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HIGHER ELEMENTARY GEOMETRY

(AN INTRODUCTION)

For

PRE-UNIVERSITY STUDENTS

VENUGOPAL

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PREFACE.

This geometry text is written for the Pre-university students and also to serve as an introduction to Higher Elementary Geometry for the three year degree students offering Geometry either as a subsidiary or as a main subject. I believe that the elementary is the most vital and the object of writing this book is to give to the students certain Elementary ideas about the subject. Endeavours have been made to make everything as simple as possible without losing rigour.

A good number of worked examples are given to illustrate the theorems. As there are only a few theorems all exercises are given together at the end. Students can try these riders after they finish reading the entire text.

I hope that this text book will lay down a good foundation for students who intend to pursue their studies in geometry. It is my sincere wish that students must acquire that taste for Geometry without which the Greeks thought—and rightly, in my opinion,—that there is no real culture. If the book is found to be suitable for the class of students for whom it is intended the author will feel amply rewarded.

For any corrections and suggestions for the improvement of the book, I shall be thankful.

VALSARAJ VILLA,
ILAKKÓOL, TELLICHERRY.
August 15, 1958.

K. C. VENUGOPAL.

NOTE.

(As per Government Regulations)

This is neither an official nor an officially-sponsored Publication. The author is working in Government Brennen College, Tellicherry, as Tutor in Mathematics.

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HIGHER ELEMENTARY GEOMETRY

(AN INTRODUCTION)

Chapter 1

PLANE FIGURES.

Plane figures are figures lying in a plane and not in a curved surface. Every plane figure with which we deal in Geometry is only an aggregate of points, though the point in itself is undefined. However a vague definition of a point can be given. A point is a geometrical entity which has position and nothing else. From points we come to curves. A curve is a collection of infinite (a very large) number of points arranged closely one after the other. We easily see that this definition of a curve also applies to circles and straight lines. So we conclude that circles and straight lines are particular cases of curves. In order to get a circle the points must be arranged in a particular order. If all the points taken are equidistant from a fixed point, we get a circle. Similarly to get a straight line the points must be arranged in another particular order. But this particular order cannot be easily described, though

we are sure that a straight line can be obtained by arranging a number of points in a particular order. As a matter of fact straight line is only a special case of a circle. If we take a piece of wire of a definite length in the form of an arc of a circle and also a number of such pieces having the same shape and size it will be possible for us to get a complete circle of which each such piece is a part, by placing the pieces suitably on a table. This is the principle adopted by masons in shaping the stones for the purpose of erecting the round walls of a well. In fact, the circle mentioned above can be generated by one piece of such wire, shifting it from place to place in a particular order, till the complete circle is obtained. Now, by decreasing the curvature (amount of bending) of this piece (i. e., by increasing the length of the bounding chord of the arc formed out of this piece of wire) we can generate a circle of radius greater than that of the previous one. Thus, as the curvature decreases, radius of the circle generated increases. Hence a circle of infinite radius can be generated with the help of the same piece of wire, by making its curvature zero i. e. by making it straight (a straight line). So this time the piece of the wire in the form of a straight line is a part of a very great circle. This shows that a straight line is a circle of infinite radius. If a mason uses

ordinary stones which are used for ordinary buildings, he will be actually erecting the round wall of a well of infinite radius. In this case, if he starts construction from a particular point, he won't be able to come back to the original position, completing the construction.

It is because a straight line is a particular case of a circle, we find certain properties, common to circles and straight lines. For example, the student can compare Apollonius' theorem mentioned elsewhere in this text, and the elementary theorem that the locus of a point equidistant from two fixed points A, B is the perpendicular bisector of AB.

If we have a piece of wire in the form of any curve, at our disposal, we can make it either in the form of a circle or in the form of a straight line according as we wish. This itself is sufficient to show that circles and straight lines are both particular cases of curves. A curve may be a closed curve (like the circle) or may not be a closed one. If it is a closed one we can talk of its area. The area of a plane closed curve is the superficial space whose boundary is the closed curve. So we can talk of the area of a circle, for it is a closed curve.

In Elementary Geometry we deal only with the two particular cases of curves, viz., circle and straight line and the compound figures formed out of these two independently or together. Straight lines independently form what is known as Polygons. We shall discuss the different types of polygons in the following paragraphs.

Polygon: — A polygon is a geometrical figure formed by any number of straight lines. The point of intersection of two adjacent straight lines forming a polygon is called a vertex of the polygon. The segment, between two consecutive vertices, of a straight line forming a polygon is called a side. Line joining any two vertices other than consecutive vertices is called a diagonal of the polygon. The area of a polygon is the superficial space whose boundaries are the sides of the polygon.

A polygon in which each angle is less than two right angles is called a convex polygon. In a polygon if any angle is greater than two right angles it is called a re-entrant polygon.

A polygon in which all the sides are equal is called an equilateral polygon. If all the angles are equal in a polygon it is called an equiangular

polygon. A regular polygon is one in which both these conditions are satisfied, viz, sides are equal and angles are equal.

Two or more polygons are said to be similar if (1) their angles are equal and (2) their corresponding sides are proportional. Two or more polygons are said to be similar and similarly situated or homothetic if (1) their angles are equal, (2) corresponding sides are proportional and (3) corresponding sides are parallel.

Polygon of sides 10, 9, 8, 7, 6, 5 are respectively called Decagon, Nonagon, Octagon, Heptagon, Hexagon and Pentagon. If these are also regular, they will be called Regular Decagon Regular Nonagon etc.

A polygon of four sides is known as a quadrilateral. If one pair of opposite sides of a quadrilateral are parallel it is called a Trapezium. If the two pairs of opposite sides are parallel, the quadrilateral is called a parallelogram. If all the sides of a parallelogram are equal it is called a Rhombus. If all the angles of a parallelogram are equal (right angles) it is called a rectangle. If all the sides of a rectangle are equal it is called a square. It is easy to see that Rhombus and

rectangle are special cases of a parallelogram and that square is a special special case of a parallelogram.

A polygon of three sides is called a Triangle. From the definition of equilateral polygons, equiangular polygons etc. it follows that a triangle is equilateral if its sides are equal, equiangular if its angles are equal etc. (Note that equilateral triangles are equiangular and that equiangular triangles are equilateral). If two sides of a triangle are equal (or if two angles of a triangle are equal) it is known as an isosceles triangle. A triangle is said to be (1) acute angled if each angle is less than a right angle (2) right angled if one of the angles is a right angle (3) obtuse angled if one of the angles is greater than a right angle.

The side facing the right angle is called the hypotenuse of the right angled triangle. Hypotenuse is the unique feature of a right angled triangle and so if mention is made of a hypotenuse of a triangle it will follow that the triangle in question is a right angled triangle, the angle opposite to the hypotenuse being necessarily a right angle.

Two or more Geometrical figures are said to be congruent if they agree in shape as well as in

size. Hence if there are two congruent figures one can be completely superposed on the other. Students will note that two similar figures agree in shape only. So if there are two similar figures one cannot be superposed on the other. But two similar figures can be placed such that their corresponding sides are parallel, if they are not already so, by rotating one of them. Then they become similar and similarly situated figures. Therefore if the positions of two similar figures are not given, they can always be made homothetic or similar and similarly situated. But the student must realise that in Geometry position is also very often important and that it is only the position that draws a line of demarcation between similar figures and homothetic figures.

As an exercise, students are advised to draw all Geometrical figures one after the other, strictly following the definition of each given above.

Chapter 2

RATIO AND PROPORTION.

Ratio.— Ratio is merely a relation between two quantities of the same kind, showing how

many times one quantity is greater than the other, both of them being measured in the same unit of measurement.

If A B and C D are two lengths, A B being equal to 4 inches and C D equal to

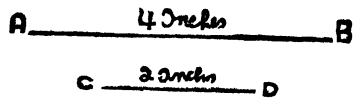


Fig. 1 (a)

2 inches the ratio of A B to C D (usually written as $\frac{AB}{CD}$) is $\frac{4}{2}$ or 4:2 i e the ratio is $\frac{2}{1}$ or 2:1. This shows that A B is two times greater than C D.

In general if the length A B is a inches and the length C D is b inches the ratio of A B to C D is $\frac{a}{b}$ or a:b. This relation shows that A B is $\left(\frac{a}{b}\right)$ times greater than C D. for $a = \left(\frac{a}{b}\right) b$ and hence $AB = \left(\frac{a}{b}\right) CD$. Here a and b are called the terms of the ratio $\frac{a}{b}$.

If a and b are both multiples of the same quantity, say, k so that $a = pk$ and $b = qk$, the ratio of A B to C D $\left(\frac{AB}{CD}\right)$ becomes $\frac{a}{b} = \frac{pk}{qk} = \frac{p}{q}$. So the ratio of A B to C D is $\frac{p}{q}$. p and q are called the terms of the ratio $\frac{p}{q}$. From this it is

clear that it is always customary to express a ratio in its simplest form.

If A B is a inches and C D is b centimeters the ratio of A B to C D is not $\frac{a}{b}$ for by definition the lengths must be measured in the same unit of measurement.

Also, if A B is a length equal to a inches and P Q R is a triangle whose area is equal to b square inches we cannot talk of the ratio between A B and Δ P Q R, since by definition for the existence of a ratio the two quantities must be of the same kind.

Proportion:— If two ratios are equal the four terms taken in order are called proportionals and are said to be in proportion.

If $\frac{a}{b} = \frac{c}{d}$ a, b, c, d are proportionals. The proportion is written as a:b::c:d and is read “a is to b as c is to d” Here b and c are called the means and a and d are called extremes of the proportion. d is called the fourth proportional to a, b and c.

If a, b, c are connected by the relation $\frac{a}{b} = \frac{b}{c}$ [$b^2 = ac$] b is called the mean proportional or geometric mean between a and c and c is called

the third proportional to a and b. Also a, b, c in this case are said to be in continued proportion.

If a, b, c, d are connected by the relation $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ then a, b, c, d are said to be in continued proportion and so on.

SIMPLE RESULTS IN RATIO AND PROPORTION.

(1) If $\frac{a}{b} = \frac{c}{d}$ each ratio is equal to $\frac{a+c}{b+d}$ Let $\frac{a}{b} = \frac{c}{d} = k$ so that $a = bk$ and $c = dk$ then to prove that $\frac{a+c}{b+d}$ is also equal to k.

$$a = bk$$

$$c = dk$$

adding $a + c = k(b + d)$

$$\therefore \frac{a+c}{b+d} = k = \frac{a}{b} = \frac{c}{d}$$

In the following results also, the method of proof will be the same and therefore the proofs are left to the students.

(2) If $\frac{a}{b} = \frac{c}{d}$ each ratio is equal to $\frac{a-c}{b-d}$

(3) If $\frac{a}{b} = \frac{c}{d}$ each ratio is equal to $\sqrt{\frac{ac}{bd}}$.

(4) If $\frac{a}{b} = \frac{c}{d}$ each ratio is equal to $\sqrt{\frac{a^2 + c^2}{b^2 + d^2}}$.

(5) If $\frac{a}{b} = \frac{c}{d}$, $\frac{a}{c} = \frac{b}{d}$ (Alternando)

for $\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$ i. e. $\frac{a}{c} = \frac{b}{d}$.

(6) If $\frac{a}{b} = \frac{c}{d}$, $\frac{b}{a} = \frac{d}{c}$ (Invertendo)

for $\frac{1}{\frac{a}{b}} = \frac{1}{\frac{c}{d}}$ i. e. $\frac{b}{a} = \frac{d}{c}$

(7) If $\frac{a}{b} = \frac{c}{d}$, $\frac{a+b}{b} = \frac{c+d}{d}$ (Componendo)

for $\frac{a}{b} + 1 = \frac{c}{d} + 1$ i. e. $\frac{a+b}{b} = \frac{c+d}{d}$

(8) If $\frac{a}{b} = \frac{c}{d}$, $\frac{a-b}{b} = \frac{c-d}{d}$ (Dividendo)

for $\frac{a}{b} - 1 = \frac{c}{d} - 1$ i. e. $\frac{a-b}{b} = \frac{c-d}{d}$.

(9) If $\frac{a}{b} = \frac{c}{d}$, $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (Componendo et Dividendo)

for $\frac{\frac{a}{b} + 1}{\frac{a}{b} - 1} = \frac{\frac{c}{d} + 1}{\frac{c}{d} - 1}$ i. e. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

All these results though simple are very important. Hence the students must always bear these results in their mind.

Definition:— $\frac{a^2}{b^2}$ is called the duplicate ratio of $\frac{a}{b}$,
 $\frac{a^3}{b^3}$ is called the triplicate ratio of $\frac{a}{b}$; $\frac{\sqrt{a}}{\sqrt{b}}$ is called
the sub-duplicate ratio of $\frac{a}{b}$.

POINTS OF DIVISION

It AB is a straight line and P any point on it (whether between A and B



fig. 1 (b)

as shown in the figure, on A B produced or on B A produced) then $\frac{AP}{PB}$ is the ratio in which the point P divides the straight line AB. Though the segments AB and BA are equal in magnitude they are opposite in sign. For, the direction AB (i. e. the direction from left to right) is taken as positive and the direction BA (consequently) as negative. Thus $AB \neq BA$, but $AB = -BA$ or $AB + BA = 0$. Again the segments AP and PB by our convention are positive and, BP and PA negative. If the point P lies within A and B on

the straight line AB by our convention AP and PB are both positive and hence the ratio $\frac{AP}{PB}$ is positive. If P lies without A and B i. e. on AB or BA produced either AP or PB will be negative and hence the ratio $\frac{AP}{PB}$ will be negative. When the ratio $\frac{AP}{PB}$ is positive (i. e. when P lies within A and B) the point P is said to divide AB internally. In this case P is called the internal point of division. If the ratio $\frac{AP}{PB}$ is negative (i. e. when the point P lies without A and B) the point P is said to divide AB externally. Here P is called the external point of division. Hence the sign of a ratio will decide whether the point of division in question is an internal or external point of division.

Note 1:— The ratio in which the point P divides the straight line BA is $\frac{BP}{PA}$ wherever may be the position of the point P on the straight line (whether between A and B , on AB produced or on BA produced)

Note 2:— If there are two points P and P^1 , P lying within A and B and P^1 lying without A and B (either on AB produced or on BA produced) $\frac{AP}{PB}$

is the ratio in which P divides A B (internally) and $\frac{AP^1}{P^1B}$ is the ratio in which P¹ divides A B (externally)

It is evident that $\frac{AP}{PB}$ is positive and that $\frac{AP^1}{P^1B}$

as such is negative. Hence $\frac{AP}{PB}$ can never be

equal to $\frac{AP^1}{P^1B}$ (for a positive quantity can never

be equal to a negative quantity). But it may

happen that $\frac{AP}{PB} = \left(-\frac{AP^1}{P^1B}\right)$ which is positive

$$\text{i. e. } \frac{AP}{PB} = -\frac{AP^1}{P^1B} = \frac{AP^1}{BP^1}$$

In this case i. e. when a point P divides a straight

line A B internally and a point P¹ divides the same

straight line externally in the same ratio, P and P¹

are said to divide A B harmonically. Also

(APBP¹) is called a harmonic range. The student

must note that if P and P¹ divide A B harmonically

the internal ratio $\frac{AP}{PB}$ and the external ratio

$\frac{AP^1}{P^1B}$ are only equal in magnitude and not in

sign. In sign also if they are equal P and P¹ will

coincide. This will be the truth of our first

theorem.

Note 3:— Approximate position of the external point of division when the internal point of division is given:

If P, the internal point of division lies between O and B, where O is the middle point of line A B, P¹, the external point of division (or the harmonic conjugate of P, as it is usually called) will lie on A B produced. For, $\frac{A P}{P B}$ in this case is clearly greater than one and $\frac{A P^1}{B P^1}$ as long as P¹ lies on B A produced will be less than one; Hence there won't be any possibility of $\frac{A P}{P B}$ and $\frac{A P^1}{B P^1}$ becoming equal if P¹ lies on B A produced. So we conclude that P¹ must be on A B produced. Conversely if the external point P¹ lies on A B produced, P the internal point of division (or the harmonic conjugate of P¹ as it is usually called) must lie between O and B where O is the middle point of A B.

Similarly it can be shown that if P the internal point of division lies between A and O where O is the middle point of A B, P¹ its harmonic conjugate must lie on B A produced. Conversely if P¹, the external point of division lies on B A produced, P its harmonic conjugate will lie between A and O where O is the middle point of A B.

Note 4:— If the harmonic conjugate of O, the middle point of A B is represented by O¹, O¹ must

be a point either on AB produced or on BA produced such that $\frac{AO^1}{BO^1} = 1$ i e. $AO^1 = BO^1$. Hence O^1 must be a point either on AB produced or on BA produced such that the distance AB becomes negligible when compared to the distance of O^1 from A and B i e. the distance of O^1 from A and B must be sufficiently great or in other words O^1 must be the point at infinity on the line AB .

In chapter I it has been already shown that a straight line is a circle of infinite radius. It is also easy to see that the tangent at any point to a straight line is itself. Hence the perpendicular erected at any point on a given straight line to itself is a radius of the straight line. Therefore the centre of the straight line (regarded as a circle of infinite radius) must be the point at infinity in a direction perpendicular to the straight line. Since the centre itself is at infinity, the other end of the diameter through the foot of any perpendicular to the given line, will also be at infinity in the same direction. So any straight line in a plane passes through the point at infinity in a direction perpendicular to the line or in other words a straight line is a circle passing through the point

at infinity. Since this is so, we note that there is only one point at infinity on a straight line. Hence the question, whether O^1 , the harmonic conjugate of O should be on AB produced or on BA produced does not arise. The two ends of the straight line AB when indefinitely produced will be coming to the point O^1 . So we simply say that the harmonic conjugate of O , the middle point of AB is the point at infinity on the line AB .

