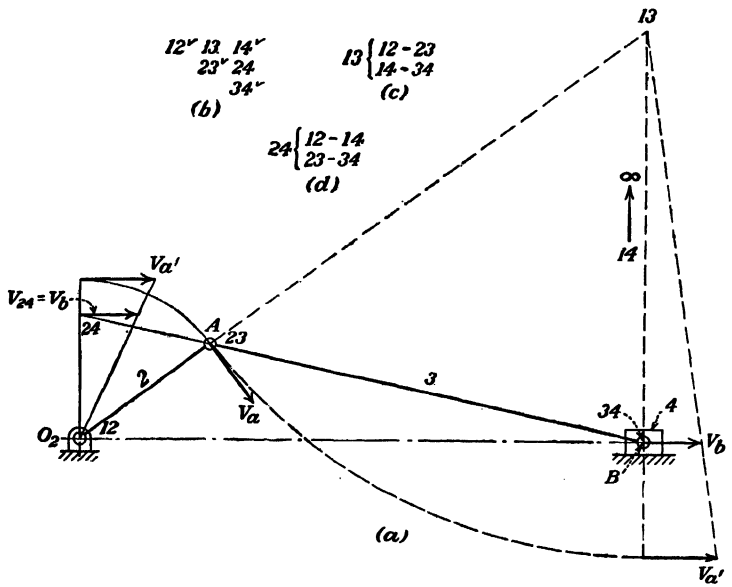


FIG. 239.

164. Application Involving Cams or Gears.—In Fig. 240 is shown a three-link cam mechanism. Link 2 turning in the sense indicated by the arrow drives link 3. Centers 12 and 13 are

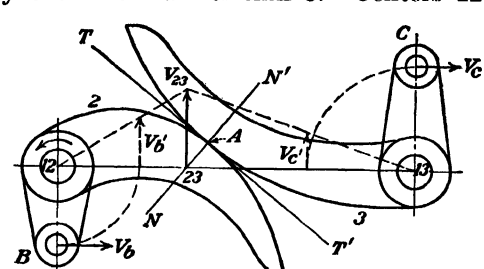


FIG. 240.

located by inspection. From the law of three centers the remaining center 23 is known to lie along the line 12-13. Its position along this line may be determined in the following manner: The

point of contact A of the links is in reality two coincident points. A_2 on link 2 and A_3 on link 3. While the cams are in driving contact, these two points cannot have relative motion in the direction of the normal NN'. It is evident, therefore, that the instantaneous relative motion of points A_2 and A_3 must be along the common tangent TT'. The instantaneous center of the relative motion of A_2 and A_3, then, must lie along a line at right angles to the common tangent, *i.e.*, along the common normal NN'. Thus, the intersection of the line of centers 12-13, and the common normal NN' locates 23, the instantaneous center of relative motion of links 2 and 3.

Now let it be assumed that the velocity of any point B on the driver is known and that the velocity of any point C on the follower is required. Since center 23 is the point of common velocity in the two links, it becomes, as in the preceding examples, the transfer point in passing from velocities on 2 to velocities on 3. The solution is shown graphically on the figure.

It is evident (Art. 162) that the ratio of the angular velocity of the follower to that of the driver is given by the expression

$$\frac{\omega_{31}}{\omega_{21}} = \frac{12 - 23}{13 - 23}$$

165. Application to a Complex Mechanism.—The solution of a problem, involving a mechanism of six links is shown in Fig. 241. Since there are six links and therefore 15 centers (See Art. 161), the problem of locating all the centers is somewhat more involved than in the preceding illustrations. Although it is seldom that all the centers are required, the complete determination will be made here because of its value as an illustrative problem. The procedure in locating the centers is the same as in Art. 161. At *b* is shown the group of 15 centers used for checking purposes, and at *c* all the possible combinations that may be used in determining one of the unknown centers, 24, for example. These combinations are found by combining with 2 and 4, one at a time, the remaining links 1, 3, 5, and 6. If there are too many unknowns in this setup for 24, try another, 13 for example, and continue until a workable setup is found. In a complex mechanism it is helpful to recognize four-member linkages, since the unknown centers in such a linkage lie at the intersections of opposite sides of the linkage (see Figs. 237 and 239).

After finding all the instantaneous centers, let it be assumed that the velocity of point A on link 2 is known and that the velocity of point D on link 6 is required. Link 6 has a motion of translation, and hence the velocities of all its points are identical. The velocity V_{26} of the point on 2 at the center 26 is easily determined, as shown. The point on 6 at this center has the

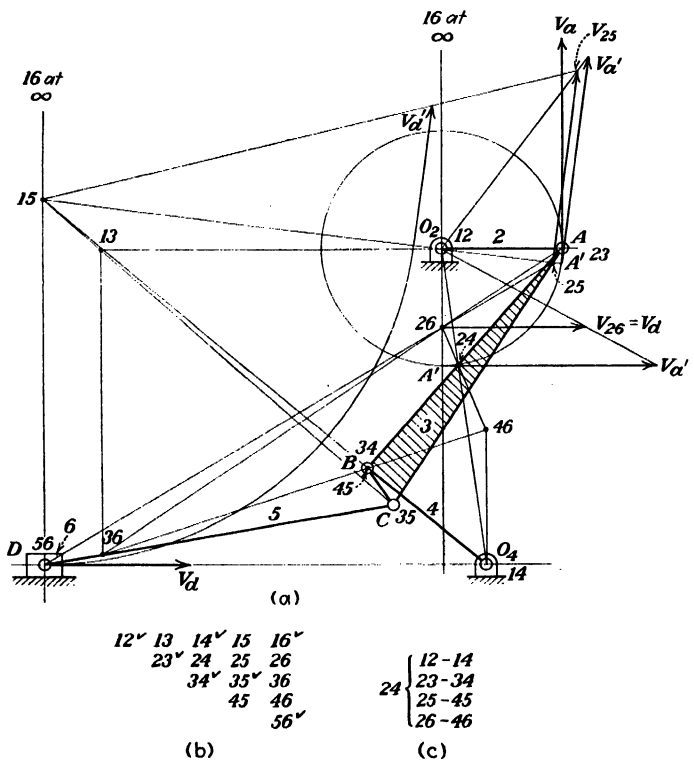


FIG. 241.

same velocity. Hence V_d the velocity of D is now known. The figure shows how V_d may be checked, using center 25. The solution by means of 25 is possible because the velocity of a point on 2 is given and the velocity of a point on 5 is required.

166. Application to Epicyclic-gear Trains.—A useful application of the instantaneous center method in the determination of velocities is that in connection with the solution of epicyclic-gear-train problems. After working out the solution of an epicyclic-gear-

train problem by the tabular method of Chap. IX, it may be desirable in certain instances to have a check on the result by some other method. It is the purpose of this article to outline a convenient graphical method of making such a check.

In the epicyclic-gear train shown in Fig. 242, the members of the train are designated in the usual way, and the letters a, b, and c represent the numbers of teeth in the gears. The pitch of the teeth is 10, and the numbers of teeth are $a = 75$, $b = 30$, and $c = 15$. Applying the tabular method of Chap. IX it will be

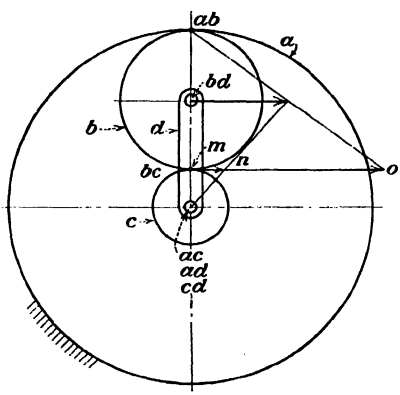


FIG. 242.

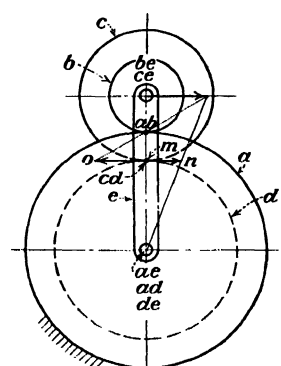


FIG. 243.

found that the ratio of the number of turns of the last gear c of the train to the number of turns of the arm d is

$$\frac{\omega_c}{\omega_d} = \frac{1 + \frac{a}{c}}{1} = \frac{1 + \frac{75}{15}}{1} = \frac{6}{1}$$

Now as a check on this solution, find the instantaneous centers, which in this case are designated by letters, and then lay off a vector of any convenient length at bd, the end of arm d. Since center bd is the point of common velocity for links b and d, its velocity is the same whether considered to rotate about center ad or center ab. It is evident from the construction shown in the figure that vector $\overline{mo}$ represents the velocity of a point on the pitch circle of gear c, and that vector $\overline{mn}$ represents the velocity of the coincident point on arm d. Since the two coincident points are rotating about the same center ac (or ad), $\overline{mo}$, and $\overline{mn}$

are proportional to the angular velocities of c and d. That is $\omega_c/\omega_d = \overline{mo}/\overline{mn} = 3.96/0.68 = 6/1$ (approximately). The numerical result was obtained by scaling vectors $\overline{mo}$ and $\overline{mn}$ from the original drawing, and it can be seen that this result checks very closely with the exact result obtained above by the tabular method.

Let the case shown in Fig. 243 be considered next. The pitch of the gears is 8, and the numbers of teeth are $a = 60$, $b = 18$, $c = 33$, and $d = 45$. Applying the tabular method of Chap. IX, it is found that $\frac{\omega_d}{\omega_e} = \frac{1 - \frac{ac}{bd}}{1} = \frac{1 - \frac{60 \times 33}{18 \times 45}}{1} = -\frac{1.444}{1}$. Applying the graphical construction in exactly the same manner as in Fig. 242, it is found that the ratio of the number of turns of the last gear d in the train to the number of turns of arm e is

$$\frac{\omega_d}{\omega_e} = \frac{mo}{mn} = \frac{1.08}{0.75} = -\frac{1.44}{1} \text{ (approximately)}$$

Again it is observed that this result is in very close agreement with the exact result obtained by the tabular method.

It is obvious that the graphical work described above could be duplicated step by step analytically. The details will not be worked out here other than to call attention to the fact that the vector $\overline{mo}$ in Fig. 242 is obviously twice the length of the vector at point bd; also that the vector $\overline{mn}$ is one-third the length of the vector at bd. Therefore, $\overline{mo}$ is *exactly* six times as long as $\overline{mn}$.

167. Centrodes.—The relative motions of any two links may be considered as a series of rotations about successive positions of their instantaneous center of relative rotation. A pair of curves in contact, one attached to and moving with each link and always containing this center, constitute what are known as the *centrodes* of relative motion of these links. It follows, therefore, that there is a pair of centrodes for each pair of links in a mechanism.

For example, if any link, such as link 3 (Fig. 244) has motion relative to the fixed link 1, then this motion may be regarded as a series of rotations about successive positions of the instantaneous center 13. If a curve is drawn connecting all the positions of the instantaneous center 13, this curve is called the *fixed centrode.* This locus of center 13 may be regarded as a curve attached to

link 1. Similarly, if a plane is attached so as to move with link 3, a second curve that always contains the center 13 can be drawn on this plane. The second curve is called the *moving centrode*. If these curves were given material backing and constrained to roll upon each other without slip they could replace the actual connecting links 2 and 4 and continue to give the same relative motion of the links 1 and 3. Centrodes such as those of Fig. 244 are mainly of theoretical interest. Centrodes of symmetrical form such as circles and ellipses may, on the other hand, be useful. For example, if a wheel rolls along a rail the instantaneous center of the wheel is always the point of contact with the rail. The top of the rail is therefore the locus of this center, *i.e.*,

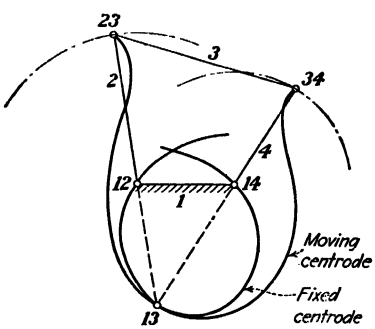


FIG. 244.

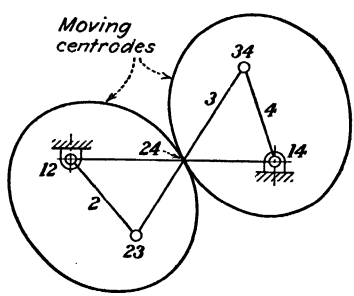


FIG. 245.

the fixed centrode. Also, the rim of the wheel always contains the instantaneous center and is, therefore, the moving centrode.

In the case of a pair of gears the instantaneous center of relative motion is always the point of contact of the pitch circles. These pitch circles, therefore, constitute a pair of centrodes of relative motion and are both moving centrodes. In the case of the crossed four-bar linkage Fig. 245, the centrodes of relative motion are a pair of ellipses. In general, it may be said that except in the case of symmetrical curves such as those mentioned above, centrodes are of little practical significance.

OTHER METHODS

168. Special Methods.—It occasionally happens that a velocity problem is encountered that cannot be solved directly by either of the two general methods thus far described. In such cases a certain amount of ingenuity must be exercised, as no general

rule can be laid down for the solution of all the special problems that may arise. The case shown in Fig. 246 will serve as an example. In this figure four links of a rather complicated mechanism have been shown, the other links having been omitted in order to avoid confusion. The velocities of the three points D, E, and F are assumed to be known, and it is required to determine the velocity of any point on link 2.

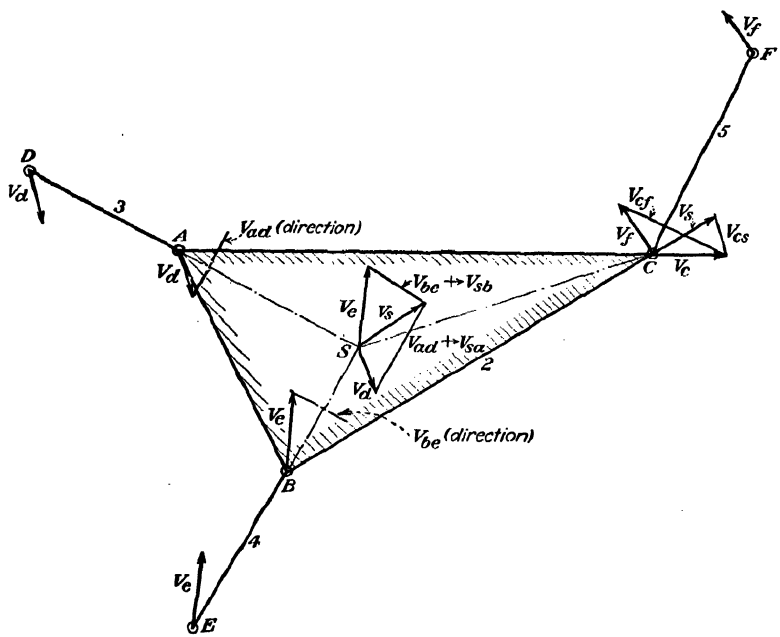


FIG. 246.

In attempting to determine the velocity of the point A by the method of relative velocities, it is at once seen that since the direction of V_a is unknown, the method breaks down. The same statement applies to the points B and C. It would seem, therefore, as though the problem could not be solved.

On link 2, however, let a line be drawn from the point A in the same direction as the line DA. In the same way let a line be drawn from B in the direction EB. The velocity of the intersection S of these two lines is given by the simultaneous solution of the two vector equations

$$V_s = V_a \mathbin{+\!\!+} V_{sa} = V_d \mathbin{+\!\!+} V_{ad} \mathbin{+\!\!+} V_{sa}$$
$$V_s = V_b \mathbin{+\!\!+} V_{sb} = V_c \mathbin{+\!\!+} V_{bc} \mathbin{+\!\!+} V_{sb}$$

Now let it be noted that the vectors V_{ad} and V_{sa} are both perpendicular to the line DAS. They must therefore have the same direction and hence their vector sum ($V_{ad} \mathbin{+\!\!+} V_{sd}$) will also have this direction. The latter, therefore, may be treated as a single vector perpendicular to the line DAS. A similar statement can be made for the vector sum ($V_{bc} \mathbin{+\!\!+} V_{sb}$). If, therefore, the vectors ($V_{ad} \mathbin{+\!\!+} V_{sb}$) and ($V_{bc} \mathbin{+\!\!+} V_{sb}$) are laid off (in the polygon at S)

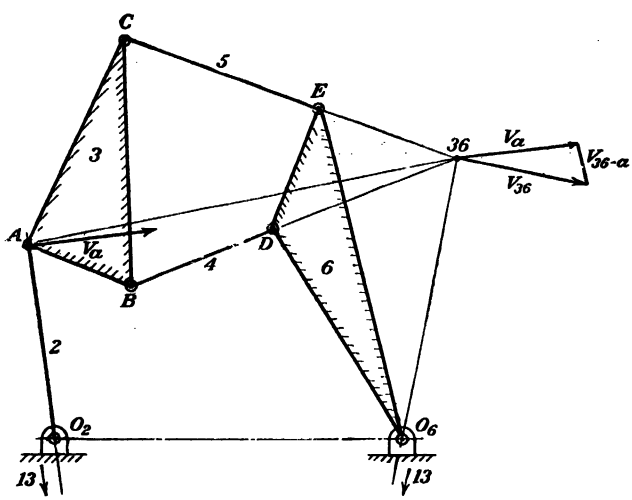


FIG. 247.

from the extremities of the vectors V_d and V_c respectively, their intersection will give the vector representing the velocity V_s. V_s having been determined, the velocity of any other point of link 2, such as C, can be determined.

It sometimes happens that a combination of the method of relative velocities and the method of instantaneous centers is required for the solution of a given problem. A case of this kind is the mechanism shown in Fig. 247. This is a purely hypothetical mechanism, its only useful purpose being to illustrate the point that it is desired to bring out. It is, however, a constrained mechanism as may readily be verified. The velocity

of the point A is given, and it is required to determine the velocity of the point E.

In attempting to solve the problem by the method of relative velocities, it is at once seen that, since the directions of the velocities of B and C are unknown, further progress is impossible. In order to apply the method of instantaneous centers, the centers 13, 16, and 36 should be known. The position of 16 is at O_6, and the position of 36 may readily be determined; the center 13, on the other hand, is situated at a point far outside the limits of the paper and hence its determination and use will be difficult, if not impossible.

In order to solve the problem, let it first be noted that 36 is the position of a pair of coincident points on 3 and 6 having the same velocity. As a point on 6 the direction of its velocity must be perpendicular to the line O_6-36. As a point on 3 the following vector equation must hold:

$$V_{36} = V_a + V_{36\text{-}a}$$

The velocity V_{36} may therefore be determined as indicated in the figure. V_{36} having been determined, the velocity of the point E may be determined by either the method of relative velocities or the method of instantaneous centers.

169. Analytical Methods.—The methods described in the preceding articles have been in the main graphical. It must not be inferred from this, however, that graphical methods are always to be preferred to analytical methods. Both methods have their particular application, and one should not be used where the other is more convenient. This article will be devoted to the solution of several velocity problems by analytical means.

a. The Slider-crank Mechanism.—As a first example let the slider-crank mechanism shown in Fig. 248 be considered. It has already been shown in Art. 23 that the displacement x of the crosshead is given by

$$\begin{aligned} x &= R(1 - \cos\theta) + L(1 - \cos\phi) \\ &= R(1 - \cos\theta) + R\left[\frac{1}{2}\frac{R}{L}\sin^2\theta + \frac{1}{2\cdot 4}\frac{R^3}{L^3}\sin^4\theta + \right. \\ &\qquad \left. \frac{1\cdot 3}{2\cdot 4\cdot 6}\frac{R^5}{L^5}\sin^6\theta + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\frac{R^7}{L^7}\sin^8\theta \cdots\right] \end{aligned}$$

If this equation is differentiated with respect to time, the velocity V_b of the crosshead will be obtained.

$$V_b = \frac{dx}{dt} = R \sin\theta \frac{d\theta}{dt} + R\left[\frac{2}{2}\frac{R}{L} \sin\theta \cos\theta \frac{d\theta}{dt} + \frac{1 \cdot 4}{2 \cdot 4}\frac{R^3}{L^3} \sin^3\theta \cos\theta \frac{d\theta}{dt} \cdots\right]$$

$$= R\omega_2\left\{\sin\theta + \frac{1}{2}\frac{R}{L} \sin 2\theta\left[1 + \frac{1}{2}\frac{R^2}{L^2} \sin^2\theta + \frac{1 \cdot 3}{2 \cdot 4}\frac{R^4}{L^4} \sin^4\theta + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{R^6}{L^6} \sin^6\theta \cdots\right]\right\} \quad (55)$$

where ω_2, the angular velocity of the crank, is written for the derivative $d\theta/dt$.

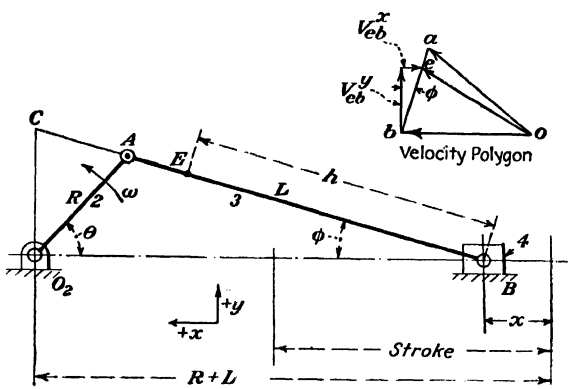


FIG. 248.

The foregoing is what might be styled the *fundamental analytical method.* It consists of first determining the displacement of one member in terms of the known displacement of another member and then in differentiating to determine the velocity of the first member in terms of the known velocity of the second. It is frequently more convenient, however, to make use of the fundamental principles that were brought out in the discussion of the two graphical methods.

As an illustration let the slider-crank mechanism in Fig. 248 again be considered. It has already been shown in Art. 163 that

$$V_b = V_a \frac{O_2C}{O_2A}$$

This expression may be put in a form suitable for calculation as follows:

$$V_b = V_a \frac{O_2C}{\overline{O_2A}} = V_a \frac{\overline{O_2B} \cdot \tan \phi}{R} = V_a \frac{(L \cos \phi + R \cos \theta) \tan \phi}{R}$$
$$= R\omega_2 \left(\frac{L}{R} \sin \phi + \frac{\sin \phi \cos \theta}{\cos \phi} \right)$$

and since

$$\sin \phi = \frac{R}{L} \sin \theta$$

$$V_b = R\omega_2 \left(\sin \theta + \frac{R}{L} \frac{\sin \theta \cos \theta}{\cos \phi} \right) = R\omega_2 \left(\sin \theta + \frac{1}{2} \frac{R \sin 2\theta}{L \cos \phi} \right) \quad (56)$$

Comparing this expression with the expression previously developed for V_b, it is evident that $1/\cos \phi$ is equal to the infinite series in the parentheses. An expression for $1/\cos \phi$ in terms of R, L, and θ may be obtained as follows:

$$\frac{1}{\cos \phi} = \frac{1}{\sqrt{1 - \sin^2 \phi}} = \frac{1}{\sqrt{1 - \frac{R^2}{L^2} \sin^2 \theta}}$$

If this expression is expanded, the infinite series obtained will be identical with the one obtained above.

The angular velocity of the connecting rod may be determined as follows:

$$\sin \phi = \frac{R}{L} \sin \theta$$
$$\cos \phi \frac{d\phi}{dt} = \frac{R}{L} \cos \theta \frac{d\theta}{dt}$$
$$\omega_3 = \frac{d\phi}{dt} = \frac{R \cos \theta}{L \cos \phi} \omega_2 \quad (57)$$

Now let it be required to determine the velocity of a point E on the connecting rod situated at a distance h from the wrist pin B.

The horizontal and vertical components of V_e are

$$V_e^x = V_b^x - V_{eb}^x = V_b - V_{eb} \sin \phi$$
$$V_e^y = V_b^y + V_{eb}^y = 0 + V_{eb} \cos \phi = V_{eb} \cos \phi$$

The relative velocity V_{eb} is given by

$$V_{eb} = h\omega_3 = h \frac{R \cos \theta}{L \cos \phi} \cdot \omega_2$$

Therefore,

$$V_e^x = R\omega_2\left(\sin\theta + \frac{1}{2}\frac{R}{L}\frac{\sin 2\theta}{\cos\phi}\right) - h\frac{R}{L}\frac{\cos\theta}{\cos\phi}\omega_2\sin\phi$$

$$= R\omega_2\left(\sin\theta + \frac{1}{2}\frac{R}{L}\frac{\sin 2\theta}{\cos\phi}\right) - R\omega_2\frac{h}{L}\cdot\frac{R}{L}\frac{\sin\theta\cos\theta}{\cos\phi}$$

$$= R\omega_2\left(\sin\theta + \left[1 - \frac{h}{L}\right]\frac{1}{2}\frac{R}{L}\frac{\sin 2\theta}{\cos\phi}\right) \qquad (58)$$

and

$$V_e^y = h\frac{R}{L}\frac{\cos\theta}{\cos\phi}\omega_2\cos\phi = \frac{h}{L}R\omega_2\cos\theta \qquad (59)$$

If h is equal to zero, the horizontal component will be equal to V_b and the vertical component will be zero. If h is equal to L, the horizontal and vertical components will be the same as those of the crankpin.

b. The Crank-shaper Mechanism.—As a further application of the analytical method let the crank shaper discussed in Art. 156 be again considered. Referring to Fig. 226, let it be assumed that the angular velocity of the crank is ω_2, and let it be required to determine the velocity V_{b_5} of the ram. This may be accomplished by means of the following analysis, which is self-explanatory:

$$V_{b_5} = \frac{V_{b_4}}{\cos\phi} = \frac{V_{a_4}\dfrac{O_4B}{O_4A}}{\cos\phi}$$

$$V_{a_4} = V_{a_2}\cos(\theta - \phi) = V_{a_2}(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$$\frac{O_4B}{O_4A} = \frac{h_2}{h_1 + R\cos\theta}$$

Therefore

$$V_{b_5} = \frac{V_{a_2}h_2}{h_1 + R\cos\theta}\cdot\frac{\cos\theta\cos\phi + \sin\theta\sin\phi}{\cos\phi}$$

$$V_{a_2} = R\omega_2 \quad \text{and} \quad \tan\phi = \frac{R\sin\theta}{h_1 + R\cos\theta}$$

Therefore

$$V_{b_5} = R\omega_2\frac{h_2}{h_1 + R\cos\theta}\left(\cos\theta + \frac{R\sin^2\theta}{h_1 + R\cos\theta}\right)$$

CHAPTER XI

ACCELERATIONS IN MACHINES

In the preceding chapter methods were outlined for the determination of velocities in machines. In this chapter similar methods will be developed for the determination of accelerations. It will be found that a velocity analysis is a necessary accompaniment of an acceleration analysis. Similarly, both the velocity and acceleration analyses are essential in an inertia-force analysis. The study of inertia forces will be taken up in Chap. XIII.

170. Acceleration of a Point.—The acceleration of a point is the time rate of its vector change of velocity. Referring to Fig. 249*a*

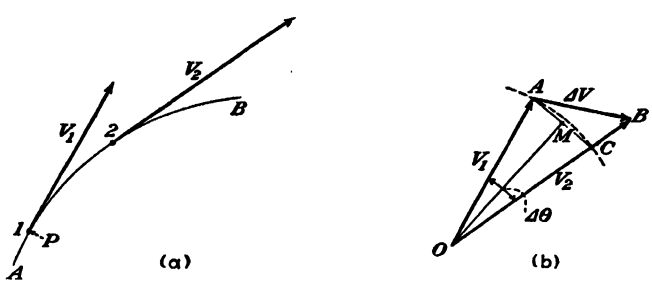


FIG. 249.

let the point P be moving along the plane curve AB, and let it be assumed that as the point moves from position 1 to position 2, its velocity changes from V_1 to V_2. This change is completely represented by the vector difference ΔV, as shown at *b* in the figure. The ratio $\Delta V/\Delta T$ of the vector change of velocity to the length of time required for the point to pass from position 1 to position 2 is the magnitude of the *average acceleration* of the point during the interval ΔT. The direction of the average acceleration is that of the vector ΔV. If now the segment 1-2 and the time required for P to traverse it are infinitesimals, then the ratio $\Delta V/\Delta T$ becomes dV/dt and is the magnitude of the instantaneous acceleration of the point. The term *acceleration* usually means *instantaneous acceleration.*

Referring again to the vector polygon shown at *b* in Fig. 249, let OC be laid off equal to OA by drawing an arc through A with O as a center and OA as a radius, and let the chord AC be drawn. It is evident from the figure that the vector change of velocity ΔV is the vector sum of AC and CB, and may be represented by the following vector equation:

$$\Delta V = \overline{AC} \mathbin{+\!\!+} \overline{CB}$$

The vector length CB obviously represents the change of the magnitude (scalar change) of the velocity of the point P as it passes from position 1 to position 2. AC is the vector that would represent the total change of velocity if the scalar change were zero, *i.e.*, if the velocity were changed in direction only.

It is evident from the figure that the above equation may be written as follows:

$$\Delta V = \overline{AC} \mathbin{+\!\!+} \overline{CB} = V_1 \cdot 2 \sin \frac{\Delta\theta}{2} \mathbin{+\!\!+} (V_2 - V_1)$$

where $\Delta\theta$ is the angle which measures the change of the direction of the velocity of the point P. If both sides of the equation are divided by the quantity ΔT, the resulting expression will be equal to the average acceleration of the point P during the interval ΔT.

$$A_{avg} = \frac{\Delta V}{\Delta T} = \frac{\overline{AC}}{\Delta T} \mathbin{+\!\!+} \frac{\overline{CB}}{\Delta T} = V_1 \frac{2 \sin \frac{\Delta\theta}{2}}{\Delta T} \mathbin{+\!\!+} \frac{V_2 - V_1}{\Delta T}$$

It should be noted that the direction of the component $\overline{AC}/\Delta T$ is perpendicular to the line OM, which bisects the angle $\Delta\theta$, and that the direction of the component $\overline{CB}/\Delta T$ is the same as that of the final velocity V_2.

If the segment 1-2 is an infinitesimal, the foregoing equation becomes

$$A = V\frac{d\theta}{dt} \mathbin{+\!\!+} \frac{dV}{dt} \tag{60}$$

where A is the instantaneous acceleration of the point P. Let it be noted that in this case the directions of V_1, OM, and V_2 coincide except for negligible quantities. It follows, therefore, that the component dV/dt is tangent to the curve at the position

of the point. Also, since the component $V \cdot \frac{d\theta}{dt}$ is perpendicular to the line OM, it is evident this component is normal to the curve at the position of the point. The normal and tangential components of the acceleration A are represented by A^n and A^t, respectively. The expression for the total acceleration may be written

$$A = V\frac{d\theta}{dt} + \frac{dV}{dt} = A^n + A^t \tag{61}$$

An inspection of Fig. 249 will bring out the fact that the component A^n is directed toward the center of curvature of the path, and that the direction of the component A^t is either the same as or opposite that of the velocity depending upon whether the magnitude of the latter is increasing or decreasing.

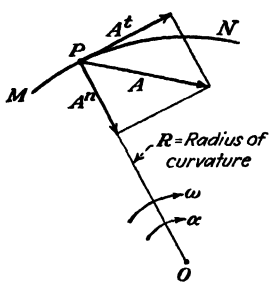


FIG. 250.

The quantity $d\theta$ is the angular displacement of the velocity vector V during the time interval dt. The quantity $d\theta/dt$, therefore, may be conveniently thought of as the angular velocity of the velocity vector V. If the radius of curvature of the path of the point is imagined to move *with* the point, then the angular velocity of the vector V is also that of the radius,

There are a number of interesting relations between the quantities V, A^n, A^t and the radius of curvature R, which will now be discussed. Referring to Fig. 250, let the point P be moving along the plane curve MN with a velocity V and an acceleration A, and let the radius of curvature OP ($= R$) be assumed to move with the point. It has been shown that the velocity V is equal to the product $R\omega$, where ω is the angular velocity of the radius of curvature. It follows that

$$A^n = V\frac{d\theta}{dt} = V\omega = R\omega \cdot \omega = R\omega^2 = \frac{V^2}{R} \tag{62}$$

and

$$A^t = \frac{dV}{dt} = \frac{d}{dt}(R\omega) = R\frac{d\omega}{dt} + \omega\frac{dR}{dt} = R\alpha + \omega\frac{dR}{dt} \tag{63}$$

where α is the angular acceleration of the velocity vector and hence of the radius of curvature. The quantity α is obviously equal to $d^2\theta/dt^2$. It is to be noted that if the radius of curvature is a constant, (*i.e.*, if the point is moving along a circular arc) the quantity $\omega \frac{dR}{dt}$ becomes equal to zero.

Since A^n and A^t are perpendicular to each other, the magnitude of the total acceleration may be expressed as follows:

$$A = \sqrt{(A^n)^2 + (A^t)^2} \tag{64}$$

171. Relative Acceleration of Independent Points.—The relative displacement of two points has been defined as the vector change of their relative position, and their relative velocity as the time rate of the vector change of their relative position. By analogy the relative acceleration of two points will be defined as the time rate of the vector change of their relative velocity.

Referring to Fig. 251*a*, let the point A traverse the segment A_1A_2 in the time interval ΔT, and let the initial and final velocities of the point be V'_a and V''_a, respectively. During the same interval of time let the point B traverse the segment B_1B_2, and let its initial and final velocities be V'_b and V''_b. The vector change of the velocity of the point A during the interval ΔT is ΔV_a, and that of the point B is ΔV_b. Now let V'_{ab}, the velocity of A relative to B at the beginning of the time interval ΔT, and V''_{ab}, the relative velocity at the end of the interval, be determined as shown by the heavy lines in the polygon at c. It is clearly brought out by the remaining lines of the polygon that the vector change ΔV_{ab} of the velocity of A relative to B is equal to the vector difference of the absolute velocity changes ΔV_a and ΔV_b. The following vector equations may therefore be written:

$$\begin{aligned}
\Delta V_{ab} &= V''_{ab} \rightarrow V'_{ab} \\
&= (V''_a \rightarrow V''_b) \rightarrow (V'_a \rightarrow V'_b) \\
&= (V''_a \rightarrow V'_a) \rightarrow (V''_b \rightarrow V'_b) \\
&= \Delta V_a \rightarrow \Delta V_b \\
\frac{\Delta V_{ab}}{\Delta T} &= \frac{\Delta V_a}{\Delta T} \rightarrow \frac{\Delta V_b}{\Delta T}
\end{aligned}$$

That is, the acceleration of A relative to B is the vector difference of the absolute accelerations of A and B. If A_a is accel-

eration of the point A, A_b that of the point B, and A_{ab} the acceleration of A relative to B, then

$$A_a = A_b \leftrightarrow A_{ab} \tag{65}$$

In Fig. 251*d* the relative velocities V'_{ab} and V''_{ab} and their vector difference ΔV_{ab} have been laid off separately, and the remaining lines of the figure have been drawn in the same manner as in

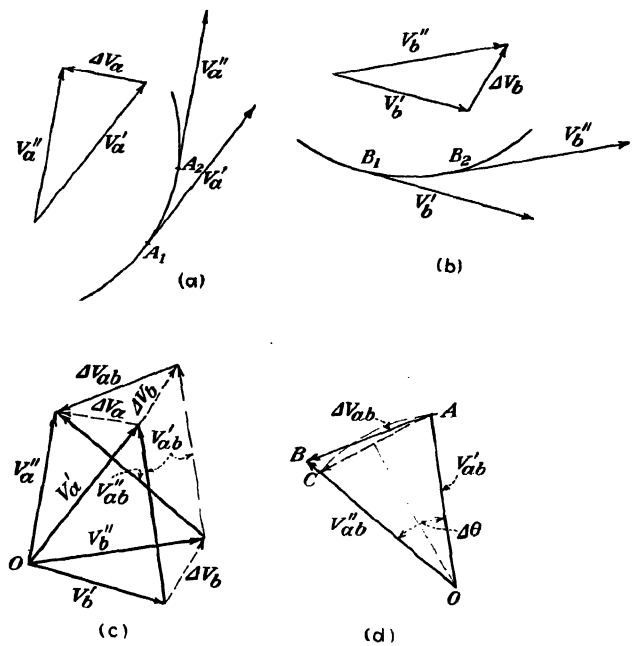


FIG. 251.

Fig. 249*b*. In a manner entirely analogous to that employed in Art. 170, it may be shown that

$$\frac{\Delta V_{ab}}{\Delta T} = \frac{AC}{\Delta T} \leftrightarrow \frac{CB}{\Delta T} = V'_{ab}\frac{2\sin\frac{\Delta\theta}{2}}{\Delta T} \leftrightarrow \frac{V''_{ab} - V'_{ab}}{\Delta T}$$

$$A_{ab} = V_{ab}\frac{d\theta}{dt} \leftrightarrow \frac{dV_{ab}}{dt} = A^n_{ab} \leftrightarrow A^t_{ab} \tag{66}$$

The normal component A^n_{ab} is perpendicular to the relative velocity V_{ab} and is directed toward the concave side of what may

be called the *relative path.* The quantity $d\theta/dt$ is the angular velocity of the relative-velocity vector. The direction of the tangential component A^t_{ab} is the same as or opposite that of the relative velocity V_{ab}, depending upon whether the magnitude of the latter is increasing or decreasing. Obviously, if A_b is given

$$A_a = A_b + A^n_{ab} + A^t_{ab} \tag{67}$$

172. Relative Acceleration of Two Points on the Same Rigid Body.—The relative acceleration of two points on the same rigid body possesses certain special characteristics that correspond in a general way to those possessed by the relative velocity of two

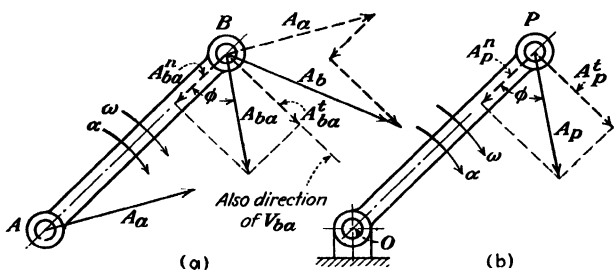


FIG. 252.

such points. In order to bring out these characteristics, let the case shown at *a* in Fig. 252 be considered.

It was shown in Art. 171 that the relative acceleration of any two points may be broken up into components perpendicular and parallel to the direction of the *relative velocity.* It was also shown that the magnitudes of these components are

$$A^n_{ba} = V_{ba}\frac{d\theta}{dt} \text{ and } A^t_{ba} = \frac{dV_{ba}}{dt}$$

where $d\theta/dt$ is the angular velocity of the relative-velocity vector V_{ba}. Since in this case the vector V_{ba} is always perpendicular to the line AB, it is evident that the direction of the normal component is along this line, from B toward A; never from B away from A.[1] Its magnitude is

$$A^n_{ba} = V_{ba}\frac{d\theta}{dt} = V_{ba}\omega = (\overline{AB} \cdot \omega)\omega = \overline{AB} \cdot \omega^2 \tag{68}$$

[1] The proof of this statement makes a good exercise. Let the rigid body be drawn in two positions, and then let the fundamental method discussed in Art. 171 be applied.

where $d\theta/dt$ has been replaced by its obvious equivalent ω, the angular velocity of the body.

The direction of the tangential component is parallel to V_{ba} and hence perpendicular to the line AB. The sense of its direction is such as to make B turn about A in the same direction as the angular acceleration of the body. Its magnitude is

$$A^t_{ba} = \frac{d(V_{ba})}{dt} = \frac{d}{dt}(\overline{AB} \cdot \omega) = \overline{AB}\frac{d\omega}{dt} = \overline{AB} \cdot \alpha \qquad (69)$$

It should be noted that the line AB enters into the expressions for A^t_{ba} and A^n_{ba} in the same way that the radius of curvature of the path of a point moving in a circle enters into the corresponding expressions for its absolute acceleration. This was to be expected, as the motion of the point B relative to the point A is a rotation of the former about the latter, the relative path, so to speak, being a circle of radius AB. Obviously, as indicated by the dashed vectors at B,

$$A_b = A_a + A^n_{ba} + A^t_{ba}$$

It has been seen that the direction of the relative velocity of two points on the same rigid body is perpendicular to the line joining the two points. A similar condition is true for the relative acceleration of two points on the same rigid body. The angle ϕ in Fig. 252*a* is given by the following relation:

$$\tan \phi = \frac{A^t_{ba}}{A^n_{ba}} = \frac{\overline{AB} \cdot \alpha}{\overline{AB} \cdot \omega^2} = \frac{\alpha}{\omega^2} \qquad (70)$$

It follows that the angle ϕ is independent of the location of the two points A and B and is dependent only upon the angular velocity and angular acceleration of the body.

When a rigid body is turning about a fixed center O as shown in Fig. 252*b*, the absolute acceleration of any point P of the body is merely its relative acceleration with respect to the fixed center O. It follows, therefore, that the normal and tangential components of the absolute acceleration of the point P are

$$A^n_p = \overline{OP} \cdot \omega^2$$
$$A^t_p = \overline{OP} \cdot \alpha$$

The total acceleration of the point is

$$A_p = A_p^n \leftrightarrow A_p^t = \overline{OP} \cdot \omega^2 \leftrightarrow \overline{OP} \cdot \alpha$$

The angle that this vector makes with the line joining the point P to the fixed center of the rotation is given by

$$\tan \phi = \frac{A_p^t}{A_p^n} = \frac{\alpha}{\omega^2}$$

Very often in acceleration problems the angular velocity ω of a link is known, but the angular acceleration is unknown. As an illustration of this type of problem, let the motion of the rigid

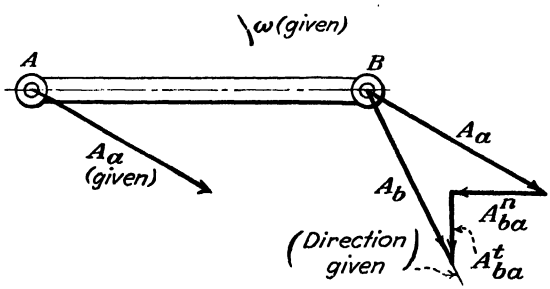


FIG. 253.

body shown in Fig. 253 be considered. The acceleration of the point A is completely known, the acceleration of the point B is known in direction only, and the angular velocity ω of the rigid body is known. It is required to determine the magnitude of A_b and the angular acceleration α. The acceleration A_b is given by the following vector equation:

$$A_b = A_a \leftrightarrow A_{ba}^n \leftrightarrow A_{ba}^t$$

The magnitude of A_{ba}^n may be calculated as follows:

$$A_{ba}^n = \overline{AB} \cdot \omega^2$$

and it may then be laid off from the extremity of A_a as shown. Although the magnitude of A_{ba}^t is unknown, its direction is known to be perpendicular to A_{ba}^n, and it may therefore be laid off from the extremity of A_{ba}^n as shown. The intersection of this line with the line representing the direction of A_b will determine the magnitudes of the vectors A_b and A_{ba}^t. The angular

acceleration of the rigid body is $\alpha = A'_{ba}/AB$. It is evident from the construction that the sense of the angular acceleration is negative, *i.e.*, clockwise.

Attention is called to the fact that in dealing with acceleration problems the instantaneous center of absolute motion must not be confused with the center of curvature. This may be understood by referring to Fig. 254. The point A of the four-link mechanism there shown moves in a circle, the center of which is O_2. A however, is also a point on link 3, the instantaneous center of which is O_3. The normal component of the acceleration of point A is directed toward the center of curvature O_2 of its path and *not* toward the instantaneous center O_3.

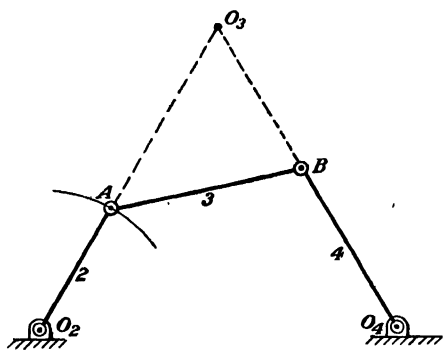


FIG. 254.

173. The Acceleration Polygon.—In the preceding article it was shown how accelerations could be determined by means of vector additions on the figure. It is the purpose of this article to outline the procedure to be followed in the construction of an acceleration polygon. In Fig. 255*a* is shown a simple four-link mechanism that has been drawn to a convenient scale. The angular velocity ω_2 and the angular acceleration α_2 of the driving link are given and the complete acceleration polygon is required. The first step in the solution is to draw the velocity polygon. Since the velocity of A is known from the relation $V_a = \overline{O_2A}\omega_2$, the vector oa representing this velocity is drawn to a convenient scale and the velocity polygon completed as shown at b in the figure.

The next step is to calculate the acceleration of A on link 2 and start the construction of the acceleration polygon at c. The

acceleration of A is

$$A_a = A_a^n \leftrightarrow A_a^t = \overline{O_2A}\omega_2^2 \leftrightarrow \overline{O_2A}\alpha_2$$

After selecting a pole o and a convenient scale the vectors A_a^n and A_a^t are drawn and the vector oa determined as shown in the figure.

Since A represents a pair of coincident points on links 2 and 3, which have the same acceleration, the acceleration of one point on link 3 has now been determined. The acceleration of point B on link 3 is, therefore,

$$A_b = A_a \leftrightarrow A_{ba} = A_a \leftrightarrow (A_{ba}^n \leftrightarrow A_{ba}^t)$$

The normal component A_{ba}^n is readily determined. Its magnitude $(V_{ba})^2/AB$ $(= \overline{AB} \cdot \omega_3^2)$ can be calculated since either V_{ba} or ω_3 can be determined from the velocity polygon. Its direction and

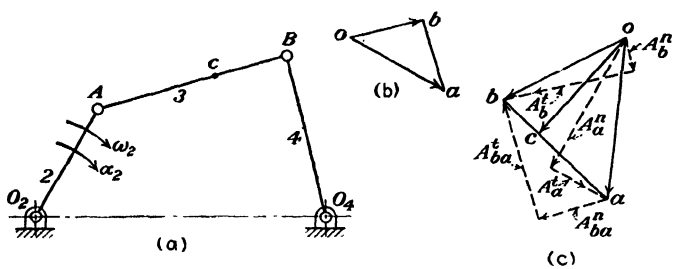


FIG. 255.

sense are BA. The tangential component A_{ba}^t is known in direction only, perpendicular to AB. Its magnitude $\overline{AB} \cdot \alpha_3$ cannot be calculated since α_3 is unknown. However, the vector A_{ba}^n can be drawn from the extremity of oa in the direction and sense BA, and A_{ba}^t can be represented in direction as shown in the figure at c.

In order to complete the acceleration polygon, it is necessary, to consider B as a point on link 4.

$$A_b = A_b^n \leftrightarrow A_b^t$$

The normal component A_b^n is known in both magnitude and direction, since its magnitude $V_b^2/\overline{O_4B}$ $(= \overline{O_4B} \cdot \omega_4^2)$ can be calculated (since V_b or ω_4 can be determined from the velocity polygon) and its direction and sense are BO_4. The tangential component A_b^t is known in direction only, perpendicular to O_4B. However, the

vector A_b^n can be drawn from the pole o (since this is an absolute acceleration) in the direction and sense BO_4. The component A_b^t is represented in direction as shown in the figure. The point b and hence the vector ob are determined by the intersection of the components A_{ba}^t and A_b^t, and the polygon is completed by the final step of drawing in the relative-acceleration vector ab.

As in the case of the velocity polygon, oa in the acceleration polygon is the image of link O_2A, ob the image of link O_4B, and ab the image of link AB. Hence, the acceleration of any other point on AB, say C, is represented by the vector oc drawn from the pole o to a point c such that ac in the polygon is proportional to AC in the figure of the mechanism.

It is observed that the acceleration polygon proper, oab, has been drawn in heavy solid lines. Obviously, as in the case of the velocity polygon, arrow heads on absolute-acceleration vectors are not essential. It is convenient, however, to have them on absolute-acceleration vectors, as shown in the figure, but it is not desirable to have them on relative-acceleration vectors. The normal and tangential components have been drawn in light broken lines and identified by the symbols and arrow heads in the manner indicated in the figure.

174. Accelerations in the Four-link Mechanism.—As an illustration of the application of the principles developed in the preceding articles, let the four-link mechanism shown in Fig. 256 be considered. The angular velocity and angular acceleration of link 2 are given as

$$\omega_2 = 20 \text{ radians/sec.} \qquad \alpha_2 = 150 \text{ radians/sec.}^2$$

and it is required to make a complete acceleration determination for the mechanism. The scale of the original drawing was 3 in. = 1 ft.

The first step in the solution is to determine the normal and tangential components of the acceleration of the point A. Since link 2 is rotating about the fixed center O_2, these components are

$$A_a^n = \overline{O_2A} \cdot \omega_2^2 = \tfrac{6}{12} \times 20^2 = 200 \text{ f.p.s.}^2$$
$$A_a^t = \overline{O_2A} \cdot \alpha_2 = \tfrac{6}{12} \times 150 = 75 \text{ f.p.s.}^2$$

The magnitude and direction of total acceleration A_a may, after a suitable scale has been chosen (a scale of 1 in. = 150 f.p.s.^{2} was

chosen for the original drawing), be determined by the vector addition of A_a^n and A_a^t.

Since A is a point on both links 2 and 3, the acceleration of one point on link 3 has been determined. The acceleration of the point B is represented by the vector equation

$$\begin{aligned} A_b &= A_a + A_{ba} \\ &= A_a + A_{ba}^n + A_{ba}^t \end{aligned}$$

The determination of A_{ba}^n requires that the angular velocity of link 3 be known. In order to determine ω_3, let the magnitude

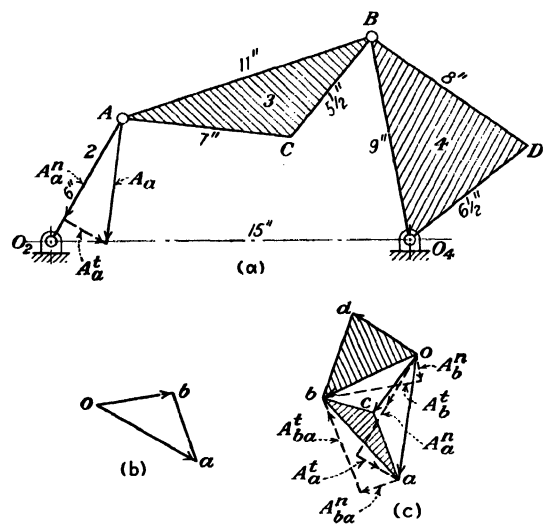


FIG. 256.

of V_a be calculated and let the partial velocity polygon shown at b in the figure be drawn.

$$V_a = \overline{O_2A} \cdot \omega_2 = {}^6/_{12} \times 20 = 10 \text{ f.p.s.}$$

The scale of the polygon in the original drawing was 1 in. = 8 f.p.s. The length of the vector ab in the polygon was 0.96 in. The relative velocity V_{bo} is therefore 7.68 f.p.s. The angular velocity of link 3 is

$$\omega_3 = \frac{V_{ba}}{AB} = \frac{7.68}{11/12} = 8.38 \text{ radians/sec.}$$

The magnitude of the normal component A^n_{ba} may now be calculated as follows:

$$A^n_{ba} = \overline{AB} \cdot \omega_3^2 = 11\frac{1}{12} \times \overline{8.38}^2 = 64.3 \text{ f.p.s.}^2$$

Now let the vector oa ($= A_a$) be constructed from the pole o in the acceleration polygon as shown at c in the figure, and from its extremity let the vector A^n_{ba} be drawn. Although the magnitude of the tangential component A^t_{ba} is unknown, its direction is known to be perpendicular to A^n_{ba}. This direction may be represented in the polygon by a line perpendicular to and through the extremity of the vector A^n_{ba}.

In order to determine the magnitude of the tangential component A^t_{ba}, the angular acceleration α_3 should be known. Since this is for the present unknown, the problem must be attacked from another angle.

It is evident that since the velocity V_b may be obtained from the velocity polygon, the angular velocity ω_4 of link 4 may be determined. Scaling the vector ob in the velocity polygon, it is found that V_b is 6.55 f.p.s. Therefore,

$$\omega_4 = \frac{V_b}{\overline{O_4B}} = \frac{6.55}{\frac{9}{12}} = 8.72 \text{ radians/sec.}$$

The angular velocity being known, the normal component of the acceleration of the point B may be determined as follows:

$$A^n_b = \overline{O_4B} \cdot \omega_4^2 = \frac{9}{12} \times \overline{8.72}^2 = 57.0 \text{ f.p.s.}^2$$

The vector A^n_b may now be laid off in the polygon. A line through the extremity of this vector and perpendicular to it will represent the direction of the tangential component A^t_b. It is evident that the extremity of the total acceleration vector A_b must be situated on this line. It must also be situated on the line representing the direction of A^t_{ba}, and hence it must be at the intersection of these two lines. The position of the point b in the acceleration polygon can, therefore, be determined and the vector ob may be scaled to determine the magnitude of the acceleration of the point B.

$$A_b = 165 \text{ f.p.s.}^2$$

The accelerations of the points C and D may be determined most conveniently by drawing figures (images) in the accelera-

tion polygon similar to links 3 and 4. The accelerations thus determined are

$$A_c = 117 \text{ f.p.s.}^2 \qquad A_d = 121 \text{ f.p.s.}^2$$

If desired, the angular accelerations of links 3 and 4 may be determined as follows:

$$\alpha_3 = \frac{A^t_{ba}}{AB} = \frac{167}{11\frac{1}{12}} = 182 \text{ radians/sec.}^2;$$

$$\alpha_4 = \frac{A^t_b}{O_4B} = \frac{159}{9\frac{1}{12}} = 212 \text{ radians/sec.}^2$$

175. Application to Slider-crank Mechanism.—As a further application of the principles developed in the foregoing articles, let the slider-crank mechanism shown in Fig. 257 be considered. The crank is assumed to have a uniform motion of 3,000 r.p.m., and it is required to determine the acceleration of the wrist pin B and the acceleration of the center of gravity G of the connecting rod. The scale of the original drawing was 6 in. = 1 ft.

Since the angular acceleration of the crank O_2A is zero, the total acceleration of the crankpin is the same as the normal acceleration. That is,

$$A_a = \overline{O_2A}\,\omega_2^2 = \frac{2.5}{12} \times \left(2\pi \times \frac{3{,}000}{60}\right)^2 = 20{,}570 \text{ f.p.s.}^2$$

The acceleration of point B is given by the following vector addition:

$$A_b = A_a \leftrightarrow A_{ba} = A_a \leftrightarrow (A^n_{ba} \leftrightarrow A^t_{ba})$$

Before A^n_{ba} can be determined, the angular velocity ω_3 of the connecting rod (link 3) must be known. But in order to determine ω_3 it will be necessary to draw the velocity polygon for the mechanism. From the given data

$$V_a = \overline{O_2A} \cdot \omega_2 = \frac{2.5}{12} \times \left(\frac{2\pi \times 3{,}000}{60}\right) = 65.5 \text{ f.p.s.}$$

A velocity scale of 1 in. = 30 f.p.s. was chosen for the original drawing and the velocity polygon drawn as shown at b in the figure. From this polygon the relative velocity V_{ba} is found to be 46.80 f.p.s. The angular velocity of link 3 is

$$\omega_3 = \frac{V_{ba}}{AB} = \frac{46.80}{10/12} = 56.16 \text{ radians/sec.}$$

The magnitude of the normal component A^n_{ba} is

$$A^n_{ba} = AB \cdot \omega_3^2 = 10/12 \times (56.16)^2 = 2{,}624 \text{ f.p.s.}^2$$

The acceleration polygon can now be drawn. Let the vector oa be laid off from a pole o as shown at c in the figure. The scale chosen for the original drawing was 1 in. = 10,000 f.p.s.^{2} Next lay off the vector A^n_{ba}, remembering that it must be drawn, not from the pole o but from point a in the polygon and that its

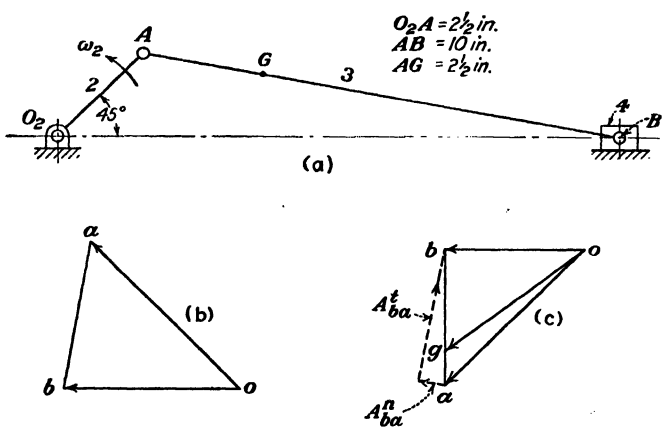


FIG. 257.

direction must be from B toward A in the figure. A line through the extremity of the vector A^n_{ba} and perpendicular to it will represent the direction of the tangential component A^t_{ba}. The direction of the acceleration of the point B is determined by the fact that this point moves in a straight line. It follows that a horizontal line through the pole o will represent its direction. The intersection of this line with the line representing the direction of A^t_{ba} determines the magnitudes of A_b and A^t_{ba}.

The line ab in the acceleration polygon is the image of the connecting rod AB. Hence the following proportion must hold:

$$\frac{ag}{ab} = \frac{AG}{AB}$$

Hence

$$ag = ab\frac{AG}{AB} = 1.44 \times \frac{2.5}{10} = 0.36 \text{ in.}$$

If, therefore, the distance ag is calculated and laid off in the acceleration polygon, the vector og representing the acceleration of point G will have been determined. Using the acceleration scale chosen above, the accelerations of points B and G were found to be

$$A_b = 14{,}400 \text{ f.p.s.}^2 \text{ and } A_g = 18{,}000 \text{ f.p.s.}^2$$

Attention is called to the fact that these accelerations are many times greater than the acceleration of gravity (32.2 f.p.s.^{2}). It is evident, therefore, that in an engine of this size running at 3,000 r.p.m. the inertia forces of its moving members must be many times greater than their weights.

176. Graphical Constructions. Scale Relations.—Although the methods employed in the preceding articles were in the main graphical, it was nevertheless necessary to make a number of intermediate calculations before a full solution of the problem could be obtained. The present article deals with a method that will materially reduce such calculations.

In any acceleration problem it is possible to choose the space, velocity, and acceleration scales so that the vector representing the normal component of the acceleration of any point may be determined by means of a graphical construction. This construction, which will be explained below, involves the normal component vector, the vector representing the velocity of the point, and the length on the drawing of the radius of curvature of the path of the point.

a. The Scales Defined.—Before proceeding further it will be necessary to make sure that a clear understanding is had as to the precise meaning of the three scales just mentioned. Velocities and accelerations are usually expressed in feet per second and feet per second^2 respectively, and in order, therefore, to make use of the various motion formulas, lengths must also be expressed in feet. The space, velocity, and acceleration scales (k_s, k_v, and k_a, respectively) used in this text will be defined as follows:

1 in. on drawing = a distance of k_s ft. on mechanism.

1 in. on drawing = a velocity of k_v f.p.s.

1 in. on drawing = an acceleration of k_a f.p.s.^{2}

For example:

For a space scale of 1 in. = ⅓ ft. (3 in. = 1 ft.), $k_s = \frac{1}{3}$.

For a velocity scale of 1 in. = 100 f.p.s., $k_v = 100$.

For an acceleration scale of 1 in. = 150 f.p.s.^{2}, $k_a = 150$.

b. Graphical Method.—The graphical method mentioned above for determining A^n will now be developed. Referring to Fig. 258, let the point P be moving along a plane curve with the velocity V. Let this velocity be represented in the figure by the vector length $\overline{PQ}$ inches, and let R, the actual radius of curvature of the path of the point in feet, be represented on the drawing by the length $\overline{OP}$ inches. The normal component of the acceleration is

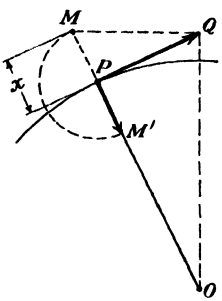


FIG. 258.

$$A^n = \frac{V^2}{R} = \frac{\overline{PQ}^2 \cdot k_v^2}{\overline{OP} \cdot k_s} = \frac{\overline{PQ}^2}{\overline{OP}} \cdot \frac{k_v^2}{k_s}$$

where k_v and k_s are the velocity and space scales, respectively. Now let it be assumed that the acceleration scale is k_a and that the vector length representing A^n is x inches. Then, from the preceding expression

$$A^n = xk_a = \frac{\overline{PQ}^2}{\overline{OP}} \cdot \frac{k_v^2}{k_s}$$

If the three scales k_s, k_v, and k_a are chosen so that

$$k_a = \frac{k_v^2}{k_s} \tag{71}$$

Then,

$$x = \frac{\overline{PQ}^2}{\overline{OP}} \text{ or } \frac{x}{\overline{PQ}} = \frac{\overline{PQ}}{\overline{OP}}$$

That is, PQ is a mean proportional between x and OP, and hence any of the various geometrical constructions pertaining to a mean proportional may be employed to determine x from PQ, or vice versa.

In the construction employed in Fig. 258 to determine x, the line OQ is drawn and the right triangle OQM constructed.

The distance PM is the desired vector length x. That is,

$$PM = x = \frac{\overline{PQ}^2}{\overline{OP}} = \frac{A^n}{k_a}$$

In order to show the vector PM in its true position, it may be rotated about P into the position PM' as shown in the figure. Of the various geometrical constructions pertaining to a mean proportional, the two shown in Fig. 259 will be found convenient for this work.

