DAMAGE BOOK

A CONCISE GEOMETRICAL CONICS

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BY

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PREFACE

THIS forms a companion volume to the author's *Projective Geometry*. It is designed primarily for those whose main study of the geometry of the conic is conducted along projective lines. A course of this character requires supplementing, since many metrical properties are best tackled without recourse to projection. But this auxiliary treatment should be as concise as possible.

It is hoped, however, that the volume may also be of service to those who have not the time to study projective methods. Alternative methods of proof of certain theorems, best treated by projection, have therefore been added in an appendix, in order to make the volume self-contained.

A large proportion of the examples are simple straightforward applications of the bookwork, but sufficient riders of a more difficult character have been included in order to make the collection representative and to give the reader the experience which is essential for success.

A volume containing hints and solutions of all the more difficult riders has been prepared (price 2s. 6d.).

Although not a complete key, it is believed that it will meet all ordinary requirements.

The author wishes to acknowledge gratefully many helpful criticisms received from Mr. R. V. H. Roseveare and Mr. C. B. G. Hunter.

C. V. D.

December, 1926.

CONTENTS

SUMMARY OF RESULTS

It is convenient to denote certain cardinal points by special letters. The reader is advised to adopt this plan and to remember properties in terms of letters, whenever practicable.

THE PARABOLA

Notation. S is the focus, A is the vertex, X is the foot of the directrix.

P is any point on the curve.

PN is the ordinate of P to the axis.

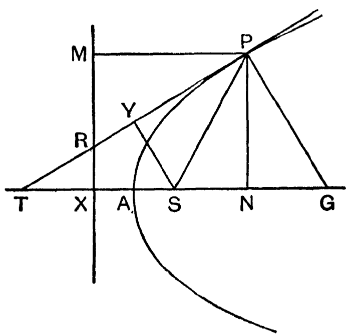

FIG. 1.

PM is the perpendicular from P to the directrix.

The tangent at P cuts the directrix at R and the axis at T.

The normal at P cuts the axis at G.

SY is the perpendicular from S to PT.

It is suggested that the reader should draw figures in the margin to illustrate the following summary.

Equation of Curve. $y^2=4ax$ or $PN^2=4AS\,.\,AN$. (p. 2.)

Theorem 9. (i) PT bisects $\angle SPM$; (ii) $SP=ST=SG$; (iii) SM and PT bisect each other at right angles; (iv) Y lies on the tangent at the vertex; (v) $AN=AT$; (vi) $AY=\frac{1}{2}NP$; (vii) $NG=2AS$; (viii) $SY^2=SA\,.\,SP$ (p. 26).

Theorem 10. (i) Tangents at the ends of a focal chord PSP' meet at right angles on the directrix.

(ii) If K is the vertex of the diameter bisecting PSP', then $PP'=4SK$ (p. 28).

Theorem 11. If the tangents at P, Q meet at O; (i) $\angle SOQ=\angle SPO=\angle PTS$; (ii) $SO^2=SP\,.\,SQ$; (iii) the exterior angle between the tangents equals the angle either subtends at the focus (p. 31).

Theorem 12. The circumcircle of a triangle circumscribing a parabola passes through the focus, and the orthocentre of the triangle lies on the directrix (p. 32).

Theorem 13. The mid-points of parallel chords of a parabola lie on a line (a diameter) parallel to the axis. If the diameter cuts the directrix at H, SH is perpendicular to each chord (p. 34).

Theorem 14. If the tangents at P, Q meet at O and if the diameter through O cuts the curve at K and PQ at V, then $OK=KV$, $PV=VQ$ and the tangent at K is parallel to PQ.

Further $\dfrac{OP}{OQ}=$ ratio of sides of triangle whose sides are parallel to OP, OQ and whose base is parallel to the axis (p. 35-6).

Theorem 15. If PV is the ordinate to the diameter KV, vertex K, then $PV^2=4SK\,.\,KV$ (p. 37).

CENTRAL CONICS

Notation for Central Conics. C is the centre.

ACA′ is the transverse axis, BCB′ is the conjugate axis.

S is one focus, X is the foot of the corresponding directrix ; S′ is the other focus, and X′ is the foot of the corresponding directrix.

LSL′ is the latus rectum.

P is any point on the curve.

PN is the ordinate of P to ACA′, PM is the perpendicular from P to the directrix XM.

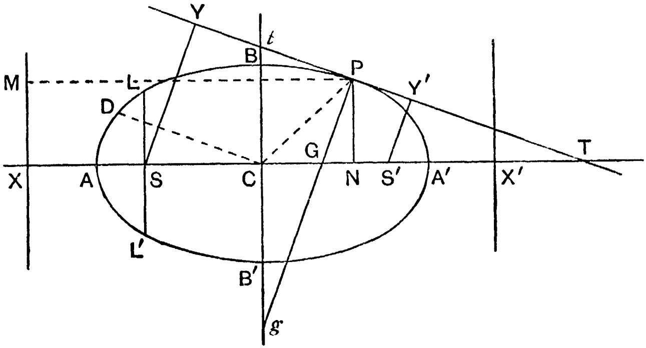

FIG. 2.

The tangent at P cuts CA, CB at T, t.

The normal at P cuts CA, CB at G, g.

SY, S′Y′ are the perpendiculars from S, S′ to PT.

CD is the semi-diameter conjugate to CP.

It is suggested that the reader should draw figures in the margin to illustrate the following summary :

Equation of Curve. *The ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{or} \quad \frac{\text{PN}^2}{\text{AN}\,.\,\text{A}'\text{N}}=\frac{\text{PN}^2}{\text{CA}^2-\text{CN}^2}=\frac{b^2}{a^2} \text{ (p. 5).}$$

The hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{or} \quad \frac{\text{PN}^2}{\text{AN}\,.\,\text{A}'\text{N}}=\frac{\text{PN}^2}{\text{CN}^2-\text{CA}^2}=\frac{b^2}{a^2} \text{ (p. 8).}$$

THE GENERAL CONIC

Theorem 1. (i) If a chord PQ cuts the directrix at R, SR is a bisector of $\angle$ PSQ.

(ii) If the tangent at P cuts the directrix at R, $\angle$ PSR $=90^\circ$.

(iii) Tangents at the ends of a focal chord meet on the directrix (p. 14).

Theorem 2. T is a point on the tangent at P; TU, TK are the perpendiculars from T to SP and the directrix, then SU $=e$. TK (p. 15).

Theorem 3. If a chord PQ cuts the directrix at R, and if the tangents at P, Q meet at T, $\angle$ TSP and $\angle$ TSQ are equal or supplementary, and $\angle$ TSR $=90^\circ$ (p. 17).

Theorem 4. SG $=e$. SP (p. 18).

Theorem 5. The projection of PG on PS equals SL (p. 18).

Theorem 6. If PSP′ is a focal chord, $\dfrac{1}{\mathrm{SP}}+\dfrac{1}{\mathrm{SP'}}=\dfrac{2}{l}$ and

$$\mathrm{SP}\,.\,\mathrm{SP'}=\tfrac{1}{2}l\,.\,\mathrm{PP'} \text{ (p. 19).}$$

Theorem 7. (i) If two lines OPQ, OP′Q′ cut a conic at P, Q and P′, Q′ and make angles θ, θ' with the transverse axis, then

$$\frac{\mathrm{OP}\,.\,\mathrm{OQ}}{\mathrm{OP'}\,.\,\mathrm{OQ'}}=\frac{1-e^2\cos^2\theta'}{1-e^2\cos^2\theta}.$$

(ii) If ESF, E′SF′ are focal chords parallel to OP, OP′, and if TZ, TZ′ are tangents parallel to OP, OP′, and if HCK, H′CK′ are diameters parallel to OP, OP′, then

$$\frac{\mathrm{OP}\,.\,\mathrm{OQ}}{\mathrm{OP'}\,.\,\mathrm{OQ'}}=\frac{\mathrm{EF}}{\mathrm{E'F'}}=\frac{\mathrm{TZ}^2}{\mathrm{TZ'}^2}=\frac{\mathrm{CH}^2}{\mathrm{CH'}^2} \text{ (p. 23).}$$

Theorem 8. If a circle cuts a conic at P, Q, H, K, then PQ and HK make equal angles with the axis (p. 24).

CENTRAL CONICS

Metrical Relations. (i) CS $=ae$; (ii) CX $=\dfrac{a}{e}$; (iii) SL $=\dfrac{b^2}{a}$; (iv) AS . A′S $=b^2$; (v) $b^2=a^2(1-e^2)$, ellipse; $b^2=a^2(e^2-1)$, hyperbola (pp. 4, 8).

Theorem 16. (i) PG, PT are the bisectors of $\angle$ SPS′ ; ,
(ii) $SP \pm S'P = 2a$; (iii) if $SP > S'P$, $SP = a + e \,.\, CN$;
(iv) $CG = e^2 \,.\, CN$; (v) the circle SPS′ passes through g, t (p. 40).

Theorem 17. If TP, TP′ are tangents $\angle STP = \angle S'TP'$ (p. 42).

Theorem 18. (i) Y, Y′ lie on the auxiliary circle ;
(ii) $SY \,.\, S'Y' = CB^2$.
(iii) If a line through C parallel to PT cuts PS at E, $PE = CA$ (p. 44).

Theorem 19. If TP, TP′ are perpendicular tangents,
$$CT^2 = a^2 \pm b^2 \text{ (p. 46).}$$

Theorem 20. The mid-points of parallel chords of a conic lie on a straight line (a diameter). If the diameter cuts the directrix at K, SK is perpendicular to each chord (pp. 49, 92).

Theorem 21. If CP bisects chords parallel to CD, then CD bisects chords parallel to CP, and the tangent at P is parallel to CD (p. 93).

Theorem 22. If the tangents at Q, Q′ meet at T, and if CT cuts QQ′ at V and the curve at P, then $QV = VQ'$ and $CV \,.\, CT = CP^2$ (p. 94).

Theorem 23. If PP′ is a diameter and Q is a point on the curve, the (supplemental) chords QP, QP′ are parallel to conjugate diameters (p. 50).

Theorem 24. If CP, CD are conjugate semi-diameters of an ellipse, and if DR is the ordinate of P to CA, then

(i) $CN \,.\, CT = CA^2$; $Cn \,.\, Ct = CB^2$.

(ii) If p, d are the points on the auxiliary circle which correspond to P, D, then $\angle pCd = 90°$.

(iii) $\triangle PNC = \triangle DRC$ or $PN \,.\, NC = DR \,.\, RC$.

(iv) $\dfrac{PN \,.\, DR}{CN \,.\, CR} = \dfrac{b^2}{a^2}$.

(v) $CN^2 + CR^2 = CA^2$; $PN^2 + DR^2 = CB^2$; $CP^2 + CD^2 = CA^2 + CB^2$.

(vi) The area of the parallelogram having CP, CD as sides equals $a \,.\, b$.

(vii) If QV is the ordinate from Q to PCP′, then
$$\frac{QV^2}{CP^2 - CV^2} = \frac{CD^2}{CP^2} \text{ (pp. 52, 95-8).}$$

Theorems 25, 32. (i) $SP \cdot S'P = CD^2$;

(ii) $\frac{SY}{SP} = \frac{CB}{CD}$ (pp. 54, 66).

Theorems 26, 33. (i) $PF \cdot PG = b^2$; (ii) $PF \cdot Pg = a^2$;

(iii) $\frac{PG}{CD} = \frac{b}{a}$; (iv) $PG \cdot Pg = CD^2$ (pp. 55, 67).

THE HYPERBOLA

If the directrix of a hyperbola cuts the auxiliary circle at Y, Z, then CY, CZ are the asymptotes and SY, SZ touch the auxiliary circle.

If the angle between the asymptotes is 2α, then $\sec \alpha = e$; $\tan \alpha = \frac{b}{a}$ (p. 58).

Theorem 28. The tangent at a point P on a hyperbola cuts the asymptotes at T, T′; a chord QQ′, parallel to TP, cuts CT, CT′ at R, R′; then (i) $TP = PT'$; (ii) $TP^2 = RQ \cdot RQ'$; (iii) $RQ = Q'R'$; (iv) $TP^2 = RQ \cdot QR'$; (v) $TP = CD = C\delta \cdot \sqrt{-1}$ (p. 59, 60, 64).

Theorem 29. The tangent at a point P on a hyperbola cuts the asymptotes at T, T′; the lines through P parallel to CT′, CT cut CT, CT′ at H, H′; then (i) area of parallelogram $CHPH' = \frac{1}{2}ab$; (ii) area of $\triangle CTT' = ab$; (iii) $CT \cdot CT' = 4CH \cdot CH' = CS^2$. (p. 62).

Theorem 30. If CP, CD are conjugate semi-diameters of a hyperbola, then (i) PD is parallel to one asymptote and is bisected by the other; (ii) $CP^2 - CD^2 = a^2 - b^2$; (iii) the area of the parallelogram having CP, CD as sides equals $a \cdot b$ (p. 65).

Theorem 31. If QV is the ordinate from Q to PCP′, and if CD is conjugate to CP, $\frac{QV^2}{CV^2 - CP^2} = \frac{CD^2}{CP^2}$ (p. 66).

THE RECTANGULAR HYPERBOLA

$e = \sqrt{2}$, $a = b$ (p. 68).

Theorem 34. If the tangent at a point P on a rectangular hyperbola cuts the asymptotes at T, T′, and if CD is conjugate to CP, then

(i) CT and CT′ are the bisectors of $\angle$PCD ; (ii) CP = CD ; (iii) if CQ, CD are perpendicular semi-diameters, CQ = CD ; (iv) supplemental chords are equally inclined to either asymptote ; (v) PG = CD = Pg (p. 68).

Theorem 35. (i) Any chord PQ of a rectangular hyperbola subtends equal or supplementary angles at the extremities of any diameter HCH′ ; (ii) if HCH′ is a fixed diameter, $\angle$PHH′ ~ $\angle$PH′H has one of two fixed (supplementary) values (p. 70).

Theorem 36. (i) If a rectangular hyperbola circumscribes a triangle, it passes through the orthocentre ; (ii) If a conic circumscribes a triangle and passes through the orthocentre, it must be a rectangular hyperbola (p. 71).

Theorem 37. If a rectangular hyperbola circumscribes a triangle, its centre lies on the nine-point circle of the triangle (p. 72).

SECTIONS OF A RIGHT CIRCULAR CONE

Theorem 38. If the semi-vertical angle of the cone is α, and if the plane of section makes an angle β with the axis of the cone, the eccentricity of the conic section equals sec α . cos β.

Further, the foci are the points of contact of the plane of section with the focal spheres, and the directrices are the lines of intersection of the plane of section with the planes of contact of the cone with the focal spheres (p. 76-9).

Theorem 39. (i) The minor axis of the conic section is a mean proportional between the radii of the focal spheres.

(ii) The latus rectum for a given cone varies as the length of the perpendicular from the vertex of the cone to the plane of section (p. 80).

CURVATURE OF A CONIC

Theorem 40. If the circle of curvature at P cuts the conic again at Q, the tangent at P and the common chord PQ make equal angles with the axis (p. 84).

Theorem 41, 42. For a central conic,

(i) the central chord of curvature at P equals $\frac{2CD^2}{CP}$;

(ii) the radius of curvature at P equals $\frac{CD^3}{CA \cdot CB}$;

(iii) the focal chord of curvature at P, and the focal chord of the conic parallel to the tangent at P, each equal $\frac{2CD^2}{CA}$ (p. 86).

Theorems 43, 44. For a parabola,

(i) the chord of curvature through P parallel to the axis, and the focal chord of curvature, each equal 4SP;

(ii) the radius of curvature at P equals $\frac{2SP^{\frac{3}{2}}}{SA^{\frac{1}{2}}}$ (p. 88-9).

CHAPTER I

STANDARD FORMS

Definitions. Given a fixed point S and a fixed line XM, if a point P moves in the plane SXM, so that its distance from S bears a constant ratio e to its perpendicular distance PM from XM, then the locus of P is called a **conic section**, or, more shortly, a **conic.**

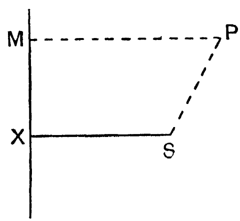

FIG. 3.

With the above notation,

$$\frac{\mathrm{SP}}{\mathrm{PM}}=e \quad \text{or} \quad \mathrm{SP}=e\,.\,\mathrm{PM}.$$

The fixed point S is called the **focus**, the fixed line XM is called the **directrix,** the constant ratio e is called the **eccentricity**.

If SX is the perpendicular from the focus S to the directrix, X is called the **foot of the directrix**, and the line SX produced both ways is called the **transverse axis,** or sometimes simply **the axis,** of the conic. It is evident from the definition that a conic is *symmetrical about its transverse axis.*

The points of intersection of a conic with its transverse axes **are** called the **vertices** of the conic.

If $e < 1$, the conic is called an **ellipse.**

If $e = 1$, the conic is called a **parabola.**

If $e > 1$, the conic is called a **hyperbola.**

Standard Equations of the Conic.

(1) **The Parabola** ($e=1$). Bisect SX at A; since SA $=$ AX, by definition A lies on the parabola.

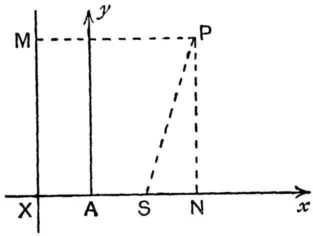

FIG. 4.

Take AS as x-axis and the line through A perpendicular to Ax as y-axis.

Let the coordinates of any point P on the locus be (x, y) and let SA $= a =$ AX.

Then $$\text{MP} = \text{XN} = \text{XA} + \text{AN} = a + x$$

and $$\text{SP}^2 = \text{SN}^2 + \text{NP}^2 = (\text{AN} - \text{AS})^2 + \text{NP}^2 = (x-a)^2 + y^2.$$

But by definition, SP $=$ PM, $\therefore\ (x-a)^2 + y^2 = (a+x)^2$;

$$\therefore\ x^2 - 2ax + a^2 + y^2 = a^2 + 2ax + x^2;$$

$$\therefore\ \mathbf{y^2 = 4ax.}$$

This is the equation of a parabola referred to its vertex A as origin and its transverse axis as x-axis.

From the equation, we see that the y-axis, $x=0$, meets the curve where $y^2 = 0$ and is therefore a tangent.

The form of the curve is shown in Fig. 5.

The curve lies entirely on one side of the directrix and is an *open* curve.

If the line through S perpendicular to the axis cuts the curve at L, L′, the chord LL′ is called the **latus rectum**, and the *length* of SL, the semi-latus rectum, is always denoted by the letter l. From the definition $l = \mathrm{SL} = \mathrm{SX} = 2a$, and $\mathrm{LL}' = 2\mathrm{SL} = 4a$.

The equation of the parabola may be interpreted geometrically, as follows:

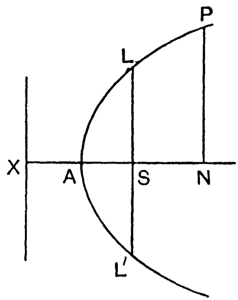

FIG. 5.

If PN is the perpendicular from any point P on a parabola to the axis, then $\mathbf{PN^2 = 4AS \cdot AN}$.

(2) **The Ellipse** ($e<1$). Divide SX internally at A and externally at A′ in the ratio e. By definition, A and A′ lie on the ellipse. Bisect AA′ at C.

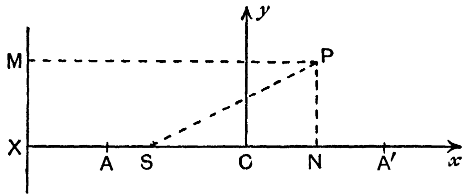

FIG. 6.

Let $\mathsf{CA}=a=\mathsf{CA}'$. Take C as origin and the transverse axis as x-axis. Let the coordinates of any point P on the locus be (x, y).

Then $$e=\frac{\mathsf{SA}}{\mathsf{AX}}=\frac{a-\mathsf{CS}}{\mathsf{CX}-a} \quad\text{and}\quad e=\frac{\mathsf{SA}'}{\mathsf{A'X}}=\frac{a+\mathsf{CS}}{\mathsf{CX}+a};$$

$$\therefore\ e=\frac{(a+\mathsf{CS})+(a-\mathsf{CS})}{(\mathsf{CX}+a)+(\mathsf{CX}-a)}=\frac{(a+\mathsf{CS})-(a-\mathsf{CS})}{(\mathsf{CX}+a)-(\mathsf{CX}-a)};$$

$$\therefore\ e=\frac{2a}{2\mathsf{CX}}=\frac{2\mathsf{CS}}{2a};$$

$$\therefore\ \mathbf{CX}=\frac{\mathbf{a}}{\mathbf{e}} \quad\text{and}\quad \mathbf{CS}=\mathbf{ae}.$$

Now $\quad \mathsf{MP}=\mathsf{XN}=\mathsf{XC}+\mathsf{CN}=\frac{a}{e}+x,$

and $\quad \mathsf{SP}^2=\mathsf{SN}^2+\mathsf{NP}^2=(\mathsf{SC}+\mathsf{CN})^2+\mathsf{NP}^2=(ae+x)^2+y^2.$

But $\mathsf{SP}^2=e^2 \,.\, \mathsf{PM}^2$;

$$\therefore\ (ae+x)^2+y^2=e^2\left(\frac{a}{e}+x\right)^2;$$

$$\therefore\ a^2e^2+2aex+x^2+y^2=a^2+2aex+e^2x^2;$$

$$\therefore\ x^2(1-e^2)+y^2=a^2(1-e^2);$$

$$\therefore\ \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1.$$

Put $\mathbf{a^2(1-e^2)=b^2}$ (Note $e<1$, $\therefore$ b is real.)

Then
$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}=\mathbf{1}.$$

This is the standard equation of an **ellipse.**

The points A, A′ are the **vertices** of the ellipse and the mid-point C of AA′ is called the **centre** of the ellipse.

The form of the equation of the curve shows that *it is symmetrical about both the x-axis and the y-axis.* The line here taken as y-axis is called the **conjugate axis** of the ellipse. The form of the curve is shown in Fig. 7.

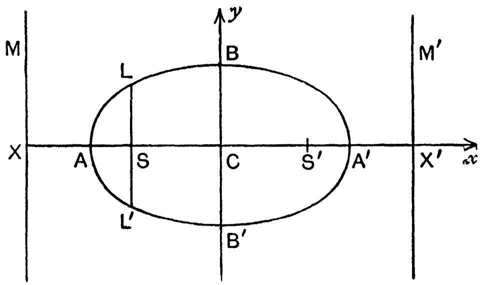

FIG. 7.

Since the curve is symmetrical about the y-axis, there exists a second focus S′ and a second corresponding directrix X′M′, such that $CS'=CS=ae$ and $CX'=CX=\frac{a}{e}$.

From the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we see that, for *real* values of x and y, $\frac{x^2}{a^2}$ can never be greater than 1, and $\frac{y^2}{b^2}$ can never be greater than 1.

$$\therefore\ -a\leqslant x\leqslant +a \quad \text{and} \quad -b\leqslant y\leqslant +b.$$

Therefore all points of the curve lie inside or on a rectangle of length $2a$ and breadth $2b$.

Let the y-axis meet the curve at B, B′; if we put $x=0$ in the equation of the curve, we obtain

$$\frac{y^2}{b^2}=1 \quad \text{or} \quad y^2=b^2 \quad \text{or} \quad y=\pm b;$$

$$\therefore \ \mathrm{CB}=\mathrm{CB}'=b.$$

The finite lines AA′ and BB′ are called the **major axis** and **minor axis** respectively of the ellipse; the lengths a, b of the semi-major axis and semi-minor axis are connected by the relation

$$\mathbf{b^2=a^2(1-e^2)},$$

so that b is real and less than a.