Students will get some more ideas about the internal and external points of division when they study inverse points with respect to a circle under properties of circles.

THEOREM 1.

A straight line cannot be divided in the same ratio in more than one point (either internally or externally.)



Fig (2)

Let P be a point on AB dividing AB internally in the ratio $\frac{l}{k}$. Then it is evident that P must lie within A and B . Since P is a point on

A B that too within A and B, $\frac{AP}{PB}$ is the ratio in which the point P divides the straight line AB internally. But this ratio is given to be $\frac{l}{k}$

$$\therefore \frac{AP}{PB} = \frac{l}{k}$$

If there is any other point dividing A B internally in the same ratio $\frac{l}{k}$, let it be P'. Then

$$\frac{AP'}{P'B} = \frac{l}{k} \quad \text{but} \quad \frac{AP}{PB} = \frac{l}{k}$$

$\therefore \frac{AP}{PB} = \frac{AP'}{P'B}$ Adding one to both sides and cancelling A B in the numerators of the two resulting ratios, we get,

$$\frac{1}{PB} = \frac{1}{P'B} \quad \text{i. e.} \quad PB = P'B \quad \text{or} \quad BP = BP'$$

$\therefore P'$ coincides with P. (\because Both of them being internal points of division lie within A and B) i. e. there is only one point dividing A B internally in the ratio $\frac{l}{k}$.

Similarly there is only one point dividing A B externally in the ratio $l:k$. Hence the theorem.

THEOREM 2.

A straight line drawn parallel to one side of a triangle cuts the other two sides or those sides produced proportionally. Let MN be parallel to the side BC of $\triangle ABC$ cutting AB at M and AC at N . Divide AM into p equal parts and MB into q such equal parts. Draw parallels to BC through these points of division. Then AN will be divided into p equal parts and NC will be divided into q such equal parts (by a theorem)

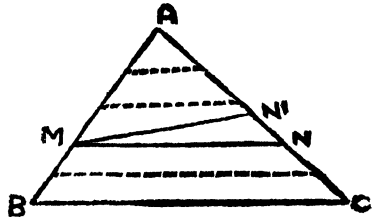


Fig. 3.

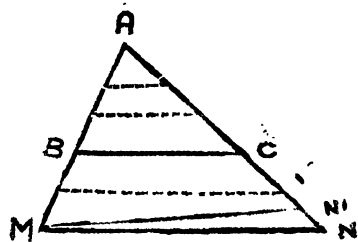


Fig. 4

$$\therefore \frac{AN}{NC} = \frac{p}{q} \text{ But } \frac{AM}{MB} = \frac{p}{q} \text{ (construction)}$$

$$\therefore \frac{AN}{NC} = \frac{AM}{MB} \text{ Hence the theorem.}$$

Conversely:— If a straight line cuts two sides of a triangle (both internally or both externally) in the same ratio it is parallel to the third side.

Let MN be a st line cutting the sides AB , AC of $\triangle ABC$ at M and N respectively such that $\frac{AM}{MB} = \frac{AN}{NC}$ Required to prove that $MN \parallel BC$.

If MN is not parallel to BC let a parallel to BC be drawn through M cutting AC at N^1 then by the previous theorem, $\frac{AM}{MB} = \frac{AN^1}{N^1C}$ But by hypothesis $\frac{AM}{MB} = \frac{AN}{NC} \therefore \frac{AN}{NC} = \frac{AN^1}{N^1C}$
 i. e. N and N^1 divide AC (both internally or both externally) in the same ratio which is impossible by theorem 1. $\therefore N^1$ must coincide with N .
 i. e. MN is parallel to BC .

CONSTRUCTION.

1. Divide a straight line AB in the ratio $l:k$ internally. Take any line through A and mark $AQ=l$ and $QP=k$ along that line in the same direction. Join P, B . Then draw QC parallel to PB to meet AB at C .

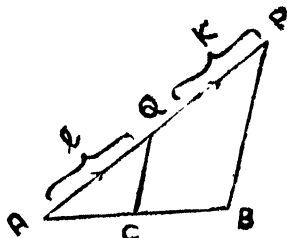


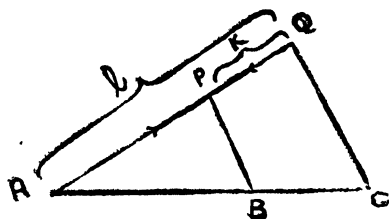
fig. 5.

Then C will be the required point. For, since $QC \parallel PB$, $\frac{AC}{CB} = \frac{AQ}{QP}$ (By theorem 2), where AQ is exactly equal to l and QP exactly equal to k (by construction)

2. Divide a straight line AB externally in the ratio $l : k$.

A slight modification is necessary in this case.

Instead of taking QP in the same direction as AQ take QP in the opposite direction and proceed as before.



Since $QC \parallel PB$

$$\frac{AC}{CB} = \frac{AQ}{QP} = \frac{l}{-k} = -\frac{l}{k}$$

(By theorem 2 as before)

Thus C divides AB externally in the ratio $\frac{l}{k}$. When the word 'externally' is removed, we will have to say that C divides AB in the ratio $-\frac{l}{k}$. The negative sign indicates that the point of division is external.

3. To find a fourth proportional to three given lengths a , b , c .

Take two straight lines ABC and APQ intersecting at A at any angle. on ABC , step off a length $AB=a$ and $BC=b$. Along APQ measure a length $AP=c$. Join B, P . Draw a line through C parallel to BP to meet APQ at Q . Then PQ is the fourth proportional to a , b , c .

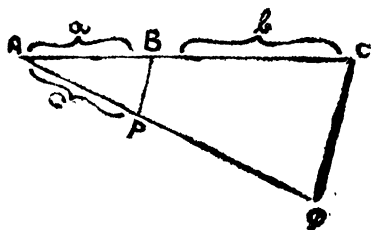


Fig. 7(a)

$$\text{Proof: } - \frac{AB}{BC} = \frac{AP}{PQ} \text{ i. e. } \frac{a}{b} = \frac{c}{PQ}.$$

Hence the result.

4. To find a third proportional to two given lengths a , b .

As in the previous case take any two straight lines ABC and APQ cutting each other at A at a convenient angle. On ABC mark off a length $AB=a$ and a length

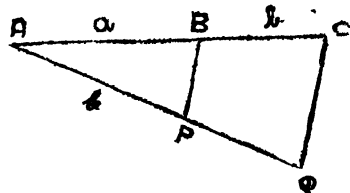


Fig. 7 (b)

$BC=b$. Also step off a length $AP=b$ on AQ . Join B, P and draw a line through C parallel to

BP to cut APQ at Q. Then PQ will be the third proportional.

$$\text{Proof:— } \frac{AB}{BC} = \frac{a}{b} \text{ (construction)}$$

$$\text{But } \frac{AP}{PQ} = \frac{AB}{BC}$$

$$\therefore \frac{AP}{PQ} = \frac{a}{b} \text{ i. e. } \frac{a}{b} = \frac{b}{PQ} \text{ Hence the result}$$

THEOREM 3.

Triangles and parallelograms of equal altitudes are to one another as their bases.

1. Triangles:—

Let Δ s ABC and PQR standing on bases BC and QR have equal altitudes h.

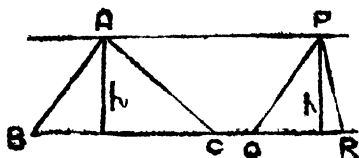


Fig. 7 (c)

$$\text{Then } \Delta ABC = \frac{1}{2} h \cdot BC$$

$$\text{" } \Delta PQR = \frac{1}{2} h \cdot QR$$

$$\therefore \frac{\Delta ABC}{\Delta PQR} = \frac{\frac{1}{2} h \cdot BC}{\frac{1}{2} h \cdot QR} = \frac{BC}{QR}$$

2. *Parallelograms:*—

Let parallelograms ABCD and PQRS standing on bases AB and PQ have equal altitudes h .

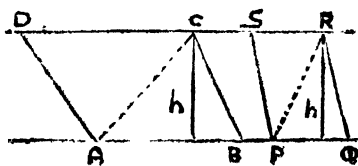


Fig 8

Join A, C; P, R

Then $\square ABCD = 2\triangle ABC$ ($\because \triangle ABC \equiv \triangle CDA$)

$\square PQRS = 2\triangle PQR$ ($\because \triangle PQR \equiv \triangle RSP$)

$$\therefore \frac{\square ABCD}{\square PQRS} = \frac{2\triangle ABC}{2\triangle PQR} = \frac{\triangle ABC}{\triangle PQR} = \frac{AB}{PQ} \text{ by 1.}$$

WORKED EXAMPLE.

Three concurrent lines through the vertices A, B, C of a $\triangle ABC$ meet the opposite sides in D, E, F respectively. Prove that $BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$ (Ceva's Theorem)

Let the three lines concur at O. The altitudes from the vertex A for $\triangle s$ BAD and DAC are the same.

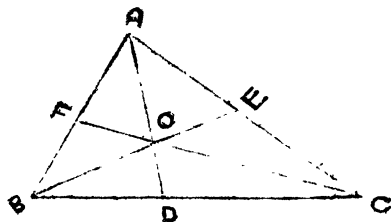


Fig. 9

Same

$$\therefore \frac{BD}{DC} = \frac{\Delta BAD}{\Delta DAC}. \text{ Also, } \Delta s \text{ BOD and DOC}$$

have a common altitude, viz, the altitude from O.

$$\begin{aligned} \therefore \frac{BD}{DC} &= \frac{\Delta BOD}{\Delta DOC} \quad \text{Thus } \frac{BD}{DC} = \frac{\Delta BAD}{\Delta DAC} = \frac{\Delta BOD}{\Delta DOC} \\ &= \frac{\Delta BAD - \Delta BOD}{\Delta DAC - \Delta DOC} = \frac{\Delta BAO}{\Delta OAC} \\ &\quad \text{(by ratio and proportion)} \end{aligned}$$

Similarly $\frac{CE}{EA} = \frac{\Delta CBO}{\Delta OBA}$ and $\frac{AF}{FB} = \frac{\Delta ACO}{\Delta OCB}$

Multiplying the three,

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{\Delta BAO}{\Delta OAC} \cdot \frac{\Delta CBO}{\Delta OBA} \cdot \frac{\Delta ACO}{\Delta OCB} = 1$$

i. e. $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$ or $BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$.

*** THEOREM 4.**

If an angle of a triangle is bisected internally (or externally), the bisector divides the opposite side internally (or externally) in the ratio of the other two sides of the triangle.

Let AD bisect $\angle A$ of ΔABC

Then to prove that $\frac{BD}{DC} = \frac{AB}{AC}$

* The method of proof adopted here for this theorem was first (i. e. in October 1956) given to Mr. Broadbent of Royal Naval college, Greenwich.

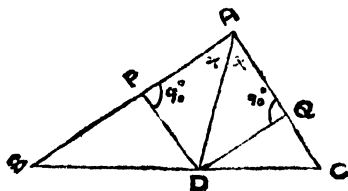


Fig. 10.

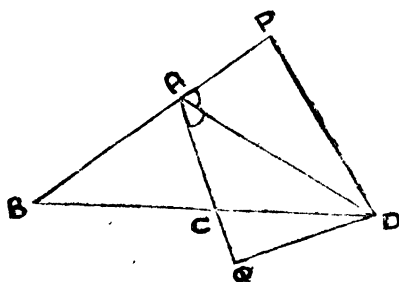


Fig. 11

Draw DP perpendicular to AB and DQ perpendicular to AC .

Then $\triangle APD \cong \triangle AQD$

($\therefore \hat{P}AD = \hat{Q}AD$; $\hat{A}PD = \hat{A}QD = 90^\circ$; AD is common)

$$\therefore DP = DQ$$

$$\therefore \frac{\triangle BAD}{\triangle DAC} = \frac{AB}{AC} \quad (\text{by theorem 3})$$

But $\frac{\triangle BAD}{\triangle DAC} = \frac{BD}{DC}$ (by theorem 3 for

the altitude from A for both \triangle s is common)

$$\therefore \frac{BD}{DC} = \frac{AB}{AC}$$

Conversely, if a side of a triangle is divided internally (or externally) in the ratio of the other two sides, then the line joining the point of division to the opposite vertex bisects the angle at that vertex internally (or externally.)

(Figure and construction the same as before)

given that $\frac{BD}{DC} = \frac{AB}{AC}$ to prove that AD is a bisector of $\angle A$.

$$\text{Now } \frac{BD}{DC} = \frac{\Delta BAD}{\Delta DAC} \quad \text{But } \frac{BD}{DC} = \frac{AB}{AC} \text{ (given)}$$

$$\therefore \frac{AB}{AC} = \frac{\Delta BAD}{\Delta DAC} = \frac{\frac{1}{2} AB \cdot DP}{\frac{1}{2} AC \cdot DQ} \therefore DP = DQ$$

or $DP = DQ$; Further AD is the common hypotenuse for Δ s APD and AQD \therefore right angled Δ s APD and AQD are congruent

$$\therefore \angle PAD = \angle QAD$$

i. e. AD is a bisector of $\angle A$ (internal bisector in figure 10 and external bisector in fig. 11)

Note 1:— If AD, AE are respectively the internal and external bisectors of $\angle A$ of ΔABC meeting the base BC at D and E then D and E divide BC in the

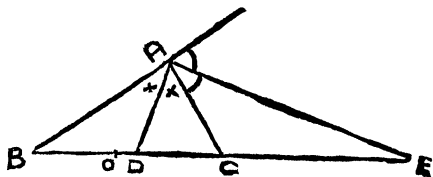


Fig. 12

same ratio $\frac{AB}{AC}$ one internally and the other externally. i. e. D and E divide BC harmonically. Hence (BDCE) is a harmonic range. D and E are called harmonic conjugates with respect to B and C. In the figure note that E, the external point of division lies on BC produced as D, the internal point of division lies between O and C where O is the middle point of BC.

Note 2:— If D and E divide BC harmonically,

(i) B and C divide DE harmonically

(use Fig. 12 where line BC is divided harmonically at D and E)

$$\frac{BD}{DC} = \frac{BE}{CE} \quad (\text{by hypothesis})$$

$$\therefore \frac{-BD}{DC} = \frac{-BE}{CE}$$

$$\frac{DB}{DC} = \frac{EB}{CE}$$

$$\therefore \frac{DB}{EB} = \frac{DC}{CE} \quad (\text{alternando})$$

i. e. B and C divide DE harmonically.

(ii) BD, BC and BE are in harmonical progression.

$$\frac{BD}{DC} = \frac{BE}{CE}$$

$$\therefore \frac{BD}{(BD + DC) - BD} = \frac{BE}{(BC + CE) - BC}$$

$$\text{i. e. } \frac{BD}{BC - BD} = \frac{BE}{BE - BC}$$

$$\text{Inverting } \frac{BC - BD}{BD} = \frac{BE - BC}{BE}$$

$$\text{i. e. } \frac{BC}{BD} - 1 = 1 - \frac{BC}{BE}$$

$$BC \left(\frac{1}{BD} + \frac{1}{BE} \right) = 2$$

$$\therefore \frac{1}{BD} + \frac{1}{BE} = \frac{2}{BC} \text{ i. e. } \frac{1}{BD} + \frac{1}{BE} = \frac{1}{BC} + \frac{1}{BC}$$

or $\frac{1}{BD} - \frac{1}{BC} = \frac{1}{BC} - \frac{1}{BE}$ Hence by definition
BD, BC, BE are in Harmonical progression.

$$\text{by (i) } \frac{DB}{EB} = \frac{DC}{CE}$$

$$\therefore \frac{DB}{EB} = \frac{-DC}{-CE} = \frac{CD}{EC}$$

$$\text{Inverting } \frac{EB}{DB} = \frac{EC}{CD} \text{ i. e. } \frac{EC}{CD} = \frac{EB}{DB}$$

\therefore C and B divide ED harmonically.

Hence as before EC, ED, EB are also in harmonical progression.

So if (BDCE) is a harmonic range BD, BC, BE are in Harmonical progression and EC, ED, EB are also in Harmonical progression.

(iii) $OD \cdot OE = OB^2 = OC^2$ where O is the midpoint of BC.

$$\frac{BD}{DC} = \frac{BE}{CE} \quad (\text{hypothesis})$$

$$\text{i. e. } \frac{BO + OD}{OC - OD} = \frac{BO + OE}{OE - OC} \quad \text{But } BO = OC$$

$$\therefore \frac{BO + OD}{BO - OD} = \frac{BO + OE}{OE - BO} \quad \text{Hence}$$

$$\frac{BO + OD + BO - OD}{BO + OD - (BO - OD)} = \frac{BO + OE + OE - BO}{BO + OE - (OE - BO)}$$

(componendo et dividendo)

$$\text{i. e. } \frac{2 BO}{2 OD} = \frac{2 OE}{2 BO} \quad \therefore BO^2 = OD \cdot OE$$

$$\text{i. e. } OD \cdot OE = BO^2 = (-OB)^2 = OB^2 = OC^2$$

conversely if O is the midpoint of BC and D, E two points on it (on the same side of O) such that $OD \cdot OE = OB^2 = OC^2$, then (BDCE) is a harmonic range.

$$OD \cdot OE = BO^2$$

$$\text{i. e. } \frac{BO}{OD} = \frac{OE}{BO}$$

$$\therefore \frac{BO + OD}{BO - OD} = \frac{OE + BO}{OE - BO}$$

$$\text{i. e. } \frac{BO + OD}{OC - OD} = \frac{OE + BO}{OE - OC} \quad (\because BO = OC)$$

$$\text{i. e. } \frac{BD}{DC} = \frac{BE}{CE}$$

\therefore D and E divide BC internally and externally in the same ratio. Hence by definition (BDCE) is a harmonic range.