The construction shown at a is for the purpose of finding the acceleration vector A^n_{po} when the velocity vector V_{po} is given. This has already been explained in connection with Fig. 258.

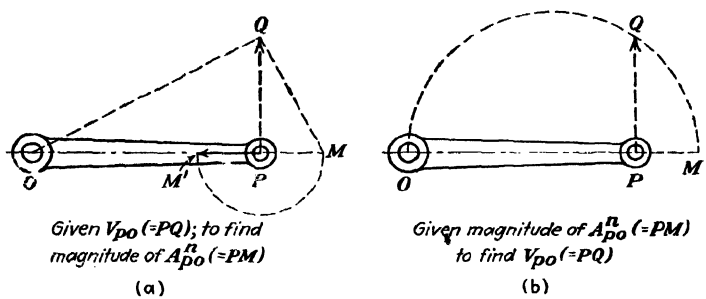


FIG. 259.

The construction at b is for the purpose of finding the velocity vector V_{po} when the acceleration vector A^n_{po} is given. The length PM representing A^n_{po} is laid off as an extension of OP, and a circle drawn on OM as a diameter. The line PQ drawn at right angles to OM is the velocity vector V_{po}. The proof is obvious from Fig. 258.

It is important to keep well in mind that the use of the above constructions (or any similar constructions) requires that the space, velocity, and acceleration scales satisfy the relation $k_a = k_v^2/k_s$. It should be noted also that two of the scales may be selected at will, but the third scale must be determined from the above relation.

c. Uniformly Rotating Crank.—When the motion of one of the members of a mechanism is uniform rotation about a fixed center, it is convenient to choose the velocity scale so that the vector representing the velocity of any point of the member is equal in

length to the radius of the point. When this is done, and when the acceleration scale is chosen in accordance with the relation $k_a = k_v^2/k_s$, the radius of the point also represents its acceleration. This statement may be proved as follows: Referring to Fig. 260 let the link 2 be rotating with uniform angular velocity ω about the fixed center O, and let the velocity scale be so chosen that the vector PQ representing the velocity of the point P is equal in length to the distance OP on the drawing. Then let the line OQ be drawn and the right angle OQM constructed. PM is the length of the vector that represents the normal component of the acceleration of the point P; and since the tangential component is zero, owing to the uniform rotation, it is evident that in this case the length PM is also that of the total acceleration vector. Now let it be noted that since $PQ = OP$, the angle PQO is 45 deg. and hence PQM is also 45 deg., from which it follows that $PQ = PM$. That is, PM is equal to the length OP on the drawing.

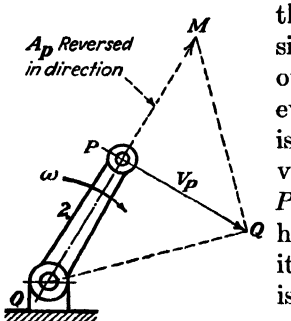


FIG. 260.

• **177. Four-link Mechanism. Complete Graphical Method.**—As an application of the methods described in the preceding article, let the four-link mechanism again be considered. Referring to Fig. 261, let the angular velocity and angular acceleration be given as follows:

$$\omega_2 = 22 \text{ radians/sec.} \qquad \alpha_2 = 180 \text{ radians/sec.}^2$$

and let it be required to determine the accelerations of the points C and D.

The first step is to calculate the magnitudes of the normal and tangential components of the acceleration of the point A.

$$A_a^n = \overline{O_2A} \cdot \omega_2^2 = \tfrac{5}{12} \times \overline{22^2} = 201.7 \text{ f.p.s.}^2$$
$$A_a^t = \overline{O_2A}\alpha_2 = \tfrac{5}{12} \times 180 = 75.0 \text{ f.p.s.}^2$$

Before proceeding with the graphical work, attention must be given to the selection of scales. The space scale, as is usually the case, has already been selected, that of the original drawing having been

$$3 \text{ in.} = 1 \text{ ft.}; \textit{ i.e.}, k_s = \tfrac{1}{3}$$

If the methods of Art. 176 are to be followed, only one of the other two scales may be selected arbitrarily, and for the purposes of the present problem it will be more convenient to select the acceleration scale. In the case of the original drawing the scale selected was

$$1 \text{ in.} = 100 \text{ f.p.s.}^2; \textit{ i.e.}, k_a = 100$$

For the purposes of the present problem it is not necessary to

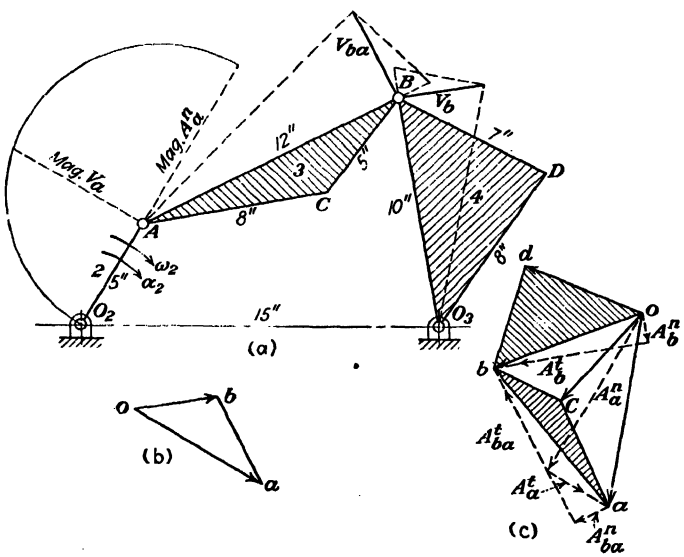


FIG. 261.

make use of the velocity scale. It could, however, be calculated in the following manner:

$$k_a = \frac{k_v^2}{k_s}$$

Therefore

$$k_v = \sqrt{k_a k_s} = \sqrt{100/3} = 5.79; \textit{ i.e.}, 1 \text{ in.} = 5.79 \text{ f.p.s.}$$

The acceleration scale having been selected, the vector A_a may be determined from its normal and tangential components and laid off in the polygon as shown at *c*.

The next step is to determine the length of the vector representing the velocity of the point A. This has been accomplished on the drawing of the mechanism itself by the methods of Art. 176. The partial velocity polygon may then be drawn as shown at b.

Using the vector length V_{ba}, the normal component A^n_{ba} may be determined by means of the construction shown on the figure of the mechanism. The vector length A^n_b may be determined in a like manner. Both of these vectors are then laid off in the polygon. Lines representing the directions of the tangential components, A^t_{ba} and A^t_b, may also be laid off in the polygon, the intersection of these lines determining the position of the extremity of the vector A_b. The acceleration images of links 3 and 4 are then drawn and the accelerations of the points C and D obtained as follows:

$$A_c = 118 \text{ f.p.s.}^2 \qquad A_d = 134 \text{ f.p.s.}^2$$

It is seen that for the solution of the foregoing problem only one calculation had to be made, namely, the calculation by means of which the acceleration of the point A was determined.

178. Slider-crank Mechanism. The Ritterhaus Construction. The slider-crank mechanism is of sufficient importance to warrant the development of special methods for its analysis. This is the purpose of the present article.

In Fig. 262 the velocity and acceleration polygons shown at b and c respectively have been drawn in accordance with the methods of the preceding articles but the construction has been omitted from the figure. The crank is assumed to be turning at uniform rate, and the velocity and acceleration scales have, therefore, been chosen so that the vectors representing the velocity and acceleration of the crankpin A are both equal in length to the crank radius OA.

Referring to the construction shown on the figure, the triangle OAM on the figure of the mechanism is equal to the triangle oab in the velocity polygon. Let the polygon oab be turned through 90 deg. and shifted so that the pole o coincides with the center O of the crank and so that the velocity vector oa falls along the line OA. It follows that OA and OM represent the magnitudes of the velocities of A and B respectively, and that AM represents the magnitude of the relative velocity of these points.

Now let the acceleration polygon at c be turned through 180 deg. and shifted so that the pole o of the polygon coincides with the center O of the crank, and so that the acceleration vector A_a falls along the line OA. It is evident that:

1. The point a of the polygon will coincide with the point A of the figure.

2. The vector ob will fall along the line OB.

3. The normal component A^n_{ba} will fall along the connecting rod AB, and the tangential component A^t_{ba} will be perpendicular to the connecting rod.

It is to be observed that the directions of the acceleration vectors in this construction are directly *opposite* their true directions.

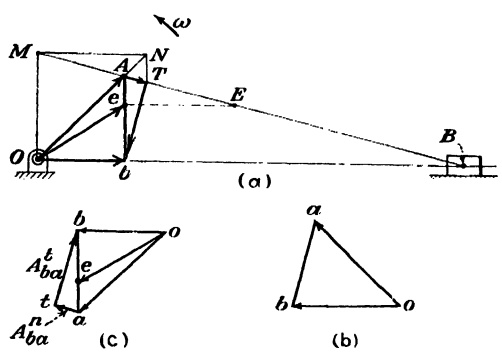


FIG. 262.

The length AT (which is equal and opposite the vector at in the acceleration polygon) may be determined more conveniently by means of the construction shown on the figure of the mechanism than by the method employed in the preceding articles. This construction is as follows: First, the line BA is extended until it intersects the vertical line OM at M. Through M a horizontal line is drawn intersecting the line OA extended at N. A vertical line through N will intersect the line AB at a point T such that the distance AT is the vector that, reversed in direction, represents the normal component of the relative acceleration of the points A and B, that is A^n_{ba}. This construction is known as the *Ritterhaus construction*. Its proof is as follows: The triangle ANT is similar to the triangle AOM; and the triangle AMN is similar to the triangle ABO. Therefore

$$\frac{AT}{AM} = \frac{AN}{AO} \text{ and } \frac{AM}{AB} = \frac{AN}{AO}$$

from which

$$\frac{AT}{AM} = \frac{AM}{AB} \text{ or } AT = \frac{\overline{AM}^2}{AB}$$

The line AM, however, is equal to the vector length V_{ba}.

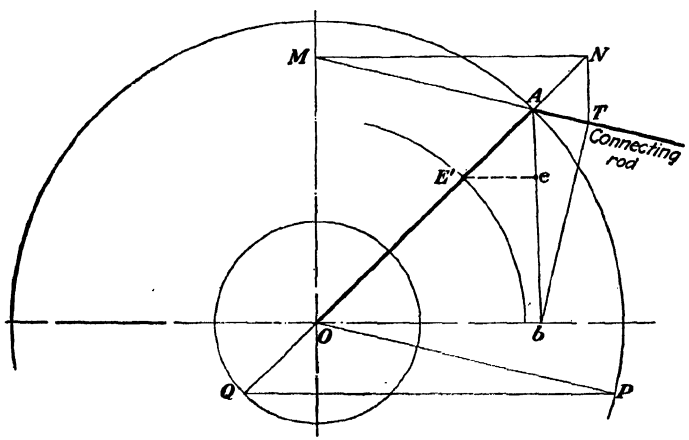


FIG. 263.

Therefore

$$AT = \frac{V_{ba}^2}{AB}$$

and, since the polygons have been drawn in accordance with the methods of Art. 176, it follows that the distance AT is equal to the vector length A_{ba}^n.

The acceleration image of the point E may be determined by drawing a line through E parallel to the horizontal line BO. The proof of this construction follows from the fact that the triangles AEe and ABb are similar.

In order to bring out the method of dealing with a problem where a number of acceleration determinations are required for different positions of the crank, let the case shown in Fig. 263 be considered. The outline of the mechanism is the same as that

of Fig. 262. In Fig. 263 half the crank circle has been drawn to an enlarged scale, and the acceleration determination has been made for one position of the crank.

The determination for the crank position shown was made as follows: First, a circle of radius OQ was drawn such that

$$\frac{OQ}{OA} = \frac{\text{length of crank}}{\text{length of connecting rod}}$$

The line AO was then extended to meet this circle at Q, and a horizontal line was drawn through Q intersecting the crank circle at P. The line OP gives the direction of the connecting rod[1] for this position of the crank. Using this direction, the vector length AT, which represents the normal component of the relative acceleration A_{ba} reversed in direction, was determined by means of the Ritterhaus construction as shown. A line through T perpendicular to the connecting rod gave the vector length ob, which represents the acceleration of the wrist pin reversed in direction. The acceleration of a point E on the connecting rod (outside the limits of the drawing) was determined as follows: First, a circle of radius OE' was drawn such that

$$\frac{OE'}{OA} = \frac{\text{distance from wrist pin to the point } E}{\text{length of connecting rod}}$$

Through the intersection E' of the line OA with this circle, a horizontal line was drawn intersecting Ab at e. The point e is the acceleration image of the point E on the connecting rod and Oe is the vector that (reversed in direction) represents the acceleration of point E.

The determinations for any other crank positions may be made in the same manner. This illustration brings out very clearly the extent to which special methods can be developed for the routine solution of certain types of problems, provided the problems are of sufficient importance to warrant such a development.

[1] This means of eliminating the full length of the connecting rod from the drawing permits of the use of a much larger scale in laying out the crank circle. This results in larger scales being used for the velocities and accelerations and thereby increases the accuracy of the determination. Note that the Ritterhaus construction does not require the full length of the connecting rod as does the usual construction given in Art. 176.

179. Relative Acceleration of a Pair of Coincident Points. Coriolis' Law.—The preceding articles have dealt with the relative acceleration of two points situated on the same rigid body. The present article deals with the relative acceleration of a pair of coincident points of two rigid bodies. The general characteristics of the *relative velocity* of two such points have already been discussed in Chap. X by means of the theory of instantaneous centers; in discussing their relative acceleration a slightly different method of attack will be employed.

Referring to Fig. 264, let links 2 and 4 of the four-link mechanism there shown be assumed extended in the manner indicated. It is evident that as the mechanism is set in motion, the point P on the extension of link 2 will trace out a curve such as MN on

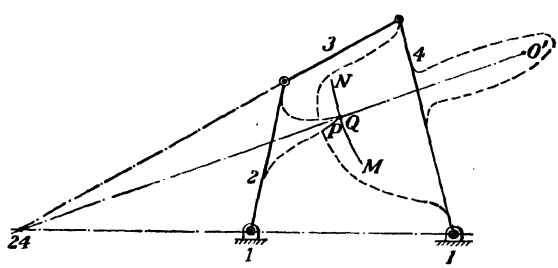


FIG. 264.

the extension of link 4. In the position shown, P is coincident with the point Q of link 4.

It was shown in Chap. X that the magnitude of the relative velocity V_{pq} is given by the product $(\overline{24\text{-}P}) \times \omega_{24}$, where ω_{24} is the angular velocity of link 2 relative to link 4, and that the direction of V_{pq} is perpendicular to the instantaneous radius $\overline{24\text{-}P}$. This method of determining V_{pq} does not involve any reference to the curve MN mentioned above.

The relative velocity V_{pq} could, however, be expressed also in terms of the curve MN traced out by P on link 4. For, let ds be the length of an infinitesimal segment of the curve and dt the time required for the point P to trace out that segment. The length of the infinitesimal segment is given by

$$ds = Rd\theta$$

where $R(= O'Q)$ is the average radius of curvature of the infini-

tesimal segment, and where $d\theta$ is the angle between the two radii drawn to the ends of the segment. If both sides of the equation are divided by dt,

$$\frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega_{r4}$$

where ds/dt is obviously the magnitude of V_{pq}, and where $d\theta/dt(= \omega_{r4})$ is the angular velocity of the radius of curvature (assumed to move with P) relative to the rigid body 4. The direction of V_{pq} is perpendicular both to the radius of curvature R, and, as seen above, to the instantaneous radius 24-P. It follows that the two radii are collinear. It should be observed, however, that the center of curvature O' and the instantaneous center 24 are separate and distinct points, and that in this case they are even situated on opposite sides of the curve MN.

The foregoing discussion has shown how the relative velocity of a pair of coincident points may be discussed in terms of the path that one of the points traces out on the rigid body of the other. The relative acceleration of a pair of coincident points will be discussed in a similar manner.

Referring to Fig. 265*a*, let the point P (shown as P_1 and P_2) be moving in a manner such that it traces out the curve AB on the *moving* body M, and at the beginning of a given time interval ΔT let P be coincident with the point A of the body, the common initial position of P and A being P_1 and A_1 in the figure. It is required to determine an expression for the acceleration of P relative to A.

At the beginning of the interval ΔT the velocity of P relative to A is $V_{p_1a_1}$. At the end of the interval the point P will be coincident with some other point B of the curve (indicated by P_2 and B_2), and the body M will have moved to some other position as indicated by the dotted lines. The velocity of P relative to A is now $V_{p_2a_2}$, but it should be noted that this is no longer the relative velocity of a pair of coincident points. The vector change of the velocity of P relative to A is given by

$$\Delta V_{pa} = V_{p_2a_2} \rightarrow V_{p_1a_1}$$

as shown by the polygon at c, and the time rate of this vector change of velocity is, at the limit, the acceleration of P relative

to A, an expression for which, containing only usually known quantities, will now be determined.

Since, at the end of the interval ΔT, P is no longer coincident with the point A, its velocity relative to A will be made up of two components, namely, $V_{p_2b_2}$, the velocity of P relative to B, (the point with which P *is* coincident), and $V_{b_2a_2}$, the velocity of B relative to A. That is,

$$V_{p_2a_2} = V_{p_2b_2} \dashrightarrow V_{b_2a_2}$$

A reference to the polygon at b in the figure will make this step clear.

Substituting in the expression for ΔV_{pa} given above,

$$\begin{aligned} \Delta V_{pa} &= (V_{p_2b_2} \dashrightarrow V_{b_2a_2}) \longrightarrow V_{p_1a_1} \\ &= V_{b_2a_2} \dashrightarrow (V_{p_2b_2} \longrightarrow V_{p_1a_1}) \end{aligned}$$

In order to express the vector difference $(V_{p_2b_2} \longrightarrow V_{p_1a_1})$ in terms usually known, let the vectors $V_{p_1a_1}$ and $V_{p_2b_2}$ be laid off from a single pole as shown at d in the figure, and let the line on be drawn making an angle $\Delta\phi$ with the vector $V_{p_1a_1}$, this being the angle between the normals drawn to the curve at A and B. It is evident from the geometry of the figure that on will make an angle $\Delta\theta$ with the vector $V_{p_2b_2}$, where $\Delta\theta$ is the angle through which the body M turns during the interval ΔT. With o as a center, let arcs be struck intersecting the line on at the points r and s, and let the vectors $\overline{qr}$, $\overline{rs}$, and $\overline{st}$ be drawn. The magnitudes of these vectors are given in the figure, the proof following directly from the geometrical relations involved. It is evident that

$$V_{p_2b_2} = V_{p_1a_1} \dashrightarrow \overline{qr} \dashrightarrow \overline{rs} \dashrightarrow \overline{st}$$

Therefore

$$(V_{p_2b_2} \longrightarrow V_{p_1a_1}) = \overline{qr} \dashrightarrow \overline{rs} \dashrightarrow \overline{st}$$

Substituting in the expression given above for ΔV_{pa}.

$$\Delta V_{pa} = V_{b_2a_2} \dashrightarrow \overline{qr} \dashrightarrow \overline{rs} \dashrightarrow \overline{st}$$

For an infinitesimal interval of time dt, these velocity vectors become *infinitesimal* velocity vectors, which, if divided by dt become acceleration vectors. The vector ΔV_{pa} obviously becomes A_{pa}, the desired relative acceleration. Before taking up the

others, let it be noted that $O'_1A_1(= O'A)$ and $O'_1B_1(= O'B)$ in the figure become, for an infinitesimal interval, the radius of

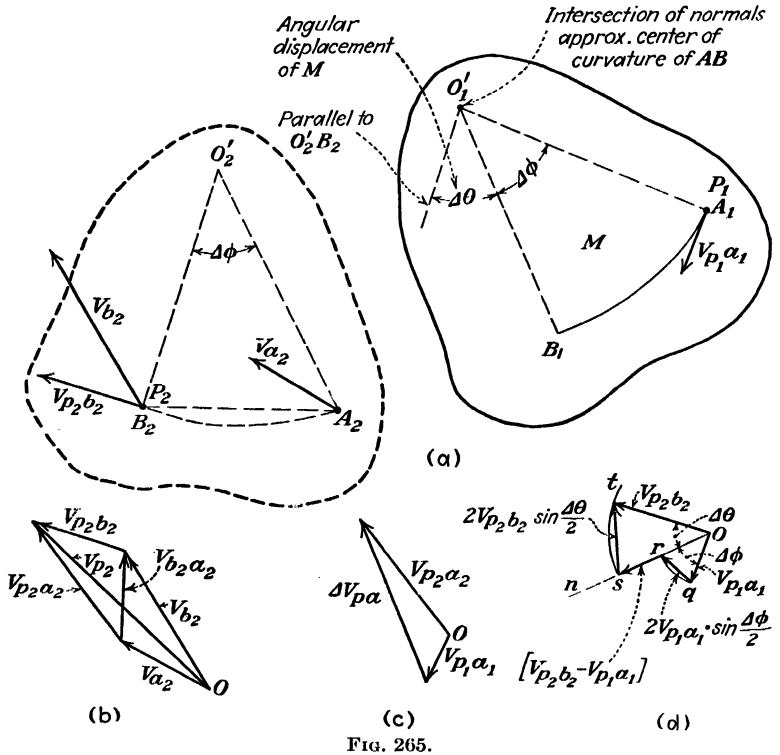


FIG. 265.

curvature R of the path AB of the moving point P; also that

$$\lim \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} = \text{angular velocity of body } M$$

$$\lim \frac{\Delta\phi}{\Delta t} = \frac{d\phi}{dt} = \text{angular velocity of } R \text{ (considered as moving with } P\text{) relative to } M.$$

$$\lim \frac{\overline{AB}}{\Delta t} = \frac{ds}{dt} = V_{pa}$$

Then, again considering the four vectors on the right-hand side of the equation

$$\lim \frac{V_{b_2a_2}}{\Delta t} = \lim \frac{\overline{AB}\cdot d\theta/dt}{\Delta t} = \frac{ds}{dt}\cdot\frac{d\theta}{dt} = V_{pa}\frac{d\theta}{dt}$$

which is normal to $\overline{AB}$ and hence to the curve; it is directed so as to satisfy the condition that B must turn about A in the same sense as that of the angular velocity $d\theta/dt$ of the body M; this is loosely equivalent to saying that it is directed so as to make the extremity of the vector V_{pa} turn about its origin in the same sense as that of $d\theta/dt$.

The vector $\overline{qr}$ when divided by Δt becomes

$$\lim \frac{\overline{qr}}{\Delta t} = \lim \frac{2V_{p_1a_1} \sin \frac{\Delta\phi}{2}}{\Delta t} = V_{pa}\frac{d\phi}{dt}$$

which is normal to the curve and directed toward the center of curvature O'.

The vector $\overline{rs}$ becomes

$$\lim \frac{\overline{rs}}{\Delta t} = \lim \frac{[V_{p_2b_2} - V_{p_1a_1}]}{\Delta t} = \frac{dV_{pa}}{dt}$$

which is tangent to the curve and directed the same as or opposite to V_{pa}, depending upon whether the latter is increasing or decreasing.

The vector $\overline{st}$ becomes

$$\lim \frac{\overline{st}}{\Delta t} = \lim \frac{2V_{p_2b_2} \sin \frac{\Delta\theta}{2}}{\Delta t} = V_{pa}\frac{d\theta}{dt}$$

which is normal to the curve and is directed the same as the component $\lim V_{b_2a_2}/\Delta t \left(= V_{pa}\frac{d\theta}{dt}\right)$ discussed above.

The vector expression for the relative acceleration A_{pa} of the coincident points P and A is, therefore,

$$\begin{aligned} A_{pa} &= V_{pa}\frac{d\theta}{dt} \leftrightarrow V_{pa}\frac{d\phi}{dt} \leftrightarrow \frac{dV_{pa}}{dt} \leftrightarrow V_{pa}\frac{d\theta}{dt} \\ &= V_{pa}\frac{d\phi}{dt} \leftrightarrow \frac{dV_{pa}}{dt} \leftrightarrow 2V_{pa}\frac{d\theta}{dt} \end{aligned}$$

since the two terms $V_{pa}\frac{d\theta}{dt}$ are the same in magnitude, direction, and sense.

Now let ω_m = angular velocity of body M.

ω_{rm} = angular velocity of the radius of curvature R (considered as moving with P) relative to body M.

α_{rm} = angular acceleration of R relative to M.

Then

$$V_{pa}\frac{d\phi}{dt} = V_{pa}\omega_{rm} = R\omega_{rm}^2$$

$$\frac{dV_{pa}}{dt} = \frac{d}{dt}(R\omega_{rm}) = R\alpha_{rm} + \omega_{rm}\frac{dR}{dt}$$

$$= R\alpha_{rm} \text{ if } R \text{ is constant, the usual case.}$$

$$2V_{pa}\frac{d\theta}{dt} = 2V_{pa}\omega_m$$

It is to be observed that $V_{pa}\frac{d\phi}{dt}$ and $\frac{dV_{pa}}{dt}$ are precisely the normal and tangential components of the acceleration which P would have if the body M were standing still and the point P were moving along the curve with a velocity $V_p(= V_{pa})$. A special notation $\mathfrak{A}_{pa}$, $\mathfrak{A}^n_{pa}$ and $\mathfrak{A}^t_{pa}$ will be used to designate this

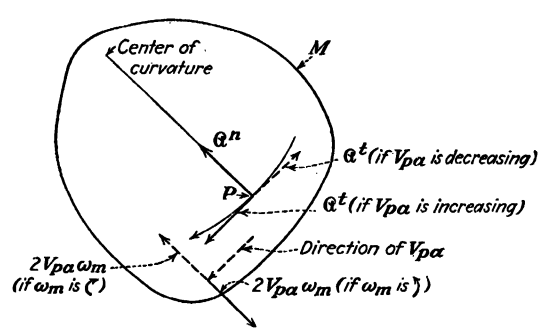


FIG. 266.

acceleration and its components. The quantity $\mathfrak{A}_{pa}$ is *not*, however, the acceleration of P relative to A, and must not be confused with A_{pa}. Evidently,

$$\mathfrak{A}^n_{pa} = V_{pa} \cdot \omega_{rm} = \frac{V^2_{pa}}{R} = R\omega^2_{rm} \tag{72}$$

$$\mathfrak{A}^t_{pa} = \frac{dV_{pa}}{dt} = R\alpha_{rm} + \omega_{rm}\frac{dR}{dt} = R\alpha_{rm} \text{ (if } R \text{ is constant)} \tag{73}$$

$$\mathfrak{A}_{pa} = \mathfrak{A}^n_{pa} + \mathfrak{A}^t_{pa} \tag{74}$$

Therefore

$$\begin{aligned} A_{pa} &= \mathfrak{A}^n_{pa} \leftrightarrow \mathfrak{A}^t_{pa} \leftrightarrow 2V_{pa}\omega_m \\ &= \mathfrak{A}_{pa} \leftrightarrow 2V_{pa}\omega_m \end{aligned} \tag{75}$$

If A_a is the acceleration of the point A, the acceleration of the point P is

$$\begin{aligned} A_p &= A_a \leftrightarrow A_{pa} \\ &= A_a \leftrightarrow \mathfrak{A}_{pa} \leftrightarrow 2V_{pa}\omega_m \end{aligned} \tag{76}$$

This is known as Coriolis' law, and the vector $2V_{pa}\omega_m$ is known as the Coriolis component.

For reference the rules regarding the directions of the various vectors of the foregoing have been illustrated in Fig. 266.

180. Applications. The Governor Mechanism.—The application of Coriolis' law will be illustrated first by considering the following simple problem: The flywheel of an engine (Fig. 267) has an angular velocity $\omega_2 = 30$ radians per sec., and an angular

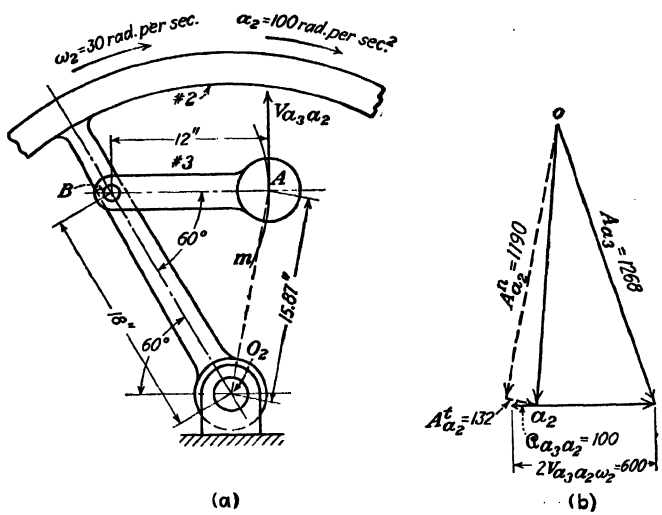


FIG. 267.

acceleration $\alpha_2 = 100$ radians/sec.^{2} The governor weight is moving outward relative to the wheel with a *uniform* velocity of 10 f.p.s. The problem is to find the acceleration of A, which is the center of gravity of the governor weight. This application

will be made clear by writing the general expression for Coriolis' law [Eq. (76)] as given in the preceding article, followed by substitution of the particular notation given in Fig. 267. That is,

$$\begin{aligned} A_p &= A_a \twoheadrightarrow A_{pa} \\ &= A_a \twoheadrightarrow \mathfrak{A}_{pa} \twoheadrightarrow 2V_{pa}\omega_m \text{ (Coriolis' law)} \end{aligned}$$

In Fig. 267 the letter A represents a pair of coincident points A_2 and A_3. The point A_2 lies on the arc m described on the wheel (link 2) by the coincident point A_3 on link 3. As applied to Fig. 267, the above general expression then becomes

$$\begin{aligned} A_{a_3} &= A_{a_2} \twoheadrightarrow A_{a_3a_2} \\ &= A_{a_2} \twoheadrightarrow \mathfrak{A}_{a_3a_2} \twoheadrightarrow 2V_{a_3a_2}\omega_2 \\ &= (A^n_{a_2} \twoheadrightarrow A^t_{a_2}) \twoheadrightarrow (\mathfrak{A}^n_{a_3a_2} \twoheadrightarrow \mathfrak{A}^t_{a_3a_2}) \twoheadrightarrow 2V_{a_3a_2}\omega_2 \end{aligned}$$

The magnitudes of the several components of the acceleration A_{a_3} are determined as follows:

$$A^n_{a_2} = \overline{O_2A}_2 \cdot \omega_2^2 = \frac{15.87}{12} \times \overline{30}^2 = 1{,}190 \text{ f.p.s.}^2 \text{ (Direction } A_2O_2)$$

$$A^t_{a_2} = \overline{O_2A}_2 \cdot \alpha_2 = \frac{15.87}{12} \times 100 = 132 \text{ f.p.s.}^2 \quad \text{(Direction perpendicular to } A_2O_2)$$

$$\mathfrak{A}^n_{a_3a_2} = \overline{BA}_3 \cdot \omega_{32}^2 = \frac{(V_{a_3a_2})^2}{BA_3} = \frac{\overline{10}^2}{1\frac{2}{12}} = 100 \text{ f.p.s.}^2 \text{ (Direction } A_3B)$$

$$\mathfrak{A}^t_{a_3a_2} = \overline{BA}_3 \cdot \alpha_{32} = 0 \quad \text{(Since } \alpha_{32} = 0)$$

$$2V_{a_3a_2}\omega_2 = 2 \times 10 \times 30 = 600 \text{ f.p.s.}^2 \quad \text{(Direction } BA_3)$$

A pole o and a convenient scale having been selected, the acceleration polygon may now be drawn as shown at b in Fig. 267. By scaling the closing vector A_{a_3} in this polygon, the required acceleration A_{a_3} is found to be 1,268 f.p.s.^{2}

181. The Slider-crank Mechanism.—As a further illustration of the application of Coriolis' law, let the slider-crank mechanism shown in Fig. 268 be considered. The normal and tangential components of the acceleration of the crankpin are as given in the figure, and it is required to determine the acceleration of the wrist pin B.

At the outset let it be understood that the solution of this problem can be obtained more conveniently by the ordinary method (see Art. 175) than by the application of Coriolis' law.

The chief purposes here are to aid the student in recognizing a situation where the law may be applicable, and to give practice in applying it.

Since the acceleration of A_2 (one point on link 2) is known, and since the link is rotating about the fixed center O_2 the acceleration of any other point on the link may easily be determined. Let link 2, therefore, be extended so as to have a point at B. The connecting rod, link 3, also has a point at B, the acceleration of which it is desired to determine. The problem is therefore

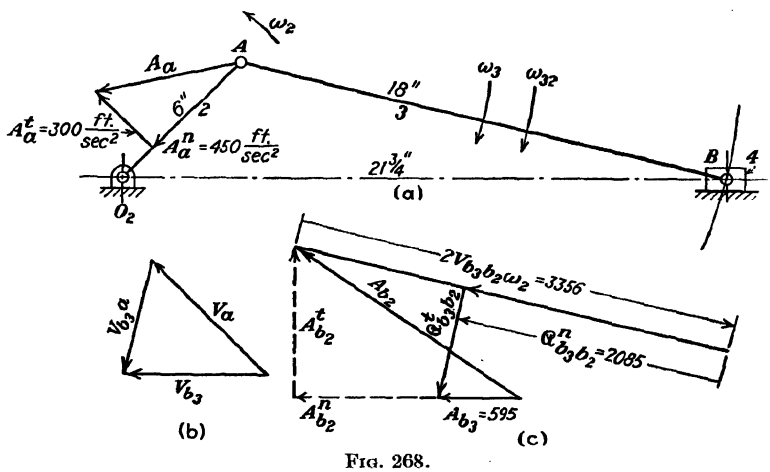


FIG. 268.

that of determining the acceleration of B_2 and the acceleration of the coincident point B_3. The point B_3 corresponds to point P, and the point B_2 to point A of Art. 179 and hence Coriolis' law applies. Therefore

$$A_{b_3} = A_{b_2} + A_{b_3b_2} = A_{b_2} + \alpha_{b_3b_2} + 2V_{b_3b_2}\omega_2$$
$$= A_{b_2} + \alpha^n_{b_3b_2} + \alpha^t_{b_3b_2} + 2V_{b_3b_2}\omega_2$$

It is evident from the figure that the curve traced out by the point B_3 on the extension of link 2 is a circle of radius AB; also that the angular velocity of the radius of curvature, relative to link 2, is ω_{32}. Therefore

$$\alpha^n_{b_3b_2} = \overline{AB} \cdot \omega_{32}^2 \text{ and } \alpha^t_{b_3b_2} = \overline{AB} \cdot \alpha_{32}$$

In order to determine the angular velocities ω_{32} and ω_2, the velocity of the crankpin A is first determined and the polygon drawn as shown at b.

$$A_a^n = 450 = \frac{V_a^2}{O_2A} = \frac{V_a^2}{6/12}, \text{ therefore } V_a = \sqrt{450 \times \frac{6}{12}} = 15 \text{ f.p.s.}$$

and the velocity V_{b_3a} is found from the polygon to be 10.92 f.p.s. The angular velocities of links 2 and 3 are then

$$\omega_2 = \frac{V_a}{O_2A} = \frac{15}{6/12} = 30 \text{ radians/sec.}$$

$$\omega_3 = -\frac{V_{b_3a}}{AB} = -\frac{10.92}{18/12} = -7.28 \text{ radians/sec.}$$

the negative sign in the case of ω_3 being required by the direction of V_{b_3a}. The relative angular velocity ω_{32} is

$$\omega_{32} = \omega_3 - \omega_2 = -7.28 - 30.00 = -37.28 \text{ radians/sec.}$$

The normal and tangential components of A_{b_2} are

$$A_{b_2}^n = A_a^n \frac{OB}{OA} = 450 \frac{21\frac{3}{4}}{6} = 1{,}631 \text{ f.p.s.}^2$$

$$A_{b_2}^t = A_a^t \frac{OB}{OA} = 300 \frac{21\frac{3}{4}}{6} = 1{,}087 \text{ f.p.s.}^2$$

The magnitudes of $\mathfrak{A}_{b_3b_2}^n$ and $2V_{b_3b_2}\omega_2$ may be calculated as follows:

$$\mathfrak{A}_{b_3b_2}^n = \overline{AB} \cdot \omega_{32}^2 = 18/12 \times \overline{37.28^2} = 2{,}085 \text{ f.p.s.}^2$$

$$2V_{b_3b_2}\omega_2 = 2\overline{AB} \cdot \omega_{32} \cdot \omega_2 = 2 \times 18/12 \times 37.28 \times 30 = 3{,}356 \text{ f.p.s.}^2$$

The magnitude of $\mathfrak{A}_{b_3b_2}^t$ is unknown since α_{32} is unknown. In order to make the vector $V_{b_3b_2}$ (which is obviously perpendicular to the connecting rod and directed in the same sense as ω_{32}) rotate about its origin in the same sense as ω_2 (see Fig. 266), the vector $2V_{b_3b_2}\omega_2$ must be directed to the right, away from A. The direction of $\mathfrak{A}_{b_3b_2}^n$ is toward the center of curvature A.

Now let the known vectors be laid off in the polygon as shown at c. The magnitude of the vector $\mathfrak{A}_{b_3b_2}^t$ is unknown as stated above, but its direction is known to be perpendicular to AB. The direction of A_{b_2} is known to be horizontal. If, therefore, these two directions are laid off in their appropriate places in the

polygon, their intersection will determine the magnitudes of $\mathfrak{A}^t_{b_3b_2}$ and A_{b_3} as shown.

182. Shaper Mechanism.—Although the problem of the preceding article could have been solved much more readily by the method of relative accelerations, there are many problems in which the applications of Coriolis' law provides the more convenient means of obtaining a solution. As a matter of fact, in

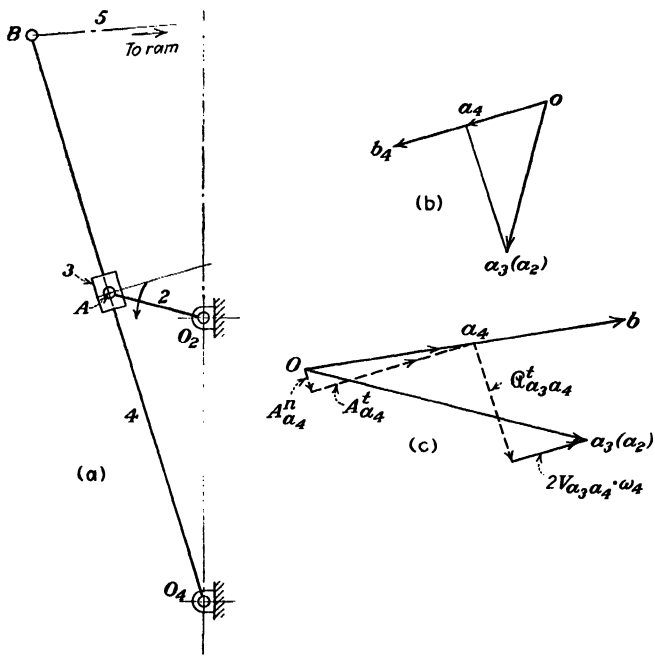


FIG. 269.

all cases where the relative motion of a pair of coincident points on different bodies must be considered in order to study the motion of the mechanism, the method of relative accelerations breaks down and Coriolis' law must of necessity be applied. A case of this kind is the shaper mechanism that has been partially shown in Fig. 269. The crank O_2A, which is 6 in. in length, makes an angle of 15 deg. with the horizontal. The crank rotates in a counterclockwise sense at a uniform rate of 10½ r.p.m. It is required to determine the acceleration of the point B for the crank position given.

The first step in the analysis is to draw the velocity polygon. The velocity of the crankpin is

$$V_{a_2} = O_2A\omega_2 = \tfrac{6}{12} \times 2\pi \times 10.5 = 33 \text{ f.p.m.}$$

After choosing a suitable scale, the velocity polygon may be drawn as shown at *b*.

Since the coincident points A_3 and A_4 are involved, it will be found necessary to apply Coriolis' law in the determination of the acceleration polygon. Observing the fact that the curve traced out by A_3 on link 4 is a straight line coincident with O_4A, it will be more convenient to consider the acceleration of A_3 relative to A_4, rather than vice versa. Therefore

$$\begin{aligned} A_{a_3} &= A_{a_1} + A_{a_3a_4} \\ &= A_{a_4} + \mathfrak{A}_{a_3a_4} + 2V_{a_3a_4}\omega_4 \text{ (Coriolis' law)} \\ &= (A^n_{a_4} + A^t_{a_4}) + (\mathfrak{A}^n_{a_3a_4} + \mathfrak{A}^t_{a_3a_4}) + 2V_{a_3a_4}\omega_4 \end{aligned}$$

The magnitudes of the several components of acceleration may now be determined as follows, the velocities being scaled from the velocity polygon as required:

$$A_{a_3} = \frac{V^2_{a_3}}{\overline{O_2A}} = \frac{(33/60)^2}{6/12} = 0.605 \text{ f.p.s.}^2$$

$$A^n_{a_4} = \frac{V^2_{a_4}}{\overline{O_4A}} = \frac{(17.2/60)^2}{20.4/12} = 0.0483 \text{ f.p.s.}^2$$

$$A^t_{a_4} = \overline{O_4A} \cdot \alpha_4 \text{ (Unknown, since } \alpha_4 \text{ is unknown, but known in direction)}$$

$$\mathfrak{A}^n_{a_3a_4} = V_{a_3a_4} \cdot \omega_{34} = 0 \text{ (since } \omega_{34} \text{ is zero)}$$

$$\mathfrak{A}^t_{a_3a_4} = \frac{dV_{a_3a_4}}{dt} \text{ (Unknown in magnitude, but known in direction)}$$

$$2V_{a_3a_4}\omega_4 = 2 \times \frac{28.2}{60} \times \frac{17.2/60}{20.4/12} = 0.1585 \text{ f.p.s.}^2$$

Proceeding now with the construction of the acceleration polygon at *c* in the figure, lay off vector oa_3 ($= A_{a_3}$) in the direction AO_2. Next lay off $A^n_{a_4}$ and the known direction of $A^t_{a_4}$. From a_3 lay off the component $2V_{a_3a_4}\omega_4$ in the position shown. Applying the rule given in Art. 179, this vector is perpendicular

to O_4A and is directed to the right. The component shown as $\mathfrak{A}^t_{a_3a_4}$ in the figure is drawn perpendicular to $2V_{a_3a_4}\omega_4$ from the origin of the latter. The intersection of the tangential components $A^t_{a_4}$ and $\mathfrak{A}^t_{a_3a_4}$ at a_4 is the image of A_4. Hence $oa_4 = A_{a_4}$; and ob, determined by proportion, is A_b. By scaling the vector ob the acceleration of point B is obtained as 0.630 f.p.s.[2]

183. Analytical Methods.—As an example of the analytical determination of the accelerations in a mechanism, let the case shown in Fig. 270 be considered. The crank is assumed to be turning with a uniform angular velocity ω_2, and it is required to determine the acceleration of the point E on the connecting rod.

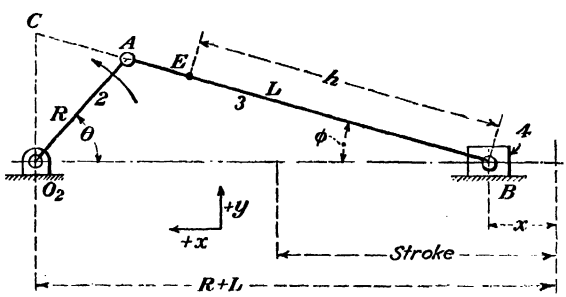


FIG. 270.

It was seen in Art. 169 of the chapter on Velocities that the x-component of the velocity of the point E is given by

$$V^x_e = R\omega_2\left[\sin\theta + \left(1 - \frac{h}{L}\right)\frac{1}{2}\frac{R}{L}\frac{\sin 2\theta}{\cos\phi}\right]$$

Differentiating with respect to time,

$$A^x_e = \frac{d}{dt}(V^x_e) = R\omega_2\left[\cos\theta\frac{d\theta}{dt} + \left(1 - \frac{h}{L}\right)\frac{1}{2}\frac{R}{L}\left(\frac{\cos 2\theta}{\cos\phi}\frac{d(2\theta)}{dt} + \frac{\sin 2\theta\tan\phi}{\cos\phi}\frac{d\phi}{dt}\right)\right]$$

In the article mentioned above it was also shown that

$$\frac{d\phi}{dt} = \frac{R}{L}\frac{\cos\theta}{\cos\phi}\omega_2 \text{ and } \sin\phi = \frac{R}{L}\sin\theta$$

Therefore

$$A_e^x = R\omega_2^2\left[\cos\theta + \left(1 - \frac{h}{L}\right)\frac{1}{2}\frac{R}{L}\left(2\frac{\cos 2\theta}{\cos\phi} + \frac{2\sin\theta\cos\theta\tan\phi}{\cos\phi}\cdot\frac{R}{L}\frac{\cos\theta}{\cos\phi}\right)\right]$$

$$= R\omega_2^2\left[\cos\theta + \left(1 - \frac{h}{L}\right)\frac{R}{L}\left(\frac{\cos 2\theta}{\cos\phi} + \frac{R^2}{L^2}\frac{\sin^2\theta\cos^2\theta}{\cos^3\phi}\right)\right] \quad (77)$$

The vertical component of the velocity of the point E is

$$V_e^y = R\omega_2\frac{h}{L}\cos\theta$$

Differentiating,

$$A_e^y = R\omega_2\frac{h}{L}(-\sin\theta)\frac{d\theta}{dt} = -R\omega_2^2\frac{h}{L}\sin\theta \quad (78)$$

If h is equal to zero, the expression A_e^x becomes the acceleration of the crosshead

$$A_h = R\omega_2^2\left[\cos\theta + \frac{R}{L}\left(\frac{\cos 2\theta}{\cos\phi} + \frac{R^2}{L^2}\cdot\frac{\sin^2\theta\cos^2\theta}{\cos^3\phi}\right)\right] \quad (79)$$

This is the exact expression. The following approximate expression is, however, sufficiently accurate for most purposes.

$$A_h = R\omega_2^2\left[\cos\theta + \frac{R}{L}\cos 2\theta\right] \text{ (approximately)} \quad (80)$$

Similarly

$$A_e^x = R\omega_2^2\left[\cos\theta + \left(1 - \frac{h}{L}\right)\frac{R}{L}\cos 2\theta\right] \text{ (approximately)} \quad (81)$$

CHAPTER XII

STATIC FORCES IN MACHINES

The analysis of the forces acting in any machine is based upon the fundamental principle which states that the system composed of all the external forces and all the inertia forces acting upon any single member of the machine is a system in equilibrium. The forces acting in machines having plane motion are for the most part situated in parallel planes. In such cases it is customary to analyze the system as though all its forces were situated in the same plane, the couples due to the offset of the forces being disregarded. This method will be employed in the following articles unless otherwise stated. In the *static force analysis* of a machine the inertia forces of the machine members are disregarded. If the inertia forces are taken into account, the analysis is called a *dynamic force analysis.* In either case the effect of friction may be taken into account or disregarded depending upon the requirements of the problem. The present chapter deals only with the static force analysis of machines.

184. Reactions between Members with Friction Disregarded.—In the present article and in the next few articles the effect of friction will be disregarded. An analysis in which friction is disregarded is useful in obtaining preliminary results, and in the case of well-lubricated machinery these results are often all that is required. In such an analysis the results obtained represent ideal conditions, *i.e.*, 100 per cent efficiency.

The forces acting upon a machine member are in general applied at its points of contact with other members of the machine, the two most important exceptions being the dead-weight forces and the inertia forces. This contact is usually made by means of turning or sliding pairs, and sometimes by means of higher pairs. Whatever the type of connection, the forces exerted by one member upon another are, if friction is disregarded, normal to the surfaces of the members at their points of contact.

When the contact between two members of a machine is made by means of a turning or sliding joint, the force exerted by one of the members upon the other is not applied at one point alone, but is the resultant of a number of elementary forces distributed over a considerable area. These elementary forces have directions normal to the surfaces at their corresponding points of contact. In the case of the turning joint the line of action of each elementary force passes through the center of the joint, and hence the resultant of all the elementary forces will also pass through the center of the joint. In the case of the sliding joint the elementary forces are all normal to the same plane, and hence their resultant will also be normal to the plane. The line of action of this resultant, however, cannot be determined from

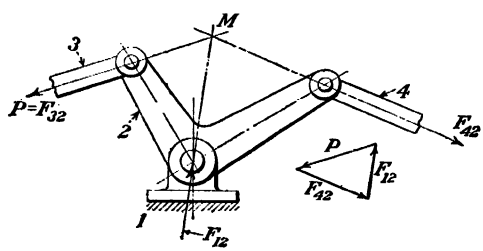


FIG. 271.

the characteristics of the joint alone. In order to simplify the analysis, the position of the line of action of the resultant is sometimes assumed to be situated at the center of gravity of the contact area, but it is usually determined from the conditions of equilibrium.

185. Analysis of a Bell Crank.—A simple example of a static force analysis is shown in Fig. 271. The bell crank 2 is part of a system of levers, and it is connected to the rest of the system by means of the rods 3 and 4. The force P is assumed to be known, and it is required to determine the other forces acting on the bell crank. The forces exerted by the rods are assumed to act along their center lines. With regard to the notation, F_{42} means the force exerted by link 4 upon link 2; F_{24} would be the equal and opposite force exerted by link 2 upon link 4. This notation will be employed very extensively in the force analyses to follow.

Three forces act upon the bell crank 2: the known force P, the force exerted by rod 4, and the force exerted by the bearing

1. The line of action of F_{42} is completely known. The resultant of P and F_{42} must pass through the intersection M of their lines of action. The remaining force F_{12}, which passes through the center of the bearing, must be equal and opposed to the resultant of P and F_{42}, and hence it also must pass through the point M. The directions of all three forces being known, the force polygon may be drawn as shown in the figure and the magnitudes of F_{12} and F_{42} thus determined.

186. Steam-engine Mechanism.—In the force analysis of a machine, it is necessary to assume either that the effort is known or that the useful resistance is known. In the case of the steam-engine mechanism shown in Fig. 272*a*, the effort P due to the

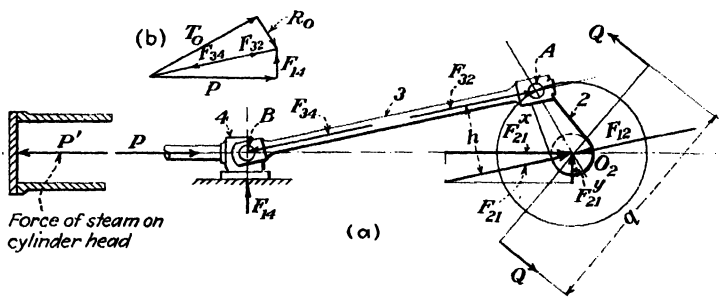


FIG. 272.

pressure of the steam against the piston is the known force. The useful resistance is the unknown couple Qq, which acts upon the crankshaft and which is equal and opposed to the torque exerted *by* the latter.

The first step in the analysis is to consider the forces acting upon the crosshead (link 4). These are as follows:

P = the known effort.
F_{14} = the reaction of the guides.
F_{34} = the force exerted by the connecting rod.

Of these three forces only the effort is known. Owing to the fact, however, that only two forces act upon the connecting rod, namely, a force at A and a force at B, it is evident that the line of action of F_{34} must be coincident with the line AB. The resultant of P and F_{34} must pass through the intersection B of their lines of action; and since only three forces act upon the crosshead F_{14} must also pass through the point B. The direc-

tion of F_{14} must be normal to the guides as shown. All three forces, therefore, being known as to direction, the force polygon can be drawn as shown at *b* in the figure and the magnitudes of F_{34} and F_{14} thus determined.

The force F_{32} exerted by the connecting rod upon the crank, is obviously equal and opposed to F_{34}. The other forces acting upon the crank are F_{12}, the main bearing pressure, and the forces of the unknown couple Qq. In order to hold the couple Qq in equilibrium, the force F_{12} must form a couple with F_{32}; that is, F_{12} must be equal and parallel to F_{32} and must have the opposite direction. The moment of the couple Qq is equal to $F_{32}h$; the latter is usually called the turning moment. If F_{32} is resolved at the crankpin into the components T_0 and R_0, perpendicular and parallel respectively to the crank, then the perpendicular component T_0 is called the turning effort. It is evident from the figure that $\frac{F_{32}}{T_0} = \frac{O_2A}{h}$; therefore $F_{32}h = T_0 \cdot \overline{O_2A}$.

During any one revolution of the crank the effort P is a constantly varying quantity, and the turning effort T_0 is subject to corresponding variations. On the other hand the actual useful resistance overcome by the engine is ordinarily fairly constant, and in order to prevent violent fluctuations in the speed of the crankshaft, it is in general necessary to make use of a flywheel. The function of the latter is to store up energy when P is too great and to deliver this energy to the crankshaft when P is too small. It is evident that where a flywheel is used, the couple Qq is not the actual useful resistance but is rather the combined effect of this resistance and the inertia of the flywheel.

For completeness an analysis of the forces acting upon the frame should be made, although this analysis is usually omitted. The forces acting upon the frame are as follows:

1. The force P', which is equal and opposed to P.
2. The force F_{21}, which is equal and opposed to F_{12}.
3. The force F_{41}, which is equal and opposed to F_{14}.
4. The forces due to the weight of the frame and foundation.

Since the vector sum of P, F_{14}, and F_{12} is zero, it is evident that the vector sum of P', F_{41}, and F_{21} will also be zero. The latter three forces, however, are not in equilibrium, as may be seen from the following analysis. Let F_{21} be resolved at the center of the main bearing into its horizontal and vertical components

F^x_{21} and F^y_{21}. The horizontal component is equal and opposed to P' and is thus neutralized, but the vertical component F^y_{21} forms a couple with the force F_{41}. This couple, which is equal to the couple Qq, tends to rotate the engine as a whole about the crankshaft and must be held in equilibrium by the forces due to the weight of the frame and the foundation.

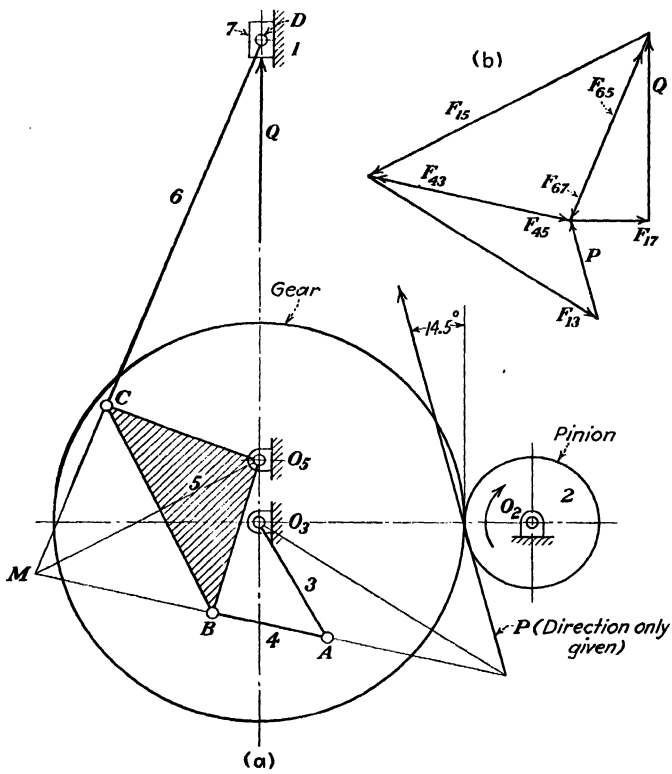


FIG. 273.

187. Drag-link Mechanism.—The static force analysis of a somewhat more complicated mechanism is shown in Fig. 273*a*. It is required to determine the effort P that must be applied in the form of pressure upon the teeth of gear 3 in order to overcome the known resistance Q. The first part of the analysis is the same as that of the steam engine mechanism discussed in the preceding article. The forces acting upon the ram (link 7) are

Q, F_{17} and F_{67}. The magnitudes of F_{17} and F_{67} may be determined from the force polygon shown at *b*. The forces acting upon link 5 are F_{65}, F_{45}, and F_{15}. The lines of action of F_{65} and F_{45} are completely known, and hence the intersection M of these lines can be determined. F_{15} must pass through this point and through the center of the bearing O_5. The directions of all three forces being known, their magnitudes may be determined from the force polygon. The forces acting upon link 3 may be determined in a similar manner, and the tooth pressure P thus obtained.

188. Friction.—Part of the energy supplied to every machine must be used in overcoming the frictional forces occurring within the machine. These frictional forces are classified according to the manner in which they are set up, the most important being sliding friction, rolling friction, and the frictional resistance that a wrapping connector offers to being bent around a sheave or pulley. In pumps and in those cases where a fluid is used for the transmission of power, part of the energy is used in overcoming fluid friction.

Sliding friction is the resistance offered to motion, or the tendency to motion, when two surfaces are rubbed on each other. This type of friction may be materially reduced by the use of lubricants, the action of the latter being partially to separate the rubbing surfaces, thus substituting fluid friction for a part of the sliding friction.

Rolling friction is the resistance offered to motion when two surfaces roll on each other. The fact that this kind of friction is much less than sliding friction is the reason for substituting ball and roller bearings for the ordinary turning joint in cases where it is desired to reduce the frictional resistance to a minimum.

The frictional resistance of a wrapping connector is an internal friction caused by the fibers of the connector sliding over one another.

189. Sliding Friction.—In dealing with sliding friction, it is customary to assume that the so-called *law of proportionality* is strictly true. This law, which is but an approximation, states that the friction is proportional to the normal force pressing the rubbing surfaces together. It may be represented by the equation $F = \mu N$, where F is the force of friction, N the normal force, and μ a quantity called the coefficient of friction. This coefficient is usually assumed to be the same for all the

points of contact of any one joint, but it is not necessarily assumed to be the same for all joints. The force of friction at any point of contact has a direction and line of action tangent to the rubbing surface at that point.

a. Sliding Joint.—A simple sliding joint consisting of a block moving along a plane surface is shown in Fig. 274. The block 2 is assumed to be under the action of a system of forces (not shown) that cause it to move in the direction indicated and that press it against the flat surface 1. The normal unit pressure exerted by the flat surface upon the block will not in general be uniform; *i.e.*, it will vary in magnitude at different points of the contact area. Let dN be the elementary normal force acting at A and μ the coefficient of friction. The elementary force of friction at A is

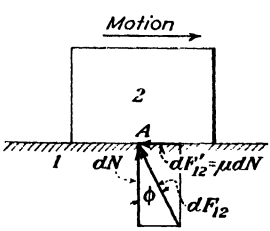


Fig. 274.

$$dF'_{12} = \mu dN$$

The resultant of the elementary force of friction and the elementary normal force dN is the total elementary force dF_{12} exerted upon the block 2 by the flat surface 1 at the point A. The angle ϕ that the direction of dF_{12} makes with the normal is called the angle of friction. This angle may be determined in the following manner:

$$\tan \phi = \frac{dF'_{12}}{dN} = \frac{\mu dN}{dN} = \mu$$

$$\phi = \text{arc tan } \mu \qquad (82)$$

It is evident from the equation that if μ is the same for all points of the contact area, the angle ϕ is also the same. It follows that the resultant of all the elementary total forces will make an angle ϕ with the resultant normal force, and that the lines of action of these two forces will pass through the same point of the contact area.

Now let the case shown in Fig. 275 be considered. The block 2 is pressed against the flat surface 1 by the known force Q and is moved along the flat surface under the influence of the force P, the latter being known as to line of action but unknown as to magnitude. For equilibrium the force exerted upon the sliding block by the flat surface must pass through the point A.

If friction is taken into account, the force F_{12} must be inclined to the normal by an amount equal to the angle of friction. If ϕ is this angle, the direction of F_{12} is as shown. The magnitudes of P and F_{12} may be determined from the force polygon shown. If F_{12} is resolved into horizontal and vertical components at its point of application, it is seen at once that the horizontal component is equal to P and has the opposite direction, and that the vertical component is equal and opposite Q. If there were no friction, the force P would be zero, and the resultant normal force F^y_{12} would be directly opposed to Q; *i.e.*, it would have the same line of action as Q. It is evident, therefore, that the effect of friction has been to shift the line of action of F^y_{12} in the same direction as that of the motion of the block.

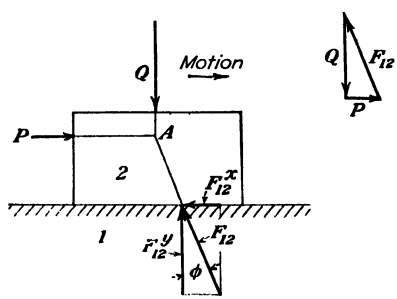


FIG. 275.

b. Turning Joint.—In a perfectly lubricated bearing, the frictional resistance offered to the turning of the journal is entirely due to the viscosity of the lubricant. In such cases the law of proportionality does not hold, even approximately, as the frictional resistance is very nearly independent of the load. Most bearings, however, are far from being perfectly lubricated, and this is particularly true of the turning joints found in linkages. The frictional resistance of these imperfectly lubricated bearings is approximately proportional to the load, and since in any case the frictional resistance is small, it is convenient to assume for the purposes of analysis that the law of proportionality holds exactly.