Since $\mathrm{CS}=ae$, this relation may be written

$$\mathbf{CB^2=CA^2-CS^2}=(\mathrm{CA}-\mathrm{CS})(\mathrm{CA}+\mathrm{CS})=\mathbf{AS\,.\,SA'}.$$

The chord LL′ through S perpendicular to AA′ is called the **latus rectum**, and the length of SL, the semi-latus rectum, is denoted by l.

We see that $l=\mathrm{SL}=e\,.\,\mathrm{SX}=e(\mathrm{CX}-\mathrm{CS})$

$$=e\left(\frac{a}{e}-ae\right)=a-ae^2=a(1-e^2)=\frac{b^2}{a}; \qquad \therefore \ \mathbf{SL=l=\frac{b^2}{a}}.$$

The equation of the ellipse may be interpreted geometrically.

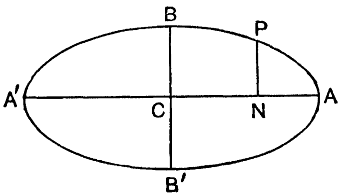

FIG. 8.

Let PN be the perpendicular from any point P on the ellipse to AA′.

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}=\frac{(a+x)(a-x)}{a^2};$$

$$\therefore \ \frac{\mathrm{PN}^2}{\mathrm{CB}^2}=\frac{(\mathrm{A'C}+\mathrm{CN})(\mathrm{CA}-\mathrm{CN})}{\mathrm{CA}^2}=\frac{\mathrm{A'N}\,.\,\mathrm{NA}}{\mathrm{CA}^2};$$

$$\therefore \ \mathbf{\frac{PN^2}{A'N\,.\,NA}=\frac{CB^2}{CA^2}=\frac{PN^2}{CA^2-CN^2}}.$$

Note. This is a special case of Theorem 7, Corollary, p. 23.

The Ellipse and its Auxiliary Circle.

The circle on AA′ as diameter is called the **auxiliary circle** of the ellipse.

Its equation is $$x^2+y^2=a^2.$$

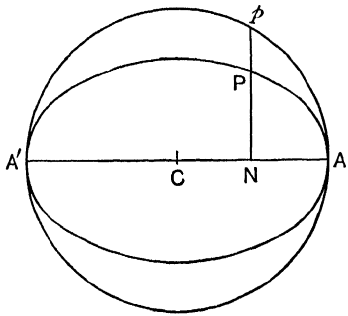

FIG. 9.

Let the perpendicular PN from a point P on the ellipse to AA′ cut AA′ at N and the auxiliary circle at p.

Then, as above, $$PN^2=\frac{b^2}{a^2}(a^2-CN^2),$$

and $$pN^2=Cp^2-CN^2=a^2-CN^2;$$

$$\therefore\ \frac{\mathbf{PN}}{\mathbf{pN}}=\frac{\mathbf{b}}{\mathbf{a}}.$$

It should be noted that if the auxiliary circle is rotated about ACA′ through an angle θ where $\cos\theta=\dfrac{b}{a}$, the new position of p is such that $\angle pPN=90°$, and therefore the ellipse is the orthogonal projection of the auxiliary circle in this position.

Hence or otherwise it is easy to see that the tangents at p and P intersect on ACA′ and to deduce a large group of properties of the ellipse from those of the circle. [See the author's *Projective Geometry*, Chap. II.]

The points P, p are called **corresponding points.**

(3) **The Hyperbola** ($e > 1$). Divide SX internally at A and externally at A′ in the ratio e. By definition, A and A′ lie on the hyperbola. Since $e > 1$, A and A′ lie on opposite sides of the directrix. Bisect AA′ at C.

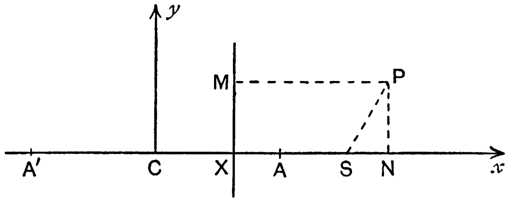

FIG. 10.

Let $CA = a = CA'$. Take C as origin and the transverse axis as x-axis. Let the coordinates of any point P on the locus be (x, y).

Then $$e = \frac{SA}{AX} = \frac{CS - a}{a - CX} \quad \text{and} \quad e = \frac{SA'}{A'X} = \frac{CS + a}{a + CX};$$

$$\therefore \ e = \frac{(CS - a) + (CS + a)}{(a - CX) + (a + CX)} = \frac{(CS + a) - (CS - a)}{(a + CX) - (a - CX)};$$

$$\therefore \ e = \frac{2CS}{2a} = \frac{2a}{2CX};$$

$$\therefore \ \mathbf{CX} = \frac{\mathbf{a}}{\mathbf{e}} \quad \text{and} \quad \mathbf{CS} = \mathbf{ae}, \text{ as for the ellipse.}$$

Now $$MP = XN = CN - CX = x - \frac{a}{e},$$

and $$SP^2 = SN^2 + NP^2 = (CN - CS)^2 + NP^2 = (x - ae)^2 + y^2.$$

But $SP^2 = e^2 . PM^2$;

$$\therefore \ (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2;$$

$$\therefore \ x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2;$$

$$\therefore \ x^2(e^2 - 1) - y^2 = a^2(e^2 - 1);$$

$$\therefore \ \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1.$$

Put $\mathbf{a^2(e^2 - 1) = b^2}$. (Note $e > 1$, $\therefore$ b is real.)

Then $$\frac{\mathbf{x^2}}{\mathbf{a^2}} - \frac{\mathbf{y^2}}{\mathbf{b^2}} = \mathbf{1}.$$

This is the standard equation of a **hyperbola**.

The points A, A′ are the **vertices** of the hyperbola, and the mid-point C of AA′ is called the **centre** of the hyperbola.

The form of the equation of the curve shows that it is *symmetrical about both the x-axis and the y-axis.* The line, here taken as y-axis, is called the **conjugate axis** of the hyperbola. The form of the curve is shown in Fig. 11.

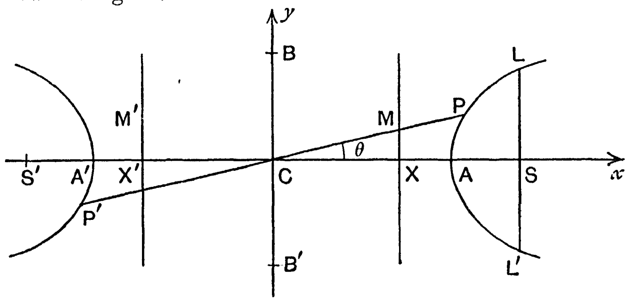

FIG. 11.

Since the curve is symmetrical about the y-axis, there exists a second focus S′ and a second corresponding directrix X′M′, such that

$$CS' = CS = ae \quad \text{and} \quad CX' = CX = \frac{a}{e}.$$

If we put $x = 0$ in the equation of the curve, we obtain

$$-\frac{y^2}{b^2} = 1 \quad \text{or} \quad y^2 = -b^2,$$

so that y is imaginary.

The hyperbola therefore does not cut the conjugate axis at real points. And, since from the equation of the curve,

$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) = \frac{b^2(x^2 - a^2)}{a^2},$$

we see that, if y is real, x cannot lie between $+a$ and $-a$, so that there are no real points of the locus in the portion of the plane between lines through A and A′ parallel to Cy.

The curve is composed of two branches extending to infinity in either direction.

If P is any point on the curve such that $CP = r$ and $\angle PCA = \theta$, the coordinates of P are given by $x = r\cos\theta$, $y = r\sin\theta$ (see Fig. 11);

$$\therefore \quad \frac{r^2\cos^2\theta}{a^2} - \frac{r^2\sin^2\theta}{b^2} = 1;$$

$$\therefore \quad r^2 = \frac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta} = \frac{a^2b^2\sec^2\theta}{b^2 - a^2\tan^2\theta}.$$

Consequently, if $\frac{b}{a} > \tan\theta > -\frac{b}{a}$, the value of r^2 is positive and the two values of r are real, equal in magnitude and opposite in sign, corresponding to the fact that the line through C cuts the curve at points P, P′ on opposite sides of C, such that $PC = CP'$.

But if $\tan\theta > \frac{b}{a}$ or if $\tan\theta < -\frac{b}{a}$, the value of r^2 is negative and the line cuts the curve at two imaginary points, say δ and δ', where $\delta C = C\delta'$ and $C\delta^2$ is a real *negative* number. In this case, the length of the actual diameter of the hyperbola is imaginary.

Take points B, B′ on the y-axis such that $CB = b = CB'$; then the finite lines AA′ and BB′ are called the **major axis** and **minor axis** respectively of the hyperbola; the lengths a, b of the semi-major axis and semi-minor axis are connected (see p. 8) by the relation

$$\mathbf{b^2 = a^2(e^2 - 1)}.$$

Since $CS = ae$, this relation may be written,

$$\mathbf{CB^2 = CS^2 - CA^2} = (CS - CA)(CS + CA) = \mathbf{AS \,.\, A'S}.$$

The following points should be noted:

(i) Since $e > 1$, b is real.

(ii) b can be less than a or equal to a or greater than a; $b = a$ if $e = \sqrt{2}$ and $b \gtreqless a$ as $e \gtreqless \sqrt{2}$.

(iii) The points B and B′ do *not* lie on the hyperbola, for the hyperbola does not cut the conjugate axis at real points.

(iv) The "b^2" of an ellipse corresponds, as defined above, to the "$-b^2$" of a hyperbola.

The chord LL′ through S perpendicular to AA′ is called the **latus rectum**, and the length of SL, the semi-latus rectum, is denoted by l.

We see that $l = \mathrm{SL} = e \,.\, \mathrm{SX} = e(\mathrm{CS} - \mathrm{CX})$

$$= e\left(ae - \frac{a}{e}\right) = ae^2 - a = a(e^2 - 1);$$

$$\therefore\ \mathbf{SL} = \mathbf{l} = \frac{\mathbf{b}^2}{\mathbf{a}}.$$

The equation of the hyperbola may be interpreted geometrically.

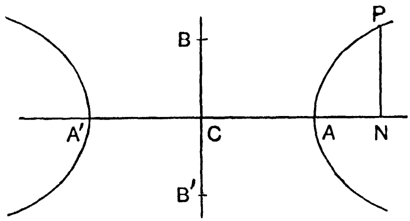

FIG. 12.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1; \quad \therefore\ \frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 = \frac{x^2 - a^2}{a^2} = \frac{(x+a)(x-a)}{a^2};$$

$$\therefore\ \frac{\mathrm{PN}^2}{\mathrm{CB}^2} = \frac{(\mathrm{CN} + \mathrm{A'C})(\mathrm{CN} - \mathrm{CA})}{\mathrm{CA}^2} = \frac{\mathrm{A'N} \,.\, \mathrm{AN}}{\mathrm{CA}^2};$$

$$\therefore\ \frac{\mathbf{PN}^2}{\mathbf{A'N} \,.\, \mathbf{AN}} = \frac{\mathbf{CB}^2}{\mathbf{CA}^2} = \frac{\mathbf{PN}^2}{\mathbf{CN}^2 - \mathbf{CA}^2}.$$

Note. (i) This is a special case of Theorem 7, Corollary, p. 23.

(ii) This result is really identical with that for the ellipse on p. 6. The apparent difference of sign is due to the fact that the "CB²" as defined for the hyperbola is *minus* the "CB²" of the ellipse.

Definitions. If P is any point on a conic and if Q is another point on the conic in the neighbourhood of P, the **tangent** at P is defined as the limiting position of the line PQ when Q tends to P along the curve, in such a way that the length of PQ tends to zero.

The line through P at right angles to the tangent at P is called the **normal** at P, and P is called the **foot** of the normal.

If the perpendicular from P to the transverse axis cuts it at N and cuts the curve again at P′, then PN is called the **ordinate** of P to the axis and PP′ is called a **principal double ordinate.**

If the tangent and normal at P meet the transverse axis at T, G, and if PN is the ordinate to the axis, then the lengths of PT and PG are called the *lengths of the tangent and normal* at P ; also their projections TN and NG on the transverse axis (see Fig. 2) are called the **subtangent** and **subnormal.**

EXERCISE I.

(For the notation, see pp. ix, xi.)

1. Prove that $CS = e^2 . CX$.

2. Prove that $SB = a$.

3. Prove that $S'L = a(1 + e^2)$.

4. Prove that (i) $SP = a \pm e . CN$, (ii) $SP + S'P = 2a$.

5. If $e = \frac{1}{2}\sqrt{2}$, prove that $\angle SBS' = 90°$.

6. Prove that $CS . SX = b^2$.

7. If the line through L parallel to CS cuts BS in K, prove that $SK = b$.

8. If LA cuts the directrix at R, prove that $\angle RSX = 45°$.

9. If SL cuts the auxiliary circle at K, prove that $SK = b$.

10. If P is a point on a parabola such that PN bisects AS, prove that $AP = 3AN$.

11. If P is a point on a parabola such that $\angle PAS = 45°$, prove that $PN = 2SL$.

12. If P is a point on a parabola, prove that the tangent from A to the circle SPN equals $\frac{1}{2}PN$.

13. If in a parabola the perpendicular bisector of AL cuts AS at O, prove that $SO = \frac{3}{2}AS$.

14. Prove that $CP^2 - e^2 . CN^2$ is constant.

15. A circle touches CA at S and passes through B; prove that its diameter equals $\frac{a^2}{b}$.

16. Two parabolas have the same focus; prove that their common chord is one of the bisectors of the angles formed by the directrices.

17. Two parabolas, foci S and S′, have the same directrix; prove that their common chord bisects SS′ at right angles.

18. The line through a point P on a parabola perpendicular to AP cuts the axis at K; prove that NK = 2SL.

19. The perpendicular from the focus of a parabola to the chord AP cuts the tangent at A in K; prove that PN = 4AK.

20. Given the directrix and two points on a conic, prove that the locus of the corresponding focus is a circle.

21. A sphere of radius R rolls between two fixed horizontal straight wires which intersect at an angle 2α. Prove that, until the sphere slips through, its centre describes an ellipse of semi-minor axis R and eccentricity $\cos \alpha$.

22. A′P meets the auxiliary circle at Q; the perpendicular from Q to AA′ cuts AP, AA′ at R, M; prove that $\frac{QM}{RM}$ is constant.

23. A chord PP′ of a parabola subtends a right angle at A; PN, P′N′ are the perpendiculars from P, P′ to the axis; prove that

$$AN . AN' = 4SL^2.$$

24. Any chord PP′ of a parabola cuts the axis at K; PN, P′N′ are the perpendiculars from P, P′ to the axis; prove that $PN . P'N' = 4SA . AK$.

CHAPTER II

THE GENERAL CONIC

Theorem 1. (i) If a chord PQ of a conic, focus S, meets the corresponding directrix at R, then SR is a bisector of $\angle$ PSQ.

(ii) If the tangent at any point P of a conic, focus S, meets the corresponding directrix at R, then $\angle \mathrm{PSR} = 90^\circ$.

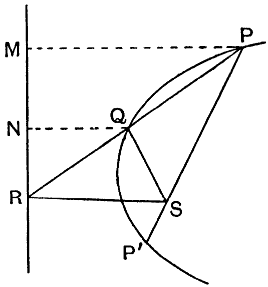

FIG. 13.

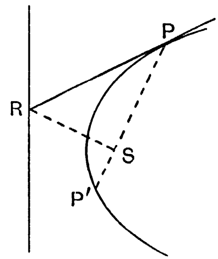

FIG. 14.

(i) Draw PM, QN perpendicular to the directrix ; let PS meet the conic again at P′.

By definition, $\mathrm{SP} = e \,.\, \mathrm{PM}$ and $\mathrm{SQ} = e \,.\, \mathrm{QN}$;

$$\therefore \frac{\mathrm{SP}}{\mathrm{SQ}} = \frac{\mathrm{PM}}{\mathrm{QN}} = \frac{\mathrm{PR}}{\mathrm{QR}} \text{ by parallels ;}$$

$\therefore$ SR is a bisector of $\angle$ PSQ.

(ii) Now suppose Q tends to P along the curve, then the limiting position of PQR is the tangent at P (see Fig. 14); an the limiting value of $\angle$ P′SQ is 180° ; but SR always bisects $\angle$ P′SQ ;

$\therefore$ in the limit, when PR is a tangent,

$$\angle \mathrm{RSP} = 90^\circ.$$

Corollary 1. If R is a point on the directrix such that $\angle PSR = 90°$, then RP is the tangent at P.

This is the converse of (ii).

Corollary 2. If PSP′ is any focal chord, the tangents at P, P′ meet at a point R on the directrix, such that SR is at right angles to PP′.

In Fig. 14, $\angle RSP = 90°$; $\therefore$ $\angle RSP' = 90°$; $\therefore$ RP′ is the tangent at P′.

Note. If P and Q are points on *different* branches of a hyperbola, SR is the *internal* bisector of $\angle PSQ$. The reader should draw a figure to illustrate this case.

Theorem 2. (Adams' Property). If from any point T on the tangent at P perpendiculars TU, TK are drawn to SP and the directrix, then $SU = e \,.\, TK$.

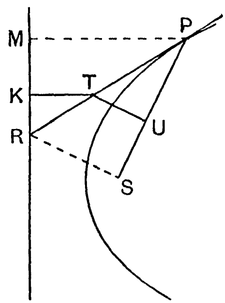

FIG. 15.

Let PT meet the directrix at R, so that $\angle RSP = 90°$; draw PM perpendicular to the directrix.

Since $\angle TUP = 90° = \angle RSP$, TU is parallel to RS; also TK is parallel to PM;

$$\therefore \frac{SU}{SP} = \frac{RT}{RP} = \frac{TK}{PM};$$

$$\therefore \frac{SU}{TK} = \frac{SP}{PM} = e;$$

$$\therefore SU = e \,.\, TK.$$

EXERCISE II. a.

(For the notation, see p. xi.)

1. Prove that $SP \sim SL = e \,.\, SN$.

2. If PA, PA′ cut the directrix at K, K′, prove that $\angle KSK' = 90°$.

3. If the chords PQ, PQ′ of a conic cut the directrix at R, R′, prove that $\angle RSR'$ is equal or supplementary to $\frac{1}{2} \angle QSQ'$.

4. The tangent at P meets the minor axis in t; tH is the perpendicular from t to SP; prove that $SH = a$.

5. The tangent at L cuts NP produced at O; prove that $ON = SP$.

6. The tangent at P and the chord QQ′ intersect on the directrix; prove that SP bisects $\angle QSQ'$.

7. The tangent at A cuts SM at H; prove that HP bisects $\angle SPM$.

8. The tangent at P cuts the directrix at R and the transverse axis at T; SM cuts PR at O; prove that $OR \,.\, OT = OS^2$.

9. PSQ is a focal chord; prove that SX bisects $\angle PXQ$.

10. PN cuts the tangent at Q in T; prove that $SP = ST \,.\, \cos QST$.

11. Prove that XY and SM make equal angles with the axis.

12. PSQ is a focal chord; the tangents at P, Q cut SL at H, K; prove that $HS = SK$.

13. PNP′ is a principal double ordinate; PX cuts the conic again at Q; prove that S lies on P′Q.

14. The tangent at P cuts the directrix at R; SL is produced to cut PR at O; prove that $SO = e \,.\, SR$.

15. Prove that $SY = e \,.\, YX$.

16. PSQ is a focal chord; the line through Q perpendicular to QP cuts the tangent at P in O; prove that the directrix bisects OQ.

17. PSQ is a focal chord; the normals at P, Q intersect at O; OH is the perpendicular to PQ; prove that $HP = QS$.

Theorem 3. The tangents from any point to a conic subtend equal or supplementary angles at a focus.

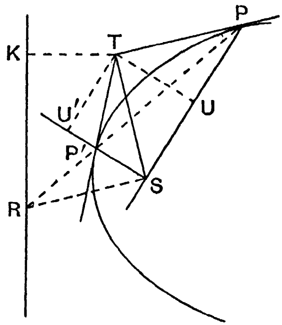

FIG. 16.

Let TP, TP′ be the tangents from any point T to the conic; TU, TU′, TK are the perpendiculars from T to SP, SP′ and the directrix.

Then $\qquad SU = e\,.\,TK \quad \text{and} \quad SU' = e\,.\,TK;$

$$\therefore\ SU = SU';$$

$\therefore$ in Δs TSU, TSU′,

$$\angle TUS = 90^\circ = \angle TU'S,$$

$$SU = SU' \text{ and } ST \text{ is common};$$

$$\therefore\ \Delta TSU \equiv \Delta TSU';$$

$$\therefore\ \angle TSU = \angle TSU', \text{ i.e. } \angle TSP = \angle TSP'.$$

Corollary. If PP′ is a chord of a conic which cuts the directrix at R, and if the tangents at P, P′ meet in T, then $\angle RST = 90^\circ$.

For SR and ST are the two bisectors of $\angle PSP'$.

Note. If TP, TP′ are tangents to two different branches of a hyperbola, $\angle$s TSP, TSP′ are supplementary. The reader should draw a figure to illustrate this case. The Corollary still holds good.

Theorem 4. If the normal at any point P on a conic cuts the transverse axis at G, then SG $= e$. SP.

Let the tangent at P cut the directrix at R; draw PM perpendicular to the directrix; join SR, SM.

Since $\angle$RSP $= 90° = \angle$RMP, the circle on RP as diameter passes through S, M; it also touches PG, since $\angle$RPG $= 90°$.

$\therefore$ $\angle$SPG $= \angle$SMP, alternate segment.

Also $\angle$PSG $= \angle$MPS, by parallels.

$\therefore$ $\triangle$s SPG, PMS are similar;

$$\therefore \frac{SG}{SP} = \frac{SP}{PM} = e;$$

$\therefore$ SG $= e$. SP.

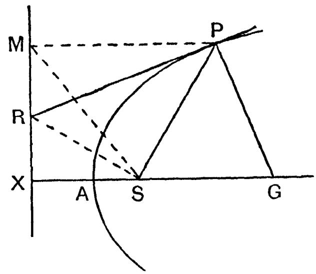

FIG. 17.

Theorem 5. The projection of the normal PG on SP is equal to the semi-latus rectum.

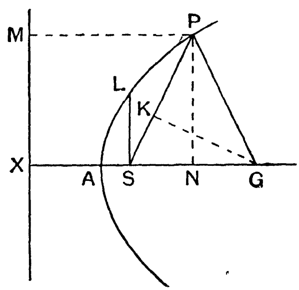

FIG. 18.

Draw GK perpendicular to SP, so that PK is the projection of PG on SP.

Draw PN, PM perpendicular to the transverse axis and the directrix. The $\triangle$s GKS, PNS are similar ;

$$\therefore \frac{SK}{SN}=\frac{SG}{SP}=e=\frac{SP}{PM}=\frac{SP-SK}{PM-SN}.$$

But $SP-SK=PK$ and $PM-SN=NX-SN=SX$;

$$\therefore \frac{PK}{SX}=e, \quad \therefore PK=e\,.\,SX=SL.$$

Theorem 6. If PSP′ is a focal chord with P, P′ on the same branch of the conic, then

$$\text{(i)}\ \frac{1}{SP}+\frac{1}{SP'}=\frac{2}{l}; \qquad \text{(ii)}\ SP\,.\,SP'=\tfrac{1}{2}l\,.\,PP'.$$

Let PP′ produced cut the directrix at R ; draw PM, P′M′ perpendicular to the directrix.