Note 3:— In fig. 12, if ABC is an isosceles triangle ($AB = AC$) D will coincide with O and AD will be perpendicular to BC. $\angle DAE = 90^\circ$ always. Hence AE will be parallel to BC. i. e. E in this case is the point at infinity on the line BC. (or more generally the point at infinity in a direction parallel to BC). So we note that the harmonic conjugate of the middle point of BC is the point at infinity on the line BC.

WORKED EXAMPLE.

1. If A, B are fixed points and P a variable point such that the ratio of PA to PB is always constant prove that the locus of P is in general a circle.

[This is called Apollonius' Theorem. The student in future when he studies the Geometry of the conic, will note that the locus of a point which moves such that the ratio of its distance from a focus to its distance from the foot of the corresponding directrix is a constant equal to the eccentricity of the conic, is its auxiliary circle. Hence auxiliary circle of a conic may be appropriately called Apollonian circle of the conic.]

Let P be a point such that $\frac{PA}{PB} = \frac{l}{k}$. Divide AB internally and externally in the same fixed ratio $\frac{l}{k}$. Let C be the internal point of division and D the external point of division.

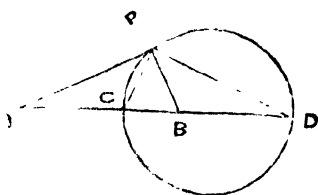


Fig. 13

$$\text{Then } \frac{PA}{PB} = \frac{l}{k} = \frac{AC}{CB} = \frac{AD}{BD}$$

i. e. $\frac{PA}{PB} = \frac{AC}{CB} \therefore PC$ is the internal bisector of $\angle APB$.

and $\frac{PA}{PB} = \frac{AD}{BD} \therefore PD$ is the external bisector of $\angle APB$

$$\text{Hence } \angle CPD = 90^\circ$$

Now if we describe a circle on CD as diameter this circle passes through the point P. ($\angle CPD = 90^\circ$). But P is any point satisfying the given condition. Thus any point satisfying the given condition (namely $\frac{PA}{PB} = \frac{l}{k}$) lies on this circle on CD as diameter and this circle is a fixed circle (\because A and B are fixed points and the points C, D which divide the line joining these fixed points

in the fixed ratio $\frac{l}{k}$ are also fixed. Consequently the circle on CD as diameter is also a fixed circle. Hence the locus of P is this circle on CD as diameter.

[Locus is by definition the aggregate of all points satisfying any geometrical condition. All loci however are found to be curves. Students must note that any point lying on a locus will satisfy the condition for the locus and that any point satisfying the condition for a locus will lie on the locus.]

Definition:— The circle on CD as diameter (fig. 13) is called the Apollonius' circle of the two fixed points A and B for the constant ratio $\frac{l}{k}$.

[Note that Apollonius' circle reduces to a straight line (a circle of infinite radius or a circle passing through the point at infinity) when the constant ratio is unity].

2. CA, CB are two tangents to a circle A and B being the points of contact. E is the foot of the perpendicular from B to AD the diameter through A. Prove that BA, BD bisect angle CBE. Deduce that CD bisects BE (March 1948)

Let BA and BE meet CD in L and K respectively. CA, CB being tangents from A are equal

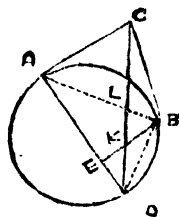


Fig. 14

$$\therefore \angle CAB = \angle CBA$$

$$\angle CAE + \angle BEA = 90^\circ + 90^\circ = 180^\circ$$

$$\therefore CA \parallel BE$$

$$\therefore \angle CAB = \angle ABE$$

Hence $\angle CBA = \angle ABE$

i. e. BA is the internal bisector of $\angle CBE$.

$\angle ABD = 90^\circ$ (since AD is a diameter)

Hence BD is the other bisector

$$\therefore \frac{KD}{CD} = \frac{LK}{CL}$$

Then to prove that $EK = KB$

Now Δ s DEK and DAC are \parallel ($\because EK \parallel AC$)

$$\therefore \frac{EK}{AC} = \frac{KD}{CD}$$

||ly Δ s KBL and CAL are similar.

$$\therefore \frac{KB}{CA} = \frac{LK}{CL}$$

$$\frac{KD}{CD} = \frac{LK}{CL}$$

$$\therefore \frac{EK}{AC} = \frac{KB}{CA} \quad \text{or} \quad EK = KB$$

3, AE bisects the angle A of a $\triangle ABC$ and meets BC in E. If O and O' be the circumcentres of \triangle s ABE and ACE prove that

$$\frac{OE}{O'E} = \frac{BE}{EC} \quad (1939 \text{ M. U.})$$

Join O, B; O, E; O'E; O'C.

In \triangle s OBE and O'CE

$$\begin{aligned} \hat{B}OE &= \hat{E}O'C \quad (\text{Since} \\ &\quad \hat{B}AE = \hat{E}AC) \\ &= \theta \quad (\text{say}) \end{aligned}$$

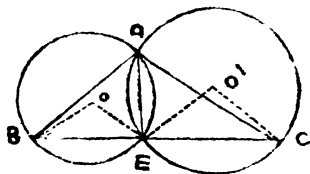


Fig. 15

$$\text{also } \hat{O}BE = \hat{O}E'B = \frac{\hat{O}BE + \hat{O}E'B}{2} = \frac{180 - \theta}{2}$$

$$\text{||| ly } \hat{O}'EC = \hat{O}'CE = \frac{180 - \theta}{2}$$

$$\therefore \hat{O}BE = \hat{O}'CE \text{ and } \hat{O}EB = \hat{O}'EC$$

Hence \triangle s OBE and O'CE are Equiangular

$\therefore \frac{BE}{EC} = \frac{OE}{O'E}$ (\because If two \triangle s are Equiangular corresponding sides are proportional).

In the figure prove that \triangle s BEO' and CEO have the same area. (Proof is left to the student).

Aliter:—Let OP and O'Q be \perp rs to BE and EC respectively from O and O'.

$$\text{Let } \angle BAE = \angle EAC = \theta$$

$$\text{Then } \angle BOE = \angle EO'C = 2\theta$$

$$\text{and } \angle POE = \angle QO'E = \theta$$

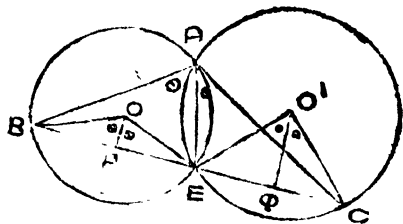


Fig 16

From rt. angled $\triangle POE$

$$PE = \frac{1}{2} BE = OE \sin \theta \text{ or } BE = 2 OE \sin \theta$$

$$\text{Similarly } EC = 2 O'E \sin \theta$$

$$\therefore \frac{BE}{EC} = \frac{OE}{O'E}$$

CHAPTER 3.

SIMILAR TRIANGLES.

Definition:— Two \triangle s are said to be similar if (1) their angles are equal and (2) their corresponding sides (sides opposite to equal angles) are proportional. But it is found that if any one of the above conditions is satisfied the other will be automatically satisfied. This is indicated in the following two theorems.

THEOREM 1.

If two \triangle s are equiangular their corresponding sides are proportional.

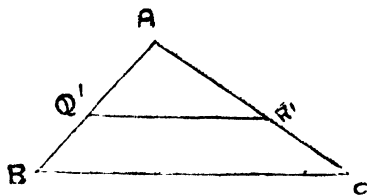


Fig. 17

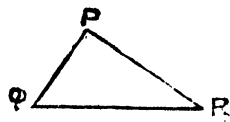


Fig. 18

In Δ s ABC and PQR, let $\angle A = \angle P$, $\angle B = \angle Q$.
(Then the third set of angles are obviously equal)

Place the Δ PQR so that $\angle P$ coincides with $\angle A$ and PQ coincides with AB. Then since $\angle P = \angle A$, PR coincides with AC.

Let AQ^1R^1 be the new position of Δ PQR
Since $\angle B = \angle Q = \angle Q^1$ (or since $\angle C = \angle R = \angle R^1$)

$$Q^1R^1 \parallel BC \quad \therefore \frac{AQ^1}{Q^1B} = \frac{AR^1}{R^1C}$$

$$\text{i. e. } \frac{Q^1B}{AQ^1} = \frac{R^1C}{AR^1} \quad \text{adding 1 to both sides;}$$

$$\frac{AB}{AQ^1} = \frac{AC}{AR^1} \quad \text{i. e. } \frac{AB}{PQ} = \frac{CA}{RP}$$

(ΔAQ^1R^1 being the new position of Δ PQR,
 $\Delta AQ^1R^1 \equiv \Delta PQR$)

Similarly by placing the Δ PQR so that $\angle R$ coincides with $\angle C$ and RP coincides with CA

we can prove that $\frac{CA}{RP} = \frac{BC}{QR}$

$$\text{But } \frac{AB}{PQ} = \frac{CA}{RP} \quad (\text{already proved})$$

$$\therefore \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$$

Note:— The student should take care in writing down the corresponding sides in the two triangles. Sides opposite to equal angles in the

two triangles are corresponding sides. Note also that the two triangles are now similar (by definition).

Conversely, if the sides of one triangle be proportional to the sides of another, the two triangles are equiangular.

Let ABC and PQR be two Δ s in which

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$$

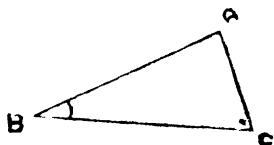


Fig. 19

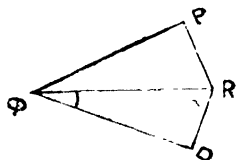


Fig. 20.

Required to prove that they are equiangular

Let D be a point on the side opposite to P , of QR such that $\angle DQR = \angle ABC$ and $\angle DRQ = \angle ACB$

Then Δ s ABC and DQR are equiangular (by construction)

$$\therefore \frac{AB}{DQ} = \frac{BC}{QR} = \frac{CA}{RD} \quad (\text{by theorem 1, Chapter 3})$$

$$\text{But } \frac{BC}{QR} = \frac{CA}{RP} = \frac{AB}{PQ} \quad (\text{Hypothesis})$$

$$\therefore \frac{AB}{DQ} = \frac{AB}{PQ} \quad \text{and} \quad \frac{CA}{RD} = \frac{CA}{RP}$$

Hence $PQ = DQ$ and $RP = RD$

QR is common to $\triangle PQR$ and $\triangle DQR$

$\therefore \triangle PQR \equiv \triangle DQR$ and hence they are equiangular.

But $\triangle s$ ABC and DQR are equiangular.

$\therefore \triangle s$ ABC and PQR are equiangular.

Note that the two triangles are now similar (by definition).

CONSTRUCTION.

Divide a straight line AB internally and externally in the ratio $l:k$.

Erect a perpendicular AP, at A to line AB such that $AP = l$ units. Also, erect a perpendicular BQ at B to line AB in the same side of AB as AP, such that $BQ = k$ units. Produce QB to Q^1 such that $Q^1B = BQ$. Join P, Q^1 to meet AB at C. Then C will be the required internal point of division. Join P, Q and produce it to meet AB or BA produced as the case may be at D. Then D will be the required external point of division.

(Draw the figure and supply proof)

The student will note that this method is very useful in dividing a straight line internally and externally in the same ratio.

WORKED EXAMPLE.

Two circles cut orthogonally at A and B. A diameter of one of the circles is drawn cutting the other in C and D. Show that $BC \cdot AD = AC \cdot BD$.

[Sept. 1950 M. U]

Def:— Two circles are said to cut orthogonally if the angle between the tangents to the two circles at a common point is a right angle (or if the radius of one through a common point is the tangent to the other at that common point).

Let a diameter of circle I cut the circle II at C and D. Let O be the centre of circle I. Join O,A; O,B. Since OA by definition of orthogonal circles is a Tangent to circle II.

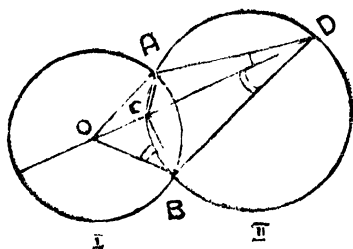


Fig. 21.

$$\angle OAC = \angle ODA.$$

Hence \triangle s OAC and ODA are |||

$$\therefore \frac{CA}{AD} = \frac{OA}{OD}$$

Similarly since \triangle s OBC and ODB are |||

$$\frac{BC}{DB} = \frac{OB}{OD}$$

But $OA = OB$

$$\therefore \frac{CA}{AD} = \frac{BC}{DB} \text{ i. e. } AC \cdot BD = AD \cdot BC.$$

THEOREM 2.

If two Δ s have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the two Δ s are similar.

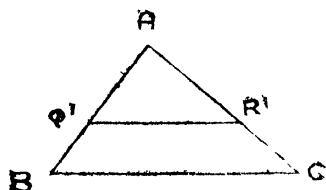


Fig. 22

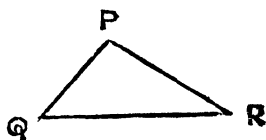


Fig. 23

In the two Δ s ABC and PQR let $\angle A = \angle P$ and $\frac{AB}{PQ} = \frac{AC}{PR}$.

Construction.— Place the ΔPQR on ΔABC such that $\angle P$ coincides with $\angle A$ and PQ falls along AB . Since $\angle A = \angle P$, PR will coincide with AC . Let Q' , R' , be the new positions of Q and R respectively.

$$\therefore AQ^1 = PQ; AR^1 = PR.$$

$$\frac{AB}{PQ} = \frac{AC}{PR} \text{ (given) i. e. } \frac{AB}{AQ^1} = \frac{AC}{AR^1}$$

$$\text{i. e. } \frac{AB}{AB - AQ^1} = \frac{AC}{AC - AR^1} \text{ i. e. } \frac{AB}{Q^1B} = \frac{AC}{R^1C} \text{ or}$$

$$\frac{AB}{BQ^1} = \frac{AC}{CR^1} \therefore Q^1R^1 \parallel BC \text{ (by Theorem 2, ch. 2.)}$$

$\therefore \angle Q^1 = \angle B; \angle R^1 = \angle C$ i. e. $\angle Q = \angle B$
and $\angle R = \angle C$.

Thus \triangle s ABC and PQR are equiangular and hence their corresponding sides are proportional (by Theorem 1, Ch. 3.)

i. e. The two \triangle s are similar.

WORKED EXAMPLE.

In a quadrilateral which is not cyclic prove that the rectangle contained by the diagonals is always less than the sum of the rectangles contained by pairs of opposite sides.

ABCD is a quadrilateral which is not cyclic. Let O be a point within the quadrilateral such that $\angle OAD = \angle CAB$ and $\angle ODA = \angle ACB$. By

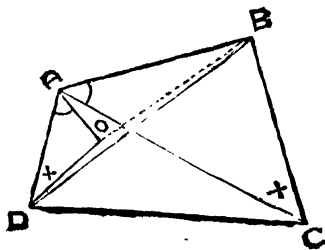


Fig. 24

construction \triangle s $AO\text{D}$ and ABC are equiangular.

$$\therefore \frac{AD}{AC} = \frac{OD}{BC} = \frac{AO}{AB} \quad (\text{A}) \quad (\text{Theorem 1. Ch. 3})$$

Hence $AD \cdot BC = OD \cdot AC$. I

Now in \triangle s DAC and OAB

Evidently $\angle DAC = \angle OAB$

Further $\frac{AD}{AC} = \frac{AO}{AB}$ by (A)

$\therefore \triangle DAC \parallel \triangle OAB$ (by Theorem 2. Ch. 3)

$$\therefore \frac{AC}{AB} = \frac{DC}{OB} \quad \text{i. e. } AB \cdot CD = AC \cdot OB. \quad \text{II.}$$

Adding I and II,

$$AB \cdot CD + AD \cdot BC = AC (OD + OB)$$

But $OD + OB > BD$

$$\therefore AB \cdot CD + AD \cdot BC > AC \cdot BD$$

When the quadrilateral is cyclic $\angle ACB = \angle ADB$
(Angles in the same segment)

But $\angle ACB = \angle ODA$ (by construction)

$\therefore \angle ADB = \angle ADO$. Hence O lies on BD

$$\therefore OD + OB = BD.$$

Thus when the quadrilateral is cyclic

$AB \cdot CD + AD \cdot BC = AC \cdot BD$ which is known as Ptolemy's Theorem.

Def:— If a, b, c are three quantities connected by the relation $\frac{a}{b} = \frac{b}{c}$ ($b^2 = ac$) the three quantities, a, b, c are said to be in continued proportion. Also, b is called the mean proportional between a and c .

THEOREM 3.

If from the right angle A of a right angled triangle ABC , AD is drawn perpendicular to BC then (i) AD is the mean proportional between BD and DC (ii) BA is the mean proportional between BD and BC (iii) CA is the mean proportional between CD and CB .

Let ABC be a \triangle right angled at A and $AD \perp$ to BC from A .