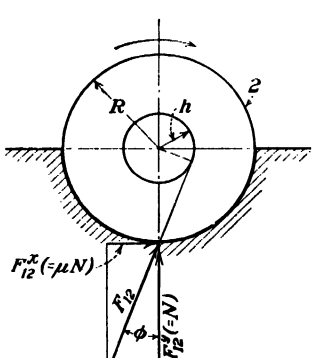


FIG. 276.

The motion of a journal in a bearing is, therefore, for each element of surface the same with respect to friction as the sliding action shown in Fig. 274. The resultant force F_{12} is laid off at

an angle ϕ with the radius as shown in Fig. 276. This force may be considered the resultant of the resultant normal force N and the resultant tangential force μN. The force F_{12} is tangent to the small circle of radius h. The radius of this circle is $h = R \sin \phi$. But since angle ϕ is very small, $\sin \phi = \tan \phi$ (approximately). Hence $h = R \tan \phi = \mu R$. This approximation (assuming the sine and tangent to be equal) is permissible in the case of small angles and is sufficiently accurate in this case because the angle ϕ is very small and also because the coefficient μ is a quantity that may vary between wide limits. It is evident from the foregoing equation that if μ remains constant in accordance with the law of proportionality, the distance h will remain constant also and will be independent of the load. It follows that the line of action of the resultant force due to a weight of any size will be tangent to a circle of radius h drawn about the center of the journal. This circle is called the *friction circle* and is dependent only upon the radius of the journal and the coefficient of journal friction. The force exerted by the journal upon the bearing is the only other force acting upon the bearing, from which it follows that this force also must be tangent to the friction circle.

The friction circle is of considerable service in a graphical analysis. Its radius may be calculated as soon as the coefficient of journal friction is known or assumed, after which it may be drawn in and used to locate the line of action of the force exerted by the journal upon the bearing, or vice versa. Whatever the direction of the force, its line of action must be tangent to the friction circle.

c. Higher Pair.—The friction analysis of a higher pair is almost identical with that of a sliding pair. An example is the locomotive driving wheel shown in Fig. 277. The normal reaction of the rail upon the wheel is directed toward the center of the wheel. The total reaction of the rail upon the wheel has a direction differing from that of the normal by an amount equal to the angle ϕ. It is evident that the total reaction of the rail exerts a turning moment on the wheel that opposes the turning moment due to the force exerted by the driving rod.

d. Magnitude of Friction.—The magnitude of the coefficient of friction in any of the three cases discussed above is very much dependent upon the conditions as to lubrication. In the case of

a well-lubricated bearing, it may be as low as 0.001, or even less, and in the case of a dry flat slide, it may be as high as 0.30. In actual practice the coefficient rarely approaches either extreme. In a well-lubricated sliding joint, the value of μ varies from about 0.02 to 0.04, while for a well-lubricated bearing it varies from about 0.01 to 0.02. In the case of gears and cams running in an oil bath, the value of μ is about the same as it is for a sliding joint. With the intermittent lubrication ordinarily found in connection with the turning and sliding joints of linkages, these values are

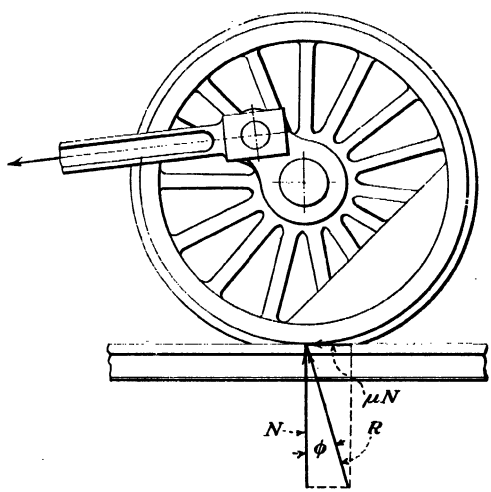


FIG. 277.

considerably higher, say about 0.09 for sliding joints and gears, and about 0.06 for bearings. The friction between a locomotive driving wheel and the rail is represented by a value of μ varying from 0.15 to 0.25. In a graphical force analysis it is desirable to be on the safe side, and for this reason the following values are given for problem work:

Sliding joints, cams, and gears	0.12
Turning joints	0.10
Dry friction	0.20

190. Friction Analysis of Bell Crank.—An application of the methods described in the preceding article is given in Fig. 278. The effort P acts through the bell crank there shown, overcoming the useful resistance Q. The only other force acting upon the

bell crank is F_{12}, the force exerted by the bearing. The line of action of F_{12} must pass through the intersection of the lines of action of P and Q. It must also be tangent to the friction circle drawn about the center of the journal. The radius of the friction circle is given by $r = \mu R$, where μ is the coefficient of journal friction and R is the radius of the journal. In order to decide on which side of the friction circle the force F_{12} should be drawn, the direction of the motion, or the tendency to motion, should be known. Then since F_{12} is the force exerted by the bearing upon the bell crank, it should be drawn so as to oppose the motion of the latter relative to the former. The direction of the three forces being known, the force polygon can be drawn and their relative magnitudes determined.

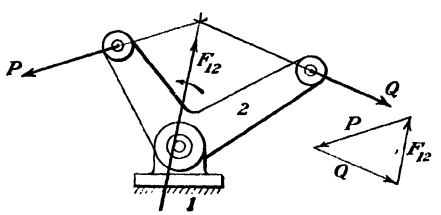


FIG. 278.

191. Efficiency.—The efficiency of a machine is defined in general as the ratio of the energy that a machine delivers in a given interval of time to the energy that it receives during the same interval of time. In many machines the motions of the members, as well as the forces acting on them, vary for different positions of the machine. In such cases the efficiency at any instant, called the instantaneous efficiency, is determined by the ratio of the energy supplied and delivered during an infinitesimal interval of time. The average efficiency is that corresponding to the time required for the machine to make one complete cycle. The word *efficiency* as ordinarily used means *average efficiency*.

The efficiency of a machine may be expressed algebraically as follows:

$$Eff. = \frac{W_1 - W_f}{W_1} = \frac{W_2}{W_2 + W_f} \tag{83}$$

where W_1 is the input, W_2 the output, and W_f the lost work, all for the same interval of time. If, for a given interval, the effort

and the useful resistance remain constant, then the quantities W_1 and W_2 may be expressed as follows:

$$W_1 = Pp \qquad W_2 = Qq$$

where P is the driving force, p is the distance through which it acts during the given interval of time, Q is the resisting force at the point where the net energy of the machine is being delivered, and q is the distance through which the force Q acts. Since the input is equal to the output plus the lost work,

$$Pp = Qq + W_f$$

The foregoing equation leads to a means of expressing efficiency in terms of forces alone. Let Q be the useful resistance overcome by the driving force P in the actual machine, and let Q_0 be the useful resistance which could be overcome by P if the machine were able to operate without losing any work. Then

$$Pp = Qq + W_f = Q_0q$$

$$Eff. = \frac{W_1 - W_f}{W_1} = \frac{Pp - W_f}{Pp} = \frac{Qq}{Q_0q} = \frac{Q}{Q_0} \tag{84}$$

Again let Q be the useful resistance overcome by the driving force P in the actual machine, and let P_0 be the force that would be required to overcome this resistance were the machine able to operate without loss of work. Then

$$P_0p = Pp - W_f = Qq$$

$$Eff. = \frac{W_2}{W_2 + W_f} = \frac{Qq}{Qq + W_f} = \frac{P_0p}{Pp} = \frac{P_0}{P} \tag{85}$$

If the forces P and Q are constantly varying, the ratios Q/Q_0 and P_0/P represent the instantaneous efficiency of the machine.

192. Effect of Friction in the Engine Mechanism.—The friction analysis of the engine mechanism is shown for one position of the mechanism in Fig. 279. Here the force P acts on the crosshead (link 4), and it is required to determine the other forces acting in the machine. The first step in the analysis is to assume a coefficient of journal friction, not necessarily the same for each joint, and to calculate the radii of the friction circles for the turning joints O, A, and B. Next let the forces acting on the connecting rod be considered. These are F_{43} and F_{23}. The angular

motion of the connecting rod relative to the crosshead is counterclockwise, and to oppose this motion the force F_{43} must be tangent to the upper part of the friction circle at B. Similarly, the motion of the connecting rod relative to the crank is counterclockwise, and hence F_{23} must be tangent to the lower part of the friction circle at A. The common line of action of F_{43} and F_{23} is thus determined.

The forces acting on the crosshead are P, F_{14}, and F_{34}. The force P and the direction of the force F_{34} are known and may be laid out in the force polygon at b. The polygon shows that the guide reaction F_{14} must be directed upward, and the direction of

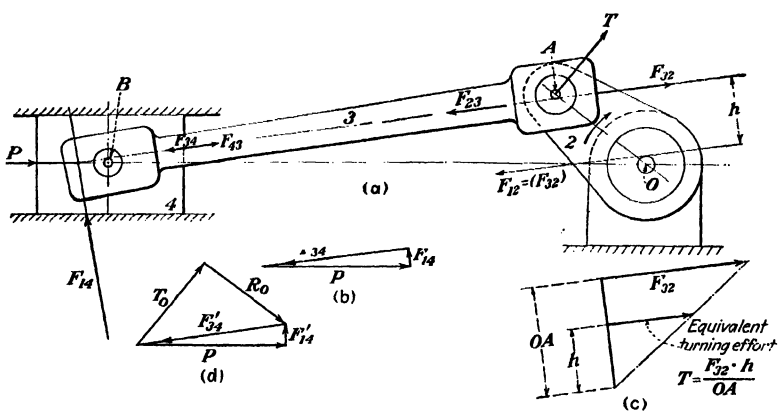


FIG. 279.

the motion of the crosshead shows that it must be inclined to the left, the angle of this inclination being determined by the value assumed for the coefficient of sliding friction. The direction of F_{14} being determined, the polygon may be completed as shown. Attention is called to the fact that the sense of the inclination of F_{14} is such as to decrease the force transmitted through the connecting rod to the crank. The line of action of F_{14} is determined by the fact that it must pass through the intersection of the lines of action of P and F_{34}.

The forces acting upon the crank are F_{32}, F_{12}, and the forces of the couple exerted on the main shaft by the useful resistance. In order to balance the couple, the force F_{12} must be equal and parallel to F_{32} and must have the opposite direction. In order to oppose the motion of the crank, F_{12} must be tangent to the

upper part of the friction circle at O. The moment arm h of the couple formed by the forces F_{32} and F_{12} is therefore determined. The equivalent turning effort is the force that, if applied at the crankpin in a direction perpendicular to the crank, would have a moment with respect to the point O equal to the moment of the couple $F_{32}h$. If T is this force and OA is the crank radius, then

$$T = \frac{F_{32}h}{OA}$$

A graphical scheme for obtaining the vector length corresponding to T is shown at c.

The instantaneous efficiency of the mechanism may be determined by making a static analysis as shown at d, disregarding friction and determining the turning effort T_0 that would be given by the force P if there were no friction. The efficiency for the position shown is then given by the ratio T/T_0. In the original drawing the length of the vector representing T was 1.18 in.; the length of the vector representing T_0 was 1.52 in. The corresponding efficiency is 77.6 per cent. The reason for the low efficiency is that the effect of friction was purposely exaggerated in order to facilitate the demonstration.

193. Stone Crusher.—The friction analysis of the stone crusher shown in Fig. 280 is very similar to that of the engine mechanism discussed in the preceding article. In this case, however, the useful resistance is known, and it is required to determine the equivalent turning effort that must be applied at the crank in order to overcome resistance Q.

The radii of the various friction circles having been determined, the lines of action of the forces acting in the mechanism may be determined as shown. The force Q may then be laid off as shown at b and the remainder of the polygon completed. The equivalent turning effort is determined by means of the construction at c.

In order to determine the efficiency, let the static force analysis, disregarding friction, be made on the skeleton outline of the mechanism as shown in Fig. 281, and the turning effort T_0 corresponding to 100 per cent efficiency thus determined. The ratio T_0/T is equal to the instantaneous efficiency. In the determination made on the original drawing this was found to

be 68 per cent. Here again the low efficiency is explained by the fact that the effect of friction has been purposely exaggerated.

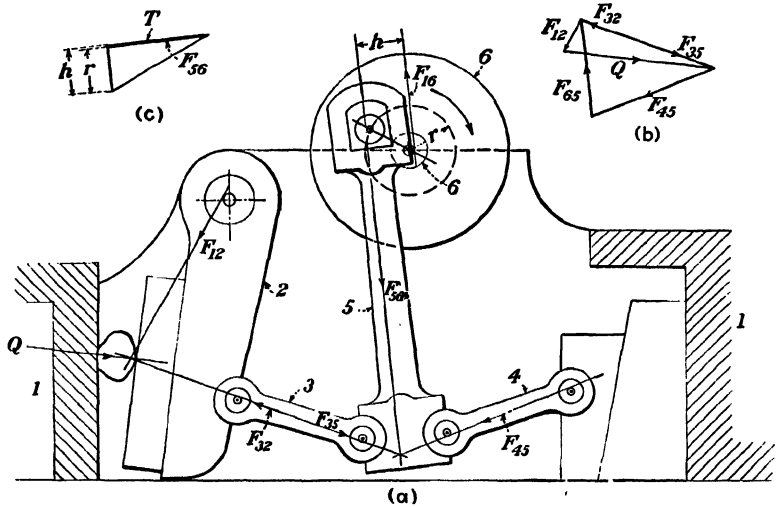


FIG. 280.

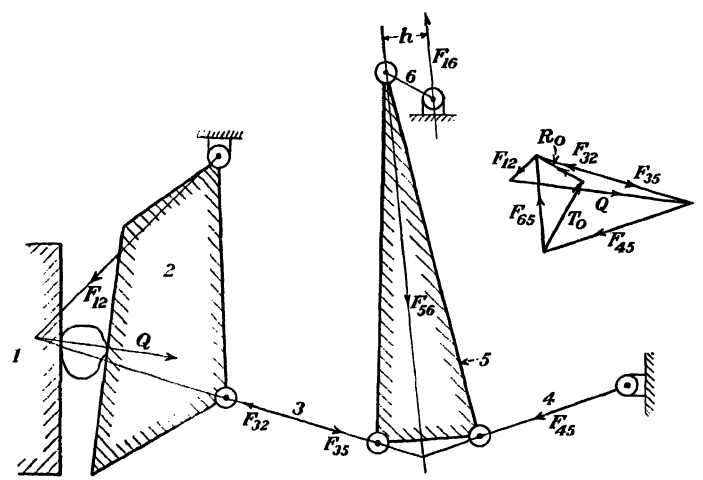


FIG. 281.

194. Cam Friction.—As stated in Art. 189 the friction analysis of a higher pair is very similar to that of a flat sliding joint. An example where the higher pair consists of the working profiles of a

pair of cams is shown in Fig. 282. The left-hand cam (link 2) drives the right-hand cam (link 3) by means of pressure exerted at the point of contact A. The total force F_{23} exerted by cam 2 upon cam 3 is inclined to the common normal by an amount equal to ϕ, the angle of friction, and the sense of this inclination is such as to oppose the relative motion of the two cams. The direction of the relative motion, which is determined from kinematic considerations, is indicated by the two arrows shown at the point A.

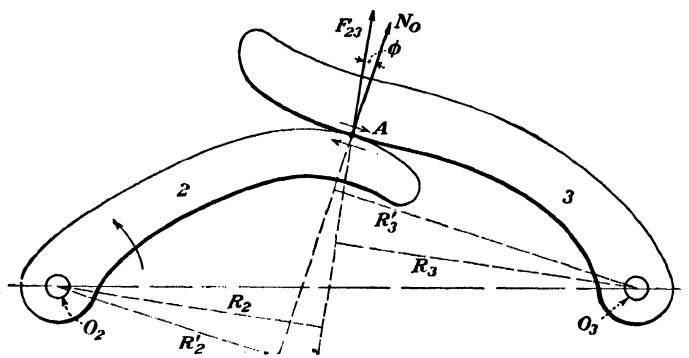


FIG. 282.

If Pl is the turning moment exerted on the driving cam by the shaft to which it is keyed, it is evident that

$$Pl = F_{23}R_2$$

Likewise, if Qh is the resisting moment exerted on cam 3,

$$Qh = F_{23}R_3$$

These two equations may be combined, thus expressing the resisting moment in terms of the effort:

$$Qh = Pl\frac{R_3}{R_2}$$

If friction were eliminated from the mechanism, the force exerted by cam 2 upon cam 3 would act along the common normal. If N_0 were this force and Q_0h the corresponding resisting moment then

$$Pl = N_0R_2' \qquad Q_0h = N_0R_3'$$

and

$$Q_0 h = Pl\frac{R'_3}{R'_2}$$

The efficiency of the mechanism is

$$Eff. = \frac{Q}{Q_0} = \frac{R_3}{R_2} \cdot \frac{R'_2}{R'_3}$$

An application of this expression to the original figure gave

$$Eff. = \frac{3.18}{2.82} \cdot \frac{2.45}{3.31} = 0.805 = 80.5 \text{ per cent}$$

Attention is called to the fact that in the above analysis the friction of the bearings has not been taken into account.

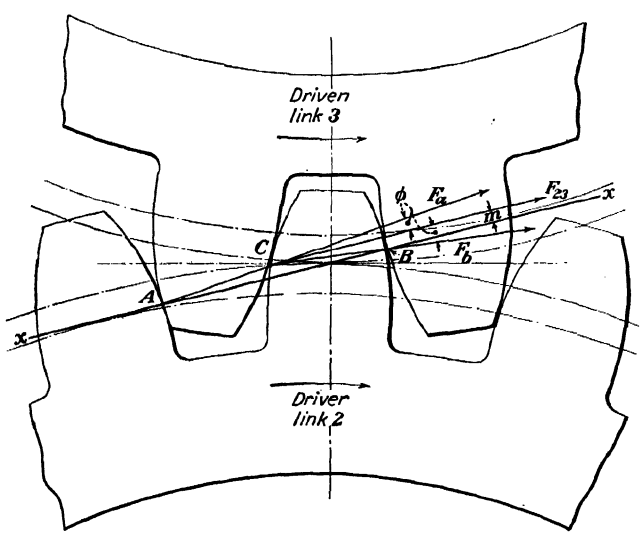


FIG. 283.

195. Spur Gears.—The friction analysis of a pair of gears is fundamentally that of a pair of cams. In Fig. 283 a pair of one-pitch, 24-tooth, 14½-deg., standard, involute spur gears have been partially drawn. In the position shown there are two pairs of teeth in contact. It will be recalled that in the case of involute gears the common normal to the profiles of a pair of teeth coincides with the path of contact for all positions of the pair, the

path of contact being, of course, a straight line. It follows that the normal forces of the driver upon the driven have at both A and B the same line of action xx. The resultant normal force is therefore simply the sum of the normal forces at A and B. The total force F_a is swung by friction from the normal through an angle ϕ; F_b is swung through the same angle but in the opposite sense. The resultant total force F_{23} must pass through the intersection C of the lines of action of F_a and F_b. If the latter are assumed to be equal, the direction of the resultant F_{23} will be parallel to the common normal. The effect of friction therefore has been to shift the line of action of the total force exerted by the driving gear upon the driven gear parallel to itself and away from the center of the driver. If Pl is the driving moment, Qh the resisting moment, r_2 and r_3 the radii of the base circles, and m the displacement of the line of action of F_{23}, then

$$Pl = F_{23}(r_2 + m) \qquad Qh = F_{23}(r_3 - m)$$

$$Qh = Pl\frac{r_3 - m}{r_2 + m}$$

Without friction m will be equal to zero and Q_0h the corresponding resisting moment, will be

$$Q_0h = Pl\frac{r_3}{r_2}$$

The efficiency will then be

$$Eff. = \frac{Q}{Q_0} = \frac{r_2}{r_3}\cdot\frac{r_3 - m}{r_2 + m} = \frac{1 - \dfrac{m}{r_3}}{1 + \dfrac{m}{r_2}} =$$

$$1 - \left(\frac{m}{r_3} + \frac{m}{r_2}\right) \text{ very nearly} \quad (86)$$

In the case shown in the figure the coefficient of friction was assumed equal to 0.1, and the resulting displacement of the line of action was 0.158 in. The radii of the base circles are each equal to 11.618 in. Hence

$$Eff. = 1 - \left(\frac{0.158}{11.618} + \frac{0.158}{11.618}\right) = 1 - 0.027 = 0.973$$

$$= 97.3 \text{ per cent}$$

It is worthy of note that even with a coefficient of friction greater than would ordinarily be expected, the efficiency of the pair of gears is high. This is in accord with numerous experiments made to determine the efficiency of spur gears.

The special case just discussed may be taken as a basis for the treatment of all the types of spur gears ordinarily encountered in practice. This treatment is justified by the fact that the efficiency of spur gearing is so high that the use of more exact methods, particularly in a graphical analysis, would give results

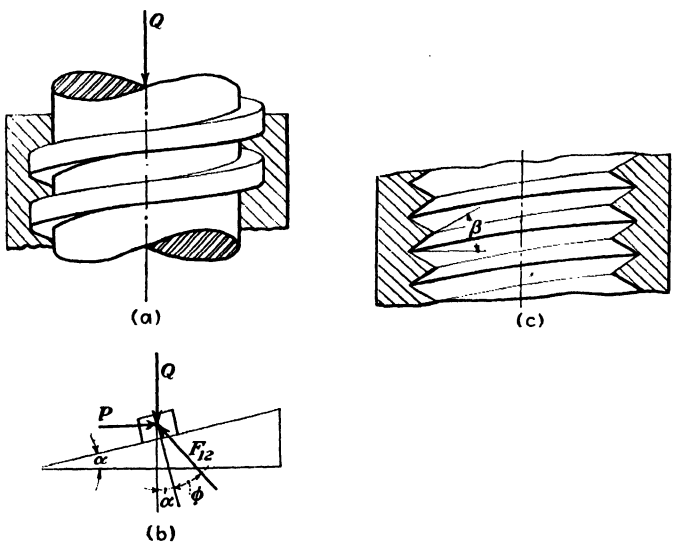


FIG. 284.

differing from those of the approximate method by a very small amount. In applying this method the fundamental assumptions are that two pairs of teeth are always in contact, that the load is equally divided between both pairs, and that the pressure angle is a constant. Gears having tooth profiles other than involute may be treated as involute gears having a pressure angle equal to the average pressure angle of the actual gears. The value of m for a given coefficient of friction may be computed as follows. Referring to Fig. 283,

$$m = \mu \frac{AB}{2}$$

The distance AB is the normal pitch of the gears and is hence equal to the circular pitch p' multiplied by the cosine of the pressure angle θ. Therefore

$$m = \frac{\mu p' \cos \theta}{2} \tag{87}$$

196. Screw Threads.—The force analysis of a square-thread screw is merely that of a block sliding on an inclined plane, the latter being wrapped around a cylinder. The screw itself is illustrated in Fig. 284*a*. The analysis is based upon the assumption that the resultant effect of the forces exerted upon the screw by the nut in which it turns is equivalent to that of a single force concentrated at the mean thread radius. A development of one thread of the nut is shown at *b*, and the block may be taken to represent the screw. The block is subjected to the vertical load Q and is moved up the inclined plane (thread of nut) under the action of the horizontal force P, which is the turning effort exerted upon the screw reduced to the mean thread radius. The force of the plane upon the block is inclined to the normal by an amount ϕ, the latter being the angle of friction. If α is the mean helix angle, then

$$P = F_{12} \sin (\alpha + \phi) \qquad Q = F_{12} \cos (\alpha + \phi)$$

$$P = Q \frac{F_{12} \sin (\alpha + \phi)}{F_{12} \cos (\alpha + \phi)} = Q \tan(\alpha + \phi) \tag{88}$$

The efficiency may be determined by assuming ϕ equal to zero as follows:

$$P_0 = Q \tan \alpha$$

$$Eff. = \frac{P_0}{P} = \frac{\tan \alpha}{\tan (\alpha + \phi)} = \frac{1 - \mu \tan \alpha}{1 + \mu \cot \alpha} \quad \text{where } \mu = \tan \phi \tag{89}$$

The corresponding expressions for the case where the threads are not *square* as, for example, as shown at *c* in the figure, are as follows:

$$P = Q \frac{\tan \alpha + \mu \sqrt{1 + \dfrac{\tan^2 \beta}{1 + \tan^2 \alpha}}}{1 - \mu \tan \alpha \sqrt{1 + \dfrac{\tan^2 \beta}{1 + \tan^2 \alpha}}} \tag{90}$$

where β is half the thread angle as indicated in the figure. A very close approximation is to disregard $\tan^2 \alpha$ under the radical. This gives

$$P = Q\frac{\cos \beta \tan \alpha + \mu}{\cos \beta - \mu \tan \alpha} \tag{91}$$

The corresponding expression for the efficiency is

$$Eff. = \frac{\cos \beta - \mu \tan \alpha}{\cos \beta + \mu \cot \alpha} \tag{92}$$

197. Rolling Friction.—The motion of a roller or wheel along a smooth surface is always accompanied by a certain amount of

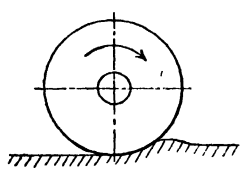


FIG. 285.

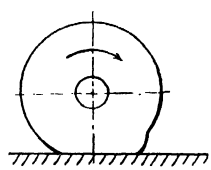


FIG. 286

friction, the effect of which is to oppose the motion of the roller. This friction is the result of the elastic or semielastic deformation of both the roller and the surface. A general idea of the effect of these two deformations may be had by considering them separately. In Fig. 285 the roller is assumed to be perfectly rigid and the surface is assumed to be elastic. Under these conditions the surface will be depressed, and a wave will be built up in front of the roller opposing its motion. If the roller is elastic and the surface rigid, the conditions shown in Fig. 286 will maintain. The roller will be deformed and will tend to bulge slightly on the leading side, again with the effect of opposing the motion.

The most convenient method of taking account of rolling friction is to consider that the normal force exerted upon the roller by the surface is shifted to one side as indicated in Fig. 287. The normal force N, after it has been thus shifted, forms a couple with the load Q upon the roller, the effect of this couple being to oppose the motion. The full force diagram is shown in Fig. 288. The force P that causes the motion is assumed to be applied at the center of the roller. The complementary static frictional force T forms a couple with P that is equal and

opposed to the couple formed by the forces Q and N. The force T is tangential to the surface; it is the tractive force required to overcome the resistance to rolling and is precisely similar to the much larger static frictional forces by means of which the

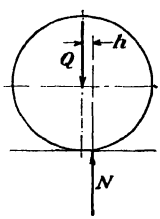


FIG. 287.

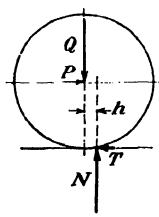


FIG. 288.

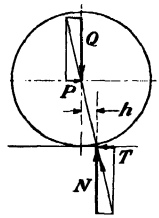


FIG. 289.

traction of a locomotive is secured. The total reaction of the surface upon the roller is the resultant of N and T. It must be equal and opposed to the resultant of Q and P as indicated in Fig. 289.

The offset h in the case of a steel roller moving on a steel surface is very small. Professor Heck, in his treatise on "Mechanics of Machinery," states that for an ordinary car wheel (about 33 in. in diameter) the value of h is likely to be about 0.02 in. In the case of a ball or roller bearing, it is very much smaller, too small in fact to be given any consideration in a graphical analysis. For an automobile wheel, on the other hand, h is apt to be fairly large, owing principally to the much larger deformation of the rubber tire.

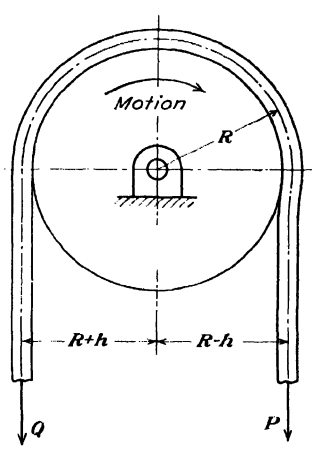


FIG. 290.

198. Friction of Wrapping Connectors.—The action of a rope in passing over a sheave is illustrated in Fig. 290. Owing to its internal friction, the rope first resists being bent around the sheave and then resists being straightened again. The net result is that the rope tends to force itself away from the running-on side of the sheave and tends to hug the running-off side. The combined effect of these two actions is to make the moment arm

of the force Q greater than that of the force P. For equilibrium, assuming that the magnitude of the offset is the same on both sides of the sheave,

$$P(R - h) = Q(R + h) \text{ Therefore } P = Q\frac{R + h}{R - h} \qquad (93)$$

The foregoing relation does not take account of the bearing friction of the sheave.

Very few data are available as to the value of the offset h. Professor Heck gives the following formula:

$$h = \left(0.10 + \frac{kd^2}{P}\right)d^2 \qquad (94)$$

where P = load in pounds.
d = diameter of rope in inches.
k = 10 for fiber rope.
= 60 for soft-iron wire rope.
= 120 for strong steel rope.

He states, however, that the frictional resistance of a rope is very much dependent upon its condition and that for a new stiff hemp rope the value of h may be 50 per cent higher than that given by the above equation, whereas for an old and soft rope it may be one-half or even one-third the value given by the equation.

Professor Leutwiler, in his text on "Machine Design," has summarized the results of some experiments on the efficiency of hoisting tackle. He has, however, included the effect of the friction of the sheave bearings, and this should be taken into account in making comparisons with the above formula. If proper allowance is made for bearing friction, it will be found that his results check the formula fairly closely.

199. Forces in Parallel Planes.—In all of the preceding articles the couples due to the offset of the forces have been disregarded. The reason for this is that the forces acting in machines are usually situated in parallel planes, and such a system may, by disregarding the couples, be analyzed as though all the forces were situated in the same plane. As a result the force analysis is very much simplified. Usually, these couples are of little importance. If the offset is of sufficient magnitude to make necessary the consideration of the couples, they may be determined by means of a very simple secondary analysis, once the coplanar analysis has been made.

CHAPTER XIII

INERTIA FORCES

Whenever the motion of a rigid body is changed in any manner whatsoever, inertia forces are set up that tend to resist the change of motion. The starting point in the discussion of these inertia forces is the inertia force of a particle, this being the resistance that the mass of the particle offers to any outside influence that tends to change its (the particle's) motion. The magnitude of the inertia force of a particle is equal to the product of its mass and acceleration. The line of action of this inertia force passes through the particle in a direction opposite that of the acceleration.

A rigid body may be considered as being made up of a large number of such particles held in fixed positions relative to each other. These particles are called the *elements* of the rigid body. The inertia force of any element is dependent upon its mass and its acceleration, and the inertia force of the body as a whole is the resultant of all its elementary inertia forces. Inertia forces are thus like gravity forces in that they belong to the body and enter into an equilibrium system with the forces impressed on the body. This conception of inertia forces is of great importance in the solution of force problems, since by introducing the inertia forces, they form, with the external forces on the body, a force system that is in equilibrium, thus reducing the kinetics problem to an equivalent statics problem.

As stated in the preceding chapter, the force analysis of a machine taking account of inertia forces is called a dynamic force analysis. In order to simplify the discussion, the effect of friction will be disregarded in the dynamic force analyses that follow. This is further justified by the fact that in most machines having inertia forces large enough to be taken into account the journals are fairly well lubricated, and in consequence the frictional forces are small.

200. Inertia Forces of a Rigid Body Having Plane Motion.— The elements of a rigid body having plane motion move in parallel

planes. Hence the inertia forces of these elements have lines of action situated in parallel planes, and, as a result, the inertia forces may conveniently be treated as though they formed a coplanar system. An analysis based on such an assumption neglects certain couples that are usually unimportant but may, if desired, be determined by means of a very simple secondary analysis. The assumption of a coplanar system is, in dealing with inertia forces, also equivalent to assuming that the mass of the body has been concentrated in the plane of motion.

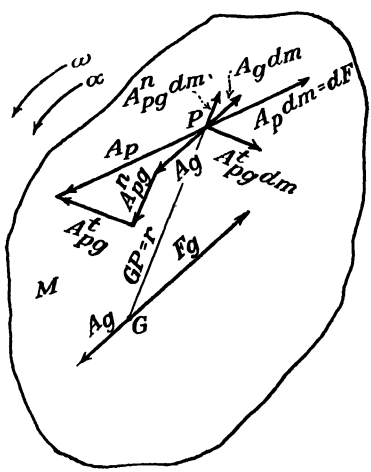


FIG. 291.

Referring to Fig. 291, let G be the center of gravity of the rigid body M, the mass of which is assumed to be concentrated in one plane, and let P be any element of the body, the mass of P being dm. The acceleration of the element P is

$$A_p = A_g + A^n_{pg} + A^t_{pg}$$
$$= A_g + r\omega^2 + r\alpha$$

where A_g is the acceleration of the center of gravity G, and ω and α are respectively the angular velocity and angular acceleration of the body. The inertia force dF of the element P has a direction opposed to that of the acceleration A_p, and a magnitude as given by

$$dF = A_p dm$$

The inertia force dF may be resolved, as shown in the figure, into the components $A_g dm$, $A^n_{pg}dm$, and $A^t_{pg}dm$. The resultant inertia force of the body as a whole is made up of:

1. The resultant of all forces like $A_g dm$.
2. The resultant of all forces like $A^n_{pg}dm$.
3. The resultant of all forces like $A^t_{pg}dm$.

The forces like $A_g dm$ are all parallel and opposite to the acceleration of the center of gravity G, and the magnitude of each is proportional to the mass of its corresponding element. They form, therefore, a system of parallel forces, the resultant of

which, F_g, passes through the center of gravity of the body as indicated in the figure, this last characteristic being due to the properties of the center of gravity. The magnitude of F_g is

$$F_g = \Sigma A_g dm = A_g \Sigma dm = MA_g \tag{95}$$

M being the mass of the body.

The forces like $A^n_{pg}dm (= r\omega^2 \cdot dm)$, all pass through the center of gravity G, and the magnitude of each is proportional to the mass of the corresponding element multiplied by the distance of the element from G, *i.e.*, proportional to rdm. The vector sum of such a concurrent system of forces is zero, again due to the properties of the center of gravity, and they cannot have a couple as a resultant (since all the forces pass through the same point). Hence the resultant of all forces like $A^n_{pg}dm$ is zero.

The magnitudes of the forces like $A^t_{pg}dm (= r\alpha \cdot dm)$ are proportional to the magnitudes of the corresponding forces like $A^n_{pg}dm$, and, since the direction of each is at right angles to the corresponding force like $A^n_{pg}dm$ it is evident that the vector sum of all forces like $A^t_{pg}dm$ is also zero. In this case, however, as the forces obviously do not pass through the same point, their resultant is a couple. Let moments be taken about the point G. Then, if T is the sum of the moments of all forces like $A^t_{pg}dm$, the moment of the resultant couple is:

$$T = -\Sigma A^t_{pg}dm \cdot r = -\Sigma r\alpha dm \cdot r = -\alpha \Sigma r^2 \cdot dm = -I\alpha \tag{96}$$

where I is the moment of inertia of the body with respect to the center of gravity G. The significance of the negative sign is merely that the sense of the suggested rotation of the couple is opposite that of the angular acceleration α.

The resultant of all the inertia forces of the rigid body is therefore the force F_g and the couple $-I\alpha$. The latter may be combined with the former as indicated at a and b in Fig. 292, giving thus a single force as the resultant inertia force of the body. The magnitude and direction of the resultant inertia force F are the same as those of $F_g (= MA_g)$, and the line of action is situated at a distance $h = I\alpha/F_g$ from the center of gravity G.

Attention is again called to the fact that the above analysis is based upon the assumption that the mass of the rigid body is concentrated in one plane, namely, a plane parallel to the plane of motion and containing the center of gravity G of the body.

Actually, however, the mass is distributed in three dimensions and in order completely to determine the net effect of the elementary inertia forces, it is necessary to take this fact into account. It can be shown that the total inertia force thus obtained is made up of the force and couple determined by the coplanar analysis described above, and a secondary couple tending to rotate the rigid body about an axis in the plane of motion. This secondary couple usually may be disregarded, in which case the analysis can be made on the coplanar basis described above. If the reference plane is a plane of symmetry and if the density of the body is uniform throughout, then the resultant secondary couple

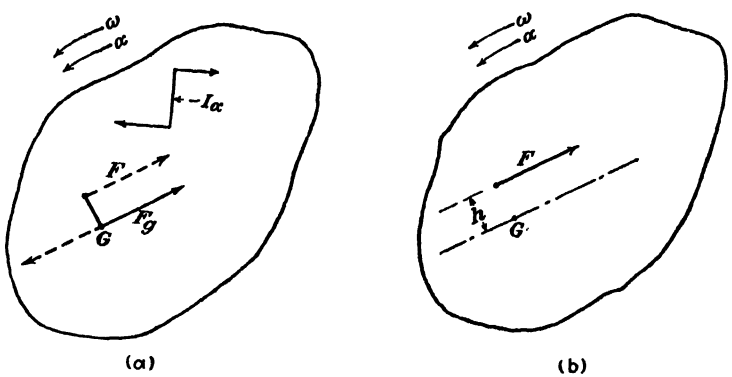


FIG. 292.

disappears. In the following articles it is always assumed that the mass is concentrated in one plane unless otherwise stated.

201. Inertia Forces of a Floating Link.—The engine connecting rod, having a motion of combined rotation and translation, is a link that affords an ideal illustration of the general case of inertia forces in a link having plane motion, as discussed in the preceding article. Moreover, the connecting rod is a typical example of what may be termed a "*floating link*" (*i.e.*, a link not fixed at any point), and the method outlined for the determination of its resultant inertia force will cover "floating members" in general.

In Fig. 293*a* is shown a connecting rod of mass M with the acceleration A_g of its center of gravity and its angular acceleration α given as indicated in the figure. From the discussion in the preceding article it is known that the resultant of the inertia forces of the link is made up of the force $F_g = MA_g$ passing

through the center of gravity G in a direction opposite that of the acceleration A_g, and a couple whose moment is $T = I\alpha$, where I is the moment of inertia with respect to G. This is indicated at b in the figure. Since the angular acceleration is clockwise, the sense of the inertia couple must be counterclockwise. The forces of the couple may be assumed, of course, to act on the link anywhere in its plane of motion, provided the moment of the couple remains constant.

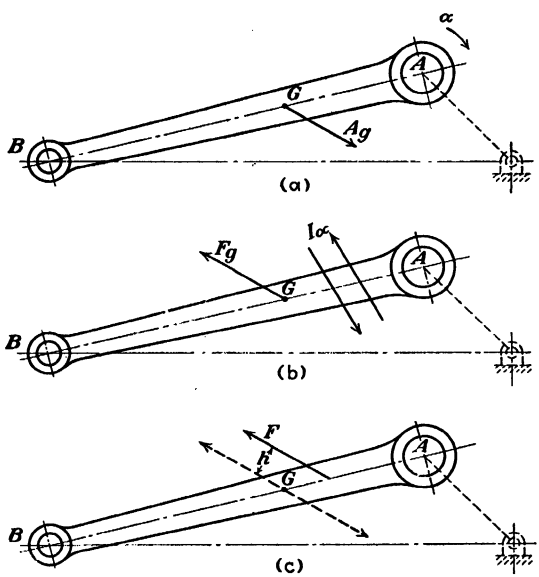


FIG. 293.

If it is desired to represent the resultant of the inertia forces as a single force, rather than a force and couple, this can be accomplished by letting the forces of the couple $= F$. Then the arm of the couple is

$$h = \frac{I\alpha}{F}$$

Placing the couple with one force F equal and opposite the force F_g through G as shown at c in the figure the resultant is the single force F, which is parallel to A_g, opposite in direction to A_g, and displaced a distance h from A_g. It is obvious that the moment Fh must oppose the angular acceleration of the link.

202. Inertia Force of a Link Rotating about a Fixed Center.—Figure 294 represents a link rotating about center O with an angular velocity ω and an angular acceleration α in the sense indicated. The mass of the link is M and its moment of inertia about its center of gravity G is I.

The resultant of the inertia forces must be a force $F = MA_g$, and a couple whose moment is $T = I\alpha$. The acceleration of the center of gravity is:

$$A_g = A_g^n \twoheadrightarrow A_g^t = \overline{OG} \cdot \omega^2 \twoheadrightarrow \overline{OG} \cdot \alpha$$

From the vector A_g the magnitude and direction of the resultant inertia force F can be determined. Its line of action is situated at a distance h such that,

$$h = \frac{I\alpha}{F}$$

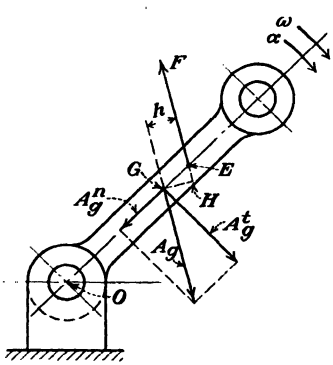


FIG. 294.

the distance h being measured from G in a direction such that the moment of F with respect to G tends to oppose α.

The inertia force F always passes through the same point E on the line OG extended regardless of the values of ω and α. This is shown by the following considerations:

$$\frac{\overline{GE}}{\overline{GH}} = \frac{A_g}{A_g^t}$$

$$\overline{GE} = \overline{GH}\frac{A_g}{A_g^t} = h\frac{A_g}{A_g^t} = \frac{I\alpha}{F} \cdot \frac{A_g}{\overline{OG}\alpha} = \frac{I}{MA_g} \cdot \frac{A_g}{\overline{OG}} = \frac{I}{M \cdot \overline{OG}}$$

From the above expression it may be observed that the distance GE is independent of ω and α and depends only on I, M, and OG. Therefore, with a *fixed* axis of rotation, the inertia force of the link always passes through a fixed point E on the link. This point E is called the center of percussion of the link with respect to the fixed axis of rotation O.

If k is the principal radius of gyration,

$$\overline{GE} = \frac{I}{M \cdot \overline{OG}} = \frac{Mk^2}{M \cdot \overline{OG}} = \frac{k^2}{\overline{OG}}, \text{ therefore } k^2 = \overline{OG} \cdot \overline{GE} \quad (97)$$

If a rigid body is permitted to swing freely as a pendulum, its period of vibration is the same as that of a simple pendulum having a length equal to the distance between the axis of oscillation and the center of percussion with respect to that axis, *i.e.*, a length equal to the distance OE for the case just discussed. This suggests an experimental method of determining the moment of inertia of a machine member. Referring to Fig. 295, let it be required to determine the moment of inertia of the connecting rod there shown. It is assumed that the position of the center of gravity G has been previously determined, that I is the moment of inertia with respect to G and that E is the center of percussion with respect to the axis of oscillation. The connecting rod is suspended on the knife-edge B, about which it is permitted to oscillate as a pendulum. The time required for a given number of complete oscillations (over and back) is noted, thus determining the time T for one complete oscillation. The center of percussion and moment of inertia of the rod are then determined from the pendulum formula (see any textbook on analytical mechanics) as follows:

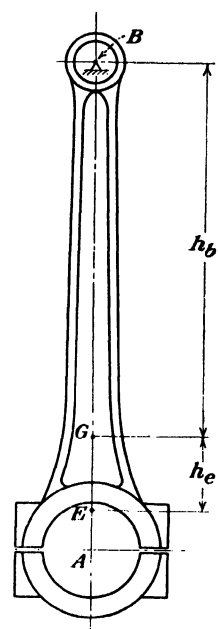


FIG. 295.

$$T = 2\pi\sqrt{\frac{BE}{g}} = 2\pi\sqrt{\frac{h_b + h_e}{g}}, \text{ where } g = 32.2 \text{ f.p.s.}^2 \quad (98)$$

The distance of the center of percussion E from the center of gravity G is then

$$h_e = \frac{T^2 g}{4\pi^2} - h_b \quad (99)$$

The moment of inertia I with respect to the center of gravity is

$$I = Mk^2 = Mh_bh_e = Mh_b\left(\frac{T^2g}{4\pi^2} - h_b\right) \qquad (100)$$

203. Transverse and Radial Components.—In problems involving inertia forces, it is often necessary to deal with a single force that acts upon a two-joint link and has a line of action crossing the link at some point between the centers of the two joints. An example is the force F_e in Fig. 296. In order to determine the other forces acting on link 3, let F_e be resolved at E into components perpendicular and parallel to the line AB. This is done in order to facilitate the taking of moments with respect to the point A. The perpendicular component F_e^{TA} is

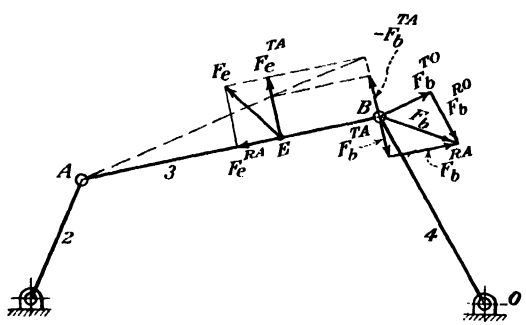


FIG. 296.

called the transverse component of F_e with respect to the point A, and F_e^{RA} is called the radial component of F_e with respect to A. It is evident that in taking moments about A the moment of the radial component F_e^{RA} is zero. The moment of the transverse component is $F_e^{TA} \cdot \overline{AE}$, and it follows from the principle of moments that this product is also the moment of the original force F_e with respect to the point A.

Besides F_e, only two other forces act upon link 3, namely, a force at A and a force at B. The moment of the force at A is zero. That of the force at B is equal, in accordance with the scheme outlined above, to the product of its transverse component F_b^{TA} and the distance AB. Since for equilibrium the sum of all moments must be zero,

$$F_e^{TA} \cdot \overline{AE} + F_b^{TA} \cdot \overline{AB} = 0$$

and

$$F_b^{TA} = -F_e^{TA}\frac{AE}{AB} \tag{101}$$

The significance of the negative sign is that the direction of F_b^{TA} is opposite that of F_e^{TA}. A graphical method of applying the moment relation to determine F_b^{TA} is shown in the figure.

The radial component F_b^{RA} is unknown, and unless something further is known regarding the total force F_b, no further progress is possible. Let it be supposed, however, that F_b^{TO}, the tangential component of the total force F_b with respect to O, has been determined from other considerations. Then, since F_b^{RO} and F_b^{RA} are known to be perpendicular to F_b^{TO} and F_b^{TA} respectively, it is evident that the extremity of the total-force vector F_b is situated at the intersection of the perpendiculars drawn through the extremities of F_b^{TO} and F_b^{TA}. After the total force F_b has been determined, the force acting at A may be determined in the usual manner.

204. Four-link Mechanism.—In the present article the dynamic force analysis of the four-link mechanism shown in Fig. 297 is discussed. The angular velocity and angular acceleration of link 2 are given as follows:

$$\omega_2 = 20 \text{ radians/sec.} \qquad \alpha_2 = 160 \text{ radians/sec.}^2$$

The necessary linear dimensions are given in the figure. The weights and moments of inertia of the various members are as follows:

$$W_2 = 4.65 \text{ lb.} \qquad W_3 = 2.17 \text{ lb.}$$
$$I_2 = 0.01379 \text{ slug-ft.}^2 \qquad I_3 = 0.00814 \text{ slug-ft.}^2$$
$$W_4 = 5.27 \text{ lb.}$$
$$I_4 = 0.02040 \text{ slug-ft.}^2$$

It is to be noted that the moments of inertia given above are in mass units where the unit of mass (W/g) is here called a *slug*.

It is required to determine (*a*) the inertia forces of the moving members, (*b*) the effort required to accelerate the mechanism, and (*c*) the effect of the inertia forces upon the frame.

a. Determination of Inertia Forces.—The first step is to make the complete acceleration determination. This, of course, would necessitate the use of the velocity polygon at *b*. The normal and

tangential components of the acceleration of the point A may be calculated as follows:

$$A_a^n = \overline{O_2A} \cdot \omega_2^2 = \frac{10}{12} \times \overline{20}^2 = 333 \text{ f.p.s.}^2$$

$$A_a^t = \overline{O_2A} \cdot \alpha_2 = \frac{10}{12} \times 160 = 133 \text{ f.p.s.}^2$$

Then, after a suitable scale has been chosen, the acceleration polygon may be drawn as shown at c by means of the methods of

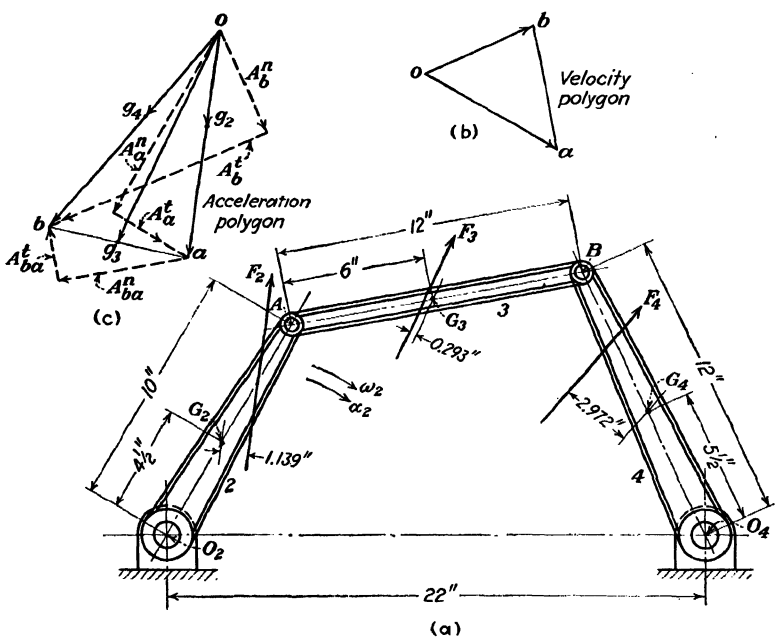


FIG. 297.

Chap. XI. The scale chosen for the original drawing was 1 in. = 150 f.p.s.^{2}. The magnitudes of the accelerations of the several centers of gravity as determined from the polygon are

$$A_{g_2} = 162 \text{ f.p.s.}^2 \qquad A_{g_3} = 371 \text{ f.p.s.}^2 \qquad A_{g_4} = 189 \text{ f.p.s.}^2$$

The magnitudes of the inertia forces are

$$F_2 = \frac{W_2}{g} A_{g_2} = \frac{4.65}{32.2} \times 162 = 23.25 \text{ lb.}$$

$$F_3 = \frac{W_3}{g} A_{g_3} = \frac{2.17}{32.2} \times 371 = 25.01 \text{ lb.}$$

$$F_4 = \frac{W_4}{g} A_{g_4} = \frac{5.27}{32.2} \times 189 = 30.89 \text{ lb.}$$

In order to determine the lines of action of the various inertia forces, it is first necessary to determine the angular accelerations of the corresponding links. The angular acceleration of link 2 is given, and those of links 3 and 4 may be determined as follows:

$$\alpha_3 = \frac{A_{ba}^t}{AB} = \frac{75}{12/12} = 75 \text{ radians/sec.}^2,$$

$$\alpha_4 = \frac{A_b^t}{O_4B} = \frac{375}{12/12} = 375 \text{ radians/sec.}^2$$

the numerical values of A_{ba}^t and A_b^t being determined from the acceleration polygon. The distances of the lines of action of the various inertia forces from the centers of gravity of their corresponding links may now be calculated as follows:

$$h_2 = \frac{I_2\alpha_2}{F_2} = \frac{0.01379 \times 160}{23.25} = 0.0949 \text{ ft.} = 1.139 \text{ in.}$$

$$h_3 = \frac{I_3\alpha_3}{F_3} = \frac{0.00814 \times 75}{25.01} = 0.0244 \text{ ft.} = 0.293 \text{ in.}$$

$$h_4 = \frac{I_4\alpha_4}{F_4} = \frac{0.02040 \times 375}{30.89} = 0.2467 \text{ ft.} = 2.972 \text{ in.}$$

The inertia forces F_2, F_3, and F_4 may then be laid off on the drawing of the mechanism as shown in Fig. 297, the direction of each force being opposite that of the corresponding acceleration and the moment of each force opposing the angular acceleration. Attention is called to the fact that the lines of action of the inertia forces of links 2 and 4 might have been located by determining the centers of percussion of these links with respect to their fixed centers of rotation.

b. Accelerating Effort.—The effort required to accelerate the mechanism may be determined by applying the conditions of equilibrium. The underlying fundamental principle is the fact that the system composed of all the inertia forces and all the external forces acting on any one member is a system in equilibrium. Let it be assumed that the accelerating effort is applied

in the form of a couple acting on the shaft to which link 2 is keyed.

Referring to Fig. 298*a* where the mechanism has been redrawn in skeleton form in order to avoid confusion of lines, let the forces acting on link 4 be first considered. These are

F_4 = the inertia force of link 4.
F_{14} = the reaction at O_4.
F_{34} = the reaction at B.

A suitable force scale having been chosen (a scale of 1 in. = 30 lb. was chosen for the original drawing), the inertia force F_4 is laid

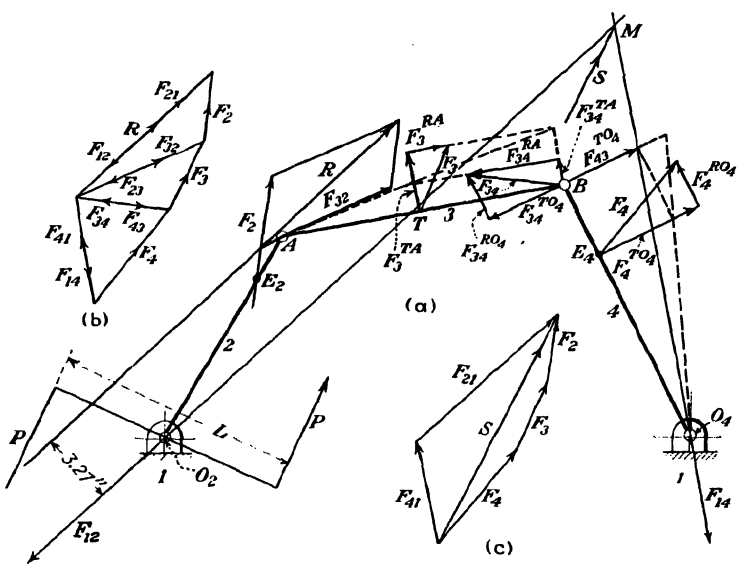


FIG. 298.

off at E_4 and resolved into transverse and radial components with respect to the point O_4. Moments are then taken about O_4, and the transverse component $F_{34}^{TO_4}$ is determined in accordance with the methods of the preceding article. No further progress is possible by considering the forces acting on link 4 alone.

The forces acting on link 3 are

F_3 = the inertia force of link 3.
F_{43} = the reaction at B.
F_{23} = the reaction at A.

Let the inertia force F_3 be resolved at T into transverse and radial components with respect to the point A. The transverse component F_{43}^{TA}, or rather its opposite F_{34}^{TA}, may then be determined by taking moments about A. The extremity of the vector representing the total force F_{34} is situated at the intersection of the perpendiculars drawn through the extremities of the vectors $F_{34}^{TO_4}$ and F_{34}^{TA}.

The reaction F_{34} having been determined, several methods of attack are open. Probably the most convenient is to lay off the forces F_3, F_4, and F_{34} in a polygon as shown at *b*. The closing sides of the two force triangles thus formed are the reactions F_{14} and F_{23}.

The forces acting on link 2 are

F_2 = the known inertia force of link 2.
F_{32} = the reaction at A that has just been determined.
F_{12} = the unknown reaction at O_2.
PL = the unknown accelerating couple.

Let the inertia force F_2 be laid off in the polygon as shown. As the couple has no effect upon the polygon, it is evident that the closing side of the latter is F_{12}. The moment of the couple PL must equal that of the couple formed by the force F_{12} and the resultant R of the forces F_{32} and F_2. The magnitude of the resultant R is determined from the polygon as 60.8 lb., and the distance between the lines of action of R and F_{12} may be scaled from the drawing as 3.27 in. The moment of the couple PL is, therefore,

$$PL = 60.8 \times 3.27 = 198.8 \text{ lb.-in.} = 16.57 \text{ lb.-ft.}$$

It is interesting to note that the accelerating effort, in this particular case, is resisting the motion of the mechanism and that the actual driving force is largely supplied by the kinetic energy of link 4.

c. Effect of Inertia Forces on the Frame.—The only forces acting on the frame of the mechanism are the reactions at O_2 and O_4, *i.e.*, the forces F_{21} and F_{41}. The resultant S of these two forces passes through the intersection M of their lines of action. The force S is called the resultant shaking force. It is the force that tends to shake or lift the frame of the machine from the founda-

tion to which it is fastened. The magnitude and direction of S are given by the vector sum of the forces F_{21} and F_{41}. An inspection of the partially redrawn polygon at c will bring out the fact that the magnitude and direction of S are also given by the vector sum of the three inertia forces F_2, F_3, and F_4. This relation is true in the case of all machines, and because of this fact the magnitude and direction of the resultant shaking force of any machine may be determined directly from the inertia forces of its members without any reference to the manner in which these inertia forces are transmitted through the machine. The position of the line of action of S, however, can be determined only by means of a complete analysis.

205. Combined Static and Inertia-force Analysis.—In the preceding article, the inertia forces alone have been considered. In most problems, however, it is necessary to consider not only the inertia forces but also the static forces and reactions by means of which the effort is transmitted through the machine. In such cases the reaction between any two members of the machine may be determined by combining the results of two separate analyses, one in which the inertia forces are taken into account and the static forces are disregarded, and another in which the inertia forces are disregarded and only the static forces are considered.

This will be discussed briefly, using as an illustration the mechanism shown in Fig. 299. For simplicity, links 3 and 4 will be assumed weightless so that their inertia forces are both zero. The known inertia force f_2 and the known resistance Q are assumed to be held in equilibrium by the unknown effort P. It is required to determine the effort P and the other forces acting in the machine. The following notation will be used:

f will denote inertia forces and the reactions that are due to inertia forces.

$\mathfrak{F}$ will denote static forces and the reactions that are due to static forces.

F will denote the forces and reactions due to the combined effect of the inertia forces and the static forces.

The static-force analysis is based upon the assumption that the inertia forces are nonexistent, and that the resistance Q is held in equilibrium by means of the static effort $\mathfrak{P}$, which is applied on link 4 in a horizontal direction. On this assumption, the only forces acting on link 2 are

Q = the known resistance.
$\mathfrak{F}_{12}$ = the reaction at O_2.
$\mathfrak{F}_{32}$ = the reaction at A.

With regard to $\mathfrak{F}_{12}$ and $\mathfrak{F}_{32}$ let it be emphasized that these forces are not the actual reactions at O_2 and A, but are merely those

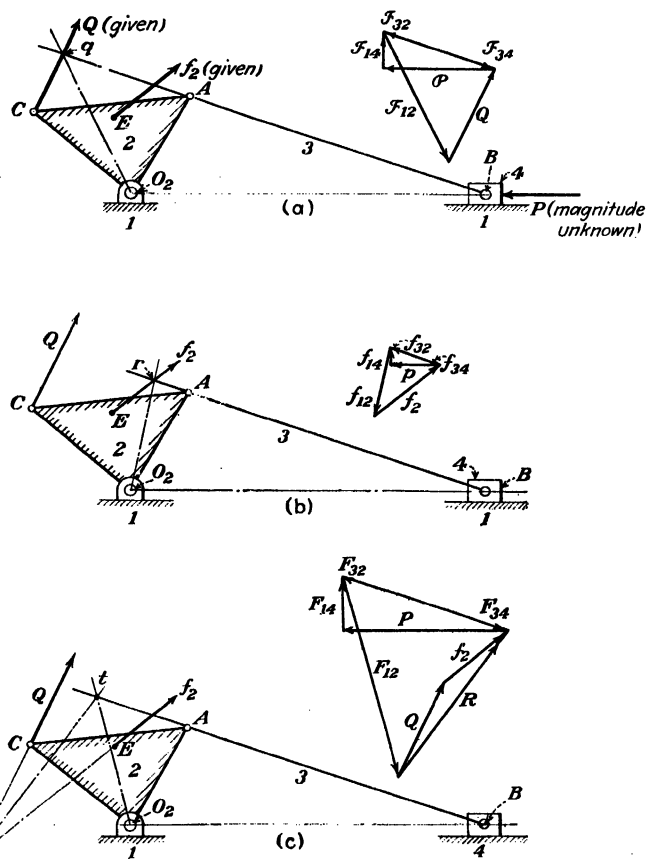


FIG. 299

parts of the actual reactions that are due to the resistance Q. The line of action of $\mathfrak{F}_{32}$ must be coincident with the line AB, since only two forces act on link 3. The third force $\mathfrak{F}_{12}$ must pass through the intersection q of the lines of action of Q and $\mathfrak{F}_{32}$. The magnitudes of $\mathfrak{F}_{12}$ and $\mathfrak{F}_{32}$ may, therefore, be determined by

means of the polygon shown at a, and the remainder of the polygon may then be drawn without special difficulty, the force $\mathfrak{F}_{14}$ being that part of the guide reaction that is due to the resistance Q and the force $\mathcal{P}$ being the static effort required to hold Q in equilibrium.

The method employed in making the inertia-force analysis is similar in every respect to that employed above, the only difference being that the inertia force f_2 is taken into account and the resistance Q is disregarded. The inertia-force analysis is shown at b in the figure. The force p is the accelerating effort required to overcome the inertia force f_2.

After the preceding two analyses have been made, the total effort P and the total reaction between any two members may be determined by combining the corresponding vectors of the polygons at a and b. The total effort, for example, is

$$P = \mathcal{P} \leftrightarrow p = \mathcal{P} + p$$

since $\mathcal{P}$ and p are parallel. Likewise, the total guide reaction is given by the vector sum of $\mathfrak{F}_{14}$ and f_{14}; the total reaction of link 1 on link 2, by the vector sum of $\mathfrak{F}_{12}$ and f_{12}; etc.