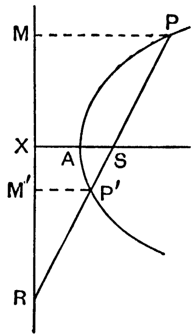

FIG. 19.

$$\text{(i)}\ \frac{SP}{SP'}=\frac{e\,.\,PM}{e\,.\,P'M'}=\frac{PM}{P'M'}=\frac{PR}{P'R};$$

$\therefore$ {RP′SP} is harmonic ;

$\therefore$ RP, RS, RP′ are in Harmonical Progression ;

$\therefore$ by proportion, PM, SX, P′M′ are in H.P. ;

$\therefore$ $e\,.\,PM$, $e\,.\,SX$, $e\,.\,P'M'$ are in H.P. ;

$\therefore$ SP, SL, SP′ are in H.P. ;

$$\therefore \frac{1}{SP}+\frac{1}{SP'}=\frac{2}{SL}=\frac{2}{l}.$$

$$\text{(ii) Now}\ \frac{1}{SP}+\frac{1}{SP'}=\frac{SP'+SP}{SP\,.\,SP'}=\frac{PP'}{SP\,.\,SP'};$$

$$\therefore SP\,.\,SP'=\tfrac{1}{2}l\,.\,PP'.$$

Note. If P, P′ lie on different branches of a hyperbola, as in Fig. 20, then

$$\text{(i) } \frac{1}{SP} - \frac{1}{SP'} = \frac{2}{l}; \quad \text{(ii) } SP \,.\, SP' = \tfrac{1}{2}l \,.\, PP'.$$

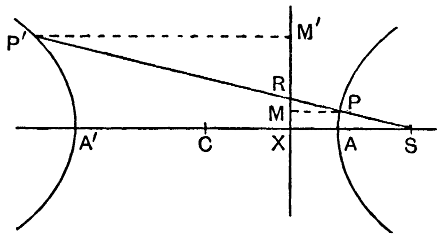

FIG. 20.

As before {RS ; PP′} is harmonic ; and it is left to the reader to deduce that in Fig. 20, $\frac{1}{PR} - \frac{1}{P'R} = \frac{2}{SR}$, from which the required results follow.

EXERCISE II. b.

(For the notation, see p. xi.)

1. PSQ is a focal chord ; the tangents at P, Q cut any other tangent at H, K ; prove that $\angle HSK = 90°$. What does this become if P is at A ?

2. The tangents at P, Q meet at T ; another tangent cuts TP, TQ at H, K ; prove that $\angle HSK$ is equal or supplementary to $\frac{1}{2}\angle PSQ$.

3. If a quadrilateral circumscribes an ellipse, prove that either pair of opposite sides subtend supplementary angles at the focus.

4. If SG = GP, prove that SP = 2SL.

5. Prove that the circle, centre G, radius GP, cuts off from SP a chord of constant length.

6. If P tends towards A along the curve, prove that the limiting value of PG equals SL.

7. If GK is the perpendicular from G to SP, prove that NK is parallel to SM.

8. H, K are points on a tangent to a conic such that $\angle HSK = 90°$, prove that the other tangents from H, K intersect on the directrix.

9. Prove that the distance of G from SP equals $e \,.\, MX$.

10. The tangent at P cuts the tangents at A, A′ in H, H′ ; A′P, AP cut the tangents at A, A′ in K, K′ ; prove that

$$AK \cdot A'K' = 4AH \cdot A'H' = 4b^2.$$

11. PSP′ is a focal chord of a parabola ; PN, P′N′ are ordinates to the axis; prove that (i) $AN \cdot AN' = AS^2$; (ii) $PN \cdot P'N' = SL^2$.

12. If, in No. 11, AP cuts SL at K, prove that SK = P′N′.

Construction. Given the focus, directrix and eccentricity of a conic, construct the points in which a given line intersects the conic.

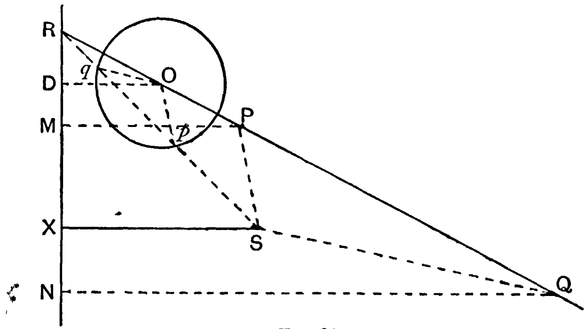

FIG. 21.

Let the given line cut the directrix at R, take any other point O on the line.

Draw OD perpendicular to the directrix. With centre O and radius $e \cdot$ OD describe a circle. Join SR and let SR cut the circle at p, q.

Draw SP, SQ parallel to Op, Oq to cut OR at P, Q.

Then P, Q are the required points of intersection of OR with the conic.

Draw PM, QN perpendicular to the directrix.

By parallels
$$\frac{SP}{Op} = \frac{PR}{OR} = \frac{PM}{OD};$$

$$\therefore \frac{SP}{PM} = \frac{Op}{OD} = \frac{e \cdot OD}{OD} = e, \text{ by construction};$$

$\therefore$ P lies on the conic.

Similarly Q lies on the conic.

Note. We may construct the tangents from any given point O to the conic as follows: From S draw the tangents Sf, Sf' to the circle and produce them to cut the directrix at R, R′; then RO, R′O when produced touch the conic, for if in Fig. 21, p coincides with q, then P coincides with Q.

Definition. If OD is the perpendicular from any point O to the directrix, the circle, centre O, radius e . OD, is called the **eccentric circle of O.**

Note. The general treatment arising from the use of the eccentric circle is not easy, and it may therefore be considered desirable to leave it for a second reading. In this case, the various forms of the Corollary of Theorem 7 should be established for the ellipse by orthogonal projection and should be assumed for the general conic.

Theorem 7. If from any point O two lines OPQ, OP′Q′ are drawn cutting a conic at P, Q and P′, Q′ and making angles θ, θ' with the transverse axis, then $\dfrac{\mathrm{OP}\,.\,\mathrm{OQ}}{\mathrm{OP'}\,.\,\mathrm{OQ'}} = \dfrac{1 - e^2\cos^2\theta'}{1 - e^2\cos^2\theta}$.

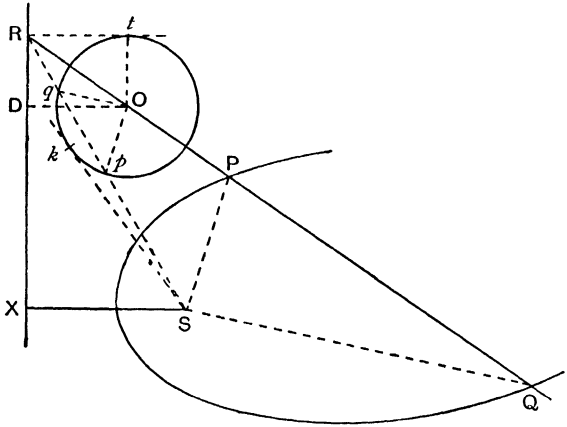

FIG. 22.

Make the same construction as on p. 21.

Draw tangents Rt, Sk from R, S to the eccentric circle of O.

By parallels $\dfrac{\mathsf{OP}}{\mathsf{OR}}=\dfrac{\mathsf{S}p}{\mathsf{R}p}$ and $\dfrac{\mathsf{OQ}}{\mathsf{OR}}=\dfrac{\mathsf{S}q}{\mathsf{R}q}$;

$$\therefore \frac{\mathsf{OP}\,.\,\mathsf{OQ}}{\mathsf{OR}^2}=\frac{\mathsf{S}p\,.\,\mathsf{S}q}{\mathsf{R}p\,.\,\mathsf{R}q}=\frac{\mathsf{S}k^2}{\mathsf{R}t^2}.$$

Now $\mathsf{R}t^2=\mathsf{OR}^2-\mathsf{O}t^2=\mathsf{OR}^2-e^2\,.\,\mathsf{OD}^2.$

But $\angle\mathsf{DOR}=\theta$, $\therefore\ \mathsf{OD}=\mathsf{OR}\cos\theta$;

$$\therefore\ \mathsf{R}t^2=\mathsf{OR}^2-e^2\,.\,\mathsf{OR}^2\cos^2\theta=\mathsf{OR}^2(1-e^2\cos^2\theta);$$

$$\therefore\ \mathsf{OP}\,.\,\mathsf{OQ}=\mathsf{S}k^2\,.\,\frac{\mathsf{OR}^2}{\mathsf{R}t^2}=\frac{\mathsf{S}k^2}{1-e^2\cos^2\theta}.$$

Similarly, $\mathsf{OP}'\,.\,\mathsf{OQ}'=\dfrac{\mathsf{S}k^2}{1-e^2\cos^2\theta'}$;

$$\therefore \frac{\mathsf{OP}\,.\,\mathsf{OQ}}{\mathsf{OP}'\,.\,\mathsf{OQ}'}=\frac{1-e^2\cos^2\theta'}{1-e^2\cos^2\theta}.$$

Corollary. If, from a *variable* point O, lines are drawn in *fixed* directions cutting a given conic in P, Q and P′, Q′, then $\dfrac{\mathsf{OP}\,.\,\mathsf{OQ}}{\mathsf{OP}'\,.\,\mathsf{OQ}'}$ is constant.

Note. There are several important special forms of this constant.

(i) Let ESF, E′SF′ be the focal chords parallel to OPQ, OP′Q′;

then $\dfrac{\mathsf{OP}\,.\,\mathsf{OQ}}{\mathsf{OP}'\,.\,\mathsf{OQ}'}=\dfrac{\mathsf{SE}\,.\,\mathsf{SF}}{\mathsf{SE}'\,.\,\mathsf{SF}'}=\dfrac{\mathsf{EF}}{\mathsf{E'F'}}$, Th. 6.

(ii) Let TZ, TZ′ be the tangents parallel to OPQ, OP′Q′; then

$$\frac{\mathsf{OP}\,.\,\mathsf{OQ}}{\mathsf{OP}'\,.\,\mathsf{OQ}'}=\frac{\mathsf{TZ}\,.\,\mathsf{TZ}}{\mathsf{TZ}'\,.\,\mathsf{TZ}'}=\frac{\mathsf{TZ}^2}{\mathsf{TZ}'^2}.$$

(iii) Let HCK, H′CK′ be the diameters (in the case of a central conic) parallel to OPQ, OP′Q′; then

$$\frac{\mathsf{OP}\,.\,\mathsf{OQ}}{\mathsf{OP}'\,.\,\mathsf{OQ}'}=\frac{\mathsf{CH}\,.\,\mathsf{CK}}{\mathsf{CH}'\,.\,\mathsf{CK}'}=\frac{\mathsf{CH}^2}{\mathsf{CH}'^2},$$

since HC = CK and H′C = CK′.

Theorem 8. If a circle and a conic intersect at four points, their common chords, taken in pairs, make equal angles with the axis of the conic.

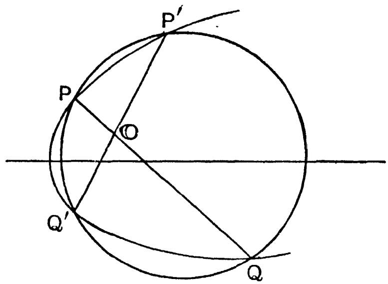

FIG. 23.

Let the common chords PQ, P′Q′ intersect at O.

Then by a property of the circle

$$OP \cdot OQ = OP' \cdot OQ';$$

∴ from note (i) above, the focal chords parallel to PQ, P′Q′ are of equal length and therefore make equal angles with the axis of the conic;

∴ PQ, P′Q′ also make equal angles with the axis.

Note. For a central conic, we could use note (iii) instead of note (i), if preferred.

EXERCISE II. c.

(For the notation, see p. xi.)

1. Prove that the auxiliary circle is the eccentric circle of C.

2. PSP′ is a focal chord, parallel to the diameter QCQ′; prove that $SP \cdot SP' = \frac{b^2}{a^2} \cdot CQ^2$.

3. With the data of No. 2, prove that $PP' = \frac{2CQ^2}{CA}$.

4. PQ is a chord of a conic; the tangents at P, Q meet at T; a line parallel to TQ cuts PT, PQ and the conic at O, R, H, K; prove that $OR^2 = OH \cdot OK$.

5. A conic touches the sides QR, RP, PQ of a triangle at X, Y, Z; prove that PY . RX . QZ = PZ . QX . RY.

6. Two conics S_1, S_2 cut at A, B, C, D; x_1, y_1 and x_2, y_2 are the lengths of the pairs of diameters of s_1 and s_2 parallel to AB, CD; prove that $x_1y_2 = x_2y_1$.

7. A circle touches a conic at P and cuts it again at Q, Q′; prove that the axis of the conic is parallel to a bisector of the angle between QQ′ and the tangent at P.

8. If PQ and VW are intersecting chords of a conic making equal angles with the axis, prove that PV and QW also make equal angles with the axis.

9. PQ, HK are two non-intersecting chords of a conic; also

$$\angle PHQ = \angle PKQ;$$

VW is a chord parallel to HK; prove that $\angle$s PVQ, PWQ are equal or supplementary.

10. If the sides BC, CA, AB of a triangle cut a conic at P_1, P_2; Q_1, Q_2; R_1, R_2 respectively; prove that

$$\frac{AR_1 \cdot AR_2}{AQ_1 \cdot AQ_2} \cdot \frac{BP_1 \cdot BP_2}{BR_1 \cdot BR_2} \cdot \frac{CQ_1 \cdot CQ_2}{CP_1 \cdot CP_2} = 1.$$

11. A circle touches a conic at P and cuts it again at V, W; lines through V, W parallel to the axis cut the circle again at V′, W′; prove that V′W′ is parallel to the tangent at P.

12. PQ is a chord of a conic, normal at P and cutting the axis at G; PK is a chord cutting the axis at H. If PG = PH; prove that

$$\angle PKQ = 90°.$$

13. PH, QK are chords of a conic, normal at P and Q; if PH is perpendicular to QK; prove that PQ is parallel to HK.

14. PQ, PR are two chords equally inclined to the tangent at P; prove that $\frac{PQ}{PR}$ equals the ratio of the parallel focal chords.

15. PQ, PQ′ are chords making angles ϕ, ϕ' with the tangent at P; prove that $\frac{PQ \cdot \sin \phi'}{PQ' \cdot \sin \phi}$ equals the ratio of the parallel focal chords. (In Theorem 7, suppose that P′ tends to P along the curve.)

16. From a point P on an ellipse, a line is drawn perpendicular to AP, cutting AA′ in R; prove that the focal chord parallel to AP equals AR.

17. HSK is a focal chord parallel to another chord PQ; the tangent at P cuts HK at T; prove that $\frac{PQ}{HK} = \frac{SP}{ST}$.

CHAPTER III

THE PARABOLA

PROPERTIES of diameters (*e.g.* Theorems 13, 14) are best treated projectively. But independent proofs are given in the text, owing to their intrinsic interest.

The notation adopted is explained on p. ix.

Theorem 9. (i) PT bisects $\angle$SPM and SP = ST = SG.

(ii) SM and PT bisect each other at right angles.

(iii) Y lies on the tangent at the vertex.

(iv) AN = AT and AY = $\frac{1}{2}$NP and NG = 2AS.

(v) SY^2 = SA . SP.

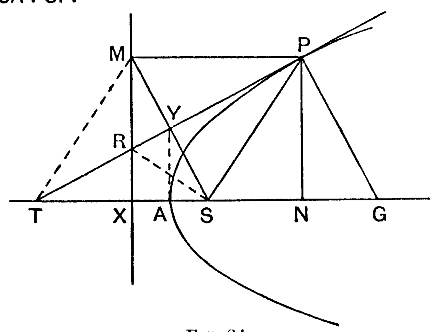

FIG. 24.

(i) By Th. 1 (ii), $\angle$RSP = 90°;

$\therefore$ in $\triangle$s RSP, RMP, $\angle$RSP = 90° = $\angle$RMP.

SP = PM, definition; RP is common;

$\therefore$ $\triangle$RSP $\equiv$ $\triangle$RMP; $\therefore$ TRP bisects $\angle$SPM.

Further, $\angle$SPT = $\angle$MPT = $\angle$PTS, alternate $\angle$s; $\therefore$ SP = ST.

Also, $\angle TPG = 90°$, $\therefore$ $ST = SP = SG$.

(ii) Since $ST = SP = PM$, and since ST is parallel to PM, TSPM is a rhombus;

$\therefore$ SM and PT bisect each other at right angles.

(iii) From (ii), Y is the point of intersection of SM, PT and $SY = YM$. But $SA = AX$;

$\therefore$ AY is parallel to XM and perpendicular to SX;

$\therefore$ AY is the tangent at A, since by (i) the tangent at A bisects the angle between SA and AX.

(iv) Since $PY = YT$, and since YA is parallel to PN,

$$AN = AT \quad \text{and} \quad AY = \tfrac{1}{2}PN.$$

Also, $\triangle$s PNG, YAS are similar, since corresponding sides are parallel;

$$\therefore \frac{NG}{AS} = \frac{PN}{AY} = 2.$$

(v) $\angle ASY = \angle SMP$, alternate $\angle$s, $= \angle MSP$, since $SP = PM$;

$$\therefore \triangle\text{s } ASY, YSP \text{ are similar,}$$

for $\angle ASY = \angle YSP$ and $\angle YAS = 90° = \angle PYS$;

$$\therefore \frac{AS}{SY} = \frac{SY}{SP} \quad \text{or} \quad SY^2 = SA \,.\, SP.$$

Theorem 10. If the tangents at the extremities of any focal chord PSP′ of a parabola meet at R, then (i) the line through R parallel to the axis bisects PP′ at V, (ii) $\angle$ PRP′ = 90°, (iii) if RV cuts the parabola at K, RK = KV and PP′ = 4SK.

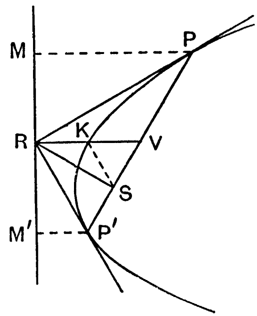

FIG. 25.

By Theorem 1, Corollary 2, R lies on the directrix and $\angle$ RSP = 90°.

Draw PM, P′M′ perpendicular to the directrix and draw RKV parallel to the axis, cutting the curve at K; join SK.

$\angle$ VRP = $\angle$ RPM, alternate angles
= $\angle$ RPV; Th. 9 (i).

$\therefore$ RV = VP; similarly RV = VP′;

$\therefore$ PV = VR = VP′, which proves (i),

and $\angle$ PRP′ = 90°, which proves (ii).

Further, SK = KR (definition) and $\angle$ RSV = 90°, by Th. 1 (ii);

$\therefore$ RK = KS = KV;

$\therefore$ PP′ = 2PV = 2VR = 4KR = 4SK.

EXERCISE III. a.

(For the notation, see pp. ix, 3.)

1. Prove that PG = 2SY.

2. Prove that AY^2 = AS . AN.

3. Prove that $PG^2 = 2SL \cdot SP$.

4. Prove that in Fig. 24, $PY \cdot YR = SA \cdot SP$.

5. Prove that $TY = YN$.

6. Prove that the line through Y, parallel to the axis, bisects SP.

7. If PSQ is a focal chord, prove that the tangent at Q is parallel to SM.

8. The tangent at P cuts SL at K and the directrix at R; prove that $SK = SR$.

9. Prove that the tangent at A touches the circle on SP as diameter.

10. If $SG = 2SX$, prove that $\triangle SPG$ is equilateral.

11. Two parabolas have a common focus and their axes in opposite directions; prove that they cut orthogonally.

12. The perpendicular through P to AP cuts the axis at K; prove that $NG = GK$.

13. SK is the perpendicular from S to PG; prove that KY is parallel to the axis.

14. PSQ is a focal chord; circles are drawn through S touching the parabola at P, Q; prove that they are orthogonal.

15. RP, RP′ are tangents to a parabola from a point R on the directrix; the lines through S parallel to RP, RP′ cut the directrix at K, K′; prove that $KR = RK'$.

16. GP is produced to cut the directrix at K; the perpendicular through S to SP cuts PG at H; prove that $KP = PH$.

17. PSQ is a focal chord; the tangents at P, Q meet at T; TQ cuts SL at K; prove that $TK = PG$.

18. If $\triangle TPS$ is equilateral, prove that $SP = \frac{4}{3}AS$.

19. PP′ is a principal double ordinate; the perpendicular from P′ to the tangent at P cuts the axis at K; prove that KG is constant.

20. If $\angle PSN = 60°$, prove that the tangent at P touches the circle having LL′ as diameter.

21. T is a point on the tangent at a variable point P of a parabola. If $\angle STP$ is constant, prove that the locus of T is a straight line.

22. If PA cuts the directrix at R, prove that $\angle MSR = 90°$.

23. PSQ is a focal chord; prove that the circle on PQ as diameter touches the directrix.

24. PN cuts the tangents at L, L′ at K, K′; prove that $SN^2 = PK \cdot PK'$.

25. If the ordinate QK from a point Q on a parabola to the axis bisects PG, prove that $QK = PG$.

26. Two parabolas having a common focus intersect at P, Q; prove that the curves cut at equal angles at P and Q.

27. The normals at the points P, P′ cut the axis at G, G′; if $PG^2 + P'G'^2$ is constant, prove that the locus of the mid-point of PP′ is a straight line.

28. PSQ is a focal chord; PA cuts the directrix at K; prove that KQ is parallel to the axis.

29. The normals at the points P, P′ cut the axis at G, G′; prove that $PG^2 \sim P'G'^2 = 4AS \,.\, GG'$.

30. AQ is a chord which cuts the tangent at P at right angles at K; prove that $AK \,.\, AQ = SL^2$.

31. PSQ is a focal chord; PK is a chord equally inclined to the axis; prove that $\angle PQK = 3\angle PKQ$.

32. PSQ is a focal chord parallel to the tangent at L; prove that $PQ = 4SL$.

33. In Fig. 24, RS cuts PG at H; HK is the perpendicular from H to the axis; prove that $SK = TN$.

34. YS is produced to Z, so that $YS = SZ$; prove that $\angle PAZ = 90°$.

35. Two parabolas, having the same axis and focus S, intersect at P; PS cuts them again at H, K; prove that $SP^2 = SH \,.\, SK$.

36. The normals at P, P′ cut the axis at G, G′; if $P'G' - PG = 2SL$, prove that $P'G' + PG = GG'$.

37. PSQ is a focal chord; the line through Q parallel to the axis cuts PN at K; prove that $\angle PGK = 90°$.

38. A chord PP′ of a parabola, when produced, passes through X; PN, P′N′ are the ordinates of P, P′ to the axis; prove that $AN \,.\, AN' = a^2$.

39. If a particle is projected in a given vertical plane from a fixed point P with velocity $\sqrt{(2gh)}$, it can be proved that it describes a parabola whose axis is vertical and whose directrix is at a height h above P. Prove that for different directions of projection the parabolic paths all touch a fixed parabola, focus P, latus rectum $4h$, and that the line joining the foci of the fixed and variable parabola passes through their point of contact.

40. With the data of No. 39, show that the greatest range on a plane through P at angle α with the horizontal is $\dfrac{2h}{1+\sin\alpha}$.

Theorem 11. If OP, OQ are any two tangents to a parabola, then (i) the triangles SOP, SQO are similar and $SO^2 = SP \cdot SQ$, (ii) the exterior angle between the tangents is equal to the angle either subtends at the focus, (iii) $\angle SOQ$ equals the angle which OP makes with the axis.

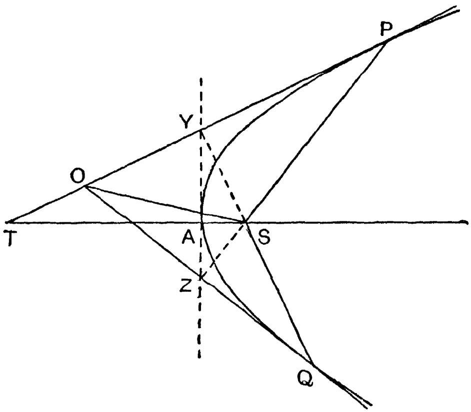

FIG. 26.

Let the tangent at the vertex A cut OP, OQ at Y, Z ; join SY, SZ ; let PO cut the axis at T.