Proof:—

$$\angle ABD + \angle BAD = 90^\circ$$

$$\angle ABC + \angle ACB = 90^\circ$$

$$\therefore \angle BAD = \angle ACB$$

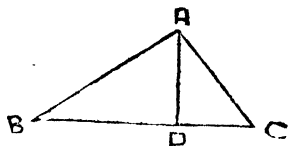


Fig. 25

Also $\angle B$ is common to \triangle s ABD and CBA

Hence \triangle s ABD and CBA are similar.

Similarly \triangle s CAD and CBA are similar.

$\therefore \triangle$ s ABD and CAD are also similar.

(i) From similar \triangle s ABD and CAD

$$\frac{AD}{DC} = \frac{BD}{AD} \quad \text{or} \quad AD^2 = BD \cdot DC.$$

(ii) From similar \triangle s ABD and CBA

$$\frac{BA}{BC} = \frac{BD}{BA} \quad \text{or} \quad BA^2 = BD \cdot BC.$$

(iii) From similar \triangle s CAD and CBA

$$\frac{CA}{CB} = \frac{CD}{CA} \quad \text{or} \quad CA^2 = CD \cdot CB.$$

[An alternate method of proof is given below for these three results so that the student may recollect them easily to his memory without any confusion].

(i) Describe a circle on BC as diameter. Since $\angle BAC = 90^\circ$ this passes through A. If AD meets this circle again at A^1 , BC the diameter bisects this perpendicular chord ADA^1 . Hence $AD = DA^1$.

By a property of the circle,

$$AD \cdot DA^1 = BD \cdot DC$$

$$\text{i. e. } AD^2 = BD \cdot DC. \quad (AD = DA^1).$$

(ii) If we describe a circle on CA as diameter this passes through D ($\angle ADC = 90^\circ$). Further since BA is perpendicular to CA, BA becomes the tangent to this circle on CA as diameter at A. Hence $BA^2 = BD \cdot BC$.

(iii) Similarly by describing a circle on BA as diameter we get $CA^2 = CD \cdot CB$.

[The student will note that Pythagoras' Theorem follows from (ii) and (iii), on addition].

Def:— If the values of two quantities vary, subject to the condition that their product is always constant those two quantities are said to be in inverse proportion.

WORKED EXAMPLE.

1. PA, PB are Tangents to a circle whose centre is O from any external point P. AB cuts OP in Q. Prove that OP and OQ are in inverse proportion.

Join O to A. Then

$$\angle OAP = 90^\circ.$$

Also AQ is \perp to OP

($\because \triangle OAP \equiv \triangle OBP$)

and hence $\triangle QAP \equiv$

$\triangle QBP$.

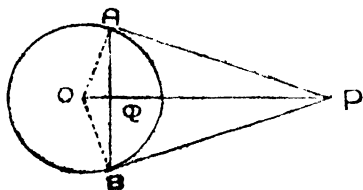


Fig. 26

$$\text{Thus } \angle AQP = \angle BQP = \frac{\hat{AQP} + \hat{BQP}}{2} = \frac{180}{2} = 90^\circ)$$

Hence by theorem 3 chapter 3,

$$OA^2 = OQ \cdot OP.$$

$$\begin{aligned} \text{i. e. } OQ \cdot OP &= (\text{radius of the circle})^2 \\ &= \text{a constant.} \end{aligned}$$

\therefore OP and OQ are in inverse proportion.

2 C is a point on the semi-circle on the line AB as diameter. Semi-circles are outwardly described on AC and BC as diameters. Prove that the sum of the crescent shaped areas lying outside the semi-circle ACB is equal to the area of the \triangle ACB. (Inter March 1956)

The area of semi-circle on AB as diameter is

$$\frac{\pi}{2} \left(\frac{AB}{2} \right)^2 = \frac{\pi}{8} AB^2$$

|||ly the Areas of semi-circles on AC and BC as

diameters are respectively $\frac{\pi}{8} AC^2$ and $\frac{\pi}{8} BC^2$

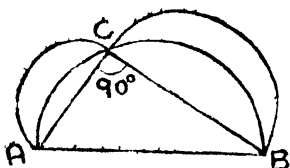
Since AB is a diameter $\angle ACB = 90^\circ$ and hence by Pythagoras' Theorem,

$$AB^2 = AC^2 + BC^2. \text{ Multiplying throughout by } \frac{\pi}{8}$$

$$\frac{\pi}{8} AB^2 = \frac{\pi}{8} AC^2 + \frac{\pi}{8} BC^2$$

i. e. Area of semi-circle on AB as diameter is equal to the sum of the areas of semi-circles on AC and BC as diameters. Now let the area of the semi-circle ACB excluding \triangle ACB be S

$$\text{Then, } \left(\frac{\pi}{8} AB^2 - S \right) = \left(\frac{\pi}{8} AC^2 + \frac{\pi}{8} BC^2 \right) - S$$



Ffg. 27

i. e. $\Delta ACB =$ Sum of the crescent shaped areas lying outside the semi-circle ACB .

3. Prove that the common tangent to two circles having external contact is a mean proportional between the diameters of the circles.

[Inter 1925 M. U.]

Let two circles, centres A and B touch externally at M. Also let FQ be one of their common tangents. (The other common tangent will also be equal in length by symmetry.)

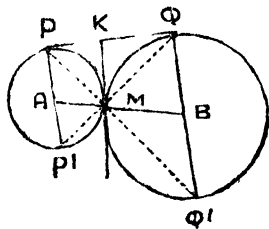


Fig. 28

Produce PA and QB to meet the circles again at P^1 and Q^1 . Then PP^1 is a diameter of the circle A and QQ^1 is a diameter of the circle B.

Join P, M; Q, M; P^1 , M and Q^1 , M.

$\angle PMP^1 = 90^\circ$. Also $\angle PMQ = 90^\circ$ (For, if we draw the common Tangent at M to the two circles, cutting PQ at K, $KP = KM = KQ$)

$\therefore P^1MQ$ is a straight line. ||| Q^1MP is a st. line.

In $\Delta s PP^1Q$ and QPQ^1

$$P^1PQ = PQQ^1 = 90^\circ$$

$$\angle PP^1Q = \angle PP^1M = \angle QPM = \angle QPQ^1$$

(Property of the circle)

$$\therefore \triangle PP^1Q \parallel \triangle QPQ^1$$

$$\text{Hence } \frac{PQ}{QQ^1} = \frac{PP^1}{PQ} \quad \text{or} \quad PQ^2 = PP^1 \cdot QQ^1$$

Aliter:—

Let the radii of the two circles centres A and B and touching externally at M, be a and b respectively;

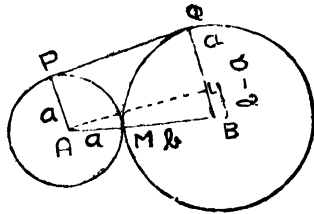


Fig. 29

Also let PQ be one of the two common tangents. Join P, A; Q, B. Draw a perpendicular from A to QB cutting it at L. Then evidently PQLA is a rectangle.

$$\therefore PQ = AL \text{ and } PA = QL = a$$

$$\therefore BL = BQ - LQ = (b - a) \quad (\because BQ = b)$$

$$AB = (a + b)$$

From right angled $\triangle ALB$,

$$AB^2 = AL^2 + BL^2$$

$$\text{i. e. } (a + b)^2 = PQ^2 + (b - a)^2$$

$$\begin{aligned} \therefore PQ^2 &= (a + b)^2 - (b - a)^2 \\ &= 4ab = (2a)(2b) \end{aligned}$$

Construction:— Find the mean proportional between two given lengths a and b .

Take any straight line AB .
 Let O be a point on it.
 Measure out $OA = a$ and $OB = b$ on opposite sides of O on the straight line AB .

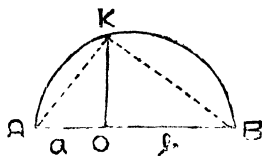


Fig. 30

Draw the circle on AB as diameter. Through O draw a perpendicular to cut the circle in K and K^1 (K^1 is not shown in the figure). Then OK or OK^1 will be the mean proportional between AO and OB . Join A, K ; B, K .

Proof:— $\angle AKB = 90^\circ$ and KO is \perp to AB .
 Hence applying Theorem 3, Chapter 3.

$$AO \cdot OB = OK^2 \quad \text{i. e. } a \cdot b = OK^2$$

Hence OK is the mean proportional between a and b .

Note 1:— This construction geometrically illustrates the algebraic proposition that, if the sum of two positive quantities is given the product of them is greatest when the two quantities are equal. For, for all positions of the straight line OK (O lying between A and B) the relation $AO \cdot OB = OK^2$ is true. The left hand side will be greatest when the right hand side is greatest. The

R. H. S evidently will be greatest when O coincides with the centre of the circle. Then $AO=OB$.

Note 2:— If OB and OA ($OA < OB$) are taken in the same side of O, the construction will be as follows:

Draw the circle on OB as diameter and erect a $\perp r$ to OB at A to meet the circle at one of the two points (say) at K. Join O. K and B, K

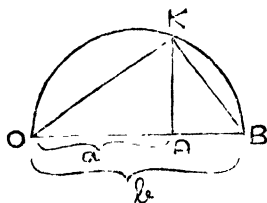


Fig. 31

Then OK will be the mean proportional between OA and OB.

Proof: — $\angle OKB = 90^\circ$ and KA is $\perp r$ to OB.

\therefore By applying Theorem 3, Chapter 3,
 $OK^2 = OA \cdot OB$.

i. e. $OK^2 = a \cdot b$

\therefore OK is the mean proportional between a and b.

THEOREM 4.

Similar \triangle s are to one another as the squares on their corresponding sides. (Areas of

two similar triangles are in the duplicate ratio of corresponding sides).

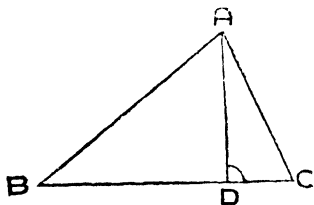


Fig. 32

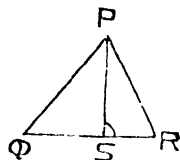


Fig. 33

Let $\triangle ABC$ be similar to $\triangle PQR$. Also let AD be \perp to BC and PS be \perp to QR . Then in the two right angled \triangle s ADC and PSR .

$$\hat{A}DC = \hat{P}SR = 90^\circ$$

$$\hat{A}CD = \hat{P}RS \quad (\angle C = \angle R)$$

$$\therefore \triangle ADC \parallel \triangle PSR$$

$$\text{So } \frac{AD}{PS} = \frac{AC}{PR} \quad \text{But } \frac{BC}{QR} = \frac{CA}{RP} = \frac{AB}{PQ}$$

$$\therefore \frac{AD}{PS} = \frac{CA}{RP} = \frac{AB}{PQ} = \frac{BC}{QR} \quad (\text{Thus alti-}$$

tudes to two corresponding sides of two $\parallel \triangle$ s are in the ratio of corresponding sides).

$$\begin{aligned} \frac{\triangle ABC}{\triangle PQR} &= \frac{\frac{1}{2} BC \cdot AD}{\frac{1}{2} QR \cdot PS} = \frac{BC}{QR} \cdot \frac{BC}{QR} \left(\because \frac{AD}{PS} = \frac{BC}{QR} \right) \\ &= \frac{BC^2}{QR^2} = \frac{CA^2}{RP^2} = \frac{AB^2}{PQ^2} \end{aligned}$$

WORKED EXAMPLE.

1. The tangent at A to the circumcircle of a triangle ABC meets BC produced at D. Show

that $\frac{BD}{CD} = \frac{AB^2}{AC^2}$ (Inter 1934 M. U.)

In Δ s ABD and CAD

$\angle D$ is common.

$\angle ABD = \angle CAD$
(angle between a tangent to a circle and any chord through the point of

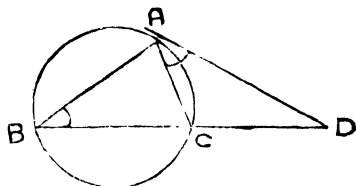


Fig. 34

contact is equal to the angle subtended by that chord in the alternate segment),

Hence the two Δ s are similar.

$$\therefore \frac{\Delta ABD}{\Delta CAD} = \frac{AB^2}{AC^2}. \text{ The altitude from A to the two } \Delta\text{s is Common.}$$

$$\therefore \frac{\Delta ABD}{\Delta CAD} = \frac{BD}{CD} \text{ Hence } \frac{BD}{CD} = \frac{AB^2}{AC^2}$$

Aliter:—Since the two Δ s ABD and CAD are equiangular, corresponding sides are proportional.

Thus, $\frac{AB}{AC} = \frac{BD}{AD} = \frac{AD}{CD} = \sqrt{\frac{BD \cdot AD}{AD \cdot CD}} = \sqrt{\frac{BD}{CD}}$

$\therefore \frac{BD}{CD} = \frac{AB^2}{AC^2}$ or taking the two values of $\frac{AB}{AC}$ and multiplying together we get the result.

2. In two similar triangles corresponding lines such as (a) *medians (b) altitudes (c) circum-radii (d) in-radii etc. are in the ratio of corresponding sides.

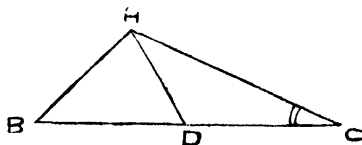


Fig. 35

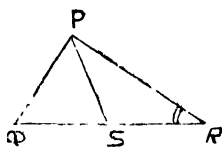


Fig 36

(a) Let ABC and PQR be two similar triangles and AD, PS medians to the sides BC and QR respectively.

Proof:— Since $\triangle ABC \sim \triangle PQR$

$$\frac{BC}{QR} = \frac{CA}{RP} \quad \therefore \quad \frac{\frac{1}{2}BC}{\frac{1}{2}QR} = \frac{CA}{RP} \quad \text{i. e.} \quad \frac{DC}{SR} = \frac{CA}{RP}$$

Further $\angle C = \angle R$

$\therefore \triangle ACD$ and $\triangle PRS$ are similar by theorem 2, chapter 3.

* For definition of medians of a plane triangle see appendix II.

$$\text{Hence } \frac{AD}{PS} = \frac{CA}{RP} = \frac{AB}{PQ} = \frac{BC}{QR}$$

Similarly the other medians are also in the ratio of corresponding sides.

(b) Already proved in theorem 3 chapter 3.

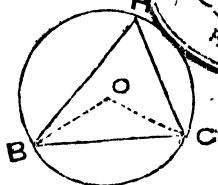


Fig. 37

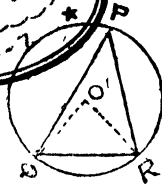


Fig. 38

(c) [Circumcentre of a triangle is the point of concurrence of the perpendicular bisectors of the sides of the Δ . Hence it is evident that the circumcentre of a triangle is equidistant from the vertices of the triangle. i. e. with the circumcentre as centre a circle can be drawn to pass through the vertices of the triangle. This circle is called the circumcircle. The radius of this circle is called the circum-radius.]

Let O and O' be the circumcentres of two similar triangles ABC and PQR.

Join B, O; C, O; Q, O'; R, O'.

Proof:— Since $\angle A = \angle P$

$$\angle BOC = \angle QO'R$$

($\because \angle BOC = 2\angle A$ and $\angle QO'R = 2\angle P$)

$$\text{Further } \frac{BO}{OC} = \frac{QO'}{O'R} = 1$$

$$\text{or } \frac{BO}{QO'} = \frac{OC}{O'R}$$

Hence $\triangle BOC \parallel \triangle QO'R$ by Theorem 2, Chapter 3.

$$\therefore \frac{OB}{O'Q} = \frac{BC}{QR} = \frac{CA}{RP} = \frac{AB}{PQ}$$

(d) [Incentre of a triangle is the point of concurrence of the internal bisectors of the angles

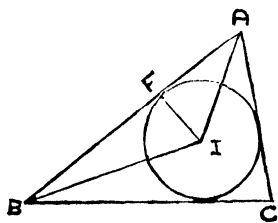


Fig. 39

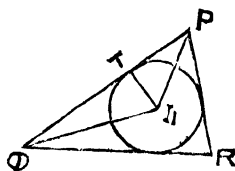


Fig. 40

of the triangle. Hence it is evident that the perpendicular distances of the incentre of a triangle from the three sides are equal. i. e. We can draw a circle with the incentre as centre to touch the three sides of the triangle. This circle is called the incircle and the radius of this circle is called the in-radius.]

Let I and I_1 be the incentres of two similar triangles ABC and PQR . Let IF be perpendicular to AB and I_1T perpendicular to PQ . Then, evidently F is the point of contact of the side AB of $\triangle ABC$ with the incircle of $\triangle ABC$ and T is the point of contact of the side PQ of $\triangle PQR$ with the incircle of $\triangle PQR$. i. e. IF and I_1T are the in-radii of the two \triangle s ABC and PQR .

Proof:— In \triangle s ABC and PQR ,

$$\angle A = \angle P \quad \therefore \angle BAI = \angle PQI_1 \text{ and}$$

similarly $\angle ABI = \angle PQI_1$

$$\therefore \triangle ABI \parallel \triangle PQI_1$$

IF and I_1T are altitudes to the corresponding sides AB and PQ of these similar \triangle s.

$$\text{Hence by (b) } \frac{IF}{I_1T} = \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$$

3. In two \triangle s if one angle of the one equals one angle of the other, their areas are in the ratio of the rectangles contained by sides about equal angles.