The total force determination might, of course, have been made in the first place as shown at c. The method would be to determine first the resultant R in the polygon, and then to draw st, the line of action of R, on the figure of the mechanism. The succeeding steps are the same as those of the above analyses. In some instances this method may be advantageous, but in general it tends to obscure the separate effects of the two classes of forces.

This is probably the proper place to call attention to the fact that, in most practical problems involving the determination of inertia forces, it is usually assumed that the motion of one member of the machine is completely known, and that either the effort or the useful resistance, but not both, is also known. This usually involves either the assumption of a very large flywheel, or the assumption that power is received from a constant-speed source.

There is one class of problems, however, in which both the effort and the useful resistance are assumed to be known, namely, problems dealing with the speed fluctuations in machinery. In such cases it is usual to assume that the velocities of the machine members are known but that the accelerations are not known.

Problems of this nature are considerably more complicated than those in which the motion of one member is assumed to be completely known.

206. Kinetically Equivalent System.—In determining the line of action of the inertia force F from the expression $Fh = I\alpha$, it is necessary to determine first the moment of inertia I and the angular acceleration α of the link under consideration. A determination wholly in terms of forces, thus avoiding the preceding moment equation, will now be developed. The procedure is to reduce the distributed mass of a link such as shown in Fig. 300 to a simple concentrated system, namely, two masses. As finally applied this procedure may be found more convenient in certain applications, such as will be discussed later, than that outlined in the preceding articles.

In Fig. 300 let G be the center of gravity of the rigid link there shown, and let the mass M be replaced by the two masses at B and E. One of the masses may be placed at any convenient point such as B, but the other must be so located as to satisfy three conditions, the first of which is that the total mass must not be changed. That is,

$$M_b + M_e = M$$

where M_b and M_e are the masses at B and E respectively. As a second condition, the center of gravity must be unchanged, and the masses at B and E must be on a line through that center at distances such as h_b and h_e from G satisfying the moment equation

$$M_b h_b = M_e h_e$$

The third condition is that the moment of inertia of the system must remain unchanged. Therefore

$$M_b h_b^2 + M_e h_e^2 = Mk^2 = I$$

and

$$M_b h_b h_b + M_e h_e h_e = M_e h_e h_b + M_b h_b h_e = Mk^2$$

Therefore

$$(M_e + M_b) h_b h_e = Mk^2, \text{ or } h_b h_e = k^2$$

The masses concentrated at B and E may be thought of as being connected by a weightless rod, and, since this system has

the same mass, same center of gravity, and same moment of inertia as that of the original body, it is known as a *kinetically equivalent system.*

In order to determine the characteristics of the kinetically equivalent system, the mass, center of gravity, and moment of inertia of the original body must be known. Then the following

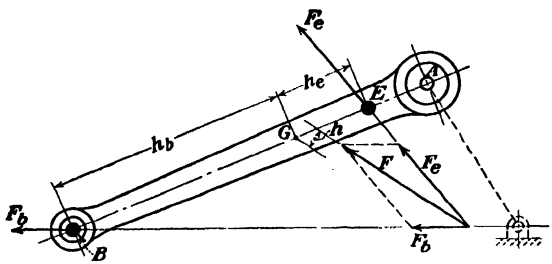


FIG. 300.

relations, Fig. 300, will determine the two-mass system to be substituted for the actual body.

$$h_b h_e = k^2 = \frac{I}{M} \tag{102}$$

$$M_e = \frac{h_b}{h_b + h_e} M \tag{103}$$

$$M_b = \frac{h_e}{h_b + h_e} M \tag{104}$$

Obviously the two systems must have a resultant inertia force F having the same magnitude, direction, and line of action. In any given case the acceleration polygon for the mechanism would be drawn. The force determination for the original system has been covered in the preceding articles. That for the kinetically equivalent system is shown in the figure. Forces F_b and F_e are determined in the usual way by means of data from the acceleration polygon. These forces are then the two components of the resultant force F, as indicated in the figure.

In the case of the connecting rod shown in Fig. 300, it is convenient to locate one of the masses at B, the axis of the wrist pin. The other mass will then be at E, the center of percussion of the rod with respect to the wrist-pin axis.

It may be well at this point to consider the advantage in using a kinetically equivalent system. There is obviously no advan-

tage if the inertia-force determination is to be made for one crank position only. The advantage becomes evident only when determinations must be made for a number of crank positions, as, for example, in the case of the gasoline engine, where it may be desirable to make a complete force analysis involving 48 crank positions in the four-stroke cycle (see Appendix). In such a case the line of action of the inertia force of the piston is definitely located in the path of reciprocation. That of the unbalanced crank, assuming the general case of angular acceleration, passes always through a fixed point on the crank, the center of percussion (see Art. 202). In the case of the connecting rod, however, the distance $h = \frac{I\alpha}{F}$, which locates the line of action of the inertia force of the rod, is different for each of the 48 crank positions, since this is a floating link. If, therefore, the kinetically equivalent system is substituted for the connecting rod, all the inertia forces in the mechanism will have their lines of action through fixed points on the links, regardless of the positions of the links. This will be found a decided convenience when making the analysis for a large number of crank positions.

207. Application to Engine Mechanism.—In order to illustrate the application of the kinetically equivalent system, let the engine mechanism shown in Fig. 301 be considered. The applied force $\mathfrak{F}_4$ is assumed to be known, and it is required to determine the reactions at O, A, and B, taking into account not only the force $\mathfrak{F}_4$ but also the inertia forces of the moving members of the mechanism. The crank is assumed to be turning counterclockwise with a uniform angular velocity of ω radians per second. The masses of the three moving members are M_2, M_3, and M_4, respectively. The center of gravity of the crank is at G_2; that of the connecting rod is at G_3. It is also assumed that E, the center of percussion of the connecting rod with respect to the point B, has been determined experimentally by the method described in Art. 202.

In determining the turning effort, the guide pressure, and the reactions at O, A, and B, the methods of Art. 205 will be employed. That is, two separate analyses will be made: a static-force analysis and an inertia-force analysis. The results of the two analyses will then be combined to determine the required forces.

a. Static-force Analysis.—In the case of the present problem, the effort $\mathfrak{F}_4$ is the known quantity and is therefore the starting point in drawing the static-force polygon shown at *b* in the figure. The static guide pressure is $\mathfrak{F}_{14}$, and the static reactions at *A* and *B* are $\mathfrak{F}_{32}(= -\mathfrak{F}_{23})$ and $\mathfrak{F}_{34}(= -\mathfrak{F}_{43})$, respectively. The static

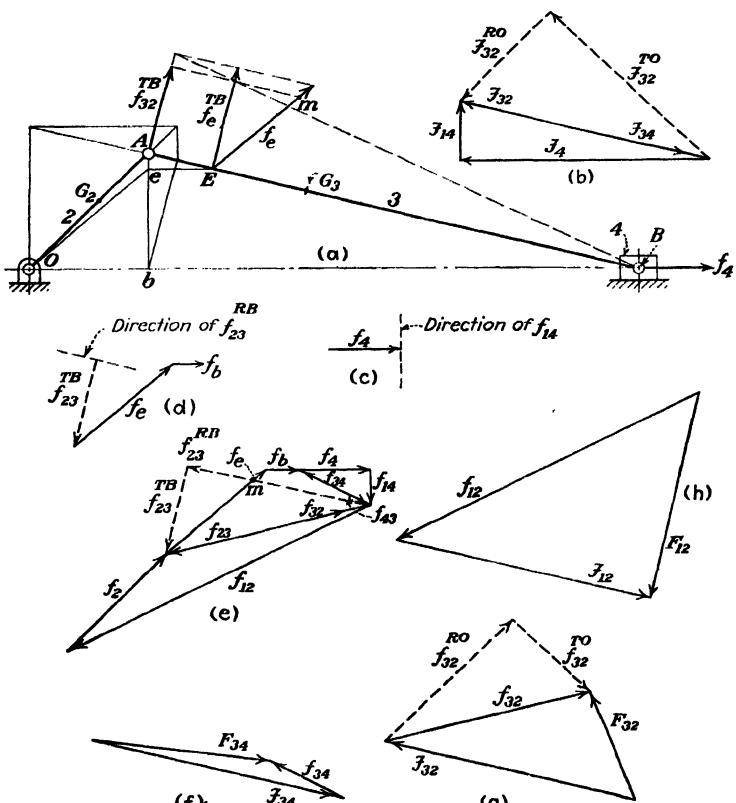


FIG. 301

force $\mathfrak{F}_{32}$ is the force that would be transmitted along the connecting rod to the crank (and thence to a useful application) if part of the effort $\mathfrak{F}_4$ were not required to overcome inertia forces. Furthermore, $\mathfrak{F}_{32}$ is the static force at *A* on the crankpin (assumed integral with the crank), and $\mathfrak{F}_{34}$ is the static force at *B* on the wrist pin (assumed fastened to the piston or crosshead). The

equal and opposite forces, $\mathfrak{F}_{23}$ and $\mathfrak{F}_{43}$, are obviously the static forces exerted on the bearings of the connecting rod.

If it is assumed that power is taken off the shaft in the form of a couple, the static reaction at O, *i.e.*, $\mathfrak{F}_{12}$ (which is not shown in the polygon at *b*), is equal, parallel, and opposite to the reaction $\mathfrak{F}_{32}$ at A. This force, $\mathfrak{F}_{12}$, is the static force exerted by the main bearing at O upon the crankshaft; the force $\mathfrak{F}_{21}$ is the equal and opposite force exerted upon the bearing. The static turning effort $\mathfrak{F}_{32}^{TO}$ is determined by resolving $\mathfrak{F}_{32}$ into components perpendicular and parallel to the crank as shown.

b. Inertia-force Analysis.—Preliminary to the inertia-force analysis, it is necessary to make the acceleration determination. Since the crank has uniform rotation, and since its axis is on the center line of the piston, this can be most easily accomplished by means of the Ritterhaus construction, as shown at *a* in Fig. 301. It should be recalled that in this construction the true directions of the accelerations are opposite those of the acceleration vectors as read from the pole O. Hence the inertia forces are in the *same* direction as the acceleration vectors as read from the pole O.

The next step is to determine a kinetically equivalent system for the connecting rod (link 3), since this member has a fairly complex motion. The most convenient system for our purposes consists of two masses, one concentrated at B and the other at E, the magnitudes of the masses being given as follows:

$$M_b = \frac{\overline{G_3E}}{\overline{BE}} M_3 \qquad M_e = \frac{\overline{BG_3}}{\overline{BE}} M_3$$

As link 2 is rotating about the fixed center O with a uniform angular velocity, there is no advantage to be gained in replacing this link by its kinetically equivalent system. The same statement applies to link 4, in this case because of the fact that the link has a motion of translation.

The various inertia forces are:

Inertia force of crank..... $f_2 = M_2 A_{g_2}$.
Inertia force of connecting-rod mass at E. $f_e = M_e A_e$.
Inertia force of connecting-rod mass at B. $f_b = M_b A_b$.
Inertia force of link 4..... $f_4 = M_4 A_b$.

It will be assumed in what follows that the inertia forces have been calculated in accordance with the above equations and that they are represented in the polygons by the vectors f_2, f_e, f_b, and f_4. The determinations of the reactions due to the inertia forces will now be discussed in detail, the analysis being based upon the fact that the inertia force of any single link taken together with the inertia reactions (not the static reactions) of the other links upon it are a system of forces in equilibrium.

Let the forces acting upon link 4 be considered first. Since only three forces (f_4, f_{14}, and f_{34}) act upon the link, they will intersect at B. As indicated at c in the figure, the first step in drawing the polygon is to lay off the known force f_4, after which the known direction of f_{14} may be laid off. As nothing is known as yet concerning the third force f_{34}, no further progress can be made in this direction.

The forces acting on link 3 are f_e, f_b, f_{23}, and f_{43}. The known forces f_e and f_b may be laid off as shown at d in Fig. 301. By taking moments about B, the transverse component of f_{23} with respect to B may be determined, the construction being made on the drawing of the mechanism, as shown at a.[1] The force f_{23}^{TB} may then be laid off in the polygon and a line drawn perpendicular to it through its origin to represent the general direction of f_{23}^{RB}. No further progress is possible by considering the forces acting on link 3 alone. Moments might be taken about A to determine the transverse component of f_{43}, but there would be no particular advantage gained by so doing.

The two solutions attempted above have both failed because of a seeming lack of known quantities. That sufficient data are given for the solution of the problem is at once evident, however, if the two polygons are combined as shown at e in the figure. There it is seen that the two vectors f_{23}^{RB} and f_{14} must intersect at the extremity of $f_{43}(= -f_{34})$ in order to satisfy the relations

$$f_{43} + f_{23}^{RB} + f_{23}^{TB} + f_e + f_b = 0$$
$$f_{34} + f_4 + f_{14} = 0$$

The vectors f_{34} and $f_{23}(= -f_{32})$ may then be drawn as shown, and the two force polygons thus completed.

[1] The letter m in the construction at a, and in the polygon at e, has no significance here, being shown in the figure only to facilitate the discussion in the next article.

Since only three forces, $f_{32}(= -f_{23})$, f_2, and f_{12}, act on link 2, and since the first two are known, the third force f_{12} forms the closing side of the triangle, and the polygon of forces at e can be completed for the entire mechanism. The force at B on the wrist pin due to inertia is f_{34}, the force at A on the crankpin is f_{32}, and the force at O on the crankshaft is f_{12}. The equal and opposite forces, namely, f_{43}, f_{23}, and f_{21}, are obviously the forces on the bearings at B, A, and O respectively. It should be observed that, although the force polygon at e closes, the inertia forces and reactions acting on the crank are not in equilibrium, since the lines of action of f_2, f_{12}, and f_{32} do not intersect in a common point. For equilibrium, a counterclockwise couple is required, which may be assumed to be supplied *substantially* by a flywheel keyed to the crankshaft.

If the force f_{32} is resolved into components perpendicular and parallel to the crank, as shown at g where f_{32} from e has been redrawn, the perpendicular component f_{32}^{TO} will represent the inertia or dynamic turning effort. The inertia force f_2 of the crank (a centrifugal force) has a line of action passing through the center of the main bearing, and has, therefore, no effect upon the turning effort. It is to be noted that the dynamic turning effort f_{32}^{TO} opposes $\mathfrak{F}_{32}^{TO}$ for the position of the mechanism shown. Later in the stroke this condition reverses itself, and the kinetic energy previously given to the connecting rod and piston is regained.

c. Total Forces.—Corresponding static- and inertia-force vectors can now be combined to give any desired total force. For example, the total force at B on the wrist pin is $F_{34} = \mathfrak{F}_{34} \leftrightarrow f_{34}$, as shown at f; the total force at A on the crankpin is $F_{32} = \mathfrak{F}_{32} \leftrightarrow f_{32}$, as shown at g; and the total force at O on the crankshaft is $F_{12} = \mathfrak{F}_{12} \leftrightarrow f_{12}$, as shown at h. The total turning effort F_{32}^{TO} is obviously the algebraic sum of components $\mathfrak{F}_{32}^{TO}$ and f_{32}^{TO}, or it may be obtained directly from the total force F_{32} shown at g.

It could be shown in a manner entirely analogous to that employed in Art. 204 that the total shaking force exerted on the frame of the machine is equal to the vector sum of the three inertia forces. The line of action of this force would pass through the intersection of the lines of action of f_{21} and f_{41}. There is also a static couple exerted on the frame, the moment of this couple being equal to the moment of the static turning effort.

208. Engine Mechanism. Special Methods.—In Art. 178 special methods were developed for dealing with the engine mechanism in cases where acceleration determinations are required for a number of different positions of the crank. In the present article similar methods will be developed for expediting the routine work of analyzing the forces in the engine mechanism for a number of crank positions. Data will be given for the solution of a numerical example, and, in order to illustrate the method, the analysis will be worked out in detail for one position of the crank.

It is required to determine the total turning effort and the various reactions of the engine mechanism (Fig. 302) covered by the following description:

Length of crank OA	4 in.
Length of connecting rod AB	12 in.
Distance of center of percussion E of connecting rod from center of wrist pin B	10¾ in.
Speed of crank, link 2 (uniform rotation)	1,800 r.p.m.
Total weight of connecting rod, link 3	3.59 lb.
Weight assumed concentrated at center of percussion	2.84 lb.
Weight assumed concentrated at wrist pin	0.75 lb.
Weight of piston, link 4	2.00 lb.
Force on piston due to gas pressure (45-deg. position)	1,800 lb.

The crank is assumed to be fully counterbalanced; *i.e.*, weights have been added to the crankshaft opposite the crankpin such that the center of gravity of the resulting system is situated at the center line of the main bearing. As a result, the inertia force of the crank is zero. It is observed that a kinetically equivalent system has been substituted for the connecting rod.

Referring to Fig. 302*a*, the first step in the solution of the problem is to make the acceleration determination, and this has been accomplished by the methods of Art. 178. In the original drawing, the crank circle was laid out full size, and the distance OA on the drawing was therefore 4 in. The acceleration scale k_a was then calculated as follows:

$$A_a = R\omega_2^2 = \frac{4}{12}\left(\frac{2\pi \times 1{,}800}{60}\right)^2 = 11{,}850 \text{ f.p.s.}^2$$

$$k_a = \frac{11{,}850}{4} = 2{,}963 \text{ f.p.s.}^2$$

The scale having been calculated, the accelerations of the wrist pin and the center of percussion are determined from the polygon to be

$$A_b = 8{,}500 \text{ f.p.s.}^2$$
$$A_c = 11{,}260 \text{ f.p.s.}^2$$

The inertia forces f_c, f_b, and f_4 are then calculated as follows:

$$f_c = \frac{W_c A_c}{g} = \frac{2.84 \times 11{,}260}{32.2} = 993 \text{ lb.}$$
$$f_b = \frac{W_b A_b}{g} = \frac{0.75 \times 8{,}500}{32.2} = 198 \text{ lb.}$$
$$f_4 = \frac{W_4 A_b}{g} = \frac{2.00}{32.2} \times 8{,}500 = 528 \text{ lb.}$$

The forces f_c, f_b, and f_4 may now be laid off in the polygon as shown. It is convenient to use O the center of the main bearing

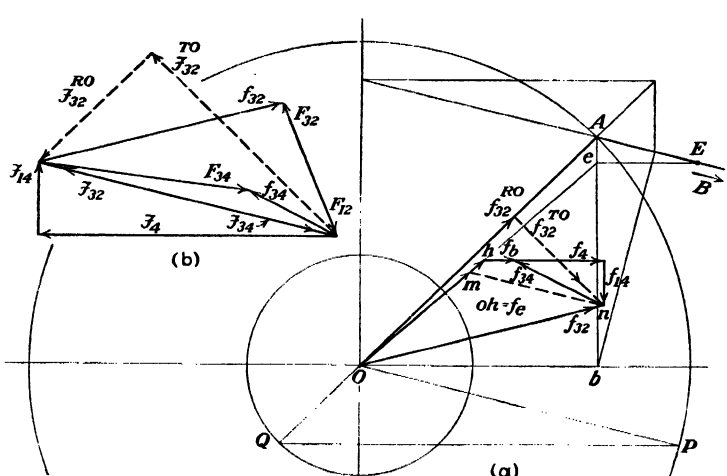


FIG. 302.

as the origin for the vector f_c when drawing the force polygon, for the reason that it will bring the vector f_c always along the line Oe and in the sense Oe. The direction of f_{14} may be laid off from the extremity of f_4 as shown. In order to determine the magni-

tude of f_{14}, let a point m be located on the vector f_e such that

$$Om = f_e \frac{BE}{BA} = 993 \times \frac{10.75}{12} = 896 \text{ lb.}$$

The line mn drawn parallel to the connecting rod will then determine the length of the vector f_{14}. The proof of this statement is as follows: Referring to Fig. 301*e*, let the intersection of the vector f_{23}^{RB} (which is parallel to the connecting rod) and the vector f_e be denoted by the letter m. The same letter is used to designate the corresponding point on the drawing of the mechanism (on the vector f_e drawn from the point E in Fig. 301*a*). It is evident from the geometry of the figure that

$$\frac{\overline{Em}}{f_e} = \frac{f_{32}^{TB}}{f_e^{TB}} = \frac{\overline{BE}}{\overline{BA}} \qquad \text{Therefore } \overline{Em} = f_e \frac{\overline{BE}}{\overline{BA}}$$

Referring again to Fig. 302, it is to be noted that the point n (intersection of f_{14} and the line mn parallel to the connecting rod) also locates the extremity of the vector f_{32} (origin of f_{23}). The force polygon can now be completed by drawing in the vectors f_{34} and f_{32}. With the exception that $f_2 = 0$, and the omission of f_{23}^{TB} and f_{23}^{RB}, it can be seen that the arrangement of vectors in this superimposed force polygon is the same as that shown at e in Fig. 301.

If the vector f_{32} is resolved into components perpendicular and parallel to the crank, the component f_{32}^{TO} will give the turning effort due to inertia. The procedure outlined above is of great convenience when determinations must be made for a large number of crank positions, as for example, in the Appendix, where, in the complete analysis of a gasoline engine, determinations are made for 48 crank positions.

The next step in the solution is to make the static-force analysis; *i.e.*, to determine the forces due to the gas pressure. These forces are represented in the triangle shown in Fig. 302*b*, in which piston pressure $\mathfrak{F}_4$ forms the base. If the static turning effort $\mathfrak{F}_{32}^{TO}$ is required, it may be determined as shown in the figure. Any total force may be determined by combining the inertia-force vector with the corresponding static-force vector, as shown in Fig. 302*b*, where, for convenience, the vector addition has been made on the figure. The solution may be completed and the required forces obtained as follows:

Total force on crankpin.....	F_{32}	$= \mathfrak{F}_{32} + f_{32} =$	890 lb.
Total turning effort.....	F_{32}^{TO}	$= \mathfrak{F}_{32}^{TO} + f_{32}^{TO} =$	810 lb.
Total force on wrist pin.....	F_{34}	$= \mathfrak{F}_{34} + f_{34} =$	1,300 lb.
Total force on crankshaft.....	F_{12}	$= \mathfrak{F}_{12} + f_{12} =$	890 lb.
Total force on cylinder wall...	F_{41}	$= \mathfrak{F}_{41} + f_{41} =$	160 lb.
Total effective piston force....	F_4	$= \mathfrak{F}_4 + f_4 =$	1,270 lb.

In the above analysis no attention has been given to signs, *i.e.*, as to whether a force is positive or negative. In general, it is a good plan to consider a force aiding the motion as positive and a force opposing the motion as negative. An example of this procedure is shown in Fig. 342 in the Appendix, where positive values of the turning effort have been plotted as ordinates above the base line and negative values as ordinates below the base line.

Force Diagrams.—In the force analysis of a machine, where the speed is such as to set up inertia forces of appreciable magnitude, it may be desirable to plot certain force diagrams. This is particularly true in the case of high-speed engines, where diagrams of piston forces, turning effort, bearing pressures, etc., for the complete cycle serve a valuable purpose. Examples of such diagrams, for a high-speed gasoline engine where the forces were determined in accordance with the principles outlined in this chapter, are given in the Appendix. It will be found that the various forces represented by the vectors in the force polygons, Fig. 301, have been plotted for the complete engine cycle. In Fig. 340, for example, the static and inertia piston forces $\mathfrak{F}_4$ and f_4, respectively, and their combined effect have been plotted for each 15-deg. crank position. Static- and inertia-force vectors $\mathfrak{F}_{32}'$ and f_{32}', representing turning effort on the crank, and their combined effect have been plotted in Fig. 342. As another example, crankpin pressures F_{32} are shown in Fig. 345. At this point the student should read over the Appendix where it will be found that all the forces represented in the static- and inertia-force polygons (Fig. 301) have been plotted in the form of diagrams for the complete cycle. For this purpose, force determinations were made for 48 crank positions (every 15 deg.) in the four-stroke cycle. The total turning-effort diagram for the six cylinders combined is shown in Fig. 343. In this figure, also, is shown the mean turning-effort diagram, which is the basis for the design of the engine flywheel.

209. Inertia Forces in a Cam Mechanism.—The inertia-force analyses thus far have dealt with linkwork. The methods out-

lined can readily be adapted to the great majority of cases that may arise. An example of the application to direct-contact mechanisms, such as cams or gears, is the object of the present article.

Let it be assumed that the cam shown in Fig. 62, Chap. III, is rotating at the rate of 180 r.p.m. and that the roller follower receives its maximum displacement of 2 in. while the cam rotates clockwise through the angle of 120 deg. Referring to Fig. 303*a*, assume that the follower has harmonic motion and let its total rise, or total displacement, be represented by h. Its displace-

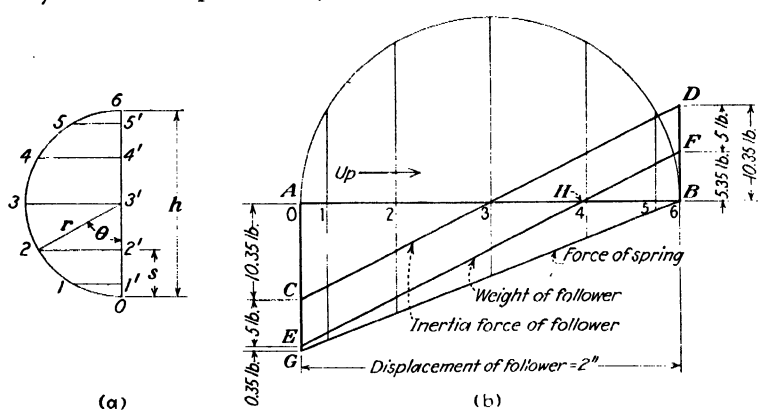


FIG. 303.

ment s at any point will be determined by the projection of the extremity of the rotating radius $r(= h/2)$.

The displacement of the follower is

$$s = r(1 - \cos \theta)$$

The velocity of the follower is

$$V = \frac{ds}{dt} = r\omega_r \sin \theta$$

The acceleration of the follower is

$$A = \frac{dV}{dt} = r\omega_r^2 \cos \theta \qquad (105)$$

where $r = h/2$ and ω_r = the angular velocity of the rotating radius r. The time for the total displacement of the follower is

$$t = {}^{60}\!/_{180} \times {}^{120}\!/_{360} = {}^{1}\!/_{9} \text{ sec.}$$

The angular velocity of the rotating radius r is

$$\omega_r = \frac{\pi}{\frac{1}{9}} \text{ radians/sec.} = 9\pi \text{ radians/sec.}$$

From Eq. (105) the accelerations of the follower at positions 0 to 6 are respectively 66.62, 57.69, 33.31, 0, −33.31, −57.69, −66.62 f.p.s.^{2}

The weight of the follower is 5 lb. From the expression $F = MA$ the inertia forces of the follower at positions 0 to 6 are respectively 10.35, 8.96, 5.18, 0, −5.18, −8.96, −10.35 lb.

In Fig. 303*b* the inertia forces for the six follower positions have been plotted to a scale of 1 in. = 10 lb. on base line AB representing the 2-in. stroke of the follower. These inertia forces are represented by line CD. The weight of the follower is plotted to the same scale and is represented by line EF in the figure; ordinates below base line AB represent vertical forces of the follower against the cam, and ordinates above AB represent vertical forces away from the cam. That is, beyond point H between positions 3 and 4 the acceleration of the follower is such that its weight is not sufficient to keep it in contact with the cam, and it becomes necessary to use a spring. At the end of the follower stroke the difference between the inertia force of the follower and its weight is 10.35 − 5 = 5.35 lb. Hence, a spring with a scale of 2½ lb. per in. set up under an initial load of 0.35 lb. will be just sufficient to keep the follower in contact with the cam when it reaches the end of its stroke, as indicated by the line GB in the figure. It should be stated, however, that the effect of friction has been neglected in the foregoing analysis.

210. Analytical Methods.—The analytical determination of the forces acting in even a simple mechanism is likely to become somewhat involved. In certain cases, however, it will be found more convenient to use analytical methods than to use the graphical methods discussed in the preceding articles. For this reason it is worth while to give some consideration to the general analytical method of attack. In the present article, a brief outline will be given of the analytical determination of the inertia forces and dynamic reactions that act in the engine mechanism.

Referring to Fig. 304, the various accelerations are given as follows (see Art. 183, chapter on Accelerations):

$$A_b = R\omega_2^2\left[\cos\theta + \frac{R}{L}\left(\frac{\cos 2\theta}{\cos\phi} + \frac{R^2}{L^2}\cdot\frac{\sin^2\theta\cos^2\theta}{\cos^3\phi}\right)\right]$$

$$A_e^x = R\omega_2^2\left[\cos\theta + \left(1-\frac{h}{L}\right)\frac{R}{L}\left(\frac{\cos 2\theta}{\cos\phi} + \frac{R^2}{L^2}\cdot\frac{\sin^2\theta\cos^2\theta}{\cos^3\phi}\right)\right]$$

$$A_e^y = -R\omega_2^2\frac{h}{L}\sin\theta$$

$$A_a^x = R\omega_2^2\cos\theta \text{ and } A_a^y = -R\omega_2^2\sin\theta$$

$$A_{g_2} = r\omega_2^2 = \frac{r}{R}\cdot R\omega_2^2 = \frac{r}{R}A_a$$

Therefore

$$A_{g_2}^x = \frac{r}{R}A_a^x \text{ and } A_{g_2}^y = \frac{r}{R}A_a^y$$

In order to reduce the algebraic work, let
$A_b' = R\omega_2^2\cos\theta$ and

$$A_b'' = R\omega_2^2\frac{R}{L}\left(\frac{\cos 2\theta}{\cos\phi} + \frac{R^2}{L^2}\cdot\frac{\sin^2\theta\cos^2\theta}{\cos^3\phi}\right) = \frac{R^2}{L}\omega_2^2\cos 2\theta \text{ (approximately)}$$

The quantities A_b' and A_b'' are called respectively the primary and secondary acceleration of the wrist pin. Substituting in the above equations,

$$A_b = A_b' + A_b'' \tag{106}$$

$$A_e^x = A_b' + \left(1-\frac{h}{L}\right)A_b'' \tag{107}$$

Also

$$A_a^x = A_b' \text{ and } A_e^y = \frac{h}{L}A_a^y \tag{108}$$

The expressions for the various inertia forces may then be written as follows, the notation being the same as in Arts. 207 and 208 except that the letters x and y are used to denote horizontal and vertical components respectively.

$$f_4 = -M_4(A_b' + A_b'') \qquad f_b = -M_b(A_b' + A_b'')$$

$$f_e^x = -M_e\left(A_b' + \left[1-\frac{h}{L}\right]A_b''\right) \qquad f_e^y = -M_e\frac{h}{L}A_a^y$$

$$f_2^x = -M_2\frac{r}{R}A_a^x \qquad f_2^y = -M_2\frac{r}{R}A_a^y$$

The inertia force of the piston is usually given special prominence because of its importance in connection with the balancing of engines. As ordinarily written, this force is

$$f_4 = -M_4 R\omega_2^2(\cos\theta + \frac{R}{L}\cos 2\theta) \qquad (109)$$

In order to determine any of the dynamic reactions, it is necessary to set up the equations for the equilibrium of one or more members of the mechanism. The method will be illustrated by determining the expression for the reaction at A. The first step

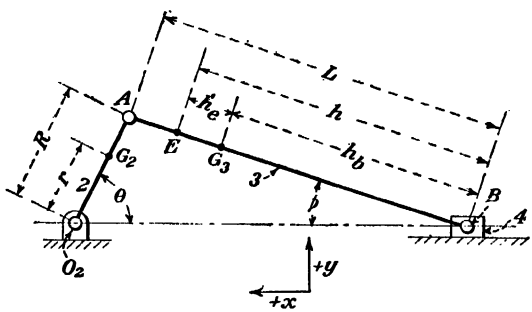


FIG. 304.

is to set the sums of the horizontal and vertical components of the forces acting on links 3 and 4 equal to zero. The equations for link 3 are

$$f_{23}^x + f_e^x + f_b + f_{43}^x = 0$$
$$f_{23}^y + f_e^y + f_{43}^y = 0 \text{ (not used)}$$

The corresponding equations for link 4 are

$$f_4 + f_{34}^x = 0$$
$$f_{14} + f_{34}^y = 0 \text{ (not used)}$$

In order to obtain a solution, one more relation is necessary, since two of the above equations are of no direct use in determining the reaction at A. The most convenient one to apply is obtained by setting the sum of the moments about B of the forces acting on link 3 equal to zero.

$$f_{23}^x L \sin\phi + f_e^x h \sin\phi - f_{23}^y L\cos\phi - f_e^y h\cos\phi = 0$$

The simultaneous solution of the preceding equations will give the following expression for the dynamic reaction at A:

$$f_{32}^x = f_e^x + f_b + f_4 \tag{110}$$

$$f_{32}^y = f_{32}^x \tan \phi - \frac{h}{L}(f_e^x \tan \phi - f_e^y) \tag{111}$$

$$= (f_e^x + f_b + f_4) \tan \phi - \frac{h}{L}(f_e^x \tan \phi - f_e^y)$$

Stated in terms of acceleration, this becomes

$$f_{32}^x = -(M_3 + M_4)\,(A_b' + A_b'') + M_e\frac{h}{L}A_b'' \tag{112}$$

$$f_{32}^y = f_{32}^x \tan \phi + M_e\frac{h}{L}\left\{\left[A_b' + \left(1 - \frac{h}{L}\right)A_b''\right]\tan \phi - \frac{h}{L}A_a^y\right\} \tag{113}$$

The reactions f_{34} and f_{14} could be determined in the same way and would be represented by similar expressions.

The dynamic turning effort T_i is given by

$$T_i = f_{32}^x \sin \theta + f_{32}^y \cos \theta \tag{114}$$

The total shaking force exerted on the frame is the resultant of the inertia forces of the members. If the shaking force is represented by f_s, then

$$f_s^x = -(M_3 + M_4)(A_b' + A_b'') + M_e\frac{h}{L}A_b'' - M_2\frac{r}{R}A_a^x \tag{115}$$

$$f_s^y = -M_e\frac{h}{L}A_a^y - M_2\frac{r}{R}A_a^y \tag{116}$$

The horizontal component passes through the center of the main bearing, and the vertical component has a line of action situated at a distance $\bar{x}$ to the right of the main bearing, where $\bar{x}$ is given by the following equation:

$$\bar{x} = \frac{f_{14}(R \cos \theta + L \cos \phi)}{f_s^y} \tag{117}$$

CHAPTER XIV

BALANCING OF MACHINERY

One of the most important applications of the study of inertia forces is in connection with the balancing of machinery. Moving parts of machines, in general, have either reciprocating motion similar to that of the piston of an engine, or rotating motion such as that of the crankshaft of an engine or the rotor of a turbine or dynamo. If the moving parts are not in perfect balance, or if the parts have variable motion or are subject to acceleration, inertia forces (also called shaking forces) are set up that tend to produce vibrations in the frame of the machine, and hence in the foundations to which the frame is attached. Such vibrations, particularly in the case of high speeds, may produce excessive noise, cause undue wear and tear on the machinery and its supports, or result in the faulty performance of the work for which the machine is designed. Furthermore, if the period of vibration of any part of the supporting framework or foundation should happen to coincide with the period of vibration of the moving part, the disturbances set up may become dangerous. The purpose of balancing, therefore, is to neutralize or minimize these unpleasant and injurious vibratory effects as far as may be practicable.

The moving parts may be in (*a*) static or standing balance or (*b*) dynamic or running balance.

Static balance exists if the parts are in equilibrium among themselves when not running, regardless of the position in which the parts may be placed, *i.e.*, if the center of gravity of all the moving parts remains in a fixed position relative to the frame of the machine regardless of the positions of the parts. A system is in dynamic balance when the inertia forces and couples exerted by the moving masses are in equilibrium among themselves.

The discussion in this chapter is confined very largely to the balancing of engines, but the methods and principles involved are fundamental and may be applied to machinery of all kinds.

The reactions between the component parts of an engine in motion may be analyzed in two main groups, namely, the forces due to the working fluid and the forces due to the accelerations of the parts. The former are static forces, and their effect upon the engine frame is dependent upon the manner in which power is taken off the crankshaft. When the useful resistance overcome by the crankshaft is in the form of a couple, there is a corresponding static couple that tends to rotate the engine frame about the crankshaft. The forces due to acceleration are, of course, the inertia forces, and these may be combined to form a force, or a couple or both, which ultimately act upon the foundation and tend to cause vibration.

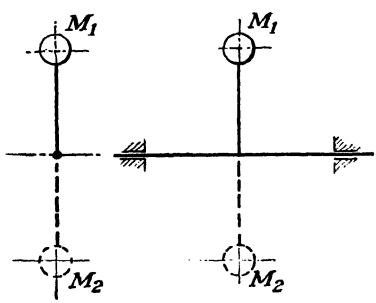


FIG. 305.

The inertia forces in high-speed engines are of great magnitude, and the balancing of these forces has received a great deal of attention in the development of marine engines, locomotives, and the automotive type of engine.

211. Balancing of Rotating Masses.—The effect of rotating masses is to produce inertia forces or kinetic loads on the shaft to which the masses are connected. The balancing of such a system consists in rearranging the masses forming the system, or in introducing into the system additional masses, so that the inertia forces acting on the shaft in consequence of rotation form a system in equilibrium.

The inertia forces set up by a system of rotating masses are always situated in planes perpendicular to the axis of the shaft to which the masses are connected. If the rotation is uniform, as will be assumed here and in the succeeding articles, the lines of actions of the inertia forces all pass through the axis of the shaft; that is, they are pure centrifugal forces.

212. Effect of a Single Rotating Mass.—If a shaft (Fig. 305) rotating at an angular velocity of ω radians per second carries a single mass M_1 (equal to the weight of the mass divided by 32.2) whose center of gravity is R_1 feet from the axis of rotation, the shaft will be subjected to a centrifugal force of magnitude $M_1R_1\omega^2$. This force, which is continually changing in direction, causes the

shaft to exert forces on the bearings, and these forces in turn are transmitted to the frame and foundation of the machine. The effect of such a kinetic load is to set up vibrations that are particularly noticeable at high speed.

The effect of the centrifugal force due to the single rotating mass may be eliminated by the addition of another mass M_2, whose center of gravity is in the plane of M_1 diametrically opposite M_1 and at a distance R_2 from the axis of rotation, such that

$$M_1R_1\omega^2 = M_2R_2\omega^2$$

It will be noted that with these two masses in the same plane, the shaft is in both static and running balance. The center of gravity is in the axis of the shaft and remains therefore in a fixed position relative to the frame of the machine regardless of whether the shaft is turning or standing still.

213. Effect of Two Rotating Masses Not in the Same Plane of Rotation.—If a shaft (Fig. 306) carries two rotating masses M_1

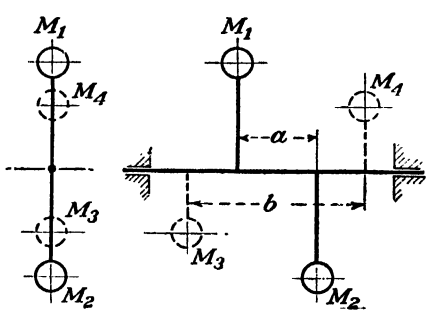


FIG. 306.

and M_2 in different planes of rotation but in the same axial plane, *i.e.*, the same plane passing through the axis of the shaft, and if, further, the centrifugal forces $M_1R_1\omega^2$ and $M_2R_2\omega^2$ are equal, then the shaft is subjected to the action of an unbalanced *centrifugal couple*. This unbalanced couple tends to turn the shaft in an axial plane and is therefore resisted by an equal couple applied at the bearings.

Although this system is obviously not in running balance, it is, nevertheless, in standing balance. That is, the center of gravity of the system is situated in the axis of the shaft and is, therefore, stationary.

The two rotating masses may be balanced by the introduction of two additional masses M_3 and M_4 in the same axial plane such that the centrifugal couple that they set up equals the centrifugal couple of M_1 and M_2. That is,

$$M_1R_1\omega^2a = M_3R_3\omega^2b, \text{ and } M_3R_3\omega^2 = M_4R_4\omega^2$$

214. Several Rotating Masses in a Single Transverse Plane.—If several masses, M_1, M_2, M_3, etc., lie in the same transverse

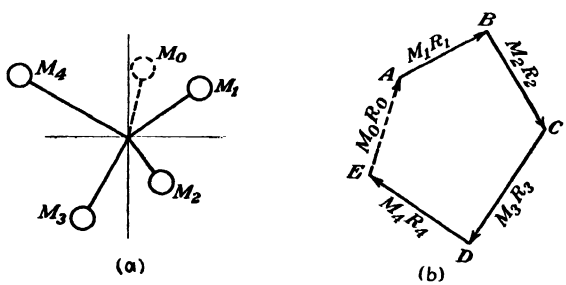


FIG. 307.

plane, as in Fig. 307 at a, the shaft is subject to a concurrent system of centrifugal forces,

$$M_1R_1\omega^2,\ M_2R_2\omega^2, \text{ etc.}$$

The condition that such a system shall balance is that

$$M_1R_1\omega^2 + M_2R_2\omega^2 + \cdots + M_nR_n\omega^2 = 0 \qquad (118)$$

It is to be noted that this equation requires that the center of gravity of the system be situated in the axis of the shaft.

Since ω^2 is a common factor in the above equation the products M_1R_1, M_2R_2, etc., may be used instead of the actual forces and the equation of equilibrium may be written

$$M_1R_1 + M_2R_2 \cdots + M_nR_n = 0 \qquad (119)$$

That is, the vectors representing M_1R_1, M_2R_2, etc., if laid off in succession, each in its proper direction, will form a closed polygon.

In Fig. 307 at b suppose that the products M_1R_1, M_2R_2, etc., representing the concurrent system of forces at a in the figure, when laid off as vectors do not form a closed polygon. In order to close the polygon and thus balance the system, a mass M_0 must be added at a distance R_0, such that the product M_0R_0 is

represented in both magnitude and direction by the closing side EA of the vector polygon. By assuming a convenient value for R_0 the value of M_0 is definitely determined. The gap EA may be closed, however, by two or more vectors chosen at will, and the system may thus be balanced by two or more masses instead of one.

215. Several Rotating Masses in Different Transverse Planes. Referring to Fig. 308, let the masses M_1 and M_2 be connected to a shaft the axis of which is XX; and let the centers of gravity of the two masses be situated in the planes A_1 and A_2. The forces set up by the rotation of the system are F_1 and F_2. At some point

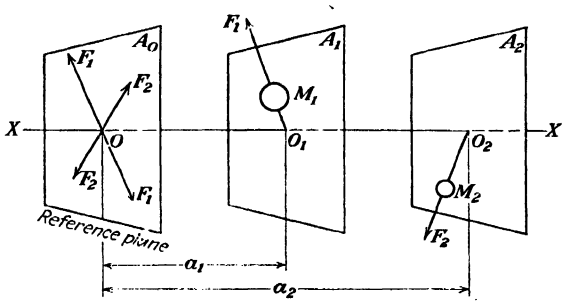


FIG. 308.

O in the shaft axis pass a transverse plane A_0 (called the reference plane, or briefly, the R.P.). At O in the reference plane introduce two equal and opposite forces F_1 parallel to the original force F_1 in plane A_1. The force F_1 at O and the opposite force F_1 at O_1 form a centrifugal couple whose moment is $M_1R_1\omega^2a_1$, where ω is the angular velocity of the shaft. Hence the single force F_1 acting at O_1 in plane A_1 may be replaced by an equal and parallel force acting at O in the reference plane and a couple whose moment is F_1a_1. Likewise the force F_2 acting at O_2 in plane A_2 may be replaced by an equal and parallel force F_2 acting in the reference plane and a couple whose moment is $F_2a_2 = M_2R_2\omega^2a_2$.

Therefore it is evident that a reference plane can be chosen arbitrarily, and the system of forces set up by the rotating system may be reduced to a system of concurrent forces acting in the reference plane and a system of couples acting in various axial planes. The forces, if not balanced, have a single resultant in the reference plane, and the couples, if not balanced, can be

reduced to a single couple in some axial plane. Hence, in general, the system of forces, if not balanced, may be reduced to a single force and a single couple. The magnitude of the couple will depend on the position chosen for the reference plane.

The conditions of balance, then, for the system of rotating masses in different transverse planes are evidently the following:

1. The vector sum of all the centrifugal forces must be zero. That is,

$$\vec{\Sigma} MR\omega^2 = 0 \tag{120}$$

2. The vector sum of the moments of all the forces with respect to any arbitrarily chosen reference plane must be zero. That is,

$$\vec{\Sigma} MR\omega^2 a = 0 \tag{121}$$

In order to balance such a system, at least two additional masses in different transverse planes are required. That is, the forces must be balanced by a mass in the reference plane, and the moments by a mass situated in some other transverse plane.

216. General Graphical Method of Balancing Any Number of Rotating Masses.—Referring to Fig. 309, let it be required to balance the system composed of the three rotating masses M_1, M_2, and M_3. The solution consists merely in applying the conditions of balance developed in the preceding article. These conditions are represented by the equations

$$\vec{\Sigma} MR\omega^2 = 0$$
$$\vec{\Sigma} MR\omega^2 a = 0$$

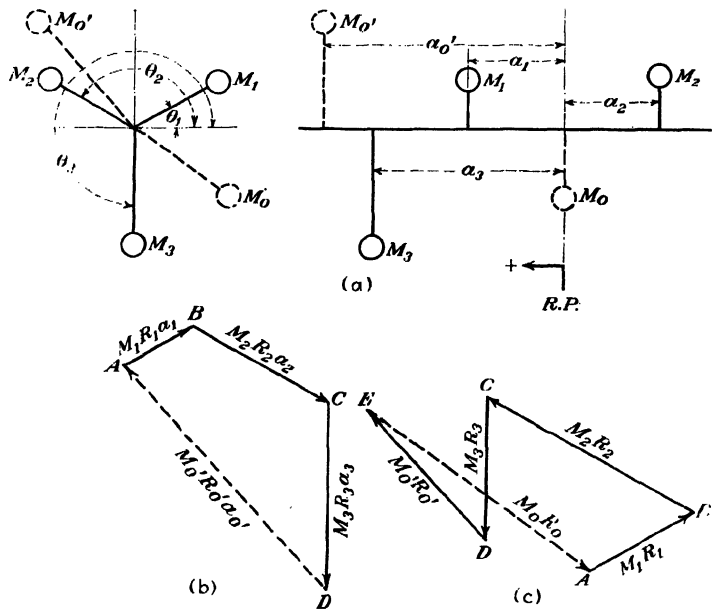
or, if the common term ω^2 be dropped,

$$\left.\begin{aligned} \vec{\Sigma} MR &= 0 \\ \vec{\Sigma} MRa &= 0 \end{aligned}\right\} \tag{122}$$

Two methods of solving the problem will be given, the first of which will be made to depend upon both the force and the moment relations, and the second of which will depend upon the moment relation alone.

a. First Method.—Let the various masses and distances of the system, "M_1, M_2, M_3," be as given in the table shown at *d* in

the figure, and let the reference plane (denoted by R.P.) be chosen as shown. Let the balancing masses be M_0 and $M_{0'}$ and let the mass M_0 be situated in the reference plane. The left side of the reference plane is taken as the positive side.



Plane	Mass	Radius	a	θ	MR	MRa
1	15	4	4	30°	60	240
2	30	4	−4	150°	120	−480
3	15	5	8	270°	75	600
	Assume		Assume			
0′	16	6	10	131°	96	960
	Assume					
0 (R.P.)	30	4.88	0	324°	146.5	0

(*d*)

FIG. 309.

The first step is to calculate the terms MR and MRa for the three masses M_1, M_2, and M_3. This has been done, and the numerical values of these quantities have been set down in the table. Let it be noted that since the mass M_2 is on the negative side of the reference plane, its moment is negative.

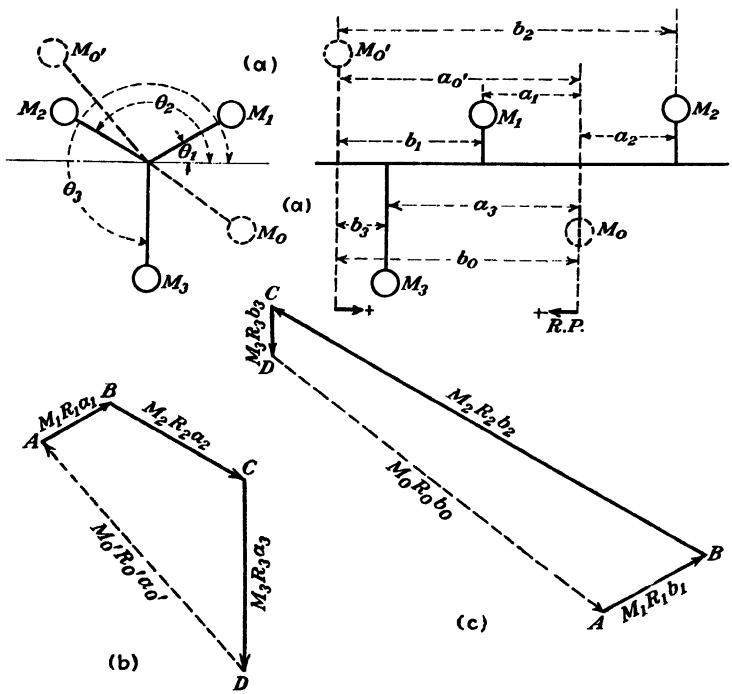
The next step is to draw the moment polygon shown at *b* in the figure. On the positive side of the reference plane, the directions of the moment vectors are the same as those of the corresponding forces, namely, outward from the axis of the shaft and parallel to the corresponding radii. The moment $M_2R_2a_2$, being negative, is represented by a vector having a direction opposite that of the centrifugal force of the mass M_2. The unknown balancing mass M_0, being situated in the reference plane, has zero for a moment. The moment $M_{0'}R_{0'}a_{0'}$ of the unknown balancing mass $M_{0'}$ is given by the closing side of the polygon. Assuming that $M_{0'}$ is situated on the left-hand (positive) side of the reference plane, the direction of its radius will be the same as that of the moment vector $M_{0'}R_{0'}a_{0'}$. Let it also be assumed that its distance to the left of the reference plane is 10 in. The quantity $M_{0'}R_{0'}$ may then be calculated as shown, and if it is decided to use a balancing mass of 16, the proper radius to use is found to be 6 in.

It is seen, therefore, that by applying the moment relation the characteristics of the balancing mass $M_{0'}$ have been determined. It remains to determine the characteristics of the balancing mass M_0 in the reference plane. This is accomplished by applying the force relation, namely, that the vector sum of *all* the centrifugal forces must be zero.

The quantities MR are first calculated for all the known masses including the balancing mass $M_{0'}$. The force polygon is then drawn as shown at *c* in the figure, the closing side of the polygon being the vector that represents the quantity M_0R_0. A convenient value of M_0 is then assumed, and the value of R_0 calculated as shown in the table.

b. Second Method.—The second method for the solution of the problem (refer to Fig. 310) is exactly like the first up to the point where the moment polygon has been drawn and the characteristics of the balancing mass $M_{0'}$ determined. The remainder of the problem consists merely of taking moments with respect to the plane of the balancing mass $M_{0'}$ and drawing a second moment polygon as shown at *c* in the figure. In this case the right side of the plane 0′ has been taken as the positive side. The closing side of the polygon is the vector $M_0R_0b_0$ from which the characteristics of the balancing mass M_0 can be determined as shown in the table at *d*.

This method is analogous to the determination of the reactions of a simple beam by taking moments first with respect to one support and then with respect to the other. It is evident by



Plane	Mass	Radius	a	b	θ	MR	MRa	MRb
1	15	4	4	6	30°	60	240	360
2	30	4	−4	14	150°	120	−480	1,680
3	15	5	8	2	270°	75	600	150
	Assume		Assume					
0′	16	6	10	0	131°	96	960	0
	Assume							
0	30	4.88	0	10	324°	146.5	0	1,465

(*d*)

FIG. 310.

comparing the tables in Figs. 309 and 310 that the results of the two methods are identical. The second method, however, has the advantage of requiring one vector less in the layout, and in

addition it eliminates the necessity of remembering whether to draw the moment polygon or the force polygon first. It also corresponds to the method used in connection with the balancing machine described later in the chapter.

217. Reduction of Masses to a Common Radius.—It has been assumed in the discussion of the preceding articles that the masses have unequal radii. The centrifugal force $MR\omega^2$ is proportional to the product MR, and the individual factors may have any value provided their product remains constant. It is therefore possible to choose some convenient radius, say that of an engine crank, or unity in a general problem, and all the masses that have different radii can be reduced to masses having this common radius. In this case the factor R can be dropped and the reduced masses M_1, M_2, etc., can be used for the sides of the force polygon, and the reduced mass moments M_1a_1, M_2a_2, etc., become the sides of the moment polygon.

218. Analytical Method of Balancing a System of Rotating Masses.—It is often convenient to use analytical methods in the solution of balancing problems instead of the graphical methods discussed in the preceding articles of this chapter. In applying the analytical method to a system of rotating masses, the forces are resolved into vertical and horizontal components as shown in Fig. 311. Each group of components, as well as the two groups formed by taking the moments of the components with respect to any arbitrarily chosen reference plane, must be balanced separately. The conditions of balance are expressed by the following equations, the masses being reduced to a common radius:

Horizontal forces:

$$(M_1 \cos \theta_1 + M_2 \cos \theta_2 \ldots\ldots\ldots + M_n \cos \theta_n)R\omega^2 = 0$$

Vertical forces:

$$(M_1 \sin \theta_1 + M_2 \sin \theta_2 \ldots\ldots\ldots + M_n \sin \theta_n)R\omega^2 = 0$$

Horizontal moments:

$$(M_1a_1 \cos \theta_1 + M_2a_2 \cos \theta_2 \ldots\ldots + M_na_n \cos \theta_n)R\omega^2 = 0$$

Vertical moments:

$$(M_1a_1 \sin \theta_1 + M_2a_2 \sin \theta_2 \ldots\ldots + M_na_n \sin \theta_n)R\omega^2 = 0$$

Dropping the common factor $R\omega^2$, the four equations may be written as follows:

$$\text{(I)} \quad \Sigma M \cos\theta = 0$$
$$\text{(II)} \quad \Sigma M \sin\theta = 0$$
$$\text{(III)} \quad \Sigma Ma \cos\theta = 0$$
$$\text{(IV)} \quad \Sigma Ma \sin\theta = 0$$

With Eqs. (I) and (II) satisfied, the system will be in static balance; but for running balance, all four equations must be satisfied. The significance of either of the moment equations not being satisfied is that a centrifugal couple exists that will

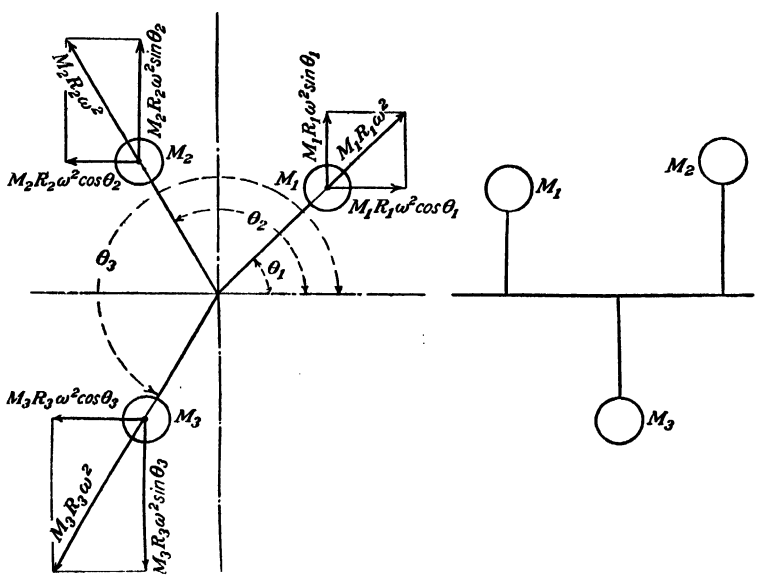
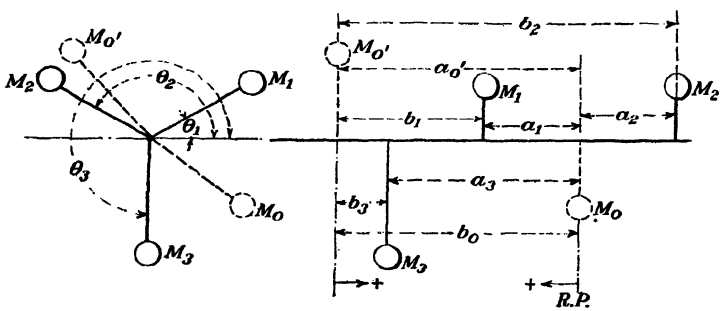


FIG. 311.

set up vibrations when the system is rotating but will disappear and not affect the equilibrium of the system when the latter is standing still.

As brought out in the preceding articles, the characteristics of the masses required to balance any rotating system may be determined either by applying both the force and moment relations, or by applying the moment relation twice; *i.e.*, either by using all four of the preceding equations or by making use of Eqs. (III) and (IV) twice. The second method is that used in connection with the balancing machine described later in this chapter.

As an example of the applications of the analytical method, let it be required to balance the system composed of the three rotating masses shown in Fig. 312. This system is the same as that shown in Fig. 309, except that the masses have been reduced to a unit radius. The second of the two schemes outlined above is used in the solution of the problem. The method of calculation is fairly well outlined in the table. The unknown



Plane	M	θ	a	Ma	Ma cos θ	Ma sin θ	b	Mb	Mb cos θ	Mb sin θ
1	60	30°	4	240	207.8	120	6	360	311.7	180
2	120	150°	−4	−480	415.7	−240	14	1,680	−1,455	840
3	75	270°	8	600	0	−600	2	150	0	−150
0′	[95.25]	[130.9°]	10	[952.5]	[−623.5]	[720]	0	0	0	0
0	[143.6]	[322.7°]	0	0	0	0	10	[1,436]	[1,143]	−870

$$\tan \theta_{0'} = \frac{720}{623.5} \qquad \tan \theta_0 = \frac{870}{1,143}$$

FIG. 312.

quantities that were determined by taking moments are enclosed in brackets. The horizontal and vertical components of $M_{0'}a_{0'}$ were obtained from

$$\begin{aligned} M_{0'}a_{0'} \cos \theta_{0'} &= -\Sigma Ma \cos \theta \\ M_{0'}a_{0'} \sin \theta_{0'} &= -\Sigma Ma \sin \theta \end{aligned} \tag{123}$$

where $\Sigma Ma \cos \theta$ is written for the algebraic sum of the moments of the horizontal components of the three original masses with respect to the plane 0; and $\Sigma Ma \sin \theta$ for the moments of the vertical components. The quadrant of the angle $\theta_{0'}$ was determined from the signs of $M_{0'}a_{0'} \cos \theta_{0'}$ and $M_{0'}a_{0'} \sin \theta_{0'}$, after which its numerical value was calculated from the tangent as

indicated. The quantity $M_{0'}a_{0'}$ was obtained from

$$M_{0'}a_{0'} = \sqrt{(M_{0'}a_{0'}\cos\theta_{0'})^2 + (M_{0'}a_{0'}\sin\theta_{0'})^2} \qquad (124)$$

The characteristics of the mass M_0 were determined in a similar manner.

219. The Balancing of Reciprocating Masses.—The discussion in the preceding articles shows that it is always possible to balance a system of rotating masses by means of two properly placed masses in any two desired planes of rotation. The following articles deal with the more complex problems of balancing masses that do not revolve in a circular path, but have either a motion of translation at variable speed, such as the piston of an engine, or else a swinging motion such as the connecting rod of an engine.

In an engine mechanism, when the motion of the crankpin is given, the corresponding acceleration of the piston can be found for any phase of the mechanism. A certain amount of force is required to produce this acceleration, *i.e.*, to overcome the inertia of the piston. This force must be taken from or added to the total fluid pressure acting on the piston, to find the force transmitted along the connecting rod to the crank, according as the acceleration is positive or negative. Hence there arises an unbalanced force (inertia force) equal and opposite to the force required to accelerate the piston, and, since this force must be absorbed by the frame and foundation, the effect is to set up vibrations.

The effect of the accelerating force on the piston may be compared with that of the motion of a shot from a gun. The acceleration of the shot is always accompanied by the recoil of the gun. Similarly, if the engine frame were free to move in the direction of the stroke, a recoil would take place just as if the piston were a shot and the cylinder a gun. It should be distinctly borne in mind that the recoil of the frame depends only upon the acceleration of the reciprocating mass and has nothing to do with the agent that causes the acceleration. That is, for a given acceleration of the reciprocating mass, the recoil of the framework is the same, whether the motion is due to the action of steam pressure on the piston, or to the explosion of a gas in the cylinder, or whether the turning effort on the crankshaft is accomplished by an outside agency as when the mechanism is that of an air compressor or pump.

220. Inertia Effects of the Reciprocating Masses in the Engine Mechanism.—It was shown in the chapter on Inertia Forces that the inertia force of the reciprocating masses of the engine mechanism is given with very fair accuracy by the relation

$$F = MA = MR\omega^2\left(\cos\theta + \frac{R}{L}\cos 2\theta\right)$$

where M and A are the mass and acceleration respectively of the reciprocating parts, and ω is the angular velocity of the crank. The various steps by means of which this relation was deduced will be repeated here in order to bring all the steps together.