(i) By Th. 9 (iii), $\angle SYO = 90° = \angle SZO$, so that OYSZ is a cyclic quadrilateral, and by Th. 9 (v), $\triangle$s ASY, YSP are equiangular;

$$\therefore \angle SPY = \angle SYA \text{ or } \angle SYZ$$
$$= \angle SOZ, \text{ since OYSZ is a cyclic quadrilateral ;}$$

$\therefore \angle SPO = \angle SOQ$. Similarly, $\angle SOP = \angle SQO$;

$\therefore \triangle$s SOP, SQO are similar and $\dfrac{SP}{SO} = \dfrac{SO}{SQ}$; $\therefore SO^2 = SP \cdot SQ$.

(ii) $\angle TOQ = 180° - (\angle SOQ + \angle SOP) = 180° - (\angle SPO + \angle SOP)$;
$= \angle OSP$ or $\angle OSQ$.

(iii) $\angle SOQ = \angle SPO$ (or $\angle SPT$) $= \angle STP$, Th. 9 (i), since $SP = ST$.

Note. The construction used for proving this theorem gives a method for drawing the tangents from any point O to a parabola, for the circle on SO as diameter cuts the tangent at A in the points Y, Z ; and the required tangents are OY, OZ.

Theorem 12. The circumcircle of a triangle circumscribing a parabola passes through the focus.

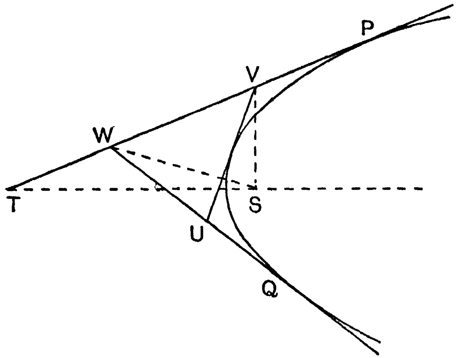

FIG. 27.

Let the three tangents form the triangle UVW and let the tangent at P cut the axis at T.

By Th. 11 (iii), $\angle SVU = \angle STP$,

and $\angle SWU = \angle STP$;

$\therefore \angle SVU = \angle SWU$;

$\therefore$ S, V, W, U lie on a circle.

Note. The following alternative method deserves notice.

By Th. 9 (iii), the feet of the perpendiculars from S to VW, WU, UV lie on the tangent at the vertex and are therefore collinear. Hence by the converse of "Simson's Line," S lies on the circumcircle of the triangle UVW and the tangent at the vertex is the pedal line of S with respect to this triangle.

From this, we may deduce Steiner's theorem.

The orthocentre of any triangle circumscribing a parabola lies on the directrix.

For if the orthocentre of $\triangle$UVW is H, it is a well-known theorem that SH is bisected by the pedal line of S; $\therefore$ SH is bisected by the tangent at the vertex; $\therefore$ H lies on the directrix.

EXERCISE III. b.

1. In Fig. 26, prove that $\frac{OP^2}{OQ^2}=\frac{SP}{SQ}$.

2. In Fig. 26, prove that OQ touches the circle OPS.

3. The tangents from O to a parabola contain an angle θ; prove that the distance of O from the directrix is SO . cos θ.

4. A variable tangent cuts two fixed tangents at V, W; prove that $\frac{SV}{SW}$ is constant.

5. PP′ is a principal double ordinate; any tangent cuts the tangents at P, P′ in O, O′; prove that SO = SO′.

6. UVW is the triangle formed by three tangents to a parabola; prove that the lines through U, V, W perpendicular to SU, SV, SW are concurrent.

7. In Fig. 26, prove that the circumcentre of △OPQ lies on the circle SPQ.

8. Through each of two points H, K is drawn a pair of lines, forming a cyclic quadrilateral, with HK as the third diagonal; prove that the focus of the parabola touching the four lines lies on HK.

9. In Fig. 26, the lines bisecting ∠POQ cut the axis at R, R′; prove that SR = SO = SR′.

10. UVW is an equilateral triangle circumscribing a parabola; prove that US cuts VW at its point of contact.

11. Given the focus of a parabola and two tangents, construct their points of contact.

12. In Fig. 26, the circles OSP, OSQ cut the axis again at H, K; prove that PH is parallel to OK.

13. Z, Z′ are points on the tangent at P such that SZ = SZ′; prove that the other tangents from Z, Z′ intersect on the axis.

14. OP, OQ are the tangents to a parabola from a point O on SL; the line through Q parallel to the axis cuts SP at K; prove that

$$\angle KOP = 90^\circ.$$

15. Prove that the four circles which circumscribe the four triangles formed by taking three out of four given straight lines have a common point.

16. If a chord PQ of a parabola cuts the directrix at R and if the tangents at P, Q cut at O; prove that RP . RQ = RO^2. [Prove that RP . RQ = RS^2 + SP . SQ, using Th. 1.]

17. Two parabolas each touch the sides of a triangle PQR and have perpendicular axes; prove that their axes intersect on the circumcircle of △PQR.

18. In Fig. 26, OS is produced to cut the circle OPQ at K; prove that OS = SK.

19. In Fig. 26, the tangents at P, Q to the circle OPQ intersect at K; prove that OS passes through K.

20. In Fig. 26, the circle, centre S, radius SO, cuts OP, OQ at H, K; prove that HK is perpendicular to the axis and that its distance from O is twice its distance from A.

Theorem 13. The locus of the mid-points of a system of parallel chords of a parabola is a straight line parallel to the axis, which cuts the directrix at a point H, such that SH is perpendicular to each chord of the system.

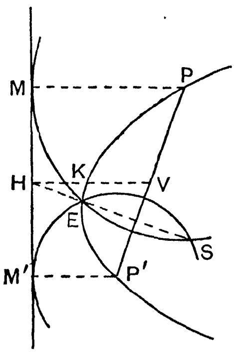

FIG. 28.

Let PP′ be any chord of the system; draw PM, P′M′ perpendicular to the directrix.

Draw two circles, centres P and P′, radii PM and P′M′; these circles pass through the focus S, since SP = PM and SP′ = P′M′; let them cut again at E. MM′ is a common tangent.

Then the common chord ES cuts the line of centres PP′ at right angles and bisects the common tangent MM′ at H, say.

Through H draw a line parallel to the axis, cutting the curve at K and PP′ at V.

Now MH = HM′ and MP, HV, M′P′ are parallel;

$$\therefore\ PV = VP';$$

∴ the mid-point of PP′ lies on the line through H parallel to the axis.

But H is a fixed point, since SH is perpendicular to PP′, which is fixed in direction;

∴ the locus of the mid-point of PP′ is a straight line which cuts the directrix at a point H such that SH is perpendicular to PP′, and therefore perpendicular to each chord of the system.

Definition. The locus of the mid-points of parallel chords of a parabola is called a **diameter** of the parabola and its point of intersection K with the curve is called the **vertex** of that diameter.

If a chord PQ cuts the diameter KV which bisects it at V (see Fig. 29), then PV is called the **ordinate** from P to the diameter KV, and KV is called the **abscissa**, K being the vertex of the diameter.

Theorem 14. If K is the vertex of the diameter KV bisecting PQ at V, then (i) the tangent at K is parallel to PQ.

(ii) The tangents at P and Q intersect at a point O on the diameter KV which bisects PQ.

(iii) OK = KV.

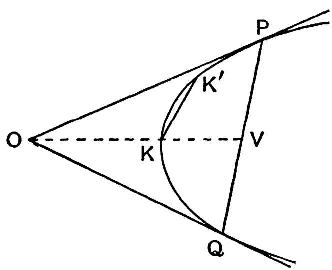

FIG. 29.

(i) Draw through K a line parallel to PQ; if this line cuts the curve again at K′, the mid-point of KK′ lies on KV, which is impossible, unless K′ coincides with K;

∴ the line through K, parallel to PQ, must touch the curve at K.

(ii) Draw any chord $P'Q'$ parallel to PQ; let PP' cut QQ' at O'; bisect PQ at V; let $O'V$ cut $P'Q'$ at V'.

Then by parallels, since $PV=VQ$, $\therefore$ $P'V'=V'Q'$;

$\therefore$ $O'V'V$ is the diameter bisecting PQ.

But when P' tends towards P along the curve, Q' tends towards Q, and in the limiting position $O'P'P$, $O'Q'Q$ become the tangents OP, OQ at P, Q;

$\therefore$ the tangents at P, Q meet on the diameter bisecting PQ.

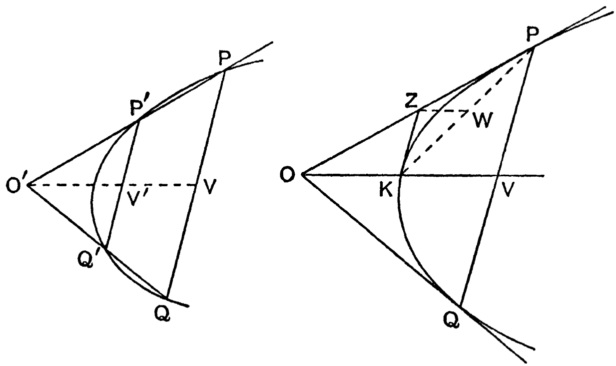

FIG. 30.

(iii) Let the tangent at K cut OP at Z; bisect PK at W; join ZW.
Then ZW is a diameter by (ii), and is therefore parallel to OK.
But $PW=WK$, $\therefore$ $PZ=ZO$.
Also, ZK is parallel to PV by (i).
But $OZ=ZP$, $\therefore$ $OK=KV$.

Corollary. The ratio of the lengths of any two tangents to a parabola is equal to the ratio of the sides of any triangle which has its sides parallel to the tangents and its base parallel to the axis.

For in Fig. 30, $ZP=ZO$;

$\therefore$ $\dfrac{ZP}{ZK}=\dfrac{ZO}{ZK}=$ ratio of sides of $\triangle ZOK$ where base OK is parallel to the axis.

Theorem 15. If PV is the ordinate from any point P on a parabola to the diameter KV, whose vertex is K, then $PV^2 = 4SK.KV$.

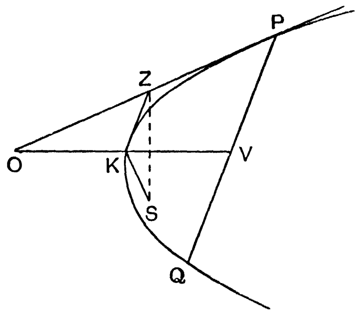

FIG. 31.

Let the tangent at P cut KV at O and let the tangent at K cut OP at Z; join SZ.

By Th. 14, KZ is parallel to PV and $OK = KV$;

$$\therefore \quad PV = 2ZK.$$

Also $\triangle OKZ$ is equiangular to $\triangle ZKS$,

since $\angle OZK = \angle ZSK$, Th. 11 (ii),

and $\angle ZOK = \angle SZK$, Th. 11 (iii);

$\therefore$ the triangles are similar and $\dfrac{OK}{KZ} = \dfrac{KZ}{KS}$;

$$\therefore \quad KZ^2 = OK.KS;$$

$$\therefore \quad PV^2 = 4ZK^2 = 4OK.KS = 4SK.KV.$$

EXERCISE III. c.

(For the notation, see p. ix.)

1. PQ, P′Q′ are parallel chords of a parabola: P′Q′ cuts the tangents at P, Q in H, K; prove that $P'H = Q'K$.

2. P, Q, R are points on a parabola on the same side of the axis. If PQ is parallel to AR, prove that the ordinate of R is equal to the sum of the ordinates of P and Q.

3. PQ is a variable chord, fixed in direction; if P and Q lie on opposite sides of the axis, prove that the difference of their ordinates is constant.

4. PSQ is a focal chord; the normal at P cuts the curve again at K; prove that the diameter through Q bisects PK.

5. PSQ is a focal chord; the tangents at P, Q meet at O; the normals at P, Q meet at K; prove that OK is parallel to the axis.

6. A chord PQ makes 45° with the axis; prove that its mid-point is at a distance $2a$ from the axis.

7. A circle cuts a parabola at P, Q, H, K; if P, Q lie on one side of the axis and H, K on the other side, prove that the sum of the ordinates of P and Q is equal to that of H and K. What happens if P, Q, H lie one side of the axis and K on the other side?

8. PSP′ is a focal chord parallel to the chord AQ PN, P′N′ are ordinates to the axis; prove that $AQ = SP \sim SP' = NN'$.

9. PQ is a chord, normal at P; the tangents at P, Q meet at O; prove that OP is bisected by the directrix.

10. PSQ is a focal chord; the tangents at P, Q meet at R; prove that the normal chord at P equals 4QR.

11. In Fig. 31, prove that the length of the focal chord parallel to PQ is $\dfrac{PV^2}{KV}$.

12. A chord PQ cuts the axis at K; the tangents at P, Q cut at O; prove that SK equals the distance of O from the directrix.

13. The perpendicular bisector of a chord QQ′ cuts QQ′ at V and the axis at K; VH is the perpendicular to the axis; prove that $HK = 2AS$.

14. PSQ is a focal chord; the perpendicular bisector of PQ cuts the axis at K; prove that $PQ = 2SK$.

15. If the diameter bisecting the chord PP′ cuts the directrix at O, and if OS cuts PP′ at K and if PM, P′M′ are the perpendiculars to the directrix, prove that $\angle POP' = \angle MKM'$.

16. A circle has three-point contact with a parabola at P and cuts it again at Q; the tangent at P cuts the axis at T; prove that $PQ = 4PT$.

17. OP, OQ are tangents at P, Q to a parabola; the line bisecting PQ at right angles meets the axis at G; prove that the focal chord parallel to PQ bisects OG.

18. Through a point P on a parabola, a line PK is drawn perpendicular to AP and cutting the axis at K; prove that the focal chord parallel to AP equals AK. What happens when P coincides with A?

19. PQ is a chord normal at P. If $\angle PSQ = 90°$, prove that

$$SQ = 2SP = 10a.$$

20. QSQ′ is a focal chord, P is the vertex of the diameter bisecting it; PK is the perpendicular to QQ′; prove that $SQ \,.\, SQ' = 4PK^2$.

21. QV is an ordinate to the diameter PV, vertex P; QK is the perpendicular to PV; prove that $QK^2 = 4AS \cdot PV$.

22. The tangents at the pairs of points P, Q; Q, R; R, P intersect at R', P', Q' respectively; prove that

$$\frac{PR'}{R'Q'} = \frac{R'Q}{QP'} = \frac{Q'P'}{P'R}.$$

(Draw diameters through R, P', Q', R' to cut PQ.)

23. OP, OP' are the tangents from O to a parabola; the diameter through O cuts the curve at Q; prove that $OP \cdot OP' = 4OS \cdot OQ$.

24. The tangent at P cuts a chord QR, when produced, at T; the diameter through P cuts QR at K; prove that $TQ \cdot TR = TK^2$.

25. The tangents at P, Q meet at T; O is the centre of the circle PTQ; prove that $\angle OST = 90°$.

26. A chord QQ' cuts a diameter PO, vertex P, in O; $QV, Q'V'$ are ordinates to PO; prove that $PV \cdot PV' = PO^2$.

27. PQ is a chord; the tangent at P cuts the diameter through Q in T; prove that $TP^2 = 4SP \cdot TQ$.

28. From a variable point T on a fixed diameter of a parabola, tangents TP, TQ are drawn to the curve; prove that the locus of the circumcentre of $\triangle TPQ$ is a straight line.

29. PQ is a chord, normal at P; the tangents at P, Q meet at T; prove that $\angle PTQ = \angle PAS$.

30. The tangents at P, Q meet at O; OK is the perpendicular from O to PQ; prove that SK equals the distance of O from the directrix.

CHAPTER IV

THE CENTRAL CONICS

Theorem 16. PG, PT are the normal and tangent at any point P on a conic, centre C; then (i) PG, PT are the bisectors of $\angle$ SPS′; (ii) SP + S′P $= 2a$ for an ellipse and SP $\sim$ S′P $= 2a$ for a hyperbola; (iii) if PN is the perpendicular from P to the transverse axis,

$$\text{CG} = e^2 \,.\, \text{CN}.$$

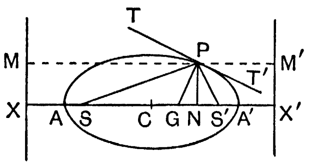

FIG. 32.

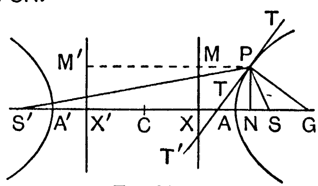

FIG. 33.

(i) By Th. 4, SG $= e$. SP and S′G $= e$. S′P;

$$\therefore \frac{\text{SG}}{\text{S}'\text{G}} = \frac{\text{SP}}{\text{S}'\text{P}};$$

$\therefore$ PG is one bisector of $\angle$ SPS′.

But PT is perpendicular to PG, $\therefore$ PT is the other bisector of $\angle$ SPS′.

(ii) SP $= e$. PM and S′P $= e$. PM′;

$\therefore$ for the ellipse,

$$\text{SP} + \text{S}'\text{P} = e\,.\,(\text{PM} + \text{PM}') = e\,.\,\text{MM}' = e\,.\,\text{XX}' = e\,.\,\frac{2a}{e} = 2a,$$

and for the hyperbola, in Fig. 33,

$$\text{S}'\text{P} - \text{SP} = e(\text{PM}' - \text{PM}) = e\,.\,\text{M}'\text{M} = 2a, \text{ as before.}$$

If P lies on the other branch of the hyperbola, $SP - S'P = 2a$.

(iii) $SG = e \cdot SP = e^2 \cdot PM = e^2 \cdot NX$ and $S'G = e \cdot S'P = e^2 \cdot NX'$.

For the ellipse, $$SG - S'G = e^2(NX - NX');$$
$$\therefore\ 2CG = e^2 \cdot 2CN.$$

For the hyperbola $$SG + S'G = e^2(NX + NX');$$
$$\therefore\ 2CG = e^2 \cdot 2CN;$$

$\therefore$ in each case, $$CG = e^2 \cdot CN.$$

Corollary. The circumcircle of $\triangle SPS'$ cuts CB at points t, g, such that Pt, Pg are the tangent and normal at P.

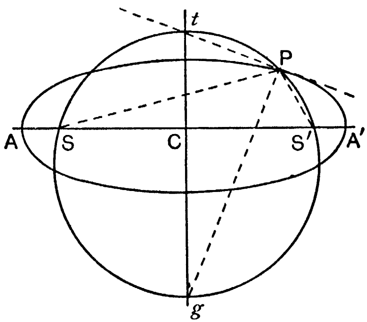

FIG. 34.

BC is the perpendicular bisector of SS'; $\therefore$ g, t are the mid-points of the two circular arcs SS';

$\therefore$ Pg, Pt are the bisectors of $\angle SPS'$ and are therefore the normal and tangent at P.

Note. (i) Th. 16 (i) is equivalent to saying that "*the tangent at any point makes equal angles with the focal distances of the point,*" *i.e.* $\angle SPT = \angle S'PT'$ (Fig. 32).

(ii) In Fig. 32, $SP = a + e \cdot CN$ and $S'P = a - e \cdot CN$.

For $$SP = e \cdot NX = e(CX + CN) = e\left(\frac{a}{e} + CN\right) = a + e \cdot CN.$$

And $$S'P = e \cdot NX' = e(CX' - CN) = e\left(\frac{a}{e} - CN\right) = a - e \cdot CN.$$

Similarly, in Fig. 33, $SP = e \cdot CN - a$ and $S'P = e \cdot CN + a$.

Theorem 17. If TP, TP′ are tangents from any point T to a conic, then $\angle STP = \angle S'TP'$.

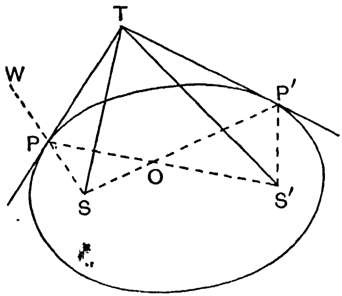

FIG. 35.

Join SP, S′P, SP′, S′P′ and let SP′ cut S′P at O.
Produce SP to W.
By Th. 16, PT bisects $\angle S'PW$.
By Th. 3, ST bisects $\angle PSP'$;

$$\therefore \angle STP = \angle WPT - \angle PST$$
$$= \tfrac{1}{2}\angle WPO - \tfrac{1}{2}\angle PSO = \tfrac{1}{2}(\angle WPO - \angle PSO)$$
$$= \tfrac{1}{2}\angle POS.$$

Similarly, $\angle S'TP' = \tfrac{1}{2}\angle P'OS'$;

$$\therefore \angle STP = \angle S'TP'.$$

EXERCISE IV. a.

(For the notation, see p. xi.)

1. With the data of Fig. 32, prove that the circles SPM, S′PM′ touch each other.

2. PP′ is a variable diameter of an ellipse, prove that $SP + SP'$ is constant.

3. If the incircle of $\Delta SPS'$ touches SP at H, prove that PH is constant.

4. Prove that PN bisects $\angle YNY'$.

5. Prove that $Cg = \frac{a^2e^2}{b^2} \cdot Cn$.

6. If the incircle of $\triangle SPS'$ touches SS' at K; prove that $AK = SP$.

7. Find the locus of the centre of a variable circle which touches two fixed circles.

8. PK is the tangent from a point P on an ellipse to the circle on BB' as diameter; prove that $PK^2 = CG \cdot CN$.

9. If the normal at a point P on an ellipse passes through B, prove that $CG^2 = a^2 - 2b^2$.

10. Lines through A, A′ parallel to the tangent at P cut PS′, PS respectively at H, K; prove that $S'H = SK$.

11. Prove that the least value of gt is $2ae$.

12. H is a point on AC such that $AH = SP$, prove that $BH = CP$.

13. If there exists a normal to an ellipse which cuts the minor axis at a point outside the ellipse, prove that $CS > CB$.

14. Prove that $gP = \frac{1}{e} \cdot gS$. (Use Ptolemy's theorem.)

15. Prove that (i) $\angle tSP = \angle PTS$,

(ii) $SP \cdot S'P = PT \cdot Pt$.

16. With the data of Fig. 35, SY, SZ are the perpendiculars from S to TP, TP′; prove that S′T is perpendicular to YZ.

17. If gK is the perpendicular from g to SP, prove that $PK = a$.

18. An ellipse, foci S, S′, is inscribed in the triangle PQR; prove that $\angle QSR + \angle QS'R = 180° + \angle QPR$.

19. The tangents at points P, Q on an ellipse meet at T; SP cuts S′Q at R; prove that TR bisects $\angle PRQ$.

20. P is a variable point on the line AB between the fixed points A, B; prove that the ellipses of the same fixed eccentricity with A, P and P, B respectively as foci intersect on a fixed ellipse of which A, B are the foci.

21. Prove that $CG \cdot CT = CS^2$.

22. If the tangents to a conic at P, P′ meet at T and if PT is produced to Q, prove that

(i) if PP′ cuts SS′ externally, $\angle PSP' + \angle PS'P' = 2\angle P'TQ$,

(ii) if PP′ cuts SS′ internally, $\angle PSP' \sim \angle PS'P' = 2\angle P'TP$.

23. The focal chord parallel to the tangent at P cuts CB at K; prove that (i) $SK = e \cdot St$; (ii) $CK^2 = e^2 \cdot tB \cdot tB'$.

24. From a fixed point T, tangents TP, TQ are drawn to any one of a given system of confocal ellipses; prove that $\angle SPS' \pm \angle SQS'$ is constant.

25. tH, tK are the perpendiculars from t to SP, S′P; prove that HKC is a straight line.

26. The tangents at P, Q cut at O; H. K are points on OP, OQ such that OH = OS and OK = OS′; prove that HK equals the transverse axis. (Produce SP to R so that PR = PS′, join OR.)

27. An endless string, forming a loop, passes round a given ellipse and is kept taut by a pencil in the loop. Prove that the pencil traces out a confocal ellipse. (*Graves*' theorem.)