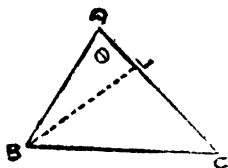


Fig. 41

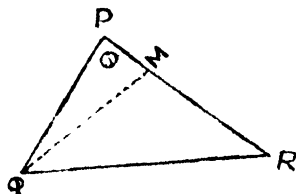


Fig. 42

Let ABC and PQR be two triangles such that $\angle BAC = \angle QPR = \theta$ (say)

Construction:— Draw BL , QM perpendiculars to AC and PR from B and Q respectively.

Then $\triangle ABC = \frac{1}{2} AC \cdot BL$; $\triangle PQR = \frac{1}{2} PR \cdot QM$.

But $BL = AB \sin \theta$; $QM = PQ \sin \theta$.

$$\therefore \frac{\triangle ABC}{\triangle PQR} = \frac{\frac{1}{2} AC \cdot AB \sin \theta}{\frac{1}{2} PR \cdot PQ \sin \theta} = \frac{AB \cdot AC}{PQ \cdot PR} \quad (\sin \theta \neq 0)$$

Since, $\sin \theta = \sin (180 - \theta)$ it follows that in two \triangle s if one angle of the one is a supplement of one angle of the other their areas are in the ratio of the rectangles contained by sides about these supplementary angles.

From this problem it easily follows that the areas of two similar triangles are in the duplicate ratio of corresponding sides. For if ABC and PQR are two similar \triangle s.

$$\begin{aligned} \frac{\triangle ABC}{\triangle PQR} &= \frac{AB \cdot AC}{PQ \cdot PR} \quad (\because \angle A = \angle P) \\ &= \frac{AB}{PQ} \cdot \frac{AB}{PQ} \quad \left(\because \frac{AB}{PQ} = \frac{AC}{PR} \right) \\ &= \frac{AB^2}{PQ^2} \end{aligned}$$

CONSTRUCTIONS.

i. Divide a triangle ABC into two parts whose areas are in the ratio $l : m$ by drawing a straight line parallel to the side BC.

It is required to draw a line which is such that it divides the \triangle into two portions whose areas are in the ratio $l : m$ and which must at the same time be parallel to BC.

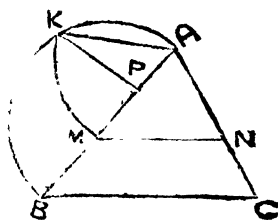


Fig. 43

To fix up a line either two points on it or any one point on it and its direction must be known. Here since the straight line must have to be parallel to BC its direction can be taken as given. Hence it is sufficient if we find a point on it.

Analysis:— Suppose MN is the required parallel line cutting AB and AC at M and N respectively.

$$\text{Then } \frac{\triangle AMN}{\square MNCB} = \frac{l}{m}$$

$$\therefore \frac{\triangle AMN}{\triangle AMN + \square MNCB} = \frac{l}{l+m}$$

$$\text{i. e. } \frac{\triangle AMN}{\triangle ABC} = \frac{l}{l+m}$$

But \triangle s AMN and ABC are also similar (MN \parallel BC)

$$\therefore \frac{\triangle AMN}{\triangle ABC} = \frac{AM^2}{AB^2}$$

$$\text{Hence } \frac{AM^2}{AB^2} = \frac{l}{l+m}$$

i. e. M is a point on AB such that $\frac{AM^2}{AB^2} = \frac{l}{l+m}$

(Similarly N will be a point on AC such that $\frac{AN^2}{AC^2} = \frac{l}{l+m}$)

Now if $\frac{l}{l+m}$ is a perfect square the ratio $\frac{AM}{AB}$ can be easily found out and hence the point M can be easily fixed up on AB. If $\frac{l}{l+m}$ is not a perfect square M will have to be found out, employing some other geometrical result. In this case if P is a point on AB such that $AM^2 = AP \cdot AB$. (i. e. AM becomes the mean proportional between AP and AB).

$$\frac{AM^2}{AB^2} = \frac{AP \cdot AB}{AB \cdot AB} = \frac{AP}{AB} = \frac{l}{l+m}$$

So if we find a point P such that $\frac{AP}{AB} = \frac{l}{l+m}$ the point M can be found out by finding, AM the mean proportional between AP and AB.

Construction:— Let P be a point on AB such that $\frac{AP}{AB} = \frac{l}{l+m}$ (i. e. $\frac{AP}{PB} = \frac{l}{m}$). Find the mean proportional between AP and AB. (use construction under Theorem 3, Chapter 3). Let it be AM. Then M is the point through which the required parallel line passes. Hence draw a line through M parallel to BC. This will be the required line.

Proof:— $\frac{\Delta AMN}{\Delta ABC} = \frac{AM^2}{AB^2}$ (Since the two Δ s are similar)

But $AM^2 = AP \cdot AB$ (construction)

$$\therefore \frac{AM^2}{AB^2} = \frac{AP \cdot AB}{AB \cdot AB} = \frac{AP}{AB} \therefore \frac{\Delta AMN}{\Delta ABC} = \frac{AP}{AB} = \frac{l}{l+m}$$

Hence $\frac{\Delta AMN}{\Delta ABC - \Delta AMN} = \frac{l}{l+m-l}$

i. e. $\frac{\Delta AMN}{\square MNCB} = \frac{l}{m}$.

2. Divide a triangle ABC into two parts whose areas are in the ratio $l:m$ by drawing a perpendicular to BC.

Analysis:— Let MN be the required perpendicular to BC. Draw AK perpendicular to BC.

$$\frac{\triangle CMN}{\square MNBA} = \frac{l}{m}$$

(Hypothesis)

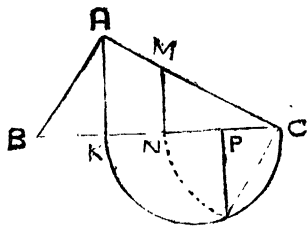


Fig. 44

$$\therefore \frac{\triangle CMN}{\triangle CAB} = \frac{l}{l+m}$$

Now $\frac{\triangle CMN}{\triangle CAK} = \frac{CN^2}{CK^2}$ (1) ($MN \parallel AK$)

$$\frac{\triangle CAK}{\triangle CAB} = \frac{CK}{CB} \quad (2) \text{ (Altitude AK is common)}$$

(1) \times (2) gives $\frac{\triangle CMN}{\triangle CAB} = \frac{CN^2}{CK \cdot CB}$

$$\therefore \frac{CN^2}{CK \cdot CB} = \frac{l}{l+m}$$

$$\frac{CN^2}{CK} = \frac{l}{l+m} \cdot CB.$$

i. e. CN is the mean proportional between CK and CP where CP is equal to $\frac{l}{l+m} \cdot CB$.

$$\frac{CP}{CB} = \frac{l}{l+m} \quad \therefore \frac{CP}{PB} = \frac{l}{m}$$

So P divides CB in the ratio $l:m$.

Construction:— Divide CB in the ratio $l:m$ at P (this is the same thing as dividing BC in the ratio $m:l$ at P). Find the mean proportional between CP and CK, where K is the foot of the perpendicular from A on BC. Let it be CN. Then N is the point at which the required perpendicular will have to be erected to BC. Hence draw a line through N perpendicular to BC, cutting AC at M. Then MN will be the required line.

$$\text{Proof: } - \frac{CP}{PB} = \frac{l}{m} \quad \therefore \frac{CP}{CB} = \frac{l}{l+m}$$

(by ratio and proportion)

$$\frac{\triangle CMN}{\triangle CAB} = \frac{\triangle CMN}{\triangle CAK} \cdot \frac{\triangle CAK}{\triangle CAB}$$

$$\text{But } \frac{\triangle CMN}{\triangle CAK} = \frac{CN^2}{CK^2} \quad (\because \text{they are similar})$$

$$\text{and } \frac{\triangle CAK}{\triangle CAB} = \frac{CK}{CB} \quad (\text{they have the same altitude})$$

$$\therefore \frac{\triangle CMN}{\triangle CAB} = \frac{CN^2}{CK^2} \cdot \frac{CK}{CB} = \frac{CN^2}{CK \cdot CB} = \frac{CP \cdot CK}{CK \cdot CB}$$

(CN is the mean proportional between CP and CK.)

$$\therefore \frac{\triangle CMN}{\triangle CAB} = \frac{CP}{CB} \quad \text{Hence } \frac{\triangle CMN}{\square MNBA} = \frac{l}{m}$$

3. Draw a triangle equal in area to a given triangle and similar to another given \triangle .

Let PQR and ABC be two given triangles. Then it is required to construct another triangle which is equal in area to $\triangle ABC$ and similar to $\triangle PQR$.

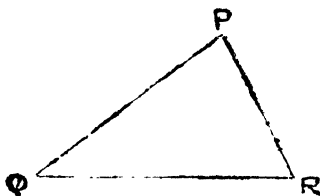


Fig. 45

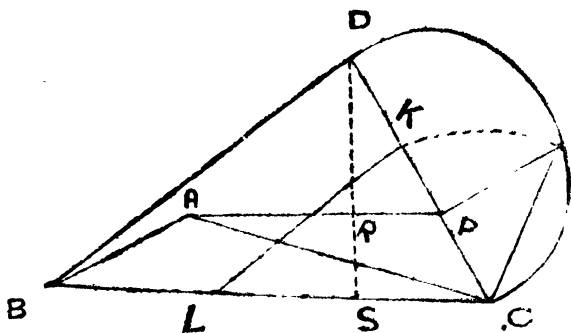


Fig. 46

Construction:— On BC construct a triangle DBC equiangular to $\triangle PQR$. Through A draw a parallel to BC , cutting DC at P . Find the mean proportional between CP and CD . Let it be CK .

Through K draw a parallel to DB, cutting BC at L.
Then $\triangle KLC$ will be the required triangle.

Proof:— $\triangle PQR$ and $\triangle DBC$ by construction are equiangular and hence similar.

Since $LK \parallel BD$, $\triangle KLC$ and $\triangle DBC$ are equiangular and hence similar.

$\therefore \triangle PQR$ and $\triangle KLC$ are also similar... (1)

If we draw $DRS \perp$ to parallel lines AP and BC cutting them at R and S respectively

$$\frac{CP}{CD} = \frac{SR}{SD} \text{ (by parallels)}$$

But $\frac{\triangle KLC}{\triangle DBC} = \frac{CK^2}{CD^2} = \frac{CP \cdot CD}{CD \cdot CD} = \frac{CP}{CD}$ (CK is the mean proportional between CP and CD)

$\therefore \frac{\triangle KLC}{\triangle DBC} = \frac{SR}{SD} = \frac{\triangle ABC}{\triangle DBC}$ ($\triangle ABC$ and $\triangle DBC$ have got the same base BC)

$\therefore \triangle KLC = \triangle ABC$ (2)

From (1) and (2) it follows that $\triangle KLC$ is the required triangle.

EXERCISES.

1. If a straight line m inches in length is divided internally and externally in the ratio $l : k$ find the lengths of the segments in each case.

2. A, B, C, D are four collinear points (points lying in a line) such that AC is divided at B and D in the ratio $k : l$. Show that DB is divided at C and A in the ratio $(k+1) : (k-1)$

3. AB is divided internally and externally in the same ratio at C and D : If O is the middle point of AB , prove that $OC \cdot OD \equiv OA^2 = OB^2$.

4. Two diameters AB and CD of a circle are divided in the same ratio internally and externally at $P, Q; R, S$ respectively, the ratio for AB and CD being the same or different. Prove that P, Q, R, S , are concyclic.

5. Show that three or more parallel lines cut any two transversals proportionally.

6. ABC is a triangle inscribed in a circle. Show that the perpendicular from A on BC is a mean proportional between the perpendiculars from B and C on the tangent at A .

[Hint. — If AP is the $\perp r$ from A to BC and BQ, CR \perp rs to the tangent at A from B and C, then figure APBQ is cyclic $\therefore \frac{AP}{BQ} = \frac{OP}{OQ}$ where O is the point of intersection of the tangent at A and BC. Similarly since figure ARCP is cyclic $\frac{CR}{AP} = \frac{OC}{OA}$. But since OA is a tangent at A, $AC \parallel QP$
 $\therefore \frac{OP}{OQ} = \frac{OC}{OA}$. Hence $\frac{AP}{BQ} = \frac{CR}{AP}$ or $AP^2 = BQ \cdot CR$].

7. ABCD is a quadrilateral; show that if the bisectors of the angles A and B meet in the diagonal BD, the bisectors of the angles B and D will meet on AC.

8. The bisector of angle A of $\triangle ABC$ meets BC in D and DE is drawn parallel to AC to cut AB at E and DF parallel to AB to cut AC at F. Prove that $\frac{BE}{CF} = \frac{AB^2}{AC^2}$

9. The median AD of a $\triangle ABC$ meets BC in D. The internal bisectors of angles ADB and ADC meet AB and AC in P and Q respectively. Prove that PQ is parallel to BC.

10. CA, CB are two tangents to a circle, A and B being the points of contact, E is the foot of the

perpendicular from B to AD, the diameter through A. Prove that BA, BD bisect angle CBE. Deduce that CD bisects BE.

11. The straight line BC is divided in the same ratio at D and E. DE subtends a right angle at P. Show that PD and PE bisect $\angle BPC$.

12. Show how to construct a triangle on a given base so as to have its vertical angle bisected by a given straight line.

[Hint:— Let the given straight line cut the given base, say BC or the base BC produced at the point D. Then D divides BC in the ratio $\frac{BD}{DC}$. Find the point E on BC or BC produced as the case may be, such that $\frac{BE}{CE} = \frac{BD}{DC}$. Draw the circle on DE as diameter to cut the straight line again at A. Then ABC will be the required triangle. The teacher is expected to explain the property of Apollonius' circle.]

13. Construct a parallelogram whose sides are 3.5", 2.7" and whose diagonals are in the ratio of 2 : 1.

14. I is the in-centre of the $\triangle ABC$. A straight line drawn through I perpendicular to AI meets AB , AC in D , E respectively. Show that $BD \cdot CE = ID^2$.

15. Two circles of radii a and b touch each other externally at the point k . AB is one of their common tangents, A and B being the points of contact. Show that AB subtends a right angle at k and that $AB^2 = 4ab$.

16. PM and QN are the perpendiculars from two given points P and Q to a given straight line AB . PN and QM intersect in R . If RS be drawn perpendicular to AB , show that PS and QS make equal angles with AB .

17. ABC is an isosceles triangle right angled at A . Any point P is taken on AB and BD is drawn perpendicular to BC on the side of BC opposite to A , of such length that $\frac{BD}{BC} = \frac{AP}{AC}$.

Prove that $\angle CPD$ is a right angle and $CP = DP$.

18. AB is a diameter of a circle. PQ is a parallel chord. The tangent at A meets BP , BQ in R and S respectively. Prove that $BP \cdot BR = BQ \cdot BS = AB^2$.

19. AB is a diameter of a circle and C is any point on the circumference. From a point on AB the perpendicular to AB is drawn cutting CA , CB and the circumference at D , E , F respectively. Prove that $PF^2 = PD \cdot PE$.

20. ABC is a triangle right angled at A . The bisector of the angle A meets the circumcircle in D . Show that $2 AD^2 = (AB + AC)^2$.

21. PA , PB are tangents from any point P on a circle to a concentric circle lying within it. PA and PB are produced to cut the outer circle in C and D . Show that $CB : CA = CD^2 : CP^2$.

[Hint:— Use that the length of a tangent drawn from any point on the outer circle to the inner concentric circle is constant]

22. Two circles intersect at A and B . The tangents at A meet the circles again in X and Y . Prove that $\triangle ABX : \triangle ABY = (AB^2 + BX^2) : (AB^2 + BY^2)$

23. If from the vertex of a \triangle , a perpendicular be drawn to the base, prove that the rectangle contained by the two sides is equal to the rectangle contained by the altitude of the base and the circumdiameter.

Hence deduce that in any triangle, $abc = 4R \Delta$

24. A transversal PQR cuts the sides BC, CA, AB of a $\triangle ABC$ in P, Q, R respectively. Prove that

$$\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} = -1 \text{ (Menelaus' Theorem)}$$

[Hint:— Draw perpendiculars from the vertices of the $\triangle ABC$ to the transversal and consider a set of similar triangles]

25. If one diagonal of a quadrilateral bisects the angle between two of the sides and be a mean proportional between them prove that the segments of the other diagonal are in the duplicate ratio of the other sides.

[Hint:— If ABCD is a quadrilateral in which AC bisects angle A and $AC^2 = AB \cdot AD$, $\triangle DAC$ and CAB are similar.

$$\text{Hence } \frac{AD}{AC} = \frac{DC}{BC} = \frac{CA}{AB} \therefore \frac{DC^2}{BC^2} = \frac{AD}{AB} = \frac{DO}{OB}$$

where O is the intersection of AC and BD]

26. ABCD is a cyclic quadrilateral. If $AB \cdot BC = CD \cdot DA$ prove that AC bisects BD.

[Hint:— Draw \perp rs BP, DQ on AC from B and D respectively. Then AB. EC = BP. 2R and CD. DA = DQ. 2R where R is the circum-radius, Since AB. BC = CD. DA, EP = DQ. Hence \triangle s CDQ and OBP are congruent, where O is the point of intersection of AC and BD.

\therefore OB = OD]

27. The tangent at A to the circumcircle of a triangle ABC meets LC produced at D. Show that $\frac{BD}{DC} = \frac{AB^2}{AC^2}$

Hence deduce that, if the tangents at A, B, C of a $\triangle ABC$ to the circumcircle meet the opposite sides in D, E, F respectively, then $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$.