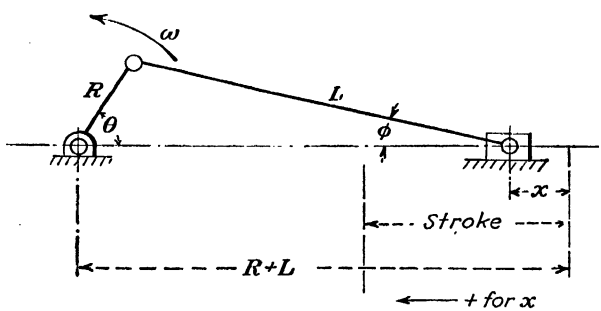


FIG. 313.

Referring to Fig. 313, the displacement of the piston is given by

$$\begin{aligned} x &= R + L - R\cos\theta - L\cos\phi \\ &= R(1 - \cos\theta) + L(1 - \cos\phi) \end{aligned} \qquad (125)$$

The cosine of the angle ϕ is given by

$$\cos\phi = \sqrt{1 - \sin^2\phi} = \sqrt{1 - \frac{R^2}{L^2}\sin^2\theta}, \text{ since } \sin\phi = \frac{R}{L}\sin\theta$$

The quantity under the radical may be expanded as follows:

$$\sqrt{1 - \frac{R^2}{L^2}\sin^2\theta} = 1 - \frac{1}{2}\left(\frac{R}{L}\sin\theta\right)^2 - \frac{1}{2\cdot 4}\left(\frac{R}{L}\sin\theta\right)^4 - \frac{1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{R}{L}\sin\theta\right)^6 \cdots \qquad (126)$$

Substituting in Eq. (125), the displacement x of the piston becomes

$$x = R(1 - \cos\theta) + R\left(\frac{1}{2}\frac{R}{L}\sin^2\theta + \frac{1}{2\cdot 4}\frac{R^3}{L^3}\sin^4\theta + \frac{1\cdot 3}{2\cdot 4\cdot 6}\frac{R^5}{L^5}\sin^6\theta \cdots\right)$$

Since R/L is usually less than $\frac{1}{4}$ the terms beyond the square may be dropped without appreciable error. Then

$$x = R(1 - \cos\theta) + \frac{1}{2}\frac{R^2}{L}\sin^2\theta \text{ (approximately)} \quad (127)$$

Differentiating with respect to time,

$$\frac{dx}{dt} = R\sin\theta\frac{d\theta}{dt} + \frac{1}{2}\frac{R^2}{L}\cdot 2\sin\theta\cos\theta\frac{d\theta}{dt}$$

$$= R\omega\left(\sin\theta + \frac{1}{2}\frac{R}{L}\sin 2\theta\right) \quad (128)$$

where ω has been written for $d\theta/dt$ and the quantity $2\sin\theta\cos\theta$ has been replaced by its equal $\sin 2\theta$.

Differentiating a second time,

$$\frac{d^2x}{dt^2} = R\omega\left(\cos\theta\frac{d\theta}{dt} + \frac{1}{2}\frac{R}{L}\cdot 2\cos 2\theta\frac{d\theta}{dt}\right) =$$

$$R\omega^2\left(\cos\theta + \frac{R}{L}\cos 2\theta\right) \quad (129)$$

The inertia force due to a mass M having this acceleration is

$$F = M\frac{d^2x}{dt^2} = MR\omega^2\left(\cos\theta + \frac{R}{L}\cos 2\theta\right) \quad (130)$$

When this expression is positive, the inertia force will be directed away from the main bearing, and when it is negative, toward the main bearing.

The expression for F may be written as follows:

$$F = MR\omega^2\cos\theta + M\left(\frac{R^2}{4L}\right)(2\omega)^2\cos 2\theta \quad (131)$$

The term $MR\omega^2\cos\theta$ is called the *primary* inertia force and would be the total inertia force if the connecting rod were infinite in length. The term $M(R^2/4L)(2\omega)^2\cos 2\theta$ is called the *secondary*

inertia force and is introduced as a result of the fact that the connecting rod is actually of finite length. It corresponds to the case of a mass M, an infinite connecting rod, and a crank of length $(R^2/4L)$ turning at a speed of 2ω.

It should be noted that both the primary and secondary inertia forces may be considered as due to a mass M having harmonic motion. In the case of the primary inertia force, the amplitude of the harmonic motion is equal to the length of the crank, and the period is the time required for the crank to turn through one complete revolution. In the case of the secondary force, the amplitude is equal to the length $R^2/4L$ of an imaginary crank

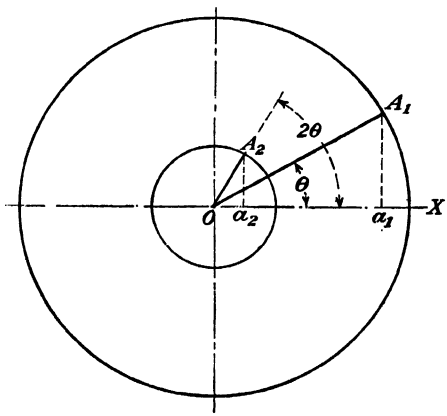


FIG. 314.

rotating at twice the speed of the actual crank; the period, therefore, is half that corresponding to the primary force.

The above inertia forces may be determined graphically for any position of the crank as follows: First, let a suitable force scale be chosen and then, referring to Fig. 314, let two concentric circles be drawn, one with a radius $OA_1 = MR\omega^2$ and the other with a radius $OA_2 = M\left(\frac{R^2}{4L}\right)(2\omega)^2$. Now let the crank angle θ be laid off as shown and let the line OA_2 be laid off making an angle 2θ with the horizontal line OX. Then if the points A_1 and A_2 are projected on the horizontal line OX the lines Oa_1 and Oa_2 will represent the primary and secondary inertia forces respectively.

The relation between primary and secondary inertia forces will be made clear by the following example: For a six-cylinder

gasoline engine the length of crank = 2⅛ in.; length of connecting rod = 8⅛ in.; weight of piston = 1.94 lb.; and r.p.m. of crankshaft = 2,500.

The primary inertia forces for a single cylinder have been calculated for each 30-deg. crank position and plotted as the

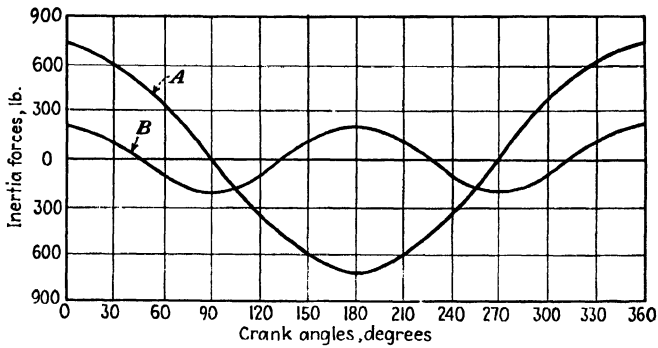


FIG. 315.

harmonic curve *A* in Fig. 315. Similarly, the secondary inertia forces have been calculated and plotted as the harmonic curve *B*. It will be observed that the two curves are in phase at the head-end dead center, but that curve *B* has twice the frequency of curve *A*.

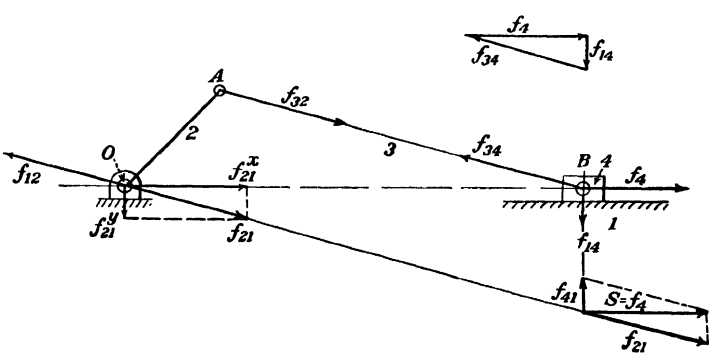


FIG. 316.

The inertia forces of the reciprocating masses are transmitted to the frame in the following manner: Referring to Fig. 316, the inertia force f_4 sets up the reactions f_{14} and f_{34}, the magnitudes of the latter being given by the polygon shown. The force f_{14} is the side pressure exerted upon the piston by the cylinder walls.

An equal and opposite force f_{41} is exerted by the piston upon the cylinder walls, and this force tends to shake the frame. The reaction f_{34} is transmitted along the connecting rod and finally appears at the main bearing as f_{21}. The resultant shaking force S is obtained by combining f_{41} and f_{21}. The magnitude and direction of the force S are the same as those of the inertia force f_4. The line of action of S, however, is offset as indicated (except for the dead-center positions of the crank).

Instead of combining f_{21} and f_{41} to determine the resultant shaking force S, let the force f_{21} be resolved into horizontal and vertical components at the center of the main bearing O. The horizontal component f_{21}^x is equal to f_4 and has the same line of

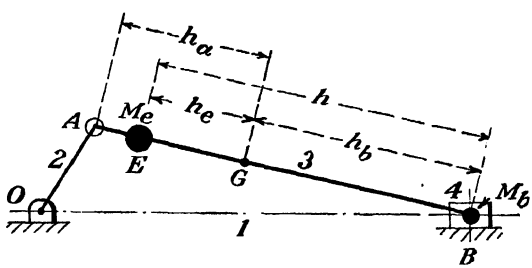


FIG. 317.

action. The vertical component f_{21}^y forms, with the force f_{41}, a couple that tends to rotate the frame of the engine about the crankshaft. The effect of this couple upon the frame is usually considered along with the other torque effects, such as that of the static turning effort, and it will therefore be disregarded in discussing the balancing of engines. This being the case, the inertia effect of the reciprocating masses upon the frame may be considered as equivalent to a force f_{21}^x, which is applied at the main bearing and has the same magnitude and direction as the inertia force f_4.

221. Inertia Effects of Crank and Connecting Rod.—The inertia effect of the crank is a pure centrifugal force (assuming a uniform speed of rotation) and it may be calculated by multiplying the mass of the crank by the acceleration of its center of gravity. In analyzing the shaking forces of the engine, it is customary to replace the mass of crank by an equivalent mass at the crankpin. Since the inertia force of the crank is a pure

centrifugal force, it may be completely balanced by placing a mass of the proper characteristics opposite the crank.

The inertia effects of the connecting rod may be studied by replacing it by a kinetically equivalent system, *i.e.*, by assuming that the mass of the connecting rod is concentrated at two points, at the wrist pin and, referring to Fig. 317, at the center of percussion E of the connecting rod with respect to the wrist-pin center. If the position of the center of gravity G of the rod is known, and if the moment of inertia of the rod with respect to G is also known, then the position of the center of percussion E may be calculated as follows:

$$h_e = \frac{I}{M_3 h_b}$$

where I and M_3 are respectively the moment of inertia and mass of the rod. The masses assumed concentrated at B and E are given by

$$M_e = \frac{h_b}{h} M_3 \text{ and } M_b = \frac{h_e}{h} M_3$$

The horizontal and vertical components of the inertia force of the mass concentrated at E are as follows (refer to Art. 210):

$$\left.\begin{aligned} F_e^x &= M_e R\omega^2 \left[\cos\theta + \left(1 - \frac{h}{L}\right)\frac{R}{L} \cos 2\theta \right] \\ &= M_3 \frac{h_b}{h} R\omega^2 \left[\cos\theta + \left(1 - \frac{h}{L}\right)\frac{R}{L} \cos 2\theta \right] \\ F_e^y &= M_e R\omega^2 \frac{h}{L} \sin\theta = M_3 \frac{h_b}{L} R\omega^2 \sin\theta \end{aligned}\right\} \quad (132)$$

The vertical component of the inertia force of the mass concentrated at B is zero. The horizontal component is

$$\begin{aligned} F_e^x &= M_b R\omega^2 \left(\cos\theta + \frac{R}{L} \cos 2\theta \right) \\ &= M_3 \frac{h_e}{h} R\omega^2 \left(\cos\theta + \frac{R}{L} \cos 2\theta \right) \end{aligned} \quad (133)$$

The inertia forces determined by means of the above expression are exact in so far as concerns mass distribution, the only error being that introduced by the use of the approximate expres-

sion for the horizontal components of the accelerations at E and B.

Although a rigid analysis would require the mass of the connecting rod to be distributed as shown in Fig. 317, a convenient and much used approximate method of taking into account the inertia effect of the connecting rod is to assume that its mass is concentrated at the crankpin and wrist pin so that

$$M_a h_a = M_b h_b; \text{ i.e., } M_a = \frac{h_b}{L} M_3 \text{ and } M_b = \frac{h_a}{L} M_3$$

Then in determining the inertia forces of the mechanism, the mass assumed concentrated at the wrist pin is simply added to the other reciprocating masses; and likewise the mass assumed concentrated at the crankpin is added to the crank mass.

The magnitude and direction of the resultant inertia force of the rod as obtained by assuming the mass concentrated at the crank- and wrist pins are identical with the magnitude and direction as obtained using the kinetically equivalent system. The line of action, however, is different.

222. The Single-cylinder Engine. Counterbalancing.—As stated in the preceding article, the inertia force of the crank and that part of the connecting-rod mass that, by the approximate method, may be considered as concentrated at the crankpin, can always be balanced by placing a mass of the proper characteristics opposite the crankpin. Consideration of these forces will therefore be omitted in this and in the following articles unless otherwise stated.

The situation in the case of the inertia forces of the reciprocating masses is altogether different; in the single-cylinder engine they cannot be balanced at all. They can be modified, however, by the process of counterbalancing. The primary force is $MR\omega^2 \cos \theta$. If, referring to Fig. 318, a counterbalancing mass M_0 is placed opposite the crank at a distance R_0 such that $M_0 R_0 = MR$, this mass will set up a centrifugal force equal to $MR\omega^2$ in a direction opposite the crank radius. If this force is resolved, at the center of the main bearing, into horizontal and vertical components, the result will be as follows:

$$\text{Horizontal force} = -MR\omega^2 \cos \theta$$
$$\text{Vertical force} = -MR\omega^2 \sin \theta$$

The primary shaking forces are thus balanced and vertical shaking forces substituted. Considering a complete revolution of the crank, the result is to shift the original shaking forces through 90 deg.

For engines of this type it is common practice to balance, by means of a revolving mass opposite the crankpin, all the revolving crank mass, including that part of the mass of the connecting rod assumed concentrated at the crankpin, and approximately two-thirds of the reciprocating mass. The balancing conditions will be made clear from the analysis of a typical case, as follows:

A single-crank gas engine of the stationary type (Fig. 319) runs at a speed of 230 r.p.m.

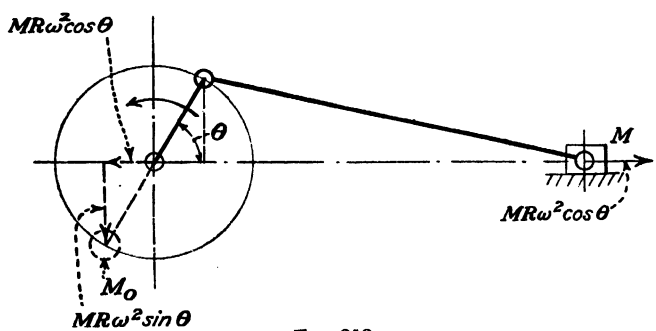


FIG. 318.

Length of crank = 10½ in.
Length of connecting rod = 52½ in.
Distance of center of gravity of connecting rod from crank end = 20.3 in.
Equivalent unbalanced weight of crank at 10½-in. radius = 162.5 lb.
Weight of piston = 244 lb.
Weight of connecting rod = 254.5 lb.
Equivalent weight of counterbalance at 10½-in. radius = 533 lb.

From Art. 221 the proportion of the weight of the connecting rod that may be assumed to be concentrated at the crankpin is $32.2/52.5 \times 254.5 = 156$ lb. The remainder, $254.5 - 156 = 98.5$ lb., is assumed to be concentrated at the crosshead pin. The total revolving crankpin weight at 10½-in. radius is then 162.5 +

156 = 318.5 lb., and the total reciprocating weight is 244 + 98.5 = 342.5 lb.

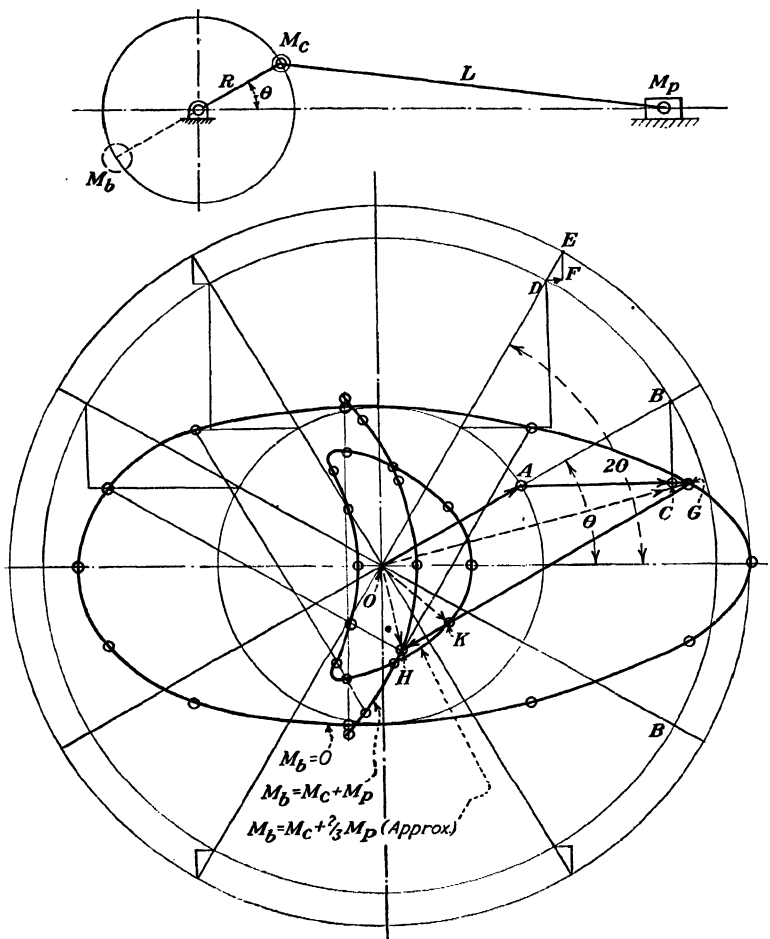


FIG. 319.

The total inertia forces are as follows:
For the revolving crank mass,

$$F_c = M_c R\omega^2 = 5{,}030 \text{ lb.}$$

For the reciprocating mass (primary),

$$F'_p = M_p R\omega^2 \cos\theta = 5{,}400 \cos\theta \text{ lb.}$$

For the reciprocating mass (secondary),

$$F''_p = M_p\left(\frac{R^2}{4L}\right)(2\omega)^2 \cos 2\theta = 1{,}080 \cos 2\theta \text{ lb.}$$

For the counterbalancing mass,

$$F_b = M_b R\omega^2 = 8{,}400 \text{ lb.}$$

It will be convenient, in studying the effect of the inertia forces, to plot curves (Fig. 319) showing the resultant inertia forces for each 30-deg. position of the crank.

From a pole O (Fig. 319), draw the vector OA, representing to scale the inertia force (= 5,030 lb.) of the crank mass. Next draw AB(= 5,400 lb.). Then AC(= 5,400 cos θ) is the primary inertia force. If DE is now drawn to represent 1,080 lb., DF will represent 1,080 cos 2θ, the secondary force. Lay off $CG = DF$. OG is the resultant inertia force for this position of the crank (θ = 30 deg.), and G is one point on the curve of inertia forces for the condition where the engine is not counterbalanced.

The effect of using a counterbalancing weight equal to the sum of the crank weight and the reciprocating weight may be shown by bringing into the force polygon $OACG$ the vector $GH = OB$, giving a new resultant OH. The point H is then one point on the curve of inertia forces for this balancing condition. The curve clearly shows that although the horizontal forces have been largely neutralized (only the secondary horizontal forces remain), rather large vertical forces have been substituted.

The effect of the actual counterbalancing weight used on the engine is shown by the curve correspondingly indicated in the figure. The point K on this curve for the 30-deg. position of the crank is obtained by laying off GK = 8,400 lb. The counterbalancing weight is approximately equal to the rotating weight plus two-thirds of reciprocating weight.

The circles of radii OA, OB, and OE have been drawn for convenience in making the graphical constructions for the remaining points on the curves that have just been discussed.

It may be observed from the figure that these curves are symmetrical about the horizontal axis. If the secondary forces were neglected, the resulting curves would be symmetrical about the vertical axis also.

223. Conditions of Balance in a Multicylinder Engine.—In an engine having more than one cylinder the rotating crank masses can easily be balanced, but the inertia forces of the reciprocating masses may or may not tend to neutralize each other, depending upon the arrangement of cranks and the proportions of the various links of the mechanism. In Fig. 320 the schematic outline of a three-cylinder engine has been shown, the inertia forces of the reciprocating masses being indicated by the arrows. These forces are transmitted through the connecting rods to the cranks

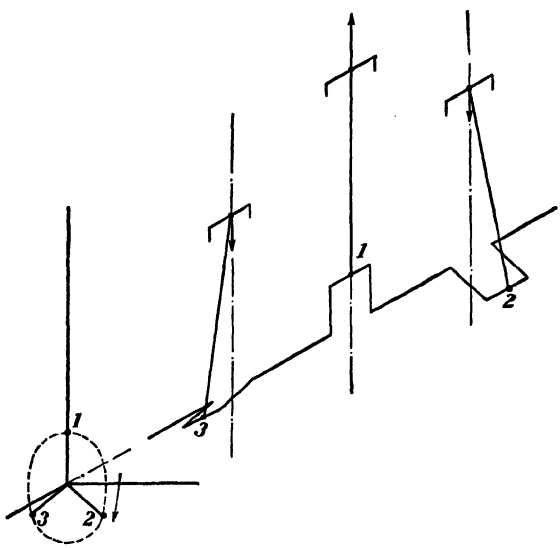


FIG. 320.

and thence to the crankshaft. If the forces acting on the crankshaft are in equilibrium, no pressure due to the inertia of the reciprocating masses will be exerted on the main bearings; otherwise a force, a couple, or both will be exerted that will tend to shake the frame of the engine.

Assuming that the resultant shaking force due to the inertia of the reciprocating masses will pass through the axis of the crankshaft (*i.e.*, disregarding the couple mentioned in Art. 220), the problem of balancing the engine is reduced to the analysis of a system of vertical forces (for the case shown in Fig. 320) all of which are situated in the same axial plane.

In order to bring out the general conditions of balance for an engine of this type, let the case shown in Fig. 321 be considered. This is assumed to be an n-cylinder engine, although only four of the units are shown. The stroke line of each cylinder is assumed to be situated in a horizontal plane passing through the axis of the crankshaft, and the inertia forces of the reciprocating masses are of course also situated in this plane. It will be convenient to consider the equilibrium of the primary and secondary forces separately. Let R represent the crank, L the connecting rod, and M the reciprocating mass. The subscripts 1, 2, 3, etc. will refer to cylinders 1, 2, 3, etc.

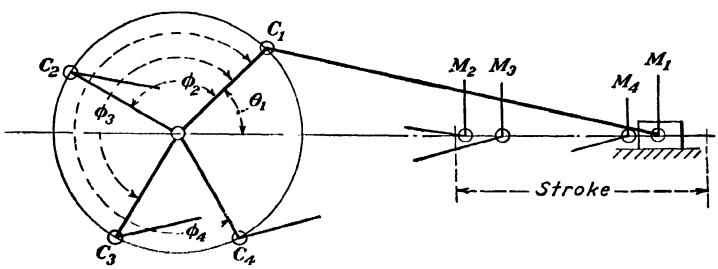


FIG. 321.

1. The primary forces will be in equilibrium if their algebraic sum and the algebraic sum of their moments with respect to any transverse plane are both equal to zero. Setting the algebraic sum of the primary forces equal to zero,

$$M_1R_1\omega^2 \cos \theta_1 + M_2R_2\omega^2 \cos (\theta_1 + \phi_2) \cdots + M_nR_n\omega^2 \cos (\theta_1 + \phi_n) = 0 \quad (134)$$

It is to be noted that the angle θ_1 is a variable whereas the values of the angles ϕ_2, ϕ_3, ϕ_4, etc., are constant, being merely the fixed angular positions of the various cranks relative to crank 1. The angle ϕ_1 is, of course, equal to zero. Therefore

$$M_1R_1\omega^2 \cos (\theta_1 + \phi_1) + M_2R_2\omega^2 \cos (\theta_1 + \phi_2) \cdots + M_nR_n\omega^2 \cos (\theta_1 + \phi_n) = \omega^2[\cos \theta_1(M_1R_1 \cos \phi_1 + M_2R_2 \cos \phi_2 \cdots + M_nR_n \cos \phi_n) - \sin \theta_1(M_1R_1 \sin \phi_1 + \cdots + M_nR_n \sin \phi_n)] = 0 \quad (135)$$

In order that the above equation may hold for all values of θ_1, each of the quantities in the parentheses must be equal to zero.

Therefore,

$$\Sigma MR \cos \phi = 0$$
$$\Sigma MR \sin \phi = 0$$

Similarly, it may be shown that

$$\Sigma MRa \cos \phi = 0$$
$$\Sigma MRa \sin \phi = 0$$

where a is the distance from the center line of any cylinder to an arbitrarily chosen reference plane, the latter being perpendicular to the axis of the crankshaft.

Any engine for which the preceding conditions are satisfied is said to be in primary balance.

It is to be noted that the preceding conditions of balance are precisely those of a rotating system, the masses of which are equal to the reciprocating masses and are situated at distances from the axis of rotation equal to the lengths of the cranks.

2. Setting the algebraic sum of the secondary forces equal to zero,

$$M_1\frac{R_1^2}{L_1}\omega^2 \cos 2(\theta_1 + \phi_1) + M_2\frac{R_2^2}{L_2}\omega^2 \cos 2(\theta_1 + \phi_2) + \cdots + M_n\frac{R_n^2}{L_n}\omega^2 \cos 2(\theta_1 + \phi_n) = 0 \quad (136)$$

Therefore

$$\omega^2\left[\cos 2\theta_1\left(M_1\frac{R_1^2}{L_1}\cos 2\phi_1 + \cdots M_n\frac{R_n^2}{L_n}\cos 2\phi_n\right) - \sin 2\theta_1\left(M_1\frac{R_1^2}{L_1}\sin 2\phi_1 + \cdots M_n\frac{R_n^2}{L_n}\sin 2\phi_n\right)\right] = 0 \quad (137)$$

In order for this equation to hold for all values of θ_1,

$$\sum M\frac{R^2}{L}\cos 2\phi = \sum M\frac{R^2}{L}(\cos^2\phi - \sin^2\phi) = 0$$
$$\sum M\frac{R^2}{L}\sin 2\phi = 0; \textit{ i.e.,} \sum M\frac{R^2}{L}\sin\phi\cos\phi = 0$$

Similarly it may be shown that

$$\sum M\frac{R^2}{L}a(\cos^2\phi - \sin^2\phi) = 0$$

$$\sum M\frac{R^2}{L}a \sin \phi \cos \phi = 0$$

where a is the distance from the center lines of the cylinders to any transverse reference plane.

Any engine for which these conditions are satisfied is said to be in secondary balance.

224. The Eight Fundamental Equations of Balance.—It is evident from the foregoing discussion that for the complete balance of the primary and secondary forces and couples, eight equations must be satisfied. Letting $x = \cos \phi$, and $y = \sin \phi$ these equations may be written as follows:

$$\left.\begin{aligned} &\text{(I)} \quad \sum MRx = 0 \\ &\text{(II)} \quad \sum MRy = 0 \end{aligned}\right\} \text{Primary forces balanced}$$

$$\left.\begin{aligned} &\text{(III)} \quad \sum MRax = 0 \\ &\text{(IV)} \quad \sum MRay = 0 \end{aligned}\right\} \text{Primary couples balanced}$$

$$\left.\begin{aligned} &\text{(V)} \quad \sum M\frac{R^2}{L}(x^2 - y^2) = 0 \\ &\text{(VI)} \quad \sum M\frac{R^2}{L}xy = 0 \end{aligned}\right\} \text{Secondary forces balanced}$$

$$\left.\begin{aligned} &\text{(VII)} \quad \sum M\frac{R^2}{L}a(x^2 - y^2) = 0 \\ &\text{(VIII)} \quad \sum M\frac{R^2}{L}axy = 0 \end{aligned}\right\} \text{Secondary couples balanced}$$

These eight fundamental equations express completely[1] the analytical conditions of primary and secondary balance among the reciprocating parts for an engine with any number of cranks, the cylinders being arranged as in the automobile type of engine, *i.e.*, all on one side of the crankshaft and their center lines all in the plane that contains the axis of the crankshaft.

Partial balance may be secured by satisfying part of the eight fundamental equations. It should be noted, however, that these equations must always be taken in pairs. That is, in order

[1] Based upon the assumption that the inertia forces of the reciprocating masses are exerted at the main bearing and that the expression $F = MR\omega^2\left(\cos \theta + \frac{R}{L}\cos 2\theta\right)$ is sufficiently exact.

that the primary forces may be in balance *both* Eqs. (I) and (II) must be satisfied, and similarly for the other forces and couples.

Applications of the method will be illustrated in the articles which follow. In these applications the lengths of the connecting rods and cranks are assumed to be the same for all cylinders; and the reciprocating masses are all assumed to be equal. In this case the equations of balance can be written in the following form:

$$\begin{aligned} \Sigma x &= 0 & \Sigma(x^2 - y^2) &= 0 \\ \Sigma y &= 0 & \Sigma xy &= 0 \\ \Sigma ax &= 0 & \Sigma a(x^2 - y^2) &= 0 \\ \Sigma ay &= 0 & \Sigma axy &= 0 \end{aligned} \tag{138}$$

If, after the equations of balance have been applied, it is desired to determine the magnitudes of the unbalanced forces and couples for any position of the mechanism, the following equations may be applied:

$$\begin{aligned} F_p &= MR\omega^2(\cos\theta_1\Sigma x - \sin\theta_1\Sigma y) \\ C_p &= MR\omega^2(\cos\theta_1\Sigma ax - \sin\theta_1\Sigma ay) \\ F_s &= M\frac{R^2}{L}\omega^2[\cos 2\theta_1\Sigma(x^2 - y^2) - 2\sin 2\theta_1\Sigma xy] \\ C_s &= M\frac{R^2}{L}\omega^2[\cos 2\theta_1\Sigma a(x^2 - y^2) - 2\sin 2\theta_1\Sigma axy] \end{aligned} \tag{139}$$

where F_p, C_p, F_s, and C_s are respectively the unbalanced primary forces, unbalanced primary couples, unbalanced secondary forces, and unbalanced secondary couples.

If an unbalanced force exists, its line of action is situated in the reference plane. If in addition, as is usually the case, an unbalanced couple also exists, it may be combined with the force to form a single force situated in some other transverse plane. It should be noted that, for an actual couple tending to tip the engine frame end over end to exist, the forces must be in balance. If the forces are not in balance, the couple equation merely shows the position of the line of action of the resultant shaking force (the force obtained by combining the unbalanced force and the unbalanced couple). In this case the unbalanced couple is simply the moment of the resultant force with respect to the reference plane. If for any position of the crank, however, the force becomes zero without the couple also becoming zero, then a true

couple is exerted on the engine frame tending to tip it end over end; this is equivalent to the resultant shaking force (the resultant of the zero force and the couple) being situated in a plane at an infinite distance from the reference plane.

225. The Two-cylinder Engine.—In order to investigate the shaking forces in the two-cylinder engine, let the case shown in Fig. 322 be considered. Here the cranks are at 180 deg. The quantity $MR\omega^2$ is assumed equal to one, and the quantity R/L is assumed equal to one-fourth. The distance between the center lines of any two adjacent cylinders will be assumed equal

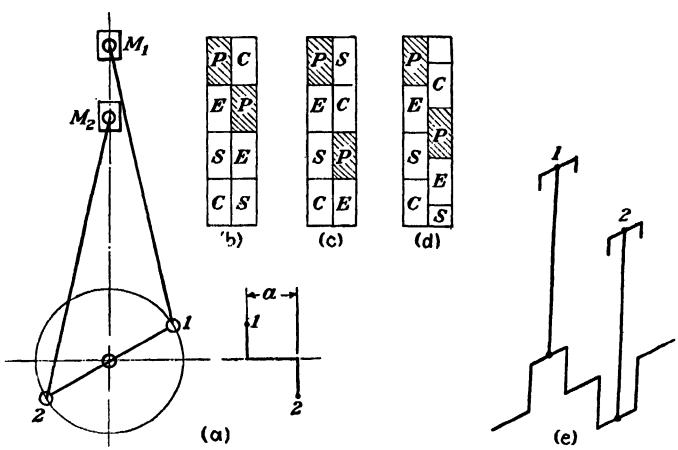


FIG. 322.

to unity, and the reference plane will be taken through the center line of the first cylinder. These same values will be maintained throughout the discussion of engine balance in order to compare the shaking forces of the various types of engines.

The first step is to apply the eight fundamental equations. A convenient tabular method of applying these equations is shown in Table 6. The results thus obtained indicate that the primary forces alone are balanced.

The unbalanced forces and couples for any crank position may be calculated as follows:

$$F_p = MR\omega^2(\cos\theta_1 \Sigma x - \sin\theta_1 \Sigma y) = \cos\theta_1 \cdot 0 - \sin\theta_1 \cdot 0 = 0$$

$$C_p = MR\omega^2(\cos\theta_1 \Sigma ax - \sin\theta_1 \Sigma ay) = \cos\theta_1(-1) - \sin\theta_1 \cdot 0 = -\cos\theta_1$$

$$F_s = MR\omega^2\frac{R}{L}[\cos 2\theta_1 \Sigma(x^2 - y^2) - 2 \sin 2\theta_1 \Sigma xy]$$
$$= \tfrac{1}{4}(\cos 2\theta_1 \cdot 2 - 2 \sin 2\theta_1 \cdot 0) = \tfrac{1}{2} \cos 2\theta_1$$
$$C_s = MR\omega^2\frac{R}{L}[\cos 2\theta_1 \Sigma a(x^2 - y^2) - 2 \sin 2\theta_1 \Sigma axy]$$
$$= \tfrac{1}{4}(\cos 2\theta_1 \cdot 1 - 2 \sin 2\theta_1 \cdot 0) = \tfrac{1}{4} \cos 2\theta_1$$

The total shaking force and the total shaking couple are, respectively,

$$F = F_p + F_s = \tfrac{1}{2} \cos 2\theta_1$$
$$C = C_p + C_s = -\cos \theta_1 + \tfrac{1}{4} \cos 2\theta_1$$

The resultant shaking force obtained by combining the force in the reference plane and the couple has a magnitude equal to

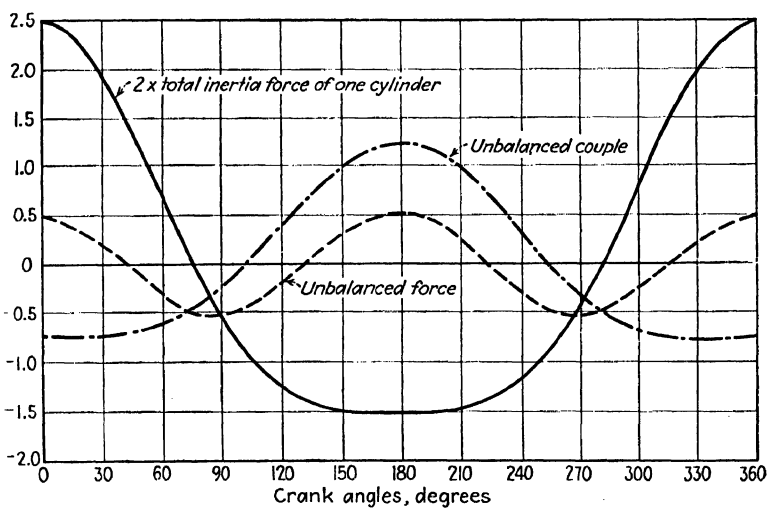


FIG. 323.

F and a line of action as given by

$$z = \frac{C}{F} = \frac{-\cos \theta_1 + \tfrac{1}{4} \cos 2\theta_1}{\tfrac{1}{2} \cos 2\theta_1} = \frac{1}{2} - \frac{2 \cos \theta_1}{\cos 2\theta_1}$$

where z is the distance of the line of action of the resultant shaking force from the reference plane (*i.e.*, from the center line of the first cylinder).

In order to study the manner in which the total unbalanced shaking forces and shaking couples vary, they may be calculated

for each 15-deg. position of the crank as shown in Table 7. These results may then be plotted as shown in Fig. 323. In this figure twice the total inertia force of a single cylinder has also been

TABLE 6.—TWO-CRANK ENGINE (CRANKS AT 180 DEG.)

Crank	ϕ	x	y	a	ax	ay	x^2	y^2	$(x^2 - y^2)$	xy	$a(x^2 - y^2)$	axy
1	0	1	0	0	0	0	1	0	1	0	0	0
2	180	−1	0	1	−1	0	1	0	1	0	1	0
Total.....	...	0	0	..	−1	0	..	..	2	0	1	0

Let $MR\omega^2 = 1$ and $\frac{R}{L} = \frac{1}{4}$ $\qquad F = F_p + F_s = 0 + \frac{1}{2}\cos 2\theta_1 = \frac{\cos 2\theta_1}{2}$

$C = C_p + C_s = -\cos\theta_1 + \frac{1}{4}\cos 2\theta_1$

TABLE 7.—TWO-CRANK ENGINE (CRANKS AT 180 DEG.)

θ	$\cos\theta$	$\cos 2\theta$	F_p	F_s	F	C_p	C_s	C
0	+1.000	+1.000	0.000	+0.500	+0.500	−1.000	+0.250	−0.750
15	+0.966	+0.866	.....	+0.433	+0.433	−0.966	+0.217	−0.749
30	+0.866	+0.500	.....	+0.250	+0.250	−0.866	+0.125	−0.741
45	+0.707	+0.000	.....	0.000	0.000	−0.707	0.000	−0.707
60	+0.500	−0.500	.....	−0.250	−0.250	−0.500	−0.125	−0.625
75	+0.259	−0.866	.....	−0.433	−0.433	−0.259	−0.217	−0.476
90	0.000	−1.000	.....	−0.500	−0.500	0.000	−0.250	−0.250
105	−0.259	−0.866	.....	−0.433	−0.433	+0.259	−0.217	+0.042
120	−0.500	−0.500	.....	−0.250	−0.250	+0.500	−0.125	+0.375
135	−0.707	0.000	.....	0.000	0.000	+0.707	0.000	+0.707
150	−0.866	+0.500	.....	+0.250	+0.250	+0.866	+0.125	+0.991
165	−0.966	+0.866	.....	+0.433	+0.433	+0.966	+0.217	+1.183
180	−1.000	+1.000	.....	+0.500	+0.500	+1.000	+0.250	+1.250
195	−0.966	+0.866	.....	+0.433	+0.433	+0.966	+0.217	+1.183
210	−0.866	+0.500	.....	+0.250	+0.250	+0.866	+0.125	+0.991
225	−0.707	0.000	.....	0.000	0.000	+0.707	0.000	+0.707
240	−0.500	−0.500	.....	−0.250	−0.250	+0.500	−0.125	+0.375
255	−0.259	−0.866	.....	−0.433	−0.433	+0.259	−0.217	+0.042
270	0.000	−1.000	.....	−0.500	−0.500	0.000	−0.250	−0.250
285	+0.259	−0.866	.....	−0.433	−0.433	−0.259	−0.217	−0.476
300	+0.500	−0.500	.....	−0.250	−0.250	−0.500	−0.125	−0.625
315	+0.707	0.000	.....	0.000	0.000	−0.707	0.000	−0.707
330	+0.866	+0.500	.....	+0.250	+0.250	−0.866	+0.125	−0.741
345	+0.966	+0.866	.....	+0.433	+0.433	−0.966	+0.217	−0.749
360	+1.000	+1.000	.....	+0.500	+0.500	−1.000	+0.250	−0.750

plotted, thus giving some idea of the extent to which the engine is balanced.

The arrangement with cranks at 180 deg. is not ideal in a gasoline engine on account of the uneven turning effort, the two

TABLE 8.—TWO-CRANK ENGINE (QUARTER CRANK)

Crank	ϕ	x	y	a	ax	ay	x^2	y^2	$(x^2 - y^2)$	xy	$a(x^2 - y^2)$	axy
1	0	1	0	0	0	0	1	0	1	0	0	0
2	90	0	1	1	0	1	0	1	−1	0	−1	0
Total.....	..	1	1	..	0	1	..	..	0	0	−1	0

Let $MR\omega^2 = 1$ and $\frac{R}{L} = \frac{1}{4}$

$F_p = \cos\theta_1 - \sin\theta_1$ $\qquad C_p = 0 - \sin\theta_1$

$F_s = \frac{1}{4}[0 - 0]$ $\qquad C_s = \frac{1}{4}[-\cos 2\theta_1 - 0]$

explosions of the complete cycle taking place in the same revolution of the crank, as shown by the chart (Fig. 322*b*). If the

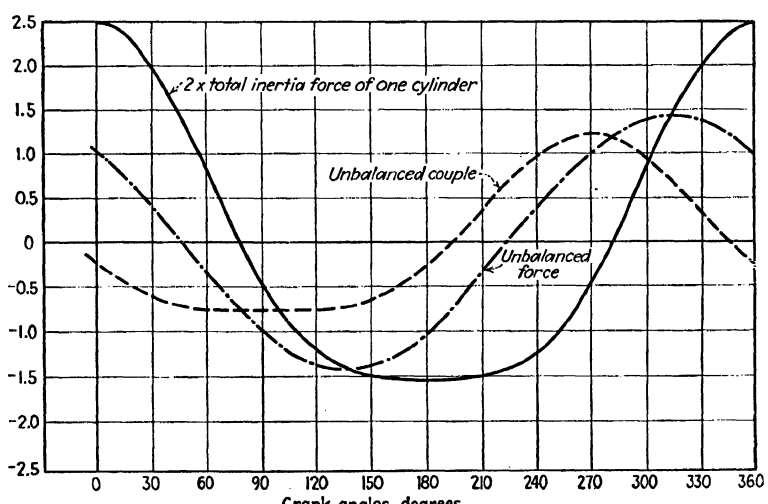


FIG. 324.

two cranks are placed on the same side of the crankshaft, the engine becomes equivalent to two single-crank engines. In this case the balance is poor but the turning effort is good. If the cranks are placed at right angles, the balance is fair and the turning effort is good. The firing-order charts (Figs. 322*c* and 322*d*) show the character of the turning effort in the last two cases. Table 8 shows the balance with cranks at right angles, and the curves in Fig. 324 show how the unbalanced forces vary.

It must be remembered, in considering the arrangements of cranks in a multicylinder engine as regards balance, that the arrangement is governed primarily by the considerations of even turning effort. That is, the power strokes are so distributed as to produce as steady a torque as possible.

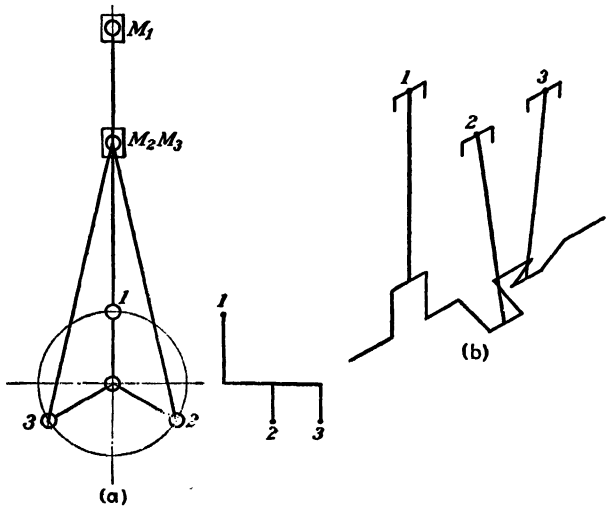


FIG. 325.

226. Three-cylinder Engine.—The cranks of the three-cylinder engine are usually placed 120 deg. apart, as shown in Fig. 325. The balance of this engine is shown by Table 9. The primary and secondary forces are in balance but the couples are not.

TABLE 9.—THREE-CRANK ENGINE

Crank	ϕ	x	y	a	ax	ay	x^2	y^2	$(x^2 - y^2)$	xy	$a(x^2 - y^2)$	axy
1	0	1.000	0	0	0	0	1.000	0	1.000	0	0	0
2	120	−0.500	+0.866	1	−0.500	+0.866	0.250	0.750	−0.500	−0.433	−0.500	−0.433
3	240	−0.500	−0.866	2	−1.000	−1.732	0.250	0.750	−0.500	+0.433	−1.000	+0.866
Total	....	0	0	..	−1.500	−0.866	.....		0	0	−1.500	+0.433

Let $MR\omega^2 = 1$ and $\frac{R}{L} = \frac{1}{4}$

$F_p = 0$
$F_s = 0$
$F = F_p + F_s = 0$

$C_p = -1.500 \cos \theta_1 + 0.866 \sin \theta_1$
$C_s = \frac{1}{4}(-1.500 \cos 2\theta_1 - 0.866 \sin 2\theta_1)$

The unbalanced couples have been plotted in Fig. 326, and a curve representing three times the inertia force of a single cylinder has also been drawn. This type of engine has been much used in marine work.

227. The Four-cylinder Automobile-type Engine.—The conventional crank arrangement of the four-cylinder automobile-type engine is shown in Fig. 327, and the balance of this type of

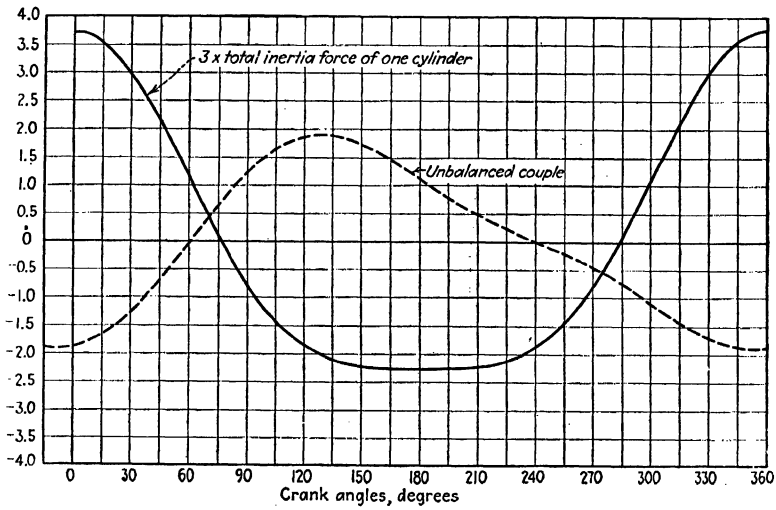


FIG. 326.

engine is shown in Table 10. The secondary forces are unbalanced, and when, as is the case here, the reference plane has been

TABLE 10.—FOUR-CRANK ENGINE

Crank	ϕ	x	y	a	ax	ay	x^2	y^2	$(x^2 - y^2)$	xy	$a(x^2 - y^2)$	axy
1	0	1	0	0	0	0	1	0	1	0	0	0
2	180	−1	0	1	−1	0	1	0	1	0	1	0
3	180	−1	0	2	−2	0	1	0	1	0	2	0
4	0	1	0	3	3	0	1	0	1	0	3	0
Total.....		0	0	..	0	0	..	..	4	0	6	0

$F_p = 0$
$F_s = \frac{1}{4}[\cos 2\theta \cdot 4 - 0]$
$= \cos 2\theta$
$F = F_p + F_s = \cos 2\theta$

$C_p = 0$
$C_s = \frac{1}{4} \cos 2\theta \cdot 6 = \frac{3}{2} \cos 2\theta$
$C = C_p + C_s = \frac{3}{2} \cos 2\theta$

chosen through the center line of the first cylinder, it appears that the secondary couples are also unbalanced. It may be shown by the following analysis, however, that there are no secondary couples in the true sense of the term. The unbalanced forces and couples are given by

$$F = F_p + F_s = 0 + \tfrac{1}{4}(\cos 2\theta_1 \cdot 4 - 2 \sin 2\theta_1 \cdot 0) = \cos 2\theta_1$$
$$C = C_p + C_s = 0 + \tfrac{1}{4}(\cos 2\theta_1 \cdot 6 - 2 \sin 2\theta_1 \cdot 0) = \tfrac{3}{2} \cos 2\theta_1$$

The line of action of the resultant shaking force, obtained by

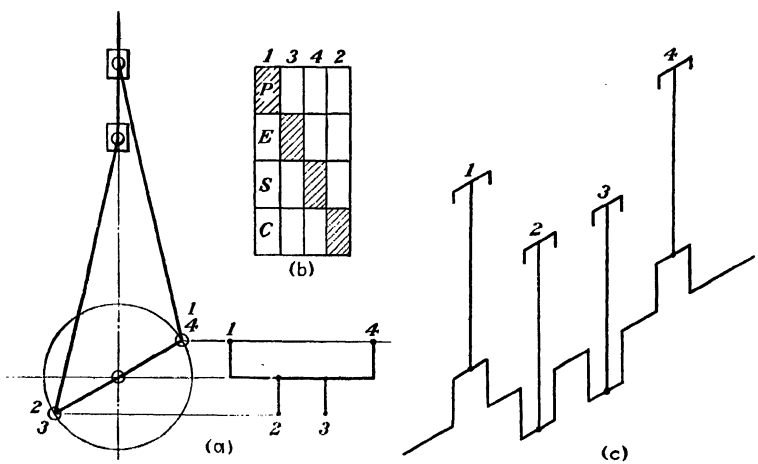


FIG. 327.

combining the force in the reference plane and the couple, is given by

$$z = \frac{C}{F} = \frac{\tfrac{3}{2} \cos 2\theta_1}{\cos 2\theta_1} = \frac{3}{2}$$

That is, for every position of the crank the line of action of the resultant inertia force is situated midway between the two middle cylinders.

The firing chart shown at *b* in Fig. 327 indicates that the turning effort with this arrangement of cranks is good; and the shaking force curves in Fig. 328 show that the amount of unbalance is not particularly great.

A simple and effective device that has been used for balancing the secondary forces in the standard four-cylinder engine is

represented kinematically in Fig. 329. Two gears driven at a speed twice that of the crankshaft are placed in an axial plane

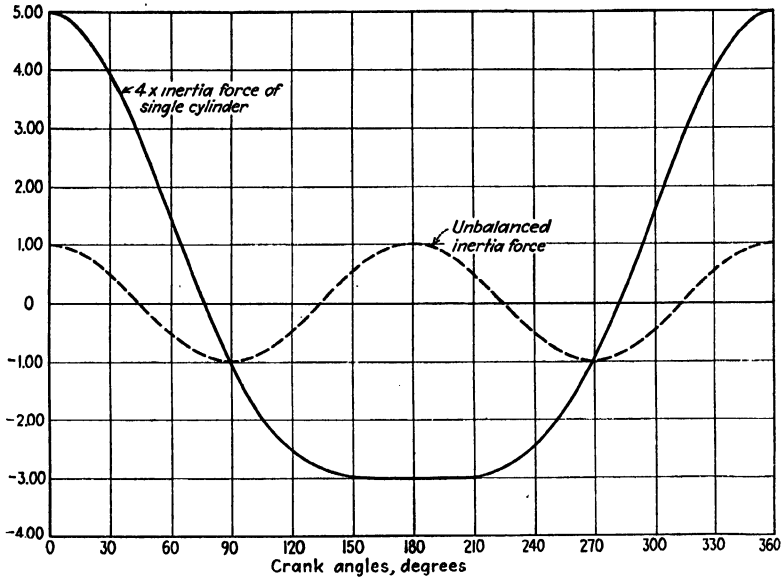


FIG. 328.

so that their pitch point is directly below the mid-point of the crankshaft, *i.e.*, so that the line of action of the resultant shaking force passes through their pitch point. On these gears are placed the equal masses M_2 and M_3, set so that the horizontal components of their centrifugal forces always neutralize each other, leaving their combined vertical components to balance the otherwise unbalanced secondary forces. Since the masses rotate at a speed twice that of the crank, it is evident that the vertical components of their centrifugal forces will vary harmonically in the same period as that of secondary inertia forces. A device designed in accord-

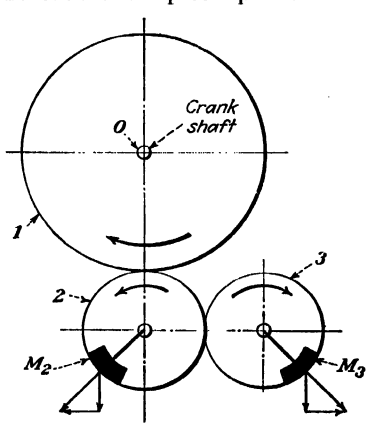


FIG. 329.

ance with the above principles was in use several years ago on certain four-cylinder automobile engines.

228. The Six-cylinder Automobile-type Engine.—The crank arrangement in this type of engine is shown in Fig. 330. The

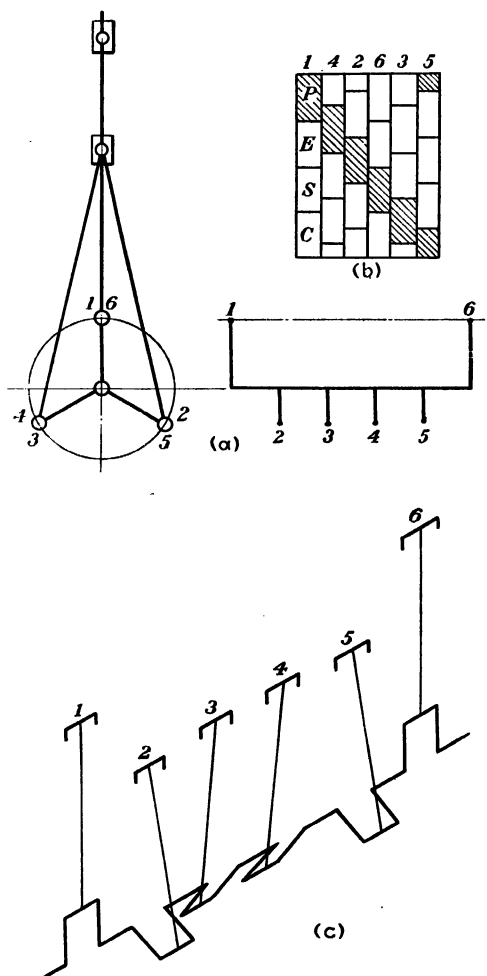


FIG. 330.

balance of the engine is shown in Table 11. As indicated in the table, the six-cylinder engine is in complete balance with regard

to both primary and secondary forces and couples. The six-cylinder engine arranged in this way is one of the most perfectly balanced engines of the usual multicylinder type that it is possible to construct.

TABLE 11.—SIX-CRANK ENGINE

Crank	ϕ	x	y	a	ax	ay	x^2	y^2	$x^2 - y^2$	xy	$a(x^2 - y^2)$	axy
1	0	+1.000	0	0	0	0	1.000	0.000	1.000	0.000	0.000	0.000
2	120	−0.500	+0.866	1	−0.500	+0.866	0.250	0.750	−0.500	−0.433	−0.500	−0.433
3	240	−0.500	−0.866	2	−1.000	−1.732	0.250	0.750	−0.500	+0.433	−1.000	+0.866
4	240	−0.500	−0.866	3	−1.500	−2.598	0.250	0.750	−0.500	+0.433	−1.500	+1.299
5	120	−0.500	+0.866	4	−2.000	+3.464	0.250	0.750	−0.500	−0.433	−2.000	−1.732
6	0	+1.000	0.000	5	+5.000	0.000	1.000	0.000	1.000	0.000	+5.000	0
Total...		0	0	..	0	0	.....	.....	0	0	0	0

$F_p = 0$ $C_p = 0$
$F_s = 0$ $C_s = 0$

229. The Eight-cylinder Automobile Engine.—In this type of engine the following constructions have been employed:

1. In the form of two units, each composed of four cylinders arranged as described in Art. 227, the axes of the cylinders in

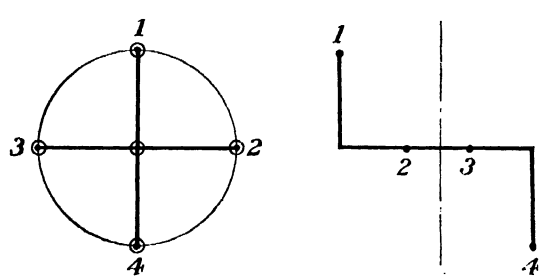


FIG. 331.

one group being inclined at some angle (usually 60 or 90 deg.) to those of the other group. Because of this arrangement the engine is referred to as the V-type.

2. A V-type differing from the conventional arrangement described under case (1) in that the four throws or cranks are at right angles to each other, as shown in Figs. 331 and 332.

3. The eight-cylinder-in-line type consisting of two units each composed of four cylinders arranged as described in Art. 227, with crankshafts 90 deg. apart and constructed in a tandem arrangement.

4. The eight-cylinder-in-line type in which the middle four cranks are arranged in an axial plane making an angle of 90 deg. with the plane of the outer four throws. The arrangement is shown in Fig. 333 and Fig. 334.

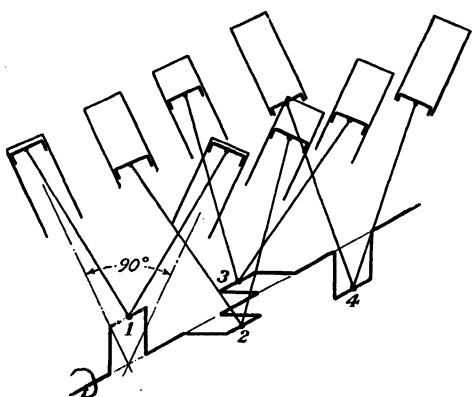


FIG. 332.

In cases (1) and (3) the arrangement is such that the unbalanced forces cannot completely neutralize each other, and, therefore, as in the case of the four-cylinder engine, the engine is not balanced as far as the secondary forces are concerned.

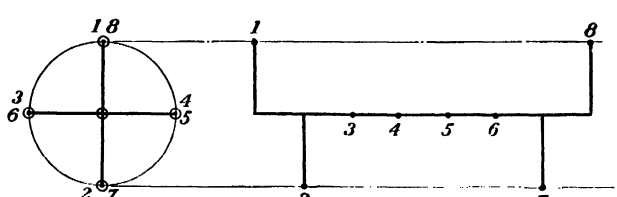


FIG. 333.

In case (2), with the crank arrangement described, only the primary couples are unbalanced, and in practice these are neutralized by the addition to the crankshaft of rotating counterweights.

In case (4) the engine is in complete balance as regards both primary and secondary forces and couples.

230. The Twelve-cylinder Automobile-type Engine.—This type of engine usually consists of two groups of six cylinders, each group being arranged as described in Art. 228. It is evident that, since each of the two groups is completely balanced in itself, the entire engine is in perfect balance as regards both primary and secondary forces and couples.

231. Balancing the Crankshaft.—The crankshaft is in general an unbalanced system of rotating masses in which the planes of rotation and the crank angles are fixed by the reciprocating

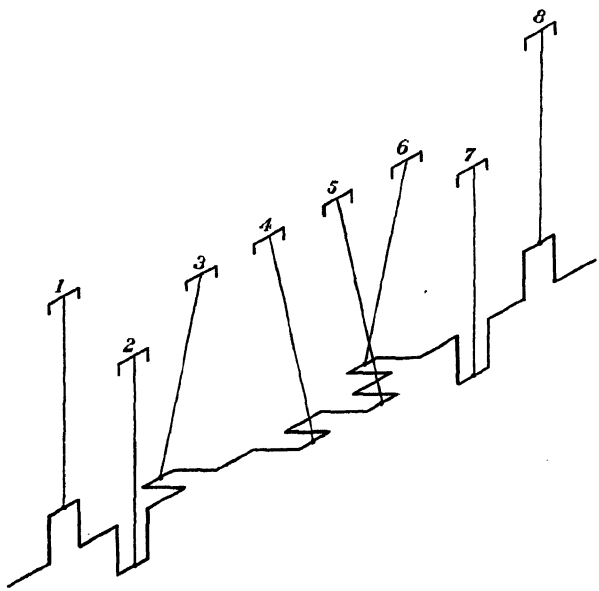


FIG. 334.

system that it operates. The crankshaft may be balanced in either of two ways:

1. By the addition of two balancing masses in two separate planes of rotation, in accordance with the method outlined in this chapter.

2. By the extension of the arms of each crank to form balancing masses for the rotating masses of that crank. In this case there are as many balancing masses as there are crank arms. This method, however, has the advantage that the intermediate parts of the crankshaft do not have to transmit force from one crank to another, since each crank is balanced in its own plane of rota-

tion. The crankshaft is thus relieved of any bending moment due to rotating masses.

In addition to the lateral bending to which the crankshaft is subjected because of unbalance or applied periodic forces, it is also susceptible of vibrating about its axis. This action is known as torsional vibration. A periodic twisting couple agreeing with the periodic time of the shaft about its axis, is capable of exciting large vibrations of that period, even though the magnitude of the couple producing it is extremely small. Automobile engines are now equipped with devices for eliminating the torsional vibration of the crankshaft.