Theorem 18. If SY, S′Y′ are the perpendiculars from the foci to the tangent at any point P, then (i) Y and Y′ lie on the auxiliary circle (*i.e.* the circle on AA′ as diameter); (ii) $SY \cdot S'Y' = CB^2$.

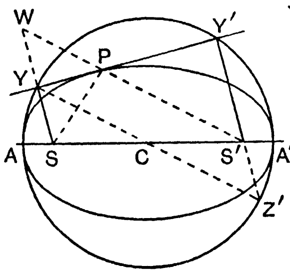

FIG. 36.

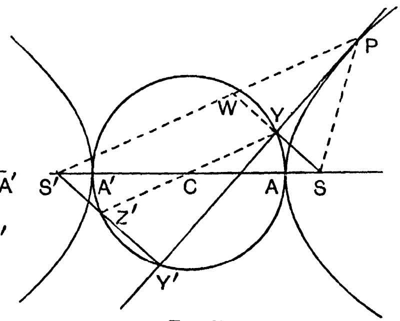

FIG. 37.

Let SY and S′P cut at W; join YC and produce it to cut S′Y′ at Z′.

(i) $\triangle SPY \equiv \triangle WPY$, for $\angle SPY = \angle WPY$, Th. 16 (i),

$\angle SYP = 90° = \angle WYP$, given, and YP is common;

$\therefore\ SP = WP$;

$\therefore$ in Fig. 36, $\quad S'W = S'P + PW = S'P + PS = 2a$,

and in Fig. 37, $\quad S'W = S'P - PW = S'P - PS = 2a$.

But $SC = CS'$ and $SY = YW$;

$\therefore$ CY is parallel to S′W and $CY = \frac{1}{2}S'W = a$;

$\therefore$ Y lies on the circle, centre C, radius CA; and in a similar way we can show that Y′ lies on this circle, which is the auxiliary circle.

(ii) $\triangle YSC \equiv \triangle Z'S'C$, for $\angle SCY = \angle S'CZ'$ and by parallels

$$\angle YSC = \angle Z'S'C, \quad \text{also} \quad SC = CS';$$

$$\therefore \; CY = CZ' \quad \text{and} \quad SY = S'Z'.$$

Since $CZ' = CY = a$, Z' also lies on the auxiliary circle;

$\therefore \; SY \,.\, S'Y' = S'Z' \,.\, S'Y'$

$= S'A \,.\, S'A'$, intersecting chords, $AY'A'Z'$ being on a circle,

$= CB^2$ (see pp. 6, 10).

Corollary 1. If a line is drawn through C parallel to the tangent at P to cut PS, PS′ at E, E′, then

$$PE = CA = PE'.$$

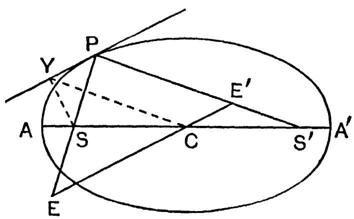

FIG. 38.

For CY is parallel to PS′, Th. 18 (i), and CE′ is drawn parallel to PY;

$\therefore$ YCE′P is a parallelogram; $\therefore \; PE' = CY = CA$.

Similarly, $PE = CY' = CA$.

Corollary 2. If Y is any point on the auxiliary circle, the line through Y perpendicular to SY touches the conic.

Corollary 3. If S′Y′, S′Z′ are the perpendiculars from S′ to two parallel tangents, then $S'Y' \,.\, S'Z' = CB^2$.

In Fig. 36, Z′ lies on the auxiliary circle; therefore the line through Z′ perpendicular to Z′S′ touches the conic; but this line is parallel to Y′PY; also $S'Z' \,.\, S'Y' = SY \,.\, S'Y' = CB^2$.

Note. If the conic is a parabola, its auxiliary circle degenerates into the tangent at the vertex. [Cf. Th. 9 (iii).]

Theorem 19. The locus of the point of intersection of two perpendicular tangents to a conic is a circle concentric with the conic, of radius $\sqrt{(CA^2+CB^2)}$ for the ellipse, and $\sqrt{(CA^2-CB^2)}$ for the hyperbola.

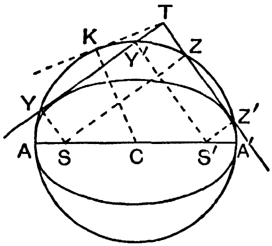

FIG. 39.

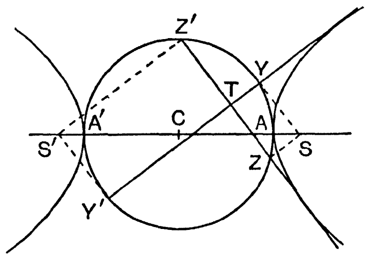

FIG. 40.

Let TY, TZ be two perpendicular tangents; draw the perpendiculars SY, SZ, $S'Y'$, $S'Z'$ from S, S' to TY, TZ.

Then Y, Y', Z, Z' lie on the auxiliary circle.

Since $\angle YTZ = 90°$, $SYTZ$ is a rectangle, so is $S'Y'TZ'$;

$$\therefore \quad TZ = SY \quad \text{and} \quad TZ' = S'Y';$$

$$\therefore \quad TZ \,.\, TZ' = SY \,.\, S'Y' = CB^2 \ \text{[Th. 18 (ii)]}.$$

For the ellipse, from T, draw the tangent TK to the auxiliary circle;

$$\therefore \quad CT^2 = CK^2 + KT^2 = CA^2 + TZ \,.\, TZ' = CA^2 + CB^2;$$

$\therefore$ T lies on a circle, centre C, radius $\sqrt{(CA^2+CB^2)}$.

For the hyperbola, T lies between Y and Y' and is therefore inside the auxiliary circle;

$$\therefore \quad ZT \,.\, TZ' = (\text{radius})^2 - CT^2;$$

$$\therefore \quad CT^2 = CA^2 - TZ \,.\, TZ' = CA^2 - CB^2;$$

$\therefore$ T lies on a circle, centre C, radius $\sqrt{(CA^2-CB^2)}$.

Definition. The locus of a point from which the tangents to a conic are at right angles is called the **director circle**.

By Theorem 10, the director circle of a parabola degenerates into the directrix.

EXERCISE IV. b.

(For the notation, see p. xi.)

1. CD is the semi-diameter parallel to the tangent at L; S′L cuts CD at K; prove that $S'K = ae^2$.

2. If SK is the perpendicular from S to PG; prove that YK passes through C.

3. The line through C parallel to the tangent at P cuts SP, S′P at K, K′; prove that $SK = S'K'$.

4. The tangent at P cuts the auxiliary circle at Y, Z; prove that (i) PN bisects $\angle YNZ$; (ii) the circle YNZ passes through C.

5. The line through C perpendicular to the tangent at P cuts SP at K; prove that $KY = CS$.

6. If the tangent at L cuts CB at t, prove that $\angle S'tL = 90°$.

7. Prove that the circle on SP as diameter touches the auxiliary circle.

8. Prove that S′Y and SY′ bisect PG.

9. The perpendicular from C to the tangent at P cuts SP, S′P at K, K′; prove that $SK = S'K' = a$.

10. Prove that the length of the tangent from any point on the director circle to the auxiliary circle of an ellipse is equal to the semi-minor axis.

11. If an ellipse of fixed size slides between two fixed perpendicular lines, prove that the locus of its centre is a circle.

12. RK is the tangent from a point R on the directrix to the director circle; prove that $RK = RS$.

13. TP, TP′ are the tangents to an ellipse from a point T on the auxiliary circle; prove that SP, CT, S′P′ are parallel.

14. Prove that $\dfrac{1}{SY^2} = \dfrac{2}{SL \cdot SP} - \dfrac{1}{b^2}$.

15. An ellipse is inscribed in a triangle with one focus at the orthocentre; prove that its centre is at the nine-point centre and that its major axis equals the circumradius.

16. The tangent at A to an ellipse cuts the director circle at P, P′; a confocal ellipse is drawn through P, P′; prove that the tangents at P, P′ meet on the director circle of the confocal.

17. HKPQ is a rectangle circumscribing an ellipse; HK meets the directrix in R; prove that $\angle RSH = \angle RKS$.

18. Prove that the focus is a limiting point of the coaxal system having the directrix as radical axis and the director circle as a circle of the system.

19. Prove that $SY = e \cdot YX$.

20. Prove that XY and $X'Y'$ intersect on CB.

21. H, K are points on the tangent at P such that $SH = SK = 2a$; prove that the circumradius of $\triangle HKS'$ equals $2a$.

22. P is a point on a hyperbola; prove that the tangent from P to the auxiliary circle is equal to the semi-minor axis of the confocal ellipse which passes through P.

23. The common tangent of an ellipse and the circle, centre S, radius SC, touches the ellipse at P and the circle at Y; prove that (i) $\sec PSY = 2e$; (ii) $S'P = 2SL$.

24. If a tangent to one of two given confocal conics is perpendicular to a tangent to the other, prove that the locus of the point of intersection of the tangents is a circle.

25. CH, CK are the perpendiculars from the centre C to two parallel tangents of two given confocal conics, prove that $CH^2 - CK^2$ is constant.

26. If in Fig. 36, the other tangents YQ, $Y'Q'$ are drawn to the ellipse from Y, Y', prove that SQ and $S'Q'$ intersect on the conic.

CHAPTER V

DIAMETERS

Diameters. The fundamental properties of diameters are best obtained by projective methods. We shall therefore assume that the following theorems have been obtained in this way. Independent proofs will, however, be found in the Appendix.

Theorem 20. The locus of the mid-points of a system of parallel chords of a central conic is a straight line passing through the centre of the conic and *this line is called a diameter*. Further, the line joining the focus to the point of intersection of the diameter and the directrix is perpendicular to the chords bisected by that diameter.

Theorem 21. If PCP′, DCD′ are two diameters of a central conic such that PCP′ bisects all chords parallel to DCD′, then DCD′ bisects

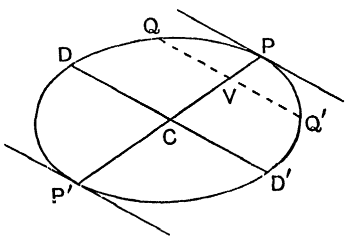

FIG. 41.

all chords parallel to PCP′, and PCP′, DCD′ are called **conjugate diameters.** Further, the tangents at P, P′ are parallel to DCD′ and the tangents at D, D′ are parallel to PCP′.

Theorem 22. If the tangents at any two points QQ′ on a conic meet at T, then (i) CT bisects QQ′; (ii) if CT cuts the conic at P, P′ and QQ′ at V, CV . CT = CP^2 or {P′VPT} is harmonic.

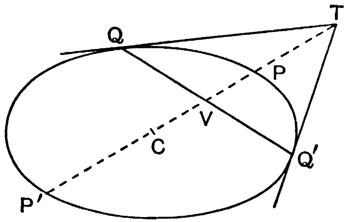

FIG. 42.

Definitions. (i) If QQ′ is any chord which is bisected at V by the diameter PCP′, then QV is called the **ordinate from R to the diameter PP′.** (See Fig. 42.)

(ii) If PCP′ is a diameter and if Q is any point on the curve, QP, QP′ are called **supplemental chords.** (See Fig. 43.)

Theorem 23. Supplemental chords of a conic are parallel to conjugate diameters.

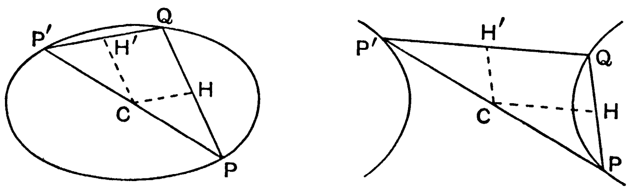

FIG. 43.

Let PCP′ be a diameter and let Q be any point on the conic.

Bisect QP, QP′ at H, H′; join CH, CH′.

Since PC = CP′ and PH = HQ, ∴ CH is parallel to P′Q.

Similarly, CH′ is parallel to PQ;

∴ CH bisects one chord parallel to CH′ and ∴ bisects all chords parallel to CH′;

∴ CH, CH′ are conjugate diameters;

∴ PQ, P′Q are parallel to conjugate diameters.

EXERCISE V. a.

(For the notation, see p. xi.)

1. TP, TQ are tangents to a conic, prove that $\triangle CPT = \triangle CQT$.

2. If the tangent at P cuts the tangents at A, A′ in K, K′, prove that $AK \cdot A'K' = b^2$.

3. The diameters CR, CR′ which bisect the chords PQ, PQ′ cut the directrix at R, R′; prove that $\angle$s RSR′, QPQ′ are equal or supplementary.

4. PCP′ is a diameter; a line through P cuts the conic at Q and the tangent at P′ in K; prove that the tangent at Q bisects P′K.

5. PP′ is a diameter; TP, T′P′ are tangents; prove that $\angle$s SPT, SP′T′ are equal or supplementary.

6. P, Q are points on an ellipse; the two circles are drawn which pass through both P and Q and touch the ellipse at other points, say H and K; prove that HK is a diameter.

7. PP′ is a chord perpendicular to CB; the tangent at P′ meets the tangent at A in T; PN is the ordinate to CA. If AT = AC, prove that PN = AN.

8. P, Q are points on a conic such that CP bisects the normal at Q; prove that CQ bisects the normal at P.

9. PQ, QP′ are supplemental chords of an ellipse and make equal angles with the tangent at Q. Prove that the tangents at P, Q intersect on the director circle and that $PQ + QP' = 2\sqrt{(a^2+b^2)}$.

10. HK is a chord of an ellipse such that the tangents at H, K cut at right angles; CD is the semi-diameter parallel to HK; prove that $\dfrac{HK}{CD^2}$ is constant.

11. A chord PQ of the director circle of an ellipse touches the ellipse; prove that CP, CQ are conjugate diameters of the ellipse.

12. PQ is a chord of an ellipse, normal at P; HCH′ is the diameter bisecting PQ; prove that PH + PH′ equals the diameter of the director circle.

13. From the mid-point V of a chord PP′ of an ellipse, VK is drawn perpendicular to PP′ and cuts the major axis at K; prove that

$$SK = \tfrac{1}{2}e(SP + SP').$$

In the case of metrical properties of diameters, it is best to use different treatments for the ellipse and hyperbola.

The Ellipse. The fundamental metrical properties of diameters of an ellipse are obtained most easily by **orthogonal projection** (see p. 7). Alternative proofs will be found in the Appendix. The results may be summarised as follows:

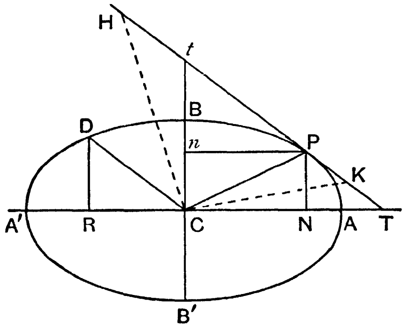

FIG. 44.

Theorem 24. If CP, CD are conjugate semi-diameters of an ellipse, and if the tangent at P meets the axes at T, t, and if PN, DR are the ordinates to CA, and if Pn is the ordinate to CB, then

(i) $\text{CN} \cdot \text{CT} = \text{CA}^2$ and $\text{C}n \cdot \text{C}t = \text{CB}^2$.

(ii) If PN, DR cut the auxiliary circle at p, d, then $\angle p\text{C}d = 90°$.

(iii) $\triangle$s PNC, DRC are equal in area or $\text{PN} \cdot \text{NC} = \text{DR} \cdot \text{RC}$.

(iv) $\dfrac{\text{PN} \cdot \text{DR}}{\text{CN} \cdot \text{CR}} = \dfrac{\text{CB}^2}{\text{CA}^2}$.

(v) $\text{CN}^2 + \text{CR}^2 = \text{CA}^2$ and $\text{PN}^2 + \text{DR}^2 = \text{CB}^2$ and hence
$$\text{CP}^2 + \text{CD}^2 = \text{CA}^2 + \text{CB}^2.$$

(vi) The area of the parallelogram having CP, CD as sides is constant and equal to CA . CB.

(vii) If any other pair of conjugate diameters cut PT at H, K, then
$$\text{PH} \cdot \text{PK} = \text{CD}^2.$$

(viii) If QV is the ordinate from any point Q to PP′ (see Fig. 41), then $\dfrac{\text{QV}^2}{\text{CP}^2 - \text{CV}^2} = \dfrac{\text{CD}^2}{\text{CP}^2}$. [This is a special case of Th. 7.]

Note. Th. 24 (i) is a special case of Th. 22 (ii).

Th. 24 (ii) supplies the basis for proving properties of conjugate diameters by orthogonal projections.

EXERCISE V. b.

(For the notation, see p. xi.)

1. Prove that $CG \,.\, CT = CS^2$.

2. Prove that $NG \,.\, CT = CB^2$.

3. Prove that the circle SLS' touches the auxiliary circle.

4. The ordinate at P cuts the auxiliary circle at p; prove that the perpendicular from S to the tangent at p to the circle equals SP.

5. If the auxiliary circle cuts CB at K, prove that KL touches the ellipse.

6. If CP, CD are conjugate semi-diameters, prove that

$$PD^2 = (SP - SD)^2 + 2CB^2.$$

7. P, Q are points on an ellipse such that the tangent at P meets the normal at Q on CB; prove that the tangent at Q meets the normal at P on CB.

8. The tangent at P makes 45° with CA; CP cuts the tangent at A in K; prove that $AK = SL$.

9. CP, CD are conjugate semi-diameters of an ellipse; H, K are points on the normal at P such that $HP = CD = PK$; prove that $CH = a + b$ and $CK = a - b$.

10. AP, BQ are parallel chords of an ellipse; prove that CP, CQ are conjugate semi-diameters.

11. CP, CQ are perpendicular semi-diameters of an ellipse; prove that $\dfrac{1}{CP^2} + \dfrac{1}{CQ^2}$ is constant.

12. EFHK is a rectangle circumscribing an ellipse; EF touches the ellipse at P; CP, CD are conjugate semi-diameters; prove that

$$EP \,.\, PF = CD^2.$$

13. PSP′, QSQ′ are perpendicular focal chords of an ellipse; prove that $\dfrac{1}{PP'} + \dfrac{1}{QQ'}$ is constant.

14. Two variable conjugate semi-diameters cut the tangent at a fixed point P on an ellipse in H, K; CP cuts the circle CHK at Q; prove that $CP \,.\, CQ$ is constant.

15. PCP', DCD' are conjugate semi-diameters of an ellipse; prove that $\angle PBP' + \angle DAD' = 180°$.

16. CP, CD are conjugate semi diameters of an ellipse; a tangent to the ellipse and a straight line through D parallel to this tangent cut CP in T, K; prove that $CT^2 - CK^2 = CP^2$.

17. pqr is an equilateral triangle inscribed in the auxiliary circle of an ellipse; P, Q, R are the corresponding points on the ellipse. Prove that the normals at P, Q, R are concurrent.

Theorem 25. If CP, CD are conjugate semi-diameters of an ellipse, then $SP \,.\, S'P = CD^2$.

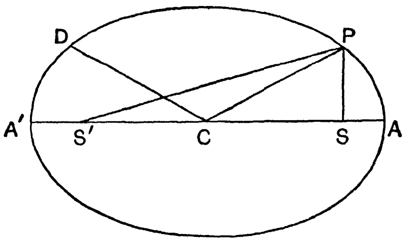

FIG. 45.

$$SP + S'P = 2a\,;$$

$$\therefore\ SP^2 + S'P^2 + 2SP \,.\, S'P = 4a^2.$$

But from $\triangle SPS'$, since $CS = CS'$

$$SP^2 + S'P^2 = 2CP^2 + 2CS^2 = 2CP^2 + 2a^2 - 2b^2\,;$$

$$\therefore\ 2SP \,.\, S'P = 4a^2 - (2CP^2 + 2a^2 - 2b^2)\,;$$

$$\therefore\ SP \,.\, S'P = a^2 + b^2 - CP^2 = CD^2 + CP^2 - CP^2, \text{ Th. 24 (v)}\,;$$

$$\therefore\ SP \,.\, S'P = CD^2.$$

Corollary. If SY is the perpendicular from S to the tangent at P, then $\dfrac{SY}{SP} = \dfrac{CB}{CD}$.

For, with the notation of Fig. 36, p. 44,

$$\frac{SY}{SP} = \frac{S'Y'}{S'P} = \sqrt{\left(\frac{SY \,.\, SY'}{SP \,.\, S'P}\right)} = \sqrt{\left(\frac{CB^2}{CD^2}\right)} = \frac{CB}{CD}.$$

Theorem 26. If CP, CD are conjugate semi-diameters of an ellipse, and if the normal at P cuts CA, CB, CD at G, g, F, then

(i) $PF \,.\, PG = CB^2$; (ii) $PF \,.\, Pg = CA^2$;

(iii) $\dfrac{PG}{CD} = \dfrac{CB}{CA}$ and $\dfrac{Pg}{CD} = \dfrac{CA}{CB}$; (iv) $PG \,.\, Pg = CD^2$.

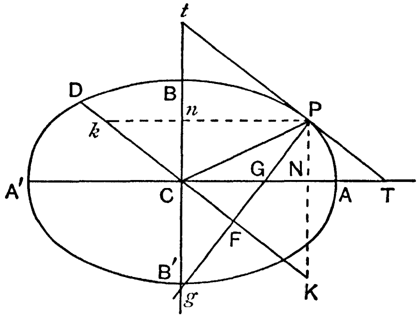

FIG. 46.

Let the tangent at P cut CA, CB at T, t; let the ordinates PN, Pn from P to CA, CB cut CD at K, k.

(i) PKCt is a parallelogram;

$$\therefore \; PK = Ct; \text{ also } PN = Cn.$$

The normal PG at P is perpendicular to CD since CD is parallel to PT; $\therefore \; \angle GFK = 90°$.

Also, $\angle GNK = 90°$; $\therefore$ GNKF is a cyclic quadrilateral;

$$\therefore \; PG \,.\, PF = PN \,.\, PK = Cn \,.\, Ct = CB^2. \quad \text{Th. 24 (i).}$$

(ii) Similarly, $\angle kng = 90° = \angle kFg$, $\therefore$ $knFg$ is a cyclic quadrilateral;

$$\therefore \; PF \,.\, Pg = Pn \,.\, Pk = CN \,.\, CT = CA^2. \quad \text{Th. 24 (i).}$$

(iii) $PF \,.\, CD$ = area of parallelogram whose sides are CP, CD

$$= CA \,.\, CB. \quad \text{Th. 24 (vi)};$$

$$\therefore \text{ from (i) } \frac{PG}{CD} = \frac{PG \,.\, PF}{CD \,.\, PF} = \frac{CB^2}{CA \,.\, CB} = \frac{CB}{CA}.$$

(iv) Similarly,

$$\frac{Pg}{CD} = \frac{CA}{CB}, \quad \therefore \; \frac{PG \,.\, Pg}{CD^2} = \frac{CB}{CA} \cdot \frac{CA}{CB} = 1; \quad \therefore \; PG \,.\, Pg = CD^2.$$

EXERCISE V. c.

(Throughout this exercise, CP and CD represent conjugate semi-diameters. For other notation, see p. xi.)

1. Prove that $Pg = \frac{a}{b} \cdot CD$.

2. Prove that $\frac{PG}{Pg} = \frac{b^2}{a^2}$.

3. Prove that $Gg = e^2 \cdot Pg$.

4. Prove that $(SP - CA)^2 + (SD - CA)^2 = CS^2$.

5. Prove that $Sg = e \cdot Pg$.

6. PG, DH are the normals at P, D, terminated by AA′; prove that $PG^2 + DH^2 = CB^2 + SL^2$.

7. Prove that the limiting value of Gg when P coincides with A is ae^2.

8. If SP cuts CD at K, prove that $PK = a$.

9. If SP cuts the circle SG g at K, prove that $PK = S'P$.

10. Any tangent to the circle on BB′ as diameter cuts the ellipse at P, P′ and the director circle at Q, Q′; prove that PQ, PQ′ are equal to SP, S′P.