28. AB is a diameter, of a circle and AM, BN are drawn perpendicular to the tangent at any point C on the circle. Show that the area ABC is the sum of the areas of ACM and CBN.

29. Prove that the opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles. Deduce the following:— The sides AB, DC of a quadrilateral ABCD meet at E and the sides AD, BC meet at F. Prove that the circles circumscribed to the four triangles

ABF, BCE, CDF and DAE have a common point, say, K. Prove further that if ABCD is cyclic, K lies on EF.

30 Draw a $\triangle ABC$, having $AB=4.2$ cm; $AC=5.3$ cm; and $\angle ABC=62^\circ$. Construct a \triangle similar to $\triangle ABC$ so as to have its greatest side less than the sum of the remaining two sides by 1.2 cm. Explain your construction.

31. Construct an equilateral triangle equal in area to a $\triangle PQR$ having $PQ=2.5''$, $QR=3''$ and the $\angle Q=40^\circ$.

32. Given that $AB=3.2''$, $C=75^\circ$ and $\frac{AC}{BC}=5:3$ show how to construct the triangle.

[Hint:— Construct a segment containing the angle 75° on AB. Divide AB internally and externally in the ratio 5 : 3 (say) at D and E and describe the circle on DE as diameter to cut the segment at C. Then ABC is the required triangle—The circle on DE as diameter becomes the Apollonius' circle of the two fixed points A and B for the ratio $\frac{5}{3}$. In numerical problems like this the alternate method is advisable. Draw a $\triangle AC_1B_1$ such that $C_1A=5''$, $C_1B_1=3''$ and $\angle AC_1B_1=75^\circ$. Then take a point B on AB_1 or AB_1 produced if necessary such that $AB=3.2''$

Through B draw a line parallel to B_1C_1 meeting AC_1 at C . Then ABC is the required triangle]

33. Draw a $\triangle ABC$ given $a : b : c = 3 : 4 : 6$ and $R = 2''$. Measure the sides and angles of the \triangle .

[Hint:— Draw a $\triangle PQR$ such that $p = 3''$, $q = 4''$ and $r = 6''$. Find the circumcentre T of $\triangle PQR$. With T as centre and $2''$ as radius draw a circle meeting TP , TQ , TR in A , B , C respectively. Then ABC is the required triangle]

34. ABC is a \triangle in which $a = 1.2''$, $b = 1.6''$ and $c = .9''$.

Draw an isosceles \triangle of equal area having the vertical angle equal to A . Measure its sides.

35. Draw a $\triangle ABC$ having $AB = 9$ cm; $BC = 8$ cm; $\angle ABC = 60^\circ$. Construct a triangle equal in area to $\triangle ABC$ and having its sides in the ratio $4 : 5 : 7$. State the construction and prove it.

[Hint:— Draw a \triangle whose sides are 4 cm, 5 cm and 7 cm and call it $\triangle PQR$. Then construct a \triangle similar to $\triangle PQR$ and equal in area to $\triangle ABC$]

36. The sides of a \triangle are 3 , 5 and 7 cm, Bisect the area of the \triangle by a line drawn (1) perpendicular to (2) parallel to the longest side.

37. Construct an equilateral Δ which is equal in area to a triangle ABC in which $a = 2.6''$, $b = 3''$, $B = 48^\circ$. Measure its sides.

38. Construct a parallelogram ABCD in which $AB = 2.5''$, $AD = 2''$, $AC = 3.4''$. Divide the parallelogram into three equal parts by straight lines parallel to the diagonal AC.

39. Construct a Δ whose sides have the ratios $5 : 7 : 9$ and whose area is 10 sq. Inches.

[Hint:— Apply $abc = 4R\Delta$, find R and proceed as in exercise No. 33].

40. Draw a ΔABC having the sides $a = 2.5''$, $b = 2''$ and $c = 1.5''$. Construct a similar triangle having two-thirds of the area of the ΔABC .

41. Construct a parallelogram of area equal to 6 square inches and having its sides in the ratio $3 : 2$ and having an angle 70° .

42. The sides of a Δ are 5, 12 and 13 cm. respectively. Show how to trisect the area of the Δ by a line drawn parallel to the longest side.

[Hint:— Divide the area of the Δ into two portions whose areas are in the ratio $1 : 2$ (this is trisection) by a straight line parallel to the longest side, as in construction 1.]

43. In a triangle ABC, AP and AQ are drawn perpendicular to the internal bisectors of the angles B and C. Prove that PQ is parallel to BC.

[Hint:— Apply angle chasing method]

44. One circle touches another internally at P. A straight line touches the inner circle at A and meets the outer circle at B and C. Prove that $PB : PC = AB : AC$

[Hint:— Apply, angle between the tangent at a point on a circle and the chord through the point of contact is equal to the angle subtended by that chord in the alternate segment of the circle, repeatedly and prove that PA is a bisector of $\angle CPB$].

45. The tangent to a circle at a point A on it, meets two parallel tangents at B and C. If O is the centre of the circle prove that $OA^2 = AB \cdot AC$.

[The points of contact of parallel tangents and the centre O are collinear. Hence OB, OC become bisectors of supplementary angles and are therefore at right angles. i. e. $\angle BOC = 90^\circ$. Further, OA is perpendicular to BC, the hypotenuse. $\therefore OA^2 = AB \cdot AC$.]

*** APPENDIX I.**

THEOREM 1.

If two triangles have one angle of the one equal to one angle of the other and the sides about one other angle in each proportional, then the third angles of the two triangles are either equal or supplementary and in the former case the two triangles are similar.

In triangles ABC and PQR let $\angle C = \angle R$ and $\frac{AB}{PQ} = \frac{CA}{RP}$. Then to prove that $\angle B$ and $\angle Q$ are either equal (in which case the two triangles will be similar) or supplementary.

One of the two things may happen for the two Δ s ABC and PQR . $\angle A$ will either be equal to $\angle P$ or will not be equal to $\angle P$. (These are the only possibilities).

If $\angle A$ and $\angle P$ are equal (as in fig. 47 and fig. 48(a)) since $\angle C$ and $\angle R$ are already equal $\angle B$ will be equal to $\angle Q$ i. e. the two Δ s become equiangular and hence similar.

* This may be omitted by the Pre-university students.

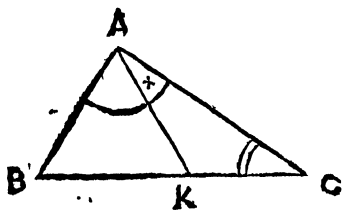


Fig. 47

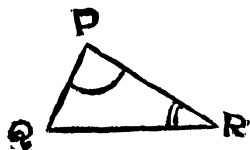


Fig. 48 (a)

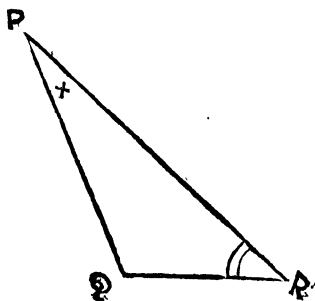


Fig. 48 (b)

If $\angle A \neq \angle P$ (as in fig. 47 and fig. 48(b)) draw a line AK through A meeting BC at K such that $\angle CAK = \angle RPQ$.

[Construction must be so effected that $\triangle CAK$ formed includes $\angle C$]

$$\angle CAK = \angle RPQ \text{ (construction).}$$

$$\angle KCA = \angle QRP \text{ (given).}$$

$$\therefore \triangle CAK \equiv \triangle RPQ$$

$$\therefore \frac{CA}{RP} = \frac{AK}{PQ}$$

But $\frac{AB}{PQ} = \frac{CA}{RP}$ (given)

$$\therefore \frac{AK}{PQ} = \frac{AB}{PQ}$$

i. e. $AK = AB$

$$\therefore \angle ABK = \angle AKB$$

i. e. $\angle ABC = \angle AKB$

$$\angle PQR = \angle AKC (\because \triangle RPQ \parallel \triangle CAK)$$

Now $\angle AKB$ and $\angle AKC$ are evidently supplementary angles.

$\therefore \angle ABC$ and $\angle PQR$ are also supplementary angles.

Note:— In fig. 47, $AB < CA$. Therefore in figures 48(a) and 48(b) PQ is less than RP in order that $\frac{AB}{PQ} = \frac{CA}{RP}$ i. e. in order that $\frac{AB}{CA} = \frac{PQ}{RP}$. This point must be borne in mind while drawing the three figures.

SECOND METHOD.

(figure is not necessary)

Given that in triangles ABC and PQR
 $\angle C = \angle R$ and $\frac{AB}{PQ} = \frac{CA}{RP}$ to prove that $\angle B$ and
 $\angle Q$ are either equal (in which case the two
triangles will be equiangular and hence similar) or
supplementary.

Proof:— Since $\angle C = \angle R$

$\frac{\triangle ABC}{\triangle PQR} = \frac{BC \cdot CA}{QR \cdot RP}$ (\because In two Δ s if an angle of
the one is equal to an angle of the other, their areas
are proportional to the rectangles contained by the
sides about those equal angles.)

$$\text{But } \frac{CA}{RP} = \frac{AB}{PQ} \text{ (given)}$$

$$\begin{aligned} \therefore \frac{\triangle ABC}{\triangle PQR} &= \frac{BC \cdot CA}{QR \cdot RP} = \left(\frac{BC}{QR} \right) \left(\frac{CA}{RP} \right) \\ &= \left(\frac{BC}{QR} \right) \left(\frac{AB}{PQ} \right) = \frac{AB \cdot BC}{PQ \cdot QR} \end{aligned}$$

$$\frac{\triangle ABC}{\triangle PQR} = \frac{\frac{1}{2} AB \cdot BC \cdot \sin \angle ABC}{\frac{1}{2} PQ \cdot QR \cdot \sin \angle PQR}$$

(From Trigonometry)

$$\therefore \frac{AB \cdot BC}{PQ \cdot QR} = \frac{AB \cdot BC \cdot \sin \angle ABC}{PQ \cdot QR \cdot \sin \angle PQR}$$

Hence $\frac{\sin \angle ABC}{\sin \angle PQR} = 1$ or $\sin \angle ABC = \sin \angle PQR$

\therefore either $\angle ABC = \angle PQR$ or $\angle ABC = 180^\circ - \angle PQR$
(each angle of a \triangle is less than 180°)

i. e. either $\angle B = \angle Q$ or $\angle B + \angle Q = 180^\circ$

If $\angle B = \angle Q$, as $\angle C = \angle R$ already, the two \triangle s ABC and PQR will be equiangular and hence similar.

EXERCISES.

1. A and B are the centres of two circles whose radii are respectively r_1 and r_2 . If S is a point dividing AB internally in the ratio $\frac{r_1}{r_2}$ prove that any secant through S cuts the circles at the extremities of parallel radii one in each circle.

[Hint:— If a secant cuts the circle A at P, Q and circle B at P^1 , Q^1 (draw a figure and in it take Q, Q^1 within P, P^1 or P, P^1 within Q, Q^1) consider \triangle s SAP and SBP^1 . To rule out the possibility of the third set of angles in the two

triangles becoming supplementary use that the base angles of any isosceles triangle are acute angles and that two angles which are both acute or both obtuse cannot be supplementary. $\triangle SAP \parallel \triangle SBP^1$. $\therefore AP \parallel BP^1$. Similarly consider $\triangle s$ SAQ and SBQ^1 . They are similar. $\therefore AQ \parallel BQ^1$. Hence the result. The point S is usually called the internal centre of similitude of the two circles A and B .]

2. Prove the above result for a point S^1 dividing AB externally in the ratio $\frac{r_1}{r_2}$.

[This point S^1 is usually called the external centre of similitude of the two circles A and B .

3. A and B are the centres of two circles whose radii are respectively r_1 and r_2 . S and S^1 are points dividing AB internally and externally in the ratio $\frac{r_1}{r_2}$. Prove that the lengths of the tangents drawn from any point on the circle on SS^1 as diameter to the two circles A and B are in the ratio $\frac{r_1}{r_2}$.

[Hint:— Use that the circle on SS^1 as diameter is the Apollonius' circle of the two fixed points

A, B for the constant ratio $\frac{r_1}{r_2}$. Now, if P is any point on this circle, $\frac{PA}{PB} = \frac{r_1}{r_2}$. Let PL, PM be the tangents from P to the two circles A and B.

Then $\frac{PA}{PB} = \frac{r_1}{r_2} = \frac{AL}{BM}$

Further $\angle PLA = \angle PMB = 90^\circ$.

$\therefore \angle APL$ and $\angle BPM$ are either equal or Supplementary. But these two angles being both acute, cannot be supplementary. Hence they are equal and therefore $\triangle ALP \parallel \triangle BMP$

$\therefore \frac{PL}{PM} = \frac{AL}{BM} = \frac{r_1}{r_2}$. The circle on SS' as diameter is usually called the circle of similitude of the two circles A and B]

The application of theorem 1 given in appendix I will also be found in the Geometry of the conic while proving the theorem that the tangent at any point on a central conic is equally inclined to the focal distances of the point.

THEOREM 2.

In equal circles, angles whether at the centres or at the circumferences, are in the ratio of the arcs on which they stand.

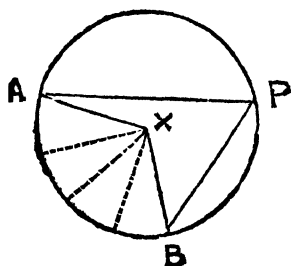


Fig. 49

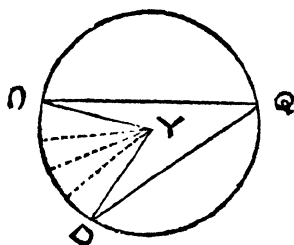


Fig. 50

Let APB and CQD be two equal circles whose centres are X and Y respectively; Also let AXB, CYD be the angles at the centres and APB, CQD those at the circumferences, standing on the arcs AB and CD respectively.

Then to prove that

$$(i) \quad \frac{\angle AXB}{\angle CYD} = \frac{\text{Arc AB}}{\text{Arc CD}}$$

$$(ii) \quad \frac{\angle APB}{\angle CQD} = \frac{\text{Arc AB}}{\text{Arc CD}}$$

Proof:— Suppose there is a common measure, say, k for the lengths of the two arcs AB and CD. [Making the two arcs straight, suppose we measure them and find AB to be 5 inches and CD to be 3 inches. This implies that Arc AB can be divided into 5 equal parts each equal to 1 inch.

and that arc CD can be divided into 3 equal parts each equal to 1 inch. If this is so, the common measure of the two arcs AB and CD is an inch and the ratio of AB to CD is $\frac{5}{3}$. Now, if AB = 4·5 inches and CD = 3 inches the common measure for the two is ·5 of an inch, for, AB in this case can be divided into 9 equal parts each equal to ·5 of an inch and CD can be divided into 6 equal parts each equal to ·5 of an inch. The ratio of AB to CD in this case is $\frac{9}{6}$ ($= \frac{4·5}{3}$). Hence whatever may be the lengths of the two arcs AB and CD it will be quite possible to find out a common measure for the two and so our supposition is justified]

Let the arc AB be divided into p equal arcs each equal to k and arc CD be divided into q such equal arcs (each equal to k).

In each circle, let radii be drawn through the points of division of the arcs AB and CD.

Then, $\angle AXB$ is divided into p equal angles each equal to θ (say) and $\angle CYD$ is divided into q such equal angles (each equal to θ). [\because In equal circles, equal arcs subtend equal angles at the centres. This theorem is also true in the same circle.]

$$\therefore \angle AXB = p\theta; \angle CYD = q\theta$$

$$\text{So } \frac{\angle AXB}{\angle CYD} = \frac{p\theta}{q\theta} = \frac{p}{q}$$

But Arc AB = pk and Arc CD = qk and hence

$$\frac{\text{Arc AB}}{\text{Arc CD}} = \frac{pk}{qk} = \frac{p}{q}$$

$$\therefore \frac{\angle AXB}{\angle CYD} = \frac{\text{Arc AB}}{\text{Arc CD}}$$

Now $\angle APB = \frac{1}{2} \angle AXB$ and $\angle CQD = \frac{1}{2} \angle CYD$.
 [∵ The angle subtended by an arc of a circle at the centre is equal to double the angle subtended by that arc at any point on the circumference.]

$$\therefore \frac{\angle APB}{\angle CQD} = \frac{\frac{1}{2} \angle AXB}{\frac{1}{2} \angle CYD} = \frac{\angle AXB}{\angle CYD} = \frac{\text{Arc AB}}{\text{Arc CD}}$$

Note:— This theorem is true in the same circle also.

WORKED EXAMPLE.

Prove that a radian is a constant angle.

Radian by definition is an angle subtended at the centre of any circle by an arc equal in length to the radius of the circle.

This definition does not explicitly tell us that radian is a constant angle. If radian is not a constant angle we won't be able to accept radian as a unit of measurement for angles. So it is imperative to prove that radian is a constant angle. It is because right angle is a constant angle (all right angles are equal) that we accept right angle as a unit of measurement for angles. Any angle can be measured in terms of right angles and their parts.

(1 degree = $\frac{1}{90}$ th of a right angle, 1 minute = $\frac{1}{60}$ th of a degree and 1 second = $\frac{1}{60}$ th of a minute). When we say that an angle is equal to (say) 240 degrees, this statement actually means that the angle measured is equal to 2 right angles plus 60 degrees or is equal to $2\frac{2}{3}$ right angles. We will have to note that radians are not divided and subdivided like right angles.