232. Balancing the Connecting Rod.—The connecting rod has two distinct effects:

1. It disturbs the simple harmonic motion of the reciprocating masses. This effect on the balance of the engine has been dealt with in Art. 223.

2. Its motion is one of periodic acceleration, and hence inertia forces are set up that have periodic variations. These forces ultimately appear as reactions on the frame or foundation and because of their periodic nature tend to cause vibration. In balancing an engine a commonly used approximate method of taking care of the inertia effect of the connecting rod is to assume the mass of the rod to be distributed between the crankpin and wrist pin inversely as the mass center divides the rod. Balancing masses are then added in accordance with this assumed distribution.

233. Vibrations Due to Impulse and to Fluctuating Pressure of the Working Fluid.—When the forces and couples due to the motion of the parts of an engine have been balanced, the engine is not necessarily free from vibration, because balance of mass does not take into account the fluctuating pressures in the cylinders, or the effort of each succeeding impulse to rotate the engine backward around the crankshaft. Hence, after the inertia forces are balanced, there still remains a couple acting on the frame equal and opposite to the turning couple on the crankshaft, and this couple may have a periodic variation sufficiently great to cause vibration. In order to have an engine entirely free from vibration, therefore, it must be arranged not only for balance among the moving parts, but so that the turning effort on the crank is uniform.

234. Vibrations of Foundations and Supports.—In the preceding articles, the links of all the mechanisms considered have been regarded as rigid. Although the forces acting on the frame of a machine due to a set of moving masses in balance may form a system of forces in equilibrium, the individual forces of the system cause elastic deformation of the frame at the places where they act. In general, machine frames are stiff enough to limit the elastic deformations to negligibly small amplitude, but in some cases, notably in revolving shafts, there may be appreciable deflections, and these deflections, being of course periodic, may become of sufficient amplitude at certain *critical speeds* to cause vibrations of a serious nature.

The foundation or support of an engine or machine is usually an elastic system susceptible of vibrating in a variety of ways. If the engine or machine is unbalanced, it applies a periodic force or couple to its supports, resulting in forced vibrations that may, in general, be of little consequence, even though a large force or couple may be acting. If, however, the support happens to have among its natural modes of vibration one whose period is equal or approximately equal to the time of revolution or other motion cycle of the machine, then the amplitude of the forced vibration may be large, even though the magnitude of the force producing it is extremely small. This phenomenon may be explained by the fact that the work done in displacing the system from its position of rest appears as the energy of the vibration. If the succeeding applications of the force are so timed as to begin to act just at the instant that the vibrations are about to repeat themselves, other small amounts of energy are communicated to the system without interfering with each other. Energy is thus gradually accumulated in the system to such an extent that the amplitude of the forced vibration is all out of proportion to the force producing it.

It is in this way that the engines of a power plant or the machines in a factory, though firmly secured to massive concrete foundations, may set up vibrations in the earth in which the foundation is embedded, which may be transmitted all around the neighborhood. Within the building itself, irrespective of neighboring properties, the vibrating action of machinery is often felt and has been known to cause a great deal of annoyance.

In addition to engines, the types of machinery usually likely to cause annoyance from vibration are printing presses, pumps, ice machines, air compressors, and various types of purely rotative machines such as blowers, centrifugal pumps, etc.

235. Critical Speeds.—In some unbalanced rotating members, notably in revolving shafts, the deflections and vibrations at certain *critical speeds* become extremely severe. The peculiar behavior of an unbalanced rotating shaft has been a matter of common knowledge for many years, and has received increasing attention as reciprocating machinery has been replaced by high-speed rotative machinery. The most striking phenomena are as follows:

1. Increasing deflection of the shaft as the speed of rotation increases from zero to the critical speed.

2. Dangerous deflection at the critical speed unless some damping influence, usually in the form of friction, comes into play.

3. Gradual return to a condition of stability as the speed increases above the critical value.

The vibration phenomena of a shaft while passing through its critical speed are very interesting from a theoretical point of view, and, at the same time, they are of great practical importance in the design of steam turbines and other high-speed machines. A comprehensive literature on the subject exists, so that detailed treatment is not possible here.

236. Balancing Machines.—It is not in general possible to secure the satisfactory balance of a machine member merely by giving it the proper proportions when it is being laid out on the drawing board; and this is particularly true of members intended to rotate at high speeds. This is due to the ordinarily permissible variations in the dimensions of the member that arise from imperfections in the manufacturing process, and also to the fact that the material of which the member is made will usually vary slightly in density from point to point. There is, therefore, a real need for a means of adjusting the balance of a rotating member after it has been made, and this need is met by the various types of balancing machines.

The function of any dynamic balancing machine is to determine where metal should be added or taken away from a rotating member in order to secure its dynamic balance. This is usually accomplished by mounting the member in two bearings on an

elastically supported frame. The member is then rotated, and any lack of balance is at once evidenced by the resulting vibration of the frame. The magnitude of this unbalance is determined by methods peculiar to the type of balancing machine being used. The body is then dynamically balanced by the addition or removal of weight in two arbitrarily selected planes of correction perpendicular to the axis of rotation.

Three types of balancing machines are in use. In one of these the work is supported and rotated in a pivoted carriage, with one of the correction planes over the pivot axis, and measurement of the amount and angular position of the unbalance of the rotor taken in the other plane of correction. In another type of machine the work is supported and rotated on light flexible bearings, and the unbalance is determined by calculation from measurements of the vibration produced at the supports. The most recent development is embodied in a machine in which the work is carried on light flexible supports and the unbalance of the rotor is determined by means of a highly developed system of electrical networks. It is claimed that this machine retains the desirable features of the other two types and avoids their inherent disadvantages.

237. Description of a Balancing Machine. *a. Principle of Operation.*—In order to illustrate the principle upon which a dynamic balancing machine operates, a machine of the pivoted-carriage type manufactured by the Gisholt Machine Company will be discussed in some detail. This machine possesses certain novel features in addition to those ordinarily encountered, which will also be described.

A schematic outline of the machine has been shown in Fig. 335*a*. In the actual machine there are, of course, many refinements not shown in the schematic outline. The latter, however, shows all the essential elements and will, therefore, suffice for the purposes of this discussion.

The rotating member to be tested (indicated in the figure by the dotted outlines) is mounted in two half bearings that in turn are mounted on the frame B. The latter rests on two knife-edge pivots P and is connected by means of the link L to the long flat spring S. One end of the member to be tested is connected to the spindle F by means of the clutch C. The latter is so constructed that it provides a positive driving connection that is

at the same time flexible. The spindle is carried by the headstock E, which is rigidly fastened to the frame B. Keyed to the opposite end of the spindle from the clutch is the drum D, and fastened to the drum is the balancing disk K. The connection between the drum and the balancing disk is provided with an adjustment T by means of which the balancing disk may be set in any desired angular position relative to the drum, and hence

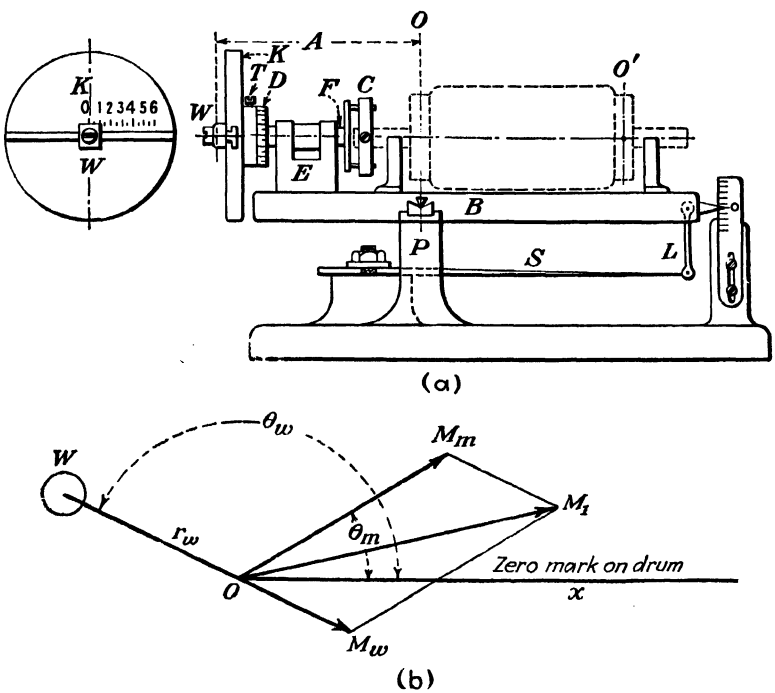


FIG. 335.

to the spindle and to the member being tested. The counterweight W is fastened to the balancing disk K in such a manner that it may be moved along the radial slot shown and set at any desired distance from the axis of the spindle. With the counterweight set at zero, the system composed of the spindle, clutch, drum, and disk is in perfect dynamic balance.

Now let it be supposed that the entire rotating system including the member being tested is set in motion. If the counterweight is set at zero, the only unbalanced centrifugal forces will

be those of the member being tested. These forces will cause the elastically supported frame to vibrate, the amplitude of the vibrations being indicated by means of the scale and pointer shown at the right. If the piece being tested is in perfect balance, no vibrations will occur.

In the actual operation of the machine, the rotating system is brought up to some convenient speed and then permitted to rotate freely under the influence of its own inertia. As the system continues to rotate, it gradually slows down until eventually it passes through a critical speed corresponding to the natural period of the elastically supported frame; at this point the amplitude of vibration is a maximum. This maximum amplitude is proportional to the amount of unbalance in the rotating system, and the balancing process for which this machine was designed is based upon this principle.

In order to bring out the full significance of the foregoing statement, let it be assumed that when the counterweight is set at zero, the unbalanced mass moment of the rotating system with respect to a transverse plane through the pivots is represented by the letter M_m. Since this unbalanced moment is due entirely to the member being tested, it is evident that

$$M_m = \vec{\Sigma} r dm \cdot a \tag{140}$$

where dm = the mass of any element of the member being tested.
r = the distance at which the element is situated from the axis of rotation.
a = the distance between the element and the transverse plane through the pivot, this distance being considered positive when measured to the right from the plane.

If the system is rotated, the amplitude of the vibration of the frame will reach a maximum S_m as the system passes through its critical speed.

Now let it be assumed that the counterweight is set at some distance r_w from the axis of rotation. The total unbalanced mass moment of the system will then be

$$M_1 = \vec{\Sigma} r dm \cdot a \twoheadrightarrow (-m_w r_w A) \tag{141}$$

where m_w is the mass of the counterweight and the distance A is as indicated in Fig. 335. The negative sign is due to the fact that the counterweight is situated on the left-hand side of the reference plane through the pivots. If the system is rotated, the maximum amplitude S_1 will be different from that obtained when the counterweight was set at zero. The statement that the amount of unbalance in the system is proportional to the maximum amplitude means that

$$\frac{M_m}{S_m} = \frac{M_1}{S_1} = k$$

The quantity k is called the calibration constant.

The maximum amplitude of the vibration of the frame depends not only upon the magnitude and position of the unbalanced centrifugal forces but also upon the size, shape, and weight of the member being tested. Strictly speaking, therefore, the calibration constant k is different for every member tested. Practically, however, the machine has been so designed that there is almost no difference between the calibration constants corresponding to two members having approximately the same proportions. Due to this fact, it is possible to calibrate the machine for one of a number of similar members and then to use the constant thus obtained in determining the unbalance of the other members.

b. Procedure in Balancing a Rotating Member.—In describing the method of testing for unbalance, it will be assumed that the calibration constant has already been determined. The member to be tested is set up in the machine and rotated with the counterweight set at zero. Let it be assumed that the observed maximum amplitude is S_m. The magnitude of the unbalanced moment of the member with respect to the transverse plane through the pivots is then given by

$$M_m = kS_m$$

The direction of the unbalanced moment M_m is as yet undetermined. Referring to Fig. 335*b*, let it be assumed that this direction makes an angle θ_m with the zero mark on the drum. Now let the counterweight W be set at some arbitrary distance r_w from the axis of rotation. The magnitude of the additional unbalanced moment thus introduced into the system is given without

regard to sign by the expression

$$M_w = m_w r_w A$$

where m_w is the mass of the counterweight and A is the distance indicated in Fig. 335*a*. Let it be assumed that the direction of the radial slot in which the counterweight is situated makes an angle θ_w with the zero mark on the drum. The direction of M_w, the moment of the counterweight, will be opposite that of the radial slot as indicated in Fig. 335*b*. The magnitude of the total unbalanced moment of the entire system is

$$M_1 = \sqrt{M_m^2 + M_w^2 - 2M_m M_w \cos(\theta_m - \theta_w)} \qquad (142)$$

In this equation the quantities θ_w, M_w, and M_m are known, the last quantity having been determined by observing the maximum amplitude S_m with the counterweight set at its zero position. The quantity M_1 may be determined by observing the maximum amplitude with the counterweight set as indicated in Fig. 335*b*. The equation may, therefore, be solved for θ_m and the direction of the unbalanced moment M_m thus determined.

In applying the foregoing method it is customary to set the counterweight so that $M_w = M_m$ and $\theta_w = 0$. Then

$$M_1 = \sqrt{M_m^2 + M_m^2 - 2M_m M_m \cos(\theta_m - 0)}$$
$$= M_m \sqrt{2 - 2\cos\theta_m} \qquad (143)$$

If S_1 is the maximum amplitude corresponding to M_1, then

$$\sqrt{2 - 2\cos\theta_m} = \frac{M_1}{M_m} = \frac{kS_1}{kS_m} = \frac{S_1}{S_m} \qquad (144)$$

from which θ_m may readily be determined.

The accuracy with which the unbalanced moment M_m and the angle θ_m have been determined can be checked by rotating the system with the counterweight set so that $M_w = M_m$ and $\theta_w = \theta_m$. If the determination has been accurately made, there will be no vibration of the frame.

The operations described have resulted in the determination of the unbalanced moment of the member being tested with respect to the plane O of the member. In order completely to balance the member, it is necessary to determine the unbalanced moment with respect to some other plane O'. This is accom-

plished by turning the member end for end and setting it up in the machine so as to bring the plane O' over the pivots. The process described above is then repeated and the unbalanced moment with respect to the plane O' thus determined. The unbalance of the member may then be corrected by means of mass adjustments made in the planes O and O' in accordance with the methods described in Art. 216. It sometimes happens that it is inconvenient to make the mass adjustments in the planes O and O'. When this occurs, the equivalent mass adjustments for any other pair of arbitrarily chosen planes may be determined by applying the fundamental principles discussed in this chapter.

238. Balancing Rotors by Means of Electrical Networks.—Typical of the balancing machines of the third type mentioned in Art. 236 is that illustrated in schematic outline in Fig. 336. This diagram and the information that follows were supplied by the Gisholt Machine Company, manufacturers of the machine. The operation of this machine is based upon a fundamental principle of mechanics that involves a very interesting application of the center of percussion.

As stated in Art. 236, a body may be dynamically balanced by the addition or removal of weight in two arbitrarily selected planes of correction perpendicular to the axis of rotation. In the schematic drawing, W_1 and W_2 represent the unbalances in the arbitrarily selected correction planes 1 and 2 that will give the same vibration effect at bearings A and B as is produced by the unbalances existing in the rotor. If it is desired that bearing vibration at A and B be reduced to zero, a means must be found for determining the magnitudes of the unbalances W_1 and W_2.

The method whereby this is accomplished may be understood by reference to the schematic diagram (Fig. 336). The rotor is carried on two independent, flexibly supported lightweight structures which permit of vibration only in a horizontal plane. The rotor is supported at the bearings A and B where it is desired to reduce vibration to zero. It is driven by the motor through a light, flexible belt.

Before proceeding further with a discussion of the elements of the balancing machine, consider the effect of a horizontal periodic force P applied to the axis L-L of the rotor in correction plane 2. This force P will cause a combined periodic translation and

rotation of the axis of the rotor in a horizontal plane (the axis of the rotor will move between the lines L_1-L_1 and L_2-L_2) so that there will be a point X_2 on the axis of the rotor that will have no motion. A vertical axis through X_2, then, will be the axis of suspension for a center of percussion at the intersection of the line of action of the force P with the axis L-L of the rotor. Then any point in a plane perpendicular to L-L containing X_2 will have no motion due to the periodic force P.

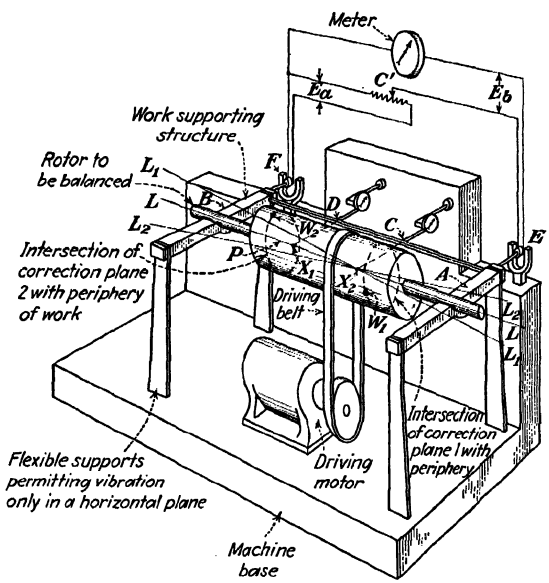


FIG. 336.

Under rotation, the unbalance W_2 in the rotor will provide the periodic force P because the supporting structure has freedom only in a horizontal plane. Further, the point C, on a bar connecting the two work-supporting structures and in a plane perpendicular to the axis of rotation that contains the point X_2, will have no motion due to the unbalance W_2 in the correction plane 2. However, this point C will have motion due to a force in any other correction plane such as the force produced by unbalance W_1 in correction plane 1. Therefore, an indicator between the machine base and point C will provide a means for measuring the unbalance effect W_1 without any effect from unbalance W_2.

In a similar manner, there can be found a point X_1, which will be on a vertical line that is the axis of suspension for the center of percussion in correction plane 1. Then there will likewise be a point D on the bar connecting the two work-supporting structures which will have no motion due to unbalance W_1 and will, therefore, serve as a means of measuring the unbalance effect W_2.

In this machine an electrical indicating means is used for obtaining the measurements of the effects of the unbalances. By attaching generating coils E and F to the work-supporting structures and permitting these coils to vibrate in the fields of powerful permanent magnets, voltages will be generated in the coils that will be proportional to the movements of the bar connecting the work-supporting structure. As the motion of point C on this bar is a function of the movements of the ends of the bar, a position C' on a potentiometer can be found, which will give a resultant voltage on the meter, which is the same function of the voltages generated at E and F. Electrical means of this sort are used because tremendous amplification of the generated voltages are possible and because it is also possible by these electrical means to determine the angular positions of the unbalances W_1 and W_2 by means of a stroboscopic lamp.

A simplified mechanical arrangement of this machine consists in the elimination of the complicated electrical equipment and the use of the ordinary indicators shown at C and D, reading in thousandths of an inch, for the purpose of measuring the unbalances W_1 and W_2. It is claimed that the advantage of this arrangement is a comparatively simple and inexpensive machine that is made available for the small manufacturing plant.

APPENDIX

FORCE ANALYSIS OF A GASOLINE ENGINE

The following analysis provides an excellent means for applying the principles of dynamics, as outlined in the foregoing chapters, in the solution of problems pertaining to high-speed machinery.

This problem may be adapted to classroom work by distributing the routine work among a group of students. In this manner the complete analysis has been made in a period of 6 weeks, working 6 hr. per week.

Figures 337 to 347 and Tables 12 to 15 give the results of an investigation of the forces acting in a standard six-cylinder engine of the automobile type.

The following data were taken from information furnished by the manufacturers and from actual measurements of the engine.

Dimensions:	
R.p.m.	2,800
Number of cylinders	6
Diameter of cylinders	3⅜ in.
Stroke	5½ in.
Length of connecting rod, center to center	11 in.
Distance of center of gravity of connecting rod from crankpin end	2.66 in.
Principal radius of gyration (to be calculated)	
Diameter of wrist pin	1 in.
Diameter of crankpin	2 in.
Diameter of main bearing	2 in.
Center of gravity of unbalanced weight of crank from center line of crankshaft	2⅝ in.
Weights:	
Connecting rod, complete (including bushings)	2.88 lb.
Piston, complete (including wrist pin, rings, etc.)	1.54 lb.
Estimated unbalanced weight of crank	4.75 lb.

Timing:

Inlet valve opens 16 deg. after head-end dead center.
Inlet valve closes 44 deg. after crank-end dead center.
Exhaust valve opens 46 deg. before crank-end dead center.
Exhaust valve closes 10 deg. after head-end dead center.

Indicator Diagram:
Compression volume, 23.7 per cent of piston displacement.
Maximum explosion pressure, 300 lb. per sq. in. absolute.
Pressure rise completed at 2 per cent of stroke.
Development of expansion curve begins at 3.2 per cent of stroke.
Mean suction pressure 12 lb. per sq. in. absolute.
Mean exhaust pressure 16 lb. per sq. in. absolute.
Atmospheric pressure 14.7 lb. per sq. in. absolute.
Pressure at beginning of compression, atmospheric.
Exponent for expansion curve, 1.3.
Exponent for compression curve, 1.28.

239. Indicator Diagram.—With the given timing data, the crank and piston positions showing the succession of events in the cylinder were laid out as shown in Fig. 337. On the stroke of the piston as a base line, the indicator diagram was next drawn. The ordinates used in constructing the expansion and compression curves, corresponding to piston positions laid out for each 15-deg. crank position, are given in Table 12. In the determination of these ordinates the equation $PV^n = C$ was used. For convenience in making calculations, the equation may be put in the form $\log P = -n \log V + \log C$. Since this is a straight-line equation, the labor in determining the points on the expansion and compression curves is considerably reduced by making use of logarithmic cross-section paper as shown in Fig. 338. Points on the expansion curve were obtained by drawing the expansion line in accordance with the form $\log P = -1.3 \log V + \log C$ where pressures are ordinates and percentages of volume are abscissas. The pressure at the beginning of expansion is given as 300 lb., and the expansion curve begins at 3.2 per cent of the stroke, or at $23.7 + 3.2 = 26.9$ per cent of volume. Thus with the coordinates for one point and the slope known, the expansion line is completely determined, and other points on the expansion curve may readily be found. Points for the compression curve are determined in like manner from the compression line in Fig. 338.

Since the ordinates in the indicator diagram represent absolute pressures in pounds per square inch, the total effective pressures on the piston (Table 12) are obtained by deducting the atmospheric pressure and multiplying by the area of the piston in square inches. In the original diagram the pressures were drawn to a scale of 1 in. = 50 lb.

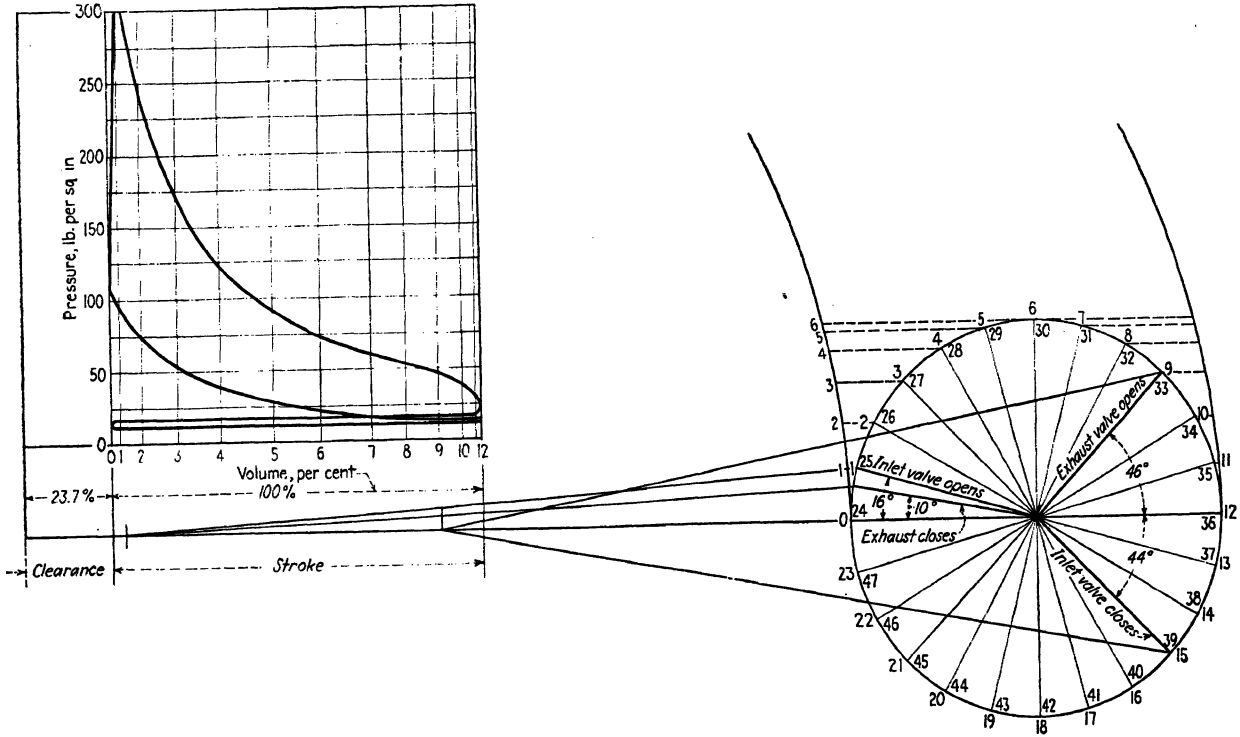

Pressure, lb. per sq. in
300
250
200
150
100
50
0
0 1 2 3 4 5 6 7 8 9 10 12
Volume, per cent
23.7%
100%
Clearance
Stroke
Exhaust valve opens
46°
25 Inlet valve opens
16°
10°
Exhaust closes
44°
Inlet valve closes

FIG. 337.

240. Inertia Forces.—Having obtained from the indicator diagram the gas pressures acting on the piston throughout the cycle, the next step in the analysis is to determine the inertia forces of the piston, connecting rod, and crank. Although these forces can be determined by analytical methods, the labor is

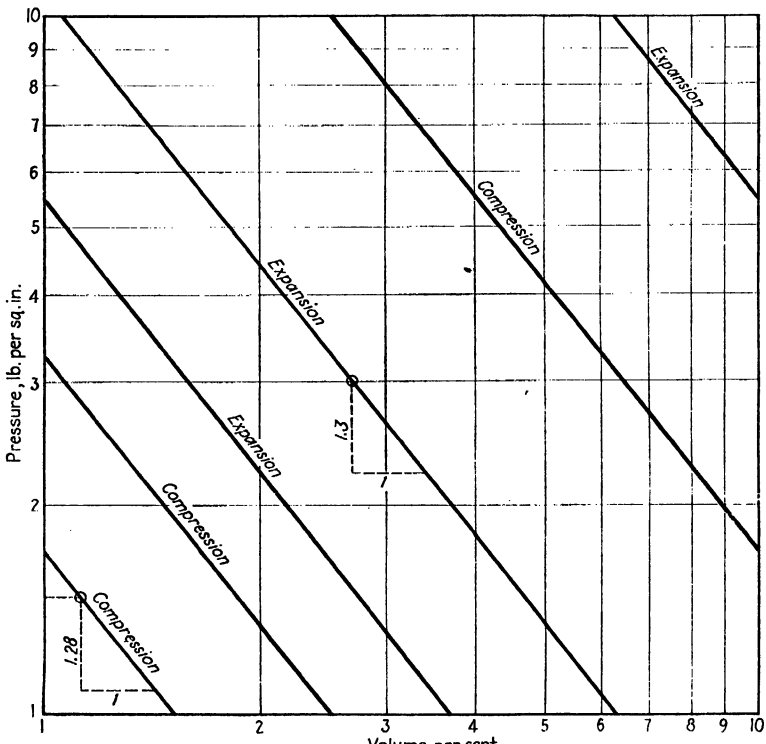


FIG. 338.

greatly reduced by graphical methods such as have been used in this analysis.

a. Piston Inertia Forces.—By means of the Ritterhaus construction (Art. 178), the vector *Ob* (Table 13), which represents to scale the acceleration of the piston, was obtained for each 15-deg. crank position. From this the actual acceleration was determined, and the piston inertia force f_4 (Table 14) was then computed.

TABLE 12.—INDICATOR-CARD DATA

Expansion curve					Compression curve				
			Gas pressure					Gas pressure	
Piston position	Crank angle, degrees	Volume ratio	Absolute pressure, pounds per square inch, P_a	Total effective pressure $(P_a - 14.7) \times A = P_g \times A$	Piston position	Crank angle	Volume ratio	Absolute pressure, pounds per square inch, P_a	Total effective pressure, $(P_a - 14.7) \times A = P_g \times A$
0	0	0.237	108	834	36	180	1.237	14.0	−6.3
1	15	0.259	300	2,550	37	195	1.224	14.7	0
e	...	0.269	300	2,550	38	210	1.185	14.7	0
2	30	0.320	239	2,005	e	...	1.127	14.7	0
3	45	0.415	171	1,370	39	225	1.122	14.8	0.9
4	60	0.535	123	950	40	240	1.034	16.4	15.2
5	75	0.667	92.4	682	41	255	0.926	18.9	37.5
6	90	0.800	72.7	518	42	270	0.800	22.8	72.4
7	105	0.926	60.1	406	43	285	0.667	29.0	127.9
8	120	1.034	52.2	327	44	300	0.535	38.3	211
9	135	1.122	46.9	288	45	315	0.415	53.1	343
10	150	1.185	42.8	251	46	330	0.320	73.5	531
11	165	1.224	35.6	187	47	345	0.258	95.7	742
12	180	1.237	26.0	101	48	360	0.237	108	834
*12 to 24	180 to 360	.....	16.0	11.6					
†24 to 36	0 to 180	.....	12.0	−24.1					

* Exhaust line.
† Intake line.

b. Connecting-rod Inertia Forces.—A kinetically equivalent system was substituted for the connecting rod as described in

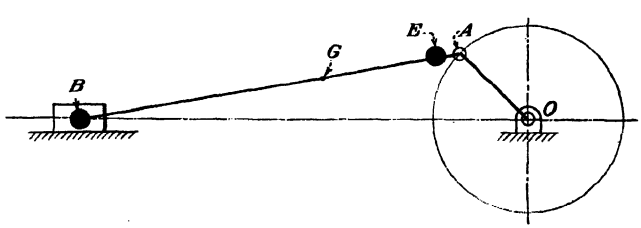


FIG. 339.

Art. 206. It was found experimentally that the time for 50 complete swings (over and back constitutes one complete swing) of the connecting rod about the center of the wrist-pin end of the rod was 52.4 sec. By the methods of Art. 206 it was found that with one mass at B (Fig. 339) the other mass at E was located

TABLE 13.—ACCELERATIONS

Crank position				Acceleration of piston $Ob \times k_a = A_b$		Acceleration of center of percussion $Oe \times k_a = A_e$		Acceleration of center of gravity $Og \times k_a = A_g$	
Number		Angle, degrees		Vector *Ob* in.	Acceleration f.p.s.^{2}	Vector *Oe* in.	Acceleration f.p.s.^{2}	Vector *Og* in.	Acceleration f.p.s.^{2}
0	24	0	360	6.875	24,630	5.53	19,840	5.83	20,900
1	23	15	345	6.56	23,500	5.53	19,840	5.70	20,430
2	22	30	330	5.46	19,600	5.46	19,600	5.34	19,150
3	21	45	315	3.90	13,980	5.43	19,500	4.88	17,520
4	20	60	300	2.07	7,420	5.36	19,220	4.46	16,000
5	19	75	285	0.21	753	5.33	19,130	4.17	14,950
6	18	90	270	1.44	5,160	5.35	19,190	4.19	15,030
7	17	105	255	2.65	9,500	5.36	19,220	4.40	15,770
8	16	120	240	3.43	12,300	5.40	19,360	4.62	16,560
9	15	135	225	3.86	13,830	5.44	19,500	4.88	17,500
10	14	150	210	4.03	14,460	5.46	19,590	5.04	18,060
11	13	165	195	4.11	14,730	5.47	19,600	5.11	18,310
12	..	180	...	4.125	14,770	5.47	19,600	5.167	18,350

TABLE 14.—INERTIA FORCES AND THEIR RESULTANTS, IN LB.

Crank position				Inertia forces				Resultant inertia forces		
Number		Angle, degrees		f_e	$f_e \frac{EB}{AB}$	f_b	f_4	Turning force f_{32}^t	Normal force crank f_{32}^n	Side thrust f_{14}
0	24	0	360	1,375	1,343	492	1,178	0	3,000	0
1	23	15	345	1,375	1,343	470	1,124	550	2,880	120
2	22	30	330	1,360	1,399	392	938	860	2,400	190
3	21	45	315	1,353	1,322	280	668	830	1,880	200
4	20	60	300	1,333	1,302	148	355	520	1,460	140
5	19	75	285	1,328	1,298	16	36.0	60	1,300	45
6	18	90	270	1,330	1,300	103	247	350	1,390	55
7	17	105	255	1,333	1,302	190	454	610	1,630	135
8	16	120	240	1,342	1,311	246	588	665	1,900	165
9	15	135	225	1,353	1,322	276	662	570	2,120	160
10	14	150	210	1,358	1,327	289	691	410	2,260	110
11	13	165	195	1,360	1,329	294	704	210	2,340	60
12	..	180	...	1,360	1,329	296	706	0	2,390	0

0.25 in. from A. By means of the Ritterhaus construction the accelerations of masses at B and E and of the entire connecting rod were determined, and Table 13 completed. The inertia forces f_e and f_b (Table 14) were next calculated and the resultant inertia forces of Table 14 determined by constructing the inertia-force polygons in accordance with the methods of Art. 207.

c. Inertia Force of Crank.—From the given data the inertia force (centrifugal force) of the crank is $MR\omega^2 = 4.75/32.2 \times 2.75/12 \times (2\pi \times 2{,}800)^2/60^2 = 2{,}915$ lb. This force exerts a pressure on the crankshaft bearing but has no effect on the piston or the connecting rod. It has therefore been omitted when drawing the inertia-force polygons such as are illustrated in Fig. 301 (Art. 207). Moreover, if balancing weights are attached to the crankshaft opposite each crank, which is common practice, this force is completely neutralized.

241. Combined or Resultant Forces.—In order to obtain the total forces acting on the several links of the mechanism, it is necessary to combine the forces due to gas pressure (static forces) and those due to inertia. The forces due to gas pressure are found by means of force polygons such as described in Art. 207, and the results are recorded in Table 15 as shown. The table is then completed by combining the gas forces with the inertia forces previously determined.

The investigation up to this point has been very largely concerned with the work of securing and recording all the forces acting on the various members for each 15-deg. crank position throughout one complete cycle (two revolutions).. The characteristics of these forces and their effects can best be understood from a study of the diagrams (Fig. 337 to 347) that have been plotted from the data recorded in Tables 12 to 15.

242. Indicator Diagram (Fig. 337).—This diagram, which has already been discussed, is the starting point in the analysis. It very closely approximates the form of the actual diagram taken from an engine of this design, and the piston pressures obtained therefrom must therefore closely approximate the pressures that actually occur.

243. Diagram of Piston Forces (Fig. 340).—The purpose of this diagram is to show for one cylinder the variation of piston forces that act in the line of stroke, during one cycle. The diagrams have been developed on a base line representing the total

motion of the piston (disregarding reversals) during a cycle. The forces due to gas pressure $\mathfrak{F}_4$ (Table 12) and the forces due to inertia f_4 (Table 14) have been plotted separately and then combined to form the total-pressure diagram $F_4 = \mathfrak{F}_4 + f_4$. Forces aiding the motion of the piston are considered positive and are plotted above the base line, and forces resisting the

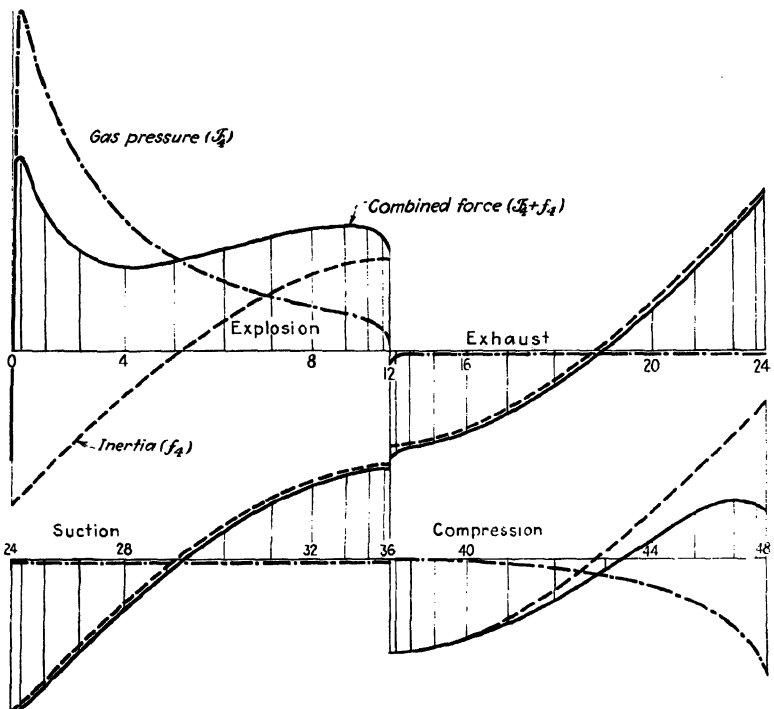


FIG. 340.

motion are considered negative and are plotted below the base line.

244. Side Pressure of the Piston (Fig. 341).—This diagram shows the variation of pressure of the piston against the wall of each cylinder during a cycle. From Tables 14 and 15 the side thrust $\mathfrak{F}_{14}$ due to gas pressure and the side thrust f_{14} due to inertia have been plotted separately and then combined to form the total side-pressure diagram F_{14}. The base line is of arbitrary length,

TABLE 15.—FORCES DUE TO GAS PRESSURE AND TOTAL FORCES, IN LB.

Position		Turning forces due to gas pressure, $\mathfrak{F}_{32}^{t}$	Total turning forces, F_{32}^{t}	Gas pressure N $\mathfrak{F}_{32}^{n}$	Total along ℄ crank, F_{32}^{n}	Total force C. rod on crank, F_{32}	Side thrust due to gas pressure, $\mathfrak{F}_{14}$	Total side thrust, F_{14}	Total force C. rod on piston F_{34}
Number	Angle								
Explosion Stroke									
0	0	0	0	848	2,155	2,155	0	0	326
1	15°	815	290	2,430	460	540	160	50	1,430
2	30°	1,240	400	1,600	800	900	260	90	1,100
3	45°	1,155	320	810	1,000	1,050	250	70	730
4	60°	950	430	300	1,170	1,250	220	80	630
5	75°	720	660	15	1,385	1,575	180	120	670
6	90°	500	860	130	1,500	1,750	130	180	780
7	105°	370	980	210	1,835	2,075	100	240	900
8	120°	250	920	235	2,120	2,310	80	250	950
9	135°	165	740	240	2,370	2,480	65	225	980
10	150°	100	510	230	2,495	2,545	35	150	950
11	165°	35	240	182	2,515	2,530	20	80	895
12	180°	0	0	110	2,400	2,400	0	0	816
Exhaust Stroke									
12	180°	.....*	0	.....*	2,400	2,400	...*	0	816
13	195°	.....	210	.....	2,340	2,350	...	60	705
14	210°	.....	410	.....	2,260	2,290	...	110	700
15	225°	.....	510	.....	2,120	2,195	...	160	675
16	240°	.....	665	.....	1,900	2,015	...	165	610
17	255°	.....	610	.....	1,630	1,740	.....	135	475
18	270°	.....	350	.....	1,390	1,440	.....	55	255
19	285°	.....	60	.....	1,300	1,300	...	45	55
20	300°	.....	520	.....	1,460	1,555	...	140	375
21	315°	.....	830	.....	1,880	2,040	...	200	690
22	330°	.....	860	.....	2,400	2,545	.....	190	945
23	345°	.....	550	.....	2,880	2,920	...	120	1,125
24	360°	.....	0	.....	3,000	3,000	...	0	1,145
Suction Stroke									
24	0°	.....*	0	.....*	3,000	3,000	.....*	0	1,145
25	15°	.....	550	.....	2,880	2,900	...	120	1,125
26	30°	.....	860	.....	2,400	2,560	...	190	945
27	45°	.....	830	.....	1,880	2,050	...	200	690
28	60°	.....	520	.....	1,460	1,555	...	140	380
29	75°	.....	60	.....	1,300	1,300	...	45	55
30	90°	.....	350	.....	1,390	1,440	.....	55	255
31	105°	.....	610	.....	1,630	1,740	...	135	475
32	120°	.....	665	.....	1,900	2,011	...	165	610
33	135°	.....	570	.....	2,120	2,195	...	160	675
34	150°	.....	410	.....	2,260	2,300	....	116	700
35	165°	.....	210	.....	2,340	2,350	...	60	705
36	180°	.....	0	.....	2,390	2,391	...	0	707
Compression Stroke									
36	180°	0	0	0	2,390	2,390	0	0	...
37	195°	.....*	210	.....*	2,340	2,340	...*	60	720
38	210°	.....	410	.....	2,260	2,290	...	110	695
39	225°	.....	570	.....	2,120	2,190	...	160	680
40	240°	.....	665	.....	1,900	2,100	...	165	650
41	255°	30	610	16	1,640	1,750	10	140	512
42	270°	70	412	19	1,400	1,450	18	65	325
43	285°	125	55	0	1,295	1,310	25	20	95
44	300°	205	315	70	1,390	1,425	50	85	180
45	315°	290	540	195	1,695	1,780	60	130	365
46	330°	320	530	410	2,015	2,090	65	130	425
47	345°	235	290	680	2,200	2,220	45	60	410
48	360°	0	0	848	2,155	2,155	0	0	326

* Negligible.

representing the 48 crank positions at 15-deg. intervals for one cycle. The scale of the original drawing was 1 in. = 500 lb.

245. Turning-effort Diagrams for One Cylinder (Fig. 342).—The tangential component of the pressure on the crankpin is the *turning effort*. On an arbitrary length of base line representing the 48 15-deg. crank intervals in a cycle, turning effort $\mathfrak{T}^t_{32}$ due

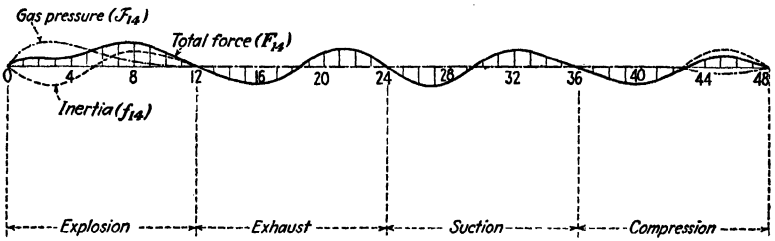


FIG. 341.

to gas pressure, turning effort f^t_{32} due to inertia, and combined turning effort F^t_{32} from Tables 14 and 15 have been plotted as shown in Fig. 342. Turning effort that aids the motion has been plotted above the base line, and that which opposes the motion has been plotted below the base line.

246. Combined Turning-effort Diagrams (Fig. 343).—The total turning effort of the engine is the result of the combined

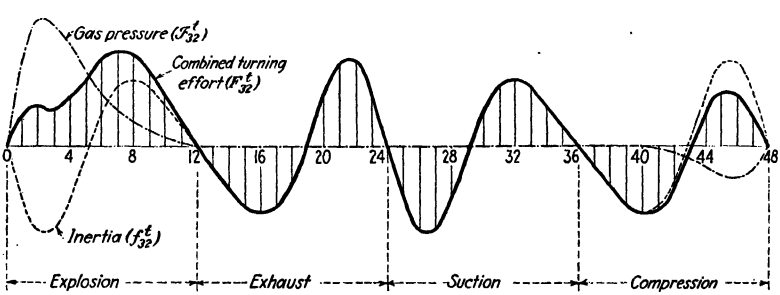


FIG. 342.

efforts of the six separate cylinders. In Fig. 330 (Art. 228) is shown the order in which the events occur in the different cylinders. With the aid of this diagram and data from Tables 14 and 15, the combined effort for the six cylinders was plotted as shown in Fig. 343. Since ordinates represent turning effort and abscissas represent movement of the crank, the area under the curve

represents the work performed. It is the function of the flywheel to maintain a uniform turning effort of the magnitude represented by the mean ordinate shown in the figure. The areas above the mean ordinate line therefore represent excess energy that is stored in the flywheel and given up in those parts of the cycle where there is a deficiency of energy, as indicated by the *negative* areas under the mean ordinate line.

Since friction has been neglected in the analysis, the horsepower of the engine when calculated from the turning-effort diagram should be equal to the horsepower calculated from the indicator diagram.

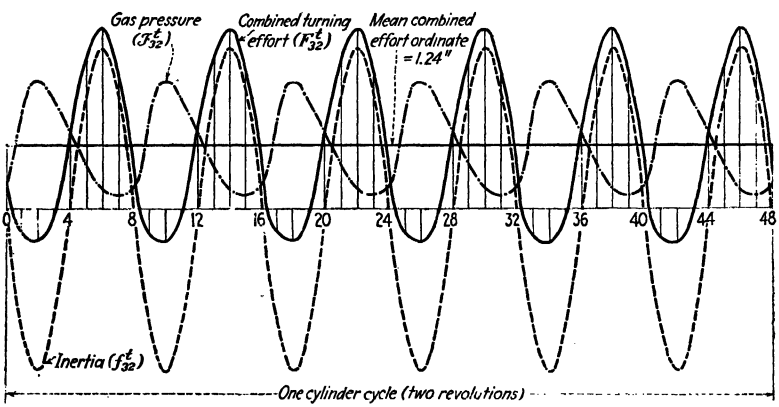


FIG. 343.

247. Pin Pressures (Figs. 344 to 346).—In the design of the wrist pin, crankpin, and crankshaft it is necessary to have information regarding the magnitude and distribution of the forces acting on these members and their bearings in order to proportion for strength and wear, and also to determine the best place for the oil to enter the bearing.

a. Wrist-pin Pressures (*Fig.* 344).—In Fig. 344 the wrist pin was drawn full size in the original drawing, and resultant pressures F_{34} from Table 15 plotted in magnitude and direction as vectors acting against the surface of the pin. The vectors thus represent forces acting on the pin, and the figure shows clearly where wear may be expected to occur on the pin. The pin was assumed to be fixed in the piston, and in the original drawing the vectors were plotted to a scale of 1 in. = 500 lb.

b. Crankpin Pressures (*Fig.* 345).—In the original drawing (Fig. 345) the crankpin was drawn in full size and the resultant forces F_{32} (Table 15) were drawn to a scale of 1 in. = 500 lb., as vectors representing forces acting against the surface of the pin. Since the crankpin is rotating, the vectors are drawn, not in their

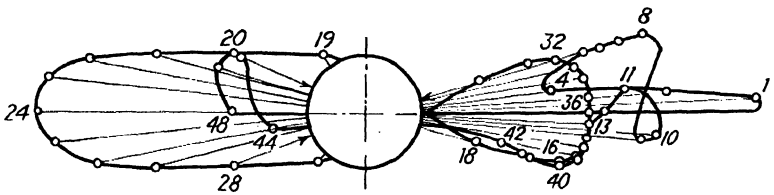


FIG. 344.

true direction, but in their direction relative to a line turning with the pin, in this case the center line of the crank. The figure clearly shows that the pressure is always on the same side of the pin, thus demonstrating the cause of uneven wear of the pin. The diagram is also of value in showing where the oil-feed hole

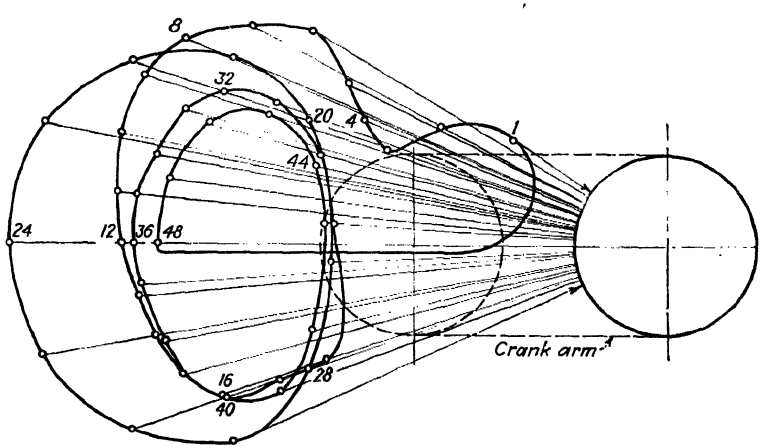


FIG. 345.

in the crankpin should be located, namely, on the side opposite that on which the pressure occurs.

c. Main-bearing Pressures (*Fig.* 346).—The pressure on the crankshaft per cylinder is evidently F_{12} ($= -F_{32}$) from Table 15. The reaction or pressure on the bearing is therefore F_{21} ($= F_{32}$). With this force must be combined the centrifugal force of the

crank in order to obtain the resultant force on the bearing. For a seven-bearing crankshaft the resultant force on each bearing may be assumed as half the forces from each of the pair of adjacent cranks.

In Fig. 346 are shown the magnitude and distribution of pressures acting on the crankshaft at the center main bearing. These pressures were obtained by combining one-half of F_{12} ($= -F_{32}$) from Table 15 for each of the adjacent cranks with one-half the centrifugal force of each of the adjacent cranks. The combined pressures are represented by vectors drawn against the surface

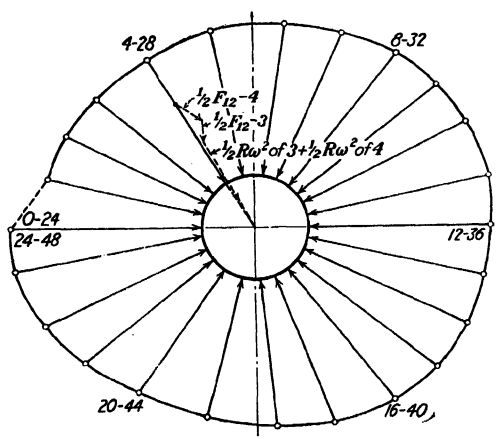


FIG. 346.

of the crankshaft. In the original drawing the vectors were drawn to a scale of 1 in. = 1,000 lb.

248. Diagram of Shaking Forces (Fig. 347).—The centrifugal force (inertia force) of the crank arm and pin (when unbalanced), the inertia force of the connecting rod, and the inertia force of the piston all tend to cause vibrations of the engine. The shaking forces for a single cylinder are shown in Fig. 347. The vectors in the original diagram were drawn to a scale of 1 in. = 1,500 lb. Force polygons have been drawn for each 15-deg. crank position, the end of the resultant vector locating a point on the curve. Referring to crank position 8, from a pole O the vector OA was laid off to represent the centrifugal force of the crank arm and pin. AB represents the inertia force of the connecting rod ($f_c \leftrightarrow f_b$) Table 14. BC represents the shaking force due

to the mass of the piston. OC represents the resultant shaking force for this crank position, and C is one point on the curve of shaking forces for one revolution of the crank. Now let it be assumed that the crank arm and pin are to be balanced by a rotating weight opposite the crankpin. This introduces a new vector CE $(= OA)$ in the force polygon $OABC$, giving a new resultant OE, the point E being a point on the curve of shaking forces with the crank arm and pin balanced. Only the shaking forces of the connecting rod and piston now remain.

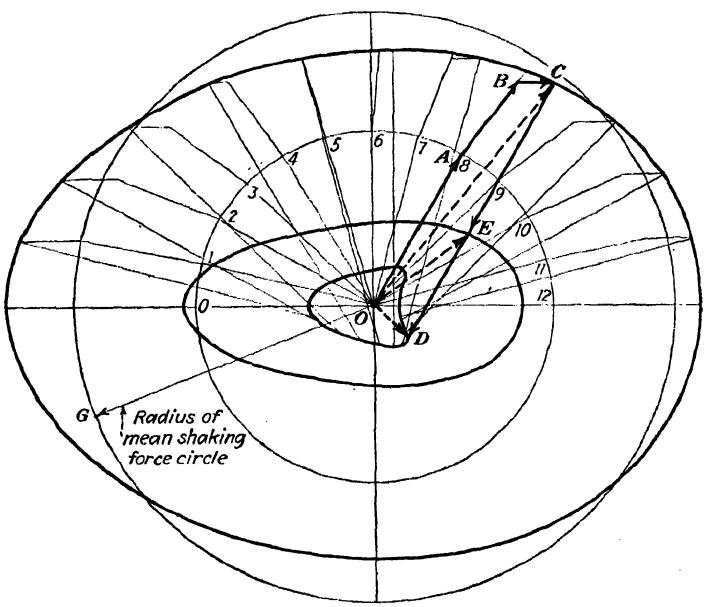


FIG. 347.

If, instead of balancing the crank arm and pin only, the rotating weight opposite the crankpin is of magnitude such that its centrifugal force is represented by the vector OG, the radius of the mean shaking-force circle, the vector $CD(= OG)$, will be brought into the original force polygon $OABC$, giving the resultant OD. D is one point on the curve of remaining shaking forces. This may be considered to represent the best balancing condition, since any further increase in the size of the counterweight, although decreasing the horizontal forces in curve D, would increase the vertical forces.

It should be kept in mind that curves *C*, *E*, and *D* represent the shaking forces for a single cylinder for the conditions stated. This would be the proper procedure in the exact analysis of the balancing conditions for a *single-cylinder* engine. In a six-cylinder engine the shaking forces for the six cylinders combined form a balanced system. In the case of this particular engine, therefore, the diagram (Fig. 347) is chiefly of interest in showing what shaking forces are set up in the plane of rotation of each crank.

PROBLEMS

With a few exceptions the problems in the following list have been arranged for solution in pencil on 8½- by 11-in. sheets. A heavy grade of white paper is recommended. The general layout of the sheet is given in Fig. 348, where the solution of Prob. 6 is shown as a typical example. In the statement of each problem the necessary scales are given. Positions of the mechanisms, diagrams, etc., on the sheet are given by X and Y coordinates measured from the edges of the sheet. For example, the coordinates designated as $O_2(6, 8)$, Prob. 6, are X and Y coordinates that locate the point O_2 in Fig. 348, 6 in. from the left-hand edge of the sheet and 8 in. from the lower edge of the sheet. Similarly, the coordinates $0(1\frac{1}{4}, 1)$ locate the point numbered 0 on the displacement diagram, $1\frac{1}{4}$ in. from the left-hand edge of the sheet and 1 in. from the lower edge of the sheet. All such locating dimensions are in inches.

If larger drawing sheets are preferred, 12 by 18 or 18 by 24 in., for example, these sheets may be blocked out for two or four problems respectively, or the scale of the drawings, as specified, may be changed as desired.

Chapter I

INTRODUCTORY CONSIDERATIONS

1. Expand pair C in Fig. 6 to form a slider-block pair in which link 3 becomes the block, as shown in Fig. 11. Assume that link 2 oscillates 30 deg. each side of its given position and determine length of slot required. Add $\frac{1}{16}$ in. on each end of slot to length as actually determined by construction. Data: AB makes an angle of 60 deg. with the horizontal; $AB = 1$ in; $BC = 3$ in.; $CD = 2$ in.; $AD = 4$ in.; scale, full size.

a. Draw the mechanism as shown in Fig. 6 in position A $(2\frac{1}{4}, 7)$.

b. Redraw in position A (2, 2) and proceed with the solution. Indicate all construction work by light continuous lines.

2. By expanding the crosshead pin of pair C (Fig. 3) to include the crank-pin at B, show how a slider-block pair similar to that in Fig. 26 may be formed. Data: AB makes an angle of 45 deg. with the horizontal; $AB = 1$ in.; $BC = 3\frac{1}{2}$ in.; scale, full size.

a. Draw the mechanism in position $A(2\frac{1}{4}, 8)$ indicating thereon letters and dimensions. Let the two circles at each joint be $\frac{1}{4}$ and $\frac{1}{8}$ in. diameter, let the width of the links 2 and 3 be $\frac{1}{8}$ in. and the dimensions of the block, link 4, $\frac{3}{8}$ by $\frac{5}{8}$ in.

b. Redraw in position $A(2\frac{1}{4}, 5\frac{1}{2})$ and proceed with the solution.

c. Show by means of a new drawing at $A(2\frac{1}{4}, 1\frac{3}{4})$ the effect of making the connecting rod of infinite length. (The resulting mechanism is known as the Scotch yoke, shown in Fig. 28.)

3. Show how a punch and shear mechanism similar to that in Fig. 27 may be derived from the slider-crank mechanism (Fig. 3) by expanding pair C to include pair B. Data: AB makes an angle of 45 deg. with the horizontal: $AB = \frac{3}{4}$ in.; $BC = 1\frac{3}{4}$ in.; scale, full size.

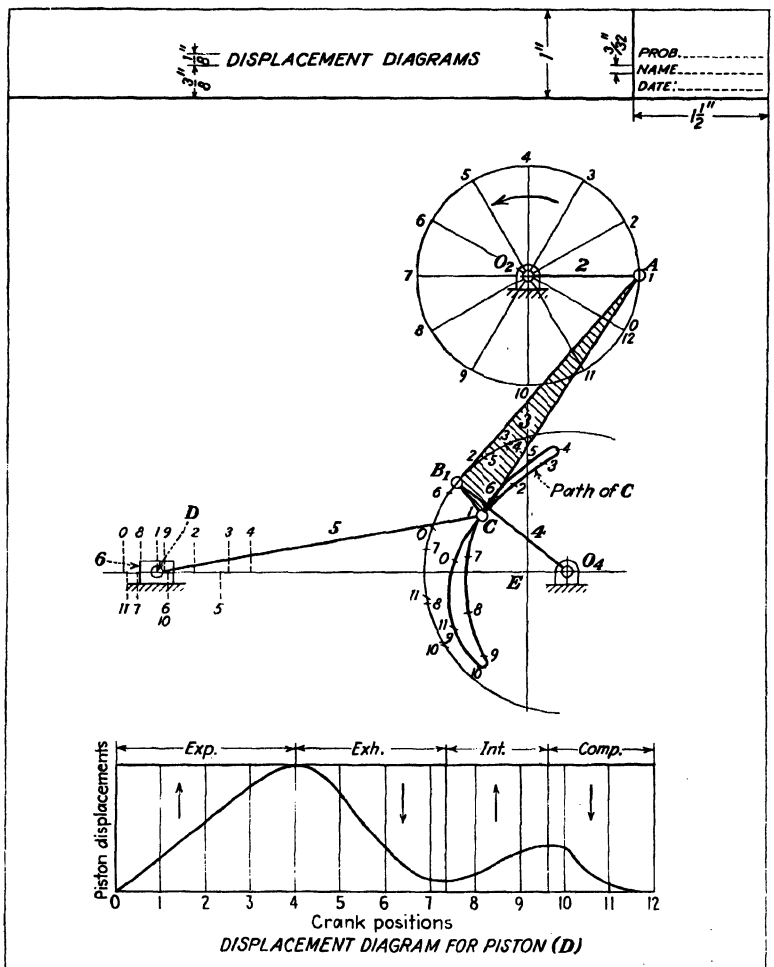


FIG. 348.

In order to avoid confusion of lines, show the construction in two stages, the first in position $A(2\frac{1}{2}, 7\frac{1}{2})$, and the second in position $A(2\frac{1}{2}, 2\frac{3}{4})$.

4. Starting with the skeleton outline of the ordinary steam-engine mechanism (Fig. 3), show with skeleton outline drawings how the oscillating engine

mechanism may be derived from it by applying the principle of inversion. Data: AB = 1 in.; BC = 3½ in.; scale, full size.

a. Draw the skeleton outline of Fig. 3 to the dimensions given, in position A(2¼, 8¼). Crank angle = 45 deg.

b. Show the inversion that represents the skeleton outline of the oscillating engine, in position A(2¼, 5½). Crank angle = 45 deg.

c. Using the same dimensions of links as in (*b*) above, make a drawing of the oscillating engine in some detail; *i.e.*, show cylinder, flywheel, frame, etc., but omit minor details, such as bolts, nuts, etc. Position A(2¼, 2½).

5. Draw the skeleton outline of the "up-and-down-cut shear," Fig. 387, to a scale of 1½ in. = 1 ft. For dimensions see Prob. 117. The numbering of the links and the lettering of the joints must be the same as those given in the figure. Suggested proportions: ⅛ in. diameter for the circles that represent the turning joints, and ⅜ by ¼ in. for the rectangles that represent the sliding members. O_2(2½, 7¼).

6. Draw the skeleton outline of the Atkinson gas engine (Fig. 15), determine paths of points B and C and construct the displacement diagram for the piston (link 6), following the general scheme shown in the figure. For scale and dimensions refer to Prob. 78. For the mechanism, O_2(6, 8); for the displacement diagram, o(1¼, 1). Length of diagram, 6 in.

7. Referring to Fig. 4, draw the skeleton outline of the shaper mechanism to a scale of 2 in. = 1 ft. with the crank (link 2) making an angle of 30 deg. with the horizontal. Starting with the oscillating arm (link 4) in its extreme right-hand position, divide the crank circle in 12 equal parts and on a base line 6 in. long draw the displacement diagram of the ram (link 6), showing the variation of its motion through the complete cycle. The requirements of the completed drawing as to numbering, lettering, etc., are indicated in Fig. 40. For dimensions of the mechanism refer to Prob. 85; for the mechanism, O_4(4½, 4½); for the displacement diagram, o(1⅛, ½).