11. HSK is a focal chord of the auxiliary circle of an ellipse; DCD′ is the diameter of the ellipse perpendicular to HK; prove that

$$HK \cdot DD' = 4ab.$$

12. Prove that $\frac{1}{SY^2} + \frac{1}{S'Y'^2} - \frac{4}{PG^2}$ is constant.

13. If $SP = PG$, prove that $\frac{SP}{S'P} = \frac{b^2}{a^2}$.

14. Prove that the tangents from D to the circle on BB′ as diameter are parallel to SP, S′P.

15. If SP cuts CD at E, prove that $CP^2 - SE^2 = b^2$.

16. DK is the perpendicular from D to a line through C parallel to SP; prove that $DK = b$.

CHAPTER VI

THE HYPERBOLA

Definition. If PT is the tangent at a point P on a hyperbola, the limiting position of PT when P tends to infinity along the curve is called an **asymptote** of the hyperbola.

A hyperbola has two asymptotes, which may be constructed as follows:

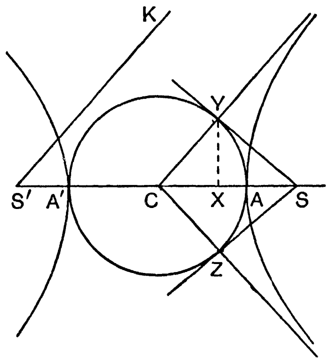

FIG. 47.

From S, draw the tangents SY, SZ to the auxiliary circle.

Then CY, CZ are the asymptotes.

By Th. 18, p. 44, if Y is a point on the auxiliary circle, the line through Y perpendicular to SY touches the conic.

Since SY is a tangent to the circle, $\angle CYS = 90°$; therefore CY is perpendicular to SY and so touches the hyperbola.

Again, by Th. 18, the point of contact P of the tangent lies on the intersection of the tangent with a line S′K through S′ parallel to CY. Since CY is the tangent, the point of contact P is therefore at infinity.

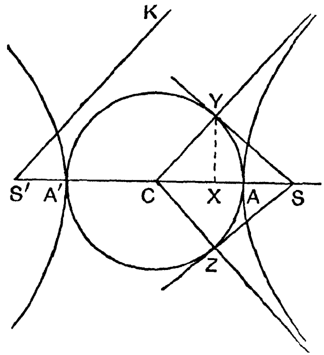

FIG. 47.

Consequently CY is an asymptote, and similarly it may be proved that CZ is also an asymptote.

If the angle between the asymptotes is 2α, we see that

$$\sec\alpha = \frac{\mathrm{CS}}{\mathrm{CY}} = \frac{ae}{a} = e.$$

Further, if YX is the perpendicular from Y to CS,

$$\mathrm{CX} = \mathrm{CY}\cos\alpha = \frac{a}{e};$$

therefore X is the foot of the directrix. Hence *the asymptotes cut the auxiliary circle at its points of intersection with the directrix.*

From the general properties of tangents, we may deduce special properties of the asymptotes by regarding the point of contact as the point at infinity on the curve. This is illustrated in the method of proof of the following theorem.

Theorem 27. If the tangent at any point P of a hyperbola cuts the asymptotes at T, T′, then TP = PT′.

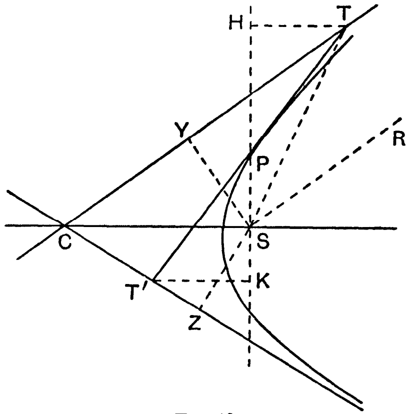

FIG. 48.

SY, SZ are the perpendiculars from S to the asymptotes; TH, T′K are the perpendiculars from T, T′ to SP; SR is drawn parallel to CY.

Since SR may be regarded as passing through the point of contact (at infinity) of CY with the curve, and since tangents subtend equal angles at the focus,

$$\angle PST = \angle TSR$$
$$= \angle CTS \text{ since SR is parallel to CT;}$$
$$\therefore \triangle HST \equiv \triangle YTS,$$

for $\angle THS = 90° = \angle SYT$ and $\angle HST = \angle YTS$, just proved, and ST is common;

$$\therefore TH = SY.$$

Similarly, T′K = SZ; but SY = SZ; $\therefore$ TH = T′K;

$$\therefore \triangle HPT \equiv \triangle KPT',$$

for $\angle HPT = \angle KPT'$ and $\angle THP = 90° = \angle T'KP$ and TH = T′K;

$$\therefore TP = PT'.$$

Note. An alternative method of proof is given in the next theorem.

Theorem 28. If the tangent at any given point P of a hyperbola cuts the asymptotes at T, T′, and if a variable chord QQ′, parallel to PT, cuts the asymptotes at R, R′, then (i) $RQ \,.\, RQ' = TP^2$, (ii) $TP = PT'$, (iii) $RQ = Q'R'$, (iv) $RQ \,.\, QR' = TP^2$.

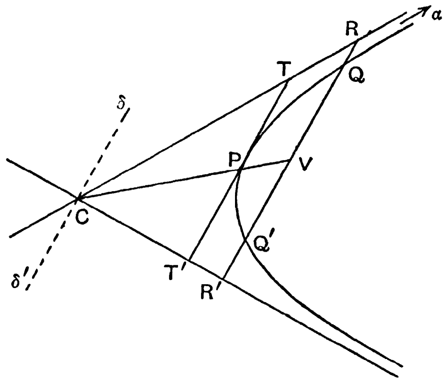

FIG. 49.

Let α be the point at infinity at which the asymptote CT touches the hyperbola.

Draw the diameter $\delta C \delta'$ parallel to PT and cutting the curve at δ, δ'. (See p. 10.)

(i) By Th. 7, $\dfrac{RQ \,.\, RQ'}{TP^2} = \dfrac{R\alpha^2}{T\alpha^2} = 1$, since α is a point at infinity;

$$\therefore\ RQ \,.\, RQ' = TP^2.$$

(ii) Similarly, $\dfrac{RQ \,.\, RQ'}{C\delta \,.\, C\delta'} = \dfrac{R\alpha^2}{C\alpha^2} = 1$, but $C\delta' = -C\delta$;

$$\therefore\ TP^2 = RQ \,.\, RQ' = C\delta \,.\, C\delta' = -C\delta^2.$$

In the same way, since C lies on the asymptote CT′R′,

$$T'P^2 = R'Q \,.\, R'Q' = C\delta \,.\, C\delta' = -C\delta^2;$$

$$\therefore\ TP^2 = T'P^2 \quad \text{or} \quad TP = PT'.$$

(iii) Since QQ′ is parallel to PT, the diameter CP, when produced, cuts QQ′ at its mid-point V.

Since $TP = PT'$, by parallels $RV = VR'$;

But $QV = VQ'$; $\therefore$ $RQ = Q'R'$.

(iv) By (iii), $RQ' = RQ + QQ' = Q'R' + QQ' = QR'$;

$$\therefore \text{ by (i), } RQ \,.\, QR' = RQ \,.\, RQ' = TP^2.$$

Corollary. If the tangent at A cuts an asymptote at E and if a chord QQ′ perpendicular to CA cuts the asymptote at R, then

(i) $RQ \,.\, RQ' = EA^2 = b^2$;

(ii) $CE = ae$.

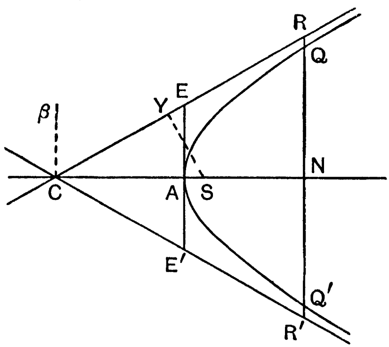

FIG. 50.

For if the conjugate axis cuts the curve at the imaginary point β,

$$C\beta^2 = -b^2; \text{ (see p. 9)}$$

$$\therefore RQ \,.\, RQ' = EA^2 = -C\beta^2 = b^2.$$

And $CE^2 = CA^2 + AE^2 = a^2 + b^2 = a^2 + a^2(e^2 - 1) = a^2e^2$.

Or we may say, if SY is the perpendicular from S to CE, since Y lies on the auxiliary circle, $CY = CA$,

$$\therefore \triangle CAE \equiv \triangle CYS; \therefore CE = CS = ae;$$

$$\therefore AE^2 = CE^2 - CA^2 = a^2e^2 - a^2 = b^2.$$

Theorem 29. (i) If lines are drawn from any point P on a hyperbola parallel to the asymptotes, they form with the asymptotes a parallelogram of constant area equal to $\frac{1}{2}ab$.

(ii) If the tangent at P cuts the asymptotes at T, T′, the triangle CTT′ is of constant area equal to ab.

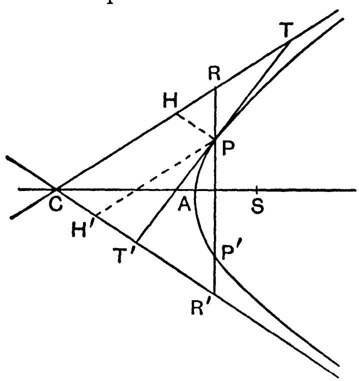

FIG. 51.

Let PH, PH′ be the lines parallel to the asymptotes.

Let the perpendicular from P to the transverse axis CA cut the asymptotes at R, R′.

(i) Let the angle between the asymptotes be 2α, so that

$$\angle \mathsf{CRP} = 90^\circ - \alpha.$$

$$\frac{\mathsf{PH}}{\mathsf{PR}} = \frac{\sin \mathsf{HRP}}{\sin \mathsf{PHR}} = \frac{\sin(90^\circ - \alpha)}{\sin 2\alpha} = \frac{\cos\alpha}{\sin 2\alpha} = \frac{\cos\alpha}{2\sin\alpha\cos\alpha} = \tfrac{1}{2}\operatorname{cosec}\alpha\,;$$

$$\therefore\ \mathsf{PH} = \tfrac{1}{2}\operatorname{cosec}\alpha \,.\, \mathsf{PR}.$$

Similarly, $$\mathsf{PH}' = \tfrac{1}{2}\operatorname{cosec}\alpha \,.\, \mathsf{PR}'\,;$$

$$\therefore\ \mathsf{PH} \,.\, \mathsf{PH}' = \tfrac{1}{4}\operatorname{cosec}^2\alpha \,.\, \mathsf{PR} \,.\, \mathsf{PR}'.$$

But by Th. 28 (i) and Coroll., $\mathsf{PR} \,.\, \mathsf{PR}' = b^2$;

$$\therefore\ \mathsf{PH} \,.\, \mathsf{PH}' = \tfrac{1}{4}\operatorname{cosec}^2\alpha \,.\, b^2 = \text{constant}\,;$$

$\therefore$ area of parallelogram $\mathsf{CH'PH} = \tfrac{1}{4}\operatorname{cosec}^2\alpha \,.\, b^2 \,.\, \sin 2\alpha$

$$= \tfrac{1}{4}b^2\operatorname{cosec}^2\alpha \,.\, 2\sin\alpha\cos\alpha = \tfrac{1}{2}b^2\cot\alpha.$$

But $b = a\tan\alpha$, $\therefore$ area of $\mathsf{CH'PH} = \tfrac{1}{2}ab$.

(ii) Since $TP = PT'$, by parallels $TH = HC$ and $T'H' = H'C$;

$\therefore\ \triangle CTT' = 4\triangle CHH' = 2$ parallelogram $CH'PH = ab$.

Corollary. $CT \,.\, CT' = CS^2$ and $CH \,.\, CH' = \frac{1}{4}CS^2$.

Since $CT \,.\, CT'$ is constant, $CT \,.\, CT' = CE \,.\, CE'$ (see Fig. 50),

$$= CE^2 = CS^2,$$

and $$CH \,.\, CH' = \tfrac{1}{4}CT \,.\, CT' = \tfrac{1}{4}CS^2.$$

EXERCISE VI. a.

(Throughout this exercise T and T′ are the intersections of the tangents at P with the asymptotes. For other notation, see p. xi.)

1. If SL cuts an asymptote at K, prove that $SK = e \,.\, b$.

2. If two hyperbolas have the same asymptotes, and if a tangent to one at P cuts the other at Q, Q′, prove that $PQ = PQ'$.

3. A line from P parallel to an asymptote cuts the directrix at K; prove that $SP = PK$.

4. If the directrix cuts an asymptote at Z and if ZP is the tangent from Z, prove that SP is parallel to CZ.

5. Prove that the distance of T from SP is constant.

6. If the normal at P is parallel to one asymptote, prove that the mid-point of Gg lies on the other asymptote.

7. If SP cuts CT at R, prove that $SR = RT$.

8. The line through T parallel to CT′ cuts the curve at Q; prove that $TQ = \frac{1}{4}CT'$.

9. P, Q are points on a hyperbola whose asymptotes are CH, CK; if PH is parallel to CK and if QK is parallel to CH, prove that PQ is parallel to HK.

10. Prove that $\angle TST'$ is equal or supplementary to $\frac{1}{2}\angle TCT'$.

11. SP, S′P cut an asymptote in R, R′; prove that the perimeter of $\triangle PRR'$ is constant.

12. Prove that the points T, T′, S, S′ lie on a circle.

13. The ordinate at a point P of a hyperbola to the axis cuts an asymptote at R; the line through R perpendicular to CR cuts the axis at K; prove that PK is the normal at P.

14. Prove that ST touches the circle SCT'.

15. Prove that $\angle PST = \angle CST'$.

16. If PS' cuts CT at K, prove that $\angle S'KC = 2\angle STP$.

17. If the line through P parallel to CT cuts the directrix at K, prove that $\angle KST = 90°$.

18. Prove that $ST \,.\, S'T = CT \,.\, TT'$.

19. The line through P parallel to one asymptote cuts the other in R; RQ is the tangent from R to the curve; PQ produced cuts the asymptotes in H, K; prove that $HK = 3PQ$.

20. The line through S parallel to one asymptote cuts the curve at Q and the other asymptote at R. Prove that (i) $4SQ$ equals the latus rectum, (ii) $4QR$ equals the transverse axis.

21. If HK is a fixed chord of a hyperbola, and if P is a variable point on the curve, prove that PH, PK intercept a constant length on an asymptote.

Conjugate Diameters of a Hyperbola. The proof of Theorem 28 shows that $C\delta^2 = -TP^2$ or that $C\delta = TP \,.\, \sqrt{-1}$, where $C\delta$ is the semi-diameter conjugate to CP. Therefore the diameter conjugate to CP cuts the curve at imaginary points and its length is imaginary.

If a line CD is drawn equal and parallel to PT, so that $CD = C\delta \,.\, \sqrt{-1}$ and $CD\delta$ is a straight line, it is customary to speak of the length of CD as the length of the semi-diameter conjugate to CP. This is simply a convention, and it makes the CD of an ellipse equivalent to the "$CD \,.\, \sqrt{-1}$" of a hyperbola, or the "CD^2" of an ellipse equivalent to the "$-CD^2$" of a hyperbola, just as we have already taken (see p. 11) the "CB^2" of an ellipse equivalent to the "$-CB^2$" of a hyperbola.

Consequently certain general theorems of the conic are enunciated for the ellipse in a form different from that for the hyperbola, because the letters have different meanings in the two cases.

Thus for the ellipse [Th. 24 (v)], $CP^2 + CD^2 = CA^2 + CB^2$.

But, for the hyperbola [Th. 30 (ii)], $CP^2 - CD^2 = CA^2 - CB^2$.

These two enunciations represent the same theorem, the difference of sign is due to the difference of meaning of the letters in the two cases.

Theorem 30. If CP, CD are conjugate diameters of a hyperbola, then (i) PD is parallel to one asymptote and is bisected by the other, (ii) $CP^2 - CD^2 = a^2 - b^2$, (iii) the area of the parallelogram having CP, CD as sides is constant and equal to ab.

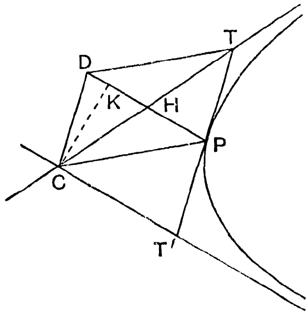

FIG. 52.

Let the tangent at P meet the asymptotes at T, T′; join TD; let PD cut CT at H.

(i) CD is equal and parallel to PT;

$\therefore$ CPTD is a parallelogram;

$\therefore$ CT and PD bisect each other.

Also since TH = HC and TP = PT′, $\therefore$ PH is parallel to CT′.

(ii) Draw CK perpendicular to PD.

$$CP^2 - CD^2 = (CK^2 + KP^2) - (CK^2 + KD^2) = PK^2 - KD^2$$
$$= (PK + KD)(PK - KD)$$
$$= 2PH \cdot 2HK = 4PH \cdot HK.$$

Let the asymptotes cut at angle 2α.

Then $\angle KHC = 2\alpha$, $\therefore$ $KH = HC \cos 2\alpha$;

$\therefore$ $CP^2 - CD^2 = 4PH \cdot HC \cdot \cos 2\alpha =$ constant. [Th. 29 (i)].

But when P is at A, $CP = a$ and $CD = b$;

$$\therefore\ CP^2 - CD^2 = a^2 - b^2.$$

(iii) The area of parallelogram $DCPT = 2\triangle CPT = \triangle T'CT$
$= ab$. [Th. 29 (ii)].

Theorem 31. If QV is the ordinate from any point Q on a hyperbola to the diameter PCP′, and if CD is the semi-diameter conjugate to CP, then

$$\frac{QV^2}{CV^2 - CP^2} = \frac{CD^2}{CP^2}.$$

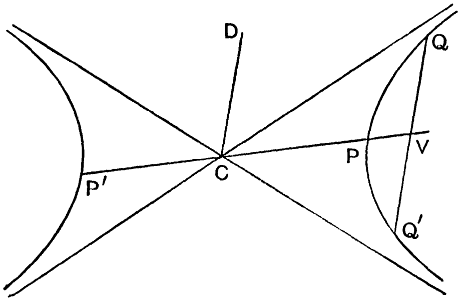

FIG. 53.

By Th. 7, if the conjugate diameter CD meets the hyperbola at δ, δ', then

$$\frac{VQ \cdot VQ'}{VP \cdot VP'} = \frac{C\delta \cdot C\delta'}{CP \cdot CP'}; \quad \therefore \frac{-QV^2}{VP \cdot VP'} = \frac{-C\delta^2}{-CP^2},$$

$$\text{but } C\delta^2 = -CD^2; \qquad \therefore \frac{QV^2}{VP \cdot VP'} = \frac{CD^2}{CP^2}.$$

Since $PC = CP'$, $VP \cdot VP' = CV^2 - CP^2$;

$$\therefore \frac{QV^2}{CV^2 - CP^2} = \frac{CD^2}{CP^2}.$$

Theorem 32. If CP, CD are conjugate semi-diameters of a hyperbola, then $SP \cdot S'P = CD^2$.

[The proof is similar to that of Th. 25.]

$$SP \sim S'P = 2a, \quad \therefore SP^2 + S'P^2 - 2SP \cdot S'P = 4a^2.$$

But from $\triangle SPS'$, $SP^2 + S'P^2 = 2CP^2 + 2CS^2$

$$= 2CP^2 + 2a^2 + 2b^2;$$

$$\therefore 2SP \cdot S'P = 2CP^2 + 2a^2 + 2b^2 - 4a^2;$$

$$\therefore SP \cdot S'P = CP^2 - (a^2 - b^2) = CP^2 - (CP^2 - CD^2)$$
$$= CD^2.$$

Theorem 33. If CP, CD are conjugate semi-diameters of a hyperbola and if the normal at P cuts CA, CB, CD at G, g, F, then

(i) $FP \cdot PG = CB^2$; (ii) $PF \cdot Pg = CA^2$;

(iii) $\dfrac{PG}{CD} = \dfrac{CB}{CA}$ and $\dfrac{Pg}{CD} = \dfrac{CA}{CB}$; (iv) $PG \cdot gP = CD^2$.

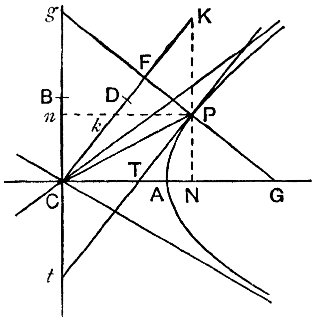

FIG. 54

The proof of Th. 26 for the ellipse applies unchanged to this theorem. A figure is given to assist the reader.

EXERCISE VI. b.

(For the notation, see p. xi.)

1. Prove that $CG = e^2 \cdot CN$.

2. Prove that $GN = \dfrac{b^2}{a^2} \cdot CN$.

3. Prove that $NG \cdot CT = b^2$.

4. Prove that $Cg \cdot Ct = a^2e^2$.

5. The tangent and normal at P cut an asymptote and the transverse axis at T, G respectively. Prove that $\angle PTG$ is constant.

6. Prove that the projection of Pg on SP equals a.

7. If PCP′, DCD′ are variable conjugate diameters and if Q is any point on the curve, prove that $QP^2 + QP'^2 - QD^2 - QD'^2$ is constant.

8. P is a point on a hyperbola such that $CP = CS$; the tangent at P cuts CG at T; prove that $\angle CPT = \angle PGC$.

9. A tangent to a hyperbola cuts two conjugate diameters in H, K; prove that the other tangents from H, K are parallel.

10. The tangent at P cuts an asymptote at T and the transverse axis in K; prove that $\angle$ GKT = $\angle$ GTC.

11. If the tangent at P cuts the asymptotes at T, T′, prove that $\angle$TGT′ is constant.

12. If the tangent and normal at P cut CB at t, g, and if CP, CD are conjugate semi-diameters, prove that $\dfrac{\mathrm{S}t}{gt} = \dfrac{\mathrm{CB}}{\mathrm{CD}}$.

13. If the tangent at P cuts the asymptotes at T, T′, prove that the circle CTT′ passes through G and g.

14. If QSQ′ is the focal chord perpendicular to PG, prove that QS . SQ′ = PG^2.

15. A circle is drawn to touch CA at C and to touch the hyperbola at P, say; if CP, CD are conjugate semi-diameters, prove that CD = CS.

The Rectangular Hyperbola.

If the asymptotes of a conic are at right angles, the conic is called a **rectangular hyperbola.**

Metrical Relations. With the previous notation, $2\alpha = 90°$;

$$\therefore \quad e = \sec \alpha = \sec 45° = \sqrt{2},$$

$$b = a \tan \alpha = a \tan 45° = a.$$

Also, $\mathrm{CS}^2 = 2a^2$; $\mathrm{CX}^2 = \frac{1}{2}a^2$; $\mathrm{SL} = \dfrac{b^2}{a} = a.$

Theorem 34. If CT, CT′ are the asymptotes of a rectangular hyperbola, and if CP, CD are conjugate semi-diameters, then (i) CT and CT′ are the bisectors of $\angle$ PCD, (ii) CP = CD, (iii) if CD and CQ are perpendicular semi-diameters, then CD = CQ.

Let the tangent at P cut the asymptotes at T, T′.

(i) Since TP = PT′ and $\angle$ TCT′ = 90°;

$\therefore$ TP = PC and $\angle$ PCT = $\angle$ PTC.

But PT is equal and parallel to CD;

$\therefore$ $\angle$ DCT = $\angle$ CTP by parallels = $\angle$ PCT;

$\therefore$ CT is one bisector of $\angle$ PCD, and CT′, being perpendicular to CT, is the other bisector.