All right angles are equal (this is axiomatic). Hence we can say that right angle is a constant angle. In the case of radians, if it is possible to prove that all radians are equal, it will follow that radian is a constant angle. Suppose OA and OB are two radii of a circle whose centre is O, such that Arc AB = OA = OB = r where r is the radius of

the circle. Producing OA and OB to meet a concentric circle of radius r_1 at A_1 and B_1 if we are able to prove that $\text{arc } A_1 B_1 = OA_1 = OB_1 = r_1$, then it will follow that radian is a constant angle. But it is not very easy to establish this result. Hence we shall proceed with a circle and find out the value of a radian defined in terms of the radius and arc, equal to the radius, of this circle, in degrees and thus see that the value of a radian has nothing to do with the radius of the circle taken to define it.

Let OA and OB be two radii of a circle, centre O and radius r , such that $\text{arc } AB = r$. (Draw a figure). Then by definition $\angle AOB$ is a radian. Produce AO to meet the circle again at the point K .

$$\text{Then, } \text{arc } AK = \frac{2\pi r}{2} = \pi r.$$

$$\text{Hence } \frac{\angle AOK}{\angle AOB} = \frac{\text{Arc } AK}{\text{Arc } AB} \quad (\text{Vide note given under theorem 2, appendix I})$$

$$\text{i. e. } \frac{\angle AOK}{\text{a radian}} = \frac{\pi r}{r} = \frac{\pi}{1}$$

$$\therefore \angle AOK = \pi \text{ of a radian or } \pi \text{ radians.}$$

$$\text{But } \angle AOK = 180^\circ.$$

$$\text{Hence } 180^\circ = \pi \text{ radians.}$$

(π represents only a number whose approximate value is $\frac{22}{7}$ and hence the student must not try to replace π by 180° . The equation $180 \text{ degrees} = \pi \text{ radians}$ implies that an angle measured in degrees and found to be 180° when measured in radians will be equal to π radians. It is important to remember that the value or magnitude of an angle does not change whether it is measured in degrees or in radians. So there must have a relationship between degrees and radians and that relationship is given above.)

$$\pi \text{ radians} = 180^\circ$$

$$\therefore \text{ a radian} = \frac{180^\circ}{\pi}$$

(||| ly starting with another circle we can prove that, a radian = $\frac{180^\circ}{\pi}$ i. e. this result is independent of the radius of the circle, taken)

Hence a radian is a constant angle.

The value of a radian is approximately equal to $57^\circ 17' 45''$ (taking the approximate value of $\pi = 3.14159$) — [The ratio of the circumference to the diameter is found to be the same for all

circles i. e. this ratio is a constant. This constant ratio (a number) though its value cannot be actually found out is represented by π . Since its value cannot be found out, this number π (this number represented by π) is called a transcendental number.

Thus we have proved that a radian is a constant angle and henceforth we can talk of “the radian”

$$\pi \text{ radian} = 180^\circ$$

$$\therefore 2\pi \text{ radians} = 360^\circ$$

So, if equal arcs each equal to the radius are cut off from the circumference of a circle we will be getting six such equal arcs each subtending a radian at the centre of the circle. The remaining portion $r(2\pi - 6)$ of the circumference will be subtending an angle equal to $(2\pi - 6)$ of a radian at the centre.

Now if OA, OB are two radii of a circle, centre O and radius r such that arc AB = r and OC is any other radius, as before,

$$\frac{\angle AOC}{\angle AOB} = \frac{\text{Arc AC}}{\text{Arc AB}}$$

$$\text{i. e.} \quad \frac{\angle \text{AOC}}{\text{a radian}} = \frac{\text{Arc AC}}{r}$$

$$\therefore \angle \text{AOC} = \left(\frac{\text{Arc AC}}{r} \right) \text{ of a radian}$$

Result 1:— From this it follows that the length of an arc of a circle divided by the radius of the circle gives the magnitude of the angle which the arc subtends at the centre in radians.

Result 2:— When two straight lines meet they are said to contain an angle. Suppose, two straight lines OX and OY meet at O and that we want to measure the angle which they contain, viz. $\angle \text{XOY}$ in radians. For this, draw a circle with O as centre and any length as radius to cut OX at A and OY at B. Measure the length of the arc AB and also the radius of the circle drawn, both in the same units of measurement.

Then $\frac{\text{the length of the arc AB}}{\text{radius of the circle drawn}}$ will give the magnitude of the angle XOY in radians.

Result 3:— The length of an arc of a circle is equal to the product of its radius and the radian measure of the angle the arc subtends at the centre of the circle.

APPENDIX II

NEDIANS OF A PLANE TRIANGLE.

Definition of medians has been recently extended by Mr. John Satterly, thus: "If the sides of a triangle are, in order, divided such that the short section of each side is $\frac{1}{n}$ th the length of the side, and the points of subdivision are joined to the opposite angular points, the joining lines may be called the Nedians of the triangle (the name recalls the n) and the triangle formed by the nedians may be called the nedian triangle". We get the medians from this definition of nedians by putting $n=2$.

Again according to Satterly if A_1, B_1, C_1 are points on the sides BC, CA, AB respectively of a $\triangle ABC$ such that $\frac{BA_1}{BC} = \frac{CB_1}{CA} = \frac{AC_1}{AB} = \frac{1}{n}$ (where BA_1, CB_1, AC_1 are short sections of the sides BC, CA, AB respectively) then AA_1, BB_1, CC_1 are the forward nedians and the triangle formed by them is the forward nedian triangle; If A_2, B_2, C_2 are points on the sides BC, CA, AB respectively such that

* This may be omitted by the Pre-university students.

$\frac{CA_2}{CB} = \frac{BC_2}{BA} = \frac{AB_2}{AC} = \frac{1}{n}$ (where CA_2, BC_2, AB_2 , are short sections of the sides CB, BA, AC respectively) then AA_2, BB_2, CC_2 are the backward medians and the triangle formed by them is the backward median triangle. When $n=2$, A_1 and A_2 both coincide with the middle point of BC . Hence AA_1, AA_2 both coincide with the median through the vertex A . Similarly BB_1, BB_2 , both coincide with the median through the vertex B and CC_1, CC_2 both coincide with the median through the vertex C . So the medians can be regarded both as forward medians and as backward medians. It is interesting to note that in two similar triangles corresponding medians are also in the ratio of corresponding sides.

THEOREM 1

(Satterly's Theorem)

If AA_1, BB_1, CC_1 are Medians (either forward medians or backward medians) to the sides BC, CA, AB respectively, of $\triangle ABC$, then

$$AA_1^2 + BB_1^2 + CC_1^2 = \left(\frac{n^2 - n + 1}{n^2} \right) (AB^2 + BC^2 + CA^2)$$

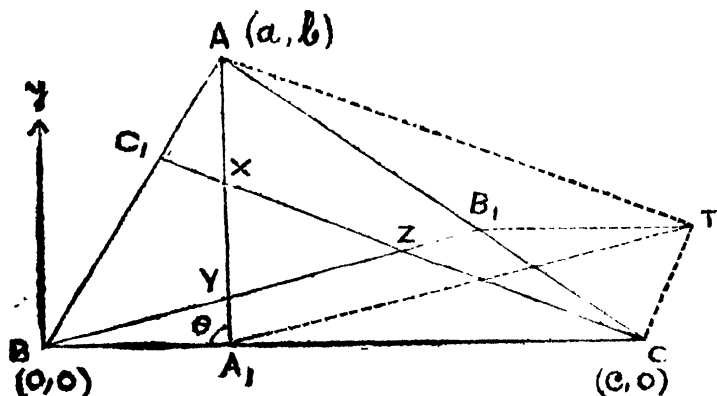


Fig. 51

Let us suppose that AA_1, BB_1, CC_1 are the forward medians so that

$$\frac{BA_1}{BC} = \frac{CB_1}{CA} = \frac{AC_1}{AB} = \frac{1}{n}$$

from this, $\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = \frac{1}{n-1}$

i. e. $(n-1) BA_1 = A_1C$; $(n-1) CB_1 = B_1A$;
 $(n-1) AC_1 = C_1B$.

If $\angle AA_1B = \theta$, $\angle AA_1C = (180 - \theta)$

$$\therefore AB^2 = AA_1^2 + BA_1^2 - 2AA_1 \cdot BA_1 \cdot \cos \theta \quad (1) \text{ and}$$

$$CA^2 = AA_1^2 + A_1C^2 + 2AA_1 \cdot A_1C \cdot \cos \theta \quad (2)$$

Multiplying (1) by $(n-1)$ and then adding to (2), we get, $(n-1) AB^2 + CA^2 = nAA_1^2 +$

$$(n-1) BA_1^2 + A_1C^2$$

But $BA_1 = \frac{BC}{n}$ and $A_1C = \frac{n-1}{n} BC$.

$$[(n-1) BA_1 = A_1C]$$

$$\therefore (n-1) AB^2 + CA^2 = nAA_1^2 +$$

$$(n-1) \frac{BC^2}{n^2} + \frac{(n-1)^2}{n^2} BC^2$$

$$= nAA_1^2 + \frac{n-1}{n^2} BC^2 [1+n-1]$$

$$= nAA_1^2 + \frac{n-1}{n} BC^2$$

$$\text{i. e. } (n-1)AB^2 + CA^2 = nAA_1^2 + \left(1 - \frac{1}{n}\right)BC^2 \quad \dots I$$

$$\text{Similarly } (n-1)BC^2 + AB^2 = nBB_1^2 + \left(1 - \frac{1}{n}\right)CA^2 \quad \dots II$$

$$\text{and } (n-1)CA^2 + BC^2 = nCC_1^2 + \left(1 - \frac{1}{n}\right)AB^2 \quad \dots III$$

I+II+III gives,

$$n[AB^2 + BC^2 + CA^2] = n[AA_1^2 + BB_1^2 + CC_1^2] + \left(1 - \frac{1}{n}\right)(AB^2 + BC^2 + CA^2)$$

$$\text{i. e. } (AB^2 + BC^2 + CA^2) \left(n + \frac{1}{n} - 1 \right) = n (AA_1^2 + BB_1^2 + CC_1^2)$$

$$\therefore AB^2 + BC^2 + CA^2 = \left(\frac{n^2}{n^2 - n + 1} \right) (AA_1^2 + BB_1^2 + CC_1^2)$$

$$\text{i. e. } AA_1^2 + BB_1^2 + CC_1^2 = \left(\frac{n^2 - n + 1}{n^2} \right) (AB^2 + BC^2 + CA^2).$$

If the lengths of the medians AA_1 , BB_1 , CC_1 are respectively represented by n_1 , n_2 , n_3 ,

$$n_1^2 + n_2^2 + n_3^2 = \left(\frac{n^2 - n + 1}{n^2} \right) (a^2 + b^2 + c^2)$$

where a , b , c as usual denote the lengths of the sides BC , CA and AB , respectively, of $\triangle ABC$

$$\text{or } \geq n_1^2 = \left(\frac{n^2 - n + 1}{n^2} \right) \geq a^2$$

As a particular case, if AA_1 , BB_1 , CC_1 are the medians ($n=2$)

$$AA_1^2 + BB_1^2 + CC_1^2 = \frac{3}{4} (AB^2 + BC^2 + CA^2)$$

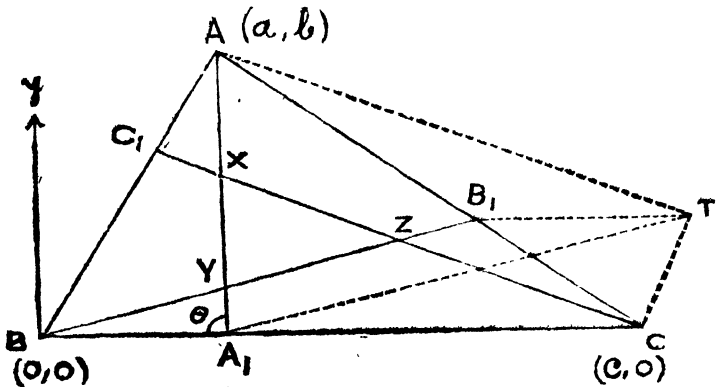
(This result can also be proved separately as in the case of medians).

Note:— The result is true in the case of backward medians also.

THEOREM 2.

(Satterly's Theorem)

The area of a triangle whose sides are equal to the lengths of the medians of a triangle is $\left(\frac{n^2 - n + 1}{n^2}\right)$ times the area of the original triangle.



ABC is the original triangle and AA_1 , BB_1 , CC_1 are its medians to the sides BC, CA, AB respectively.

Draw a line through B_1 , parallel to BC and another line through C parallel to AB. Let these two lines meet at the point T. Join A, T; A_1 , T. Then we shall first prove that $\triangle AA_1T$ is a

triangle whose sides are equal to the lengths of the medians and further that the area of $\triangle AA_1T$ is equal to $\left(\frac{n^2 - n + 1}{n^2}\right)$ times the area of the triangle ABC.

Proof (Analytical proof):— Let line BC be taken as the x-axis and the perpendicular at B to BC be taken as the y-axis. Also let the x and y co-ordinates of the point A be a and b respectively and the x co-ordinate of the point C be c. (The y co-ordinate of the point C is evidently zero).

The point A_1 divides BC in the ratio 1:n-1
 $\left(\frac{BA_1}{BC} = \frac{1}{n}\right) \quad \therefore \text{The point } A_1 \text{ is } \left(\frac{c}{n}, 0\right)$

||ly the point B_1 is $\left[\frac{a+c(n-1)}{n}, \frac{b}{n}\right]$ and the point C_1 is $\left[-\frac{a(n-1)}{n}, \frac{b(n-1)}{n}\right]$

Equation of the line B_1T which is passing through the point B_1 $\left[\frac{a+c(n-1)}{n}, \frac{b}{n}\right]$ and is parallel to the x-axis is $y = \frac{b}{n} \dots\dots I$

Slope of line CT is the same as that of AB (CT || AB) and is equal to $-\frac{b}{a}$

Hence equation of the line CT is $\frac{y-0}{x-c} = \frac{b}{a}$ or

$$ay = bx - bc$$

$$\therefore bx = ay + bc \dots\dots II$$

Solving I and II we get the co-ordinates of the point T

$$bx = \frac{ab}{n} + bc$$

$$\therefore x = \frac{a + cn}{n}$$


$$y = \frac{b}{n}$$

\therefore The point T is $\left(\frac{a + cn}{n}, \frac{b}{n} \right)$

The slope of TA is equal to $\frac{\frac{b}{n} - b}{\frac{a + cn}{n} - a} = \frac{b - nb}{a + n(c - a)}$

The slope of $CC_1 = \frac{\left[\frac{b(n-1)}{n} - 0 \right]}{\left[\frac{a(n-1)}{n} - c \right]} = \frac{nb - b}{n(a - c) - a}$
 $= \frac{b - nb}{a + n(c - a)}$

$\therefore TA \parallel CC_1$. But $CT \parallel AB$ (Construction)

Hence $TA C_1 C$ is a .

$$\therefore TA = CC_1$$

$$\text{The slope of } A_1T = \frac{\left(\frac{b}{n} - 0\right)}{\left(\frac{a+cn}{n} - \frac{c}{n}\right)} = \frac{b}{a+c(n-1)}$$

$$\begin{aligned} \text{The slope of } BB_1 &= \frac{\left(\frac{b}{n} - 0\right)}{\left\{\frac{a+c(n-1)}{n} - 0\right\}} \\ &= \frac{b}{a+c(n-1)} \end{aligned}$$

$\therefore A_1T \parallel BB_1$. But $B_1T \parallel BC$ (construction)

Hence TB_1BA_1 is a \square $\therefore A_1T = BB_1$

The median AA_1 is already a side of the $\triangle AA_1T$. Hence the sides of the $\triangle AA_1T$ are equal to the lengths of the medians of the $\triangle ABC$.

By the formula, $\Delta = \frac{1}{2} \sum x_1 (y_2 - y_3)$,

we get, area of the $\triangle AA_1T =$

$$\begin{aligned} &\frac{1}{2} \left[a \left(0 - \frac{b}{n}\right) + \frac{c}{n} \left(\frac{b}{n} - b\right) + \left(\frac{a+cn}{n}\right) (b-0) \right] \\ &= \frac{1}{2} \left[-\frac{ab}{n} + \frac{bc}{n^2} - \frac{bc}{n} + \frac{ab}{n} + bc \right] \\ &= \frac{bc}{2} \left[\frac{n^2 - n + 1}{n^2} \right] \end{aligned}$$

But area of $\triangle ABC = \frac{1}{2} \times BC \times$ (altitude to the base BC)

$$= \frac{1}{2} \times c \times b = \frac{bc}{2} .$$

Hence, area of $\triangle AA_1T = \left(\frac{n^2 - n + 1}{n^2}\right)$ (area of the $\triangle ABC$)

Note the common factor $\frac{n^2 - n + 1}{n^2}$ in the two theorems

As a particular case, the area of a \triangle whose sides are equal to the lengths of the medians of a triangle is $\frac{3}{4}$ times the area of the original triangle, for the medians AA_1, BB_1, CC_1 become medians when $n=2$.

(This result can also be separately proved as in the previous case).