8. In Fig. 23 the drag-link quick-return mechanism of a slotting machine is shown in skeleton outline. This mechanism is designed to give a slow motion to the cutting tool on the down stroke, or working stroke, and a quick return of the tool on the up stroke, or idle stroke. Draw the mechanism to a scale of 3 in. = 1 ft. with the crank (link 2) in the position shown in Fig. 23 (30-deg. angle with the vertical center line). Data: O_2O_4 = 2⅝ in.; O_2A = 5⅝ in.; AB = 4⅞ in.; O_4B = 6¾ in.; O_4C = 6¾ in.; BC = 9¾ in.; CD = 16⅞ in.

Starting with the ram (link 6) in its upper extreme position and using 12 equally spaced divisions on the crank circle (since the crank has uniform motion), draw the displacement diagram on a base line 4½ in. long. For the mechanism, O_2(6½, 2½); for the displacement diagram, o(¾, —).

Refer to Fig. 23 for the requirements of the completed drawing as to numbering, lettering, etc.

Chapter II

LINKWORK

9. Figure 349 shows the skeleton outline of an air-pump mechanism designed to secure a piston stroke equal to four times the length of crank

instead of twice the length, as would be the case when using the ordinary slider-crank mechanism. Plot the paths of points E and C, and draw a displacement diagram 4½ in. long for the piston (link 5) for one complete rotation of the crank (link 2).

In the figure the relative positions of the links are shown when the piston is in its upper extreme position. Draw this configuration of the mechanism to the dimensions given and in the solution of the problem use 12 equal divisions on the crank circle. Data: AB = 1 in.; $BC = BE$ = 1⅛ in.; FE = 3⅛ in.; DC = 4¾ in.; scale, full size. For the mechanism, $A(2\frac{3}{8}, 2\frac{1}{2})$; for the displacement diagram, $o(3\frac{5}{8}, —)$.

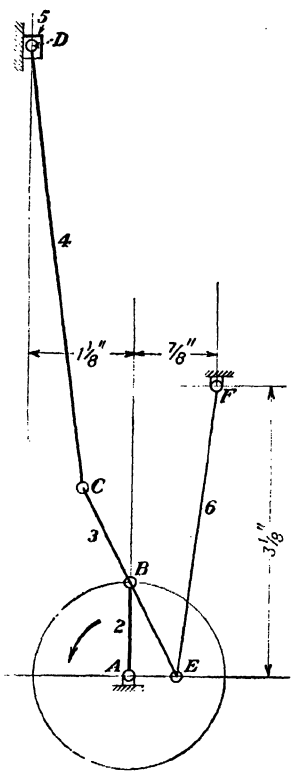


Fig. 349.

10. In Fig. 350 is shown in skeleton outline a claw mechanism at one time applied in motion-picture machinery to give the film an intermittent motion as it passed in front of the lens. The claws engaged the perforations on both edges of the film and thus operated in pairs. In addition to the specific purpose mentioned here, this mechanism has other applications. It is used, for example, in obtaining an intermittent advancement of a transport for conveying materials.

The claws are operated by means of two equal gears A and B in mesh. The gears are represented in skeleton outline by their pitch circles. The claw is pivoted to gear A at C, and its motion is further controlled by the pin D, which is fastened to gear B and moves in the slotted end of the claw. Draw the mechanism as outlined in the figure to the dimensions given (scale, full size) and plot the path of end E of the claw for one complete rotation of the gears A and B. $C(5\frac{1}{2}, 4\frac{1}{2})$ (Note: These coordinates are based on the assumption that the long side of sheet is placed in the horizontal position).

11. In Fig. 351 is shown the mechanism for operating the steam valve of a steam engine equipped with the Corliss nonreleasing type of valve gear. It is that part of the valve gear shown at C in Fig. 55.

The angular motion imparted to link 6 by the reach rod 7 during one complete rotation of the eccentric is indicated in the figure, the unequal angular divisions shown corresponding to equal angular divisions on the eccentric circle. The steam valve rotates through the same angle as link 3 to which it is rigidly connected.

a. Determine the angular positions of link 3 corresponding to the given angular positions of link 6.

b. Draw a diagram of displacements of B (valve displacements) in Fig. 351, corresponding to equal angular positions of the eccentric Fig. 55. (Corresponding angular positions of O_6E are shown in Fig. 351). Data: Draw

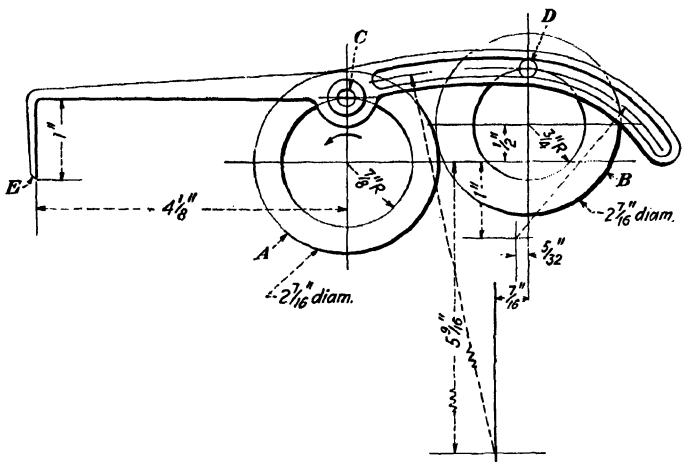


FIG. 350.

the mechanism to the following dimensions: $O_2O_6 = 6\frac{3}{8}$ in.; $O_2A = 3\frac{1}{4}$ in.; $O_2B = 2\frac{5}{8}$ in.; $BC = 3$ in.; $AC = 1\frac{3}{4}$ in.; $CD = 2\frac{1}{4}$ in.; $O_6D = 4\frac{1}{4}$ in.;

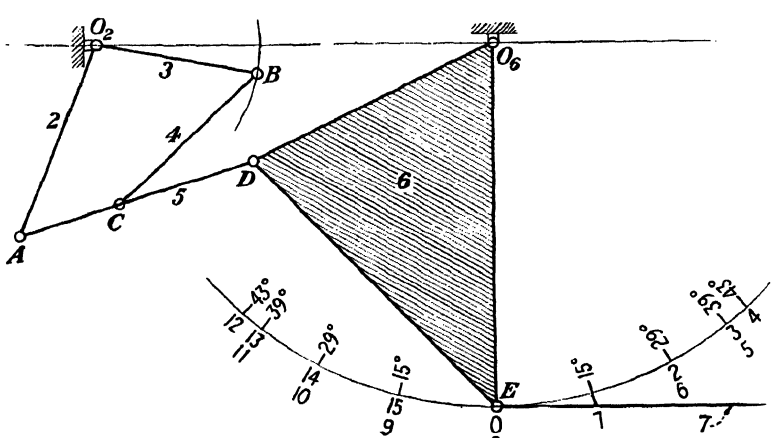


FIG. 351.

$DE = 5\frac{5}{8}$ in.; $O_6E = 6$ in. and is in a vertical position. Note: For an 8½- by 11-in. sheet, draw the mechanism to a scale two-thirds of the dimensions given. Let O_2O_6 run parallel to long edge of sheet. For the mecha-

nism, $O_2(2\frac{1}{4}, 6\frac{3}{4})$; for the displacement diagram, $o(\frac{3}{4}, \frac{3}{8})$; length of diagram = 8 in.

12. Figure 352 shows the skeleton outline of the Whitworth quick-return mechanism as used in a slotting machine of recent design.

a. Draw the skeleton outline to a scale of 3 in. = 1 ft. and make a diagram showing ram displacements corresponding to equal angular displacements of the driving link 3. Take ram displacements as ordinates and equal angular displacements of link 3 as abscissas.

b. Determine the time ratio of the mechanism. Data: $O_3O_5 = 3\frac{3}{4}$ in.; $O_3A = 7\frac{1}{2}$ in., and it makes an angle of 60 deg. with the horizontal; O_5B =

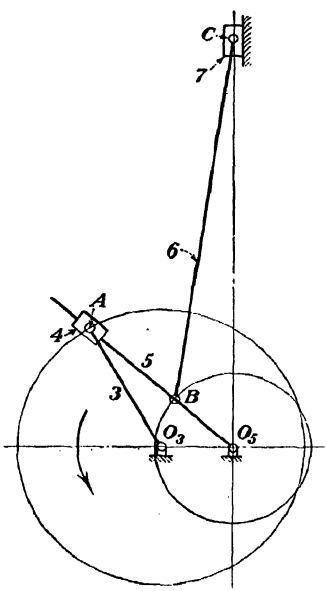


FIG. 352.

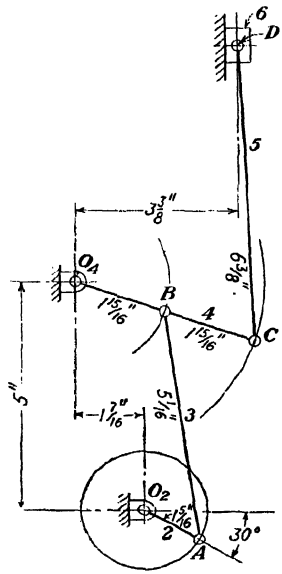


FIG. 353.

4 in.; BC = 20 in. For the mechanism, $O_3(6\frac{1}{8}, 3)$; for the displacement diagram, $o(5\frac{5}{8}, —)$. Length of displacement diagram = 6 in.

13. The skeleton outline of a Powell lever engine is shown in Fig. 353. Starting with the piston (link 6) in the upper extreme position draw its displacement diagram for the complete cycle on a base line 6 in. long. Draw the mechanism in the position shown to a scale of 6 in. = 1 ft. For the mechanism, $O_2(6\frac{1}{2}, 2)$; for the displacement diagram, $o(\frac{3}{4}, —)$.

14. The Andreau variable-stroke engine is represented in skeleton outline in Fig. 354. The circles M and N represent the pitch circles of a pair of Citroën gears (see Art. 93) in mesh at P. Links 2 and 5 are rigidly connected to gears M and N, respectively. Diam. of $M = 2\frac{1}{16}$ in.; Diam. of $N = 4\frac{1}{8}$ in.

(In the design of a machine or in a study of its operation, particularly where the motions of the members are as complex as in the present case, it

is essential to have diagrams showing paths of various points and variations in the piston stroke during a cycle.)

Starting with the member O_5C in the position shown [corresponding to the lowest positions of the piston (link 7)] and placing it successively in 24 equally spaced positions in the cycle (one revolution), plot the path of point B and draw the displacement diagram for the piston (point D). The length of the displacement diagram is 6 in. The first upward stroke of the piston is the exhaust stroke. Indicate along the upper edge of the diagram the distances that represent the exhaust, intake, compression, and power strokes, as in Fig. 15. For the mechanism, $O_2(3\frac{7}{8}, 1\frac{1}{2})$; for the displacement diagram, $o(2\frac{1}{4}, —)$. Make the drawing to the dimensions given in the figure.

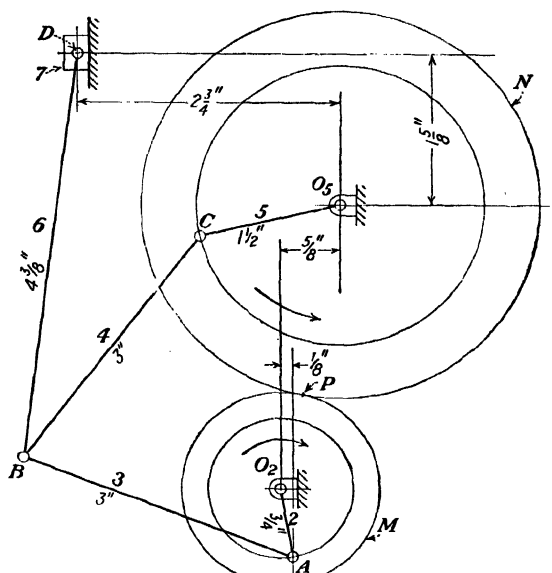


FIG. 354.

15. In Fig. 355 is shown a skeleton outline of the Zoller double-piston engine. Draw the mechanism to the dimensions given. Starting with the crank (link 2) in the position indicated in the figure, *i.e.*, with piston 4 in its extreme upper position, and using 12 divisions in the cycle, plot the path of point B. Also, draw the displacement diagrams of the two pistons (points C and D) on the same base line for comparison. For the mechanism, $O_2(5\frac{1}{4}, 1\frac{3}{4})$; for the displacement diagrams, $o(3\frac{1}{4}, 3\frac{1}{4})$. (Note: Let the base line of the displacement diagrams run vertically on the sheet. Length = 6 in.)

16. The linkage outlined in Fig. 356 is that of the Wellman geared bucket, used in excavating work. The outline of the bucket (one half) in its closed position is shown in dotted lines. Plot the path of the cutting edge of the bucket (point C) between its closed position, where operating link 2 makes

an angle of 27 deg. with the vertical line through O_2, and its extreme open position, after link 2 has moved through an angle of 105 deg. Draw the mechanism to a scale of 1½ in. = 1 ft.: $O_2(3\frac{1}{2}, 6\frac{3}{8})$.

17. Figure 357 shows in skeleton outline a straight-line mechanism for an oil circuit breaker. The point C moves in a straight line through its operating range.

a. Draw the mechanism to the dimensions given in the figure, $O_2(\frac{1}{2}, 3)$, and plot the paths of points A, B, and C using 1-in. displacements (approximately) of point B, starting with O_2A at an angle of 60 deg. with

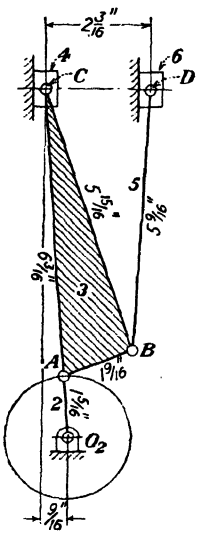


FIG. 355.

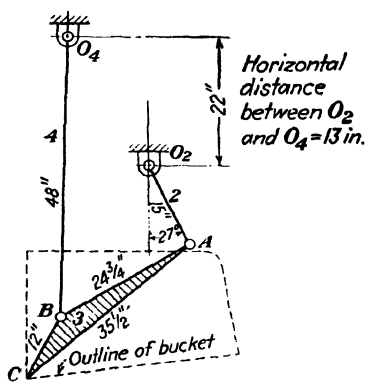


FIG. 356.

the horizontal. Adopt a numbering scheme to indicate intermediate positions of all points. Determine the complete path of point C, which is a closed curve.

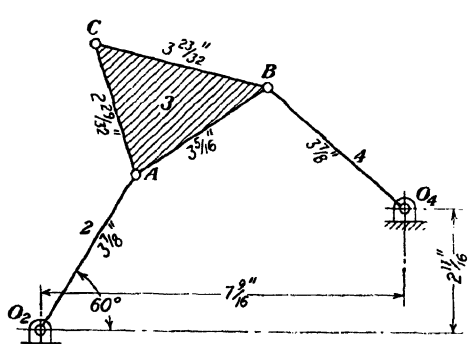


FIG. 357.

b. Indicate clearly (by showing the mechanism in dotted lines) the dead-point positions when 2 is the driving link and when 4 is the driving link.

18. Draw the Crosby indicator mechanism (Fig. 358) to a scale of 1½ in. = 1 in. $O_2(1\frac{1}{2}, 4)$; $O_2O_4 = \frac{5}{8}$ in.; O_2A (vertical) = $1\frac{5}{16}$ in.; $O_4D = \frac{7}{32}$ in., $BD = 1\frac{7}{32}$ in.; $BC = 3\frac{1}{8}$ in.; $AB = 2\frac{1}{32}$ in.

Starting with point C in its lowest position, move it vertically in its straight-line path, and at ¾-in. intervals of C plot the path of D. Find the arc of a circle that will pass through the several positions of D (approximately) and thus determine the position of O_5 and the length of link 5.

19. The skeleton outline of the Holst pumping engine, with dimensions, is shown in Fig. 359. The purpose of this design is to give the pump piston (slide F) a reciprocating motion that is approximately uniform while the crank O_2A is rotating with uniform motion.

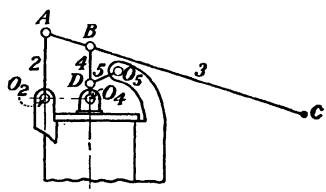


FIG. 358.

The result required is two displacement diagrams on the same base line for comparison; one of piston 4 (point B or C) showing the deviation from uniform motion due to the angularity of the connecting rod; and one of piston 8 (point F) showing how closely uniform motion is obtained by the use of this mechanism. Use 12 crank positions and a length of 6 in. for the base line of the diagrams. Position of diagrams, $o(1\frac{1}{4}, \frac{3}{4})$.

In position $O_2(6\frac{1}{2}, 8\frac{1}{4})$ draw the part of the linkage O_2AB to a scale of 4 in. = 1 ft. with O_2A vertically upward and find the displacement of link 4. In position $O_6(4\frac{1}{4}, 4\frac{1}{4})$ draw the remainder of the mechanism to the same scale and find the displacement of piston 8. While determining the displacements of point F, plot the path of point E.

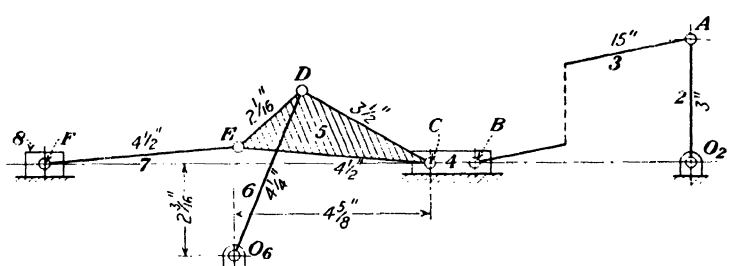


FIG. 359.

20. Draw the skeleton outline of the "up-and-down-cut shear" shown in Fig. 387 to a scale of 1½ in. = 1 ft. and make a displacement diagram for the upper shear member (link 9). For dimensions see Prob. 117. (For the purpose of this problem the lower shear member (link 8) and therefore links 4 and 6, will be assumed stationary. Now change the numbering and lettering so that the skeleton outline will appear as follows: Numbers 2 and 3 as shown in figure; triangular link 5 in figure will appear as number 4, with fixed center of rotation O_4, number 7 will appear as number 5; number 9 as number 6; and numbers 4, 6, and 8 in the figure as a part of the fixed link, number 1.)

Make determinations for every 45 deg. of motion of the crank O_2A, numbering positions from 0 to 8. Let zero position of crank be that which will bring link 6 to its upper extreme position. Draw the displacement diagram with crank positions represented on a 6-in. base line and let the ordinates that represent displacements of link 6 be drawn to twice the length scaled from the drawing. Refer to Fig. 15 for the general plan to be followed in numbering points, etc. For the mechanism, $O_2(2\frac{1}{2}, 7\frac{1}{4})$; for the displacement diagram, $o(1\frac{1}{4}, 1)$.

Chapter III

CAMS

21. Construct the outline of a disk cam that will transmit harmonic motion to a roller follower in the following manner: Up 1 in. while the cam turns

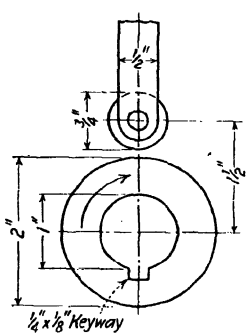


Fig. 360.

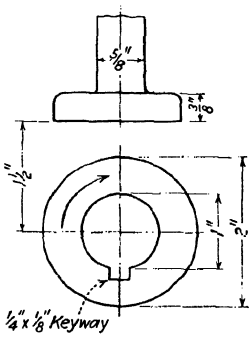


Fig. 361.

through 120 deg.; rest while the cam turns through 60 deg.; down 1 in. while the cam turns through 90 deg.; rest while the cam turns through 90 deg. The initial position of the follower relative to the axis of the camshaft and other information regarding the general layout are shown in Fig. 360. Coordinates of axis of cam, $(4\frac{1}{4}, 4\frac{3}{4})$.

22. Same as Prob. 21 except that follower has parabolic motion.

23. Same as Prob. 21 except that follower has modified straight-line motion.

24. Same as Prob. 21 except that path of roller follower is in vertical line $\frac{5}{8}$ in. to left of vertical line through axis of cam.

25. Same data as in Prob. 21, except that: diameter of shaft = $\frac{1}{2}$ in.; diameter of hub = 1 in.; keyway = $\frac{1}{8}$ by $\frac{1}{16}$ in.; distance that axis of roller follower is above axis of cam = 1 in.; travel of roller follower = $\frac{3}{4}$ in.

a. Construct the outline of the cam and show on the drawing the pitch circle and the maximum pressure angle (refer to Fig. 69). Coordinates of axis of cam, $(4\frac{1}{4}, 4\frac{1}{4})$.

b. Draw a displacement diagram of the follower (refer to Fig. 69). Coordinates of displacement diagram, $o(\frac{1}{2}, 8\frac{1}{4})$.

26. *a.* Construct the outline of a disk cam that will transmit harmonic motion to a flat-faced follower in the following manner: up 1 in. while the cam turns through 90 deg.; rest while the cam turns through 90 deg.; down 1 in. while the cam turns through 90 deg.; rest while the cam turns through 90 deg.

b. Determine the required length of face of the follower, and then add ⅛ in. on each end. Information regarding the general layout is given in Fig. 361. Coordinates of axis of cam, (4¼, 4¾).

27. Same as Prob. 26, except that the face of the follower as shown in Fig. 361 is to be tipped clockwise 15 deg. about its point of intersection with the vertical center line.

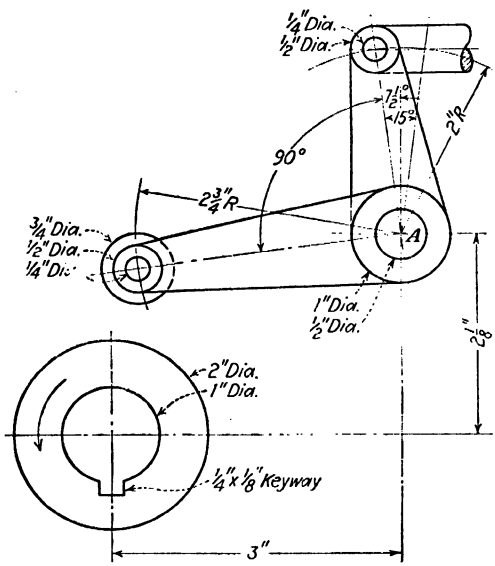


FIG. 362.

28. The roller follower (Fig. 362) is mounted on the end of an arm that is pivoted at *A*. Construct the cam outline that will impart motion to the follower in accordance with the following specifications: Harmonic-motion base curve. Up 15 deg. while the cam turns through 120 deg.; rest while the cam turns through 150 deg.; down 15 deg. while the cam turns through 90 deg. The lowest position of the follower and other general information regarding the layout are given in the figure. Coordinates of axis of cam, (4¼, 4½).

29. The flat-faced follower, which is pivoted at *A* in Fig. 363, is required to move in accordance with the following specifications: Harmonic-motion base curve. Up 15 deg. while the cam turns through 120 deg.; rest while the cam turns through 90 deg.; down 15 deg. while the cam turns through 90 deg.; rest while the cam turns through 60 deg.

a. Determine the required cam outline.

b. Determine the limits of contact along the face of the follower. The lowest position of the follower and other general information regarding the layout are given in the figure. Coordinates of axis of cam, (4¼, 5).

30. Determine the shape of the groove in the cylindrical cam in Fig. 364, that will impart motion to the arm that swings about axis *A-A* in the following manner: Outward 15 deg. with harmonic motion in one-third turn of cam. Rest for one-sixth turn of cam. Return with harmonic motion in one-third turn of cam. The roller is to be ⅝ in. in diameter at large end, ⅜ in. in length, and of such form as to give pure rolling contact. Coordi-

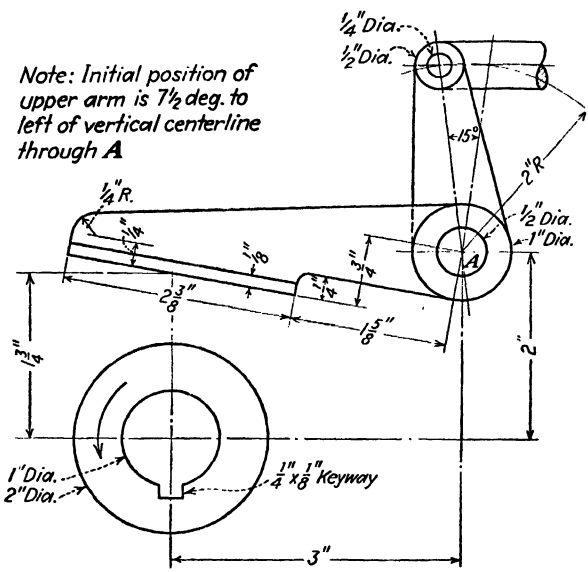


FIG. 363.

nates for the axis of the cam: (6¼, 3). For the displacement diagram, *i.e.*, the developed cylindrical surface, which will run parallel to left-hand edge of sheet, *o*(2½, ¼).

31. In the toe-and-wiper cam mechanism, Fig. 84, the face of the follower is shown in its lowest position, which is 1 in. above the axis of the cam. The follower is to have a vertical movement of 1¼ in. while the cam turns counterclockwise through 45 deg. at a uniform rate. The positions of the follower corresponding to equal angular positions of the cam are to be ⅛, 7/16, 13/16, and 1¼ in., respectively, above its initial position. Determine the working outline of the cam and draw, in good proportion, the complete outline of both cam and follower. Axis of cam, (2¾, 4¼).

32. The dimensions of a stamp-mill cam similar to that shown in Fig. 85 are as follows: diameter of camshaft, 5 in.; diameter of hub, 12 in.; center line of stamp shaft, 5 in. to the right of center line of camshaft; the lifting

plate is 16 in. above the center line of the camshaft when in its upper extreme position and from this position it has a fall of 9 in. to its extreme lower position; diameter of lifting plate = 9 in.; height of lifting plate = 10½ in.; diameter of stamp shaft = 3 in. The contact surface of the cam is 2 in. wide, and it passes ¼ in. in front of the stamp shaft. Draw the cam and follower to a scale of 3 in. = 1 ft. Coordinates of axis of camshaft, (4¼, 4⅛). The cam has a counterclockwise rotation of 55 r.p.m.

a. Assuming 24 time periods per revolution, that the stamp is to rise with uniform motion and rest two-thirds of a time period after it has fallen to its

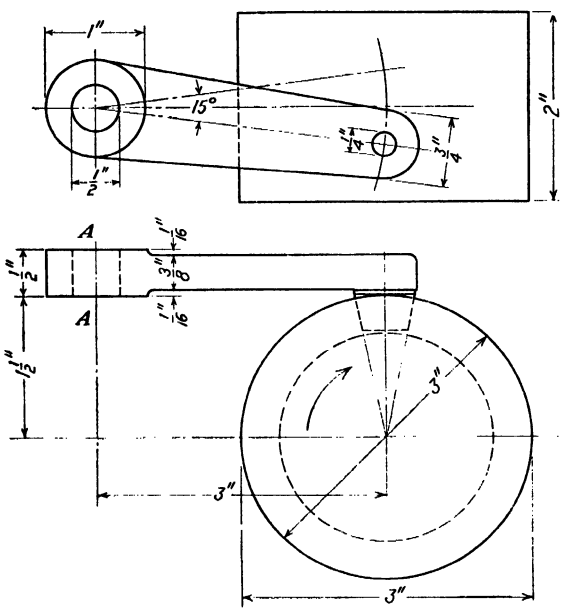


FIG. 364.

lower extreme position, make a displacement diagram of the follower motion through its complete cycle on a base line 6 in. long. Coordinates, *o*(1¼, ¼).

b. Determine the required outline of the cam.

Chapter IV

TOOTHED GEARING—SPUR GEARS

33. An involute gear with 14½-deg. full-depth involute teeth of 1 diametral pitch has a pitch diameter of 12 in. Axis of gear, *o*(4¼, ½).

Using the method shown in Fig. 97, Art. 68, lay out full size one complete tooth. (Refer to Arts. 98 and 101 for tooth proportions.)

34. Lay out the teeth of a pair of involute spur gears to conform to the following specifications: 14½-deg. composite system (Arts. 98 and 101);

diametral pitch = 2½; pitch diameter of pinion = 6 in.; pitch diameter of gear = 12.800 in.; backlash = 0. The pinion is the driver, and its direction of rotation is clockwise. Axis of pinion, (4¼, ½) (refer to Art. 72).

a. Show not more than four or five teeth on each gear, having one pair of teeth in contact at the pitch point.

b. Indicate on the drawing the following: (1) Angles of approach, α_p and α_g; (2) angles of recess β_p and β_g; (3) line of action; (4) path of contact (indicate by heavy line); and (5) pressure angle, θ.

c. Examine the teeth for interference and, if there is any, indicate it in the manner shown in Fig. 103.

35. Same as Prob. 34 except that rotation of pinion is counterclockwise.

36. Statement same as for Prob. 34 with the exception that the pitch diameter of the gear is 12 in.

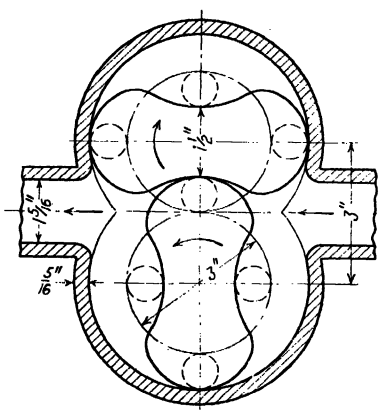


FIG. 365.

37. Same as Prob. 36, except that rotation of pinion is counterclockwise.

38. A cycloidal gear of 1 diametral pitch has a pitch diameter of 12 in. Axis of gear, $O(4¼, ½)$.

a. Using the method given in Art. 76, lay out full size one complete tooth. (Refer to Art. 98 for tooth proportions.)

39. Lay out the teeth of a pair of interchangeable cycloidal spur gears to conform to the following specifications: diametral pitch = 2; number of teeth in pinion = 12; number of teeth in gear = 25; backlash = 0. The pinion is the driver, and its direction of rotation is clockwise. Axis of pinion (4¼, ½). (Refer to Art. 98 for tooth proportions.)

a. Show not more than four or five teeth in each gear, having one pair of teeth in contact at the pitch point.

b. Indicate on the drawing the following: (1) angles of approach, α_p and α_g; (2) angles of recess, β_p and β_g; (3) path of contact (indicate by heavy line); and (4) maximum pressure angle, θ.

40. A drawing and a description of the Root positive blower are given under Art. 31. In Fig. 365 some general dimensions for such a blower are given. Lay out the cycloidal outlines of the rotors, or impellers, by the methods outlined in Chap. IV. The diameter of the generating circle is one-fourth that of the pitch circle. (For a three-lobed wheel it would be one-sixth, etc.) It can be seen that, with the complete curves, the impellers act like a pair of mating teeth on cycloidal gears. There can, however, be real tooth action (in the sense of effective driving) over only certain parts of the revolution. The maintenance of proper rolling is insured by an external pair of equal gears having the pitch circles indicated in the figure. Axis of lower impeller, (4¼, 3¾). Scale, full size.

41. Statement same as for Prob. 40, with the exception that the impellers are to have involute outlines, and the diameter of the pitch circles of the mating gears is 4 in. The pressure angle is 35 deg. Start construction with the long axis of the lower impeller in a 45-deg. position in the second quadrant. The end of each impeller is to be rounded off with an arc of a circle having a $1\frac{3}{32}$-in. radius, the center of this arc to be located on the pitch circle. The same arc, with its center on the pitch circle, will also connect the involute curves of the two lobes inside the base circle. Axis of lower impeller, ($4\frac{1}{4}$, $2\frac{7}{8}$). Scale, full size.

Chapter V

BEVEL, HELICAL, AND WORM GEARING

42. Lay out a pair of involute bevel gears to meet the following requirements: $14\frac{1}{2}$-deg. composite system (refer to Art. 98 for proportions); angle between shafts = 75 deg.; velocity ratio = 2:1; diametral pitch = 4; pitch diameter of gear = 8 in.; length of face = $1\frac{1}{2}$ in. Lay out at least two teeth on the developed back cones of each gear, and show sectional view of the gears as in Fig. 115. Assume proportions for bores, hubs, webs, etc., using proportions shown in Fig. 115 as a guide. Axes of pinion and gear intersect at ($4\frac{1}{4}$, 2, with long edge of sheet horizontal).

43. Same as Problem 42 except that the teeth are to conform to the Gleason System for Bevel Gears (Art. 102).

44. Make a layout, similar to that shown in Fig. 126, of a worm and worm wheel having a velocity ratio of 18:1. The worm is right handed and double threaded and has an axial pitch of $\frac{3}{4}$ in.

The distance between centers is 5.747 in. The bore of the worm is $1\frac{3}{8}$ in., and that of the worm wheel $1\frac{3}{4}$ in. Refer to Art. 103 for proportions, and use proportions shown in Fig. 126 as a guide in drawing hub, web, rim, etc. Coordinates of axis of worm wheel, ($4\frac{1}{4}$, $1\frac{1}{2}$). Note: The design illustrated in Fig. 126 conforms to the standards of the American Gear Manufacturer's Association (Art. 103).

45. The purpose of this problem is to make a comparison of the tooth forms of the $14\frac{1}{2}$-deg. composite system, the Fellows stub-tooth system and the Gleason system for bevel gears (unequal addendum), using the following data and making the drawings to a scale of 2 in. = 1 in.

The $14\frac{1}{2}$-deg. composite system: pitch diameter = 4 in.; pitch = 4; radius of fillet = $\frac{1}{10}$ in.; axis of gear, ($1\frac{1}{2}$, 2) (Arts. 98 and 101).

The Fellows stub-tooth system: pitch diameter = 4 in.; pitch = $\frac{4}{5}$; involute outline of tooth to be approximated by an arc of $\frac{3}{4}$-in. radius; radius of fillet = $\frac{1}{16}$ in.; axis of gear, ($4\frac{1}{4}$, 2) (Arts. 99 and 101).

The Gleason system for bevel gears: pitch diameter = 4 in.; pitch = 4; addendum = 0.3375 in.; dedendum = 0.2095 in.; tooth thickness = 0.4424 in.; involute tooth outline to be approximated by an arc of $\frac{7}{8}$-in. radius; radius of fillet $\frac{1}{16}$ in.; axis of gear, (7, 2) (Art. 102).

Lay out one tooth of the $14\frac{1}{2}$-deg. composite system, one of the Fellows system, and one of the Gleason system. Show on each as dimensions in

inches, the addendum, dedendum, and tooth thickness. Also indicate the pressure angle in degrees.

Chapter VII

BELTS, ROPES, AND CHAINS

46. A pair of three-step cone pulleys connect a driving and a driven shaft (42 in. apart) by means of a crossed belt. The driving shaft runs at 275 r.p.m., and the smallest step on the driving pulley is 12 in. in diameter. The driven shaft is to run at speeds of 175, 225, and 300 r.p.m.

a. Determine the size of all steps on both pulleys.

b. Determine the length of belt required.

47. *a.* Solve the preceding problem if an open belt is used instead of a crossed belt.

b. Check the results found in (*a*), using Burmester's graphical method.

48. In a belt-driven slotting machine, countershaft A running at 200 r.p.m. is connected with drive shaft B by means of a pair of four-step cone pulleys and a crossed belt. The distance between shafts is 8 ft., and the diameter of the largest step on the countershaft pulley is 15 in. The maximum and minimum speeds of the drive shaft are 544 and 68 r.p.m., respectively, and the intermediate speeds must be in geometric ratio. Calculate the length of the belt required and also the diameters of all the steps of both pulleys.

49. *a.* Solve the preceding problem if an open belt is used instead of a crossed belt.

b. Check the results found in (*a*) by means of Burmester's graphical method.

50. In a block and tackle similar to that shown in Fig. 162, the upper block has two sheaves and the lower block has three sheaves. Show the mechanism in diagrammatic form as in Fig. 163 and determine the mechanical advantage.

51. Make a diagrammatic representation (as in Fig. 163) of the hydraulic elevator system shown in Fig. 164 and give the mechanical advantage of the system. What is the ratio of the speed of the cage to that of the plunger?

52. In a differential chain hoist similar to that shown in Fig. 166, the upper block consists of a 22-pocket chain sheave and a 20-pocket chain sheave fastened to the same shaft. The lower, or hook, block is a 21-pocket chain sheave.

a. What is the mechanical advantage of this hoist?

b. If the chain is pulled over the upper sheave at the rate of 50 f.p.m., at what speed will the load be lifted?

Chapter VIII

INTERMITTENT-MOTION MECHANISMS

53. Lay out a pair of intermittent gears, similar to the pair shown in Fig. 169, in which the driven gear advances one-fourth turn for every turn of the driving gear.

54. Lay out a Geneva wheel combination in which the driven wheel makes one-sixth turn for one turn of the driver. Distance between centers = 4 in. Axis of driven wheel ($4\frac{1}{4}$, $3\frac{5}{8}$) when directly below driving wheel.

55. A circular plate is required to have 17 accurately spaced notches cut in its periphery. Describe fully the method of performing this operation on a milling machine, using the index plate shown in Fig. 172.

56. Draw to a scale four times that shown in the figure the skeleton outline of the toggle mechanism used to operate the friction clutch (Fig. 183) and show the positions of all the links when in their extreme *in* and *out* positions. With the long edge of the paper horizontal, the intersection of the center line of the shaft with the center line of the pulley is ($1\frac{1}{4}$, $\frac{3}{4}$).

Chapter IX

TRAINS OF MECHANISM

57. Given the gear train shown in Fig. 366 and the following data: gear *A*, 20 teeth; gear *B*, 12 teeth, 10 pitch; pinion *D* $1\frac{3}{4}$ in. in pitch diameter,

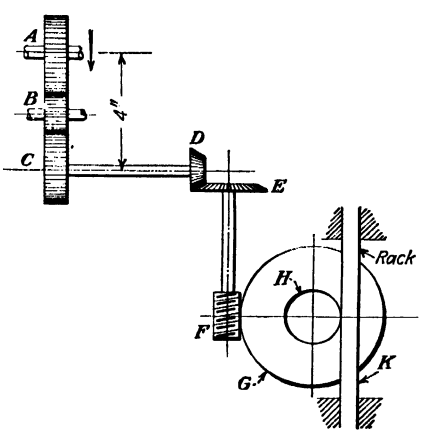


FIG. 366.

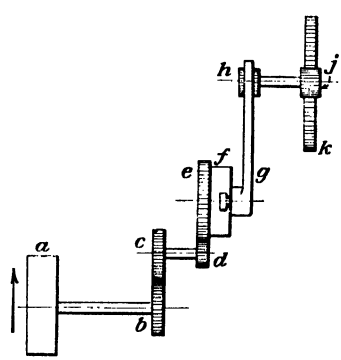


FIG. 367.

8 pitch; gear *E*, 30 teeth; worm *F* triple-threaded, right-hand type; worm wheel *G*, 45 teeth; gear *H*, 15 teeth, 6 pitch.

a. How many revolutions of gear *A*, in the direction indicated by arrow, are necessary to move the rack *K* a distance of 1 in.?

b. Will the rack move up, or down?

58. In Fig. 367 is shown a train of mechanism through which power is transmitted to the cutting tools of a bevel-gear-cutting machine. *a* is the driving pulley; *b* and *c* are two of a set of simple change gears by means of which the speed of the tools is controlled; member *f* is a crank plate attached to gear *e*; *g* is a rack in mesh with pinion *h* at one end and pivoted on an adjustable stud in the crank plate on the other end. The adjustable stud provides means for varying the length of tool stroke. The cutting tool is carried on a reciprocating slide to which rack *k* is fastened. Data: drive

pulley a runs at 500 r.p.m.; the stud in crank plate f is set for a throw of 2½ in.; gear b, 24 teeth, 8 pitch; gear c, 56 teeth, 8 pitch; gear d, 15 teeth, 8 pitch; gear e, 100 teeth, 8 pitch; gear h, 25 teeth, 6 pitch; gear j, 25 teeth, 6 pitch.

a. How many strokes per minute does the tool make?

b. What is the cutting speed in feet per minute?

c. If the tool must run at 50 strokes per minute, figure out the pair of change gears necessary at b and c.

59. A motor running at 1,800 r.p.m. drives a conveyor through the double-stage spur-gear speed reducer shown in Fig. 368.

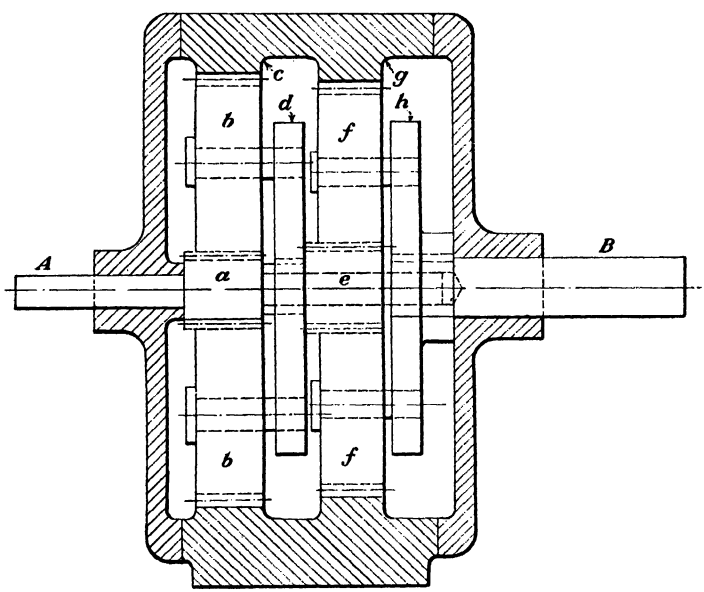


FIG. 368.

a. Find the total speed reduction from motor to conveyor.

b. Find the r.p.m. of the conveyor shaft.

c. Do the motor and conveyor shafts run in the same direction?

Data: A is the shaft to the motor; B is the shaft to the conveyor; gear a, 12 teeth; gear b, 24 teeth; gear c, 60 teeth; gear e, 14 teeth; gear f, 18 teeth; gear g, 50 teeth.

60. *a.* In the triplex hoist (Fig. 198), find the number of turns hand sprocket a must make in order to produce one turn of hoisting sprocket g.

b. How many feet of hand chain must be moved to raise weight W 1 ft.?

Data: gear b, 13 teeth; gear c, 31 teeth; gear d, 12 teeth; gear e, 49 teeth. Pitch diameter of hand sprocket = 9¾ in. Pitch diameter of hoisting sprocket = 3⅛ in.

61. A good example of a spur-gear differential train is shown in Fig. 369. The driver (link 2) consists of the shaft and gear a, which is keyed to it. The

planetary gears *b* and *c*, which are keyed together, constitute link 3. Link 4 consists of gear *d* and the shaft on which it is keyed. The spider that carries the planetary gears *b* and *c* is link 5. The motion of the driven shaft (link 4)

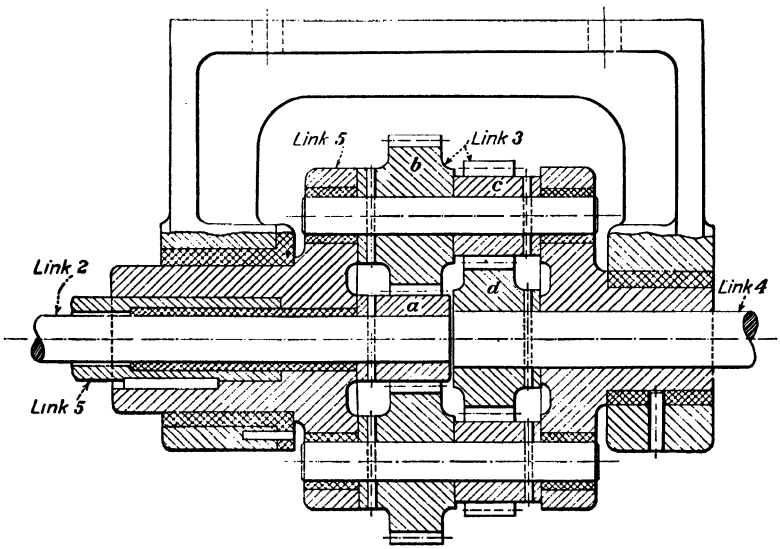


FIG. 369.

is the result not only of the continuous motion of the main driving shaft (link 2) but also of the continuous or intermittent motion of link 5, which is driven through the sleeve shown at the left in the figure.

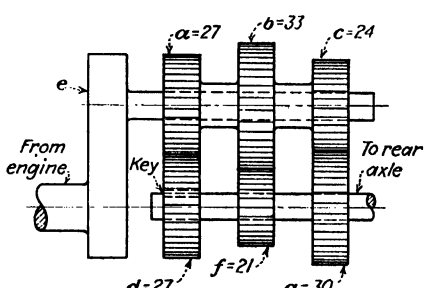


FIG. 370.

All the gears have the same pitch, and their number of teeth are indicated by the letters *a*, *b*, *c*, and *d*. Determine the angular velocity ω_4 of the driven member in terms of the angular velocities ω_2 and ω_5 of the driving members.

62. In Fig. 370 is shown a diagram of the arrangement of the gears in the old Ford transmission (Model T).

a. For low speed, gear *f* must be held stationary. In this case, find number of turns of gear *d* (and hence rear-axle shaft) for one turn of engine drive shaft *e*.

b. For reverse, gear *g* must be held stationary. In this case, find number of turns of *d* for one turn of *e*.

c. For high speed, how does the mechanism operate?

63. The train of gears shown in Fig. 371 includes an epicyclic-bevel-gear train in the form sometimes used in indexing and differential mechanisms of machine tools. The drive is from *a* and *b* through the epicyclic train to *d*. Data: gear *a*, 60 teeth, 12 pitch; gear *b*, 60 teeth, 12 pitch; gear *c*, 52 teeth, 8 pitch; gear *d*, 26 teeth, 8 pitch; gear *f*, 24 teeth, 8 pitch; gear *g*, 20 teeth, 8 pitch.

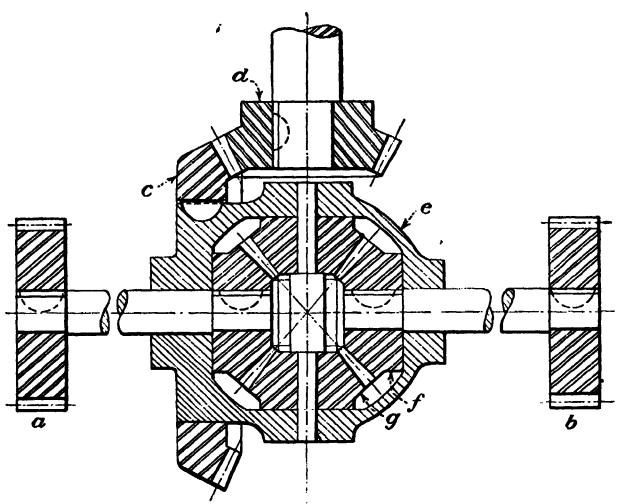


FIG. 371.

a. If *a* and *b* are running in the same direction at the same speed, how many revolutions does *d* make for one revolution of *a*?

b. If *a* and *b* are running in opposite directions at the same speed, what is the speed of *d*?

c. If *b* is held stationary, how many revolutions does *d* make for one revolution of *a*?

d. If *a* runs at 50 r.p.m. and *b* at 25 r.p.m. in the same direction, what is the r.p.m. of *d*?

64. In Fig. 372 is shown an epicyclic boring-bar feed train. Such a train is sometimes used in lathes to cut internal threads. The piece to be bored or threaded is fastened to the bed of the lathe concentric with the boring bar, which is mounted between the lathe centers. The bar rotates with the face plate of the lathe and drives its own feed train as shown in the figure. The boring bar *e* and the fixed gear *a*, which is fastened to the tailstock are

concentric with the center line of the lathe. The compound gears *b* and *c* rotate on a pin in a lug of the boring bar. Gear *d* is keyed to the feed screw *s*, which lies in a horizontal groove in the bar and passes through a threaded lug on the tool head. The direction and magnitude of the feed of the tool head along the bar for each revolution of the bar will depend on the direction and number of revolutions of the feed screw relative to the bar. Assuming that the bar rotates forward and that the lead screw has 8 right-hand threads per inch, determine the number of teeth for gears *a* and *b* to cut 6 internal

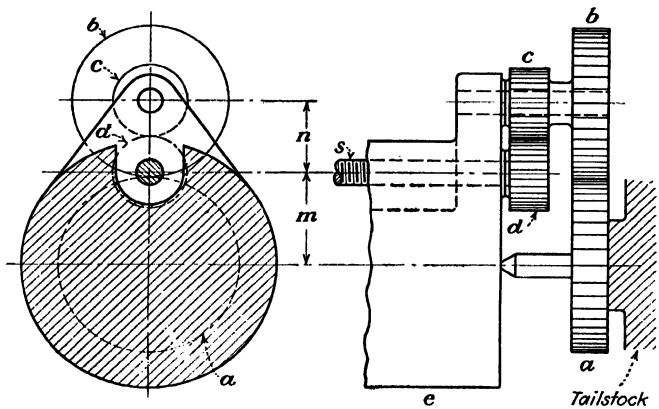


FIG. 372.

right-hand threads per inch. Gears *c* and *d* each have 24 teeth. The ratio of distance *n* to *m* + *n* is 3 to 7.

Chapter X

VELOCITIES IN MACHINES

65. Draw the oil-circuit-breaker mechanism (Fig. 357) to the dimensions given. $O_2(\frac{5}{8}, 2\frac{1}{2})$. Assuming that link 2 has clockwise rotation for the instant and that the velocity of A is represented by a vector $2\frac{1}{2}$ in. long, draw the velocity polygon for the mechanism. $o(3\frac{1}{2}, 3)$.

66. In the tool-slide mechanism of a bevel-gear-cutting machine (Fig. 373), link 2 is a rotating driving link that imparts a reciprocating motion to a slide attached to link 6. The slide carries an adjustable tool head. Draw the mechanism to a scale of 6 in. = 1 ft. $O_4(3, 5\frac{1}{2})$. Assuming counterclockwise rotation and letting the velocity of point A be represented by a vector $2\frac{1}{2}$ in. long, draw the velocity polygon. $o(3, 3)$.

67. Draw the mechanism of the Andreau variable-stroke engine to the dimensions given in Fig. 354. $O_2(4, 1\frac{1}{2})$. Refer to Prob. 14 for description of the mechanism. Assuming the velocity of crankpin A to be represented by a vector 2 in. long, draw the velocity polygon. $o(5, 8)$.

68. Figure 374 represents in skeleton outline the mechanism of a toggle press. Link 2 is the driving link, and link 8 is the last driven link. Draw

the mechanism to a scale of 1½ in. = 1 ft. Data: O_2A = 10 in.; AB = 32 in.; BC = 26½ in.; CD = 40½ in.; $O_6D = O_6E = DE$ = 9 in.; EF = 17¾ in.; O_4C = 14 in.; $O_2(6, 7½)$. O_6D is vertical.

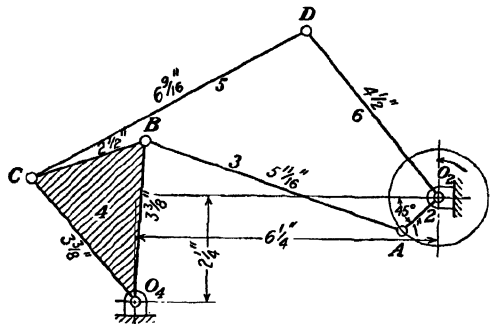


FIG. 373.

If the velocity of A is represented by a vector 1½ in. long, find the length of vector that represents the velocity of F, by means of the velocity polygon. $o(3, 3)$.

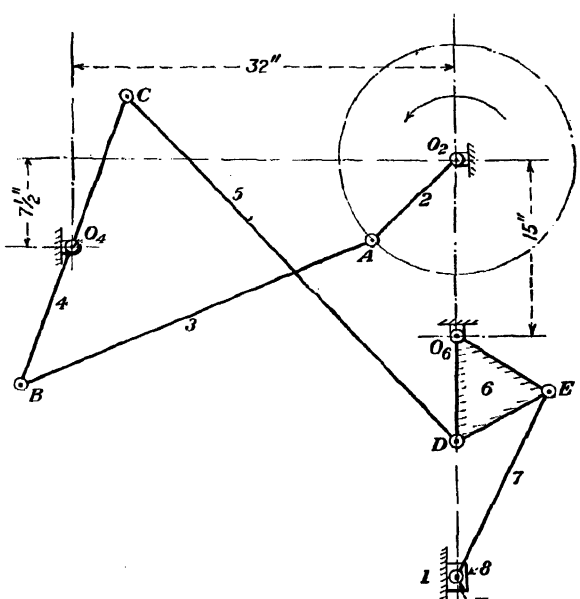


FIG. 374.

69. The crank of the lever engine, Fig. 353, is running at 2,000 r.p.m. Draw the mechanism in accordance with the specifications of Prob. 13. By means of the velocity polygon, determine the numerical value of the velocity

of the piston (link 6) by means of the velocity polygon. Velocity scale, 1 in. = 15 f.p.s. $o(3, 7)$. Rotation of crank, link 2, is clockwise.

70. Draw the drag-link quick-return mechanism (Fig. 23) to the dimensions and scale given in Prob. 8. $O_2(3\frac{1}{4}, 3\frac{3}{4})$.

The angular velocity of link 2 is 50 r.p.m. By means of the velocity polygon, find the velocity of the ram (link 6). The velocity scale is 1 in. = 1 f.p.s. $o(5\frac{1}{2}, 7)$.

71. Given the steam-valve mechanism of the Corliss nonreleasing type of valve gear (Fig. 351). Draw the mechanism to the dimensions and scale given in Prob. 11. $O_6(6\frac{1}{2}, 8)$.

Assuming that the velocity of E is represented by a vector 4 in. long (sense to the right), find the length of vector that represents the velocity of B by means of the velocity polygon. $o(1, 4)$. What is the mechanical advantage?

72. The steam-valve mechanism of the Corliss nonreleasing type of valve gear is shown in Fig. 351. Draw the mechanism to the dimensions and scale given in Prob. 11. $O_6(6\frac{1}{2}, 9)$.

a. Assuming vector $V_e = 3$ in. (sense to the right), determine vector V_b by means of the velocity polygon. $o(\frac{3}{4}, 5\frac{1}{4})$.

b. Assume that O_6E swings 45 deg. each way from its vertical position. Point E will, therefore, have a total travel over an arc of 180 deg. during the cycle. Starting toward the left from E, divide this arc through which E travels into 16 equal parts. With displacements of E represented on a base line 8 in. long, draw a diagram that will show the corresponding displacements of B for the entire cycle. $o(\frac{1}{4}, 1)$.

73. Draw the skeleton outline of the Atkinson gas-engine mechanism, given in Fig. 15, to a scale of $1\frac{1}{2}$ in. = 1 ft., and by the application of the law of three centers find all the instantaneous centers. Refer to Prob. 78 for dimensions. $O_2(6, 7)$.

74. Determine all the instantaneous centers of the crank-shaper mechanism referred to in Prob. 85. In this case, the crank (link 2) is to make an angle of 15 deg. with a horizontal line through O_2 to the right. $O_2(5, 4\frac{1}{2})$.

75. Determine all the instantaneous centers of the mechanism of the Corliss valve gear shown in Fig. 351. Refer to Prob. 11 for dimensions and scale of drawing. $O_2(2\frac{1}{4}, 8)$.

76. Find all the instantaneous centers of the drag-link mechanism shown in Fig. 23. Draw the mechanism to the scale and dimensions given in Prob. 8. $O_4(4\frac{1}{4}, 4\frac{1}{4})$.

77. The Whitworth quick-return mechanism of a slotting machine is shown in Fig. 352. Draw this mechanism to the dimensions and scale given in Prob. 12 and find all the instantaneous centers. $O_3(6, 3\frac{1}{2})$.

78. Given: the skeleton outline of the mechanism of Fig. 15 drawn to a scale of $1\frac{1}{2}$ in. = 1 ft. Data: $O_2A = 10$ in.; $O_4B = 12\frac{3}{4}$ in.; $AB = 24\frac{3}{4}$ in.; $AC = 25\frac{3}{4}$ in.; $BC = 3\frac{3}{4}$ in.; $CD = 29\frac{1}{2}$ in.; $O_2E = 26\frac{3}{4}$ in.; $O_4E = 3\frac{1}{2}$ in. $O_2(6, 7\frac{1}{2})$.

a. Find all the instantaneous centers.

b. Draw the velocity polygon. Assume that the velocity of A equals 330 f.p.m. and use a scale of 1 in. = 150 f.p.m. $o(5\frac{1}{4}, 1\frac{1}{2})$.

c. Check the velocity of D as found in *b* using centers 12, 26, and 16.

d. Same as *c* except that centers 12, 25, and 15 are to be used.

79. Determine all the instantaneous centers of the shaper mechanism referred to in Prob. 85. In this case the crank (link 2) makes an angle of 15 deg. with a horizontal line through O_2 to the right. $O_2(5, 4\frac{1}{2})$. Gear 2 has a uniform velocity of 22 r.p.m. Draw the skeleton outline to a scale of 2 in. = 1 ft. Note: velocity scale, 1 in. = 70 f.p.m.

a. Find all the instantaneous centers.

b. Find V_c, using centers 12, 25, 15.

c. Find V_c, using centers 12, 26, 16.

d. Find V_c, by means of the velocity polygon. $o(7\frac{1}{4}, 2)$.

80. Draw the Corliss valve gear in accordance with the dimensions and scale given in Prob. 11. $O_2(2\frac{1}{4}, 8)$.

a. Find all the instantaneous centers.

b. If the velocity of point E on link 6 is represented by a vector 4 in. long, find the vector length that represents the velocity of point B on link 3.

c. Check the velocity found in (*b*) above, by means of the velocity polygon. $o(1, 4)$.

81. The drag-link mechanism is to be drawn to the scale and dimensions given in Prob. 8. $O_4(4\frac{1}{4}, 4\frac{1}{4})$.

a. Find all the instantaneous centers.

b. If the crank (link 2) rotates at 50 r.p.m., find the velocity of the ram (link 6), by means of the velocity polygon. Scale, 1 in. = 1 f.p.s. $o(5\frac{1}{2}, 7\frac{1}{2})$.

c. Check the velocity of the ram found in (*b*), using center 26.

82. Given the quick-return mechanism of the slotting machine (Fig. 352). Draw the skeleton outline of the mechanism to the dimensions and scale given in Prob. 12. $O_3(6, 3\frac{1}{2})$.

a. Find all the instantaneous centers.

b. If link 3, the driving link, rotates at a uniform velocity of 84 r.p.m., and the velocity scale is 1 in. = 150 f.p.m., find the velocity of C by means of instantaneous centers, using (1) centers 13, 36, and 16 and (2) centers 13, 37, and 17.

c. Check the above result by means of the velocity polygon. $o(2\frac{1}{2}, 2\frac{1}{2})$.

83. Given the quick-return mechanism of a slotting machine drawn to the same dimensions and scale as in Prob. 12. Driving link 3 rotates with a uniform velocity of 80 r.p.m. $O_3(2\frac{3}{4}, 3)$.

a. Construct a velocity-displacement diagram on the path of the ram (link 7) as a base line. Note: velocity scale, 1 in. = 150 f.p.m. Start with the ram in the upper extreme position and make velocity determinations for 12 equally spaced crank positions.

b. Construct a velocity-time diagram on a base line $4\frac{1}{2}$ in. long. $o(6, 4)$. Base line vertical.

84. Draw the drag-link mechanism to the dimensions and scale given in Prob. 8. $O_4(2, 3)$.

a. Construct a velocity-space diagram for the slide (link 6), using its path as a base line. The crank (link 2) has a uniform velocity of 50 r.p.m. Velocity scale, 1 in. = 1.6 f.p.s. Start with the slide in its upper extreme position and make velocity determinations for 12 equally spaced crank posi-

tions. Determinations may be made for intermediate positions where desirable.

b. Construct a velocity-time diagram on a base line 6 in. long. $o(5\frac{1}{2}, 2)$. Let this diagram run vertically on the sheet.

85. In the crank-shaper mechanism (Fig. 4), the gear 2 has a uniform velocity of 22 r.p.m. Draw the skeleton outline of the mechanism to a scale of 2 in. = 1 ft. $O_2A = 5\frac{1}{4}$ in.; $O_4B = 27\frac{7}{8}$ in.; $BC = 6$ in.; $O_2O_4 = 14\frac{3}{4}$ in.; distance from O_2 to path of $C = 12\frac{1}{4}$ in.; $O_2(4\frac{1}{2}, 6\frac{1}{2})$. In the configuration shown, O_2A makes an angle of 30 deg. with the horizontal.

a. Construct a velocity-space diagram on the path of the ram C as a base line. In order to simplify velocity determinations, refer to construction outlined in Art. 156.

Note: Velocity scale, 1 in. = 70 f.p.m.

Start with the ram in the right-hand extreme position and make velocity determinations for 12 equally spaced crank positions.

b. Construct a velocity-time diagram on a base line 6 in. long. $o(1\frac{1}{4}, 3)$.

86. An 18-tooth pinion rotating at 500 r.p.m. in a clockwise direction drives a 32-tooth gear. The teeth are of four pitch and conform to the $14\frac{1}{2}$-deg. involute system. By the method of relative velocities, find the rate of sliding between a pair of mating teeth at the last point of contact. Use a graphical method. Velocity scale, 1 in. = 300 f.p.m. Note: It is not necessary to draw in accurate tooth outlines. Axis of pinion, $(4\frac{1}{4}, 2)$.