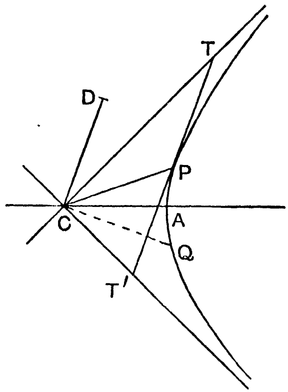

FIG. 55.

(ii) CD $=$ PT $=$ CP.

(iii) $\angle$ DCQ $= 90° = \angle$ TCT′, $\therefore$ $\angle$ DCT $= \angle$ QCT′;

$\therefore$ $\angle$ TCP $= \angle$ TCD $= \angle$ QCT′.

But CA bisects $\angle$ TCT′, $\therefore$ $\angle$ PCA $= \angle$ QCA;

$\therefore$ CP $=$ CQ, being equally inclined to CA;

$\therefore$ CD $=$ CQ.

Corollary 1. Any chord PQ of a rectangular hyperbola and the diameter CV which bisects it are equally inclined to either asymptote.

For PQ is parallel to the diameter conjugate to CV.

Corollary 2. Any pair of supplemental chords of a rectangular hyperbola are equally inclined to either asymptote.

By Th. 23, the chords are parallel to conjugate diameters.

Corollary 3. If QV is an ordinate to the diameter PCP′ of a rectangular hyperbola, then $QV^2 = CV^2 - CP^2$ (see Fig. 53).

This follows from Th. 31, since CD $=$ CP.

Corollary 4. If the normal at any point P on a rectangular hyperbola cuts CA, CB at G, g, then PG $=$ CD $=$ Pg (see Fig. 54).

This follows from Th. 33, since CB $=$ CA.

Theorem 35. (i) Any chord PQ of a rectangular hyperbola subtends equal or supplementary angles at the extremities of any diameter HCH′.

(ii) If HCH′ is a fixed diameter, $\angle PHH' \sim \angle PH'H$ has one of two fixed values, which are supplementary;

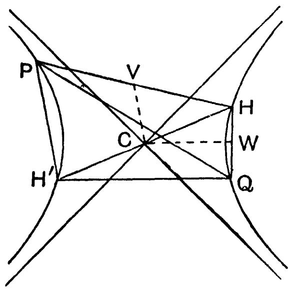

FIG. 56.

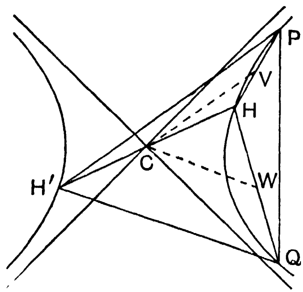

FIG. 57.

(i) Join C to the mid-points V, W of HP, HQ.

By Th. 34, Coroll. 1, CV and PH are equally inclined to the asymptotes; so also are CW and QH, for the same reason;

$$\therefore\ \angle PHQ = \angle VCW, \text{ in Fig. 56,}$$

and $$\angle PHQ = 180^\circ - \angle VCW, \text{ in Fig. 57.}$$

But CV, CW are parallel to H′P, H′Q;

$$\therefore\ \angle PHQ = \angle PH'Q, \text{ in Fig. 56,}$$

and $$\angle PHQ = 180^\circ - \angle PH'Q, \text{ in Fig. 57.}$$

(ii) In Fig. 56, $\angle PH'H + \angle QH'H = \angle PHH' + \angle QHH'$, just proved;

$$\therefore\ \angle PH'H - \angle PHH' = -(\angle QH'H - \angle QHH').$$

In Fig. 57, $\angle PHQ = 180^\circ - \angle PH'Q$;

$$\therefore\ 360^\circ - \angle PHH' - \angle QHH' = 180^\circ - \angle PH'H - \angle QH'H;$$

$$\therefore\ \angle PHH' - \angle PH'H = 180^\circ - (\angle QHH' - \angle QH'H);$$

$\therefore$ in all cases $\angle PHH' \sim \angle PH'H$ has one of two fixed supplementary values.

Note. The converse of Th. 35 (ii) is also true:

If H, H′ are fixed points, and if a point P moves so that the difference of the angles PHH′, PH′H is constant, the locus of P is a rectangular hyperbola having HH′ as a diameter.

This is an immediate deduction from Chasles' theorem on homographic pencils. It can be deduced from Th. 35 (ii) by using the fact that a rectangular hyperbola is determined uniquely if its centre and two points on it are given.

Theorem 36. If a rectangular hyperbola circumscribes a triangle it passes through the orthocentre.

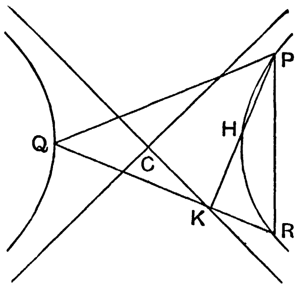

FIG. 58.

Let PQR be the triangle; draw the altitude PK and let it cut the curve in H.

Then $\frac{KH \cdot KP}{KQ \cdot KR}$ = ratio of squares of the semi-diameters parallel to PK, QR; but these are at right angles;

∴ by Th. 34 (iii), $KH \cdot KP = QK \cdot KR$.

But the orthocentre of $\triangle QPR$ is a point O on PK such that

$$KO \cdot KP = QK \cdot KR;$$

∴ H and O coincide.

Corollary. If a conic circumscribes a triangle and passes through the orthocentre, it must be a rectangular hyperbola.

Theorem 37. If a rectangular hyperbola circumscribes a triangle, its centre lies on the nine-point circle of the triangle.

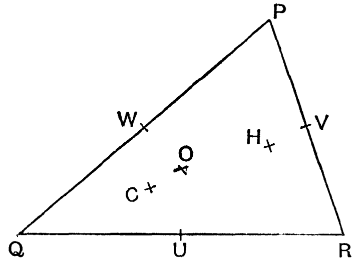

FIG. 59.

Let H be the orthocentre of the given triangle PQR.

By Th. 36, any rectangular hyperbola through PQR must also pass through H.

Let U, V, W be the mid-points of QR, RP, PQ.

Let C be the centre of the hyperbola.

By Th. 34, Coroll. 1, CW and PQ are equally inclined to an asymptote; so also are CV and PR for the same reason;

$\therefore$ $\angle VCW$ is equal or supplementary to $\angle VPW$.

But $\angle VPW = \angle VUW$;

$\therefore$ either $\angle VCW = \angle VPW = \angle VUW$ or

$$\angle VCW = 180^\circ - \angle VPW = 180^\circ - \angle VUW;$$

$\therefore$ C lies either on the circle WPV or on the circle WUV.

Similarly, C lies either on the circle WQU or on the circle WVU, and C lies either on the circle VRU or on the circle VWU.

Now the circles WPV, VRU, UQW intersect at the circumcentre O of $\triangle PQR$, for $\angle OWP = 90^\circ = \angle OVP$, etc.;

$\therefore$ either C must coincide with O or C must lie on the circle WUV, which is the nine-point circle of $\triangle PQR$. If C coincides with O, the circle PQR and the hyperbola are concentric, and so one pair of their common chords bisect each other at C, consequently C is the mid-point of one of the sides of $\triangle PQR$ and $\therefore$ still lies on the nine-point circle of $\triangle PQR$.

EXERCISE VI. c.

(All examples in this Exercise refer to a *rectangular* hyperbola. For the notation, see p. xi.)

1. Prove that $CN = NG$.

2. Prove that $Cg \cdot Ct = 2a^2$.

3. Prove that the tangent from N to the auxiliary circle equals PN.

4. PP′ is a principal double ordinate; prove that the tangent at P′ is perpendicular to CP.

5. If a tangent cuts the asymptotes at T, T′, prove that $\angle$s TST′, TS′T′ are 45°, 135°.

6. CK is the perpendicular from C to the tangent at P; prove that CA bisects $\angle$PCK.

7. PCP′ is a diameter; Q is any point on the curve; prove that the bisectors of $\angle$PQP′ are parallel to the asymptotes.

8. The tangent at P cuts a diameter QCQ′ at O; prove that

$$\angle OPQ = \angle CPQ'.$$

9. Two hyperbolas are such that the axes of either are the asymptotes of the other; prove that they cut orthogonally.

10. The tangents at P, Q cut CQ, CP at H, K and cut each other at O; prove that C, H, O, K lie on a circle.

11. PCP′ is a diameter; Q is any point on the curve; QP′ cuts the asymptotes at R, R′; prove that $PQ = RR'$.

12. A circle cuts a rectangular hyperbola at P, Q, V, W: prove that the diameter of the hyperbola perpendicular to PQ bisects VW.

13. PH, PK are the perpendiculars from a point P on the curve to two conjugate diameters; prove that HK is parallel to the normal at P.

14. If an equilateral triangle is inscribed in a rectangular hyperbola, prove that its circumcentre lies on the curve.

15. The tangents at P, Q cut at O; prove that $\angle OCP = \angle OPQ$.

16. The perpendicular from C to a chord QR cuts the curve at P; prove that the circle PQR touches the hyperbola at P.

17. PQR is a triangle inscribed in a rectangular hyperbola. If $\angle QPR = 90°$, prove that the normal at P is parallel to QR.

18. PQ and VW are perpendicular chords; prove that $\angle$s PVQ, PWQ are equal or supplementary.

19. PCP', QCQ' are any two diameters ; V is any point on the curve ; prove that $\angle PVQ = \angle P'VQ'$.

20. KCK' is a diameter ; the circle, centre K, radius KK', cuts the hyperbola again at P, Q, R ; prove that $\triangle PQR$ is equilateral.

21. If the normal at P cuts the curve again at Q, and if K is the mid-point of PQ, prove that $\angle KCP = 90°$.

22. PCP' is a diameter ; any circle through P, P' cuts the curve again at V, W ; prove that VW is a diameter of the circle.

23. PQ is a chord, normal at P ; prove that $PQ^2 = 3CP^2 + CQ^2$.

24. P is any point on a rectangular hyperbola ; the lines through P perpendicular to AP, $A'P$ cut AA' at H, H' ; prove that $PH = PA'$ and that the normal at P bisects HH'.

SECTIONS OF A CONE

Historical Note. The discovery of the conic is attributed to *Menaechmus* (350-330 B.C.) who made use of the curves in his solution of the Delian problem, the duplication of the cube; it is impossible to say whether he regarded it as the locus of a point in a plane or as a section of a cone. But there seems no doubt that *Aristaeus*, who lived shortly after Menaechmus, based his investigations on the cone. The plane of section was drawn at right angles to the generator of the cone, and different species of conics were obtained by altering the vertical angle of the cone: the acute-angled cone giving an ellipse, the right-angled cone giving a parabola and the obtuse-angled cone giving a hyperbola. The names are due to Apollonius (247-205 B.C.), who showed that the different species could all be obtained from the same cone by altering the slope of the plane of section. Apollonius deduced the property, $\dfrac{PN^2}{AN \, . \, NA'} = \text{constant}$, from the geometry of the cone, and made this the basis of his researches. The focus-directrix property is mentioned by *Pappus* (300 A.D.), but little use was made of it till *Newton* (1642-1727) called attention to it in the *Principia*; the theory of the real foci was worked out by Kepler (1571-1630), to whom the term "focus" is due. But the method which introduces the focal spheres is a recent discovery and is due to Dandelin (1822) and Morton (1825).

CHAPTER VII

SECTIONS OF A CONE

Theorem 38. If the semi-vertical angle of a right circular cone is α and if a plane is drawn making an angle β with the axis of the cone, then the plane section is a conic of eccentricity $\sec \alpha \cos \beta$.

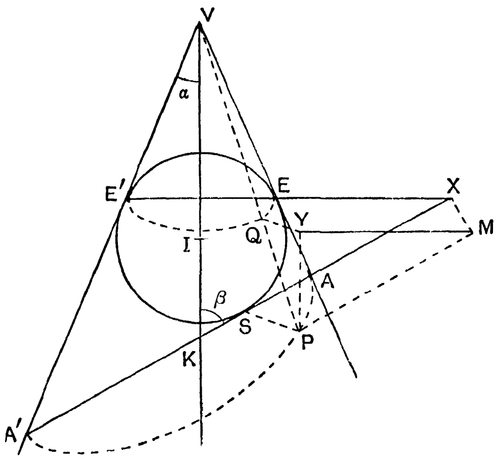

FIG. 60.

Inscribe a sphere in the cone, touching it along the circle EQE′ and touching the plane of section at S. The centre I of the sphere lies on the axis VK of the cone, which cuts the plane of section at K.

Let the plane of the paper be the plane VKS which cuts the cone in the lines VA, VA′. Let P be any point on the section of the given plane with the cone.

Let the plane EQE′ cut the plane of section APA′ in the line MX.

Let PY be the perpendicular from P to the plane EQE′ and let VP cut the plane EQE′ at Q. Join YQ and draw YM perpendicular to XM. Join PM.

Every line through S in the plane APA′ is a tangent to the sphere and is therefore perpendicular to SI, which lies in the plane of the paper; $\therefore$ the plane APA′ is perpendicular to the plane of the paper; but so also is the plane EQE′;

$\therefore$ the line of intersection XM of the planes EQE′, APA′ is also perpendicular to the plane of the paper and is therefore perpendicular to the line AA′.

Since PY is perpendicular to plane EQE′, PY is perpendicular to XM, also YM is perpendicular to XM;

$\therefore$ XM, being perpendicular to PY and YM, is perpendicular to the plane PYM and $\therefore$ to the line PM;

$$\therefore \text{ PM is parallel to AA}'.$$

But PY is parallel to VK, since each is perpendicular to plane EQE′;

$$\therefore \angle YPM = \angle VKA = \beta.$$

Also $\angle QPY = \angle QVK =$ semi-vertical angle of cone $= \alpha$.

In $\triangle QPY$, $\angle QYP = 90^\circ$, $\therefore$ $QP = PY \sec \alpha$.

In $\triangle PYM$, $\angle PYM = 90^\circ$, $\therefore$ $PY = PM \cos \beta$;

$$\therefore QP = PM \sec \alpha \cos \beta.$$

But $PQ = PS$, tangents to a sphere;

$$\therefore SP = PM \sec \alpha \cos \beta;$$

$$\therefore \frac{SP}{PM} = \sec \alpha \cos \beta = \text{constant};$$

$\therefore$ the locus of P is a conic, eccentricity $\sec \alpha \cos \beta$, with S as focus and XM as directrix.

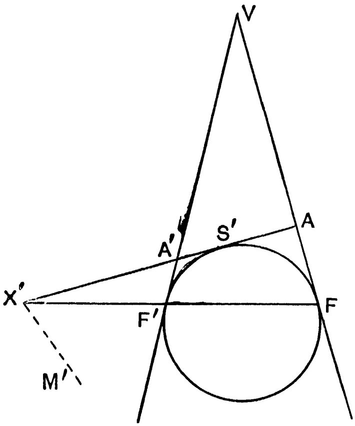

FIG. 61.

If a sphere is inscribed in the cone touching the plane on the other side at S′, and if the plane of section cuts the plane of contact of the sphere with the cone in the line X′M′, it may be proved in the same way that S′ is the second focus and M′X′ the corresponding directrix.

Corollary. For different sections of the same cone, the eccentricity e varies as $\cos\beta$.

Note. If $\beta < \alpha$, $e = \sec\alpha\cos\beta > 1$; the plane of section in this case cuts both branches of the cone and we obtain a hyperbola (Fig. 62).

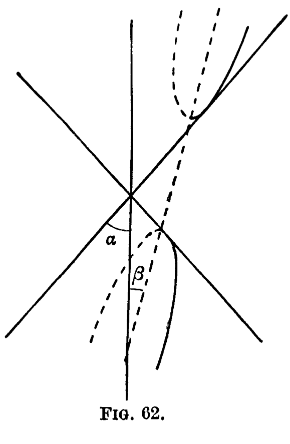

FIG. 62.

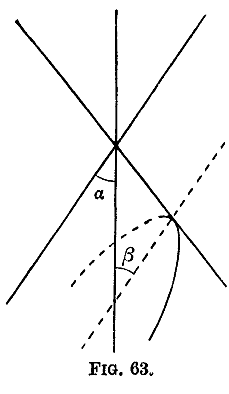

FIG. 63.

If $\beta = \alpha$, $e = \sec\alpha\cos\beta = 1$; the plane of section in this case is parallel to a generator of the cone and we obtain a parabola (Fig. 63).

We also can prove that the section is a conic in the following way:

Let the plane of section touch the inscribed and escribed spheres at S, S′. Take any point P on the plane section and let VP cut the planes of contact EQE′, FQ′F′ at Q, Q′.

Then in Fig. 64,

$$PS = PQ, \text{ tangents to a sphere,}$$

$$PS' = PQ', \text{ tangents to a sphere;}$$

$$\therefore \ SP + S'P = PQ + PQ' = QQ' = EF = \text{constant};$$

$\therefore$ the locus of P is an ellipse with S, S′ as foci.

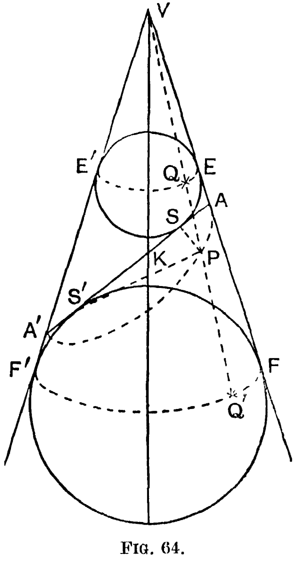

FIG. 64.

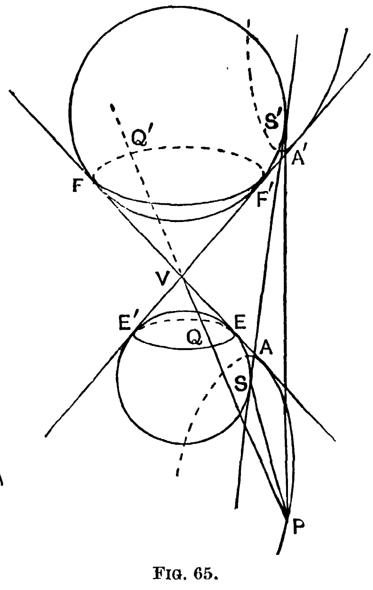

FIG. 65.

The reader should now consider the case of the hyperbola, shown in Fig. 65, and prove that $SP \sim S'P = QQ' = EF = \text{constant}$.

This method fails in the case of the parabola.

Definition. The spheres inscribed in the cone and touching the plane of section are called the **focal spheres** corresponding to the plane of section.

Theorem 39. (i) The minor axis of the conic section is a mean proportional between the radii r, r_1 of the focal spheres.

(ii) The latus rectum of the conic section, for a given cone, varies as the length of the perpendicular from the vertex of the cone to the plane of section.

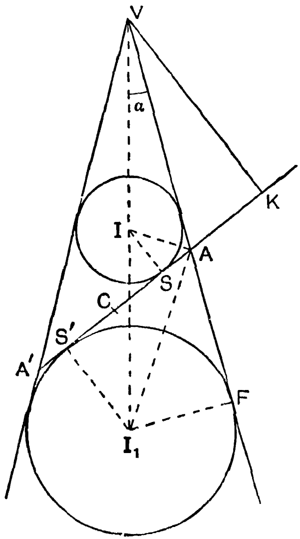

FIG. 66.

(i) Let I, I_1 be the centres of the focal spheres; let C be the centre of the conic-section; IA, I_1A are the bisectors of $\angle VAA'$, and are therefore at right angles.

Also $\angle ISA = 90° = \angle I_1S'A$;

$$\therefore \triangle ISA \text{ is similar to } \triangle AS'I_1;$$

$$\therefore \frac{SA}{SI} = \frac{S'I_1}{S'A};$$

$\therefore$ SA . S′A $=$ SI . S′$I_1 = rr_1$; but SA . S′A $= b^2$ (see p. 6);

$$\therefore b^2 = rr_1.$$

(ii) Let VK be perpendicular to AA′

$$\mathsf{VK}\,.\,\mathsf{AA'}=2\triangle\,\mathsf{VAA'}=r\,.\,(\mathsf{VA}+\mathsf{AA'}+\mathsf{A'V})$$
$$=r\,.\,2\mathsf{VF}=r\,.\,2r_1\cot\alpha$$
$$=2b^2\cot\alpha\,;$$
$$\therefore\ \mathsf{VK}=\frac{2b^2\cot\alpha}{2a}=\frac{b^2}{a}\,.\cot\alpha.$$

But $\dfrac{b^2}{a}=$semi-latus rectum (see p. 6);

$\therefore$ VK varies as the semi-latus rectum.

EXERCISE VII.

(In this Exercise, the word "cone" is used as meaning a *right circular cone.*)

1. The vertical angle of a cone is 120° ; prove that a section which makes 45° with the axis is a rectangular hyperbola.

2. Prove that the latus rectum of a parabola cut from a given cone varies as the distance between the vertices of the cone and the parabola.

3. Two parallel sections of a cone touch the same focal sphere. Prove that the geometric mean of the minor axes equals the diameter of the sphere.

4. The major axis of a section of a cone is of constant length ; prove that the square of the minor axis varies as the distance of the vertex of the cone from the plane of section.

5. Prove that the sphere whose diameter is the join of the centres of the focal spheres passes through the auxiliary circle of the section.

6. Two elliptic sections of a cone are similar and their transverse axes lie in a plane ; prove that the quadrilateral whose corners are the ends of the transverse axes is either cyclic or is a trapezium.

7. If in Fig. 66, AN, A′N′ are the perpendiculars from A, A′ to VI, prove that AN . A′N′ $=b^2$.

8. If in Fig. 66, △AVA′ is of constant area, prove that the volume of the portion of the given cone cut off by the plane of section is constant.

9. PP′ is a variable diameter of a given elliptic section of a given cone, vertex V ; prove that VP + VP′ is constant.

10. Prove that the locus of the vertices of all cones having a given ellipse, eccentricity e, as section is a hyperbola of eccentricity $\frac{1}{e}$, with its foci at the ends of the major axis of the ellipse.

11. Prove that the tangent to a section of a cone at a point P makes equal angles with PS and the generator through P of the cone.

12. Prove that the director circle of the section of a cone lies on the sphere through the two circles in which the focal spheres touch the cone.

CHAPTER VIII

CURVATURE OF A CONIC

IF O is any point on the normal at P to a conic, the circle, centre O, radius OP, *touches* the conic at P and cuts it again at two points Q, R (real or imaginary).

If now R is made to move along the conic, towards P, the centre O of the circle moves along the normal at P.

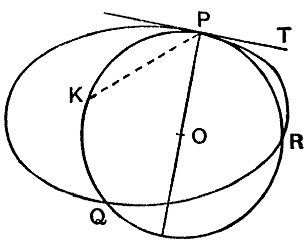

FIG. 67.

The limiting position of the circle when R coincides with P is called the **circle of curvature** at P and its centre is called the **centre of curvature** at P and its radius is called the **radius of curvature** at P. If PK is any chord of the circle it is called a **chord of curvature.**

Since the contact at P of the conic and the circle of curvature is the limit of **three** points of intersection, the circle of curvature touches the conic at P and *crosses* it.

Theorem 40. If the circle of curvature at P cuts the conic again at Q, then the tangent PT at P and the chord PQ make equal angles with the axis of the conic.

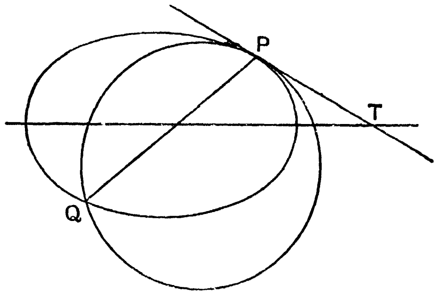

FIG. 68.

By Th. 8 (p. 24), if a circle and a conic intersect at (say) P, P′, Q′, Q, the common chords PP′, QQ′ make equal angles with the axis.

If P′ and Q′ move along the conic towards P, in the limit when they coincide with P, the circle becomes the circle of curvature at P and the common chords become the tangent PT and the chord PQ;

∴ they make equal angles with the axis.