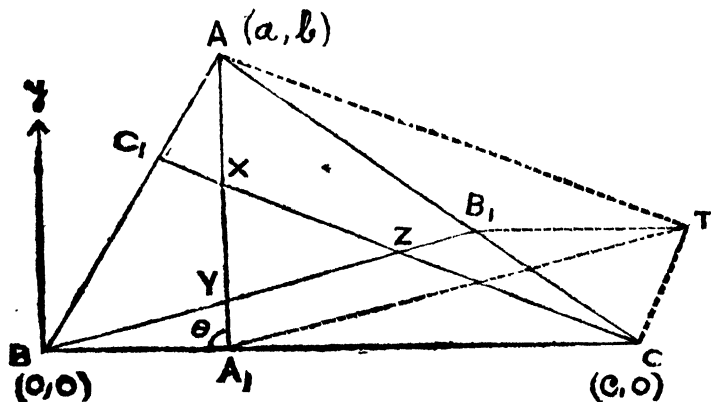
Def:— The triangle formed by the medians of a triangle is called the median triangle of that triangle.

In the figure, XYZ is the triangle formed by the medians AA_1, BB_1, CC_1 of $\triangle ABC$ and hence is the median triangle (Strictly speaking it is the forward median triangle, as AA_1, BB_1, CC_1 are forward medians.) of $\triangle ABC$

THEOREM 3

(Satterly's Theorem)

The sum of the squares of the sides of the median triangle of a triangle is equal to $\frac{(n-2)^2}{n^2-n+1}$ times the sum of the squares of the sides of the original triangle.



Equation of line AA_1 is, $nbx - (na - c)y = bc$...I

Equation of line BB_1 is, $y = \left(\frac{b}{a + cn - c} \right) x$ II

Equation of line CC_1 is, $x(nb - b) + y(a + cn - an) = nbc - bc$...III

Solving I and III we get the co-ordinates of the point X.

∴ The point X is $\left[\frac{a(n-1)^2 + c}{n^2 - n + 1}, \frac{b(n-1)^2}{n^2 - n + 1} \right]$

Solving I and II we get the co-ordinates of the point Y

∴ The point Y is $\left[\frac{a+c(n-1)}{n^2 - n + 1}, \frac{b}{n^2 - n + 1} \right]$

Now AX, AY and AA₁, are in the ratio of the perpendicular distances of the points X, Y and A₁ from the line through A perpendicular to BC. (∵ Corresponding sides in similar triangles are proportional)

$$\begin{aligned} \text{i. e. } AX:AY:AA_1 &:: \left(\frac{a(n-1)^2 + c}{n^2 - n + 1} - a \right) \\ &:: \left(\frac{a+c(n-1)}{n^2 - n + 1} - a \right) : \left(\frac{c}{n} - a \right) \end{aligned}$$

$$\text{i. e. } AX:AY:AA_1 :: \frac{1}{n^2 - n + 1} : \frac{n-1}{n^2 - n + 1} : \frac{1}{n}$$

$$\therefore \frac{AY}{AX} = \frac{n-1}{1} \quad \text{Hence} \quad \frac{XY}{AX} = \frac{n-2}{1} \quad \dots \dots (1)$$

$$\frac{AX}{AA_1} = \frac{n}{n^2 - n + 1} \quad \dots \dots \dots (2)$$

$$(1) \times (2) \text{ gives } \frac{XY}{AA_1} = \frac{n(n-2)}{n^2 - n + 1}$$

If AA_1 , BB_1 , CC_1 are represented by n_1 , n_2 , n_3 and YZ , ZX , XY by x , y , z .

$$A_1T = BB_1 = n_2; TA = CC_1 = n_3$$

Now $\triangle s$ XYZ and AA_1T are similar.

$$\therefore \frac{XY}{AA_1} = \frac{YZ}{A_1T} = \frac{ZX}{TA}$$

$$\text{i. e. } \frac{z}{n_1} = \frac{x}{n_2} = \frac{y}{n_3} = \frac{n(n-2)}{n^2-n+1}$$

$$\therefore x^2 + y^2 + z^2 = \left\{ \frac{n(n-2)}{n^2-n+1} \right\}^2 (n_1^2 + n_2^2 + n_3^2)$$

$$\text{But } n_1^2 + n_2^2 + n_3^2 = \left(\frac{n^2 - n + 1}{n^2} \right) \times (a^2 + b^2 + c^2)$$

where a , b , c represent the sides BC , CA , AB of $\triangle ABC$. (by theorem I Appendix II)

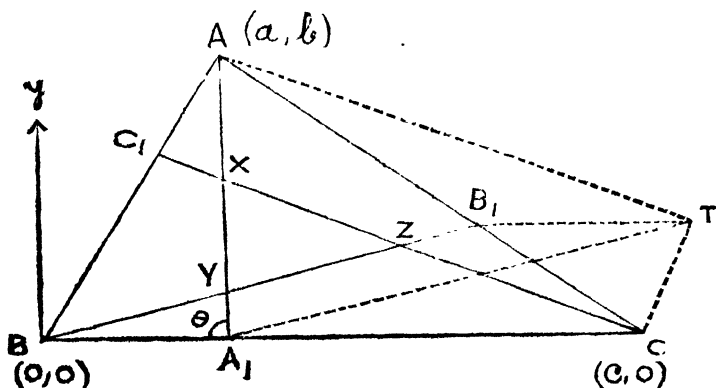
$$\begin{aligned} \therefore x^2 + y^2 + z^2 &= \left\{ \frac{n(n-2)}{n^2-n+1} \right\}^2 \times \frac{(n^2-n+1)}{n^2} \times (a^2 + b^2 + c^2) \\ &= \left\{ \frac{(n-2)^2}{n^2-n+1} \right\} (a^2 + b^2 + c^2) \end{aligned}$$

Note:— When AA_1 , BB_1 , CC_1 become the medians ($n=2$). $x^2 + y^2 + z^2 = 0 \quad \therefore x=0, y=0, z=0$. Hence the medians form a 'point triangle' or in other words, the medians of a triangle are concurrent.

THEOREM 4

(Satterly's Theorem)

The area of the median triangle of a triangle is equal to $\frac{(n-2)^2}{n^2-n+1}$ times the area of the original triangle.



$$\frac{\Delta XYZ}{\Delta AA_1T} = \frac{z^2}{n_1^2} = \frac{x^2}{n_2^2} = \frac{y^2}{n_3^2} = \frac{x^2 + y^2 + z^2}{n_1^2 + n_2^2 + n_3^2} = \left\{ \frac{n(n-2)}{n^2-n+1} \right\}^2$$

$$\therefore \text{Area of } \Delta XYZ = \left\{ \frac{n(n-2)}{n^2-n+1} \right\}^2 (\text{area of } \Delta AA_1T)$$

$$\text{But area of } \Delta AA_1T = \left(\frac{n^2-n+1}{n^2} \right) (\text{area of triangle ABC})$$

Hence, area of $\triangle XYZ$

$$= \left\{ \frac{n(n-2)}{n^2-n+1} \right\}^2 \left(\frac{n^2-n+1}{n^2} \right) \cdot (\text{area of triangle } ABC)$$

$$= \left\{ \frac{(n-2)^2}{n^2-n+1} \right\} (\text{Area of triangle } ABC)$$

Note:— When AA_1, BB_1, CC_1 , become the medians ($n=2$), area of the triangle formed by these three lines i. e. area of $\triangle XYZ=0$ i. e. the medians of a \triangle are concurrent.

Note the common factor $\frac{(n-2)^2}{n^2-n+1}$ in the two theorems.

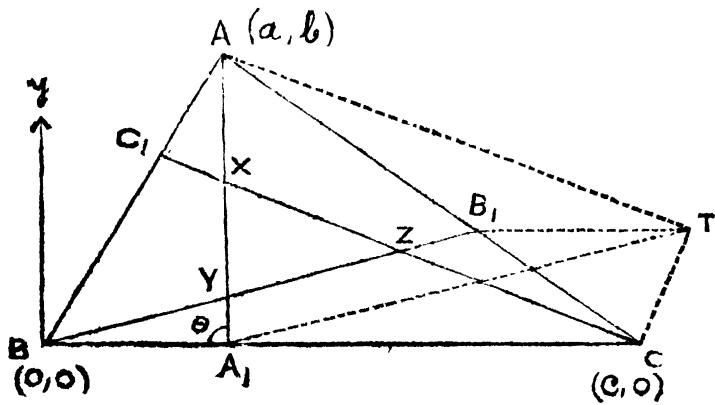
Definition:— If A_1, B_1, C_1 are points on the sides BC, CA, AB respectively of a $\triangle ABC$ such that $\frac{BA_1}{BC} = \frac{CB_1}{CA} = \frac{AC_1}{AB} = \frac{1}{n}$ the triangle formed by the lines joining the points A_1, B_1 and C_1 is called the Aliquot triangle or the $\frac{1}{n}$ th point-division triangle of $\triangle ABC$.

THEOREM 5.

(Satterly's Theorem)

The sum of the squares of the sides of the aliquot triangle of a triangle is equal to

$\left(\frac{n^2 - 3n + 3}{n^2}\right)$ times the sum of the squares of the sides of the original triangle.



Join $A_1, B_1; B_1, C_1$ and C_1, A_1 .

From ΔAB_1C_1 we get

$$B_1C_1^2 = AB_1^2 + AC_1^2 - 2AB_1 \cdot AC_1 \cdot \cos A.$$

If a_1, b_1, c_1 denote the lengths of the sides B_1C_1, C_1A_1 and A_1B_1 respectively of the Aliquot triangle $A_1B_1C_1$,

$$a_1^2 = \frac{(n-1)^2}{n^2} b^2 + \frac{c^2}{n^2} - \frac{2(n-1)}{n^2} bc \cos A \dots\dots\dots(1)$$

$$\left(\because \frac{AB_1}{AC} = \frac{n-1}{n}, \frac{AC_1}{AB} = \frac{1}{n}\right)$$

$$\text{||ly } b_1^2 = \frac{(n-1)^2}{n^2} c^2 - \frac{a^2}{n^2} - \frac{2(n-1)}{n^2} ca \cos B \dots(2)$$

$$\text{and } c_1^2 = \frac{(n-1)^2}{n^2} a^2 + \frac{b^2}{n^2} - \frac{2(n-1)}{n^2} ab \cos C \dots (3)$$

(1) + (2) + (3) gives,

$$\begin{aligned} a_1^2 + b_1^2 + c_1^2 &= (a^2 + b^2 + c^2) \left\{ \frac{(n-1)^2}{n^2} + \frac{1}{n^2} \right\} - \\ &\quad \frac{(n-1)}{n^2} (2bc \cos A + 2ca \cos B + 2ab \cos C) \\ &= (a^2 + b^2 + c^2) \left\{ \frac{(n-1)^2}{n^2} + \frac{1}{n^2} \right\} - \\ &\quad \frac{(n-1)}{n^2} (a^2 + b^2 + c^2) \\ &\quad (\because a^2 = b^2 + c^2 - 2bc \cos A \text{ etc.}) \end{aligned}$$

$$\text{i. e. } a_1^2 + b_1^2 + c_1^2$$

$$\begin{aligned} &= (a^2 + b^2 + c^2) \left\{ \frac{(n-1)^2}{n^2} + \frac{1}{n^2} - \frac{(n-1)}{n^2} \right\} \\ &= \left(\frac{n^2 - 3n + 3}{n^2} \right) (a^2 + b^2 + c^2) \end{aligned}$$

As a particular case, AA_1 , BB_1 , CC_1 become medians ($n=2$),

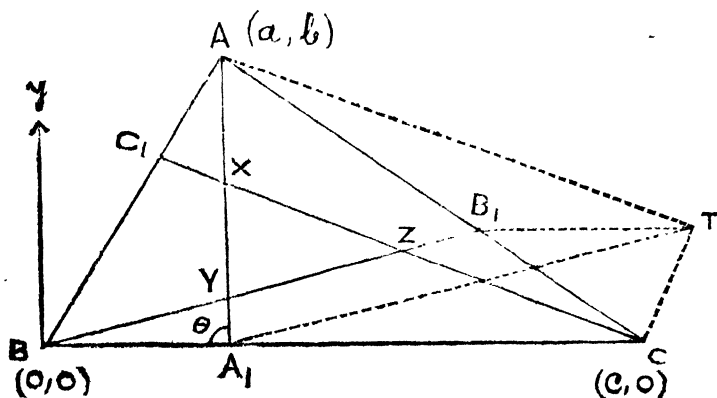
$$a_1^2 + b_1^2 + c_1^2 = \frac{1}{4} (a^2 + b^2 + c^2)$$

i. e. the sum of the squares of the sides of the medial triangle of a triangle is $\frac{1}{4}$ th of the sum of the squares of the sides of the original triangle. This result is even otherwise evident. (The Δ formed by the lines joining the mid-points of the sides of a Δ is called the medial triangle of that triangle.)

THEOREM 6

(Satterly's Theorem)

The area of the aliquot triangle of a triangle is equal to $\left(\frac{n^2 - 3n + 3}{n^2}\right)$ times the area of the original triangle.



Area of $\triangle C_1B_1A = \frac{1}{2} AB_1 \cdot AC_1 \cdot \sin A$

$$= \frac{1}{2} \frac{(n-1)}{n} \cdot b \cdot \frac{c}{n} \sin A$$

$$= \frac{(n-1)}{n^2} \cdot \frac{1}{2} bc \sin A \dots (1)$$

Similarly Area of $\triangle A_1C_1B = \frac{(n-1)}{n^2} \cdot \frac{1}{2} ca \sin B \dots (2)$

and Area of $\triangle B_1A_1C = \frac{(n-1)}{n^2} \cdot \frac{1}{2} ab \sin C \dots (3)$

(1) + (2) + (3) gives

Area of $\triangle ABC$ — Area of $\triangle A_1B_1C_1$

$$= \frac{(n-1)}{n^2} \left\{ \frac{1}{2} bc \sin A + \frac{1}{2} ca \sin B + \frac{1}{2} ab \sin C \right\}$$

But $\frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C =$ area of $\triangle ABC$

\therefore Area of $\triangle ABC$ — Area of $\triangle A_1B_1C_1$

$$= \frac{3(n-1)}{n^2} (\text{area of } \triangle ABC)$$

$$\begin{aligned} \therefore \text{Area of } \triangle A_1B_1C_1 &= \left\{ 1 - \frac{3(n-1)}{n^2} \right\} (\text{area of } \triangle ABC) \\ &= \left(\frac{n^2 - 3n + 3}{n^2} \right) (\text{area of } \triangle ABC) \end{aligned}$$

As a particular case, the area of the medial triangle of a triangle is $\frac{1}{4}$ th of the area of the original \triangle (Put $n=2$). This result is even otherwise evident.

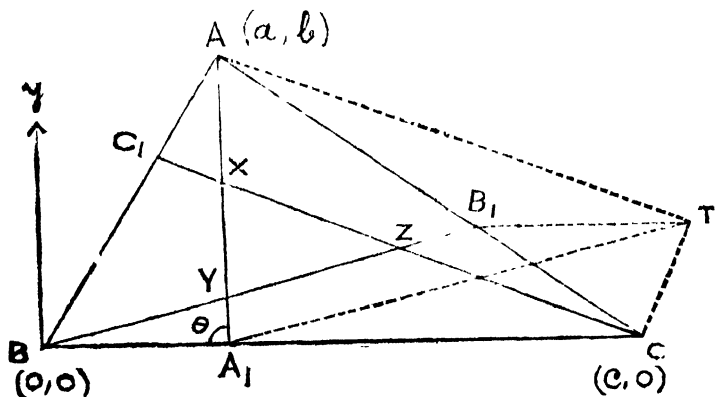
Note the common factor $\frac{n^2 - 3n + 3}{n^2}$ in the above two theorems.

THEOREM 7.

(Pappus' Theorem)

If A_1, B_1, C_1 are points on the sides BC, CA, AB respectively of $\triangle ABC$ such that $\frac{BA_1}{BC} = \frac{CB_1}{CA} = \frac{AC_1}{AB} = \frac{1}{n}$, the centroids of the two triangles $A_1B_1C_1$ and ABC are one and the same point.

(Note that $A_1B_1C_1$ by definition, becomes the aliquot triangle of $\triangle ABC$)



As the centroid of a triangle is a point of trisection of each median, if the vertices of a triangle are given to be the points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , its centroid will be the point $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3} \right)$.

Applying this, we easily see that the centroid of $\triangle ABC$ is the point $\left[\frac{a+c}{3}, \frac{b}{3} \right]$

Similarly the centroid of the $\triangle A_1B_1C_1$ is the point

$$\left[\frac{\frac{c}{n} + \frac{a+c(n-1)}{n} + \frac{a(n-1)}{n}}{3}, \frac{0 + \frac{b}{n} + \frac{b(n-1)}{n}}{3} \right]$$

i. e. it is the point $\left[\frac{a+c}{3}, \frac{b}{3} \right]$

which is the same as the centroid of $\triangle AB'C'$

THEOREM 8

(Satterly's Theorem)

The centroids of the median triangle and the aliquot triangle of a triangle and that of the original triangle are one and the same point.

By Pappus' theorem, the centroids of the aliquot triangle and the original triangle are one and the same point. We have found that this common centroid for the $\triangle ABC$ and its aliquot triangle $A_1B_1C_1$ is $\left[\frac{a+c}{3}, \frac{b}{3} \right]$

So prove that the same point $\left[\frac{a+c}{3}, \frac{b}{3} \right]$ is also the centroid of $\triangle XYZ$.

(Proof is left to the student)

For more details about medians, median triangle, aliquot triangle etc., vide *Mathematical Gazette* Vol. XXXVIII, No. 324 (May 1954) and Vol. XL ~~No. 332~~ (May 1956).