Chapter XI

ACCELERATIONS IN MACHINES

87. The table of a hydraulically reciprocated grinder is arranged to operate with a $\frac{3}{4}$-in. stroke and makes 200 complete cycles (over and back) per minute. Except during a very short period at each end of the stroke, the table moves with a uniform velocity of 8 in. per sec. Assuming that during the period of reversal the table is accelerated at a uniform rate, what is the magnitude of the acceleration in feet per second^2?

88. At a given instant a 12-in. pulley is turning at a rate of 600 r.p.m., and its speed is increasing at a rate of 1 per cent per revolution.

a. Determine the angular velocity and angular acceleration of the pulley in radians per second and radians per second per second respectively.

b. Determine the magnitude and direction of the acceleration of a point on the face of the pulley.

c. Determine the magnitude of the acceleration of a point on the straight portion of the belt 8 in. from the point of tangency of the belt and the pulley. Disregard the thickness of the belt.

d. Determine the acceleration of this point relative to the point on the face of the pulley with which it was previously in contact.

89. A four-link mechanism similar to that of Fig. 16 is to be drawn full size to the following dimensions: link 2 (O_2A) is 2 in. in length and makes an angle of 60 deg. with the horizontal. $O_2(2, 6)$. Distance $O_2O_4 = 4\frac{1}{2}$ in. Link 4(O_4B), which is in a vertical position, is $2\frac{1}{2}$ in. in length.

a. Draw the velocity polygon. The angular velocity of link 2 is ω_2 = 60 radians/sec. (clockwise). Velocity scale, 1 in. = 5 f.p.s. $o(1, 4\frac{1}{2})$.

b. Draw the acceleration polygon. The angular acceleration of link 2 is α_2 = 900 radians/sec.^{2} (clockwise). Acceleration scale, 1 in. = 200 f.p.s.^{2} $o(7, 5)$.

c. What is the velocity of *B* in feet per second? What is the acceleration of *B* in feet per second^2?

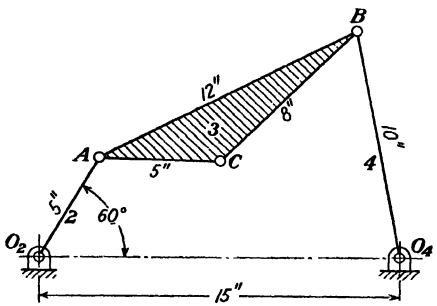


FIG. 375.

90. The four-link mechanism of Fig. 375 is to be drawn to a scale of 3 in. = 1 ft. Dimensions are shown in the figure. $O_2(2\frac{1}{4}, 6)$. The angular velocity of link 2 is ω_2 = 25 radians/sec. (clockwise), and the angular acceleration is α_2 = 180 radians/sec.^{2} (clockwise).

a. Draw the velocity polygon; scale, 1 in. = 5 f.p.s. $o(1, 4\frac{1}{2})$.

b. Draw the acceleration polygon; scale, 1 in. = 100 f.p.s.^{2} $o(7, 5)$.

c. What is the acceleration of point *C* in feet per second^2?

91. Lay out the mechanism of Fig. 376 to the dimensions given and draw the acceleration polygon, using the *analytical method*, as in Art. 174. The angular velocity and angular acceleration of link 2 are ω_2 = 40 radians/sec. and α_2 = 400 radians/sec.^{2} respectively, both in the counterclockwise sense.

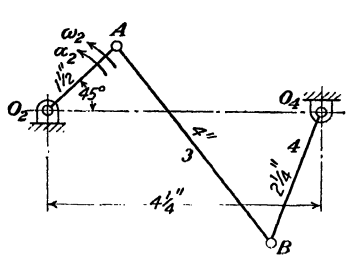


FIG. 376.

For the mechanism: scale, full size; $O_2(2\frac{1}{4}, 7)$. For the velocity polygon: scale, 1 in. = 2½ f.p.s.; $O(7\frac{3}{4}, 2\frac{1}{2})$. For the acceleration polygon: scale, 1 in. = 100 f.p.s.^{2}; $o(3, 4)$.

92. Same as Prob. 91 except that the *graphical method* is to be employed, as in Art. 177. In this case, let the velocity scale be determined from the conditions of the problem.

93. Lay out a four-link mechanism to the following dimensions, using a scale of 3 in. = 1 ft.; $O_2(2, 8)$; $O_4(7, 5)$. Link 2 (O_2A) makes an angle of 45 deg. with the horizontal in the second quadrant. Link 3 (AB) is 20 in. long, and link 4 (O_4B) is 13½ in. long. The angular velocity and angular acceleration of link 2 are respectively ω_2 = 60 radians/sec. and α_2 = 900 radians/sec.^{2}, both in clockwise sense. Link 2 (O_2A) is 4 in. in length.

a. Draw the velocity polygon; scale, 1 in. = 10 f.p.s. $o(3, 5\frac{1}{4})$.

b. Draw the acceleration polygon; scale, 1 in. = 300 f.p.s.^{2} $o(1, 3\frac{1}{2})$.

94. In the Roberts straight-line mechanism (Fig. 377), the path of point C is a straight line. Make a full-size drawing of the mechanism, using the dimensions given on the figure. $O_2(1\frac{1}{2}, 6\frac{1}{2})$. The angular velocity of link 2 is $\omega_2 = 20$ radians/sec., and the angular acceleration is $\alpha_2 = 100$ radians/sec.^{2}, both clockwise.

a. Draw the velocity polygon. $o(5, 5)$. Scale, 1 in. = 3 f.p.s.

b. Draw the acceleration polygon. $o(4\frac{1}{4}, 4\frac{1}{2})$. Scale, 1 in. = 50 f.p.s.^{2}

95. Draw the oil-circuit-breaker mechanism to the dimensions given in Fig. 357, letting link 2 make an angle of 45 deg. with the horizontal. $O_2(\frac{1}{2}, 3\frac{1}{2})$. Scale, full size. The angular velocity and acceleration of link 2 are, respectively, $\omega_2 = 18$ radians/sec. and $\alpha_2 = 160$ radians/sec.^{2}, both in clockwise sense.

a. Draw the velocity polygon. $o(5\frac{1}{2}, 6)$. Scale, 1 in. = 2.5 f.p.s.

b. Draw the acceleration polygon using the analytical method as in Art. 174. $o(4\frac{1}{4}, 6)$. Scale, 1 in. = 50 f.p.s.^{2}

96. Same as Prob. 95, except that the graphical method of Art. 177 is to be used in the determination of the acceleration polygon. The space scale and the velocity scale are to remain as in Prob. 95. Acceleration polygon, $o(4\frac{1}{4}, 5)$.

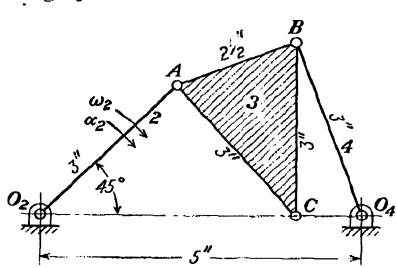


FIG. 377.

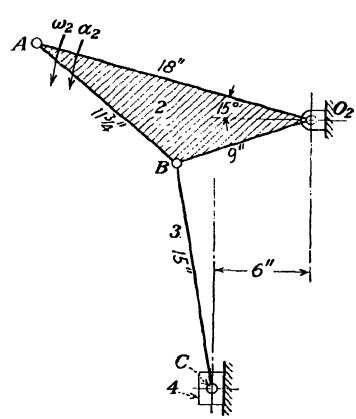


FIG. 378.

97. Lay out the control mechanism (Fig. 378) to the dimensions given. $O_2(7\frac{1}{4}, 8\frac{1}{4})$. Scale, 3 in. = 1 ft. The angular velocity and angular acceleration of link 2 are, respectively, $\omega_2 = 15$ radians/sec. and $\alpha_2 = 100$ radians/sec.^{2}, both in a counterclockwise sense.

a. Draw the velocity polygon. $o(2\frac{3}{4}, 3\frac{1}{2})$. Scale, 1 in. = 10 f.p.s.

b. Draw the acceleration polygon. $o(1, 6)$. Scale, 1 in. = 100 f.p.s.^{2}

c. What is the numerical value of the acceleration of C?

98. In the circuit-breaker mechanism represented in Fig. 379, the point C moves in a vertical straight line. Draw the mechanism to a scale of 6 in. = 1 ft., using the dimensions given. $O_2(1\frac{1}{2}, 7\frac{1}{2})$. Link 4 is in a vertical position. The angular velocity and angular acceleration of link 2 are respectively $\omega_2 = 45$ radians/sec. and $\alpha_2 = 300$ radians/sec.^{2}, both in clockwise sense.

a. Draw the velocity polygon. Scale, 1 in. = 10 f.p.s. $o(6\frac{1}{2}, 9)$. Find the velocity of C in f.p.s.

b. Draw the acceleration polygon employing the *analytical method*, as in Art. 174. Scale, 1 in. = 400 f.p.s.^{2} $o(3\frac{7}{8}, 5)$.

c. Draw the acceleration polygon, employing the *graphical method*, as in Art. 177. Scale (to be computed). $o(7\frac{1}{2}, 4)$. Find the magnitude of the acceleration of C in feet per second^2. Note: Arrange on the sheet, in neat

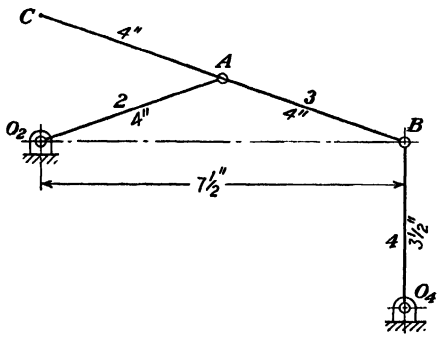


FIG. 379.

form, all the computations made in connection with the solution of this problem.

99. The drag-link mechanism of Fig. 23 is to be drawn to the dimensions given in Prob. 8, with link 2 making an angle of 30 deg. with the vertical center line, in the fourth quadrant. Scale, 3 in. = 1 ft. $O_2(2\frac{3}{4}, 3\frac{7}{8})$. Link 2 has a uniform angular velocity of $\omega_2 = 50$ r.p.m., counterclockwise.

a. Draw the velocity polygon. Scale, 1 in. = 1.25 f.p.s. $o\ (5\frac{1}{4}, 8\frac{1}{2})$.

b. Draw the acceleration polygon. Scale, 1 in. = 5 f.p.s.^{2} $o(6, 4)$.

c. What is the velocity of the slide (link 6) in feet per second? What is the acceleration in feet per second^2?

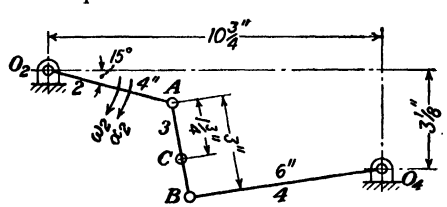


FIG. 380.

d. Assuming that the space and velocity scales are the same as in the analytical solution above, draw the acceleration polygon by means of the graphical method, as in Art. 177. $o(6, 1)$.

100. Using the dimensions given, draw the Watt straight-line mechanism (Fig. 380) to a scale of 6 in. = 1 ft. $O_2(1\frac{1}{2}, 7\frac{3}{4})$. The angular velocity and angular acceleration of link 2 are, respectively, $\omega_2 = 60$ radians/sec., and $\alpha_2 = 900$ radians/sec.^{2} both in a clockwise sense.

a. Draw the velocity polygon. Scale, 1 in. = 10 f.p.s. $o(2, 6)$.

b. Draw the acceleration polygon. Scale, 1 in. = 600 f.p.s.^{2} $o(6, 3)$.

c. Find acceleration of point C in feet per second^2.

101. The Scott-Russell straight-line mechanism is represented in Fig. 381. Draw the mechanism to the dimensions given and to a scale of 6 in. = 1 ft. $O_2(7, 6)$. If the angular velocity of link 2 is $\omega_2 = 40$ radians/sec., and the angular acceleration is $\alpha_2 = 400$ radians/sec.^{2}, both in clockwise sense, determine the following:

a. The velocity polygon. Scale, 1 in. = 10 f.p.s. $o(1\frac{1}{4}, 1\frac{1}{4})$.

b. The acceleration polygon. Scale, 1 in. = 400 f.p.s.^{2} $o(3\frac{3}{4}, 4\frac{3}{4})$.

102. Figure 356 represents the operating mechanism of the Wellman geared bucket. Link 3 as shown is a skeleton outline representation of the excavating bucket, the point C representing the cutting edge of the bucket. The mechanism is to be drawn to a scale of $1\frac{1}{2}$ in. = 1 ft., with link 2 in the 45-deg. position shown. $O_2(3\frac{1}{2}, 6)$. The angular velocity of link 2 is

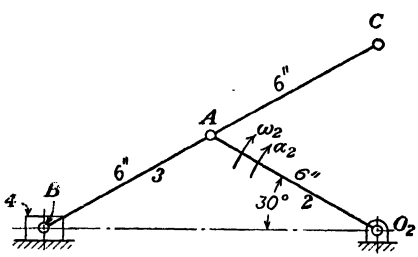


FIG. 381.

$\omega_2 = 40$ radians/sec., clockwise, and the angular acceleration is $\alpha_2 = 400$ radians/sec.^{2}, counterclockwise.

a. Draw the velocity polygon. Scale, 1 in. = 25 f.p.s. $o(7\frac{1}{2}, 3)$.

b. Draw the acceleration polygon. Scale, 1 in. = 500 f.p.s.^{2} $o(6\frac{1}{2}, 4\frac{3}{4})$.

103. Using the dimensions given in Fig. 353, draw the lever engine to a scale of 6 in. = 1 ft. $O_2(6\frac{1}{4}, 2\frac{1}{2})$. The crank, which in the position shown makes an angle of 30 deg. with the horizontal, is rotating at a uniform rate of 2,000 r.p.m., counterclockwise.

a. Find the velocity of the piston (link 6) by means of the velocity polygon. Scale, 1 in. = 20 f.p.s. $o(1\frac{5}{8}, 6\frac{1}{4})$.

b. Find the acceleration of the piston by means of the acceleration polygon. Scale, 1 in. = 2,000 f.p.s.^{2} $o(4\frac{1}{4}, 1\frac{1}{2})$.

104. In the tool-slide mechanism of Fig. 373, link 2 is the driving member that operates a reciprocating tool block attached to a sliding member that receives its motion through oscillating link 4. Draw the mechanism to the dimensions given with link 2 in the 45-deg. position shown. Scale, 6 in. = 1 ft. $O_2(6, 7)$.

The angular velocity of link 2 is $\omega_2 = 96$ radians/sec., and the angular acceleration is $\alpha_2 = 2{,}400$ radians/sec.^{2}, both counterclockwise.

a. Draw the velocity polygon. Scale, 1 in. = 4 f.p.s. $o(1, 3\frac{1}{2})$.

b. Draw the acceleration polygon. Scale, 1 in. = 200 f.p.s.^{2} $o(4\frac{1}{4}, 3\frac{1}{2})$.

105. In Fig. 382 is shown the skeleton outline of the operating lever of an air compressor. Draw the mechanism to a scale of 3 in. = 1 ft. $O_2(3, 5\frac{1}{2})$. The angular velocity of link 2 is $\omega_2 = 40$ radians/sec., and the angular acceleration is $\alpha_2 = 400$ radians/sec.^{2}; both clockwise.

a. Draw the velocity polygon. Scale, 1 in. = 10 f.p.s. $o(5\frac{1}{2}, 6\frac{1}{2})$.

b. Draw the acceleration polygon. Scale, 1 in. = 400 f.p.s.^{2} $o(5\frac{1}{2}, 3)$.

106. A gasoline engine runs at a speed of 2,800 r.p.m. Length of crank $OA = 2\frac{3}{4}$ in. clockwise rotation. Length of connecting rod $AB = 11$ in.

a. Draw the mechanism to a scale 6 in. = 1 ft., with the crank in the 30-deg. position before crank-end dead center, and construct the velocity and acceleration polygons by means of the Ritterhaus construction. $O_2(6, 7\frac{1}{2})$.

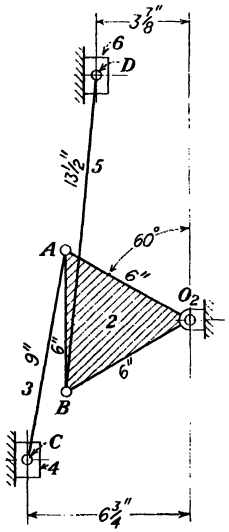


Fig. 382.

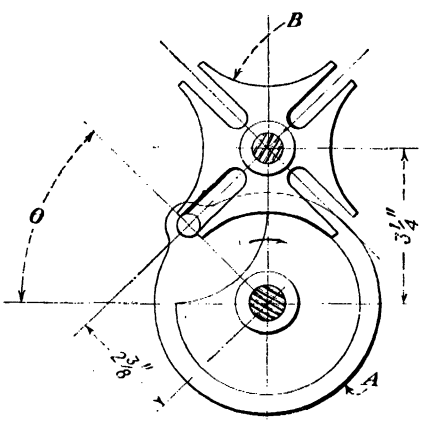


Fig. 383.

b. Determine the velocity and acceleration scales.

c. Construct the velocity and acceleration polygons, using the scales determined in (*b*) and the graphical method of Art. 177. Compare these polygons with those obtained in (*a*). Velocity polygon, $o(2\frac{1}{4}, 2)$. Acceleration polygon, $o(6\frac{1}{2}, 1\frac{1}{2})$. Redraw mechanism for part *c*. $O_2(6, 2\frac{3}{4})$.

107. The flywheel of an engine (Fig. 267) rotates with an angular velocity $\omega_2 = 30$ radians/sec. and an angular acceleration $\alpha_2 = 20$ radians/sec.^{2} The governor weight is moving outward relative to the wheel with a uniform velocity of 10 f.p.s. Find the acceleration of G, the center of gravity of the governor weight.

Calculate each component of the acceleration and find the resultant graphically. Note: Apply Coriolis' law.

108. Wheel A of the Geneva motion shown in Fig. 383 rotates at a uniform rate of 100 r.p.m. Determine the angular acceleration of wheel B. Note: Apply Coriolis' law.

109. The vane of a centrifugal pump (Fig. 384), rotates with a uniform angular velocity of 30 radians/sec. Using the dimensions given in the figure, find the acceleration of a particle of water P, which moves with a

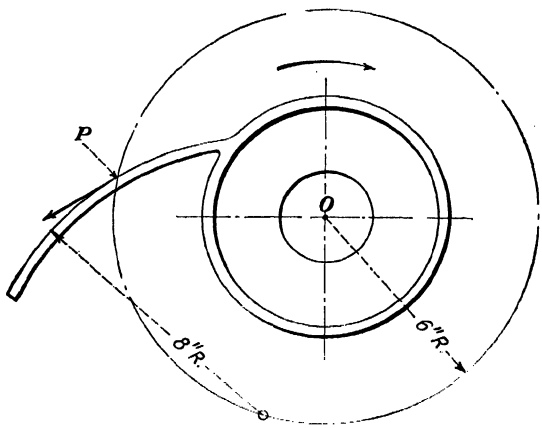


FIG. 384.

velocity along the vane of 20 f.p.s. and an acceleration along the vane of 100 f.p.s.^{2} Calculate each component of the acceleration and find the resultant graphically. Note: Apply Coriolis' law.

110. One of the buckets of a Pelton water wheel is represented in Fig. 385. Using the dimensions given, make a drawing of the bucket to a suitable scale. The velocity of the jet is 180 f.p.s. in a horizontal direction and the velocity of the bucket is assumed to be 0.4 of the velocity of the jet. Find the acceleration of the water at point P, using Coriolis' law. Note: Assume $A^t_{pa} = 600$ f.p.s.^{2}

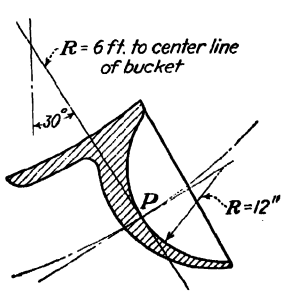


FIG. 385.

111. The driving link 3 of the Whitworth quick-return mechanism (Fig. 352) runs at a uniform speed of 80 r.p.m. Draw the mechanism to a scale of 3 in. = 1 ft., using the dimensions given in Prob. 12.

By the application of Coriolis' law, find the acceleration of the piston (link 7).

Chapter XII

STATIC FORCES IN MACHINES

112. In the Atkinson gas-engine mechanism, Fig. 15 (see Prob. 78 for dimensions) the force on the piston (link 6) is represented by a vector 2 in. long directed to the right. By means of the force polygon, determine the

vector length that represents the tangential force on the crankpin at A. Scale of drawing, 1½ in. = 1 ft. O_2(6¼, 7¼). Origin of vector P in force polygon, (1½, 6). Friction is to be neglected.

113. Draw the drag-link mechanism Fig. 23 to the dimensions given in Prob. 8, with the crank (link 2) making an angle of 30 deg. with the vertical, in the fourth quadrant, and having counterclockwise rotation. Scale, 3 in. = 1 ft. O_2(3¼, 4). The gear driving O_2A has its axis at O_2 and a diameter of 16½ in. The pinion has a diameter of 5½ in. and its axis lies on a horizontal line to the right of O_2. The pressure angle is 14½ deg.

If the force Q on the ram is represented by a vector 2 in. long, determine the length of vector that represents the normal force on the teeth of the driving pinion. Origin of vector Q in force polygon, (7¼, 7½). Neglect friction. Note: Draw the complete force polygon showing all forces properly labeled.

114. Draw the oil-circuit-breaker mechanism (Fig. 357) to the dimensions given. O_2A makes an angle of 45 deg. with the horizontal. O_2(½, 2½). Scale, full size. Assuming that vector Q (= 2 in.) directed upward represents the resistance at C, determine the length of vector P that represents the force applied in a horizontal direction at A. Origin of vector Q in force polygon, (6,2). Neglect friction.

115. For the steam-engine mechanism shown in Fig. 272 the following data are given: piston pressure P = 7,500 lb.; length of crank = 6 in.; length of connecting rod = 30 in.; diameter of crankshaft bearing = 5 in.; diameter of crankpin bearing = 3½ in.; diameter of wrist-pin bearing = 2½ in.

Using the methods of Arts. 186 and 192, determine the following for the 60-deg. crank position, shown in the figure (scale, 1 in. = 5 in.):

a. The turning effort T_o, disregarding friction.

b. The turning effort T, when friction is taken into account. Select coefficients of friction from Art. 189*d*.

c. The efficiency of the mechanism for the position shown.

Note: On account of the small size of friction circles, it is desirable to work with a full-size drawing in the solution of part (*b*) of this problem.

116. In Fig. 386 is shown the drawing of a 20-ton air riveter for which the following data are given: AB = 35 in.; AD = 6 in.; DE = 3¼ in.; DC = 17¼ in.; AC = 18 in.; BC = 21¼ in.; stroke of ram = 2½ in. The diameters of the bearings are as follows: at A, 2½ in.; at D and E, 2 in.; at C and B, 1½ in.

a. When the ram is 1½ in. from the upper end of its stroke, find by means of a graphical analysis, the piston pressure necessary to produce a pressure of 22½ tons at the end of the ram, neglecting friction. Scale, 2 in. = 1 ft. O_2(1½, 6). Scale of forces, 1 in. = 10,000 lb. Origin of vector representing ram pressure, (3, 2).

b. Make the force analysis outlined in (*a*) above when friction is taken into account. Select coefficients of friction from Art. 189*d*.

c. Determine the approximate efficiency of the mechanism.

Note: On account of the small size of friction circles, it is desirable to work with a full-size drawing in the solution of part (*b*) of this problem.

117. The following data apply to the "up-and-down-cut shear" shown in Fig. 387: O_2A = 7 in. and is inclined at an angle of 45 deg. with the horizontal. Point C is 2½ in. vertically below a horizontal line through O_2. AB = 26¾ in.; BC = 17 in.; CD = 9½ in.; BD = 25½ in.; CE = 18 in.; DF = 21¾ in. Scale, 1½ in. = 1 ft.

a. The resisting forces exerted by the billet upon links 8 and 9 are assumed to be represented by two equal and opposite vectors Q, 3 in. long, acting

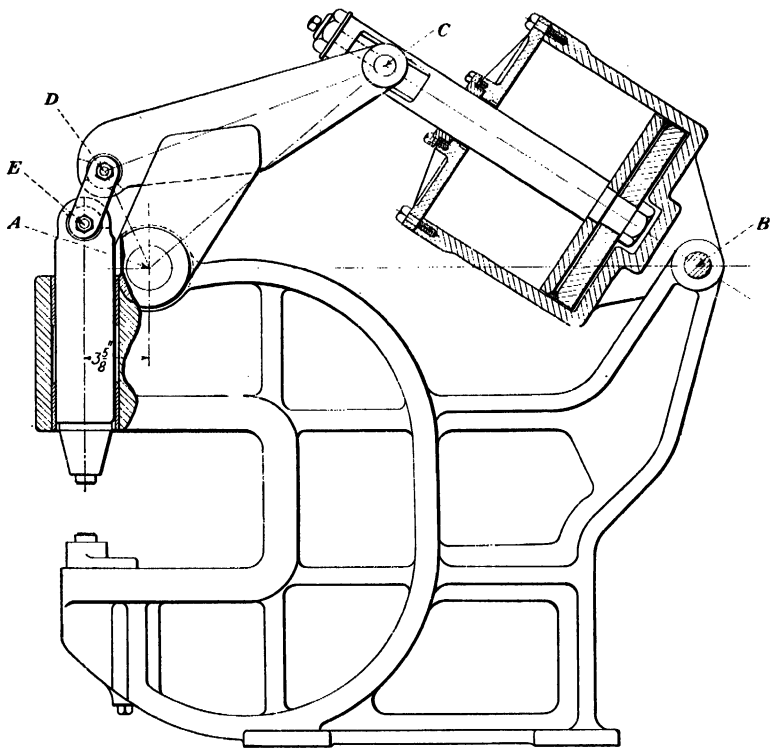


FIG. 386.

along the center line of the slide. Neglecting friction, determine the vector length that for equilibrium must be applied at A in a direction perpendicular to the crank O_2A. Origin of vector Q, (2¾, 2¼).

In the solution of the problem, the lower slide 8 together with links 4 and 6 can be assumed to be stationary. Therefore, at O_2(2, 7) draw the skeleton outline accordingly, and renumber the links, so that number 5 becomes number 4, number 7 becomes number 5, number 9 becomes number 6, and numbers 4, 6, and 8 become number 1.

b. Make the force analysis outlined in (*a*) above when friction is taken into account. Select coefficients of friction from Art. 189*d*. The diameter

of the bearings are as follows (Fig. 387): at O_2, A and E, 5 in.; at B, 3 in.; at C, D, and F, 6 in.

c. Determine the approximate efficiency of the mechanism.

Note: On account of the small size of friction circles, it is desirable to work with a full-size drawing in the solution of part (*b*) of this problem.

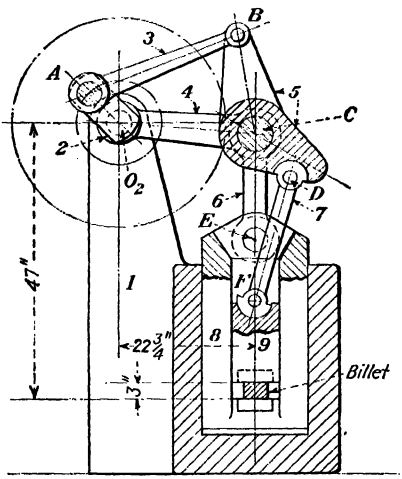


FIG. 387.

Chapter XIII

INERTIA FORCES

118. In this problem an inertia-force analysis of the four-link mechanism of Fig. 297 is required. Link 2 makes an angle of 60 deg. with the horizontal, $\omega_2 = 24$ radians/sec. and $\alpha_2 = 240$ radians/sec.^{2}, clockwise. The data are otherwise the same as given in Art. 204. Draw the skeleton outline of the mechanism to a scale of 3 in. = 1 ft., using the dimensions given in the figure. $O_2(1\frac{3}{4}, 4)$. In making this analysis, the following procedure is suggested: (*a*) Draw the velocity polygon, (scale, 1 in. = 20 f.p.s.). $o(6, 8\frac{1}{2})$. (*b*) Draw the acceleration polygon, (scale, 1 in. = 200 f.p.s.^{2}). $o(3, 9\frac{1}{2})$. (*c*) Determine the inertia forces and draw force polygon with origin of F_4 at $(2\frac{1}{2}, 1)$. Scale, 1 in. = 40 lb. See that all inertia forces and reactions are set up on the figure in magnitude, direction, and line of action, and that necessary computations are indicated in the lower right-hand part of sheet.

119. A six-cylinder gasoline engine is running at a uniform speed of 2,500 r.p.m. Length of crank $O_2A = 2\frac{1}{8}$ in.; length of connecting rod $AB = 8\frac{1}{8}$ in.; distance from A to center of gravity G of connecting rod = $1\frac{3}{16}$ in.; weight of piston = 1.94 lb.; weight of connecting rod = 1.75 lb.; equivalent unbalanced weight of crank at crank-pin radius = 3.25 lb.; time for 68 complete swings (over and back constitute one complete swing) of connecting rod about wrist-pin axis is 60 sec.

a. Make an inertia-force analysis of the mechanism when the crank is in the 60-deg. position. All inertia forces and reactions are to be determined, as in Art. 204. Procedure: Use two sheets. On sheet 1, make a drawing of the mechanism and show thereon all dimensions, weights, etc. Scale, ¾ in. = 1 in. $O_4(8, 7)$, piston to the left. Place all necessary computations on this sheet. On sheet 2 make a drawing of the mechanism and indicate thereon the inertia forces f_2, f_3 and f_4 in magnitude, direction and line of action. Scale of drawing, ¾ in. = 1 in. O_2 (8, 7), piston to left. Locate the several polygons on the sheet as follows: velocity polygon, $o(\frac{1}{2}, 3)$, scale 1 in. = 30 f.p.s.; acceleration polygon, $o(\frac{1}{2}, 6)$, 1 in. = 5,700 f.p.s.^{2}; inertia-force polygon, origin of vector $f_2(7, 1\frac{1}{2})$, scale, 1 in. = 300 lb.

b. Make a static-force analysis of the mechanism. The gas pressure on the piston at this position is 750 lb. Origin of vector $\mathfrak{F}_4$, $(\frac{1}{2}, 1\frac{1}{2})$. Scale, 1 in. = 300 lb.

c. By combining the inertia forces and static forces, determine the magnitudes and directions of the following total forces: force on *wrist pin;* force on *crankshaft;* force on *crankpin; turning effort.*

120. Same as Prob. 119, except that the solution is to be made for some other crank position.

121. A six-cylinder gasoline engine is running at a uniform speed of 2,500 r.p.m. Length of crank O_2A = 2⅛ in.; length of connecting rod AB = 8⅛ in.; distance from A to center of gravity of connecting rod = 1 13/16 in.; weight of piston = 1.94 lb.; weight of connecting rod = 1.75 lb.; equivalent unbalanced weight of crank at crank-pin radius = 3.25 lb.; time for 68 complete swings (over and back constitute one complete swing) of the connecting rod about the wrist-pin axis is 60 sec. (Note: The data given and the results required in this problem are the same as in Prob. 119. The procedure involves the substitution of a kinetically equivalent system for the connecting rod.)

a. Make an inertia-force analysis of the mechanism when the crank is in the 60-deg. position. All inertia forces and reactions are to be determined as in Art. 207. Procedure: Use two sheets. On sheet 1 make a drawing of the mechanism and show thereon all dimensions, weights, etc. Scale, ¾ in. = 1 in., O_2 (8, 7), piston to left. Place all necessary computations on this sheet. On sheet 2 make a drawing of the mechanism and indicate thereon the inertia forces f_2, f_c, f_b, and f_4 in magnitude, direction, and line of action. Scale of drawing, ¾ in. = 1 in., O_2 (8, 7), piston to the left. Locate the several polygons on the sheet as follows: velocity and acceleration polygons (use Ritterhaus construction); inertia-force polygon, origin of vector f_2 $(7, 1\frac{1}{2})$, scale, 1 in. = 300 lb.

b. Make a static-force analysis of the mechanism. The force on the piston due to gas pressure is 750 lb. Origin of vector $\mathfrak{F}_4$, $(\frac{1}{2}, 5)$. Scale 1 in. = 300 lb.

c. By combining the inertia forces and static forces, determine the magnitudes and directions of the following total forces: force on *wrist pin;* force on *crankshaft;* force on *crankpin; turning effort.*

122. Same as Prob. 121, except that the solution is to be made for some other crank position.

123. A six-cylinder gasoline engine is running at a speed of 2,500 r.p.m. Length of crank O_2A = 2⅛ in.; length of connecting rod AB = 8⅛ in., distance from A to center of gravity of connecting rod = 1 3/16 in.; weight of piston = 1.94 lb.; weight of connecting rod = 1.75 lb.; the crank is balanced; time for 68 complete swings (over and back constitute one complete swing) of the connecting rod is 60 sec. (Note: The data given in this problem and the results required are the same as in Problems 119 and 121. In this case, the crank is balanced. The inertia-force polygon will be superimposed on the Ritterhaus construction on an enlarged drawing of the mechanism.)

a. Make an inertia-force analysis of the mechanism when the crank is in the 60-deg. position. This analysis is to be similar to that in Art. 208, where the inertia-force polygon is superimposed on the Ritterhaus construction. Procedure: Use two sheets. On sheet 1 make a drawing of the mechanism and show thereon all dimensions, weights, etc. Scale, ¾ in. = 1 in. O_2 (8, 7), piston to the left. Place all necessary computations on this sheet. On sheet 2 make a drawing of the mechanism twice full size with the crank in the 60-deg. position. O_2 (6¼, 4¾), piston to the left. Determine the inertia forces and reactions by means of the force polygon superimposed on the Ritterhaus construction. Scale of forces, 1 in. = 300 lb.

b. Make a static-force analysis of the mechanism. The force on the piston due to gas pressure is 750 lb. Origin of vector $\mathfrak{F}_4$ is at A on the mechanism. Scale 1 in. = 300 lb.

c. By combining the inertia forces and static forces, determine the magnitudes and directions of the following total forces: force on *wrist pin;* force on *crankshaft;* force on *crankpin; turning effort.*

124. Same as Prob. 123, except that the solution is to be made for some other crank position.

125. *a.* Using the data and methods of Problem 123, plot a diagram of inertia forces of the piston on a base line 6 in. long, located 6⅝ in. from lower edge of sheet. Start with zero position of crank when piston is to the extreme left and make the determinations for each 30-deg. crank position during one revolution. For the end positions (0, 6, and 12) use the analytical methods of Art. 210. Scale, 1 in. = 300 lb.

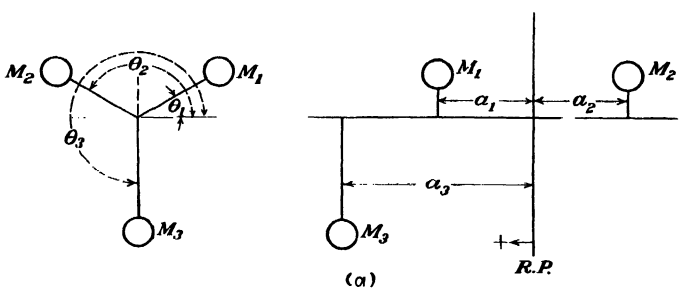
b. On a base line 6 in. long, located 2½ in. from lower edge of sheet, plot a diagram showing the variation of the turning effort at the crankpin during one revolution. (Note: These diagrams are similar to those shown in Figs. 340 and 342.) Procedure: Use two sheets. On sheet 1 make a drawing of the mechanism with piston to the left. O_2 (4⅝, 5¼). Scale, 1½ in. = 1 in. Draw in the mechanism for crank positions 1, 3, 5, 8, and 10 and determine the inertia forces and reactions for these positions by means of the force polygon superimposed on the Ritterhaus construction. Scale of forces, 1 in. = 300 lb. (It will be observed that the magnitudes of the inertia forces for crank positions 11, 9, 7, 4, and 2 will be the same as for positions 1, 3, 5, 8, and 10 respectively.)

126. In addition to the diagrams plotted in Prob. 125, other diagrams such as those described in the Appendix may be assigned for problem work as desired.

Chapter XIV

THE BALANCING OF MACHINERY

127. Three masses M_1, M_2, and M_3 (Fig. 388*a*), which revolve in the transverse planes 1, 2, and 3, are to be balanced by the addition of two revolving masses. Plane 3 is chosen as the plane of one of the masses $M_{o'}$, and the plane of the other mass M_o, is to be arbitrarily selected as the reference plane. Determine by graphical method values of M_o and $M_{o'}$ and the lengths and directions of the corresponding radii for kinetic balance. The accompanying table (Fig. 388*b*) gives the values of the masses in pounds, the length in inches, the directions of their radii, and their distances in inches from the reference plane. Scale of drawing, ¼ in. = 1 in. End elevation of axis, $o(2\frac{1}{2}, 8)$. Scale of couple polygon, 1 in. = 300 lb.-in. Scale of force polygon, 1 in. = 50 lb.



(*a*)

Plane	Mass	Radius	*a*	*θ*	*MR*	*MRa*
1	15	4	4	30°		
2	30	4	−4	150°		
3	15	5	8	270°		
	Assume		Assume			
0′	20		8			
	Assume					
0(R.P.)	30					

(*b*)

FIG. 388.

128. Same as Prob. 127 except that the solution is to be accomplished by means of two couple polygons.

129. Same as Prob. 127, except that the solution is to be accomplished by the application of the analytical method.

130. Same as Prob. 127, except that mass $M_2 = 20$, $R_2 = 5$, and $R_3 = 6$.

131. Same as Prob. 130, except that the solution is to be accomplished by means of two couple polygons.

132. Same as Prob. 130 except that the solution is to be accomplished by means of the analytical method.

133. The crankshaft of a horizontal stationary gas engine of the center-crank type carries two flywheels A and B, the planes of revolution of which are 3 ft. 8 in. apart. The plane of revolution of the crank is 18 in. from the plane of flywheel A. The equivalent weight of the crank arm at a radial distance of 10 in. from the crankshaft and in the plane of revolution of the crank is 110 lb. The crank is to be balanced by weights placed at a radial distance of 24 in. from the crankshaft, one in each flywheel. Find the magnitudes of the weights. Make a diagram of the general setup to a scale of 1½ in. = 1 ft. Start center line of crankshaft at (1½, 6).

134. A single-cylinder horizontal gas engine runs at a speed of 250 r.p.m. Length of crank = 10¼ in. Length of connecting rod = 51¼ in. Distance of center of gravity of connecting rod from crank end = 19.8 in. Equivalent unbalanced weight of crank at 10¼ in. = 159.5 lb. Weight of piston = 234 lb. Weight of connecting rod = 251 lb. Equivalent weight of counterbalance at 10¼ in. radius = 522 lb. Draw the skeleton outline of the mechanism to a scale ¼₀ size and indicate thereon the dimensions and weights. O_2(1½, 8½). Place all computations on this sheet. Using the methods outlined in Art. 222, plot to a scale of 1 in. = 3,000 lb. (on another sheet) the following diagrams of inertia forces. o(4¼, 5).

a. Assuming that no counterbalancing weight is used.

b. Assuming that a counterbalancing weight is used that is equal to the sum of all the rotating weight concentrated at the crankpin and all the reciprocating weight concentrated at the piston, the counterbalancing weight to be placed opposite the crank at a distance equal to the crank radius.

c. For the actual conditions existing in this engine, *i.e.*, when the given counterbalancing weight is used.

135. A single-cylinder horizontal oil engine runs at a speed of 250 r.p.m. Length of crank = 9 in.; length of connecting rod = 47⅞₆ in.; distance of center gravity of connecting rod from crank end = 16 in.; equivalent unbalanced weight of crank at 9-in. radius = 150 lb.; weight of piston = 198.5 lb.; weight of connecting rod = 194.6 lb.; equivalent weight of counterbalance at 9-in. radius = 410 lb.

Draw the skeleton outline of the mechanism to a scale of 1½ in. = 1 ft. and indicate thereon the dimensions and weights. O_2(7¼, 8½). Place all computations on this sheet.

Using the methods outlined in Art. 222, plot to a scale of 1 in. = 3,000 lb. (on another sheet) the following diagrams of inertia forces:

a. Assuming that no counterbalancing weight is used. o(4¼, 5).

b. Assuming that a counterbalancing weight is used that is equal to the sum of the rotating weight concentrated at the crankpin and the reciprocating weight concentrated at the piston, the counterbalancing weight to be placed opposite the crank at a distance equal to the crank radius.

c. For the actual conditions existing in this engine; *i.e.*, when the given counterbalancing weight is used.

136. A two-cylinder 80-hp. vertical Diesel engine running at 327 r.p.m. has the two cranks in the same axial plane on the same side of the crankshaft.

Length of crank = 6⅞ in.; length of connecting rod = 31¾ in., distance of center of gravity of connecting rod from crank end = 11.32 in.; equivalent unbalanced weight of crank at 6⅞ in. = 133 lb.; weight of piston = 194 lb.; weight of connecting rod = 146 lb.; equivalent weight of counterbalance at 6⅞-in. radius = 215 lb.

Draw the skeleton outline of the engine to a scale 2 in. = 1 ft. and indicate thereon the dimensions and weights. $O_2(2, 2\frac{1}{2})$. Draw crank in 45-deg. position. Place all computations on this sheet.

Using the methods outlined in Art. 222, plot to a scale of 1 in. = 3,000 lb. (on another sheet) the following diagrams of inertia forces per cylinder:

a. Assuming that no counterbalancing weight is used. $o(4\frac{1}{4}, 4\frac{3}{4})$.

b. Assuming that a counterbalancing weight is used that is equal to the sum of the rotating weight concentrated at the crankpin and the reciprocating weight concentrated at the piston, the counterbalancing weight to be placed opposite the crank at a distance equal to the crank radius.

c. For the actual conditions existing in this engine; *i.e.*, when the given counterbalancing weight is used.

137. The arrangement of cranks in an eight-cylinder 16 by 20 vertical Diesel engine is shown diagrammatically in Fig. 389, with crank 1 making any angle θ with the vertical center line.

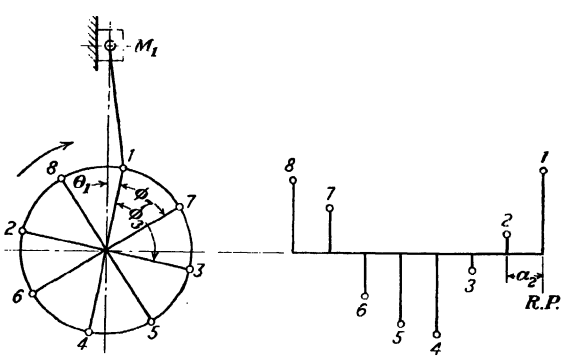


FIG. 389.

a. Assuming ω constant and M, R, and a each equal to unity, determine by the methods illustrated in Arts. 223 to 230 what forces or couples are unbalanced.

b. The angular velocity of this engine is 300 r.p.m. Total reciprocating weight per cylinder = 984 lb. The distance between cranks is 30 in. Determine the magnitude of the unbalanced couple for positions of $\theta_1 = 0$, 45, 90, etc., to 360 deg. and plot a graph showing its variation in magnitude. Base line of graph, $0(2\frac{1}{4}, 1\frac{1}{4})$; length 4 in.; scale of ordinates 1 in. = 25,000 lb.-ft. (Length of connecting rod = 54 in.).

138. Applying the methods of Arts. 223 to 230, make a study of the balance of the 90-deg. V-8 engine referred to in Fig. 332, showing what forces or couples are unbalanced and what may be done to put the engine in

balance. For the purpose of this analysis, it will be convenient to assume that one of the groups of cylinders is in a vertical position on the sheet.

139. The data given below apply to a six-cylinder gasoline engine of the automotive type. A complete force analysis of this engine is required, in accordance with the method outlined in the Appendix.

Dimensions:		
R.p.m. at maximum horsepower (80 hp.)	3,300	
Number of cylinders	6	
Diameter of cylinders	$3\frac{5}{16}$	in.
Stroke	4	in.
Length of connecting rod, center to center	$7\frac{1}{2}$	in.
Distance of center of gravity of connecting rod from crank-pin end	$2\frac{3}{8}$	in.
Diameter of wrist pin	1	in.
Diameter of crankpin	$2\frac{1}{8}$	in.
Diameter of main bearing	$2\frac{3}{16}$	in.
Center of gravity of unbalanced rotating weight of each crank from center line of crankshaft (when cranks are not balanced)	2	in.
Time for 50 complete swings (over and back constitutes one complete swing) of connecting rod about center of wrist-pin end is	43	sec.
Weights:		
Connecting rod, complete (including bushings)	1.99	lb.
Piston, complete (including wrist pin, rings, etc.)	2.36	lb.
Estimated unbalanced weight of each crank (when cranks are not balanced)	2.25	lb.
Timing:		
Inlet valve opens 4 deg. before head-end dead center.		
Inlet valve closes 34 deg. after crank-end dead center.		
Exhaust valve opens 47 deg. before crank-end dead center.		
Exhaust valve closes 4 deg. after head-end dead center.		
Indicator diagram:		
Compression volume, 22.5 per cent of piston displacement.		
Maximum explosion pressure, 375 lb. per sq. in. abs.		
Pressure rise completed at 2 per cent of stroke.		
Development of expansion curve begins at 5 per cent of stroke.		
Mean suction pressure, 12 lb. per sq. in. abs.		
Mean exhaust pressure, 16 lb. per sq. in. abs.		
Pressure at beginning of compression, 12 lb. per sq. in.		
Exponent for expansion curve	1.3	
Exponent for compression curve	1.3	
Atmospheric pressure 14.7 lb. per sq. in. abs.		

140. The data given below apply to an eight-cylinder gasoline engine of the automotive type. A complete force analysis of this engine is required, in accordance with the method outlined in the Appendix.

Dimensions		
R.p.m.	3,000	
Number of cylinders	8	
Diameter of cylinders	$3\frac{3}{16}$	in.
Stroke	$3\frac{1}{2}$	in.
Length of connecting rod, center to center	$7\frac{11}{16}$	in.
Distance of center of gravity of connecting rod from crankpin end	1.75	in.
Diameter of wrist pin	$1\frac{5}{16}$	in.
Diameter of crankpin	2	in.
Diameter of main bearing	$2\frac{5}{16}$	in.
Center of gravity of unbalanced rotating weight of each crank from center line of crankshaft (when cranks are not balanced)	$1\frac{3}{4}$	in.
Time for 50 complete swings (over and back constitutes one complete swing) of connecting rod about center of wrist-pin end is	42	sec.
Weights:		
Connecting rod, complete (including bushings)	2.03	lb.
Piston, complete (including wrist pin, rings, etc.)	2.13	lb.
Estimated unbalanced weight of each crank (when cranks are not balanced)	2.00	lb.
Timing:		
Inlet valve opens 5 deg. after head-end dead center.		
Inlet valve closes 39 deg. after crank-end dead center.		
Exhaust valve opens 45 deg. before crank-end dead center.		
Exhaust valve closes 15 deg. after head-end dead center.		
Indicator diagram:		
Compression volume, 21.2 per cent of piston displacement.		
Maximum explosion pressure, 375 lb. per sq. in. abs.		
Pressure rise completed at 2 per cent of stroke.		
Development of expansion curve begins at 5 per cent of stroke.		
Mean suction pressure, 12 lb. per sq. in. abs.		
Mean exhaust pressure, 16 lb. per sq. in. abs.		
Pressure at beginning of compression, 12 lb. per sq. in.		
Exponent for expansion curve	1.3	
Exponent for compression curve	1.3	
Atmospheric pressure 14.7 lb. per sq. in. abs.		

141. Secure data similar to that in Prob. 140 for an eight-cylinder two-cycle Diesel engine, and make a complete force analysis of the engine in accordance with the method outlined in the Appendix.

INDEX

A

Acceleration, analysis of four-link mechanism, 278, 286
analysis of slider-crank mechanism, 281, 288, 299, 304
normal and tangential components of, 270
of a point, 268
polygon, 276
relative, 271–276, 291–298
Ritterhaus construction, 289
scale relations for graphical work, 283
Accelerations, 418
analytical methods, 304
in machines, 268–305, 453
American standard gear tooth forms (A.S.A.; A.G.M.A.), 135–140
Analytical methods, 264, 304, 357 370
Angular motion, 225, 231
Angular velocity, 226
by method of instantaneous centers, 255
by method of relative velocities, 232
relative, 227
Angular-velocity ratio, 80
of belt drive, 153–155
of gear drive, 90
of pair of curves in contact, 81
Appendix, 413–427
Automobile rear-axle drive, 211
Automobile transmission, 205–207

B

Balancing, of connecting rod, 401
general analytical method, 370
general graphical method, 366
Balancing, of machinery, 361–411. 465
of reciprocating masses, 373–402
of rotating masses in different transverse planes, 365
of single rotating mass, 362
Balancing machines, 403–411
Base curves for cams, 58–60
Bell crank, friction analysis, 315
static-force analysis, 307
Belt, cone pulleys, 158–160, 444
connections between nonparallel shafts, 160–161
drive, velocity ratio of, 153–154
length of, 156–158
open and crossed belt reversing mechanism, 185
tensions, 155
V-belt drives, 162
Bevel gears, 115–123, 443
angular, 115
classification of, 115
Gleason system, 120, 134, 140–143
hypoid, 115, 122
layout of, 118–120
nomenclature, 117, 118
as a reversing mechanism, 185
skew, 115, 122
spiral (which see), 115, 120–122 (*See also* Spiral bevel gears)
straight-tooth, 115, 118
teeth, form of, 116
Tredgold's approximation, 117
Brakes, 189
Brown and Sharpe gear-tooth system, 131–133
Burmester's method, 160

C

Cams, 53–77, 438
base curves, 58–60, 64

Cams, cylindrical, 74
disk, 53–75
displacement diagrams, 56–58
friction analysis, 320
inertia-force analysis, 355
layout, fundamental procedure, 54–56
minimum size of, 63, 68
miscellaneous types of, 73–77
pivoted flat-faced follower, 71, 72
pivoted roller follower, 69–71
sliding flat-faced follower, 56, 67–69
sliding roller follower, 54, 61–67, 75
spherical, 77
velocity analysis, 235, 256
Centrodes, 260
Chains, kinematic, 6
hoisting and conveying, 166, 444
power-transmission, 166–168, 444
Chamber trains, 35, 36
Chamber-wheel mechanisms, 35–36
Clutches, 186–189
Cone pulleys, 158–160
Connecting rod, balancing of, 401
of finite length, 26
inertia effects of, 378
inertia forces, 417
of infinite length, 25
Coriolis' law, 291–298
governor mechanism, 298
shaper mechanism, 302
slider-crank mechanism, 299
Corliss valve gear, 20, 46
nonreleasing, 46, 432, 433
Crank, balancing of, 400
inertia effect of, 378
Crankpin pressures, 424
Crankshaft, balancing of, 400
inertia effect of, 378
Crank-shaper mechanism, 8, 33
acceleration and inertia force of ram, 244
acceleration analysis of, 302
velocity analysis of, 238–244, 267
velocity-space diagram of, 244
Crank-shaper mechanism, velocity-time diagram of, 243
Critical speeds, 403
Curves, in direct contact, 78–86
Cycloid, 103, 104
Cycloidal gears, 103–114
advantages of, 113
character of action, 105–107
construction of curves, 103, 104
describing circles, 103, 106, 109
interchangeable, 110
internal, 111
layout of, 107–109
pin-gearing, 112
rack and pinion, 111

D

Dead points, 18
Definitions, 1, 2
Diagram of shaking forces, 425
Differential chain hoist, 171
Displacement, angular, 226
combined, 221, 222
of point, 219
relative, of two points, 223
of rigid body, 229
Displacement diagrams, 13–15, 56–58, 430
for cams, 56–58
for Geneva wheel, 181
Drag-link mechanism, 21–23
displacement diagram, 22
static-force analysis, 310

E

Efficiency, 316
Epicyclic gears, bevel, 211, 448
compound-train, 209, 447–449
as reversing mechanism, 184
simple-train, 207
triplex-hoist, 210
velocity-analysis, by instantaneous center method, 258–260
by tabular method, 207–215
Escapements, 175

F

Fellows stub-tooth gears, 133
Force analysis of gasoline engine, 413–427
Four-bar mechanism, 17–21
  acceleration analysis, 278, 286
  dead points, 18
  inertia-force analysis, 377
  velocity analysis, method of instantaneous centers, 254
    method of relative velocities, 232–234
Friction, coefficients of, 315
  rolling, 326
  sliding, 311
  turning, 313
  wrapping connectors, 327
Friction gearing, 86, 87

G

Gear teeth, action of, fundamental law of, 91
  bevel, form of, 116
  comparison of forms, 113
  definitions, 88–91
  methods of producing, 145–152
  usual forms, 91
Gear-tooth systems, 131–145
  American standard (A.S.A.; A.G.M.A.), 135–140
  Brown and Sharpe, 131–133
  Fellows stub-tooth, 133
  Gleason, 134
  Maag unequal addendum, 134
Gear trains, 191–215
  automobile transmission, 205–207
  change gears, 202–205
  direction of rotation, 197
  epicyclic, 207–212, 258–260
  hunting-tooth, 200
  lathe, 191, 203–205
  nonparallel axes, 198
  selection of gears for, 199, 200–202
  speed reducers, 212–215
  velocity ratio of, 195–197
Gears and gearing, bevel, 115–123
  Citröen, 126, 434
  classification of, 88
  cycloidal, 103–114
    (*See also* Cycloidal gears)
  definitions, 88–91
  efficiency of, 323
  epicyclic, 184, 207–212, 215, 258
  friction, 86, 87
  friction analysis of, 320
  helical, 123–127
    (*See also* Helical gears)
  intermittent, 175–177
  involute, 92–103
    (*See also* Involute gears)
  miter, 115
  spur, 78–114, 441
  valve, 20, 21, 45–48
  velocity analysis of, 235, 236
  worm, 127–130
    (*See also* Worm gearing)
  (*See also* Bevel gears)
Geneva wheel, 176, 180–183, 458
Gleason bevel-gear system, 134, 140–143
Governors, 48–52
  application of Coriolis' law to, 298

H

Harmonic motion, 25
Helical gears, 123–127, 443
  advantages of, 126
  herringbone, 125
  on nonparallel, nonintersecting shafts, 126, 127
  normal pitch, 125
  on parallel shafts, 123–126
  tooth proportions, 139
Herringbone gears, 125
  tooth proportions of, 139
Hoisting tackle, 170–172
Hunting tooth, 200
Hyperboloids, 85
Hypoid gears, 122

I

Idler gear, 197, 198
Indexing devices, 177–183

Indicator diagram, 414, 415, 419
Indicator mechanisms, 41, 437
Inertia forces, 329–332, 416, 462
  analytical methods, 357
  in cam mechanism, 355
  combined static and inertia-force analysis, 342
  connecting-rod, 417
  of floating link, 332
  in four-link mechanism, 337
  of link rotating about fixed center, 334
  piston, 416
  in slider-crank mechanism, 347, 352, 358
  transverse and radial components, 336
Instantaneous acceleration, 268
  angular velocity, 226
  centers, location of, 251, 253
    method of, 244–261
    of pair of links, 248
    of rotation, 246
  velocity, 220
Interchangeable gears, cycloidal, 110
  involute, 101
Interference, 98
Intermittent gearing, 175–177, 444
Inversion, 11
  of pairs, kinematic chains and mechanisms, 12
  of slider-crank mechanism, 30–33
Inversions, 11, 55
Involute, 92–94
Involute gears, 92–103
  advantages of, 113
  character of action, 96, 97
  construction of involute, 93, 94
  interchangeable, 101
  interference in, 98
  internal, 102
  layout of, 99–101
  rack and pinion, 101
Isosceles linkage, 29, 30

K

Kinematic chain, 6
  inversion of, 11
Kinetically equivalent system, 345
  application to slider-crank mechanism, 347

L

Lathe gear train, 191–193
Law of three centers, 252
Linkage, isosceles, 29, 30
  parallel, 41, 42
Links, 5
Linkwork, 17–52, 431

M

Machine, 1, 6
  motion, 2–4
  representation, 7
Main-bearing pressures, 424
Mechanical advantage, 243
Mechanism, 1, 6
  drag-link, 21–23
  four-bar, 17, 19, 20
  four-link, 17
  inversion of, 11
  trains of, 191–215, 445
Mechanisms, indicator, 41
  intermittent motion, 173–190
  quick-return, 33, 34
  link, 17–52
  slider-crank, 23–26
  straight-line, 37, 38, 40
Method, of instantaneous centers, 244–261
    cams and gears, 256
    epicyclic gear trains, 258
    four-link mechanism, 254
    law of three centers, 252
    location of centers, 251, 253
    slider-crank mechanism, 255
  of relative velocities, 229–244
    cams and gears, 235
    crank shaper, 238–244
    determination of angular velocity by, 232
    four-link mechanism, 232
    points on same link, 231
    toggle press, 235

Moment of inertia, experimental determination of, 335
Motion, continuous, intermittent, or reciprocating, 4
cycle, period, and phase of, 3
harmonic, 25
plane, helical, and spherical, 2, 3

O

Oldham coupling, 34, 35

P

Pairing elements, 4
Pairs, 4
expansion of, 9–11
inversion of, 11
lower and higher, 4, 5
Pantograph, 38–40
Peaucellier mechanism, 37
Pin gearing, 112
Pin pressures, 423
Planetary (epicyclic) gearing, 207–215, 446–449
Point paths, 13
Position, definition, 219
Positive blowers, 36, 442, 443
Problems, 429–469
Pulleys, cone, 158
crowning of, 161
for quarter-turn drive, 160
for V-belt, 163
Pure rolling, 80
centrodes, 260, 261
condition for, 79, 80
curves that give, 82–86

Q

Quick-return mechanisms, 33, 34
crank-shaper, 33
drag-link, 21–23
Whitworth, 32, 33

R

Ratchet mechanisms, 174
Reducing motions, 35
Relative acceleration, of independent points, 271
of two points on same rigid body, 273
of pair of coincident points on two rigid bodies, 291
Relative angular velocity, 227
Relative velocities, method of, 229–244
Relative velocity of two points, 224
Reversing mechanisms, 183–186
bevel-gear, 184
epicyclic-gear, 184
idler-gear, 183
open and crossed belts, 185
Revolved velocity polygon, 240
Ritterhaus construction, 289, 416
Rolling cones, 83–85
Rolling cylinders, 83
Rolling hyperboloids, 85, 86
Root positive blowers, 36, 442, 443
Rope drives, 163–165

S

Scale relations, for graphical acceleration analyses, 283
Scotch yoke, 25, 27
Screw cutting, selection of gears for, 204
Screw threads, friction in, and efficiency of, 325
Shaking forces, diagram of, 425
Skeleton outline, 7–9
Slider-crank mechanism, 23–28
acceleration analysis of, 281, 288, 299, 304
balancing of, 373–402
combined static- and inertia-force analysis, 342, 347
connecting rod of infinite length, 25
force analysis, 342, 347
friction analysis, 317
inertia forces in, 347, 352, 358
inversions of, 30–33
static-force analysis, 308
toggle mechanisms, 28

Slider-crank mechanism, variations of, 24
velocity analysis of, 255, 264
Speed reducers, 212–215, 446
Spiral bevel gears, 120–122
Spur gears, 78–114, 441
Static forces, 306–328, 459
analysis of, 306
combined static- and inertia-force analysis, 342
effect of friction, 311
transverse and radial components, 336
Steam-engine mechanism, 4, 5
friction analysis, 317
static-force analysis, 308
Stephenson valve gear, 47
Stone crusher, 29, 319
friction analysis, 319
Straight-line mechanisms, 37, 38
applications of, 40
Peaucellier, 37
Roberts, 455
Scott-Russel, 457
Watt, 37
Stub-tooth gears, 133

T

Time ratio, of quick-return mechanisms, 34
Toggle mechanisms, 28
Toggle press, velocity analysis of, 235, 236
Trains of mechanism, 191–215, 445
Transverse and radial forces, 336
Tredgold's approximation, 117
Triplex hoist, 210
Tumbler gears, 198

U

Universal joint, 42–45

V

Valve gear (or gears), 45–48
Baker, 48
Corliss, 20, 21, 46
locomotive, 47–48
Southern, 48
steam-engine, 45, 46
Stephenson, 47, 48
Walschaert, 47, 48
Young, 48
Variable-speed transmissions, 168–170
V-belt drives, 162
Vectors, 216–218
Velocities, in machines, 216–267, 449
method of relative, 229–244
Velocity, analytical methods, 264
combined, 222
instantaneous, 220
of a point, 220
polygon, 233, 240
relative, of two points, 224
space diagram, 242
time diagram, 243
Velocity ratio, angular, 80
of belt drive, 153
of cone pulley drive, 158
of epicyclic gear trains, 207–212
of gear trains, 195
instantaneous center method, 255–257
of pair of gears, 90
Vibrations, 401, 402

W

Walschaert valve gear, 47
Watt straight-line mechanism, 37
Whitworth quick-return mechanism, 32, 434
Worm gearing, 127–130, 443
A.G.M.A. standard, 140, 144
Wrist-pin pressures, 423