Theorem 41. If CP, CD are conjugate semi-diameters of a conic, the chord of curvature PCH through C of the circle of curvature at P is of length $\dfrac{2CD^2}{CP}$.

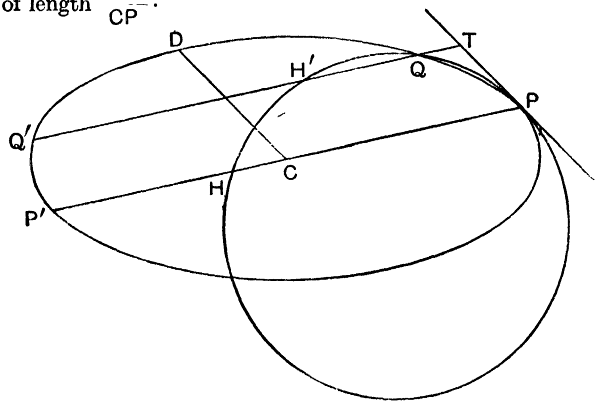

FIG. 69.

Take a point Q on the conic near P and draw a chord QQ′ of the conic parallel to CP and let it cut the tangent at P in T.

Draw the circle which touches PT at P and passes through Q; Let TQ and PC cut the circle at H′, H.

By Th. 7, $$\frac{TP^2}{TQ \cdot TQ'} = \frac{CD^2}{CP^2}.$$

But from the circle, $TP^2 = TQ \cdot TH'$;

$$\therefore \frac{TQ \cdot TH'}{TQ \cdot TQ'} = \frac{CD^2}{CP^2}; \quad \therefore \frac{TH'}{TQ'} = \frac{CD^2}{CP^2}.$$

Now suppose Q moves along the curve towards P; in the limiting position when Q coincides with P, the circle PQH′ becomes the circle of curvature at P.

Also, T coincides with P, the chord QQ′ of the conic becomes the diameter PCP′ of the conic, and H′ coincides with H;

$\therefore$ in the limit, $$\frac{PH}{PP'} = \frac{CD^2}{CP^2};$$

$\therefore$ the central chord of curvature

$$PH = \frac{CD^2}{CP^2} \cdot 2CP = \frac{2CD^2}{CP}.$$

Theorem 42. If CP, CD are conjugate semi-diameters of a conic then (i) the radius of curvature at P is equal to $\frac{CD^3}{CA \cdot CB}$, (ii) the focal chord of curvature at P is equal to $\frac{2CD^2}{CA}$.

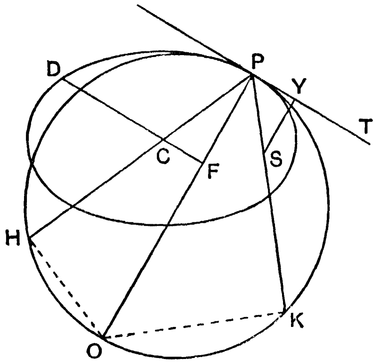

FIG. 70.

Let the normal at P cut CD at F and the circle at O; let PC, PS cut the circle at H, K.

(i) The normal at P is perpendicular to CD;

$$\therefore \angle PFC = 90°.$$

By Th. 41, $$PH = \frac{2CD^2}{CP}.$$

Now $\angle PFC = 90° = \angle PHO$, $\angle$ in semicircle;

$$\therefore \text{ HCFO is a cyclic quadrilateral;}$$

$$\therefore PO \cdot PF = PH \cdot PC \quad \text{or} \quad PO = PH \cdot \frac{PC}{PF}.$$

But $PF \cdot CD = CA \cdot CB$, Th. 24 (vi);

$$\therefore PO = \frac{2CD^2}{CP} \cdot \frac{CP}{PF} = \frac{2CD^2}{PF} = \frac{2CD^3}{CA \cdot CB};$$

$$\therefore \text{ the radius of curvature} = \tfrac{1}{2}PO = \frac{CD^3}{CA \cdot CB}.$$

(ii) Draw SY perpendicular to the tangent PT at P.

Since PO is a diameter of the circle and PY is a tangent, $\triangle$PKO is similar to $\triangle$SYP;

$$\therefore \frac{PK}{PO}=\frac{SY}{SP}=\frac{CB}{CD} \text{ (Th. 25 Coroll.)};$$

$$\therefore PK=\frac{2CD^3}{CA . CB}\cdot\frac{CB}{CD}=\frac{2CD^2}{CA}.$$

Corollary. The focal chord of curvature PK is equal to the focal chord of the conic parallel to the tangent at P.

For by Th. 7, Coroll. (i) and (iii),

$$\frac{\text{focal chord of conic, parallel to PT}}{\text{major axis}}=\frac{CD^2}{CA^2};$$

$$\therefore \text{ focal chord of conic, parallel to PT}=\frac{CD^2}{CA^2}\cdot 2CA=\frac{2CD^2}{CA}=PK.$$

EXERCISE VIII. a.

(For the notation, see p. xi.)

1. Prove that the radius of curvature at A equals $a(1-e^2)$.

2. What is the radius of curvature at B?

3. If $a^2=2b^2$, prove that the circle of curvature at L passes through S′.

4. Prove that the radius of curvature at P varies as PG^3.

5. If $e=\frac{1}{2}$, prove that the circle of curvature at A passes through S′.

6. The tangents at P and Q meet at T; prove that the radii of curvature at P and Q are in the ratio $TP^3 : TQ^3$.

7. Prove that the centre of curvature at B lies on the circle SBS′.

8. PCP′, DCD′ are conjugate diameters. If the circle of curvature at P touches the tangent at P′, prove that $CD^2=a . b$.

9. If CP, CD are conjugate semi-diameters, prove that the sum of the focal chords of curvature at P and D is constant.

10. If PCP′ is a diameter of a rectangular hyperbola, prove that the central chord of curvature at P equals PP′.

11. Prove that the normal chord which divides an ellipse into two portions, the difference of whose areas is a maximum, makes 45° with the axis of the ellipse.

12. If PQ is a chord of a rectangular hyperbola, normal at P, prove that the diameter of curvature at P equals PQ.

13. P is a point on an ellipse such that CP=PX; prove that S′ lies on the circle of curvature at P.

Theorem 43. If P is any point on a parabola, the chord of curvature through P parallel to the axis is of length 4SP.

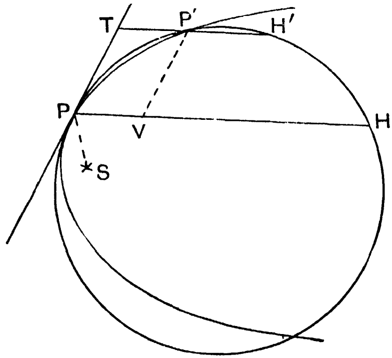

FIG. 71.

Take a point P′ on the parabola near P and draw a diameter through it, cutting the tangent at P in T.

Draw the circle which touches PT at P and passes through P′, and let it cut TP′ and the diameter of the parabola through P in H′, H.

Draw the ordinate P′V to PH, so that P′V is parallel to TP.

From the circle, $TP^2 = TP' . TH'$.

By construction, TP′VP is a parallelogram ;

$$\therefore\ TP' = PV \quad \text{and} \quad TP^2 = P'V^2 = 4SP . PV \quad \text{(Th. 15)};$$

$$\therefore\ 4SP . PV = TP^2 = TP' . TH' = PV . TH';$$

$$\therefore\ TH' = 4SP.$$

Now suppose P′ moves along the curve towards P ; in the limiting position when P′ coincides with P, the circle PP′H′ becomes the circle of curvature at P ; also T coincides with P, and H′ coincides with H ;

∴ in the limit, the chord of curvature PH through P parallel to the axis = 4SP.

Corollary. The focal chord of curvature through P is equal to 4SP.

For SP and PH make equal angles with the tangent to the circle ; therefore the circle cuts off from PH and PS equal chords.

Note. Th. 43, Corollary, and so Th. 43, may be deduced from Th. 42, Corollary, since by Th. 10 (iii) the focal chord of the parabola parallel to PT equals 4SP.

Theorem 44. The radius of curvature at any point P of a parabola is equal to $\frac{2SP^{\frac{3}{2}}}{SA^{\frac{1}{2}}}$.

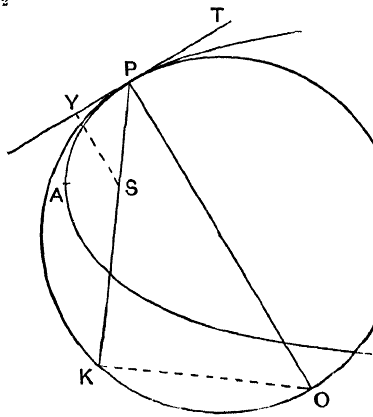

FIG. 72.

Let PS and the normal at P cut the circle of curvature in K and O. Draw SY perpendicular to the tangent at P.

Since $\angle PYS = 90° = \angle PKO$, $\angle$ in semicircle,

and $\angle YPS = \angle POK$, alternate segment,

$\triangle PYS$ is similar to $\triangle OKP$;

$$\therefore \frac{PO}{PK} = \frac{SP}{SY}.$$

But $PK = 4SP$, Th. 43, Coroll.,

and $SY^2 = SA \cdot SP$ or $SY = \sqrt{SA \cdot SP}$ [Th. 9 (v)];

$$\therefore PO = \frac{4SP^2}{\sqrt{SA \cdot SP}} = \frac{4SP^{\frac{3}{2}}}{SA^{\frac{1}{2}}};$$

$\therefore$ radius of curvature at $P = \frac{1}{2}PO = \frac{2SP^{\frac{3}{2}}}{SA^{\frac{1}{2}}}$.

Note. $PO = \frac{4SP^2}{SY}$, $\therefore$ radius of curvature $= \frac{2SP^2}{SY}$.

EXERCISE VIII. b.

(All Examples in this Exercise refer to a *parabola*. For the notation, see pp. ix, 3.)

1. Prove that the radius of curvature at L equals twice the normal at L.

2. O is the centre of curvature at P ; OH is the perpendicular from O to the diameter through P ; prove that $PH = 2SP$.

3. If the circle of curvature at L cuts the curve again at K, prove that LK is a diameter of the circle.

4. If the common chord of a parabola and the circle of curvature at P passes through S, prove that $\angle ASP = 60°$.

5. PK is the tangent from P to the circle of curvature at A ; prove that $PK = AN$.

6. If the normal at P cuts the directrix at H, prove that the radius of curvature at P equals 2PH.

7. Prove that the chord of curvature at P through the vertex equals $\dfrac{4SP \cdot AN}{AP}$.

8. If the circle of curvature at P cuts the curve again at K, and if the tangent at P cuts the axis at T, prove that $PK = 4PT$.

9. O is the centre of curvature at P ; the circle on OP as diameter cuts the diameter through P of the parabola in K ; prove that $\angle PGK = 90°$.

10. O is the centre of curvature at P ; prove that the distance of O from the directrix equals 3NX.

11. Prove that the normal chord which cuts off a segment of minimum area from a parabola is the normal at L.

12. PQ is the common chord of a parabola and its circle of curvature at P ; prove that the ordinate of Q is three times that of P.

13. O is the centre of curvature at P ; K is the mid-point of OP ; prove that $\angle PSK = 90°$.

14. With the data of No. 13, prove that $OP^2 = OS^2 + 3SP^2$.

APPENDIX

The fundamental properties of diameters may be established without using projective methods. The object of this appendix is to show how this can be done. The Theorems are numbered in the same way as in the text.

Theorem 20. The locus of the middle points of any system of parallel chords of a conic is a straight line (called a diameter) which meets the directrix on the line through the focus at right angles to the chords.

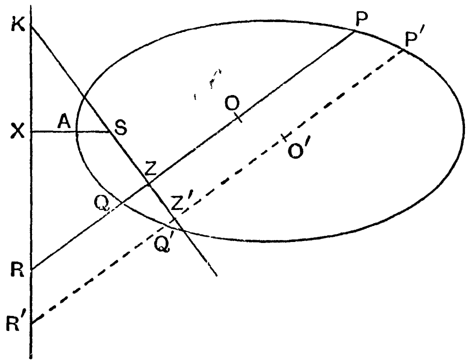

FIG. 73.

Let PQ be one of the system of chords and O its mid-point. Produce PQ to cut the directrix at R. Draw SZ perpendicular to PQ from the focus S and produce it to cut the directrix at K.

As in Th. 1, $\frac{SP}{PR}=\frac{SQ}{QR}$; $\therefore \frac{SP^2-SQ^2}{PR^2-QR^2}=\frac{SP^2}{PR^2}$.

But $SP^2-SQ^2=(SZ^2+ZP^2)-(SZ^2+ZQ^2)=ZP^2-ZQ^2$
$=(OP+OZ)^2-(OQ-OZ)^2$
$=4OZ\,.\,OP$, since $QO=OP$.

And $PR^2-QR^2=(RO+OP)^2-(RO-QO)^2$
$=4RO\,.\,OP$, since $QO=OP$;

$$\therefore \frac{4ZO\,.\,OP}{4RO\,.\,OP}=\frac{SP^2}{PR^2} \quad \text{or} \quad \frac{ZO}{RO}=\frac{SP^2}{PR^2}.$$

Let P′Q′ be any other chord of the system; let its mid-point be O′ and let it cut SK and the directrix at Z′, R′.

Then as before, $\frac{Z'O'}{R'O'}=\frac{SP'^2}{P'R'^2}$.

But $\frac{SP}{PR}=\frac{SP'}{P'R'}$, since PR is parallel to P′R′;

$$\therefore \frac{ZO}{RO}=\frac{Z'O'}{R'O'}.$$

But RR′ cuts ZZ′ at K ; ∴ O′ lies on KO ;

∴ the mid-points of all chords of the system lie on the line KO.

Q.E.D.

Corollary 1. In a central conic, the line KO passes through the centre, since every chord through the centre is bisected at the centre.

Corollary 2. If the diameter KO which bisects PQ cuts the curve at D, the tangent at D is parallel to PQ and the tangents at P, Q intersect on KO.

The proof is identical with that used for Theorem 14 (i), (ii), p. 35.

Theorem 21. If PCP′, DCD′ are two diameters of a central conic such that PCP′ bisects chords parallel to DCD′, then DCD′ bisects chords parallel to PCP′.

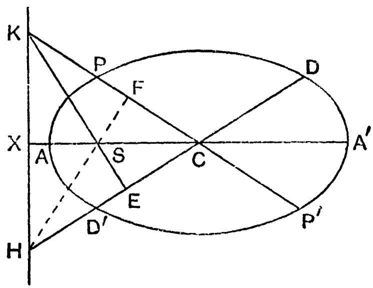

FIG. 74.

Let CP, CD cut the directrix at K, H.

Since PCP′ is the locus of mid-points of chords parallel to DCD′, KS is perpendicular to DCD′. (Th. 20.)

But CS is perpendicular to KH ;

∴ S is the orthocentre of △CKH ;

∴ SH is perpendicular to PCP′ ;

∴ CH or DCD′ is the locus of mid-points of chords parallel to PCP′. (Th. 20.)

Corollary. The tangents at P, P′ are parallel to DCD′ and the tangents at D, D′ are parallel to PCP′.

This follows from the definition of a tangent (see p. 12), and Th. 20, Coroll. 2.

Theorem 22. (i) If QV is the ordinate of any point Q to the diameter PCP′ which cuts the curve at real points P, P′ and cuts the tangent at Q in T, then $CV \cdot CT = CP^2$.

(ii) If Qv is the ordinate of any point Q to the diameter DCD′ which does not cut the curve at real points and cuts the tangent at Q in t, then $Cv \cdot Ct = CD^2$.

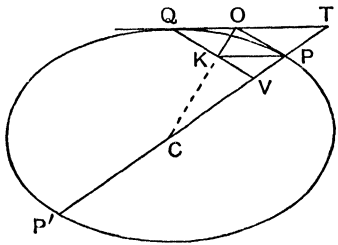

FIG. 75.

(i) Let the tangent at P cut QT at O; draw PK parallel to OQ to cut QV at K; join OK.

Then POQK is a parallelogram; ∴ OK bisects PQ;

∴ OK passes through C;

∴ by parallels,
$$\frac{CT}{CP} = \frac{CO}{CK} = \frac{CP}{CV};$$
$$\therefore\ CV \cdot CT = CP^2.$$

(ii) Let CP be the semi-conjugate diameter to CD; draw the ordinate QV to CP so that QV is equal and parallel to Cv.

By parallels,
$$\frac{Ct}{CT} = \frac{QV}{VT};$$
$$\therefore\ \frac{QV \cdot Ct}{CV \cdot CT} = \frac{QV^2}{CV \cdot VT} = \frac{QV^2}{CV^2 - CV \cdot CT} = \frac{QV^2}{CV^2 - CP^2} = \frac{CD^2}{CP^2}.$$

But $\quad CV . CT = CP^2, \therefore QV . Ct = CD^2.$
And $\quad QV = Cv, \therefore Cv . Ct = CD^2.$

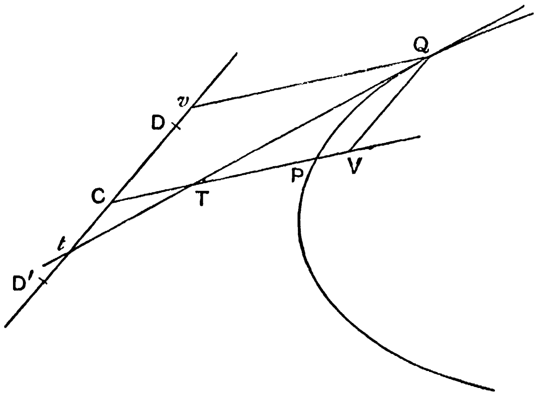

FIG. 76.

Theorem 24 (iv). If CP, CD are conjugate semi-diameters of an ellipse, and if PN, DR are ordinates to ACA′, then

$$\frac{PN . DR}{CN . CR} = \frac{CB^2}{CA^2}.$$

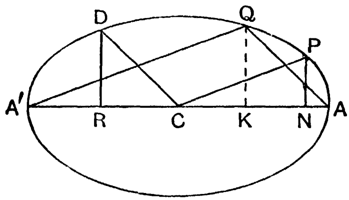

FIG. 77.

Draw A′Q parallel to CP to cut the curve at Q ; join AQ.

Since A′Q, QA are supplemental chords, QA is parallel to CD ; draw the ordinate QK of Q to AA′.

By similar triangles, $\frac{PN}{CN} = \frac{QK}{A'K}$ and $\frac{DR}{CR} = \frac{QK}{AK}$;

$$\therefore \frac{PN . DR}{CN . CR} = \frac{QK^2}{AK . KA'} = \frac{CB^2}{CA^2}. \quad \text{(Th. 7, Corollary.)}$$

Theorem 24 (v). If CP, CD are conjugate semi-diameters of an ellipse, and if PN, DR are ordinates to ACA′, then $CN^2+CR^2=CA^2$ and $PN^2+DR^2=CB^2$ and $CP^2+CD^2=CA^2+CB^2$.

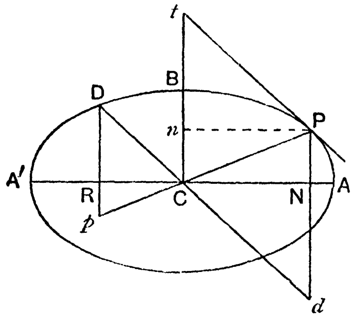

FIG. 78.

Let the tangent at P cut CB at t and let Pn be the ordinate to CB. Let PN cut DC at d and PC cut DR at p.

$$PdCt \text{ is a parallelogram, } \therefore Pd=Ct;$$

$$\therefore PN \,.\, Pd=Cn \,.\, Ct=CB^2. \quad \text{(Th. 22.)}$$

Similarly, $DR \,.\, Dp=CB^2$; $\therefore PN \,.\, Pd=DR \,.\, Dp$;

$$\therefore \frac{PN}{DR}=\frac{Dp}{Pd}=\frac{CD}{Cd}=\frac{CR}{CN};$$

$$\therefore \frac{PN^2}{PN \,.\, DR}=\frac{PN}{DR}=\frac{CR}{CN}=\frac{CR^2}{CN \,.\, CR};$$

$$\therefore \frac{PN^2}{CR^2}=\frac{PN \,.\, DR}{CN \,.\, CR}=\frac{CB^2}{CA^2}\,[\text{Th. 24 (iv)}]=\frac{PN^2}{CA^2-CN^2} \quad \text{(Th. 7, Coroll.)};$$

$$\therefore CR^2=CA^2-CN^2;$$

$$\therefore CN^2+CR^2=CA^2.$$

Similarly, $PN^2+DR^2=CB^2$;

$\therefore$ by addition, $CP^2+CD^2=CN^2+NP^2+CR^2+RD^2$
$=CA^2+CB^2$.

Corollary 1. $\frac{PN}{CR}=\frac{CB}{CA}=\frac{DR}{CN}$.

For we have just proved that

$$\frac{PN^2}{CR^2}=\frac{CB^2}{CA^2}, \text{ and by Th. 24 (iv) } \frac{PN}{CR}\cdot\frac{DR}{CN}=\frac{CB^2}{CA^2}.$$

[This proves Theorem 24 (iii).]

Theorem 24 (vi). If CP, CD are conjugate semi-diameters of an ellipse, the area of the parallelogram having CP, CD as sides is constant and equal to CA . CB.

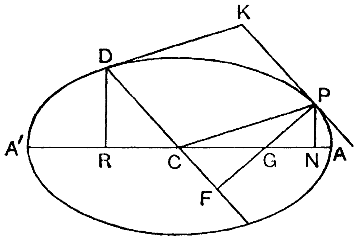

FIG. 79.

PN and DR are the ordinates of P and D to AA′; let the normal at P cut CA at G and CD at F.

The sides of $\triangle$PGN are respectively perpendicular to those of $\triangle$CDR; $\therefore$ the triangles are similar.

$$\therefore \frac{PG}{CD}=\frac{PN}{CR}=\frac{CB}{CA} \quad \text{[Th. 24 (v) Coroll. just proved]};$$

$$\therefore \text{ area of parallelogram PCDK} = PF . CD = PF . PG . \frac{CA}{CB}.$$

But $\qquad PF . PG = CB^2$ (see p. 55);

$$\therefore \text{ area of PCDK} = CB^2 . \frac{CA}{CB} = CA . CB.$$

Note. The above proof supplies an alternative method of proving that $\frac{PG}{CD}=\frac{CB}{CA}$; the proof on p. 55 assumes that the result of Th. 24 (vi) has been obtained by orthogonal projection.

Theorem 24 (vii). If PCP′, DCD′ are conjugate diameters, and if any other pair of conjugate diameters cut the tangent at P in H, K, then $PH \cdot PK = CD^2$.

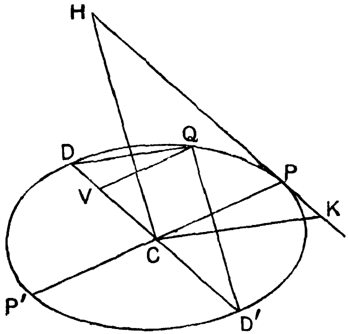

FIG 80.

Draw D′Q parallel to CH to cut the curve at Q ; join QD.
Then QD′, QD are supplemental chords ; ∴ QD is parallel to CK.
Draw the ordinate QV to DD′.
$\triangle$CPK is similar to $\triangle$QVD, since corresponding sides are parallel ;

$$\therefore \frac{KP}{PC} = \frac{DV}{VQ}.$$

Similarly,
$$\frac{HP}{PC} = \frac{VD'}{VQ};$$

$$\therefore \frac{PH \cdot PK}{CP^2} = \frac{DV \cdot VD'}{VQ^2} = \frac{CD^2}{CP^2} \quad \text{(Th. 7, Coroll.)};$$

$$\therefore PH \cdot PK = CD^2.$$

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